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Representations of Galois Groups Associated to Modular Forms

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Introduction. We will describe in a special case the conjectural relationship among automorphic forms, l-adic representations, and motives. To make the discussion concrete, we shall restrict ourselves to weight 2 modular forms for GL_2. In this case the modular forms can be thought of either as certain harmonic forms on products of the upper half complex planes and hyperbolic three spaces or as cohomology classes for certain quotients of these products. As such, they are relatively concrete and often computable topological objects. Similarly, we shall restrict attention to irreducible two-dimensional l-adic representations that are de Rham with Hodge-Tate numbers 0 and -1, and to certain abelian varieties.

In the first section, we shall describe in some detail exactly what modular forms we wish to consider. In the second, we shall describe the conjectural relationship to Galois theory and algebraic geometry. Finally, in the third, we shall describe what is currently known about these conjectures.

Of course the situation we consider is very special and the conjectures admit enormous generalization (see for example [Cl]). We concentrate on this very special case because it is more concrete, yet the conjectures are already extraordinary and the difficulties seem immense.

Modular Forms. Let $\mathcal{Z}_2^{\pm}$ denote $\mathbb{C} - \mathbb{R}$ and let $\mathcal{Z}_3$ denote a hyperbolic three space, that is the set of quaternions $z = x + jy \in \mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ for which $y \in \mathbb{R}_{>0}$. Then $\mathcal{Z}_2^{\pm}$ has an action of $GL_2(\mathbb{R})$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto (az+b)(cz+d)^{-1}.$$

The same formula defines an action of $SL_2(\mathbb{C})$ on $\mathcal{Z}_3$ and we can extend this to an action of $GL_2(\mathbb{C})$ by letting the center act trivially. We will let c denote complex conjugation.

Now let K be a number field. We decompose the set of embeddings $K \to \mathbb{C}$ as $I_{\mathbb{R}} \cup I_{\mathbb{C}} \cup cI_{\mathbb{C}}$, where $I_{\mathbb{R}}$ denotes those embeddings with image in $\mathbb{R}$, $I_{\mathbb{C}}$ denotes half the remaining embeddings, and $cI_{\mathbb{C}}$ denotes the set of composites $c \circ \tau$ with $\tau \in I_{\mathbb{C}}$. We let $r_{\mathbb{R}}$ and $r_{\mathbb{C}}$ denote the cardinalities of $I_{\mathbb{R}}$ and $I_{\mathbb{C}}$. We let $d = r_{\mathbb{R}} + 2r_{\mathbb{C}}$

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denote the degree of K over $\mathbb{Q}$. We set $\mathcal{Z}_K = (\mathcal{Z}_2^{\pm})^{I_{\mathbb{R}}} \times \mathcal{Z}_3^{I_{\mathbb{C}}}$, so that $GL_2(K)$ acts on $\mathcal{Z}_K$ via

$$\gamma((z_\tau)_{\tau\in I_{\mathbb{R}}\cup I_{\mathbb{C}}}) = (({}^{\tau}\gamma)(z_\tau))_{\tau\in I_{\mathbb{R}}\cup I_{\mathbb{C}}}.$$

We let $\widehat{\mathcal{O}_K}$ denote the direct product of all completions of the ring of integers $\mathcal{O}_K$ of K at finite primes and $\mathbb{A}_K^{\infty}$ denote the ring of finite adeles of K, i.e. $\widehat{\mathcal{O}_K}\otimes_{\mathcal{O}_K} K$. If $\mathfrak{n}$ is an ideal of $\mathcal{O}_K$ then we define $U_1(\mathfrak{n})$ to be the subgroup of $GL_2(\widehat{\mathcal{O}_K})$ consisting of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with c and $d-1 \in \mathfrak{n}\widehat{\mathcal{O}_K}$. We will write $\Gamma_1(\mathfrak{n})$ for the intersection $U_1(\mathfrak{n})\cap GL_2(\mathcal{O}_K)$. Our main object of study will be the orbifold

$$Y_1(\mathfrak{n}) = GL_2(K)\backslash((GL_2(\mathbb{A}_K^{\infty})/U_1(\mathfrak{n}))\times\mathcal{Z}_K).$$

It is an orbifold of dimension $2d - r_{\mathbb{C}}$. If K has strict class number 1, it is nothing other than $\Gamma_1(\mathfrak{n})\backslash\mathcal{Z}_K$, in general, it is a finite union of quotients of $\mathcal{Z}_K$ by discrete groups. If $K=\mathbb{Q}$, it is the complex points of a modular curve; if K is totally real, it is the complex points of a Hilbert-Blumenthal variety; whereas if K is imaginary quadratic it is an arithmetic hyperbolic 3-orbifold.

We shall be interested in the cohomology groups $H^d(Y_1(\mathfrak{n}),\mathbb{C})$, which are finite-dimensional complex vector spaces. Analytically, $H^d(Y_1(\mathfrak{n}),\mathbb{C})$ can be identified with a space of certain harmonic forms on $(GL_2(\mathbb{A}_K^{\infty})/U_1(\mathfrak{n}))\times\mathcal{Z}_K$ that are invariant under the action of $GL_2(K)$ (see [H]). This provides the link with more usual definitions of modular forms. For example if $K=\mathbb{Q}$ then we obtain

$$H^1(Y_1(\mathfrak{n}),\mathbb{C}) \cong M_2(\Gamma_1(\mathfrak{n}))\oplus\overline{S_2(\Gamma_1(\mathfrak{n}))},$$

where M_2 (resp. S_2) denotes the classical elliptic modular forms (resp. cusp forms).

It is not simply the finite-dimensional vector space $H^d(Y_1(\mathfrak{n}),\mathbb{C})$ that is of so much interest, but more importantly, it comes equipped with a natural set of linear operators. Suppose that $\wp$ is a prime of $\mathcal{O}_K$ not dividing $\mathfrak{n}$. Let $\varpi_\wp$ denote a uniformizer of $K_\wp$. Then there are two finite maps π_1 and $\pi_2 : Y_1(\mathfrak{n}\wp)\to Y_1(\mathfrak{n})$ induced by the maps Id and $\eta_\wp\times\mathrm{Id} : GL_2(\mathbb{A}_K^{\infty})/U_1(\mathfrak{n})\times\mathcal{Z}_K \to GL_2(\mathbb{A}_K^{\infty})/U_1(\mathfrak{n}\wp)\times\mathcal{Z}_K$, where $\eta_\wp$ denotes right multiplication by the element

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi_\wp \end{pmatrix} \in GL_2(K_\wp)\subset GL_2(\mathbb{A}_K^{\infty}).$$

We let $T_\wp$ denote the composite

$$\pi_{1*}\circ\pi_2^* : H^d(Y_1(\mathfrak{n}),\mathbb{C}) \to H^d(Y_1(\mathfrak{n}),\mathbb{C}).$$

Similarly, there is a map $\sigma : Y_1(\mathfrak{n})\to Y_1(\mathfrak{n})$, which is induced by right multiplication by $\varpi_\wp 1_2\in GL_2(K_\wp)$. We let $S_\wp$ denote σ^*. The linear operators $T_\wp$ and $S_\wp$ for $\wp\nmid\mathfrak{n}$ commute among themselves; they are called Hecke operators. We let $\mathbb{T}_1(\mathfrak{n})$ denote the $\mathbb{Z}$-algebra generated by these Hecke operators in the endomorphisms of $H^d(Y_1(\mathfrak{n}),\mathbb{C})$. As the Hecke operators preserve $H^d(Y_1(\mathfrak{n}),\mathbb{Z})$, $\mathbb{T}_1(\mathfrak{n})$ is a finitely generated $\mathbb{Z}$-module. Also $\mathbb{T}_1(\mathfrak{n})\otimes\mathbb{C}$ embeds in $\mathrm{End}_{\mathbb{C}}(H^d(Y_1(\mathfrak{n}),\mathbb{C}))$.

It is the eigenvectors of $\mathbb{T}_1(N)$ on $H^d(Y_1(\mathfrak{n}),\mathbb{C})$, or more precisely the corresponding sets of eigenvalues, that will be our main object of study. Equivalently we shall be interested in the ring homomorphisms (characters)

$$\theta : \mathbb{T}_1(\mathfrak{n}) \longrightarrow \mathbb{C}.$$

We shall call θ *trivial* if $\theta(T_{(\varpi)}) = 1 + \#\mathcal{O}_K/(\varpi)$ for every totally positive prime element $\varpi\in\mathcal{O}_K$ with $\varpi-1\in\mathfrak{n}$. It is known that the generalised eigenspaces of nontrivial θ are in fact eigenspaces. We call θ and θ' of levels $\mathfrak{n}$ and $\mathfrak{n}'$ *equivalent* if for all primes $\wp$ not dividing $\mathfrak{n}\mathfrak{n}'$ we have $\theta(T_\wp)=\theta'(T_\wp)$.

We remark that if $\theta : \mathbb{T}_1(\mathfrak{n})\to\mathbb{C}$ then its image generates a number field that we shall denote E_θ.

Conjectures. The expectation is that these characters θ encode subtle arithmetic information of a completely different nature. We will give some of the standard conjectures below. Before doing so we need a couple of definitions.

DEFINITION 1. *If L and F are finite extensions of $\mathbb{Q}_l$ we say that a continuous representation*

$$\rho : \mathrm{Gal}(\overline{L}/L) \longrightarrow GL_2(F)$$

has weight two *if either (or both) of the following conditions hold.*

(1) *There is a finite extension L'/L and an l-divisible group $A/\mathcal{O}_{L'}$ such that $\rho|_{\mathrm{Gal}(\overline{L'}/L')}$ is equivalent to the Tate module $((\varprojlim A[l^n](\overline{L'}))\otimes_{\mathbb{Z}_l}\mathbb{Q}_l)$ of A as a $\mathbb{Q}_l[\mathrm{Gal}(\overline{L'}/L')]$-module.*

(2) *There is a finite extension L'/L and an unramified character χ of $\mathrm{Gal}(\overline{L'}/L')$ such that $\rho|_{\mathrm{Gal}(\overline{L'}/L')}\otimes\chi$ is of the form*

$$\begin{pmatrix} \chi_l & * \\ 0 & 1 \end{pmatrix}$$

where χ_l denotes the cyclotomic character.

DEFINITION 2. *Let K be a field and E a number field.*

(1) *By a* generalized elliptic curve *over K with multiplication by E we mean an abelian variety A/K of dimension $[E:K]$ together with an embedding $E\hookrightarrow\mathrm{End}_K^0(A)$.*

(2) *By a* false generalized elliptic curve *over K with multiplication by E we mean an abelian variety A/K of dimension $2[E:K]$, a quaternion algebra (possibly split) D with center E and an embedding $D\hookrightarrow\mathrm{End}_K^0(A)$.*

(3) *We shall say that an abelian variety A/K has* CM over K *if there is a number field of degree $2\dim A$ that embeds in $\mathrm{End}_K^0(A)$.*

We note that there is a natural injection from generalized elliptic curves over K with multiplication by E to false generalized elliptic curves over K with multiplication by E, which sends (A,i) to $(A^2, M_2(E), M_2(i))$. We shall use this without mentioning it.

We are now in a position to state the conjectures that will interest us.

CONJECTURE 1 (Generalized Ramanujan-Peterson Conjecture).
If $\theta : \mathbb{T}_1(\mathfrak{n}) \longrightarrow \mathbb{C}$ is a nontrivial character then

$$|\theta(T_\wp)| \leq 2(\#\mathcal{O}_K/\wp)^{1/2}$$

for all $\wp \not| \mathfrak{n}$.

CONJECTURE 2. *If $\theta : \mathbb{T}_1(\mathfrak{n}) \longrightarrow \mathbb{C}$ is a nontrivial character then there is a finite extension F_θ/E_θ and for each prime λ of F_θ there is a continuous irreducible representation*

$$\rho_{\theta,\lambda} : \mathrm{Gal}(\overline{K}/K) \longrightarrow GL_2(F_{\theta,\lambda}),$$

such that if $\wp \not| \mathfrak{n}l$ (with l the residue characteristic of λ) then $\rho_{\theta,\lambda}$ is unramified at $\wp$ and $\rho_{\theta,\lambda}(\mathrm{Frob}_\wp)$ (which is a well-defined conjugacy class in $GL_2(F_{\theta,\lambda})$) has characteristic polynomial

$$X^2 - \theta(T_\wp)X + \theta(S_\wp)(\#\mathcal{O}_K/\wp).$$

Moreover if $\wp | l$ then $\rho_{\theta,\lambda}|_{\mathrm{Gal}(\overline{K_\wp}/K_\wp)}$ has weight 2.

We remark that in fact for any prime $\wp \not| l$, $\rho_{\theta,\lambda}|_{\mathrm{Gal}(\overline{K}_\wp/K_\wp)}$ should be completely describable in terms of θ as is described in [Ca].

CONJECTURE 3. *If $\theta : \mathbb{T}_1(\mathfrak{n}) \longrightarrow \mathbb{C}$ is a nontrivial character then there is a false generalized elliptic curve A/K with multiplication by E_θ such that for all $\wp \not| \mathfrak{n}$ we have*

$$\#A(\mathcal{O}_K/\wp) = \#\mathcal{O}_E/(1 - \theta(T_\wp) + \theta(S_\wp)(\#\mathcal{O}_K/\wp))^2.$$

Moreover A does not have CM over K. The quaternion algebra D implicit here can be taken to be split by $E.K$. If K has a real place the false generalised elliptic curve arises from a true one.

Note that Conjecture 3 implies Conjectures 1 and 2. Conjecture 1 follows by using the theorems of Hasse and Weil. Conjecture 2 follows on looking at the Tate module.

Further, it is now standardly conjectured that the constructions implicit in Conjectures 2 and 3 give rise to bijections between the following classes of objects:

(1) Equivalence classes of nontrivial characters $\theta : \mathbb{T}_1(\mathfrak{n}) \to \mathbb{C}$, for variable $\mathfrak{n}$, but for fixed K;
(2) Isogeny classes of false generalized elliptic curves A/K which do not have CM over K;
(3) Continuous irreducible representations

$$\rho : \mathrm{Gal}(\overline{K}/K) \longrightarrow GL_2(\overline{\mathbb{Q}}_l)$$

that are unramified outside a finite set of primes and that have the property that whenever $\wp$ is a prime of K above l then $\rho|_{\mathrm{Gal}(\overline{K}_\wp/K_\wp)}$ has weight 2.

Results. If $K = \mathbb{Q}$, Conjectures 1, 2, and 3 are all theorems. The seminal step was taken by Eichler in 1954 [E]. He proved Conjecture 1, but for $\Gamma_0(\mathfrak{n})$ not $\Gamma_1(\mathfrak{n})$, and he only checked it outside an unknown finite set of primes. Shimura generalized this to $\Gamma_1(\mathfrak{n})$ [S1] and Igusa determined the possible bad primes [I]. Although in retrospect Eichler's paper contains most of the ingredients to prove Conjecture 3 (and hence also Conjecture 2), it seems that the idea of decomposing under the action of the Hecke algebra was a long time in arising. To the best of our knowledge, Conjecture 3 was proved by Shimura in his book of 1971 [S4]. We note that Ribet [R] proved the irreducibility of the representations. Carayol [Ca], following Deligne and Langlands, determined the restriction of the l-adic representation to the decomposition group at any prime not dividing l.

All this work relied on realizing the modular curve $Y_1(\mathfrak{n})$ over the rational numbers and finding the desired abelian variety as a factor of the Jacobian of its completion. The hard part is to calculate the action of Frobenius on the Tate module of this Jacobian. Eichler pioneered the so-called Eichler-Shimura congruence relation to do this. Langlands later developed a second approach based on the Selberg trace formula (see [La]).

We note that given an elliptic curve $A/\mathbb{Q}$ one can in practice usually check that A arises from some $\theta : \mathbb{T}_1(\mathfrak{n}) \longrightarrow \mathbb{C}$. Indeed one can take $\mathfrak{n}$ to be the conductor of A, then one can compute the first few values $\theta(T_\wp)$ for all θ with that value of $\mathfrak{n}$ and this should allow one to rule out all but one θ. To show that A is in fact associated to this θ it suffices to show that $\rho_{\theta,2}$ is isomorphic to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the 2-adic Tate module of A. According to the Faltings-Serre method [Li] it suffices to check that the traces agree on an explicit finite set of Frobenius elements. This is often a rather easy calculation. Whenever one makes such a calculation for an elliptic curve $A/\mathbb{Q}$, one proves that its L-function is entire and satisfies a functional equation.

Over a totally real field the situation is nearly as good. Conjectures 1 and 2 are known, except that the l-adic representations are not always known to be of weight 2 at primes dividing l. Carayol [Ca] has computed the restriction of these l-adic representations to the decomposition groups at all primes not dividing l. Conjecture 3 is known in many cases, but not all. Specifically, it is known if K has odd degree or if θ is discrete series at some finite prime, a condition that we will not make explicit here (see [Ca]).

In the cases where K is of odd degree or where θ is a discrete series at some finite place, Conjecture 3 (and hence Conjectures 1 and 2) follow on combining the results of [S3], [JL], and [A]. Indeed Jacquet and Langlands prove that in these cases θ can be realized as a character of a Hecke algebra for a Shimura curve (part of their argument was only sketched and was completed by Arthur), whereas Shimura had proved the analogue of Conjecture 3 for characters of Hecke algebras on Shimura curves by an analogue of Eichler's method. In the remaining cases Conjecture 1 was proved by Brylinski and Labesse [BL] except that the set of bad primes was not known exactly. They used Langlands' method to analyze the intersection cohomology of a certain compactification of $Y_1(\mathfrak{n})$ defined over $\mathbb{Q}$. Wiles and the author [Wi], [T1] settled Conjecture 2 in the remaining cases, except that they did not show that the l-adic representations were of weight 2.

This was done by constructing congruences to characters that were discrete series at some finite place and then piecing together the l-adic representations already constructed. Combined with [BL] this proves Conjecture 1 at all good primes.

Whenever K is not totally real there is no known direct link to algebraic geometry. The orbifolds $Y_1(\mathfrak{n})$ do not have a complex structure. The only positive results in these cases (except for CM forms or base changed forms) are when K is imaginary quadratic. In this case there has been, for some time, considerable computational evidence to support Conjecture 3 (see [GHM], [EGM] and [Cr]). With Harris and Soudry we have recently proved (see [HST], [T1], and [FH]) the following slightly weakened version of Conjecture 2 in this case.

THEOREM 1. *Suppose that K is an imaginary quadratic field. If $\theta : \mathbb{T}_1(\mathfrak{n}) \to \mathbb{C}$ is a nontrivial character such that $\theta(S_\wp) = \theta(S_{\wp^c})$ for all $\wp$, then there is a finite extension F_θ/E_θ, and for each prime λ of F_θ there is a continuous irreducible representation*

$$\rho_{\theta,\lambda} : \mathrm{Gal}(\overline{K}/K) \longrightarrow GL_2(F_{\theta,\lambda}),$$

such that $\rho_{\theta,\lambda}$ is unramified outside $l\mathfrak{n}\mathfrak{n}^c$ and the discriminant of $K/\mathbb{Q}$. Moreover for all primes outside an explicit set of Dirichlet density zero $\rho_{\theta,\lambda}(\mathrm{Frob}_\wp)$ has characteristic polynomial

$$X^2 - \theta(T_\wp)X + \theta(S_\wp)(\#\mathcal{O}_K/\wp).$$

Combining this theorem with the Faltings-Serre method, one can prove for many explicit pairs θ, A/K that

$$\#A(\mathcal{O}_K/\wp) = 1 + \#(\mathcal{O}_K/\wp) - \theta(T_\wp)$$

for all $\wp$ outside a set of Dirichlet density zero. With Cremona, we carried this out in the case $K = \mathbb{Q}(\sqrt{-3})$, θ the unique homomorphism $\mathbb{T}((17+\sqrt{-3})/2) \to \mathbb{C}$, and A the elliptic curve $y^2 + xy = x^3 + (3+\sqrt{-3})x^2/2 + (1+\sqrt{-3})x/2$. From this one can deduce that for this θ and for all $\wp$ outside a set of Dirichlet density zero we have that

$$|\theta(T_\wp)| \leq 2(\#\mathcal{O}_K/\wp)^{1/2}.$$

To prove the above theorem one considers a set of twists $\theta \otimes \eta$ as η runs over quadratic characters of K and where $\theta \otimes \eta(T_\wp) = \theta(T_\wp)\eta(\wp)$ and $\theta \otimes \eta(S_\wp) = \theta(S_\wp)$. The fact that $GL_2(K)$ is closely related to a fourvariable orthogonal group and the theta lift from that orthogonal group to $GSp_4(\mathbb{Q})$ are used to construct from many of these $\theta \otimes \eta$ a character of the Hecke algebra of a space of holomorphic Siegel modular forms of genus 2 and weight 2 (see [HST] and [FH] for a refinement). Essential use is made of the disconnectedness of the orthogonal group to get a holomorphic lift. To ensure that the lift is nonzero for many η we require a nonvanishing theorem for L-functions. We originally used a result of Waldspurger [Wa] and but for the result quoted above we need a stronger result of Friedberg and Hoffstein [FH]. Congruences between these characters and characters of higher weight Siegel modular forms (which also occur on the l-adic cohomology of Siegel threefolds), the Eichler-Shimura congruence relation for Siegel threefolds [S2], [D], [CF], and the method of pseudo-representations [T2] allow one to construct for many η an l-adic representation R_η of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that for almost all rational primes p that split $p = \wp\wp^c$ in K, $R_\eta(\mathrm{Frob}_p)$ has eigenvalues contained in the set of roots of $(X^2 - \theta(T_\wp)X + \theta(S_\wp)p)(X^2 - \theta(T_{\wp^c})X + \theta(S_\wp)p)$, whereas for almost all inert primes p, $R_\eta(\mathrm{Frob}_p)$ has eigenvalues contained in the set of roots of $X^4 - \theta(T_p)X^2 + \theta(S_p)p^2$. See [T2] and [T3] for details. One can then show that there is a two-dimensional l-adic representation r of $\mathrm{Gal}(\overline{K}/K)$ such that for many η we have $R_\eta = r \otimes \eta \oplus (r \otimes \eta)^c$ (see [T4]). It is then not hard to see that r is the desired representation.

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Einstein Metrics and Metrics with Bounds on Ricci Curvature

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There is a well-developed theory of the behavior of Riemannian metrics on smooth manifolds, which have uniform bounds on the sectional curvature K. The compactness theorem of Cheeger-Gromov [Ch], [Gr] implies that the space of metrics satisfying the bounds

$$|K| \leq \Lambda, \ \mathrm{vol} \geq v, \ \mathrm{diam} \leq D, \tag{0.1}$$

is $C^{1,\alpha}$ precompact. Thus, given any sequence of metrics g_i satisfying the bounds (0.1), there is a subsequence $\{i'\}$ and a sequence of diffeomorphisms $\phi_{i'}$ of M, such that the isometric metrics $g_i' = (\phi_{i'})^* g_{i'}$ converge, in the $C^{1,\alpha'}$ topology on M, to a $C^{1,\alpha}$ metric g_∞ on M, $\forall \alpha' < \alpha < 1$. If the volume or diameter bounds are removed in (0.1), one no longer has such compactness, but the degeneration of the sequence $\{g_i\}$ is well understood, through the works of Cheeger-Gromov [CG$_1$], [CG$_2$] and Fukaya [F], cf. also [CGF]. The manifolds (M, g_i) divide into two regions, the thick part M^ε and the thin part M_ε. Roughly speaking, on M^ε the metrics converge, as above, to a limit $C^{1,\alpha}$ metric, while the complement M_ε is ε-collapsed along a well-defined topological structure, called an F-structure, or more generally an N-structure.

Here, we are basically concerned with the possible extensions of such a theory to spaces of metrics with bounds imposed on the Ricci curvature, in place of the full curvature. There are several (related) reasons why it is important to consider such extensions. First, both the metric and the Ricci curvature are symmetric bilinear forms. Thus, the problem of controlling the behavior of metrics with bounds on Ricci curvature is, roughly speaking, a determined problem. Assuming bounds on the full curvature corresponds to a highly overdetermined problem. To illustrate this, there are very few manifolds, in general dimensions, that admit metrics of constant curvature. It is not unreasonable to expect that most manifolds admit metrics, or possibly metrics with mild singularities, of constant Ricci curvature, i.e. Einstein metrics. Second, one of the main applications of an understanding of convergence and degeneration of sequences of metrics would be to establish an existence theory for canonical or distinguished metrics on compact manifolds, as for

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