Abelian Varieties of GL_n-type and Galois Representations

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Introduction

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- Recall: an abelian variety A is a projective algebraic variety that is also an algebraic group.
- We will consider A/k, k a number field.
- $\mathrm{End}(A)$ = endomorphisms of A that are defined over k.
- $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ (we work up to isogeny).
  - $\mathrm{End}^0(A)$ is a semisimple $\mathbb{Q}$-algebra
  - $\mathrm{End}^0(A) \simeq \prod_{i=1}^{s} \mathrm{M}_{r_i}(D_i)$.
  - Each D_i is a division algebra over a totally real or a CM field. Albert classification.
- For every prime ℓ, we have a Galois representation

$$V_\ell(A) := (\varprojlim_n A[\ell^n]) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell^{2\dim A}$$

Rational ℓ-adic Tate module of A $\rightsquigarrow \rho_{A,\ell} : \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}_{2\dim A}(\mathbb{Q}_\ell)$

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- We say A/k is **of** GL_2**-type** if there is a number field $E \subseteq \mathrm{End}^0(A)$ with $[E : \mathbb{Q}] = \dim A$. Includes all elliptic curves.
- For each prime $\mathfrak{L}$ of E: $\rho_{A,\mathfrak{L}} : \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}_2(E_{\mathfrak{L}})$,

$$\rho_{A,\ell} = \bigoplus_{\mathfrak{L}|\ell} \mathrm{Res}_{E_{\mathfrak{L}}/\mathbb{Q}_\ell}\, \rho_{A,\mathfrak{L}}.$$

Theorem (Eichler-Shimura, Ribet, Khare-Wintenberger)

$$\left\{ \begin{array}{c} \textit{isogeny classes} \\ \textit{of } A/\mathbb{Q} \\ \textit{of } \mathrm{GL}_2\textit{-type} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{normalized} \\ \textit{classical modular} \\ \textit{eigenforms of} \\ \textit{weight 2} \end{array} \right\}$$

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Abelian surfaces and the Paramodular conjecture

- The first abelian varieties not of GL_2-type are abelian surfaces $A/\mathbb{Q}$ with $\mathrm{End}^0(A) = \mathbb{Q}$.
- Brumer-Kramer Conjecture: $\rho_{A,\ell} \simeq \rho_{f,\ell}$, for some $f : \mathcal{H}_2 \to \mathbb{C}$ weight 2 Siegel paramodular eigenform.
  - $\mathcal{H}_2 = \{Z \in \mathrm{M}_2(\mathbb{C}) \mid Z^\top = Z,\ \mathrm{Im}\, Z > 0\}$.
  - f must be holomorphic + invariance property.
  - Attached to f: $\rho_{f,\ell} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}_4(\bar{\mathbb{Q}}_\ell)$.
- Weil pairing on $V_\ell(A) \rightsquigarrow \rho_{A,\ell}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})) \subset \mathrm{GSp}_4(\mathbb{Q}_\ell)$.
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QM abelian fourfolds

- Calegari's observation: if f is a paramodular eigenform with rational coefficients, then it could also correspond to some abelian fourfold $A/\mathbb{Q}$ with quaternionic multiplication.

Theorem (Chi)

Let $A/\mathbb{Q}$ be an abelian fourfold with $\mathrm{End}^0(A) =$ indefinite quaternion algebra over $\mathbb{Q}$. Then

$$\rho_{A,\ell} \simeq \sigma_{A,\ell} \oplus \sigma_{A,\ell},$$

where $\sigma_{A,\ell} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}_4(\mathbb{Q}_\ell)$.

- Boxer-Calegari-Gee-Pilloni (under several conjectures): given automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with rational coefficients, $\rho_{\pi,\ell} \simeq \rho_{A,\ell}$, where $A/\mathbb{Q}$ and
  1. $\dim A = 2$, $\mathrm{End}^0(A) = \mathbb{Q}$,
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Larger coefficients

Question

Suppose the field of coefficients of π is $\supsetneq \mathbb{Q}$, and $\rho_{\pi,\lambda} \simeq \rho_{A,\lambda}$.
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Splitting behaviour: QM abelian surfaces I

- The splitting "$\rho_{A,\ell} \simeq \sigma_{A,\ell} \oplus \sigma_{A,\ell}$" already occurs in dimension 2.
- Let A/k be an abelian surface with $\mathrm{End}^0(A) \simeq D$ quaternion algebra.
- A priori, $V_\ell(A) \rightsquigarrow \rho_{A,\ell} : \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$.
- We have $\mathrm{CharPoly}(\rho_{A,\ell}(s)) = (L_s(T))^2$ for some $L_s(T) \in \mathbb{Q}_\ell[T]$ of degree 2.
- Why? there exists a representation

$$\sigma_{A,\ell} : \mathrm{Gal}(\bar{k}/k) \to (D \otimes \mathbb{Q}_\ell)^\times,$$

and for $\ell \notin \mathrm{Ram}(D)$, we have $(D \otimes \mathbb{Q}_\ell)^\times \simeq \mathrm{GL}_2(\mathbb{Q}_\ell)$; and moreover

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Splitting behaviour: QM abelian surfaces II

- A/k with QM $\rightsquigarrow \rho_{A,\ell} \simeq \sigma_{A,\ell} \oplus \sigma_{A,\ell}$.
- There is a geometric counterpart to this fact.
- Given $\mathfrak{p}$ prime of good reduction, let $A_{\mathfrak{p}} := A \mod \mathfrak{p}$.

Theorem (Morita and Yoshida, 1970s)

Let A/k be a QM abelian surface. For all but finitely primes $\mathfrak{p}$, $A_{\mathfrak{p}} \sim E^2$.

Questions

Suppose $\dim A > 2$ and $\mathrm{End}^0(A)$ is non-commutative.

- Does $V_\ell(A)$ split into some copies of the same subrepresentation?
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Organization of the talk

1. Introduction
2. Local conditions on endomorphism algebras (Chapters 2, 3)
3. Abelian varieties of GL_n-type (Chapters 4, 5)
4. Examples of abelian varieties genuinely of GSp_4-type (Chapter 6)
5. k-varieties and Galois representations (Chapter 7)

Local conditions on endomorphism algebras (Chapters 2, 3)

Conditions on $\mathrm{End}^0(A)$

- Let A a simple abelian variety over a number field k. Want to relate $\mathrm{End}^0(A)$ to other algebras.
- If $\mathfrak{p}$ is a prime of good reduction, we have embedding to algebra of $A_{\mathfrak{p}} := A \mod \mathfrak{p}$,
$$\mathrm{End}^0(A) \to \mathrm{End}^0(A_{\mathfrak{p}}).$$
- If K/k is an extension, we have an embedding to algebra of base change A_K,
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- Assume $\mathrm{End}^0(A_{\mathfrak{p}}), \mathrm{End}^0(A_K)$ are simple algebras. Can we relate the Brauer classes $[\mathrm{End}^0(A)]$ and $[\mathrm{End}^0(A_{\mathfrak{p}})], [\mathrm{End}^0(A_K)]$?
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A theorem on split reductions

Theorem (Achter '09, Zywina '14)

Let A/k be an abelian variety with non-commutative $\mathrm{End}^0(A)$.
For every prime $\mathfrak{p}$ out of a set of density zero, $A_{\mathfrak{p}}$ splits.

Theorem (F.)

Let k number field, A/k simple abelian variety with non-commutative $\mathrm{End}^0(A)$. Let $\mathfrak{p}$ be a prime of good reduction.

- If $A_{\mathfrak{p}}$ is simple, then $\mathrm{End}^0(A)$ ramifies at a place over p.
- $\rightsquigarrow S = \{\mathfrak{p} \text{ good prime } : A_{\mathfrak{p}} \text{ simple}\}$ is **finite**. "$\subset \mathrm{Ram}(\mathrm{End}^0(A))$"

Idea of proof:

- Prove an equality $t[\mathrm{End}^0(A) \otimes_F F(\pi)] = t[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} F(\pi)]$ in $\mathrm{Br}(F(\pi))$, for some $t > 0$.
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The set of simple reductions

- $S = \{\mathfrak{p} \text{ good prime} : A_{\mathfrak{p}} \text{ simple}\}$ "$\subset \mathrm{Ram}(\mathrm{End}^0(A))$".

Question

- Can we say anything more about the set of simple reductions?
- Is it ever nonempty?

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Let k number field, A/k a simple abelian fourfold such that $\mathrm{End}^0(A)$ is a **quaternion algebra over Q**.

- For every good $\mathfrak{p}$, $A_{\mathfrak{p}}$ is split or supersingular.
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Proposition

There exists a simple abelian fourfold A over $K = \mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{13})$, given as a quotient of $J_1(156)$, such that

1. $\mathrm{End}^0(A)$ is a **quaternion algebra over** $\mathbb{Q}(\sqrt{17})$, and
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Abelian varieties of GL_n-type (Chapters 4, 5)

The Tate module

- A abelian variety over number field k, ℓ prime.
- Let $V_\ell(A) := \varprojlim_n A[\ell^n] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell^{2\dim A}$.
- $G_k := \mathrm{Gal}(\bar{k}/k)$ acts on $V_\ell(A) \rightsquigarrow$ continuous Galois representation

$$\rho_{A,\ell} : G_k \to \mathrm{GL}_{2\dim A}(\mathbb{Q}_\ell).$$

Properties of interest

- Irreducible Subrepresentations $\rho_{A,\lambda}$ of $\rho_{A,\ell}$.
- Relation of $\rho_{A,\ell}$ with features of $\mathrm{End}^0(A)$.
- Galois-equivariant pairings to restrict image $\rho_{A,\lambda}(G_k)$.

Introducing coefficients

- Suppose we have a field $E \subseteq \mathrm{End}^0(A)$.
- Ribet '75: $V_\ell(A)$ is a free $E \otimes \mathbb{Q}_\ell$-module, of rank $\frac{2\dim A}{[E:\mathbb{Q}]}$.
- By considering the projection $E \otimes \mathbb{Q}_\ell \simeq \prod_{\mathfrak{L}'|\ell} E_{\mathfrak{L}'} \to E_{\mathfrak{L}}$, can define

$$V_{\mathfrak{L}}(A) := V_\ell(A) \otimes_{E\otimes\mathbb{Q}_\ell} E_{\mathfrak{L}},$$

of $E_{\mathfrak{L}}$-dimension $\frac{2\dim A}{[E:\mathbb{Q}]}$.

Definition

Let A/k be an abelian variety. We say A is **of** GL_n**-type** if

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Properties of GL_2-types over $\mathbb{Q}$

Theorem (Ribet)

Let $A/\mathbb{Q}$ be simple of GL_2-type, $E \subseteq \mathrm{End}^0(A)$ maximal, and suppose $A_{\bar{\mathbb{Q}}}$ has no isogeny factor with CM.

1. $A_{\bar{\mathbb{Q}}} \sim B^r$ for some simple B of GL_2-type without CM.
2. $E = \mathrm{End}^0(A)$.
3. $E = \mathbb{Q}(\{\mathrm{Tr}(\rho_{A,\mathfrak{L}}(\mathrm{Frob}_p))\}_{p\nmid \ell N})$.
4. $\mathrm{End}^0(B) = \begin{cases} \text{a totally real field } F, \text{ or} \\ \text{a totally indefinite quaternion algebra over } F. \end{cases}$
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- Ribet used the hypothesis "A of GL_2-type, $A_{\bar{k}}$ has no CM isogeny factor".
- Suppose A/k is simple of GL_n-type.

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We say A is **genuinely of GL_n-type** if $A_{\bar{k}}$ has no isogeny factor of GL_m-type, for $m < n$.

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- B enjoys many properties found in the theory of building blocks by Ribet, Pyle, Quer and Guitart.

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- $V_\ell(A) \simeq \bigoplus_{\lambda|\ell} \mathrm{Res}_{H_\lambda/\mathbb{Q}_\ell} W_\lambda(A)^{\oplus t_A}$, where $t_A^2 = \dim_H \mathrm{End}^0(A)$.

The compatible system of representations

Theorem

Let A/k be simple of GL_n-type,
H the center of $\mathrm{End}^0(A)$, and $t_A^2 = \dim_H \mathrm{End}^0(A)$.

1. For every $\lambda \notin \mathrm{Ram}(\mathrm{End}^0(A))$, $W_\lambda(A) \simeq H_\lambda^n$.
2. For $\lambda \in \mathrm{Ram}(\mathrm{End}^0(A))$,

$$\rho_{A,\lambda} : \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}_{n/t_A}((\mathrm{End}^0(A) \otimes_H H_\lambda)^{op}).$$

Example

If $A/\mathbb{Q}$ is a fourfold with QM by D, then have

$$\rho_{A,\ell} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2((D \otimes \mathbb{Q}_\ell)^{op})$$

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Pairings

- For $A/\mathbb{Q}$ of GL_2-type, we had $\det \rho_{A,\mathfrak{L}} = \varepsilon\chi_\ell$.
- To generalize this to larger n, we see the determinant as a (Galois-equivariant) pairing

$$W_\lambda(A) \times W_\lambda(A) \to H_\lambda,$$

observe that $\mathrm{GL}_2 = \mathrm{GSp}_2$.

Goal

- Want to find a Galois-equivariant pairing

$$\psi_\lambda : W_\lambda(A) \times W_\lambda(A) \to H_\lambda(\chi).$$

- $\psi_\lambda({}^s v, {}^s w) = \chi(s)\psi_\lambda(v, w)$.
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The Nebentype

- Let $F = Z(\mathrm{End}^0(B))$. We have $F \subseteq H$.
- For every $s \in G_k$ and $\varphi \in \mathrm{End}^0(A_{\bar{k}})$, there exists some $\alpha(s) \in H^\times/F^\times$ such that ${}^s\varphi = \alpha(s)\varphi\alpha(s)^{-1}$.
- Assuming F is totally real, we may define the **Nebentype** of A as the character

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\varepsilon : \mathrm{Gal}(\bar{k}/k) &\to H^\times \\
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Definition

We say A is **geometrically of the first kind** if F is totally real.

Proposition

If $a_{\mathfrak{p}} = \mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}} \mid W_\lambda(A))$, then $a_{\mathfrak{p}} = \bar{a}_{\mathfrak{p}} \cdot \varepsilon(\mathrm{Frob}_{\mathfrak{p}})$.

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Characterising AVs geometrically of the first kind I

Theorem (Fité-F.-Guitart, F.)

Let A/k genuinely of GL_n-type. The following are equivalent:

1. A is geometrically of the first kind. (Center F of $\mathrm{End}^0(B)$ is tot. real)
2. There exists a finite character $\xi : \mathrm{Gal}(\bar{k}/k) \to H^\times$, such that for all but finitely many λ there exists a nondegenerate H_λ-bilinear, $\mathrm{Gal}(\bar{k}/k)$-equivariant pairing

$$\psi_\lambda : W_\lambda(A) \times W_\lambda(A) \to H_\lambda(\xi\chi_\ell).$$

$$\psi_\lambda({}^s v, {}^s w) = \xi(s)\chi_\ell(s)\psi_\lambda(v, w)$$

In that case, $\xi = \varepsilon$, and ψ_λ is unique up to scalars.

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Theorem (Fité-F.-Guitart)

Let A be genuinely of GL_n-type and geometrically of the first kind. Then:

- The pairings $\psi_\lambda : W_\lambda(A) \times W_\lambda(A) \to H_\lambda(\varepsilon\chi_\ell)$ are:
  - Alternating, if

$$\mathrm{End}^0(B) = \begin{cases} \text{a totally real field } F, \text{ or} & \text{Type I} \\ \text{a totally indefinite quaternion algebra over } F. & \text{Type II} \end{cases}$$

  - Symmetric, if

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Galois image restrictions

Definition

Let A be genuinely of GL_n-type and geometrically of the first kind. Let B be the building block associated to A.

- We say A is **genuinely of** GSp_n**-type**, if B has Albert type I or II.
- We say A is **genuinely of** GO_n**-type**, if B has Albert type III.

- We can now answer the question: if A is Siegel-modular, what conditions do we have on A?
- Let π cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$.
- $\rightsquigarrow$ Galois rep. $\rho_{\pi,\ell} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\bar{\mathbb{Q}}_\ell)$.
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- Let π cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$.
- $\rightsquigarrow$ Galois rep. $\rho_{\pi,\ell} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\bar{\mathbb{Q}}_\ell)$.
- Say $A/\mathbb{Q}$ is Siegel-modular if $\rho_{A,\lambda} \simeq \rho_{\pi,\ell}$ for some λ and ℓ.

Siegel-modular abelian varieties

Theorem (F.)

Let $A/\mathbb{Q}$ be an abelian variety genuinely of GL_4-type that is Siegel-modular and corresponds to π. Then A is genuinely of GSp_4-type.

Corollary

Suppose π has field of coefficients $\mathbb{Q}$. Then, $A/\mathbb{Q}$ is either

- An abelian surface with $\mathrm{End}^0(A) = \mathbb{Q}$, or
- An abelian fourfold with $\mathrm{End}^0(A) =$ indefinite quaternion algebra over $\mathbb{Q}$.

$\rightsquigarrow$ We rule out the definite case, confirming the expectation of [BCGP21].

Proof. $\mathrm{End}^0(A) \to \mathrm{M}_r(\mathrm{End}^0(B))$, $F = H = \mathbb{Q}$;

$$[\mathrm{End}^0(A)] = [\mathrm{End}^0(B) \otimes_F H] = [\mathrm{End}^0(B)],$$

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Examples of abelian varieties genuinely of GSp_4-type (Chapter 6)

A recipe to construct AVs genuinely of GSp_4-type

- In the literature, we find some families of abelian varieties genuinely of GSp_n and GO_n-type, for even $n \geq 4$.
- There were no constructions of abelian varieties genuinely of GSp_n-type with nontrivial Nebentype, or even $[H : F] > 1$.
- Suppose we have:
  - $K/\mathbb{Q}$ quadratic, $\mathrm{Gal}(K/\mathbb{Q}) = \langle \sigma \rangle$.
  - B/K abelian surface with $\mathrm{End}^0(B_{\bar{K}}) = \mathbb{Q}$.
  - K-isogeny ${}^{\sigma}B \to B$.

  B is a GL_4-type abelian $\mathbb{Q}$-variety.
- Then: $\mathrm{Res}_{K/\mathbb{Q}}(B)$ is genuinely of GSp_4-type, and H/F is quadratic.

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Theorem (Fité-F.-Guitart)

Let $K=\mathbb{Q}(\sqrt{\Delta})$, $\langle\sigma\rangle=\mathrm{Gal}(K/\mathbb{Q})$, $\alpha\in K^\times$ with $-2\,\mathrm{Nm}(\alpha)\in(K^\times)^2$.

Let $r,s,t\in\mathbb{Q}$, and define

$$F_1(X)=X^2+(r+s\sqrt{\Delta})X+\left(\frac{r^2-\Delta s^2+1}{4}+t\sqrt{\Delta}\right)$$

$$F_2(X)=\sqrt{\Delta}X+\left(-\frac{s\Delta}{2}+\frac{r}{2}\sqrt{\Delta}\right)$$

$$F_3(X)=-\sqrt{\Delta}X^2-(s\Delta+r\sqrt{\Delta})X+\left(-\Delta t+\left(\frac{-r^2+\Delta s^2+1}{4}\right)\sqrt{\Delta}\right).$$

Let $C:\alpha Y^2=F_1(X)F_2(X)F_3(X)$. Then

$$\mathrm{Jac}(C)\sim_K {}^{\sigma}\mathrm{Jac}(C).$$

If $\mathrm{End}^0(\mathrm{Jac}(C)_{\bar{K}})=\mathbb{Q}$, then $\mathrm{Res}_{K/\mathbb{Q}}(\mathrm{Jac}(C))$ is an abelian fourfold genuinely of GSp_4-type.

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Proposition

Let $\alpha \in K^\times$ be such that $-2\,\mathrm{Nm}(\alpha) \in (K^\times)^2$, $r, s, t \in \mathbb{Q}$, and

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as above. Let $A := \mathrm{Res}_{K/\mathbb{Q}}(\mathrm{Jac}(C))$. Then

$$\mathrm{End}^0(A) \simeq \begin{cases} \mathbb{Q}(\sqrt{+2}), & \text{if } \mathrm{Nm}(\alpha) \in -2\Delta(\mathbb{Q}^\times)^2, \\ \mathbb{Q}(\sqrt{-2}), & \text{if } \mathrm{Nm}(\alpha) \in -2(\mathbb{Q}^\times)^2. \end{cases}$$

In particular:

- $\varepsilon_A =$ trivial, if $\mathrm{Nm}(\alpha) \in -2\Delta(\mathbb{Q}^\times)^2$.
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An example

- Over $K = \mathbb{Q}(\sqrt{17})$, let $(r, s, t) = (1, -3, 1)$, and

$$C : \alpha Y^2 = F_1(X)F_2(X)F_3(X)$$

with $\alpha = -\frac{3+\sqrt{17}}{2}$.

- Let $A := \mathrm{Res}_{K/\mathbb{Q}}(\mathrm{Jac}(C))$.
- Since $\mathrm{Nm}(\alpha) = -2$, we have

$$\mathrm{End}^0(A) = \mathbb{Q}(\sqrt{-2}).$$

- $\rightsquigarrow \rho_{A,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{Q}(\sqrt{-2})_\lambda)$.
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Local factors

- 31 is inert in $\mathbb{Q}(\sqrt{17})$, $\eta_{K/\mathbb{Q}}(31) = -1$.

$$T^4 - 2\sqrt{-2}T^3 - 26T^2 + 62\sqrt{-2}T + 961.$$

- Roots: $\pi_1 \approx -5.04 - 2.38i$, $\pi_2 \approx -4.08 + 3.79i$, $\frac{-31}{\pi_1}$, $\frac{-31}{\pi_2}$.
- 43 splits in $\mathbb{Q}(\sqrt{17})$, $\eta_{K/\mathbb{Q}}(43) = +1$.

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k-varieties and Galois representations (Chapter 7)

Weak abelian k-varieties

- So far: A/k genuinely of GL_n-type.
- It remains to consider A/k of GL_n-type, but *not genuinely*.
- Suppose L/k finite Galois, $A_L \sim B^r$ for a simple B/L of GL_m-type, $m < n$.
- $B \sim {}^sB$ for all $s \in G_k$.

We say B is a **weak abelian k-variety**.

- From the representation $\rho_{B,\ell} : G_L \to \mathrm{GL}_m(\bar{\mathbb{Q}}_\ell)$, we want to build a representation of G_k **of dimension** $\leq n$.

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- It remains to consider A/k of GL_n-type, but *not genuinely*.
- Suppose L/k finite Galois, $A_L \sim B^r$ for a simple B/L of GL_m-type, $m < n$.
- $B \sim {}^sB$ for all $s \in G_k$.

We say B is a **weak abelian k-variety**.

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Abelian surfaces with potential QM

Example

Let $A/\mathbb{Q}$ be an abelian surface with

- $\mathrm{End}^0(A) = \mathbb{Q}$.
- $\mathrm{End}^0(A_{\bar{\mathbb{Q}}}) =$ an indefinite quaternion algebra with center $\mathbb{Q}$.

Then A is of GL_4-type, but A_L is of GL_2-type (for some finite $L/\mathbb{Q}$).
This setting also covers the case $A_{\bar{\mathbb{Q}}} \sim E^2$.

- Consider the two representations

$$\rho_{A,\ell} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{Q}_\ell), \quad \sigma_{A_L,\ell} : G_L \to \mathrm{GL}_2(\mathbb{Q}_\ell).$$

- There exists a finite character $\xi : G_L \to \bar{\mathbb{Q}}^\times$ and a representation

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  such that $\tilde{\sigma}_{A_L,\ell}|G_L \simeq \sigma_{A_L,\ell} \otimes \xi$.
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Propagating modularity from GL_2 to GSp_4

- The representation $\tilde{\sigma}_{A_L,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ is modular.
- $\rightsquigarrow$ we can use Langlands base change and automorphic induction to show $\rho_{A,\ell} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{Q}_\ell)$ is Siegel-modular.

Theorem (F.-Pacetti)

Let $D =$ indefinite quaternion $\mathbb{Q}$-algebra or $\mathrm{M}_2(\mathbb{Q})$. Let $A/\mathbb{Q}$ be an abelian surface with $\mathrm{End}^0(A) = \mathbb{Q}$ and $\mathrm{End}^0(A_{\bar{\mathbb{Q}}}) = D$. Then

- A is Siegel-modular: there exists a cuspidal automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ such that $\rho_{A,\ell} \simeq \rho_{\pi,\ell}$.

Under additional hypotheses we can show paramodularity.

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Preprints and articles

- *Abelian varieties genuinely of* GL_n*-type*, with F. Fité and X. Guitart. Submitted. arXiv:2412.21183
- *k-varieties and Galois representations*, with A. Pacetti. Submitted. arXiv:2412.03184
- *Abelian varieties that split modulo all but finitely many primes*. Proc. Amer. Math. Soc. 153 (2025), 1491-1499

Abelian Varieties of GL_n-type and Galois Representations

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