Modularity of abelian surfaces with potential QM

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Modularity conjecture for abelian surfaces

Theorem (Taylor, Taylor-Wiles, BCDT)

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. There exists a classical eigenform $f_E \in S_2(\Gamma_0(N))$ with rational eigenvalues such that $L(E, s) = L(f_E, s)$.

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Theorem (Ribet, Khare-Wintenberger)

Let $A/\mathbb{Q}$ be an abelian variety of conductor N and such that $E = \mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$ is a number field with $[E : \mathbb{Q}] = \dim A$. There exists a classical eigenform $f_A \in S_2(\Gamma_1(N))$ such that $L(A, s) = \prod_{\sigma : E \to \mathbb{C}} L({}^{\sigma} f_A, s)$.

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Conjecture (Brumer-Kramer)

Let $A/\mathbb{Q}$ be an abelian surface with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and conductor N. Then there exists a Siegel paramodular newform $f_A \in S_2(K(N))$ with rational eigenvalues such that $L(A, s) = L(f_A, s, spin)$.

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*Automorphy is proven, without the paramodular level.

Siegel Paramodular Forms

Let $N \in \mathbb{Z}_{\geq 1}$. Our substitute for $\Gamma_0(N)$ is the level N paramodular group

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Q})$$

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where $* \in \mathbb{Z}$. Let $\mathcal{H}_2 = \{Z \in \mathrm{Mat}_2(\mathbb{C}) \mid \mathrm{Im}(Z) > 0\}$. A Siegel paramodular form of level N and weight 2 is a holomorphic $f : \mathcal{H}_2 \to \mathbb{C}$ satisfying

- $f(MZ) = \det(CZ + D)^2 f(Z)$ for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K(N)$.
- $f(Z)$ satisfies a boundedness condition.

Modular Galois representations

Theorem (Taylor, Sorensen, Mok)

Let π *be a cuspidal automorphic representation of* $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ *of weight* $(2,2)$ *which is not CAP or endoscopic. Suppose the coefficient field of* π *is* $\mathbb{Q}$ *and* π *has trivial central character. Then for every* ℓ*, exists a continuous semisimple Galois representation*

$$\rho_{\pi,\ell} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GSp}_4(\bar{\mathbb{Q}}_\ell)$$

such that:

1. $\rho_{\pi,\ell}$ *is unramified at all* $p \notin S \cup \{\ell\}$.
2. *For all* $p \notin S \cup \{\ell\}$, $\mathrm{Tr}\,\rho_{\pi,\ell}(\mathrm{Frob}_p) = a_p, \quad \mathrm{sim}\,\rho_{\pi,\ell} = \chi_\ell$.
3. $\rho_{\pi,\ell} \simeq \rho_{\pi,\ell}^\vee \otimes \mathrm{sim}\,\rho_{\pi,\ell}$.
4. *The Hodge-Tate-Sen weights of* $\rho_{\pi,\ell}|_{\mathbb{Q}_\ell}$ *are* $\{0,0,1,1\}$.
5. *Local-global compatibility.*

Main result

Theorem (F.-Pacetti)

Let $D =$ indefinite quaternion $\mathbb{Q}$-algebra or $\mathrm{Mat}_2(\mathbb{Q})$. Let $A/\mathbb{Q}$ be an abelian surface with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and $\mathrm{End}_{\bar{\mathbb{Q}}}(A) \otimes \mathbb{Q} = D$.

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- A is Siegel modular: there exists a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ whose L-series matches that of A.
- If A is principally polarizable and $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$ is an hereditary quaternion order, then A is paramodular.

Abelian surfaces with potential QM

- Let $A/\mathbb{Q}$ be an abelian surface with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$ an order in an indefinite quaternion algebra $D_{/\mathbb{Q}}$.

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- Let $A/\mathbb{Q}$ be an abelian surface with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$ an order in an indefinite quaternion algebra $D_{/\mathbb{Q}}$.
- Let L be the smallest field with $\mathrm{End}_L(A) = \mathrm{End}_{\bar{\mathbb{Q}}}(A)$.
  Dieulefait-Rotger: $\mathrm{Gal}(L/\mathbb{Q}) \simeq C_2 \times C_2, D_{2\cdot 3}, D_{2\cdot 4}$ or $D_{2\cdot 6}$.

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- Since $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$, the action of $\mathrm{Gal}_{\mathbb{Q}}$ on $V_\ell(A) \simeq T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ gives a continous representation

$$\rho_{A,\ell}^{(\mathbb{Q})} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{Q}_\ell) \subset \mathrm{GL}_4(\mathbb{Q}_\ell) \simeq \mathrm{Aut}(V_\ell(A))$$

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  which is absolutely irreducible for all ℓ.
- Suppose $\ell \nmid \operatorname{disc} D$, so that $D \otimes \mathbb{Q}_\ell \simeq \mathrm{Mat}_2(\mathbb{Q}_\ell)$. Then

$$V_\ell(A_L) \simeq W_\ell(A) \oplus W_\ell(A),$$

  where $W_\ell(A)$ is an abs. irreducible $\mathbb{Q}_\ell[\mathrm{Gal}_L]$-submodule of $V_\ell(A_L)$.

$$\rightsquigarrow \rho_{A,\ell}^{(L)} : \mathrm{Gal}_L \to \mathrm{Aut}(W_\ell(A)) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell).$$

Intermediate quadratic fields

Lemma

There exists a quadratic subextension $K/\mathbb{Q}$ of $L/\mathbb{Q}$ such that $E = \mathrm{End}_K(A) \otimes \mathbb{Q}$ is a quadratic field. □

- Fix K and E. Let $\ell = \lambda\lambda'$ in $\mathcal{O}_E$. Then

$$V_\ell(A_K) \simeq W_\lambda(A) \oplus W_{\lambda'}(A),$$

let $\rho^{(K)}_{A,\lambda} : \mathrm{Gal}_K \to \mathrm{Aut}(W_\lambda(A)) \simeq \mathrm{GL}_2(E_\lambda)$.

- We have $\rho^{(K)}_{A,\lambda}|\,\mathrm{Gal}_L \simeq \rho^{(L)}_{A,\ell}$.
- Let ${}^s\rho^{(K)}_{A,\lambda}(t) := \rho^{(K)}_{A,\lambda}(sts^{-1})$. Then

$${}^s\rho^{(K)}_{A,\lambda} \simeq \rho^{(K)}_{A,\lambda'}.$$

- $\implies \rho^{(\mathbb{Q})}_{A,\ell} = \mathrm{Ind}^{\mathbb{Q}}_K \rho^{(K)}_{A,\lambda}.$

Extension of $\rho_{A,\ell}^{(L)}$ to $\mathrm{Gal}_{\mathbb{Q}}$

Since $Z(\mathrm{End}(A_L) \otimes \mathbb{Q}) = \mathbb{Q}$, A_L is a (strong) $\mathbb{Q}$-variety:

1. For all $s \in \mathrm{Gal}_{\mathbb{Q}}$ there is an isogeny $\mu_s : {}^sA \to A$.
2. For all $\varphi \in \mathrm{End}_L(A)$ and $s \in \mathrm{Gal}_{\mathbb{Q}}$, $\mu_s {}^s\varphi = \varphi\mu_s$.

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We can apply the following result.

Lemma (Ribet, Guitart)

There exists a finite order character $\xi : \mathrm{Gal}_L \to \bar{\mathbb{Q}}^\times$ *and a representation* $\tilde{\rho} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ *such that* $\tilde{\rho}|\,\mathrm{Gal}_L \simeq \rho_{A,\ell}^{(L)} \otimes \xi$. □

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- By Serre's modularity conjecture, $\tilde{\rho}$ is modular: $\tilde{\rho} \simeq \rho_{f,\mathfrak{l}}$ for some classical modular eigenform f.

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- By Serre's modularity conjecture, $\tilde{\rho}$ is modular: $\tilde{\rho} \simeq \rho_{f,\mathfrak{l}}$ for some classical modular eigenform f.
- There is a finite order character $\chi : \mathrm{Gal}_K \to \bar{\mathbb{Q}}^\times$ such that $\rho_{f,\mathfrak{l}}|\,\mathrm{Gal}_K \simeq \rho_{A,\lambda}^{(K)} \otimes \chi$.

Sketch of proof

Theorem (F.-Pacetti)

Let $D =$ indefinite quaternion $\mathbb{Q}$-algebra or $\mathrm{Mat}_2(\mathbb{Q})$. Let $A/\mathbb{Q}$ be an abelian surface with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and $\mathrm{End}_{\bar{\mathbb{Q}}}(A) \otimes \mathbb{Q} = D$. Then

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- Since $\rho_{A,\ell}^{(\mathbb{Q})}$ is attached to an abelian surface, Π must be the transfer of an automorphic rep'n π of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$. $\square$

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  - $F/\mathbb{Q}$ imaginary quadratic by Berger, Dembélé, Pacetti and Sengün.

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- $\det \rho_{A,\lambda}^{(K)} = \chi_\ell$, the cyclotomic character.
- Hence the representation π of $\mathrm{GL}_2(\mathbb{A}_K)$ has trivial central character.

Paramodularity II

Theorem (F.-Pacetti)

Let $D =$ indefinite quaternion $\mathbb{Q}$-algebra or $\mathrm{Mat}_2(\mathbb{Q})$. Let $A/\mathbb{Q}$ be an abelian surface with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and $\mathrm{End}_{\bar{\mathbb{Q}}}(A) \otimes \mathbb{Q} = D$. Then

- If A is principally polarizable and $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$ is an hereditary quaternion order, then A is paramodular.

- Dieulefait-Rotger showed that $L/\mathbb{Q}$ is biquadratic.
- We may pick $K/\mathbb{Q}$ quadratic subextension so that $\mathrm{End}_K(A) \otimes \mathbb{Q}$ is a real quadratic field.
- $\det \rho_{A,\lambda}^{(K)} = \chi_\ell$, the cyclotomic character.
- Hence the representation π of $\mathrm{GL}_2(\mathbb{A}_K)$ has trivial central character.
- $\implies$ the induction of π to $\mathbb{Q}$ is paramodular. $\square$

An example

- Consider the following genus 2 curve from Dieulefait-Rotger
$$C : y^2 = (x^2+7)(83/30x^4 + 14x^3 - 1519/30x^2 + 49x - 1813/120),$$
and let $A = \mathrm{Jac}\, C$.
- $\mathrm{End}_L(A)$ is a maximal order in B_6 (the indefinite quaternion algebra of discriminant 6), for $L = \mathbb{Q}(\sqrt{-6}, \sqrt{-14})$.
- $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.
- $\mathrm{End}_{\mathbb{Q}(\sqrt{-14})}(A) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{2})$.
- $\mathrm{End}_{\mathbb{Q}(\sqrt{21})}(A) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{3})$.
- $\mathrm{End}_{\mathbb{Q}(\sqrt{-6})}(A) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-6})$.

Paramodularity in general?

- If $\mathrm{Gal}(L/\mathbb{Q}) = C_2 \times C_2$, then for some $K/\mathbb{Q}$ we have $\mathrm{End}_K(A) \otimes \mathbb{Q}$ real quadratic, and $\det \rho_{A,\lambda}^{(K)} = \chi_\ell$.

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- The Gal_K-equivariant determinant form $W_\lambda(A) \wedge W_\lambda(A) \to E_\lambda(\chi_\ell)$ extends to a $\mathrm{Gal}_{\mathbb{Q}}$-equivariant alternating pairing

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- If $\mathrm{Gal}(L/\mathbb{Q}) \simeq D_{2\cdot n}$, $n = 3, 4, 6$, then there is a single quadratic subextension K of L, and $\mathrm{End}_K(A) \otimes \mathbb{Q}$ is always imaginary quadratic.

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- Moreover, $\det \rho_{A,\lambda}^{(K)} = \varepsilon\chi_\ell$, where ε is a nontrivial character of $\mathrm{Gal}(L/K)$ which usually does not extend to $\mathrm{Gal}(L/\mathbb{Q})$.
- Can't extend determinant form to a $\mathrm{Gal}_\mathbb{Q}$-equivariant pairing on $\mathrm{Ind}_K^\mathbb{Q} W_\lambda(A)$.