Properties of Gaussian Distributions

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Part III Astrostatistics

1 Joint, Conditional, and Marginal Properties of Multivariate Gaussian Random Variables

1.1 The Joint Probability

Suppose $\boldsymbol{f}$ is an n-dimensional (column) vector with a multivariate Gaussian distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Then $\boldsymbol{f}$ is said to be a multivariate Gaussian random vector:

$$\boldsymbol{f} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{1}$$

A proper covariance matrix $\boldsymbol{\Sigma}$ must be *positive-definite* (implying that $|\boldsymbol{\Sigma}| = \det \boldsymbol{\Sigma}$ is positive), symmetric ($\boldsymbol{\Sigma}^T = \boldsymbol{\Sigma}$), and invertible ($\boldsymbol{\Sigma}^{-1}$ exists). Its probability density is:

$$P(\boldsymbol{f}) = N(\boldsymbol{f}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \equiv [\det(2\pi\boldsymbol{\Sigma})]^{-1/2} \exp\left[-\frac{1}{2}(\boldsymbol{f}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{f}-\boldsymbol{\mu})\right]. \tag{2}$$

Suppose this vector comprises a d-dimensional vector $\boldsymbol{U}$ and and $(n-d)$-dimensional vector $\boldsymbol{V}$. The mean vector and covariance matrix of f can be partitioned accordingly:

$$\boldsymbol{f} = \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_U \\ \boldsymbol{\mu}_V \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_U & \boldsymbol{\Sigma}_{UV} \\ \boldsymbol{\Sigma}_{VU} & \boldsymbol{\Sigma}_V \end{bmatrix} \right). \tag{3}$$

Note that the symmetric nature of $\boldsymbol{\Sigma}$ requires that $\boldsymbol{\Sigma}_U$, $\boldsymbol{\Sigma}_V$ be symmetric. The matrices $\boldsymbol{\Sigma}_{UV} = \boldsymbol{\Sigma}_{VU}^T$ need not be square if $d \neq n/2$.

1.2 Marginal Probabilities

The marginal probability density of $\boldsymbol{U}$ (integrating out $\boldsymbol{V}$) is:

$$P(\boldsymbol{U}) = \int P(\boldsymbol{U}, \boldsymbol{V})\, d\boldsymbol{V} = N(\boldsymbol{U}|\, \boldsymbol{\mu}_U, \boldsymbol{\Sigma}_U), \tag{4}$$

so marginally, $\boldsymbol{U}$ is a d-dim multivariate Gaussian vector $\boldsymbol{U} \sim N(\boldsymbol{\mu}_U, \boldsymbol{\Sigma}_U)$. Similarly, the $\boldsymbol{V}$, integrating out $\boldsymbol{U}$, is marginally a $(n-d)$-dimensional multivariate Gaussian vector $\boldsymbol{V} \sim N(\boldsymbol{\mu}_V, \boldsymbol{\Sigma}_V)$:

$$P(\boldsymbol{V}) = \int P(\boldsymbol{U}, \boldsymbol{V}))\, d\boldsymbol{U} = N(\boldsymbol{V}|\, \boldsymbol{\mu}_V, \boldsymbol{\Sigma}_V). \tag{5}$$

1.3 Conditional Probabilities

The joint probability density can be expressed as the product of a marginal and a conditional: $P(\boldsymbol{U}, \boldsymbol{V}) = P(\boldsymbol{U}|\boldsymbol{V}) \times P(\boldsymbol{V})$, where $P(V)$ is given in Eq. 5. If we observe the values of $\boldsymbol{V}$, then we can compute the conditional probability density of $\boldsymbol{U}$ given $\boldsymbol{V}$. The conditional probability of $\boldsymbol{U}|\boldsymbol{V}$ is also multivariate Gaussian,

$$\boldsymbol{U}|\,\boldsymbol{V} \sim N\left(\mathbb{E}[\boldsymbol{U}|\boldsymbol{V}],\, \mathrm{Var}[\boldsymbol{U}|\boldsymbol{V}]\right) \tag{6}$$

with a conditional expectation,

$$\mathbb{E}[\boldsymbol{U}|\boldsymbol{V}] = \boldsymbol{\mu}_U + \boldsymbol{\Sigma}_{UV}\, \boldsymbol{\Sigma}_V^{-1}(\boldsymbol{V} - \boldsymbol{\mu}_V) \tag{7}$$

and a conditional variance,

$$\mathrm{Var}[\boldsymbol{U}|\boldsymbol{V}] = \boldsymbol{\Sigma}_U - \boldsymbol{\Sigma}_{UV}\, \boldsymbol{\Sigma}_V^{-1}\, \boldsymbol{\Sigma}_{VU}. \tag{8}$$

1.4 Constructing the Joint from a Marginal and Conditional

We saw in the previous section that the joint multivariate Gaussian probability density for random vector $\boldsymbol{f}$ can be decomposed into the marginal and conditional densities of its subvectors $\boldsymbol{U}$ and $\boldsymbol{V}$. Now, we show that can construct the joint density of $\boldsymbol{f}$, if we specify the marginal and conditional densities of the subvectors. Suppose we specify the marginal distribution of V:

$$\boldsymbol{V} \sim N(\boldsymbol{V}_0, \boldsymbol{\Sigma}_V) \tag{9}$$

and a conditional distribution of $\boldsymbol{U}|\boldsymbol{V}$:

$$\boldsymbol{U}|\boldsymbol{V} \sim N(\boldsymbol{U}_0 + \boldsymbol{X}\boldsymbol{V}, \boldsymbol{\Sigma}_{U|V}) \tag{10}$$

for some matrix $\boldsymbol{X}$ of the appropriate dimensionality Then their joint distribution is also multivariate Gaussian

$$\begin{pmatrix}\boldsymbol{U}\\ \boldsymbol{V}\end{pmatrix} \sim N\left(\begin{pmatrix}\boldsymbol{U}_0 + \boldsymbol{X}\boldsymbol{V}_0\\ \boldsymbol{V}_0\end{pmatrix}, \begin{pmatrix}\boldsymbol{X}\boldsymbol{\Sigma}_V\boldsymbol{X}^T + \boldsymbol{\Sigma}_{U|V} & \boldsymbol{X}\boldsymbol{\Sigma}_V\\ \boldsymbol{\Sigma}_V\boldsymbol{X}^T & \boldsymbol{\Sigma}_V\end{pmatrix}\right). \tag{11}$$

The marginal probability density of the vector $\boldsymbol{U}$, integrating out, $\boldsymbol{V}$ is also multivariate Gaussian:

$$P(\boldsymbol{U}) = \int P(\boldsymbol{U}, \boldsymbol{V})\, d\boldsymbol{V} = N(\boldsymbol{U}|\,\boldsymbol{U}_0 + \boldsymbol{X}\boldsymbol{V}_0, \boldsymbol{X}\boldsymbol{\Sigma}_V\boldsymbol{X}^T + \boldsymbol{\Sigma}_{U|V}) \tag{12}$$

with a marginal mean and marginal variance that can be read off Eq. 11.