DigSig1 formulary

Dozent: G.Schuster, Buch: Introduction to Signal Processing, Orfanidis

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Contents

1 Random Signals Chapter 13 S713-719

probability distribution	$F_x(\alpha) = P(x \leqslant \alpha)$
probability density	$f_x(\alpha) = \frac{d}{d\alpha} F_x(\alpha)$
mean / expected value	$E(x) = \sum_k \alpha_k P(x = \alpha_k) = \int_{-\infty}^{\infty} \alpha \cdot f_x(\alpha) d\alpha$ $E(y) = E(g(x)) = \int_{-\infty}^{\infty} g(\alpha) f_x(\alpha) d\alpha$
signal power	$E(x^2) = \sum_k \alpha_k^2 P(x = \alpha_k) = \int_{-\infty}^{\infty} \alpha^2 f_x(\alpha) d\alpha$
average absolute value	$E(\|x\|) = \int_{-\infty}^{\infty} \|\alpha\| f_x(\alpha) d\alpha = \int_0^{\infty} \alpha[f_x(\alpha) + f_x(-\alpha)] d\alpha$
variance σ^2	$\sigma_x^2 = E\{[x - E(x)]^2\} = E(x^2) - E(x)^2 = \int_{-\infty}^{\infty} [\alpha - E(x)]^2 f_x(\alpha) d\alpha$
Joint distribution function	$F_{x(1),x(2)}(\alpha_1, \alpha_2) = Pr\{x(1) \leqslant \alpha_1, x(2) \leqslant \alpha_2\}$
Joint density function	$f_{x(1),x(2)}(\alpha_1, \alpha_2) = \frac{\partial^2}{\partial\alpha_1 \partial\alpha_2} F_{x(1),x(2)}(\alpha_1, \alpha_2)$
covariance	$c_{xy} = Cov(x, y) = E\{(x - E(x))(y - E(y))^*\} = E(xy*) - E(x)E(y^*)$
correlation	$r_{xy} = E(xy^*)$ $\quad r_{xy} = 0 \rightarrow$ orthogonal; if zero mean $\rightarrow$ uncorrelated
correlation coefficient	$\rho_{xy} = \frac{c_{xy}}{\sigma_x \sigma_y}$ $\quad -1 \leqslant \rho_{xy} \leqslant 1$ for $\rho_{xy} = 0 \rightarrow$ uncorrelated
autocorrelation function	$R_{XX}(k) = E(x(n + k)x(n))$ $\hat{R}_{XX}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n + k)x(n)$ (sample autocorrelation) $R_{XX}(k) = R_{XX}(-k)$
power	$\sigma_x^2 = R_{XX}(0) = E(x(n)^2) = \int_{-\pi}^{\pi} S_{XX}(\omega) \frac{d\omega}{2\pi}$ $\sigma_y^2 = \int_{-\pi}^{\pi} S_{YY}(\omega) \frac{d\omega}{2\pi} = \sigma_x^2 \cdot \int_{-\pi}^{\pi} \|H(\omega)\|^2 \frac{d\omega}{2\pi}$
power spectrum	$S_{XX}(\omega) = \sum_{k=-\infty}^{\infty} R_{XX}(k) e^{-j\omega k} = \lim_{N\to\infty} E(\hat{S}_{XX}(\omega))$ $\quad \omega = \frac{2\pi f}{f_s}$ $S_{YY}(\omega) = \|H(\omega)\|^2 S_{XX}(\omega)$
periodogram spectrum	$\hat{S}_{XX}(\omega) = \frac{1}{N} \|X_N(\omega)\|^2$ with $X_N(\omega) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$ $\hat{S}(\omega) = \frac{1}{K}[\hat{S}_1(\omega) + \ldots + \hat{S}_K(\omega)] = \frac{1}{KN}[\|X_1(\omega)\|^2 + \ldots + \|X_K(\omega)\|^2]$ (with x_i as block of length N)
Noise reduction ratio (NRR)	$NRR = \frac{\sigma_y^2}{\sigma_x^2} = \int_{-\pi}^{\pi} \|H(\omega)\|^2 \frac{d\omega}{2\pi} = \sum_n h(n)^2$

2 Sampling and Reconstruction Chapter 1

2.1 Analog Signals S2

$\Omega = 2\pi f \qquad [\Omega] = \frac{rad}{s}$

$\omega = \Omega T = \frac{2\pi f}{f_s} \qquad [\omega] = \frac{rad}{sample}$

Fourier transform	$X(\Omega) = \int\limits_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$
inverse Fourier transform	$x(t) = \int\limits_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}\frac{d\Omega}{2\pi}$

2.2 Digital Signals

2.2.1 Sampling Theorem S4-6

Sampling means that the analog signal is periodically measured with a sampling interval T. The discrete index n, relates to the time t as follows:

$$t = nT \qquad n = 0, 1, 2, \ldots$$

The sampling frequency relates to the sampling interval as follows:

$$f_s = \frac{1}{T}$$

Sampling Theorem (Nyquist rate):

$$f_s \geqslant 2f_{max} \qquad \text{or} \qquad T \leqslant \frac{1}{2f_{max}}$$

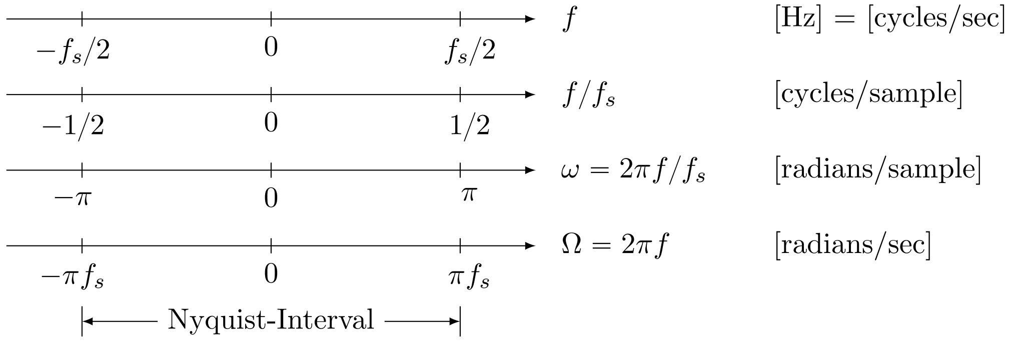
Nyquist interval:

$$\left[-\frac{f_s}{2}, \frac{f_s}{2}\right]$$

2.2.2 DSP Frequency Units S29-30

A sampled sinusoid takes the form in these units:

$$e^{2\pi jfTn} = e^{2\pi j(f/f_s)n} = e^{j\Omega Tn} = e^{j\omega n}$$



2.2.3 Flat-top sampling S30

In practical sampling each sample is held for a short period of time (τ).

$$x_{flat}(t) = \sum_{n=-\infty}^{\infty} x(nT)p(t - nT) \qquad p(t)\text{: flat-top pulse with duration } \tau$$

This is equivalent to filtering the perfectly sampled signal $\hat{x}$ with a linear filter with the impulse response $p(t)$. The spectrum of the filter looks like this

$$|P(f)| = \tau \left|\frac{sin(\pi f\tau)}{\pi f\tau}\right|$$

2.2.4 Discrete-Time Fourier Transform (DTFT) S31

$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(nT)e^{-2\pi jfTn}$$

$$x(nT) = \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} \hat{X}(f)e^{2\pi jfTn}df = \int_{-\pi}^{\pi} \hat{X}(\omega)e^{j\omega n}\frac{d\omega}{2\pi}$$

A sampled signal has always a periodic spectrum, with its spectrum center at the multiples of the sampling frequency.

$\hat{X}(f) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(f - mf_s)$ $\quad T\hat{X}(f) = X(f)$ for $-\frac{f_s}{2} \leqslant f \leqslant \frac{f_s}{2}$

2.2.5 Aliasing S10, 38

If the signal frequency f is outside the Nyquist interval, the signal will be aliased with $f \pm n \cdot f_{sampling}$.

Example: $sin(8\pi t)$ (signal frequency $f = 4$) sampled at a rate of $f_s = 5Hz$ will be aliased to $sin(2\pi(f - f_s)t) = sin(2\pi(-1)t)$

$$f_{ia} = f_i + nf_s$$

n must be selected such that f_{ia} is in the Nyquist intervall. n can also be negative.

Is the signal frequency f inside the Nyquist interval $[-f_s/2, +f_s/2]$, no aliasing will be perceived.

2.2.6 Antialiasing Prefilter S38

The stop frequency from the antialiasing prefilter is defined as:

$$f_{stop} = f_s - f_{pass}$$

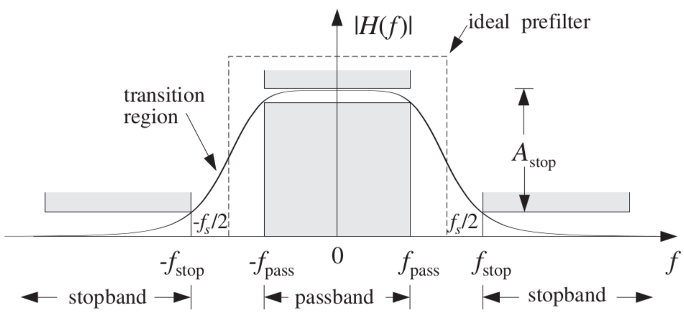
The attenuation of the antialiasing prefilter is:

$$A(f) = -20 \cdot \log_{10}\left|\frac{H(f)}{H(f_0)}\right| \quad [dB]$$

$$A_{dB} = \alpha \log_{10}\left(\frac{f}{f_{cutoff}}\right) \qquad \alpha = \frac{dB}{dek}$$

$$A_{dB} = \beta \log_{2}\left(\frac{f}{f_{cutoff}}\right) \qquad \beta = \frac{dB}{okt}$$

$$A_X(f) = \underbrace{A(f)}_{Prefilter} + \underbrace{A_{X_{in}}(f)}_{Inputspectrum}$$



f_0: reference frequency (typ. DC)

2.3 Reconstructors S42

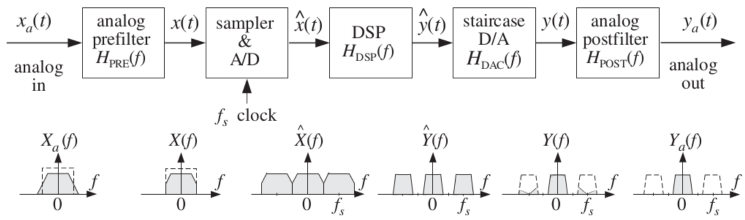
2.3.1 Ideal reconstructor S43

$$H(f) = \begin{cases} T & |f| \leqslant \frac{f_s}{2} \\ 0 & else \end{cases} \qquad h(t) = \frac{\sin(\pi f_s t)}{\pi f_s t} \qquad y(t) = \sum_{n=-\infty}^{\infty} y(nT)h(t - nT)$$

2.3.2 Staircase reconstructor S45

$$h(t) = u(t) - u(t-T) = \begin{cases} 1 & 0 \leqslant t \leqslant T \\ 0 & else \end{cases}$$

$$H(f) = \frac{1}{2\pi j f}(1-e^{-2\pi j f T}) = T \cdot \frac{\sin(\pi f T)}{\pi f T} \cdot e^{-\pi j f T}$$



The spectral images at higher frequencies are not well suppressed, therefore an **anti-image postfilter** is needed.

3 Quantization Chapter 2

3.1 Quantization process S62

R	full-scale range	$R = Q \cdot 2^B$
B	bits	$B = log_2\left(\frac{R}{Q}\right)$
Q	quantization width	$Q = \frac{R}{2^B}$
e	quantization error (quantization noise)	$e_Q(nT) = x_Q(nT) - x(nT)$
e_{RMS}	root-mean-square error	$e_{RMS} = \frac{Q}{\sqrt{12}}$
r	dynamic range	$r = 20log_{10}\left(2^B\right) \approx 6dB \cdot B$
SNR	signal-to-noise ratio (with uniform white noise)	$SNR = 20log_{10}\left(\frac{R}{Q}\right) = 6B\,dB$
σ_e^2	average power / variance of quantization error	$\sigma_e^2 = E[e^2(n)] = \frac{Q^2}{12}$

3.2 Oversampling and noise shaping S66-70

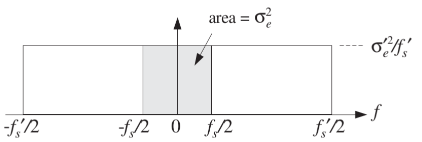
L	oversampling ratio	$L = \frac{f_s'}{f_s}$ with f_s' as higher sampling rate
ΔB	saved bits without noise shaping saved bits with noise shaping	$\Delta B = 0.5 \cdot log_2(L)$ $\Delta B = (p+0.5) \cdot log_2(L) - 0.5 \cdot log_2\left(\frac{\pi^{2p}}{2p+1}\right)$ $L = \left(\frac{2^{2\Delta B}\pi^{2p}}{2p+1}\right)^{\frac{1}{2p+1}}$

p = order of the noise shaping filter

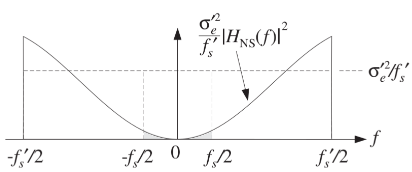
Performance of oversampling noise shaping quantizers:

p	L	4	8	16	32	64	128
0	$\Delta B \approx 0.5 \cdot log_2(L)$	1.0	1.5	2.0	2.5	3.0	3.5
1	$\Delta B \approx 1.5 \cdot log_2(L) - 0.86$	2.1	3.6	5.1	6.6	8.1	9.6
2	$\Delta B \approx 2.5 \cdot log_2(L) - 2.14$	2.9	5.4	7.9	10.4	12.9	15.4
3	$\Delta B \approx 3.5 \cdot log_2(L) - 3.55$	3.5	7.0	10.5	14.0	17.5	21.0
4	$\Delta B \approx 4.5 \cdot log_2(L) - 5.02$	4.0	8.5	13.0	17.5	22.0	26.5
5	$\Delta B \approx 5.5 \cdot log_2(L) - 6.53$	4.5	10.0	15.5	21.0	26.5	32.0

Oversampled quantization noise power, **without noise shaping.**



Spectrum of oversampling **noise shaping quantizer**.



3.3 D/A converters S71-73

Name	Output calculation	Min	Max
natural binary	$x_Q = R(b_1 2^{-1} + b_2 2^{-2} + \ldots + b_B 2^{-B})$	0	$R - Q$
offset binary	$x_Q = R(b_1 2^{-1} + b_2 2^{-2} + \ldots + b_B 2^{-B} - 0.5)$	$-\frac{R}{2}$	$\frac{R}{2} - Q$
two's complement	$x_Q = R(\overline{b_1} 2^{-1} + b_2 2^{-2} + \ldots + b_B 2^{-B} - 0.5) \qquad (\overline{b_1} = 1 - b_1)$	$-\frac{R}{2}$	$\frac{R}{2} - Q$

3.4 A/D converters S75

3.4.1 Successive Approximation Converters S76

Conversion algorithms	
Natural and offset binary	**Two's complement**

if mode = rounding:
    $y = x + \frac{Q}{2}$
else if mode = truncation:
    $y = x$
for each x to be converted, do:
    initialize $\mathbf{b} = [0, 0, \ldots, 0]$
    for $\mathbf{i} = 1, 2, \ldots, B$ do:
        $b_i = 1$
        $x_Q = \text{dac}(\mathbf{b}, B, R)$
        $b_i = u(y - x_Q)$
	if mode = rounding:
	    $y = x + \frac{Q}{2}$
	else if mode = truncation:
	    $y = x$
	for each x to be converted, do:
	    initialize $\mathbf{b} = [0, 0, \ldots, 0]$
	    $b_1 = 1 - u(y)$
	    for $\mathbf{i} = 2, 3, \ldots, B$ do:
	        $b_i = 1$
	        $x_Q = \text{dac}(\mathbf{b}, B, R)$
	        $b_i = u(y - x_Q)$

3.5 Analog and digital dither S84-86

Dither is a small white noise signal that is added to the input signal befor quantization. The variance of the noise is σ_v^2 and the variance of the quantization is σ_e^2. The total variance is :

$$\sigma_\epsilon^2 = \sigma_e^2 + \sigma_v^2 = \frac{1}{12}Q^2 + \sigma_v^2$$

$$\sigma_\epsilon^2 = \begin{cases} \frac{Q^2}{12} & \text{undithered} \\ \frac{2Q^2}{12} & \text{rectangular dither} \\ \frac{3Q^2}{12} & \text{triangular dither} \\ \frac{4Q^2}{12} & \text{gaussian dither} \end{cases}$$

Goal of the dither is to eliminate quantization distortion and granulation and force the quantization error to look more like white noise.

4 Discrete-Time Systems Chapter 3

4.1 Linearity and time invariance S100-102

Linearity: $x(n) = ax_1(n) + bx_2(n) \xrightarrow{H} y(n) = ay_1(n) + by_2(n)$

Time invariance: If $x(n) \xrightarrow{H} y(n)$ then $x(n+\delta) \xrightarrow{H} y(n+\delta)$

testing: $x_D(n) = x(n-D)$ and $y_D(n) = y(n-D)$

4.2 Impulse response S103-105

LTI form: $y(n) = \sum_m x(m)h(n-m)$

direct form: $y(n) = \sum_m h(m)x(n-m)$

4.3 Finite and Infinite Impulse Response filters S105,106

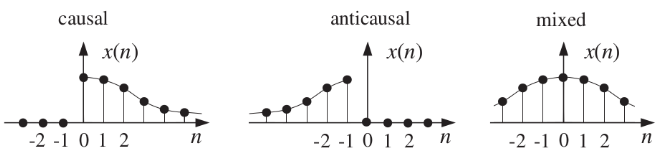
M	filter order	
h	filter impulse response	$\{h_0, h_1, h_2, \dots, h_M, 0, 0, \dots\}$
L_h	length of h	$L_h = M + 1$
FIR	FIR filtering equation	$y(n) = \sum_{m=0}^{M} h(m)x(n-m)$
IIR	IIR filtering equation	$y(n) = \sum_{m=0}^{\infty} h(m)x(n-m)$

4.4 Causality S112

causal right sided signals, they are non-zero for $n >= 0$

anticausal left sided signals, they are non-zero for $n <= -1$

mixed signals double-sided signals



4.4.1 Anticausal to Causal S113

Delay the system by D to move the negative time from $n = -D$ to 0

$$
\begin{aligned}
h_D(n) &= h(n-D) \\
y(n) &= \sum_m h(m)x(n-m) \\
y_D(n) &= \sum_m h_D(m)x(n-m) = \sum_m h(m-D)x(n-m) \\
&\xrightarrow{m:=k+D} \sum_k h(k)x(n-k-D) = y(n-D)
\end{aligned}
$$

4.5 Stability S115

Stability Condition $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

An LTI system is stable, if a bounded input can only generate bounded outputs. Always prefer stability over causality!

5 FIR Filtering and Convolution Chapter 4

5.1 Block Processing Methods

5.1.1 Convolution S121

duration of data record	$T_L = LT$	Signal Samples	$L = T_L f_s$
$x(n)$	$n = 0, 1, \dots, L-1$	sampling Time	T

$y(n)$	direct and LTI forms of convolution	$y(n) = \sum_m h(m)x(n-m) = \sum_m x(m)h(n-m)$
$y(n)$	convolution table form	$y(n) = \sum_{\substack{i,j \\ i+j=n}} h(i)x(j)$

5.1.2 Direct Form S123

h	$h = [h_0, h_1, \dots, h_M]$
L_h	$L_h = M + 1$
L_x	$L_x = L$
L_y	$L_y = L + M = L_x + L_h - 1$
$y(n)$	$y(n) = \sum_{m=max(0,n-L+1)}^{min(n,M)} h(m)x(n-m)$

5.1.3 Convolution Table S126

	x_0	x_1	x_2	x_3	x_4
h_0	h_0x_0	h_0x_1	h_0x_2	h_0x_3	h_0x_4
h_1	h_1x_0	h_1x_1	h_1x_2	h_1x_3	h_1x_4
h_2	h_2x_0	h_2x_1	h_2x_2	h_2x_3	h_2x_4
h_3	h_3x_0	h_3x_1	h_3x_2	h_3x_3	h_3x_4

y_0 y_1 y_2 y_3 y_4 y_5 y_6 y_7

$y = [y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7]$

5.1.4 LTI Form S127

	h_0	h_1	h_2	h_3	0	0	0	0
x_0	x_0h_0	x_0h_1	x_0h_2	x_0h_3	0	0	0	0
x_1	0	x_1h_0	x_1h_1	x_1h_2	x_1h_3	0	0	0
x_2	0	0	x_2h_0	x_2h_1	x_2h_2	x_2h_3	0	0
x_3	0	0	0	x_3h_0	x_3h_1	x_3h_2	x_3h_3	0
x_4	0	0	0	0	x_4h_0	x_4h_1	x_4h_2	x_4h_3
y_n	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

$$y(n) = \sum_{m=max(0,n-M)}^{min(n,L-1)} x(m)h(n-m)$$

5.1.5 Matrix Form S129

The convolutional equations can also be written in the linear matrix form:

$$y = Hx \qquad \text{or} \qquad y = Xh$$

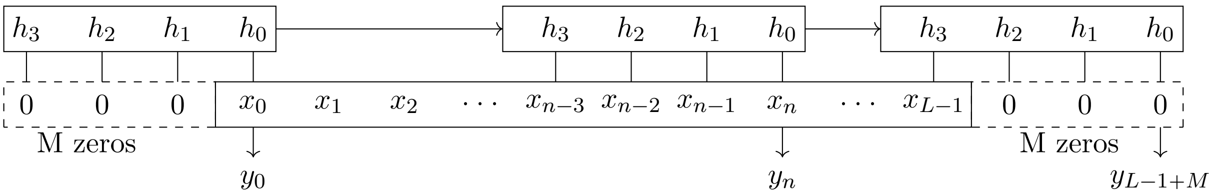
where H (Toeplitz Matrix) is built out of the filter's impulse response h or the signal matrix X is built out of the input signal. The filter matrix H, respectively the signal matrix X, must be rectangular with dimensions

$$\underbrace{L_y \times L_x = (L+M) \times L}_{\text{dimension of H}} \quad \text{or} \quad \underbrace{L_y \times L_h = (L+M) \times (M+1)}_{\text{dimension of X}}$$

Example:

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 & 0 \\ h_3 & h_2 & h_1 & h_0 & 0 \\ 0 & h_3 & h_2 & h_1 & h_0 \\ 0 & 0 & h_3 & h_2 & h_1 \\ 0 & 0 & 0 & h_3 & h_2 \\ 0 & 0 & 0 & 0 & h_3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = Hx$$

5.1.6 Flip-and-Slide Form S131



$$y(n) = h_0 x_n + h_1 x_{n-1} + \cdots + h_M x_{n-M}$$

5.1.7 Overlap-Add Form S143,144

1. Divide input x into smaller blocks $x_0, x_1, \ldots$ of length L. If the input is not long enough for a last complete block, the last block is filled up with zeros
2. Calculate the output of the convolution of block x_0 with h, resulting in y_0
3. Repeat step 2 for all blocks, resulting in $y_1, y_2, \ldots$
4. Add $y_0, y_1, \ldots$ up using the following table. y_{n+1} is always moved to the right with an offset of L compared to y_n.

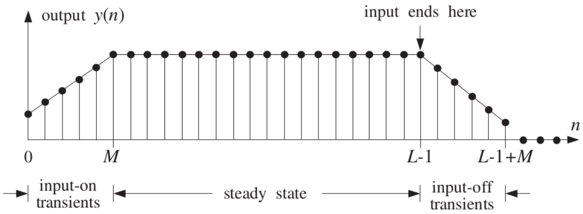
example: with $L = 3$ and $M = 3$

$$y_0 = h * x_0 \qquad y_1 = h * x_1 \qquad y_2 = h * x_2 \qquad x = \underbrace{n_0, n_1, n_2}_{x_0}, \underbrace{n_3, n_4, n_5}_{x_1}, \underbrace{n_6, n_7, n_8}_{x_2}$$

n	0	1	2	3	4	5	6	7	8	9	10	11
y_0	$y_{0,0}$	$y_{0,1}$	$y_{0,2}$	$y_{0,3}$	$y_{0,4}$	$y_{0,5}$						
y_1				$y_{1,0}$	$y_{1,1}$	$y_{1,2}$	$y_{1,3}$	$y_{1,4}$	$y_{1,5}$			
y_2							$y_{2,0}$	$y_{2,1}$	$y_{2,2}$	$y_{2,3}$	$y_{2,4}$	$y_{2,5}$
y	Σn_0	Σn_1	Σn_2	Σn_3	Σn_4	Σn_5	Σn_6	Σn_7	Σn_8	Σn_9	Σn_{10}	Σn_{11}

5.1.8 Transient and Steady-State Behaviour S132,133

input-on transient	$0 \leqslant n < M$
steady state	$M \leqslant n \leqslant L-1$
input-off transient	$L-1 < n \leqslant L-1+M$



Therefore, the direct form takes the following different forms depending on the value of the output index n:

$$y_n = \begin{cases} \sum\limits_{m=0}^{n} h_m x_{n-m} & \text{if } 0 \leqslant n < M & \text{input-on} \\ \sum\limits_{m=0}^{M} h_m x_{n-m} & \text{if } M \leqslant n \leqslant L-1 & \text{steady state} \\ \sum\limits_{m=n-L+1}^{M} h_m x_{n-m} & \text{if } L-1 < n \leqslant L-1+M & \text{input-off} \end{cases}$$

The DC gain of a stable filter is the steady-state value to which the output converges when the input is a unit step

$$y_{dc} = \sum_{m} h(m) \qquad y_{dc} = \sum_{m=0}^{\infty} h(m)$$

5.1.9 Convolution of Infinite Sequences S134

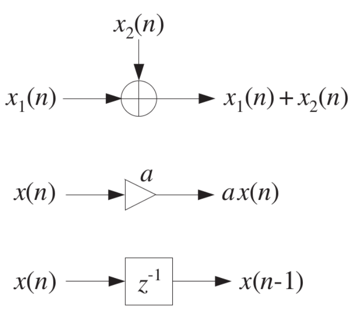
Three cases:

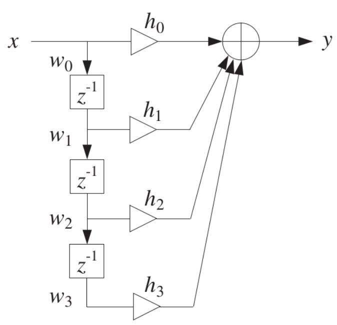
1. Infinite filter, finite input; i.e., $M = \infty$, $L < \infty$
2. Finite filter, infinite input; i.e., $M < \infty$, $L = \infty$
3. Infinite filter, infinite input; i.e., $M = \infty$, $L = \infty$

Therefore, the direct form takes the following different forms (See also 6.2 Region of Convergence (ROC) S186):

$$y_n = \begin{cases} \sum\limits_{m=max(0,n-L+1)}^{n} h_m x_{n-m} & \text{if } M = \infty,\ L < \infty \\ \sum\limits_{m=0}^{min(n,M)} h_m x_{n-m} & \text{if } M < \infty,\ L = \infty \\ \sum\limits_{m=0}^{n} h_m x_{n-m} & \text{if } M = \infty,\ L = \infty \end{cases}$$

5.2 Sample Processing Methods S146





5.2.1 Pure Delays S147-151

5.2.2 FIR Filtering in Direct Form S152-156

5.2.3 Hardware realizations and circular buffers S162

6 z-Transform Chapter 5

6.1 Basic Properties S183

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

linearity	$a_1 x_1(n) + a_2 x_2(n)$	$\xrightarrow{Z}$	$a_1 X_1(z) + a_2 X_2(z)$
delay	$x(n-D)$	$\xrightarrow{Z}$	$z^{-D} X(z)$
convolution	$y(n) = h(n) * x(n)$	$\xrightarrow{Z}$	$Y(z) = H(z) X(z)$
modulation	$a^n g(n)$	$\xrightarrow{Z}$	$G(\frac{z}{a})$
time inversion	$g(-n)$	$\xrightarrow{Z}$	$G(z^{-1})$

6.2 Region of Convergence (ROC) S186

$$\{z \in \mathbb{C} \quad | \quad X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \neq \pm\infty\}$$

infinite geometric series 1	$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$	for $\lvert x\rvert < 1$
infinite geometric series 2	$x + x^2 + x^3 + \ldots = \sum_{m=1}^{\infty} x^m = \frac{x}{1-x}$	for $\lvert x\rvert < 1$

If there is no ROC specified, we assume that the system is causal.

6.3 Causality and Stability S193

causal signals	ROC $\lvert z\rvert > \max_i \lvert p_i\rvert$
mixed signals	ROC $\min_i \lvert p_i\rvert < \lvert z\rvert < \max_i \lvert p_i\rvert$
anticausal signals	ROC $\lvert z\rvert < \min_i \lvert p_i\rvert$
stable signals	$\{z \big\rvert (\lvert z\rvert = 1)\} \in$ ROC

For a signal or system to be **simultaneously stable and causal**, it is necessary that all its poles lie strictly **inside** the unit circle in the z-plane.

$$1 > \max_i |p_i|$$

A signal or system can also be **simultaneously stable and anticausal**, but in this case all its poles must lie strictly **outside** the unit circle.

$$1 < \min_i |p_i|$$

Marginally stable signals have poles, that fall exactly onto the unit circle!

6.4 Frequency Spectrum S196-210

$$z = e^{j\omega} \qquad \omega = \frac{2\pi f}{f_s}$$

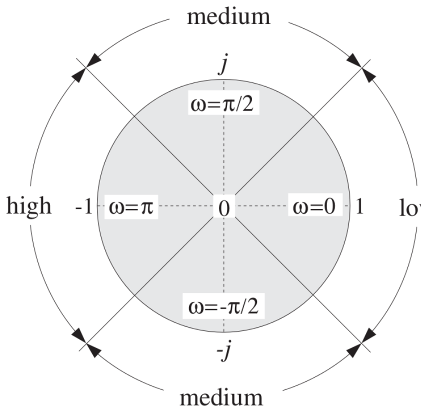
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad \text{(DTFT)}$$

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \qquad \text{(frequency response)}$$

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

$$-\pi \leqslant \omega \leqslant \pi \qquad \text{nyquist interval}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \qquad \text{(inverse DTFT)}$$



Another useful relationship is Parseval's equation, which relates the total energy of a sequence to its spectrum:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \qquad \text{(Parseval)}$$

For real valued discrete time sequences:
$X(\omega)^* = X(-\omega)$
$|X(\omega)| = |X(-\omega)|$
$argX(\omega) = -argX(-\omega)$

Some DTFT-Transforms:

$\delta[n]$	$X_{2\pi}(\omega) = 1$
$\delta[n-M]$	$X_{2\pi}(\omega) = e^{-i\omega M}$
$u[n]$	$X(\omega) = \frac{1}{1-e^{-i\omega}} + \pi \cdot \delta(\omega)$
$e^{-i\omega_0 n}$	$X(\omega) = 2\pi \cdot \delta(\omega+\omega_0), -\pi \leqslant \omega_0 < \pi$
$\cos(\omega_0 n)$	$X(\omega) = \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)], -\pi < \omega_0 < \pi$
$\sin(\omega_0 n)$	$X(\omega) = -j\pi\delta(\omega-\omega_0) + j\pi\delta(\omega+\omega_0)$

Some z-Transforms:

x(n)	X(z)	ROC
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$(-1)^n u(n)$	$\frac{1}{1+z^{-1}}$	$\lvert z\rvert > 1$
$-(-1)^n u(-n-1)$	$\frac{1}{1+z^{-1}}$	$\lvert z\rvert < 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$ (causal)
$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$ (anticausal)
$u(n)e^{\alpha n}$	$\frac{1}{1-e^{\alpha}z^{-1}}$	$\lvert z\rvert > \lvert e^{\alpha}\rvert$
$u(n)\cos(\omega n)$	$\frac{1-\cos(\omega)z^{-1}}{1-2\cos(\omega)z^{-1}+z^{-2}} = \frac{1}{2}\left[\frac{1}{1-e^{j\omega}z^{-1}} + \frac{1}{1-e^{-j\omega}z^{-1}}\right]$	$\lvert z\rvert > 1$
$u(n)\sin(\omega n)$	$\frac{\sin(\omega)z^{-1}}{1-2\cos(\omega)z^{-1}+z^{-2}} = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega}z^{-1}} - \frac{1}{1-e^{-j\omega}z^{-1}}\right]$	$\lvert z\rvert > 1$
$A\delta(n)$	A	all z
$A\delta(n-D)$	Az^{-D}	$z \neq 0$

6.5 Inverse z-Transform S202-204

$$X(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{(1-p_1z^{-1})(1-p_2z^{-1})\dots(1-p_Mz^{-1})} = A_0 + \frac{A_1}{1-p_1z^{-1}} + \dots + \frac{A_M}{1-p_Mz^{-1}}$$

with $A_0 = X(z)|_{z=0}$ otherwise $z = p_i$

Partial fraction expansion S203**:** for Order of $N(z) \leqslant$ Order of $D(z)$

$$A_i = [(1-p_iz^{-1})X(z)]_{z=p_i} = \left[\frac{N(z)}{\prod\limits_{j\neq i}(1-p_jz^{-1})}\right]_{z=p_i}$$

for $A_0 \to z = 0$

Euler:

$$\cos(\alpha) = \frac{e^{j\alpha}+e^{-j\alpha}}{2} \qquad \sin(\alpha) = \frac{e^{j\alpha}-e^{-j\alpha}}{2j} \qquad e^{\pm jn\omega} = \cos n\omega \pm j\sin n\omega$$

$$e^{j\frac{\pi}{2}n} = j^n \qquad e^{-j\frac{\pi}{2}n} = (-j)^n \qquad e^{jn\pi} = e^{jn\pi} = (-1)^n$$

$$\sqrt[n]{1} = e^{j\cdot 2\pi\frac{k}{n}} \quad k \in [1, n]$$

Complex valued poles: $(1-ae^{j\omega}z^{-1})(1-ae^{-j\omega}z^{-1}) = 1 - 2a\cos(\omega)z^{-1} + a^2z^{-2}$

7 Transfer Functions Chapter 6

7.1 Equation description S215,216

transfer function:	$H(z) = \frac{5+2z^{-1}}{1-0.8z^{-1}}$
impulse response:	$h(n) = -2.5\delta(n) + 7.5(0.8)^n u(n)$
impulse response coefficient:	$h(n) = [1 \quad 0 \quad 1 \quad 0]$
difference equation:	$h(n) = 0.8h(n-1) + 5\delta(n) + 2\delta(n-1)$
I/O difference equation:	$y(n) = 0.8y(n-1) + 5x(n) + 2x(n-1)$
frequency response:	$H(\omega) = \frac{5+2e^{-j\omega}}{1-0.8e^{-j\omega}}$
magnitude response:	$\lvert H(\omega)\rvert = \sqrt{H(\omega)\cdot H(\omega)^*}$

7.2 IIR-Form: S223,224

$$H(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1z^{-1} + b_2z^{-2} + \ldots + b_Nz^{-N}}{1 + a_1z^{-1} + a_2z^{-2} + \ldots + a_Mz^{-M}} \qquad a_0 = 1 \text{ normalize to } 1$$

if $D(z) = 1$, the IIR Form can be reduced to a FIR Filter:

$$H(z) = N(z) = b_0 + b_1z^{-1} + b_2z^{-2} + \ldots + b_Nz^{-N}$$

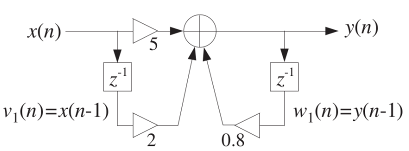
Example: find $H(z)$ for $h(n) = [1,3,4,5]$

$$H(z) = 1 + 3z^{-1} + 4z^{-2} + 5z^{-3}$$

Example: $y(n) = 0.25\cdot y(n-2) + x(n)$ (I/O difference equation)

$$Y(z) = 0.25z^{-2}Y(z) + X(z)$$

direct form S217

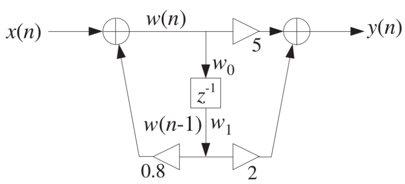


$H(z) = \frac{Y(z)}{X(z)} = \frac{5+2z^{-1}}{1-0.8z^{-1}}$

$Y(z)(1-0.8z^{-1}) = X(z)(5+2z^{-1})$

$Y(z) = 5X(z) + 2z^{-1}X(z) + 0.8z^{-1}Y(z)$

canonical form S220



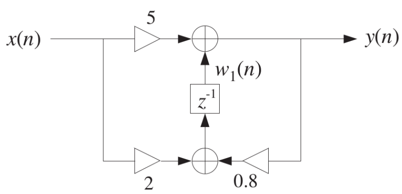
$H(z) = \frac{Y(z)}{X(z)} = \frac{5+2z^{-1}}{1-0.8z^{-1}}$

$W(z) = \frac{1}{1-0.8z^{-1}}X(z)$

$W(z) = X(z) + 0.8z^{-1}W(z)$

$Y(z) = (5+2z^{-1})W(z)$

transposed form S222



Transposition rules:
replace adders by nodes,
nodes by adders,
reversing all flows and
exchanging input with output

7.3 Steady state response S229-232

$$\cos(\omega_0 n) \xrightarrow{H} |H(\omega_0)| \cos(\omega_0 n + \arg(H(\omega_0))) \qquad \sin(\omega_0 n) \xrightarrow{H} |H(\omega_0)| \sin(\omega_0 n + \arg(H(\omega_0)))$$

$$e^{j\omega_0 n} \xrightarrow{H} H(\omega_0) e^{j\omega_0 n} = |H(\omega_0)| e^{j\omega_0 n + j\arg H(\omega_0)}$$

phase delay	$d(\omega) = -\frac{\arg(H(\omega))}{\omega}$ $\quad \arg H(\omega) = -\omega d(\omega)$
group delay	$d_g(\omega) = -\frac{d}{d\omega}(\arg(H(\omega)))$

7.4 Transient Response S232

Input sine: $x(n) = e^{j\omega_0 n} \cdot u(n) \Rightarrow X(z) = \frac{1}{1-e^{j\omega_0} z^{-1}}$ ROC $|z| > |e^{j\omega_0}| = 1$

time until the output is stable: $n_{eff} = \frac{\ln \epsilon}{\ln \rho}$ [sample] typ. $\epsilon = 1\%$

$\rho = \max_i |p_i|$ $\quad |p_i|$ is the magnitude of the pole

time constant: $\tau = n_{eff} \cdot T$

$H(\omega) = \frac{b}{1-ae^{-j\omega}} \Rightarrow |H(\omega)| = \frac{b}{\sqrt{1-2a\cos(\omega)+a^2}}$

$|1 - ae^{-j\omega}| = \sqrt{1 - 2a\cos(\omega) + a^2}$

7.5 Unit Step Response S239

DC-Gain: $H(0) = H(z)|_{z=1} = \sum_{n=0}^{\infty} h(n)$

AC-Gain: $H(\pi) = H(z)|_{z=-1} = \sum_{n=0}^{\infty} (-1)^n h(n)$

7.6 Pole/Zero Design S242-258

7.6.1 First-Order Filters S242

Transfer function: $H(z) = \frac{G(1+bz^{-1})}{1-az^{-1}}$ $\quad a = \epsilon^{1/n_{eff}}$

b can be calculated from: $\frac{H(\pi)}{H(0)} = \frac{\text{AC Gain}}{\text{DC Gain}} \begin{cases} > 1 & HP \\ < 1 & LP \end{cases}$ $\quad a, b \leqslant 1$ and $G =$ gain

7.6.2 2 pole conjugate filter S244-246

poles: $p = Re^{j\omega_0}$ and $p^* = Re^{-j\omega_0}$

Transfer function: $H(z) = \frac{G}{(1-Re^{-j\omega_0}z^{-1})(1-Re^{j\omega_0}z^{-1})} = \frac{G}{1+a_1z^{-1}+a_2z^{-2}}$

Parameter: $a_1 = -2R\cos(\omega_0)$; $a_2 = R^2$

filter impulse Response $h(n) = \frac{G}{\sin(\omega_0)}R^n \sin(\omega_0 n + \omega_0)$

$G = (1-R)\sqrt{1-2R\cos(2\omega_0)+R^2}$ only for $|H(\omega_0)| = 1$

3dB width $\Delta\omega \simeq 2(1-R)$ =: R is the magnitude of the pole

full width at half maximum of the magnitude squared response $|H(\omega)|^2 = \frac{1}{2}|H(\omega_0)|^2 = \frac{1}{2}$

7.6.3 2 pole 2 zero filter S228,249

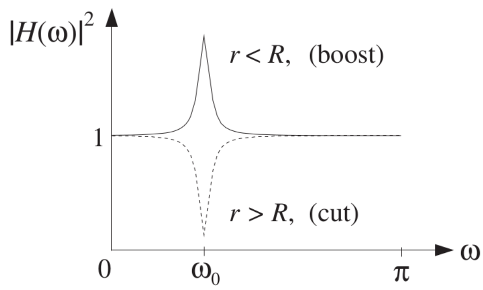
poles: $p = Re^{j\omega_0}$ and $p^* = Re^{-j\omega_0}$

zeros: $z_1 = re^{j\omega_0}$ and $z_1^* = re^{-j\omega_0}$

Transfer function: $H(z) = \frac{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}{(1-Re^{j\omega_0}z^{-1})(1-Re^{-j\omega_0}z^{-1})} = \frac{1+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$

Parameter: $a_1 = -2R\cos(\omega_0)$; $a_2 = R^2$

$b_1 = -2r\cos(\omega_0)$; $b_2 = r^2$



7.6.4 Notch and Comb Filter S249-251

The zeros of the filters located on the unit circle and the poles are in the unit circle.

Transfer function: $H(z) = \frac{N(z)}{D(z)}$

$D(z) = N(\rho^{-1}z) = \prod_{i=1}^{M}(1-e^{j\omega_i}\rho z^{-1})$ ρ = Radius

Notch filter: $N(z) = \prod_{i=1}^{M}(1-e^{j\omega_i}z^{-1})$

$H(z) = \frac{N(z)}{(N\rho^{-1}z)} = \frac{1+b_1z^{-1}+b_2z^{-2}+...+b_Mz^{-M}}{1+\rho b_1z^{-1}+\rho^2b_2z^{-2}+...+\rho^Mb_Mz^{-M}}$ with $0 < |\rho| < 1$

$a_i = \rho^i b_i$ mit $i = 1, 2, \dots, M$

Comb filter: $N(z) = \prod_{i=1}^{M}(1-e^{j\omega_i}rz^{-1})$

$H(z) = \frac{N(r^{-1}z)}{N(\rho^{-1}z)} = \frac{1+rb_1z^{-1}+...+r^Mb_Mz^{-M}}{1-\rho b_1z^{-1}+...+\rho^Mb_Mz^{-M}}$ with $|r| < |\rho| < 1$

7.7 Deconvolution, Inverse Filters and Stability S254-259

$$H_{inv}(z) = \frac{1}{H(z)} = \frac{D(z)}{N(z)}$$

Deconvolution: $x(n) = h_{inv}(n) * y(n)$

$$\hat{x}(n) = h_{inv}(n) * y(n) = x(n) + \hat{\nu}(n)$$

Filtered noise: $\hat{\nu}(n) = h_{inv}(n) * \nu(n)$

$$\tilde{h}_{inv}(n) = \begin{cases} h_{inv}(n) & \text{if } n \geqslant -D \\ 0 & \text{if } n < -D \end{cases}$$

Because $H_{inv}(z)$ can have poles outside the unit circle, the stable inverse z-transform $h_{inv}(n)$ will necessarily be anticausal. To make a causal system, by clipping of the anticausal tail of the impulse response by a time $n = -D$ and delayed by D time units.

$y(n)$ is bounded by some maximal value $|y(n)| \leqslant A$, so the deconvolution error can be calculated by
$|x(n) - \tilde{x}(n)| \leqslant A \sum_{m=-\infty}^{-D-1} |h_{inv}(m)|$

8 Idiotenseite

8.1 Funktionswerte für Winkelargumente

deg	rad	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

deg	rad	sin	cos
90°	$\frac{\pi}{2}$	1	0
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

deg	rad	sin	cos
180°	π	0	-1
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

deg	rad	sin	cos
270°	$\frac{3\pi}{2}$	-1	0
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

8.2 Periodizität

$\cos(a + k \cdot 2\pi) = \cos(a) \qquad \sin(a + k \cdot 2\pi) = \sin(a) \qquad (k \in \mathbb{Z})$

8.3 Additionstheoreme

$$\sin(a \pm b) = \sin(a) \cdot \cos(b) \pm \cos(a) \cdot \sin(b)$$
$$\cos(a \pm b) = \cos(a) \cdot \cos(b) \mp \sin(a) \cdot \sin(b)$$
$$\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a) \cdot \tan(b)}$$

8.4 Doppel- und Halbwinkel

$$\sin(2a) = 2\sin(a)\cos(a)$$
$$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$$
$$\cos^2\left(\tfrac{a}{2}\right) = \tfrac{1+\cos(a)}{2} \qquad \sin^2\left(\frac{a}{2}\right) = \tfrac{1-\cos(a)}{2}$$

8.5 Summe und Differenz

$$\sin(a) + \sin(b) = 2 \cdot \sin\left(\tfrac{a+b}{2}\right) \cdot \cos\left(\tfrac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2 \cdot \sin\left(\tfrac{a-b}{2}\right) \cdot \cos\left(\tfrac{a+b}{2}\right)$$
$$\cos(a) + \cos(b) = 2 \cdot \cos\left(\tfrac{a+b}{2}\right) \cdot \cos\left(\tfrac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2 \cdot \sin\left(\tfrac{a+b}{2}\right) \cdot \sin\left(\tfrac{a-b}{2}\right)$$
$$\tan(a) \pm \tan(b) = \frac{\sin(a \pm b)}{\cos(a)\cos(b)}$$

8.6 Produkte

$$\sin(a)\sin(b) = \tfrac{1}{2}(\cos(a-b) - \cos(a+b))$$
$$\cos(a)\cos(b) = \tfrac{1}{2}(\cos(a-b) + \cos(a+b))$$
$$\sin(a)\cos(b) = \tfrac{1}{2}(\sin(a-b) + \sin(a+b))$$

8.7 Reihenentwicklungen

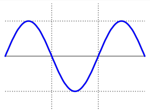
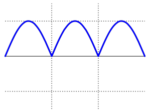
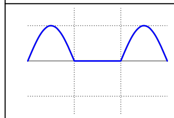
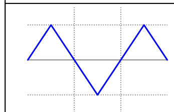
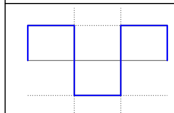
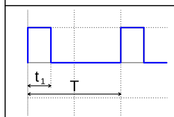
Geometrische Reihe $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x| < 1$

$\displaystyle\sum_{k=0}^{\infty} k\,x^k = x\sum_{k=1}^{\infty} k\,x^{k-1} = \frac{x}{(1-x)^2} \qquad x \neq 1$

Binominalreihe $\displaystyle\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha \qquad x \in (-1, 1)$

E-Funktion $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$

8.8 Eigenschaften unterschiedlicher Schwingungsformen

Form	Funktion	Gleichrichtwert	Formfaktor	Effektivwert	Scheitelfaktor	X_0	X^2	**var(X)**
Formel		$\overline{\lvert x\rvert} = \frac{1}{T}\int_0^T \lvert x(t)\rvert \, dt$	$\frac{X}{\overline{\lvert x\rvert}}$	$X = \sqrt{X^2} = \sqrt{\frac{1}{T}\int\limits_{t_0}^{t_0+T} x^2(t)dt}$	$k_s = \frac{X_{\max}}{X_{\text{eff}}}$			
	$A \cdot \sin(t)$	$\frac{2}{\pi} \approx 0.637$	$\frac{\pi}{2\sqrt{2}} \approx 1.11$	$\frac{1}{\sqrt{2}} \approx 0.707$	$\sqrt{2} \approx 1.414$	0	$\frac{A^2}{2}$	$\frac{A^2}{2}$
	$A \cdot \lvert\sin(t)\rvert$	$\frac{2}{\pi} \approx 0.637$	$\frac{\pi}{2\sqrt{2}} \approx 1.11$	$\frac{1}{\sqrt{2}} \approx 0.707$	$\sqrt{2} \approx 1.414$	$\frac{2A}{\pi}$	$\frac{A^2}{2}$	$\frac{A^2}{2} - \frac{4A^2}{\pi^2}$
	$\begin{cases} A \cdot \sin(t) & 0 < t < \pi \\ 0 & \text{True} \end{cases}$	$\frac{1}{\pi} \approx 0.318$	$\frac{\pi}{2} \approx 1.571$	$\frac{1}{2} = 0.5$	2	$\frac{A}{\pi}$	$\frac{A^2}{4}$	$\frac{A^2}{4} - \frac{A^2}{\pi^2}$
	$A \cdot \Lambda(t)$	$\frac{1}{2} = 0.5$	$\frac{2}{\sqrt{3}} \approx 1.155$	$\frac{1}{\sqrt{3}} \approx 0.557$	$\sqrt{3} \approx 1.732$	0	$\frac{A^2}{3}$	$\frac{A^2}{3}$
	$\begin{cases} A & 0 < x < t \\ 0 & \text{True} \end{cases}$	1	1	1	1	0	A^2	A^2
DC	1	1	1	1	1	-	-	-
		$\frac{t_1}{T}$	$\sqrt{\frac{T}{t_1}}$	$\sqrt{\frac{t_1}{T}}$	$\sqrt{\frac{T}{t_1}}$	$A\frac{t}{T}$	$A^2\frac{t}{T}$	$\frac{A^2 t}{T} - \frac{A^2 t^2}{T^2}$