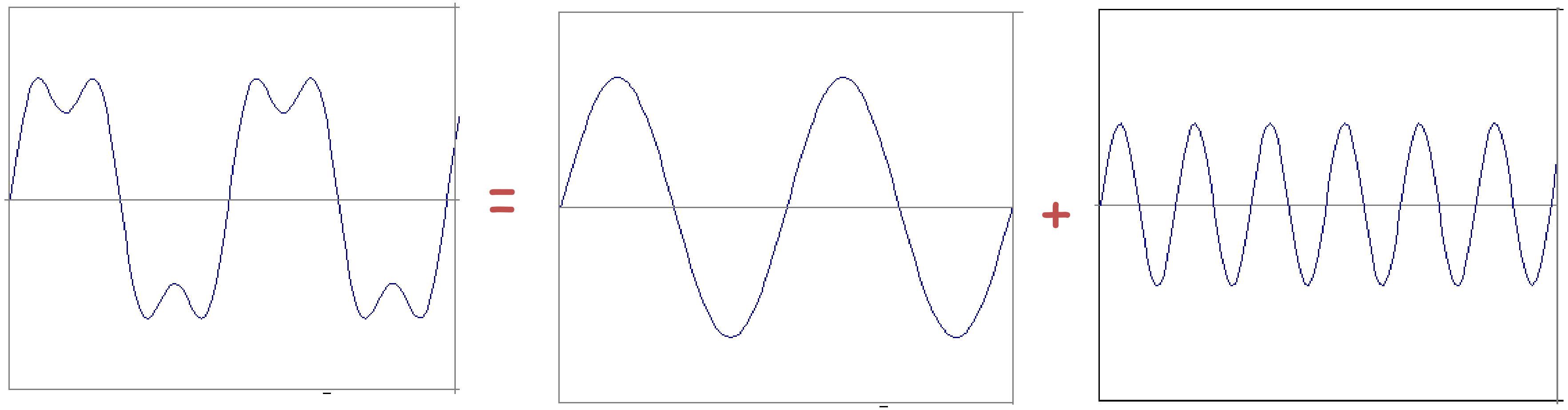
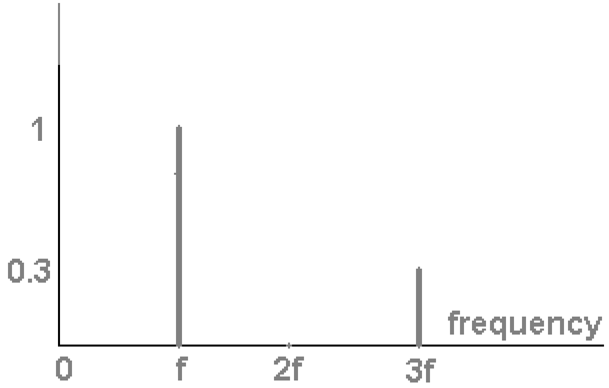
SAMPLING THEORY

Le Thanh Ha, Ph.D
Assoc. Prof. at University of Engineering and Technology,
Vietnam National University
ltha@vnu.edu.vn; lthavnu@gmail.com; 0983 692 592

Frequency Spectra

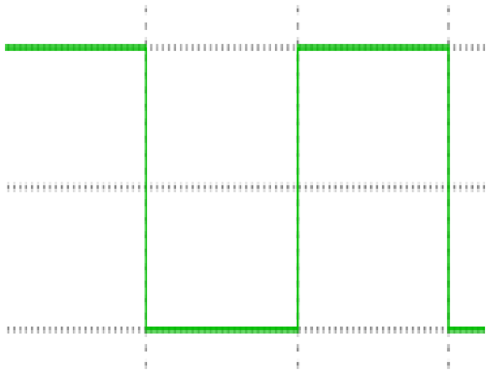
- example : $g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$



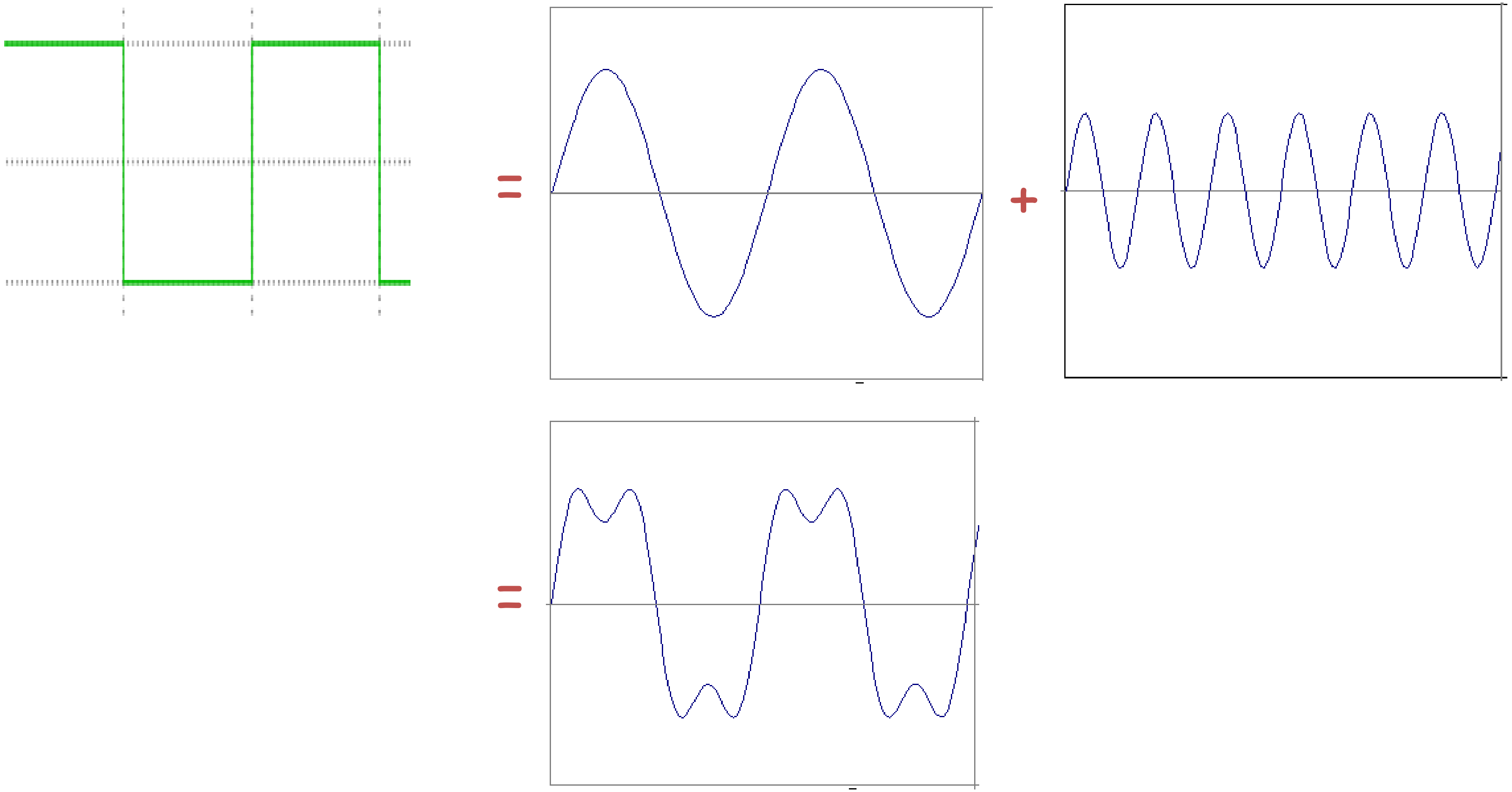


Slides: Efros

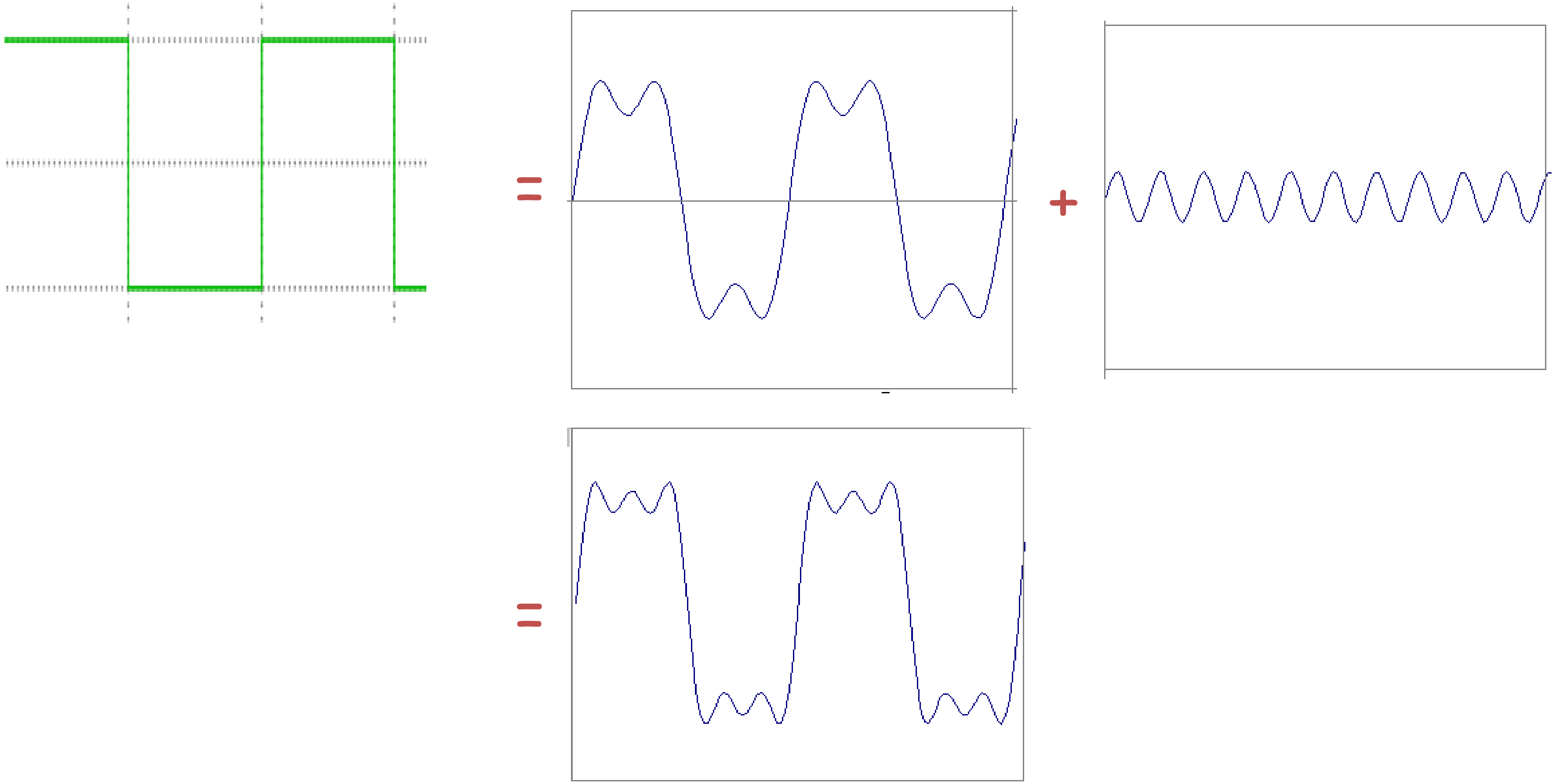
Frequency Spectra



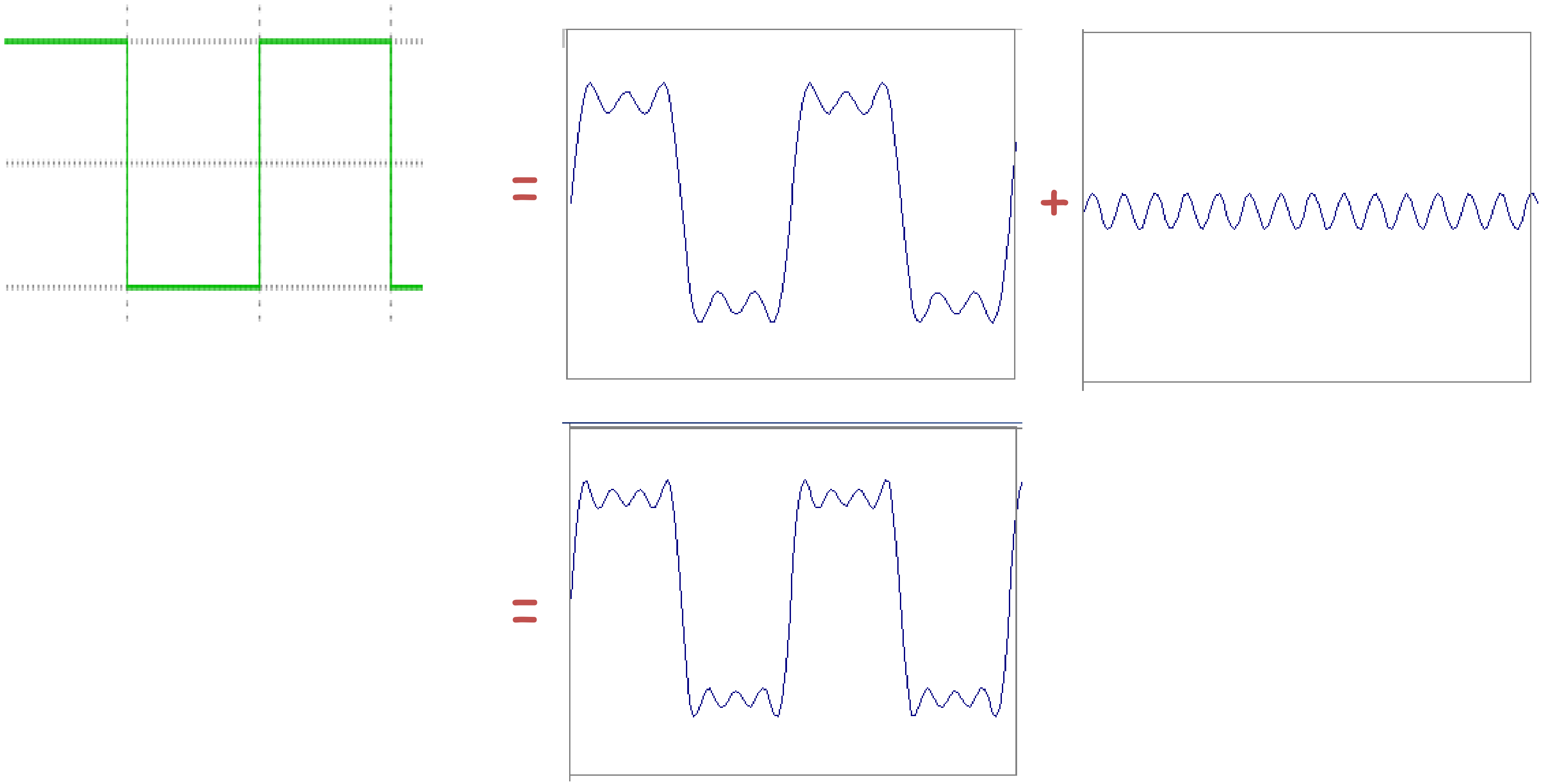
Frequency Spectra



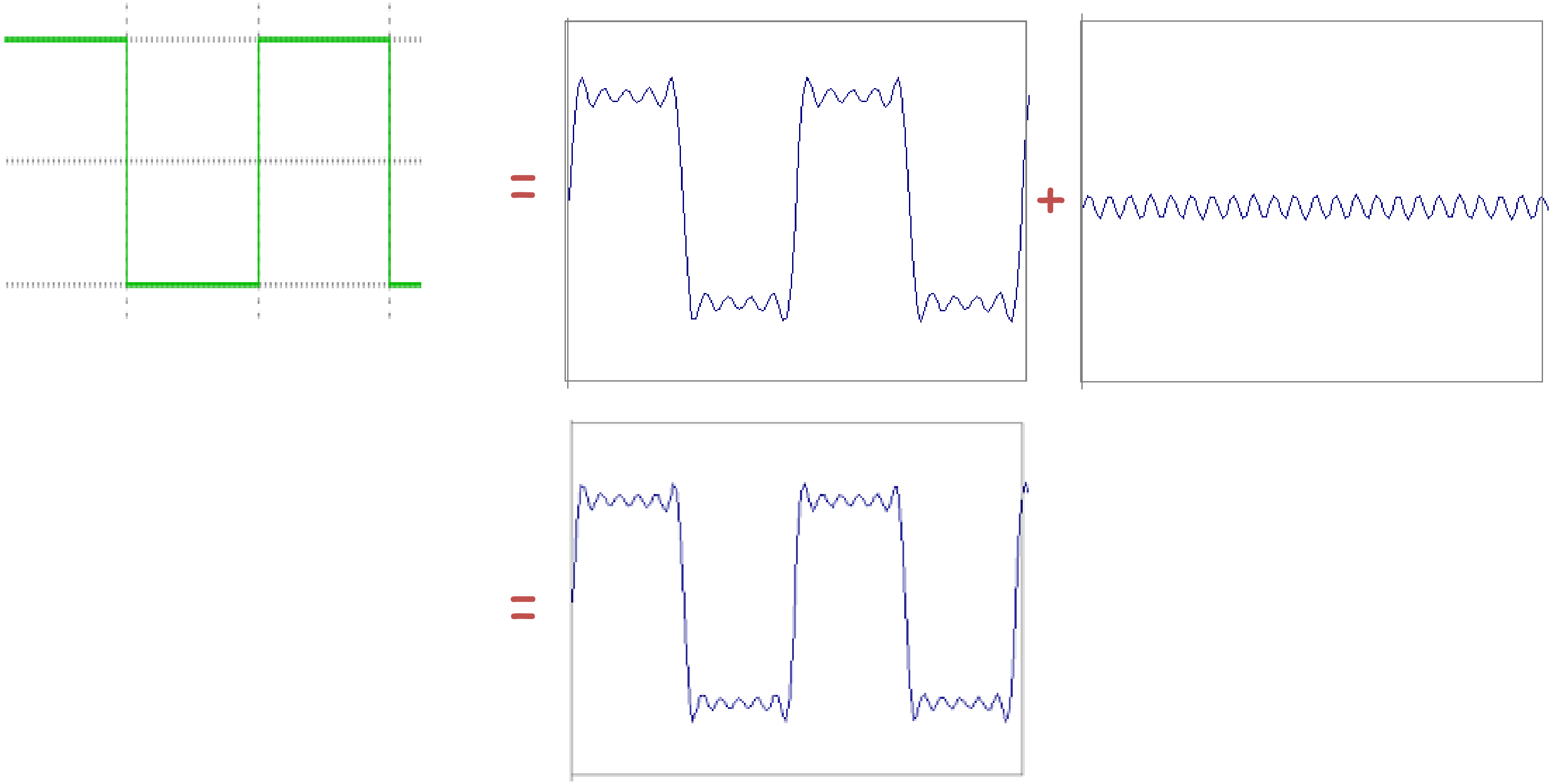
Frequency Spectra



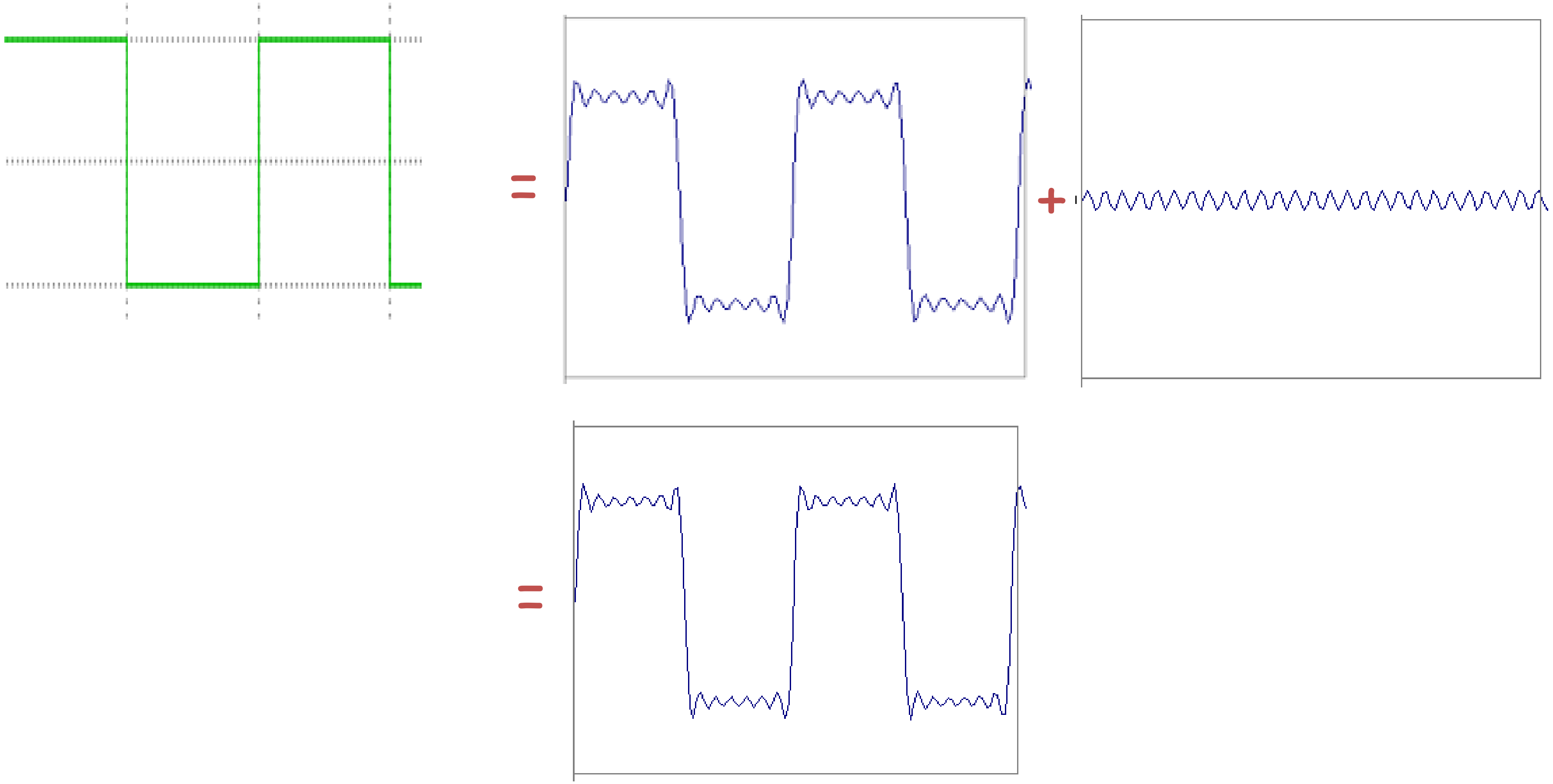
Frequency Spectra



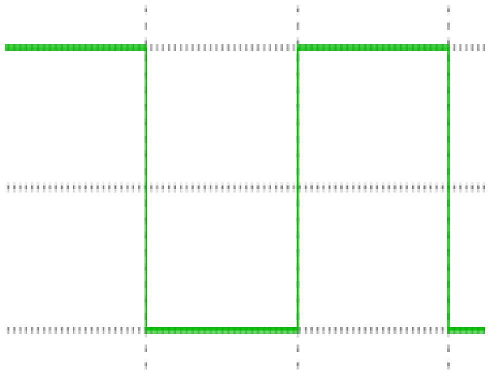
Frequency Spectra



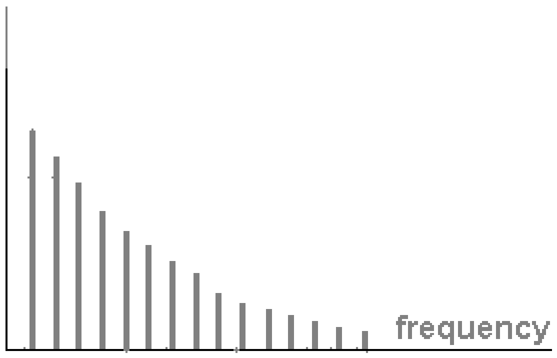
Frequency Spectra



Frequency Spectra



$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



Review: 1D Fourier Transform

A function f(x) can be represented as a weighted combination of phase-shifted sine waves

$$f(x) = \int_{-\infty}^{+\infty} F(u) e^{i2\pi ux} du$$

Inverse Fourier Transform

How to compute *F(u)*?

$$F(u) = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi ux} dx$$

Fourier Transform

Review: 1D Fourier Transform

- Trigonometric identities

$$e^{ix} = \cos(x) + i\sin(x)$$

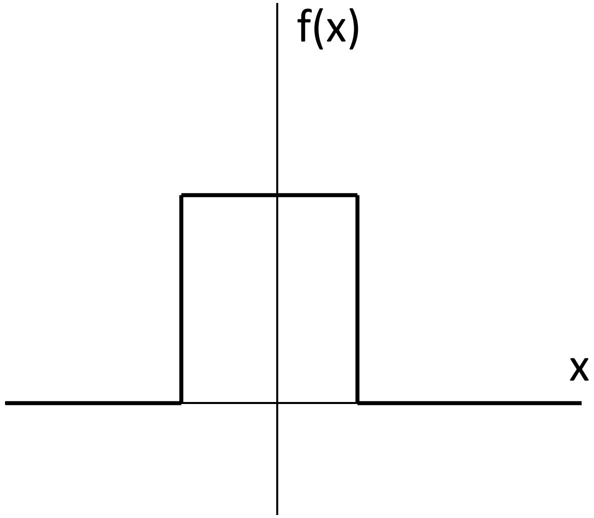
$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

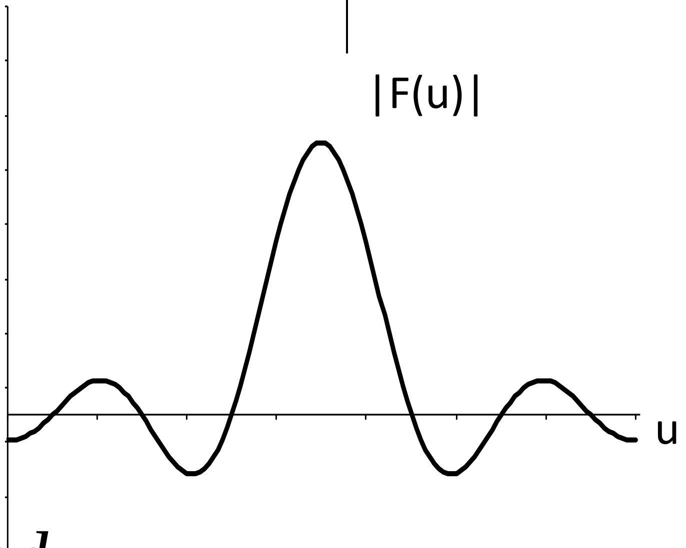
- Home work: Calculate Fourier transform of $\cos(2\pi sx)$

Review: Box Function

$$f(x) = \begin{cases} 1 & |x| \le \tfrac{1}{2} \\ 0 & |x| > \tfrac{1}{2} \end{cases}$$



$$F(u) = \frac{\sin \pi u}{\pi u} = \sin c(u)$$

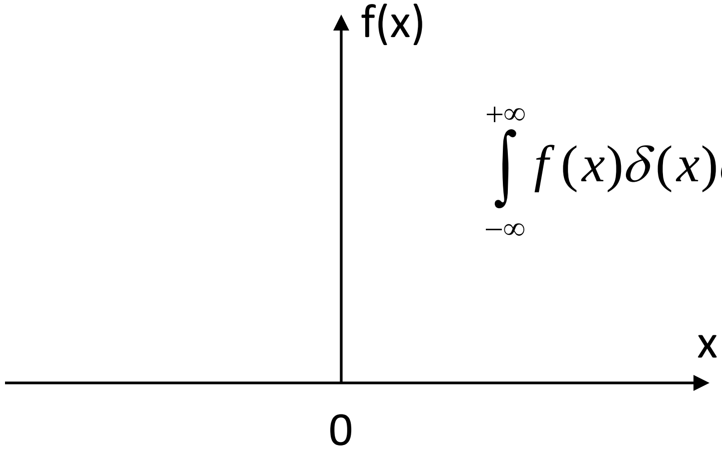


If *f(x)* is bounded, *F(u) is unbounded*

Review: Dirac Delta and its Transform

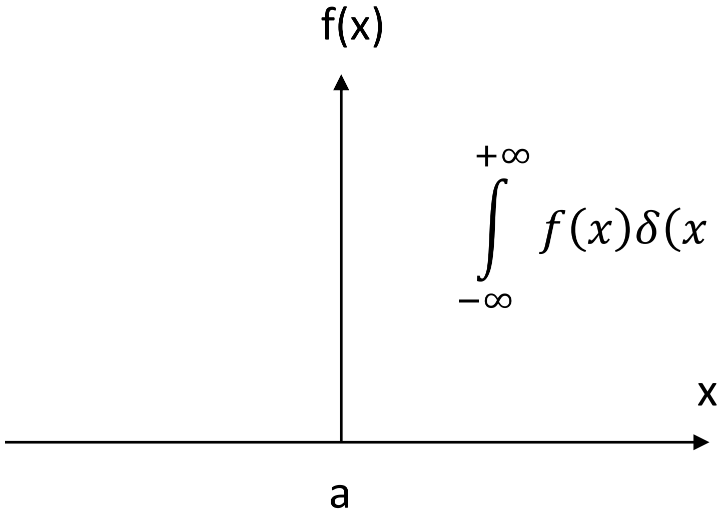
$$\delta(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$$

$$\int_{-\infty}^{+\infty}\delta(x)dx=1$$



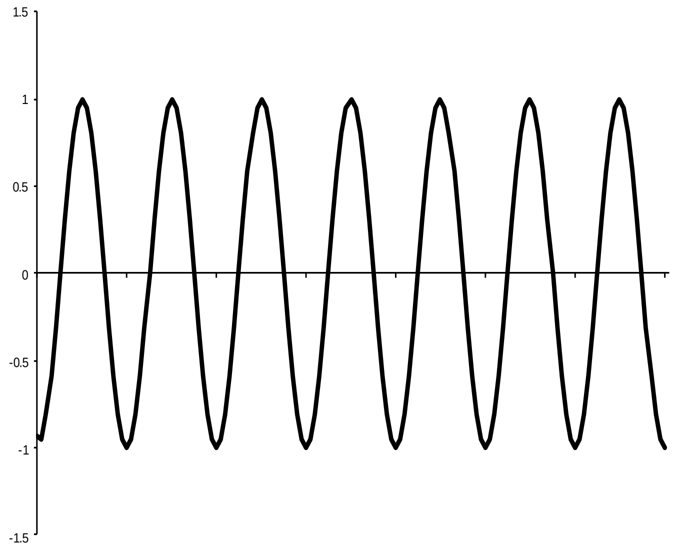
$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)$$

$$F\{\delta(x-a)\}=e^{-i2\pi ua}$$

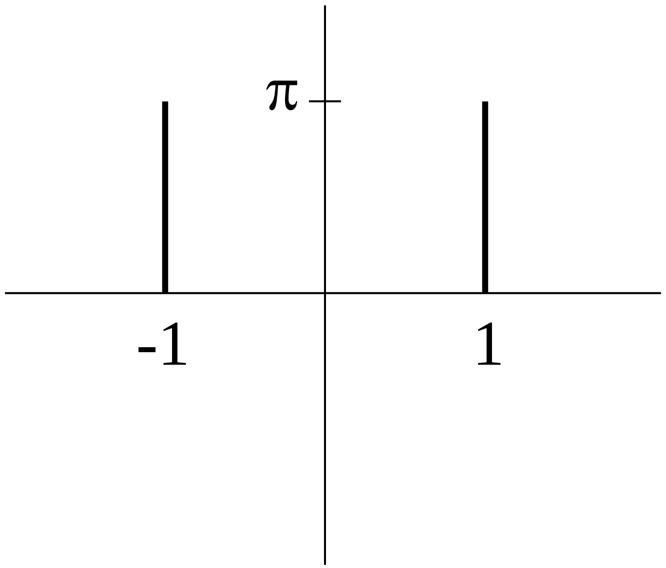


$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)$$

Review: Cosine



$\cos(2\pi x)$



If $f(x)$ is even, so is $F(u)$

Review: Convolution

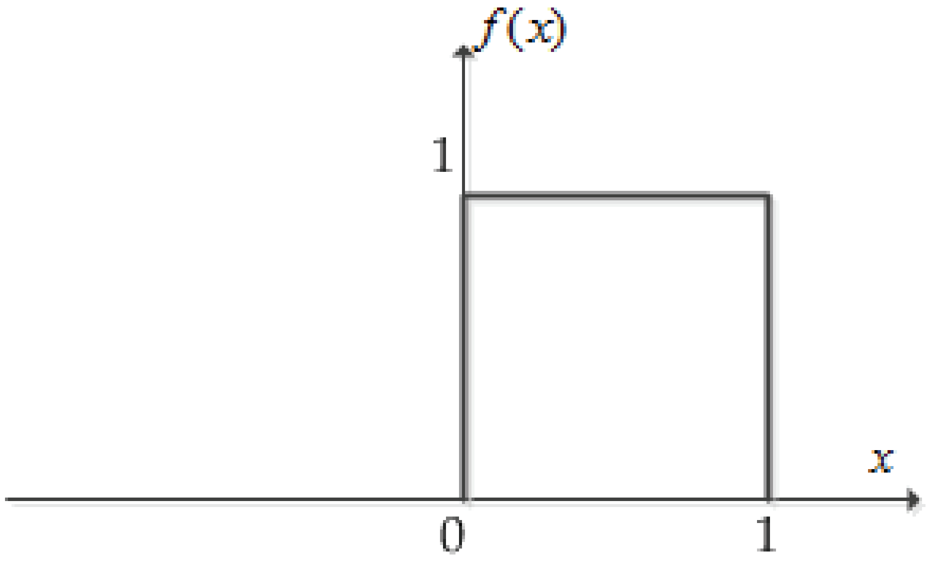
- A mathematical operator which computes the “amount of overlap” between two functions. Can be thought of as a general moving average

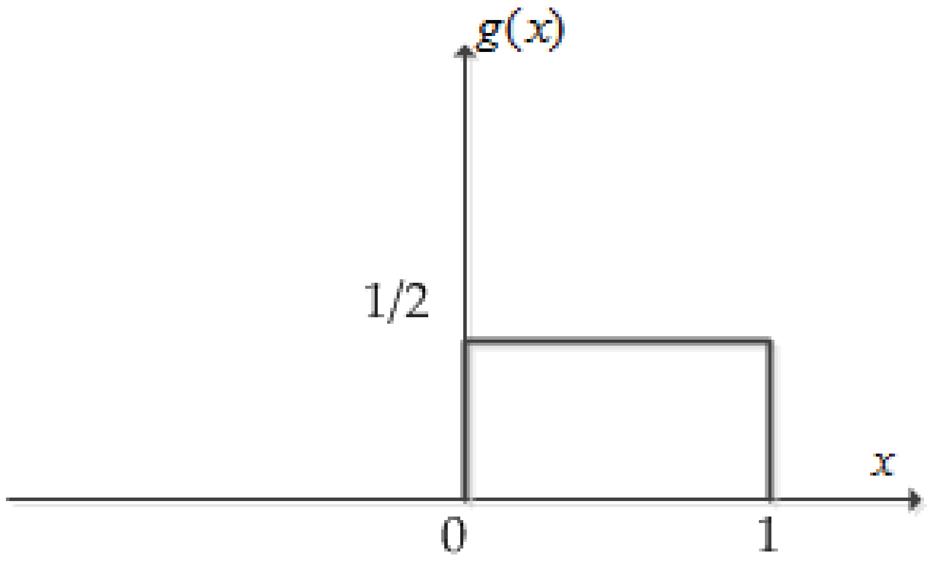
- Continuous domain:

$$f(x) \otimes g(x) = \int f(a) g(x-a) da$$

Conv. Example

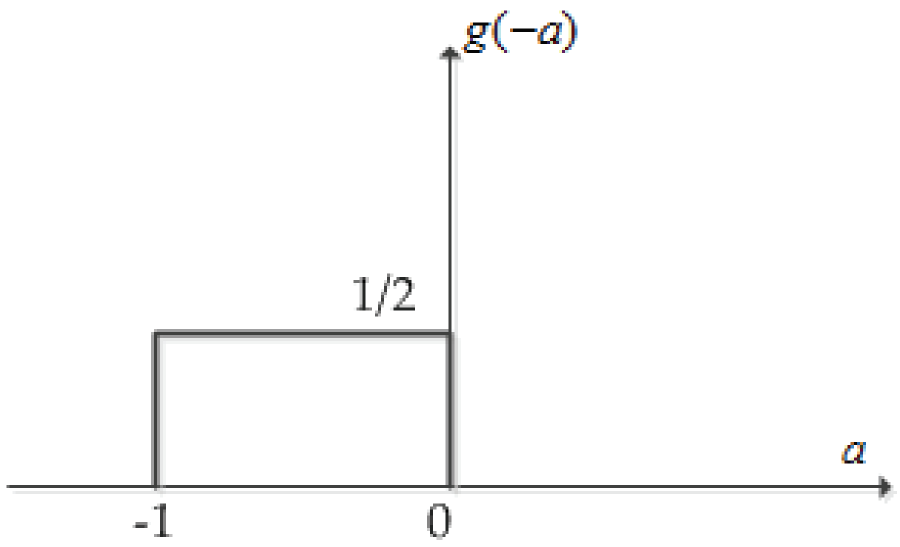
- Suppose we want to compute the convolution of the following two functions:



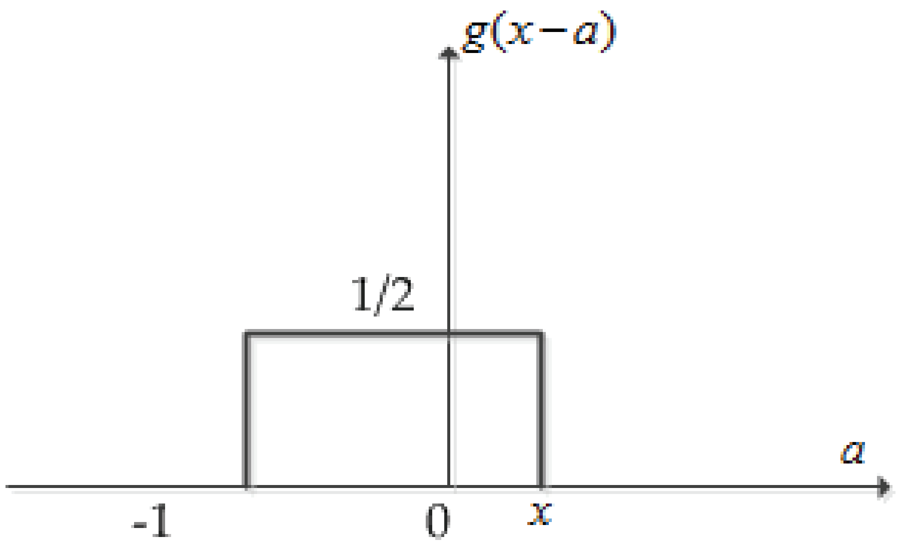


Conv. Example

Step 1: find g(-a)

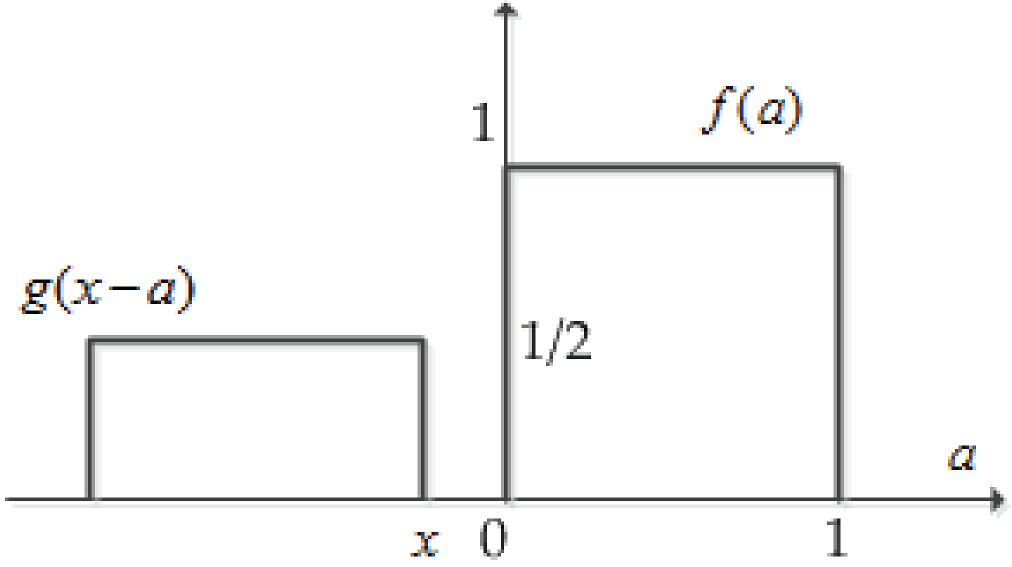


Step 2: find g(x-a)



Conv. Example

- **Step 3: Shift the impulse response function over target function and take integral at every position**
  - **Case 1 (x<0): no overlap.**

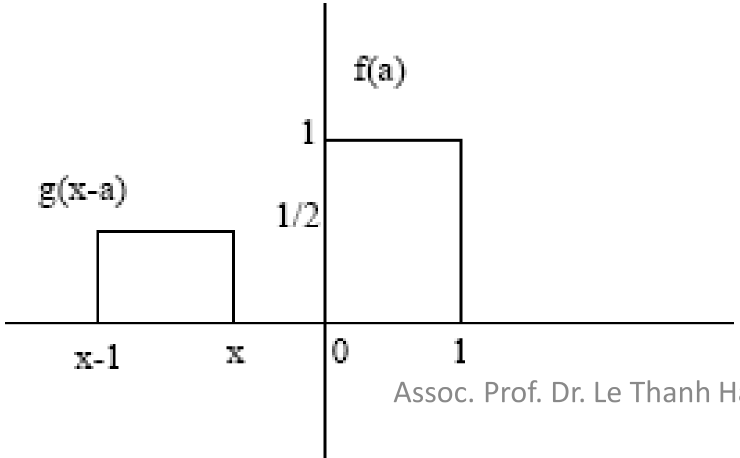


Conv. Example

- Case 1:

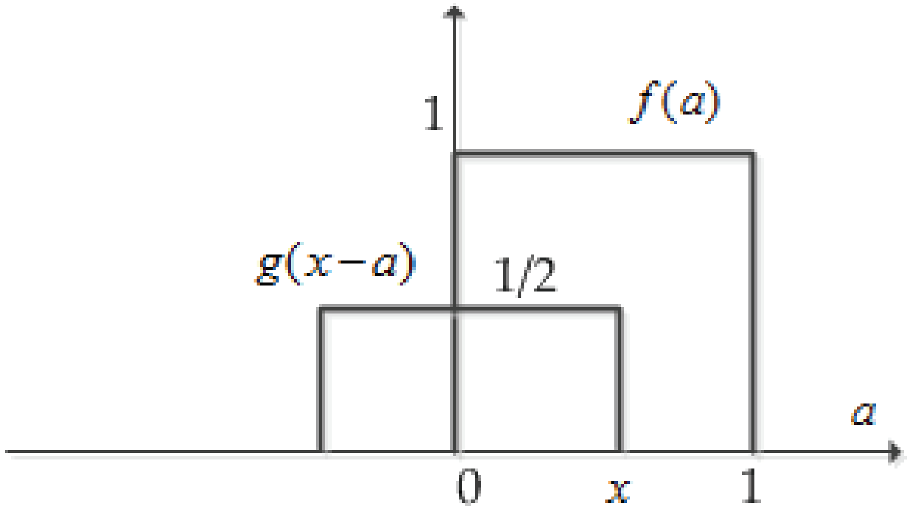
Step 3: consider all possible cases for x:

<u>Case 1:</u> $x < 0$



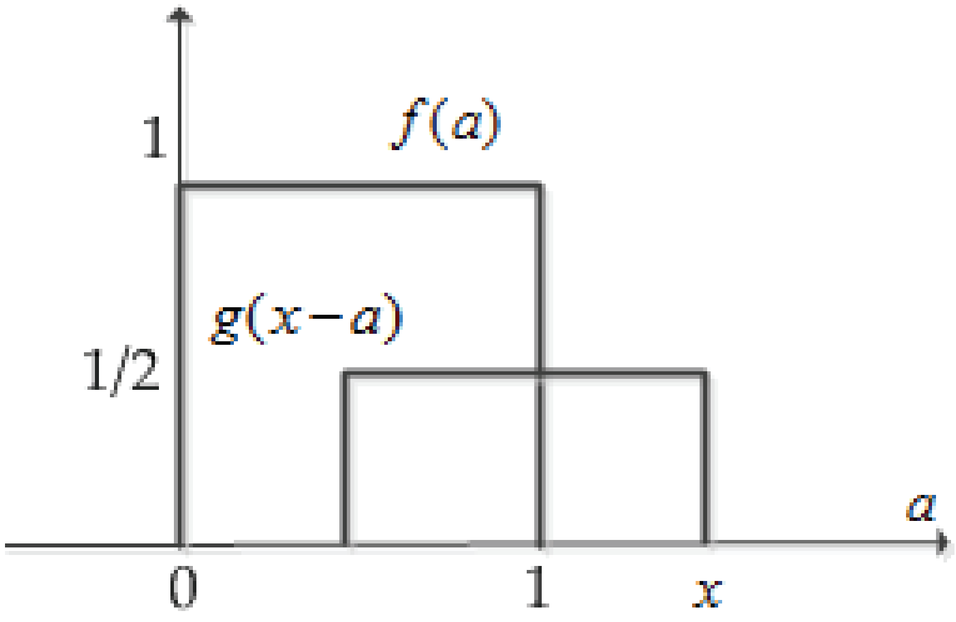
Conv. Example

- Case 2 ($0 \leq x < 1$):



$$\int_{-\infty}^{\infty} f(a)g(x-a)da = \int_{0}^{x} 1\,\frac{1}{2}\,da = \frac{x}{2}$$

Conv. Example

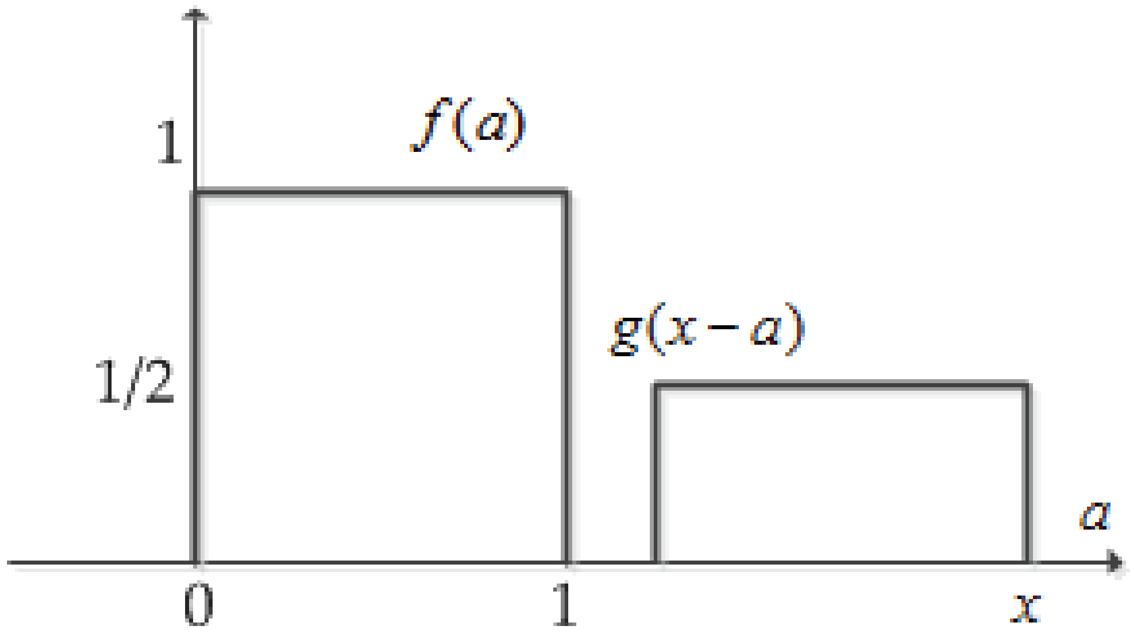


Case 3: $1 \le x \le 2$

$$\int_{-\infty}^{\infty} f(a)g(x-a)da = \int_{x-1}^{1} 1\,\frac{1}{2}\,da = 1 - \frac{x}{2}$$

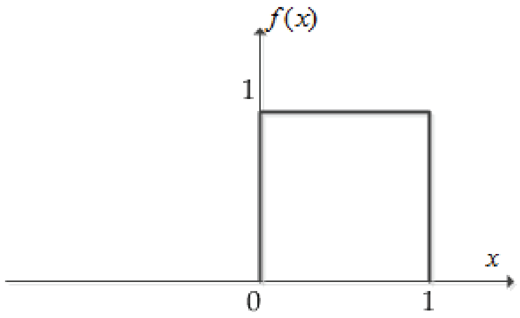
Conv. Example

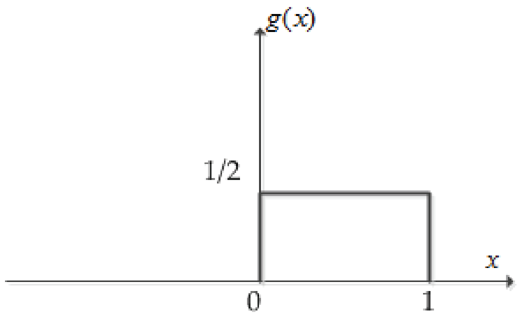
- Case 4: $x > 2$

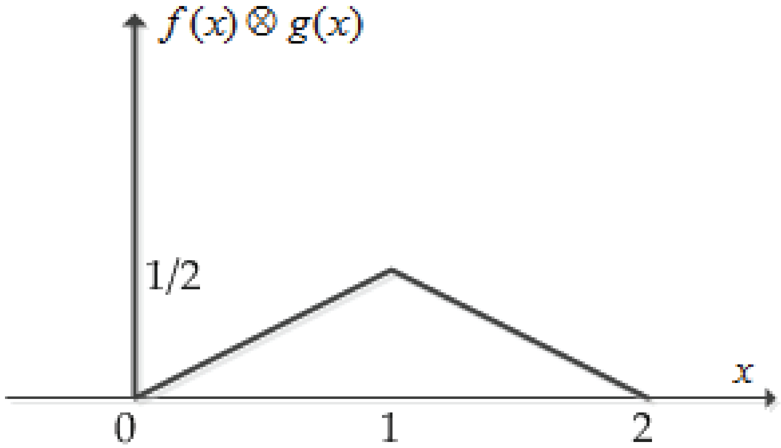


$$\int_{-\infty}^{\infty} f(a)g(x-a)da = 0$$

Conv. Example

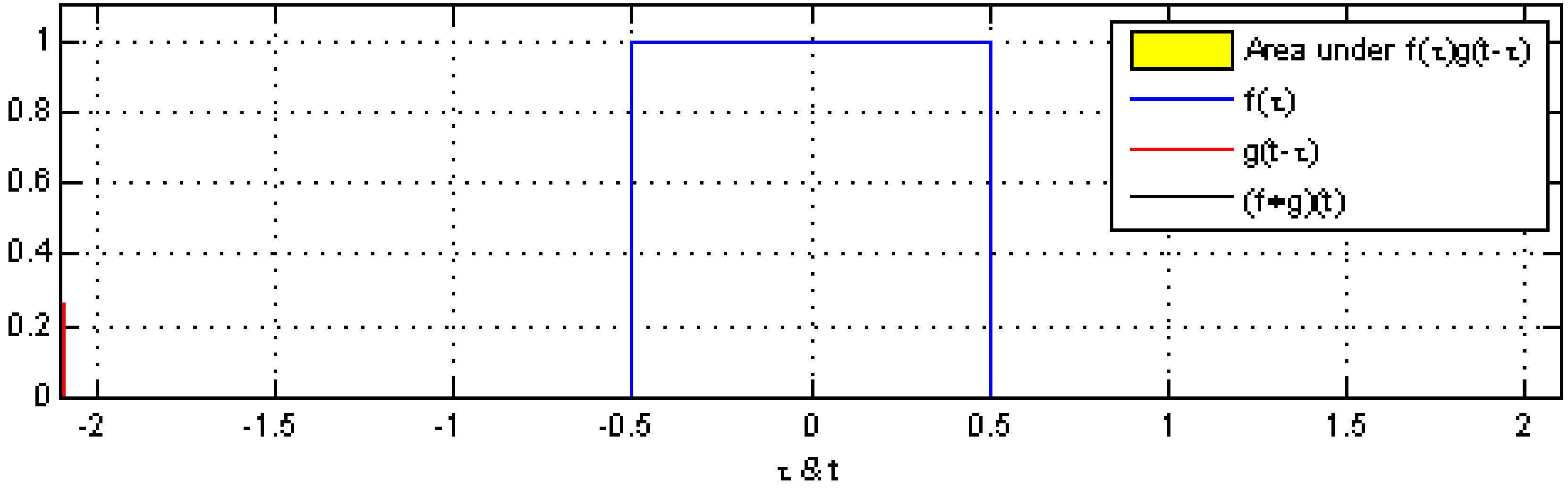






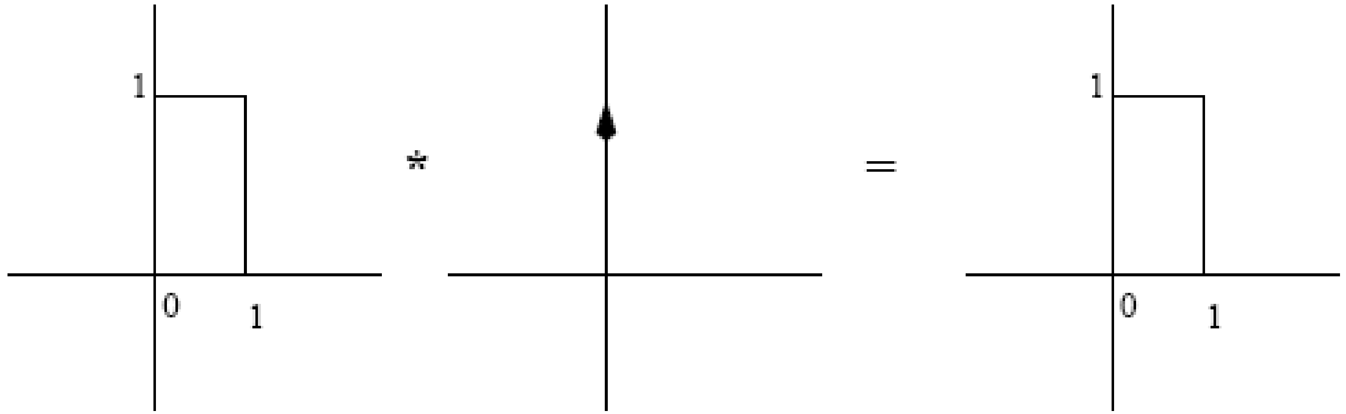
$$f(x) * g(x) = \begin{cases} x/2 & 0 \le x \le 1 \\ 1 - x/2 & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$$

Conv. Example (Wiki)

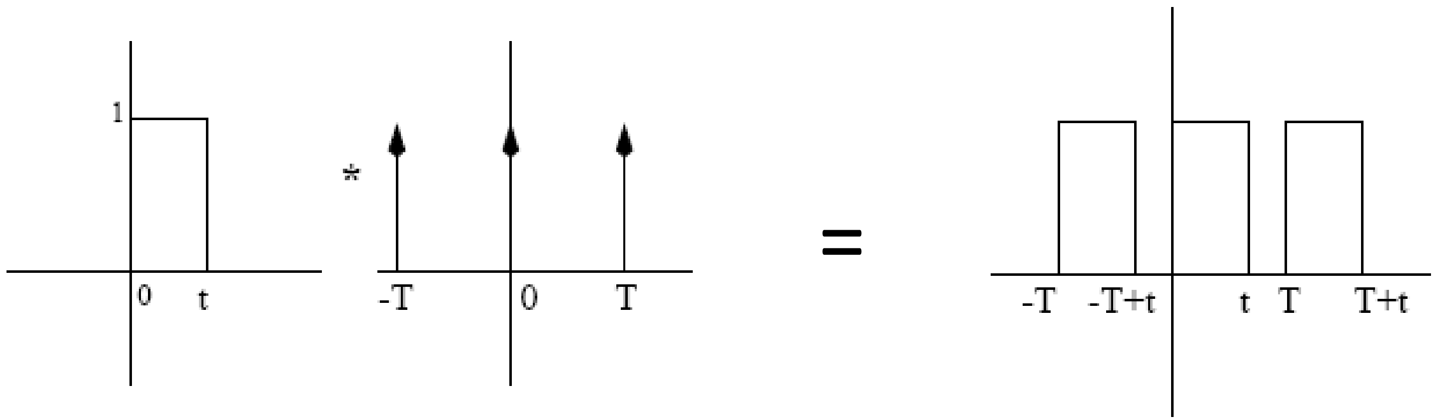


Convolution with Dirac Delta function

$$f(x) * \delta(x) = \int_{-\infty}^{\infty} f(a)\delta(x-a)da = f(x)$$



Convolution with an “train” of impulses



Review: Properties

Linearity: $af(x)+bg(x) \Leftrightarrow aF(u)+bG(u)$

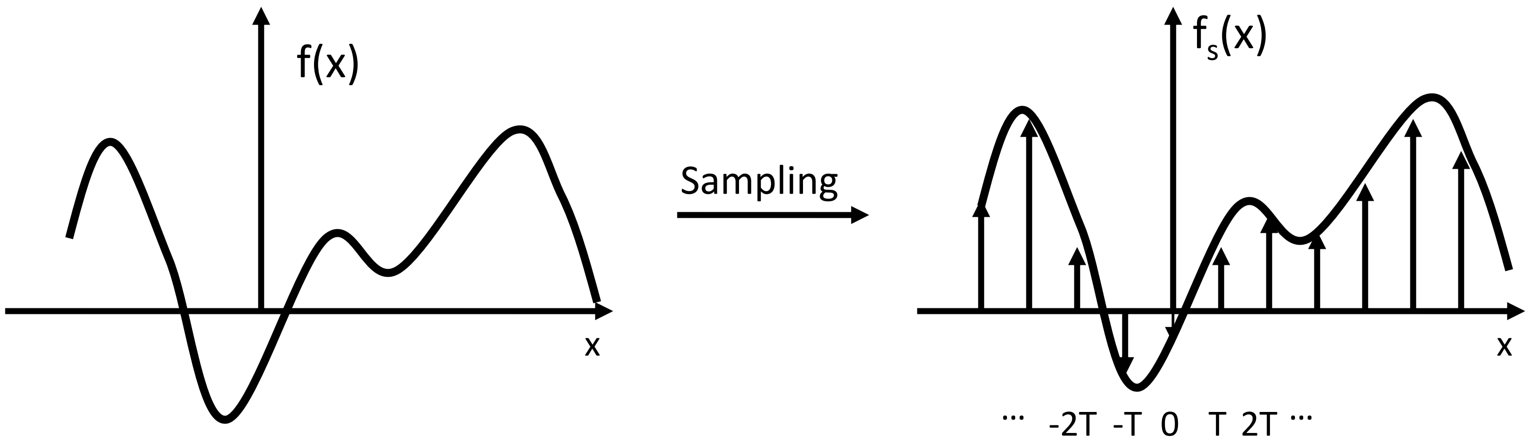
Time shift: $f(x-x_0) \Leftrightarrow e^{-i2\pi u x_0} F(u)$

Derivative: $\dfrac{df(x)}{x} \Leftrightarrow uF(u)$

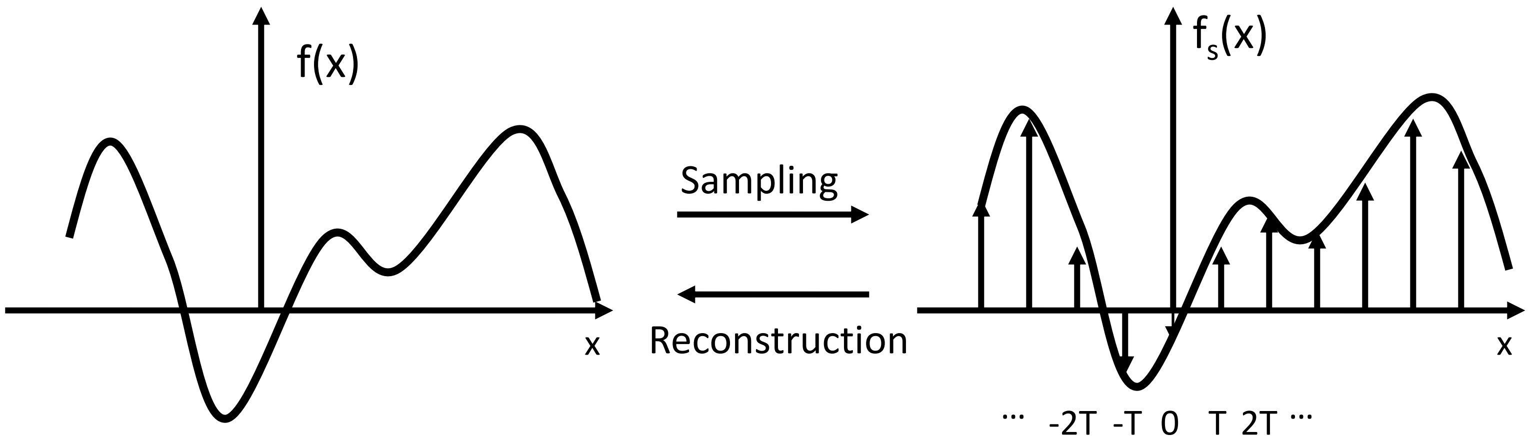
Integration: $\int f(x)dx \Leftrightarrow \dfrac{F(u)}{u}$

Convolution: $f(x) \otimes g(x) \Leftrightarrow F(u)G(u)$

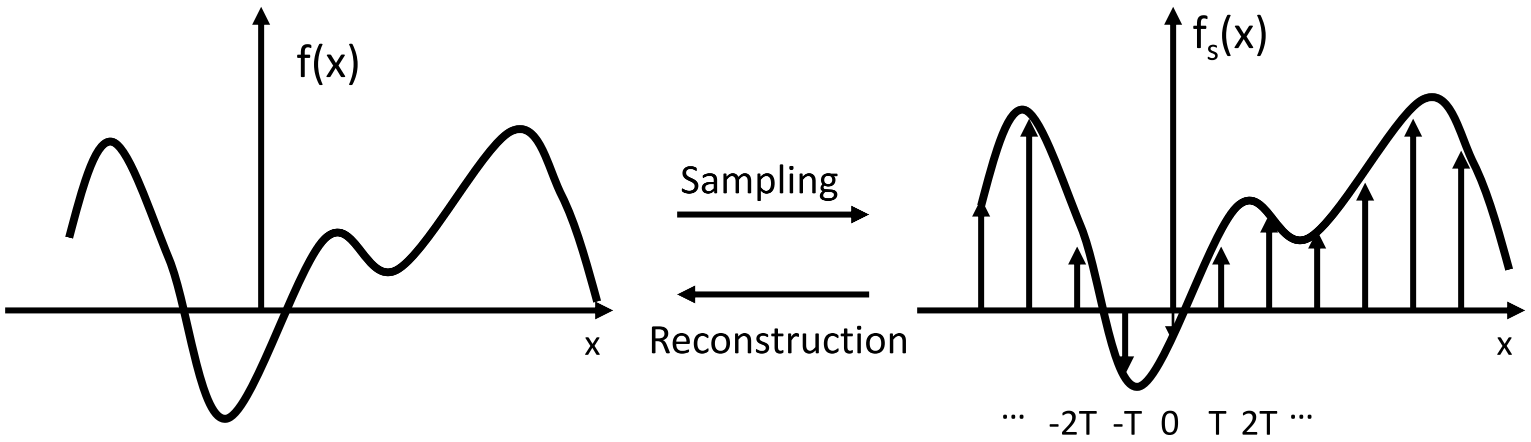
Sampling Analysis



Sampling Analysis



Sampling Analysis



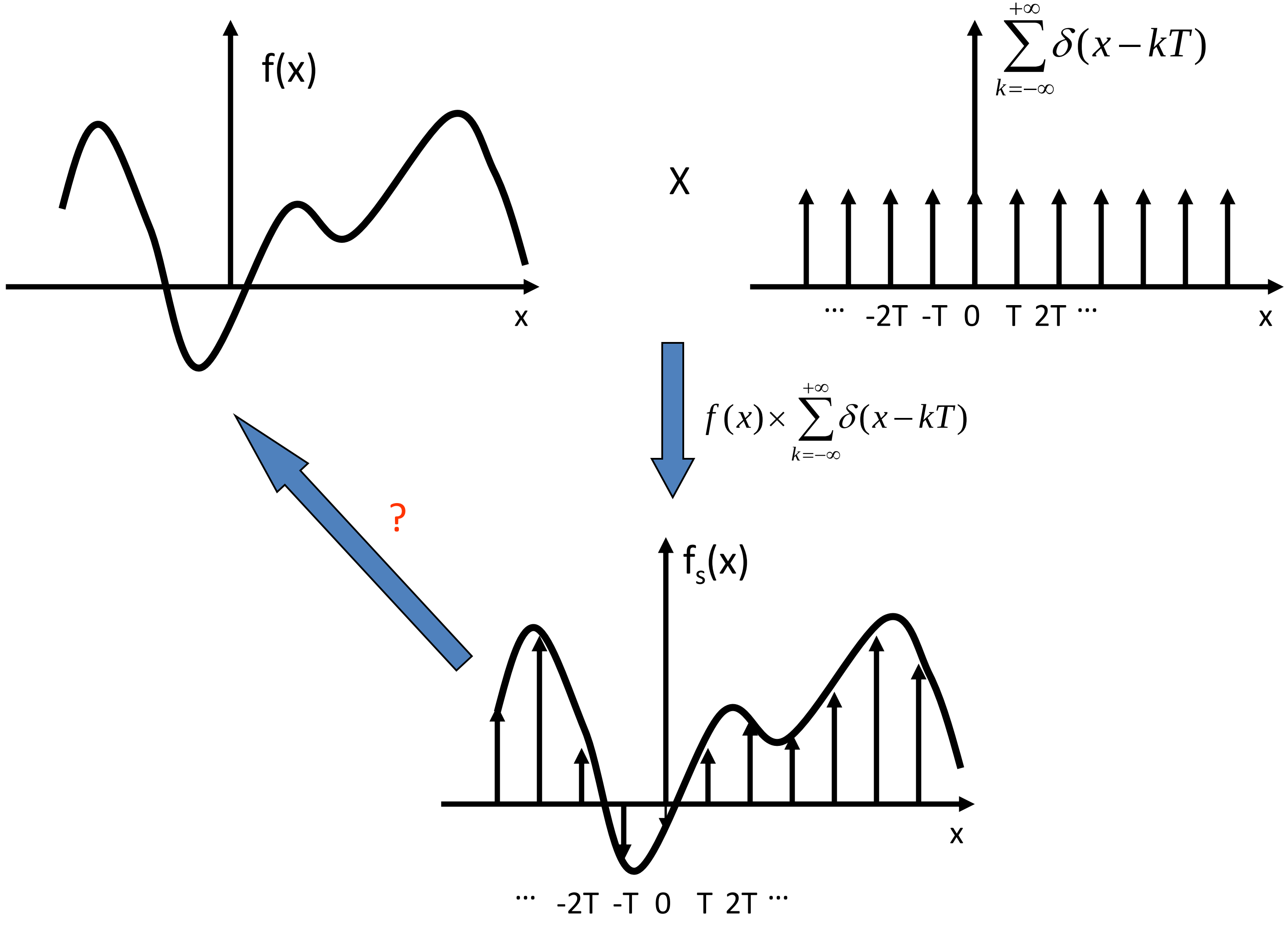
What sampling rate (T) is sufficient to reconstruct the continuous version of the sampled signal?

Sampling Theory

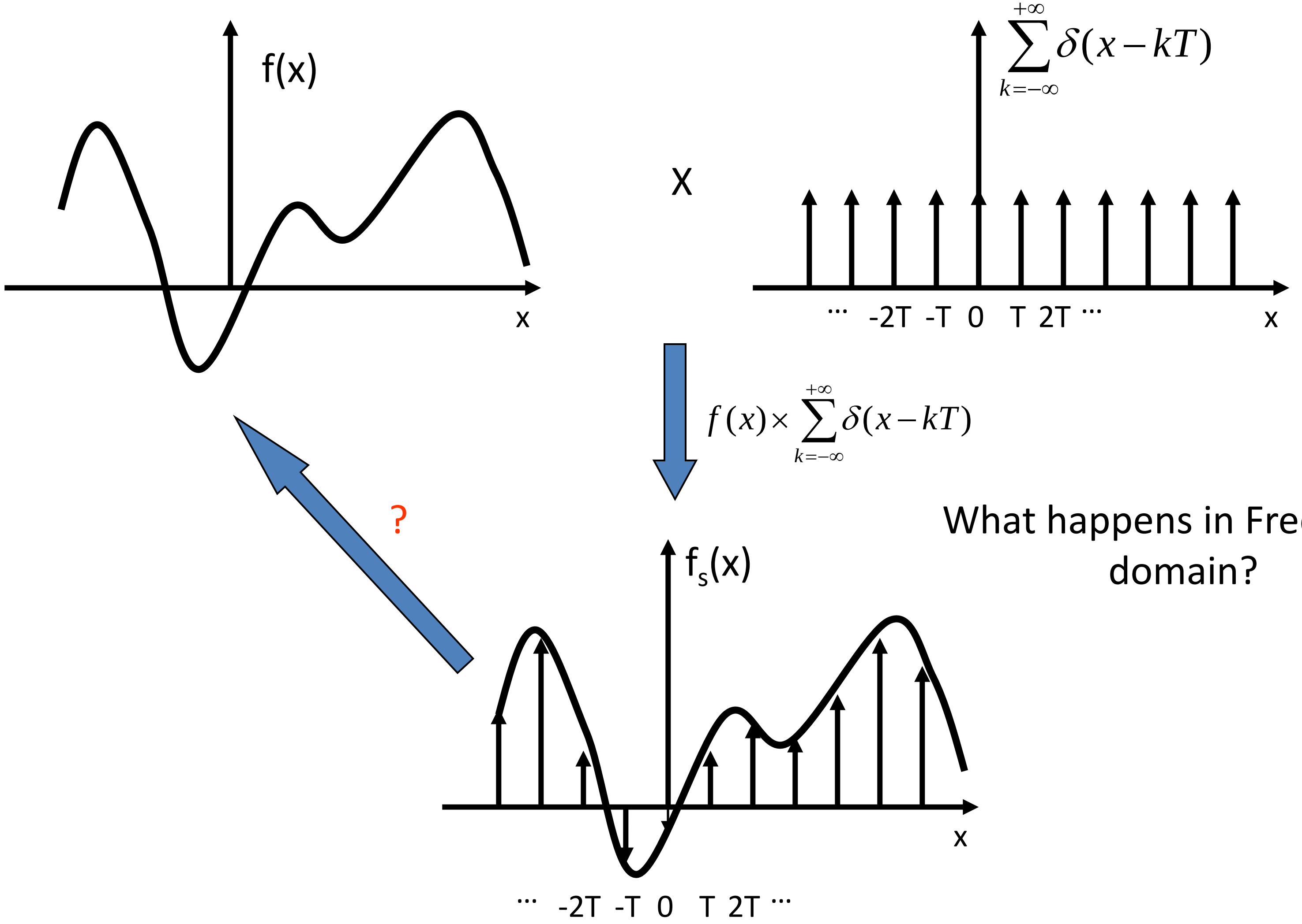
- How many samples are required to represent a given signal without loss of information?

- What signals can be reconstructed without loss for a given sampling rate?

Sampling Analysis: Spatial Domain



Sampling Analysis: Spatial Domain



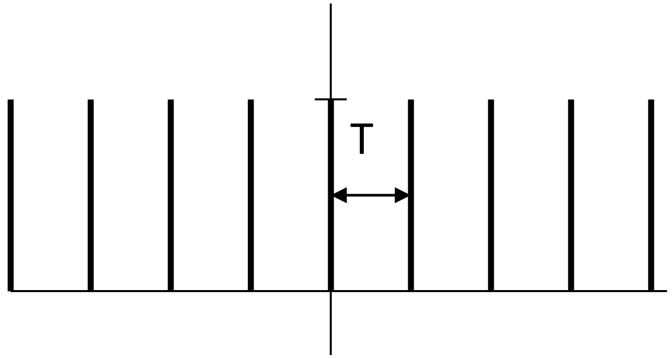
What happens in Frequency domain?

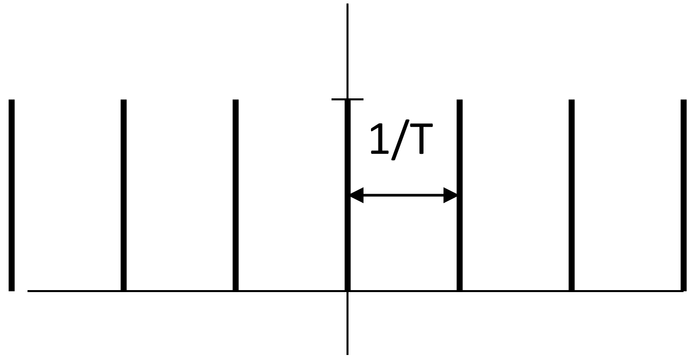
Fourier Transform of Dirac Comb

$$\Delta_T(x) = \sum_{-\infty}^{\infty} \delta(x - kT),$$

$$\begin{aligned}
\mathcal{F}\{\Delta_T(x)\} &= \sum_{k=-\infty}^{\infty} \mathcal{F}\{(\delta(x - kT)\} \\
&= \sum_{k=-\infty}^{\infty} e^{-i2\pi ukT} \\
&= \sum_{k=-\infty}^{\infty} \delta(u - k/T) \\
&= \Delta_{1/T}(x).
\end{aligned}$$

Fourier Transform of Dirac Comb





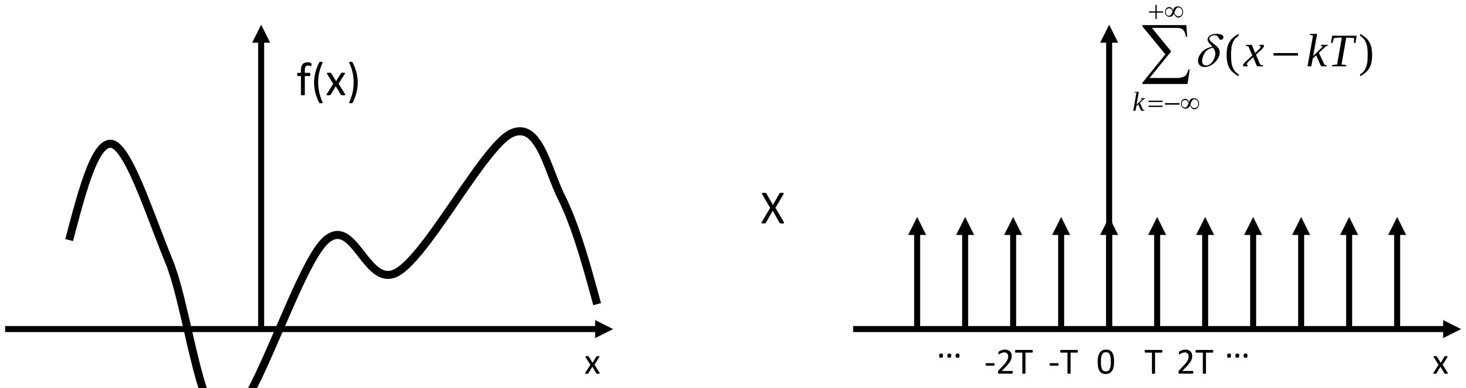
$$\sum_{k=-\infty}^{+\infty} \delta(x - kT)$$

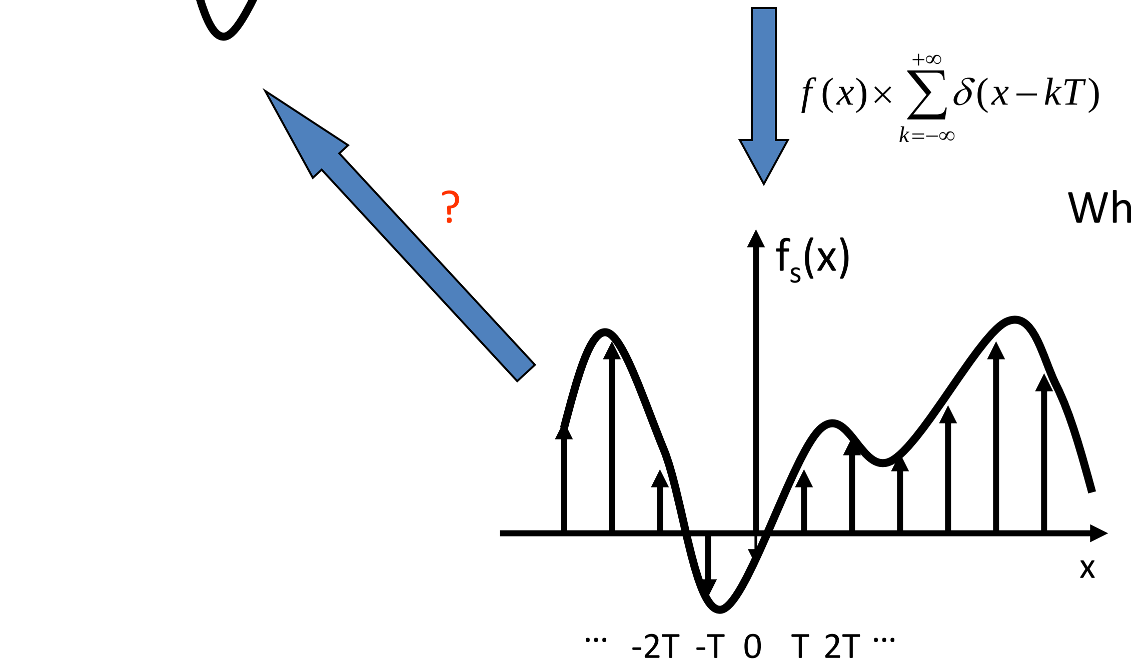
$$\sum_{k=-\infty}^{+\infty} \delta(u - \frac{k}{T})$$

$$\sum_{k=-\infty}^{+\infty} e^{-i2\pi ukT}$$

Moving the spikes closer together in the spatial domain moves them farther apart in the frequency domain!

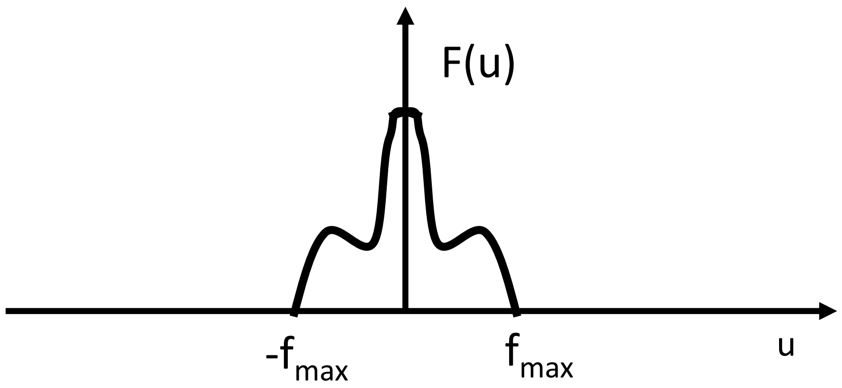
Sampling Analysis: Spatial Domain





What happens in Frequency domain?

Sampling Analysis: Freq. Domain



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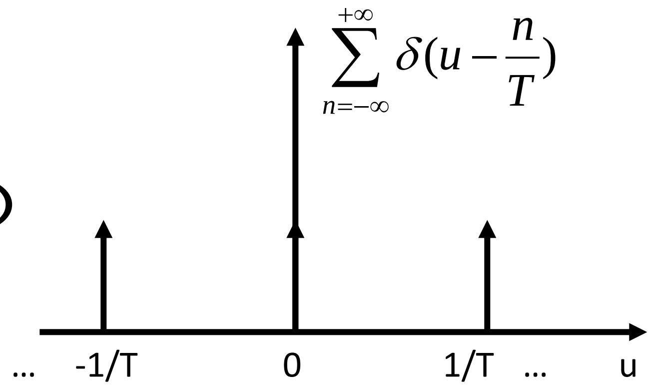
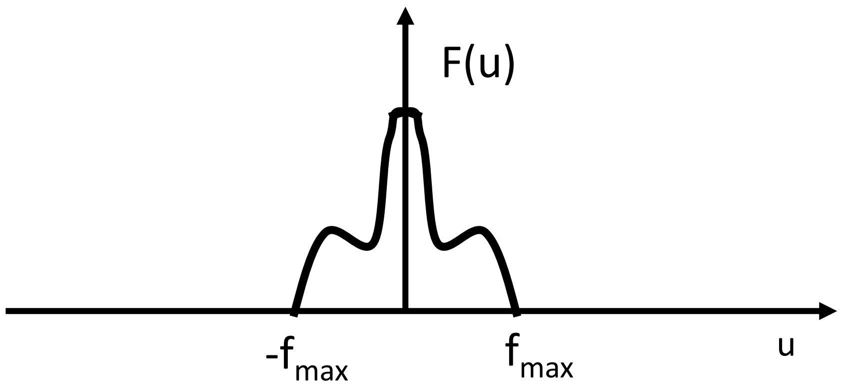


Sampling Analysis: Freq. Domain



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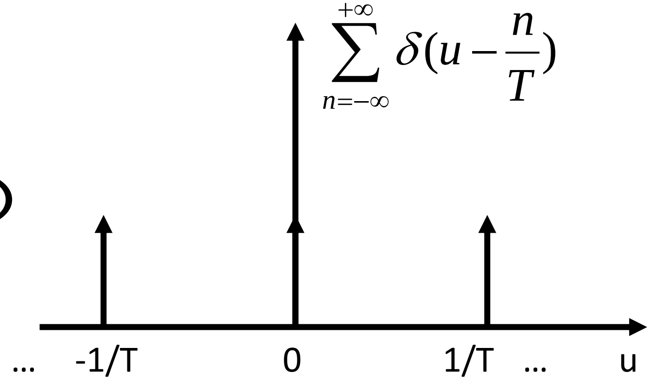
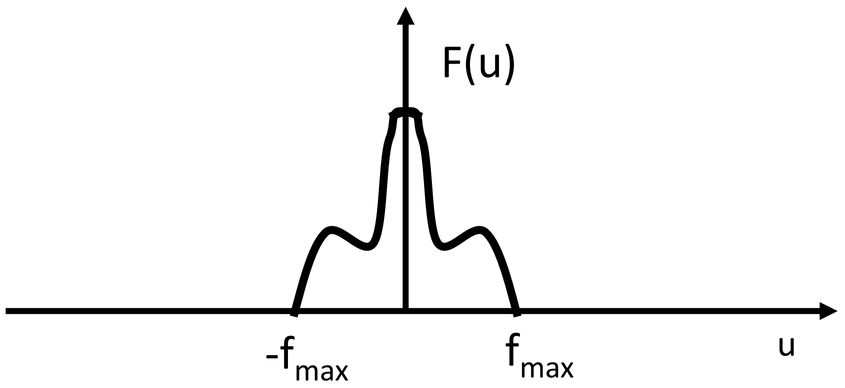


How does the convolution result look like?

Sampling Analysis: Freq. Domain



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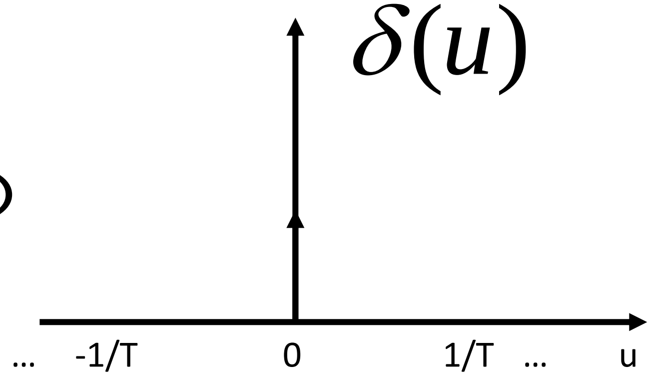
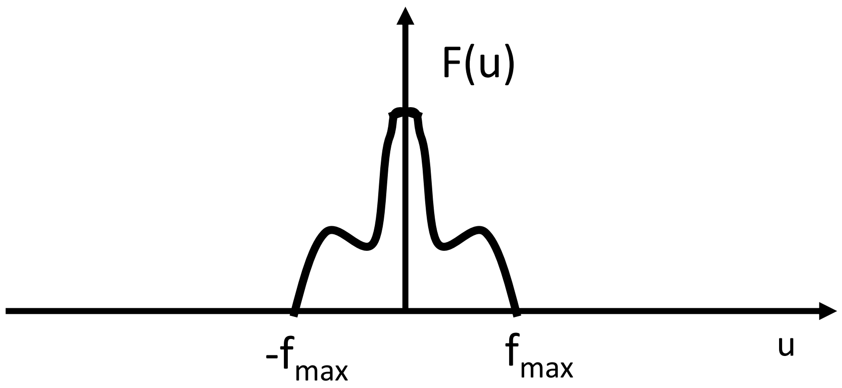


Sampling Analysis: Freq. Domain



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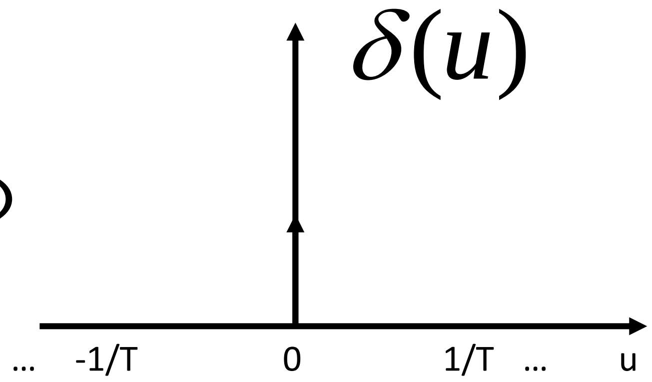
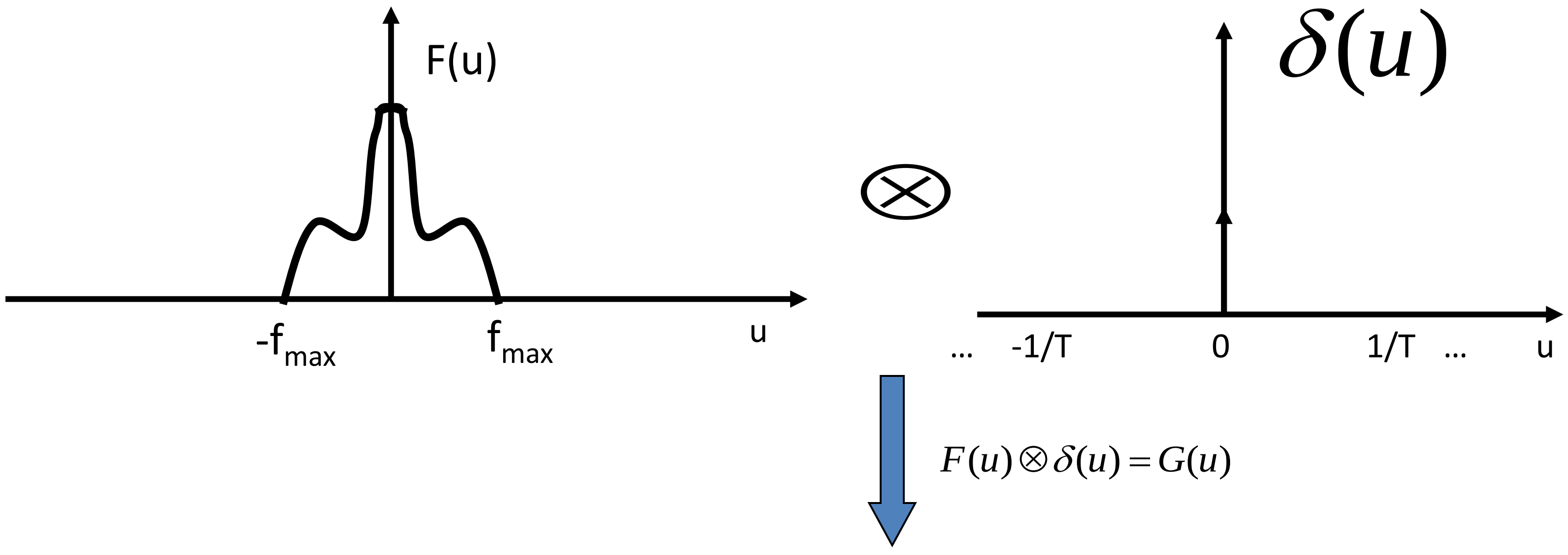


$$F(u) \otimes \delta(u)$$

Sampling Analysis: Freq. Domain



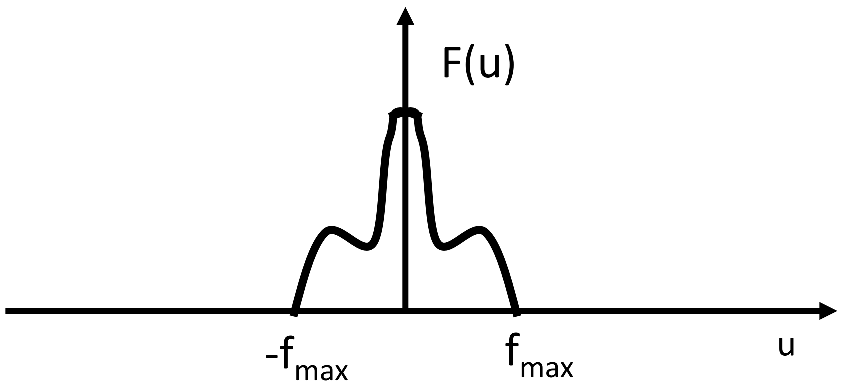
$F(u) \otimes \delta(u) = G(u)$

G(0)?

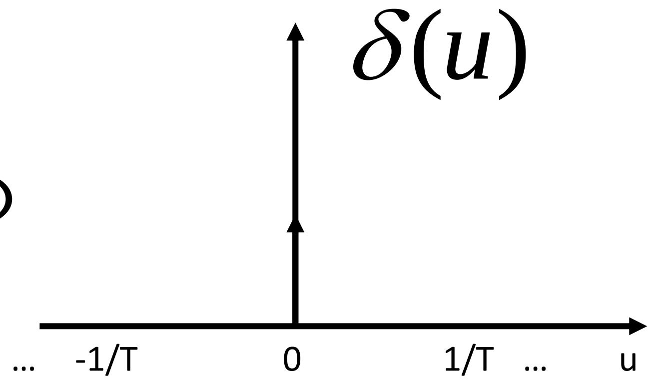
G(f_{max})?

G(u)?

Sampling Analysis: Freq. Domain



$\otimes$



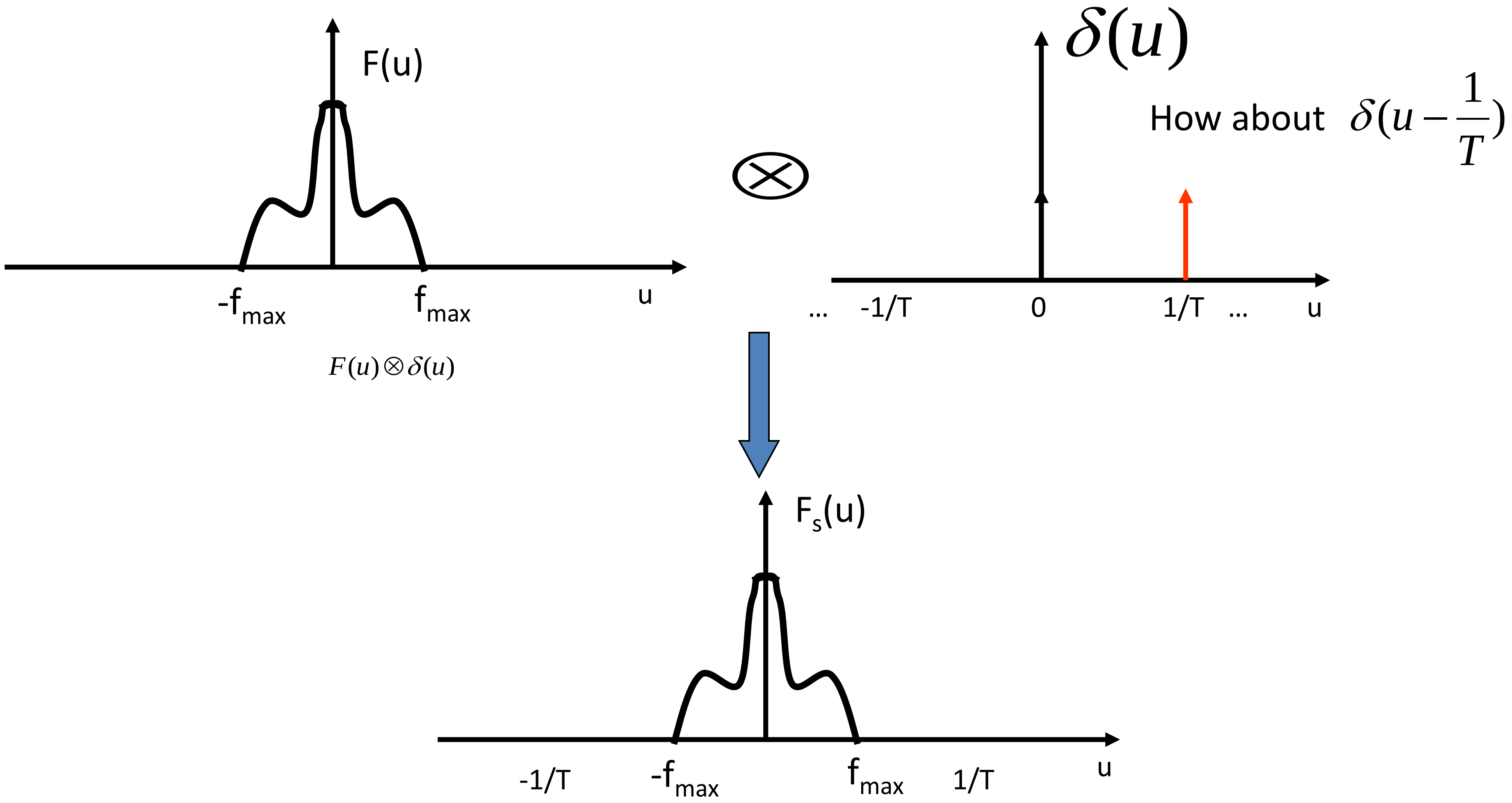
$$F(u) \otimes \delta(u) = G(u)$$

$G(0) = F(0)$

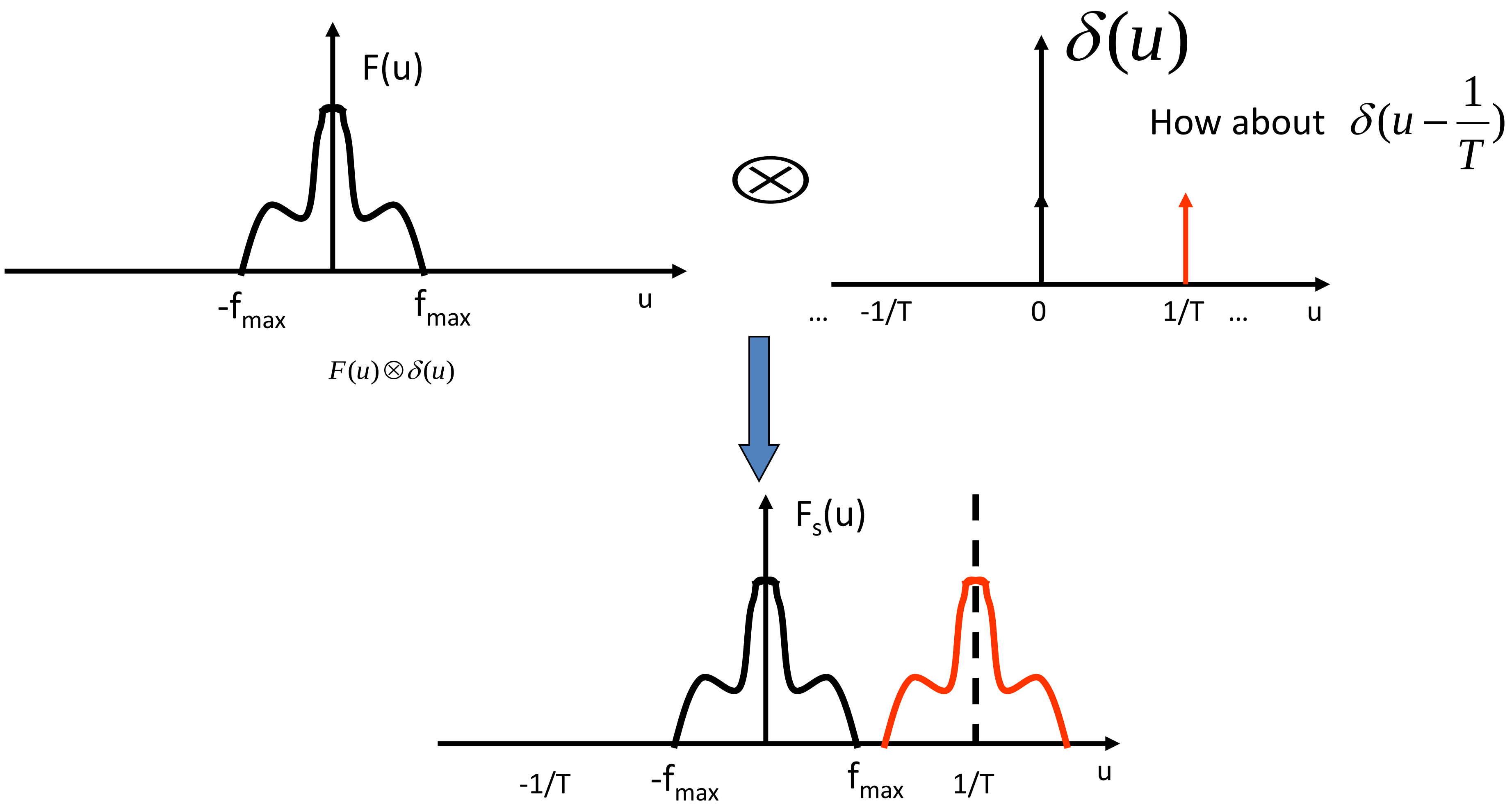
$G(f_{max}) = F(f_{max})$

$G(u) = F(u)$

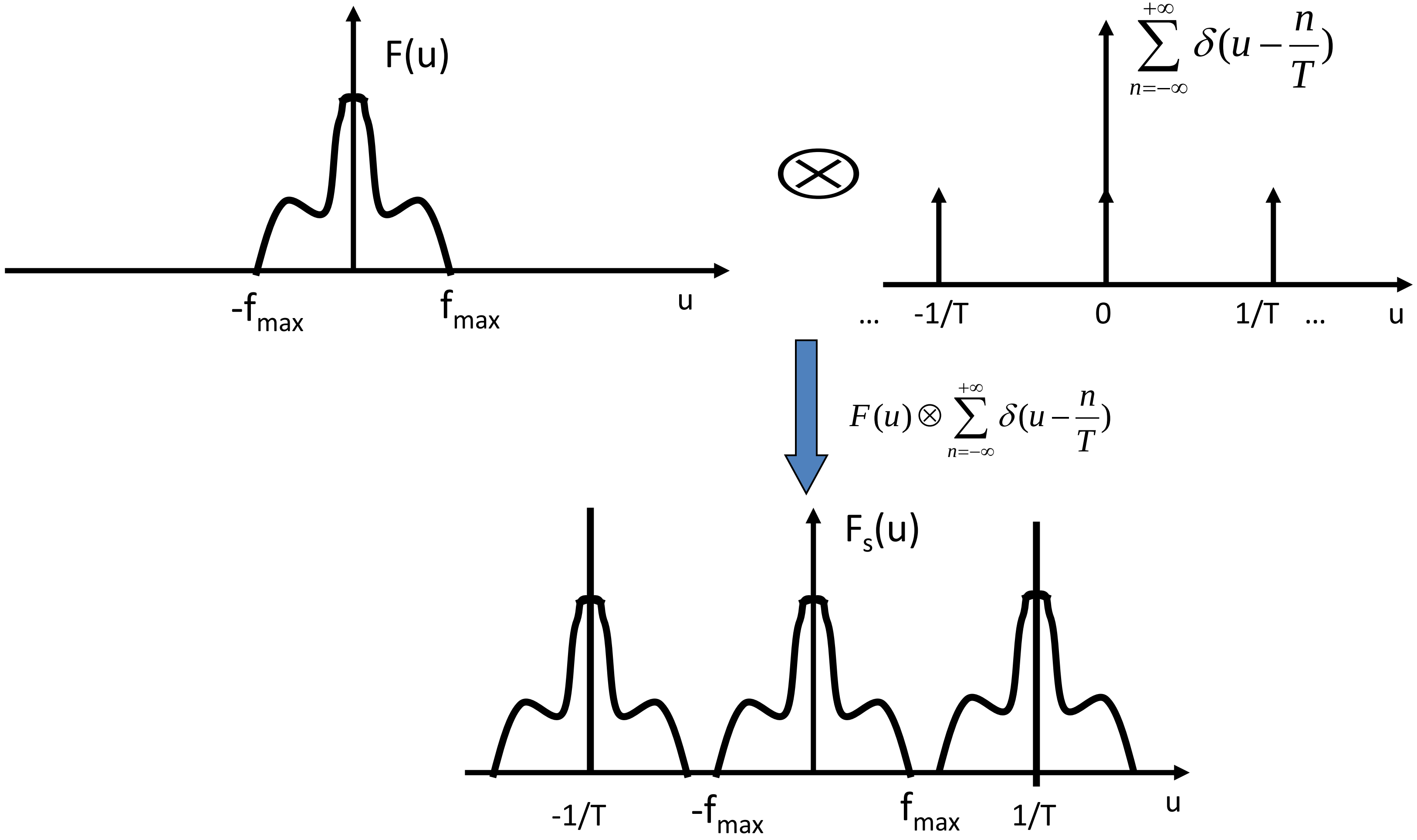
Sampling Analysis: Freq. Domain



Sampling Analysis: Freq. Domain



Sampling Analysis: Freq. Domain

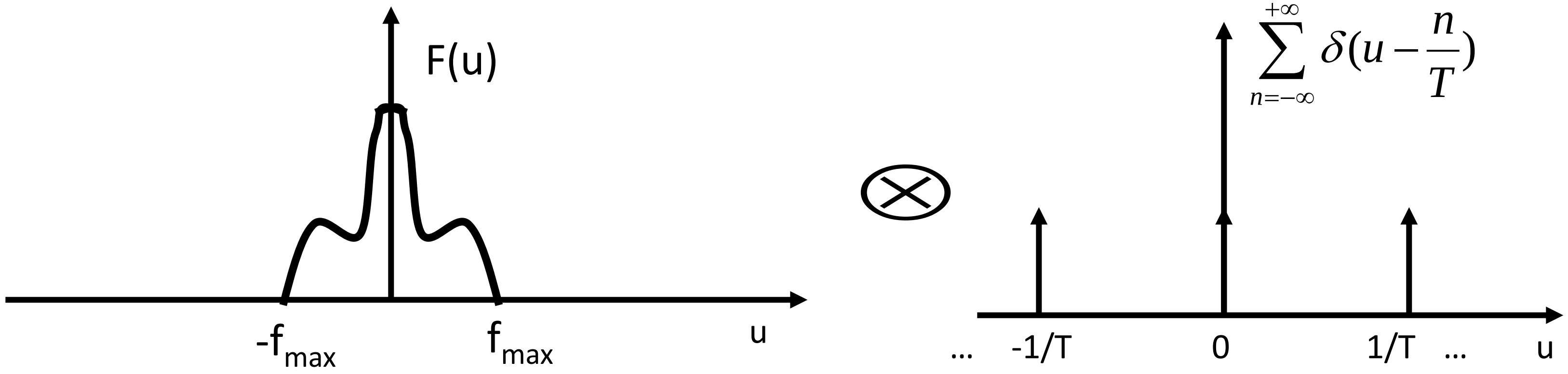


Sampling Theory

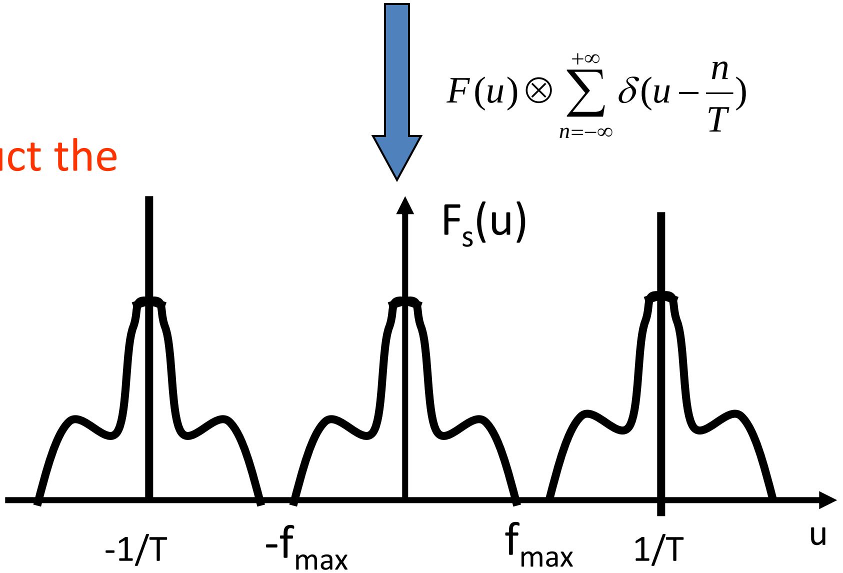
- How many samples are required to represent a given signal without loss of information?

- What signals can be reconstructed without loss for a given sampling rate?

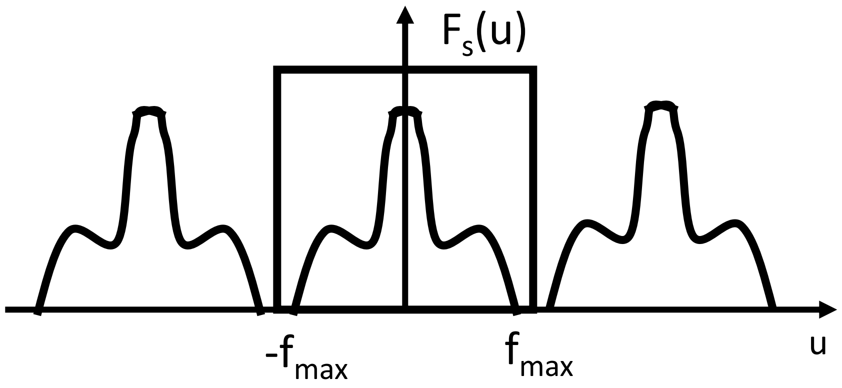
Sampling Analysis: Freq. Domain



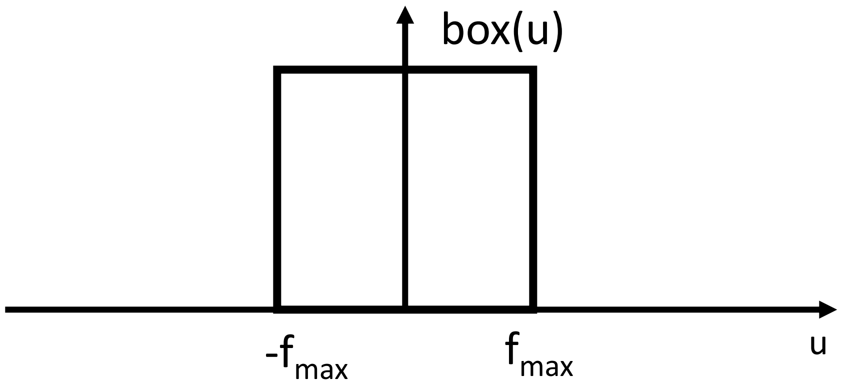
How can we reconstruct the original signal?



Reconstruction in Freq. Domain



$\times$



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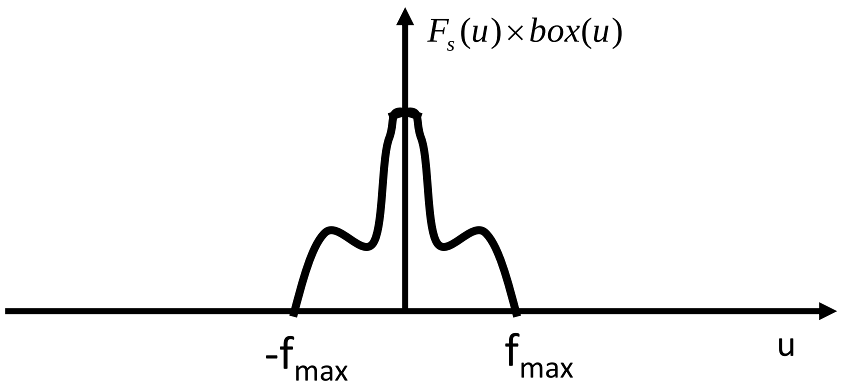
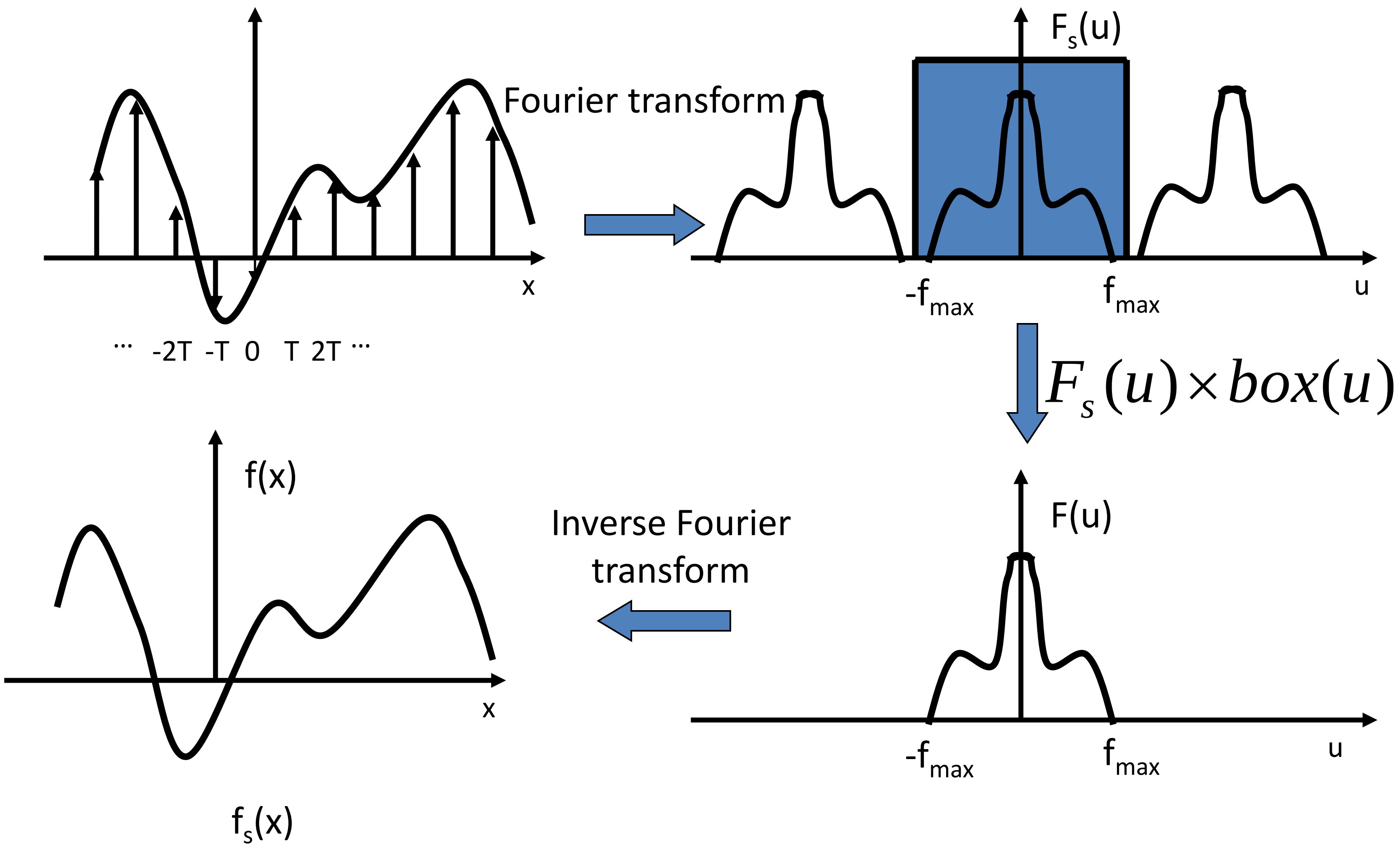
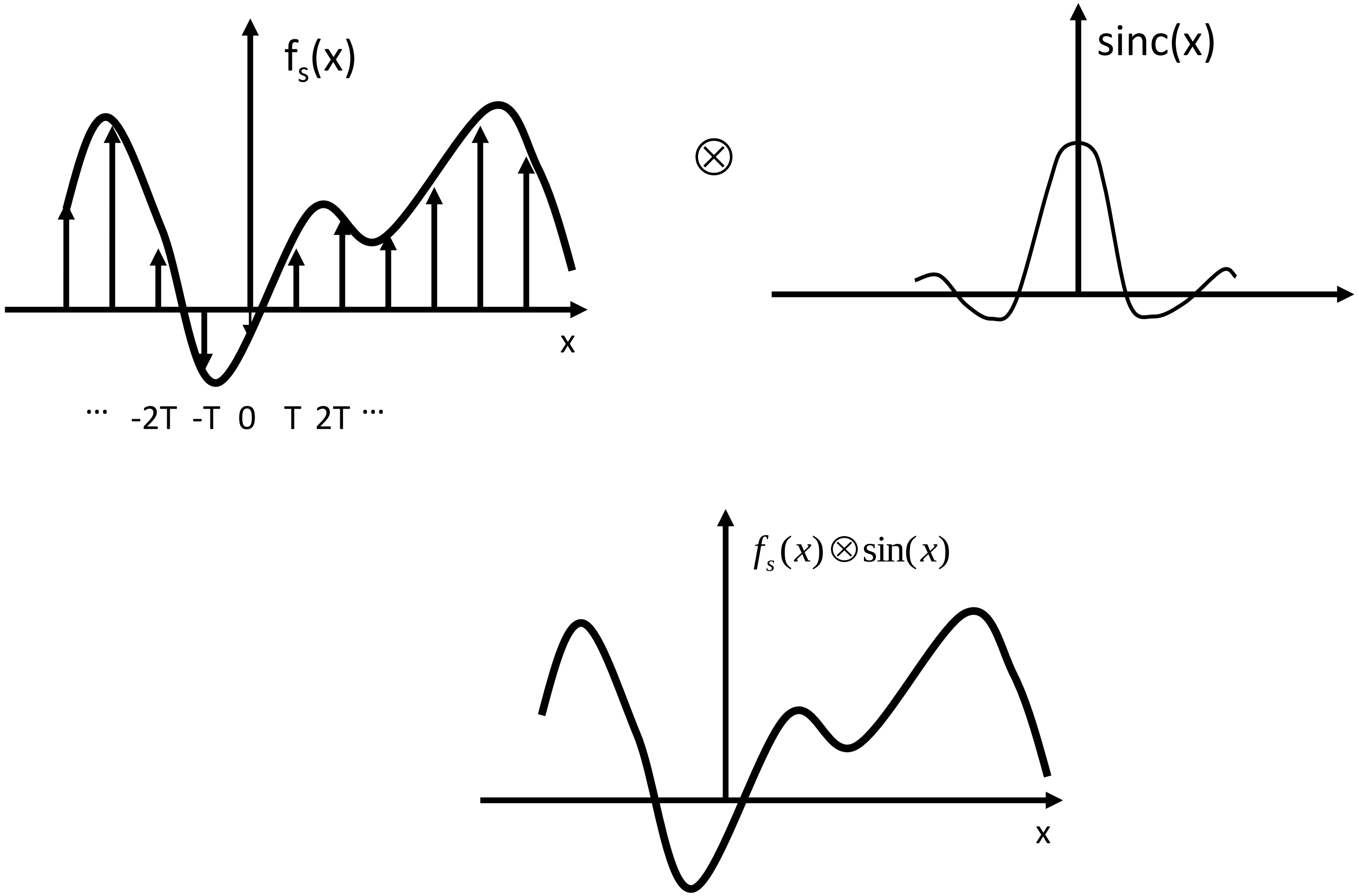


Signal Reconstruction in Freq. Domain

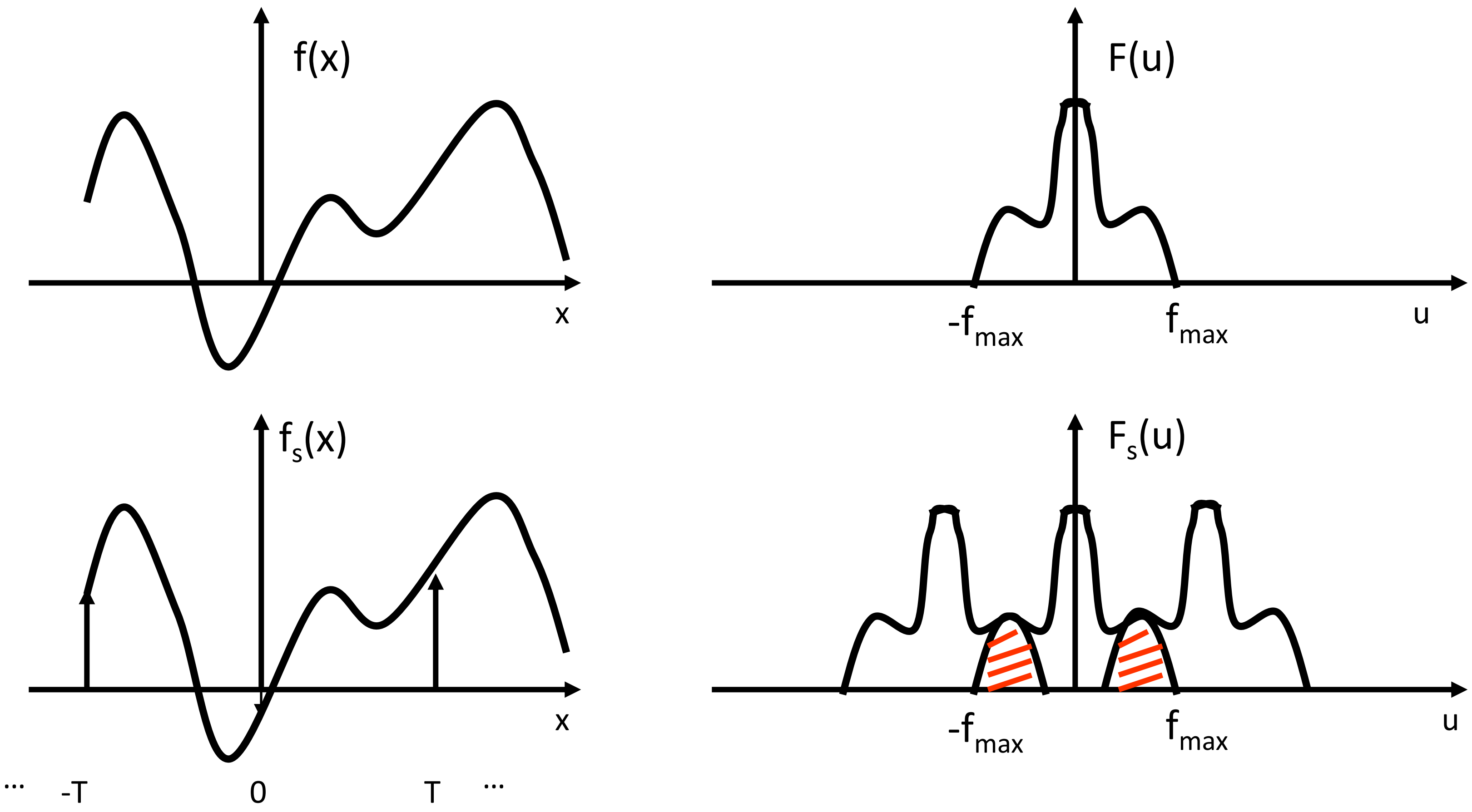


Signal Reconstruction in Spatial Domain



Sampling Analysis

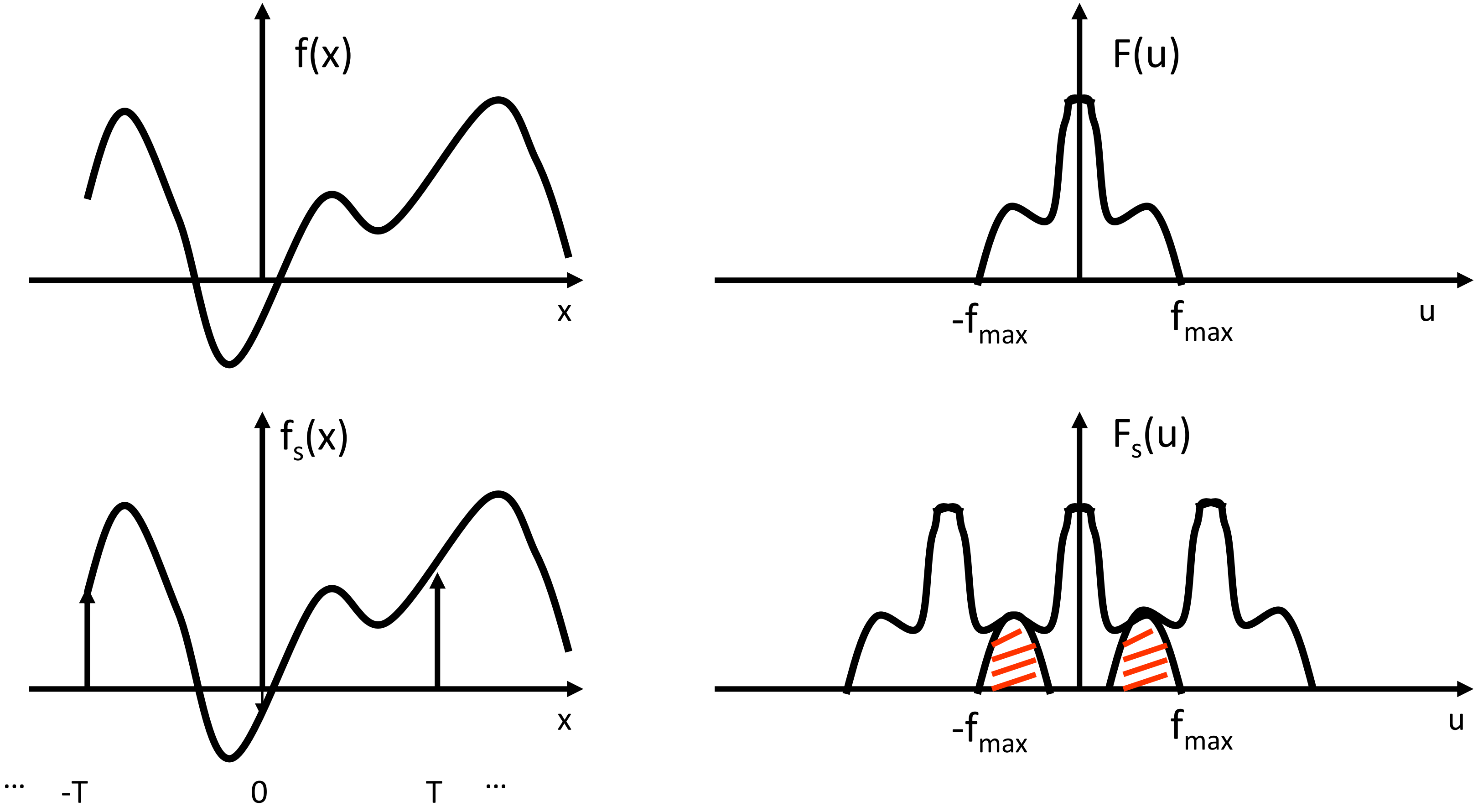
When does aliasing happen?



Sampling Analysis

When does aliasing happen?

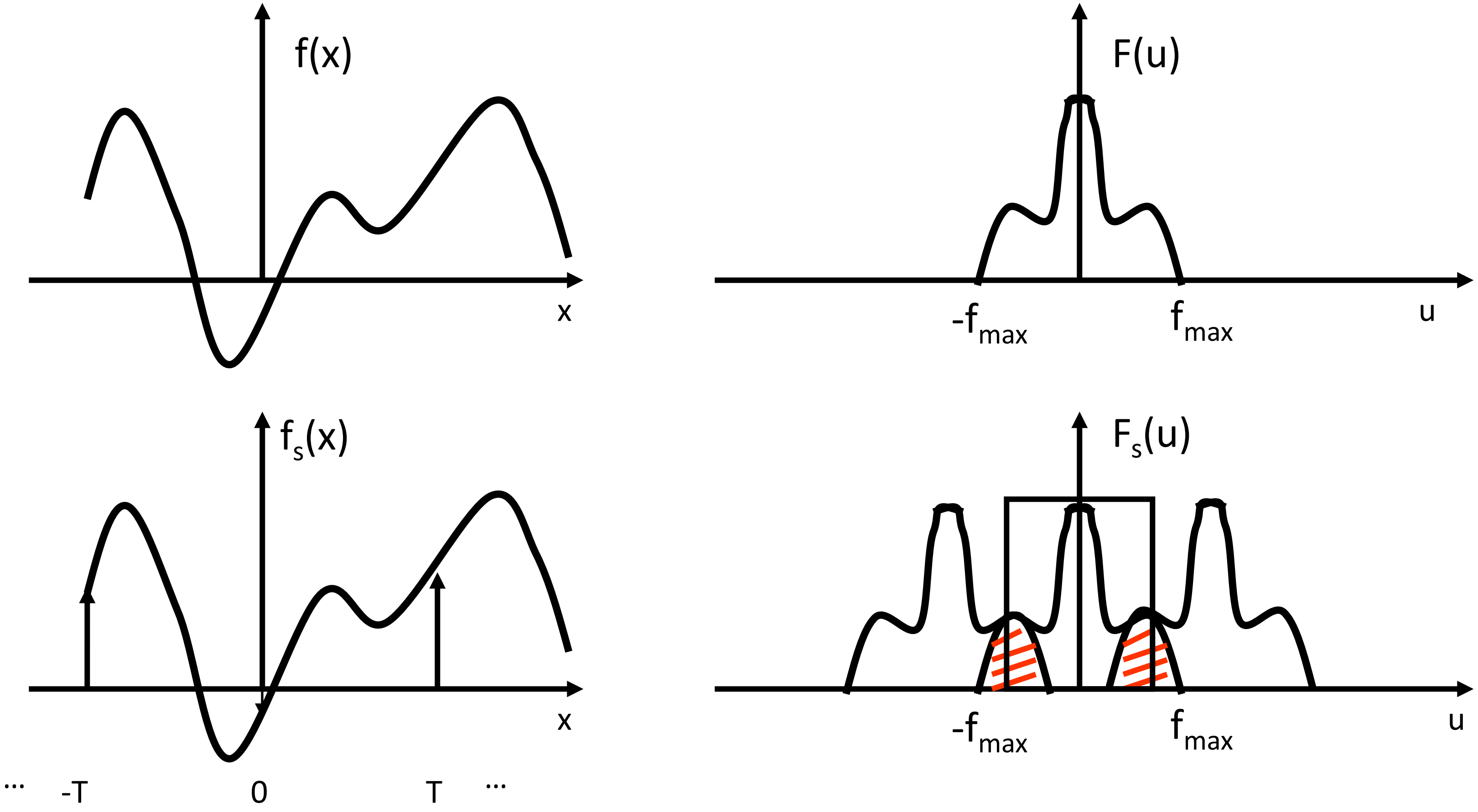
$$\frac{1}{T} < 2f_{\max}$$



Sampling Analysis

When does aliasing happen?

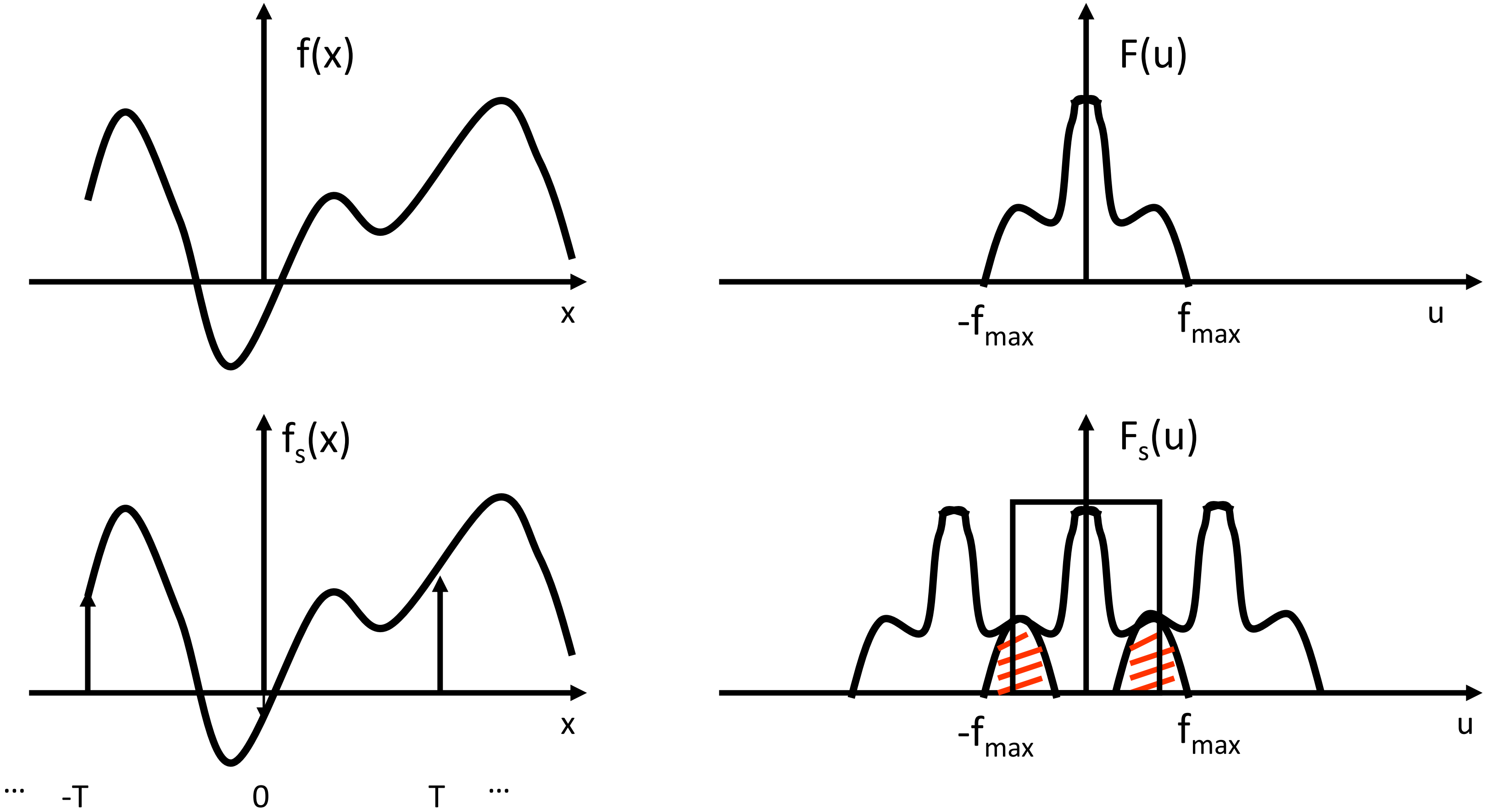
$$\frac{1}{T} < 2f_{\max}$$



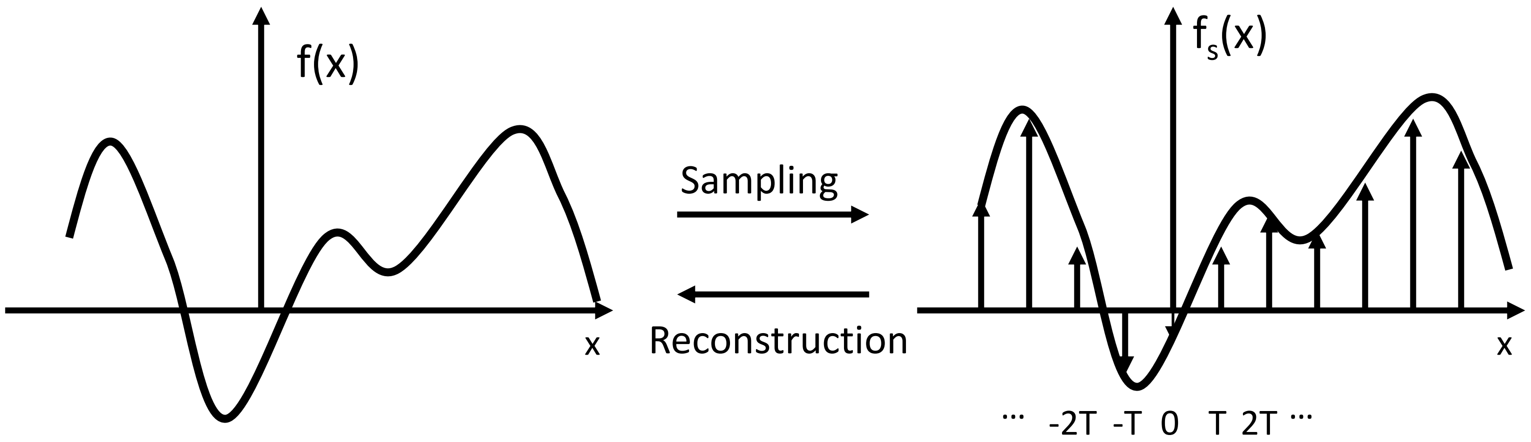
Sampling Analysis

When does aliasing happen?

$$\frac{1}{T} < 2f_{\max}$$

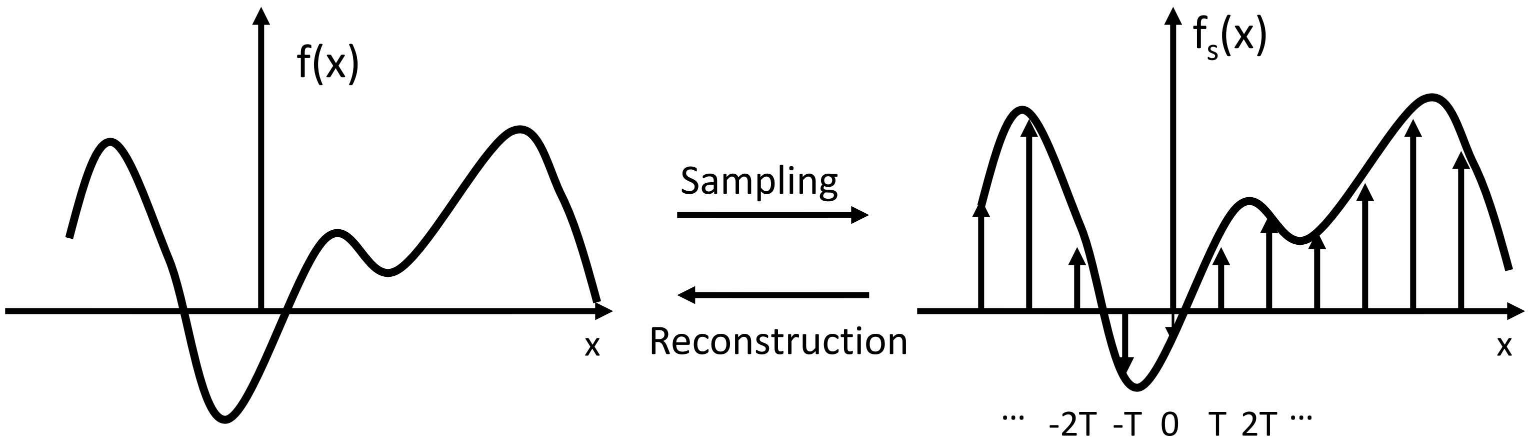


Sampling Analysis



What sampling rate (T) is sufficient to reconstruct the continuous version of the sampled signal?

Sampling Analysis



What sampling rate (T) is sufficient to reconstruct the continuous version of the sampled signal?

Sampling Rate ≥ 2 * max frequency in the signal

- this is known as the Nyquist Rate