Finding Dense Subgraphs via Low-Rank Bilinear Optimization

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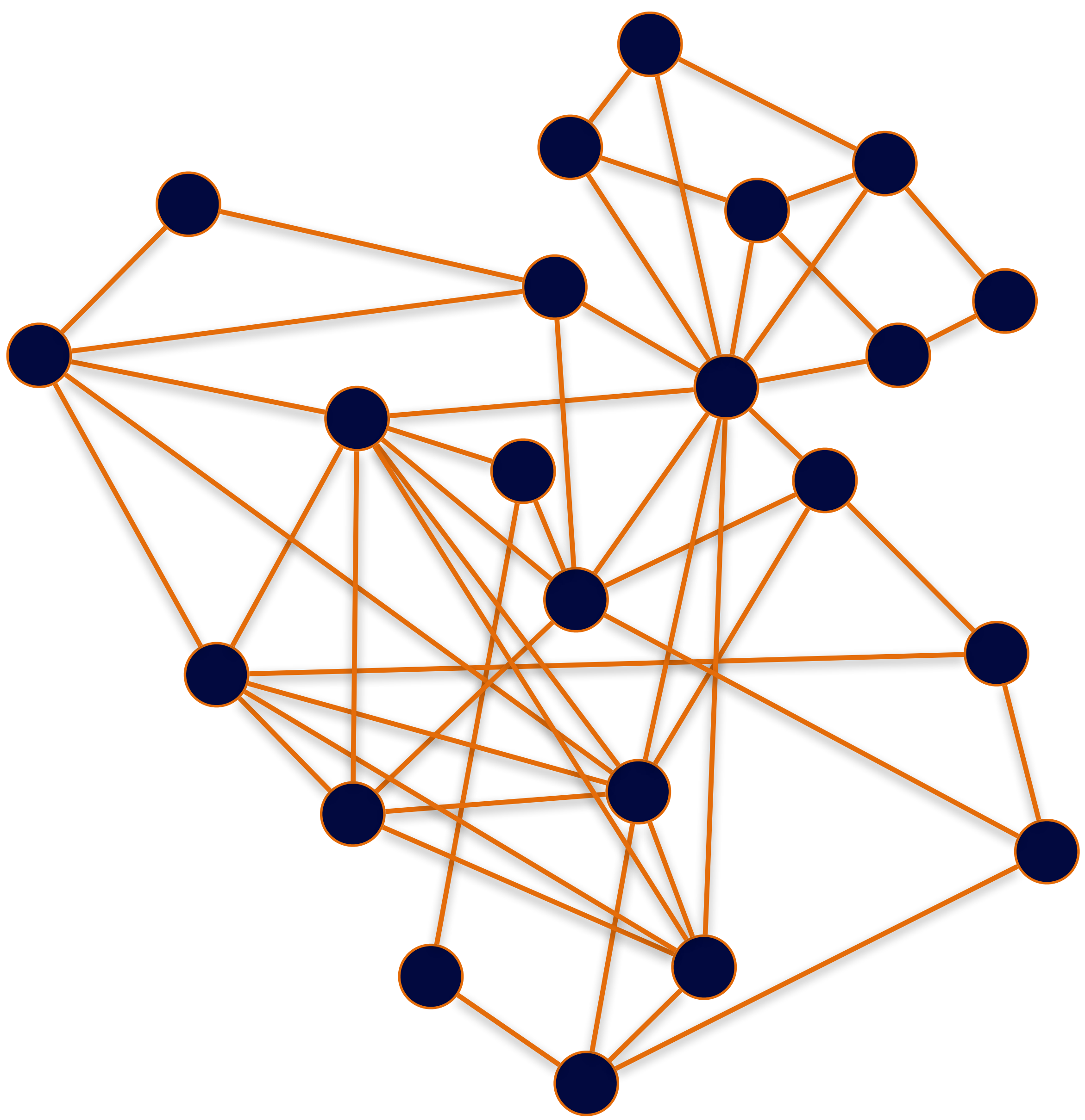
Alex Dimakis
Constantine Caramanis

Densest k-Subgraph (DkS)



Given

graph and a **parameter k**

Find

k vertices containing **most edges**

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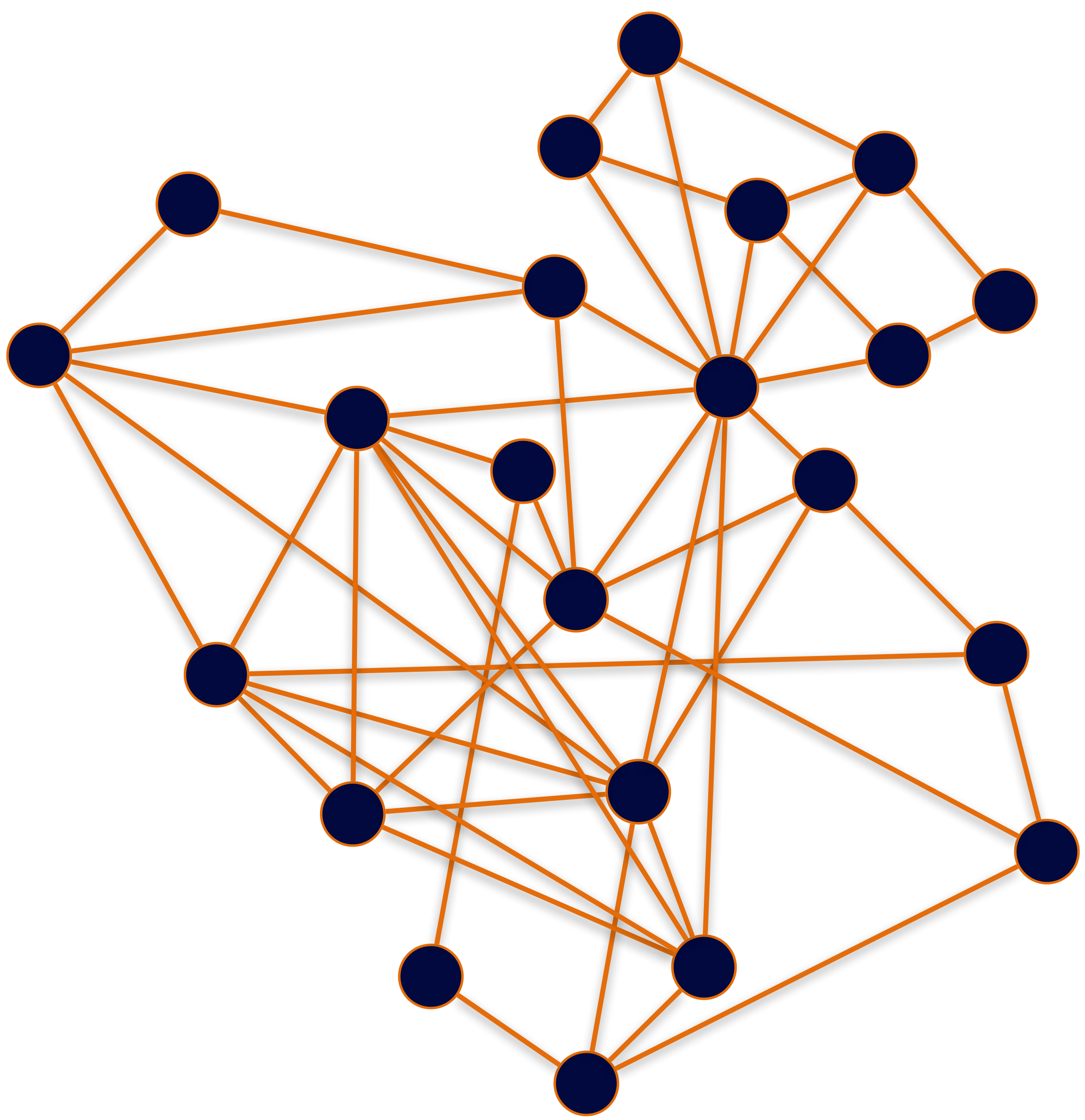

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Applications

Community Mining

communities = large dense components

Link Spam Detection

dense parts of web: ***spam***

Computational biology

complex patterns in gene annotation graphs

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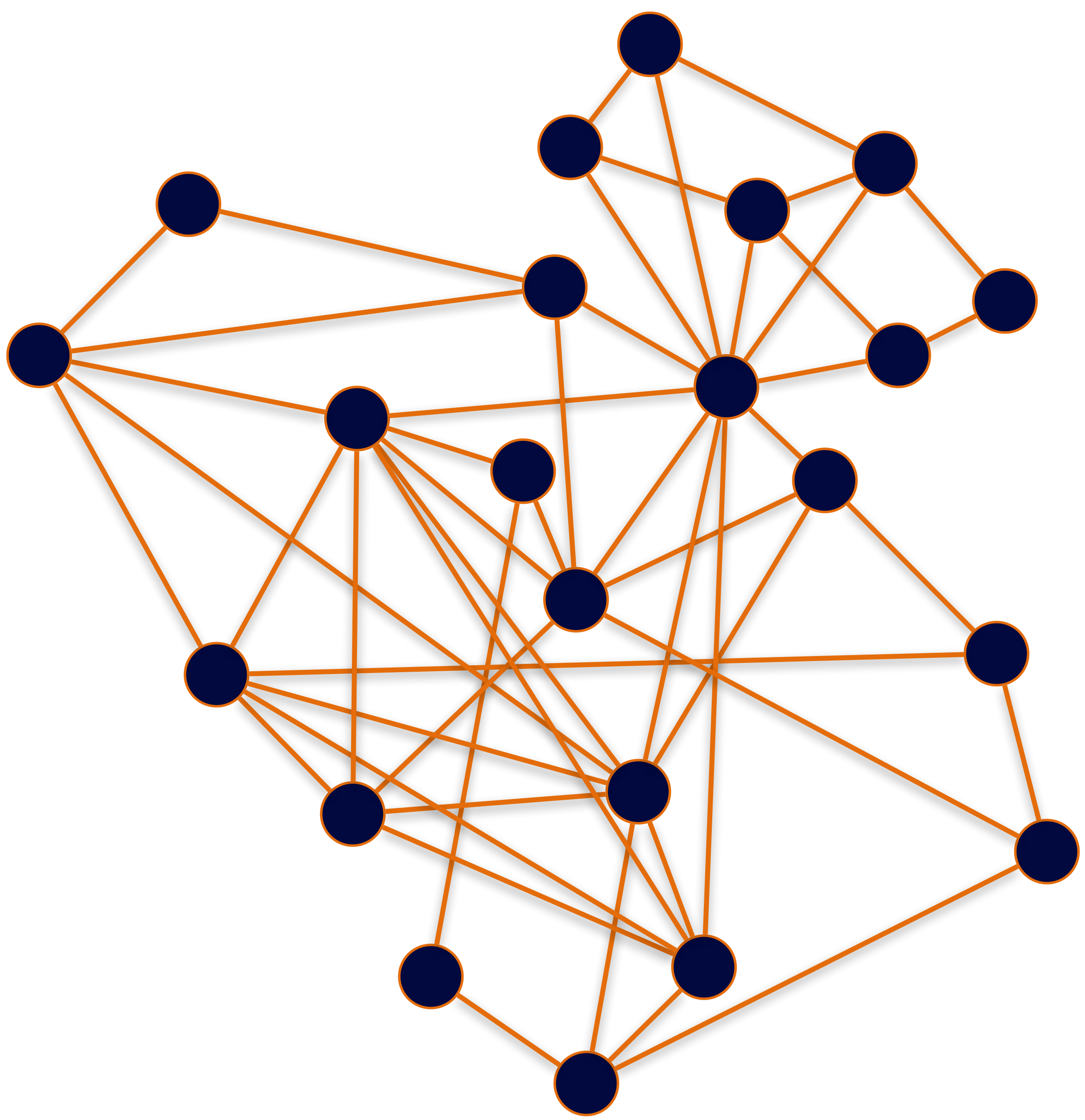

There is a

5-subgraph with 10 edges

Q: Can you find it?

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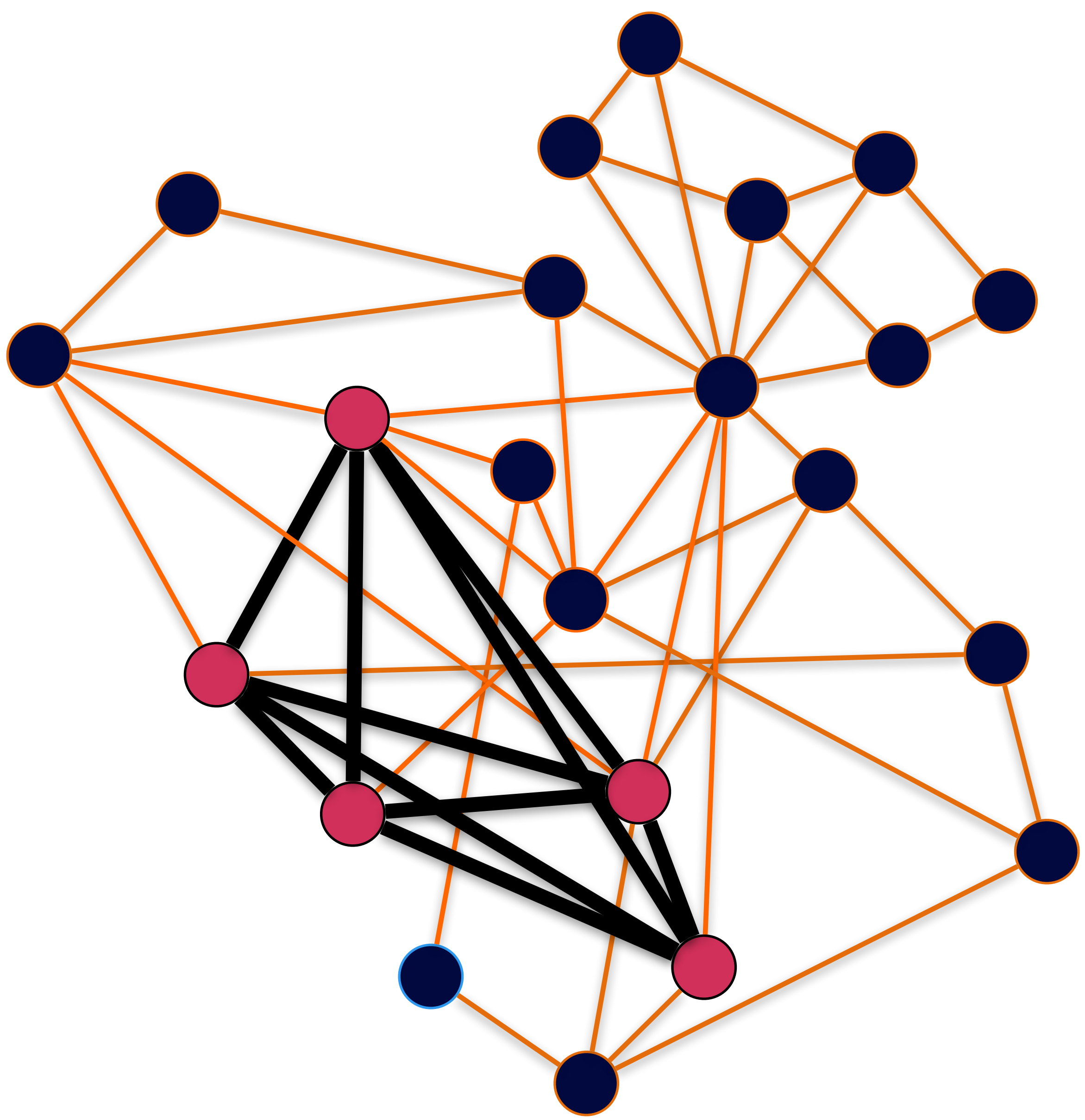

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Hard to approximate

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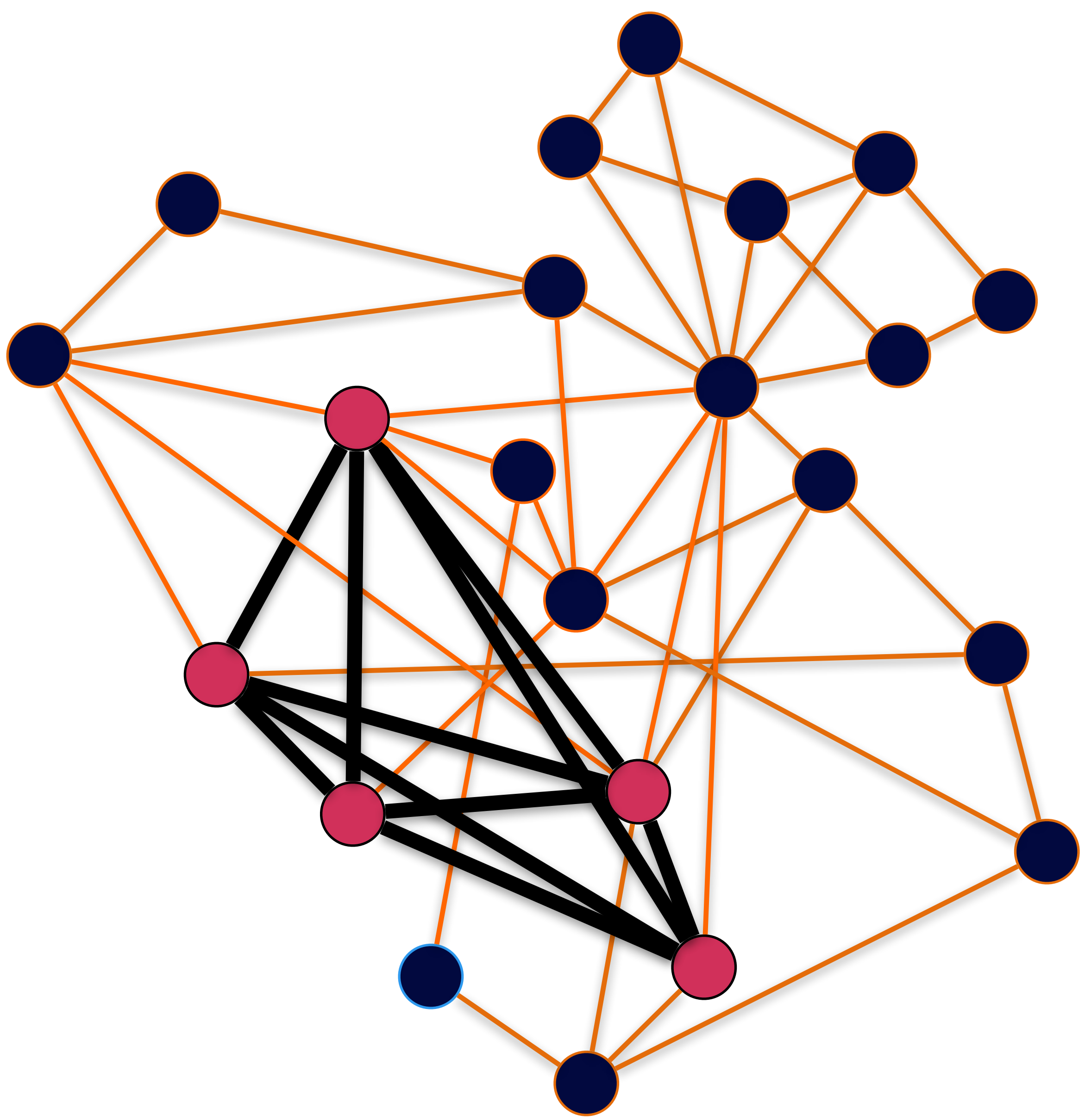

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[Khot, 2004]

*Except in specific cases: [Arora et al 95]
(1+ε) approx. for linear subgraphs of dense graphs

Worst-Case Analysis

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After long effort, [Feige, 2001], [Bhaskara et al., STOC ’10]

Best known **ratio**

$$\text{Approx} \geq \frac{\text{OPT}}{n^{0.25}}$$

10-factor approx. for graphs with **10K nodes**

100-factor approx. for graphs with **100 Million nodes**

Known DkS guarantees are not useful in practice…

under worst case analysis

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Q1: Provable, graph-dependent bounds?

Q2: DkS on billion-scale graphs?

Beyond the Worst Case

New DkS algorithm:

Graph-dependent bounds

In practice: $\text{Approx} \geq 0.7 \cdot \text{OPT}$

Scalable

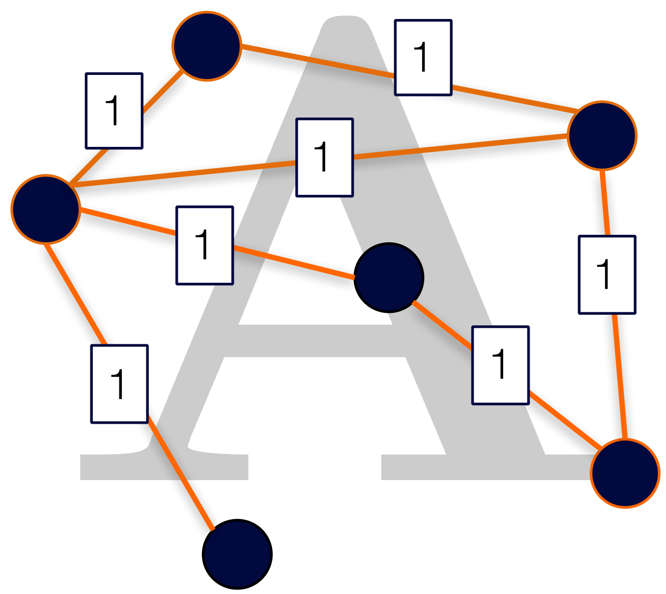
nearly-linear times for many real-world graphs

Parallelizable

implementation in **MapReduce+Python**

up to **billion-edge** graphs on **800 cores on Amazon EC2**

Our Low-Rank Framework



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Low-rank DkS is related to original DkS

Results: Theory

Graph-dependent Guarantees

$$\text{density} = \frac{2 \cdot \#\ \text{edges in subgraph}}{k} \qquad \text{(av.degree)}$$

Theorems:

Algorithm computes in **time O(n^{d+2}/δ)** a *k*-subgraph with **density**

$$\mathrm{OPT}_d \geq \mathrm{OPT} \cdot 0.5 \cdot (1-\delta) - 2|\lambda_{d+1}|$$

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Our algorithm computes in **time** $O(|E| \cdot \log n + \frac{n}{\epsilon^d})$

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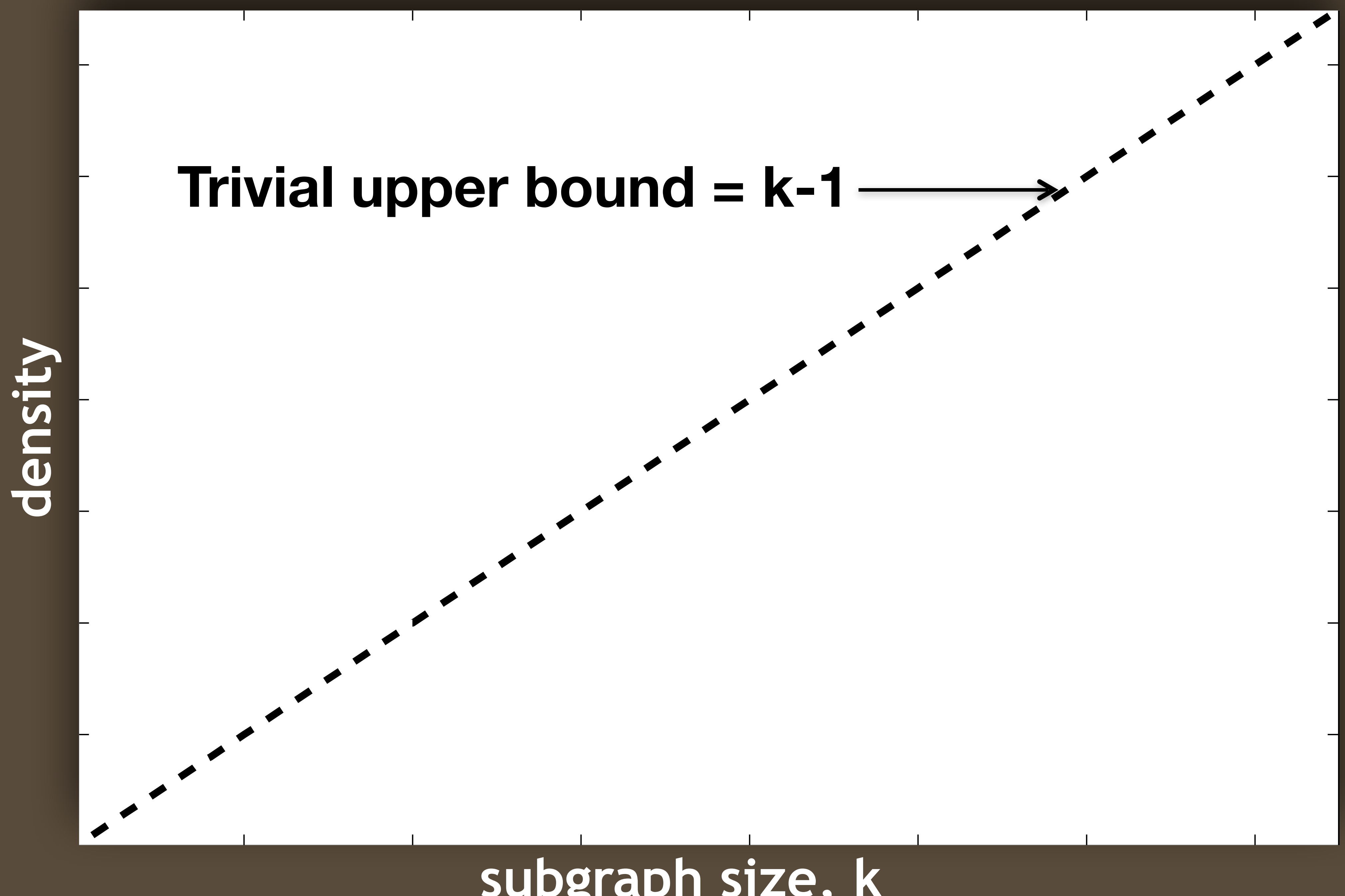
larger *d* => **better** approximation, **slower** computation

Performance in Practice

com-LiveJournal graph

4M nodes, 35M edges



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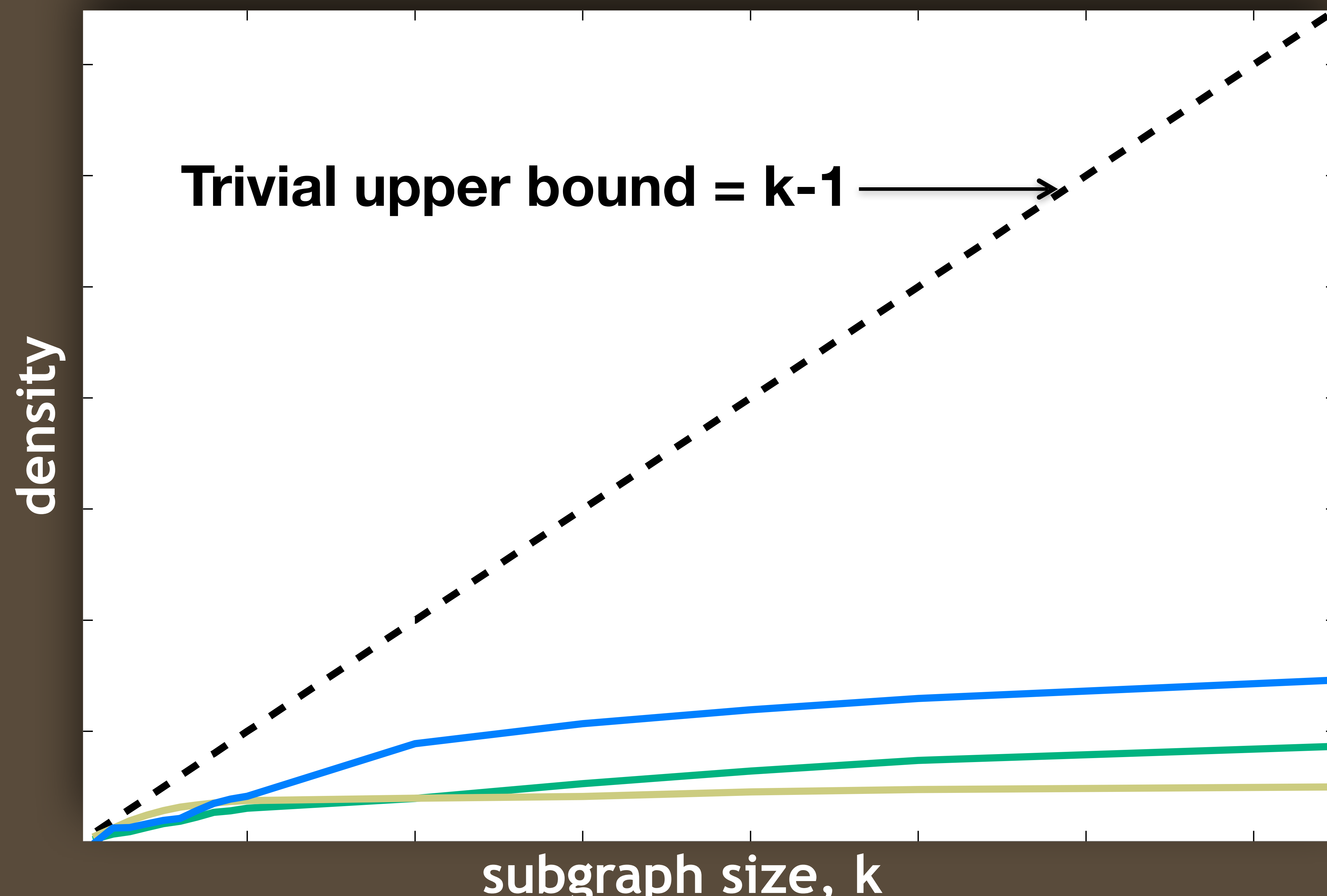

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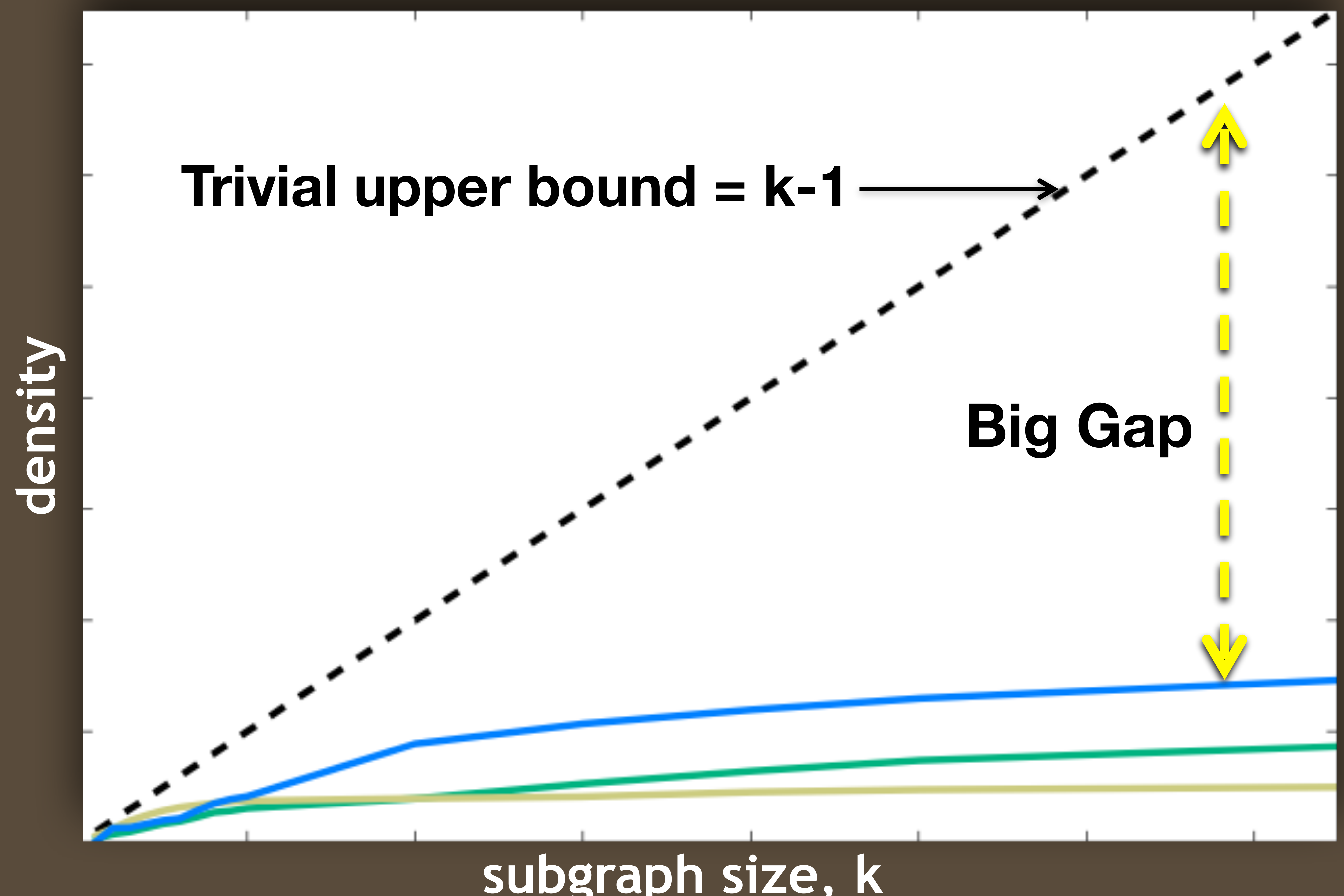

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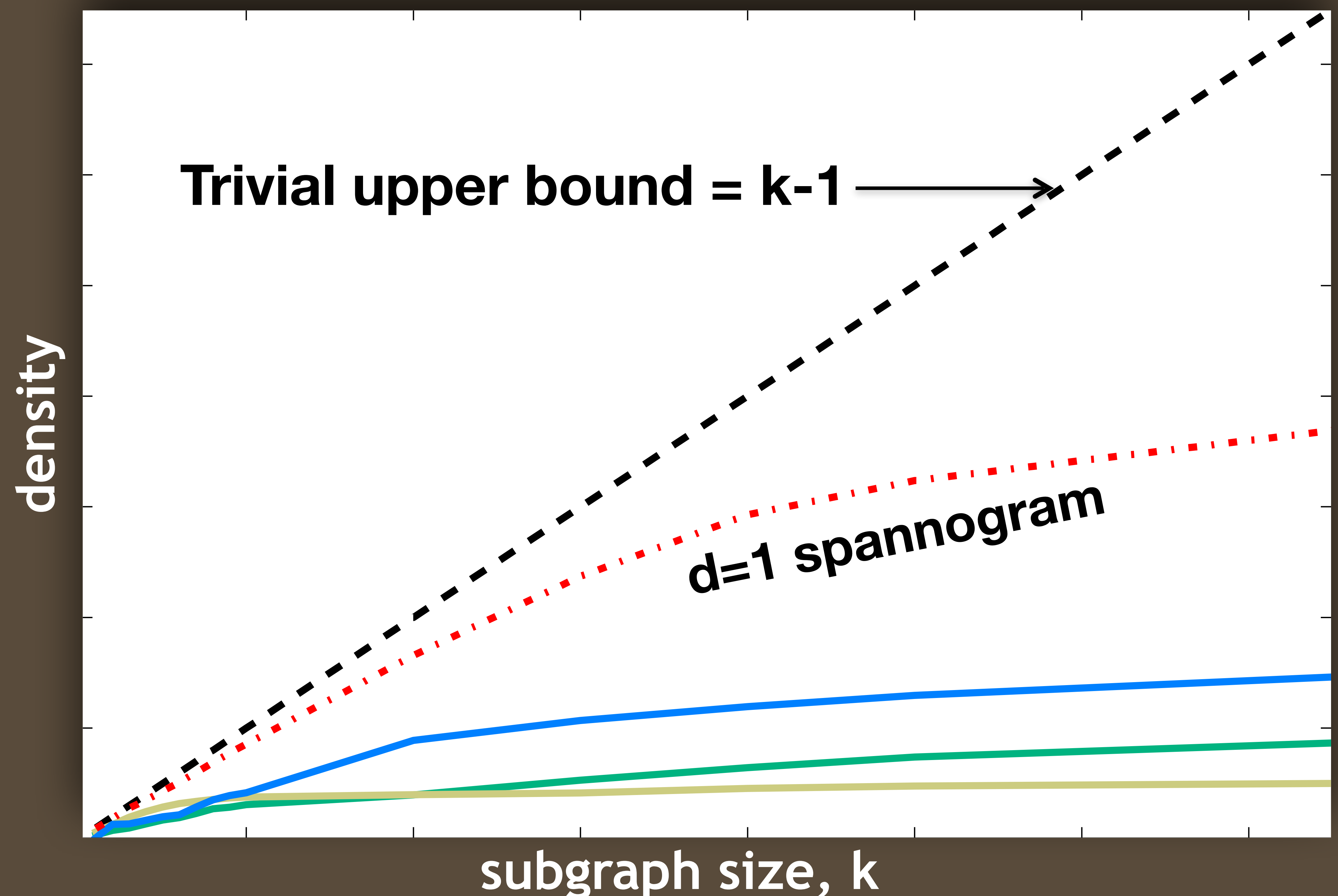

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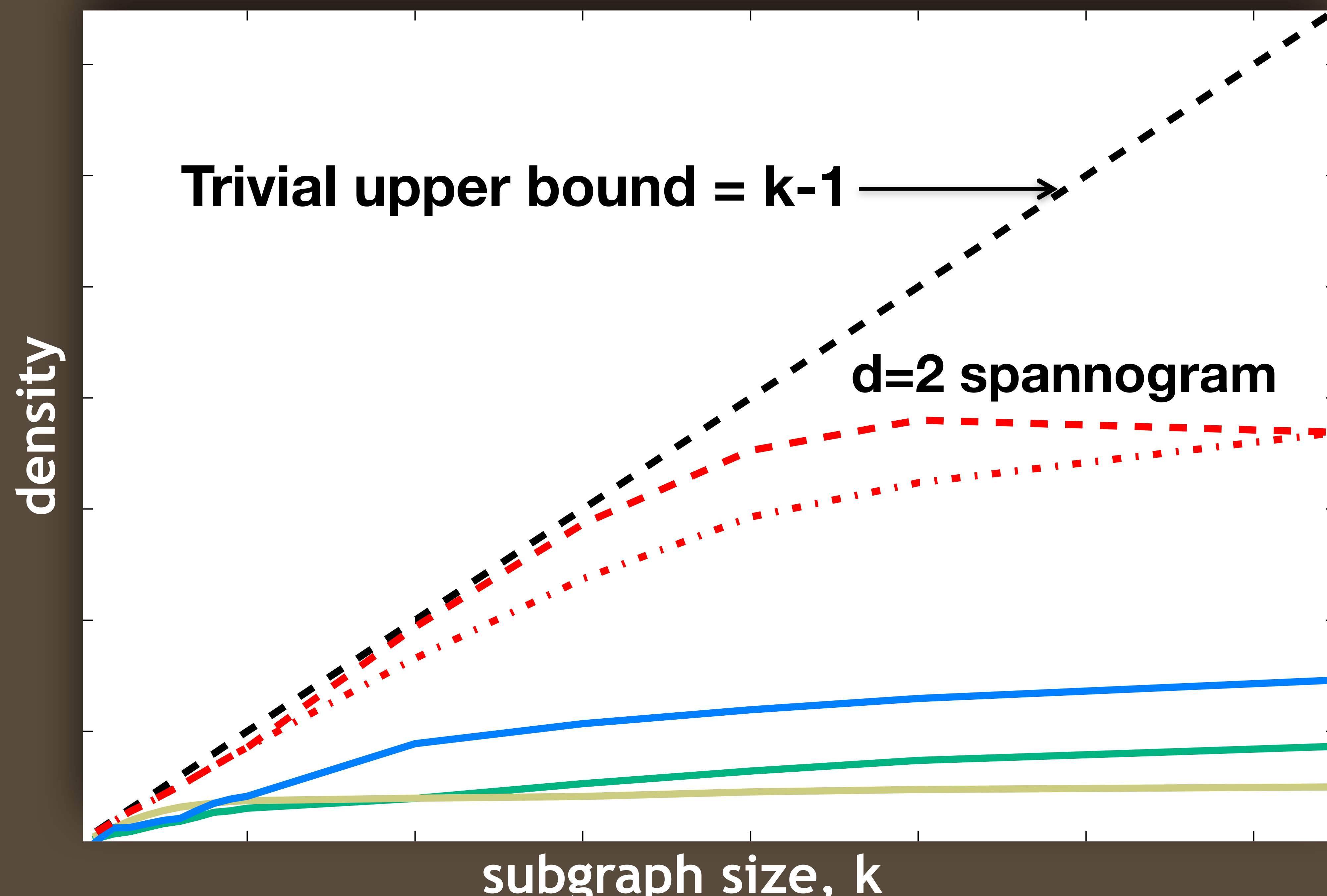

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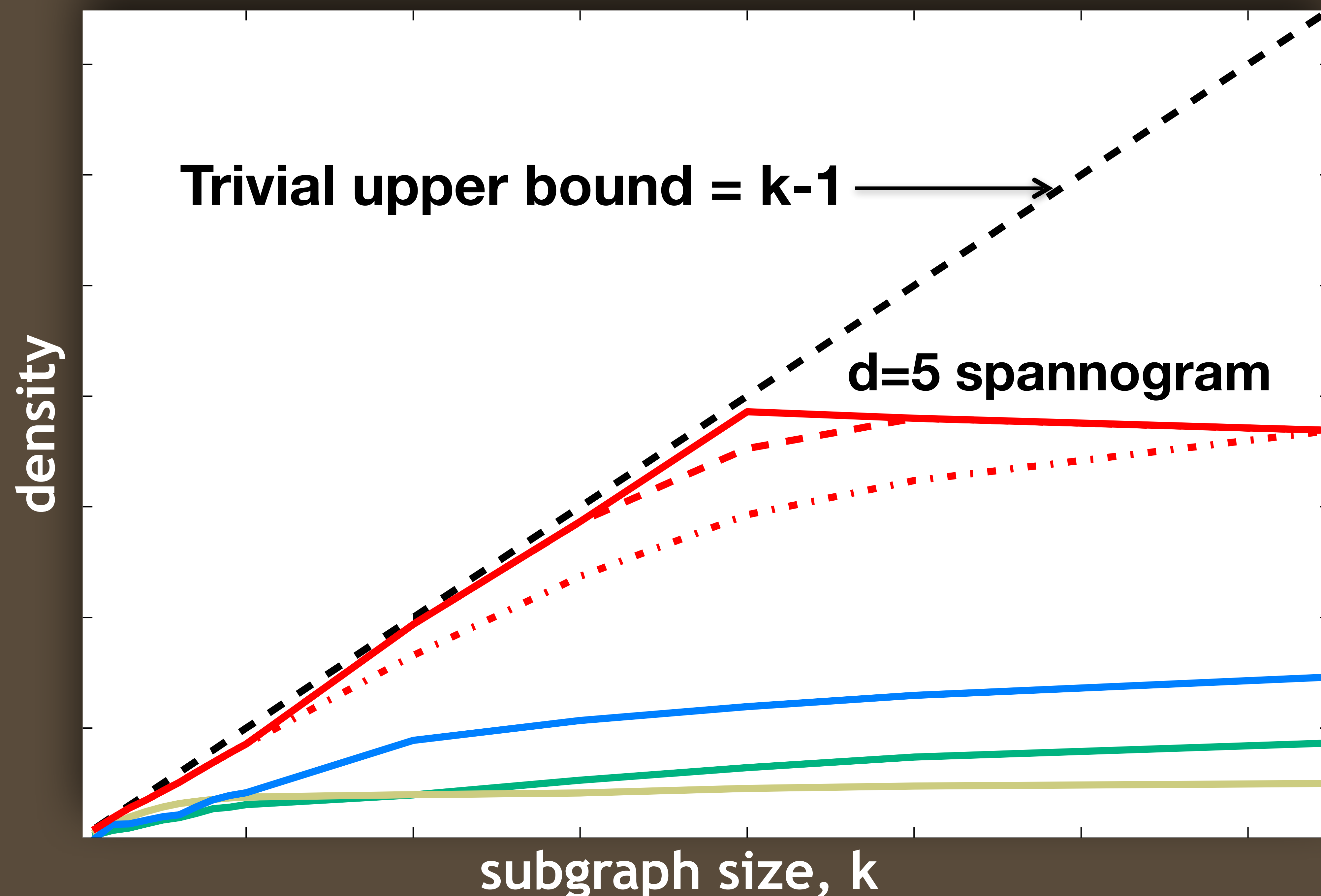

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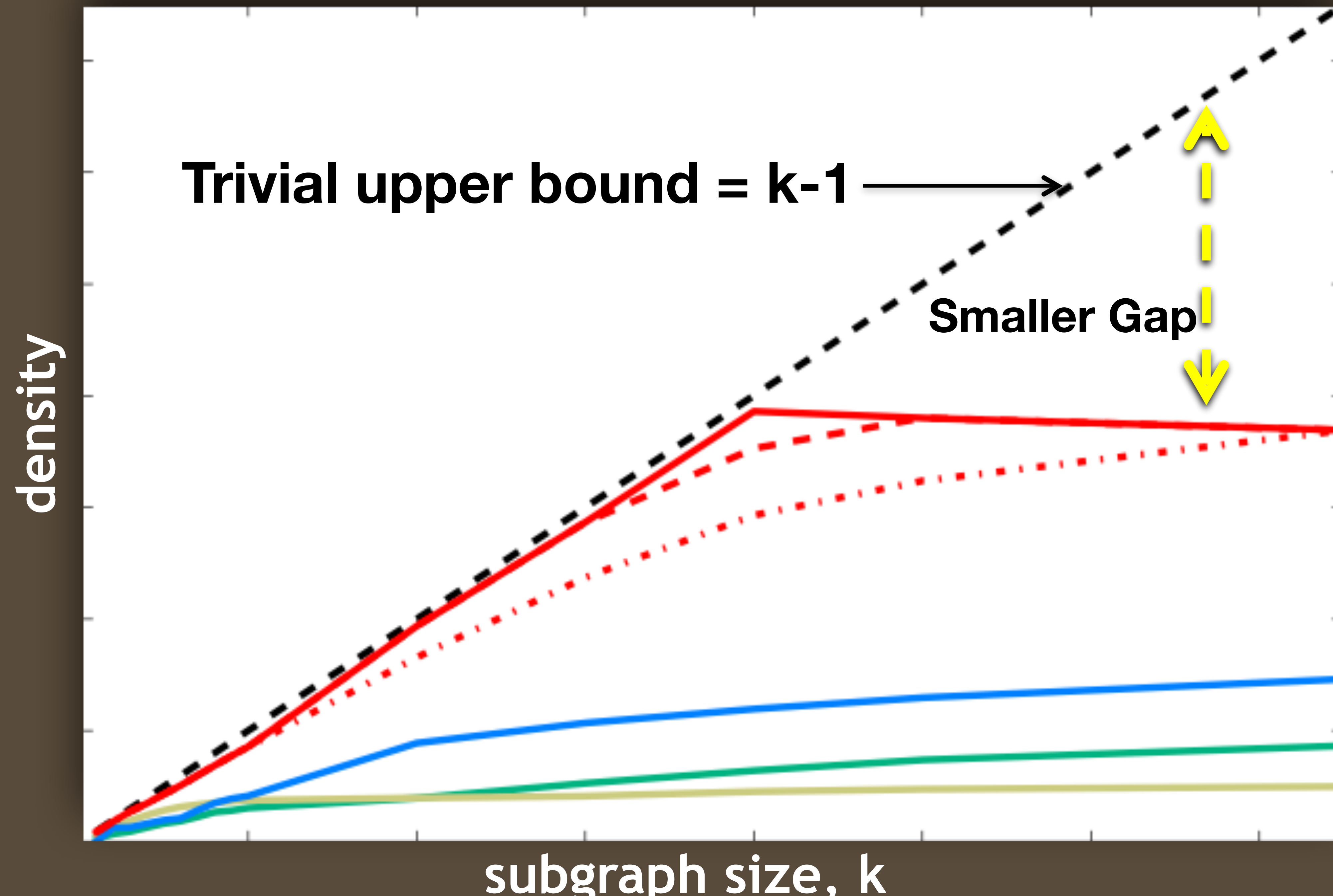

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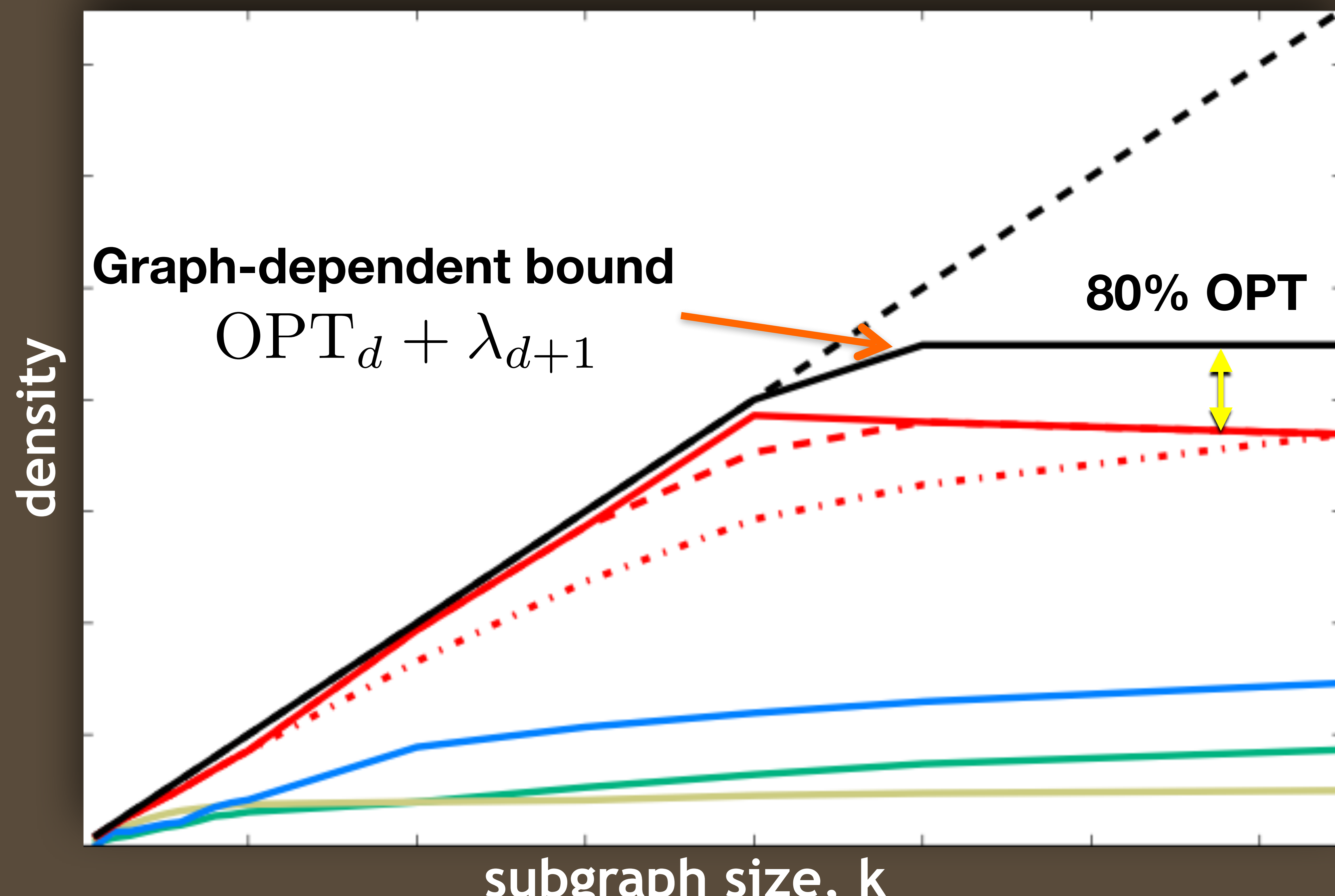

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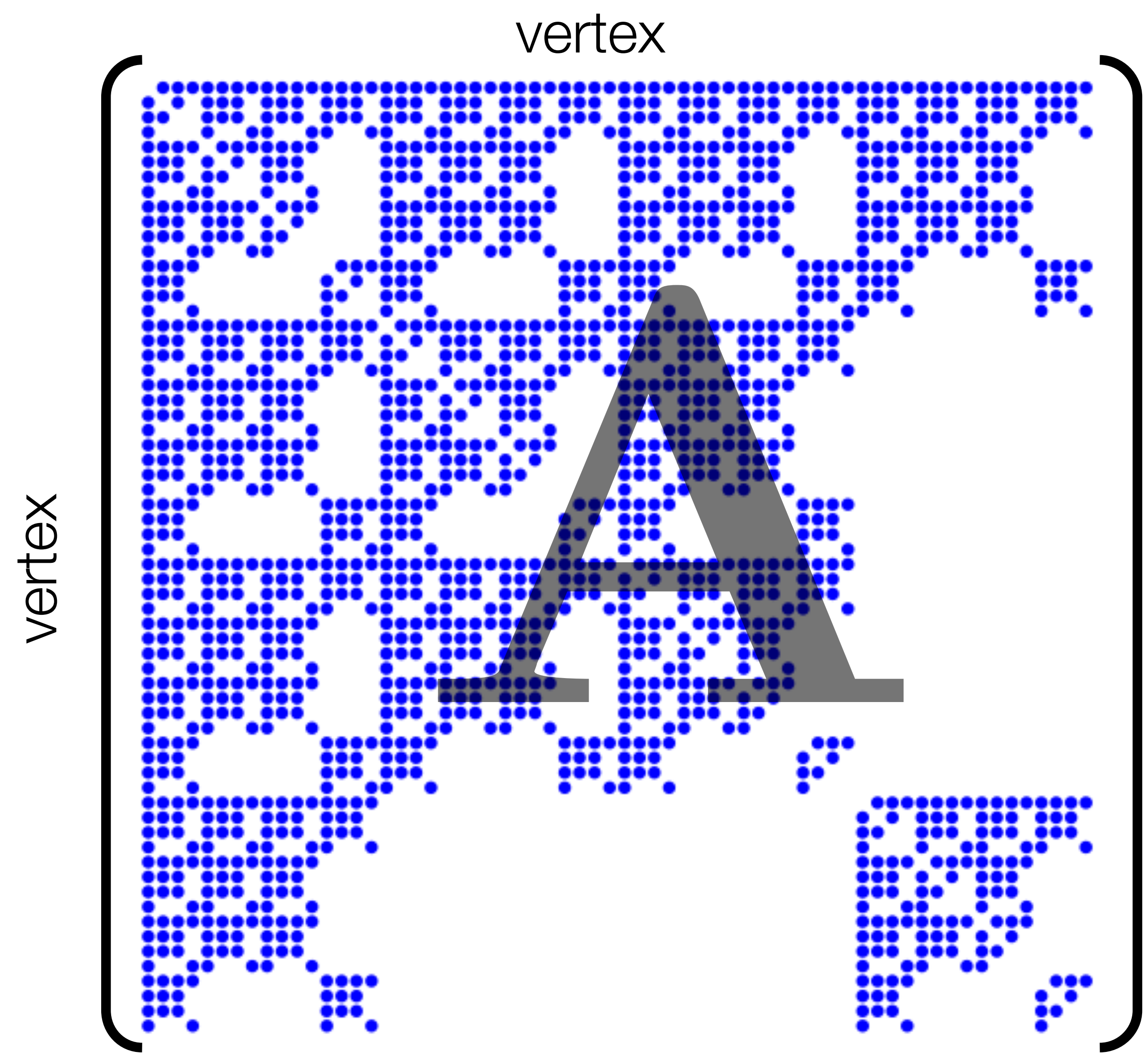
How we do it

DkS via Quadratic Optimization



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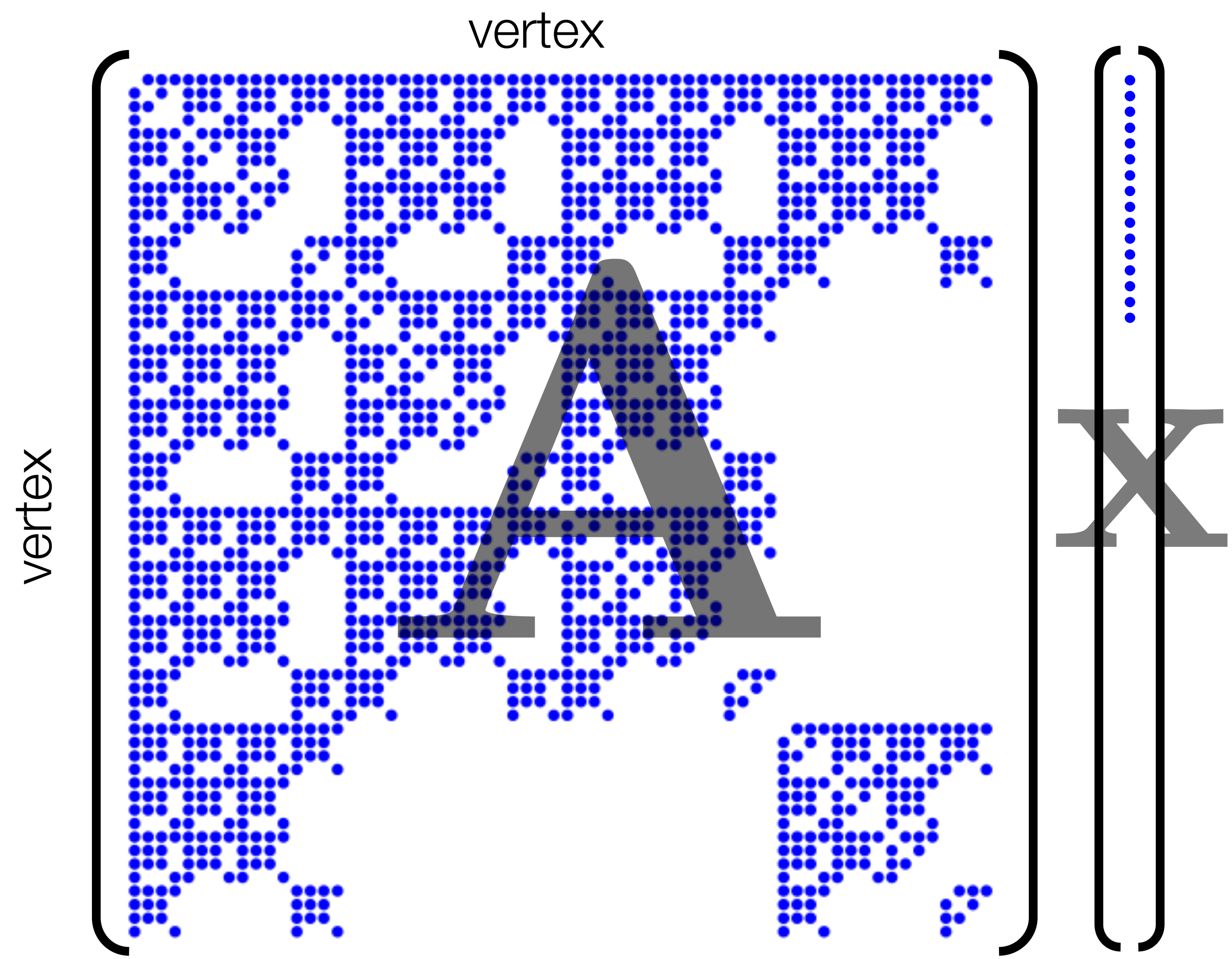

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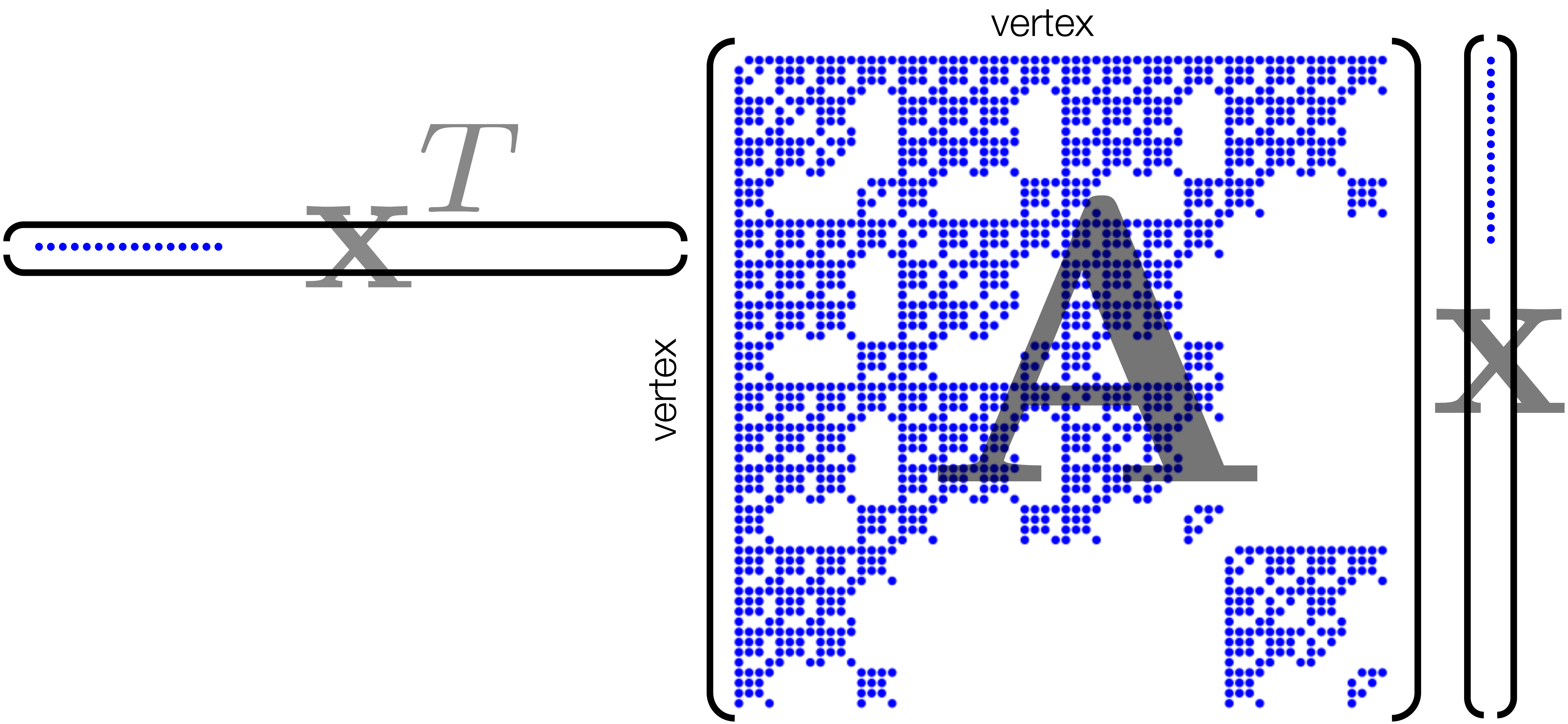

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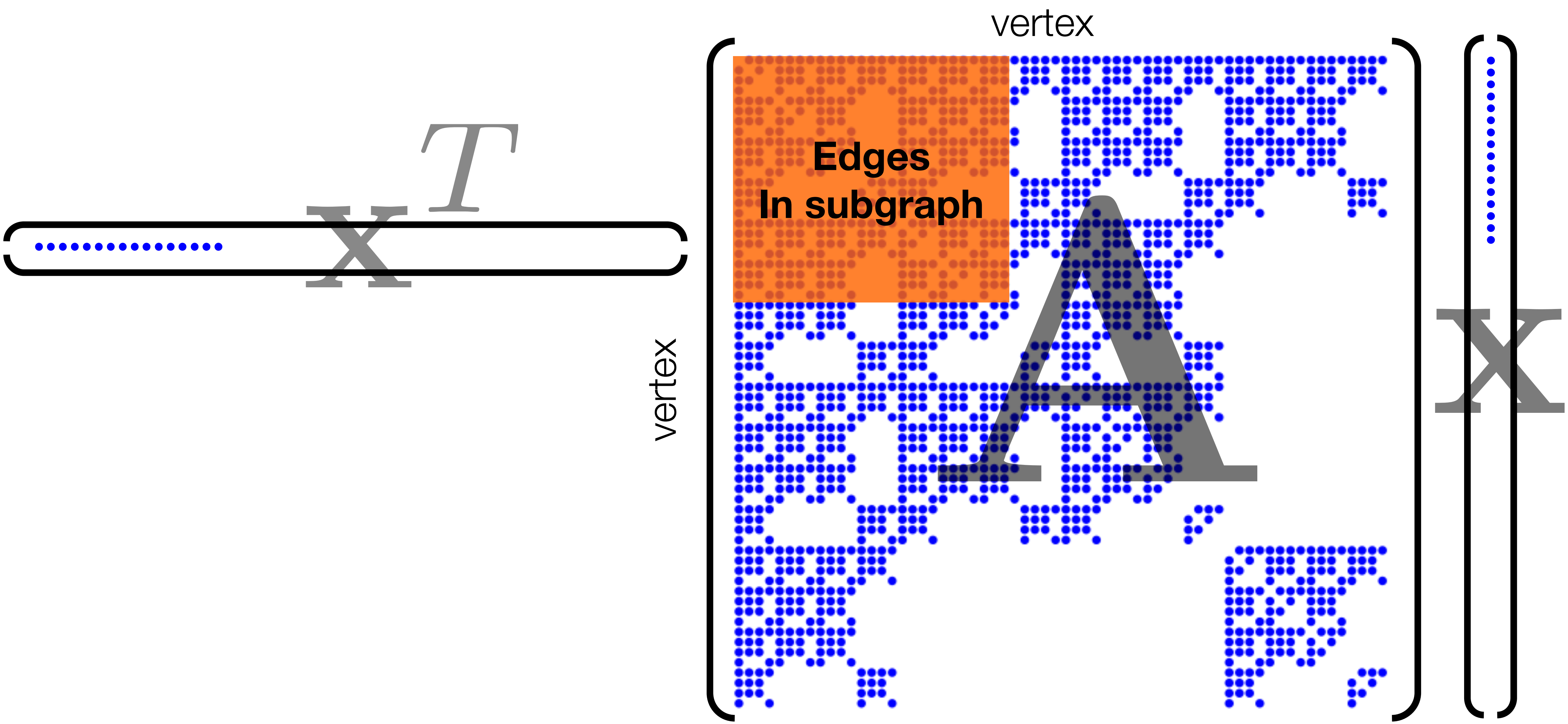

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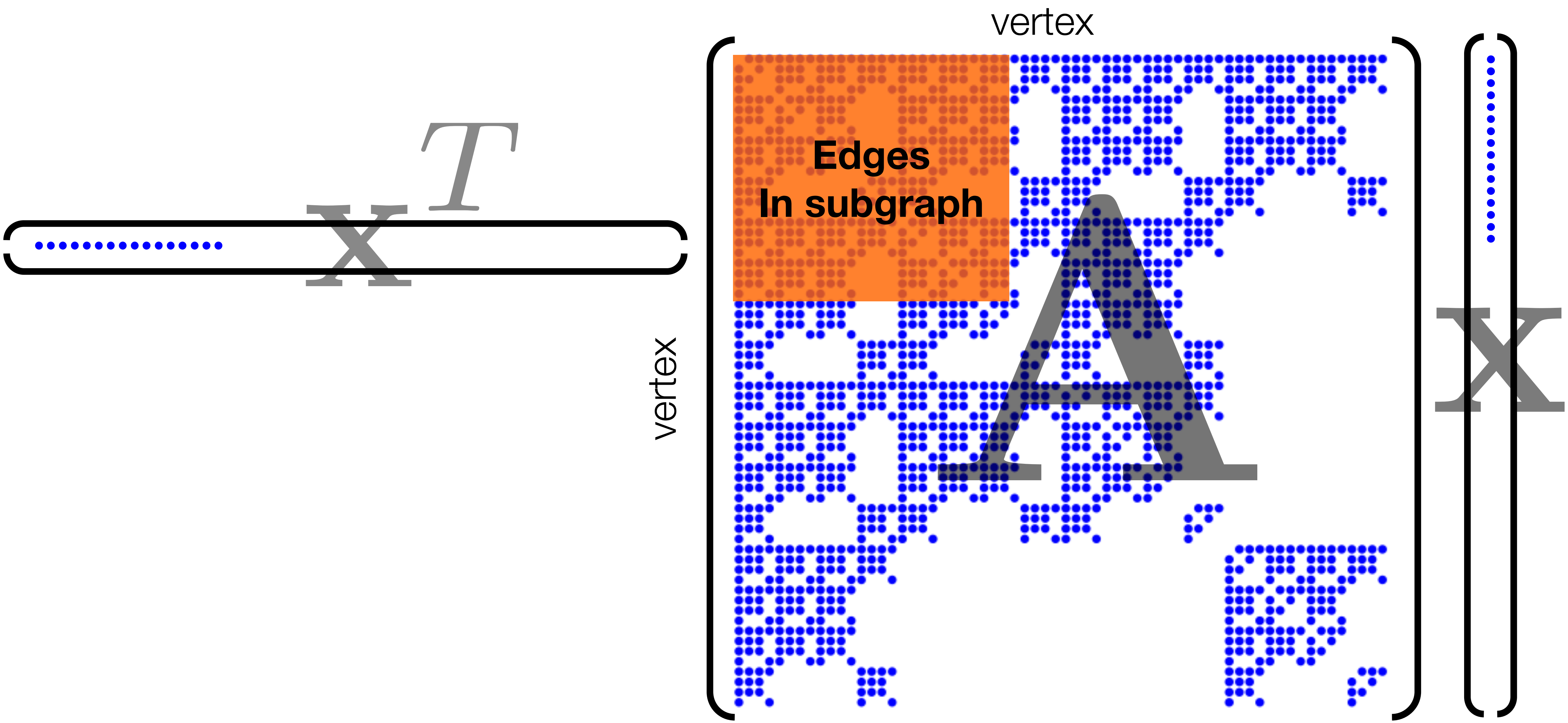

DkS: $$\mathrm{OPT} = \max_{\substack{\mathbf{x} \in \{0, 1/\sqrt{k}\}^n \\ \|\mathbf{x}\|_0 = k}} \mathbf{x}^T \mathbf{A} \mathbf{x}$$

DkS via Bilinear Optimization

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Lemma:
ρ-approximation for DBkS = ½ρ-approximation for DkS

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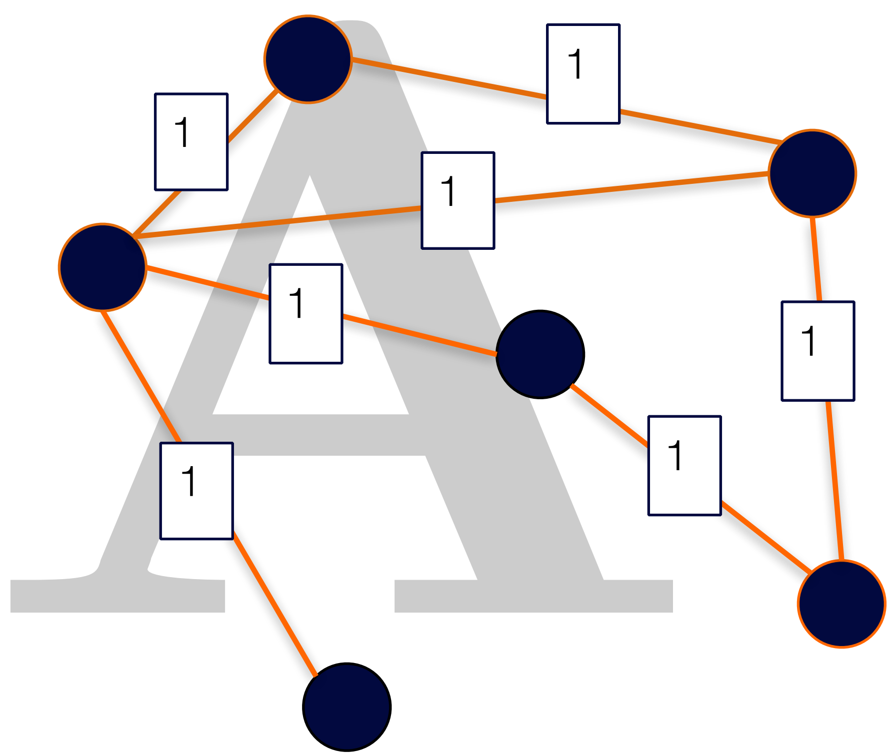

Low-Rank Approximation

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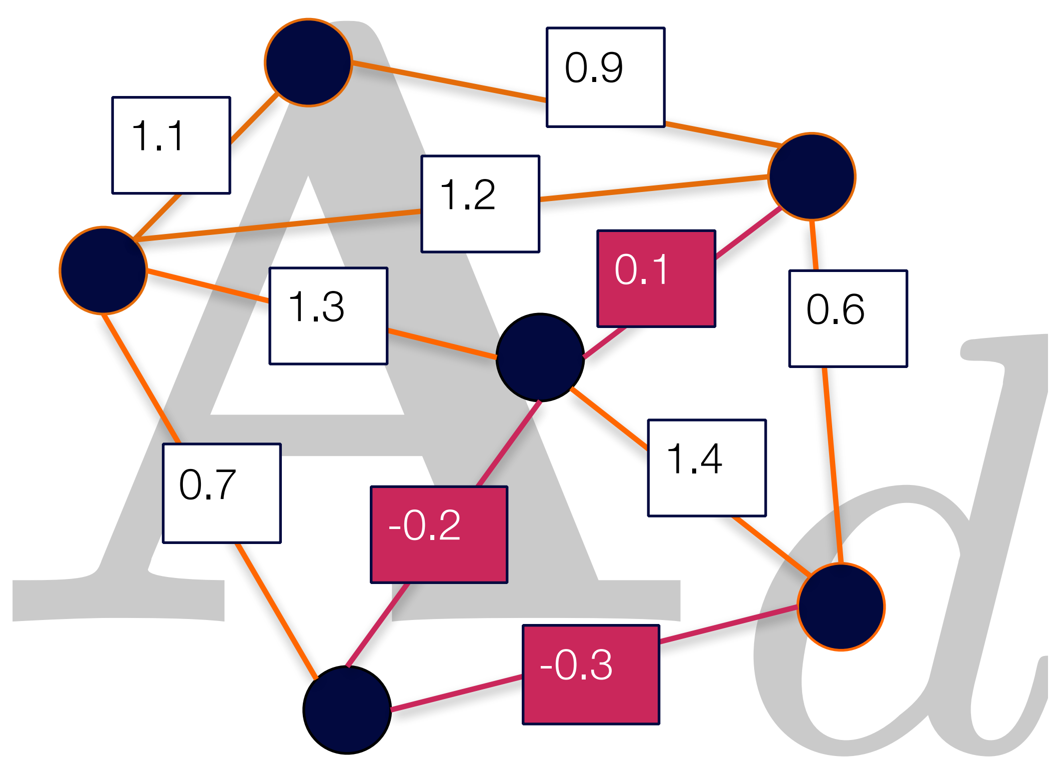

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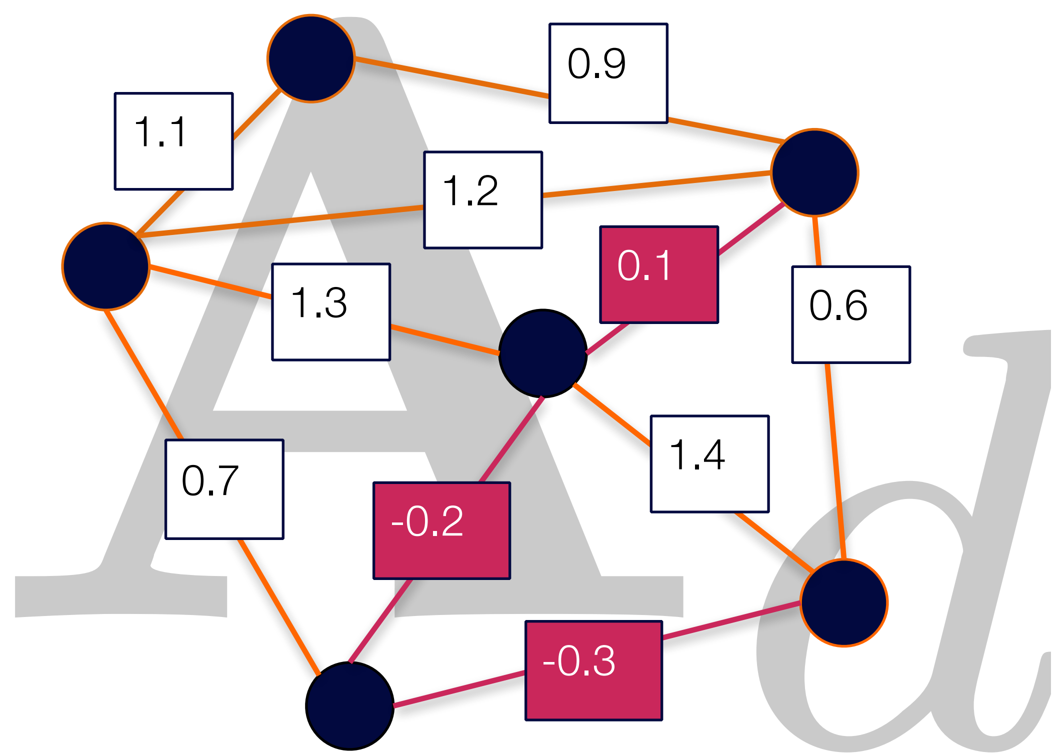

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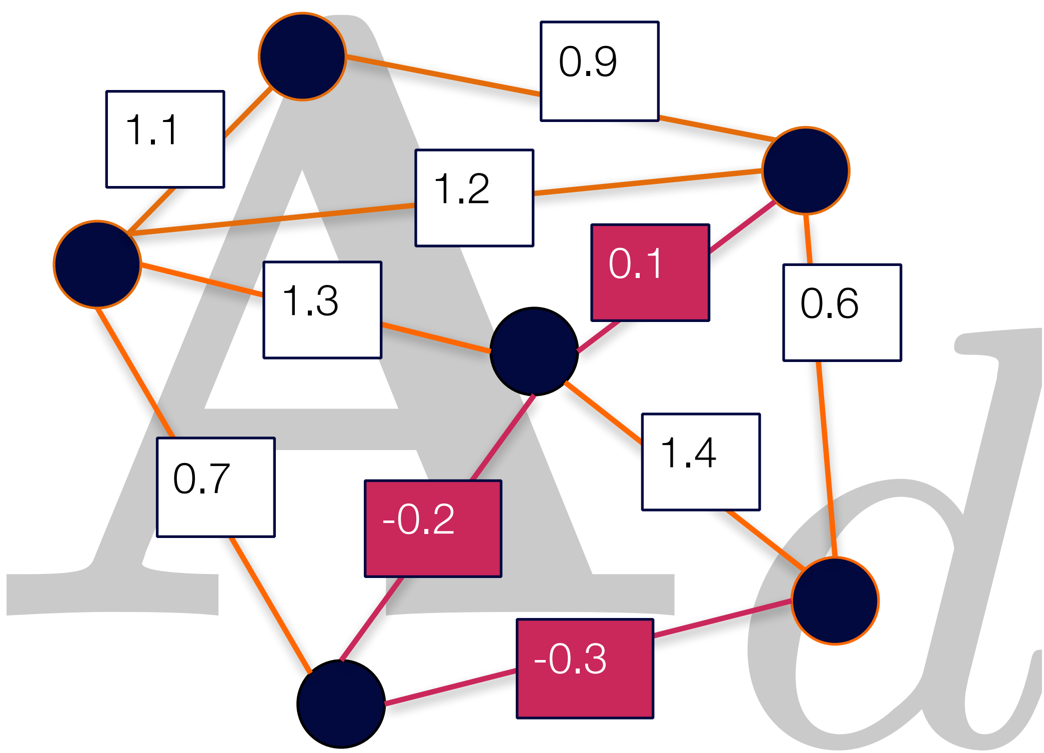

Efficiently solvable

How the Low-Rank Solver Works

Naïvely: Check all $\binom{n}{k}$ subgraphs

Rank-1 case:

$$\mathbf{A}_1 = \mathbf{v}\mathbf{v}^T$$

Q: Maximize the product of two numbers

$$\max_{\substack{\mathbf{x},\mathbf{y} \in \{0,1\}^n \\ \|\mathbf{x}\|_0 = \|\mathbf{y}\|_0 = k}} (\mathbf{x}^T \mathbf{v}) \cdot (\mathbf{v}^T \mathbf{y})$$

A: Maximize each number **individually**

How the Rank-1 Solver Works

$$\max_{\substack{\mathbf{x},\mathbf{y}\in\{0,1\}^n \\ \|\mathbf{x}\|_0=\|\mathbf{y}\|_0=k}} (\mathbf{x}^T \quad) \cdot (\quad ^T \mathbf{y})$$

top-k set: the k-largest coordinates of a vector,
e.g., if k =2, then top-2 set = {3,4}

Intuition: *x*, *y* pick the top-k set of *v*.

How the Rank-2 Solver Works

$$\max\left(\mathbf{x}^T \begin{bmatrix} 1 & 5 \\ 2 & 2 \\ 3 & 7 \\ 4 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 5 \\ 2 & 2 \\ 3 & 7 \\ 4 & 0 \end{bmatrix}\right)^T \mathbf{y}$$

Intuition: *x*, *y* pick the top-*k* set of a vector from a 2-dimensional span.

Q: How many top-k sets are there in a 2-dimensional span?

Based on **Spannogram** [Asteris, Papail., Karystinos, ISIT2011]

Theorem: # top-*k* sets in a d-dimensional span: $\binom{d}{\frac{d}{2}}\binom{n}{d} = O(n^d)$

Spannogram: Traverses all of them efficiently

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Randomized algorithm

Take random points: $s_1, \ldots, s_{1/\epsilon^d} \in \mathrm{span}(v_1, \ldots, v_d)$

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Practically linear time

Implementation

MapReduce Implementation

```
def spannogram_mapper(self, coordinate, values):
    i = coordinate
    for j in range(int(self.options.rowcount)):
        yield j, values

def spannogram_reducer(self, coordinate, values):
    k = 5
    V = list(values)
    i = coordinate
    Vc = []
    opt_support = [];
    opt_metric = 0;
    for j in range(i+1, int(self.options.rowcount)):
        x = []
        Vc = []
        Vtemp = []
        # compute c_ij intersection vector
        x = [V[i][l]-V[j][l] for l in range(2)]
        # cumpute v_ij = Vc_ij
        Vc = [V[l][0]*x[1]-V[l][1]*x[0] for l in range(int(self.options.rowcount))]
        # find top and bottom support
        top_support_pos = zip(*heapq.nlargest(k, enumerate(Vc), key=operator.itemgetter(1)))[0]
        top_support_neg = zip(*heapq.nsmallest(k, enumerate(Vc), key=operator.itemgetter(1)))[0]
        # compute support metric
        Vtemp = [V[s] for s in top_support_pos]
        metric_pos = sum([x**2 for x in [sum(a) for a in zip(*Vtemp)]])
        Vtemp = []
        Vtemp = [V[s] for s in top_support_neg]
        metric_neg = sum([x**2 for x in [sum(a) for a in zip(*Vtemp)]])
        # find locally optimal support
        metric_list = [opt_metric, metric_pos, metric_neg]
        metric_index = metric_list.index(max(metric_list))
        opt_support = [opt_support, top_support_pos, top_support_neg][metric_index]
        opt_metric = max(metric_list)

        yield i, [opt_metric, opt_support]
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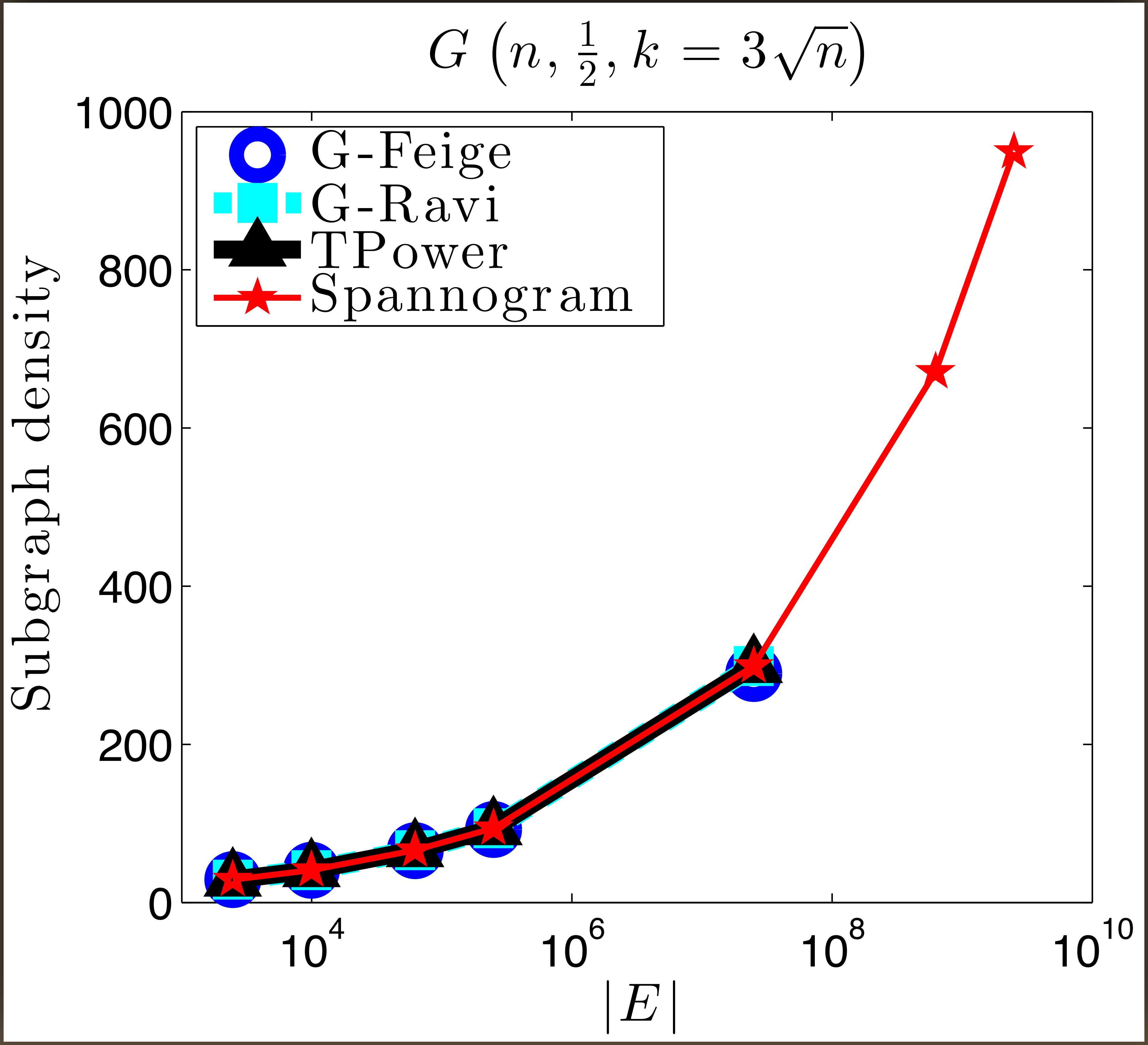
git.io/spannogram

Billion-scale Graphs



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- Empirically outperforms previous state of the art
- Highly scalable implementation

Thank you

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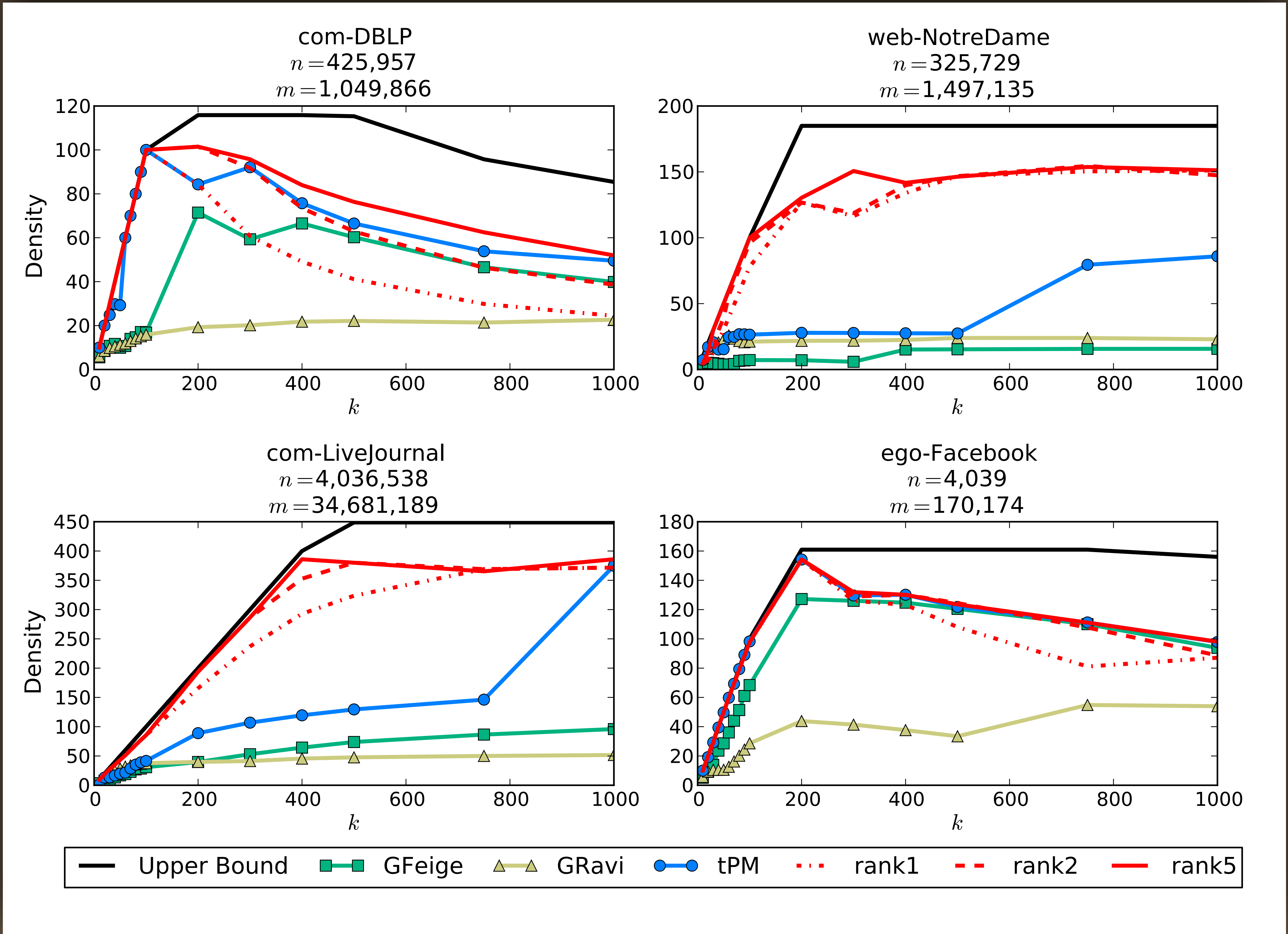
Backup slides

Other experiments



Randomized Algorithm

Step 1

Take random points: $s_1, \ldots, s_{1/\epsilon^d} \in \mathrm{span}(v_1, \ldots, v_d)$

Step 2

Find largest k entries: $\mathrm{top}_k(\mathbf{s}_i)$

Step 3

Compute density of corresponding subgraph

Randomized Algorithm

Step 1

Take random points: $s_1, \ldots, s_{1/\epsilon^d} \in \text{span}(v_1, \ldots, v_d)$

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Compute density of corresponding subgraph

Practically **linear time**