Numerical Analysis Project
MATH 5600
Homework 2

Authors:
Dane Gollero · Ike Griss Salas · Magon Bowling

Due: March 3, 2021

Contents

1 Homework Problems

-1- (Taylor Series.) Let

$$f(x) = e^x \text{ and } g(x) = ln(x+1) \tag{1}$$

and let p_n and q_n be the Taylor polynomials of degree n for f and g, respectively, about

$$x_0 = 0. \tag{2}$$

Plot the graphs of f, g, p_n and q_n, for some small values of n, and comment on you results. Discuss in particular how well f and g are approximated by their Taylor polynomials. Explain your observations in terms of a suitable expression for the error in the approximation.

Recall the Taylor Series

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(\alpha) \cdot \frac{(x-\alpha)^i}{i!}.$$

We know that $\forall\ i \in \mathbb{N}$,

$$f(x) = e^x \Rightarrow f^{(i)}(0) = 1.$$

Thus,

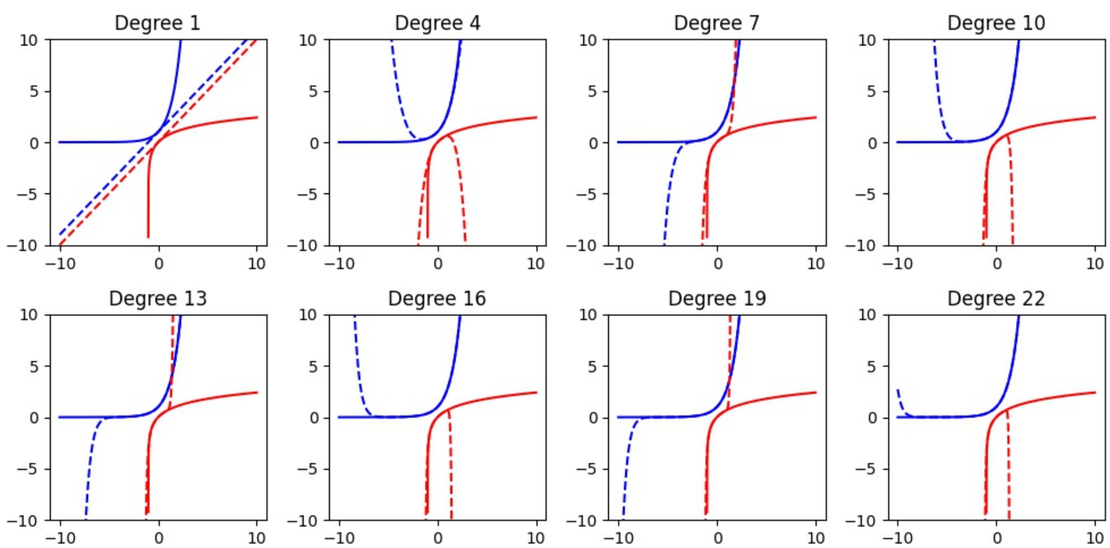
$$f(x) = e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

We also have

$$
\begin{aligned}
g(x) &= \ln(x+1) = 0 \quad \text{for } x = 0 \\
g'(x) &= \frac{1}{x+1} = 1 = 1! \\
g''(x) &= \frac{-1}{(x+1)^2} = -1 = -1! \\
g'''(x) &= \frac{2}{(x+1)^3} = 2 = 2! \\
g^{(4)}(x) &= \frac{-6}{(x+1)^4} = -6 = -3! \\
&\vdots \\
g^{(n)}(x) &= \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} = (-1)^{n-1}(n-1)! \quad \text{for } x = 0, n > 0
\end{aligned}
$$

$$
\implies \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)! \cdot \frac{x^n}{n!}.
$$

Below, we will illustrate some Taylor polynomial approximations for f and g, respectively.

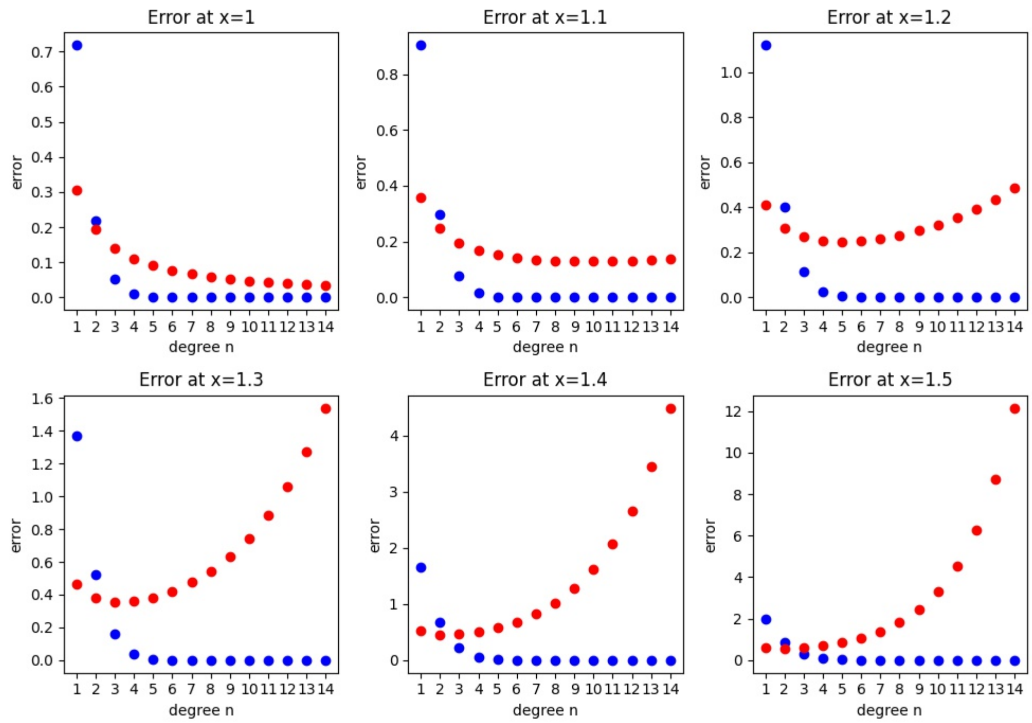


Qualitatively we can see the Taylor polynomial, p_n, begins to better approximate e^x as n gets larger. However, the Taylor polynomial, q_n, tends to be problematic for larger n. It seems that q_n doesn't approximate $g(x)$ very well for $x > 1$, despite increasing values of n. We define the suitable error functions below.

$$
E_n^f(x) = \left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right|
$$

$$
\begin{aligned}
E_n^g(x) &= \left| \ln(x+1) - \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \cdot \frac{x^k}{k!} \right| \\
&= \left| \ln(x+1) - \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} \right|
\end{aligned}
$$

We show the graphs for our suitable errors as a function n for fixed $x \geq 1$.



As we can see increasing our polynomial degree for q_n does not imply a better approximation everywhere.

-2- (A "simple" program.) Write a program that reads n and the entries x_1, x_2, ..., x_n of a vector $x \in \mathbb{R}^2$ from standard input and prints

$$||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

to standard output. Mail me your code before the lecture on March 3.

```python
from math import sqrt

def norm(vec):
    square = [i**2 for i in vec]
    square_sum = sum(square)
    size = sqrt(square_sum)
    return size
```

-3- (Some Iteration.) Consider the iteration

$$x_{n+1} = F(x_n) = \sin x_n, \quad x_0 = 1 \tag{3}$$

(where of course the angle is measured in radians). What does our theory tell us about convergence? Show that the iteration does converge! What is the limit? How fast does the iteration converge? Carefully explain the effects of rounding errors.

We have the fixed point iteration $x_{n+1} = g(x_n)$ converges if we have $|g'(\alpha)| < 1$ where $g(\alpha) = \alpha$, and x_0 is sufficiently close to α. In our case we have $g'(\alpha) = \cos(0) = 1$. On its own tells us nothing, thus our convergence theory is not particularly helpful. However, we can say that it indeed converges with $\sin(x)$ monotonically decreasing and bounded below by 0. Therefore, it must converge. In fact, because $\sin(x)$ is continuous, then as $x_n \to 0$ as $n \to \infty$. We can simply say the limit is zero by

$$\lim_{x \to 0^+} \sin(x) = \sin(0) = 0.$$

This converges much slower than linearly because $g'(\alpha) \neq 0$. Below we show a table of runtimes for achieving certain error tolerances for this iteration. [1]

Error	Steps	Runtime (seconds)
0.1	295	5.39×10^{-5}
0.01	29992	0.005967
0.001	2999989	0.677399
0.0001		

We can see that $x_{n+1} = \sin(x_n)$ will take a substantial amount of time to converge.
The issues of rounding errors can have catastrophic effects on the ability for our sequence to converge. If we have machine that only computes to say first 3 decimal points (i.e., it rounds), then the sequence will never converge. The iteration is so slow to converge that 3 decimal places of information is much too small to maintain accuracy. In fact, this becomes a constant when running it in Python. Letting $x_0 = 1$ we have

n	$\sin(x_n) = x_{n+1}$ (rounded)
0	0.841
1	0.745
2	0.678
$\vdots$	$\vdots$
125	0.145
126	0.144
127	0.144
128	0.144
$\vdots$	$\vdots$

The iteration settles on 0.144 after 126 steps.[2] Now of course, we know that $\sin(x_n) = x_{n+1}$ for $x_0 = 1$ does in fact converges to zero from our deeper analysis. If we simply went to iteration via the computer, we may get misleading results.

-4- Newton's Method Suppose f has a root of multiplicity $p > 1$ at $x = \alpha$, i.e.,

$$f(\alpha) = f'(\alpha) = ... = f^{(p-1)}(\alpha) = 0, \quad f^{(p)}(\alpha) \neq 0. \tag{4}$$

[1] Runtime will differ for different computers. Code included in Appendix.
[2] Code in Appendix.

a. Show that Newton's method applied to $f(x) = 0$ converges linearly to α.

b. Show that this modification of Newton's Method:

$$x_{k+1} = x_k - p\frac{f(x_k)}{f'(x_k)} \tag{5}$$

converges quadratically to α. **Hint:** You probably are thinking of using the Rule of L'Hopital, but the problem is much easier if you think of f as being defined by $f(x) = (x-\alpha)^p F(x)$ where $F(\alpha) \neq 0$.

(a)

$$\begin{aligned} g = x - \frac{f}{f'} &= x - \frac{(x-\alpha)^p F(x)}{p(x-\alpha)^{p-1}F(x) + (x-\alpha)^p F'(x)} \Longleftrightarrow g(\alpha) = \alpha \\ &= x - \frac{(x-\alpha)F(x)}{pF(x) + (x-\alpha)F'(x)} \end{aligned}$$

$$g' = 1 - \frac{\big(pF(x) + (x-\alpha)F'(x)\big)\big(F(x) + (x-\alpha)F'(x)\big) - (x-\alpha)F(x)\big(pF(x) + (x-\alpha)F'(x)\big)'}{\big(pF(x) + (x-\alpha)F'(x)\big)^2}$$

$$g'(\alpha) = 1 - \frac{p \cdot F(\alpha) \cdot F(\alpha)}{p^2 \cdot F(\alpha)^2} = 1 - \frac{1}{p} = \left|\frac{p-1}{p}\right| < 1, \quad \forall\, p > 1$$

We have the absolute value of $g'(\alpha)$ converges linearly.

(b)

$$\begin{aligned} g(x) &= x - p\frac{f}{f'} \\ g'(x) &= 1 - p\left(\frac{f}{f'}\right)' \Longrightarrow g'(\alpha) = 1 - p \cdot \frac{1}{p} = 0 \end{aligned}$$

This converges quadratically.

-5- (Division without division.) Suppose you have a computer or calculator that has no built-in division. Come up with a fixed point iteration that converges to $1/r$ for any given non-zero number r, and that only uses addition, subtraction, and multiplication. **Hint:** Write down an equation satisfied by $1/r$, apply Newton's method to that equation, and then modify Newton's method so that it doesn't use division. Your resulting method should converge of order 2.

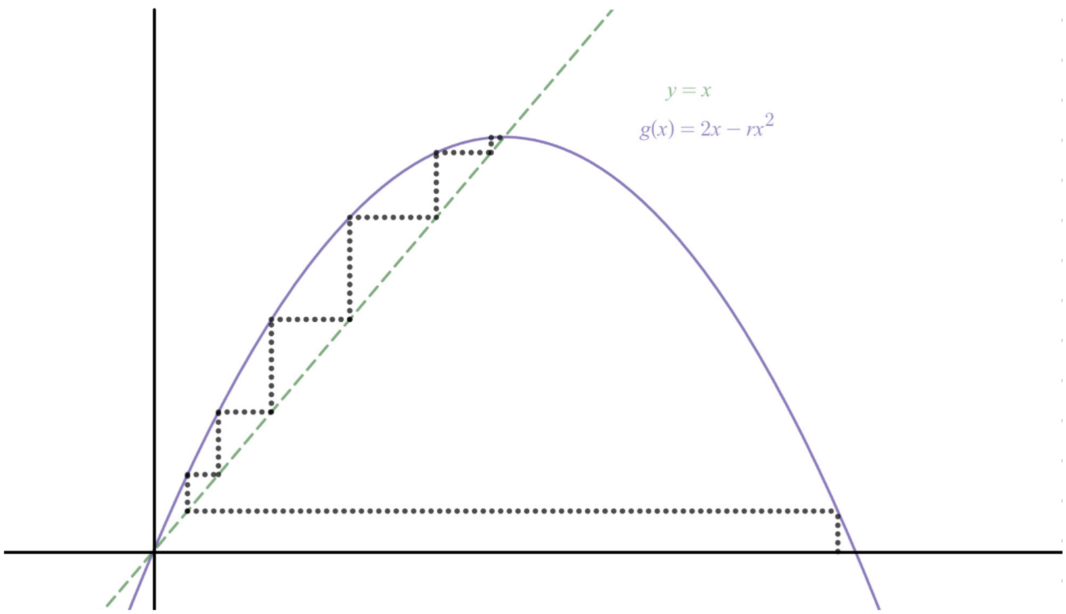
Let $f(x) = \frac{1}{x} - r$. Now f is satisfied by $x = \frac{1}{r}$ for $r \neq 0$. We have

$$\begin{aligned} f(x) &= \frac{1}{x} - r \\ f'(x) &= \frac{-1}{x^2} \\ f''(x) &= \frac{2}{x^3} \end{aligned}$$

Both f' and f'' are non-zero at $x = \frac{1}{r}$ meaning the Newton's method converges quadratically. We now look at Newton's method:

$$\begin{aligned} g(x) &= x - \frac{f}{f'} = x - \frac{\frac{1}{x} - r}{\frac{-1}{x^2}} \\ &= x - (-x + rx^2) \\ &= 2x - rx^2. \end{aligned}$$

Thus we have a fixed point iteration that does not use division. Below we show a hypothetical convergence for some r.



-6- (A cubically convergent method.) Consider the iteration

$$x_{k+1} = g(x_k) \text{ where } g(x) = x - \frac{f(x)}{f'(x)} - \frac{1}{2}\frac{f^2(x)f''(x)}{(f'(x))^3}.$$

(We assume f is sufficiently differentiable, and $f'(x) \neq 0$.) Suppose that $g(\alpha) = \alpha$. Show that

$$g'(\alpha) = g''(\alpha) = 0.$$

(Thus the fixed point method will converge of order at least 3 if we start sufficiently close to α.)

$$
\begin{aligned}
g &= x - \frac{f}{f'} - \frac{1}{2}\frac{f^2 f''}{f'^3} \\
g' &= 1 - \frac{f'^2 - f f''}{f'^2} - \frac{1}{2}\frac{f'^3(2 f f' f'' + f^2 f''') - 3 f'^2 f'' f^2 f''}{f'^6} \\
&= \frac{f f''}{f'^2} - \frac{1}{2}\frac{2 f f'^4 f''}{f'^6} - \frac{1}{2}\frac{f^2 f'^3 f'''}{f'^6} + \frac{1}{2}\frac{3 f^2 f'^2 f''^2}{f'^6} \\
&= \frac{f f''}{f'^2} - \frac{f f''}{f'^2} - \frac{1}{2}\frac{f^2 f'''}{f'^3} + \frac{1}{2}\frac{3 f^2 f''^2}{f'^4} \\
&= \frac{1}{2} f^2 \left(\frac{f''^2}{f'^4} - \frac{f'''}{f'^3} \right) = 0 \quad \text{at } x = \alpha \text{ since } f(\alpha) = 0 \text{ and } f'(\alpha) \neq 0 \\
g'' &= f f' \left(\frac{f''^2}{f'^4} - \frac{f'''}{f'^3} \right) + \frac{1}{2} f^2 \left(\frac{f''^2}{f'^4} - \frac{f'''}{f'^3} \right)' \\
&= 0 \quad \text{since each term has a multiple of } f \text{ which is 0 at } \alpha.
\end{aligned}
$$

$\Longrightarrow$ Thus our fixed point iteration converges of order at least 3.

-7- (Polynomial Interpolation.) Suppose you want to interpolate to the data $(x_i, y_i), i = 0, ..., n$ by a polynomial of degree n. Recall that the interpolating polynomial p can be written in its Lagrange form as

$$
p(x) = \sum_{i=0}^{n} y_i L_i(x) \text{ where } L_i(x) = \frac{\prod_{i \neq j}(x - x_j)}{\prod_{i \neq j}(x_i - x_j)}. \tag{6}
$$

Show that

$$
\sum_{i=0}^{n} x_i^j L_i(x) = x^j \text{ for } j = 0, ..., n. \tag{7}
$$

We first note that we have $n+1$ data points, thus any polynomial interpolation up to degree n will be unique. We have the points $\left(x_i, x_i^\delta\right)$ for $i = 0, 1, 2, ..., n$ and where δ is a fixed power that can take on the values $\delta = 0, 1, 2, ..., n$, which let's us interpolate uniquely. Also, we assume all the points are distinct. By the construction of these points, the polynomial x^δ interpolates our data points since it is unique.

$$
P(x) = \sum_{i=0}^{n} x_i^\delta L_i(x) = x^\delta \quad \text{for } \delta = 0, 1, ..., n
$$

-8- (Uniqueness of the interpolating polynomial.) Assume you are given the data

$$
\begin{array}{c|cccc}
x_i : & 1 & 2 & 4 & 8 \\
y_i : & 1 & 2 & 3 & 4
\end{array} \tag{8}
$$

Construct the interpolating polynomial using

a. the power form obtained by solving the Vandermonde system,

b. the Lagrange form,

c. the Newton form,
and show that they all yield the same polynomial.

(a) Vandermonde System: We have 4 nodes, thus we interpolate with polynomial of degree 3:

$$a + bx + cx^2 + dx^3 = y$$

$$\begin{aligned} a+b+c+d &= 1 \\ a+2b+4c+8d &= 2 \\ a+4b+16c+64d &= 3 \\ a+8b+64c+512d &= 4 \end{aligned} \iff \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 24 & & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 8 & 64 & 512 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Solving the corresponding augmented matrix reveals

$$\begin{bmatrix} a & b & c & d \end{bmatrix}^T = \begin{bmatrix} -\frac{10}{21} & \frac{7}{4} & -\frac{7}{24} & \frac{1}{56} \end{bmatrix}^T$$

$$\implies P(x) = -\frac{10}{21} + \frac{7}{4}x - \frac{7}{24}x^2 + \frac{1}{56}x^3$$

(b) Lagrange Form:

$$\begin{aligned} L_0(x) &= \frac{(x-2)(x-4)(x-8)}{(-1)(-3)(-7)} = \frac{x^3 - 14x^2 + 56x - 64}{-21} \\ L_1(x) &= \frac{(x-1)(x-4)(x-8)}{(1)(-2)(-6)} = \frac{x^3 - 13x^2 + 44x - 32}{12} \\ L_2(x) &= \frac{(x-1)(x-2)(x-8)}{(3)(2)(-4)} = \frac{x^3 - 11x^2 + 26x - 16}{-24} \\ L_3(x) &= \frac{(x-1)(x-2)(x-4)}{(7)(6)(4)} = \frac{x^3 - 7x^2 + 14x - 8}{168} \end{aligned}$$

$$\begin{aligned} \implies P(x) = \sum_{i=0}^{3} y_i L_i(x) &= \frac{1}{-21}(x^3 - 14x^2 + 56x - 64) + \frac{2}{12}(x^3 - 13x^2 + 44x - 32) \\ &\quad - \frac{3}{24}(x^3 - 11x^2 + 26x - 16) + \frac{4}{168}(x^3 - 7x^2 + 14x - 8) \\ &= \frac{1}{169}(-8x^3 + 112x^2 - 448x + 512 + 28x^3 - 364x^2 + 1232x - 896 \\ &\quad - 21x^3 + 231x^2 - 546x + 336 + 4x^3 - 28x^2 56x - 32) \\ &= \frac{1}{168}(3x^3 - 49x^2 + 294x - 80) \\ &= \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21} \end{aligned}$$

(c) Newton Form:

x	f	$F(x_i, x_j)$	$F(x_i, x_j, x_k)$	$F(x_i, x_j, x_k, x_\ell)$
1	1	$F(x_0, x_1) = \frac{f_1 - f_0}{x_1 - x_0} = 1$	$F(x_0, x_1, x_2) = \frac{F(x_1,x_2) - F(x_0,x_1)}{x_2 - x_0} = -\frac{1}{6}$	$F(x_0, x_1, x_2, x_3) = \frac{F(x_1,x_2,x_3) - F(x_0,x_1,x_2)}{x_3 - x_0} = \frac{1}{56}$
2	2	$F(x_1, x_2) = \frac{f_2 - f_1}{x_2 - x_1} = \frac{1}{2}$	$F(x_1, x_2, x_3) = \frac{F(x_2,x_3) - F(x_1,x_2)}{x_3 - x_1} = -\frac{1}{24}$	
4	3	$F(x_2, x_3) = \frac{f_3 - f_2}{x_3 - x_2} = \frac{1}{4}$		
8	4			

$$
\begin{aligned}
P(x) &= f_0 + (x - x_0)F(x_0, x_1) + (x - x_0)(x - x_1)F(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)F(x_0, x_1, x_2, x_3) \\
&= 1 + (x-1)\cdot 1 + (x-1)(x-2)\left(-\frac{1}{6}\right) + (x-1)(x-2)(x-4)\left(\frac{1}{56}\right) \\
&= 1 + x - 1 - \frac{1}{6}(x^2 - 3x + 2) + \frac{1}{56}(x^3 - 7x^2 + 14x - 8) \\
&= x - \frac{1}{6}x^2 + \frac{1}{2}x - \frac{1}{3} + \frac{1}{56}x^3 - \frac{1}{8}x^2 + \frac{1}{4}x - \frac{1}{7} \\
&= \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21}
\end{aligned}
$$

All of these interpolating methods yield the same polynomial.

-9- (The infamous Runge-Phenomenon.) It is not generally true that higher degree interpolation polynomials yield more accurate approximations. This is illustrated in this problem. Let

$$f(x) = \frac{1}{1+x^2} \quad \text{and} \quad x_j = -5 + jh, \quad j = 0, 1, ..., n, \quad h = \frac{10}{n}.$$

For

$$n = 1, 2, 3, ..., 20$$

plot the graph (in the interval $[-5, 5]$) of the interpolant.

$$p(x) = \sum_{i=0}^{n} \alpha_i x^i$$

defined by

$$p(x_i) = f(x_i), \quad i = 0, 1, ..., n.$$

Also list the approximate maximum error in the interval $[-5, 5]$ for each polynomial degree. To approximate the maximum error sample the error at 200 evenly spaced points (at least!) in the interval.

-10- (Judicious interpolation.) Repeat the above except that you interpolate at the roots of the Chebycheff polynomials, i.e.,

$$x_i = 5 \cos \frac{i\pi}{n}, \quad i = 0, 1, ..., n. \tag{9}$$

-11- (Least Squares approximation of functions.) Find a linear function $\ell(x)$ such that

$$\int_0^1 (e^x - \ell(x))^2 dx = \min. \tag{10}$$

If we let $\ell(x) = mx + b$ for $m, b \in \mathbb{R}$, then we can think of equation (10) as a function of m and b

$$F(m, b) = \int_0^1 (e^x - mx - b)^2 dx.$$

We want to minimize $F(m, b)$, so we take partial derivatives

$$
\begin{aligned}
\frac{\partial F}{\partial m} &= \frac{\partial}{\partial m}\int_0^1 (e^x - mx - b)^2 dx \\
&= \int_0^1 \frac{\partial}{\partial m}(e^x - mx - b)^2 dx \\
&= \int_0^1 2(e^x - mx - b)(-x)dx \\
&= 2\int_1^0 x(e^x - mx - b)dx \\
&= 2\left[xe^x - \frac{m}{2}x^3 - bx^2\Big|_1^0 - \left(e^x - \frac{m}{6}x^3 - \frac{b}{2}x^2\right)\Big|_1^0\right] \\
&= 2\left[-\left(e - \frac{m}{2} - b\right) - \left(1 - \left(e - \frac{m}{6} - \frac{b}{2}\right)\right)\right] \\
&= 2\left(-e + \frac{m}{2} + b - 1 + e - \frac{m}{6} - \frac{b}{2}\right) \\
&= 2\left(-1 + \frac{m}{3} + \frac{b}{2}\right) \\
&= b + \frac{2m}{3} - 2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial b} &= \frac{\partial}{\partial b}\int_0^1 (e^x - mx - b)^2 dx \\
&= \int_0^1 \frac{\partial}{\partial b}(e^x - mx - b)^2 dx \\
&= \int_0^1 2(e^x - mx - b)(-x)dx \\
&= 2\int_1^0 (e^x - mx - b)dx \\
&= 2\left[e^x - \frac{m}{2}x^2 - bx\Big|_1^0\right] \\
&= 2\left[1 - \left(e - \frac{m}{2} - b\right)\right] \\
&= 2\left(\frac{m}{2} + b + 1 - e\right) \\
&= m + 2b + 2 - 2e
\end{aligned}
$$

To minimize, we set partial derivatives equal to zero and solve for m and b.

$$m + 2b + 2 - 2e = 0$$

$$b + \frac{2m}{3} - 2 = 0$$

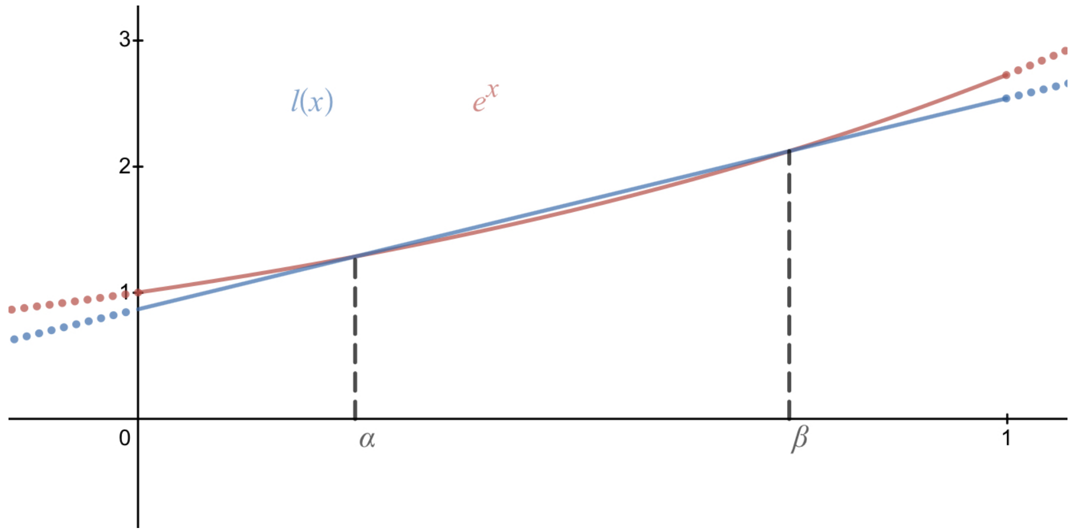
$$\left[\begin{array}{cc|c} 1 & 2 & 2(e-1) \\ \frac{2}{3} & 1 & 2 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 2 & 2(e-1) \\ 2 & 3 & 6 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 2 & 2(e-1) \\ 0 & -1 & 10-4e \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 0 & 18-6e) \\ 0 & 1 & 4e-10 \end{array}\right]$$

We have $b = 4e - 10$ and $m = 18 - 6e$. Thus, $\ell(x) = (18 - 6e)x + (4e - 10)$ minimizes equation (10).

-12- (An alternative approximation problem.) Find a linear function $\ell(x)$ such that

$$\int_0^1 |e^x - \ell(x)|dx = \min . \tag{11}$$

We know that our linear function $\ell(x)$ must actually intersect e^x twice on the interval $[0, 1]$. The graph illustrates the points of intersection α and β, some arbitrary points.



We want to minimize the following

$$I = \int_0^1 |e^x - \ell(x)|dx = \int_0^\alpha (e^x - \ell(x))dx + \int_\alpha^\beta (\ell(x) - e^x)dx + \int_\beta^1 (e^x - \ell(x))dx.$$

If we let $\ell(x) = mx + b$, the following evaluates to

$$\begin{aligned}
I &= \left[e^x - \frac{m}{2}x^2 - bx\right]\Big|_0^\alpha + \left[\frac{m}{2}x^2 + bx - e^x\right]\Big|_\alpha^\beta + \left[e^x - \frac{m}{2}x^2 - bx\right]\Big|_\beta^1 \\
&= \left(e^\alpha \frac{\alpha^2}{2}m - \alpha b - 1\right) + \left(\frac{\beta^2}{2}m + \beta b - e^\beta\right) - \left(\frac{\alpha^2}{2}m + \alpha b - e^\alpha\right) \\
&\qquad + \left(e - \frac{1}{2}m - b\right) - \left(e^\beta - \frac{\beta^2}{2}m - \beta b\right) \\
&= e^\alpha - \frac{\alpha^2}{2}m - \alpha b - 1 + \frac{\beta^2}{2}m + \beta b - e^\beta - \frac{\alpha^2}{2}m - \alpha b + e^\alpha \\
&\qquad + e - \frac{1}{2}m - b - e^\beta + \frac{\beta^2}{2}m + \beta b
\end{aligned}$$

$$F(m, b) = \left(\beta^2 - \alpha^2 - \frac{1}{2}\right)m + (2\beta - 2\alpha - 1)\, b + \left(2e^\alpha - 2e^\beta + e - 1\right)$$

$$\frac{\partial F}{\partial m} = \beta^2 - \alpha^2 - \frac{1}{2} = 0 \Longrightarrow (\beta - \alpha)(\beta + \alpha) = \frac{1}{2}$$

$$\frac{\partial F}{\partial b} = 2\beta - 2\alpha - 1 = 0 \Longrightarrow \beta - \alpha = \frac{1}{2}$$

Therefore we have,

$$\alpha = \frac{1}{4} \text{ and } \beta = \frac{3}{4}.$$

Now

$$m = \frac{e^\beta - e^\alpha}{\beta - \alpha} = 2\left(e^{3/4} - e^{1/4}\right)$$

which means that

$$y = 2\left(e^{3/4} - e^{1/4}\right)x + b \Longrightarrow e^{1/4} = \frac{1}{2}e^{3/4} - \frac{1}{2}e^{1/4} + b, \text{ and}$$

$$b = \frac{1}{2}\left(3e^{1/4} - e^{3/4}\right).$$

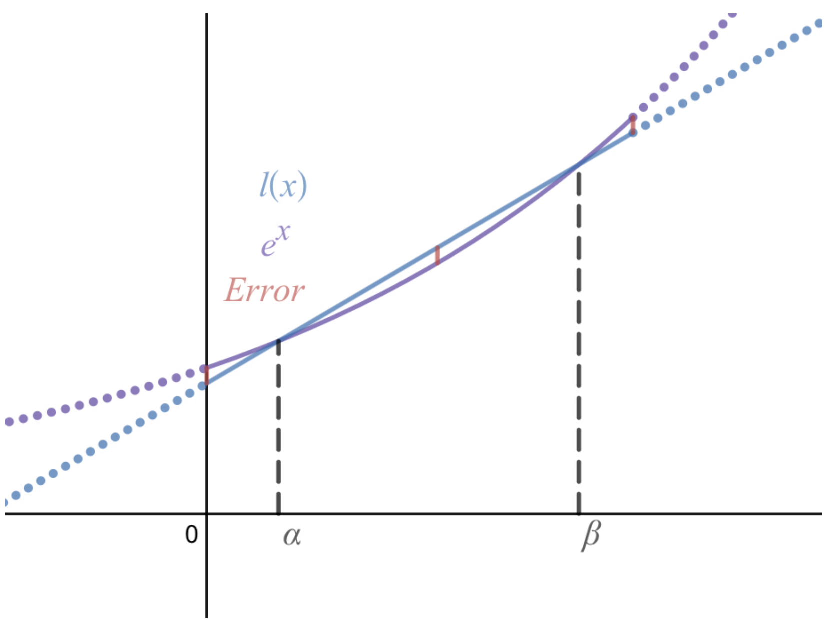
Therefore,

$$\ell(x) = 2\left(e^{3/4} - e^{1/4}\right)x + \frac{1}{2}\left(3e^{1/4} - e^{3/4}\right) \quad \text{minimizes (11).}$$

-13- (Another alternative approximation problem.) Find a linear function $\ell(x)$ such that

$$\max_{0 \le x \le 1} |e^x - \ell(x)| = \min. \tag{12}$$

We know that our function $\ell(x)$ will intersect e^x twice, say α and β. However, we won't focus on these values all that much. The idea is that the maximum error will occur at one of these intervals. We wish to minimize the maximum error, to do this the maximum error on each interval $[0, \alpha], [\alpha, \beta], \text{and}[\beta, 1]$ must be identical. We can always adjust the time to reduce the error. We can see what this looks like, qualitatively, below. Note that E denotes our minimized maximum error.



We know that based on the shape of e^x on the interval, the minimized maximum error will occur at $x = 0, x = 1$ and some $\hat{x}$ on the interval $[\alpha, \beta]$. If we let $\ell(x) = mx + b$ then we have the following system,

$$e^0 - (m(0) + b) = E$$

$$e^{\hat{x}} - (m\hat{x} + b) = -E$$
$$e^1 - (m(1) + b) = E$$

We also note that $E'(\hat{x}) = e^{\hat{x}} - m = 0 \Leftrightarrow e^{\hat{x}} = m \Leftrightarrow \hat{x} = \ln(m)$. Solving the system we have,

$$\begin{aligned} e - m - b &= E \\ (-) \quad 1 - b &= E \end{aligned}$$

$$\implies \quad e - 1 - m = 0 \quad \iff \quad m = e - 1$$

$$\begin{aligned} e - m - b &= E \\ (+) \quad e^{\hat{x}} - m\hat{x} - b &= -E \end{aligned}$$

$$\implies \quad e + e^{\hat{x}} - m - m\hat{x} - 2b = 0$$

$$\begin{aligned} \implies \quad 2b &= e + e^{\hat{x}} - m - m\hat{x} \\ b &= \frac{1}{2}\left(e - m\hat{x}\right) \\ &= \frac{1}{2}\left(e - (e-1)\ln(e-1)\right) \\ &= \frac{1}{2}\left(e - e\ln(e-1) + \ln(e-1)\right) \end{aligned}$$

Thus, $\ln(x) = (e-1)x + \frac{1}{2}\left(e - e\ln(e-1) + \ln(e-1)\right)$ minimizes our function.

2 Appendix

-A- Python Code for runtimes in problem 3.

```
from math import sin

#First program to test how many steps it takes to get epsilon
#distance from 0.
i = 0
x = 1
while True:
    i += 1
    x = sin(x)
    if x < 0.001:
        break
print(x, i)

#Code to look at how at effects of rounding errors.
i = 0
x = 1
n = 10000
rounding = 3
```

```python
for step in range(n):
    i += 1
    x = round(sin(x), rounding)
    print(f'{step+1}: {x}')
```