1

Let $\alpha \in \mathbb{R}^n$ where $\alpha = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_n]^T$ and $\alpha \neq \vec{0}$. Now consider

$$\alpha^T A \alpha$$

where $A = [(\phi_i, \phi_j)]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$. We have

$$\begin{aligned}
\alpha^T A \alpha &= \left[\sum_{i=1}^{n} \alpha_i (\phi_i, \phi_1) \quad \sum_{i=1}^{n} \alpha_i (\phi_i, \phi_2) \quad \dots \quad \sum_{i=1}^{n} \alpha_i (\phi_i, \phi_n) \right] \alpha \\
&= \sum_{i=1}^{n} \alpha_1 \alpha_i (\phi_i, \phi_1) + \sum_{i=1}^{n} \alpha_2 \alpha_i (\phi_i, \phi_2) + \dots + \sum_{i=1}^{n} \alpha_n \alpha_i (\phi_i, \phi_n) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_j \alpha_i (\phi_j, \phi_i) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} (\alpha_j \phi_j, \alpha_i \phi_i) \\
&= \left(\sum_{j=1}^{n} \alpha_j \phi_j , \sum_{i=1}^{n} \alpha_i \phi_i \right) \\
&= (s, s)
\end{aligned}$$

And since s is a non-trivial linear combo of basis function than $s \neq 0$

Which in turn gives us,

$$\alpha^T A \alpha = (s, s) > 0 \qquad \text{for all } \alpha \in \mathbb{R}^n$$

A is a positive definite matrix. $\square$

2

The Gram-Schmidt process, have basis ϕ_0, ϕ_1, ϕ_2 & want to construct orthonormal basis q_0, q_1, q_2 from basis.

Let $q_0 = \dfrac{\phi_0}{\|\phi_0\|}$ $\quad z_1 = \phi_1 - (\phi_1, q_0) q_0 \quad z_2 = \phi_2 - (\phi_2, q_1) q_1 - (\phi_2, q_0) q_0$

$q_1 = \dfrac{z_1}{\|z_1\|} \qquad q_2 = \dfrac{z_2}{\|z_2\|}$

So, $q_0 = \dfrac{1}{\left[\int_0^1 1^2 dx\right]^{1/2}} = 1$

$$z_1 = x - (x, 1)\cdot 1 = x - \int_0^1 x \cdot 1 dx \cdot 1$$

$$= x - \tfrac{1}{2}$$

$$q_1 = \frac{x - \frac{1}{2}}{\left[\int_0^1 (x - \frac{1}{2})^2 dx\right]^{1/2}} = \frac{x - \frac{1}{2}}{\left[\frac{1}{3}(x - \frac{1}{2})^3 \Big|_0^1\right]^{1/2}} = 2\sqrt{3}x - \sqrt{3}$$

$$z_2 = e^x - (e^x, 2\sqrt{3}x - \sqrt{3})(2\sqrt{3}x - \sqrt{3}) - (e^x, 1)\cdot 1$$

$$= e^x - \sqrt{3}\int_0^1 (2e^x \cdot x - e^x) dx \, (2\sqrt{3}x - \sqrt{3}) - \int_0^1 e^x \cdot 1 dx \cdot 1$$

$$e^x + \sqrt{3}(e-3)(2\sqrt{3}x - \sqrt{3}) - e + 1$$

$$e^x + (e-3)(6x-3) - e + 1$$

$$e^x + 6ex - 3e - 18x + 9 - e + 1$$

$$e^x + (6e - 18)x + (10 - 4e)$$

$$q_2 = \frac{e^x + (6e-18)x + (10-4e)}{\left[\int_0^1 \left(e^x + (6e-18)x + (10-4e)\right)^2 dx\right]^{1/2}} = \frac{e^x + (6e-18)x + (10-4e)}{\sqrt{-4e^2 + 20e + \frac{e^2-1}{2} - 28}}$$

$$-4e^2 + 20e + \tfrac{1}{2}e^2 - \tfrac{1}{2} - 28$$

$$-\tfrac{7}{2}e^2 + 20e - \tfrac{57}{2}$$

$$= -\tfrac{1}{2}(7e^2 - 40e + 57)$$

3

Have $Q_0 = \underline{1}$

$$a_1 = \frac{(x,1)}{(1,1)} = \frac{\int_{-1}^{1} x \cdot 1 \, dx}{\int_{-1}^{1} 1 \cdot 1 \, dx} = 0$$

$$Q_1 = x - a_1 = \underline{x}$$

$$a_2 = \frac{(x \cdot Q_1, Q_1)}{(Q_1, Q_1)} = \frac{(x^2, x)}{(x, x)} = \frac{\int_{-1}^{1} x^3 \, dx}{\int_{-1}^{1} x^2 \, dx} = 0$$

$$b_2 = \frac{(x \cdot Q_1, Q_0)}{(Q_0, Q_0)} = \frac{(x^2, 1)}{(1, 1)} = \frac{\int_{-1}^{1} x^2 \, dx}{\int_{-1}^{1} 1 \, dx} = \frac{1}{3}$$

$$Q_2 = (x - a_2) Q_1 - b_2 Q_0 = (x - 0)x - \frac{1}{3} \cdot 1 = \underline{x^2 - \frac{1}{3}}$$

$$a_3 = \frac{(x \cdot Q_2, Q_2)}{(Q_2, Q_2)} = \frac{(x(x^2 - \frac{1}{3}), x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} = \frac{\int_{-1}^{1} x(x^2 - \frac{1}{3})^2 \cdot dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 \, dx} = 0$$

$$b_3 = \frac{(x \cdot Q_2, Q_1)}{(Q_1, Q_1)} = \frac{(x(x^2 - \frac{1}{3}), x)}{(x, x)} = \frac{\int_{-1}^{1} x^2(x^2 - \frac{1}{3}) \, dx}{\int_{-1}^{1} x^2 \, dx} = \frac{\frac{8}{45}}{\frac{2}{3}} = \frac{4}{15}$$

$$Q_3 = (x - a_3) Q_2 - b_3 Q_1 = x(x^2 - \tfrac{1}{3}) - \tfrac{4}{15} x$$

$$= \underline{x^3 - \tfrac{3}{5} x}$$

$$a_4 = \frac{(x Q_3, Q_3)}{(Q_3, Q_3)} = \frac{(x(x^3 - \frac{3}{5}x), (x^3 - \frac{3}{5}x))}{(x^3 - \frac{3}{5}x, x^3 - \frac{3}{5}x)} = \frac{\int_{-1}^{1} x(x^3 - \frac{3}{5}x)^2 \, dx}{\int_{-1}^{1} (x^3 - \frac{3}{5}x)^2 \, dx} = 0$$

$$b_4 = \frac{(x \cdot Q_3, Q_2)}{(Q_2, Q_2)} = \frac{(x \cdot (x^3 - \frac{3}{5}x), x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} = \frac{9}{35}$$

$$Q_4 = (x - a_4) Q_3 - b_4 Q_2 = x \cdot (x^3 - \tfrac{3}{5}x) - \tfrac{9}{35}(x^2 - \tfrac{1}{3}) = \underline{x^4 - \tfrac{6}{7}x^2 + \tfrac{3}{35}}$$

$$a_5 = \frac{\left(x\cdot\left(x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right),\ x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right)}{\left(x^4 - \frac{6}{7}x^2 + \frac{3}{35},\ x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right)} = 0$$

$$b_5 = \frac{\left(x\left(x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right),\ x^3 - \frac{3}{5}x\right)}{\left(x^3 - \frac{3}{5}x,\ x^3 - \frac{3}{5}x\right)} = \frac{128}{11025}$$

$$Q_5 = x\left(x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right) - \frac{128}{11025}\left(x^3 - \frac{3}{5}x\right)$$

4

Defining the inner product as

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,dx + f'(x)g'(x)$$

With this it is not generally true that $(x\cdot f, g) = (f, xg)$.
Let $f(x) = x$ and $g(x) = 1$.

We have,

$$(x\cdot f, g) = \int_{-1}^{1} x^2 + (x^2)'(1)'\,dx = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$(f, xg) = \int_{-1}^{1} x^2 + x'x'\,dx = \frac{2}{3} + 2$$

Let $Q_0 = 1$

$$a_1 = \frac{(x\cdot 1, 1)}{(1,1)} = \frac{\int_{-1}^{1} x\,dx}{\int 1\,dx} = 0$$

$Q_1 = x$

$$a_2 = \frac{(x\cdot x, x)}{(x,x)} = \frac{\int_{-1}^{1} x^3 + 2x\,dx}{\int_{-1}^{1} x^2 + 1\,dx} = 0$$

$$b_2 = \frac{(x\cdot x, 1)}{(1,1)} = \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} = \frac{1}{3}$$

$$Q_2 = x^2 - \frac{1}{3}$$

$$a_3 = \frac{(x^3 - \frac{1}{3}x, x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} = 0$$

$$b_3 = \frac{(x^3 - \frac{1}{3}x, x)}{(x,x)} = \frac{68}{45}\cdot\frac{3}{8} = \frac{17}{30}$$

$$Q_3 = x(x^2 - \tfrac{1}{3}) - \frac{17}{30}x$$

$$Q_3 = x^3 - \frac{27}{30}x$$

$$\Longrightarrow \quad (Q_3, Q_1) = -\frac{1}{5} \neq 0$$

$$\cos(x+1) = \sum_{m=1}^{\infty} a_m \cos(mx)$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\cos(x+1) = \cos(x)\cos(1) - \sin(x)\sin(1)$$

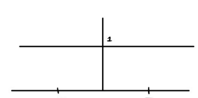
$$\cos(n$$

① $f(t) = 1 \;, \quad 0 < t < \pi$

Fourier Cosine Series

$f(-t) = 1 \qquad f_E(t) = \begin{cases} 1 & 0 < t < \pi \\ 1 & -\pi < t < 0 \end{cases}$

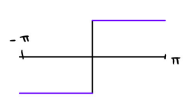
$f(t) = 1$



$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} dt = \frac{t}{\pi}\Big|_0^{\pi} = 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos(nt)\,dt = \frac{2}{n\pi}\sin(nt)\Big|_0^{\pi} = 0$$

Fourier Sine Series



$-f(-t) = -1 \qquad f_O(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nt)\,dt = -\frac{2}{n\pi}\cos(nt)\Big|_0^{\pi} = \frac{2}{n\pi}(1 - \cos(n\pi)) = \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f(t) = \sum_{n \text{ odd}}^{\infty} \frac{2}{n\pi}\sin(nt) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)}\sin((2n-1)t)$$

5

Consider Fourier Sine Series

$$f_o(x) = \begin{cases} -1 & 0 < x < \pi \\ 1 & -\pi \le x \le 0 \end{cases} \qquad -f(-x) = 1$$

$a_n = 0$

$$b_n = -\frac{2}{\pi}\int_0^\pi \sin(nx)\,dx = \frac{2}{\pi}\cos(nx)\Big|_0^\pi = \frac{-2}{n\pi}\left(1-\cos(n\pi)\right)$$

$$= \begin{cases} -\dfrac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\Longrightarrow \quad f(x) = \sum_{n \text{ odd}} -\frac{4}{n\pi}\sin(nx) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left((2n-1)x\right)$$

6

Forrier series of $\cos(x)$ on $[-\pi, \pi]$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x+1)\,dx = \frac{1}{2\pi}\left[\sin(x+1)\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\sin(\pi+1) - \sin(1-\pi)\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x+1)\cdot\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos((n+1)x+1) + \cos((n-1)x-1)\,dx \xRightarrow{n=1} \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2x+1) + \cos(1)\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\sin(2x+1) + \cos(1)x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin((n+1)x+1)}{n+1} + \frac{\sin((n-1)x-1)}{n-1}\right]_{-\pi}^{\pi} \qquad a_1 = \frac{1}{2\pi}\left(\sin(2\pi+1) - \sin(-2\pi+1)\right) + 2\cos(1)$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n+1}\left(\sin((n+1)\pi+1) + \sin((n+1)\pi-1)\right) + \frac{1}{n-1}\left(\sin((n-1)\pi-1) + \sin((n-1)\pi+1)\right)\right]$$

$n>1$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x+1)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin((n+1)x+1) + \sin((n-1)x-1)\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1}\sin((n+1)x+1) + \frac{1}{n-1}\sin((n-1)x-1)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1}\left[\sin((n+1)\pi+1) - \sin(-(n+1)\pi+1)\right)\right.$$

$$\left. + \frac{1}{n-1}\left[\sin((n-1)\pi-1) + \sin((n-1)\pi+1)\right]\right.$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(2x+1) - \sin(1)\,dx$$

$$= \frac{1}{\pi}\left(\frac{1}{2}\cos(2x+1) + \sin(1)x\right)_{\pi}^{-\pi}$$

$$= \frac{1}{\pi}\left(\frac{1}{2}\left(\cos(-2\pi+1) - \cos(2\pi+1)\right) + 2\sin(1)\pi\right)$$

$$b_1 = \frac{1}{2\pi}\left(\cos(-2\pi+1) - \cos(2\pi+1)\right) + 2\sin(1)$$

$$-\sin(-\pi+1) = \sin(\pi-1)$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x+1)\,dx = \frac{1}{2\pi}\Big[\sin(x+1)\Big]_{-\pi}^{\pi} = \frac{1}{2\pi}\Big(\sin(\pi+1) + \sin(\pi-1)\Big)$$

$$= \frac{1}{\pi}\sin(\pi)\cos(1) = 0$$

$\underline{a_0 = 0}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x+1)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos((n+1)x+1) + \cos((n-1)x-1)\,dx$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1}\sin\big((n+1)x+1\big) + \frac{1}{n-1}\sin\big((n-1)x-1\big)\right)\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\Bigg(\frac{1}{n+1}\sin\big((n+1)\pi+1\big) + \frac{1}{n+1}\sin\big((n+1)\pi-1\big) + \frac{1}{n-1}\sin\big((n-1)\pi-1\big) + \frac{1}{n-1}\sin\big((n-1)\pi+1\big)\Bigg)$$

$$= \frac{2}{\pi}\left(\frac{1}{n+1}\sin\big((n+1)\pi\big)\cos(1) + \frac{1}{n-1}\sin\big((n-1)\pi\big)\cos(1)\right)$$

$\underline{a_n = 0 \quad \text{for} \quad n \geq 2}$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\big)$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x+1)\cos(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2x+1) + \cos(1)\,dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{2}\sin(2x+1) + \cos(1)x\right)\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\Big(\sin(2\pi+1) + \sin(2\pi-1)\Big) + \frac{1}{2\pi}\Big(2\pi\cos(1)\Big)$$

$\underline{a_1 = \cos(1)}$

Lets NOT use any this unless we really want

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x+1)\sin(x)\,dx \quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(2x+1) - \sin(1)\,dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{2}\cos(2x+1) + \sin(1)x\right)\Big|_{\pi}^{-\pi}$$

$$= \frac{1}{4\pi}\left(\cos(2\pi-1) - \cos(2\pi+1)\right)$$

$$-\frac{1}{2\pi}2\pi\sin(1)$$

$$\underline{b_1 = -\sin(1)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x+1)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin((n+1)x+1) + \sin((n-1)x-1)\,dx$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1}\cos((n+1)x+1) + \frac{1}{n-1}\cos((n-1)x-1)\right)\Big|_{\pi}^{-\pi}$$

$$= \frac{1}{(n+1)\pi}\left(\cos((n+1)\pi+1) + \cos((n+1)\pi-1)\right)$$

$$+ \frac{1}{(n-1)\pi}\left(\cos((n-1)\pi-1) + \cos((n-1)\pi+1)\right)$$

$$= \frac{1}{(n+1)\pi}\left(\cos(1+(n+1)\pi) - \cos(1-(n+1)\pi)\right)$$

$$+ \frac{1}{(n-1)\pi}\left(\cos(-1+(n-1)\pi) - \cos(-1-(n-1)\pi)\right)$$

$$= \frac{1}{(n+1)\pi}\sin(1)\sin(-(n+1)\pi) + \frac{1}{(n-1)\pi}\sin(-1)\sin(-(n-1)\pi)$$

$$= \frac{-1}{(n+1)\pi}\sin(1)\sin((n+1)\pi)$$

$$+ \frac{1}{(n-1)\pi}\sin(1)\sin((n-1)\pi)$$

$$\underline{b_n = 0,}$$

So $\underline{\cos(x+1) = \cos(1)\cos(x) - \sin(1)\sin(x)}$

product sum identity

7)

$$S(x) = \begin{cases} (\gamma-1)(x^3-x^2)+x+1 & x\in[0,1] \\ \gamma x^3 - 5\gamma x^2 + 8\gamma x - 4\gamma + 2 & x \in [1,2] \end{cases}$$

$$S'(x) = \begin{cases} (\gamma-1)(3x^2-2x)+1 & x\in[0,1] \\ 3\gamma x^2 - 10\gamma x + 8\gamma & x \in [1,2] \end{cases}$$

a). $S(0) = (\gamma-1)(0)+0+1=1$

$S(1) = (\gamma-1)(1-1)+1+1=2$

$S(1) = \gamma - 5\gamma + 8\gamma - 4\gamma + 2 = 2$

$S(2) = 8\gamma - 20\gamma + 16\gamma - 4\gamma + 2 = 20$

$S'(0) = (\gamma-1)(0)+1=1$

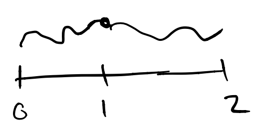
$S'(1) = (\gamma-1)(3-2)+1 = \gamma$

$S'(1) = 3\gamma - 10\gamma + 8\gamma = \gamma$

$S'(2) = 12\gamma - 20\gamma + 8\gamma = 0$

b). Need $S''(x)$ to be continuous thus thus pick γ s.t $S''(1^-) = S''(1^+)$

$$S''(x) = \begin{cases} (\gamma-1)(6x-2) & x\in[0,1] \\ 6\gamma x - 10\gamma & x\in[1,2] \end{cases}$$



$S''(1^-) = 4\gamma - 4$

$S''(1^+) = -4\gamma$

$\implies$ pick $\gamma = \frac{1}{2}$

8

Need to show for any degree d that

$$P(x) = \sum_{i=0}^{d} a_i x^i$$

Can be written as $\sum_{i+j=d} c_{ij} \frac{d!}{i!j!} b_1^i b_2^j$

So essentially before we had

$$\{1, x, \dots, x^d\} \quad \text{Basis}$$

as a basis.

$$\left\{ \frac{d!}{i!(d-i)!} b_1^i b_2^{d-i} : i = 0, 1, \dots, d \right\} \quad \text{Basis}$$

$b_1 =$

$b_2 = 1 - b_1$

To show linear independent

$$\sum_{i=0}^{d} c_i \frac{d!}{i!(d-i)!} b_1^i b_2^{d-i} = \vec{0}$$

iff

$c_i = 0$ for all $i = 0, \dots, d$

$[V_1 \quad , \quad V_2]$

poly of deg n

$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ for some a_i

Basis $\underbrace{\{1, x, x^2, \dots, x^n\}}_{n+1 \text{ vectors}}$

Bernstein

$\underbrace{C_{0,n}}\binom{n}{0} \cdot b_1^0 b_2^n + \underbrace{C_{1,n-1}}\binom{n}{1} b_1^1 b_2^{n-1} + \dots + \underbrace{C_{n,0}}\binom{n}{n} b_1^n b_2^0$

$b_1 + b_2 = 1$

$x \in [V_1, V_2] \quad \exists \underset{t}{b_1}, \underset{1-t}{b_2}$ such that

$x = b_1 V_1 + b_2 V_2$

$x = V_1 t + V_2 (1-t)$

$d = n$

Poly

B-form

$$\sum_{i=0}^{n} a_i \underbrace{x^i} \xrightarrow{A} \sum_{i+j=n} C_{ij} \underbrace{\binom{n}{i} t^i (1-t)^j}$$

$$\xleftarrow{A^{-1}}$$

Can say basis

- spans
- linearly Indep.

$$\sum_{i=0}^{d} a_i x^i \overset{?}{=} \sum_{i+j=d} c_{ij} \frac{d!}{i!\,j!} b_1^i b_2^j$$

$$j = d - i$$

$$\sum_{i+j=d} c_{ij} \frac{d!}{i!\,j!} b_1^i b_2^j = \sum_{i=0}^{d} c_{ij} \binom{d}{i} b_1^i (1-b_1)^{d-i}$$

$$= \sum_{i=0}^{d} c_{ij} \binom{d}{i} b_1^i \cdot \sum_{k=0}^{d-i} \binom{d-i}{k} b_1^k$$

$$= \sum_{i=0}^{d} \sum_{k=0}^{d-i} c_{ij} \frac{d!}{i!(d-i)!} \frac{(d-i)!}{k!(d-i-k)!} b_1^{i+k}$$

$$= \sum_{i=0}^{d} \sum_{k=0}^{d-i} c_{ij} \frac{d!}{i!\,k!\,(d-i-k)!} b_1^{i+k}$$

-9- (The interpolant to symmetric data is symmetric.) Suppose you are given symmetric data

$$(x_i, y_i), \quad i = -n, -n+1, \cdots, n-1, n, \tag{6}$$

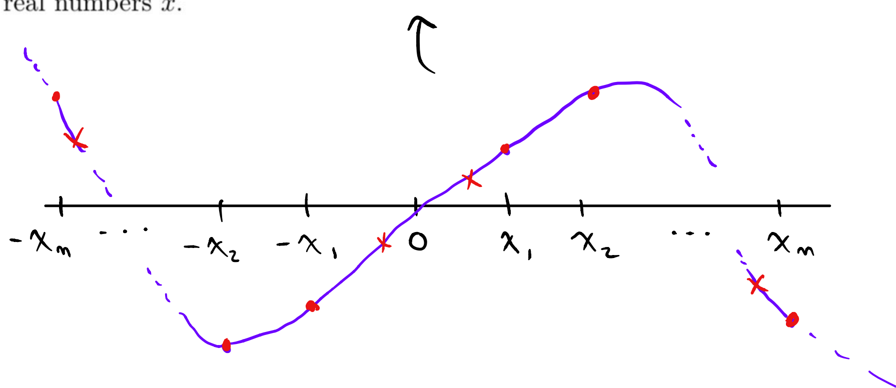
such that

$$x_{-i} = -x_i, \quad \text{and} \quad y_{-i} = -y_i \quad i = 0, 1, \cdots, n. \tag{7}$$

What is the required degree of the interpolating polynomial p? Show that the interpolating polynomial is odd, i.e.,

$$\underline{p(x) = -p(-x)} \tag{8}$$

for all real numbers x.



$2n+1$ points

<u>Goal</u>

$g(x) = \underbrace{p(x) + p(-x)}_{\text{polynomial of deg } 2n \text{ at most}} \overset{?}{=} 0 \quad \forall x$

$g(x_i) = p(x_i) + p(-x_i) = 0$

How many roots

$2n+1$ roots

$g(x)$ is $\overset{\text{Poly}}{}$ at Most degree $2n$

w/ $2n+1$ roots ???

$g(x) = x^2$ $\quad$ 2

$g(x) = x^3$ $\quad$ 3

$\vdots$

$g(x) = x^{2m}$ $\quad$ $2m$

$g(x) = 0$ $\quad$ for all x

$g(x) = p(x) + p(-x) = 0$

$p(x) = -p(-x)$.

$\Rightarrow$ Def of odd

If deg at $\cancel{2m}$ $\quad$ $2n-1$