1).

Let A be a positive definite matrix, that is for any $x \in \mathbb{R}^n$ & $x \neq \vec{0}$, then we have

$$x^T A x > 0$$

We note that any principle submatrix must be square since ... (give argument)

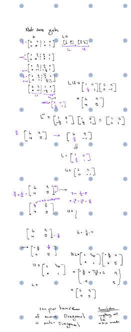
Now take a P.S.M $A_{m \times m}$ where $m < n$. This matrix discludes some set of rows & columns from A. SPIC that $\exists y \in \mathbb{R}^m$ s.t

$$y^T A_{m \times m}\, y \leq 0$$

Then we construct $\tilde{y} \in \mathbb{R}^n$, isomorphic to $y \in \mathbb{R}^m$, by adding zeroes to the indices of the rows/columns removed to construct the P.S.M $A_{m \times m}$. Then it must be the case that

$$\tilde{y}^T A\, \tilde{y} \leq 0$$

Which contradicts A being positive definite. Therefore $A_{m \times m}$ must also be positive definite. $\square$



2).

Let A be an $n \times n$ matrix. We additionally assume A requires no pivoting.

To find the U & L matrices s.t $A = UL$ we apply Gaussian elimin from the bottom to top – which we will call BT-Gauss elimination. We also work right to left. We apply BT-Gauss elim to A, where we store the multipliers in the upper triangular portion.

$$A = \begin{bmatrix} * & * & * & & * & * & * \\ * & * & * & \cdots & * & * & * \\ * & * & * & & * & * & * \\ & \vdots & & & & \vdots & \\ * & * & * & & * & * & * \\ * & * & * & \cdots & * & * & * \\ * & * & * & & * & * & * \end{bmatrix} \rightsquigarrow \begin{bmatrix} f & m_{1,2} & m_{1,3} & & m_{1,n-2} & m_{1,n-1} & m_{1,n} \\ f & f & m_{2,3} & \cdots & m_{2,n-2} & m_{2,n-1} & m_{2,n} \\ f & f & f & m_{3,4} & m_{3,n-2} & m_{3,n-1} & m_{3,n} \\ & \vdots & & & & \vdots & \\ f & f & f & & f & m_{n-2,n-1} & m_{n-2,n} \\ f & f & f & \cdots & f & f & m_{n-1,n} \\ f & f & f & & f & f & f \end{bmatrix} \quad \cdot \; r_3(L)$$

L is the working array & U is the upper Δ matrix w/ the multipliers & 1's in the main diagonal. We let $r_i(A)$ denote the i^{th} row of A. Now we take a i^{th} row of L & note

$$r_i(L) = r_i(A) - \sum_{j=i+1}^{n} m_{ij} \, r_j(L) \qquad (\ast)$$

Rearranging $(\ast)$ to solve for $r_i(A)$ yields

$\boxed{\text{Let } n=4}$

Dane's last words

$$r_i(A) = \sum_{j=i+1}^{n} m_{ij} \, r_j(L) + r_i(L)$$

We note that this equation is precisely the i^{th} row of UL. Therefore, $A = UL$. □

$$6 - 4 \cdot \frac{9}{5}$$

$$\frac{30}{5} - \frac{36}{5} = -\frac{6}{5}$$

$$\underset{A}{\begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}} \rightsquigarrow \begin{bmatrix} -\frac{6}{5} & \frac{9}{5} \\ 4 & 5 \end{bmatrix} \cdot \frac{9}{5} \quad m_{12}$$

$$\Longrightarrow U = \begin{bmatrix} 1 & \frac{9}{5} \\ 0 & 1 \end{bmatrix} \qquad A = UL$$

$$L = \begin{bmatrix} -\frac{6}{5} & 0 \\ 4 & 5 \end{bmatrix}$$

3)

No it does not. Recall the spectral radius of a matrix is the largest of the absolute value of the eigen values.

In order for the S.R. to be a norm it must satisfy the properties. However it fail prop 2.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{note that} \quad \rho(A) = 0$$

$$\text{but} \quad A \neq 0_{2\times 2}$$

Among these properties it fails the triangle inequality - so we cannot call it a semi-norm either.

- Compare this with the 2-norm of a matrix. It is given by

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sqrt{\rho(A^T A)}$$

where $\rho(B)$ is the **spectral radius** of a square matrix B. The spectral radius is the maximum of the absolute values of the eigenvalues.

4) **-4- (Inequalities are sharp.)** Explain the meaning of

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|} \le \|A\|\|A^{-1}\|\frac{\|r\|}{\|b\|} \qquad (3)$$

In inequality in (3) is useful in error analysis—specifically backward error analysis. It allows one to create upper & lower bounds on the relative error, namely $\frac{1}{\|x\|}\cdot\|e\|$. We note that in backwards error analysis we do not know e or x.

Now derive (3) for a system $Ax=b$ where A & b are known. Let $\hat{x}$ be an approx solution with some error, e. In other words.

$$\hat{x} = x - e \iff e = x - \hat{x}$$

where we then define the residual, r, as

$$r = b - A\hat{x}$$

well it follows from this that

$$Ae = A(x-\hat{x}) = Ax - A\hat{x} = b - A\hat{x} = r \qquad (\star)$$

We note that by properties of matrix & vector norms we obtain

$$b = Ax \implies \|b\| \le \|A\|\cdot\|x\| \qquad (1)$$

Pick $\|Ax\| =$
Find $\hat{x}$ based on A

$$x = A^{-1}b \implies \|x\| \le \|A^{-1}\|\cdot\|b\| \qquad (2)$$

$$r = Ae \implies \|r\| \le \|A\|\cdot\|e\| \qquad (3)$$

∞ Nor Row-Sum

$$e = A^{-1}r \implies \|e\| \le \|A^{-1}\|\|r\| \qquad (4)$$

(2) can be written as $\frac{1}{\|A^{-1}\|\|b\|} \le \frac{1}{\|x\|}$, and (3) as $\frac{\|r\|}{\|A\|} \le \|e\|$. Multiplying together we obtain

$$\frac{1}{\|A\|\cdot\|A^{-1}\|}\cdot\frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|}$$

Next, (1) as $\frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$, and multiply by (4) we have

$$\frac{\|e\|}{\|x\|} \le \|A\|\|A^{-1}\|\cdot\frac{\|r\|}{\|b\|}$$

We are done. □

As a general comment on matrix norms, for a general matrix A in a finite dimensional space, the norm is given as

$$\|A\| : \max_{\|x\|=1} \|Ax\|.$$

Due to the finite dimension of the space $\exists$ unit vector x that achieves this maximum. When A is invertible the same can be said of A^{-1}. The choice of x is contigent the matrix norm being used — i.e in this case the 2-norm & ∞-norm.

Now in order to show the upper & lower equality of (eqn #) it is suff. to show equality for eqn (1), (4) & eqn (2),(3), respectively. Equality for (eqn #) follows based on the derivation in the proof above.

upper equality We assume A is $m \times m$.

a). Let $\|\cdot\| := \|\cdot\|_2$. For the upper inequality we find x & e s.t (1) & (4) hold. I.e

$$\|b\| = \|A\| \|x\| \quad \& \quad \|e\| = \|A^{-1}\| \|r\|.$$

Finding x (Equal of (1))

We note that $A = U \Sigma V^T$ where U & V are orthogonal matrices. Meaning that

$\|A\| = \|\Sigma\| = \sigma_1$ where σ_1 is the largest singular value. So we simply want to pick x to be the unit vector s.t $\|Ax\| = \sigma_1$. Pick $x = v_1$ where v_1 is the right singular vector corresponding to σ_1. We have

$$\|Av_1\| = \|U \Sigma V^T v_1\| = \|U \Sigma e_1\|$$

$$= \sigma_1 \|U e_1\|$$

$$= \sigma_1$$

since v_1 is a unit vector then

$$\|A\| \|v_1\| = \sigma_1 \cdot 1$$

Finding e (Equality of (4))

Now for A^{-1} we write $A^{-1} = (U \Sigma V^T)^{-1} = V \overset{\Sigma^{-1} \text{ exist since } A \text{ invertible}}{\Sigma^{-1}} U^T$, thus $\|A^{-1}\| = \|\Sigma^{-1}\| = \frac{1}{\sigma_m}$ where σ_m is the smallest singular value. To find e we simply pick a unit vector r s.t $\|A^{-1} r\| = \frac{1}{\sigma_m}$ and let $e = A^{-1} r$.

Let $r = u_m$ where u_m is unit column vector in U. Then we have,

$$\|A^{-1} u_m\| = \|V \Sigma^{-1} U^T u_m\|$$

$$= \|V \Sigma^{-1} e_m\|$$

$$= \frac{1}{\sigma_m} \|V e_m\|$$

$$= \frac{1}{\sigma_m} \cdot 1$$

and since u_m is a unit vector then,

$$\|A^{-1}\| \|u_m\| = \frac{1}{\sigma_m} \cdot 1.$$

Lower Equality

Without a loss in generality, picking x & e to show equality in equations (2) & (3) follow the same argument. Thus showing the lower inequality. $\square$

b). Let $\|\cdot\| := \|\cdot\|_\infty$. Again we assume A is an $n \times n$ invertible matrix. Recall the ∞-norm of A is the maximum absolute row sum. i.e.

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \max_{i=1\ldots n} \sum_{j=1}^{n} |a_{ij}|$$

Again it suff. to pick x & e s.t equality in eqn (1), (4) & similarly for (2), (3).

These are eqn I am referring too

$\|b\| \leq \|A\| \cdot \|x\|$ (1)

$\|x\| \leq \|A^{-1}\| \cdot \|b\|$ (2)

$\|r\| \leq \|A\| \cdot \|e\|$ (3)

$\|e\| \leq \|A^{-1}\| \|r\|$ (4)

Upper Equality

Finding x (Equality of (1))

Let

$$x = s = [\mathrm{sign}(a_{ij})] \quad \text{where} \quad \mathrm{sign}(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \\ 0 & z = 0 \end{cases}.$$

Now observe,

$$\|As\| = \left\| \left[\sum_{j=1}^{n} |a_{1j}| \ \cdots \ \sum_{j=1}^{n} |a_{nj}| \right]^T \right\|_\infty$$

$$= \max_{i=1\ldots n} \sum_{j=1}^{n} |a_{ij}| .$$

We note that $\|s\|_\infty = 1$, thus

$$\|A\| \|s\| = \max_{i=1\ldots n} \sum_{j=1}^{n} |a_{ij}| \cdot 1 .$$

Finding e (Equality of (4))

We let $r = s$ as above and note again that

$$\|A^{-1}s\| = \|A^{-1}\| \|s\|$$

Thus let $e = A^{-1}s$.

Lower Equality

The argument is precisely the same for eqn (2)(3).

We are done. $\square$

-5- (Backward Error Analysis.) This problem explores the effects of a perturbation in the coefficient matrix (rather than the right hand side) of the linear system

$$Ax = b \tag{4}$$

Suppose we solve instead of (4) the system

$$(A - E)(x - e) = b \tag{5}$$

where E is a perturbation of A that causes an error e in the solution x. Show that

$$\frac{\|e\|}{\|x-e\|} \leq \|A\|\|A^{-1}\|\frac{\|E\|}{\|A\|} \tag{6}$$

We assume $\|A\| \neq 0$ & $\|x-e\| \neq 0$

Take $(A-E)(x-e) = b$

$$A(x-e) - E(x-e) = b$$

$$Ax - Ae - E(x-e) = b$$

$$-Ae - E(x-e) = b - Ax = 0$$

$$Ae = -E(x-e)$$

$$e = -A^{-1}E(x-e)$$

$$\|e\| \leq \|A^{-1}\| \|E\| \|x-e\|$$

$$\frac{\|e\|}{\|x-e\|} \leq \|A^{-1}\| \|E\| = \|A\|\|A^{-1}\| \frac{\|E\|}{\|A\|} \quad \square$$

-6- (First Order Systems.) Write the second order initial value problem

$$y'' = \underbrace{xy^2}_{u=y'}, \qquad y(0) = 1, \qquad y'(0) = 2 \tag{7}$$

as an autonomous first order system $\quad u' = y'' = xy^2$

$$y' = f(y), \qquad y(a) = y_0. \tag{8}$$

(In other words, specify y, f, a, and y_0 such that the two problems are equivalent. Of course, y will have different meanings for the two problems.)

Let $u = y'$ & $x = t$, then we obtain the following autonomous first order system

$$\begin{cases} y' = u & y(0) = 1 \\ u' = xy^2 & u(0) = 2 \\ x' = 1 & x(0) = 0 \end{cases}$$

7 & 8

$1 - \varepsilon - (-1 + \varepsilon)$

$2 - 2\varepsilon$

$2(1 - \varepsilon)$

$$\frac{\ln(x^2+1)}{\sqrt{1-x^2}}$$

Need to program.