## Problem 6

We that this problem is quite easily solvable by using the sum-to-product identity for cosine.

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Giving us,

$$\cos(x + 1) = \cos(1)\cos(x) - \sin(1)\sin(x)$$

which fits the form of the Fourier series.

## Problem 8

In order to show that any polynomial in power form can be uniquely written in B-form, we can simply show that the Bezier polynomials form a basis for the degree $d$ space. Let

$$B_i^d = \binom{d}{i} b_1^1 b_2^{d-i} \quad \text{for} \quad i = 0, 1, \cdots, d$$

It suffices to show the $B_i^d$ are linearly independent polynomials with respect to $b_1$, where $b_2 = 1 - b_1$. This is sufficient since we have the correct number of polynomials to form a basis for this space. Using the Binomial expansion theorem and the fact that $b_2 = 1 - b_1$ we have,

$$\begin{aligned} B_i^d &= \binom{d}{i} b_1^i \cdot (1 - b_1)^{d-i} \\ &= \binom{d}{i} b_1^i \sum_{k=0}^{d-i} (-1)^k \binom{d-i}{k} b_1^k \end{aligned}$$

Applying a change of index, and some algebraic manipulation gives us,

$$\begin{aligned} &= \sum_{k=i}^{d} (-1)^{k-i} \binom{d}{i} \binom{d-i}{k-i} b_1^{k+i-i} \\ &= \sum_{k=i}^{d} (-1)^{k-i} \binom{d}{k} \binom{k}{i} b_1^k \end{aligned}$$

The last equality comes from the fact that $\binom{d}{i}\binom{d-i}{k-i} = \binom{d}{k}\binom{k}{i}$ which is verified at the end. So we can write our Bezier polynomials in the form derived above.

$$B_i^d = \sum_{k=i}^{d} (-1)^{k-i} \binom{d}{k} \binom{k}{i} b_1^k$$

Now for showing linear independence if we have,

$$\sum_{i=0}^{d} \alpha_i B_i^d = 0$$

for some $\alpha_i$ coefficients, we show that all $\alpha_i$ are zero. Expanding this sum out we have:

$$\alpha_0 \sum_{k=0}^{d} \binom{d}{k} \binom{k}{0} (-1^k) b_1^k + \alpha_1 \sum_{k=1}^{d} \binom{d}{k} \binom{k}{1} (-1)^{k-1} b_1^k + \cdots + \alpha_d \sum_{k=d}^{d} \binom{d}{k} \binom{k}{d} (-1)^{k-d} b_1^k = 0 \quad (1)$$

We can see the only constant term, with respect to $b_1$ as our variable, is $\alpha_0$. Meaning that $\alpha_0 = 0$ as there are no other constant terms to cancel out with. So we can simplify our equation (1) to

$$\alpha_1 \sum_{k=1}^{d} \binom{d}{k} \binom{k}{1} (-1)^{k-1} b_1^k + \cdots + \alpha_d \sum_{k=d}^{d} \binom{d}{k} \binom{k}{d} (-1)^{k-d} b_1^k = 0$$

Again we note there is now only one $b_1$ term. This term has a coefficient of $\alpha_1$. Meaning that $\alpha_1 = 0$. Continuing this process inductively we see that

$$\alpha_0 = \alpha_1 = \cdots = \alpha_d = 0$$

So $\{B_i^d\}_{i=0}^d$ forms a basis for our degree $d$ polynomial space. Thus we can write any polynomial in power form uniquely into B-form.

Lastly verifying $\binom{d}{i}\binom{d-i}{k-i} = \binom{d}{k}\binom{k}{i}$.

$$\begin{aligned}
\binom{d}{i}\binom{d-i}{k-i} &= \frac{d!}{i!(d-i)!}\frac{(d-i)!}{(k-i)!(d-k)!} \\
&= \frac{d!}{i!(k-i)!(d-k)!} \\
&= \frac{d!k!}{i!(k-i)!(d-k)!k!} \\
&= \frac{d!}{k!(d-k)!}\frac{k!}{i!(k-i)!} \\
&= \binom{d}{k}\binom{k}{i}
\end{aligned}$$

## Problem 9

We note that are interpolating at $2n+1$ distinct nodes thus we make the primitive assertion our polynomial $p$ is at most degree $2n$. We define the following function

$$g(x) = p(x) + p(-x)$$

where $g$ is a polynomial of degree at most $2n$. We note that $g(x)$ has $2n+1$ distinct roots, namely $x_i$ for $i = -n, -n+1, \cdots, n-1, n$. However the only polynomial with a larger number of roots then the degree is infact the zero polynomial, or the zero function. So we have,

$$g(x) = p(x) + p(-x) = 0 \quad \text{for all } x \in \mathbb{R}$$

Thus we have,

$$p(x) = -p(-x)$$

for all real numbers $x$. This of course means that our polynomial is an odd function, allowing us to mend our primitive answer before—$p$ can have a degree at most $2n-1$.