Ex. 10 $\|\vec{x}_v - \vec{x}_s\| = c(t_v - t_s)$

$$\|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_2}\| = c(t_{s_2} - t_{s_1}) \quad (1)$$
$$\|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_3}\| = c(t_{s_3} - t_{s_1}) \quad (2)$$
$$\|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_4}\| = c(t_{s_4} - t_{s_1}) \quad (3)$$
$$\|\vec{x}_v - \vec{x}_{s_1}\| = c(t_v - t_{s_1}) \quad (4)$$

11.

Let $\|\vec{x}_v - \vec{x}_{s_i}\| = c(t_v - t_{s_i}) = \Delta_i$

$$\left\{\begin{array}{l} \Delta_1 - \Delta_2 = \delta_{12} \\ \vdots \\ \Delta_1 - \Delta_{m-1} = \delta_{1m-1} \end{array}\right.$$ ← use least squares

$\Delta_i = c(t_v - t_{s_1})$ ← use to compute t_v

let, $X = \begin{bmatrix} \delta_{12} \\ \vdots \\ \delta_{1,m-1} \end{bmatrix}$, then our system of non-linear equations (1) becomes:

(1)
$$\begin{bmatrix} \delta_{12} - c(t_{s_2} - t_{s_1}) = 0 \\ \vdots \\ \delta_{1m-1} - c(t_{s_{m-1}} - t_{s_1}) = 0 \\ \|\vec{x}_v - \vec{x}_{s_1}\| - c(t_v - t_{s_1}) = 0 \end{bmatrix}$$

$$\boxed{F(x) = 0, \text{ where } F(x) = (1)}$$

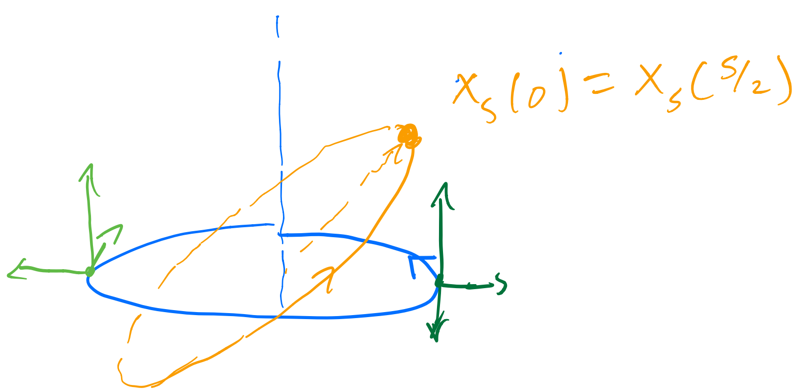
then we can form a scalar f to be minimized by: $\boxed{f(x) \equiv F(x)^T F(x)} = \boxed{F^T F}$

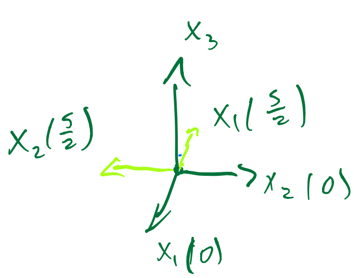
$$\boxed{\underbrace{[\delta_{12} - c(t_{s_2} - t_{s_1})]^2 + \dots + [\delta_{1,m-1} - c(t_{s_{m-1}} - t_{s_1})]^2}_{} = f(x)}$$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \sum_{n=2}^{m-1} 2[\delta_{1n} - c(t_{s_n} - t_{s_1})] \frac{\partial}{\partial x_1}[\|\vec{x}_v - \vec{x}_1\| - \|\vec{x}_v - \vec{x}_n\|] \\ \sum_{n=2}^{m-1} 2[\delta_{1n} - c(t_{s_n} - t_{s_1})] \frac{\partial}{\partial x_2}[\|\vec{x}_v - \vec{x}_1\| - \|\vec{x}_v - \vec{x}_n\|] \\ \sum_{n=2}^{m-1} 2[\delta_{1n} - c(t_{s_n} - t_{s_1})] \frac{\partial}{\partial x_3}[\|\vec{x}_v - \vec{x}_1\| - \|\vec{x}_v - \vec{x}_n\|] \end{bmatrix} = \vec{0}$$

soln $\Rightarrow \vec{x}_v \Rightarrow \boxed{\|\vec{x}_v - \vec{x}_{s_1}\| = c(t_v - t_{s_1})}$

Ex. 12



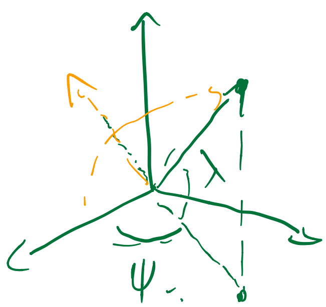


after $T = \frac{S}{2}$ x_s has returned to its starting value while our rotating coordinate system has rotated by 180° about the z-axis.

Start in the rotating frame of the Earth.

We'll use a rotation matrix.

In the rotating frame of reference the period of the satellite is s not s/2 since from the perspective of the Earth's coordinate axis it takes s seconds to get back to its original position according to the rotating frame's coordinates.



rotating frame

$$\vec{X}_{rot} = R(t)\vec{X}_{fixed}, \text{ where}$$

$$\vec{X}_{fixed} = (R)\left[\vec{u}\cos\left(\frac{2\pi t}{p}+\theta\right) + \vec{v}\sin\left(\frac{2\pi t}{p}+\theta\right)\right]$$

$\theta = 0$ for satellite 1

Apply a rotation matrix to x-fixed to get x-rot. From here we'll convert into geographic coordinates using the formulas from problem 5.

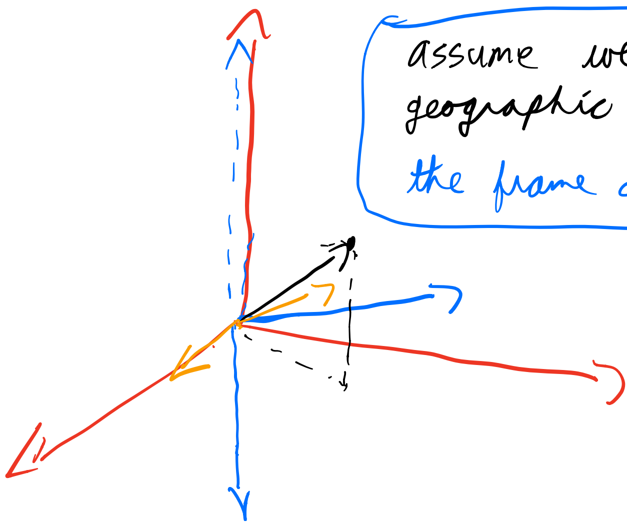
$$\vec{X}_{rot}(t) = R\left[\cos\left(\frac{2\pi}{p}t\right)\vec{u}_{rot} + \sin\left(\frac{2\pi}{p}t\right)\vec{v}_{rot}\right]$$

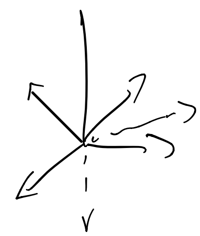
Below is the explicit derivation of x-rot with all of the numbers for u and v and the rotation matrix thrown in.

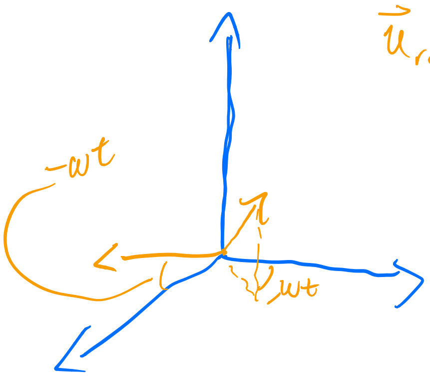
$$x(t) = R\cos\left(\tfrac{2\pi}{5}t\right) \quad (1)$$
$$y(t) = R\sin\left(\tfrac{2\pi}{5}t\right) \quad ($$

- Earth frame
- fixed Cartesian frame
- ground track of satellite
- $\vec{u}$, - $\vec{v}$ } fixed

assume we are measuring geographic coordinates in the frame of the Earth.







$$\vec{u}_{rot} = R\,\vec{u} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ +\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\omega t \\ +\sin\omega t \\ 0 \end{bmatrix}$$

$$\vec{V}_{rot} = R\,\vec{V} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ +\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 5.73\times10^{-1} \\ 8.19\times10^{-1} \end{pmatrix}$$

$$= \begin{bmatrix} -0.573\sin\omega t \\ 0.573\cos\omega t \\ 0.819 \end{bmatrix} \Rightarrow \vec{X}_{rot} = R\left[\cos\left(\tfrac{2\pi}{P}t\right)\begin{bmatrix} +\cos\omega t \\ +\sin\omega t \\ 0 \end{bmatrix} + \sin\left(\tfrac{2\pi}{P}t\right)\begin{bmatrix} \vdots \\ \vdots \\ \cdots \end{bmatrix}\right]$$

I used the transformation equations from 5. This is because instead of rotating backwards in time to the point where rotating coordinates and fixed Cartesian coordinates are the same, I rotated the position vector in fixed Cartesian into the frame rotating with the earth, which is the frame in which geographic coordinates are defined. Thus the transformation is justified. I am working on proving that this is equivalent to the method in problem 6 where we unrotated then transformed to geographic. I am pretty sure they are equivalent but I'll be checking this weekend.

Note I did this because the method in 6 I think works here but I wasn't sure until pretty far into the derivation haha.

$$\rho = \sqrt{x^2+y^2+z^2} = R\sqrt{\left\{\left[\cos\left(\frac{2\pi}{P}t\right)\cos\left(\frac{2\pi}{S}t\right)\right]^2 + \left[0.573\sin\left(\frac{2\pi}{P}t\right)\cos\left(\frac{2\pi}{S}t\right) - \cos\left(\frac{2\pi}{P}t\right)\sin\left(\frac{2\pi}{S}t\right)\right]^2 + \sin^2\left(\frac{2\pi}{P}t\right)(0.819)^2\right\}} \hat{=} R$$

I know that this is R because Rho is the square root of the norm of the rotating satellite vector which I checked was R.

Note that I have not included the h here because the ground track of the satellite is at R. In other words set h = 0

$$\varphi = \left|\arcsin\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right|$$

$$WE = \begin{cases} 1 & \text{if } 0.573\sin\left(\frac{2\pi}{P}t\right)\cos\left(\frac{2\pi}{S}t\right) - \cos\left(\frac{2\pi}{P}t\right)\sin\left(\frac{2\pi}{S}t\right) \geq 0 \\ -1 & \text{if } \quad \text{''} \quad < 0 \end{cases}$$

$$\lambda = \left\{ \text{see problem 5} \right.$$

The significance of the the period being half a sidereal day is that the satellite is in the horizon of every point of the earth at least once throughout its cycle. I believe this is done because the earth is rotating at half its speed and therefore any speed less than the one the satellite is at would leave a blind spot on the earth once per day.

13.

$$\|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_2}\| = c(t_{s_2} - t_{s_1}) \quad (1)$$

$$\|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_3}\| = c(t_{s_3} - t_{s_1}) \quad (2)$$

$$\|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_4}\| = c(t_{s_4} - t_{s_1}) \quad (3)$$

$$\|\vec{x}_v - \vec{x}_{s_1}\| = c(t_v - t_{s_1}) \quad (4)$$

Let $\vec{x}_v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Image 1

- For the receiver program in the term project we have equations like

$$z = \|x_V - x_S\| - c(t_V - t_S) = 0$$

where

x_V is the unknown position of the vehicle

x_S is the known position of the satellite

t_V is the unknown time at which the vehicle receives the signals, and

t_S is the known time at which the satellite broadcasts.

- Let's say

$$x_V = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \quad \text{and} \quad x_S = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}.$$

Then

$$z = \left(\sum_{i=1}^{3}(\xi_i - \sigma_i)^2\right)^{\frac{1}{2}} - c(t_V - t_s),$$

and, for example,

$$\frac{\partial z}{\partial \xi_i} = \frac{2(\xi_i - \sigma_i)}{2\left(\sum_{i=1}^{3}(\xi_i - \sigma_i)^2\right)^{\frac{1}{2}}} = \frac{(\xi_i - \sigma_i)}{\left(\sum_{i=1}^{3}(\xi_i - \sigma_i)^2\right)^{\frac{1}{2}}}.$$

$$y_1 = \|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_2}\| - c(t_2 - t_1)$$

$$y_2 = \|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_3}\| - c(t_3 - t_1)$$

$$y_3 = \|\vec{x}_v - \vec{x}_{s_1}\| - \|\vec{x}_v - \vec{x}_{s_4}\| - c(t_4 - t_1)$$

$$y_4 = \|\vec{x}_v - \vec{x}_{s_1}\| - c(t_v - t_{s_1})$$

Need to compute $\frac{\partial y_i}{\partial x_j}$ where $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$ and

$x_4 = t_v$

i.e we need the Jacobian J

Expand y_ℓ $\ell = 1, 2, 3$

$$y_l = \sqrt{\sum_{k=1}^{3}(x_k - \sigma_k^{(1)})^2} - \sqrt{\sum_{k=1}^{3}(x_k - \sigma_k^{(l)})^2} - c(t_l - t_1) \quad (1)$$

* note I'm using σ_k for the components of the relevant $\vec{x}_{s_i}$ e.g. if $i=1$, then σ_k would be the kth component of $\vec{x}_{s_1}$. Likewise for γ_k for $\vec{x}_{s_2}$

$$(2) \quad \frac{\partial y_l}{\partial x_j} = \begin{cases} \dfrac{x_j - \sigma_j^{(1)}}{\sqrt{\sum_{k=1}^{3}(x_k - \sigma_k)^2}} - \dfrac{x_j - \sigma_j^{(l)}}{\sqrt{\sum_{k=1}^{3}(x_k - \gamma_k)^2}} & \text{if } j \neq 4 \\ 0 & \text{if } j = 4 \end{cases}$$

$$y_4 = \sqrt{\sum_{k=1}^{3}(x_k - \sigma_k^{(1)})^2} - c(x_4 - t_{s_1})$$

$$\frac{\partial y_4}{\partial x_j} = \begin{cases} \dfrac{x_j - \sigma_j^{(1)}}{\sqrt{\sum_{k=1}^{3}(x_k - \sigma_k^{(1)})^2}} & \text{if } j \neq 4 \\ -c & \text{if } j = 4 \end{cases} \quad (3)$$

For Newton's method we iteratively solve:

$$J(\vec{x}^{(k)})\vec{s}^{(k)} = -F(\vec{x}^{(k)}), \quad \vec{x}^{(0)} = \text{guess}$$

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} + \vec{s}^{(k)}$$

Note that superscripts now denote steps in the Newton's method iteration.

14) From (10) or 11? ...

$$\left\{\begin{aligned}
&\sum_{n=2}^{m-1} 2[\delta_{1n} - c(t_{s_n} - t_{s_1})]\frac{\partial}{\partial x_1}\left[\|\vec{x}_v - \vec{x}_1\| - \|\vec{x}_v - \vec{x}_n\|\right] = 0 \\
&\sum_{n=2}^{m-1} 2[\delta_{1n} - c(t_{s_n} - t_{s_1})]\frac{\partial}{\partial x_2}\left[\|\vec{x}_v - \vec{x}_1\| - \|\vec{x}_v - \vec{x}_n\|\right] = 0 \\
&\sum_{n=2}^{m-1} 2[\delta_{1n} - c(t_{s_n} - t_{s_1})]\frac{\partial}{\partial x_3}\left[\|\vec{x}_v - \vec{x}_1\| - \|\vec{x}_v - \vec{x}_n\|\right] = 0 \\
&\|\vec{x}_v - \vec{x}_{s_1}\| - c(t_v - t_{s_1}) = 0
\end{aligned}\right\}$$

* apply Newton's method to this system