Numerical Analysis Project
MATH 5600
Homework 1

Authors:
Dane Gollero
Ike Griss Salas
Magon Bowling

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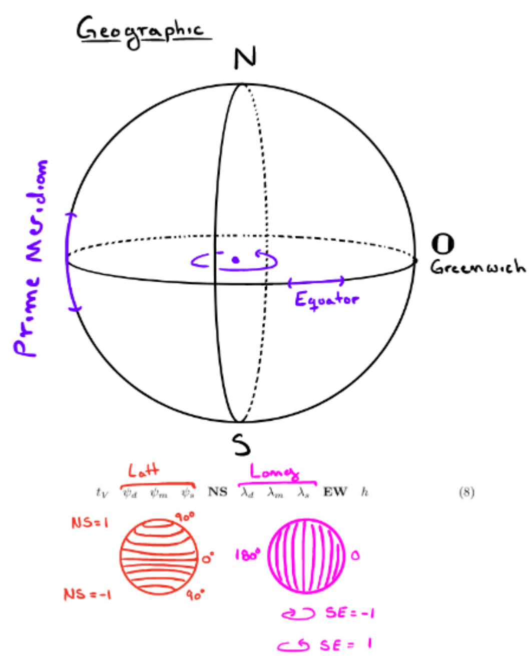
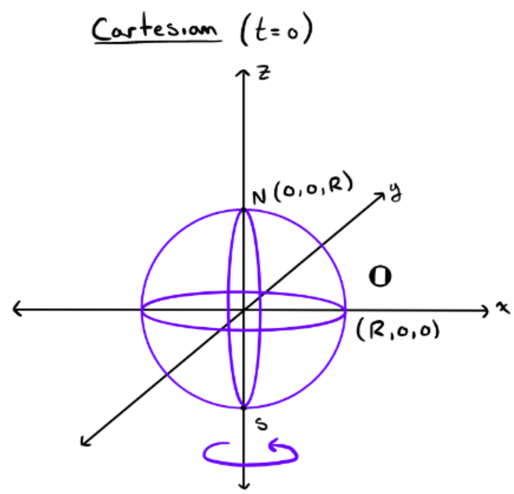
Models of the Earth with Respect to Coordinate Systems

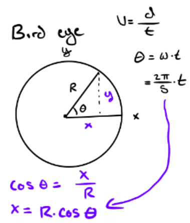
In our work through the term project, we utilize two coordinate systems: Geographical and Cartesian. Geographical coordinates are used to identify a location on the Earth in terms of Longitude and Latitude. In relationship with Cartesian coordinates, Geographic coordinates rotate around the z-axis with respect to time. At time = 0, the x-axis intersects the globe at the Equator and Prime Meridian. The y-axis therefore perpendicular to the xz-plane.





Exercise 1: Find a formula that describes the trajectory of the point **O** in Cartesian coordinates as a function of time.

From Physics, we know that distance is equal to the rate of an object multiplied by the time traveled. We can apply this to trajectory with the equation $\theta = \omega \cdot t$, where ω is angular velocity. Angular velocity is the distance around the globe divided by one sidereal day. Thus we obtain the angle $\theta = \frac{2\pi}{s} \cdot t$. The formula that describes the trajectory of the point $\mathbf{O}$ as a function of time in Cartesian coordinates is:



$$\mathbf{O}_{car}(t) = \begin{bmatrix} R\cos\left(\frac{2pi}{s}\right) \cdot t \\ R\sin\left(\frac{2pi}{s}\right) \cdot t \\ 0 \end{bmatrix}$$

Exercise 2: Write a program that converts angles from degrees, minutes, and seconds to radians, and vice versa.

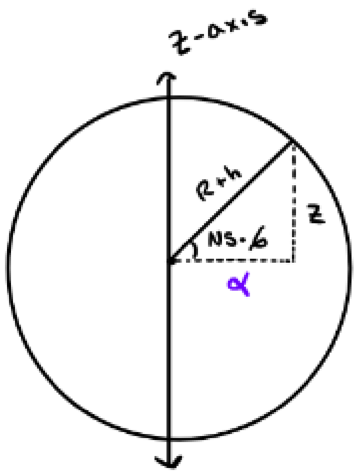
```
import math
pi = math.pi

def deg_to_rad(deg, min, sec):
    rad = (deg + min / 60 + sec / 3600) / 180 * pi
    return rad

# Need to find a way to separate the whole number.
def rad_to_deg(rad):
    dummy = rad * 180 / pi
    deg = floor(dummy) # Just use the whole number
    min = floor((dummy - floor(dummy)) * 60) # Multiply the decimals by 60
    sec = floor(((dummy - floor(dummy)) * 60 - min) * 60)
    return deg, min, sec

```

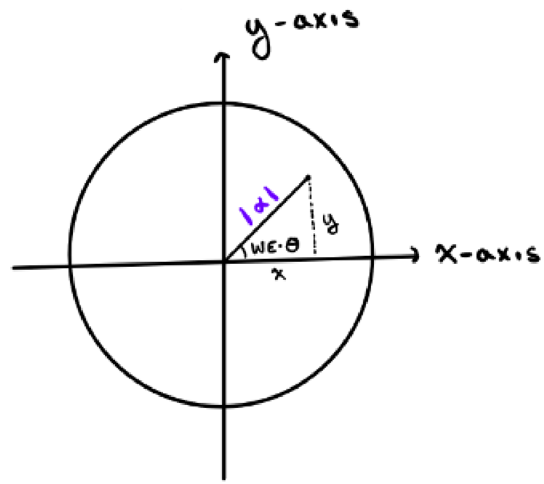
Exercise 3: Find a formula that converts position as given in (8) at time $t = 0$ into Cartesian coordinates.

It is important to note the parameters for the following variables:
$t_V = 0, \quad \psi_d, \psi_m, \psi_s \in \left[0, \frac{pi}{2}\right], \quad \text{NS} \in \{-1, 1\}, \quad \lambda_d, \lambda_m, \lambda_s \in [o, \pi], \quad \text{EW} \in \{-1, 1\}, \quad h = h$

Step 1) Using the program from **Exercise 2**, we will do the following conversions:
ψ_d, ψ_m, ψ_s to $\phi \in \left[0, \frac{pi}{2}\right]$ and $\lambda_d, \lambda_m, \lambda_s$ to $\theta \in [o, \pi]$

Step 2) We will solve for x, y, and z using trigonometry functions. Note the following diagrams, where $\mathbf{R}$ is the radius of the Earth, α is the perpendicular distance of the position, and the rest have been identified above:





We arrive at the following three equations:

- $z(\phi, \mathrm{NS}, h) = (R+h)\sin(\mathrm{NS}\cdot\phi)$
- $\alpha(\phi, \mathrm{NS}, h) = (R+h)\cos(\mathrm{NS}\cdot\phi)$
- $x(\theta, \mathrm{EW}, \alpha) = |\alpha|\cos(\mathrm{EW}\cdot\theta)$
- $y(\theta, \mathrm{EW}, \alpha) = |\alpha|\sin(\mathrm{EW}\cdot\theta)$

Thus Given we know ϕ and θ using the program from **Exercise 2**, we have at $t = 0$,

$$(8) := \mathbf{X}_o = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} |\alpha|\cos(\mathrm{EW}\cdot\theta) \\ |\alpha|\sin(\mathrm{EW}\cdot\theta) \\ (R+h)\sin(\mathrm{NS}\cdot\phi) \end{bmatrix}$$

Exercise 4: Find a formula that converts position and general time t as a given in (8) into Cartesian coordinates.

We use our formula from **Exercise 3** to find the Cartesian coordinates at $t = 0$ and then use a rotation matrix to find position at time t. It is important to note that the rotation is around the z-axis.

$$\text{Rotation Matrix:} \quad R(t) = \begin{bmatrix} \cos\left(\frac{2\pi}{s}\cdot t\right) & -\sin\left(\frac{2\pi}{s}\cdot t\right) & 0 \\ \sin\left(\frac{2\pi}{s}\cdot t\right) & \cos\left(\frac{2\pi}{s}\cdot t\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We compure the following matrix multiplication,

$$\mathbf{X(t)} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = R(t)\cdot\mathbf{x}_o$$

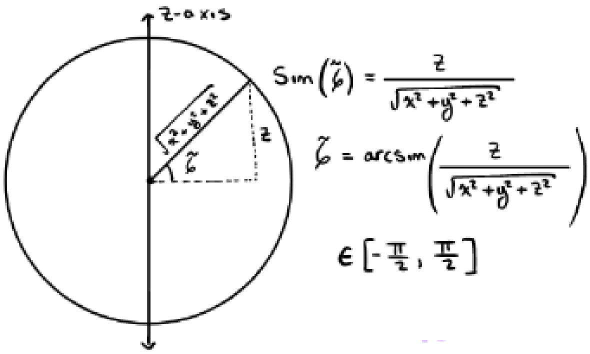
$$\mathbf{X(t)} = \begin{bmatrix} \cos\left(\frac{2\pi}{s}\cdot t\right) & -\sin\left(\frac{2\pi}{s}\cdot t\right) & 0 \\ \sin\left(\frac{2\pi}{s}\cdot t\right) & \cos\left(\frac{2\pi}{s}\cdot t\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} |\alpha|\cos(\mathrm{EW}\cdot\theta) \\ |\alpha|\sin(\mathrm{EW}\cdot\theta) \\ (R+h)\sin(\mathrm{NS}\cdot\phi) \end{bmatrix}$$

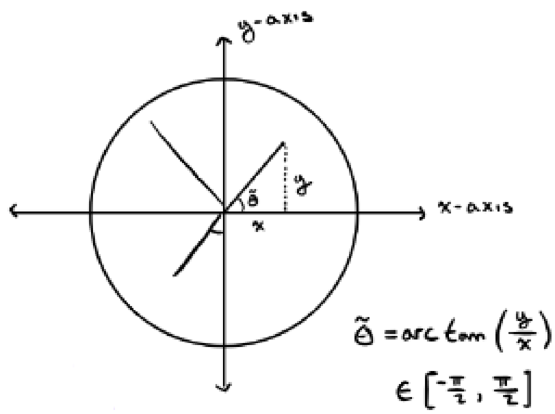
Now we arrive at,

$$\mathbf{X(t)} = \begin{bmatrix} |\alpha|\cos\left(\frac{2\pi}{s}\cdot t\right)\cos(\mathrm{EW}\cdot\theta) - |\alpha|\sin\left(\frac{2\pi}{s}\cdot t\right)\sin(\mathrm{EW}\cdot\theta) \\ |\alpha|\sin\left(\frac{2\pi}{s}\cdot t\right)\cos(\mathrm{EW}\cdot\theta) + |\alpha|\cos\left(\frac{2\pi}{s}\cdot t\right)\sin(\mathrm{EW}\cdot\theta) \\ (R+h)\sin(\mathrm{NS}\cdot\phi) \end{bmatrix}$$

Exercise 5: Find a formula that converts a position given in Cartesian coordinates at time $t = 0$ into a position of the form (8).

Given Cartesian coordinates $[x, y, z]^T$, we need to find Geographical coordinates in terms of $t_V \quad \psi_d \quad \psi_m \quad \psi_s \quad \mathrm{NS} \quad \lambda_d \quad \lambda_m \quad \lambda_s \quad \mathrm{EW} \quad h$. Using the following diagrams, we can identify Geographic variables to assist in forming the formula for conversion.

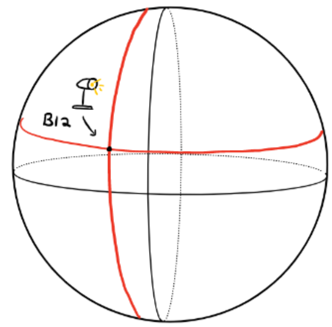




- $t_V = 0$
- $h = \sqrt{x^2+y^2+z^2} - R$
- $\mathrm{NS} = \begin{cases} 1, & z \geq 0 \\ -1, & z > 0 \end{cases}$
- $\psi_d, \psi_m, \psi_s = \left| \arcsin\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \right|$ *Use conversion program $\rightarrow [\psi_d, \psi_m, \psi_s]$
- $\mathrm{EW} = \begin{cases} 1, & y \geq 0 \\ -1, & y > 0 \end{cases}$
- $\lambda_d, \lambda_m, \lambda_s := \begin{cases} \frac{\pi}{2}, & x = 0 \\ 0, & x > 0 \text{ and } y = 0 \\ \pi, & x < 0 \text{ and } y = 0 \\ \arctan\left(\frac{y}{x}\right), & x > 0 \text{ and } y > 0 \\ \left|\arctan\left(\frac{y}{x}\right)\right|, & x > 0 \text{ and } y < 0 \\ \pi + \arctan\left(\frac{y}{x}\right), & x < 0 \text{ and } y > 0 \\ \pi - \arctan\left(\frac{y}{x}\right), & x < 0 \text{ and } y < 0 \end{cases}$ *Use conversion program $\rightarrow [\lambda_d, \lambda_m, \lambda_s]$

Exercise 6: Find a formula that converts general time t and a position given in Cartesian coordinates into a position of the form (8).

Exercise 7: Find a formula that describes the trajectory of lamp post B12 in Cartesian coordinates as a function of time.



Illustrated in the diagram at the right, we have a fixed point where street light B12 is located at the following Geographic coordinates:

$$t \quad 40 \quad 45 \quad 55.0 \quad 1 \quad 111 \quad 50 \quad 58.0 \quad -1 \quad 1372.00$$

We use the conversion program from **Exercise 2** to define ψ and θ, and then use the formula from **Exercise 4** to generate the formula we need.
Define:

$$\begin{aligned}\phi &= \phi(40, 45, 55.0)\\ &= \left(40 + \frac{45}{60} + \frac{55.0}{3600}\right) \cdot \frac{\pi}{180}\end{aligned}$$

$$\begin{aligned}\theta &= \theta(111, 50, 58.0)\\ &= \left(111 + \frac{50}{60} + \frac{58.0}{3600}\right) \cdot \frac{\pi}{180}\end{aligned}$$

Exercise 4 formula:

$$\mathbf{X}_{Lamp}(t+t_V) = \begin{bmatrix} |\alpha|\cos\left(\frac{2\pi}{s}\cdot(t+t_V)\right)\cos(\mathrm{EW}\cdot\theta) - |\alpha|\sin\left(\frac{2\pi}{s}\cdot(t+t_V)\right)\sin(\mathrm{EW}\cdot\theta) \\ |\alpha|\sin\left(\frac{2\pi}{s}\cdot(t+t_V)\right)\cos(\mathrm{EW}\cdot\theta) + |\alpha|\cos\left(\frac{2\pi}{s}\cdot(t+t_V)\right)\sin(\mathrm{EW}\cdot\theta) \\ (R+h)\sin(\mathrm{NS}\cdot\phi) \end{bmatrix}$$

where $\alpha = (R+h)\cos(\mathrm{NS}\cdot\phi)$.