1).

Let A be a positive definite matrix, that is for any $x \in \mathbb{R}^n$ & $x \neq \vec{0}$, then we have

$$x^T A x > 0$$

We note that any principle submatrix must be square since ... (give argument)

Now take a P.S.M $A_{m \times m}$ where $m < n$. This matrix discludes some set of rows & columns from A. SPIC that $\exists y \in \mathbb{R}^m$ s.t

$$y^T A_{m \times m} y \leq 0$$

Then we construct $\tilde{y} \in \mathbb{R}^n$, isomorphic to $y \in \mathbb{R}^m$, by adding zeroes to the indices of the rows/columns removed to construct the P.S.M $A_{m \times m}$. Then it must be the case that

$$\tilde{y}^T A \tilde{y} \leq 0$$

Which contradicts A being positive definite. Therefore $A_{m \times m}$ must also be positive definite. $\square$

$\{1, 2\} - \tilde{A}$

$$\overset{\hat{A}}{}$$

$$[y_1 \ \ y_2]^T \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq 0$$

$$= [y_1 \ \ y_2]^T \underbrace{\begin{bmatrix} ay_1 + by_2 \\ dy_1 + ey_2 \end{bmatrix}}$$

$$= y_1(ay_1 + by_2) + y_2(dy_1 + ey_2) \leq 0$$

$$\tilde{y} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$$

$$[y_1 \ \ y_2 \ \ 0] \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & I \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}}$$

$$[y_1 \ \ y_2 \ \ 0] \begin{bmatrix} ay_1 + by_2 \\ dy_1 + ey_2 \\ 0 \end{bmatrix}$$

$$= y_1(ay_1 + by_2) + y_2(dy_1 + ey_2) \leq 0$$

2). Not sore yet.

$$\begin{bmatrix} 6 & 4 & 4 \\ 9 & 5 & 6 \\ 1 & 3 & 2 \end{bmatrix} \quad \Big[$$

$$\begin{bmatrix} 1 & a & b \\ & 1 & c \\ & c & 1 \end{bmatrix}\Big[$$

[A | I]

U | L

3)

No it does not. Recall the spectral radius of a matrix is the largest of the absolute value of the eigen values.

In order for the S.R. to be a norm it must satisfy the properties. However it fail prop 2.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{note that} \quad \rho(A) = 0$$

$$\text{but} \quad A \neq 0_{2\times 2}$$

Among these properties it fails the triangle inequality - so we cannot call it a semi-norm either.

- Compare this with the 2-norm of a matrix. It is given by

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sqrt{\rho(A^T A)}$$

where $\rho(B)$ is the **spectral radius** of a square matrix B. The spectral radius is the maximum of the absolute values of the eigenvalues.

1 $\|e\|$ $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\frac{\|e\|}{\|x\|} = c\,\frac{\|r\|}{\|b\|} = \frac{c}{c}\cdot\frac{\|r\|}{\|x\|}$$

$\|e\| = \|r\|$

4) **-4- (Inequalities are sharp.)** Explain the meaning of

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|r\|}{\|b\|} \tag{3}$$

In inequality in (3) is useful in error analysis—specifically backward error analysis. It allows one to create upper & lower bounds on the relative error, namely $\frac{1}{\|x\|}\cdot\|e\|$. We note that in backwards error analysis we do not know e or x.

Now derive (3) for a system $Ax=b$ where A & b are known. Let $\hat{x}$ be an approx solution with some error, e. In other words.

$$\hat{x} = x - e \iff e = x - \hat{x}$$

where we then define the residual, r, as

$$r = b - A\hat{x}$$

well it follows from this that

$$Ae = A(x-\hat{x}) = Ax - A\hat{x} = b - A\hat{x} = r \tag{$\star$}$$

We note that by properties of matrix & vector norms we obtain

$$b = Ax \implies \|b\| \leq \|A\|\cdot\|x\| \tag{1}$$

$$x = A^{-1}b \implies \|x\| \leq \|A^{-1}\|\cdot\|b\| \tag{2}$$

$$r = Ae \implies \|r\| \leq \|A\|\cdot\|e\| \tag{3}$$

$$e = A^{-1}r \implies \|e\| \leq \|A^{-1}\|\|r\| \tag{4}$$

(2) can be written as $\frac{1}{\|A^{-1}\|\|b\|} \leq \frac{1}{\|x\|}$, and (3) as $\frac{\|r\|}{\|A\|} \leq \|e\|$. Multiplying together we obtain

$$\frac{1}{\|A\|\cdot\|A^{-1}\|}\cdot\frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|}$$

Next, (1) as $\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$, and multiply by (4) we have

$$\frac{\|e\|}{\|x\|} \leq \|A\|\|A^{-1}\|\cdot\frac{\|r\|}{\|b\|}$$

We are done.

For showing Cauchy-Schwarz. Let $x, y \in V$ & $\lambda \in \mathbb{R}$

$$0 \leq \langle x+\lambda y, x+\lambda y\rangle$$
$$= \langle x,x\rangle + 2\lambda\langle x,y\rangle + \lambda^2\langle y,y\rangle$$

non-negative polynomial in terms of λ

Quad-formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

A a non-negative polynomial must have

$$b^2 - 4ac \leq 0$$
$$(2\langle x,y\rangle)^2 - 4\cdot\langle y,y\rangle\cdot\langle x,x\rangle \leq 0$$
$$(\langle x,y\rangle)^2 \leq \langle y,y\rangle\cdot\langle x,x\rangle$$
$$|\langle x,y\rangle| \leq \|y\|\cdot\|x\|.$$

a). We let $\|\cdot\| := \|\cdot\|_2$. Find A, b, x, & r s.t

$$\frac{\|e\|}{\|x\|} = \|A\| \|A^{-1}\| \cdot \frac{\|r\|}{\|b\|}$$

$$\|e\| \|b\| = \|A\| \cdot \|A^{-1}\| \|r\| \|x\|$$

$$\|x - \hat{x}\| \cdot \|b\| =$$

-5- (Backward Error Analysis.) This problem explores the effects of a perturbation in the coefficient matrix (rather than the right hand side) of the linear system

$$Ax = b \tag{4}$$

Suppose we solve instead of (4) the system

$$(A - E)(x - e) = b \tag{5}$$

where E is a perturbation of A that causes an error e in the solution x. Show that

$$\frac{\|e\|}{\|x - e\|} \leq \|A\|\|A^{-1}\|\frac{\|E\|}{\|A\|} \tag{6}$$

We assume $\|A\| \neq 0$ & $\|x-e\| \neq 0$

Take $(A-E)(x-e) = b$

$A(x-e) - E(x-e) = b$

$Ax - Ae - E(x-e) = b$

$-Ae - E(x-e) = b - Ax = 0$

$Ae = -E(x-e)$

$e = -A^{-1}E(x-e)$

$\|e\| \leq \|A^{-1}\| \|E\| \|x-e\|$

$$\frac{\|e\|}{\|x-e\|} \leq \|A^{-1}\| \|E\| = \|A\|\|A^{-1}\| \frac{\|E\|}{\|A\|} \quad \square$$

-6- (First Order Systems.) Write the second order initial value problem

$$y'' = xy^2, \qquad y(0) = 1, \qquad y'(0) = 2 \tag{7}$$

as an autonomous first order system

$$y' = f(y), \qquad y(a) = y_0. \tag{8}$$

(In other words, specify y, f, a, and y_0 such that the two problems are equivalent. Of course, y will have different meanings for the two problems.)

Let $u = y'$ & $x = t$, then we obtain the following autonomous first order system

$$\begin{cases} y' = u & y(0) = 1 \\ u' = x u^2 & u(0) = 2 \\ x' = 1 & x(0) = 0 \end{cases}$$

7 & 8

Need to program.