**-6- (More on Fourier Series.)** Calculate the Fourier series of

$$f(x) = \cos(x+1).$$

Hint: Before you embark on the computation of a bunch of integrals think about what you would expect the Fourier series to be. Perhaps you can find it without doing any integrals!

Recall the sum to product formula from Trigonometry:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{1}$$

Expanding $f(x) = \cos(x+1)$ with the above identity gives:

$$f(x) = \cos(x)\cos(1) - \sin(x)\sin(1) \tag{2}$$

Since $\cos(1)$ and $\sin(1)$ are constants and using the fact that the Fourier series on the interval $[-\pi, \pi]$ for $\sin(x)$ and $\cos(x)$ are just themselves we are done.

**-7- (Spline versus Cubic Hermite Interpolation.)** Let the function $s(x)$ be defined by

$$s(x) = \begin{cases} (\gamma-1)(x^3 - x^2) + x + 1 & \text{if } x \in [0,1] \\ \gamma x^3 - 5\gamma x^2 + 8\gamma x - 4\gamma + 2 & \text{if } x \in [1,2] \end{cases}$$

**a.** Show that $s$ is the piecewise cubic Hermite interpolant to the data:

$$s(0) = 1, \quad s(1) = s(2) = 2, \quad s'(0) = 1, \quad s'(1) = \gamma, \quad s'(2) = 0$$

To prove this we'll show that plugging in the values above into $s(x)$ gives back the required data. Then by the uniqueness of the interpolating polynomial on an interval we will have our Hermite Interpolant.

$$s(0) = 0 + 1 = 1 \tag{3}$$
$$s(1) = (\gamma - 1)(1-1) + 1 + 1 = 2 \tag{4}$$
$$s(2) = 8\gamma - 20\gamma + 16\gamma - 4\gamma + 2 = 2 = s(1) \tag{5}$$

The derivatives of the two cubics on their respective intervals are:

$$s(x) = \begin{cases} (\gamma-1)(3x^2 - 2x) + 1 & \text{if } x \in [0,1] \\ 3\gamma x^2 - 10\gamma x + 8\gamma & \text{if } x \in [1,2] \end{cases}$$

Plugging in the endpoints gives:

$$s'(0) = 1 \tag{6}$$
$$s'(1) = (\gamma - 1) + 1 = \gamma = 3\gamma - 10\gamma + 8\gamma = \gamma \tag{7}$$

$$s'(2) = 3\gamma(4) - 10\gamma(2) + 8\gamma = 0 \tag{8}$$

Note that from plugging 1 into our definition of the derivatives of $s(x)$ shows that the function is continuous at the point 1 justifying it being a cubic interpolant.

**b.** For what value of $\gamma$ does $s$ become a cubic spline? To become a cubic spline the second derivative of $s(x)$ needs to exist everywhere on the interval $[0, 2]$. The second derivatives of the piecewise cubics above are:

$$s''(x) = \begin{cases} (\gamma-1)(6x - 2) & \text{if } x \in [0,1] \\ 6\gamma x - 10\gamma + & \text{if } x \in [1,2] \end{cases}$$

Plug in $x = 1$ into both piece-wise second derivatives and set them equal to obtain:

$$4(\gamma - 1) = -4\gamma \tag{9}$$

Which upon solving for $\gamma$ yields $\gamma = \frac{1}{2}$. Thus $\gamma$ must be $\frac{1}{2}$ in order for $s(x)$ to become a cubic spline.

## Problem 8

In order to show that any polynomial in power form can be uniquely written in B-form, we can simply show that the Bezier polynomials form a basis for the degree $d$ space. Let

$$B_i^d = \binom{d}{i} b_1^1 b_2^{d-i} \quad \text{for} \quad i = 0, 1, \cdots, d$$

It suffices to show the $B_i^d$ are linearly independent polynomials with respect to $b_1$, where $b_2 = 1 - b_1$. This is sufficient since we have the correct number of polynomials to form a basis for this space. Using the Binomial expansion theorem and the fact that $b_2 = 1 - b_1$ we have,

$$\begin{aligned} B_i^d &= \binom{d}{i} b_1^i \cdot (1 - b_1)^{d-i} \\ &= \binom{d}{i} b_1^i \sum_{k=0}^{d-i} (-1)^k \binom{d-i}{k} b_1^k \end{aligned}$$

Applying a change of index, and some algebraic manipulation gives us,

$$\begin{aligned} &= \sum_{k=i}^{d} (-1)^{k-i} \binom{d}{i} \binom{d-i}{k-i} b_1^{k+i-i} \\ &= \sum_{k=i}^{d} (-1)^{k-i} \binom{d}{k} \binom{k}{i} b_1^k \end{aligned}$$

The last equality comes from the fact that $\binom{d}{i}\binom{d-i}{k-i} = \binom{d}{k}\binom{k}{i}$ which is verified at the end. So we can write our Bezier polynomials in the form derived above.

$$B_i^d = \sum_{k=i}^{d} (-1)^{k-i} \binom{d}{k} \binom{k}{i} b_1^k$$

Now for showing linear independence if we have,

$$\sum_{i=0}^{d} \alpha_i B_i^d = 0$$

for some $\alpha_i$ coefficients, we show that all $\alpha_i$ are zero. Expanding this sum out we have:

$$\alpha_0 \sum_{k=0}^{d} \binom{d}{k} \binom{k}{0} (-1^k) b_1^k + \alpha_1 \sum_{k=1}^{d} \binom{d}{k} \binom{k}{1} (-1)^{k-1} b_1^k + \cdots + \alpha_d \sum_{k=d}^{d} \binom{d}{k} \binom{k}{d} (-1)^{k-d} b_1^k = 0 \tag{10}$$

We can see the only constant term, with respect to $b_1$ as our variable, is $\alpha_0$. Meaning that $\alpha_0 = 0$ as there are no other constant terms to cancel out with. So we can simplify our equation (10) to

$$\alpha_1 \sum_{k=1}^{d} \binom{d}{k} \binom{k}{1} (-1)^{k-1} b_1^k + \cdots + \alpha_d \sum_{k=d}^{d} \binom{d}{k} \binom{k}{d} (-1)^{k-d} b_1^k = 0$$

Again we note there is now only one $b_1$ term. This term has a coefficient of $\alpha_1$. Meaning that $\alpha_1 = 0$. Continuing this process inductively we see that

$$\alpha_0 = \alpha_1 = \cdots = \alpha_d = 0$$

So $\{B_i^d\}_{i=0}^d$ forms a basis for our degree $d$ polynomial space. Thus we can write any polynomial in power form uniquely into B-form.

Lastly verifying $\binom{d}{i}\binom{d-i}{k-i} = \binom{d}{k}\binom{k}{i}$.

$$\begin{aligned}
\binom{d}{i}\binom{d-i}{k-i} &= \frac{d!}{i!(d-i)!}\frac{(d-i)!}{(k-i)!(d-k)!} \\
&= \frac{d!}{i!(k-i)!(d-k)!} \\
&= \frac{d!k!}{i!(k-i)!(d-k)!k!} \\
&= \frac{d!}{k!(d-k)!}\frac{k!}{i!(k-i)!} \\
&= \binom{d}{k}\binom{k}{i}
\end{aligned}$$

## Problem 9

We note that are interpolating at $2n+1$ distinct nodes thus we make the primitive assertion our polynomial $p$ is at most degree $2n$. We define the following function

$$g(x) = p(x) + p(-x)$$

where $g$ is a polynomial of degree at most $2n$. We note that $g(x)$ has $2n+1$ distinct roots, namely $x_i$ for $i = -n, -n+1, \cdots, n-1, n$. However the only polynomial with a larger number of roots then the degree is infact the zero polynomial, or the zero function. So we have,

$$g(x) = p(x) + p(-x) = 0 \quad \text{for all } \ x \in \mathbb{R}$$

Thus we have,

$$p(x) = -p(-x)$$

for all real numbers $x$. This of course means that our polynomial is an odd function, allowing us to mend our primitive answer before—$p$ can have a degree at most $2n-1$.