CS495
Optimiztaion

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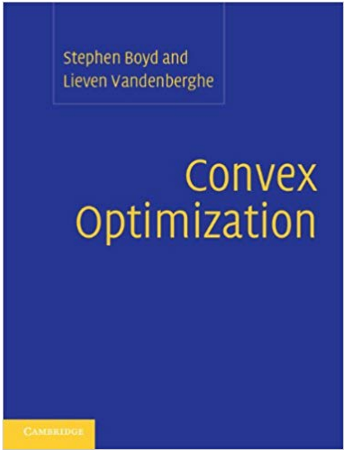
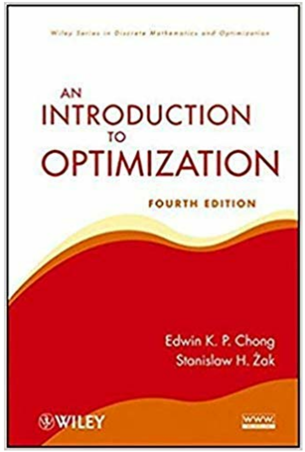
Lectures follow:
Boyd and Vandenberghe (2004)



Boyd, S., & Vandenberghe, L. (2004). Convex Optimization. Cambridge: Cambridge University Press.

Book and Stanford course: http://web.stanford.edu/~boyd/cvxbook/

Some examples from:
Chong and Zak (2001)



Chong, E. K., & Zak, S. (2001). An introduction to optimization: Wiley-Interscience.

Course Objectives

- Developing rigorous mathematical treatment for mathematical optimization.
- Building intuition, in particular to practical problems.
- Developing computer practice to using optimization SW.

Prerequisites

Calculus (both single and multivariable) and Linear Algebra.

Chapter 1:
Introduction
Snapshot on Optimization

Contents

Chapter 1

Introduction

1.1 Mathematical Optimization

Definition 1 *A mathematical optimization problem or just optimization problem, has the form (Boyd and Vandenberghe, 2004):*

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to:} & f_i(x) \le 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p, \end{array}$$

$$x = (x_1, \dots, x_n) \in \mathbf{R}^n, \qquad (\textit{optimization variable})$$

$$f_0 : \mathbf{R}^n \mapsto \mathbf{R}, \qquad (\textit{objective (cost/utility) function})$$

$$f_i : \mathbf{R}^n \mapsto \mathbf{R}, \qquad (\textit{inequality constraints (functions)})$$

$$h_i : \mathbf{R}^n \mapsto \mathbf{R}, \qquad (\textit{equality constraints (functions)})$$

$$\mathcal{D} : \bigcap_{i=1}^{m} \mathbf{dom}\, f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\, h_i \qquad (\textit{feasible set})$$

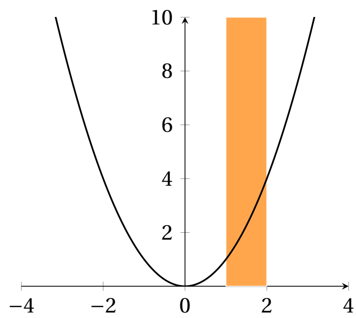
$$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \le 0 \wedge h_i(x) = 0\}$$

$$x^* : \{x \mid x \in \mathcal{D} \wedge f_0(x) \le f_0(z)\ \forall z \in \mathcal{D}\} \qquad (\textit{solution})$$

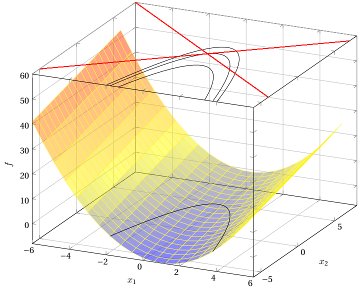
- minimize $f_0 \equiv$ maximize $-f_0$.
- $f_i \le 0 \equiv -f_i \ge 0$.
- 0s can be replaced of course by constants b_i, c_i
- unconstrained problem when $m = p = 0$.

Example 2 *:*

$$\begin{array}{ll} \underset{x}{\text{minimize}} & x^2 \\ \text{subject to:} & x \le 2 \wedge x \ge 1. \end{array}$$



$x^* = 1$.
If the constraints are relaxed, then $x^* = 0$.



$$
\begin{array}{lll}
\underset{x}{\text{minimize}} & f_0(x) & \\
\text{subject to:} & f_i(x) \le 0, & i = 1, \ldots, m \\
 & h_i(x) = 0, & i = 1, \ldots, p,
\end{array}
$$

$x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, (*optimization variable*)

$f_0 : \mathbf{R}^n \mapsto \mathbf{R}$, (*objective (cost/utility) function*)

$f_i : \mathbf{R}^n \mapsto \mathbf{R}$, (*inequality constraints (functions)*)

$h_i : \mathbf{R}^n \mapsto \mathbf{R}$, (*equality constraints (functions)*)

$\mathcal{D} : \bigcap_{i=1}^{m} \mathbf{dom}\, f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\, h_i$ (*feasible set*)

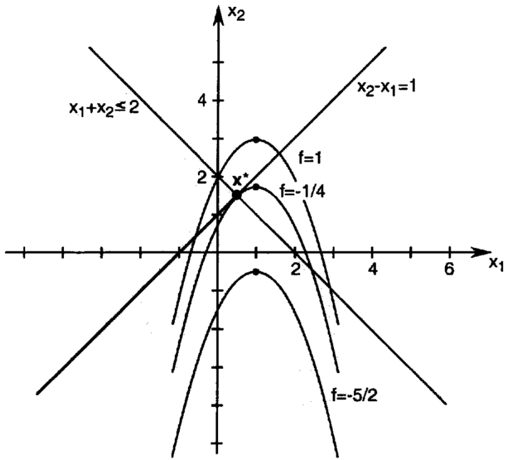
$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \le 0 \wedge h_i(x) = 0\}$

$x^* : \{x \mid x \in \mathcal{D} \wedge f_0(x) \le f_0(z)\ \forall z \in \mathcal{D}\}$ (*solution*)

Example 3 (*Chong and Zak, 2001, Ex. 20.1, P. 454*):

$$
\begin{array}{ll}
\underset{x}{\text{minimize}} & (x_1 - 1)^2 + x_2 - 2 \\
\text{subject to:} & x_2 - x_1 = 1 \\
 & x_1 + x_2 \le 2.
\end{array}
$$

No global minimizer: $\partial z / \partial x_2 = 1 \neq 0$. *However,* $z|_{(x_2 - x_1 = 1)} = (x_1 - 1)^2 + (x_1 - 1)$, *which attains a minima at* $x_1 = 1/2$.



$x* = (1/2, 3/2)'$. (Let's see animation)

1.1.1 Motivation and Applications

- *optimization problem* is an abstraction of how to make "best" possible choice of $x \in \mathbf{R}^n$.
- *constrains* represent trim requirements or specifications that limit the possible choices.
- *objective function* represents the *cost* to minimize or the *utility* to maximize for each x.

Examples:

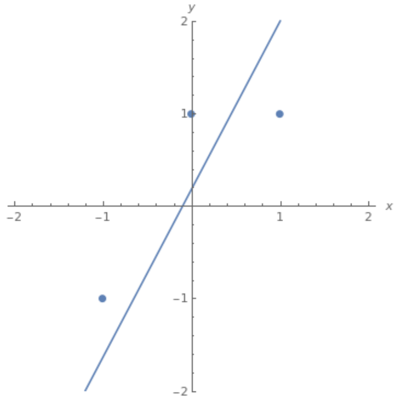
	Any problem	*Portfolio Optimization*	*Device Sizing*	*Data Science*
$x \in \mathbf{R}^n$	choice made	investment in capitals	dimensions	parameters
$f_i,\ h_i$	firm requirements /conditions	overall budget	engineering constraints	regularizer
f_0	cost (or utility)	overall risk	power consumption	error

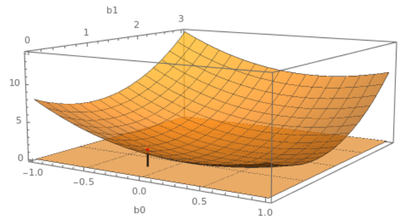
- Amazing variety of practical problems. In particular, data science: two sub-fields: construction and assessment.
- The construction of: Least Mean Square (LMS), Logistic Regression (LR), Support Vector Machines (SVM), Neural Networks(NN), Deep Neural Networks (DNN), etc.
- Many techniques are for solving the optimization problem:
  - Closed form solutions: convex optimization problems
  - Numerical solutions: Newton's methods, Gradient methods, Gradient descent, etc.
  - "Intelligent" methods: particle swarm optimization, genetic algorithms, etc.

Example 4 (Machine Learning: construction) *:*
Let's suppose that the best regression function is $Y = \beta_0 + \beta_1 X$, then for the training dataset (x_i, y_i) we need to minimize the MSE.

- Half of ML field is construction: NN, SVM, etc.
- In DNN it is an optimization problem of millions of parameters.
- Let's see animation.
- Where are Probability, Statistics, and Linear Algebra here? Let's re-visit the chart.
- Is the optimization problem solvable:
  - closed form? (LSM)
  - numerically and guaranteed? (convex and linear)
  - numerically but not guaranteed? (non-convex):
    - numerical algorithms, e.g., GD,
    - local optimization,
    - heuristics, swarm, and genetics,
    - brute-force with exhaustive search

$$\underset{\beta_o,\beta_1}{\text{minimize}} \sum_i (\beta_o + \beta_1 x_i - y_i)^2$$





1.1.2 Solving Optimization Problems

- A *solution method* for a class of optimization problems is an algorithm that computes a solution.
- Even when the *objective function* and constraints are smooth, e.g., polynomials, the solution is very difficult.
- There are three classes where solutions exist, theory is very well developed, and amazingly found in many practical problems:

Linear ⊂ Quadratic ⊂ Convex ⊂ Non-linear (not linear and not known to be convex!)

- For the first three classes, the problem can be solved very reliably in hundreds or thousands of variables!

1.2 Least-Squares and Linear Programming

1.2.1 Least-Squares Problems

A *least-squares* problem is an optimization problem with no constraints (i.e., $m = p = 0$), and an objective in the form:

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^{k} (a_i' x - b_i)^2 = \| A_{k\times n} x_{n\times 1} - b_{k\times 1} \|^2.$$

The solution is given in **closed form** by:

$$x = (A'A)^{-1} A'b$$

- Good algorithms in many SC SW exist; it is a very mature technology.
- Solution time is $O(n^2 k)$.
- Easily solvable even for hundreds or thousands of variables.
- More on that in the Linear Algebra course.
- Many other problems reduce to typical LS problem:
  - Weighted LS (to emphasize some observations)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^{k} w_i (a_i' x - b_i)^2.$$

  - Regularization (to penalize for over-fitting)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^{k} (a_i' x - b_i)^2 + \rho \sum_{j=1}^{n} x_j^2.$$

1.2.2 Linear Programming

A *linear programming* problem is an optimization problem with objective and all constraint functions are linear:

$$
\begin{aligned}
&\underset{x}{\text{minimize}} && f_0(x) = C'x \\
&\text{subject to:} && a_i'x \le b_i, && i = 1, \ldots, m \\
& && h_i'x = g_i, && i = 1, \ldots, p,
\end{aligned}
$$

- **No** closed form solution as opposed to LS.
- Very robust, reliable, and effective set of methods for numerical solution; e.g., Dantzig's simplex, and interior point.
- Complexity is $\simeq O(n^2 m)$.
- Similar to LS, we can solve a problem of thousands of variables.
- Example is *Chebyshev minimization* problem:

$$\underset{x}{\text{minimize}} f_0(x) = \max_{i=1,\ldots,k} |a_i'x - b_i|,$$

- The objective is different from the LS: minimize the maximum error. **Ex:**
- After some tricks, requiring familiarity with optimization, it is equivalent to a LP:

$$
\begin{aligned}
&\underset{x}{\text{minimize}} && t \\
&\text{subject to:} && a_i'x - t \le b_i, && i = 1, \ldots, k \\
& && -a_i'x - t \le -b_i, && i = 1, \ldots, k
\end{aligned}
$$

1.3 Convex Optimization

A *convex optimization* problem is an optimization problem with objective and all constraint function are convex:

$$
\begin{array}{llll}
\underset{x}{\text{minimize}} & f_0(x) & & \\
\text{subject to:} & f_i(x) \le 0, & & i = 1, \ldots, m \\
& h_i(x) = 0, & & i = 1, \ldots, p, \\
& f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y), \quad \alpha + \beta = 1, & 0 \le \alpha,\ 0 \le \beta, & 0 \le i \le m \\
& h_i(x) = a_i' x + b_i & 0 \le i \le p &
\end{array}
$$

- The LP and LS are special cases; however, only LS has closed-form solution.
- Very robust, reliable, and effective set of methods, including *interior point methods.*
- Complexity is almost: $O\left(\max(n^3, n^2 m, F)\right)$, where F is the cost of evaluating 1st and 2nd derivatives of f_i and h_i.
- Similar to LS and LP, we can solve a problem of thousands of variables.
- However, it is not as very mature technology as the LP and LS yet.
- There are many practical problems that can be re-formulated as convex problem **BUT** requires mathematical skills; but once done the problem is solved. **Hint:** realizing that the problem is convex requires more mathematical maturity than those required for LP and LS.

1.4 Nonlinear Optimization

A *non-linear optimization* problem is an optimization problem with objective and constraint functions are non-linear **BUT** not known to be convex (**so far**). Even simple-looking problems in 10 variables can be extremely challenging. Several approaches for solutions:

Local Optimization : starting at initial point in space, using differentiablity, then navigate

- does not guarantee global optimal.
- affected heavily by initial point.
- depends heavily on numerical algorithm and their parameters.
- More art than technology.
- In contrast to convex optimization, where a lot of art and mathematical skills are required to formulate the problem as convex; then numerical solution is straightforward.

Global Optimization : the true global solution is found; the compromise is complexity.

- The complexity goes exponential with dimensions.
- Sometimes it is worth it when: the cost is huge, not in real time, and dimensionality is low.

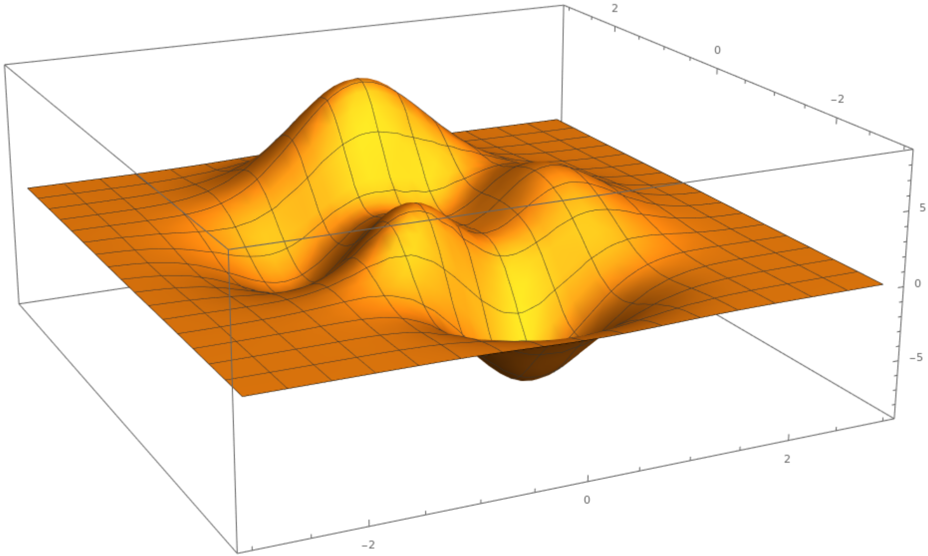
Role of Convex Optimization :

- Approximate the non-linear function to a convex one, finding the exact solution, then using it as a starting point for the original problem. (Also does not guarantee optimality)
- Setting bounds on the global solution.

Evolutionary Computations : Genetic Algorithm (GA), Simulated Annealing (SA), Particle Swarm Optimization (PSO), etc.

Example 5 (Nonlinear Objective Function) : *(Chong and Zak, 2001, Ex. 14.3)*

$$f(x,y) = 3(1-x)^2 e^{-x^2-(y+1)^2} - 10e^{-x^2-y^2}\left(-x^3+\frac{x}{5}-y^5\right) - \frac{1}{3}e^{-(x+1)^2-y^2}$$



Part I

Theory

Chapter 2

Convex sets

2.1 Affine and convex sets

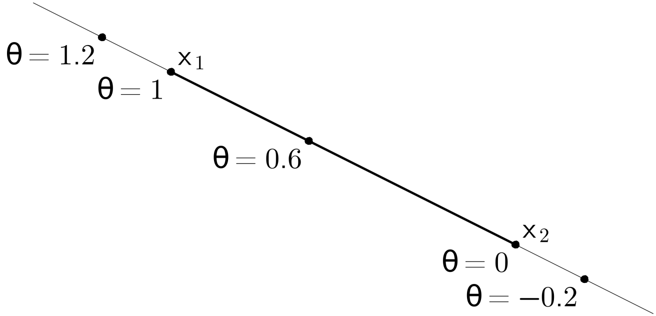
2.1.1 Lines and line segments

Definition 6 (line and line segment) *Suppose* $x_1 \neq x_2 \in \mathbf{R}^n$*. Points of the form*

$$y = \theta x_1 + (1-\theta)x_2$$
$$= x_2 + \theta(x_1 - x_2),$$

where $\theta \in \mathbf{R}$*, form the line passing through* x_1 *and* x_2*.*

- As usual, this is a definition for high dimensions taken from a proof for $n \leq 3$.
- We have done it many times: angle, norm, cardinality of sets, etc.
- if $0 \leq \theta \leq 1$ this forms a line segment.



2.1.2 Affine sets

Definition 7 (Affine sets) *A set $C \subset \mathbf{R}^n$ is affine if the line through any two distinct points in C lies in C. I.e., $\forall x_1, x_2 \in C$ and $\theta \in \mathbf{R}$, we have $\theta x_1 + (1-\theta)x_2 \in \mathbf{R}^n$. In other words, C contains any linear combination of any two points in C provided the coefficients sum to one.*

Examples: what about line, line segment, circle, disk, strip?

Corollary 8 *Suppose C is an affine set, and $x_1, \ldots, x_k \in C$, then C contains every general affine combination of the form $\theta_1 x_1 + \ldots + \theta_k x_k$, where $\theta_1 + \ldots + \theta_k = 1$.*

Proof. trivial by induction or by summation (same) ■

Definition 9 (Subspace from Linear Algebra) *a set $V \subset \mathbf{R}^n$ of vector (here points) is a subspace if it is closed under sums and scalar multiplication. I.e., $\forall v_1, v_2 \in V$ and $\forall \alpha, \beta \in \mathbf{R}$ we have $\alpha v_1 + \beta v_2 \in V$. Hint: Vimp this implies that $\mathbf{0} \in V$.*

Corollary 10 *If C is affine set and $x_0 \in C$, then the set $V = C - x_0 = \{x - x_0 | x \in C\}$ is a subspace; and then the dimension (rank) of C is defined to be the same as the dimension (rank) of V.*

Proof. Suppose $v_1, v_2 \in V$; then $v_1 + x_0, v_2 + x_0 \in C$.

$$
\begin{aligned}
c &= \alpha v_1 + \beta v_2 + x_0 \\
&= \alpha(v_1 + x_0) + \beta(v_2 + x_0) + (1 - \alpha - \beta)x_0 \in C
\end{aligned}
$$

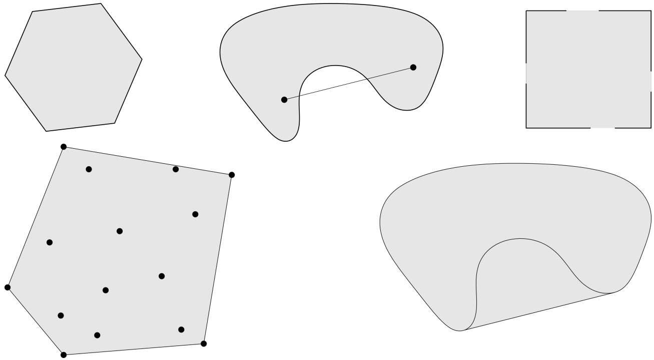
Then $c - x_0 = \alpha v_1 + \beta v_2 \in V$; and hence V is a subspace. ■

- This is true for any x_0.
- *affine* is a *subspace* plus offset.
- every subspace is affine but not the vice versa.
- subspace is a special case of affine.

Example 11

2.1.3 Affine dimension and relative interior

2.1.4 Convex sets



2.1.5 Cones

Part II

Applications

Part III

Algorithms

Bibliography

Boyd, S. and Vandenberghe, L. (2004), *Convex Optimization*, Cambridge: Cambridge University Press.

Chong, E. K. and Zak, Stanislaw, H. (2001), *An Introduction to Optimization*, Wiley-Interscience, 4th ed.