MA214:

Mathematics and Contemplations On Linear Algebra and Its Applications

towards building a "Data Scientist"

Waleed A. Yousef, Ph.D.,

Human Computer Interaction Lab.,
Computer Science Department,
Faculty of Computers and Information,
Helwan University,
Egypt.

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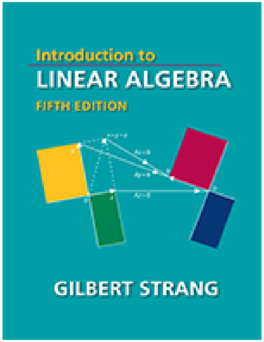
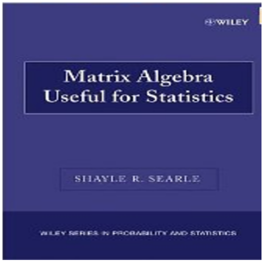
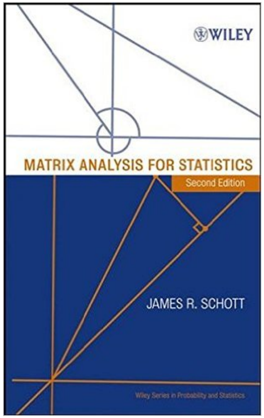
Lectures Notes (http://www.helwan.edu.eg/university/staff/Dr.WaleedYousef/HTML/Home.html) follows:



Strang, G., 2016. Introduction to linear algebra, 5th Edition. Wellesley-Cambridge Press



Searle, S. R., 1982. Matrix algebra useful for statistics. Wiley, New York



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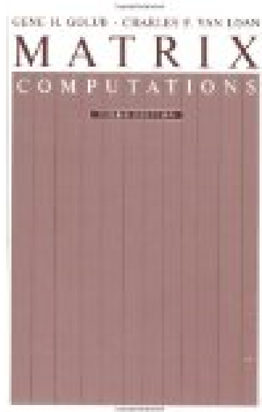

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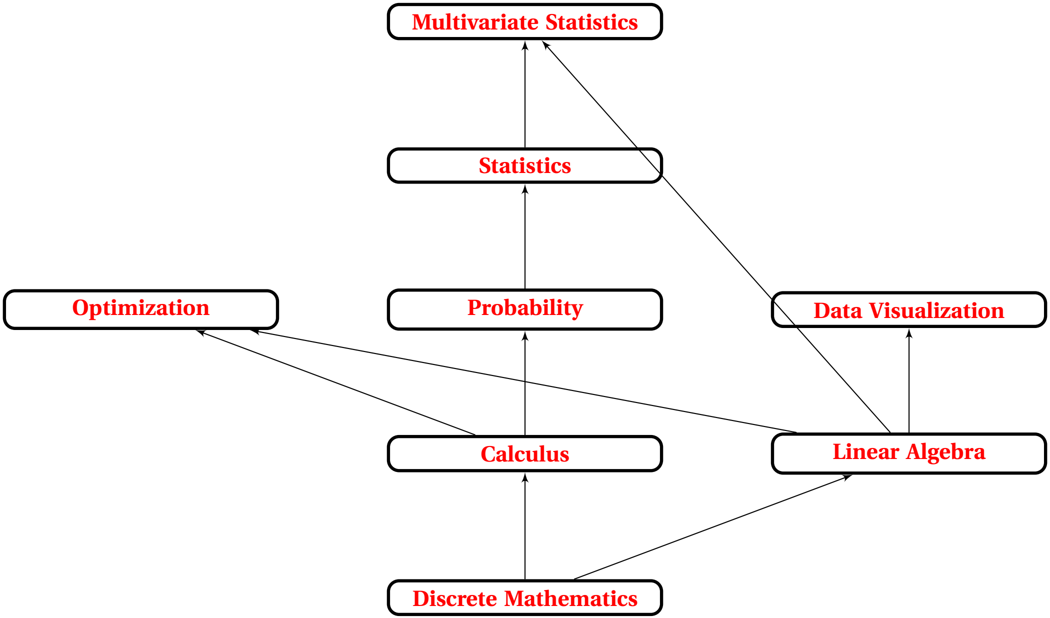
Linear Algebra: FCIHOCW vs. MITOCW

- Arabic vs. English.
- More rigorous treatment.
- Teaching, while “Data Science” in mind.

Course Objectives

- Developing rigorous treatment.
- Developing mathematical foundations to many courses and areas, in particular “Data Science”
- Building intuition.
- Linking to CS applications (e.g., Pattern Recognition, Image Processing, etc.)

Linear Algebra, Prerequisites, and Applications



- Some prerequisites are not so strict; others are possible, e.g., GPU, Algorithms, etc.
- It differs from researchers to practitioners; See pattern recognition course and big picture talk.

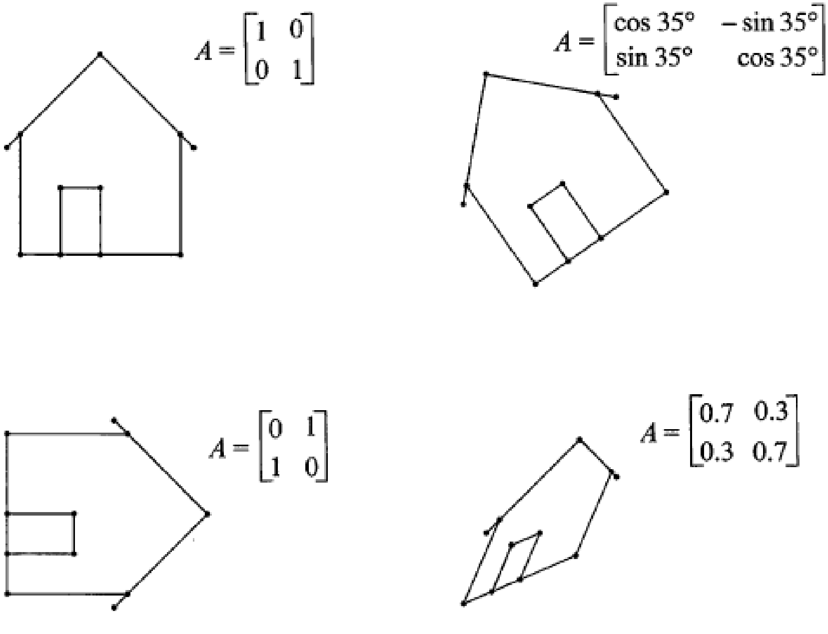


Figure 7.2: Linear transformations of a house drawn by **plot2d**$(A * H)$.

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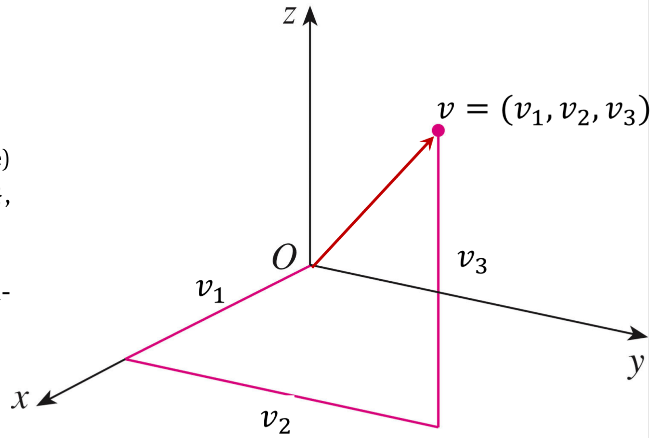
Chapter 1

Introduction

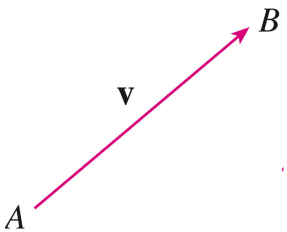
1.0 Back to School: visual space!

- We locate a point in a 3D space by three numbers.
- The coordinates are perpendicular.
- The order of the axes X , Y , Z: "right-hand" rule.
- The 3-tuple (3 ordered elements, or triple) $(v_1, v_2, v_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$, the set of all points.
- The following are equivalent (some books differentiate; we do not):
    - the 3-tuple $v = (v_1, v_2, v_3)$.
    - the point $v = (v_1, v_2, v_3)$.
    - the arrow connecting O to v, i.e., the vector $v = \overrightarrow{Ov} = (v_1, v_2, v_3)$.
- The line segment $\overline{Ov}$ consists of **all** points, not only v.

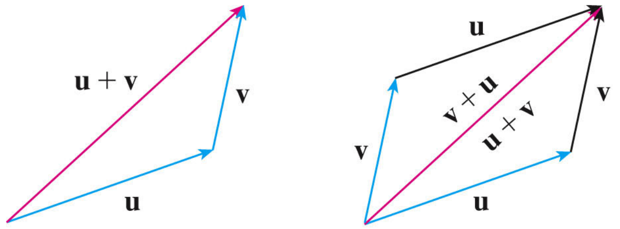


Definition 1 (Geometric Manipulation) .

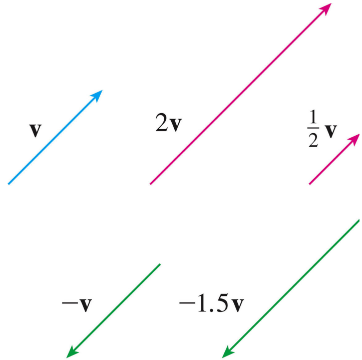
- *A vector is used to indicate a* ***displacement*** *in some* ***direction****;* ***starting point is not important***
- *Start at any point A, move a distance in the direction of $\overrightarrow{Ov}$, and end at B. Then, $\overrightarrow{AB} = \overrightarrow{Ov} = v$. ($B \neq \overrightarrow{AB}$; but $v = \overrightarrow{Ov}$)*



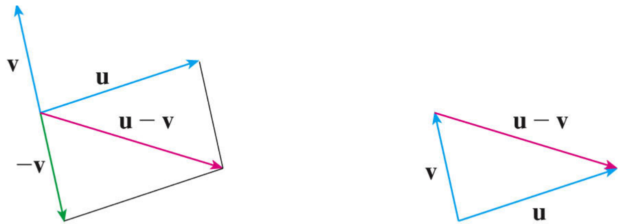
- *Addition: $u + v$*



- *Scalar Multiplication: If c is a scalar, then $u = cv$ is a vector whose length is $|c|\times$ length of v and direction:*

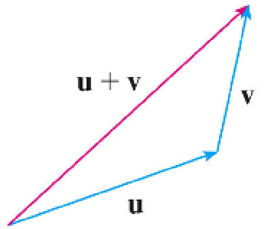


- *Scalar and Addition:*

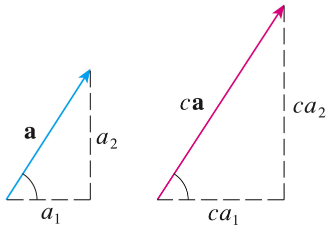


Definition 2 (Algebraic Treatment) . *Addition and Scalar: if* $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$*:*

$$\begin{aligned} a + b &= (a_1 + b_1, a_2 + b_2, a_3 + b_3), \\ ca &= (ca_1, ca_2, ca_3). \end{aligned}$$

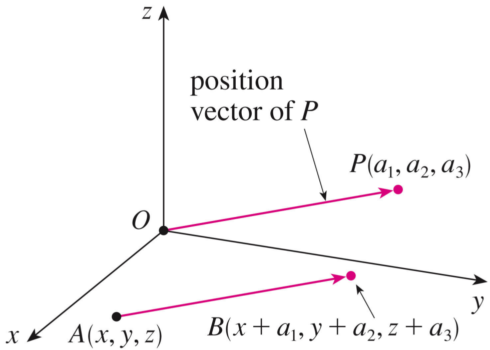


Proof of equivalence. Trivial. ■



Hint: The displacement is added algebraically: given $P = (a_1, a_2, a_3)$, and any $A = (x, y, z)$. Then:

$$\begin{aligned} P &= \overrightarrow{OP} = \overrightarrow{AB}, \\ B &= A + P = (x + a_1, y + a_2, z + a_3). \end{aligned}$$



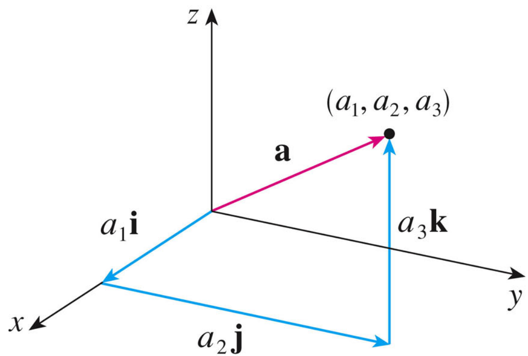
Lemma 3 (Properties of Vectors) . *For any two vectors a and b,*

$$
\begin{aligned}
a+b &= b+a\\
a+(b+c) &= (a+b)+c\\
a+\mathbf{0} &= a\\
a+(-a) &= \mathbf{0}\\
c(a+b) &= ca+cb\\
(c+d)a &= ca+da
\end{aligned}
$$

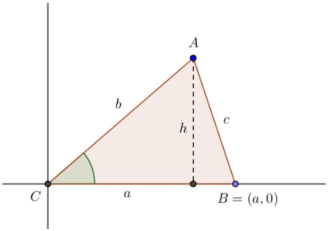
Proof. It is quite straight forward to prove **(HW)** ∎

Example 4 *Consider the vector a,*

$$
\begin{aligned}
a &= (a_1, a_2, a_3)\\
&= a_1 i + a_2 j + a_3 k,\\
i &= (1,0,0),\\
j &= (0,1,0),\\
k &= (0,0,1).
\end{aligned}
$$



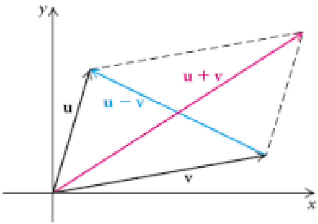
1.1 Angle, Lengths, and Dot Products (visual space and school again)



- Notation: the vector u, with 3-tuple (u_1, u_2, u_3) is written as: $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ or $u' = (u_1, u_2, u_3)$.

- it is a school business to prove that (whether 2D or 3D):

$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos\theta \\
\|u - v\|^2 &= \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta \\
2\|u\|\|v\|\cos\theta &= (u_1^2 + u_2^2) + (v_1^2 + v_2^2) - (u_1 - v_1)^2 - (u_2 - v_2)^2 = 2u_1v_1 + 2u_2v_2 \\
\cos\theta &= \frac{u_1v_1 + u_2v_2}{\|u\|\|v\|}.
\end{aligned}$$



- This is why we defined the dot product to be: $u'v = u_1v_1 + u_2v_2 = \|u\|\|v\|\cos\theta$.

- When $u'v$ is zero we say they are orthogonal.

- If $u = v$, then $\theta = 0$, $u'u = u_1u_1 + u_2u_2 = \|u\|^2$.

- u is **unit vector** if $\|u\| = 1$. Then $\forall u$, $u/\|u\|$ is a unit vector.

$$\begin{aligned}
u'v = &\ \|v\|\|u\|\cos\theta = \|v\| \times \texttt{Projection Length of } u \texttt{ on } v \\
u'(v/\|v\|) = &\ \texttt{Projection Length of } u \texttt{ on } v \\
v'(u/\|u\|) = &\ \texttt{Projection Length of } v \texttt{ on } u.
\end{aligned}$$

Lemma 5 (Properties) .

- ***Basic properties:***

$$
\begin{aligned}
u'v &= v'u \\
\|au\| &= |a|\|u\| \\
a(u'v) &= (au)'v = au'v \\
(au+bv)'w &= au'w + bv'w \\
(u+v)'(u+v) &= u'u + 2u'v + v'v.
\end{aligned}
$$

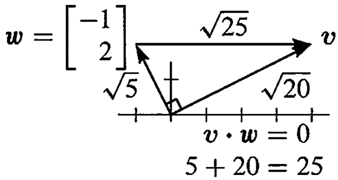
- ***Cauchy-Shwartz inequality:*** $-\|u\|\|v\| \le u'v \le \|u\|\|v\|$
  Proof. *immediate from both:* $-1 \le \cos\theta \le 1 \quad \textit{and} \quad u'v = \|u\|\|v\|\cos\theta.$ ■

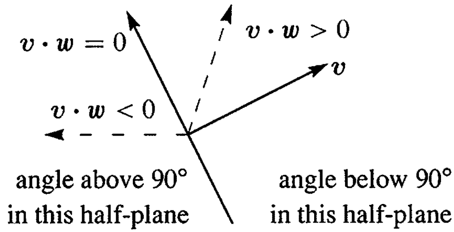
- ***Traingular inequality:*** $\|u+v\| \le \|u\| + \|v\|$
  Proof. $\|u+v\|^2 = (u+v)'(u+v) = u'u + 2u'v + v'v \le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2.$ ■

- *Then, we can generalize this definition in higher dimensions, and define the angle between two vectors for* $p > 3$.

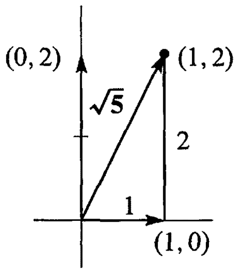
Example 6 $w = (-1,2)'$, $v = (4,2)'$, *then*

$$\cos\theta = \frac{w'v}{\|w\|\|v\|} = \frac{(-1)(4)+(2)(2)}{\sqrt{(-1)^2+(2)^2}\sqrt{(4)^2+(2)^2}} = \frac{0}{\sqrt{5}\sqrt{20}} = 0$$

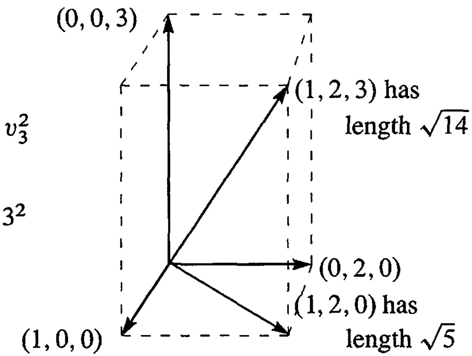




Example 7 (3D) .

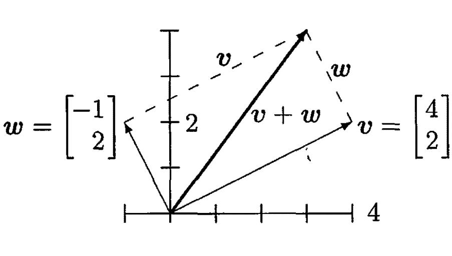


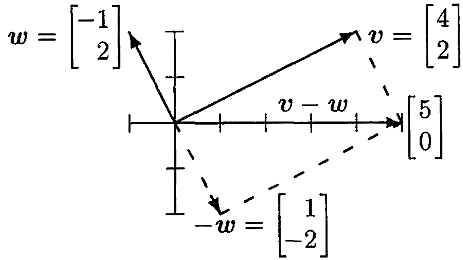
$$\begin{aligned} v \cdot v &= v_1^2 + v_2^2 + v_3^2 \\ 5 &= 1^2 + 2^2 \\ 14 &= 1^2 + 2^2 + 3^2 \end{aligned}$$



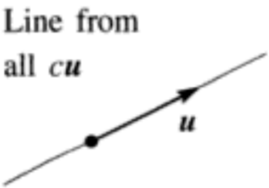
Hint: *To save space, we write, e.g.,* $v = (4,2)'$. *Sometimes, we drop the prime if there is no confusion.*

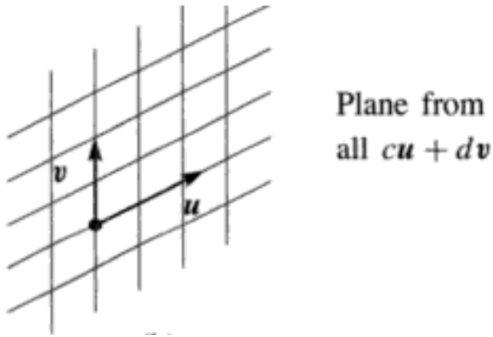
Example 8 (Linear Combination) .





$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$





- We can generalize to p-dimensions, although we cannot visualize.
- What is the picture for **ALL** linear combinations? “spanning” the space, independence ...

1.2 Extension and Abstraction: Vectors and Linear Combinations

Extension in both: meaning and number of components to treat applications.

Definition 9 (Vector) *The ordered p-tuple $(v_1, v_2, \cdots, v_p)$, $v_i \in \mathcal{R}$, is called a p-dimensional vector.*

Definition 10 (dot product (inner product), length, angle)

$$\begin{aligned}
\langle u, v\rangle = u.v = u'v &= \begin{pmatrix} u_1, & \cdots, & u_p \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix} \\
&= u_1 v_1 + \cdots + u_p v_p = \sum_{i=1}^{p} u_i v_i \\
\|u\| &= \sqrt{u'u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_p^2} \\
\cos\theta &= \frac{u'v}{\|u\|\|v\|}.
\end{aligned}$$

Now, we have to reprove Cauchy-Schwartz inequality; then triangular inequality follows directly!
Proof. $\forall \lambda \in \mathcal{R}$, (we will put it later as $\frac{u'v}{\|v\|^2}$)

$$0 \le \|u - \lambda v\|^2 = \|u\|^2 - 2\lambda u'v + \|\lambda v\|^2 = \|u\|^2 - 2\frac{(u'v)^2}{\|v\|^2} + \frac{(u'v)^2}{\|v\|^4}\|v\|^2 = \|u\|^2 - \frac{(u'v)^2}{\|v\|^2}$$

$$(u'v)^2 \le \|u\|^2\|v\|^2 \quad \Longrightarrow \quad -\|u\|\|v\| \le u'v \le \|u\|\|v\|$$

∎

Definition 11 (Linear Combination: generalization to adding vectors; this is the abstraction) .
Consider the two p-dimensional vectors v and w, and $c,\ d \in R$. We call $cv + dw$ a linear combination.

$$c\begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix} + d\begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix} = \begin{pmatrix} cv_1 + dw_1 \\ \vdots \\ cv_p + dw_p \end{pmatrix}.$$

Chapter 2

Solving Linear Equations

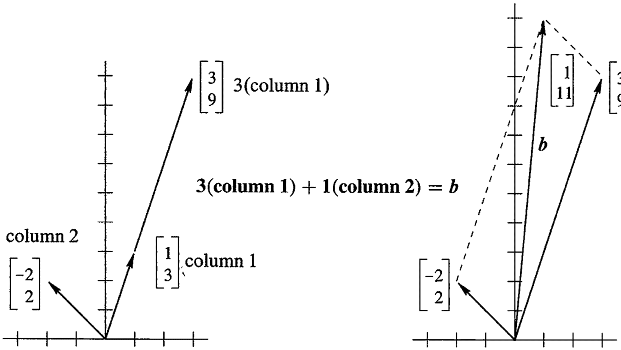
2.1 Vectors and Linear Equations

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \quad \equiv \quad \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix} \quad \equiv \quad A\mathbf{x} = \mathbf{b}$$

Column picture (linear combination)

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} x + \begin{pmatrix} -2 \\ 2 \end{pmatrix} y = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} 3 + \begin{pmatrix} -2 \\ 2 \end{pmatrix} 1 = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

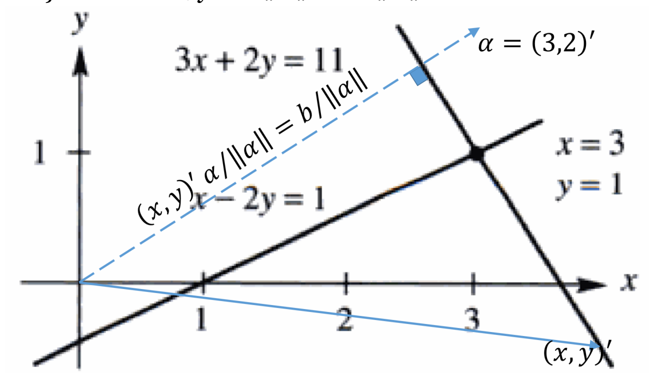


Row picture (vector equation of line intersection)

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 11$$

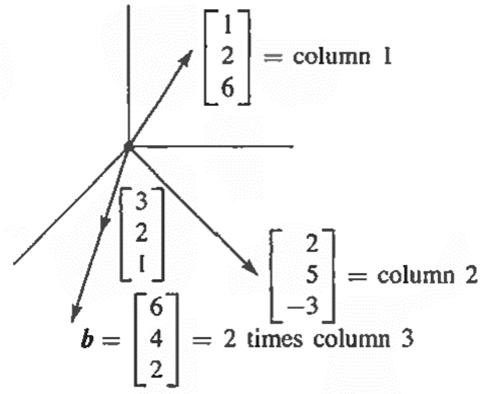
Projection $= (x, y)(\alpha / \|\alpha\|) = b / \|\alpha\| = 11/\sqrt{13}$.

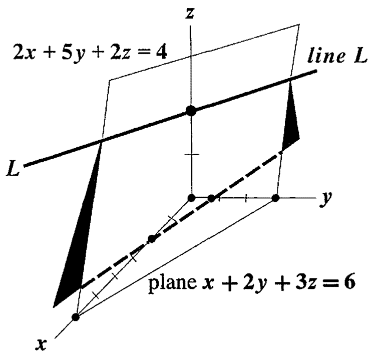


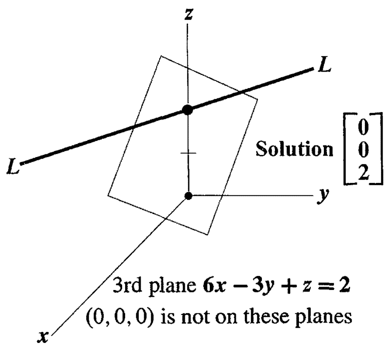
2.1.1 Three Equations in Three Unknowns

$$\begin{array}{c} x+2y+3z=6 \\ 2x+5y+2z=4 \\ 6x-3y+z=2 \end{array} \quad \equiv \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \quad \equiv \quad A\mathbf{x} = b \quad \equiv \quad C_1 x + C_2 y + C_3 z = \begin{pmatrix} R_1\mathbf{x} \\ R_2\mathbf{x} \\ R_3\mathbf{x} \end{pmatrix} = b,$$

where $\mathbf{x} = (x, y, z)'$.







2.2 The Idea of Elimination

Systematic way to solve linear equations

- Find the **pivot** (1 in this example)
- Form the uper triangle system of equations.
- **Backsubstitution**.

Example 12 (2 equations)

$$\begin{aligned} x-2y &= 1 \\ 3x+2y &= 11 \end{aligned} \tag{Before}$$

$$\begin{aligned} x-2y &= 1 \\ 8y &= 8 \end{aligned} \tag{After}$$

Example 13 (3 equations)

$$\begin{array}{ccc} \begin{aligned} 2x+4y-2z&=2 \\ 4x+9y-3z&=8 \\ -2x-3y+7z&=10 \end{aligned} & \begin{aligned} 2x+4y-2z&=2 \\ 1y+1z&=4 \\ 1y+5z&=12 \end{aligned} & \begin{aligned} \mathbf{2}x+4y-2z&=2 \\ \mathbf{1}y+1z&=4 \\ \mathbf{4}z&=8 \end{aligned} \\ \textbf{\textit{Step 0}} & \textbf{\textit{Step 1}} & \textbf{\textit{Step 2}} \end{array}$$

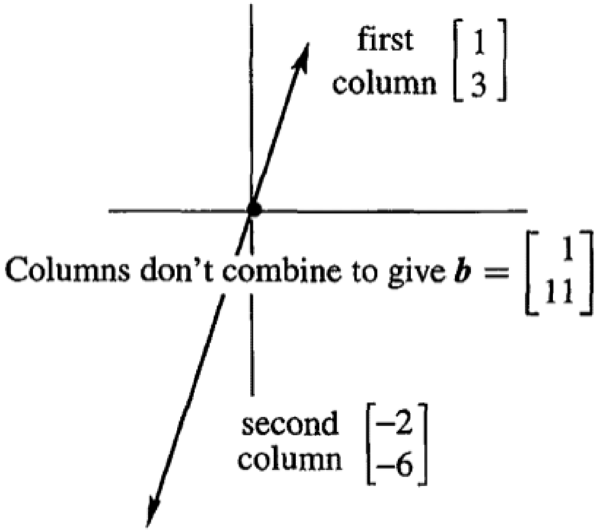
The solution is: $z=2,\ y=2,\ x=-1$; *i.e.,* $(-1,2,4)$

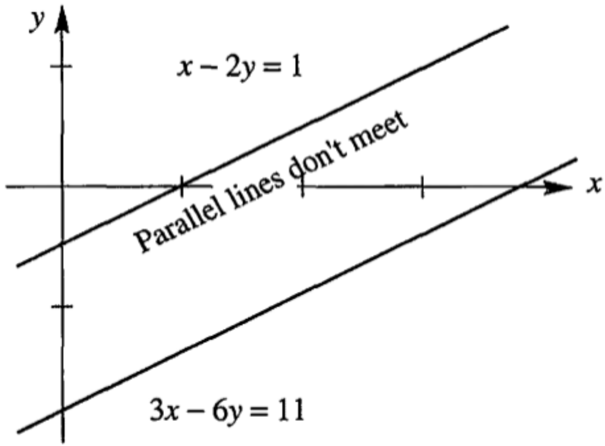
Failure 1: no solution

$$\begin{aligned} x-2y &= 1 \\ 3x-6y &= 11 \end{aligned} \qquad \text{(Before)}$$

$$\begin{aligned} x-2y &= 1 \\ 0y &= 8 \end{aligned} \qquad \text{(After)}$$

Interpretation from two perspectives



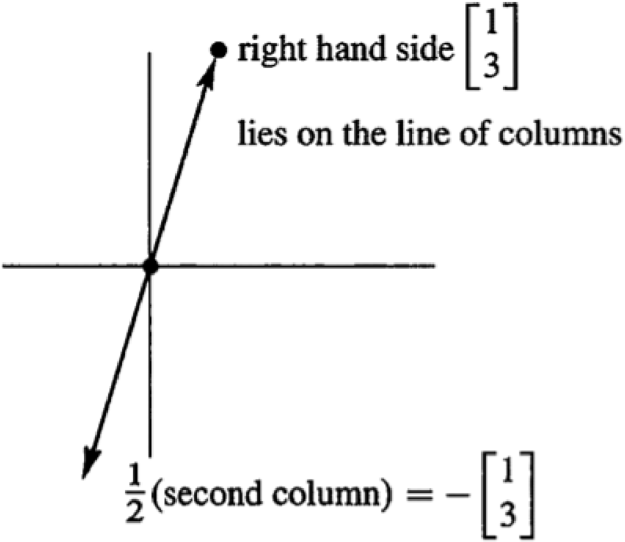


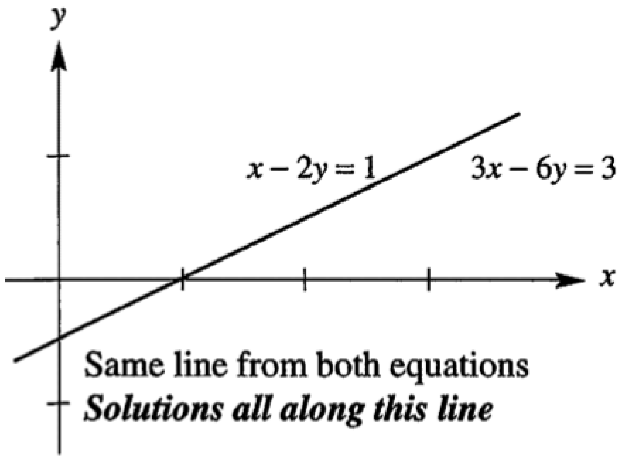
Failure 2: infinite solutions

$$\begin{aligned} x-2y &= 1 \\ 3x-6y &= 3 \end{aligned} \tag{Before}$$

$$\begin{aligned} x-2y &= 1 \\ 0y &= 0 \end{aligned} \tag{After}$$

Interpretation from two perspectives





2.3 Rules for Matrix Operations

Definition 14 (Matrix) *: A matrix $A_{m\times n}$ is a square array (of size $m \times n$) of "objects" (could be numbers could be other blocks of matrices). The element a_{ij} is located in row i and column j respectively. We say $A = (a_{ij})$ or in some books $A = ((a_{ij}))$ to denote:*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- Languages handle matrices differently; e.g., Matlab images, C (row-wise), Fortran (column-wise), etc.

- Traversing matrices is $\Theta(m \times n)$.

2.3.1 Matrix Transpose

Definition 15 *The transpose of the matrix* $A_{m\times n}$ *is* $\left(A'\right)_{n\times m}$, *where* $A_{ij} = \left(A'\right)_{ji}$

Example 16

$$A = \begin{pmatrix} 18 & 17 & 11 \\ 19 & -4 & 0 \end{pmatrix},\ A' = \begin{pmatrix} 18 & 19 \\ 17 & -4 \\ 11 & 0 \end{pmatrix}$$

Notice:

- $\left(A'\right)' = A$.
- For vectors:

$$x = \begin{pmatrix} 19 \\ -4 \\ 0 \end{pmatrix},\ x' = \begin{pmatrix} 19 & -4 & 0 \end{pmatrix}.$$

  we usually write $x = \begin{pmatrix} 19 & -4 & 0 \end{pmatrix}'$, or $x' = \begin{pmatrix} 19 & -4 & 0 \end{pmatrix}$ to save vertical space.

Definition 17 (Symmetric Matrices (around diagonal)) *A square matrix* $A_{m\times m}$ *is called symmetric if* $A_{ij} = A_{ji}$ *; i.e.,* $A = A'$.

Example 18 (write a SW to check the symmetry of) : $A = \begin{pmatrix} 18 & 17 & 11 \\ 17 & -4 & 0 \\ 11 & 0 & 2 \end{pmatrix}$

2.3.2 Matrix Partitioning

Definition 19 *A matrix $A_{p\times q}$ is said to be partitioned into $r \le p$ rows and $c \le q$ columns if it is written in the form*

$$A = \begin{array}{c} \left(\begin{array}{cccc} A_{11} & A_{12} & \cdots & A_{1c} \\ A_{21} & A_{22} & \cdots & A_{2c} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{array}\right) \begin{array}{l} \updownarrow p_1 \\ \updownarrow p_2 \\ \vdots \\ \updownarrow p_r \end{array} \\ \begin{array}{cccc} \overset{\longleftrightarrow}{q_1} & \overset{\longleftrightarrow}{q_2} & \cdots & \overset{\longleftrightarrow}{q_c} \end{array} \end{array} \tag{2.1}$$

where the block (submatrix) A_{ij} is a matrix $\left(A_{ij}\right)_{p_i q_j}$, and of course

$$A_{ij} = (A_{ij})_{p_i \times q_j}$$

$$\sum_{i=1}^{r} p_i = p$$

$$\sum_{j=1}^{c} q_j = q.$$

Example 20

$$A = \begin{pmatrix} 1 & 6 & 8 & 9 & 3 & 8 \\ 2 & 4 & 1 & 6 & 1 & 1 \\ 3 & 3 & 6 & 1 & 2 & 1 \\ 9 & 1 & 4 & 6 & 8 & 7 \\ 6 & 8 & 1 & 4 & 3 & 2 \end{pmatrix} = \left(\begin{array}{cccc|cc} 1 & 6 & 8 & 9 & 3 & 8 \\ 2 & 4 & 1 & 6 & 1 & 1 \\ 3 & 3 & 6 & 1 & 2 & 1 \\ \hline 9 & 1 & 4 & 6 & 8 & 7 \\ 6 & 8 & 1 & 4 & 3 & 2 \end{array}\right)$$

$$= \begin{pmatrix} (A_{11})_{3\times 4} & (A_{12})_{3\times 2} \\ (A_{21})_{2\times 4} & (A_{22})_{2\times 2} \end{pmatrix}$$

From the definition, this partitioning is not allowed:

$$\left(\begin{array}{cccccc} 1 & 6 & \multicolumn{1}{|c}{8} & 9 & 3 & 8 \\ \cline{3-6} 2 & 4 & \multicolumn{1}{|c}{1} & 6 & \multicolumn{1}{|c}{1} & 1 \\ 3 & 3 & \multicolumn{1}{|c}{6} & 1 & \multicolumn{1}{|c}{2} & 1 \\ \hline 9 & 1 & 4 & 6 & \multicolumn{1}{|c}{8} & 7 \\ 6 & 8 & 1 & 4 & \multicolumn{1}{|c}{3} & 2 \end{array}\right)$$

Application: dividing vectors in regression.

Transposing Partitioned Matrix

It is quite easy to show that, the transpose of a partitioned matrix (2.1) is given by

$$A' = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1c} \\ A_{21} & A_{22} & \cdots & A_{2c} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{pmatrix}' = \begin{pmatrix} A'_{11} & A'_{21} & \cdots & A'_{r1} \\ A'_{12} & A'_{22} & \cdots & A'_{r2} \\ \vdots & \vdots & & \vdots \\ A'_{1c} & A'_{2c} & \cdots & A'_{rc} \end{pmatrix}$$

Example 21

$$A = \left(\begin{array}{c|cc} 2 & 8 & 9 \\ 3 & 7 & 4 \end{array} \right) = \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$$

$$A' = \begin{pmatrix} A'_{11} \\ A'_{12} \end{pmatrix} = \left(\begin{array}{cc} 2 & 3 \\ \hline 8 & 7 \\ 9 & 4 \end{array} \right)$$

Paritioning into Vectors

Suppose that a_j is the jth column of $A_{r\times c}$. Then

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_c \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1c} \end{pmatrix},$$

where each submatrix is just a $r \times 1$ vector.

Similarly, it can be partitioned into r rows where α_i' is the ith row:

$$A = \begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{r1} \end{pmatrix} = \begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_r' \end{pmatrix}$$

2.3.3 Matrix Trace

Definition 22 *For a square matrix $A_{m\times m}$, the trace,* $\mathrm{trace}(A)$*, (for short* $\mathrm{tr}(A)$*), is defined as the sum of diagonal elements; i.e.,*

$$\mathrm{tr}(A) = \sum_{i=1}^{m} A_{ii}.$$

HW: *write a C function to calculate the trace. (of course $\Theta(m)$)*

Corollary 23

$$\mathrm{tr}(A) = \mathrm{tr}\left(A'\right).$$

$$\mathrm{tr}(x) = x \ \forall x \in \mathbb{R}.$$

Proof.

$$\mathrm{tr}(A) = \sum_i A_{ii} = \sum_i (A')_{ii} = \mathrm{tr(A')}.$$

■

Example 24

$$A = \begin{pmatrix} 1 & 7 & 6 \\ 8 & 3 & 9 \\ 4 & -2 & -8 \end{pmatrix} \Longrightarrow \mathrm{tr}(A) = -4.$$

2.3.4 Addition, Subtraction, and Scaling

Definition 25 *For equal size matrices $A_{m\times n}$ and $B_{m\times n}$, and for a scalar λ:*

- *the matrix $C = A \pm B$ is defined as*
$$C_{ij} = A_{ij} \pm B_{ij},$$
- *the matrix $D = \lambda A$ is defined as*
$$D_{ij} = \lambda A_{ij},$$
- *we say that $A = B$ if $A_{ij} = B_{ij}\ \forall i, j$.*
- *and a matrix, all of whose components are zeros, is written as $\mathbf{0}_{m\times n}$.*
- *Of course, $A + \mathbf{0} = A$*

Corollary 26 *It is quite easy to show that*

$$(A+B)' = A' + B'$$
$$\mathrm{tr}\,(A+B) = \mathrm{tr}\,(A) + \mathrm{tr}\,(B)$$

Proof. Show that the general element ij of LHS equal to that of RHS.

$$\left((A+B)'\right)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A')_{ij} + (B')_{ij} = (A'+B')_{ij}.$$
$$\mathrm{tr}(A+B) = \sum_i (A+B)_{ii} = \sum_i (A_{ii} + B_{ii}) = \sum_i A_{ii} + \sum_i B_{ii} = \mathrm{tr}(A) + \mathrm{tr}(B).$$

■

2.3.5 Matrix Multiplication

$$C = A_{m\times n}B_{n\times p} = \begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & & b_{np} \end{pmatrix} = C_{m\times p}$$

The general element C_{ij} is the dot product of Row_i and Col_j:

$$C_{ik} = a_i' b_k = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}.$$

However, we can partition either (or both) $A_{m\times n}$ and $B_{n\times p}$ as rows and/or columns to see the multiplication differently. This has a great value in mathematical treatments and semantics. We have only 4 ways to do that:

1. $A_{m\times 1}$, $B_{1\times p}$.

2. $A_{1\times n}$, $B_{n\times p}$.

3. $A_{1\times n}$, $B_{n\times 1}$.

4. $A_{m\times n}$, $B_{n\times 1}$.

Now, we will treat each case in detail.

1- As dot products

$$C = \begin{pmatrix} a_1' \\ \vdots \\ a_m' \end{pmatrix} \begin{pmatrix} b_1 & \cdots & b_p \end{pmatrix} \qquad (A_{m\times 1} B_{1\times p} \text{ partitioning})$$

$$= \begin{pmatrix} a_1' b_1 & & a_1' b_p \\ \vdots & \ddots & \vdots \\ a_m' b_1 & & a_m' b_p \end{pmatrix} \begin{pmatrix} \sum_{j=1}^n a_{1j} b_{j1} & & \sum_{j=1}^n a_{1j} b_{jp} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^n a_{mj} b_{j1} & & \sum_{j=1}^n a_{mj} b_{jp} \end{pmatrix}$$

$$C_{ik} = a_i' b_k = \sum_{j=1}^n a_{ij} b_{jk} = a_{i1} b_{1k} + \cdots + a_{in} b_{nk}$$

Example 27

$$= \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+4 & 2+16 & 0-4 \\ 3+5 & 2+20 & 0-5 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 18 & -4 \\ 8 & 22 & -5 \end{pmatrix}$$

2- As linear combinations of columns of A

$$\begin{aligned}
C &= \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_{11} & & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & & b_{np} \end{pmatrix} \qquad (A_{1\times n} B_{n\times p} \text{ partitioning}) \\
&= \begin{pmatrix} b_{11}a_1 + \cdots + b_{n1}a_n & \cdots & b_{1p}a_1 + \cdots + b_{np}a_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_j b_{j1}a_j & \cdots & \sum_j b_{jp}a_j \end{pmatrix} \\
&= \begin{pmatrix} c_1 & \cdots & c_p \end{pmatrix} \\
C_{ik} &= (c_k)_i = \Big(\sum_j b_{jk}a_j\Big)_i = \sum_j (b_{jk}a_j)_i = \sum_j b_{jk}a_{ij}.
\end{aligned}$$

Example 28

$$\begin{aligned}
C &= \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & -1 \end{pmatrix} \\
&= \left(3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + -1\begin{pmatrix} 4 \\ 5 \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} 3+4 \\ 3+5 \end{pmatrix} \quad \begin{pmatrix} 2+16 \\ 2+20 \end{pmatrix} \quad \begin{pmatrix} 0-4 \\ 0-5 \end{pmatrix} \right) = \begin{pmatrix} 7 & 18 & -4 \\ 8 & 22 & -5 \end{pmatrix}
\end{aligned}$$

3- As linear combinations of rows of B

$$C = \begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix} \begin{pmatrix} b'_1 \\ \vdots \\ b'_n \end{pmatrix} \quad (A_{m\times n} B_{n\times 1} \text{ partitioning})$$

$$= \begin{pmatrix} a_{11} b'_1 + \cdots + a_{1n} b'_n \\ \vdots \\ a_{m1} b'_1 + \cdots + a_{nm} b'_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^n a_{1j} b'_j \\ \vdots \\ \sum_{j=1}^n a_{mj} b'_j \end{pmatrix}$$

$$= \begin{pmatrix} c'_1 \\ \vdots \\ c'_m \end{pmatrix}$$

$$C_{ik} = (c'_i)_k = \big(\sum_j a_{ij} b'_j\big)_k = \sum_j \big(a_{ij} b'_j\big)_k = \sum_j a_{ij} b_{jk}.$$

Example 29

$$= \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\begin{pmatrix} 3 & 2 & 0 \end{pmatrix} + 4\begin{pmatrix} 1 & 4 & -1 \end{pmatrix} \\ 1\begin{pmatrix} 3 & 2 & 0 \end{pmatrix} + 5\begin{pmatrix} 1 & 4 & -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 3+4 & 2+16 & 0-4 \end{pmatrix} \\ \begin{pmatrix} 3+5 & 2+20 & 0-5 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 7 & 18 & -4 \\ 8 & 22 & -5 \end{pmatrix}$$

4- As summation of outer products, each is a matrix

$$C = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1' \\ \vdots \\ b_n' \end{pmatrix} \quad (A_{1\times n} B_{n\times 1} \text{ partitioning})$$

$$= a_1 b_1' + \cdots + a_n b_n'$$

$$= \sum_{j=1}^{n} a_j b_j',$$

$$a_j b_j' = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} b_j' = \begin{pmatrix} a_{1j} b_j' \\ \vdots \\ a_{nj} b_j' \end{pmatrix},$$

$$C_{ik} = \left(\sum_j a_j b_j'\right)_{ik} = \sum_j \left(a_j b_j'\right)_{ik} = \sum_j a_{ij} b_{jk}.$$

Example 30

$$= \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} \begin{pmatrix} 1 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ 3 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 16 & -4 \\ 5 & 20 & -5 \end{pmatrix} = \begin{pmatrix} 7 & 18 & -4 \\ 8 & 22 & -5 \end{pmatrix}$$

Partitioned Matrices and Multiplication (general case)

Subdevide each matrix to conforming number of blocks, e.g.

$$
\begin{aligned}
AB &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} \\
&= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{pmatrix} \qquad \text{(Must Conform)}
\end{aligned}
$$

In general: $A_{m\times n}B_{n\times p}$

$$
= \begin{array}{c}
\begin{array}{cccc} \overset{\longleftrightarrow}{n_1} & \overset{\longleftrightarrow}{n_2} & \cdots & \overset{\longleftrightarrow}{n_c} \end{array} \\
\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1c} \\ A_{21} & A_{22} & \cdots & A_{2c} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{pmatrix}_{r\times c}
\end{array}
\qquad
\begin{array}{c} n_1 \updownarrow \\ n_2 \updownarrow \\ \vdots \\ n_c \updownarrow \end{array}
\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & & \vdots \\ B_{c1} & B_{c2} & \cdots & B_{ck} \end{pmatrix}_{c\times k}
$$

$n_1 + \cdots + n_c = n.$

Product with Diagonal Matrix

Definition 31 *A matrix D is diagonal if $D_{ij} = 0\ \forall i \neq j$; i.e.,*

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_m \end{pmatrix}.$$

Since there is no confusion, we subscript d_i instead of d_{ii}. We also, for short, write $D = \operatorname{diag}(d_1, \ldots, d_n)$

Row scaling:

$$D_{m\times m} A_{m\times n} = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} \begin{pmatrix} a'_1 \\ \vdots \\ a'_m \end{pmatrix} = \begin{pmatrix} d_1 a'_1 \\ \vdots \\ d_m a'_m \end{pmatrix}$$

Column scaling:

$$A_{m\times n} D_{n\times n} = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_n \end{pmatrix} = \begin{pmatrix} a_1 d_1 & \cdots & a_n d_n \end{pmatrix}$$

Definition 32 *The identity matrix I is a special case diagonal matrix and defined as*

$$I_{m\times m} = \operatorname{diag}(1, \ldots, 1)$$

It is obvious that $IA = AI = A$.

Transpose of a Product

Lemma 33 *For conforming matrices $A_{m\times n}$ and $B_{n\times p}$,*

$$(AB)' = B'A',$$

and more general

$$(A_1 \cdots A_n)' = A'_n \cdots A'_1.$$

Proof. The general element AB_{ik} is given by

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk} = \sum_{j=1}^{n} \left(A'\right)_{ji} \left(B'\right)_{kj} = \sum_{j=1}^{n} \left(B'\right)_{kj} \left(A'\right)_{ji} = \left(B'A'\right)_{ki}.$$

Proving the second part is immediate by induction. ■

Trace of a Product

The trace is defined only for a square matrix; hence, for a product to have a trace it must be $A_{m\times n}B_{n\times m}$.

Lemma 34 *For two-side conforming matrices $A_{m\times n}$ and $B_{n\times m}$,*

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)\,,$$

and more general

$$\mathrm{tr}\,(A_1 \cdots A_n) = \mathrm{tr}\,(A_n \cdots A_1)\,.$$

Proof. $A_{m\times n}B_{n\times m} = C_{m\times m}$, $B_{n\times m}A_{m\times n} = D_{n\times n}$:

$$\mathrm{tr}(AB) = \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{m} B_{ji}A_{ij} = \sum_{j=1}^{n} (BA)_{jj} = \mathrm{tr}(BA).$$

■

Remark 1 *From the proof above, we see that*

$$\mathrm{tr}\,(AB) = \sum_{j=1}^{n}\sum_{i=1}^{m} B_{ji}A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} \left(B'\right)_{ij} A_{ij},$$

i.e., it is the sum of products of each element of A multiplied by the corresponding element of B'. And if $B = A'$

$$\mathrm{tr}\left(AA'\right) = \mathrm{tr}\left(A'A\right) = \sum_{j=1}^{n}\sum_{i=1}^{m} A_{ij}A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} A_{ij}^2,$$

i.e., it is the sum squares of all elements.

Example 35

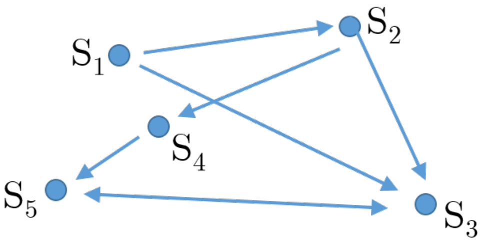
$$A = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 3 & 0 \end{pmatrix},$$

$$\operatorname{tr}\left(AA'\right) = 1^2 + 2^2 + 3^2 + (-4)^2 + 3^2 + 0^2 = 39$$

Power of a Matrix

$$A^k = AA\cdots A,\ k \text{ times} \qquad (A \text{ must be square; why?})$$

Example 36 (Graph Theory) :



- *The traffic is represented as a matrix T, where a path from S_i to S_j exists if $T_{ij} = 1$.*

$$T = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- *Number of ways of getting from S_i to S_k in exactly 2 steps is* $\sum_j T_{ij} T_{jk} = \left(T^2\right)_{ik}$.
- *Number of ways of getting from S_i to S_k in exactly 3 steps is* $\sum_j T_{ij} \left(T^2\right)_{jk} = \left(T^3\right)_{ik}$.

$$T^2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, T^3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- *Number of ways of getting from S_i to S_k in exactly r steps is* $\sum_j T_{ij} \left(T^{r-1}\right)_{jk} = (T^r)_{ik}$.
- *There is no path from S_i to S_k only if* $\sum_{r=1}^{\infty} (T^r)_{ik} = 0$.
- *What is $\sum_{r=1}^{\infty} T^r$?*

2.3.6 The Laws of Algebra

Theorem 37 *$\forall A_{m\times n}$, $B_{m\times n}$, $C_{m\times n}$, c scalar, we have*

$$A+B=B+A \tag*{(commulative)}$$

$$c\,(A+B)=cA+cB \tag*{(distributive)}$$

$$A+(B+C)=(A+B)+C, \tag*{(associative)}$$

and

$$C\,(A+B)=CA+CB, \tag*{$(\forall A_{m\times n}, B_{m\times n}, C_{k\times m})$}$$

$$(A+B)\,C=AC+BC, \tag*{$(\forall A_{m\times n}, B_{m\times n}, C_{n\times p})$}$$

$$A\,(BC)=(AB)\,C \tag*{$(\forall A_{m\times n}, B_{n\times p}, C_{p\times q})$}$$

$$A_{m\times n}B_{n\times m}\neq B_{n\times m}A_{m\times n}$$

$$A_{m\times m}B_{m\times m}=\neq B_{m\times m}A_{m\times m}$$

Example 38 (Counter Example for $AB\neq BA$)

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}0 & 2\\ 3 & 3\end{pmatrix}=\begin{pmatrix}6 & 8\\ 12 & 18\end{pmatrix}=\begin{pmatrix}0 & 2\\ 3 & 3\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

$$\underset{A}{\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}}\underset{B}{\begin{pmatrix}0 & 1\\ 3 & 5\end{pmatrix}}=\begin{pmatrix}6 & 11\\ 12 & 23\end{pmatrix}\neq\begin{pmatrix}3 & 4\\ 18 & 26\end{pmatrix}=\underset{B}{\begin{pmatrix}0 & 1\\ 3 & 5\end{pmatrix}}\underset{A}{\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}}$$

Proof of multiplication associative rule. $A_{m\times n}B_{n\times p}C_{p\times q}$

$$
\begin{aligned}
(AB)_{ij} &= \sum_{r=1}^{n} A_{ir}B_{rj}\\
((AB)\,C)_{ik} &= \sum_{j=1}^{p} (AB)_{ij}\,C_{jk}\\
&= \sum_{j=1}^{p}\sum_{r=1}^{n} A_{ir}B_{rj}C_{jk}\\
&= \sum_{r=1}^{n} A_{ir}\sum_{j=1}^{p} B_{rj}C_{jk}\\
&= \sum_{r=1}^{n} A_{ir}\,(BC)_{rk}\\
&= (A\,(BC))_{ik}
\end{aligned}
$$

■

Example 39 *Factor $Y = XPX + QX^2 + X$ and find the constraints on the order of matrices.*
It is clear that all matrices will be of order $m \times m$.

$$
\begin{aligned}
Y &= XPX + QX^2 + X \\
&= (XP + QX + I)\, X
\end{aligned}
$$

$$XPX + QXX + X$$

Product with Scalar and Quadratic Forms

Back to Definition 2 it is very important, sometimes, to make sure of conforming even for scalars; i.e., we write

$$y_{m\times 1} a_{1\times 1} \textbf{ NOT } ay.$$

This is because, sometimes, $a_{1\times 1}$ itself is a matrix multiplication that if dissembled it should conform with the remaining of equation

$$\begin{aligned} a_{1\times 1} &= x'_{1\times m} A_{m\times m} x_{m\times 1} \\ y_{n\times 1} a_{1\times 1} &= \underline{y_{n\times 1} x'_{1\times m}} A_{m\times m} x_{m\times 1} \\ a_{1\times 1} y_{n\times 1} &= x'_{1\times m} A_{m\times m} \underline{x_{m\times 1} y_{n\times 1}} && \text{(WRONG!)} \end{aligned}$$

Example 40 (Quadratic Form) *For any square matrix A, the form $y_{1\times 1} = x'_{1\times n} A_{n\times n} x_{n\times 1}$ is called quadratic form; it contains all quadratic and bilinear terms. (For scalar case, simply it is $y = xax = ax^2$).*

$$
\begin{aligned}
y &= \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_i x_i a_{i1} & \sum_i x_i a_{i2} & \dots & \sum_i x_i a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\
&= \sum_j \left(\sum_i x_i a_{ij} \right) x_j \\
&= \sum_j \sum_i a_{ij} x_j x_i \\
&= \sum_i a_{ii} x_i^2 + \sum_{i \neq j} \sum_{\text{all off diagonal}} a_{ij} \quad x_i x_j \\
&= \sum_i a_{ii} x_i^2 + \sum_{i>j} \sum \Big(\underset{LT}{a_{ij}} + \underset{UT}{a_{ji}} \Big) x_i x_j,
\end{aligned}
$$

this is because, e.g., $a_{13} x_1 x_3 + a_{31} x_3 x_1 = (a_{13} + a_{31}) x_1 x_3$.

Complexity of Ver. 1: *we sum $(n^2 - n)$ off-diagonal terms, each term is 2 multiplications $(a_{ij} x_i x_j)$; therefore* *total steps is given by:*

$$\begin{aligned}
\#\, of\ steps &= \left(n^2-n\right)(2M) + \left(n^2-n-1\right)(S)\\
&= 2\left(n^2-n\right)M + \left(n^2-n-1\right)S
\end{aligned}$$

Complexity of Ver. 2:, *we sum* $\left(n^2-n\right)/2$ *lower triangular term, each term is one summation and 2 multiplications; therefore*

$$\begin{aligned}
\#\, of\ steps &= \frac{\left(n^2-n\right)}{2}(2M+1S) + \left(\frac{\left(n^2-n\right)}{2}-1\right)S\\
&= \left(n^2-n\right)M + \left(n^2-n-1\right)S.
\end{aligned}$$

Ver. 2 is half the number of multiplications of Ver. 1; it is almost double speed gained by a simple trick.

For the following quadratic form y, expand column wise $\left(\sum_j \sum_i\right)$:

$$\begin{aligned}
y &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3\\ 4 & 7 & 6\\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1\\ x_2\\ x_3 \end{pmatrix}\\
&= x_1^2 + 4x_2x_1 + 2x_3x_1 + 2x_1x_2 + 7x_2^2 - 2x_3x_2 + 3x_1x_3 + 6x_2x_3\\
&= x_1^2 + (2+4)\,x_1x_2 + (3+2)\,x_1x_3 + 7x_2^2 + (6-2)\,x_2x_3\\
&= x_1^2 + 6x_1x_2 + 5x_1x_3 + 7x_2^2 + 4x_2x_3.
\end{aligned}$$

Without expansion, it is obvious that, e.g.,

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 6 \\ 2 & -2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 5 & 7 & 3 \\ 4 & 1 & 0 \end{pmatrix}}_{B} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

because $\left(A_{ij} + A_{ji}\right) = \left(B_{ij} + B_{ji}\right)\ \forall i, j.$

Hence, we can replace the matrix A *in any quadratic form* $y = x'Ax$ *by a symmetric matrix* $\Sigma = \left(A + A'\right)/2$ *whose diagonals and off-diagonals have:*

$$\begin{aligned} \sigma_{ii} &= (a_{ii} + a_{ii})/2 = a_{ii} \\ \left(\sigma_{ij} + \sigma_{ji}\right) &= \left(a_{ij} + a_{ji}\right)/2 + \left(a_{ji} + a_{ij}\right)/2 \\ &= a_{ij} + a_{ji} \\ x'Ax &= x'\Sigma x \end{aligned}$$

Example 41 *Expand and simplify $y = (x-\mu)' \Sigma (x-\mu)$, where x and μ are vectors and Σ is a symmetric matrix.*

$$
\begin{aligned}
y &= (x-\mu)' \Sigma (x-\mu) \\
&= (x'-\mu') \Sigma (x-\mu) \\
&= x'\Sigma x - x'\Sigma\mu - \mu'\Sigma x + \mu'\Sigma\mu \\
&= x'\Sigma x - x'\Sigma\mu - \left(\mu'_{1\times p}\Sigma_{p\times p}x_{p\times 1}\right)' + \mu'\Sigma\mu && (scalar' = scalar) \\
&= x'\Sigma x - x'\Sigma\mu - x'\Sigma\mu + \mu'\Sigma\mu \\
&= x'\Sigma x - 2x'\Sigma\mu + \mu'\Sigma\mu
\end{aligned}
$$

2.4 Elimination Using Matrices

Back to the linear system of equations (Ex., 13) (and using **pivots** for elemination):

$$\begin{array}{r} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{array} \implies \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$(\text{col1})\, x + (\text{col2})\, y + (\text{col3})\, z = \begin{pmatrix} (\text{row1})\,.x \\ (\text{row2})\,.y \\ (\text{row3})\,.z \end{pmatrix} = b$$

To eliminate: $R_2^{\text{new}} = R_2 + (-2) \times R_1$, which can be accomplished by the matrix multiplication:

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}.$$

We denote the elimination matrix $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by $E_{21}(-2)$.

Definition 42 *The elimination matrix $E_{ij}(l)$ is an identity matrix except the element $e_{ij} = l$ to perform: $R_i^{new} = R_i + l \times R_j$. If not ambiguous we write E_{ij}*

$$E_{31}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 12 \end{pmatrix}$$

Finally,

$$E_{32}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix},$$

whose solution is $z = 2,\ y = 2,\ x = -1$.
The summary of that is

$$E_{32}(-1)E_{31}(1)E_{21}(-2)AX = E_{32}(-1)E_{31}(1)E_{21}(-2)b.$$

Just for simpler notation (with same everything), we could have made up the augmented matrix

$$(A|\ b) = \left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right)$$

,then

$$E_{32}E_{31}E_{21}\,(A|\ b)$$

Definition 43 *The permutation matrix P_{ij} is an identity matrix except that in Rows i and j (to be permuted) the ones are located in p_{ij}, p_{ji} respectively; e.g.,*

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Of course, $P_{ij} = P_{ji}$.

This is needed to swap equations when the pivot is zero.

Example 44

$$\begin{aligned} x+2y+2z&=1\\ 4x+8y-9z&=3\\ 3y+2z&=1 \end{aligned} \implies (A|\ b) = \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1\\ 4 & 8 & 9 & 3\\ 0 & 3 & 2 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 0\\ -4 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} \swarrow = \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1\\ 0 & 0 & 1 & -1\\ 0 & 3 & 2 & 1 \end{array}\right) \tag{$E_{21}(-4)$}$$

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix} \swarrow = \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1\\ 0 & 3 & 2 & 1\\ 0 & 0 & 1 & -1 \end{array}\right). \tag{P_{23}}$$

This is called Gauss elimination, and by back-substitution, $z=-1,\ y=1,\ x=1.$ *Jordan would go further to get pivots on diagonal and zeros elsewhere.*

$$
\begin{aligned}
&= \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \swarrow &= \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) && (E_{23}(-2)) \\
\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \swarrow &= \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) && (E_{13}(-2)) \\
\begin{pmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \swarrow &= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) && (E_{12}(\tfrac{-2}{3})) \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \swarrow &= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) && (D(1, \tfrac{1}{3}, 1)) \\
&= (I|\ solution),
\end{aligned}
$$

where the solution is : $x = 1, y = 1, z = -1$.
The summary of that is:

$$DE_{12}E_{13}E_{23}P_{23}E_{21}\,(A|\ b)$$

Solution of system or linear equations is nothing but multiplication by E***s,*** P***s, and finally*** D

Example 45 (Elimination by blocks) *Using the first pivot, we can eliminate all elements underneath using a single matrix. Write the matrix A as*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

In general we eliminate by:

$$(E_{n\ n-1})\ldots(E_{n2}\ldots E_{42}E_{32})\,(E_{n1}\ldots E_{31}E_{21})$$

$$\begin{aligned} E &= E_{31}E_{21} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_{31}/a_{11} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a_{21}/a_{11} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -a_{21}/a_{11} & 1 & 0 \\ -a_{31}/a_{11} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ -A_{21}/a_{11} & I \end{pmatrix} \end{aligned}$$

The power of block treatment, allow us to write

$$\begin{aligned} EA &= \begin{pmatrix} 1 & \mathbf{0} \\ -A_{21}/a_{11} & I \end{pmatrix} \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & A_{12} \\ \mathbf{0} & A_{22} - A_{21}A_{12}/a_{11} \end{pmatrix}. \end{aligned}$$

Of course A can be replaced by $(A|\, b)$. For this example

$$
\begin{aligned}
(A|\, b) &= \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{array}\right) \\
A_{22} - A_{21} A_{12} / a_{11} &= \begin{pmatrix} 8 & 9 & 3 \\ 3 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \end{pmatrix} / 1 \\
&= \begin{pmatrix} 8 & 9 & 3 \\ 3 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 8 & 8 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}
\end{aligned}
$$

This gives

$$
\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{array}\right),
$$

which would be obtained of course having multiplied by $E_{21}(-4)$

2.5 Inverse Matrices

Definition 46 *The **square** matrix $A_{p\times p}$ is invertible if there exists a matrix A^{-1} such that*

$$A_l^{-1}A = AA_r^{-1} = I_{p\times p}$$

Hint: *we will show soon that $A_l^{-1} = A_r^{-1} = A^{-1}$. But we have to be cautious and **rigorous**, since $AB \neq BA$ in general.* ■

Motivation: for scalar a

$$\begin{aligned} aX &= b \\ a^{-1}aX &= a^{-1}b \\ 1\,X &= a^{-1}b \\ X &= a^{-1}b \end{aligned}$$

Analogously, what is A^{-1} such that

$$\begin{aligned} AX &= b \\ A^{-1}AX &= A^{-1}b \\ IX &= A^{-1}b \\ X &= A^{-1}b, \end{aligned}$$

although finding A^{-1} is more computational expensive than solving by elimination as we will see.

Lemma 47 *If both left and right inverses exist they are equal*

Proof. Suppose the left and right inverses of A are A_l^{-1} and A_r^{-1} (so that $A_l^{-1}A = AA_r^{-1} = I$); then consider $A_l^{-1}AA_r^{-1}$

$$A_r^{-1} = \left(A_l^{-1}A\right)A_r^{-1} = \underleftrightarrow{A_l^{-1}AA_r^{-1}} = A_l^{-1}\left(AA_r^{-1}\right) = A_l^{-1}$$

■

This Lemma is different from the last two statements in Lemma 51 (will be proven shortly), from which we can say:

1. If left (or right) inverse exists the right (or left) exists and equals it. Stated differently, if $AB = I$ then $BA = I$.

2. If the inverse exists it is unique. So, we cannot find $B_1A = I$ and $B_2A = I$ with $B_1 \neq B_2$

Therefore,
Either: the square matrix A has no inverse
Or: the left and right inverses are identical and unique.

Lemma 48 (inverse of special matrices) *:*

1. *Any* 2×2 *matrix:*
$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$$

2. *Any* $n\times n$ *diagonal matrix:*
$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n\end{pmatrix}^{-1} = \begin{pmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n\end{pmatrix}$$

3. *Any pivot cancellation matrix:*
$$E_{ij}^{-1}(l) = E_{ij}(-l)$$

4. *Any permutation matrix:*
$$\left(P_{ij}\right)^{-1} = P_{ij}$$

Proof. The proof is by direct multiplication from both sides; it is obvious for 1 and 2. For 3,
$$\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & & \vdots\\ \vdots & -l & \ddots & 0\\ 0 & \cdots & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & & \vdots\\ \vdots & l & \ddots & 0\\ 0 & \cdots & 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & & \vdots\\ \vdots & -l\times 1+l & \ddots & \\ 0 & \cdots & 0 & 1\end{pmatrix}.$$
Proving 4 follows exactly the same line. In few words, since P_{ij} is I with rows i and j swapped then $P_{ij}P_{ij}$ swaps again the same rows to bring it back to I. ∎

Example 49 *Consider $E_{21}(-5)$, then*

$$E_{21}(-5)\,A = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 - 5R_1 \\ R_3 \end{pmatrix},$$

$$E_{21}(5)\,(E_{21}(-5)\,A) = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 - 5R_1 \\ R_3 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix},$$

i.e., it subtracts what E added. Of course, $EE^{-1} = E^{-1}E = I$.

Calculating A^{-1} by Gauss-Jordan Elimination

Consider $A_{n\times n}$ and its right inverse exists: $A_r^{-1} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$. Then,

$$A\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} = I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}$$

$$\begin{pmatrix} Ax_1 & \cdots & Ax_n \end{pmatrix} = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}$$

$$Ax_1 = e_1$$

$$\vdots$$

$$Ax_n = e_n$$

Then, finding A^{-1} is nothing but solving by elimination n systems of equations, each is $n \times n$:

$$Ax_i = e_i, i = 1, \ldots, n.$$

Example 50 .

$$A=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Find A^{-1} *using the augmented matrix* $(A|\ I) =$

$$= \left(\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \qquad (E_{21}(\tfrac{1}{2}))$$

$$\to \left(\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right) \qquad (E_{32}(\tfrac{2}{3});\ \text{Gauss stops here})$$

$$\to \left(\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right) \qquad (E_{23}(\tfrac{3}{4}))$$

$$\to \left(\begin{array}{rrr|rrr} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right) \qquad (E_{12}(\tfrac{2}{3}))$$

$$\to \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right) \qquad (D(\tfrac{1}{2},\tfrac{2}{3},\tfrac{3}{4}))$$

$$= (I|\ A_r^{-1}) \qquad \text{(reduced echelon form)}$$

In summary:

$DE_{12}E_{23}E_{32}E_{21}\,(A|\ I) = (I|\ A_r^{-1})$.

Lemma 51 (Connection between A^{-1} and pivots) :

1. *If A has n pivots A^{-1} exists and*
$$M = A_l^{-1} = A^{-1} = A_r^{-1} = X,$$
where M is the multiplication of the elemination matrices and X is the solution of $AX = I$.

2. *If either inverse exists A has n pivots and hence A^{-1} exists. (This means if $AB = I$ then $BA = I$)*

3. *If the inverse exists then it is unique along with pivots and the solution to $Ax = b$.*

Proof. 1: If the pivots exist then this has been produced to initially solve the problem $AX = I$, and the solution X went to the right side; therefore the solution X is A_r^{-1}. In parallel, the solution is nothing but a series of matrix multiplications:

$$\underleftarrow{D(E_{12})\ldots(E_{1\ n-1}\ldots E_{n-3\ n-1}E_{n-2\ n-1})(E_{1n}\ldots E_{n-2\ n}E_{n-1\ n})}\cdot$$
$$\underleftarrow{(E_{n\ n-1})\ldots(E_{n2}\ldots E_{42}E_{32})(E_{n1}\ldots E_{31}E_{21})}A = I,$$

in the form $MA = I$; hence, M is A_l^{-1}. Since both inverses exist, they are equal (Lemma 47).

2: If A_r^{-1} exists ($AX = I$**)** we will prove A has n pivots by contradiction. Assume that A does not have n pivots (the elimination matrices MA produces a matrix with zero row):

$$\text{zero row mat.} = \underbrace{(MA)}_{\text{zero row mat.}} X = \underleftrightarrow{MAX} = M(AX) = MI = M.$$

However, M cannot have a zero row; otherwise it would produce a zero row matrix; while **by construction**, it should produce n pivots not a zero row; a contradiction. Hence, A has n pivots and from 1 above $M = A_l^{-1} = A^{-1} = A_r^{-1} = X$ (which means $XA = I$).

If A_l^{-1} exists ($XA = I$), then $AX = I$ with which we have just started lines above.

3: Assume that A has two inverses A_1 and A_2, so that $A_1A = AA_1 = I$ and $A_2A = AA_2 = I$.

$$\begin{aligned} A_1A &= I \\ A_1AA_2 &= A_2 \\ A_1 &= A_2 \end{aligned}$$

Since the inverse is unique, the elimination process cannot produce different pivots; hence they are unique too and the solution to $Ax = b \;\forall b$ will be unique as well and equals to $A^{-1}b$. ■

Lemma 52 *If A is symmetric, then its inverse is symmetric.*

Proof. Suppose that B is an inverse then

$$\begin{aligned} BA &= I \\ A'B' &= I \\ AB' &= I \\ BAB' &= B \\ B' &= B. \end{aligned}$$

■

Lemma 53

1. *Suppose that A, B, are invertible,* $(AB)^{-1} = B^{-1}A^{-1}$.
2. *And in general* $(A_1 \cdots A_n)^{-1} = A_n^{-1} \cdots A_1^{-1}$.

Proof. For the first part,

$$\begin{aligned}
(AB)^{-1}(AB) &= I \\
(AB)^{-1}(AB)\,B^{-1}A^{-1} &= B^{-1}A^{-1} \\
(AB)^{-1} &= B^{-1}A^{-1} \\
(AB)(AB)^{-1} &= I \\
B^{-1}A^{-1}(AB)(AB)^{-1} &= B^{-1}A^{-1} \\
(AB)^{-1} &= B^{-1}A^{-1}.
\end{aligned}$$

The proof of part 2 is immediate by induction. ■

Lemma 54 *If* $AX = \mathbf{0}$ *and* $X \neq \mathbf{0}$ *then* A *is not invertible.*

Proof. Given $X \neq \mathbf{0}$, suppose that A^{-1} exists;

$$\begin{aligned}
AX &= \mathbf{0} \\
A^{-1}AX &= A^{-1}\mathbf{0} \\
X &= \mathbf{0},
\end{aligned}$$

a contradiction; hence A^{-1} does not exist. ■

2.6 Elimination Using Matrices is $A = LU$ Factorization

Back to Sec. 2.4

Intuitively:

$$U_3 = A_3 - l_{31}U_1 - l_{32}U_2$$
$$A_3 = U_3 + l_{31}U_1 + l_{32}U_2$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2&4&-2\\4&9&-3\\-2&-3&7\end{pmatrix}=\begin{pmatrix}2&4&-2\\0&1&1\\0&0&4\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\-2&1&0\\3&-1&1\end{pmatrix}\begin{pmatrix}2&4&-2\\4&9&-3\\-2&-3&7\end{pmatrix}=\begin{pmatrix}2&4&-2\\0&1&1\\0&0&4\end{pmatrix}$$

$$E_{32}(-1)E_{31}(1)E_{21}(-2)\ A = M_L\ A = U$$

$$\begin{aligned}A &= M_L^{-1}\,U = \left(E_{32}(-1)E_{31}(1)E_{21}(-2)\right)^{-1}\,U\\ &= \left(E_{21}(2)E_{31}(-1)E_{32}(1)\right)U\\ A &= L\,U\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}2&4&-2\\4&9&-3\\-2&-3&7\end{pmatrix} &= \begin{pmatrix}1&0&0\\2&1&0\\-1&1&1\end{pmatrix}\begin{pmatrix}2&4&-2\\0&1&1\\0&0&4\end{pmatrix}\\ &= \begin{pmatrix}1&0&0\\2&1&0\\-1&1&1\end{pmatrix}\begin{pmatrix}2&0&0\\0&1&0\\0&0&4\end{pmatrix}\begin{pmatrix}1&2&-1\\0&1&1\\0&0&1\end{pmatrix}\end{aligned}$$

$$A = L\,D\,U.$$

Remark: L stores the **Gauss-elimination steps** on A, which ends up to U.

Example 55 (Using $L\,U$ in solving equations:)

$$AX = b \qquad \equiv \qquad L\,(U\,X) = b$$

Then, solve $L\,C = b$ to find C, then solve $U\,X = C$ to find X.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} \qquad \text{(Gauss-elimination for } b\text{)}$$

$$c_1 = 2 \qquad 2c_1 + c_2 = 8 \longrightarrow c_2 = 4 \qquad -c_1 + c_2 + c_3 = 10 \longrightarrow c_3 = 8$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} \qquad \text{(same obtained with augmenting)}$$

$$4z = 8 \longrightarrow z = 2 \qquad y + z = 4 \longrightarrow y = 2 \qquad 2x + 4y - 2z = 2 \longrightarrow x = -1$$

Lemma 56 **(**$A = L\,U$ **factorization)** *: for the case of no permutation, we get*

$$\begin{aligned}
\left(E_{n\;n-1}\right)\cdots\left(E_{n2}\cdots E_{42}E_{32}\right)\left(E_{n1}\cdots E_{31}E_{21}\right)A &= U\\
M_L\;A &= U\\
A &= M_L^{-1}\;U\\
&= \left(E_{21}^{-1}E_{31}^{-1}\cdots E_{n1}^{-1}\right)\left(E_{32}^{-1}E_{42}^{-1}\cdots E_{n2}^{-1}\right)\cdots\left(E_{n\;n-1}^{-1}\right)U\\
&= L\,U,
\end{aligned}$$

where: (1) both M_L*,* $M_L^{-1}(=L)$ *are LTMs, and (2)* L *has* L_{ij} *equals directly to the element of* E_{ij} *as opposed to* M_L*. The proof is immediate from the following two more general lemmas.* ***Hint:*** *to prove that the elements of* M_L *are not directly the elements of* E_{ij} *a single counter example is enough.*

Lemma 57 *Multiplication of two lower (or upper) triangular matrices is a lower (or upper) triangular matrix. The diagonal will be one if* $A_{ii}B_{ii} = 1$ *(*$A_{ii} = B_{ii} = 1$ *is a special case).*

Proof. Suppose A, B are LTMs; i.e., $A_{ij} = B_{ij} = 0\;\forall i < j$. Then, the element $C_{ij}, i \le j$ will be

$$C_{ij} = \sum_k A_{ik}B_{kj} = \sum_{k<i} A_{ik}B_{kj} + A_{ii}B_{ij} + \sum_{k>i} A_{ik}B_{kj} = \sum_{k<i} A_{ik}\,0 + A_{ii}B_{ij} + \sum_{k>i} 0\;B_{kj} = A_{ii}B_{ij}$$

which is 0 for $i < j$ and $A_{ii}B_{ii}$ for $i = j$. Hence, it is obvious that M_L is LTM with ones on the diagonal ■

Lemma 58 *Consider any two LTMs $A,\ B$ with the following properties*

$$
\begin{aligned}
A_{ij} &= B_{ij} = 0 && \forall i < j \\
A_{ii} &= B_{ii} = 1 && \\
A_j &= e_j && \forall j > J \\
B_j &= e_j && \forall j < J \\
A_{iJ} &= 0 && \forall I < i \\
B_{iJ} &= 0 && \forall J < i \leq I.
\end{aligned}
$$

Since $C_j = \sum_i B_{ij} A_i$, we get:

$$
\begin{aligned}
C_j &= \sum_{i \neq j} B_{ij} A_i + B_{jj} A_j = 0 + A_j = A_j, && (\forall j < J) \\
C_j &= \sum_{i} B_{ij} A_i = \sum_{i<j} B_{ij} A_i + \sum_{j \leq i} B_{ij} A_i = 0 + \sum_{j \leq i} B_{ij} e_i = B_j, && (\forall j > J) \\
C_J &= \sum_{i<J} B_{iJ} A_i + B_{JJ} A_J + \sum_{J<i\leq I} B_{iJ} A_i + \sum_{I<i} B_{iJ} A_i = 0 + A_J + 0 + \sum_{I<i} B_{iJ} e_i. && (j = J)
\end{aligned}
$$

Hence, each element of A and B goes to C directly in the same position.

Example 59 (Common Mistake:) *Do the elements of Es go directly to M_L and hence:*

$$\begin{aligned}
L^{-1}_{i,j} &= -L_{i,j}, && i > j\\
L^{-1}_{i,j} &= L_{i,j} = 1, && i = j\\
L^{-1}_{i,j} &= L_{i,j} = 0, && i < j.
\end{aligned}$$

Lemma 60 *If A has a row starting with zero, so does the same row in L; and when a column in A starts with zero, so does the same column in U*

Proof. If $A_{i1} = 0$, then $L_{i1} = 0$ is immediate from

$$0 = A_{i1} = \sum_k L_{ik} U_{k1} = L_{i1} U_{11} + \sum_{k>1} L_{ik} 0;$$

Also, it could be immediate from the fact that if a row in A has zero, it does not need elimination and hence the element of its E matrix will be zero. This saves computer time.
On the other hand, if $A_{1j} = 0$, then $U_{1j} = 0$ is immediate from

$$0 = A_{1j} = \sum_k L_{1k} U_{kj} = 1 U_{1j} + \sum_{k>1} 0 U_{kj},$$

which completes the proof. ■

2.7 Computational Issues:
Scientific Computing Environments (SCEs), Examples, and Complexity

2.7.1 On Scientific Computing Environments and Libraries

EISPACK : early 1970s, for solving symmetric, unsymmetric, and generalized eigenproblems.

LINPACK : late 1970s for solving linear equations and least squares problems.

BLAS (**B**asic **L**inear **A**lgebra **S**ubprograms): very efficiently preforming common linear algebra problem.

ATLAS (**A**utomatically **T**uned **L**inear **A**lgebra **S**oftware): BLAS implementation with higher performance.

LAPACK (**L**inear **A**lgebra **PACK**age): stands on EISPAC and LINPACK and heavily on BLAS (all written in Fortran) to make them run efficiently on shared-memory vector and parallel processors.

Matlab : is a commercial SW:

- late 1970s, written to access to EISPACK and LINPACK without learning Fortran.
- Then was written in C.
- Then, in 2000, rewritten to use LAPACK.

Mathematica : Commercial SW for symbolic (and numeric of course) mathematical computations.

R : free software environment for statistical computing and graphics.

Python : is a widely used high-level, general-purpose, interpreted, dynamic programming language

Sage : SageMath is a free open-source mathematics software system licensed under the GPL.

- It builds on top of many existing open-source packages: NumPy, SciPy, matplotlib, Sympy, Maxima, GAP, FLINT, R and many more.
- Access their combined power through a common, Python-based language or directly via interfaces or wrappers.
- Mission: Creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

- **Examples and Sage cheat sheet:**

2.7.2 Issues on Complexity*

To measure algorithm complexity we need to define a **step**; We adopt the definition of **FLOP** (Floating Point Operation) from the great and very mature reference for matrix computations (Golub and Van Loan, 1996, Sec. 1.2.4): $\square \times \square \quad + \quad \square$ **(almost the inner loop)**

Example 61 (LU factorization) *:*

***LU factorization steps* =**

$$
\begin{aligned}
&= (n)(n-1) + (n-1)(n-2) + \cdots + 2\cdot 1 \\
&= \sum_{i=1}^{n} (n-i+1)(n-i) \\
&= \sum_{i} \left(i^2 - (2n+1)i + n(n+1)\right) \\
&= \left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right) - (2n+1)\left(\frac{1}{2}n(n+1)\right) + n^2(n+1) \\
&= \frac{1}{3}n^3 - \frac{1}{3}n = O(n^3).
\end{aligned}
$$

***steps of side of* b =**

$$
\begin{aligned}
&= (n-1) + (n-2) + \cdots + 1 \\
&= \frac{1}{2}(n-1)n = \frac{1}{2}n^2 - \frac{1}{2}n = O(n^2).
\end{aligned}
$$

```
sage: var('i,j,k,n');
sage: sum((n-i+1)*(n-i),i,1,n)
1/3*n^3 - 1/3*n
```

Example 62 (Elaboration on Lemma 57 and looping over LT (or UT)) .

Multiplication $A_{m\times n}B_{n\times p}$, $C_{ij} = \sum_k A_{ik}B_{kj}$, mnp *(or* n^3*) steps:*

```
C = 0
for i=1:m
    for j=1:p
        for k=1:n
            C(i,j) = A(i,k)B(k,j) + C(i,j)
```

If both A, B *are LT:* $C_{ij} = 0,\ \forall i < j$, $C_{ij} = \sum_{k=j}^{i} A_{ik}B_{kj},\ \forall j \le i$

```
C = 0
for i=1:n
    for j=1:i // (to access the UT j=i:n)
        for k=j:i //B=0 for k<j, A=0 for i<k
            C(i,j) = A(i,k)B(k,j) + C(i,j)
```

no. of steps =

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{i} 1$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i} (i+1-j)$$

$$= \sum_{i=1}^{n}\Big((i+1)i - \frac{1}{2}i(i+1)\Big) = \frac{1}{2}\sum_{i=1}^{n}(i+i^2)$$

$$= \frac{1}{2}\Big(\frac{1}{2}n(n+1) + \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\Big)$$

$$= \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n.$$

```
sage: var('i,j,k,n');
sage: sum(sum(sum(1, k, j, i), j, 1, i), i, 1, n)
1/6*n^3 + 1/2*n^2 + 1/3*n
```

Example 63 (Matrix round off error and LU partial permutation) *(Golub and Van Loan, 1996, Sec. 3.3). Suppose the PC has a floating point arithmetic with $t = 3$ digits; what is the LU factorization/solution to:*

$$\begin{pmatrix} .001 & 1.00 \\ 1.00 & 2.00 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.00 \\ 3.00 \end{pmatrix}$$

Infinite precision solution (exact):

$$L = \begin{pmatrix} 1 & 0 \\ 1000 & 1 \end{pmatrix}, \; U = \begin{pmatrix} .001 & 1 \\ 0 & -998 \end{pmatrix}, \; LU = \begin{pmatrix} .001 & 1.00 \\ 1 & 2.00 \end{pmatrix} = A, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 500/499 \\ 997/998 \end{pmatrix} = \begin{pmatrix} 1.002004 \\ 0.998998 \end{pmatrix}$$

3-digit precision:

$$L = \begin{pmatrix} 1.00 & 0 \\ 1000 & 1.00 \end{pmatrix}, \; U = \begin{pmatrix} .001 & 1.00 \\ 0 & -1000 \end{pmatrix}, \; LU = \begin{pmatrix} .001 & 1.00 \\ 1.00 & 0.00 \end{pmatrix} = A, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 1.00 \end{pmatrix}$$

some calculation steps:

$$-1000 \times 1 + 2 = -1.00 \times 10^3 + 0.002 \times 10^3 = (-1.00 + 0.00) \times 10^{-3} = -1000.$$

$$1 \times c_1 = 1 \rightarrow c_1 = 1; \quad 1000c_1 + c_2 = 3 \rightarrow c_2 = -1000; \quad -1000x_2 = -1000 \rightarrow x_2 = 1; \quad .001x_1 + x_2 = 1 \rightarrow x_1 = 0.$$

3-digit precision with partial pivoting:

$$L = \begin{pmatrix} 1.00 & 0 \\ .001 & 1.00 \end{pmatrix}, \; U = \begin{pmatrix} 1.00 & 2.00 \\ 0 & 1.00 \end{pmatrix}, \; LU = \begin{pmatrix} 1 & 2.00 \\ .001 & 1.00 \end{pmatrix} = A, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.00 \\ .996 \end{pmatrix}$$

Chapter 3

Vector Spaces and Subspaces

3.1 Spaces of Vectors

Definition 64 (A Real Vector Space) *is a set $\mathcal{V}$ of vectors (each is $n-$tuple) over $\mathcal{R}$ with an addition and scalar multiplication on $\mathcal{V}$ such that:*

commutativity $u+v=v+u\in\mathcal{V} \quad \forall u,v\in\mathcal{V}$.

associativity $(u+v)+w=u+(v+w)\in\mathcal{V}$ *and* $(ab)v=a(bv)\in\mathcal{V} \quad \forall u,v,w\in\mathcal{V},\ a,b\in\mathcal{R}$.

additive identity $\exists\,\mathbf{0}\in\mathcal{V}$ *such that* $v+\mathbf{0}=v, \quad \forall v\in\mathcal{V}$.

additive inverse $\forall v\in\mathcal{V}\ \exists w\in\mathcal{V}$ *such that* $v+w=\mathbf{0}$. *(we may denote w by $-v$)*

multiplicative identity $1v=v \quad \forall v\in\mathcal{V}$.

distributive properties $a(u+v)=au+av\in\mathcal{V}$ *and* $(a+b)u=au+bu\in\mathcal{V} \quad \forall u,v\in\mathcal{V},a,b\in\mathcal{R}$.

Hint:

- *Informally: it is the set of vectors which all additions and scalars lay in the set as well.*
- *any linear combination lie in the subspace (from first and last identity)* ■

Example 65 ($\mathcal{R}^2=\{(x_1,x_2)|x_1,x_2\in\mathcal{R}\}$ vs. $\mathcal{V}=\{(x_1,x_2)|-a\le x_1,x_2\le a\}$ is NOT an example) .

Hint:

- We expanded from the visual $n = 3$ to general n.
- We can expand from the set R to any $\mathcal{F}$; the vector space will be defined then over this $\mathcal{F}$. ■
- $x = (x_1, x_2) \in \mathcal{R}^2$ is a point, vector, 2-tuple, element in $\mathcal{R}^2$.
- We can generalize for $\mathcal{R}^n$, or even $\mathcal{C}^n$, or polynomial, or others.
- The human brain cannot visualize or provide geometric models of $\mathcal{R}^n,\ n \geq 4$.
- Edwin A. Abbott, 1884, “Flatland: a romance of many dimensions”: can help creatures living in three-dimensional space, such as ourselves, imagine a physical space of four or more dimensions.
- However, we can do mathematics defined $\forall n$ which complies with geometry of $1 \leq n \leq 3$.

Example 66 (Many other spaces of) :

Real: $p = (1, 4, \sqrt(3), -1, 0) \in \mathcal{R}^5$

Complex: $p = (1 + i, -2i, -\sqrt(2) + 3i) \in \mathcal{C}^3$.

Polynomial: $p(z) = a_o + a_1 z + \cdots + a_m z^m$.

3.1.1 Properties of Vector Spaces (seems trivial for R but deep for others!)*

Proposition 67 *For ANY vector space satisfying definition 64 we have the following properties:*

1. *the additive identity is unique.*
2. *the additive inverse of every element is unique.*
3. $0v = \mathbf{0} \quad \forall v \in \mathcal{V}$.
4. $a\mathbf{0} = \mathbf{0} \quad \forall a \in F$.
5. $-1v \; \forall v \in \mathcal{V}$ *is the additive inverse of* v, $(-v)$.

The proof is very trivial:
Proof.

$$\mathbf{0}' = \underleftrightarrow{\mathbf{0}' + \mathbf{0}} = \mathbf{0}$$

$$w = w + \mathbf{0} = \underleftrightarrow{w + (v + w') = (w + v) + w'} = \mathbf{0} + w' = w'.$$

$$0v = (0+0)v = 0v + 0v \longrightarrow 0v = \mathbf{0}$$

$$a\mathbf{0} = a(\mathbf{0}+\mathbf{0}) = a\mathbf{0} + a\mathbf{0} \longrightarrow a\mathbf{0} = \mathbf{0}$$

$$v + (-1)v = (1)v + (-1)v = (1-1)v = 0v = \mathbf{0} \longrightarrow (-1)v \text{ is the additive inverse of } v$$

■

3.1.2 Subspaces

Definition 68 *A subset $\mathcal{U}$ of $\mathcal{V}$ is called a subspace of $\mathcal{V}$ if $\mathcal{U}$ is also a vector space (of course using the same addition and scalar as $\mathcal{V}$).*

Example 69 $\mathcal{U} = \left\{(x_1, x_2, 0) | x_1, x_2 \in \mathcal{R}\right\}$ *is a subspace of $\mathcal{R}^3$ since it satisfies all the properties of a space.* ■

Proposition 70 *For any space $\mathcal{V}$ and a subset $\mathcal{U} \subset \mathcal{V}$, $\mathcal{U}$ is a space (or a subspace) if the following hold:*

additive identity $\mathbf{0} \in \mathcal{U}$.

closed under addition $\forall u, v \in \mathcal{U},\ u + v \in \mathcal{U}$.

closed under scalar multiplication $\forall a \in \mathcal{R},\ au \in \mathcal{U}$.

Proof. The proof is obvious since other properties are satisfied immediately on the subset as long as they are satisfied on the whole set. ■

Corollary 71 *The smallest subspace over $\mathcal{R}^n$ is* $\mathbf{0}$.

Example 72 *Which of the following is a subspace (draw):*

- $\mathcal{U} = \left\{(x_1, x_2) | x_1 \in \mathcal{R}, x_2 = ax_1 + b,\ a \neq 0\right\}$, *compare it to* $\mathcal{R}^2$, *then set a condition to be a subspace of* $\mathcal{R}^2$.
  1. $(0,0) \in \mathcal{U} \longrightarrow (0, a \times 0 + b) \in \mathcal{U} \longrightarrow b = 0.$
  2. $(x_1, ax_1) + (x_2, ax_2) = ((x_1 + x_2), a(x_1 + x_2)) \in \mathcal{U}.$
  3. $k(x_1, ax_1) = ((kx_1), a(kx_1))$

- $\mathcal{U} = \left\{(x_1, x_2) | 0 \leq x_1, x_2\right\}.$

- $\mathcal{U} = \left\{(x_1, x_2) | x_1 \in \mathcal{R}, x_2 = ax_1^2,\ a \neq 0\right\}$
  1. $(0,0) \in \mathcal{U}.$
  2. $(x_1, ax_1^2) + (x_2, ax_2^2) = \left((x_1 + x_2), a(x_1^2 + x_2^2)\right) \neq \left((x_1 + x_2), a(x_1 + x_2)^2\right)$

3.1.3 The column space of the matrix A

Definition 73 (Column Space) *of a matrix $A_{m\times n}$, denoted by $\mathcal{C}(A)$, is the vector subspace of $\mathcal{R}^m$ (or probably the whole $\mathcal{R}^m$) consisting of all linear combinations of the matrix columns; i.e., Ax. Said differently:*

$$\mathcal{C}(A) = \left\{Ax \mid \forall x \in \mathcal{R}^n\right\}.$$

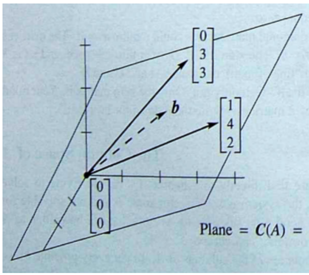
$\mathcal{C}(A)$ ***is the span of the columns of*** A

proof of $\mathcal{C}(A)$ is really a subspace. : $\mathcal{C}(A)$ is really a subspace since $\mathbf{0} \in \mathcal{C}(A)$ by choosing $x = 0$; $Ax_1 + Ax_2 = A(x_1 + x_2) \in \mathcal{C}(A)$; and $a(Ax_1) = A(ax_1) \in \mathcal{C}(A)$. ■

Remark 2 *This recalls the solution of $Ax = b$ exists? b must be in the column space of A.*

Example 74 *What is the column space of the matrix $A = \begin{pmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{pmatrix}$?*

It is the set $\mathcal{C}(A) = Ax = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} x_2 \quad \forall x_1, x_2$, which is actually a plane passing through zero.



Example 75 *Describe the column spaces of each of the following:*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

It is obvious that all are subspaces of $\mathcal{R}^2$ *(probably* $\mathcal{R}^2$ *itself).*

3.2 The Nullspace of A: Solving Ax = 0 and Rx = 0

It is natural to define the row space of a matrix analogously to the column space; **but nothing new!**.

Definition 76 (Row Space $\mathcal{R}(A) \subseteq \mathcal{R}^n$)

$$\begin{aligned}\mathcal{R}(A) &= \left\{x'A \mid \forall x \in \mathcal{R}^m\right\} = \left\{(A'x)' \mid \forall x \in \mathcal{R}^m\right\}\\ \mathcal{R}(A) &= \mathcal{C}(A'),\end{aligned}$$

with no distinction between x and x' (both are in $\mathcal{R}^m$).

Now: it is natural to define a space from ONLY x (not Ax or $x'A$), under some constraint.

Definition 77 (Null Space $\mathcal{N}(A) \subseteq \mathcal{R}^n$) *, and is constructed such that $\mathcal{N}(A) \perp \mathcal{R}(A)$.*

$$\mathcal{N}(A) = \left\{x \mid Ax = \mathbf{0},\ x \in \mathcal{R}^n\right\}.$$

Proof of $\mathcal{N}(A)$ is really a subspace.

$$\begin{aligned} x = \mathbf{0} &\longrightarrow A\mathbf{0} = \mathbf{0}\\ x_1, x_2 \in \mathcal{N}(A) &\longrightarrow \quad Ax_1 = Ax_2 = \mathbf{0} = Ax_1 + Ax_2 = A(x_1 + x_2)\\ x_1 \in \mathcal{N}(A) &\longrightarrow \quad Ax_1 = \mathbf{0} = aAx_1 = A(ax_1).\end{aligned}$$

■

Remark 3 .

1. *It is impossible to have a subspace of $\left\{x \mid Ax = b,\ x \in \mathcal{R}^n\right\}$ except for $b = 0$; why?*

2. *$Ax = 0$ means both: $x \in \mathcal{R}(A)$ and x is a zero linear combination in $\mathcal{C}(A)$.*

Example 78 *What is the null space of* $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. *It is of course the solution to* $Ax = 0$ *(by def.):*

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \equiv 1x_1 + 2x_2 = 0 \longrightarrow x = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \forall x_2 \in \mathcal{R}. \qquad (E_{21}(-3))$$

$x \perp (1, 2)'$ *ONLY. The null space of A is the set of vectors constituting the line.*

Example 79 *What is the null space of the matrix:* $x_1 + 2x_2 + 3x_3 = 0$. *Here:* $A = (1, 2, 3)$. *No pivot cancellation:*

$$x = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

The solution is the set of all linear combinations of these two $(2 = 3 - 1)$ *simple vectors;* ***A PLANE:*** *let's draw it.*

Example 80 *Suppose* $A =$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -7 \end{pmatrix} \longrightarrow x_2 = -7x_3, \quad x_1 = 11x_3 \longrightarrow x = x_3 \begin{pmatrix} 11 \\ -7 \\ 1 \end{pmatrix}.$$

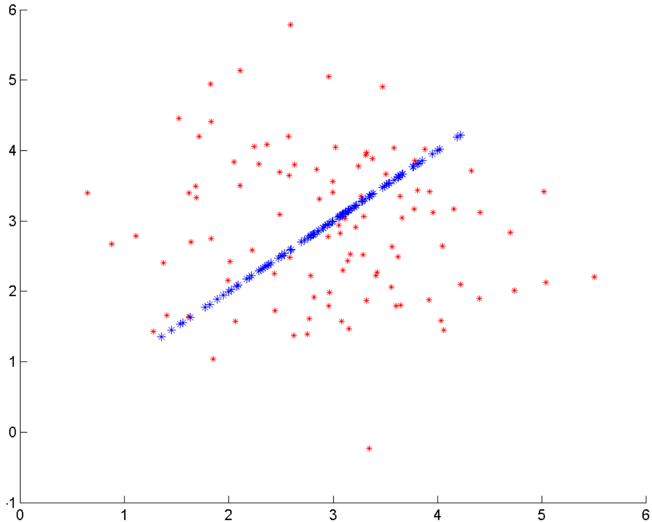
Much easier to continue from U *to the reduced echelon form* R*:*

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -11 \\ 0 & -1 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -11 \\ 0 & 1 & 7 \end{pmatrix}$$

So, the solution is the set of all linear combination of this single $(1 = 3 - 2)$ *vector;* ***A Line:*** *let's see Sage.*

Example 81 (Motivation from data science) :

- *Data reduction and compression.*
- *Data Interpretation.*
- *Data modeling and prediction.*



3.2.1 Systematic solution using pivot columns, free columns, and reduced echelon form

Example 82 *Find $\mathcal{N}(A)$:*

$$\begin{pmatrix}1&2&2&4\\3&8&6&22\end{pmatrix}\longrightarrow\begin{pmatrix}1&2&2&4\\0&2&0&10\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&2&-6\\0&2&0&10\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&2&-6\\0&1&0&5\end{pmatrix}.$$

In reduced echelon form, we get r pivot variables p and $n-r$ free variables f; in the form of $p=-\sum\alpha f$:

$$x_1=-2x_3+6x_4$$

$$x_2=0x_3-5x_4$$

$$x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=x_3\begin{pmatrix}-2\\0\\1\\0\end{pmatrix}+x_4\begin{pmatrix}6\\-5\\0\\1\end{pmatrix}$$

Example 83 *After pivot cancellation of A:*

$$\begin{pmatrix}1&0&0&a&c\\0&1&0&b&d\\0&0&1&0&e\\0&0&0&0&0\end{pmatrix}\longrightarrow\quad x=x_4\begin{pmatrix}-a\\-b\\0\\1\\0\end{pmatrix}+x_5\begin{pmatrix}-c\\-d\\-e\\0\\1\end{pmatrix}\qquad\begin{pmatrix}1&0&a&0&c\\0&1&b&0&d\\0&0&0&1&e\\0&0&0&0&0\end{pmatrix}\longrightarrow\quad x=x_3\begin{pmatrix}-a\\-b\\1\\0\\0\end{pmatrix}+x_5\begin{pmatrix}-c\\-d\\0\\-e\\1\end{pmatrix}$$

3.2.2 Gauss Elimination Algorithm: revisited and detailed

```
i = 0; j = 0; //previous pivot location
while( (i<m) && (j<n) ){
    i++; j++;
    do{
        I = argmax( |A(k,j)|, i<=k<=m);
        j+= !A(I,j);
    }while( !A(I,j) && (j<=n) );

    if(A(I,j)){//pivot or reached boundary
        Swap(R_i, R_I);
        PivotElimination(i,j);
    }
}
```

$$\begin{pmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix} \quad \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \quad \begin{pmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

$$\begin{pmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix} \quad \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \quad \begin{pmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

Corollary 84 (Gauss elimination algorithm) .

For $A_{m\times n}$ that produces r pivots:

1. $R_{ij} = 0,\ \forall i > I,\ j < J,\ R(I,J)$ *is a pivot.*
2. *the number of pivots r, number of column pivots, and number of row pivots are all equal.*
3. $r \le m, n \quad \equiv \quad r \le min(m,n)$.
4. *the $m - r$ non-pivot rows are all zeros and are deferred to the end of R.*
5. *the $n - r$ non-pivot columns have zeros under the previous pivot.*

Proof. It is trivial and is already a bi-product from the construction of elimination!

■

Lemma 85 *In pivot cancellation, a column will have no pivots if and only if it is a linear combination from preceding columns. A row will be zero if and only if it is a linear combination of preceding rows.*

Proof.
For columns:

$$\begin{pmatrix} 1 & \mathbf{0} \\ -A_1/a_{11} & \mathbf{I}. \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & \alpha a_{11} + \beta a_{21} & \cdots \\ A_1 & A_2 & \cdots & \alpha A_1 + \beta A_2 & \cdots \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & \cdots & \alpha a_{11} + \beta a_{21} & \cdots \\ \mathbf{0} & A_2 - A_1 a_{21}/a_{11} & \cdots & \beta(A_2 - A_1 a_{21}/a_{11}) & \cdots \end{pmatrix}$$

second pivot cancellation will not provide pivots in the linear combination column.

For rows:

$$\begin{pmatrix} a_{11} & R_1 \\ a_{21} & R_2 \\ \vdots & \vdots \\ \alpha a_{11} + \beta a_{21} & \alpha R_1 + \beta R_2 \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & R_1 \\ 0 & R_2 - R_1(a_{21}/a_{11}) \\ \vdots & \vdots \\ 0 & (\alpha R_1 + \beta R_2) - R_1\big(\alpha a_{11} + \beta a_{21}\big)/a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & R_1 \\ 0 & R_2 - R_1(a_{21}/a_{11}) \\ \vdots & \vdots \\ 0 & \beta\big(R_2 - R_1(a_{21}/a_{11})\big) \end{pmatrix}$$

■

Definition 86 (Rank of a matrix) *is defined as the number of its pivots r.*
Later, an equivalent definition is provided and we will show that r is the number of independent columns, independent rows, etc. ■

Lemma 87 *After Gauss elimination of A to produce the echelon (or reduced echelon) form R:*

1. *All pivot columns of R are linearly independent; their corresponding columns of A are linearly independent as well.*
2. *All non-pivot columns of R are linear combination of preceding columns; same apply for matrix A.*
3. *All pivot rows of R are linearly independent. their corresponding rows of A are linearly independent as well.*
4. *All non-pivot rows of R (the zero rows) are linear combination from the pivot rows; same apply for matrix A*

Proof. We arrange

1. We assume that $\exists\ \alpha$, a linear combination of pivot columns, such that $R\alpha = \mathbf{0}$

$$R\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_r\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix} = \mathbf{0} \quad \rightarrow \alpha \in \mathcal{N}(R) \quad \rightarrow \alpha = x_{r+1}\begin{pmatrix}*\\ \vdots\\ *\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix} + x_{r+2}\begin{pmatrix}*\\ \vdots\\ *\\ 0\\ 1\\ \vdots\\ 0\end{pmatrix} + \cdots + x_n\begin{pmatrix}*\\ \vdots\\ *\\ 0\\ 0\\ \vdots\\ 1\end{pmatrix} \quad \rightarrow x_i = 0,\ r+1 \le i \le n.$$

   which means $\alpha = \mathbf{0}$; a contradiction.

2. Since we have just proven that $\alpha \notin \mathcal{N}(A)$; then $A\alpha \neq \mathbf{0}$. Therefore, the corresponding columns of A are linearly independent as well.

■

Remark 4 *:*

- $\mathcal{N}(A) = \mathcal{N}(U) = \mathcal{N}(R)$ *of course, since pivot cancellation will not change the* $\mathbf{0}$ *vector at the R.H.S.*

- $\mathcal{C}(A) \neq \mathcal{C}(U) \neq \mathcal{C}(R)$*; simply:*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \mathcal{C}(A) = a \begin{pmatrix} 1 \\ 2. \end{pmatrix} \qquad , R = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \ \mathcal{C}(R) = a \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

  We will come later to how to find exactly $\mathcal{C}(A)$ *and* $\mathcal{R}(\mathcal{A})$.

- *The number of vectors in* $\mathcal{N}(A)$ *is itself the number of linear combinations of columns of* A *that gives* $\mathbf{0}$.

3.3 3.3 The Complete Solution to $\mathbf{Ax} = \mathbf{b}$

3.4 3.4 Independence, Basis and Dimension

3.5 3.5 Dimensions of the Four Subspaces

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