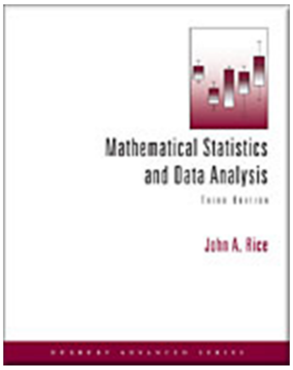
ST121:
Probability and Statistics I

Solutions to Selected Problems
&
Some Extra Materials For:



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Chapter 1

Probability

Problem 34

Prove the following identity

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m-n}{n-k} = \binom{m}{n}.$$

Proof.

$$\begin{aligned}
(x+1)^m &= (1+x)^n (x+1)^{m-n} \\
\binom{m}{n}x^n &= 1 \cdot \binom{m-n}{n}x^n + \binom{n}{1}x^1\binom{m-n}{n-1}x^{n-1} + \cdots \\
&= \sum_{k=0}^{n}\binom{n}{k}\binom{m-n}{n-k}.
\end{aligned}$$

Or, it is easier to notice that

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}\binom{m-n}{n-k} &= \sum_{k=0}^{n}\binom{n}{n-k}\binom{m-n}{n-k} \\
&= \sum_{i=n}^{0}\binom{n}{i}\binom{m-n}{i} \\
&= \sum_{i=0}^{n}\binom{n}{i}\binom{m-n}{i},
\end{aligned}$$

Then

$$\begin{aligned}
(x+1)^m &= (x+1)^n (x+1)^{m-n} \\
\binom{m}{n}x^n &= \binom{n}{0}x^n\binom{m-n}{0}x^0 + \binom{n}{1}x^{n-1}\binom{m-n}{1}x^1 + \cdots \\
\binom{m}{n} &= \sum_{i=0}^{n}\binom{n}{i}\binom{m-n}{i}.
\end{aligned}$$

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Chapter 2

Random Variables

Negative Binomial: $NBinomial(r, p)$

Prove that

$$\sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r} = 1$$

Proof.

$$\sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r} = \sum_{k'=0}^{\infty} \binom{k'+r-1}{k'} p^r (1-p)^{k'}.$$

From Taylor series

$$\frac{1}{(1-x)^r} = \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} x^r.$$

Substituting back with $x = 1-p$, the result is immediate. ■

Hypergeometric: $Hypergeometric(n, r, m)$

Prove that

$$\sum_{k=0}^{m} \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}} = 1$$

Proof.

$$\begin{aligned}
(x+1)^n &= (1+x)^r (x+1)^{n-r} \\
\binom{n}{m} x^m &= \binom{r}{m} x^m \cdot \binom{n-r}{0} x^0 + \binom{r}{m-1} x^{m-1} \binom{n-r}{1} x^1 + \cdots \\
&= \sum_{k=0}^{m} \binom{r}{m-k} \binom{n-r}{k} x^m \\
&= \sum_{k'=0}^{m} \binom{r}{k'} \binom{n-r}{k'-m} x^m.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{m} P(X=k) &= \sum_{k=0}^{m} \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}} \\
&= 1.
\end{aligned}$$

This is more general than Problem 34 in Ch.1, since in the one above we expanded in terms of the general term x^m, where $0 \le m \le r$, not necessarily $m = r$ as in Problem 34. ■

Chapter 3

Joint Distributions

Integrating pdf over an area.

Rigorously, the cdf is defined first then the pdf is defined as the derivative of the cdf. Then, prove that

$$P(X \in A) = \int_A f_X(x)\ dx$$

Proof. It can be shown that an area A can be represented as a union of disjoint rectangles

$$\begin{aligned} A &= \bigcup_{i=1}^{\infty} R_i, \\ P(X \in A) &= P\left(X \in \bigcup_{i=1}^{\infty} R_i\right) \\ &= \sum_i P(X \in R_i), \end{aligned}$$

Each probability $P(X \in R_i)$ can be expressed as summation of CDFs (each is an integration over the pdf by definition) to cover the whole rectangle as

$$P\left(x_1 < X \le x_2, y_1 < Y \le y_2\right) = F\left(x_2, y_2\right) - F\left(x_2, y_1\right) - F\left(x_1, y_2\right) + F\left(x_1, y_1\right).$$

Then

$$\begin{aligned} P(X \in A) &= \sum_i \int_{R_i} f_X(x)\ dx \\ &= \int_A f_X(x)\ dx \end{aligned}$$

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Probability of independent r.v.

Prove that

$$P(X_1 \in A, X_2 \in B) = P(X_1 \in A)\, P(X_2 \in B)$$

Proof (using cdf):. Since each p-dimensional area (even in one dimension) can be represented as a union of rectangles, then

$$\begin{aligned} \{X_1 \in A, X_2 \in B\} &= \left\{\left\{X_1 \in \bigcup_i A_i\right\} \cup \left\{X_2 \in \bigcup_j B_j\right\}\right\} \\ &= \left\{\bigcup_i \bigcup_j \{X_1 \in A_i \cup X_2 \in B_j\}\right\} \\ P\{X_1 \in A, X_2 \in B\} &= \sum_i \sum_j P\left(X_1 \in A_i, X_2 \in B_j\right) \end{aligned}$$

For 2 independent r.v., a rectangle has a probability

$$\begin{aligned}
P\left(\{X \in R\}\right) &= P\left(x_1 < X \le x_2, y_1 < Y \le y_2\right) \\
&= F\left(x_2, y_2\right) - F\left(x_2, y_1\right) - F\left(x_1, y_2\right) + F\left(x_1, y_1\right) \\
&= F_X\left(x_2\right) F_Y\left(y_2\right) - F_X\left(x_2\right) F_Y\left(y_1\right) - F_X\left(x_1\right) F_Y\left(y_2\right) + F_X\left(x_1\right) F_Y\left(y_1\right) \\
&= \left(F_X\left(x_2\right) - F_X\left(x_1\right)\right)\left(F_Y\left(y_2\right) - F_Y\left(y_1\right)\right) \\
&= P\left(x_1 < X \le x_2\right) P\left(y_1 < Y \le y_2\right)
\end{aligned}$$

Substituting above

$$\begin{aligned}
P\left\{X_1 \in A, X_2 \in B\right\} &= \sum_i \sum_j P\left(X_1 \in A_i, X_2 \in B_j\right) \\
&= \sum_i P\left(x_{1_i} < X \le x_{2_i}\right) \sum_j P\left(y_{1_j} < Y \le y_{2_j}\right) \\
&= P\left(X \in A\right) P\left(X \in B\right)
\end{aligned}$$

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Proof (using pdf):.

$$\begin{aligned}
P\left(X_1 \in A, X_2 \in B\right) &= \int_B \int_A f_{X_1 X_2}\left(x_1, x_2\right)\ dx_1\ dx_2 \\
&= \int_B \int_A f_{X_1}\left(x_1\right) f_{X_2}\left(x_2\right) dx_1\ dx_2 \\
&= \int_A f_{X_1}\left(x_1\right) dx_1 \int_B f_{X_2}\left(x_2\right)\ dx_2 \\
&= P\left(X_1 \in A\right) P\left(X_2 \in B\right)
\end{aligned}$$

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Bibliography