

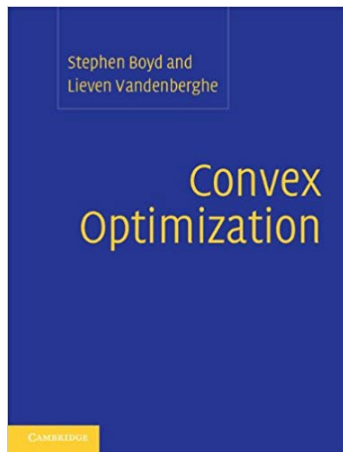
**CS495**  
**Optimiztaion**

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Lectures follow:

Boyd and Vandenberghe (2004)



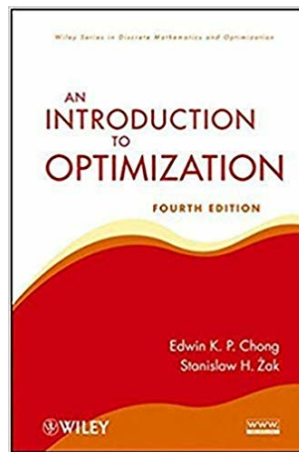
Boyd, S., & Vandenberghe, L. (2004). Convex Optimization. Cambridge: Cambridge University Press.

Book and Stanford course:

<http://web.stanford.edu/~boyd/cvxbook/>

Some examples from:

Chong and Zak (2001)



Chong, E. K., & Zak, S. (2001). An introduction to optimization: Wiley-Interscience.

# Course Objectives

- Developing rigorous mathematical treatment for mathematical optimization.
- Building intuition, in particular to practical problems.
- Developing computer practice to using optimization SW.

## Prerequisites

Calculus (both single and multivariable) and Linear Algebra.

# **Chapter 1:**

## **Introduction**

### **Snapshot on Optimization**

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## **Bibliography**



# **Chapter 1**

## **Introduction**

# 1.1 Mathematical Optimization

**Definition 1** A mathematical optimization problem or just optimization problem, has the form (Boyd and Vandenberghe, 2004):

$$\begin{aligned} &\underset{x}{\text{minimize}} && f_0(x) \\ &\text{subject to:} && f_i(x) \leq 0, && i = 1, \dots, m \\ &&& h_i(x) = 0, && i = 1, \dots, p, \end{aligned}$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$ , (optimization variable)

$f_0: \mathbf{R}^n \mapsto \mathbf{R}$ , (objective (cost/utility) function)

$f_i: \mathbf{R}^n \mapsto \mathbf{R}$ , (inequality constraints (functions))

$h_i: \mathbf{R}^n \mapsto \mathbf{R}$ , (equality constraints (functions))

$$\mathcal{D}: \bigcap_{i=1}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i \quad (\text{feasible set})$$

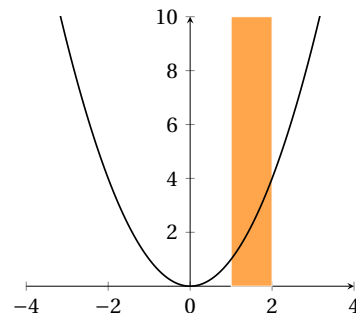
$$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \leq 0 \wedge h_i(x) = 0\}$$

$$x^*: \{x \mid x \in \mathcal{D} \wedge f_0(x) \leq f_0(z) \forall z \in \mathcal{D}\} \quad (\text{solution})$$

- minimize  $f_0 \equiv \text{maximize } -f_0$ .
- $f_i \leq 0 \equiv -f_i \geq 0$ .
- 0s can be replaced of course by constants  $b_i, c_i$
- unconstrained problem when  $m = p = 0$ .

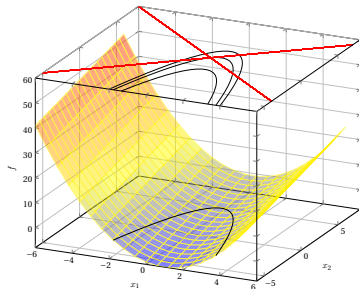
**Example 2 :**

$$\begin{aligned} &\underset{x}{\text{minimize}} && x^2 \\ &\text{subject to:} && x \leq 2 \wedge x \geq 1. \end{aligned}$$



$$x^* = 1.$$

If the constraints are relaxed, then  $x^* = 0$ .



$$\begin{aligned} &\underset{x}{\text{minimize}} && f_0(x) \\ &\text{subject to:} && f_i(x) \leq 0, && i = 1, \dots, m \\ &&& h_i(x) = 0, && i = 1, \dots, p, \end{aligned}$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$ , (optimization variable)

$f_0: \mathbf{R}^n \mapsto \mathbf{R}$ , (objective (cost/utility) function)

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$$\mathcal{D}: \bigcap_{i=1}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i \quad (\text{feasible set})$$

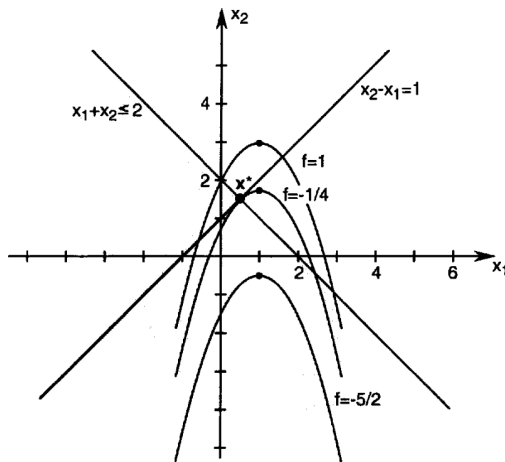
$$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \leq 0 \wedge h_i(x) = 0\}$$

$$x^*: \{x \mid x \in \mathcal{D} \wedge f_0(x) \leq f_0(z) \forall z \in \mathcal{D}\} \quad (\text{solution})$$

**Example 3** (*Chong and Zak, 2001, Ex. 20.1, P. 454*):

$$\begin{aligned} &\underset{x}{\text{minimize}} && (x_1 - 1)^2 + x_2 - 2 \\ &\text{subject to:} && x_2 - x_1 = 1 \\ &&& x_1 + x_2 \leq 2. \end{aligned}$$

No global minimizer:  $\partial z / \partial x_2 = 1 \neq 0$ . However,  $z|_{(x_2-x_1=1)} = (x_1 - 1)^2 + (x_1 - 1)$ , which attains a minimum at  $x_1 = 1/2$ .



$x^* = (1/2, 3/2)'$ . (Let's see animation)

### 1.1.1 Motivation and Applications

- *optimization problem* is an abstraction of how to make “best” possible choice of  $x \in \mathbf{R}^n$ .
- *constrains* represent trim requirements or specifications that limit the possible choices.
- *objective function* represents the *cost* to minimize or the *utility* to maximize for each  $x$ .

#### Examples:

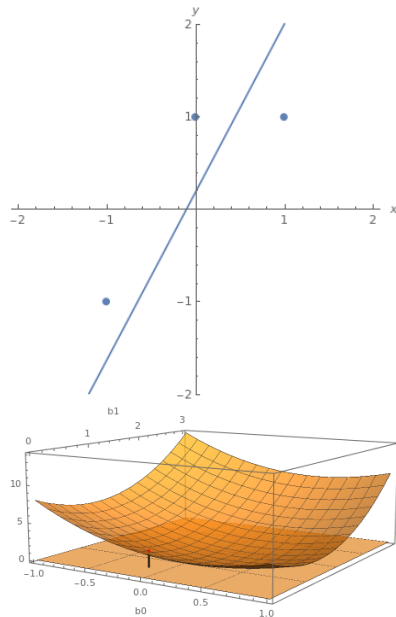
	<i>Any problem</i>	<i>Portfolio Optimization</i>	<i>Device Sizing</i>	<i>Data Science</i>
$x \in \mathbf{R}^n$	choice made	investment in capitals	dimensions	parameters
$f_i, h_i$	firm requirements /conditions	overall budget	engineering constraints	regularizer
$f_0$	cost (or utility)	overall risk	power consumption	error

- Amazing variety of practical problems. In particular, data science: two sub-fields: construction and assessment.
- The construction of: Least Mean Square (LMS), Logistic Regression (LR), Support Vector Machines (SVM), Neural Networks(NN), Deep Neural Networks (DNN), etc.
- Many techniques are for solving the optimization problem:
  - Closed form solutions: convex optimization problems
  - Numerical solutions: Newton’s methods, Gradient methods, Gradient descent, etc.
  - “Intelligent” methods: particle swarm optimization, genetic algorithms, etc.

#### Example 4 (Machine Learning: construction) :

Let's suppose that the best regression function is  $Y = \beta_0 + \beta_1 X$ , then for the training dataset  $(x_i, y_i)$  we need to minimize the MSE.

$$\underset{\beta_0, \beta_1}{\text{minimize}} \sum_i (\beta_0 + \beta_1 x_i - y_i)^2$$



- Half of ML field is construction: NN, SVM, etc.
- In DNN it is an optimization problem of millions of parameters.
- Let's see animation.
- Where are Probability, Statistics, and Linear Algebra here? Let's re-visit the chart.
- Is the optimization problem solvable:
  - closed form? (LSM)
  - numerically and guaranteed? (convex and linear)
  - numerically but not guaranteed? (non-convex):
    - \* numerical algorithms, e.g., GD,
    - \* local optimization,
    - \* heuristics, swarm, and genetics,
    - \* brute-force with exhaustive search

### 1.1.2 Solving Optimization Problems

- A *solution method* for a class of optimization problems is an algorithm that computes a solution.
- Even when the *objective function* and constraints are smooth, e.g., polynomials, the solution is very difficult.
- There are three classes where solutions exist, theory is very well developed, and amazingly found in many practical problems:

Linear  $\subset$  Quadratic  $\subset$  Convex  $\subset$  Non-linear (not linear and not known to be convex!)

- For the first three classes, the problem can be solved very reliably in hundreds or thousands of variables!

## 1.2 Least-Squares and Linear Programming

### 1.2.1 Least-Squares Problems

A *least-squares* problem is an optimization problem with no constraints (i.e.,  $m = p = 0$ ), and an objective in the form:

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k (a'_i x - b_i)^2 = \|A_{k \times n} x_{n \times 1} - b_{k \times 1}\|^2.$$

The solution is given in **closed form** by:

$$x = (A' A)^{-1} A' b$$

- Good algorithms in many SC SW exist; it is a very mature technology.
- Solution time is  $O(n^2 k)$ .
- Easily solvable even for hundreds or thousands of variables.
- More on that in the Linear Algebra course.
- Many other problems reduce to typical LS problem:
  - Weighted LS (to emphasize some observations)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k w_i (a'_i x - b_i)^2.$$

- Regularization (to penalize for over-fitting)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k (a'_i x - b_i)^2 + \rho \sum_{j=1}^n x_j^2.$$

## 1.2.2 Linear Programming

A *linear programming* problem is an optimization problem with objective and all constraint functions are linear:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) = C'x \\ \text{subject to:} & a'_i x \leq b_i, \quad i = 1, \dots, m \\ & h'_i x = g_i, \quad i = 1, \dots, p, \end{array}$$

- **No** closed form solution as opposed to LS.
- Very robust, reliable, and effective set of methods for numerical solution; e.g., Dantzig's simplex, and interior point.
- Complexity is  $\simeq O(n^2 m)$ .
- Similar to LS, we can solve a problem of thousands of variables.
- Example is *Chebyshev minimization* problem:

$$\underset{x}{\text{minimize}} f_0(x) = \max_{i=1, \dots, k} |a'_i x - b_i|,$$

- The objective is different from the LS: minimize the maximum error. **Ex:**
- After some tricks, requiring familiarity with optimization, it is equivalent to a LP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & t \\ \text{subject to:} & a'_i x - t \leq b_i, \quad i = 1, \dots, k \\ & -a'_i x - t \leq -b_i, \quad i = 1, \dots, k \end{array}$$



## 1.3 Convex Optimization

A *convex optimization* problem is an optimization problem with objective and all constraint function are convex:

$$\begin{array}{llll} \underset{x}{\text{minimize}} & f_0(x) & & \\ \text{subject to:} & f_i(x) \leq 0, & i = 1, \dots, m & \\ & h_i(x) = 0, & i = 1, \dots, p, & \\ & f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y), & \alpha + \beta = 1, & 0 \leq \alpha, 0 \leq \beta, \quad 0 \leq i \leq m \\ & h_i(x) = a'_i x + b_i & & 0 \leq i \leq p \end{array}$$

- The LP and LS are special cases; however, only LS has closed-form solution.
- Very robust, reliable, and effective set of methods, including *interior point methods*.
- Complexity is almost:  $O(\max(n^3, n^2 m, F))$ , where  $F$  is the cost of evaluating 1st and 2nd derivatives of  $f_i$  and  $h_i$ .
- Similar to LS and LP, we can solve a problem of thousands of variables.
- However, it is not as very mature technology as the LP and LS yet.
- There are many practical problems that can be re-formulated as convex problem **BUT** requires mathematical skills; but once done the problem is solved. **Hint:** realizing that the problem is convex requires more mathematical maturity than those required for LP and LS.

## 1.4 Nonlinear Optimization

A *non-linear optimization* problem is an optimization problem with objective and constraint functions are non-linear **BUT** not known to be convex (**so far**). Even simple-looking problems in 10 variables can be extremely challenging. Several approaches for solutions:

**Local Optimization** : starting at initial point in space, using differentiability, then navigate

- does not guarantee global optimal.
- affected heavily by initial point.
- depends heavily on numerical algorithm and their parameters.
- More art than technology.
- In contrast to convex optimization, where a lot of art and mathematical skills are required to formulate the problem as convex; then numerical solution is straightforward.

**Global Optimization** : the true global solution is found; the compromise is complexity.

- The complexity goes exponential with dimensions.
- Sometimes it is worth it when: the cost is huge, not in real time, and dimensionality is low.

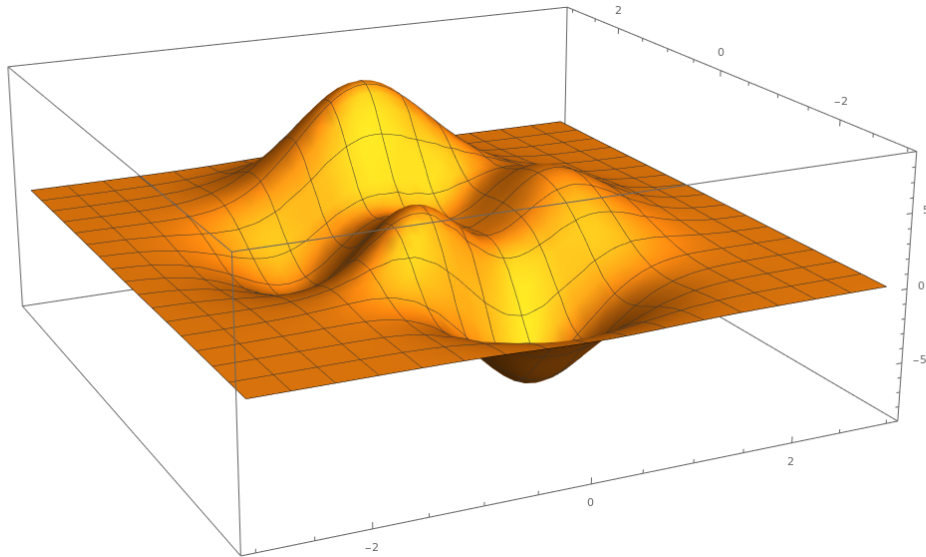
**Role of Convex Optimization** :

- Approximate the non-linear function to a convex one, finding the exact solution, then using it as a starting point for the original problem. (Also does not guarantee optimality)
- Setting bounds on the global solution.

**Evolutionary Computations** : Genetic Algorithm (GA), Simulated Annealing (SA), Particle Swarm Optimization (PSO), etc.

**Example 5 (Nonlinear Objective Function)** : (*Chong and Zak, 2001, Ex. 14.3*)

$$f(x, y) = 3(1 - x)^2 e^{-x^2 - (y+1)^2} - 10e^{-x^2 - y^2} \left( -x^3 + \frac{x}{5} - y^5 \right) - \frac{1}{3} e^{-(x+1)^2 - y^2}$$



# **Part I**

# **Theory**

# **Part II**

# **Applications**

# **Part III**

# **Algorithms**

# Bibliography

Boyd, S. and Vandenberghe, L. (2004), *Convex Optimization*, Cambridge: Cambridge University Press.

Chong, E. K. and Zak, Stanislaw, H. (2001), *An Introduction to Optimization*, Wiley-Interscience, 4th ed.