Image Processing

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Ch. 4

Inner product and orthogonalization

For vectors in p dimensional space, the inner product $\langle x, y\rangle$ is the dot product $x'y$,

$$\begin{aligned}\langle x, y\rangle &= x'y \\ &= \begin{pmatrix} x_1 & \dots & x_p \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \\ &= \sum_{i=1}^{p} x_i y_i.\end{aligned}$$

when $x'y$ is zero we say they are orthogonal. It can be shown that, for $p \leq 3$,

$$\begin{aligned}\cos\theta &= \frac{x'y}{\|x\| \cdot \|y\|} \\ &= \frac{x'y}{\sqrt{x'x} \cdot \sqrt{y'y}}\end{aligned}$$

Then, we can generalize this definition in higher dimensions, and define the angle between two vectors for $p > 3$.

Simple example (a single projection): $\mathbf{X}_{p\times 1} = X$
if $X' = (1, 0, \dots, 0)$, we simply get the first component Y_1.

if $X' = (1/\sqrt{2}, 1/\sqrt{2})$, we project on the $\pi/4$ direction; this will be the value of the new vector in the direction of X.

$$
\begin{aligned}
X'Y &= \|X\| \, \|Y\| \cos\left(Y, X\right) \\
&= \|X\| \times \text{ Projected Length}
\end{aligned}
$$

If we need the projected length only, then project on a unit vector $\frac{X'}{\|X\|}$, then

$$
\frac{X'}{\|X\|} Y = \|Y\| \cos\left(Y, X\right).
$$

Multiply this scalar in the direction of the projection to get the new component in the direction X

$$
\begin{aligned}
\widehat{Y} &= \left(\frac{X}{\|X\|}\right)\left(\frac{X'}{\|X\|} Y\right) \\
&= X\widehat{\beta} \\
&= \frac{XX'}{(X'X)} Y \\
&= P^{(X)}_{p\times p} Y_{p\times 1},
\end{aligned}
$$

where we call P the projection matrix of the direction X.

For decomposition (not projection yet) on set of vectors constituting the columns of $\mathbf{X} = (X_1, \ldots, X_n)$,

$$
\begin{aligned}
&= X_1\widehat{\beta}_1 + \ldots + X_n\widehat{\beta}_n \\
&= \mathbf{X}\widehat{\beta}
\end{aligned}
$$

We need to minimize the remaining error

$$
\begin{aligned}
e &= Y - \mathbf{X}\widehat{\beta}, \\
\|e\|^2 &= \left\langle \left(Y - \mathbf{X}\widehat{\beta}\right), \left(Y - \mathbf{X}\widehat{\beta}\right)\right\rangle \\
&= \langle Y, Y\rangle - 2\beta' \begin{pmatrix} \langle X_1, Y\rangle \\ \vdots \\ \langle X_n, Y\rangle \end{pmatrix} + \beta' \begin{pmatrix} \langle X_1, X_1\rangle & \dots & \langle X_1, X_n\rangle \\ \vdots & \ddots & \vdots \\ \langle X_n, X_1\rangle & \dots & \langle X_n, X_n\rangle \end{pmatrix} \beta \\
&= Y'Y - 2\beta'\mathbf{X}'Y + \beta'\mathbf{X}'\mathbf{X}\beta
\end{aligned}
$$

$$
\begin{aligned}
\nabla \|e\|^2 &= -2\mathbf{X}'\mathbf{y} + \mathbf{2X}'X\beta \\
&= -2\mathbf{X}'\left(\mathbf{y} - \mathbf{X}\beta\right),
\end{aligned}
$$

which means that the error is perpendicular to each component X_i; i.e.,

$$
\begin{aligned}
\langle X_i, e\rangle &= 0, \text{ or} \\
X_i'e &= 0
\end{aligned}
$$

The solution will be

$$
\widehat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'Y
$$

Therefore, the projection $\widehat{Y}$ is

$$
\begin{aligned}
\widehat{Y} &= \mathbf{X}\widehat{\beta} \\
&= \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'Y
\end{aligned}
$$

and the projection matrix $\mathbf{P}$ is

$$\mathbf{P} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'$$

The very interesting thing is

$$\begin{aligned}
\mathbf{X}'\mathbf{X} &= \begin{pmatrix} X_1' \\ \vdots \\ X_n' \end{pmatrix} \begin{pmatrix} X_1 & \dots & X_n \end{pmatrix} \\
&= \begin{pmatrix} X_1'X_1 & X_1'X_2 & \dots & X_1'X_n \\ X_2'X_1 & X_2'X_2 & \dots & X_2'X_n \\ \vdots & \vdots & \vdots & \vdots \\ X_n'X_1 & X_n'X_2 & \dots & X_n'X_n \end{pmatrix}
\end{aligned}$$

if we choose the basis X_i orthogonal

$$\begin{aligned}
\mathbf{X}'\mathbf{X} &= \begin{pmatrix} X_1'X_1 & & & \\ & X_2'X_2 & & \\ & & \ddots & \\ & & & X_n'X_n \end{pmatrix} \\
&= \operatorname{diag}\left(\|X_1\|^2, \dots, \|X_n\|^2\right),
\end{aligned}$$

$$
\begin{aligned}
\mathbf{P} &= \begin{pmatrix} X_1 & \dots & X_n \end{pmatrix} \begin{pmatrix} \frac{1}{\|X_1\|^2} & & \\ & \ddots & \\ & & \frac{1}{\|X_n\|^2} \end{pmatrix} \begin{pmatrix} X_1' \\ \vdots \\ X_n' \end{pmatrix} \\
&= \frac{X_1 X_1'}{X_1' X_1} + \dots + \frac{X_n X_n'}{X_n' X_n} \\
&= \frac{X_1}{\|X_1\|} \frac{X_1'}{\|X_1\|} + \dots + \frac{X_n}{\|X_n\|} \frac{X_n'}{\|X_n\|} \\
&= \mathbf{P}_1 + \dots + \mathbf{P}_n
\end{aligned}
$$

The projection will be

$$\widehat{Y} = \mathbf{P}_1 Y + \dots \mathbf{P}_n Y$$

The orthogonality of the basis made it possible to project on each and sum up the projections. If the set of basis X_i span the whole space, then they are complete and $\widehat{Y} = Y$ and the error is zero; which means we can express Y in terms of the new subspace $\mathbf{X}$. And for **orthonormal** basis, where $\|X_i\| = 1$

$$
\begin{aligned}
\mathbf{P}_i &= X_i X_i', \\
\widehat{\beta}_i &= X_i' Y
\end{aligned}
$$

Function Spaces: nothing new from vectors. A function is a vector with infinite dimension

instead of vector$Y$, we have the function $f(x)$, and for two basis functions $X_1(x), X_2(x)$, there inner product is defined by

$$
\begin{aligned}
\langle X_1, X_2 \rangle &= \int_{-\infty}^{\infty} X_1(x)\, X_2(x)\, dx., \\
\|X_i\|^2 &= \langle X_i, X_i \rangle
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\widehat{f}(x) &= X_1\widehat{\beta}_1 + \ldots + X_n\widehat{\beta}_n \\
&= \mathbf{X}\widehat{\beta} \\
&= \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'f \\
e &= f - \widehat{f}.
\end{aligned}
$$

for a set of orthogonal functions

$$
\begin{aligned}
\widehat{\beta}_i &= \langle \frac{X_i}{\langle X_i, X_i \rangle}, f \rangle \\
&= \int X_i(x)\, f(x)\, dx, \\
\widehat{f} &= \sum_i \widehat{\beta}_i X_i(x)
\end{aligned}
$$

Fourier expansion

choose the set X_i to be the trigonometric functions in the interval $[0, 2\pi]$, and approximate f by $\widehat{f}$ as

$$
\widehat{f} = \widehat{\beta}_0 + \sum_{n=1}^{N} \widehat{\beta}_n^{(e)} \cos(nt) + \sum_{n=1}^{N} \widehat{\beta}_n^{(o)} \sin(nt),
$$

Therefore, the basis is

$$1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots, \sin Nt, \cos Nt.$$

it is easy to show that

$$\begin{aligned}
\langle \cos(nt), \cos(mt) \rangle &= \pi \delta_{nm}, \\
\langle \sin(nt), \sin(mt) \rangle &= \pi \delta_{nm}, \\
\langle \sin(nt), \cos(mt) \rangle &= 0
\end{aligned}$$

This means that all basis are orthogonal, and

$$\begin{aligned}
\langle 1, 1 \rangle &= 2\pi, \\
\langle X_i, X_i \rangle &= \pi, \ \forall i \neq 1
\end{aligned}$$

Obviously, from before, we have

$$\begin{aligned}
\widehat{\beta}_0 &= \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f, \\
\widehat{\beta}_n^{(e)} &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt)\, dt, \\
\widehat{\beta}_n^{(o)} &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt)\, dt
\end{aligned}$$

The restriction $t \in [0, 2\pi]$ can be immediately removed by setting the basis as

$$\begin{aligned}
\widehat{f} &= \widehat{\beta}_0 + \sum_{n=1}^{N} \widehat{\beta}_n^{(e)} \cos\left(\frac{2\pi}{T} nt\right) + \sum_{n=1}^{N} \widehat{\beta}_n^{(o)} \sin\left(\frac{2\pi}{T} nt\right), \\
\widehat{\beta}_0 &= \frac{1}{T} \int_T f, \\
\widehat{\beta}_n^{(e)} &= \frac{2}{T} \int_T f(t) \cos\left(\frac{2\pi}{T} nt\right) dt, \\
\widehat{\beta}_n^{(o)} &= \frac{2}{T} \int_T f(t) \sin\left(\frac{2\pi}{T} nt\right) dt.
\end{aligned}$$

Example

$$f(t) = 1, \quad -\pi < t < \pi$$

Then

$$\begin{aligned}
\widehat{\beta}_0 &= \frac{1}{2\pi}, \\
\widehat{\beta}_n^{(e)} &= 0, \\
\widehat{\beta}_n^{(o)} &= 0,
\end{aligned}$$

of course; but

$$f(t) = \begin{cases} 1 & -\pi/2 < t < \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

It is more convinient to employ an expansion in terms of the complex notation

$$e^{i2\pi nt/T} = \cos\left(2\pi nt/T\right) + i\sin\left(2\pi nt/T\right)$$

In that sense we have to define the inner product in terms of the comlex notation to be

$$\langle X_1, X_2\rangle = \int X_1^* X_2.$$

We can see directly that the basis $e^{i2\pi nt/T}$ are orthogonal. For $n \neq m$

$$\begin{aligned}\int_0^T e^{i2\pi nt/T} e^{-i2\pi mt/T} dt &= \frac{1}{i2\pi\left(n-m\right)/T} e^{i2\pi(n-m)t/T}\Big|_0^{2\pi} \\ &= 0;\end{aligned}$$

and for $n = m$, clearly

$$\int_0^T e^{i2\pi(n-m)t/T} dt = T$$

More compactly

$$\begin{aligned}\left\langle e^{i2\pi nt/T}, e^{i2\pi mt/T}\right\rangle &= T\delta_{nm}, \\ \widehat{f} &= \sum_{n=-N}^{n=N} \widehat{\beta}_n e^{i2\pi nt/T}, \\ \widehat{\beta}_n &= \frac{\left\langle f, e^{i2\pi nt/T}\right\rangle}{\left\langle e^{i2\pi nt/T}, e^{i2\pi nt/T}\right\rangle} \\ &= \frac{1}{T}\int_T f\left(t\right) e^{-i2\pi nt/T} dt.\end{aligned}$$

Take care of two things: (1) the ugly $\frac{2}{T}$ went away; (2) the different sign of the exponents above.

It can be shown that the bases $e^{i2\pi nt/T}$ are complete for the piecewise continuous functions $f(t)$; i.e.,

$$f(t) = \sum_{n=-\infty}^{n=\infty} \widehat{\beta}_n e^{i2\pi nt/T}$$

Now, we can see Fourier series in two different ways: (1) a summation of sinosoidal components representing a finite support function $f(t)$, $0 < t < T$; (2) since $e^{i2\pi nt/T} = e^{i2\pi n(t+T)/T}$,we have

$$f(t) = f(t+T),$$

which is a representation of a periodic function of a period T.Said differently, the reconstruction $\Sigma_{n=-\infty}^{n=\infty} \widehat{\beta}_n e^{i2\pi nt/T}$ will produce our original function $f(t)$ repeated infinitly with a period T.

Delta Function

We wish to have the identity operator (for many mathematical and physical convinience shown later) to have an operator $\delta(t)$ such that at $t = 0$, we can pick up only the value $f(0)$, of the function f at its argument t.

Such a approach can be pursued by four routes:

1- Elementray and very non-rigorous

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases}, \quad \text{or}$$

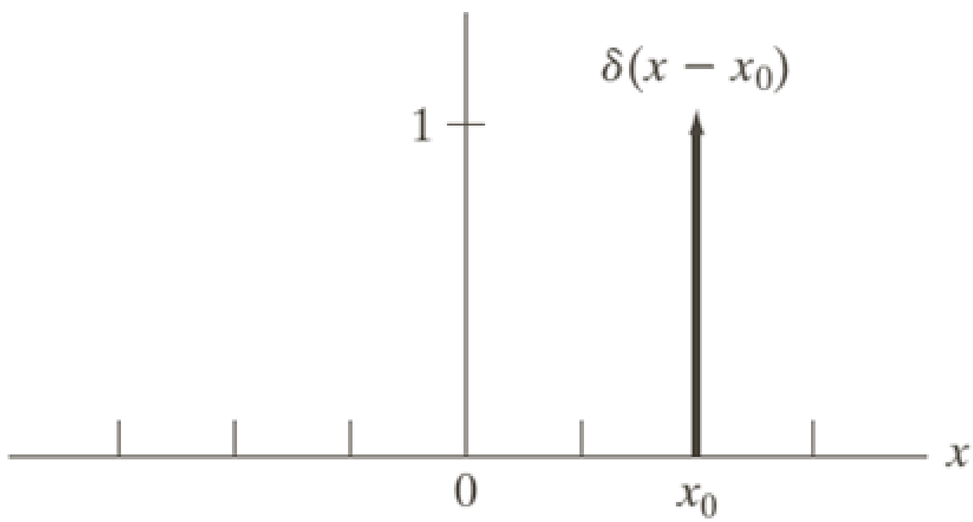
$$\delta(t - t_0) = \begin{cases} 0 & \text{if } t \neq t_0 \\ \infty & \text{if } t = t_0 \end{cases},$$

$$\int_{-\infty}^{\infty} \delta(t) = 1.$$

Therefore, for any smooth (continuous and defrentiable) function, it is straightforward to show

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\, dt = f(t_0)$$



Corollary

$$\sum_{x=-\infty}^{\infty} f(x)\,\delta(x-x_0) = f(x_0)$$

2- Rigorous approach for most practical purposes (limiting process):

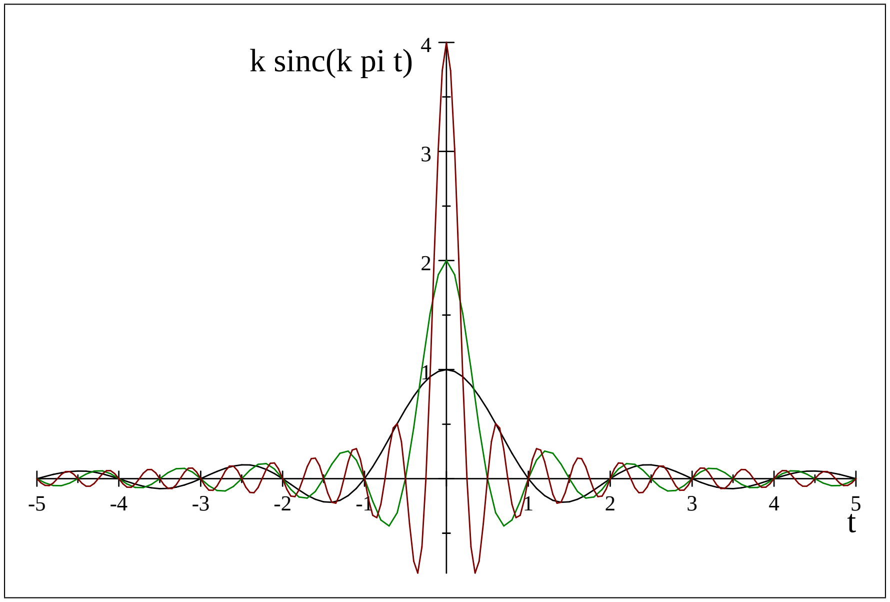
$$\delta(t-t_0) = \lim_{k} \psi_k(t-t_0)$$

For example:

$$\delta(t) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \operatorname{rect}\left(\frac{t}{\epsilon}\right),$$
$$\operatorname{rect}(t) = \begin{cases} 1 & -1/2 \le t \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Also,

$$\begin{aligned} \delta(t) &= \lim_{k\to\infty} k\ \operatorname{sinc}(kt) \\ &= \lim_{k\to\infty} k \left[\frac{\sin(\pi k t)}{\pi k t}\right] \\ &= \lim_{k\to\infty} \frac{\sin(\pi k t)}{\pi t} \end{aligned}$$



The zero-crossing always occurs at

$$\pi k t = m\pi,$$
$$t = m/k$$

The figure above is for $k = 1, 2, 4$.
It can be proven that for any of the delta representation above

$$\lim_{\epsilon \to 0} \int f(t) \frac{1}{\epsilon} \operatorname{rect}\left(\frac{t}{\epsilon}\right) dt = f(0),$$
$$\lim_{k \to \infty} \int f(t) \frac{\sin(\pi k t)}{\pi t} dt = f(0).$$

Therefore, both are valid representation of delta function.

It is also beneficial to notice that

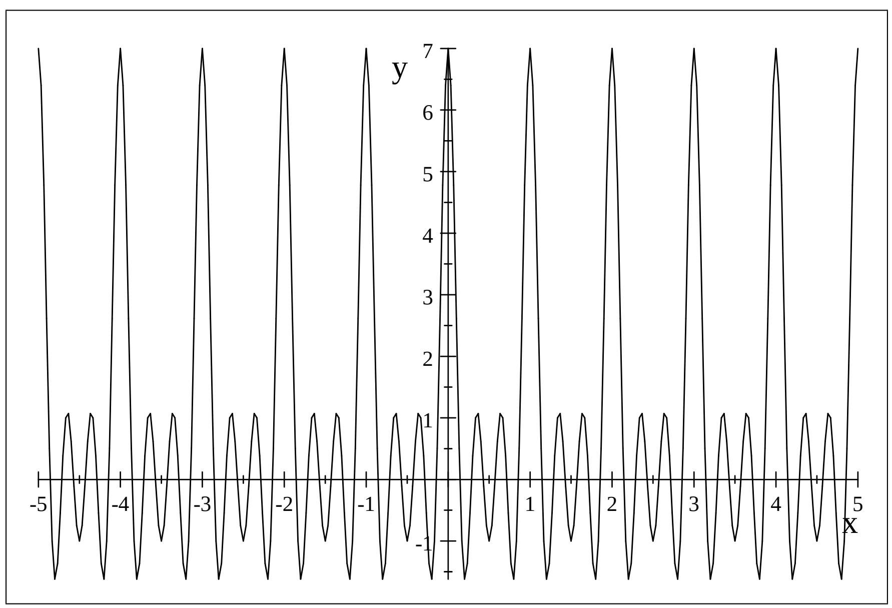
$$\begin{aligned}\int_{-k/2}^{k/2} e^{2\pi i\nu t}d\nu &= \frac{e^{2\pi i\nu t}\Big|_{-k/2}^{k/2}}{2\pi it}\\ &= \frac{2i\sin\left(\pi kt\right)}{2\pi it}\\ &= k\,\mathrm{sinc}\left(kt\right)\end{aligned}$$

Threfore,

$$\begin{aligned}\delta\left(t\right) &= \lim_{k\to\infty}\int_{-k/2}^{k/2} e^{2\pi i\nu t}d\nu\\ &= \int_{-\infty}^{\infty} e^{2\pi i\nu t}d\nu\end{aligned}$$

Another helpful form is

$$\mathrm{comb}\left(t\right) = \lim_{k\to\infty}\frac{\sin\left(\pi\left(2k+1\right)t\right)}{\sin\left(\pi t\right)}$$



It is a series of delta functions. It is easy to see that at small t, $\sin(\pi t) \approx \pi t$ and therefore $\mathrm{comb}(t) \approx \lim_{k\to\infty} k\,\mathrm{sinc}(kt)$, which is $\delta(t)$ for large k. Also, since $\mathrm{comb}(t)$ is periodic with period 1, $\delta(t)$ is repeated.

Another helpful form is to notice that (it would be nice, but not mandatory, if you try to prove)

$$\sum_{n=-k}^{n=k} e^{2\pi int} = \frac{\sin(\pi(2k+1)t)}{\sin(\pi t)}$$

Therefore

$$\begin{aligned}\mathrm{comb}(t) &= \sum_{n=-\infty}^{n=\infty} e^{2\pi int} \\ &= \sum_{n=-\infty}^{n=\infty} \delta(t-n)\end{aligned}$$

Of course,

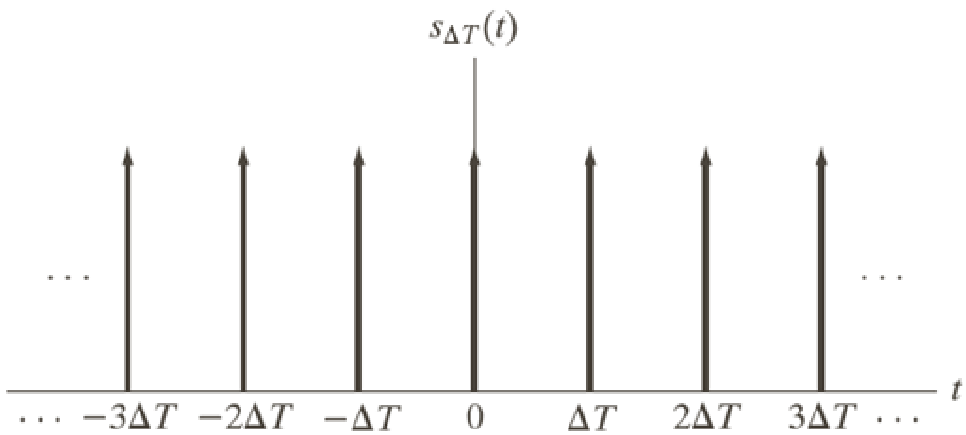
$$\begin{aligned}\mathrm{comb}\left(\frac{t}{\Delta T}\right) &= \sum_{n=-\infty}^{n=\infty} e^{2\pi in\frac{t}{\Delta T}} \\ &= \sum_{n=-\infty}^{n=\infty} \delta\left(\frac{t}{\Delta T}-n\right) \\ &= \Delta t \sum_{n=-\infty}^{n=\infty} \delta(t-n\Delta T).\end{aligned}$$

Observe that:

$$\int_{-\infty}^{\infty} \delta(at)\,dt = \frac{1}{a}\int_{-\infty}^{\infty} \delta(at)\,d(at)$$
$$= \frac{1}{a}$$

Therefore

$$\delta(at) = \frac{1}{a}\delta(t)$$



3- Very Rigorous approach of mathematicians: (theory of distribution) which is out of scope.

Fourier Integraion (Fourier Transform)

There are many routes to puruse this topic; the simplist is what was pursued by Fourier himself, (and we will explain it with overlooking many of mathematical regularity conditions). But let's first review Rieman-sum integral. For a function $F(\zeta)$, defined on $0 \leq \xi \leq L$, the area under this function can be approximated by

$$\begin{aligned} Area &\cong \sum_{n=0}^{N-1} F\left(n\frac{L}{N}\right)\left(\frac{L}{N}\right) \\ &= \sum_{n=0}^{N-1} F(n\Delta\xi)\,\Delta\xi \end{aligned}$$

and if the limit exists, this is the integration

$$\int_0^L F(\xi)\,d\xi = \lim_{N\to\infty} \sum_{n=0}^{N-1} F(n\Delta\xi)\,\Delta\xi, \quad \Delta\xi = \frac{L}{N}$$

Now, consider our periodic function $f(t)$, whose periode is T, and fourier coefficients are F_n; then

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} F_n e^{2\pi int/T} \\ &= \sum_{n=-\infty}^{\infty} \Delta\xi \frac{F_n}{\Delta\xi} e^{2\pi itn\Delta\xi}, \quad \Delta\xi = \frac{1}{T} \\ &= \int_{-\infty}^{\infty} \left[\lim_{T\to\infty} \frac{F_n}{\Delta\xi}\right] e^{2\pi it\xi} d\xi \end{aligned}$$

Now, if we allow $T \to \infty$, i.e., $\Delta\xi \to 0$,

$$\begin{aligned} F(\xi) &= \lim_{T\to\infty} \frac{F_n}{\Delta\xi} \\ &= \lim_{T\to\infty} \int_{-T/2}^{T/2} f(t)\, e^{-2\pi i n t/T} \\ &= \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t\xi} dt, \\ f(t) &= \int_{-\infty}^{\infty} F(\xi)\, e^{2\pi i t\xi} d\xi \end{aligned}$$

Convolution

$$\begin{aligned} [f * h](x) &= \int_{-\infty}^{\infty} f(x')\, h(x - x')\, dx' \\ &= \int_{-\infty}^{\infty} h(x')\, f(x - x')\, dx' \\ &= [h * f](x) \end{aligned}$$

Correlation

$$\begin{aligned} [f \star h](x) &= \int_{-\infty}^{\infty} f(x')\, h(x' - x)\, dx' \\ &= \int_{-\infty}^{\infty} h(x')\, f(x' + x)\, dx' \\ &\neq [h \star f](x) \end{aligned}$$

If the kernel h is symetric, then convolution is the same as correlation (prove).
Meaning of Correlation. (Figure 12.9)
Fourier Transform of

Bibliography