Helwan University
Faculty of Computers and Information

# Assignment

Department of Computer Science
Pattern Recognition

## Linear Methods for Classification

1. Consider the 3 class problem, with multinormal distributions, with: $f_i \sim \mathcal{N}(\mu_i, \Sigma_i),\ \Sigma_i = I,\ i = 1,2,3$
   $\mu_1 = (-a, 0),\ \mu_2 = (a, 0),\ \mu_3 = (0, \sqrt{3}a)$
   $L_{ii} = 0,\ L_{12} = L_{21} = L_{23} = L_{32} = 1,\ L_{13} = L_{31} = L,$
   $\pi_i = 1/3$. Suppose $L = 1$; prove that there is a linear decision surface between each pair of classes. Moreover, prove that the three decision surfaces intersect at $(0, a/\sqrt{3})$. **Hint:** find the pair-wise decision surfaces, the risk at these surfaces and the risk of the third class.

2. Prove that, for all values of costs and priors, the class-conditional densities (PDFs) are linear functions of predictors iff the risks $\lambda_i$ (or the scores $1 - \lambda_i$) are linear functions of predictors.

3. Prove that in the general case of priors and costs the risks may not be linear functions of predictors when the class-conditional densities are not explicit linear functions of predictors.

4. Given two sets of points: $\mathbf{X} = \{x_i,\ i = 1 \ldots, n_1\}$ and $\mathbf{Y} = \{y_j,\ j = 1 \ldots, n_2\}$; each point in these sets is in general $p$-dimensional. The *convex hull* $\mathbf{C_X}$ is defined as $\mathbf{C_X} = \{x |\ x = \sum \alpha_i x_i,\ 0 \le \alpha_i \le 1,\ \sum_i \alpha_i = 1,\ i = 1 \ldots, n_1\}$. Similarly, the *convex hull* $\mathbf{C_Y}$ is defined on the other set of points. It is known that any two sets of points are considered perfectly linearly separated (or classified with zero error) if there exists a hyper-plan (surface) $\delta(x) = w'x + w_o$ such that $\delta(x) > 0$ & $\delta(y) < 0\ \forall x, y$ belonging to the first and second set respectively.

   (a) Draw (in 2 dimensions) an example for the two sets of points **X** and **Y**, with their convex hulls $\mathbf{C_X}$ and $\mathbf{C_Y}$.

   (b) Draw an example to two sets **X** and **Y**, where the sets intersect and their convex hulls intersect (of course). Can the two set intersect while their convex hulls do not intersect (which is very naive question)?

   (c) Draw another example for two sets where $\exists x_i \in \mathbf{C_Y}$ whereas $!\exists y_j \in \mathbf{C_X}$.

   (d) Phrase the above introduction in terms of vector-matrix notation.

   (e) Prove that if the two convex hulls do not intersect iff they are linearly separated.

5. Consider solving the classification problem by regressions of indicators. Suppose you have only 2 classes; each is one dimensional. The dataset is $\mathbf{X}_1 = \{-4, -3, -1\}$, $\mathbf{X}_2 = \{-2, 0, 1\}\}$. Find the two regressions $y_1$ and $y_2$ and the final decision boundary.

6. Consider solving the classification problem by regression of indicators. Suppose you have 3 classes that are fully separable: $\mathbf{X}_1 = \{-4, -3, -2\}$, $\mathbf{X}_2 = \{-1, 0, 1\}$, $\mathbf{X}_3 = \{2, 3, 4\}$. Solve the problem and comment on the masking problem. Resolve the problem with extending the feature space to $(X_1, X_1^2)$.