

# **CS495**

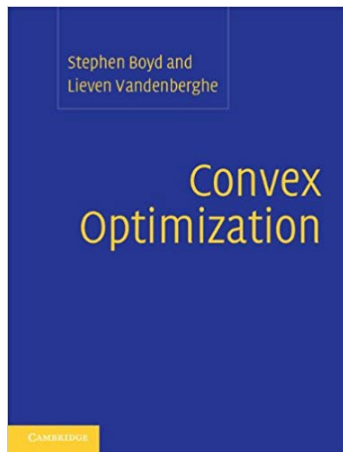
## **Optimiztaion**

Waleed A. Yousef, Ph.D.,  
Human Computer Interaction Lab.,  
Computer Science Department,  
Faculty of Computers and Information,  
Helwan University,  
Egypt.

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Lectures follow:

Boyd and Vandenberghe (2004)



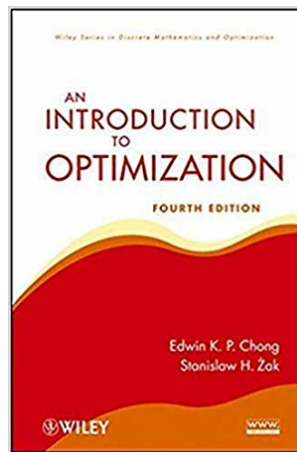
Boyd, S., & Vandenberghe, L. (2004). Convex Optimization. Cambridge: Cambridge University Press.

Book and Stanford course:

<http://web.stanford.edu/~boyd/cvxbook/>

Some examples from:

Chong and Zak (2013)



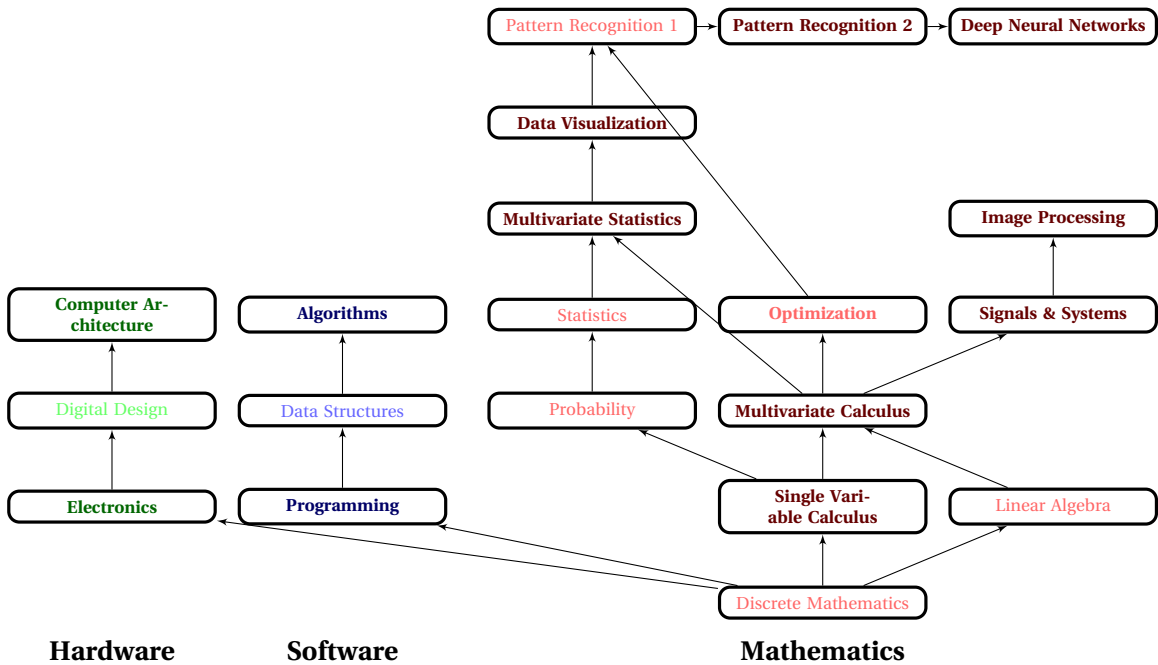
Chong, E. K., & Zak, S. (2001). An introduction to optimization: Wiley-Interscience.

# Course Objectives

- Developing rigorous mathematical treatment for mathematical optimization.
- Building intuition, in particular to practical problems.
- Developing computer practice to using optimization SW.

## Prerequisites

1. Discrete Mathematics
  2. Calculus (single variable)
  3. Calculus (multi variable)
  4. Linear Algebra
- 
5. Some Real Analysis and Topology



# **Chapter 1**

## **Introduction**

### **Snapshot on Optimization**

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# **Chapter 1**

## **Introduction**



# 1.1 Mathematical Optimization

**Definition 1** A mathematical optimization problem or just optimization problem, has the form (Boyd and Vandenberghe, 2004):

$$\begin{aligned} &\underset{x}{\text{minimize}} && f_0(x) \\ &\text{subject to:} && f_i(x) \leq 0, && i = 1, \dots, m \\ &&& h_i(x) = 0, && i = 1, \dots, p, \end{aligned}$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$ , (optimization variable)

$f_0: \mathbf{R}^n \mapsto \mathbf{R}$ , (objective (cost/utility) function)

$f_i: \mathbf{R}^n \mapsto \mathbf{R}$ , (inequality constraints (functions))

$h_i: \mathbf{R}^n \mapsto \mathbf{R}$ , (equality constraints (functions))

$$\mathcal{D}: \bigcap_{i=1}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i \quad (\text{feasible set})$$

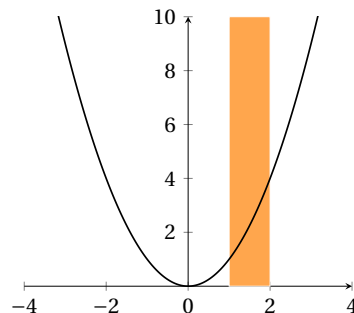
$$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \leq 0 \wedge h_i(x) = 0\}$$

$$x^*: \{x \mid x \in \mathcal{D} \wedge f_0(x) \leq f_0(z) \forall z \in \mathcal{D}\} \quad (\text{solution})$$

- minimize  $f_0 \equiv \text{maximize } -f_0$ .
- $f_i \leq 0 \equiv -f_i \geq 0$ .
- 0s can be replaced of course by constants  $b_i, c_i$
- unconstrained problem when  $m = p = 0$ .

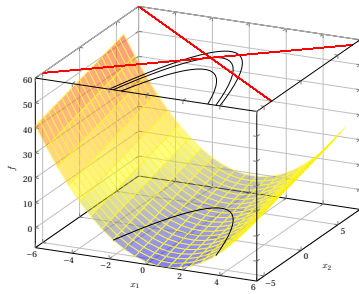
**Example 2 :**

$$\begin{aligned} &\underset{x}{\text{minimize}} && x^2 \\ &\text{subject to:} && x \leq 2 \wedge x \geq 1. \end{aligned}$$



$$x^* = 1.$$

If the constraints are relaxed, then  $x^* = 0$ .



$$\begin{aligned} &\underset{x}{\text{minimize}} && f_0(x) \\ &\text{subject to:} && f_i(x) \leq 0, && i = 1, \dots, m \\ &&& h_i(x) = 0, && i = 1, \dots, p, \end{aligned}$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$ , (optimization variable)

$f_0: \mathbf{R}^n \mapsto \mathbf{R}$ , (objective (cost/utility) function)

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$$\mathcal{D}: \bigcap_{i=1}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i \quad (\text{feasible set})$$

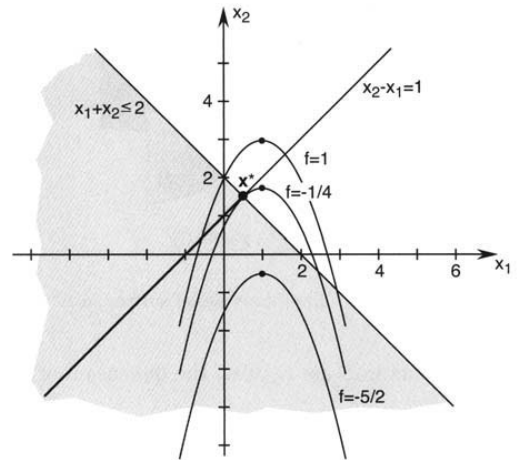
$$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \leq 0 \wedge h_i(x) = 0\}$$

$$x^*: \{x \mid x \in \mathcal{D} \wedge f_0(x) \leq f_0(z) \forall z \in \mathcal{D}\} \quad (\text{solution})$$

**Example 3** (*Chong and Zak, 2013, Ex. 20.1, P. 454*):

$$\begin{aligned} &\underset{x}{\text{minimize}} && (x_1 - 1)^2 + x_2 - 2 \\ &\text{subject to:} && x_2 - x_1 = 1 \\ &&& x_1 + x_2 \leq 2. \end{aligned}$$

No global minimizer:  $\partial z / \partial x_2 = 1 \neq 0$ . However,  $z|_{(x_2-x_1=1)} = (x_1 - 1)^2 + (x_1 - 1)$ , which attains a minimum at  $x_1 = 1/2$ .



$x^* = (1/2, 3/2)'$ . (Let's see animation)

### 1.1.1 Motivation and Applications

- *optimization problem* is an abstraction of how to make “best” possible choice of  $x \in \mathbf{R}^n$ .
- *constrains* represent trim requirements or specifications that limit the possible choices.
- *objective function* represents the *cost* to minimize or the *utility* to maximize for each  $x$ .

**Examples:**

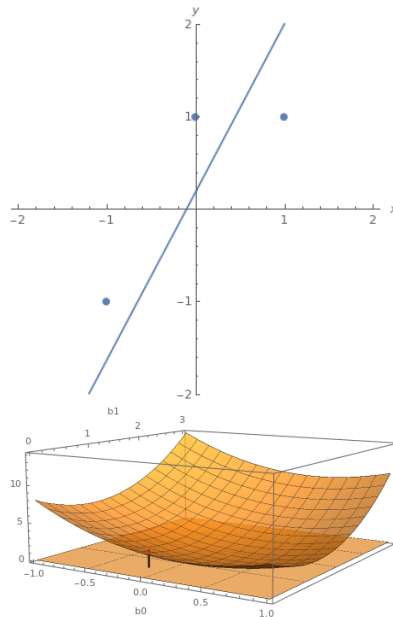
	<i>Any problem</i>	<i>Portfolio Optimization</i>	<i>Device Sizing</i>	<i>Data Science</i>
$x \in \mathbf{R}^n$	choice made	investment in capitals	dimensions	parameters
$f_i, h_i$	firm requirements /conditions	overall budget	engineering constraints	regularizer
$f_0$	cost (or utility)	overall risk	power consumption	error

- Amazing variety of practical problems. In particular, data science: two sub-fields: construction and assessment.
- The construction of: Least Mean Square (LMS), Logistic Regression (LR), Support Vector Machines (SVM), Neural Networks(NN), Deep Neural Networks (DNN), etc.
- Many techniques are for solving the optimization problem:
  - Closed form solutions: convex optimization problems
  - Numerical solutions: Newton’s methods, Gradient methods, Gradient descent, etc.
  - “Intelligent” methods: particle swarm optimization, genetic algorithms, etc.

#### Example 4 (Machine Learning: construction) :

Let's suppose that the best regression function is  $Y = \beta_0 + \beta_1 X$ , then for the training dataset  $(x_i, y_i)$  we need to minimize the MSE.

$$\underset{\beta_0, \beta_1}{\text{minimize}} \sum_i (\beta_0 + \beta_1 x_i - y_i)^2$$



- Half of ML field is construction: NN, SVM, etc.
- In DNN it is an optimization problem of millions of parameters.
- Let's see animation.
- Where are Probability, Statistics, and Linear Algebra here? Let's re-visit the chart.
- Is the optimization problem solvable:
  - closed form? (LSM)
  - numerically and guaranteed? (convex and linear)
  - numerically but not guaranteed? (non-convex):
    - \* numerical algorithms, e.g., GD,
    - \* local optimization,
    - \* heuristics, swarm, and genetics,
    - \* brute-force with exhaustive search

### 1.1.2 Solving Optimization Problems

- A *solution method* for a class of optimization problems is an algorithm that computes a solution.
- Even when the *objective function* and constraints are smooth, e.g., polynomials, the solution is very difficult.
- There are three classes where solutions exist, theory is very well developed, and amazingly found in many practical problems:

Linear  $\subset$  Quadratic  $\subset$  Convex  $\subset$  Non-linear (not linear and not known to be convex!)

- For the first three classes, the problem can be solved very reliably in hundreds or thousands of variables!

## 1.2 Least-Squares and Linear Programming

### 1.2.1 Least-Squares Problems

A *least-squares* problem is an optimization problem with no constraints (i.e.,  $m = p = 0$ ), and an objective in the form:

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k (a'_i x - b_i)^2 = \|A_{k \times n} x_{n \times 1} - b_{k \times 1}\|^2.$$

The solution is given in **closed form** by:

$$x = (A' A)^{-1} A' b$$

- Good algorithms in many SC SW exist; it is a very mature technology.
- Solution time is  $O(n^2 k)$ .
- Easily solvable even for hundreds or thousands of variables.
- More on that in the Linear Algebra course.
- Many other problems reduce to typical LS problem:
  - Weighted LS (to emphasize some observations)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k w_i (a'_i x - b_i)^2.$$

- Regularization (to penalize for over-fitting)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k (a'_i x - b_i)^2 + \rho \sum_{j=1}^n x_j^2.$$

## 1.2.2 Linear Programming

A *linear programming* problem is an optimization problem with objective and all constraint functions are linear:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) = C'x & \\ \text{subject to:} & a'_i x \leq b_i, & i = 1, \dots, m \\ & h'_i x = g_i, & i = 1, \dots, p, \end{array}$$

- **No** closed form solution as opposed to LS.
- Very robust, reliable, and effective set of methods for numerical solution; e.g., Dantzig's simplex, and interior point.
- Complexity is  $\simeq O(n^2 m)$ .
- Similar to LS, we can solve a problem of thousands of variables.
- Example is *Chebyshev minimization* problem:

$$\underset{x}{\text{minimize}} f_0(x) = \max_{i=1, \dots, k} |a'_i x - b_i|,$$

- The objective is different from the LS: minimize the maximum error. **Ex:**
- After some tricks, requiring familiarity with optimization, it is equivalent to a LP:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & t & \\ \text{subject to:} & a'_i x - t \leq b_i, & i = 1, \dots, k \\ & -a'_i x - t \leq -b_i, & i = 1, \dots, k \end{array}$$

## 1.3 Convex Optimization

A *convex optimization* problem is an optimization problem with objective and all constraint function are convex:

$$\begin{array}{llll} \underset{x}{\text{minimize}} & f_0(x) & & \\ \text{subject to:} & f_i(x) \leq 0, & i = 1, \dots, m & \\ & h_i(x) = 0, & i = 1, \dots, p, & \\ & f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y), & \alpha + \beta = 1, & 0 \leq \alpha, 0 \leq \beta, \quad 0 \leq i \leq m \\ & h_i(x) = a'_i x + b_i & & 0 \leq i \leq p \end{array}$$

- The LP and LS are special cases; however, only LS has closed-form solution.
- Very robust, reliable, and effective set of methods, including *interior point methods*.
- Complexity is almost:  $O(\max(n^3, n^2 m, F))$ , where  $F$  is the cost of evaluating 1st and 2nd derivatives of  $f_i$  and  $h_i$ .
- Similar to LS and LP, we can solve a problem of thousands of variables.
- However, it is not as very mature technology as the LP and LS yet.
- There are many practical problems that can be re-formulated as convex problem **BUT** requires mathematical skills; but once done the problem is solved. **Hint:** realizing that the problem is convex requires more mathematical maturity than those required for LP and LS.



## 1.4 Nonlinear Optimization

A *non-linear optimization* problem is an optimization problem with objective and constraint functions are non-linear **BUT** not known to be convex (**so far**). Even simple-looking problems in 10 variables can be extremely challenging. Several approaches for solutions:

**Local Optimization** : starting at initial point in space, using differentiability, then navigate

- does not guarantee global optimal.
- affected heavily by initial point.
- depends heavily on numerical algorithm and their parameters.
- More art than technology.
- In contrast to convex optimization, where a lot of art and mathematical skills are required to formulate the problem as convex; then numerical solution is straightforward.

**Global Optimization** : the true global solution is found; the compromise is complexity.

- The complexity goes exponential with dimensions.
- Sometimes it is worth it when: the cost is huge, not in real time, and dimensionality is low.

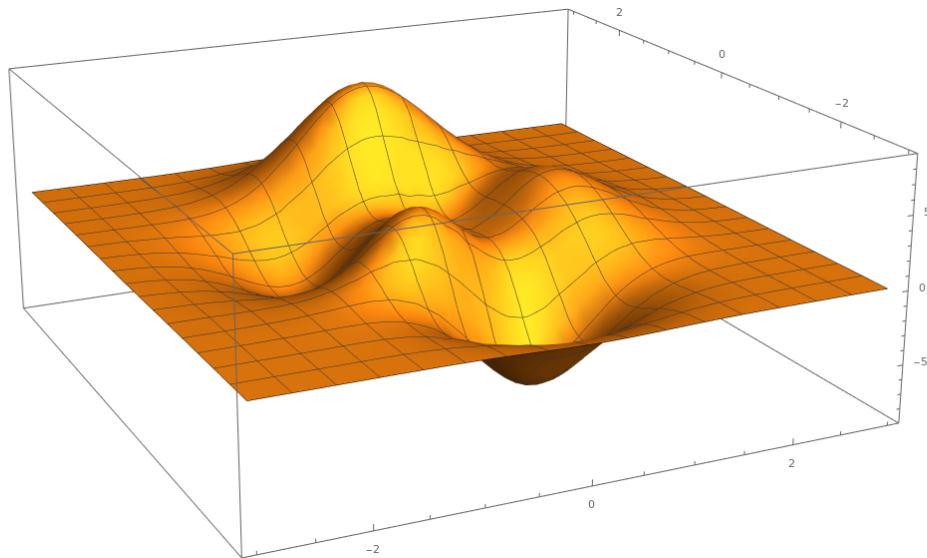
**Role of Convex Optimization** :

- Approximate the non-linear function to a convex one, finding the exact solution, then using it as a starting point for the original problem. (Also does not guarantee optimality)
- Setting bounds on the global solution.

**Evolutionary Computations** : Genetic Algorithm (GA), Simulated Annealing (SA), Particle Swarm Optimization (PSO), etc.

**Example 5 (Nonlinear Objective Function) :** (*Chong and Zak, 2013, Ex. 14.3, P.290*)

$$f(x, y) = 3(1 - x)^2 e^{-x^2 - (y+1)^2} - 10e^{-x^2 - y^2} \left( -x^3 + \frac{x}{5} - y^5 \right) - \frac{1}{3} e^{-(x+1)^2 - y^2}$$



# **Part I**

# **Theory**

## **Chapter 2**

# **Convex sets**

## 2.1 Affine and convex sets

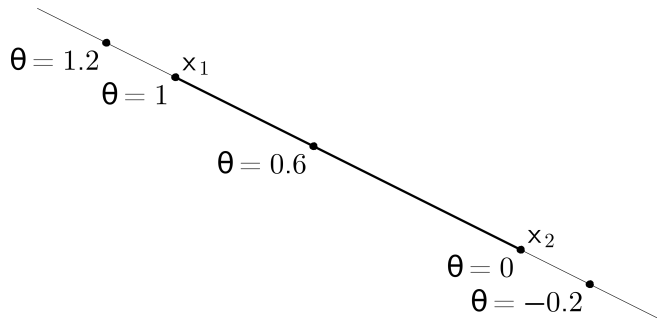
### 2.1.1 Lines and line segments

**Definition 6 (line and line segment)** Suppose  $x_1 \neq x_2 \in \mathbf{R}^n$ . Points of the form

$$\begin{aligned} y &= \theta x_1 + (1 - \theta)x_2 \\ &= x_2 + \theta(x_1 - x_2), \end{aligned}$$

where  $\theta \in \mathbf{R}$ , form the line passing through  $x_1$  and  $x_2$ .

- As usual, this is a definition for high dimensions taken from a proof for  $n \leq 3$ .
- We have done it many times: angle, norm, cardinality of sets, etc.
- if  $0 \leq \theta \leq 1$  this forms a line segment.



## 2.1.2 Affine sets

**Definition 7 (Affine sets)** A set  $C \subset \mathbf{R}^n$  is affine if the line through any two distinct points in  $C$  lies in  $C$ . I.e.,  $\forall x_1, x_2 \in C$  and  $\theta \in \mathbf{R}$ , we have  $\theta x_1 + (1 - \theta)x_2 \in C$ . In other words,  $C$  contains any linear combination (summing to one) of any two points in  $C$ .

**Examples:** what about line, line segment, circle, disk, strip, first quadrant?

**Corollary 8** Suppose  $C$  is an affine set, and  $x_1, \dots, x_k \in C$ , then  $C$  contains every general affine combination of the form  $\theta_1 x_1 + \dots + \theta_k x_k$ , where  $\theta_1 + \dots + \theta_k = 1$ .

**Wrong Proof.** Suppose  $y_1, y_2 \in C$ , then

$$x = \sum_{i=1}^k \theta_i x_i = \sum_{i=1}^k \theta_i (\alpha_i y_1 + (1 - \alpha_i) y_2);$$

and the summation of the coefficients will be

$$\sum_{i=1}^k \theta_i \alpha_i + \sum_{i=1}^k \theta_i (1 - \alpha_i) = \sum_{i=1}^k \theta_i (\alpha_i + 1 - \alpha_i) = \sum_{i=1}^k \theta_i = 1.$$

Where is the bug?

**Correct Proof. base:**  $k = 3$ .

$$\begin{aligned} x &= \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \\ &= (1 - \theta_3) \left( \frac{\theta_1}{1 - \theta_3} x_1 + \frac{\theta_2}{1 - \theta_3} x_2 \right) + \theta_3 x_3. \\ &= (1 - \theta_3)(\cdot \in C) + \theta_3(\cdot \in C). \end{aligned}$$

**induction:** suppose it is true for some  $k \geq 3$ ; i.e.,  $\sum_{i=1}^k \theta_i x_i \in C$  when  $\sum_{i=1}^k \theta_i = 1$ . Then

$$\begin{aligned} x &= \sum_{i=1}^{k+1} \theta_i x_i \\ &= \sum_{i=1}^k \theta_i x_i + \theta_{k+1} x_{k+1} \\ &= (1 - \theta_{k+1}) \sum_{i=1}^k \theta_i' (1 - \theta_{k+1}) x_i + \theta_{k+1} x_{k+1} \\ &= (1 - \theta_{k+1})(\cdot \in C) + \theta_{k+1}(\cdot \in C), \\ &\quad \text{(from the induction hypothesis)} \end{aligned}$$

■ which completes the proof. ■

**Definition 9 (Subspace from Linear Algebra)** A set  $V \subset \mathbf{R}^n$  of vector (here points) is a subspace if it is closed under sums and scalar multiplication. I.e.,  $\forall v_1, v_2 \in V$  and  $\forall \alpha, \beta \in \mathbf{R}$  we have  $\alpha v_1 + \beta v_2 \in V$ .

**Remember:**

- $\alpha + \beta$  not necessarily equals 1
- $\alpha = 0, \beta = 0 \rightarrow \mathbf{0} \in V$ .
- Any subspace  $V$  is the solution set of  $A_{m \times n} x_{n \times 1} = \mathbf{0}$ , which is  $\mathcal{N}(A)$  (the null space of  $A$ ). Geometry? I.e.,  $V = \{x \mid Ax = \mathbf{0}\}$
- **rank**( $A$ ) =  $n - \dim(V)$ .

**Corollary 10 .**

1. If  $C$  is affine, then  $V = C - x_0 = \{x - x_0 \mid x, x_0 \in C\}$  is a subspace.
2. If  $V$  is a subspace, then  $C = V + x_0 = \{x + x_0 \mid x \in V\}$  is affine  $\forall x_0$ .
3. An affine set  $C$  can be represented as the solution set of a nonhomogeneous linear system  $Ax = b$ , where  $V = C - x_0$  is  $\mathcal{N}(A)$ .
4. The solution set of any nonhomogeneous system is an affine set. (Ex. 2.1)

**Proof.**

1. Suppose  $x_1, x_2, x_0 \in C$ , an affine set. Both  $x_1 - x_0$  and  $x_2 - x_0$ , by construction,  $\in V$ ; then

$$\begin{aligned} x &= \alpha(x_1 - x_0) + \beta(x_2 - x_0) + x_0 \\ &= \alpha x_1 + \beta x_2 + (1 - \alpha - \beta)x_0 \in C \end{aligned}$$

Then  $x - x_0 = \alpha(x_1 - x_0) + \beta(x_2 - x_0) \in V$ ; hence  $V$  is a subspace.

2. Suppose  $x_1, x_2 \in V$ , a subspace. Both  $x_1 + x_0$  and  $x_2 + x_0$ , by construction,  $\in C$ ; then

$$\begin{aligned} x &= \theta(x_1 + x_0) + (1 - \theta)(x_2 + x_0) \\ &= \theta x_1 + (1 - \theta)x_2 + x_0 = (\cdot \in V) + x_0 \in C \end{aligned}$$

3. If  $C$  is affine and  $x_0 \in C$ , then

$$\begin{aligned} C - x_0 &= \{x \mid Ax = \mathbf{0}\} \quad (\text{since it is a subspace}) \\ C &= \{x + x_0 \mid A(x + x_0) = Ax_0\} \\ C &= \{c \mid Ac = b\}. \end{aligned}$$

4.  $C = \{x \mid Ax = b\}$ ; if  $x_0 \in C$  then  $Ax_0 = b$  and

$$C - x_0 = \{x - x_0 \mid A(x - x_0) = b - Ax_0 = \mathbf{0}\}.$$

Hence,  $C - x_0$  is a subspace and  $C$  is affine. ■

**Proof of the book.** Suppose  $x_1, x_2 \in C$ , where  $C = \{x \mid Ax = b\}$ . Then

$$A(\theta x_1 + (1 - \theta)x_2) = \theta Ax_1 + (1 - \theta)Ax_2 = \theta b + (1 - \theta)b = b,$$

which means  $\theta x_1 + (1 - \theta)x_2 \in C$  as well. ■

**Remark 1 :**

- *The dimension of affine is defined to be the dimension of the associate subspace.*
- *affine is a subspace plus offset.*
- *every subspace is affine but not the vice versa; i.e., subspace is a special case of affine.*



**Definition 11 (affine hull)** The “smallest” set of all affine combinations of some set  $C$  (not necessarily affine) is called the affine hull (**aff**  $C$ ):

$$\mathbf{aff} C = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in C, \sum_i \theta_i = 1 \right\}.$$

**Corollary 12** **aff**  $C$  is affine.

**Proof.** For  $x_1 = \sum_i \alpha_i x_i$ ,  $\sum_i \alpha_i = 1$ , and  $x_2 = \sum_i \beta_i x_i$ ,  $\sum_i \beta_i = 1$ , we have

$$\theta x_1 + (1 - \theta)x_2 = \theta \sum_i \alpha_i x_i + (1 - \theta) \sum_i \beta_i x_i = \sum_i (\theta \alpha_i + (1 - \theta) \beta_i) x_i$$

$$\sum_i (\theta \alpha_i + (1 - \theta) \beta_i) = \theta \sum_i \alpha_i + (1 - \theta) \sum_i \beta_i = \theta + (1 - \theta) = 1.$$

Hence, **aff**  $C$  is affine as well. ■

**Example 13** Construct the affine hull of the set  $C = \{(-1, 0), (1, 0), (0, 1)\}$

$$\begin{aligned} \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 &= (1 - \theta_3) \left( \frac{\theta_1}{1 - \theta_3} x_1 + \frac{\theta_2}{1 - \theta_3} x_2 \right) + \theta_3 x_3 \\ &= (1 - \alpha_3) \left( (1 - \alpha_2) x_1 + \alpha_2 x_2 \right) + \alpha_3 x_3 = (1 - \alpha_2)(1 - \alpha_3) x_1 + \alpha_2(1 - \alpha_3) x_2 + \alpha_3 x_3, \end{aligned}$$

$$\begin{array}{lll} \theta_3 = \alpha_3 & \theta_2 = \alpha_2(1 - \alpha_3) & \theta_1 = 1 - \theta_2 - \theta_3 = (1 - \alpha_2)(1 - \alpha_3) \\ \alpha_3 = \theta_3 & \alpha_2 = \theta_2 / (1 - \theta_3) & \alpha_1 = 1 - \alpha_2 = \theta_1 / (1 - \theta_3). \end{array}$$

**HW:** Derive expressions for  $\alpha_i$  and  $\theta_i$  for  $n$ -point combination.

## 2.1.3 Affine dimension and relative interior

**Definition 14 (some basic topology in  $\mathbf{R}^n$ ) :**

1. The ball of radius  $r$  and center  $x$  in the norm  $\|\cdot\|$ .

$$B(x, r) = \{y \mid \|y - x\| \leq r\}.$$

2. An element  $x \in C \subseteq \mathbf{R}^n$  is called an interior point of  $C$  if  $\exists \varepsilon > 0$  for which

$$B(x, \varepsilon) = \{y \mid \|y - x\|_2 \leq \varepsilon\} \subseteq C.$$

I.e.,  $\exists$  a ball centered at  $x$  that lies entirely in  $C$ .

3. The set of all points interior to  $C$  is called the interior of  $C$  and is denoted  $\text{int } C$ .
4. A set  $C$  is open if  $\text{int } C = C$ . I.e., every point in  $C$  is an interior point.
5. A set  $C$  is closed if its complement is open

$$\mathbf{R}^n \setminus C = \{x \in \mathbf{R}^n \mid x \notin C\}$$

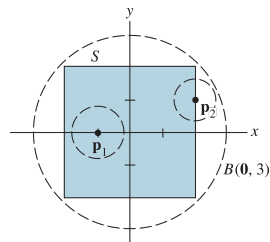
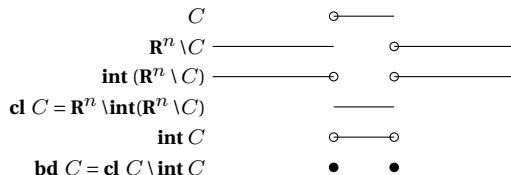
6. The closure of a set  $C$  is defined as

$$\text{cl } C = \mathbf{R}^n \setminus \text{int}(\mathbf{R}^n \setminus C).$$

7. The boundary  $C$  is defined as

$$\text{bd } C = \text{cl } C \setminus \text{int } C.$$

**Corollary 15** A boundary point (a point  $x \in \text{bd } C$ ) satisfies:  $\forall \varepsilon > 0, \exists y \in C$  and  $z \notin C$  s.t.  $y, z \in B(x, \varepsilon)$ .



**Definition 16 (alter. equiv. def.) :**

- $\text{int } C$  and  $\text{bd } C$  are defined as 2,3, corollary. (It is obvious that:  $\text{int } C \cap \text{bd } C = \phi$ .)
- $C$  is open if  $\text{int } C = C \iff C \cap \text{bd } C = \phi$ .
- $C$  is closed if  $\text{bd } C \subseteq C$ .
- $\text{cl } C = \text{bd } C \cup \text{int } C$ .

**Definition 17** We define the affine dimension of a set  $C$  as the dimension of its affine hull.

**Example 18** The unit circle in  $\mathbf{R}^2$ , i.e.,  $\{x \mid x_1^2 + x_2^2 = 1\}$  has affine hull of whole  $\mathbf{R}^2$ . So its affine dimension is 2. However, it has a dimensionality of 1 since it is parametric in just one parameter (manifold).

**Definition 19** We define the relative interior of the set  $C$ , denoted **relint**  $C$ , as its interior relative to **aff**  $C$

$$\mathbf{relint} \ C = \{x \in C \mid B(x, r) \cap \mathbf{aff} \ C \subseteq C \text{ for some } r > 0\},$$

and its relative boundary, denoted **relbd**  $C$  is defined as

$$\mathbf{relbd} \ C = \mathbf{cl} \ C \setminus \mathbf{relint} \ C.$$

**Example 20** Consider a square in the  $(x_1, x_2)$ -plane in  $\mathbf{R}^3$ , defined as:

$$C = \{x \in \mathbf{R}^3 \mid -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_3 = 0\}.$$

Then:

$$\mathbf{int} \ C = \Phi$$

$$\mathbf{cl} \ C = \mathbf{R}^n \setminus \mathbf{int}(\mathbf{R}^n \setminus C) = C$$

$$\mathbf{bd} \ C = \mathbf{cl} \ C \setminus \mathbf{int} \ C = C$$

$$\mathbf{aff} \ C = \{x \in \mathbf{R}^3 \mid x_3 = 0\}$$

$$\mathbf{relint} \ C = \{x \in \mathbf{R}^3 \mid -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\}$$

$$\mathbf{relbd} \ C = \{x \in \mathbf{R}^3 \mid \max\{|x_1|, |x_2|\} = 1, x_3 = 0\}$$

## 2.1.4 Convex sets

**Definition 21 (convex set)** A set  $C$  is convex if the line segment between any two points in  $C$  lies in  $C$ ; i.e., if for any  $x_1, x_2 \in C$  and any  $\theta$  with  $0 \leq \theta \leq 1$ , we have

$$\theta x_1 + (1 - \theta)x_2 \in C.$$

**Corollary 22** Suppose  $C$  is convex set, and  $x_1, \dots, x_k \in C$ , then  $C$  contains every general convex combination (also called mixture); i.e.,

$$\sum_i \theta_i x_i \in C, \sum_i \theta_i = 1, \theta_i \geq 0.$$

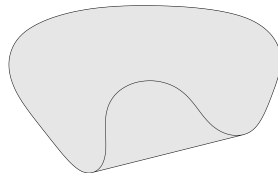
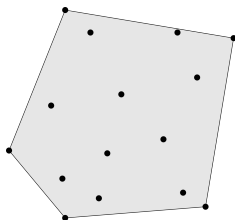
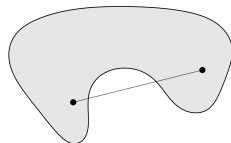
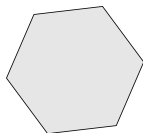
**Proof.** identical to proof of corollary 8.

**Definition 23 (convex hull)** The “smallest” set of all convex combinations of some set  $C$  (not necessarily convex) is called the convex hull (**conv** $C$ )

$$\text{conv } C = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in C, \sum_i \theta_i = 1, \theta_i \geq 0 \right\}.$$

**Corollary 24** **conv**  $C$  is convex.

**Proof.** identical to proof of corollary 12.



■ **Example 25** Revisit example 13.

**Example 26 (Applications)** : Suppose  $X \in C$  is a r.v.,  $C$  is convex. Then  $\mathbb{E}X \in C$  if it exists:

$$\mathbb{E}X = \sum_{i=1}^n p_i x_i$$

$$\mathbb{E}X = \sum_{i=1}^{\infty} p_i x_i$$

$$\mathbb{E}X = \int_C f_X(x) x \, dx \quad (\text{Riemann sum})$$

## 2.1.5 Cones

**Definition 27** A set  $C$  is called a cone (or nonnegative homogeneous) if  $\forall x \in C, \theta \geq 0$  we have  $\theta x \in C$ ; and it is a convex cone if it is convex in addition to being a cone.

**Definition 28** A point of the form  $\sum_{i=1}^k \theta_i x_i, \theta_i \geq 0$  is called a conic combination.

**Corollary 29** A set  $C$  is a convex cone if and only if it contains all conic combinations of its elements; i.e.,

$$\sum_i \theta_i x_i \in C \quad \forall x_i \in C \text{ and } \theta_i \geq 0.$$

**Proof.**

**Sufficiency:** is obvious. Choosing  $\sum_i \theta_i = 1$  implies  $C$  is convex; and setting  $\theta_i = 0 \quad \forall i > 1$  implies  $C$  is cone.

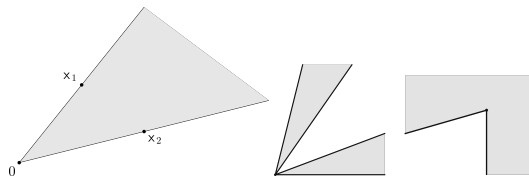
**Necessity:** Since  $C$  is convex cone, then  $\forall x_i \in C, \theta_i \geq 0$  we have:

$$\theta_i x_i \in C \quad (\text{cone})$$

$$\sum_i (1/n)(\theta_i x_i) \in C \quad (\text{convex})$$

$$n \sum_i (1/n)(\theta_i x_i) = \sum_i \theta_i x_i \in C \quad (\text{cone})$$

■

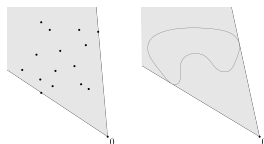


**Definition 30** A conic hull of a set  $C$  is the minimum set of all conic combination:

$$\text{cone } C = \left\{ \sum_i \theta_i x_i \mid x_i \in C, \theta_i \geq 0, i = 1, \dots, n \right\}.$$

**Corollary 31** cone  $C$  is convex cone.

**Proof.** If  $y \in \text{cone } C, \alpha \geq 0$ , then  $\alpha y = \alpha \sum_i \theta_i x_i = \sum_i (\alpha \theta_i) x_i \in \text{cone } C$ . And if  $y_1, y_2 \in \text{cone } C$  then  $\alpha y_1 + (1 - \alpha) y_2 = \alpha \sum_i \theta_i x_i + (1 - \alpha) \sum_i \mu_i x_i = \sum_i (\alpha \theta_i + (1 - \alpha) \mu_i) x_i \in \text{cone } C$  ■



## 2.2 Some important examples

### Fast Revision

- Each of the sets:  $\phi$ ,  $x_0$  (a singleton),  $\mathbf{R}^n$  are affine and convex.
- Any line is affine. If it passes through zero, it is a subspace and a convex cone.
- Any subspace is convex cone but not vice versa.
- A line segment is convex, but not affine (unless it reduces to a singleton).
- A ray,  $\{x_0 + \theta v \mid \theta \geq 0, v \neq 0\}$  is convex but not affine. It is convex cone if  $x_0 = 0$ .

## 2.2.1 Hyperplanes and halfspaces

**Definition 32** A hyperplane is a set of the form

$$\begin{aligned} \mathcal{S} &= \{x \mid a^T x = b\}, & a, b \in \mathbf{R}^n, a \neq 0 \\ &\equiv \{x \mid a^T(x - x_0) = 0\}, & a^T x_0 = b. \end{aligned}$$

- Vectors with inner product with  $a$  is  $b$ :  $\frac{a^T}{\|a\|}x = \frac{b}{\|a\|}$ .  
I.e., from  $\mathbf{0}$ , walk a distance  $\frac{b}{\|a\|}$  (either + or -) in the direction of  $a$ , then draw perpendicular line.

**Definition 33** A closed halfspace is the region generated by the hyperplane and defined as:

$$\begin{aligned} \mathcal{H} &= \{x \mid a^T x \leq b\}, & a, b \in \mathbf{R}^n, a \neq 0 \\ &\equiv \{x \mid a^T(x - x_0) \leq 0\}, & a^T x_0 = b. \end{aligned}$$

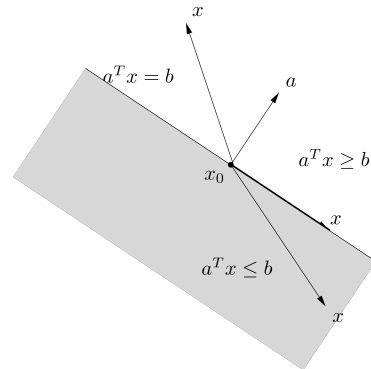
- region of all vectors with projection  $< b/\|a\|$ .
- Vectors with obtuse angle with  $a$ : ( $\cos \theta = \frac{a^T x}{\|a\|\|x\|}$ ).
- Line passing with  $p_0$  and  $\perp$  on  $\mathcal{S}$ :

$$x = p_0 + \theta \bar{a} \quad (\text{parametric eq.})$$

$$a^T x_0 = a^T p_0 + \theta_0 \|a\|$$

$$\theta_0 = (b - a^T p_0) / \|a\| \quad (x_0 \text{ pt. of intersection.})$$

$$x_0 - p_0 = \frac{(b - a^T p_0)}{\|a\|} \bar{a}.$$



**Corollary 34**  $\mathcal{S}$  is affine,  $\mathcal{H}$  is convex and not affine,  $\text{int } \mathcal{H} = \mathcal{H} \setminus \mathcal{S}$ , and  $\text{bd } \mathcal{H} = \mathcal{S}$ .

**Proof.**  $\mathcal{S}$  is affine done.  $\mathcal{H}$  is convex: take  $0 \leq \theta \leq 1$   
 $\theta a^T x_1 + (1 - \theta) a^T x_2 \leq \theta b + (1 - \theta)b = b$ . (why not affine?!)

$$y = x + ru, \quad 0 \leq \|u\| \leq 1 \quad (y \in B(x, r))$$

$$a^T y = a^T x + r a^T u = b - (b - a^T x) + r \|a\| \|u\| \cos(a, u)$$

If  $b = a^T x$ , i.e.,  $x \in \mathcal{S}$ ,  $a^T u > 0$  or  $< 0$  (depending on the angle) and hence  $a^T y > b$  or  $< b$ . Then  $\mathcal{S} \subseteq \text{bd } \mathcal{H}$ .

If  $a^T x < b$ , i.e.,  $x \in \mathcal{H} \setminus \mathcal{S}$ ,  $\exists r < \frac{b - a^T x}{\|a\|}$ , s.t.  $a^T y < b$ . Hence:  
**int**  $\mathcal{H} = \mathcal{H} \setminus \mathcal{S}$  and **bd**  $\mathcal{H} = \mathcal{S}$ .

## 2.2.2 Euclidean balls and ellipsoids

**Definition 35** A Euclidean ball in  $\mathbb{R}^n$  is the set:

$$\begin{aligned} B(x_c, r) &= \{x = x_c + ru \mid \|u\|_2 \leq 1\} \\ &= \{x \mid \|x - x_c\|_2 / r \leq 1\} \\ &= \{x \mid (x - x_c)'(x - x_c) / r^2 \leq 1\}. \end{aligned}$$

**Definition 36** Ellipsoid in  $\mathbb{R}^n$  is the set:

$$\begin{aligned} \mathcal{E} &= \{x = x_c + Au \mid \|u\|_2 \leq 1, A \succ 0\} \\ &= \{x \mid \|A^{-1}(x - x_c)\| \leq 1, A \succ 0\} \\ &= \{x \mid (x - x_c)'(A^{-1})'A^{-1}(x - x_c) \leq 1\} \end{aligned}$$

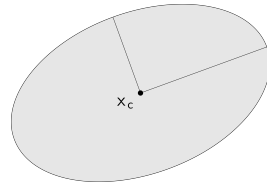
**Spectral decomposition for  $A = A'$ .**

$$\begin{aligned} Au &= (\lambda_1 v_1 v_1' + \lambda_2 v_2 v_2' + \cdots + \lambda_n v_n v_n')u \\ &= \lambda_1 v_1 (v_1' u) + \lambda_2 v_2 (v_2' u) + \cdots + \lambda_n v_n (v_n' u), \end{aligned}$$

which reduces to a Ball when  $\lambda_i = r$ .

**Remark 2**  $A$  does not have to be symmetric, since  $(A^{-1})'A^{-1} = P^{-1}$  is symmetric either way and:

$$\begin{aligned} P^{1/2}u_2 &= Au_1 && \text{is bijection} \\ \|u_2\|^2 &= u_1' A' P^{-1/2} P^{-1/2} A u_1 = \|u_1\|^2 \end{aligned}$$



**Remark 3** (Contours of  $\mathcal{N}(\mu, \Sigma)$ ) :

$$f_X(x) = \frac{1}{((2\pi)^p |\Sigma|)^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

**Corollary 37** An ellipsoid, hence a ball, is convex

**Proof.** For  $x_1, x_2 \in \mathcal{E}, 0 \leq \theta \leq 1$ ,

$$\begin{aligned} x_1 &= x_c + Au_1, \|u_1\| \leq 1 \\ x_2 &= x_c + Au_2, \|u_2\| \leq 1 \\ x &= \theta(x_c + Au_1) + (1 - \theta)(x_c + Au_2) \\ &= x_c + A(\theta u_1 + (1 - \theta)u_2) \\ \|u\| &= \|\theta u_1 + (1 - \theta)u_2\| \\ &\leq \theta\|u_1\| + (1 - \theta)\|u_2\| \\ &\leq \theta + (1 - \theta) = 1. \end{aligned}$$



## 2.2.3 Norm balls and norm cones

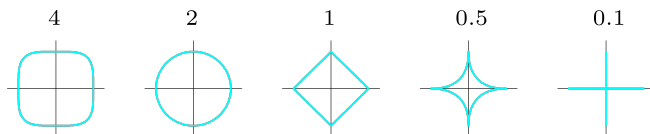
**Definition 38** Let  $x, y \in \mathbb{R}^n$ ,  $t \in \mathbb{R}$ ; a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$  with **dom**  $f = \mathbb{R}^n$  is called a norm if

1.  $f(x) = 0 \rightarrow x = 0$  (definite)
2.  $f(tx) = |t|f(x)$  (homogeneous)
3.  $f(x + y) \leq f(x) + f(y)$  (triangle inequality)

**Remark 4 :**

- norm is defined on the Euclidean vector space.
- $f(0) = 0$  is implied from (2)
- $\text{dist}(x, 0) = f(x)$
- $\text{dist}(x, y) = f(x - y) = f(y - x)$
- $\text{dist}(x, 0) = f(x)$  is a metric, but not the vice versa.

**HW:** verify that  $L^p$ -norm is a norm.



**Definition 39** ( $L^p$ -norm  $(\|\cdot\|_p)$ ) is defined as

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$

- $L_1$ -norm, Manhattan distance, Taxicab, absolute value

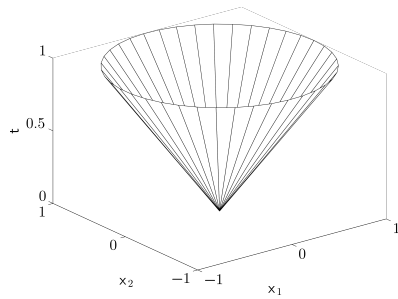
$$\|x\|_1 = \left( \sum_{i=1}^n |x_i| \right).$$

- $L_2$ -norm, Euclidean distance (most meaningful)

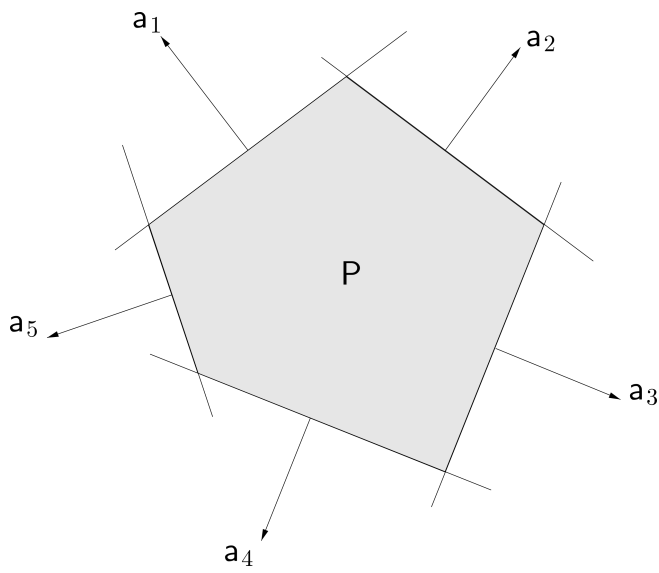
$$\|x\|_2 = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2}.$$

- $L_\infty$ -norm

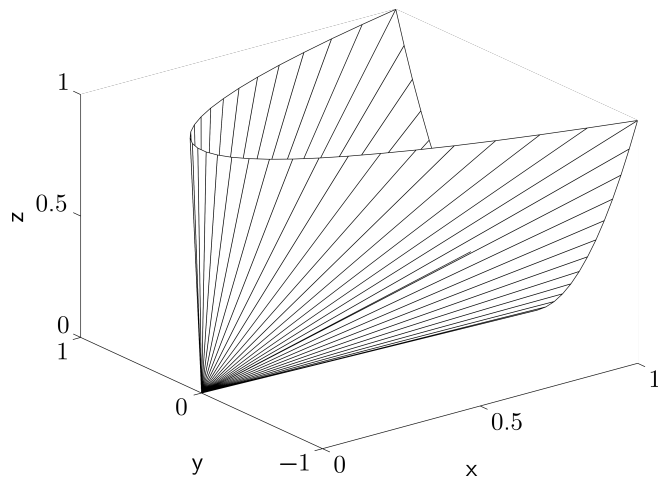
$$\|x\|_\infty = \lim_{p \rightarrow \infty} \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} = \max_i |x_i|$$



2.2.4 Polyhedra



## 2.2.5 The positive semidefinite cone



## 2.3 Operations that preserve convexity

## 2.4 Generalized inequalities

## 2.5 Separating and supporting hyperplanes

## 2.6 Dual cones and generalized inequalities



# **Part II**

# **Applications**

# **Part III**

# **Algorithms**

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