ST122:
Probability and Statistics II

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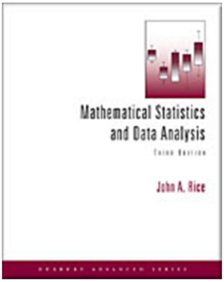
Lectures follow Rice, "*Mathematical Statistics and Data Analysis*", 3rd edition, Duxbury:



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Course Objectives

- Developing rigorous treatment.
- Building intuition and insight.
- Linking to real life problems.
- Coding and scientific computing.

Contents

Introduction: Statistical Inference in a Nutshell

Point estimate - different estimators - assessing estimators - large sample theory
Hypothesis testing.
Interval estimation.
Bayesian approach vs. Frequentist approach

Chapter 6

Distributions Derived from the Normal Distribution

6.1 Introduction

This Chapter discusses 3 probability distributions that frequently occur in Statistics: χ^2, t, and F Distributions.

Remember that if $V \sim Gamma(\alpha, \lambda)$, then

$$
\begin{aligned}
f(v) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} v^{\alpha-1} e^{-\lambda v}, \ v \geq 0, \\
M(t) &= (1 - t/\lambda)^{-\alpha}, \\
E[V] &= \alpha/\lambda, \\
\mathrm{Var}[V] &= \alpha/\lambda^2.
\end{aligned}
$$

And if $V_1, \ldots, V_n$ are i.i.d $Gamma(\alpha, \lambda)$, then

$$
\begin{aligned}
M_{\Sigma_i V_i}(t) &= (1 - t/\lambda)^{-n\alpha}, \\
\Sigma_i V_i &\sim Gamma(n\alpha, \lambda).
\end{aligned}
$$

6.2 χ^2, t, and F Distributions

Definition 1 *If $Z \sim N(0,1)$, then $U = Z^2$ is called chi-square distribution with 1 degree of freedom; i.e., $U \sim \chi_1^2$. It is easy to show that (see Lec. notes Ch. 2):*

$$f_U(u) = \frac{1}{\sqrt{2\pi}} u^{-1/2} e^{-u^2/2}.$$

Notice that:

$$\chi_1^2 \equiv Gamma\left(\frac{1}{2}, \frac{1}{2}\right),$$

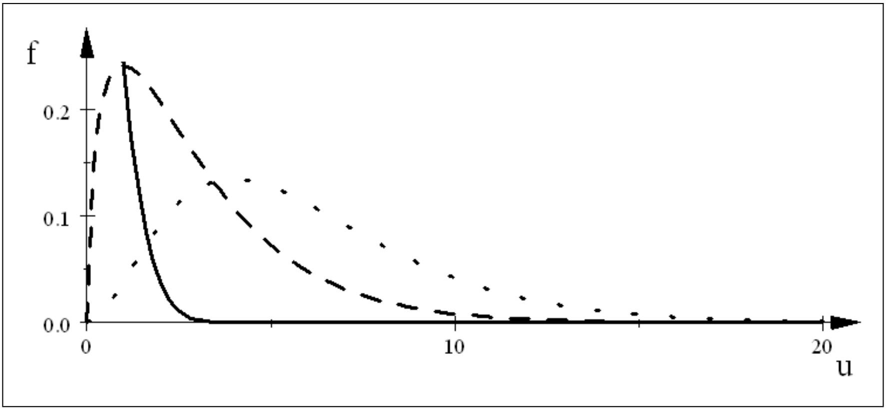
Also:

$$\begin{aligned} X &\sim N\left(\mu, \sigma^2\right), \\ \frac{X-\mu}{\sigma} &\sim N(0,1), \\ \left(\frac{X-\mu}{\sigma}\right)^2 &\sim \chi_1^2. \end{aligned}$$

Definition 2 *If $U_1, \ldots, U_n$ are i.i.d χ_1^2 r.v. then $V = \sum_i U_i$ is called chi-squre distribution with n degrees of freedom; i.e., $V \sim \chi_n^2$.*

Notice that $U_i \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$, then

$$V \sim Gamma\left(n/2, 1/2\right),$$
$$f_V(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{n/2-1} e^{-v/2},$$
$$E[V] = n, \ \mathrm{Var}[V] = 2n.$$



solid: $n = 1$, dashed: $n = 3$, dotted: $n = 6$

Suppose that U and V are indep, and

$$W = U + V.$$

If $U \sim \chi^2_m,\ V \sim \chi^2_n$ then (obviously)

$$W = \chi^2_m + \chi^2_n = \chi^2_{m+n},$$

Also, if $W \sim \chi^2_k$ and $V \sim \chi^2_n$ then

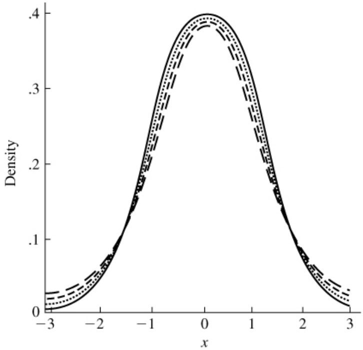
$$\begin{aligned}
\chi^2_k &= U + \chi^2_n \\
M_{\chi^2_k} &= M_U M_{\chi^2_n}, \\
M_U &= \frac{M_{\chi^2_k}}{M_{\chi^2_n}} \\
&= \frac{(1-2t)^{-k/2}}{(1-2t)^{-n/2}} = (1-2t)^{-(k-n)/2} \\
U &\sim \chi^2_{(k-n)}.
\end{aligned}$$

Definition 3 (Student's t Distribution) *:*
If $Z \sim N(0,1)$, $U \sim \chi^2_n$, and Z, U are indep. then $T = Z/\sqrt{U/n}$ is called t distribution with n degrees of freedom; i.e., $T \sim t_n$. (prove that:)

$$f_T(t) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1+\frac{t^2}{n}\right)^{-(n+1)/2},$$

$$E[T] = 0,\ n \geq 2,$$

$$\mathrm{Var}[T] = \frac{n}{n-2},\ n \geq 3.$$



- The smaller n the thicker tail.
- The figure shows $t_5, t_{10}, t_{30}\ (\approx N(0,1))$
- $t_1 \equiv Cauchy(0,1)$.

Definition 4 (Snedecor's F Distribution) *: Let $U \sim \chi^2_m$ and $V \sim \chi^2_n$, and U, V are indep. Then, $W = (U/m)/(V/n)$ is called F distribution with m, n degrees of freedom; i.e., $W \sim F_{m,n}$. (prove that:)*

$$f_W(w) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1}\left(1+\frac{m}{n}w\right)^{-\frac{(m+n)}{2}},$$

$$E[W] = n/(n-2), \ n \geq 3.$$

$$\text{Var}[W] = 2\left(\frac{n}{n-2}\right)^2 \frac{(m+n-2)}{m(n-2)}, \ n \geq 5.$$

It is obvious that if $U \sim t_n$, then $U^2 \sim F_{1,n}$.

Also, if $U \sim F_{n,m}$ then $U^{-1} \sim F_{m,n}$.

Summary (with terse notation):

$$N(0,1)^2 \sim \chi_1^2,$$
$$\sum_{i=1}^{n} N(0,1)^2 \sim \chi_n^2,$$
$$\chi_m^2 + \chi_n^2 \sim \chi_{m+n}^2,$$
$$N(0,1) / \sqrt{\chi_n^2 / n} \sim t_n,$$
$$\left(\chi_m^2 / m\right) / \left(\chi_n^2 / n\right) \sim F_{m,n},$$
$$t_n^2 \sim F_{1,n.}.$$

Example 5 *If X_1, X_2, X_3 are iid $N(0,1)$, what is the dist. of*

$$\frac{X_1}{\sqrt{\left(X_1^2 + X_2^2 + X_3^2\right) / 3}}$$

6.3 Sample Mean, Sample Variance, and Sampling from Normal Distribution

6.3.1 Basic Concepts of Random Samples

Definition 6 *The r.v.* $X_1, \ldots, X_n$ *are called a* ***random sample of size*** n ***from the population*** F *if* $X_1, \ldots, X_n$ *are i.i.d from* F*; and hence:*
$f_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \prod_i f(x_i)$.

$$
\begin{array}{llllll}
 & & X_1 & X_2 & \ldots & X_n \\
F & \xrightarrow{Sample_1} & x_1, & x_2, & \ldots & x_n \\
F & \xrightarrow{Sample_2} & x_1, & x_2, & \ldots & x_n \\
 & \vdots & & & &
\end{array}
$$

We focus in our study on infinite populations; Ch. 7 is about finite populations.

Definition 7 *Let $X_1, \ldots, X_n$ be a random sample of size n, and $T(x_1, \ldots, x_n)$ be a real- (or vector-) valued function whose domain includes the sample space of $(X_1, \ldots, X_n)$. Then the r.v. $Y = T(X_1, \ldots, X_n)$ is called a statistic.*

Definition 8 *The sample mean, sample variance, and sample standard deviations are statistics defined as:*

$$\overline{X} = \frac{1}{n}\sum_i X_i$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2,$$

$$S = \sqrt{S^2},$$

Observed values will be denoted by $\overline{x}$, s^2, and s.

$$\begin{array}{lllllll}
 & & X_1 & X_2 & \ldots & X_n & \overline{X} = \frac{1}{n}\sum_i X_i \\
F & \xrightarrow{Sample_1} & x_1, & x_2, & \ldots & x_n & \overline{x} = \frac{1}{n}\sum_i x_i \\
F & \xrightarrow{Sample_2} & x_1, & x_2, & \ldots & x_n & \overline{x} = \frac{1}{n}\sum_i x_i \\
 & \vdots & & & & &
\end{array}$$

Lemma 9 *For any numbers $x_1, \ldots, x_n$:*

$$\min_a \sum_i (x_i - a)^2 = \sum_i \left(x_i - \overline{x}\right)^2,$$
$$\sum_i \left(x_i - \overline{x}\right)^2 = \sum_i x_i^2 - n\overline{x}^2.$$

Proof. : is identical to $\arg\min_c E(Y-c)^2 = E[Y]$.

$$\begin{aligned}
\sum_i (x_i - a)^2 &= \sum_i \left(\left(x_i - \overline{x}\right) + \left(\overline{x} - a\right)\right)^2 \\
&= \sum_i \left(x_i - \overline{x}\right)^2 + \sum_i \left(\overline{x} - a\right)^2 \\
&\quad + 2\sum_i \left(x_i - \overline{x}\right)\left(\overline{x} - a\right) \qquad \left(\textstyle\sum_i x_i = n\overline{x}\right) \\
&= \sum_i \left(x_i - \overline{x}\right)^2 + \sum_i \left(\overline{x} - a\right)^2,
\end{aligned}$$

which is minimized by choosing $a = \overline{x}$.

$$\sum_i (x_i - a)^2 = \sum_i \left(x_i - \overline{x}\right)^2 + \sum_i \left(\overline{x} - a\right)^2$$
$$\sum_i \left(x_i - \overline{x}\right)^2 = \sum_i x_i^2 - n\overline{x}^2. \qquad \left(a \overset{set}{=} 0\right)$$

Notice that: both forms are $O(n)$; however this form requires only one for loop for execution! ■

HW: Write a computer program, and find its complexity (where a step is a multiplication), for calculating

$$S_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j,$$

$$S_2 = \sum_{i=1}^{n} \sum_{j \neq i} x_i x_j.$$

Can you do a mathematical trick to reduce their complexities to $O(n)$. !!!

Theorem 10 (Distribution-Free Properties) :

1. $E\left[\overline{X}\right] = \mu$,

2. $\text{Var}\left[\overline{X}\right] = \sigma^2/n$,

3. $E\left[S^2\right] = \sigma^2$.

Proof. 1 and 2 are proven before. For 3,

$$\begin{aligned}
E\left[S^2\right] &= E\left[\frac{1}{n-1}\sum_i \left(X_i - \overline{X}\right)^2\right] \\
&= \frac{1}{n-1} E\left[\sum_i X_i^2 - n\overline{X}^2\right] \\
&= \frac{1}{n-1}\left(\sum_i E\left[X_i^2\right] - nE\left[\overline{X}^2\right]\right) \\
&= \frac{1}{n-1}\left(n\left(\sigma^2+\mu^2\right) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right) = \sigma^2,
\end{aligned}$$

which is **unbiased estimator** for σ^2. ■

Lemma 11 *Let $X_1, \ldots, X_n$ be a r.s. from a population with mgf $M(t)$, then*

$$M_{\overline{X}}(t) = [M(t/n)]^n.$$

Proof. done before in CLT (just 2 lines). ■

Example 12 *Let $X_1, \ldots, X_n$ be a r.s. from $N\left(\mu, \sigma^2\right)$, then*

$$M(t) = \exp\left(\mu t + \sigma^2 t^2/2\right),$$

$$M_{\overline{X}}(t) = \left[\exp\left(\mu \frac{t}{n} + \sigma^2 \left(\frac{t}{n}\right)^2 /2\right)\right]^n,$$

$$= \exp\left(\mu t + \frac{\sigma^2}{n} t^2/2\right),$$

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

We know that $E\left[\overline{X}\right] = \mu$ and $\mathrm{Var}\left[\overline{X}\right] = \sigma^2/n$. But what is new is that $\overline{X}$ is itself Normal. ***We could have found it by transformation:*** *$Z = X_1 + X_2$. If $X_i \sim Cauchy(0,1)$, prove that $\overline{X} \sim Cauchy(0,1)$ as well!!*

6.3.2 Sampling from the Normal Distribution

Theorem 13 *Let $X_1, \ldots, X_n$ be r.s. form $N\left(\mu, \sigma^2\right)$*

1. $\overline{X} \sim N\left(\mu, \sigma^2/n\right)$,
2. $\overline{X}$ *and* $\left(X_2 - \overline{X}, \ldots, X_n - \overline{X}\right)$ *are indep,*
3. $\overline{X}$ *and* S^2 *are indep,*
4. $(n-1)\, S^2/\sigma^2 \sim \chi^2_{n-1}$.

Intuition before proof:
Meaning of $\overline{X}$ and $\left(X_2 - \overline{X}, \ldots, X_n - \overline{X}\right)$ are indep?

Suppose $X_i \sim Bernouli\,(1/2)$, and we get a sample where $\overline{X}_{10} = 1$. Obviously, $X_i = 1$.

Aside from normality, observe that

$$\sum_i \left(X_i - \overline{X}\right) = 0,$$

which means we have only $(n-1)$ differences:

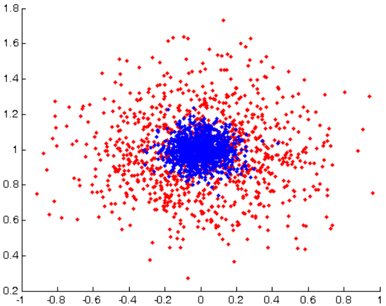
$$\begin{aligned}
\left(X_1 - \overline{X}\right) &= -\sum_{i=2}^{n} \left(X_i - \overline{X}\right), \\
S^2 &= \frac{1}{(n-1)} \sum_i \left(X_i - \overline{X}\right)^2 \\
&= \frac{1}{(n-1)} \left[\left(X_1 - \overline{X}\right)^2 + \sum_{i=2}^{n} \left(X_i - \overline{X}\right)^2\right] \\
&= \frac{1}{(n-1)} \left[\left(\sum_{i=2} \left(X_i - \overline{X}\right)\right)^2 + \sum_{i=2}^{n} \left(X_i - \overline{X}\right)^2\right]
\end{aligned}$$

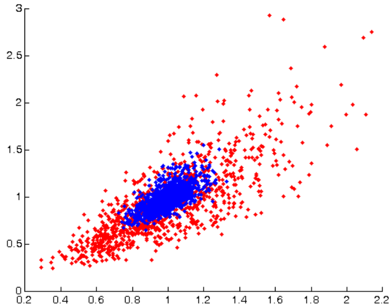
Matlab Code 6.1:

```
figure; hold on;
%Change 'Normal' to 'Exp'

x=random('Normal', 0, 1, 1000, 10);
xbar=mean(x, 2);
s=std(x, 0, 2);
plot(xbar, s, '.r')

x=random('Normal', 0, 1, 1000, 100);
xbar=mean(x, 2);
s=std(x, 0, 2);
plot(xbar, s, '.b')
```





Proof. the mgf is given by

$$
\begin{aligned}
&= M(s, t_2, \ldots, t_n) \\
&= E\left[\exp\left(s\overline{X} + t_2\left(X_2 - \overline{X}\right) + \ldots + t_n\left(X_n - \overline{x}\right)\right)\right] \\
&= E\left[\exp\left(\sum_{i=1}^{n} \frac{s}{n} X_i + \sum_{i=2}^{n} t_i\left(X_i - \overline{X}\right)\right)\right] \\
&= E\left[\exp\left(\sum_{i=1}^{n} \left(\frac{s}{n} + \left(t_i - \overline{t}\right)\right) X_i\right)\right] && (t_1 = 0) \\
&= E\left[\exp\left(\sum_{i=1}^{n} a_i X_i\right)\right] && \left(a_i = \tfrac{s}{n} + \left(t_i - \overline{t}\right)\right) \\
&= \prod_i M_{X_i}(a_i) \\
&= \prod_i \exp\left(\mu a_i + \frac{\sigma^2}{2} a_i^2\right) \\
&= \exp\left[\mu \sum_i a_i + \frac{\sigma^2}{2} \sum_i a_i^2\right] \\
&= \exp\left[\mu s + \frac{\sigma^2}{2}\left(\frac{s^2}{n} + \Sigma_i \left(t_i - \overline{t}\right)^2\right)\right] \\
&= \exp\left(\mu s + \frac{\sigma^2}{2n} s^2\right) \exp\left(\frac{\sigma^2}{2} \sum_i \left(t_i - \overline{t}\right)^2\right),
\end{aligned}
$$

the two factors are the mgf of $\overline{X}$ and $\left(X_2-\overline{X},\ldots,X_n-\overline{X}\right)$. Hence they are independent and since $S=S\left(X_2-\overline{X},\ldots,X_n-\overline{X}\right)$: $\overline{X}$ and S are independent.

Now

$$\begin{aligned}
\sum_i\left(\frac{X_i-\mu}{\sigma}\right)^2 &= \frac{1}{\sigma^2}\sum_i\left[\left(X_i-\overline{X}\right)+\left(\overline{X}-\mu\right)\right]^2 \\
&= \frac{1}{\sigma^2}\sum_i\left(X_i-\overline{X}\right)^2+\frac{1}{\sigma^2}\sum_i\left(\overline{X}-\mu\right)^2 \\
&= \frac{1}{\sigma^2}\sum_i\left(X_i-\overline{X}\right)^2+\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \\
W &= U+V && (U,V \text{ indep.}) \\
\chi_n^2 &= U+\chi_1^2 \\
U &\sim \chi_{n-1}^2. && (n-1 \text{ df})
\end{aligned}$$

■

Lemma 14

$$\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}.$$

Proof.

$$\begin{aligned}
\frac{\overline{X}-\mu}{S/\sqrt{n}} &= \frac{\left(\overline{X}-\mu\right)/\left(\sigma/\sqrt{n}\right)}{\left(S/\sqrt{n}\right)/\left(\sigma/\sqrt{n}\right)} \\
&= \frac{\left(\overline{X}-\mu\right)/\left(\sigma/\sqrt{n}\right)}{S/\sigma} \\
&= \frac{\left(\overline{X}-\mu\right)/\left(\sigma/\sqrt{n}\right)}{\sqrt{\left((n-1)\,S^2/\sigma^2\right)/(n-1)}} \\
&= \frac{N(0,1)}{\sqrt{\chi^2_{n-1}/(n-1)}} = t_{n-1},
\end{aligned}$$

used for inference about μ when σ is unkown.

$$\frac{\overline{X}-\mu}{\sigma} \sim N(0,1)$$

used for inference about μ when σ is known. ■

Lemma 15 *If $X \sim N\left(\mu_X, \sigma_X\right)$, $Y \sim N\left(\mu_Y, \sigma_Y\right)$, and we have two samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$*

$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F_{m-1,n-1}.$$

Proof.

$$\begin{aligned}
\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} &= \frac{\left((m-1)\,S_X^2/\sigma_X^2\right)/\,(m-1)}{\left((n-1)\,S_Y^2/\sigma_Y^2\right)/\,(n-1)} \\
&= \frac{\chi^2_{m-1}/\,(m-1)}{\chi^2_{n-1}/\,(n-1)} && \text{(Indep.)} \\
&= F_{m-1,n-1},
\end{aligned}$$

used for inference about σ_X^2/σ_Y^2. ∎

Chapter 8

Estimation of Parameters and Fitting of Probability Distributions

8.1 Introduction: Estimation in a Nutshell

- Distributions depend on some population parameters; e.g., $N\left(\mu,\sigma^2\right)$, $Exp\,(\lambda)$, etc. Generally, we should write (e.g.,):

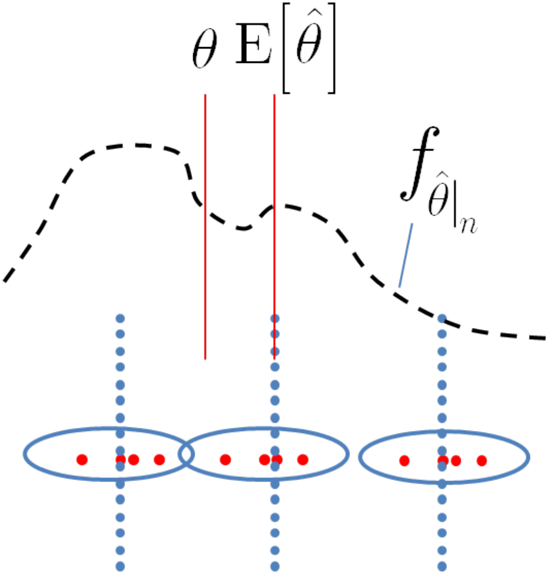
$$f_X\left(x|\mu,\sigma\right) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{-1}{2}\left(x-\mu\right)^2/\sigma^2\right]$$

- Obtaining data (values of a random sample) allows "estimating" these parameters.

Definition 16 *A* point estimator *is any function* $W(X_1,\ldots,X_n)$ *of a sample; i.e., any statistic is a point estimator.*

- We can choose, e.g., $\widehat{\sigma}^2 = \frac{1}{n}\sum_i\left(X_i-\overline{X}\right)^2$ to be an estimator for σ^2.

- $\frac{1}{n}\sum_i\left(x_i-\overline{x}_i\right)^2$ is an estimate (realization).



- How to estimate θ "well" ($\widehat{\theta}$)?
- What is $f_{\widehat{\theta}}$ (**sampling distribution**)?
- What is $\mathrm{E}\left[\widehat{\theta}\right], \mathrm{SD}\left[\widehat{\theta}\right]$ (**standard error**),...?
- How to estimate $\tau(\theta)$, e.g.:
  - σ^2, the variance, for $N\left(\mu, \sigma^2\right)$.
  - $\alpha\lambda$, the mean, for $Gamma(\alpha, \lambda)$.

How to decide F_X before estimation?

- From the physics of the problem. E.g., given number of calls in time units, the distribution is known to be $Poisson(\lambda)$.

- Assumption; you need to validate it latter.

Why do we estimate parameters?

- Understanding (interpretation).

- Prediction.

- Simulation and data generation.

How do we choose estimators?

8.2 The Method of Moments

We estimate k^{th} moment by **sample moment**

$$\mu_k = \mathrm{E}\left[X^k\right]$$
$$\widehat{\mu}_k = \frac{1}{n}\sum_i X_i^k.$$

Then for population parameters θ_i, we have

$$\begin{aligned}
\mu_1 &= \mu_1\left(\theta_1, \ldots, \theta_r\right), \\
&\vdots \\
\mu_r &= \mu_r\left(\theta_1, \ldots, \theta_r\right).
\end{aligned}$$

We solve

$$\begin{aligned}
\theta_1 &= \theta_1\left(\mu_1, \ldots, \mu_r\right), \\
&\vdots \\
\theta_r &= \theta_r\left(\mu_1, \ldots, \mu_r\right).
\end{aligned}$$

And

$$\begin{aligned}
\widehat{\theta}_1 &= \widehat{\theta}_1\left(\widehat{\mu}_1, \ldots, \widehat{\mu}_r\right), \\
&\vdots \\
\widehat{\theta}_r &= \widehat{\theta}_r\left(\widehat{\mu}_1, \ldots, \widehat{\mu}_r\right).
\end{aligned}$$

Motivation behind method of moments

$$\widehat{\mu}_k \xrightarrow{p} \mu_k.$$

Definition 17 *An estimator* $\widehat{\theta} = \widehat{\theta}(n)$*, which estimates* θ*, from a sample of size* n *is said to be consistent in probability if*

$$\widehat{\theta} \xrightarrow{p} \theta.$$

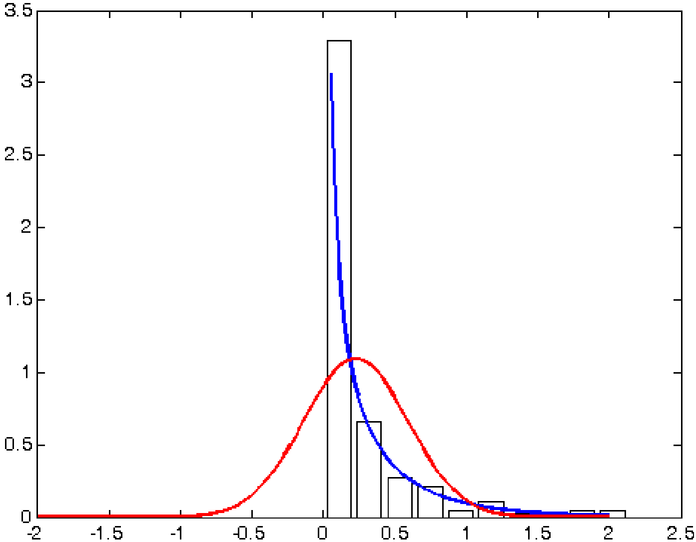
Example 18 *$N(\mu, \sigma^2)$, and the mean and variance of any other distribution:*

$$
\begin{aligned}
\hat{\mu}_1 &= \frac{1}{n}\sum_i X_i = \overline{X}, \\
\hat{\mu}_2 &= \frac{1}{n}\sum_i X_i^2, \\
\mu_1 &= \mathrm{E}[X] = \mu, \\
\mu_2 &= \mathrm{E}\left[X^2\right] = \mu^2 + \sigma^2, \\
\mu &= \mu_1, \\
\sigma^2 &= \mu_2 - \mu_1^2, \\
\hat{\mu} &= \hat{\mu}_1 = \overline{X}, \\
\hat{\sigma}^2 &= \hat{\mu}_2 - \hat{\mu}_1^2 = \frac{1}{n}\sum_i X_i^2 - \overline{X}^2 \qquad (\widehat{\sigma^2}) \\
&= \frac{1}{n}\left(\sum_i X_i^2 - n\overline{X}^2\right) = \frac{1}{n}\sum_i \left(X_i - \overline{X}\right)^2 \\
&= \frac{n-1}{n} S^2, \\
\hat{\mu} &\sim N\left(\mu, \sigma^2/n\right), \\
\frac{n\hat{\sigma}^2}{\sigma^2} &\sim \chi^2_{n-1}.
\end{aligned}
$$

Example 19 *: Analyzing real dataset for average amount of storms rainfall in Illinois.*

Let's draw data points and normalized histogram (divide by its area):

$$Area = \sum_i \Delta N_i$$
$$= \Delta \sum_i N_i = \Delta n.$$



From the mgf of Gamma we obtained

$$
\begin{aligned}
\mathrm{E}[X] = \mu_1 &= \frac{\alpha}{\lambda}, \\
\mathrm{E}\left[X^2\right] = \mu_2 &= \frac{\alpha(\alpha+1)}{\lambda^2},
\end{aligned}
$$

Solve both equations for α and λ,

$$
\begin{aligned}
\alpha &= \lambda\mu_1 \\
\mu_2 &= \frac{\lambda^2\mu_1^2 + \lambda\mu_1}{\lambda^2}, \\
&= \mu_1^2 + \mu_1/\lambda, \\
\lambda &= \frac{\mu_1}{\mu_2 - \mu_1^2}, \\
\alpha &= \frac{\mu_1^2}{\mu_2 - \mu_1^2}, \\
\widehat{\mu}_1 &= \frac{1}{n}\sum x_i = 0.2244, \\
\widehat{\mu}_2 &= \frac{1}{n}\sum x_i^2 = 0.1836, \\
\widehat{\lambda} &= 1.6842, \\
\widehat{\alpha} &= 0.3779
\end{aligned}
$$

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$
$$= 0.5178 x^{-0.6221} e^{-1.6842x}, \ x \geq 0$$

What would happen have if we fit $N\left(\mu, \sigma^2\right)$?

Matlab Code 8.1:

```
x=[];
x=[x; csvread('illinois60.txt')];
x=[x; csvread('illinois61.txt')];
x=[x; csvread('illinois62.txt')];
x=[x; csvread('illinois63.txt')];
x=[x; csvread('illinois64.txt')];

n=length(x) %will be 227
plot(x, zeros(length(x)), '.r')
[N, xout]=hist(x);
bar(xout, N/(n*(xout(2)-xout(1))), 'w'
    ); %normalize
hold on;
```

```
mu1   = sum(x)/n              % .2244
mu2   = sum(x.^2)/n           % .1836
alpha= mu1^2/(mu2-mu1^2)      % .3779
lmda  = mu1/(mu2-mu1^2)       % 1.6842

z=0.05:.01:2;
y1=(lmda^alpha)/gamma(alpha)  *  z.^(
    alpha-1)  .*  exp(-lmda*z);
plot(z, y1, 'b', 'LineWidth', 2);

z=-2:.01:2;
y2=1/(sqrt(2*pi*(mu2-mu1^2)))  *exp(-(z
    -mu1).^2  /  (2*(mu2-mu1^2)));
plot(z, y2, 'r', 'LineWidth', 2);
```

Example 20 ($Binomial(n,p)$)

$$
\begin{aligned}
\mu_1 &= np, \\
\mu_2 &= np(1-p) + (np)^2, \\
p &= \frac{\mu_1}{n}, \\
\mu_2 &= \mu_1\left(1 - \frac{\mu_1}{n}\right) + \mu_1^2 \\
n &= \frac{\mu_1^2}{\mu_1 - (\mu_2 - \mu_1^2)} \\
p &= \frac{\mu_1 - (\mu_2 - \mu_1^2)}{\mu_1}, \\
\widehat{n} &= \frac{\overline{X}^2}{\overline{X} - \frac{1}{n}\sum_i \left(X_i - \overline{X}\right)^2}, \\
\widehat{p} &= \frac{\overline{X} - \frac{1}{n}\sum_i \left(X_i - \overline{X}\right)^2}{\overline{X}}.
\end{aligned}
$$

- Sometimes the estimate will be negative!!
- In general, method of moments is a good start.

Example 21 ($\mathrm{Cov}(X, Y)$) :

$$\begin{aligned}
\sigma_X^2 &= E\left(X - \mu_X\right)^2 \\
&= E\left(X^2\right) - \mu_X^2 \\
&= \mu_{2X} - \mu_{1X}^2. \\
\mathrm{Cov}(X, Y) &= \mathrm{E}\left(X - \mu_X\right)\left(Y - \mu_Y\right) \\
&= \mathrm{E}[XY] - \mu_X \mu_Y \\
&= \mu_{11} - \mu_{1X}\mu_{1Y}
\end{aligned}$$

$$\begin{aligned}
\widehat{\sigma}_X^2 &= \frac{1}{n}\sum_i X_i^2 - \overline{X}^2 \\
&= \frac{1}{n}\sum_i \left(X_i - \overline{X}\right)^2. \\
\widehat{\sigma}_{XY} &= \frac{1}{n}\sum_i X_i Y_i - \overline{XY}. \\
&= \frac{1}{n}\sum_i \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right).
\end{aligned}$$

Given $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_m$, *what is* $\widehat{\sigma}_{XY}$*?*
What is right $\left(x_i, y_i\right)$.

$$
\begin{aligned}
\mathrm{E}\left[X_i Y_i\right] &= \mathrm{Cov}(X, Y) + \mu_X \mu_Y \\
\mathrm{E}\left[\overline{XY}\right] &= \mathrm{Cov}\left(\overline{X}, \overline{Y}\right) + \mathrm{E}\left[\overline{X}\right] \mathrm{E}\left[\overline{Y}\right] \\
&= \mathrm{Cov}\left(\frac{1}{n}\sum_i X_i, \frac{1}{n}\sum_i Y_i\right) + \mu_X \mu_Y \\
&= \frac{1}{n^2}\sum_i \sum_j \mathrm{Cov}\left(X_i, Y_j\right) + \mu_X \mu_Y \\
&= \frac{1}{n}\mathrm{Cov}(X, Y) + \mu_X \mu_Y
\end{aligned}
$$

$\mathrm{E}\sum_i \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right) =$

$$
\begin{aligned}
&= \mathrm{E}\left[\sum_i X_i Y_i - n\overline{XY}\right] \\
&= n\,\mathrm{E}[XY] - n\,\mathrm{E}\left[\overline{XY}\right]. \\
&= n\sigma_{XY} + n\mu_X\mu_Y - \sigma_{XY} - n\mu_X\mu_Y \\
&= (n-1)\,\sigma_{XY}.
\end{aligned}
$$

Therefore, $\frac{1}{n}\sum_i \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)$ is biased for σ_{XY}.

Another proof for $\mathrm{E}\left[\overline{XY}\right]$:

$$
\begin{aligned}
\mathrm{E}\left[\overline{XY}\right] &= \mathrm{E}\left[\left(\frac{1}{n}\sum_i X_i\right)\left(\frac{1}{n}\sum_i Y_i\right)\right] \\
&= \mathrm{E}\left[\frac{1}{n^2}\sum_i\sum_j X_i Y_j\right] \\
&= \frac{1}{n^2}\mathrm{E}\left[\sum_i X_i Y_i + \sum\sum_{i\neq j} X_i Y_j\right] \\
&= \frac{1}{n^2}\left(n\,\mathrm{E}\left[XY\right] + n\left(n-1\right)\mathrm{E}\left[X_i Y_j\right]\right) \\
&= \frac{1}{n}\left(\mathrm{E}\left[XY\right] + \left(n-1\right)\mathrm{E}\left[X_i Y_j\right]\right) \\
&= \frac{1}{n}\left(\mathrm{Cov}\left(X, Y\right) + \mu_X\mu_Y + \left(n-1\right)\mu_X\mu_Y\right) \\
&= \frac{1}{n}\mathrm{Cov}\left(X, Y\right) + \mu_X\mu_Y.
\end{aligned}
$$

8.3 The Method of Maximum Likelihood

Likelihood is a function of parameters:

$$
\begin{aligned}
lik(\theta) &= f_{X_1 \ldots X_n}(x_1, \ldots, x_n | \theta) \\
&= \prod_{i=1}^{n} f(x_i | \theta). && \text{(i.i.d.)}
\end{aligned}
$$

- For given data $x_1, \ldots, x_n$, what is the value of θ that maximizes $lik(\theta)$.
- Remember Example 15, Page 19 in Lecture Notes.
- Much easier, in many cases, to deal with the **log likelihood** :

$$
l(\theta) = \sum_{i=1}^{n} \log f(x_i | \theta).
$$

Example 22 ($Poisson(\lambda)$)

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!},\ 0 \leq x.$$

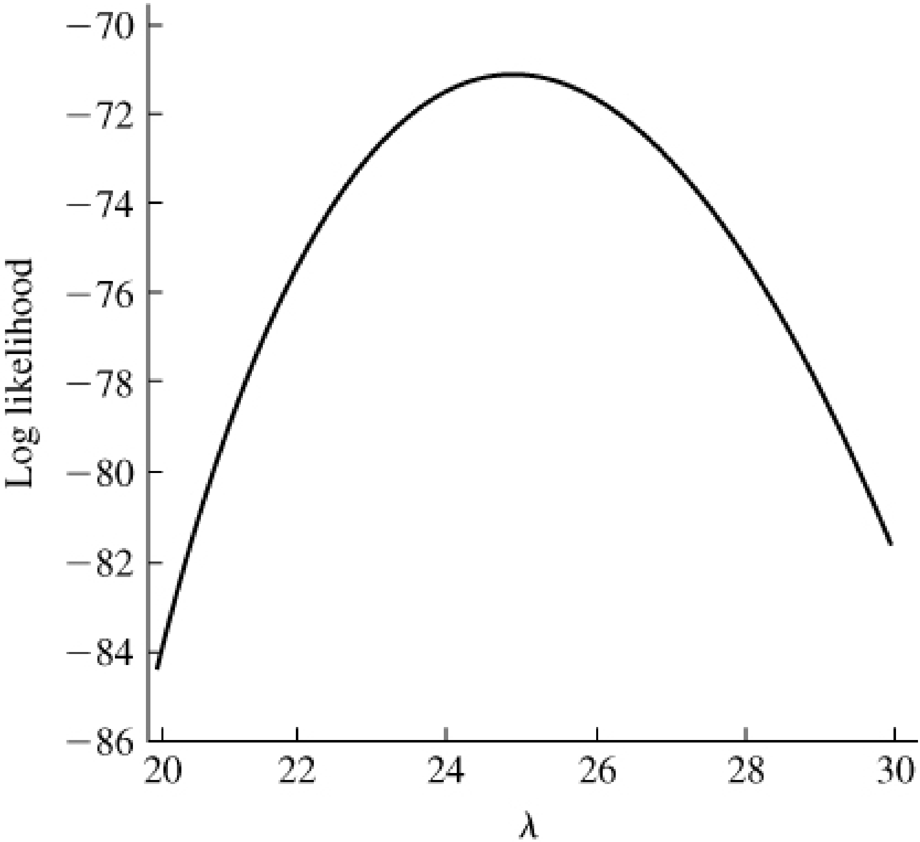
$$\begin{aligned}
lik(\lambda) &= p(x_1, \ldots, x_x) = \prod_{i=1}^{n} \left(\frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right), \\
l(\lambda) &= \sum_{i=1}^{n} \log \left(\frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right) \\
&= \sum_i \left[x_i \log \lambda - \lambda - \log(x_i!) \right] \\
&= \log(\lambda) \sum_i x_i - n\lambda - \sum_i \log(x_i!) && (8.1) \\
l'(\lambda) &= \frac{\sum_i x_i}{\lambda} - n, && (l'(\lambda) \overset{\text{set}}{=} 0) \\
\widehat{\lambda} &= \frac{1}{n} \sum x_i = \overline{X}, && \text{(MoM)} \\
l''(\lambda) &= \frac{-\sum_i x_i}{\lambda^2} \leq 0. && (x_i \geq 0)
\end{aligned}$$

Therefore, $\widehat{\lambda} = \overline{X}$ is a point of local maxima; and

$$\lim_{\lambda \to \infty} l(\lambda) = -\infty,$$

then, $\widehat{\lambda} = \overline{X}$ is a global maximum as well.

What does (8.1) mean for `asbestos` dataset?



Example 23 ($N\left(\mu,\sigma^2\right)$, both are unkown)

$$f\left(x|\mu,\sigma\right) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[\frac{-1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

$$\begin{aligned}
l\left(\mu,\sigma\right) &= \sum_{i=1}^{n}\log f\left(x_i|\mu,\sigma\right)\\
&= \sum_i\left[-\log\sigma-\log\sqrt{2\pi}-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]\\
&= -n\log\sigma-n\log\sqrt{2\pi}-\frac{1}{2\sigma^2}\sum_i\left(x_i-\mu\right)^2\\
\frac{\partial l}{\partial\mu} &= \frac{1}{\sigma^2}\sum_i\left(x_i-\mu\right) && \left(\frac{\partial l}{\partial\mu}\overset{\text{set}}{=}0\right)\\
0 &= \sum_i x_i-n\widehat{\mu},\\
\widehat{\mu} &= \frac{1}{n}\sum_i x_i=\overline{X}. && \text{(MoM)}\\
\frac{\partial l}{\partial\sigma} &= \frac{-n}{\sigma}+\frac{1}{\sigma^3}\sum_i\left(x_i-\mu\right)^2 && \left(\frac{\partial l}{\partial\sigma}\overset{\text{set}}{=}0\right)\\
\widehat{\sigma}^2 &= \frac{1}{n}\sum_i\left(x_i-\overline{X}\right)^2. && \text{(MoM)}
\end{aligned}$$

To verify that $\left(\widehat{\mu},\widehat{\sigma}\right)$ is a point of global maxima through calculus we have to satisfy:

First: it is a point of local maxima

- $\frac{\partial l}{\partial \mu}|_{\widehat{\mu}} = \frac{\partial l}{\partial \sigma}|_{\widehat{\sigma}} = 0$ *(satisfied)*

- $\frac{\partial^2 l}{\partial \mu^2}|_{\widehat{\mu}} = 0$ *or* $\frac{\partial^2 l}{\partial \sigma^2}|_{\widehat{\sigma}} = 0$ *(satisfied)*

- $\left| \begin{array}{cc} \frac{\partial^2 l}{\partial \mu^2} & \frac{\partial^2 l}{\partial \mu \partial \sigma} \\ \frac{\partial^2 l}{\partial \mu \partial \sigma} & \frac{\partial^2 l}{\partial \sigma^2} \end{array} \right|_{\widehat{\mu},\widehat{\sigma}} > 0$ *(needs work).*

Second: there is no maximum at infinity (messy).

Instead, we can use a trick:

$$l(\mu,\sigma) = -n \log \sigma - n \log \sqrt{2\pi} - \frac{1}{2\sigma^2} \sum_i (x_i - \mu)^2$$

is maximized for

$$\sum_i (x_i - \mu)^2 = \sum_i \left(x_i - \overline{X}\right)^2.$$

Then $l\left(\overline{X},\sigma\right)$ is a function in single variable σ,

$$\frac{\partial l}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_i \left(x_i - \overline{X}\right)^2, \qquad \left(\frac{\partial l}{\partial \sigma} \overset{set}{=} 0\right)$$

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_i \left(x_i - \overline{X}\right)^2$$

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_i \left(x_i - \overline{X}\right)^2$$

$$= \frac{n}{\sigma^2}\left(1 - \frac{3}{n\sigma^2}\sum_i \left(x_i - \overline{X}\right)^2\right),$$

$$\left.\frac{\partial^2 l}{\partial \sigma^2}\right|_{\widehat{\sigma}} = \frac{n}{\widehat{\sigma}^2}(1-3) < 0,$$

which gives a local maximum for $l(\sigma)$. And

$$\lim_{\sigma\to\infty} l(\sigma) = -\infty.$$

Hence, $\widehat{\sigma}$ attains a global maxima.

Example 24 ($Gamma(\alpha, \lambda)$) **:**

$$f(x) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x},\ 0 \le x < \infty$$

$$\begin{aligned}
l(\alpha, \lambda) &= \sum_{i=1}^{n} \left(\alpha \log \lambda + (\alpha - 1) \log x_i - \lambda x_i - \log \Gamma(\alpha)\right) \\
&= n\alpha \log \lambda + (\alpha - 1) \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i \\
&\quad - n \log \Gamma(\alpha)
\end{aligned}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i \qquad \left(\frac{\partial l}{\partial \lambda} \overset{set}{=} 0\right)$$

$$0 = \frac{n\widehat{\alpha}}{\widehat{\lambda}} - \sum_{i=1}^{n} x_i$$

$$\widehat{\lambda} = \frac{\widehat{\alpha}}{\overline{X}}.$$

$$\frac{\partial l}{\partial \alpha} = n \log \lambda + \sum_{i=1}^{n} \log x_i - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \qquad \left(\frac{\partial l}{\partial \alpha} \overset{set}{=} 0\right)$$

$$0 = n \log\left(\frac{\widehat{\alpha}}{\overline{X}}\right) + \sum_{i=1}^{n} \log x_i - n \frac{\Gamma'(\widehat{\alpha})}{\Gamma(\widehat{\alpha})}$$

$$0 = n \log \widehat{\alpha} - n \log \overline{X} + \sum_{i=1}^{n} \log x_i - n \frac{\Gamma'(\widehat{\alpha})}{\Gamma(\widehat{\alpha})},$$

- no closed-form solution.
- solution has to be found either by numerical methods or bootstrap (later)
- more complications for checking the second derivatives.

Example 25

$$
\begin{aligned}
f(x) &= \frac{1}{\theta},\ 0 \le x \le \theta \\
&= \frac{1}{\theta} I_{(0 \le x \le \theta)} \\
l(\theta) &= \sum_{i=1}^{n} -\log\theta,\ x_i \le \theta \\
&= -n\log\theta,\ x^{(n)} \le \theta \\
\widehat{\theta} &= x^{(n)}.
\end{aligned}
$$

- *Intuitively, this is clear.*
- *We know $f_{X^{(n)}}(x)$ for $X \sim Uniform(0,\theta)$.*
- *Compare to MoM:*

$$
\begin{aligned}
\mu_1 &= \frac{\theta}{2} \\
\widehat{\theta} &= 2\overline{X}.
\end{aligned}
$$

Example 26 ($Multinomial(p_1, \ldots, p_m)$) :
$\sum_{i=1}^m p_i = 1,\ \sum_{i=1}^m x_i = n$

$$f(x_1, \ldots, x_m) = \frac{n!}{x_1! \ldots x_m!} p_1^{x_1} \ldots p_m^{x_m}$$

$$l\left(p_1, \ldots, p_m\right) = \log n! - \sum_{i=1}^m \log x_i! + \sum_{i=1}^m x_i \log p_i$$

Using Lagrange multiplier

$$L\left(p_1, \ldots, p_m, \lambda\right) = \log n! - \sum_{i=1}^m \log x_i! + \sum_{i=1}^m x_i \log p_i + \lambda \left(\sum_{i=1}^m p_i - 1 \right)$$

$$\frac{\partial L}{\partial p_i} = \frac{x_i}{p_i} + \lambda \qquad \left(\frac{\partial L}{\partial p_i} \overset{set}{=} 0\right)$$

$$\widehat{p}_i = \frac{-x_i}{\lambda},$$

$$1 = \sum_i \widehat{p}_i = \sum_{i=1}^m \frac{-x_i}{\lambda} = \frac{-n}{\lambda},$$

$$\lambda = -n,$$

$$\widehat{p}_i = \frac{x_i}{n} \qquad \text{(intuitive)}$$

- *A special case is* $Binomial(n, p)$, *where* $m = 2$, $p_1 = p$, $x_1 = x$, *n is known*

$$\widehat{p} = \frac{x}{n},$$

- *n above is a parameter; the number of observations is 1, which is the vector* $(x_1, \ldots, x_m)$.

For K observations: $(x_{11},\dots x_{1m}),\dots,(x_{K1},\dots x_{Km})$.

$$f(x_1,\dots,x_K)=\prod_{k=1}^{K}\frac{n!}{x_{k1}!\dots x_{km}!}p_1^{x_{k1}}\dots p_m^{x_{km}}$$

$$L\left(p_1,\dots,p_m,\lambda\right)=\log(n!)^K-\sum_{i=1}^{m}\sum_{k=1}^{K}\log x_{ki}!$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{K}x_{ki}\log p_i+\lambda\left(\sum_{i=1}^{m}p_i-1\right)$$

$$\frac{\partial L}{\partial p_i}=\frac{\sum_{k=1}^{K}x_{ki}}{p_i}+\lambda,$$

$$\widehat{p}_i=\frac{-\sum_{k=1}^{K}x_{ki}}{\lambda}$$

$$1=\frac{-\sum_{i=1}^{m}\sum_{k=1}^{K}x_{ki}}{\lambda}=\frac{-nK}{\lambda}$$

$$\widehat{p}_i=\frac{\sum_{k=1}^{K}x_{ki}}{nK}=\frac{\overline{X_i}}{n},$$

which for $Binomial\left(n,p\right)$ *will be*

$$\widehat{p}=\frac{\overline{X}}{n},$$

which is very intuitive.

8.3.1 Large Sample Theory for MLE

Reminder:

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad (\overline{X})$$

$$\widehat{\mu} \xrightarrow{p} \mathrm{E}[X] \qquad \text{(WLLN)}$$

$$\sqrt{n}\frac{\widehat{\mu} - \mu}{\sigma} \xrightarrow{d} N(0,1) \qquad \text{(CLT)}$$

$$\lim_{n\to\infty} \Pr\left(\sqrt{n}\frac{\widehat{\mu} - \mu}{\sigma} \le x\right) = \Pr(N(0,1) \le x)$$

$$\begin{aligned}\lim_{n\to\infty} \Pr\left(\sqrt{n}\left(\widehat{\mu} - \mu\right) \le \sigma x\right) &= \Pr(\sigma N(0,1) \le \sigma x) \\ &= \Pr\left(N\left(0,\sigma^2\right) \le \sigma x\right)\end{aligned}$$

$$\sqrt{n}\left(\widehat{\mu} - \mu\right) \xrightarrow{d} N\left(0,\sigma^2\right) \qquad \text{(CLT')}$$

Definition 27 (Asymptotic Mean and Variance) *: For any statistic (or estimator) T_n, if*

$$k_n \frac{T_n - \mu}{\sigma} \xrightarrow{d} N(0,1), \qquad (k_n \text{ can be } \sqrt{n})$$

we call μ and σ^2 the asymptotic mean and variance (even if $\mathrm{E}[T_n] \neq \mu$ and $\mathrm{Var}[T_n] \neq \sigma^2$).

MoM:

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{$\overline{X}$}$$

$$\widehat{\mu} \xrightarrow{p} \mathrm{E}\left[X\right] \tag{WLLN}$$

$$\sqrt{n}\frac{\widehat{\mu} - \mathrm{E}\left[X\right]}{\sqrt{\mathrm{Var}\left[X\right]}} \xrightarrow{d} N(0,1) \tag{CLT}$$

$$\widehat{\mu}_r = \frac{1}{n}\sum_{i=1}^{n} X_i^r, \tag{MoM}$$

$$\widehat{\mu}_r \xrightarrow{p} \mathrm{E}\left[X^r\right] \qquad (\mathrm{E}\left[\widehat{\mu}_r\right] \overset{always}{=} \mathrm{E}\left[X^r\right])$$

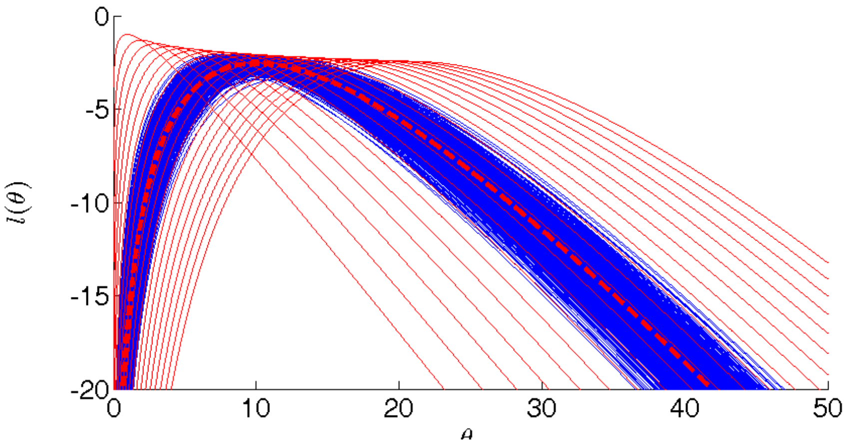
$$\sqrt{n}\frac{\widehat{\mu}_r - \mathrm{E}\left[X^r\right]}{\sqrt{\mathrm{Var}\left[X^r\right]}} \xrightarrow{d} N(0,1)$$

Notice that:

- $\mathrm{E}\left[\widehat{\mu}_r\right] = \mathrm{E}\left[X^r\right]$ (always unbiased $\forall\, n$)
- the estimated parameters, e.g., $\widehat{\sigma}^2$, may be biased for finite n.

Some Intuition First:

$$l(\theta|X) = X\log\theta - \theta - \log(X!)$$
$$\mathrm{E}[l(\theta|X)] = \mathrm{E}[X]\log\theta - \theta - \mathrm{E}\left[\log(X!)\right]$$
$$l(\theta|X_1,\ldots,X_n) = \sum_i X_i\log\theta - n\theta - \sum_i \log(X_i!)$$
$$\frac{1}{n}l(\theta) \xrightarrow{p} \mathrm{E}\left[\log f(X|\theta)\right]$$



- **Take care:** $\mathrm{E}[X]$ above is $\mathrm{E}_{X|\theta_0}[X]$.
- Why curves are less than zero?
- We simulated 1000 curves, why few are there

Matlab Code 8.2:

```
theta0=10; theta = (0:.01:50)';
C = 1000;
ltheta = zeros(length(theta), C);

figure1 = figure; fs=20;
set(gcf, 'Units', 'inches');
haxes=axes('Parent',figure1,'YLim'
    ,[-20 0],'XLim',[0 50],'FontSize',
    fs);
xlabel('$\theta$','Interpreter','latex
    ','FontSize',fs, 'Units', '
    normalized');
ylabel('$l(\theta)$','Interpreter','
    latex','FontSize',fs, 'Units', '
    normalized');

hold all;
```

```
n=10;
for c=1:C
    x=random('Poisson',theta0,[n,1]);
    ltheta(:, c)=mean(x)*log(theta)-
      theta-sum(log(factorial(x)))/n;
    plot(theta, ltheta(:, c), 'b');
end;

n=1;
for c=1:C
    x=random('Poisson',theta0,[n,1]);
    ltheta(:, c)=x*log(theta)-theta-
      sum(log(factorial(x)));
    plot(theta, ltheta(:, c), 'r');
end;
plot(theta, mean(ltheta, 2), 'r--', '
  LineWidth', 4);
```

Theorem 28 *Under regularity conditions on f, the MLE estimator is consistent*

Semi-Proof. :Under regularity conditions

$$
\begin{aligned}
l(\theta) &= \sum_{i=1}^{n} \log f(X_i|\theta), \\
\frac{1}{n} l(\theta) &\xrightarrow{p} \mathrm{E}\left[\log f(X|\theta)\right], \qquad (\mathrm{E}_{X|\theta_0}) \\
\arg\max l(\theta) &= \arg\max \frac{1}{n} l(\theta) \quad \text{(of course)} \\
&\overset{I\ hope}{=} \arg\max \mathrm{E}\left[\log f(X|\theta)\right] \\
\frac{\partial}{\partial\theta} \mathrm{E}\left[\log f(X|\theta)\right] &= \frac{\partial}{\partial\theta} \int \log f(x|\theta)\ f(x|\theta_0)\ dx \\
&= \int \frac{\partial}{\partial\theta} \log f(x|\theta)\ f(x|\theta_0)\ dx \\
&= \int \frac{\frac{\partial}{\partial\theta} f(x|\theta)}{f(x|\theta)} f(x|\theta_0)\ dx \\
\left.\frac{\partial}{\partial\theta} \mathrm{E}\left[\log f(X|\theta)\right]\right|_{\theta_0} &= \left.\int \frac{\partial}{\partial\theta} f(x|\theta)\, dx\right|_{\theta_0} \\
&= \left.\frac{\partial}{\partial\theta} \int f(x|\theta)\, dx\right|_{\theta_0} \\
&= \left.\frac{\partial}{\partial\theta} 1\right|_{\theta_0} = 0
\end{aligned}
$$

■

Lemma 29 *Under regularity conditions:*

$$\mathrm{E}\left[\frac{\partial}{\partial\theta}\log f\left(X|\theta\right)\right] = 0 \qquad (\mathrm{E}_{X|\theta})$$

$$\mathrm{E}\left[\left(\frac{\partial}{\partial\theta}\log f\left(X|\theta\right)\right)^2\right] = -\,\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\log f\left(X|\theta\right)\right],$$

which is called $I(\theta)$, the Fisher information (information number) of one observation.

- What is the meaning of "Information" here? Let's see on the figure.
- Meaning of both equations.

Proof.

$$f(x|\theta)\frac{\partial}{\partial\theta}\log f(x|\theta) = f(x|\theta)\frac{\frac{\partial}{\partial\theta}f(x|\theta)}{f(x|\theta)} = \frac{\partial}{\partial\theta}f(x|\theta)$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial\theta}(1) = \frac{\partial}{\partial\theta}\int f(x|\theta)\,dx = \int \frac{\partial}{\partial\theta}f(x|\theta)\,dx \\
&= \int f(x|\theta)\frac{\partial}{\partial\theta}\log f(x|\theta)\,dx && (\mathrm{E}_{X|\theta_0}) \\
&= \frac{\partial}{\partial\theta}\int f(x|\theta)\frac{\partial}{\partial\theta}\log f(x|\theta)\,dx \\
&= \int \frac{\partial}{\partial\theta}f(x|\theta)\frac{\partial}{\partial\theta}\log f(x|\theta)\,dx + \\
&\int f(x|\theta)\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\,dx \\
&= \int f(x|\theta)\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2 dx + \\
&\int f(x|\theta)\left(\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right) dx \\
&= \mathrm{E}\left[\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right] + \mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right]
\end{aligned}$$

■

Theorem 30 *Let* $X_1,\ldots,X_n \overset{iid}{\sim} f(X|\theta)$, $\widehat{\theta}$ *is the MLE of* θ. *Then, under regularity conditions*

$$\sqrt{n}\frac{\widehat{\theta}-\theta}{1/\sqrt{I(\theta)}} \xrightarrow{d} N(0,1),$$

$$\sqrt{n}\frac{\tau\left(\widehat{\theta}\right)-\tau(\theta)}{1/\sqrt{I(\theta)}} \xrightarrow{d} N(0,1).$$

That is, any estimator $\tau\left(\widehat{\theta}\right)$ *(or* $\widehat{\theta}$*) is asymptotically unbiased for* $\tau(\theta)$ *(or* θ*) with asymptotic variance of* $1/I(\theta)$. *So, we have* $\xrightarrow{d} N(0,1)$ *in addition to* $\xrightarrow{p} \theta$.

Proof. Suppose that the true value of θ is θ_0

$$\begin{aligned} l(\theta) &= \sum_{i=1}^{n} \log f(X_i|\theta) \\ l'(\theta) &= l'(\theta_0) + (\theta-\theta_0)\, l''(\theta_0) + \cdots \\ l'\left(\widehat{\theta}\right) &= l'(\theta_0) + \left(\widehat{\theta}-\theta_0\right) l''(\theta_0) + \cdots \\ \left(\widehat{\theta}-\theta_0\right) &\approx -l'(\theta_0)/l''(\theta_0) \qquad \text{(MLE def.)} \\ \sqrt{n}\frac{\left(\widehat{\theta}-\theta_0\right)}{\sqrt{1/I(\theta_0)}} &\approx \frac{\sqrt{n}\frac{1}{n}l'(\theta_0)/\sqrt{I(\theta_0)}}{\frac{-1}{n}l''(\theta_0)/I(\theta_0)}. \end{aligned}$$

$$
\begin{aligned}
\frac{1}{n}l'(\theta_0) &= \frac{1}{n}\sum_i \frac{\partial}{\partial\theta}\log f(X_i|\theta)\bigg|_{\theta_0} \\
\mathrm{E}\left[\frac{\partial}{\partial\theta}\log f(X_i|\theta)\bigg|_{\theta_0}\right] &= 0 && (\mathrm{E}_{X|\theta_0}) \\
\mathrm{Var}\left[\frac{\partial}{\partial\theta}\log f(X_i|\theta)\bigg|_{\theta_0}\right] &= \mathrm{E}\left[\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right)^2\bigg|_{\theta_0}\right] \\
&= I(\theta_0) \\
\sqrt{n}\frac{\frac{1}{n}l'(\theta_0)-0}{\sqrt{I(\theta_0)}} &\xrightarrow{d} N(0,1) && (\mathrm{CLT}) \\
\frac{-1}{n}l''(\theta_0) &= \frac{-1}{n}\sum_i \frac{\partial^2}{\partial\theta^2}\log f(X_i|\theta) \\
\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\bigg|_{\theta_0}\right] &= -I(\theta_0) \\
\frac{-1}{n}l''(\theta_0) &\xrightarrow{p} I(\theta_0) \\
\frac{-1}{n}l''(\theta_0)/I(\theta_0) &\xrightarrow{p} 1 \\
\sqrt{n}\frac{\left(\widehat{\theta}-\theta_0\right)}{\sqrt{1/I(\theta_0)}} &\xrightarrow{d} N(0,1)\,.
\end{aligned}
$$

■

Said differently

$$
\begin{aligned}
\sqrt{n}\frac{\widehat{\theta}-\theta_0}{\sqrt{1/I(\theta_0)}} &\xrightarrow{d} N(0,1)\,,\\
\sqrt{n}\left(\widehat{\theta}-\theta_0\right) &\xrightarrow{d} N(0,1/I(\theta_0))\,,
\end{aligned}
$$

which means that the MLE $\widehat{\theta}$

- Asymptotically unbiased
- Asymptotic variance $= 1/I(\theta_0)$
- Asymptotically normally distributed.

Why variance decreases with $I(\theta_0)$?

$$
I(\theta_0) = -\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\bigg|_{\theta_0}\right]
$$

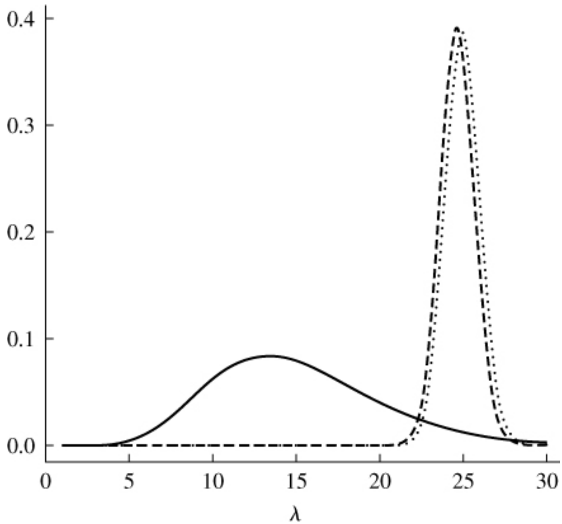
High $I(\theta_0)$ means very sharp curve at θ_0, which means very probable θ_0, which means less likely that the next dataset will not support that inference; and hence less variable the next estimator is.

8.4 The Bayesian Approach to Parameter Estimation

- We treat θ as r.v. with **subjective** prior knowledge f_Θ; as opposed to “Frequentist (or Classical) Approach”

- Data $\mathbf{x} = x_1, \ldots, x_n$ for $\mathbf{X} = X_1, \ldots, X_n$ modifies our belief and produces the posterior $f_{\Theta|\mathbf{X}}$?

- We estimate θ by many criteria; e.g.,:



1. Posterior Mode/Max. A Posteriori (MAP):

$$\widehat{\theta} = \arg\max_{\theta} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})$$

2. Posterior Mean:

$$\widehat{\theta} = \mathop{\mathrm{E}}_{\Theta}[\theta] = \int \theta f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta$$

3. Posterior loss function optimization:

$$\begin{aligned}\widehat{\theta} &= \arg\min_{\eta} \mathop{\mathrm{E}}_{\Theta}\left[L\left(\eta,\theta\right)\right] \\ &= \arg\min_{\eta} \int L\left(\eta,\theta\right) f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta\end{aligned}$$

General Framework:

$$\begin{aligned}f_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) &= \frac{f_{\mathbf{X},\Theta}(\mathbf{x},\theta)}{f_{\mathbf{X}}(\mathbf{x})} \\ &= \frac{f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\, f_{\Theta}(\theta)}{\int f_{\mathbf{X},\Theta}(\mathbf{x},\theta)\ d\theta} \\ &= \frac{f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\, f_{\Theta}(\theta)}{\int f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\, f_{\Theta}(\theta)\ d\theta} \\ &= Const(\mathbf{x})\, f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\, f_{\Theta}(\theta)\end{aligned}$$

$$Posterior \propto Likelihood \times Prior.$$

Connection to MLE:

$$\begin{aligned} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) &= Const(\mathbf{x})\, f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\, f_{\Theta}(\theta) \\ &\propto Likelihood \times Prior \end{aligned}$$

if we choose an uninformative prior $\Theta \sim U$ to let data speak for themselves:

$$\begin{aligned} f_{\Theta|X}(\theta|x) &= Const(x)\, f_{X|\Theta}(x|\theta) \\ &\propto Likelihood \end{aligned}$$

Then, if we choose MAP criterion

$$\widehat{\theta} = \arg\max l(\theta), \tag{MLE}$$

Example 31 (Poisson) **X** *denotes* $X_1, \ldots, X_n$:

$$
\begin{aligned}
f_{\mathbf{X}|\Lambda} &= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!},\ 0 \le x_i, \\
&= \frac{\lambda^{\sum_i x_i} e^{-n\lambda}}{\prod_i x_i!} \\
f_{\Lambda|\mathbf{X}} &= \frac{f_{\mathbf{X}|\Lambda}(\mathbf{x}|\lambda)\, f_\Lambda(\lambda)}{\int f_{\mathbf{X}|\Lambda}(\mathbf{x}|\lambda)\, f_\Lambda(\lambda)\ d\lambda} \\
&= \frac{\lambda^{\sum_i x_i} e^{-n\lambda} f_\Lambda(\lambda) / \prod_i x_i!}{\int \lambda^{\sum_i x_i} e^{-n\lambda} f_\Lambda(\lambda)\ / \prod_i x_i! d\lambda} \\
&= \frac{\lambda^{\sum_i x_i} e^{-n\lambda} \frac{1}{100}}{\int \lambda^{\sum_i x_i} e^{-n\lambda} \frac{1}{100}\ d\lambda} && (\Lambda \sim U(0, 100)) \\
&= \frac{\nu^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\nu\lambda} && (Gamma(\alpha, \nu)) \\
&\sim Gamma(S_n + 1, n) \\
\widehat{\lambda} &= \mathrm{E}[\Lambda] = \frac{S_n + 1}{n} = \overline{X} + \frac{1}{n} && (\text{Post. Mean}) \\
\frac{\partial f_{\Lambda|\mathbf{X}}}{\partial \lambda} &= \frac{\nu^\alpha}{\Gamma(\alpha)} \left((\alpha - 1)\lambda^{\alpha-2} e^{-\nu\lambda} - \nu\lambda^{\alpha-1} e^{-\nu\lambda}\right) \\
\widehat{\lambda} &= \frac{\alpha - 1}{\nu} = \frac{S_n}{n} = \overline{X} && (\text{MAP}\equiv\text{MLE}) \\
\frac{S_n}{n} &= \frac{573}{23} = 24.9, \quad \frac{S_n + 1}{n} = 25
\end{aligned}
$$

On the other hand, if we have the prior knowledge that Λ *has* $\mu = 15$ *and* $\sigma = 5$ *then, we can assume that* $\Lambda \sim Gamma(\alpha, \nu)$ *with*

$$\begin{aligned}
\mu &= \alpha/\nu, \\
\sigma^2 &= \alpha/\nu^2, \\
\nu &= \frac{\mu}{\sigma^2} = 0.6 << n & (n = 23) \\
\alpha &= \nu\mu = 9 << S_n, & (S_n = 573) \\
f_{\Lambda|\mathbf{x}} &= \frac{\lambda^{\sum_i x_i} e^{-n\lambda} f_\Lambda(\lambda)}{\int \lambda^{\sum_i x_i} e^{-n\lambda} f_\Lambda(\lambda)\ d\lambda} \\
&= \frac{\lambda^{\sum_i x_i} e^{-n\lambda} \frac{\nu^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\nu\lambda}}{\int \lambda^{\sum_i x_i} e^{-n\lambda} \frac{\nu^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\nu\lambda}\ d\lambda} \\
&= \frac{\lambda^{(S_n+\alpha-1)} e^{-(n+\nu)\lambda}}{\int \lambda^{(S_n+\alpha-1)} e^{-(n+\nu)\lambda} d\lambda} \\
&\sim Gamma(S_n + \alpha, n + \nu) \\
\widehat{\lambda} &= \frac{S_n + \alpha}{n + \nu} = \frac{573 + 9}{23 + .6} = 24.7 & \text{(Post. Mean)} \\
\widehat{\lambda} &= \frac{S_n + \alpha - 1}{n + \nu} = \frac{573 + 9 - 1}{23 + .6} = 24.6 & \text{(MAP)}
\end{aligned}$$

Example 32 ($Ber(p)$) *: n obs., then*

$$\mu_1 = p,$$
$$\widehat{p} = \overline{X} = \frac{\sum_i x_i}{n} = \frac{\#Heads}{n}, \tag{MoM}$$
$$p_X(x) = p^x \left(1-p\right)^{1-x}, \; x = 0,1$$
$$l\left(p\right) = \sum_i x_i \log p + \sum_i (1-x_i) \log\left(1-p\right)$$
$$l'\left(p\right) = \frac{\sum_i x_i}{p} - \frac{\sum_i (1-x_i)}{1-p} \qquad (l'\left(p\right) \overset{set}{=} 0)$$
$$\widehat{p} = \overline{X} = \frac{\sum_i x_i}{n} = \frac{\#Heads}{n}. \tag{MLE}$$

Now, if we get 5 heads in 5 trials $\widehat{p}$ *will be 1 !!!!*

Let's see the Bayesian approach.

$$f_{\mathbf{X}|P} = \prod_{i=1}^{n} p^{x_i} \left(1-p\right)^{1-x_i} = p^{\sum_i x_i} \left(1-p\right)^{\sum_i (1-x_i)}$$

$$f_P\left(p\right) = \frac{\Gamma\left(a+b\right)}{\Gamma\left(a\right)\Gamma\left(b\right)} p^{a-1} \left(1-p\right)^{b-1} \quad (\sim Beta\left(a,b\right))$$

$$\begin{aligned} f_{P|\mathbf{X}} &= \frac{f_{\mathbf{X}|P}\left(\mathbf{x}|p\right) f_P\left(p\right)}{\int f_{\mathbf{X}|P}\left(\mathbf{x}|p\right) f_P\left(p\right)\, dp} \\ &\propto p^{a-1+S} \left(1-p\right)^{b-1+(n-S)} \\ &\sim Beta\left(a+S, b+n-S\right). \end{aligned}$$

$$\begin{aligned} \widehat{p} &= \frac{A-1}{A+B-2} = \frac{a+S-1}{a+b+n-2} && \text{(MAP)} \\ &= \frac{a+S-1}{2a+n-2} && \text{(Symmetric Prior)} \end{aligned}$$

$a = 1$: $U\left(0,1\right)$, $\widehat{p} = \frac{S}{n} \equiv MLE$.

$a = 2$: *not uniform but spread.* $\widehat{p} = \left(S+1\right)/\left(n+2\right)$.

- $S = n$: $\widehat{p} = \left(n+1\right)/\left(n+2\right) \to 1$.
- $S = n/2$: $\widehat{p} = 1/2$ *(of course).*

$a >>$: *insisting on fair coin,* $\widehat{p} \approx a/\left(2a\right) = \frac{1}{2}$

$$
\begin{aligned}
f_{P|\mathbf{X}} &\sim Beta(a+S, b+n-S) \\
\widehat{p} &= \frac{A}{A+B} \\
&= \frac{a+S}{a+b+n} \qquad \text{(Posterior Mean)}
\end{aligned}
$$

8.4.1 Large Sample Theory of Bayesian Inference

X and **x** denote $X_1, \ldots, X_n$ and $x_1, \ldots, x_n$, respectively, to simplify notation.

$$f_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) \propto f_{\Theta}(\theta) f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta),$$

which is dominated by $f_{\mathbf{X}|\Theta}$ as $n \to \infty$.

$$\begin{aligned}
f_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) &\propto f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta) && (\text{as } n \to \infty) \\
&= \exp\left[\log f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\right] \\
&= \exp\left[l(\theta)\right] \\
&= \exp[l\left(\widehat{\theta}\right) + \left(\theta - \widehat{\theta}\right) l'\left(\widehat{\theta}\right) \\
&+ \frac{1}{2}\left(\theta - \widehat{\theta}\right)^2 l''\left(\widehat{\theta}\right) + \cdots] \\
&\propto \exp\left[-\frac{1}{2}\frac{\left(\theta - \widehat{\theta}\right)^2}{-1/l''\left(\widehat{\theta}\right)}\right] && \left(l'\left(\widehat{\theta}\right) = 0\right) \\
&\sim N\left(\widehat{\theta}, -1/l''\left(\widehat{\theta}\right)\right).
\end{aligned}$$

Do not confuse it with the MLE asymptotic normality.

8.5 Assessing Estimators, Efficiency, and the Cramér-Rao Lower Bound

8.5.1 Mean Squared Error (MSE) Criterion

$$
\begin{aligned}
MSE\left(\widehat{\theta}\right) &= \underset{\mathbf{X}}{\mathrm{E}}\left[\left(\widehat{\theta}-\theta\right)^2\right] \\
&= \underset{\mathbf{X}}{\mathrm{Var}}\left[\widehat{\theta}\right] + \left(\underset{\mathbf{X}}{\mathrm{E}}\,\widehat{\theta}-\theta\right)^2 \\
&= Variance\left(\widehat{\theta}\right) + \left(Bias\left(\widehat{\theta}\right)\right)^2.
\end{aligned}
$$

- Since $MSE = MSE(\theta)$ no best estimator; e.g. $\widehat{\theta} = 12.3$ is the best when $\theta = 12.3$ but terrible otherwise.

- If $Bias\left(\widehat{\theta}\right) = 0$, $\widehat{\theta}$ is unbiased for θ.

- Tradeoff exists between Bias and Variance.

- A biased estimator may has lower MSE.

Example 33 ($\widehat{\sigma}^2$ vs. S^2 for $N\left(\mu,\sigma^2\right)$) :

$$\begin{aligned}
\widehat{\sigma}^2 &= \frac{1}{n}\sum_i \left(X_i - \overline{X}\right)^2, \\
S^2 &= \frac{1}{n-1}\sum_i \left(X_i - \overline{X}\right)^2 \\
\mathrm{E}\left[S^2\right] &= \sigma^2 && \text{(unbiased)} \\
\mathrm{Var}\left[S^2\right] &= \frac{2\sigma^4}{n-1} && \text{(see Extra Materials)} \\
MSE\left(S^2\right) &= \frac{2\sigma^4}{n-1} + \left(\sigma^2 - \sigma^2\right)^2 = \frac{2\sigma^4}{n-1} \\
\mathrm{E}\left[\widehat{\sigma}^2\right] &= \frac{n-1}{n}\sigma^2 && \text{(biased)} \\
\mathrm{Var}\left[\widehat{\sigma}^2\right] &= \mathrm{Var}\left[\frac{n-1}{n}S^2\right] = \left(\frac{n-1}{n}\right)^2 \mathrm{Var}\left[S^2\right] \\
&= \left(\frac{n-1}{n}\right)^2 \left(\frac{2\sigma^4}{n-1}\right) = \frac{2\left(n-1\right)\sigma^4}{n^2} \\
MSE\left(\widehat{\sigma}^2\right) &= \frac{2\left(n-1\right)\sigma^4}{n^2} + \left(\frac{n-1}{n}\sigma^2 - \sigma^2\right)^2 \\
&= \frac{2n-1}{n^2}\sigma^4 < \frac{2\sigma^4}{n-1} \ \ \forall \sigma, n.
\end{aligned}$$

Remarks:

- Although S^2 is unbiased, $\widehat{\sigma}^2$ has less MSE.
- MSE, for scale parameter, may not be reasonable since $\sigma^2 > 0$.
- $\widehat{\theta}_1$ may be better than $\widehat{\theta}_2$ under some criterion and the other way around and another criterion.

Example 34 ($\widehat{p}$ of $Ber(p)$) :

$$\widehat{p}_M = \overline{X} \qquad \text{(MLE)}$$

$$\mathrm{E}\left[\widehat{p}_M\right] = p$$

$$\mathrm{Var}\left[\widehat{p}_M\right] = \frac{1}{n}p(1-p)$$

$$MSE\left(\widehat{p}_M\right) = \frac{1}{n}p(1-p)$$

$$\widehat{p}_B = \frac{S+a}{a+b+n} \qquad \text{(Posterior Mean)}$$

$$\mathrm{E}\left[\widehat{p}_B\right] = \frac{np+a}{a+b+n}$$

$$\mathrm{Var}\left[\widehat{p}_B\right] = \frac{np(1-p)}{(a+b+n)^2}$$

$$MSE\left(\widehat{p}_B\right) = \frac{np(1-p)}{(a+b+n)^2} + \left(\frac{np+a}{a+b+n} - p\right)^2$$

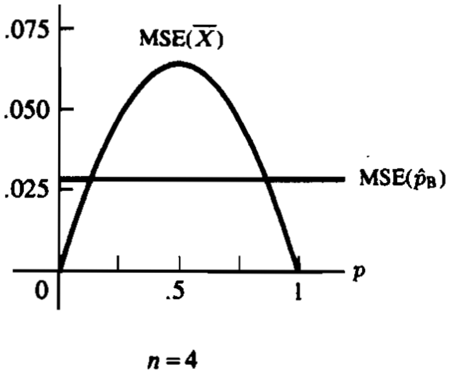
Choosing $a = b = \sqrt{n}/2$ relaxes dependence on p:

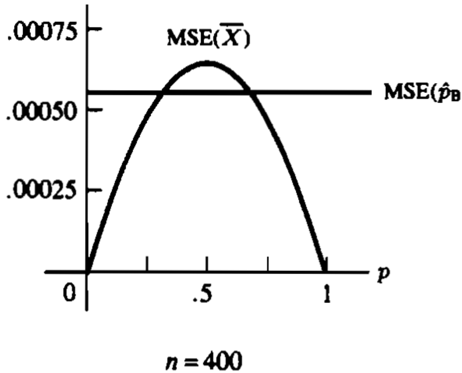
$$\widehat{p}_B = \frac{S+\sqrt{n}/2}{n+\sqrt{n}},$$

$$MSE\left(\widehat{p}_B\right) = \frac{n}{4\left(n+\sqrt{n}\right)^2}.$$

$$MSE\left(\widehat{p}_M\right) = \frac{1}{n} p\left(1 - p\right)$$

$$MSE\left(\widehat{p}_B\right) = \frac{n}{4\left(n + \sqrt{n}\right)^2}$$





- *For small n, $\widehat{p}_B$ is better unless p is on the boundary.*
- *For large n, $\widehat{p}_M$ is better unless p is in the middle.*
- *Having knowledge about the problem allows choosing the right estimator.*

8.5.2 Best Unbiased Estimator

Definition 35 (UMVUE) *: An estimator* $\widehat{\theta}^*$*, for* θ*, is a best unbiased estimator or uniform minimum variance unbiased estimator (UMVUE) if it satisfies* $\mathrm{E}\left[\widehat{\theta}^*\right] = \theta \;\forall\theta$ *and for any other estimator* $\widehat{\theta}$ *we have* $\mathrm{Var}\left[\widehat{\theta}^*\right] \leq \mathrm{Var}\left[\widehat{\theta}\right]$.

Theorem 36 (Cramér-Rao Inequality) *: Let* $X_1, \ldots, X_n \overset{i.i.d}{\sim} f(x|\theta)$ *with regularity condition. The for any estimator* $T = T(X_1, \ldots, X_n) = T(\mathbf{X})$

$$\mathrm{Var}(T) \geq \frac{\left(\frac{d}{d\theta}\mathrm{E}[T]\right)^2}{nI(\theta)},$$

$$\mathrm{Var}(T) \geq \frac{1}{nI(\theta)}. \qquad \text{(if } T \text{ is unbiased)}$$

- For all estimators with particular bias: the higher the *information number* the lower the *lower bound.*

- An estimator *attains* (*attainment*) the lower bound is called *efficient.*

Proof. :Since $1 \leq \rho = \mathrm{Cov}(T,Z)/\sqrt{\mathrm{Var}(T)\,\mathrm{Var}(Z)}$

$$\mathrm{Var}[T] \geq (\mathrm{Cov}(T,Z))^2/\mathrm{Var}(Z)$$

$$\begin{aligned}
Z &= \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(X_i|\theta) \\
\mathrm{Var}[Z] &= n\,\mathrm{Var}\left[\frac{\partial}{\partial\theta}\log f(X_i|\theta)\right] \\
&= nI(\theta) && \text{(Proof of Th. 30)} \\
\sigma_{TZ} &= \mathrm{E}(Z-\mathrm{E}[Z])(T-\mathrm{E}[T]) = \mathrm{E}[T(Z-\mathrm{E}[Z])] \\
&= \mathrm{E}[ZT] && (\mathrm{E}[Z]=0) \\
&= \mathrm{E}\left[T\frac{\partial}{\partial\theta}\log\prod_i f(X_i|\theta)\right] \\
&= \mathrm{E}\left[T\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta)\right] && (\mathbf{X} = X_1,\ldots,X_n) \\
&= \int T(\mathbf{x})\,\frac{\frac{\partial}{\partial\theta}f(\mathbf{x}|\theta)}{f(\mathbf{x}|\theta)}\,f(\mathbf{x}|\theta)\,d\mathbf{x} \\
&= \frac{\partial}{\partial\theta}\int T(\mathbf{x})\,f(\mathbf{x}|\theta)\,d\mathbf{x} \\
&= \frac{\partial}{\partial\theta}\mathop{\mathrm{E}}_{\mathbf{x}}[T(\mathbf{X})]
\end{aligned}$$

■

Example 37 (Poisson) :

$$
\begin{aligned}
I(\lambda) &= \mathrm{E}\left[\left(\frac{\partial}{\partial\lambda}\log\frac{\lambda^X e^{-\lambda}}{X!}\right)^2\right] \\
&= \mathrm{E}\left[\left(\frac{\partial}{\partial\lambda}\left(X\log\lambda-\lambda-\log X!\right)\right)^2\right] \\
&= \mathrm{E}\left[\left(\frac{X}{\lambda}-1\right)^2\right] \\
&= -\mathrm{E}\left[\frac{\partial^2}{\partial\lambda^2}\log\frac{\lambda^X e^{-\lambda}}{X!}\right] && \text{(easier)} \\
&= -\mathrm{E}\left[\frac{-X}{\lambda^2}\right] = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}, \\
\mathrm{Var}[T] &\geq \frac{\left(\frac{\partial}{\partial\lambda}\mathrm{E}[T]\right)^2}{nI(\lambda)} \\
&= \frac{\lambda}{n} && \text{(for unbiased estimators)} \\
\widehat{\lambda} &= \overline{X} && \text{(MLE)} \\
\mathrm{E}\left[\widehat{\lambda}\right] &= \lambda && \text{(unbiased)} \\
\mathrm{Var}\left[\widehat{\lambda}\right] &= \mathrm{Var}\left[\overline{X}\right] = \frac{1}{n}\mathrm{Var}[X] = \frac{\lambda}{n}, && \text{(attainment)}
\end{aligned}
$$

Example 38 ($U(0,\theta)$) *:$f(x|\theta) = 1/\theta$, then*

$$
\begin{aligned}
I(\theta) &= \mathrm{E}\left[\left(\frac{\partial}{\partial\theta}\log(1/\theta)\right)^2\right] \\
&= \mathrm{E}\left[\left(-\frac{\partial}{\partial\theta}\log\theta\right)^2\right] = 1/\theta^2, \\
\mathrm{Var}\left[\widehat{\theta}\right] &\geq \frac{\left(\frac{\partial}{\partial\theta}\mathrm{E}[T]\right)^2}{nI(\theta)} \\
&= \frac{\theta^2}{n}, && \text{(for unbiased estimators)} \\
\widehat{\theta} &= 2\overline{X}, && \text{(MoM)} \\
\mathrm{E}\left[\widehat{\theta}\right] &= \theta && \text{(unbiased)} \\
\mathrm{Var}\left[\widehat{\theta}\right] &= \frac{4}{n}\mathrm{Var}[X] = \frac{4}{n}\frac{\theta^2}{12} \\
&= \frac{\theta^2}{3n} < \frac{\theta^2}{n}. && \text{(!!!where is the problem?)}
\end{aligned}
$$

The regularity condition assumes $(n = 1)$*:*

$$\frac{\partial}{\partial\theta}\mathrm{E}[T] = \frac{\partial}{\partial\theta}\int T f(x|\theta)\,dx \qquad\qquad (\mathbf{x} = x)$$
$$= \int T\frac{\partial}{\partial\theta}f(x|\theta)\,dx$$

Let's see

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{E}[T] &= \frac{\partial}{\partial\theta}\int_0^\theta T\frac{1}{\theta}dx \\
&= \frac{\partial}{\partial\theta}\left(\frac{1}{\theta}\int_0^\theta T\,dx\right) \\
&= \left(\frac{\partial}{\partial\theta}\frac{1}{\theta}\right)\int_0^\theta T\,dx + \frac{1}{\theta}\frac{\partial}{\partial\theta}\int_0^\theta T\,dx \\
&= \left(\frac{\partial}{\partial\theta}\frac{1}{\theta}\right)\int_0^\theta T\,dx + \frac{T(\theta)}{\theta} \\
\int_0^\theta T\frac{\partial}{\partial\theta}f(x|\theta)\,dx &= \left(\frac{\partial}{\partial\theta}\frac{1}{\theta}\right)\int_0^\theta T\,dx, \\
&\neq \frac{\partial}{\partial\theta}\mathrm{E}[T],
\end{aligned}$$

unless $T(\theta) = 0\ \forall\theta$.

Homework: repeat with the MLE estimator, scale it to be unbiased, then find its variance.

Loss Function

- Not only for assessment and comparison,
- but also for designing and optimization!

The loss function:

$$
\begin{aligned}
L(\theta, T(\mathbf{X})) &= |\theta - T(\mathbf{X})| \qquad \text{(absolute error (AE))} \\
L(\theta, T(\mathbf{X})) &= (\theta - T(\mathbf{X}))^2 \qquad \text{(squared error (SE))} \\
&\vdots
\end{aligned}
$$

expresses how the estimate $T(\mathbf{X})$ deviates from θ.

The risk:

$$R(\theta, T) = \mathop{\mathrm{E}}_{\mathbf{X}} L(\theta, T(\mathbf{X}))$$

is a function of θ. $R(\theta, T_1)$ may cross with $R(\theta, T_2)$.

MSE (special case):

$$
\begin{aligned}
MSE(\theta) &= R(\theta, T) \\
&= \mathop{\mathrm{E}}_{\mathbf{X}} [L(\theta, T(\mathbf{X}))], \\
L(\theta, T(\mathbf{X})) &= (\theta - T(\mathbf{X}))^2.
\end{aligned}
$$

Example 39 (Risk of σ^2 Est.) :

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2, \quad (R\left(\sigma^2, S^2\right) = \tfrac{2\sigma^4}{n-1})$$

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2, \quad (R\left(\sigma^2, \widehat{\sigma}^2\right) = \tfrac{2n-1}{n^2}\sigma^4)$$

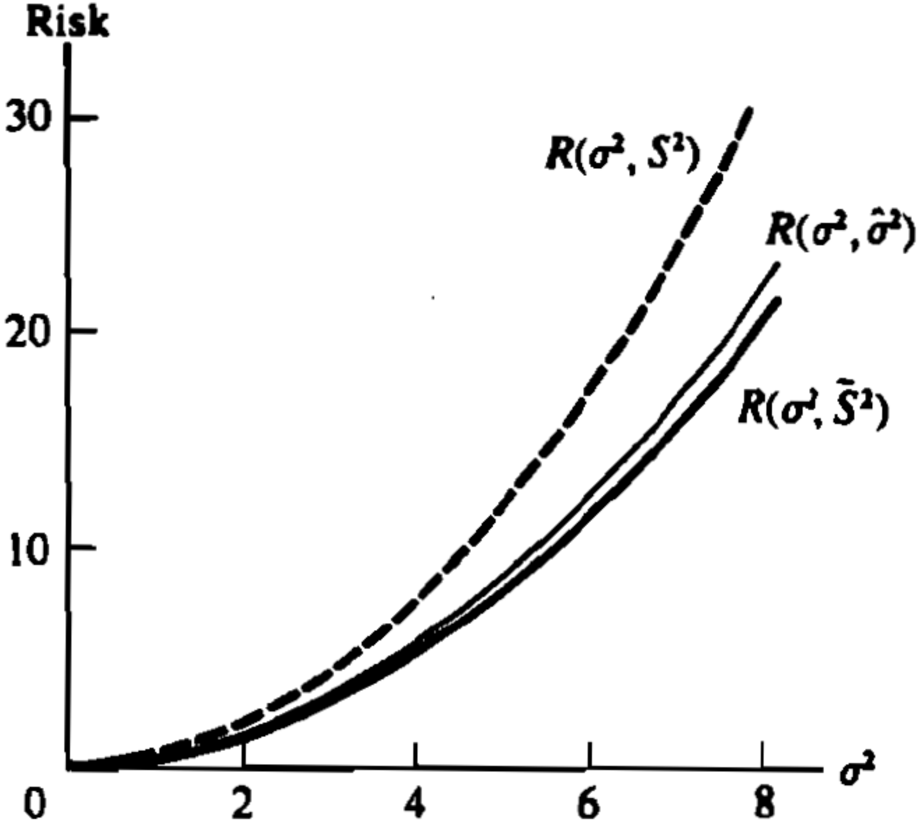
$$\widetilde{S}^2 = b\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \qquad (R\left(\sigma^2, \widetilde{S}^2\right)?)$$

$$\begin{aligned} R\left(\sigma^2, \widetilde{S}^2\right) &= \operatorname{Var}\left[b\,(n-1)\,S^2\right] \\ &\quad + \left(\operatorname{E}\left[b\,(n-1)\,S^2\right] - \sigma^2\right)^2 \\ &= b^2\,(n-1)^2\,\frac{2\sigma^4}{n-1} + (b\,(n-1)-1)^2\,\sigma^4 \\ &= \left(2b^2\,(n-1) + (b\,(n-1)-1)^2\right)\sigma^4, \\ &= c\sigma^4, \end{aligned}$$

$$c_{\min} = \frac{2}{n+1} \qquad (\text{at } b = \tfrac{1}{n+1})$$

$$\widetilde{S}^2 = \frac{1}{n+1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

$$(R\left(\sigma^2, \widetilde{S}^2\right) = \tfrac{2}{n+1}\sigma^4)$$


Risk
30
20
10
0
$R(\sigma^2, S^2)$
$R(\sigma^2, \hat{\sigma}^2)$
$R(\sigma^2, \tilde{S}^2)$
2
4
6
8
σ^2

Connection to Cramér-Rao Inequality

$$f\left(x|\mu,\sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\frac{\left(x-\mu\right)^2}{\sigma^2}\right)$$

$$l\left(\theta\right) = -\log\sqrt{2\pi} - \frac{1}{2}\log\theta - \frac{1}{2\theta}\left(x-\mu\right)^2$$

$$(\theta = \sigma^2)$$

$$l'\left(\theta\right) = \frac{-1}{2\theta} + \frac{\left(x-\mu\right)^2}{2\theta^2}$$

$$l''\left(\theta\right) = \frac{1}{2\theta^2} - \frac{\left(x-\mu\right)^2}{\theta^3}$$

$$\mathrm{E}\left[l''\left(\theta\right)\right] = \frac{1}{2\theta^2} - \frac{\theta}{\theta^3} = \frac{-1}{2\theta^2}$$

$$I\left(\theta\right) = -\mathrm{E}\left[\frac{\partial^2 l\left(\theta\right)}{\partial\theta^2}\right] = \frac{1}{2\sigma^4}$$

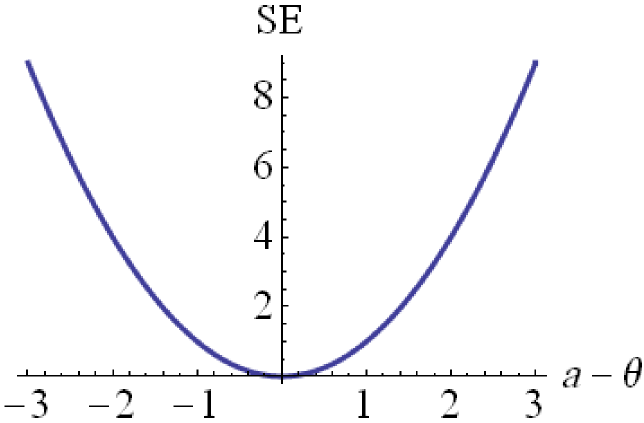
$$\mathrm{Var}\left[T\right] \geq \frac{1}{nI\left(\theta\right)} = \frac{2\sigma^4}{n},$$

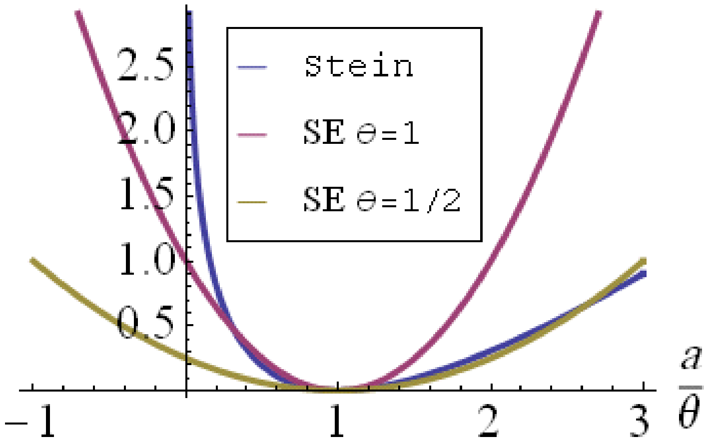
- lower bound of any unbiased estimator of σ^2.
- not attainable by the unbiased version above

Assessing with different Loss Function:

$$L(\theta, a) = (a-\theta)^2 = \theta\left(\frac{a}{\theta} - 1\right)^2 \qquad \text{(SE loss)}$$

$$L(\theta, a) = \frac{a}{\theta} - 1 - \log\left(\frac{a}{\theta}\right) \qquad \text{(Stien's loss)}$$





$$\widetilde{S}^2 = b\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

$$L(\theta, a) = \frac{a}{\theta} - 1 - \log\left(\frac{a}{\theta}\right)$$

$$R\left(\sigma^2, \widetilde{S}^2\right) = \mathrm{E}\left[b(n-1)\frac{S^2}{\sigma^2} - 1 - \log\frac{b(n-1)S^2}{\sigma^2}\right]$$

$$= b\,\mathrm{E}\left[\chi^2_{n-1}\right] - 1 - \log b - \mathrm{E}\log\chi^2_{n-1}$$

$$\frac{\partial R}{\partial b} = \mathrm{E}\left[\chi^2_{n-1}\right] - \frac{1}{b} \qquad (\overset{set}{=} 0)$$

$$b = \frac{1}{\mathrm{E}\left[\chi^2_{n-1}\right]} = \frac{1}{n-1}$$

$$\widetilde{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 = S^2.$$

“Better” in which sense?

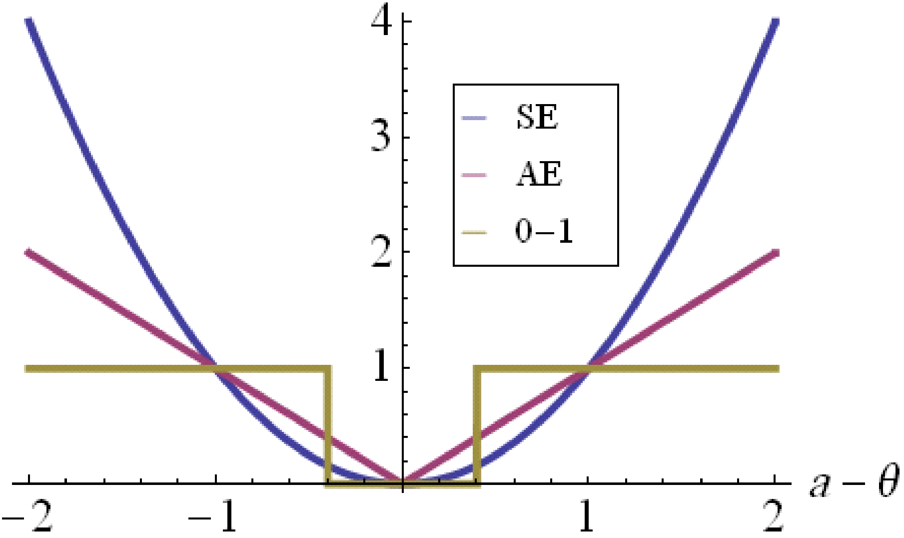
Obtaining Bayesian's Estimator by Loss Function Optimization!

$$
\begin{aligned}
R(\theta, T) &= \underset{\mathbf{X}}{\mathrm{E}}\, L(\theta, T(\mathbf{X})) \\
&= \int L(\theta, T(\mathbf{x}))\, f_{\mathbf{X}}(\mathbf{x}|\theta)\, d\mathbf{x}
\end{aligned}
$$

- no uniformly "best" estimator.
- $R(\theta, T_1)$ may cross with $R(\theta, T_2)$.

$$
\begin{aligned}
\underset{\Theta}{\mathrm{E}}\, R(\theta, T) &= \int_\theta R(\theta, T)\, f_\Theta(\theta)\, d\theta \\
&= \int_\theta \left[\int_{\mathbf{x}} L(\theta, T(\mathbf{x}))\, f_{\mathbf{X}}(\mathbf{x}|\theta)\, d\mathbf{x}\right] f_\Theta(\theta)\, d\theta \\
&= \int_{\mathbf{x}} \left[\int_\theta L(\theta, T(\mathbf{x}))\, f_{\mathbf{X}}(\mathbf{x}|\theta)\, f_\Theta(\theta)\, d\theta\right] d\mathbf{x} \\
&= \int_{\mathbf{x}} \left[\int_\theta L(\theta, T(\mathbf{x}))\, f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta\right] f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \\
T &= \arg\min_T \int_\theta L(\theta, T(\mathbf{x}))\, f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta
\end{aligned}
$$

Solutions under different loss functions:



$$T_1 = \arg\min_T \int_\theta (T-\theta)^2 f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta \qquad \text{(SE loss)}$$

$$= \int_\theta \theta f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta \qquad \text{(Posterior mean)}$$

$$T_2 = \arg\min_T \int_\theta |T-\theta|\, f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta \qquad \text{(AE loss)}$$

$$\begin{aligned}
R &= \int_\theta |T-\theta|\, f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta \\
&= \int_{-\infty}^{T} (T-\theta)\, f(\theta)\, d\theta + \int_T^\infty -(T-\theta)\, f(\theta)\, d\theta \\
&= T\int_{-\infty}^{T} f(\theta)\, d\theta - \int_{-\infty}^{T} \theta f(\theta)\, d\theta - \\
&\quad T\int_T^\infty f(\theta)\, d\theta + \int_T^\infty \theta f(\theta)\, d\theta \\
\frac{\partial R}{\partial T} &= \left(\int_{-\infty}^{T} f(\theta)\, d\theta + T f(T)\right) - T f(T) - \\
&\quad \left(\int_T^\infty f(\theta)\, d\theta - T f(T)\right) - T f(T) \\
&= \int_{-\infty}^{T} f(\theta)\, d\theta - \int_T^\infty f(\theta)\, d\theta \qquad (\overset{set}{=} 0) \\
0 &= F^{-1}_{\Theta|\mathbf{X}}(T) - \left(1 - F^{-1}_{\Theta|\mathbf{X}}(T)\right) \\
0.5 &= F^{-1}_{\Theta|\mathbf{X}}(T) \\
T_2 &= F^{-1}_{\Theta|\mathbf{X}}(0.5) \qquad \text{(Posterior median)}
\end{aligned}$$

$$
\begin{aligned}
T_3 &= \arg\min_T \int_\theta I_{0\le|T-\theta|} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\,d\theta \quad (0-1\text{ loss})\\
R &= \int_\theta I_{a\le|T-\theta|} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\,d\theta\\
&= \int_{a\le|T-\theta|} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\,d\theta\\
&= 1-\int_{|T-\theta|<a} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\,d\theta\\
&= 1-\int_{T-a}^{T+a} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\,d\theta\\
&= 1-\Pr_{\Theta|\mathbf{X}}[|\theta-T|<a]
\end{aligned}
$$

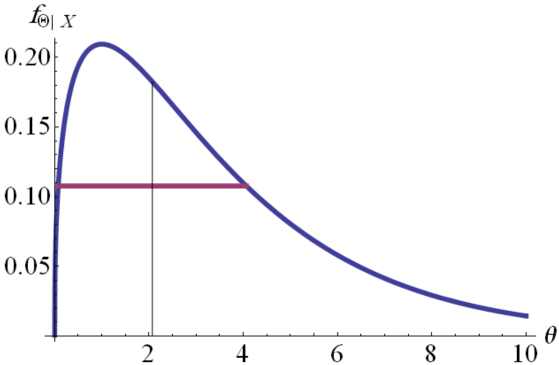
Notice that: we have to maximize the probability $\int_{T-a}^{T+a} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\,d\theta$. The period $[T-a, T+a]$ has

- a length of $(T+a)-(T-a)=2a$
- mid point of $\frac{1}{2}[(T+a)+(T-a)] = T$.
- T and mode do not necessarily coincide.,

which means that T_3 is mid-point of $2a$ modal interval.

$$\frac{\partial R}{\partial T} = f_{\Theta|\mathbf{X}}(T+a|\mathbf{x}) - f_{\Theta|\mathbf{X}}(T-a|\mathbf{x}), \qquad (\overset{set}{=} 0)$$

$$f_{\Theta|\mathbf{X}}(T+a|\mathbf{x}) = f_{\Theta|\mathbf{X}}(T-a|\mathbf{x}).$$



For unimodal symmetric $f_{\Theta|\mathbf{X}}$:
$f_{\Theta|\mathbf{X}}(\theta - M) = f_{\Theta|\mathbf{X}}(\theta + M)$. Therefore,

$$T_3 = Mode. \qquad \text{(MAP)}$$

For $a \to 0$

$$
\begin{aligned}
R &\approx 1 - f_{\Theta|\mathbf{X}}(T|\mathbf{x}) \cdot 2a, \\
T_3 &= \arg\max_{T} f_{\Theta|\mathbf{X}}(T|\mathbf{x}) = Mode \qquad \text{(MAP)}
\end{aligned}
$$

Of course T_3 could have been any point if we starte minimizing the risk from begining not by obtaining the limit:

$$
\begin{aligned}
R &= 1 - \int_{T-a}^{T+a} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta \\
&= 1 - \int_{T}^{T} f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\, d\theta \\
&= 1,
\end{aligned}
$$

unless Θ is discrete or categorical as in Pattern Recognition.

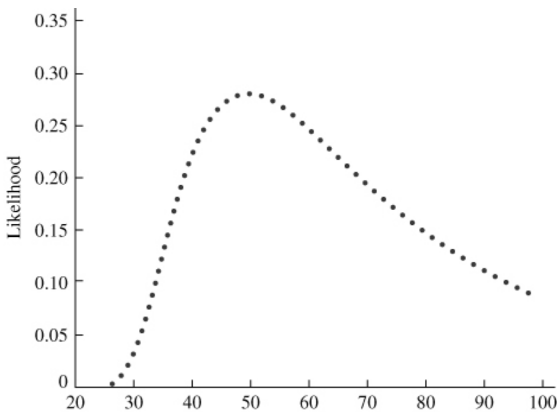
Estimation for Discrete Θ

MLE, Bayesian, Loss Functions have same treatment. However, maximization, expectation,..etc are taken over discrete space. Also, Cramér-Rao Lower Bound is derived for continuous case!

Example 40 (Capture Recapture Method) *: as in Example 15, page 19, first course. x captured animal in a population of θ animals. x was found to be 4 (we renamed variables):*

$$L(\theta) = P(x|\theta) = \frac{\binom{10}{4}\binom{\theta-10}{20-4}}{\binom{\theta}{20}}, \qquad \text{(Likelihood)}$$

$$\widehat{\theta}_{MLE} = 50$$



- maximization is obtained by $L_\theta / L_{\theta+1}$ not by $\frac{\partial L}{\partial \theta}$.

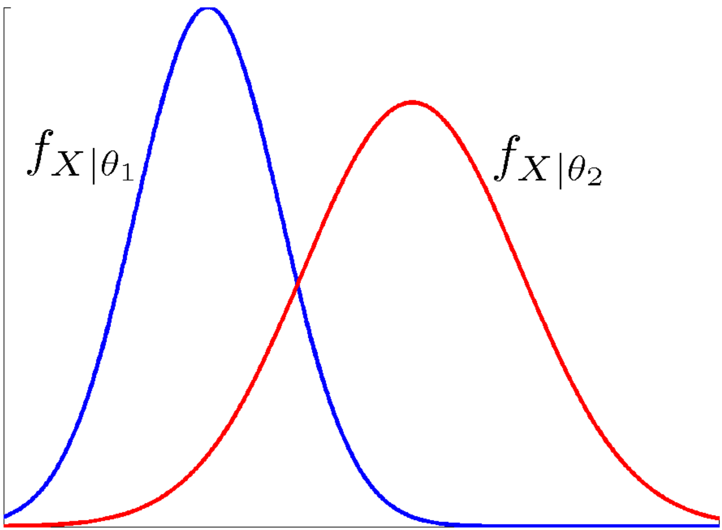
- Bayesian estimation is exactly the same throu defining $f_\Theta(\theta)$.

- However, $f_{\Theta|\mathbf{X}}(\theta|\mathbf{x})$ will be discrete.

Estimation for Categorical Θ (basis for Pattern Recognition)

- $\Theta = \{\theta_1, \ldots, \theta_K\}$, with K categories (classes).
- E.g., $\Theta = \{Male, Female\}$

$$X|\theta_1 \sim N(2,1),$$
$$X|\theta_2 \sim N(5,1).$$



8.5.3 Asymptotic Relative Efficiency (ARE)

Definition 41 *The (sequence of) estimator T_n is said to be asymptotically efficient for θ if*

$$\sqrt{n}\,(T_n - \theta) \xrightarrow{d} N\left(0, \sigma^2\right),$$
$$\sigma^2 = \frac{1}{I(\theta)},$$

which is Cramér-Rao Lower Bound.
It is clear that MLE is asymptotically efficient.

Bibliography