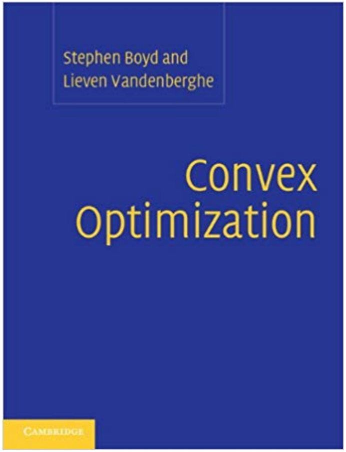
CS495
Optimiztaion

Waleed A. Yousef, Ph.D.,

Human Computer Interaction Lab.,
Computer Science Department,
Faculty of Computers and Information,
Helwan University,
Egypt.

March 13, 2019

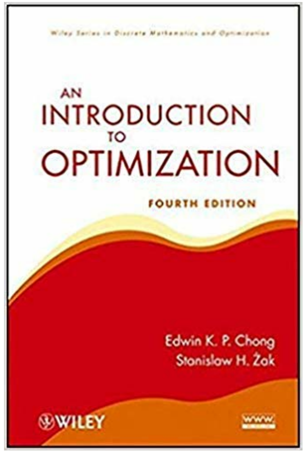
Lectures follow:
Boyd and Vandenberghe (2004)



Boyd, S., & Vandenberghe, L. (2004). Convex Optimization. Cambridge: Cambridge University Press.

Book and Stanford course: http://web.stanford.edu/~boyd/cvxbook/

Some examples from:
Chong and Zak (2001)



Chong, E. K., & Zak, S. (2001). An introduction to optimization: Wiley-Interscience.

Course Objectives

- Developing rigorous mathematical treatment for mathematical optimization.
- Building intuition, in particular to practical problems.
- Developing computer practice to using optimization SW.

Prerequisites

1. Discrete Mathematics
2. Calculus (single variable)
3. Calculus (multi variable)
4. Linear Algebra

Chapter 1
Introduction
Snapshot on Optimization

Contents

II Applications 109

III Algorithms 191

Chapter 1

Introduction

1.1 Mathematical Optimization

Definition 1 *A mathematical optimization problem or just optimization problem, has the form (Boyd and Vandenberghe, 2004):*

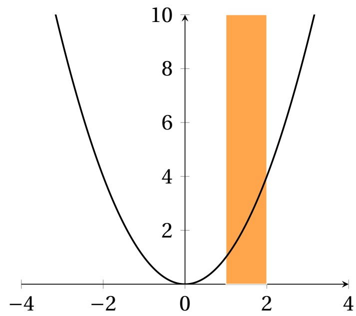
$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f_0(x) & \\
\text{subject to:} \quad & f_i(x) \le 0, & i = 1, \ldots, m \\
& h_i(x) = 0, & i = 1, \ldots, p,
\end{aligned}
$$

$$
\begin{aligned}
x = (x_1, \ldots, x_n) \in \mathbf{R}^n, & \quad (\textit{optimization variable}) \\
f_0 : \mathbf{R}^n \mapsto \mathbf{R}, & \quad (\textit{objective (cost/utility) function}) \\
f_i : \mathbf{R}^n \mapsto \mathbf{R}, & \quad (\textit{inequality constraints (functions)}) \\
h_i : \mathbf{R}^n \mapsto \mathbf{R}, & \quad (\textit{equality constraints (functions)}) \\
\mathcal{D} : \bigcap_{i=1}^{m} \mathbf{dom}\, f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\, h_i & \quad (\textit{feasible set}) \\
= \left\{ x \mid x \in \mathbf{R}^n \wedge f_i(x) \le 0 \wedge h_i(x) = 0 \right\} & \\
x^* : \left\{ x \mid x \in \mathcal{D} \wedge f_0(x) \le f_0(z)\ \forall z \in \mathcal{D} \right\} & \quad (\textit{solution})
\end{aligned}
$$

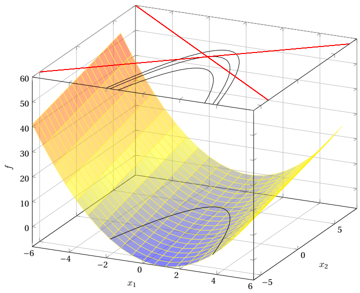
- minimize $f_0 \equiv$ maximize $-f_0$.
- $f_i \le 0 \equiv -f_i \ge 0$.
- 0s can be replaced of course by constants b_i, c_i
- unconstrained problem when $m = p = 0$.

Example 2 *:*

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & x^2 \\
\text{subject to:} \quad & x \le 2 \wedge x \ge 1.
\end{aligned}
$$



$x^* = 1$.
If the constraints are relaxed, then $x^* = 0$.



$$\underset{x}{\text{minimize}} \quad f_0(x)$$
$$\text{subject to:} \quad f_i(x) \le 0, \quad i = 1, \dots, m$$
$$h_i(x) = 0, \quad i = 1, \dots, p,$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$, (*optimization variable*)

$f_0 : \mathbf{R}^n \mapsto \mathbf{R}$, (*objective (cost/utility) function*)

$f_i : \mathbf{R}^n \mapsto \mathbf{R}$, (*inequality constraints (functions)*)

$h_i : \mathbf{R}^n \mapsto \mathbf{R}$, (*equality constraints (functions)*)

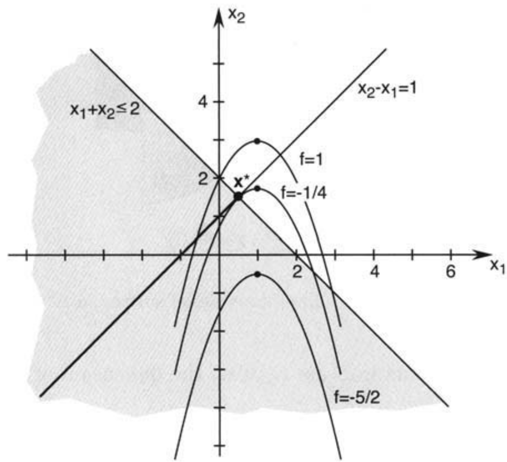
$$\mathcal{D} : \bigcap_{i=1}^{m} \textbf{dom}\, f_i \cap \bigcap_{i=1}^{p} \textbf{dom}\, h_i \qquad (\textit{feasible set})$$
$$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \le 0 \wedge h_i(x) = 0\}$$

$x^* : \{x \mid x \in \mathcal{D} \wedge f_0(x) \le f_0(z)\ \forall z \in \mathcal{D}\}$ (*solution*)

Example 3 (*Chong and Zak, 2001, Ex. 20.1, P. 454*):

$$\underset{x}{\text{minimize}} \quad (x_1 - 1)^2 + x_2 - 2$$
$$\text{subject to:} \quad x_2 - x_1 = 1$$
$$x_1 + x_2 \le 2.$$

No global minimizer: $\partial z / \partial x_2 = 1 \ne 0$. *However,* $z|_{(x_2 - x_1 = 1)} = (x_1 - 1)^2 + (x_1 - 1)$, *which attains a minima at* $x_1 = 1/2$.



$x* = (1/2, 3/2)'$. (Let's see animation)

1.1.1 Motivation and Applications

- *optimization problem* is an abstraction of how to make "best" possible choice of $x \in \mathbf{R}^n$.
- *constrains* represent trim requirements or specifications that limit the possible choices.
- *objective function* represents the *cost* to minimize or the *utility* to maximize for each x.

Examples:

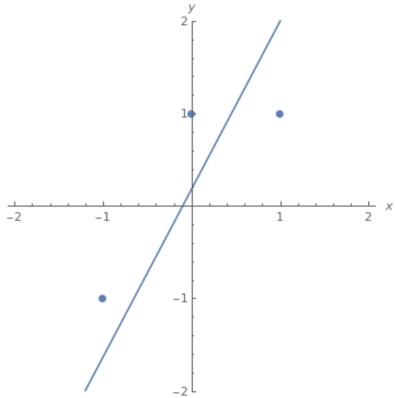
	Any problem	*Portfolio Optimization*	*Device Sizing*	*Data Science*
$x \in \mathbf{R}^n$	choice made	investment in capitals	dimensions	parameters
$f_i,\ h_i$	firm requirements /conditions	overall budget	engineering constraints	regularizer
f_0	cost (or utility)	overall risk	power consumption	error

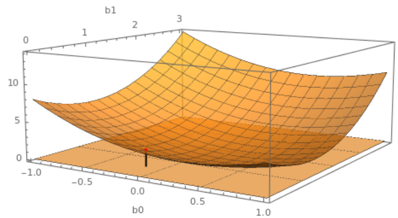
- Amazing variety of practical problems. In particular, data science: two sub-fields: construction and assessment.
- The construction of: Least Mean Square (LMS), Logistic Regression (LR), Support Vector Machines (SVM), Neural Networks(NN), Deep Neural Networks (DNN), etc.
- Many techniques are for solving the optimization problem:
  - Closed form solutions: convex optimization problems
  - Numerical solutions: Newton's methods, Gradient methods, Gradient descent, etc.
  - "Intelligent" methods: particle swarm optimization, genetic algorithms, etc.

Example 4 (Machine Learning: construction) *:*
Let's suppose that the best regression function is $Y = \beta_0 + \beta_1 X$, then for the training dataset (x_i, y_i) we need to minimize the MSE.

- Half of ML field is construction: NN, SVM, etc.
- In DNN it is an optimization problem of millions of parameters.
- Let's see animation.
- Where are Probability, Statistics, and Linear Algebra here? Let's re-visit the chart.
- Is the optimization problem solvable:
  - closed form? (LSM)
  - numerically and guaranteed? (convex and linear)
  - numerically but not guaranteed? (non-convex):
    - ∗ numerical algorithms, e.g., GD,
    - ∗ local optimization,
    - ∗ heuristics, swarm, and genetics,
    - ∗ brute-force with exhaustive search

$$\underset{\beta_o,\beta_1}{\text{minimize}} \sum_i (\beta_o + \beta_1 x_i - y_i)^2$$





1.1.2 Solving Optimization Problems

- A *solution method* for a class of optimization problems is an algorithm that computes a solution.
- Even when the *objective function* and constraints are smooth, e.g., polynomials, the solution is very difficult.
- There are three classes where solutions exist, theory is very well developed, and amazingly found in many practical problems:

  Linear $\subset$ Quadratic $\subset$ Convex $\subset$ Non-linear (not linear and not known to be convex!)

- For the first three classes, the problem can be solved very reliably in hundreds or thousands of variables!

1.2 Least-Squares and Linear Programming

1.2.1 Least-Squares Problems

A *least-squares* problem is an optimization problem with no constraints (i.e., $m = p = 0$), and an objective in the form:

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^{k} (a_i' x - b_i)^2 = \| A_{k\times n} x_{n\times 1} - b_{k\times 1} \|^2.$$

The solution is given in **closed form** by:

$$x = (A'A)^{-1} A' b$$

- Good algorithms in many SC SW exist; it is a very mature technology.
- Solution time is $O(n^2 k)$.
- Easily solvable even for hundreds or thousands of variables.
- More on that in the Linear Algebra course.
- Many other problems reduce to typical LS problem:
  - Weighted LS (to emphasize some observations)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^{k} w_i (a_i' x - b_i)^2.$$

  - Regularization (to penalize for over-fitting)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^{k} (a_i' x - b_i)^2 + \rho \sum_{j=1}^{n} x_j^2.$$

1.2.2 Linear Programming

A *linear programming* problem is an optimization problem with objective and all constraint functions are linear:

$$
\begin{array}{lll}
\underset{x}{\text{minimize}} & f_0(x) = C'x & \\
\text{subject to:} & a_i'x \leq b_i, & i = 1, \ldots, m \\
& h_i'x = g_i, & i = 1, \ldots, p,
\end{array}
$$

- **No** closed form solution as opposed to LS.
- Very robust, reliable, and effective set of methods for numerical solution; e.g., Dantzig's simplex, and interior point.
- Complexity is $\simeq O(n^2 m)$.
- Similar to LS, we can solve a problem of thousands of variables.
- Example is *Chebyshev minimization* problem:

$$\underset{x}{\text{minimize}}\, f_0(x) = \max_{i=1,\ldots,k} |a_i'x - b_i|,$$

- The objective is different from the LS: minimize the maximum error. **Ex:**
- After some tricks, requiring familiarity with optimization, it is equivalent to a LP:

$$
\begin{array}{lll}
\underset{x}{\text{minimize}} & t & \\
\text{subject to:} & a_i'x - t \leq b_i, & i = 1, \ldots, k \\
& -a_i'x - t \leq -b_i, & i = 1, \ldots, k
\end{array}
$$

1.3 Convex Optimization

A *convex optimization* problem is an optimization problem with objective and all constraint function are convex:

$$
\begin{aligned}
&\underset{x}{\text{minimize}} && f_0(x) && && \\
&\text{subject to:} && f_i(x) \le 0, && && i = 1, \ldots, m \\
& && h_i(x) = 0, && && i = 1, \ldots, p, \\
& && f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y), && \alpha + \beta = 1, && 0 \le \alpha,\ 0 \le \beta, \qquad 0 \le i \le m \\
& && h_i(x) = a_i' x + b_i && && 0 \le i \le p
\end{aligned}
$$

- The LP and LS are special cases; however, only LS has closed-form solution.
- Very robust, reliable, and effective set of methods, including *interior point methods.*
- Complexity is almost: $O\big(\max(n^3, n^2 m, F)\big)$, where F is the cost of evaluating 1st and 2nd derivatives of f_i and h_i.
- Similar to LS and LP, we can solve a problem of thousands of variables.
- However, it is not as very mature technology as the LP and LS yet.
- There are many practical problems that can be re-formulated as convex problem **BUT** requires mathematical skills; but once done the problem is solved. **Hint:** realizing that the problem is convex requires more mathematical maturity than those required for LP and LS.

1.4 Nonlinear Optimization

A *non-linear optimization* problem is an optimization problem with objective and constraint functions are non-linear **BUT** not known to be convex (**so far**). Even simple-looking problems in 10 variables can be extremely challenging. Several approaches for solutions:

Local Optimization : starting at initial point in space, using differentiablity, then navigate

- does not guarantee global optimal.
- affected heavily by initial point.
- depends heavily on numerical algorithm and their parameters.
- More art than technology.
- In contrast to convex optimization, where a lot of art and mathematical skills are required to formulate the problem as convex; then numerical solution is straightforward.

Global Optimization : the true global solution is found; the compromise is complexity.

- The complexity goes exponential with dimensions.
- Sometimes it is worth it when: the cost is huge, not in real time, and dimensionality is low.

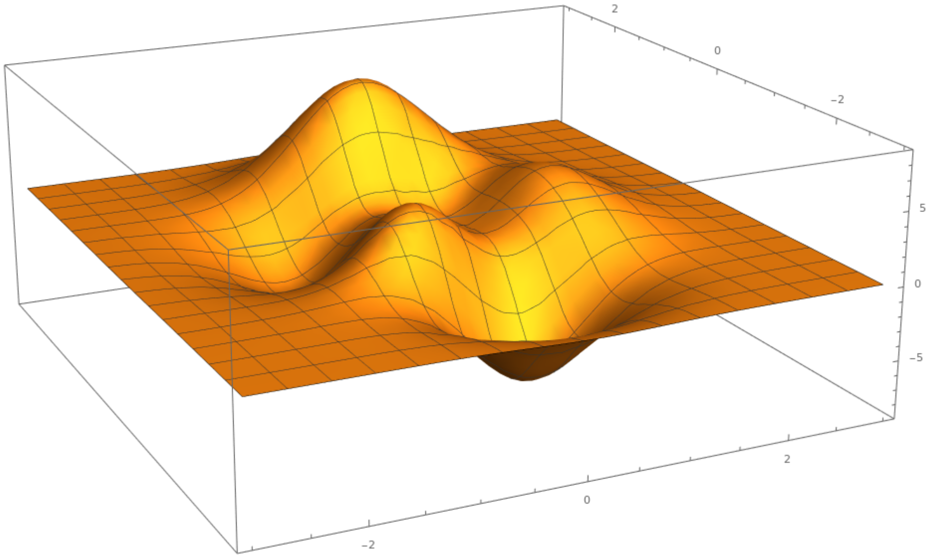
Role of Convex Optimization :

- Approximate the non-linear function to a convex one, finding the exact solution, then using it as a starting point for the original problem. (Also does not guarantee optimality)
- Setting bounds on the global solution.

Evolutionary Computations : Genetic Algorithm (GA), Simulated Annealing (SA), Particle Swarm Optimization (PSO), etc.

Example 5 (Nonlinear Objective Function) *: (Chong and Zak, 2001, Ex. 14.3, P.290)*

$$f(x,y) = 3(1-x)^2 e^{-x^2-(y+1)^2} - 10e^{-x^2-y^2}\left(-x^3 + \frac{x}{5} - y^5\right) - \frac{1}{3}e^{-(x+1)^2-y^2}$$



Part I

Theory

Chapter 2

Convex sets

2.1 Affine and convex sets

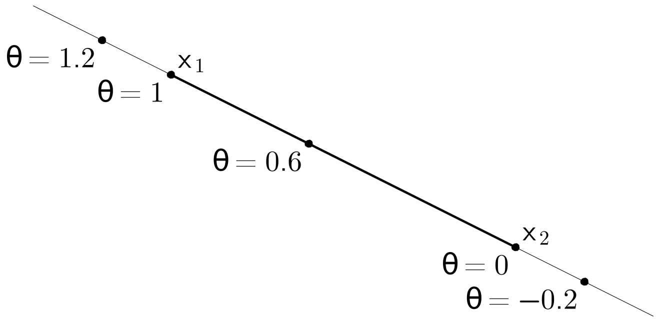
2.1.1 Lines and line segments

Definition 6 (line and line segment) *Suppose $x_1 \neq x_2 \in \mathbf{R}^n$. Points of the form*

$$y = \theta x_1 + (1-\theta)x_2$$
$$= x_2 + \theta(x_1 - x_2),$$

where $\theta \in \mathbf{R}$, form the line passing through x_1 and x_2.

- As usual, this is a definition for high dimensions taken from a proof for $n \leq 3$.
- We have done it many times: angle, norm, cardinality of sets, etc.
- if $0 \leq \theta \leq 1$ this forms a line segment.



2.1.2 Affine sets

Definition 7 (Affine sets) *A set $C \subset \mathbf{R}^n$ is affine if the line through any two distinct points in C lies in C. I.e., $\forall x_1, x_2 \in C$ and $\theta \in \mathbf{R}$, we have $\theta x_1 + (1-\theta)x_2 \in C$. In other words, C contains any linear combination (summing to one) of any two points in C.*

Examples: what about line, line segment, circle, disk, strip, first quadrant?

Corollary 8 *Suppose C is an affine set, and $x_1, \ldots, x_k \in C$, then C contains every general affine combination of the form $\theta_1 x_1 + \ldots + \theta_k x_k$, where $\theta_1 + \ldots + \theta_k = 1$.*

Wrong Proof. Suppose $y_1, y_2 \in C$, then

$$x = \sum_{i=1}^{k} \theta_i x_i = \sum_{i=1}^{k} \theta_i(\alpha_i y_1 + (1-\alpha_i)y_2);$$

and the summation of the coefficients will be

$$\sum_{i=1}^{k} \theta_i \alpha_i + \sum_{i=1}^{k} \theta_i(1-\alpha_i) = \sum_{i=1}^{k} \theta_i(\alpha_i + 1 - \alpha_i) = \sum_{i=1}^{k} \theta_i = 1.$$

Where is the bug? ∎

Correct Proof. **base:** $k = 3$.

$$\begin{aligned} x &= \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \\ &= (1-\theta_3)\left(\frac{\theta_1}{1-\theta_3}x_1 + \frac{\theta_2}{1-\theta_3}x_2\right) + \theta_3 x_3. \\ &= (1-\theta_3)(\cdot \in C) + \theta_3(\cdot \in C). \end{aligned}$$

induction: suppose it is true for some $k \geq 3$; i.e., $\sum_{i=1}^{k} \theta_i x_i \in C$ when $\sum_{i=1}^{k} \theta_i = 1$. Then

$$\begin{aligned} x &= \sum_{i=1}^{k+1} \theta_i x_i \\ &= \sum_{i=1}^{k} \theta_i x_i + \theta_{k+1} x_{k+1} \\ &= (1-\theta_{k+1}) \sum_{i=1}^{k} \theta_i / (1-\theta_{k+1}) x_i + \theta_{k+1} x_{k+1} \\ &= (1-\theta_{k+1})(\cdot \in C) + \theta_{k+1}(\cdot \in C), \\ &\qquad \text{(from the induction hypothesis)} \end{aligned}$$

which completes the proof. ∎

Definition 9 (Subspace from Linear Algebra) *a set $V \subset \mathbf{R}^n$ of vector (here points) is a subspace if it is closed under sums and scalar multiplication. I.e., $\forall v_1, v_2 \in V$ and $\forall \alpha, \beta \in \mathbf{R}$ we have $\alpha v_1 + \beta v_2 \in V$.*

Remember:

- $\alpha + \beta$ not necessarily equals 1
- $\alpha = 0, \beta = 0 \to \mathbf{0} \in V$.
- Any subspace V is the solution set of $A_{m\times n} x_{n\times 1} = 0$, which is $\mathcal{N}(A)$ (the null space of A). Geometry? I.e., $V = \{x \mid Ax = 0\}$
- $\mathbf{rank}(A) = n - \dim(V)$.

Corollary 10 .

1. *If C is affine, then $V = C - x_0 = \{x - x_0 \mid x, x_0 \in C\}$ is a subspace.*
2. *If V is a subspace, then $C = V + x_0 = \{x + x_0 \mid x \in V\}$ is affine $\forall x_0$.*
3. *An affine set C can be represented as the solution set of a nonhomogeneous linear system $Ax = b$, where $V = C - x_0$ is $\mathcal{N}(A)$.*
4. *The solution set of any nonhomogeneous system is an affine set. (Ex. 2.1)*

Proof.

1. Suppose $x_1, x_2, x_0 \in C$, an affine set. Both $x_1 - x_0$ and $x_2 - x_0$, by construction, $\in V$; then

$$\begin{aligned} x &= \alpha(x_1 - x_0) + \beta(x_2 - x_0) + x_0 \\ &= \alpha x_1 + \beta x_2 + (1 - \alpha - \beta)x_0 \in C \end{aligned}$$

   Then $x - x_0 = \alpha(x_1 - x_0) + \beta(x_2 - x_0) \in V$; hence V is a subspace.

2. Suppose $x_1, x_2 \in V$, a subspace. Both $x_1 + x_0$ and $x_2 + x_0$, by construction, $\in C$; then

$$\begin{aligned} x &= \theta(x_1 + x_0) + (1 - \theta)(x_2 + x_0) \\ &= \theta x_1 + (1 - \theta)x_2 + x_0 = (\cdot \in V) + x_0 \in C \end{aligned}$$

3. If C is affine and $x_0 \in C$, then

$$\begin{aligned} C - x_0 &= \{x \mid Ax = 0\} \qquad \text{(since it is a subspace)} \\ C &= \{x + x_0 \mid A(x + x_0) = Ax_0\} \\ C &= \{c \mid Ac = b\}. \end{aligned}$$

4. $C = \{x \mid Ax = b\}$; if $x_0 \in C$ then $Ax_0 = b$ and

$$C - x_0 = \{x - x_0 \mid A(x - x_0) = b - Ax_0 = 0\}.$$

   Hence, $C - x_0$ is a subspace and C is affine.

■

Proof of the book. Suppose $x_1, x_2 \in C$, where $C = \{x \mid Ax = b\}$. Then

$$A(\theta x_1 + (1-\theta)x_2) = \theta A x_1 + (1-\theta)Ax_2 = \theta b + (1-\theta)b = b,$$

which means $\theta x_1 + (1-\theta)x_2 \in C$ as well. ■

Remark 1 *:*

- *The dimension of affine is defined to be the dimension of the associate subspace.*
- *affine is a subspace plus offset.*
- *every subspace is affine but not the vice versa; i.e., subspace is a special case of affine.*

Definition 11 (affine hull) *The "smallest" set of all affine combinations of some set C (not necessarily affine) is called the affine hull (**aff** C):*

$$\textbf{aff } C = \{\sum_{i=1}^{k} \theta_i x_i \mid x_i \in C,\ \sum_i \theta_i = 1\}.$$

Corollary 12 **aff** C *is affine.*

Proof. For $x_1 = \sum_i \alpha_i x_i$ and $x_2 = \sum_i \beta_i x_i$ we have

$$\theta x_1 + (1-\theta)x_2 = \theta \sum_i \alpha_i x_i + (1-\theta)\sum_i \beta_i x_i = \sum_i (\theta\alpha_i + (1-\theta)\beta_i) x_i$$

$$\sum_i (\theta\alpha_i + (1-\theta)\beta_i) = \theta \sum_i \alpha_i + (1-\theta)\sum_i \beta_i = \theta + (1-\theta) = 1.$$

Hence, **aff** C is affine as well. ∎

Example 13 *Construct the affine hull of the set* $C = \{(-1,0),(1,0),(0,1)\}$

$$\begin{aligned}\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 &= (1-\theta_3)\Big(\frac{\theta_1}{1-\theta_3} x_1 + \frac{\theta_2}{1-\theta_3} x_2\Big) + \theta_3 x_3 \\ &= (1-\alpha_3)\big((1-\alpha_2)x_1 + \alpha_2 x_2\big) + \alpha_3 x_3 \qquad = (1-\alpha_2)(1-\alpha_3)x_1 + \alpha_2(1-\alpha_3)x_2 + \alpha_3 x_3,\end{aligned}$$

$$\begin{array}{lll} \theta_3 = \alpha_3 & \theta_2 = \alpha_2(1-\alpha_3) & \theta_1 = 1-\theta_2-\theta_3 = (1-\alpha_2)(1-\alpha_3) \\ \alpha_3 = \theta_3 & \alpha_2 = \theta_2/(1-\theta_3) & \alpha_1 = 1-\alpha_2 = \theta_1/(1-\theta_3). \end{array}$$

HW: Derive expressions for α_i and θ_i for n-point combination.

2.1.3 Affine dimension and relative interior

Definition 14 (some basic topology in $\mathbf{R}^n$) *:*

1. *The ball of radious r and center x in the norm $\|\cdot\|$.*

$$B(x,r)=\{y \mid \|y-x\|\le r\}.$$

2. *An element $x\in C\subseteq \mathbf{R}^n$ is called an interior point of C if $\exists\ \varepsilon>0$ for which*

$$B(x,\epsilon)=\{y \mid \|y-x\|_2\le\varepsilon\}\subseteq C.$$

   I.e., $\exists$ a ball centered at x that lies entirely in C.

3. *The set of all points interior to C is called the interior of C and is denoted* **int** C*.*
4. *A set C is open if* **int** $C=C$*. I.e., every point in C is an interior point.*
5. *A set C is closed if its complement is open*

$$\mathbf{R}^n\setminus C=\{x\in\mathbf{R}^n \mid x\notin C\}$$

6. *The closure of a set C is defined as*

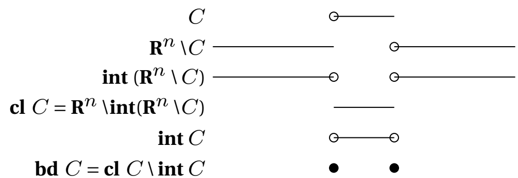
$$\mathbf{cl}\ C=\mathbf{R}^n\setminus\mathbf{int}(\mathbf{R}^n\setminus C).$$

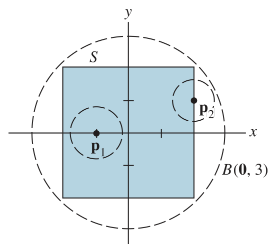
7. *The boundary C is defined as*

$$\mathbf{bd}\ C=\mathbf{cl}\ C\setminus\mathbf{int}\ C.$$

Corollary 15 *A boundary point (a point $x\in\mathbf{bd}\,C$) satisfies: $\forall\epsilon>0,\exists\ y\in C$ and $z\notin C$ s.t. $y,z\in B(x,\epsilon)$.*

I.e., there exists arbitrarily close points in C, and also arbitrarily close point not in C.





Definition 16 *We define the affine dimension of a set C as the dimension of its affine hull.*

Example 17 *The unit circle in $\mathbf{R}^2$, i.e., $\{x \mid x_1^2 + x_2^2 = 1\}$ has affine hull of whole $\mathbf{R}^2$. So its affine dimension is 2. However, it has a dimensionality of 1 since it is parametric in just one parameter (manifold).*

Definition 18 *We define the relative interior of the set C, denoted* **relint** C*, as its interior relative to* **aff** C

$$\mathbf{relint}\ C = \{x \in C \mid B(x,r) \cap \mathbf{aff}\ C \subseteq C \text{ for some } r > 0\},$$

and its relative boundary, denoted **relbd** C *is defined as*

$$\mathbf{relbd}\ C = \mathbf{cl}\ C \setminus \mathbf{relint}\ C.$$

Example 19 *Consider a square in the (x_1, x_2)-plane in $\mathbf{R}^3$, defined as:*

$$C = \{x \in \mathbf{R}^3 \mid -1 \le x_1 \le 1,\ -1 \le x_2 \le 1,\ x_3 = 0\}.$$

Then:

$$\begin{aligned}
\mathbf{int}\ C &= \Phi \\
\mathbf{cl}\ C &= \mathbf{R}^n \setminus \mathbf{int}\left(\mathbf{R}^n \setminus C\right) = C \\
\mathbf{bd}\ C &= \mathbf{cl}\ C \setminus \mathbf{int}\ C = C \\
\mathbf{aff}\ C &= \{x \in \mathbf{R}^3 \mid x_3 = 0\} \\
\mathbf{relint}\ C &= C = \{x \in \mathbf{R}^3 \mid -1 < x_1 < 1,\ -1 < x_2 < 1,\ x_3 = 0\} \\
\mathbf{relbd}\ C &= \{x \in \mathbf{R}^3 \mid \max\{|x_1|, |x_2|\} = 1,\ x_3 = 0\}
\end{aligned}$$

2.1.4 Convex sets

Definition 20 (convex set) *A set C is convex if the line segment bbetween any two points in C lies in C; i.e., if for any $x_1, x_2 \in C$ and any θ with $0 \le \theta \le 1$, we have*

$$\theta x_1 + (1-\theta)x_2 \in C.$$

Corollary 21 *Suppose C is convex set, and $x_1, \ldots, x_k \in C$, then C contains every general convex combination (also called mixture); i.e.,*

$$\sum_i \theta_i x_i \in C,\ \sum_i \theta_i = 1,\ \theta_i \ge 0.$$

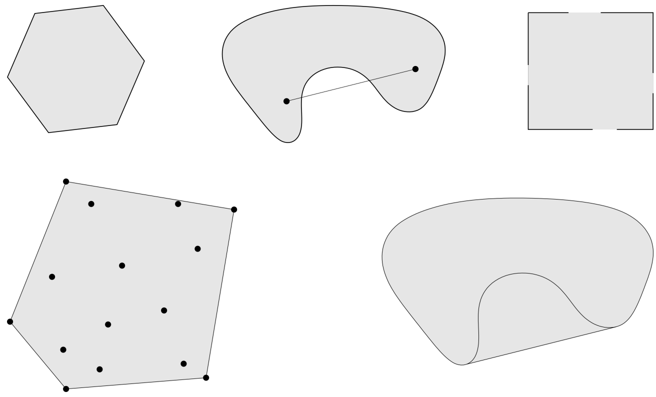
Proof. identical to proof of corollary 8. ■

Definition 22 (convex hull) *The "smallest" set of all convex combinations of some set C (not necessarily convex) is called the convex hull (**conv** C)*

$$\mathbf{conv}\, C = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in C,\ \sum_i \theta_i = 1,\ \theta_i \ge 0 \right\}.$$

Corollary 23 **conv** *C is convex.*

Proof. identical to proof of corollary 12. ■



Example 24 *Revisit example 13.*

Example 25 (Applications) *: Suppose $X \in C$ is a r.v., C is convex. Then $\mathrm{E}\,X \in C$ if it exists:*

$$\mathrm{E}\,X = \sum_{i=1}^{n} p_i x_i$$

$$\mathrm{E}\,X = \sum_{i=1}^{\infty} p_i x_i$$

$$\mathrm{E}\,X = \int_C f_X(x) x \, dx \qquad \text{(Riemann sum)}$$

2.1.5 Cones

Definition 26 *A set C is called a cone (or nonnegative homogeneous) if $\forall\ x \in C,\ \theta \geq 0$ we have $\theta x \in C$; and it is a convex cone if it is convex in addition to being a cone.*

Definition 27 *A point of the form $\sum_{i=1}^{k} \theta_i x_i,\ \theta_i \geq 0$ is called a conic combination.*

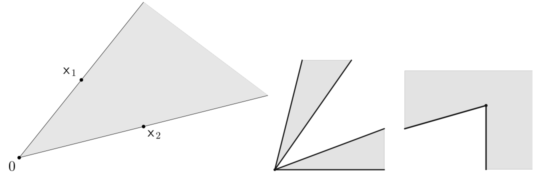
Corollary 28 *A set C is a convex cone if and only if it contains all conic combinations of its elements; i.e.,*

$$\sum_i \theta_i x_i \in C\ \forall x_i \in C \text{ and } \theta_i \geq 0.$$

Proof.
Sufficiency: is obvious. Choosing $\sum_i \theta_i = 1$ implies C is convex; and setting $\theta_i = 0\ \forall i > 1$ implies C is cone.
Necessity: Since C is convex cone, then $\forall x_i \in C, \theta_i \geq 0$ we have:

$$\theta_i x_i \in C \qquad \text{(cone)}$$
$$\sum_i (1/n)(\theta_i x_i) \in C \qquad \text{(convex)}$$
$$n \sum_i (1/n)(\theta_i x_i) = \sum_i \theta_i x_i \in C \qquad \text{(cone)}$$

■

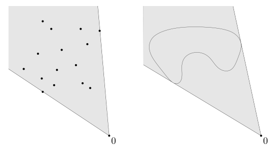


Definition 29 *A conic hull of a set C is the minimum set of all conic combination:*

$$\textbf{cone } C = \{\sum_i \theta_i x_i \mid x_i \in C,\ \theta_i \geq 0,\ i = 1, \ldots, n\}.$$

Corollary 30 **cone** *C is convex cone.*

Proof. If $y \in$ **cone** C, then $\alpha y = \alpha \sum_i \theta_i x_i = \sum_i (\alpha\theta_i) x_i \in$ **cone** C. And if $y_1, y_2 \in$ **cone** C then $\alpha y_1 + (1-\alpha) y_2 = \alpha \sum_i \theta_i x_i + (1-\alpha) \sum_i \mu_i x_i = \sum_i (\alpha\theta_i + (1-\alpha)\mu_i) x_i \in$ **cone** C ■



2.2 Some important examples

Fast Revision

- Each of the sets: ϕ, x_0 (a singleton), $\mathbf{R}^n$ are affine and convex.
- Any line is affine. If it passes through zero, it is a subspace and a convex cone.
- Any subspace is convex cone but not vise versa.
- A line segment is convex, but not affine (unless it reduces to a singleton).
- A *ray*, $\{x_0 + \theta v \mid \theta \geq 0, v \neq 0\}$ is convex but not affine. It is convex cone if $x_0 = 0$.

2.2.1 Hyperplanes and halfspaces

Definition 31 *A hyperplane is a set of the form*

$$\begin{aligned} \mathcal{S} &= \{x \mid a'x = b\}, & a, b \in \mathbf{R}^n,\ a \neq 0 \\ &\equiv \{x \mid a'(x - x_0) = 0\}, & a'x_0 = b. \end{aligned}$$

- Vectors with inner product with a is b: $\frac{a'}{\|a\|}x = \frac{b}{\|a\|}$. I.e., from $\mathbf{0}$, walk a distance $\frac{b}{\|a\|}$ (either + or -) in the direction of a, then draw perpendicular line.

Definition 32 *A closed halfspace is the region generated by the hyperplane and defined as:*

$$\begin{aligned} \mathcal{H} &= \{x \mid a'x \leq b\}, & a, b \in \mathbf{R}^n,\ a \neq 0 \\ &\equiv \{x \mid a'(x - x_0) \leq 0\}, & a'x_0 = b. \end{aligned}$$

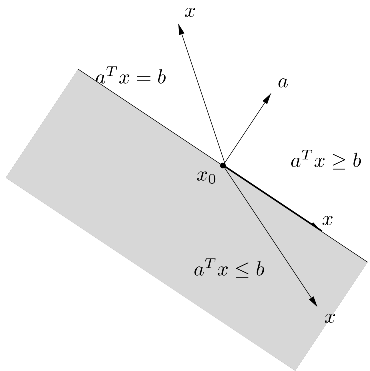
- region of all vectors with projection $< b/\|a\|$.
- Vectors with obtuse angle with a: ($\cos\theta = \frac{a'x}{\|a\|\|x\|}$).
- Line passing with p and $\perp$ on $\mathcal{S}$:

$$x - p = \theta\overline{a} \qquad \text{(parametric eq.)}$$

$$a'x - a'p = \theta\|a\|$$

$$\theta_0 = (b - a'p)/\|a\| \qquad (x_0 \text{ pt. of intersection.})$$

$$x_0 - p = \frac{(b - a'p)}{\|a\|}\overline{a}.$$



Corollary 33 *$\mathcal{S}$ is affine, $\mathcal{H}$ is convex and not affine,* **int** *$\mathcal{H} = \mathcal{H} \setminus \mathcal{S}$, and* **bd** *$\mathcal{H} = \mathcal{S}$.*

Proof. $\mathcal{S}$ is affine done. $\mathcal{H}$ is convex: $\theta a'x_1 + (1-\theta)a'x_2 \leq \theta b + (1-\theta)b = b$. (why not affine?!)

$$y = x + ru, \quad 0 \leq \|u\| \leq 1 \qquad (y \in B(x, r))$$

$$a'y = a'x + ra'u = b - (b - a'x) + r\|a\|\|u\|\cos(a, u)$$

If $b = a'x$, i.e., $x \in \mathcal{S}$, $a'u > 0$ or < 0 (depending on the angle) and hence $a'y > b$ or $< b$. Then $\mathcal{S} \subseteq$ **bd** $\mathcal{H}$.

If $a'x < b$, i.e., $x \in \mathcal{H} \setminus \mathcal{S}$, $\exists\, r < \frac{b - a'x}{\|a\|}$, s.t. $a'y < b$. Hence: **int** $\mathcal{H} = \mathcal{H} \setminus \mathcal{S}$ and **bd** $\mathcal{H} = \mathcal{S}$. ■

2.2.2 Euclidean balls and ellipsoids

Definition 34 *A Euclidean ball in* $\mathbf{R}^n$ *is the set:*

$$\begin{aligned} B(x_c, r) &= \left\{x = x_c + ru \mid 0 \le \|u\|_2 \le 1\right\} \\ &= \left\{x \mid \|x - x_c\|_2 \le r\right\} \\ &= \left\{x \mid (x - x_c)'(x - x_c) \le r^2\right\}. \end{aligned}$$

Corollary 35 *A ball is convex*

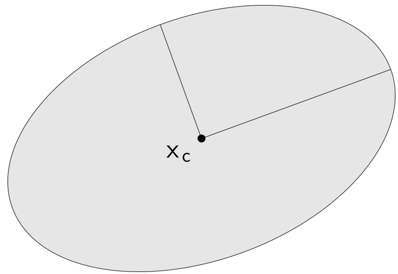
Proof. $x = \theta x_1 + (1-\theta)x_2,\ x_1, x_2 \in B(x_c, r),\ 0 \le \theta \le 1$

$$\begin{aligned} \|x - x_c\| &= \|\theta(x_1 - x_c) + (1-\theta)(x_2 - x_c)\| \\ &\le \theta\|(x_1 - x_c)\| + (1-\theta)\|(x_2 - x_c)\| \\ &\le \theta r + (1-\theta) r = r \end{aligned}$$

■

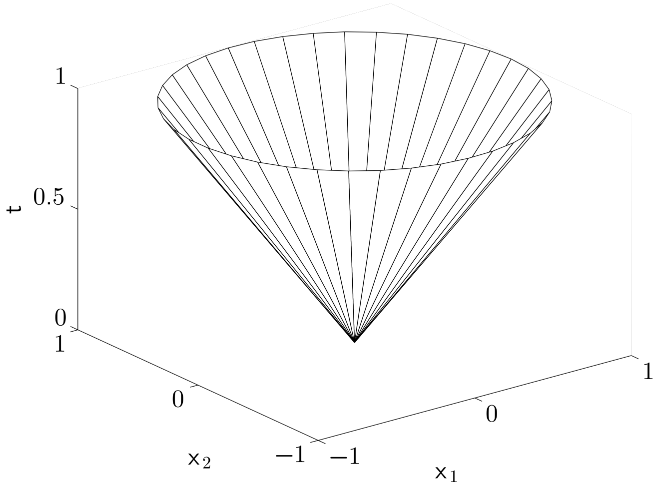
Definition 36 *Ellipsoid in* $\mathbf{R}^n$ *is the set:*

$$\mathcal{E} = \left\{x = x_c + Au \mid \|u\|_2 \le 1,\ A = A' \succ 0\right\}$$

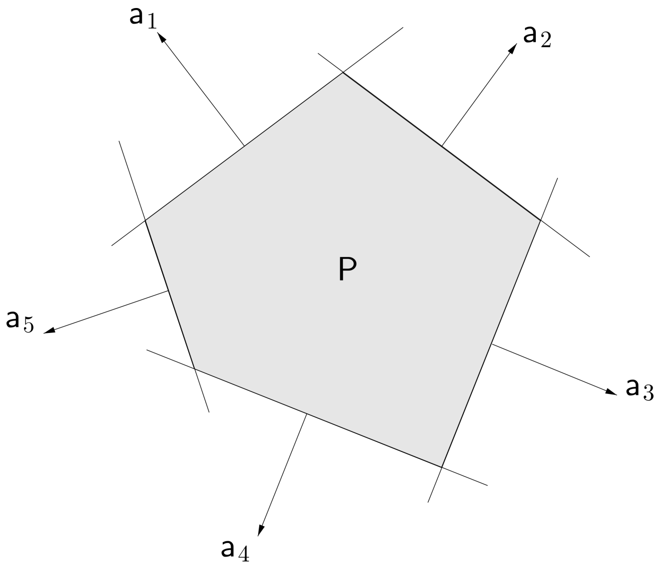


Prove/disprove that if $C = C_1 \bigcup C_2,\ C_1 \bigcap C_2 = \phi$, i.e., union of 2 disjoint sets, then C is convex.

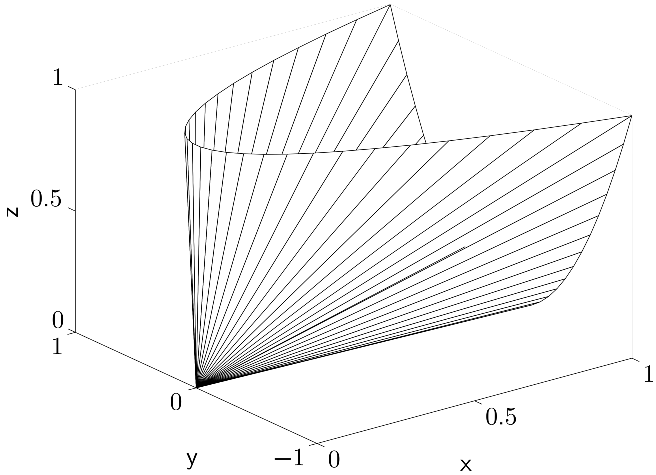
2.2.3 Norm balls and norm cones



2.2.4 Polyhedra



2.2.5 The positive semidefinite cone



2.3 Operations that preserve convexity

2.4 Generalized inequalities

2.5 Separating and supporting hyperplanes

2.6 Dual cones and generalized inequalities

Part II

Applications

Part III

Algorithms

Bibliography

Boyd, S. and Vandenberghe, L. (2004), *Convex Optimization*, Cambridge: Cambridge University Press.

Chong, E. K. and Zak, Stanislaw, H. (2001), *An Introduction to Optimization*, Wiley-Interscience, 4th ed.