A single number such as 2, 6.4 or -4 is called a *scalar*; the elements of a matrix are (usually) scalars although, as discussed in Section 2.2 and in Chapter 10, a matrix can be expressed as a matrix of smaller matrices. Sometimes it is convenient to think of a scalar as a matrix of order 1×1 (see Section 2.3).

8. GENERAL NOTATION

A well-recognized notation is that of denoting matrices by uppercase letters and their elements by the lowercase counterparts with appropriate subscripts. Vectors are denoted by lowercase letters, often from the end of the alphabet, using the prime superscript to distinguish a row vector from a column vector. Thus $\mathbf{x}$ is a column vector and $\mathbf{x}'$ is a row vector. The lowercase Greek lambda, λ, is often used for a scalar.

Throughout this book the notation for displaying a matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \end{bmatrix},$$

enclosing the array of elements in square brackets. Among the variety of forms that can be found in the literature are

$$\begin{pmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \end{pmatrix}, \quad \begin{Bmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \end{Bmatrix} \quad \text{and} \quad \begin{Vmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \end{Vmatrix}.$$

Single vertical lines are seldom used, since they are usually reserved for determinants (see Chapter 4).

Another useful notation that has already been described is

$$\mathbf{A} = \{a_{ij}\} \text{ for } i = 1,2,\ldots,r \text{ and } j = 1,2,\ldots,c.$$

The curly brackets indicate that a_{ij} is a typical term of the matrix $\mathbf{A}$ for all pairs of values of i and j from unity up to the limits shown, in this case r and c; i.e., $\mathbf{A}$ is a matrix of r rows and c columns. This is by no means a universal notation and several variants of it can be found in the literature. Furthermore, there is nothing sacrosanct about the repeated use in this book of the letter $\mathbf{A}$ for a matrix. Any letter may be used.

9. ILLUSTRATIVE EXAMPLES

A text on matrix algebra designed for pure mathematicians would deal with many topics that do not appear in this book because they have little direct connection with practical applied problems. The text for mathematicians might also have few numerical illustrations. This will not be the case here, however, and indeed, simple illustrations have already been used to introduce the concept of a matrix. The advantage of such illustrations is that for many people arithmetic involving actual numbers is easier to follow than long expositions in algebra. For this reason, mathematical results are frequently illustrated with numerical examples.

The mathematical discussion in the ensuing chapters is relatively informal. Proofs and demonstrations of general results are not omitted, however, for to do so would result in a lifeless "cookbook" presentation. Once a general mathematical result has been established it is usually illustrated by a numerical example, with additional examples available as exercises at the end of most chapters. Only a small proportion of these is algebraic; many are numerical.

Numerical illustrations come in two styles. One style draws on real-life backgrounds like the examples of sterile cultures and taxicab fares already used. These two illustrations are solely for demonstrating an idea, but later in the book when matrix tools of sufficient complexity have been developed, illustrations from real life show how matrix algebra can help solve genuine problems. Before reaching that stage, however, there are intermediate mathematical steps for which few real-world problems can be used as illustrations. For this part of the work we rely heavily on the second style of illustration, which is nothing more than numbers "pulled out of a hat" for the sole purpose of illustrating a mathematical procedure. Such numbers have no use other than the very good one of assisting the reader to understand the procedures being demonstrated. Numerical illustrations created in this manner are labeled "Example"; those with a background in real life are labeled "Illustration."

EXERCISES

1. For $a_{11} = 17 \quad a_{12} = 31 \quad a_{13} = 26 \quad a_{14} = 11$
   $a_{21} = 19 \quad a_{22} = 27 \quad a_{23} = 16 \quad a_{24} = 14$
   $a_{31} = 21 \quad a_{32} = 23 \quad a_{33} = 15 \quad a_{34} = 16$

   show that

   (a) $a_{1.} = 85$, $a_{2.} = 76$, and $a_{3.} = 75$;

   (b) $a_{.1} = 57$, $a_{.2} = 81$, $a_{.3} = 57$, and $a_{.4} = 41$;

   (c) $a_{..} = 236$;

   (d) $\sum_{i=1}^{3} a_{ii} = 59$;

(e) $\sum_{\substack{j=1\\j\neq 2}}^{4} a_{ij} = 54$, 49 and 52 for $i = 1$, 2 and 3, respectively;

(f) $\sum_{\substack{i=1\\i\neq 2}}^{3} \sum_{\substack{j=1\\j\neq 3}}^{4} a_{ij} = 119$.

2. For the elements of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 17 & 9 & -2 & 3 \\ 3 & 13 & 10 & 2 & 6 \\ 11 & -9 & 0 & -3 & 2 \\ -6 & -8 & 1 & 4 & 5 \end{bmatrix} = \{a_{ij}\}$$

for $i = 1, 2, \ldots, 4$ and $j = 1, 2, \ldots, 5$ show that

(a) $a_{1.} = 26$;

(b) $a_{.3} = 20$;

(c) $\sum_{i=1}^{3} a_{i2} = 21$;

(d) $\sum_{\substack{i=1\\i\neq 2}}^{4} a_{i5} = 10$;

(e) $\sum_{i=3}^{4} a_{i.}^2 = \sum_{i=3}^{4} (a_{i.})^2 = 17$;

(f) $a_{..} = 57$;

(g) $\sum_{i=1}^{4} \dfrac{\sum_{j=1}^{5} a_{ij}^2}{a_{i.}} = 203\dfrac{275}{442}$;

(h) $\sum_{\substack{i=1\\i\neq 2}}^{4} \sum_{\substack{j=2\\j\neq 4}}^{5} a_{ij} = 20$;

(i) $\prod_{j=1}^{3} a_{4j} = \prod_{i=1}^{4} a_{i4}$;

(j) $\sum_{i=1}^{4} a_{i1}a_{i4} = -49$;

(k) $\sum_{i=1}^{4} \sum_{j=1}^{5} a_{..} = 1140$;

(l) $\prod_{i=1}^{3} a_{i4} = 12$;

(m) $\prod_{j=1}^{5} a_{2j}a_{3j} = 0$;

(n) $\prod_{i=1}^{4} 2^{a_{i4}} = 2$;

(o) $\sum_{i=1}^{4} \sum_{j=1}^{5} (-1)^j a_{ij} = -29$;

(p) for $i = 2$, $\sum_{\substack{j=1\\j\neq 3}}^{5} (a_{ij} - a_{i+2,j})^2 = 527$;

(q) $\prod_{j=1}^{5} 2^{(-1)^j a_{2j}} = 0.0625$.

3. From $\mathbf{A}$ of Exercise 2, write down the following matrices:

$$\mathbf{B} = \{a_{i+1,j+2}\} \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad j = 1, 2, 3;$$

$$\mathbf{C} = \{a_{2i,2j-1}\} \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad j = 1, 2, 3;$$

and

$$\mathbf{D} = \{a_{i,j} + i - j\} \quad \text{for} \quad i = 2, 3 \quad \text{and} \quad j = 1, \ldots, 4.$$

4. Prove the following identities, and demonstrate their validity using $\mathbf{A}$ of Exercise 2.

(a) $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}^2$.

(b) $\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)^2 = \sum_{i=1}^{m} a_{i.}^2$.

(c) $\sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij}a_{hk} = a_{i.}a_{h.}$, for $i \neq h$.

(d) $\sum_{i=1}^{m} \sum_{j=1}^{n} 4a_{ij} = 4a_{..}$.

(e) $$\left(\sum_{j=1}^{n} a_{ij} \right)^2 = \sum_{j=1}^{n} a_{ij}^2 + 2 \sum_{j=1}^{n-1} \sum_{p=j+1}^{n} a_{ij}a_{ip}$$
$$= \sum_{j=1}^{n} a_{ij}^2 + 2 \sum_{j=1}^{n-1} \sum_{p>j}^{n} a_{ij}a_{ip}$$
$$= \sum_{j=1}^{n} (a_{ij}^2 + \sum_{j=1}^{n} \sum_{\substack{p=1\\p\neq j}}^{n} a_{ij}a_{ip}.$$

(f) $$a_{..}^2 = \sum_{i=1}^{n} a_{i.}^2 + 2 \sum_{i=1}^{n-1} \sum_{h=i+1}^{n} a_{i.}a_{h.}$$
$$= \sum_{i=1}^{n} a_{i.}^2 + 2 \sum_{i=1}^{n-1} \sum_{h>i}^{n} a_{i.}a_{h.}$$
$$= \sum_{i=1}^{n} a_{i.}^2 + \sum_{i=1}^{n} \sum_{\substack{h=1\\h\neq i}}^{n} a_{i.}a_{h.} \; .$$

(g) $\sum_{\substack{i=1\\i\neq p}}^{m} \sum_{\substack{j=1\\j\neq q}}^{n} a_{ij} = a_{..} - a_{p.} - a_{.q} + a_{pq}$.

(h) $\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} - 1) = a_{..} - mn.$

(i) $\sum_{j=1}^{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{ij} a_{ik} = a_{i\cdot}^2 - \sum_{j=1}^{n} a_{ij}^2.$

5. Show that

(a) $\sum_{i=3}^{5} 3^i = 351;$ (b) $\sum_{k=2}^{7} 2^k = 252;$

(c) $\sum_{r=1}^{5} r = 15;$ (d) $\sum_{\substack{s=1 \\ s \neq 2}}^{5} s(s+1) = 106;$

(e) $\prod_{i=1}^{4} 2^i = 1024;$ (f) $\sum_{i=1}^{3} \prod_{j=1}^{2} i^{2j-1} = 98.$

6. Write down the following matrices:

$$\mathbf{A} = \{a_{ij}\} \quad \text{for } a_{ij} = i + j \quad \text{for } i = 1, 2, 3 \quad \text{and} \quad j = 1, 2.$$

$$\mathbf{B} = \{b_{kt}\} \quad \text{for } b_{kt} = k^{t-1} \quad \text{for } k = 1, \ldots, 4 \quad \text{and} \quad t = 1, \ldots, 3.$$

$$\mathbf{C} = \{c_{rs}\} \quad \text{for } c_{rs} = 3r + 2(s-1) \quad \text{for } r = 1, \ldots, 4$$

and $s = 1, \ldots, 5$.

7. Write down the matrices

$$\mathbf{D}_1 = \mathbf{D}\{1, 2, 3, 4\}, \quad \mathbf{D}_2 = \mathbf{D}\{3^{i-2}\} \quad \text{for} \quad i = 1, \ldots, 4$$

and

$$\mathbf{D}_3 = \mathbf{D}\{i + 3^{i-2}\} \quad \text{for} \quad i = 1, \ldots, 4.$$

8. When $x_i = \sum_{j=1}^{n} \lambda_{ij} v_j$ where $\sum_{j=1}^{n} \lambda_{ij} = 1$ for all $i = 1, 2, \ldots, m$, show that the coefficients of the v_j's in $\sum_{i=1}^{m} \mu_i x_i$ sum to unity when $\sum_{i=1}^{m} \mu_i = 1$.

9. Show that

$$\sum_{i=1}^{a} \frac{1}{(1+n_i t)^2} \sum_{i=1}^{a} \frac{n_i}{1+n_i t} - \sum_{i=1}^{a} \frac{n_i}{(1+n_i t)^2} \sum_{i=1}^{a} \frac{1}{1+n_i t}$$

$$= t\left[\sum_{i=1}^{a} \frac{1}{(1+n_i t)^2} \sum_{i=1}^{a} \frac{n_i^2}{(1+n_i t)^2} - \left\{\sum_{i=1}^{a} \frac{n_i}{(1+n_i t)^2}\right\}^2\right].$$

10. Prove that

$$1 + \sum_{j=1}^{b} \frac{1}{n_{1j}} \sum_{i=2}^{a} \left(\sum_{j=1}^{b} \frac{1}{n_{ij}}\right)^{-1} = \sum_{j=1}^{b} \frac{1}{n_{1j}} \sum_{i=1}^{a} \left(\sum_{j=1}^{b} \frac{1}{n_{ij}}\right)^{-1}.$$

11. Suppose that for families with the husband aged 20 through 30 family size (number of children) is recorded each year, as of September 30, for 10 years. From these data, the frequencies f_{ij} are calculated of annual changes in family size for $i, j = 0, 1, 2, 3$ and 4, where 4 represents 4 or more children. Ignoring deaths, explain why $\mathbf{F} = \{f_{ij}\}$ is a transition probability matrix, why it is triangular, and why $f_{44} = 1$. Comment on other features of the values of f_{ij} that you think might be apparent.