Techniques of Integration

Dr Rami Younes

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GUC
German University in Cairo
الجامعة الألمانية بالقاهرة

Lecture Outline

1. Integration of Rational Functions
   - The Method of Partial Fractions

Recommended Reading

- Stewart's Caluculus: sections 7.4. Or, equally
- Thomas' Caluculus: sections 8.5
- Stewart's Calculus: section 7.5 is a highly recommended reading for those who wish to widen their integration perspective.

The Method of Partial Fractions

Objective **Evaluate** $\int \frac{f(x)}{g(x)}dx$**, where** $f(x)$ **and** $g(x)$ **are polynomials.**

Examples

How to evaluate the following integrals?

i. $\int \frac{5x-3}{x^2-2x-3}dx$

ii. $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$

iii. $\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$

iv. $\int \frac{-2x+4}{x(x^2+1)^2}dx$

v. $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx$

Remarks

- For example,

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)}$$

  The denominator of the fraction suggests that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3} \quad (A \text{ and } B \text{ constants})$$

  If A and B could be found, $\int \frac{5x-3}{x^2-2x-3}dx$ can be easily evaluated as the sum of two simpler integrals.
- **Decomposition of a rational function into a sum of "simpler" rational functions is called the method of partial fractions.**
- The method of partial fractions is a good approach to integrate rational fractions.

The Method of Partial Fractions

Example

Since $x^2 - 2x - 3 = (x-3)(x+1)$

$$\begin{aligned} \frac{5x-3}{x^2-2x-3} &= \frac{A}{x+1} + \frac{B}{x-3} \\ \Leftrightarrow \frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)+B(x+1)}{(x-3)(x+1)} \\ \Leftrightarrow \frac{5x-3}{x^2-2x-3} &= \frac{(A+B)x-3A+B}{(x-3)(x+1)} \end{aligned}$$

By comparison of numerators, one concludes that $\begin{cases} A+B &= 5 \\ -3A+B &= -3 \end{cases}$. Therefore,

$$A = 2 \text{ and } B = 3 \text{ and } \frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}.$$

Remark

$$\begin{aligned} \int \frac{5x-3}{x^2-2x-3}\,dx &= 2\int \frac{dx}{x+1} + 3\int \frac{dx}{x-3} \\ &= 2\ln|x+1| + 3\ln|x-3| + C \end{aligned}$$

Question

How to go about in order to decompose **systematically** a rational function in partial fractions?

Remarks

- A rational function $\frac{f(x)}{g(x)}$ is said to be **proper** if the degree of $f(x)$ is less than the degree of $g(x)$.
- **There exists a systematic approach to the method of partial fractions only for proper fractions.** This approach is described on the succeeding slides. If the degree of $f(x)$ is larger than that of $g(x)$ then one must perform *euclidean division* for polynomials of $f(x)$ by $g(x)$, that is, write

$$f(x) = q(x)g(x) + r(x)$$

with degree $r(x)$ less than degree $g(x)$ from which follows

$$\frac{f(x)}{g(x)} = \frac{q(x)g(x)+r(x)}{g(x)} = q(x) + \underbrace{\frac{r(x)}{g(x)}}_{\text{proper fraction}}$$

The Method of Partial Fractions

Question

How to go about in order to decompose **systematically** a rational function in partial fractions?

Example

A non-proper fraction:

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} \overbrace{=}^{\text{Polynomial Division}} \frac{2x(x^2-2x-3)+5x-3}{x^2-2x-3}$$
$$= 2x + \underbrace{\frac{5x-3}{x^2-2x-3}}_{\text{proper fraction}} = 2x + \frac{2}{x+1} + \frac{3}{x-3}.$$

Note how the method of partial fractions allows the integration of the intial rational function

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int \left[2x + \frac{2}{x+1} + \frac{3}{x-3}\right] dx$$
$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

Remarks

- The method of partial fractions for $\dfrac{f(x)}{g(x)}$ requires the factors of $g(x)$.
- Finding factors of a polynomial $g(x)$ is generally difficult.
- In theory, there is a guarantee that any polynomial can be written as the product of a constant and linear factors $x-r$ and **irreducible** quadratic factors x^2+px+q. **The factors may be repeated**. (Irreducible quadratic polynomials are those with no real roots, that is with negative discriminant)

  Example of such product

  $$2(x-1)(\underbrace{x+3}_{\textit{repeated}})^3(x^2+1)(\underbrace{x^2-x+1}_{\textit{repeated}})^3$$

- Factoring a polynomial into a product of linear factors and irreducible quadratic polynomials is much like factoring integres into products of prime numbers!

The Method of Partial Fractions

Question

How to go about in order to decompose **systematically** a rational function in partial fractions?

Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. If $x - r$ is a factor of $g(x)$ repeated m times, then it is assigned the sum of m partial fractions in the decpmposition of $f(x)/g(x)$
$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$
2. If $x^2 + px + q$ is an irreducible quadratic factor of $g(x)$ repeated n times, then it is assigned the sum of n partial fractions in the decpmposition of $f(x)/g(x)$
$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}$$
3. Having assigned each factor of $g(x)$ its own share of partial fractions in the overall decomposition, one adds the partial fractions then compares the resulting numerator with $f(x)$ to set up as system of equations in A_i's, B_i's and C_i's (in a fashion similar to that in the first example in this section).

The Method of Partial Fractions

Example

$$\frac{f(x)}{g(x)} = \frac{-2x+4}{(\underbrace{x^2+1}_{\text{irreducible}})(\underbrace{x-1}_{\text{repeated}})^2}$$
$$= \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$
$$= \frac{(Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)}{(x^2+1)(x-1)^2}$$

Therefore,

$$-2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$
$$= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D)$$

Comparing the coefficients of powers of x on both sides of the equation, one concludes that

$$\begin{cases} A+C &= 0 \\ -2A+B-C+D &= 0 \\ A-2B+C &= -2 \\ B-C+D &= 4 \end{cases}$$

The system of equations can be solved to obtain

$$A = 2,\ B = 1,\ C = -2 \text{ and } D = 1.$$

Therefore,

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

Remark

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{2x+1}{x^2+1}\,dx - 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$$
$$= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C$$

The Method of Partial Fractions

Example

Decompose in partial fractions the rational function

$$\frac{f(x)}{g(x)} = \frac{1}{x(x^2+1)^2}$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $g(x)$ both sides of the above equation, one obtains

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

Comparing the coefficients of powers of x on both sides of the equation, one concludes that

$$\begin{cases} A+B &= 0 \\ C &= 0 \\ 2A+B+D &= 0 \\ C+E &= 0 \\ A &= 1 \end{cases}$$

The system of equations can be solved to obtain

$$A = 1,\ B = -1,\ C = 0,\ D = -1,\ \text{and } E = 0.$$

Therefore,

$$\frac{1}{x(x^2+1)^2} = \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}$$

Remark

$$\begin{aligned} \int \frac{1}{x(x^2+1)^2}\,dx &= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx \\ &\quad - \int \frac{x}{(x^2+1)^2}\,dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C \\ &= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + C. \end{aligned}$$

The Method of Partial Fractions

The Heaviside "Cover-up" Method for Linear Factors

When the denominator of a proper fraction $f(x)/g(x)$ is the product of **distinct linear factors**, that is,

$$g(x) = (x - r_1)(x - r_2)\dots(x - r_n) \qquad \text{(with } r_i \neq r_j \text{ for } i \neq j)$$

there is a quick way, known as the Heaviside "cover-up" method, to decompose $f(x)/g(x)$ in partial fractions. The method is described in the fowllowing example.

Example

$$\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

To find A**multiply by the equation by its "cover" term** $x - 1$

$$\frac{x^2 + 4x + 1}{(x+1)(x+3)} = A + \frac{B(x-1)}{x+1} + \frac{C(x-1)}{x+3}$$

then set $x = 1$ **to obtain**

$$A = \frac{1^2 + 4(1) + 1}{(1+1)(1+3)} = \frac{3}{4}$$

To find B **multiply by the equation by its "cover" term** $x + 1$

$$\frac{x^2 + 4x + 1}{(x-1)(x+3)} = \frac{A(x+1)}{x-1} + B + \frac{C(x+1)}{x+3}$$

then set $x = -1$

$$B = \frac{(-1)^2 + 4(-1) + 1}{(-1-1)(-1+3)} = \frac{1}{2}$$

To find C **multiply by the equation by its "cover" term** $x + 3$

$$\frac{x^2 + 4x + 1}{(x-1)(x+1)} = \frac{A(x+3)}{x-1} + \frac{B(x+3)}{x+1} + C$$

then set $x = -3$ **to obtain**

$$C = \frac{(-3)^2 + 4(-3) + 1}{(-3-1)(-3+1)} = -\frac{1}{4}$$

Conclusion

$$\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{3}{4(x-1)} + \frac{1}{2(x+1)} - \frac{1}{4(x+3)}$$