Lattice discretisation of the Green-Schwarz superstring and AdS/CFT

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Outline

Study the AdS/CFT duality numerically at hand of the cusp anomaly function from a string theory perspective

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- Motivation
- Cusp anomaly function
- AdS/CFT duality
- String theory framework
- Numerical approach
- Results and conclusion

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- Good predictions on the real world via QCD
- Fundamental aim: particle masses as functions of parameters

$$m_{\mathrm{p}} = f(\alpha_{\mathrm{s}}, \alpha, \mu_{\mathrm{reg}}, \dots)$$

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- $\mathcal{N} = 4$ SYM:
  - no massive particles
  - measure scaling dimensions of local operators and Wilson loops

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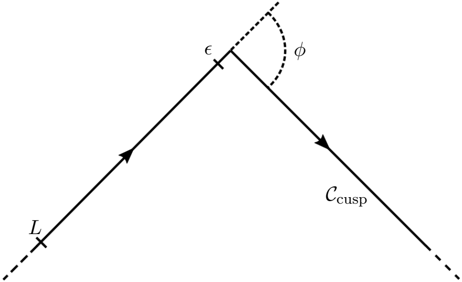
- $\mathcal{N} = 4$ SYM:
  - no massive particles
  - measure scaling dimensions of local operators and Wilson loops
- Twist two operators:
  scaling dimension $\Delta_S = 2 + S + \gamma_S$
  twist $\Delta_S - S = 2 + \gamma_S$

Wilson loop with cusp $\Leftrightarrow$ anomalous dimension γ_S

$$\langle \mathcal{W}[\mathcal{C}_{\text{cusp}}] \rangle \sim e^{-\frac{f(g)}{2}|\phi| \ln \frac{L}{\epsilon}}$$

$$\gamma_S \simeq f(g) \ln S$$



Wilson loop with cusp

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$\Leftrightarrow$

anomalous dimension γ_S

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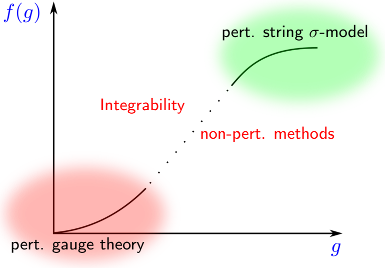
via cusp anomaly function



Here $g \equiv \frac{\sqrt{\lambda}}{4\pi}$

t'Hooft coupling

$$\lambda = g_{\text{YM}}^2 N = \frac{R^4}{\alpha'^2}$$

AdS/CFT duality

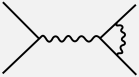
conformal field theory (CFT)

$\uparrow$ *dynamically equivalent to* $\downarrow$

string theory on background containing Anti-de Sitter (AdS) space as a factor

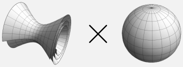
Most symmetric setting similar to QCD:

$\mathcal{N} = 4$ **SYM**
in 4d, (g_{YM}, N)



$\uparrow$ *dynamically equivalent to* $\downarrow$

Type IIB superstring theory
on $AdS_5 \times S^5$, (g_{s}, R)

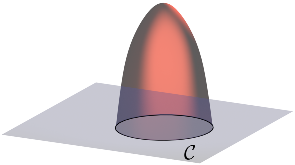


$\mathcal{N} = 4$
SYM

Wilson loop

$$\langle \mathcal{W}[\mathcal{C}] \rangle = \frac{1}{N} \mathrm{Tr}\, \mathcal{P}\, e^{\oint \left(i A_\mu \dot{x}^\mu + \phi_i \dot{y}^i \right) \mathrm{d}s}$$

$\Uparrow$
AdS/CFT
$\Downarrow$

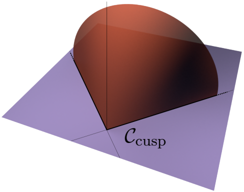


Type IIB strings

$$Z_{\text{string}}[\mathcal{C}] = \int \mathcal{D}X \mathcal{D}h\, e^{-S_{\text{string}}(X,h)} \sim e^{-A(\mathcal{C})}$$

Cusp anomaly of $\mathcal{N} = 4$ SYM from string theory

vev of light-like cusped Wilson loop



$$\langle \mathcal{W}[\mathcal{C}_{\text{cusp}}] \rangle \sim e^{-\frac{f(g)}{2}|\phi| \ln \frac{L}{\epsilon}}$$

$\Updownarrow$ AdS/CFT

$$Z_{\text{cusp}} = \int \mathcal{D}\delta X \, \mathcal{D}\delta \Psi \; e^{-S_{\text{IIB}}(X_{\text{cl}}+\delta X, \delta \Psi)} = e^{-\frac{f(g)}{2}\frac{V_2}{4}}$$

String partition function with cusp vacuum $\left(V_2 = \int \mathrm{d}t\mathrm{d}s\right)$

Green-Schwarz string in AdS light-cone gauge

- Sigma-model in $AdS_5 \times S^5$ with RR flux

  $$S = g \int \mathrm{d}\tau \, \mathrm{d}\sigma \, \left[G \partial X \cdot \partial X + \overline{\Theta} \Gamma (D + F_5) \Theta \partial X + \ldots \right]$$

  [Metsaev Tseytlin 1998]

  - X^μ - coordinates of 10d target space
  - Θ^1, Θ^2 - anti-commuting Majorana spinors

- Fix κ-symmetry and apply bosonic light-cone gauge
  $\Rightarrow$ action at most quartic in complex Grassmann fields θ^i, η^i
  (inherited from Θ)

Green-Schwarz string in null-cusp background

- In Poincaré patch

$$\mathrm{d}s^2_{AdS_5} = \frac{\mathrm{d}z^2 + \mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}x^*\,\mathrm{d}x}{z^2}, \quad x^\pm = x^3 \pm x^0, \quad x = x^1 + ix^2$$

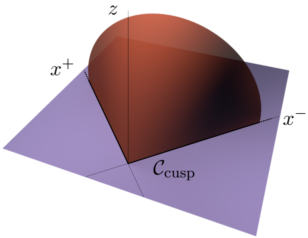
classical solution $\big(\tau, \sigma \in (0,\infty)\big)$: surface

$$z = \sqrt{\tfrac{\tau}{\sigma}}, \quad x^+ = \tau, \quad x^- = -\tfrac{1}{2\sigma}$$

bounded by a null cusp
(AdS_5 boundary at $0 = z^2 = -2x^+x^-$)

- expand around classical solution +
$(\tau, \sigma) \to (t, s) = (\ln\tau, \ln\sigma)$

$$S_{\text{cusp}} = g \int \mathrm{d}t\,\mathrm{d}s\,\mathcal{L}_{\text{cusp}}$$



Green-Schwarz string in null-cusp background

linearising quartic fermion contributions

$$\mathcal{L}_{\text{cusp}} = \left|\partial_t x + \tfrac{m}{2} x\right|^2 + \tfrac{1}{z^4}\left|\partial_s x - \tfrac{m}{2} x\right|^2 + \left(\partial_t z^M + \tfrac{m}{2} z^M\right)^2 + \frac{1}{z^4}\left(\partial_s z^M - \tfrac{m}{2} z^M\right)^2 + \phi^2 + (\phi_I)^2 + \Psi^T \mathcal{O}_{\text{F}}\, \Psi$$

- 8 bosonic coordinates: x, x^*, z^M $(M = 1, \ldots, 6)$, $z = \sqrt{z_M z^M}$
- 17 auxiliary fields ϕ, ϕ_I $(I = 1, \ldots, 16)$
- 8 fermionic variables $\Psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$, $\theta^i = \theta_i^\dagger$, $\eta^i = \eta_i^\dagger$, $(i = 1, \ldots, 4)$

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$$\mathcal{O}_{\text{F}} = \begin{pmatrix} 0 & i\mathbb{1}_4\partial_t & -i\rho^M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} & 0 \\ i\mathbb{1}_4\partial_t & 0 & 0 & -i\rho^\dagger_M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} \\ i\frac{z^M}{z^3}\rho^M\left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) & i\mathbb{1}_4\partial_t - A^T \\ 0 & i\frac{z^M}{z^3}\rho^\dagger_M\left(\partial_s - \frac{m}{2}\right) & i\mathbb{1}_4\partial_t + A & -2\frac{z^M}{z^4}\rho^\dagger_M\left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

$$A = -\frac{\sqrt{6}}{z}\phi\mathbb{1}_4 + \frac{1}{z}\widetilde{\phi} + \frac{1}{z^3}\rho^*_N\widetilde{\phi}^T\rho^L z^N z^L + \mathrm{i}\frac{z^N}{z^2}\rho^{MN}\partial_t z^M$$

Numerical approach

requires calculating expectation values

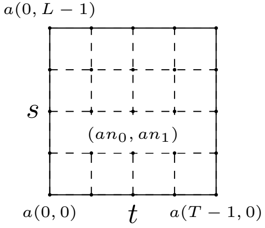
$$Z_{\text{cusp}} = \int \mathcal{D}\delta X \mathcal{D}\delta \varPsi \; e^{-S_{\text{cusp}}} = e^{-\frac{f(g)}{2}\frac{V_2}{4}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{\langle S_{\text{cusp}} \rangle} = \frac{1}{Z_{\text{cusp}}} \int \mathcal{D}\delta X \mathcal{D}\delta \varPsi \; S_{\text{cusp}} e^{-S_{\text{cusp}}}$$
$$= -g \frac{\mathrm{d} \ln Z_{\text{cusp}}}{\mathrm{d}g} = \boxed{g f'(g) \frac{V_2}{8}}$$

Lattice discretisation

$$\langle A\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, A[\phi] e^{-S[\phi]}, \quad Z = \int \mathcal{D}\phi\, e^{-S[\phi]}$$



discretise the worldsheet with const. lattice spacing a
$\Lambda = \{(n_0, n_1) | n_\alpha = 0, \ldots, (N_\alpha - 1)\}$ so that $\xi^\alpha = (\tau, \sigma) \equiv (an_0, an_1) \equiv an$

- **PI measure:** discretise fields $\phi \to \phi(n)$
  $\mathcal{D}\phi \to \prod_n \mathrm{d}\phi(n)$
- **Operators:** $\partial_\alpha \phi(n) \to \frac{1}{a}[\phi(n+\widehat{\alpha}) - \phi(n)] \Rightarrow S \to S_{\text{disc}}$

$\Rightarrow Z_{\text{disc}} = \int \prod_n \mathrm{d}\phi(n)\, e^{-S_{\text{disc}}[\phi]} \quad \sim$ multidimensional integral treatable via MC techniques

Monte Carlo simulations in QFT

generate ensemble of field configurations $\{\phi_1, \ldots, \phi_N\}$ distributed according $P[\phi] = e^{-S_{\mathrm{E}}[\phi]}/Z$

Ensemble average

$$\langle A \rangle = \int \mathcal{D}\phi \, A[\phi] P[\phi] = \frac{1}{N} \sum_{i=1}^{N} A[\phi_i] + \mathcal{O}(1/\sqrt{N})$$

- Grassmann fields are integrated out

$$\int \mathcal{D}\Psi \, e^{-\Psi^{\mathrm{T}} \widehat{\mathcal{O}}_{\mathrm{F}} \Psi} \sim \sqrt{\det \widehat{\mathcal{O}}_{\mathrm{F}}} \overset{!}{\to} \sqrt[4]{\det(\widehat{\mathcal{O}}_{\mathrm{F}} \widehat{\mathcal{O}}_{\mathrm{F}}^{\dagger})} \sim \int \mathcal{D}\zeta \mathcal{D}\zeta^{\dagger} \, e^{-\zeta^{\dagger} (\widehat{\mathcal{O}}_{\mathrm{F}} \widehat{\mathcal{O}}_{\mathrm{F}}^{\dagger})^{-\frac{1}{4}} \zeta}$$

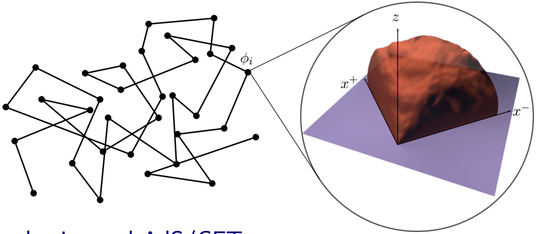
- Pseudofermionic contribution included into $P[\phi]$
- Include Wilson term to overcome fermion doubling
- Use rational hybrid Monte Carlo (RHMC) algorithm to generate ensembles

Monte Carlo simulations in QFT

generate ensemble of field configurations $\{\phi_1, \ldots, \phi_N\}$ distributed according $P[\phi] = e^{-S_{\rm E}[\phi]}/\widetilde{Z} \cdot \sqrt{\det \widehat{\mathcal{O}}_{\rm F}}$

Ensemble average

$$\langle S_{\rm cusp} \rangle = \int \mathcal{D}\phi \mathcal{D}\zeta \mathcal{D}\zeta^\dagger \, S_{\rm cusp}[\phi] P[\phi] = \frac{1}{N} \sum_{i=1}^{N} S_{\rm cusp}[\phi_i] + \mathcal{O}(1/\sqrt{N})$$



Simulation parameters and continuum limit

- Parameters in the continuum: g, m
- Dimensionless parameters on the lattice:

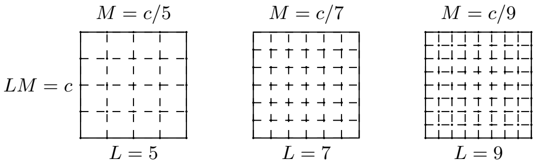
$$g,\ L\ (T \equiv 2L)\ ,\ M \equiv ma$$

- Any observable on lattice is function of input parameters

$$\langle F_{\mathrm{LAT}} \rangle = \langle F_{\mathrm{LAT}}(g, L, M) \rangle = \langle F(g) \rangle + \mathcal{O}(L^{-1}) + \mathcal{O}(e^{-LM})$$

- continuum limit $\langle F(g) \rangle$ obtained via extrapolation to infinite L $(a \to 0)$

- Take cont. limit in controlled way - Line of constant Physics:
  - Renormalised "effective" mass in continuum
    $$m_x^2(g) = \frac{m^2}{2}\left(1 - 1/(8g) + \mathcal{O}(g^{-2})\right) \quad (*)$$
  - Keep dimensionless physical quantities constant while $a \to 0$
    $$\frac{V_2 m_x^2}{2} \quad \xrightarrow{g \text{ fixed}} \quad \frac{V_2 m^2}{2} = (LM)^2 = \text{const.}$$
    if $(*)$ is also true on the lattice
- Procedure:
  - fix g
  - fix LM large enough to keep finite volume effects small
  - evaluate $\langle F_{\text{LAT}} \rangle$ for $L = 8, 10, 12, 16, \ldots$
  - extrapolate to $L \to \infty$ to obtain $\langle F(g) \rangle$

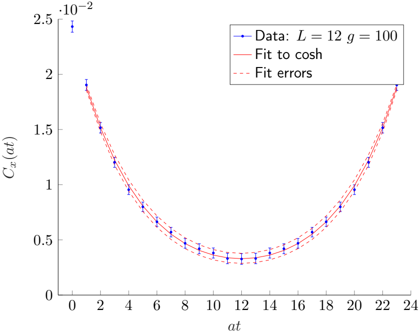


Mass of the x field

Fit timeslice correlators

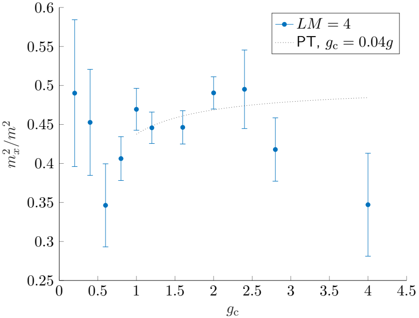
$$C_x(t) = \frac{1}{L} \sum_{s_1,s_2} \langle x(t,s_1) x^*(0,s_2) \rangle$$

$$\sim \cosh\Big((\tfrac{T}{2} - t) m_{x\mathrm{LAT}} \Big)$$



Extract mass via continuum limit

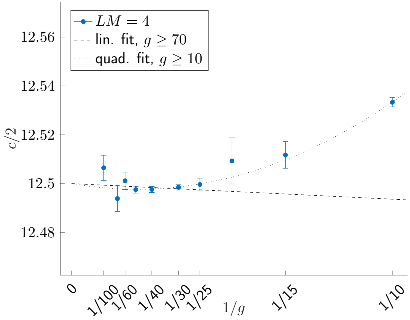
$$\frac{m^2_{x\mathrm{LAT}}(L,g)}{M^2} = \frac{m_x^2(g)}{m^2} + \mathcal{O}(L^{-1})$$

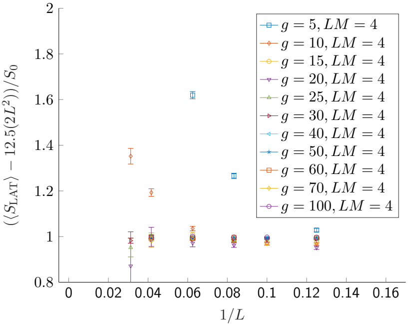


The cusp action

- Observe quadratic divergences $\langle S^{(2)}_{\text{cusp}} \rangle \sim \frac{V_2}{2}(N_{\text{B}} - N_{\text{F}})$
- On the lattice study only bosonic part of the action

$$\langle S_{\text{LAT}} \rangle = g\frac{(LM)^2}{4} f'(g) + \frac{c(g)}{2}(2L^2), \quad c(g) = N_{\text{B}} + \mathcal{O}(g^{-1})$$





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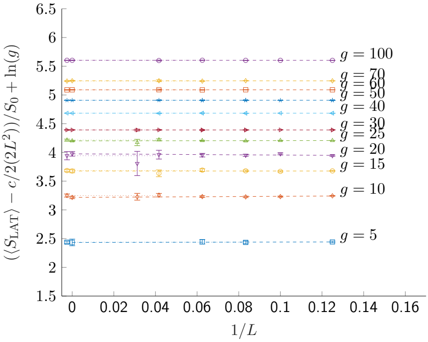
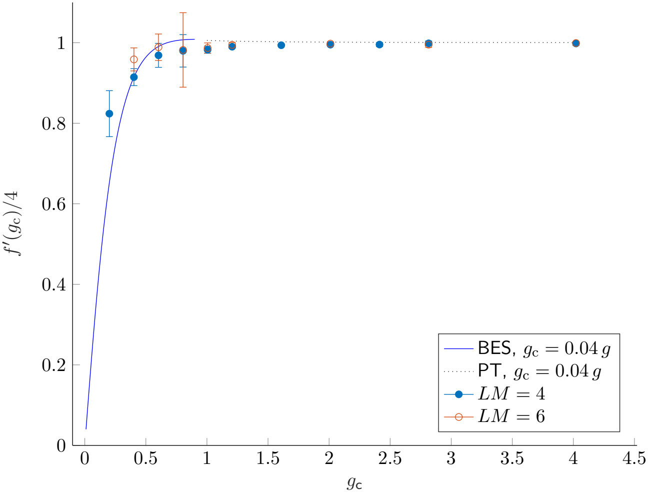


Subtract divergences and match lattice data to continuum plot via $g_{\mathrm{c}} = b \cdot g$, $f'(g) = f'(g_{\mathrm{c}})_{\mathrm{c}}$



Conclusion

- Investigate cusp anomaly of $\mathcal{N} = 4$ SYM from string theory perspective

  solve non-trivial 4d QFT with SUSY $\xrightarrow{AdS/CFT}$ solve non-trivial 2d QFT

  - More economic memory consumption
  - Green-Schwarz (GS) approach inherits supersymmetry
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  - Green-Schwarz (GS) approach inherits supersymmetry
  - Only scalar fields involved
- Lattice simulation (gauge-fixed GS string, Wilson term, RHMC):
  - Measured observables seem to be in good agreement with expectation at large g
  - At small g, $\widehat{\mathcal{O}}_{\mathrm{F}}$ has small eigenvalues $\rightarrow$ sign problem, Qualitative agreement to expectation derived from integrability

Thank you for the attention!