Lattice discretisation of the Green-Schwarz superstring and AdS/CFT

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Outline

Study the AdS/CFT duality numerically at hand of the cusp anomaly function from a string theory perspective

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- Motivation
- Cusp anomaly function
- AdS/CFT duality
- String theory framework
- Numerical approach

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- Good predictions on the real world via QCD
- Fundamental aim: particle masses as functions of parameters

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- $\mathcal{N} = 4$ SYM:
  - no massive particles
  - measure scaling dimensions of local operators and Wilson loops

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- $\mathcal{N} = 4$ SYM:
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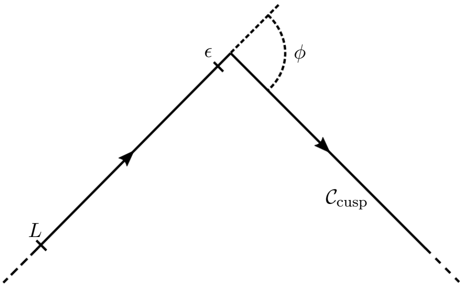
  - measure scaling dimensions of local operators and Wilson loops
- Twist two operators:
  scaling dimension $\Delta_S = 2 + S + \gamma_S$

Wilson loop with cusp

anomalous dimension γ_S

$$\langle \mathcal{W}[\mathcal{C}_{\text{cusp}}] \rangle \sim e^{-\frac{f(g)}{2}|\phi| \ln \frac{L}{\epsilon}} \quad \Leftrightarrow \quad \gamma_S \simeq f(g) \ln S$$



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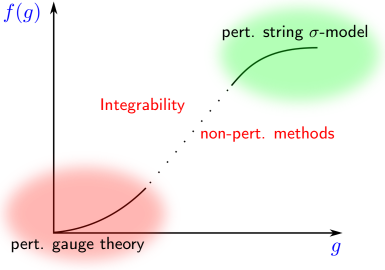
via cusp anomaly function



Here $g \equiv \frac{\sqrt{\lambda}}{4\pi}$

$$\lambda = g_{\text{YM}}^2 N = \frac{R^4}{\alpha'^2}$$

AdS/CFT duality

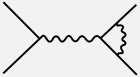
conformal field theory (CFT)

$\uparrow$ *dynamically equivalent to* $\downarrow$

string theory on background containing Anti-de Sitter (AdS) space as a factor

Most symmetric setting similar to QCD:

$\mathcal{N} = 4$ **SYM**
in 4d, $(g_{\rm YM}, N)$



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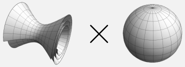
Type IIB superstring theory
on $AdS_5 \times S^5$, $(g_{\rm s}, R)$

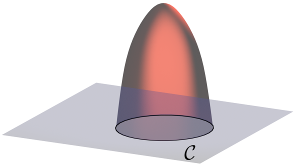


$\mathcal{N} = 4$
SYM

Wilson loop

$$\langle \mathcal{W}[\mathcal{C}] \rangle = \frac{1}{N} \mathrm{Tr}\, \mathcal{P}\, e^{\oint \left(i A_\mu \dot{x}^\mu + \phi_i \dot{y}^i \right) \mathrm{d}s}$$

$\Uparrow$
AdS/CFT
$\Downarrow$

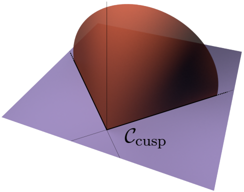


Type IIB strings

$$Z_{\mathrm{string}}[\mathcal{C}] = \int \mathcal{D}X \mathcal{D}h\, e^{-S_{\mathrm{string}}(X,h)} \sim e^{-A(\mathcal{C})}$$

Cusp anomaly of $\mathcal{N} = 4$ SYM from string theory

vev of light-like cusped Wilson loop



$$\langle \mathcal{W}[\mathcal{C}_{\text{cusp}}] \rangle \sim e^{\frac{f(g)}{2}|\phi| \ln \frac{L}{\epsilon}}$$

$\Updownarrow$ AdS/CFT

$$Z_{\text{cusp}} = \int \mathcal{D}\delta X \, \mathcal{D}\delta \Psi \; e^{-S_{\text{IIB}}(X_{\text{cl}}+\delta X, \delta \Psi)} = e^{-\frac{f(g)}{2}\frac{V_2}{4}}$$

String partition function with cusp vacuum $\left(V_2 = \int \mathrm{d}t\mathrm{d}s\right)$

Green-Schwarz string in AdS light-cone gauge

- Sigma-model in $AdS_5 \times S^5$ with RR flux

  $$S = g \int \mathrm{d}\tau \, \mathrm{d}\sigma \, \left[G_{\mu\nu} \partial X^\mu \partial X^\nu + \overline{\Theta} \Gamma (D + F_5) \Theta \partial X + \dots \right]$$

  [Metsaev Tseytlin 1998]

- Fix κ-symmetry and apply bosonic light-cone gauge
  $\Rightarrow$ action at most quartic in complex Grassmann fields θ^i, η^i

Green-Schwarz string in null-cusp background

- In Poincaré patch

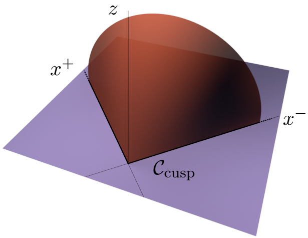
  $$\mathrm{d}s^2_{AdS_5} = \frac{\mathrm{d}z^2 + \mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}x^*\,\mathrm{d}x}{z^2}, \quad x^\pm = x^3 \pm x^0, \quad x = x^1 + ix^2$$

  classical solution $\big(\tau, \sigma \in (0, \infty)\big)$: surface

  $$z = \sqrt{\frac{\tau}{\sigma}}, \quad x^+ = \tau, \quad x^- = -\frac{1}{2\sigma}$$

  bounded by a null cusp
  (AdS_5 boundary at $0 = z^2 = -2x^+x^-$)
- expand around classical solution +
  $(\tau, \sigma) \to (t, s) = (\ln\tau, \ln\sigma)$

  $$S_{\text{cusp}} = g \int \mathrm{d}t\,\mathrm{d}s\,\mathcal{L}_{\text{cusp}}$$



Green-Schwarz string in null-cusp background

linearising quartic fermion contributions

$$\mathcal{L}_{\rm cusp} = \left|\partial_t x + \tfrac{m}{2} x\right|^2 + \tfrac{1}{z^4}\left|\partial_s x - \tfrac{m}{2} x\right|^2 + \left(\partial_t z^M + \tfrac{m}{2} z^M\right)^2 + \frac{1}{z^4}\left(\partial_s z^M - \tfrac{m}{2} z^M\right)^2 + \phi^2 + (\phi_I)^2 + \Psi^T \mathcal{O}_{\rm F}\, \Psi$$

- 8 bosonic coordinates: x, x^*, z^M $(M = 1, \ldots, 6)$, $z = \sqrt{z_M z^M}$
- 17 auxiliary fields ϕ, ϕ_I $(I = 1, \ldots, 16)$
- 8 fermionic variables $\Psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$, $\theta^i = \theta_i^\dagger$, $\eta^i = \eta_i^\dagger$, $(i = 1, \ldots, 4)$

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$$\mathcal{O}_{\text{F}} = \begin{pmatrix} 0 & i\mathbb{1}_4\partial_t & -i\rho^M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} & 0 \\ i\mathbb{1}_4\partial_t & 0 & 0 & -i\rho^\dagger_M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} \\ i\frac{z^M}{z^3}\rho^M\left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) & i\mathbb{1}_4\partial_t - A^T \\ 0 & i\frac{z^M}{z^3}\rho^\dagger_M\left(\partial_s - \frac{m}{2}\right) & i\mathbb{1}_4\partial_t + A & -2\frac{z^M}{z^4}\rho^\dagger_M\left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

$$A = -\frac{\sqrt{6}}{z}\phi\mathbb{1}_4 + \frac{1}{z}\widetilde{\phi} + \frac{1}{z^3}\rho^*_N\widetilde{\phi}^T\rho^L z^N z^L + \mathrm{i}\frac{z^N}{z^2}\rho^{MN}\partial_t z^M$$

Numerical approach

requires calculating expectation values

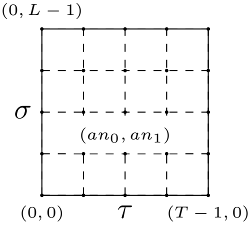
$$Z_{\text{cusp}} = \int \mathcal{D}\delta X \mathcal{D}\delta \Psi \; e^{-S_{\text{cusp}}} = e^{-\frac{f(g)}{2}\frac{V_2}{4}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{aligned} \boxed{\langle S_{\text{cusp}} \rangle} &= \frac{1}{Z_{\text{cusp}}} \int \mathcal{D}\delta X \mathcal{D}\delta \Psi \; S_{\text{cusp}} e^{-S_{\text{cusp}}} \\ &= -g \frac{\mathrm{d} \ln Z_{\text{cusp}}}{\mathrm{d} g} = \boxed{g f'(g) \frac{V_2}{8}} \end{aligned}$$

Lattice simulations

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, A[\phi] e^{-S[\phi]}, \quad Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$



discretise the worldsheet with const. lattice spacing a

$\Lambda = \{(n_0, n_1) | n_\alpha = 0, \ldots, (N_\alpha - 1)\}$ so that $\xi^\alpha = (\tau, \sigma) \equiv (an_0, an_1) \equiv an$

- **PI measure:** discretise fields $\phi \to \phi(n)$
  $\mathcal{D}\phi \to \prod_n \mathrm{d}\phi(n)$
- **Operators:** $\partial_\alpha \phi(n) \to \frac{1}{a}[\phi(n+\widehat{\alpha}) - \phi(n)] \Rightarrow S \to S_{\text{disc}}$

$\Rightarrow Z_{\text{disc}} = \int \prod_n \mathrm{d}\phi(n) \, e^{-S_{\text{disc}}[\phi]} \quad \sim$ multidimensional integral treatable via MC techniques