1 Basics on superstring theory

In this chapter we want to give a brief introduction to the basic concepts of string theory, starting from its simplest possible form, the bosonic string and further develop the subject to a supersymmetric version including fermions. Since we want to apply the concepts of AdS/CFT, we are mostly interested in strings propagating in AdS backgrounds. The following assertions are mainly based in the remarks in [1, 2].

1.1 The bosonic string

The basic idea of string theory is that the fundamental objects described by the theory are extended from point-like particles to one-dimensional strings propagating through spacetime and sweeping out a $(1+1)$-dimensional surface $\widetilde{\Sigma}$ called the string worldsheet. The string can be parameterized by two coordinates, the proper time τ and the spatial extent σ of the string which we will also denote by $(\sigma^0, \sigma^1) = (\tau, \sigma)$. We can then apply a homeomorphism $\big(\Sigma, \phi\big)$ that maps the 2d sheet Σ into the target space which is a D-dimensional Minkowski space[1] $\mathbb{R}^{D-1,1}$ with embedding coordinates $X^\mu(\tau, \sigma)$ and therefore

$$\begin{aligned} \phi: \quad \Sigma = (\tau_{\text{in}}, \tau_{\text{fin}}) \times (0, \sigma) \quad &\longrightarrow \quad \widetilde{\Sigma} \\ (\tau, \sigma) \quad &\longrightarrow \quad X^\mu(\tau, \sigma). \end{aligned} \tag{1.1}$$

The simplest possible action is proportional to the area of the surface of $\widetilde{\Sigma}$, given by

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int\limits_{\Sigma} \mathrm{d}^2\sigma \, \sqrt{-\gamma}, \tag{1.2}$$

where γ is the Gramian determinant of the embedding

$$\gamma = \det\big(\gamma_{\alpha\beta}\big), \qquad\qquad \gamma_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N \eta_{MN}, \tag{1.3}$$

where the induced metric $\gamma_{\alpha\beta}$ is the pull-back of the flat Minkowski metric η_{MN}. We also defined $\mathrm{d}^2\sigma = \mathrm{d}\sigma^0 \mathrm{d}\sigma^1$ and the parameter in front of the integral is the inverse string tension with α' being related to the string length l_s by $\alpha' = l_\text{s}^2$. Due to the square root this so called Nambu-Goto action is not very suitable for a path integral quantization. To get rid of the square root one can introduce an auxiliary metric $h_{\alpha\beta}(\sigma)$ with $h_{\alpha\beta} h^{\beta\rho} = \delta_\alpha^\rho$ and define the Polyakov action by

$$S_{\text{P}} = -\frac{1}{4\pi\alpha'} \int\limits_{\Sigma} \mathrm{d}^2\sigma \, \sqrt{-h} h^{\alpha\beta} \gamma_{\alpha\beta}. \tag{1.4}$$

This action is equivalent to the Nambu-Goto action on the classical level which can be proved by deriving the equations of motion for $h_{\alpha\beta}$, $\delta S / \delta h^{\alpha\beta} =$

[1] We will utilize the metric $\eta_{MN} = \text{diag}(-,+,+,\dots,+) with M, N = 0, 1, 2, \dots, D-1$.

0. We can use this to define the corresponding energy-momentum tensor

$$T_{\alpha\beta} \equiv -4\pi\alpha' \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} = \gamma_{\alpha\beta} - \frac{1}{2} h_{\alpha\beta} h^{\rho\sigma} \gamma_{\rho\sigma} = 0. \tag{1.5}$$

From the constraint that $T_{\alpha\beta}$ has to vanish we can derive the classical equivalence of the actions S_{NG} and S_{P} with

$$\sqrt{-\gamma} = \frac{1}{2}\sqrt{-h} h^{\rho\sigma} \gamma_{\rho\sigma}. \tag{1.6}$$

We will thus use the the Polyakov action in the following, since it is easier to handle. But before we continue, we want to analyse the symmetries preserved by the Polyakov action.

- *Global D-dimensional Poincarè invariance*

  $$X^M \to \widetilde{X}^M = \Lambda^M_N X^N + a^M, \qquad \delta h_{\alpha\beta} = 0, \tag{1.7}$$

  with Λ^M_N and a^M being D-dimensional Lorentz transformations and spacetime translations.

- *Reparametrisation invariance of the worldsheet (or diffeomorphism invariance)*
  We can choose a reparametrisation of the worldsheet coordinates $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$, where the fields X^M and the 2d metric transform according

  $$\begin{aligned} X^M(\sigma) &\to \widetilde{X}^M(\tilde{\sigma}) = X^M(\sigma), \\ h_{\alpha\beta}(\sigma) &\to \tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\alpha}{\partial\tilde{\sigma}^\gamma}\frac{\partial\sigma^\beta}{\partial\tilde{\sigma}^\delta} h_{\gamma\delta}(\sigma). \end{aligned} \tag{1.8}$$

  This is a gauge symmetry on the worldsheet.

- *Weyl invariance*

  $$\widetilde{X}^M(\sigma) = X^M(\sigma), \qquad\qquad \tilde{h}_{\alpha\beta}(\sigma) = \Omega^2 h_{\alpha\beta}(\sigma). \tag{1.9}$$

References

[1] J. Polchinski, *"String theory. Vol. 1: An introduction to the bosonic string"*, Cambridge University Press (2007).

[2] J. Polchinski, *"String theory. Vol. 2: Superstring theory and beyond"*, Cambridge University Press (2007).