1 Towards the lattice simulation

With the current status of the action (**??**) we could almost start to discretize the operators and fields, at least for the bosonic part this would not be a problem. For the fermions however this is not so straight forward. In order to include the fermionic contribution into the weight factor of the path integral like explained in section **??** one needs to integrate out the GRASSMANN variables to result into a PFAFFIAN or determinant of a fermionic operator. As presented in Appendix **??** this is only possible if the fermions appearing are of quadratic order. But in the fluctuation action (**??**) also quartic contributions of fermions appear, which have to be linearized with help of a HUBBARD-STATONOVICH transformation.[1]

1.1 Linearization of fermionic contributions

1.1.1 Naive approach and sign problem

The only quartic interactions are coming from the η fields and we can write this part of the action as

$$S_4^{\mathrm{F}}[\eta_i, \eta^i] = g \int \mathrm{d}t\mathrm{d}s \left[-\frac{1}{z^2} \left(\eta^2\right)^2 + \left(\frac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j \right)^2 \right]. \tag{1.1}$$

In the path integral representation the euclidean action contributes within an exponential $e^{-S_{\mathrm{E}}}$. By performing a naive HUBBARD-STATONOVICH transformation to this exponential we can reduce the four-fermion contributions to quadratic YUKAWA terms whereas we have to introduce 7 bosonic real auxiliary fields ϕ and ϕ^N

$$\begin{aligned}
&\exp\left\{ -g \int \mathrm{d}t\mathrm{d}s \left[-\tfrac{1}{z^2} \left(\eta^2\right)^2 + \left(\tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j \right)^2 \right] \right\} \\
&\quad \sim \int \mathcal{D}\phi \mathcal{D}\phi^{\mathcal{M}} \, \exp\left\{ -g \int \mathrm{d}t\mathrm{d}s \left[\tfrac{1}{2}\phi^2 + \tfrac{\sqrt{2}}{z}\phi\eta^2 + \tfrac{1}{2}(\phi^M)^2 \right.\right. \\
&\qquad\qquad \left.\left. - i\tfrac{\sqrt{2}}{z^4}\phi_M \left(i\eta_i (\rho^{MN})^i{}_j \eta^j \right) z_N \right] \right\}.
\end{aligned} \tag{1.2}$$

Here we can notice that the second term appears to be complex, since the $SO(6)$ matrix in parenthesis is hermitian (with respect to the indices M, N)

$$\left(i\, \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j \right)^\dagger = i\, \eta_j \left(\rho^{MN}\right)^j{}_i \eta^i, \tag{1.3}$$

where we have used (**??**). As discussed in section **??** this complex phase in the weight function is potentially leading to a non treatable sign problem. We therefore chose to make a field redefinition that circumvents the appearance of a complex phase during the HS transformation.

[1] See Appendix **??** for details

1.1.2 Alternative field redefinition

By using the identities for the $SO(6)$ matrices stated in Appendix **??** we can rewrite the second term in the LAGGRANGIAN of (1.1) as

$$\left(i\,\eta_i(\rho^{MN})^i{}_j n^N \eta^j\right)^2 = -3(\eta^2)^2 + 2\eta_i(\rho^N)^{ik} n_N \eta_k \eta^j (\rho^L)_{jl} n_L \eta^l, \tag{1.4}$$

where we defined $n^N = \frac{z^N}{z}$. This leads to the LAGRANGIAN

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-4\,(\eta^2)^2 + 2\left|\eta_i(\rho^N)^{ik} n_N \eta_k\right|^2\right). \tag{1.5}$$

In order to circumvent the sign problem the second term needs to be negative. To achieve this we define new fields[2]

$$\Sigma_i^j = \eta_i \eta^j \qquad \tilde{\Sigma}_j^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l. \tag{1.6}$$

with this new definitions it is simple to check that

$$\Sigma_i^j \Sigma_j^i = -(\eta^2)^2 \qquad \tilde{\Sigma}_i^j \tilde{\Sigma}_j^i = -(\eta^2)^2 \qquad \Sigma_j^i \tilde{\Sigma}_i^j = -\left|\eta_i(\rho^N)^{ik} n_N \eta_k\right|^2. \tag{1.7}$$

With this we now define

$$\Sigma_{\pm}{}_i^j = \Sigma_i^j \pm \tilde{\Sigma}_i^j \tag{1.8}$$

and find

$$\Sigma_{\pm}{}_i^j \Sigma_{\pm}{}_j^i = -2(\eta^2)^2 \mp 2\left|\eta_i(\rho^N)^{ik} n_N \eta_k\right|^2. \tag{1.9}$$

We can now substitute the new fields into the LAGRANGIAN and obtain

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-4\,(\eta^2)^2 \mp 2(\eta^2)^2 \mp \Sigma_{\pm}{}_i^j \Sigma_{\pm}{}_j^i\right), \tag{1.10}$$

where we only need to select the right sign in the field definition to overcome the sign problem, which is leading to

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-6\,(\eta^2)^2 - \Sigma_{+}{}_i^j \Sigma_{+}{}_j^i\right). \tag{1.11}$$

If we now perform a HS transformation there will be no complex phase. The HS transformation yields

$$-\frac{6}{z^2}(\eta^2)^2 \to \frac{12}{z}\eta^2\phi + 6\phi^2, \tag{1.12}$$

where a single bosonic field was introduced like in the naive case. And further

$$-\frac{1}{z^2}\Sigma_{+}{}_j^i \Sigma_{+}{}_i^j \to \frac{2}{z}\Sigma_{+}{}_j^i \phi_i^j + \phi_j^i \phi_i^j \qquad \text{with} \qquad \left(\phi_j^i\right)^* = \phi_i^j\,. \tag{1.13}$$

Here the collection of fields ϕ_j^i can be thought of as a complex hermitian matrix with 16 real free parameters. We find it convenient to rescale the

[2]Where we actually set $\Sigma_i{}^j = \eta_i\eta^j$, then defined $\Sigma^i{}_j \equiv (\Sigma_i{}^j)^* = \Sigma_j{}^i$ to emphasize the notation Σ_i^j and equivalent for $\tilde{\Sigma}$.

field $\phi \to \phi/\sqrt{6}$, to get rid of the pre factor of 6 in (1.12). After reinserting the old fields for Σ_+ we can conclude that

$$\mathcal{L}_4 \to \frac{12}{z}\eta^2\phi + \phi^2 + \frac{2}{z}\eta_j\phi_i^j\eta^i + \frac{2}{z}(\rho^N)^{ik}n_N\eta_k\phi_i^j(\rho^L)_{jl}n_L\eta^l + \phi_j^i\phi_i^j\,. \quad (1.14)$$

So now we can write the full LAGRANGIAN as

$$\begin{aligned}\mathcal{L}_{\text{cusp}} = &\left|\partial_t x + \frac{m}{2}x\right|^2 + \frac{1}{z^4}\left|\partial_s x - \frac{m}{2}x\right|^2 + \left(\partial_t z^M + \frac{m}{2}z^M\right)^2 \\ &+ \frac{1}{z^4}\left(\partial_s z^M - \frac{m}{2}z^M\right)^2 + \phi^2 + \operatorname{Tr}\left(\tilde{\phi}\,\tilde{\phi}^\dagger\right) + \mathit{\Psi}^T\mathcal{O}_{\text{F}}\,\mathit{\Psi}\,.\end{aligned} \quad (1.15)$$

Hereby we defined the fermionic vector $\mathit{\Psi} \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$ as well as the auxiliary matrix $\tilde{\phi} = (\tilde{\phi}_{ij}) \equiv \phi_j^i$. We used partial integration and the properties of the GRASSMANN numbers and $SO(6)$ matrices to write the fermionic contribution in a matrix-vector notation. The 16×16 fermionic operator is hereby represented as 4×4 block matrix

$$\mathcal{O}_{\text{F}} = \begin{pmatrix} 0 & i\mathbb{1}_4\partial_t & -i\rho^M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} & 0 \\ i\mathbb{1}_4\partial_t & 0 & 0 & -i\rho_M^\dagger\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} \\ i\frac{z^M}{z^3}\rho^M\left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) & i\mathbb{1}_4\partial_t - A^T \\ 0 & i\frac{z^M}{z^3}\rho_M^\dagger\left(\partial_s - \frac{m}{2}\right) & i\mathbb{1}_4\partial_t + A & -2\frac{z^M}{z^4}\rho_M^\dagger\left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}, \quad (1.16)$$

where

$$A = -\frac{\sqrt{6}}{z}\phi\mathbb{1}_4 + \frac{1}{z}\tilde{\phi} + \frac{1}{z^3}\rho_N^*\tilde{\phi}^T\rho^L z^N z^L + \mathrm{i}\frac{z^N}{z^2}\rho^{MN}\partial_t z^M. \quad (1.17)$$

The auxiliary matrix $\tilde{\phi}$ is constructed from 16 real auxiliary fields ϕ_I ($I = 1, \ldots, 16$) in the following way

$$\tilde{\phi} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\phi_{13} & \phi_1 + i\phi_2 & \phi_3 + i\phi_4 & \phi_5 + i\phi_6 \\ \phi_1 - i\phi_2 & \sqrt{2}\phi_{14} & \phi_7 + i\phi_8 & \phi_9 + i\phi_{10} \\ \phi_3 - i\phi_4 & \phi_7 - i\phi_8 & \sqrt{2}\phi_{15} & \phi_{11} + i\phi_{12} \\ \phi_5 - i\phi_6 & \phi_9 - i\phi_{10} & \phi_{11} - i\phi_{12} & \sqrt{2}\phi_{16} \end{pmatrix}, \quad (1.18)$$

so that we have the simple expression for

$$\operatorname{Tr}\left(\tilde{\phi}\,\tilde{\phi}^\dagger\right) = \sum_{I=1}^{16}(\phi_I)^2 \equiv (\phi_I)^2\,. \quad (1.19)$$

1.1.3 Matrix properties

The fermion matrix obeys some fundamental symmetries that shall be summarized in this section. Since $\mathcal{O}_{\text{F}}$ is acting as a bilinear together with the anticommuting GRASSMANN fields, it is clear that the matrix representation of $\mathcal{O}_{\text{F}}$ needs to be skew-symmetric

$$\mathcal{O}_{\text{F}}^{\text{T}} = -\mathcal{O}_{\text{F}}. \quad (1.20)$$

It further possesses a global $U(1)$ and $SO(6)$ symmetry. The $U(1)$ symmetry manifests itself through the fact that certain blocks of $\mathcal{O}_\mathrm{F}$ are zero. As we saw from (**??**) fermions transform under $U(1)$ according to a certain charge q

$$\psi \to e^{iq\alpha}\psi. \tag{1.21}$$

Therefore only fermions with complementary charges are allowed to couple, in order to respect the $U(1)$ symmetry and the blocks that lead to other couplings need to be zero. The $SO(6)$ symmetry requires the 4×4 block structure and that each block is built from the $SO(6)$ invariant structures: $\mathbb{1}_4$, $\rho^M u^M$, $\rho_M^\dagger u^M$. The fermion matrix obeys also another constraint which is reminiscent of the γ_5-hermiticity in lattice QCD [1]

$$\mathcal{O}_\mathrm{F}^\dagger = \Gamma_5 \mathcal{O}_\mathrm{F} \Gamma_5, \tag{1.22}$$

where Γ_5 is the following unitary, antihermitian matrix

$$\Gamma_5 = \begin{pmatrix} 0 & \mathbb{1}_4 & 0 & 0 \\ -\mathbb{1}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_4 \\ 0 & 0 & -\mathbb{1}_4 & 0 \end{pmatrix}, \qquad \Gamma_5^\dagger\Gamma_5 = \mathbb{1}_{16}, \quad \Gamma_5^\dagger = -\Gamma_5. \tag{1.23}$$

A general skew-symmetric block matrix M that is Γ_5-hermitian needs to have the structure

$$M = \begin{pmatrix} d_1 & a & b & c \\ -a^\mathrm{T} & -d_1^\dagger & -c^* & b^* \\ -b^\mathrm{T} & c^\dagger & d_2 & f \\ -c^\mathrm{T} & -b^\dagger & -f^\mathrm{T} & -d_2^\dagger \end{pmatrix}, \qquad d_i = -d_i^\mathrm{T}, \quad a = a^\dagger, \quad f = f^\dagger. \tag{1.24}$$

We can use the two properties (1.20) and (1.22) to ensure the absence of a complex phase

$$\begin{aligned} \det\left(\mathcal{O}_\mathrm{F}\right)^* &= \det\left(\mathcal{O}_\mathrm{F}^\dagger\right) = \det\left(\Gamma_5\mathcal{O}_\mathrm{F}\Gamma_5\right) \\ &= \det\left(\Gamma_5\right)^2 \det\left(\mathcal{O}_\mathrm{F}\right) \\ &= \det\left(\mathcal{O}_\mathrm{F}\right) \end{aligned} \tag{1.25}$$

and thereby follows that $\det\mathcal{O}_\mathrm{F} \in \mathbb{R}$. For the PFAFFIAN of $\mathcal{O}_\mathrm{F}$ to be nonnegative we require that $\det\mathcal{O}_\mathrm{F}$ is positive and factorizable into two equivalent terms, so that

$$\det\mathcal{O}_\mathrm{F} = \mathrm{Pf}\left(\mathcal{O}_\mathrm{F}\right)^2. \tag{1.26}$$

Therefore we want to examine the spectrum of $\mathcal{O}_\mathrm{F}$. Assuming that λ is an eigenvalue of $\mathcal{O}_\mathrm{F}$ and $P(\lambda)$ is the characteristic polynomial we can prove

$$\begin{aligned} P(\lambda) &= \det(\mathcal{O}_\mathrm{F} - \lambda\mathbb{1}) = \det(\Gamma_5(\mathcal{O}_\mathrm{F} - \lambda\mathbb{1})\Gamma_5) \\ &= \det(\mathcal{O}_\mathrm{F}^\dagger + \lambda\mathbb{1}) = \det(\mathcal{O}_\mathrm{F} + \lambda^*\mathbb{1})^* \\ &= P(-\lambda^*)^*, \end{aligned} \tag{1.27}$$

and thereby see that if λ is an eigenvalue, then also $-\lambda^*$ is an eigenvalue. Since $\mathcal{O}_\mathrm{F}$ is skew-symmetric also $-\lambda$ and λ^* must be eigenvalues. So if for a discretized version $\hat{\mathcal{O}}_\mathrm{F}$ all these eigenvalues would be complex with non-vanishing real and imaginary part we could write the determinant as

$$\det \hat{\mathcal{O}}_\mathrm{F} = \prod_{i=1}^{N} |\lambda_i|^2 |\lambda_i|^2 \tag{1.28}$$

and therefore

$$\mathrm{Pf}\, \hat{\mathcal{O}}_\mathrm{F} = \pm \prod_{i=1}^{N} |\lambda_i|^2 , \tag{1.29}$$

where $N = |\Lambda|/4$. Since the eigenvalues should behave like continuous functions along a HMC trajectory it should not be possible for the Pfaffian to change its sign throughout a trajectory. So if one chooses a starting configuration with positive Pfaffian it should remain nonnegative during the whole simulation and the Pfaffian should be a valid probability distribution. Yet it is not quite clear what happens if there are also purely real or imaginary eigenvalues appearing. If they come in the same kind of quartets then there should not be a problem, otherwise the behaviour is not fully understood yet.

1.1.4 Pseudofermionic weight function

Now since we have linearised fermionic contributions to quadratic order, we are able to integrate over the Grassmann fields in the partition function

$$Z = \int \mathcal{D}x\, \mathcal{D}x^*\, \mathcal{D}z^N\, \mathcal{D}\phi\, \mathcal{D}\phi_I\, \mathcal{D}\Psi\; e^{-S} , \tag{1.30}$$

where we will split $S = S_\mathrm{B} + S_\mathrm{F}$ into its bosonic (S_B) and fermionic (S_F) contributions with

$$\begin{aligned} S_\mathrm{B} = g \int \mathrm{d}t\mathrm{d}s\, & \left|\partial_t x + \frac{m}{2} x\right|^2 + \frac{1}{z^4}\left|\partial_s x - \frac{m}{2} x\right|^2 + \left(\partial_t z^M + \frac{m}{2} z^M\right)^2 \\ & + \frac{1}{z^4}\left(\partial_s z^M - \frac{m}{2} z^M\right)^2 + \phi^2 + (\phi_I)^2 , \end{aligned} \tag{1.31}$$

$$S_\mathrm{F} = g \int \mathrm{d}t\mathrm{d}s\; \Psi^T \mathcal{O}_\mathrm{F} \Psi .$$

As motivated in section ?? the Grassmann integral over Ψ will result in a Pfaffian $\mathrm{Pf}\,\mathcal{O}_\mathrm{F}$. To include the Pfaffian into the weight function we have to rewrite it in terms of pseudofermions ξ as emphasized in section ??. To legitimately apply this procedure we need the Pfaffian to be real and non-negative. By our alternative approach to linearization we made sure to exclude any potential phase ambiguities. As we could see from the previous section the Pfaffian is definitely a real quantity. So the only problem that we might have to face is if the it is negative. As discussed before this is not quite clear yet. For now we will assume the PFAFFIAN to be positive since the eigenvalue spectrum suggests it and it can also be observed to be the case for all simulations at least for large values of g. For this reason, we proceed as in [2] and introduce pseudofermions ξ via

$$\int \mathcal{D}\varPsi\, e^{-S_{\mathrm{F}}} = \mathrm{Pf}\, \mathcal{O}_{\mathrm{F}} = \left(\det \mathcal{O}_{\mathrm{F}}\, \mathcal{O}_{\mathrm{F}}^{\dagger}\right)^{\frac{1}{4}} = \int \mathcal{D}\xi\, \mathcal{D}\xi^{\dagger}\, e^{-S_{\xi}}\,, \tag{1.32}$$

where

$$S_{\xi} = g \int \mathrm{d}t\mathrm{d}s\, \xi^{\dagger} \left(\mathcal{O}_{\mathrm{F}}\, \mathcal{O}_{\mathrm{F}}^{\dagger}\right)^{-\frac{1}{4}} \xi\,, \tag{1.33}$$

and ξ is a collection of 16 complex bosonic scalar fields. But before we can go any further we need to discretize the action with help of the methods introduced in **??**.

1.2 Discretizing the action

In the previous steps we have constructed a LAGRANGIAN fitting to our problem and modified terms to be able to apply a RHMC algorithm. To proceed with this task we need a discretized version of the bosonic action and the fermionic operator $\mathcal{O}_{\mathrm{F}}$.

1.2.1 Bosonic action

For the bosonic term this is quite easy. First we need to do a dimensional analysis of the fields in the action. In the simulation we can only deal with dimensionless fields. Since the action also needs to be dimensionless we can see that the fields x, x^* and z^M are dimensionless as preferred but the auxiliary fields are of dimension $[\phi] = 1/[a]$. We therefore do a redefinition of the discretized fields

$$a\phi(n) \to \phi(n), \qquad a\phi_I(n) \to \phi_I(n) \tag{1.34}$$

in order to have dimensionless quantities as well. We can now write the discretized version of the bosonic action in (1.31) as

$$\begin{aligned} \hat{S}_{\mathrm{B}} = g \sum_{n\in\varLambda} \Bigg[& \left|x(n+\hat{0}) + \left(\tfrac{M}{2}-1\right) x(n)\right|^2 + \frac{1}{[z(n)]^4} \left|x(n+\hat{1}) - \left(\tfrac{M}{2}+1\right) x(n)\right|^2 \\ & + \sum_{M=1}^{6} \Bigg\{ \left(z^M(n+\hat{0}) + \left(\tfrac{M}{2}-1\right) z^M(n)\right)^2 \\ & \qquad + \frac{1}{[z(n)]^4} \left(z^M(n+\hat{1}) - \left(\tfrac{M}{2}+1\right) z^M(n)\right)^2 \Bigg\} \\ & + \phi^2(n) + \sum_{I=1}^{16} [\phi_I(n)]^2 \Bigg], \end{aligned} \tag{1.35}$$

where we applied a simple forward derivative to the x, x^* and z^M fields and introduced the dimensionless lattice mass parameter $M = ma$.

1.2.2 Wilson term

Before discretizing the fermionic operator we have to think about the doubling problem arising through naively discretized first derivatives, discussed in **??**. The free fermionic operator (evaluated in the bosonic vacuum) represented in a momentum basis reads

$$K_{\mathrm{F}} = \begin{pmatrix} 0 & -p_0 \mathbb{1}_4 & \left(p_1 - i\frac{m}{2}\right)\rho^M u^M & 0 \\ -p_0 \mathbb{1}_4 & 0 & 0 & \left(p_1 - i\frac{m}{2}\right)\rho^\dagger_M u^M \\ -\left(p_1 + i\frac{m}{2}\right)\rho^M u^M & 0 & 0 & -p_0 \mathbb{1}_4 \\ 0 & -\left(p_1 + i\frac{m}{2}\right)\rho^\dagger_M u^M & -p_0 \mathbb{1}_4 & 0 \end{pmatrix}, \tag{1.36}$$

and has determinant

$$\det K_{\mathrm{F}} = \left(p_0^2 + p_1^2 + \frac{m^2}{4}\right)^8 . \tag{1.37}$$

The fermionic propagators are proportional to the inverse of dynamic operators. It is therefore reasonable that the naive discretization (like in (**??**))

$$p_i \to \mathring{p}_i \equiv \frac{1}{a}\sin(p_i a) \tag{1.38}$$

will give rise to doublers. For this reason we would like to introduce a Wilson-like term that cancels the additional poles in the fermionic propagator. It should obey the following conditions:

- preserve the maximal amount of symmetries and relevant matrix properties,
- give the correct continuum limit for $a \to 0$,
- should not give rise to a complex phase.

Due to the properties of $\mathcal{O}_{\mathrm{F}}$ presented in subsubsection 1.1.3 we can see that there is only a small margin of variations that we can apply to the fermion matrix as a Wilson term and that respects the $U(1)$ and $SO(6)$ symmetries. In fact it was not possible to construct such an operator that also preserves relevant matrix properties like skew- and Γ_5-symmetry and also leads to the correct perturbative 1-loop coefficient in (**??**).
We therefore chose to explicitly break the $U(1)$ symmetry and introduce a Wilson-like operator on the diagonal blocks of $\mathcal{O}_{\mathrm{F}}$. In terms of the free fermion operator in momentum space this takes the form

$$\hat{K}_{\mathrm{F}} = \begin{pmatrix} W_+ & -\mathring{p}_0 \mathbb{1}_4 & \left(\mathring{p}_1 - i\frac{m}{2}\right)\rho^M u^M & 0 \\ -\mathring{p}_0 \mathbb{1}_4 & -W_+^\dagger & 0 & \left(\mathring{p}_1 - i\frac{m}{2}\right)\rho^\dagger_M u^M \\ -\left(\mathring{p}_1 + i\frac{m}{2}\right)\rho^M u^M & 0 & W_- & -\mathring{p}_0 \mathbb{1}_4 \\ 0 & -\left(\mathring{p}_1 + i\frac{m}{2}\right)\rho^\dagger_M u^M & -\mathring{p}_0 \mathbb{1}_4 & -W_-^\dagger \end{pmatrix}, \tag{1.39}$$

where

$$W_\pm = \frac{ra}{2}\left(\hat{p}_0^2 \pm i\hat{p}_1^2\right)\rho^M u^M, \qquad |r| = 1, \tag{1.40}$$

and similar to (??)

$$\hat{p}_i \equiv \frac{2}{a}\sin\frac{p_i a}{2}\,. \tag{1.41}$$

This leads to the analogue expression of (1.37)

$$\det \hat{K}_\mathrm{F} = \left(\mathring{p}_0^2 + \mathring{p}_1^2 + \frac{r^2a^2}{4}\left(\hat{p}_0^4 + \hat{p}_1^4\right) + \frac{m^2}{4}\right)^8\,. \tag{1.42}$$

If we substitute this into the denominator of (??) and apply the replacement $p_i^2 \to \hat{p}_i^2$ in the numerator, the discretized equivalent of the 1-loop free energy can be defined by

$$\Gamma^{(1)}_\mathrm{LAT} = -\ln Z^{(1)}_\mathrm{LAT} = \mathcal{I}(a). \tag{1.43}$$

With $r = 1$ and rescaled momentum integration over the first Brillouine zone this results in

$$\begin{aligned}
\mathcal{I}(a) = \frac{V_2}{2a^2}\int_{-\pi}^{\pi}\frac{\mathrm{d}^2p}{(2\pi)^2}\Bigg\{&5\ln\left[4\left(\sin^2\frac{p_0}{2} + \sin^2\frac{p_1}{2}\right)\right]\\
&+2\ln\left[4\left(\sin^2\frac{p_0}{2} + \sin^2\frac{p_1}{2} + \frac{M^2}{8}\right)\right]\\
&+\ln\left[4\left(\sin^2\frac{p_0}{2} + \sin^2\frac{p_1}{2} + \frac{M^2}{4}\right)\right]\\
&-8\ln\left[4\sin^4\frac{p_0}{2} + \sin^2 p_0 + 4\sin^4\frac{p_1}{2} + \sin^2 p_1 + \frac{M^2}{4}\right]\Bigg\}.
\end{aligned} \tag{1.44}$$

For a consistent discretization this should lead to the same result as (??) in the $a \to 0$ limit. And indeed a numerical integration of (1.44) yields

$$\Gamma^{(1)} = -\ln Z^{(1)} = \lim_{a\to 0}\mathcal{I}(a) = -\frac{3\ln 2}{8\pi}|\Lambda|M^2, \tag{1.45}$$

where we used that $V_2 = a^2|\Lambda| = a^2LT$ and we are left with the expected result. Given the structure of the Wilson term in the vacuum it is quite natural to generalize it to the interacting case. The discretized momentum space operator therefore reads

$$\widetilde{\mathcal{O}}_\mathrm{F} = \begin{pmatrix}
W_+ & -\mathring{p}_0\mathbb{1}_4 & \left(\mathring{p}_1 - i\frac{m}{2}\right)\rho^M u^M & 0\\
-\mathring{p}_0\mathbb{1}_4 & -W_+^\dagger & 0 & \left(\mathring{p}_1 - i\frac{m}{2}\right)\rho_M^\dagger u^M\\
-\left(\mathring{p}_1 + i\frac{m}{2}\right)\rho^M u^M & 0 & 2\frac{z^M}{z^4}\rho^M(\partial_s x - \frac{m}{2}x) + W_- & -\mathring{p}_0\mathbb{1}_4\\
0 & -\left(\mathring{p}_1 + i\frac{m}{2}\right)\rho_M^\dagger u^M & -\mathring{p}_0\mathbb{1}_4 & -2\frac{z^M}{z^4}\rho_M^\dagger(\partial_s x^* - \frac{m}{2}x^*) - W_-^\dagger
\end{pmatrix}, \tag{1.46}$$

with

$$W_\pm = \frac{ra}{2z^2}\left(\hat{p}_0^2 \pm i\hat{p}_1^2\right)\rho^M z^M, \tag{1.47}$$

where an additional $1/z$ factor is present for the purpose of improved stability during simulations. From (1.24) one can see that the added WILSON term respects the Γ_5-hermiticity and skew-symmetry which ensures the determinant to be real and positive.

1.2.3 Fermionic operator

By knowing the structure of the WILSON term we can finally discretize the fermionic operator and write it in terms of a single matrix by using the lexicographic index notation introduced in ??. The discretized fermion matrix $\hat{\mathcal{O}}_\mathrm{F}$ is of size $16V \times 16V$ and we are going to subdivide it into 4 by 4 blocks of size $4V \times 4V$ as

$$\left(\hat{\mathcal{O}}_\mathrm{F}\right)_{16V\times 16V} = \left(\begin{array}{c|c|c|c} O_{4V\times 4V} & \cdots & & \\ \hline \vdots & \ddots & & \\ \hline & & & \\ \hline & & & \end{array}\right). \tag{1.48}$$

To build the block matrices we assume that for every lattice index $l, p = 0, \dots, (V-1)$ we have a 4×4 matrix with indices $i, j = 1, \dots, 4$ and emphasize the structure

$$O_{4\times 4}(V \times V) \longleftrightarrow O_{ij}(l, p). \tag{1.49}$$

Now one can map this to a $4V \times 4V$ matrix with global indices like

$$O_{4V\times 4V} \longleftrightarrow O_{AB}, \qquad A = 4l + i, \quad B = 4p + j. \tag{1.50}$$

The discretized fermion matrix is now given by

$$\hat{\mathcal{O}}_\mathrm{F} = \begin{pmatrix} \hat{W}_+ & i\bar{\Delta}_0^A & -i\left(\vec{\Delta}_1^Z + \frac{M}{2}\bar{Z}\right) & 0 \\ i\bar{\Delta}_0^A & -\hat{W}_+^\dagger & 0 & -i\left(\vec{\Delta}_1^{Z^\dagger} + \frac{M}{2}\bar{Z}^\dagger\right) \\ i\left(\overleftarrow{\Delta}_1^Z - \frac{M}{2}\bar{Z}\right) & 0 & 2\left(\bar{\Delta}_1^x - \frac{M}{2}\bar{Z}^x\right) + \hat{W}_- & i\bar{\Delta}_0^A - \hat{A}^\mathrm{T} \\ 0 & i\left(\overleftarrow{\Delta}_1^{Z^\dagger} - \frac{M}{2}\bar{Z}^\dagger\right) & i\bar{\Delta}_0^A + \hat{A} & 2\left(\bar{\Delta}_1^{x^*} - \frac{M}{2}\bar{Z}^{x^*}\right) - \hat{W}_-^\dagger \end{pmatrix}. \tag{1.51}$$

References

[1] I. Montvay and G. Münster, *“Quantum fields on a lattice”*, Cambridge University Press (1994).

[2] R. McKeown and R. Roiban, *“The quantum $AdS_5 \times S^5$ superstring at finite coupling”*, `arXiv:1308.4875`.