Q2ai: First note that by previous results, we have that $|M^{\times}| = p^{\deg(s(x))} - 1$. Hence the order of any element can not exceed $p^{\deg(s(x))} - 1$. Since we have that $M^{\times}$ is cyclic there must exist some element of order exactly $p^{\deg(s(x))} - 1$.

2aii: Let α be a root of $\Phi_{p^m-1}(x)$. Then by HW8Q2a, we have that $\alpha \in M^{\times}$ and therefore by HW8Q3 we have that the order of α is $p^m - 1$. It follows by Lagranges theorem that $p^m - 1 | p^{\deg(s(X))} - 1$ and clearly

2aiii: It has been shown that $\gcd(p^a - 1, p^b - 1) = p^{\gcd(a,b)} - 1$. The euclidean algorithim then gives us that $\gcd(a, b) = \gcd(b, r)$. Hence we have that

$$gcd(p^a - 1, p^b - 1) = p^{\gcd(b,r)} - 1 \leq p^r - 1$$

Since $\gcd(b, r) \leq r$. It follows that if $p^b - 1 | p^a - 1$, then $\gcd(a, b) = \gcd(b, r) = b$ but by assumption $r < b$. Hence $p^a - 1 \nmid p^b - 1$. Reasoning contrapositively, by above if $m \nmid \deg(s(x))$ then $p^m - 1 \nmid p^{\deg(s(x))} - 1$.

Q2iv: Since $alpha$ is a root of Φ_{p^m-1} it must have an order of $p^m - 1$. Therefore, $\alpha^{p^m-1} = 1$ and so $\alpha^{p^m} = \alpha$.