Q4a: Assume wlog that $Q(z) = z^n + \dots c_0 = (z - a_1)\dots(z - a_n)$. By partial fraction decomposition there exists $d_1 \dots d_n$ which satisfy

$$\frac{P(z)}{Q(z)} = \frac{d_1}{(z - a_1)} + \cdots + \frac{d_n}{(z - a_n)}$$

The Heaviside Cover up Method (MAT157) tells us that each d_i is determined in the following way.

$$d_i = \frac{P(a_i)}{\Pi_{j=1,j\neq i}^n (a_i - a_j)}$$

Therefore we see that

$$\begin{aligned}\frac{P(z)}{Q(z)} &= \sum_{i=1}^n \frac{d_i}{(z - a_i)} \\ &= \sum_{i=1}^n \frac{P(a_i)}{(z - a_i)\Pi_{j=1,j\neq i}^n (a_i - a_j)}\end{aligned}$$

Since $Q'(z) = \sum_{i=1}^n \Pi_{j=1,j\neq i}^n (z - a_j)$ from the product rule, we have that $Q'(a_i) = \Pi_{j=1,j\neq i}(a_i - a_j)$ Therefore we get that

$$\frac{P(z)}{Q(z)} = \sum_{i=1}^n \frac{P(a_i)}{Q'(a_i)(z - a_i)}$$

Q4b: We define our polynomial $P(z)$ as

$$P(z) = Q(z) \sum_{j=1}^n \frac{b_j}{Q'(a_j)(z - a_j)}$$

We claim that this indeed satisfies $P(a_k) = b_k$ and it is uniquely determined by $Q(z)$. We see that

$$P(a_k) = \sum_{j=1}^n \frac{b_j \cdot \Pi_{l=1}^n (a_k - a_l)}{\Pi_{i=1,i\neq j}(a_i - a_j) \cdot (a_k - a_j)}$$

At the index $j = k$ we have that polynomial evaluates to b_k, at every other point the numerator evaluates to 0. Hence $P(a_k) = b_k$. We now claim uniqueness. From linear algebra, we know that any polynomial of degree n is completely determined by its values on at least $n + 1$ points. Hence we have that $P_1(z) = P(z)$.