Problem 1.

(a) By the Cauchy-Riemann equations, we have

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = v_{xy} - v_{yx} = 0.$$

Similarly, for v we have

$$v_{xx} + v_{yy} = -v_{xy} + v_{xy} = 0.$$

(b) Given a harmonic $u(x, y)$ we can define $v(x, y) = \int_0^y u_x(x, t)dy$. We compute that

$$v_y = \frac{\partial}{\partial y}\int_0^y u_x(x, t)dt = u_x.$$

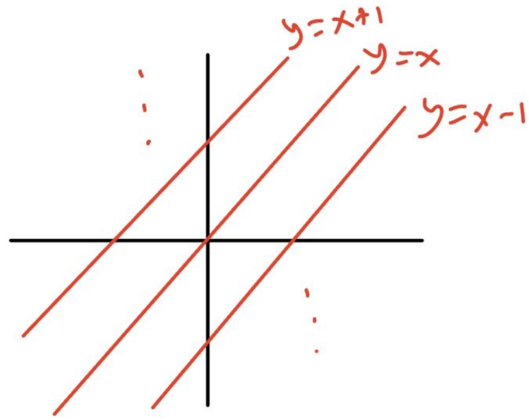
Similarly, we have

$$v_x = \int_0^y u_{xx}(x, t)dt = \int_0^y -u_{yy}(x, t)dt = -u_y.$$

Thus we can find the conjugate harmonic function to u.

Problem 2.

(a) We have shown in class that the characteristics of this equation are lines of the form $y = x + c$ for $c \in \mathbb{R}$.



(b) We solve the inhomogenous PDE now, with the boundary value $u(x,0) = f(x)$. We write $u(x(t), y(t))$, and get the following system of ODE's from out PDE through any point $(x_0, 0)$:

$$\begin{cases} \dot{x} = 1 & x(0) = x_0 \\ \dot{y} = 1 & y(0) = 0 \\ \dot{u} = 1 & u(x,0) = f(x) \end{cases}$$

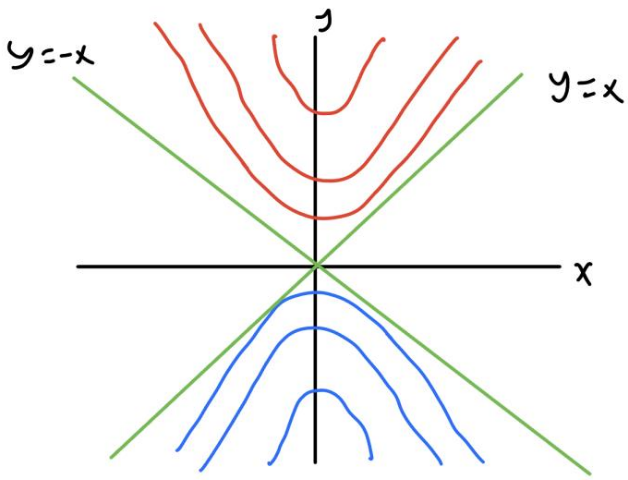
Solving this, we get $x(t) = t + x_0, y(t) = t$ and $u(x,y) = t + f(x_0)$. Since $x_0 = x - y$, and $y = t$ we have that $u(x,y) = y + f(x-y)$

Problem 3.

We parametrize the characteristic curves as $(x(t), y(t))$, through the point $(0, y_0)$, and get the system of ODE's from $u(x(t), y(t))$:

$$\begin{cases} \dot{x}(t) = y(t) & x(0) = 0 \\ \dot{y}(t) = x(t) & y(0) = y_0 \\ \dot{z}(s) = 0 & z(0) = \cos y_0 \end{cases}$$

We solve for $x(t), y(t)$ as $x(t) = Ce^t + De^{-t}, y(t) = Ce^t + De^{-t}$. With initial conditions we get that $x(t) = y_0 \sinh t, y(t) = y_0 \cosh t$. We can rewrite x, y as $y + x = y_0 e^t, y - x = y_0 e^{-t}$. Thus we have that $y_0^2 = y^2 - x^2$. Therefore $u(x, y) = \cos\left(\sqrt{y^2 - x^2}\right)$.



We have unique solutions on the regions above $y = x, y = -x$, and below $y = x, y = -x$. If we expand the region to include the union of the lines $y = x, y = -x$, we will lose uniqueness. If we take the initial value at $(0, 0)$, we will have two solutions, namely $y = x, y = -x$ passing through this point.

Problem 4.

(a) Consider the coordinate change of $\xi = \frac{t+x}{2}, \eta = \frac{t-x}{2}$. Using the chain rule, we compute that

$$u_t = \frac{1}{2}u_\xi + \frac{1}{2}u_\eta$$

and

$$u_{tt} = \frac{1}{4}u_{\xi\xi} + \frac{1}{2}u_{\xi\eta} + \frac{1}{4}u_{\eta\eta}.$$

Similarly, we have that

$$u_{xx} = \frac{1}{4}u_{\xi\xi} - \frac{1}{2}u_{\eta\xi} + \frac{1}{4}u_{\eta\eta}.$$

Therefore we have that

$$0 = u_{tt} - u_{xx} = u_{\eta\xi}.$$

(b) For $c \neq 0$, we make the coordinate change $\xi = \frac{t+cx}{2}, \eta = \frac{t-cx}{2}$. We compute that

$$u_{xx} = c^2\left(\frac{1}{4}u_{\xi\xi} + \frac{1}{2}u_{\xi\eta} + \frac{1}{4}u_{\eta\eta}\right),$$

and u_{tt} remains unchanged from a). We have once again that this coordinate change yields $u_{\eta\xi} = 0$.

(c) Given that $u_{\eta\xi} = 0$, we have that $u_\eta = f(\eta)$ for some f, and similarly. $u_\xi = g(\xi)$ for some g. Therefore we have that

$$u = G(\xi) + F(\eta) = F\left(\frac{t+x}{2}\right) + G\left(\frac{t-x}{2}\right) = G(t+x) + F(t-x),$$

for arbitrary functions F, G. Given the boundary values $u(0,x) = f(x), u_t(0,x) = 0$, we have that

$$F(x) + G(x) = f(x), F'(x) - G'(x) = 0. \implies G = F + c.$$

Therefore we can write $2F(x) + c = f(x)$ which implies that $F(x) = \frac{f(x)-c}{2}, G(x) = \frac{f(x)+c}{2}$. Thus we have that

$$u(x,y) = \frac{f(t+x) + f(t-x)}{2}.$$

(d) Yes this is unique. Suppose u, v solve the boundary value problem given in c). Then $u-v$ satisfies the same boundary value problem with $f(x) = 0$. The same equations hold, writing $(u-v) = F(t+x) + G(x-t)$, we get that $F'(x) = G'(x)$, and $F(x) + G(x) = 0$. This implies that $F = G + c$ and so $2G + c = 0$. So $G = c$, similarly $F = 2c$. The initial value tells us that $F = G = 0$. Therefore $u = v$.

Problem 5.

By the previous question, we have that the solution u takes the form

$$u(x,y) = F(x+t) + G(x-t).$$

The boundary values give us that

$$f(x) = u(x,x) = F(2x) + G(0) \implies F(x+t) = f\left(\frac{x+t}{2}\right)$$

The normal vector to $t = x$ is $(-1,1)$, so we can compute the normal derivative as:

$$u_\nu = \begin{bmatrix} F'(x+t) + G'(x-t) & F'(x+t) - G'(x-t) \end{bmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0.$$

This is a problem for existence unless $g(x) = 0$. If $g = 0$, uniqueness becomes a problem. Since G is constant on the boundary, we can choose an arbitrary G of the form $G(x-t)$, with $G(0) = 0$, and add it to our solution $f\left(\frac{x+2}{2}\right)$. Therefore this is not a well posed boundary value problem.