Q1a: Note since
$$(1-x)(1+x+\cdots+x^{q-1})=(1-x^q)$$
Any root of the cyclotomic polynomial is also root of $1-x^q$. We see that $\Phi_q(0)\equiv 1 \text{mod (p)}$. Therefore by FlT the roots of $\Phi_q(x)$ must satisfy $x^{p-1}=1$. We now check cases when $p\equiv 1$ mod (q), $p=q$ or if neither are true. First consider the case where $p\equiv 1$ mod (1), then there is some k such that $kq=p-1$
$$\begin{aligned} x^{p-1}-1 &= x^{kq-1}-1\\ &=(x^q)^k-1\\ &=(x^q-1)(x^{kq-q}+x^{kq-2q}+\cdots+1)\\ &=(x^q-1)(x^{p-1-q}+\cdots+1)\end{aligned}$$
Hence by Lagranges theorem the first term must have at most q roots and the second term must have at most $p-1-q$ roots. By Fermats little theorem, $x^{p-1}-1$ has $p-1$ roots mod p, so therefore x^q-1 has q roots and $x^{p-1-q}+\cdots+1$ has $p-1-q$ roots. Now since $(x-1)\Phi_q(x)$ has p roots, and $\Phi_q(1)\equiv q$ mod(p) which is not 0, so we must have that $\Phi_q(x)$ has $p-1$ roots. If $p\not\equiv 1$ mod (q), so $\gcd(q,p-1)=1$. Thus by bezouts identity, there exists integers u,v with $u(p-1)+v(q)=1$. Therefore, if x is a root of $\Phi_q(x)$, then $x^1=x^{u(p-1)+v(q)}=(x^{p-1})^u\cdot(x^q)^v$. Now if $p=q$ then $\Phi_q(1)=0$ mod (p) so it has 1 root. Otherwise, $\Phi_q(1)\neq 0$ so $\Phi_q(x)$ has no roots.

Q1b: By the chinese remainder theorem, any solution to $x^{18}+4x^{14}+3x+10\equiv 0$ mod (21) must also be a solution to $x^{18}+4x^{14}+3x+10\equiv 0$ mod (3) and $x^{18}+4x^{14}+3x+10\equiv 0$ mod (7). By corollary 4.4, $x^{3k}\equiv x$ mod (3) for all $k\in\mathbb{Z}$. Therefore we can reduce our polynomials to
$$x^{18}+4x^{14}+3x+10\equiv(x^{3\cdot 3})^2+x^2+1\equiv(2x^2-1)\equiv 1-x^2\equiv(1-x)(1+x) \bmod (3)$$
which has a solution of $x\equiv 1$ mod (3) and $x\equiv 2$ mod (3). Now for the mod (7) polynomial,
$$x^{18}+4x^{14}+3x+10\equiv(x^7)^2\cdot x^4+4(x^7)^2+3x+10\equiv x^6+4x^2+3x+3\equiv 0 \bmod (7)$$
By checking $x\in\{0,1,2,3,4,5,6\}$, we see that $x\equiv 3$ mod (7) and $x\equiv 5$ mod (7) are both solutions. By chinese remainder theorem, the solutions are $x\equiv 5,10,17,19$ mod (21).

Q1c: We wish to solve $x^{59}+2x^{40}+5x^{25}+x^{15}+17\equiv 0$ mod (221). By the chinese remainder theorem, any solution to this will also be a solution to $x^{59}+2x^{40}+5x^{25}+x^{15}+17\equiv 0$ mod (13) and $x^{59}+2x^{40}+5x^{25}+x^{15}+17\equiv 0$ mod (17). We will first proceed with the first equivalency. Using corollary 4.4, we have that
$$x^{59}+2x^{40}+5x^{25}+x^{15}+17\equiv(x^{13})^4\cdot x^7+2(x^{13})^3\cdot x+5(x^{13})x^{12}+x^{13}\cdot x^2+17\equiv x^{11}+2x^4+x^3+5x+4\equiv 0 \bmod (13)$$
Taking $x\equiv 1$ mod (13) will satisfy this. For mod (17), we have that
$$x^{59}+2x^{40}+5x^{25}+x^{15}+17\equiv x^{59}+2x^{40}+5x^{25}+x^{15}\equiv 0 \bmod (17)$$
We see taking $x\equiv 0$ mod (17) will satisfy. Therefore by the chinese remainder theorem, a solution to the polynomial mod (221) is $x\equiv 170$ mod (221).

Q1d: To find a solution to $55x^{19}+3x^{14}+x^2+55\equiv 0$ mod (66). By the Chinese remainder theorem, any solution to this must be a simultanous solution to $55x^{19}+3x^{14}+x^2+55\equiv 0$ mod (2), $55x^{19}+3x^{14}+x^2+55\equiv 0$ mod (3), $55x^{19}+3x^{14}+x^2+55\equiv 0$ mod (11). We will first look for solutions in the mod(2) case. We see that
$$55x^{19}+3x^{14}+x^2+55\equiv x^{19}+x^{14}+x^2+1\equiv 0 \bmod (2)$$
The only solution is $x\equiv 1$ mod (2) is a solution. Now we look at the mod (3) case.
$$55x^{19}+3x^{14}+x^2+55\equiv x^{19}+x^2+1\equiv x^2+x+1\equiv 0 \bmod (3)$$

We see that the only solution is $x \equiv 1 \bmod (3)$. Finally, we check mod (11).

$$55x^{19} + 3x^{14} + x^2 + 55 \equiv 3x^{11}x^3 + x^2 \equiv 3x^4 + x^2 \equiv x^2(3x^2 + 1) \equiv 0 \bmod (11)$$

By checking each $x \in \mathbb{Z}/11\,\mathbb{Z}$ we see that $3x^2 + 1 \not\equiv 0$ for all x. Thus we conclude that the only solution mod 11 is $x \equiv 0 \bmod (11)$. Therefore, by the chinese remainder theorem, the solution will be $x \equiv 55 \bmod (66)$.