

Q6i: We claim that  $G' \trianglelefteq G$ . Any element of  $G'$  must be of the form  $b = ghg^{-1}h^{-1}$ . We claim that for any  $a \in G$ ,  $aba^{-1} \in G'$ . Observe:

$$aba^{-1} = aghg^{-1}h^{-1}a = ageheg^{-1}eh^{-1}a^{-1} = (aga^{-1})(aha^{-1})(ag^{-1}a^{-1})(ah^{-1}a^{-1})$$

We notice that this is in the desired form, since the first and third terms are inverses of each other, and the second and fourth terms are inverses of each other. We conclude that  $G' \trianglelefteq G$ .

Q6ii: We claim that  $G/G'$  is an abelian group. It is equivalent to show that for any  $a, b \in G$ ,

$$abG' = baG' \iff ab(ba)^{-1}G' = G' \iff aba^{-1}b^{-1}G' = G'$$

The last equality is clearly true since  $aba^{-1}b^{-1} \in G'$ .

Q6iii: Let  $N \trianglelefteq G$  be such that  $G/N$  is abelian. We claim that  $G' \subseteq N$ . Let  $a \in G'$ . It must take the form  $a = ghg^{-1}h^{-1}$  for some  $g, h \in G$ . We evaluate that

$$\begin{aligned} aN &= (ghg^{-1}h^{-1})N \\ &= (gN)(hN)(g^{-1}N)(h^{-1}N) \\ &= (gN)(g^{-1}N)(hN)(h^{-1}N) \\ &= (gg^{-1})N(hh^{-1}N) \\ &= (eN)(eN) \\ &= eeN \\ &= eN \\ &= N \end{aligned}$$

Therefore we have that  $a \in N$  and we conclude that  $G' \subset N$ .