Q5a: We define

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z-n)^2}$$

and

$$g(z) = \frac{\pi^2}{(\sin \pi z)(\tan \pi z)}.$$

Observe that both f and g have simple removable poles of order 2 on the integers, which we know from the power series expansion of sin and tan. We also know that both f and g are periodic, with a period of 1. For all poles n, we can write

$$f(z) = \frac{(-1)^n}{(z-n)^2} + \tilde{f}(z)$$

for a holomorphic $\tilde{f}$ in some neighbourhood of n. Similarly, using the laurent series expansion of g, we can write

$$g(z) = \frac{(-1)^n}{(z-n)^2} + \tilde{g}(z)$$

for some holomorphic $\tilde{g}(z)$. Similarly to $Q4$, if we write $z = x + iy$ we have that $f(z), g(z) \to 0$ as $|y| \to \infty$. Thus on any strip $a_1 \leq x \leq a_2$, for $|y| \leq b$, the holomorphic function $f - g$ will attain a maximum. The limit behaviour tells us that for $|y| > b$, $f - g$ is bounded as well. Therefore for $a_1 \leq x \leq a_2$, $f - g$ is bounded. Extending by periodicity tells us that $f - g$ is bounded and hence constant by Liouvilles theorem. The limit behaviour tells us that $f - g = 0$. As desired.

Q5b: Let

$$f(z) = \frac{1}{z} + \sum_{n\geq 1}^{\infty} (-1)^n \frac{2z}{(z^2 - n^2)} = \frac{1}{z} + \sum_{n\geq 1}^{\infty} (-1)^n \Big[\frac{1}{(z-n)} + \frac{1}{(z+n)}\Big].$$

This is a series of meromorphic functions, which is normally convergent so we can take the derivative term by term to get

$$f'(z) = -\frac{1}{z^2} + \sum_{n\geq 1} (-1)^n \Big[\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2}\Big] = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z-n)^2} = \frac{\pi^2}{(\sin \pi z)(\tan \pi z)} = \frac{d}{dz}\Big[\frac{\pi}{\sin \pi z}\Big].$$

Therefore $f(z)$ and $\frac{\pi}{\sin \pi z}$ differ by a constant. Since they are both odd they must be equal.