Q5: We first claim that any mobius transformation as given must map only the real line to the real line. Suppose there was some complex number z such that $f(z) \in \mathbb{R}$. By properties of f we have that in $\hat{\mathbb{C}}$, $f(\mathbb{R})$ is a circle. Since f^{-1} is also a mobius transformation we have that $f^{-1}(\mathbb{R})$ will be disconnected, since we can write it as a disjoint union $(-\infty, f^{-1}(z)) \sqcup (f^{-1}(z), \infty)$. This is a contradiction, since continuous maps take connected sets to connected sets. Thus no such point z can exist, and f maps the real line exactly to the real line. Therefore, we can consider the real values z_1, z_2, z_3 which get mapped to $0, 1, \infty$ respectively. Since

$$f(z) = \frac{az+b}{cz+d}$$

we can deduce that

$$0 = f(z_1) = \frac{az_1+b}{cz_1+d} \implies z_1 = -\frac{b}{a} \in \mathbb{R}$$

and,

$$\infty = f(z_3) = \frac{az_3+b}{cz_3+d} \implies z_3 = -\frac{d}{c} \in \mathbb{R}$$

And so

$$1 = f(z_2) = \frac{az_2+b}{cz_2+d}$$

Together these imply that

$$\frac{a}{c} = \frac{z_2 - z_3}{z_2 - z_1}$$

We get that

$$f(z) = \frac{az+b}{cz+d} = \frac{a(z-z_1)}{c(z-z_3)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{z(z_2-z_3)+z_1(z_3-z_2)}{z(z_2-z_1)+z_3(z_1-z_2)}$$

Since each z_i is real we reach the desired result.