

Q3: We first claim that every nontrivial subgroup of  $\mathbb{Z}$  must be infinite. If  $H \leq G$  with  $|H| > 1$ , then take  $a \in H, a \neq 0$ . We must have for any  $n \in \mathbb{Z}$ ,  $an \in H$  since multiplication with integers is equivalent to repeated addition or subtraction. Therefore  $H$  must be infinite. We claim that the only subgroups of  $\mathbb{Z}$  are  $n\mathbb{Z}$  for  $n \in \{0, 1, 2, \dots\}$ . Let  $H$  be a nontrivial subgroup of  $\mathbb{Z}$ . We define  $Y = \{\gcd(|g|, |h|) : g, h \in H\}$ .  $Y$  is a nonempty set and this is bounded below by 0, hence we can apply the well ordering principle. There must exist a minimal element  $d \in Y$ . We now claim that  $H = d\mathbb{Z}$ . Note that by Bezout's identity there exists  $a, b \in \mathbb{Z}$  such that  $d = ag + bh$  for some  $g, h \in \mathbb{Z}$ , namely the  $g, h$  satisfying  $\gcd(g, h) = d$ . We now claim that  $H = d\mathbb{Z}$ . Suppose that there is some  $a \in H$  that cannot be written as  $dz = a$  for some  $z \in \mathbb{Z}$ . This would imply that  $\gcd(d, a) < d$  contradicting minimality of  $d$ . Hence we have that any subgroup of  $\mathbb{Z}$  must be of the form  $d\mathbb{Z}$ .