5a: First note that if we substitute $x = \frac{z}{y}$,

$$\int K(xy)x^{-\frac{1}{q}}dx = \int K(z)(\frac{z}{y})^{-\frac{1}{q}}\frac{1}{y}dz = y^{\frac{1}{q}-1}\int K(z)z^{\frac{1}{p}-1}dz = y^{-\frac{1}{p}}\phi(p^{-1}).$$

We now see that

$$\begin{aligned}\Big(\int K(xy)f(x)dx\Big)^p &\leq \Big(\int K(xy)x^{-\frac{1}{q}}dx\Big)^{\frac{p}{q}}\int x^{\frac{p}{q^2}}K(xy)f(x)^p dx && \text{(by Holders inequality)}\\ &= y^{-\frac{1}{q}}\phi(p^{-1})^{\frac{p}{q}}\int x^{\frac{p}{q^2}}K(xy)f(x)^p dx && \text{(by above)}\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}\int\Big(\int K(xy)f(x)dx\Big)^p dy &\leq \int y^{-\frac{1}{q}}\phi(p^{-1})^{\frac{p}{q}}dy\int x^{\frac{p}{q^2}}K(xy)f(x)^p dx && \text{(by tonelli and above)}\\ &= \phi(p^{-1})^{\frac{p}{q}}\int\int y^{-\frac{1}{q}}x^{\frac{p^2}{q}}K(xy)f(x)^p dxdy\\ &= \phi(p^{-1})^{\frac{p}{q}}\int x^{\frac{p^2}{q}}f(x)^p\int K(xy)y^{-\frac{1}{q}}dydx\\ &= \phi(p^{-1})^p\int x^{p-2}f(x)^p dx\end{aligned}$$

Therefore by Holders inequality,

$$\int\int K(xy)f(x)g(y)dxdy \leq \|g\|_q\,\phi(p^{-1})\int x^{p-2}f(x)^p dx.$$

Q5b: Using 5a, we get that

$$\|Tf(x)\|_2^2 = \int\Big|\int K(xy)f(y)dy\Big|^2 dx \leq \phi(\frac{1}{2})^2\int|f(x)|^2 dx.$$

Therefore this operator is bounded and maps L^2 into L^2.