Q1a:
Let P be a partition of A. We have that

$$\begin{aligned}
&m_S(f) + m_S(g) \\
&= \inf_{x\in S} f(x) + \inf_{x\in S} g(x) \\
&\leq \inf_{x\in S} (f+g)(x) && \text{by discussion in lecture 25} \\
&= m_S(f+g)
\end{aligned}$$

And similiarly, $M_S(f+g) \leq M_S(f) + M_S(g)$. Therefore we have that

$$\begin{aligned}
&L(f,P) + L(g,P) \\
&= \sum_{S\in P} \inf_{x\in S} f(x)\cdot vol(S) + \sum_{S\in P} \inf_{x\in S} g(x)\cdot vol(S) \\
&\leq \sum_{S\in P} \inf_{x\in S} (f+g)\cdot vol(S) \\
&= L(f+g,P)
\end{aligned}$$

Similiarly we have $U(f+g,P) \leq U(f,P) + U(g,P)$.
1b:
Choose partitions P_1 and P_2 such that $U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$ and $U(g,P_2) - U(g,P_2) < \frac{\varepsilon}{2}$. Let P_3 be a partition which refines P_1 and P_2, now by 1a we have that

$$U(f+g,P_3) - L(f+g,P_3) \leq U(f,P_1) + U(g,P_2) - L(f,P_1) - L(g,P_2) < \varepsilon$$

Thus if f and g are integrable so is $f+g$. We also have that

$$L(f) + L(g) \leq L(f+g) \leq U(f+g) \leq U(f) + U(g)$$

which implies that $\int_A (f+g) = \int_A f + \int_A g$
1c:
Let $c \in \mathbb{R}$. It suffices to check 3 cases, $c<0, c=0, c>0$. When $c=0$ the statement is trivially true, since $\int_A 0\cdot f = 0 = 0\cdot\int_A f$. If $c>0$, let $\varepsilon>0$. Choose partition P such that $U(f,P) - L(f,p) < \frac{\varepsilon}{c}$. We compute

$$\begin{aligned}
&U(cf,P) - L(cf,P) \\
&= \sum_{S\in P} [M_S(cf) - m_S(cf)]\cdot vol(S) \\
&= \sum_{s\in P} c[M_S(f) - m_S(f)]\cdot vol(S) \\
&= c[U(f,P) - L(f,P)] \\
&< \varepsilon
\end{aligned}$$

Now suppose that $c<0$, Let $\varepsilon>0$. Choose a partition P such that $U(cf,P) - L(cf,P) < -\frac{\varepsilon}{c}$. we compute

$$\begin{aligned}
&U(cf,P) - L(cf,P) \\
&= \sum_{S\in P} [M_S(cf) - m_S(cf)]\cdot vol(S) \\
&= \sum_{S\in P} [c\cdot m_S(f) - c\cdot M_s(f)]\cdot vol(S) && \text{since multiplying by c below 0 swaps sup and inf} \\
&= \sum_{S\in P} -c\cdot[M_S(f) - m_S(f)]\cdot vol(S) \\
&< \varepsilon
\end{aligned}$$

Hence for any constant c we have that if f integrable, so is cf. By above we know that

$$c[U(f,P) - L(f,p)] = U(cf,P) - L(cf,P) < \varepsilon$$

Since this is true for all $\varepsilon>0$, it must be that $\int_A cf = c\int_A f$