Problem 1.

(a) Consider the heat equation on the real line, with initial datum $\phi(x)$ even. The solution $u(x,t)$ is given by:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4kt}} \phi(y)dy.$$

We compute:

$$\begin{aligned}
u(-x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{|-x-y|^2}{4kt}} \phi(y)dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{|x+y|^2}{4kt}} \phi(y)dy \\
&= -\frac{1}{\sqrt{4\pi kt}} \int_{\infty}^{-\infty} e^{-\frac{|x-y'|^2}{4kt}} \phi(-y')dy' && \text{sub } y = -y' \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{|x-y'|^2}{4kt}} \phi(-y')dy' && \text{(*) flipping bounds} \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{|x-y'|^2}{4kt}} \phi(y')dy' && \phi \text{ even} \\
&= u(x,t)
\end{aligned}$$

Therefore u is even. If our initial datum ϕ is odd, then the calculation above is identical until $(*)$, at which point we have:

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{|x-y'|^2}{4kt}} \phi(-y')dy' = -\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{|x-y'|^2}{4kt}} \phi(y')dy' = -u(x,t).$$

Thus the solutions to the heat equation with even or odd initial datum is also even or odd.

(b) We now prove the analogous result for the wave equation. Let $\phi(x), \psi(x)$, be even initial conditions. Let $u(x,t)$ be the unique solution. By D'Alemberts formula, we have:

$$\begin{aligned}
u(-x,t) &= \frac{1}{2}[\phi(-x+ct) + \phi(-x-ct)] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(y)dy \\
&= \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] - \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-y')dy' && (\phi \text{ even, substitute } y = -y') \\
&= \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y')dy' && \text{(flip bounds, } \psi \text{ even.)} \\
&= u(x,t)
\end{aligned}$$

Similarly with initial data ϕ, ψ odd, we get instead:

$$u(-x,t) = -\frac{1}{2}[\phi(x+ct) + \phi(x-ct)]\frac{1}{2c}\int_{x-ct}^{x+ct} \phi(-y')dy' = -u(x,t).$$

Problem 2.

(a) We compute the time derivative of the energy functional:

$$\begin{aligned}
\dot{E}(t) &= 2\int_U \nu \cdot \nu_t \, dx \\
&= 2\int_u \nu \cdot (-\nabla p - (\nu \cdot \nabla)\nu)\, dx \\
&= -2\int_U \nu \cdot \nabla p \, dx - 2\int_u \nu \cdot (\nu \cdot \nabla)\nu \, dx \\
&= -2\int_U \nabla \cdot (p\nu)\, dx - 2\int_U \nabla \cdot (X\nu)\, dx && \text{(vector calc identities, for some matrix X)} \\
&= -2\int_{\partial U} p\nu \cdot \nu \, da - 2\int_{\partial U} X\nu \cdot \nu \, da \\
&= 0 && \text{(since v=0 on } \partial U \text{ and is perpendicular along it.)}
\end{aligned}$$

Such a matrix X must exist since we can write a system of ODE's where $\nabla \cdot X\nu = \nu \cdot (\nu \cdot \nabla)\nu$ has a unique solution. Therefore energy is conserved.

(b) For the NS equation, we compute the time derivative of the energy functional as:

$$\begin{aligned}
\dot{E}(t) &= 2\int_U \nu \cdot (-\nabla p - (\nu \cdot \nabla)\nu + \mu \Delta \nu)\, dx \\
&= 2\mu \int_U \nu \cdot \Delta \nu \, dx && \text{(by a))} \\
&= 2\mu \int_U \nu \cdot (\nabla \times (\nabla \times \nu))\, dx && \text{(vector laplace definition and } \nabla \cdot \nu = 0.) \\
&= -2\mu \int_U \nabla \cdot ((\nabla \times \nabla \times \nu) \times \nu)\, dx - 2\mu \int_U (\nabla \times \nu) \cdot (\nabla \times \nu)\, dx && \text{(vector calv identity)} \\
&= -2\mu \int_{\partial U} (\nabla \times \nabla \times \nu) - 2\mu \int_U \|\nabla \times \nu\|^2 \, dx && \text{(divergence theorem)} \\
&= -2\mu \int_U \|\nabla \times \nu\|^2 \, dx && \text{(since } \nu \text{ vanishes on boundary)}
\end{aligned}$$

Thus the energy decreases for all time t. We can write

$$E(t) = E(0) - 2\mu \int_0^t \int_U \|\nabla \times \nu\|^2 \, dx dt$$

Problem 3.

(a) Apply the curl operator to both sides of the PDE, use vector calculus identities to get that:

$$\nabla \times (v_t - \mu \Delta v + \nabla p) = \nabla \times \nabla f \implies \nabla \times v_t - \mu \nabla \times \Delta v \implies \omega_t = \mu \Delta \omega.$$

Thus ω's compononents satisfy the heat equation.

(b) We compute that

$$\omega_i(x,t) = \frac{1}{\sqrt{4\pi t\mu}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t\mu}} (\nabla \times v_0(y))_i \, dy$$

Problem 4.

Define the function $u = v - h$ so that u satisfies:

$$\begin{cases} u_t - u_{xx} = (f - h_t) \\ u(t, 0) = 0 \\ u(x, 0) = \phi(x) \end{cases}$$

on the half line. Let $\Phi(x)$ be the odd extension of $\phi(x)$. We now wish to solve the PDE:

$$\begin{cases} u_t - u_{xx} = (f - h_t) \\ u(x, 0) = \Phi(x) \end{cases}$$

We know that this will be solved by

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\Phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s)(f - h - t)(y, s)dyds,$$

Where $S(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}$. We now set $v = u + h(t)$, and restrict to the half line to get the desired solution.

Problem 5.

(a) Define the energy functional $E(t) = \frac{1}{2}\int_{\mathbb{R}} |u_t|^2 dx + K(t)$. We wish to determine $K(t)$ so that $\dot{E}(t) = 0$. We see that:

$$\begin{aligned}
-\dot{K}(t) &= \int u_t u_{tt} \\
&= \int u_t(c^2 u_{xx} - m^2 u)dx \\
&= c^2 \int u_t u_{xx} dx - m^2 \int u_t u dx \\
&= c^2 u_x u_t \Big|_{-\infty}^{\infty} - c^2 \int u_{tx} u_x dx - m^2 \int u_t u dx && \text{(integrating by parts)} \\
&= -c^2 \int u_{tx} u_x dx - m^2 \int u_t u
\end{aligned}$$

Therefore $K(t) = \frac{c^2}{2}\int u_x^2 + \frac{m^2}{2}\int u^2 dx$. By construction this choice of $E(t)$ will be conserved for solutions of the klein gordon equation.

(b) Suppose that u_1, u_2 are two solutions with identical initial conditions, $\phi(x), \psi(x)$. Using conservation of energy on $u_1 - u_2$, we see:

$$E(0) = \frac{1}{2}\int |u_1(x,0) - u_2(x,0)|^2 dx + \frac{c^2}{2}\int |[u_1(x,0) - u_2(x,0)]_t|^2 dx + \frac{m^2}{2}\int [u_1(x,0) - u_2(x,0)]^2 dx = 0 = E(t).$$

Since $E(t) = 0$ for all t, we have that $u_1 = u_2$.