

5.5.56: Note that the smallest closed subspace that contains E is by definition $\overline{\text{span}(E)}$. We claim that $\overline{\text{span}(E)} = E^{\perp\perp}$. If $v \in E^{\perp\perp}$ then for all $u \in E^\perp$, $\langle v, u \rangle = 0$. Therefore $v \in \text{span}(E)$ and so it belongs to the closure. Now suppose that $v \in \text{span}(E)$. Let $\{v_n\}$ be a sequence in $\text{span}(E)$ converging to v in the closure. We have that for all $u \in E^\perp$, $\langle v_n, u \rangle = 0$. Since the inner product is continuous, we have that $\langle v, u \rangle = 0$. Thus $v \in E^{\perp\perp}$.