Q4a: Note that for any for any $\lambda \in (0,1), x_1 < x_3$, let $x_2 = (1-\lambda)x_1 + \lambda x_3$. By convexity we have that

$$f(x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_3) = f(x_1) - \lambda f(x_1) + \lambda f(x_3).$$

Rearranging yields

$$f(x_2) - f(x_1) \leq \lambda[f(x_3) - f(x_1)].$$

Since

$$\lambda = \frac{x_2 - x_1}{x_3 - x_1},$$

we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1},$$

As desired.

Q4b: Let $[a,b]$ be an interval. Let $x, y \in [a,b], x < y$. For any $c < a, b < d$ by convexity we have that

$$\frac{f(x) - f(c)}{x - c} \leq \frac{f(x) - f(y)}{x - y} \leq \frac{f(y) - f(d)}{y - d}.$$

Therefore if we take $L = \max\left\{\left|\frac{f(y)-f(d)}{y-d}\right|, \left|\frac{f(x)-f(c)}{x-c}\right|\right\}$, we get that

$$\Big|\frac{f(x) - f(y)}{x - y}\Big| \leq L$$

i.e. $f$ is locally lipschitz. We now claim that $f$ is differentiable a.e. with respect to $m$. It is enough to show that it is absolutely continuous, then the claim follows from Folland Theorem 3.35. Let $\{(a_i, b_i)\}$ be a finite collection of intervals. Let $L$ be the maximum lipschitz constant on $(a_i, b_i)$. Let $\varepsilon > 0$. Take $\delta < \frac{\varepsilon}{L}$. Then we have that

$$\sum_i^N |b_i - a_i| < \frac{\varepsilon}{L} \implies \sum_{i=1}^N L|b_i - a_i| < \varepsilon \implies \sum_i^N |f(b_i) - f(a_i)| < \varepsilon\,.$$

Hence $f$ absolutely continuous and hence differentiable almost everywhere. Note that for any $x < y$ and a sufficiently small $h$, we have that

$$\frac{f(x+h) - f(x)}{h} \leq \frac{f(x) - f(y)}{x - y}.$$

This implies that $f$ is right differentiable at every $x$. By $4b$ this is an increasing function hence differentiable almost everywhere. Convexity also implies that

$$\frac{f(x+h) - f(x)}{h} \leq \frac{f(y+h) - f(y)}{h},$$

so we have that $f'(x+)$ is increasing.

Q4c: We claim that $f$ convex is twice differentiable. Let $g(x) = f'(x+)$. We evaluate that

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{f'((x+h)+) - f'(x+)}{h} = \lim_{h\to 0^+} \frac{f'(x+h) - f'(x)}{h} = f''(x)$$

We now claim that the limit

$$\lim_{h\to\infty} \frac{2}{h^2}(f(x+h) - f(x) - f'(x)h)$$

exists. Using the fundamental theorem of calculus, we write

$$f(x+h) - f(x) - f'(x)h = \int_x^{x+h} f'(t) - f'(x)dt = \int_x^{x+h} g'(t) - g'(x)dt.$$

Since $g$ is differentiable, we can take $\varepsilon > 0, \delta > 0$ such that $\frac{g(x+h)-g(x)}{h} \in (g'(x) - \varepsilon, g'(x) + \varepsilon)$ for $h < \delta$. Equivalently, we have that

$$g(x+h) - g(x) \in (h(g'(x) - \varepsilon), h(g'(x) + \varepsilon)),$$

or

$$h(g'(x) - \varepsilon) < g(x+h) - g(x) < h(g'(x) + \varepsilon).$$

Therefore

$$\int_x^{x+h} (g'(x) - \varepsilon)(t - x)dt < \int_x^{x+h} g(t) - g(x)dt < \int_x^{x+h} (g'(x) + \varepsilon)(t - x)dt.$$

Which yields

$$\frac{h^2}{2}(g'(x) - \varepsilon) < \int_x^{x+h} g(t) - g(x) < \frac{h^2}{2}(g'(x) + \varepsilon)$$

Therefore

$$\lim_{h \to 0} \frac{2}{h^2}(f(x+h) - f(x) - f'(x)h) = \lim_{h \to 0} \frac{2}{h^2} \cdot \frac{h^2}{2} g'(x) = g'(x) = f''(x)$$