Q2a: By question 1, it is enough to show that V is a $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^{2n}$. By Assignment 1, Q1, it is equivalent to finding a $\mathcal{C}^\infty$ function f that is 0 exactly on a neighborhood of a point $a \in V$ intersected with V. Note that points in V satisfy

$$|(x,y)|^2 = |x|^2 + 2\langle x, y\rangle + |y|^2 = 2.$$

Define the function $f : \mathbb{R}^n_x \times \mathbb{R}^n_y \to \mathbb{R}^3$ as

$$f(x,y) = (\langle x, y\rangle, |x|^2 - 1, |y|^2 - 1).$$

We see that for any $a \in V$, and any neighborhood U of a, $f^{-1}(\{0\}) = U \cap M$ since f is defined to be 0 exactly on V. As well note that $f \in \mathcal{C}^\infty$ since arithmetic is smooth. We compute the jacobian of f as

$$f'(x,y) = \begin{bmatrix} y_1 & \cdots & y_n & x_1 & \cdots & x_n \\ 2x_1 & \cdots & 2x_n & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 2y_1 & \cdots & 2y_n \end{bmatrix}$$

We observe that this matrix has a rank of 3, since the rows are linearly independant in $\mathbb{R}^{2n}$. Therefore V is a submanifold of $\mathbb{R}^{2n}$ and hence a $\mathcal{C}^\infty$ manifold.

Q2b: Using the analysis from A1Q1, we have that V is a submanifold of dimension $2n - 3$.

Q2c: We write $z_k = u_k + iv_k$, and we identify $\mathbb{C}^n$ with $\mathbb{R}^n_u \times \mathbb{R}^n_v$. We can rewrite our given constraints as

$$0 = \sum_k z_k^2 \iff 0 = \sum_k u_k^2 + 2iu_kv_k - v_k^2 \iff \sum_k u_kv_k = 0 \text{ and } \sum_k u_k^2 = \sum_k v_k^2,$$

and

$$1 = \sum_k |z_k|^2 \iff 1 = \sum_k u_k^2 + \sum_k v_k^2$$

We get that $\sum_k u_k^2 = \sum_k v_k^2 = \frac{1}{2}$. Therefore $W = \{(u,v) \in \mathbb{R}^n_u \times \mathbb{R}^n_v : \langle u, v\rangle = 0, |u|^2 = |v|^2 = \frac{1}{2}\}$. Similarly to $2a$, we have that W is the 0 level set of the function

$$g(u,v) = \left(\langle u, v\rangle, |u|^2 - \frac{1}{2}, |v|^2 - \frac{1}{2}\right).$$

The proof that W is a $\mathcal{C}^\infty$ manifold is idential to f as defined in $2a$ moduluo variable names. Therefore W is a $\mathcal{C}^\infty$ manifold. Define the map $h : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as $h(x,y) = (\sqrt{2}x, \sqrt{2}y)$. We claim that h is a diffeomorphism from W to V. First note that h is an invertible linear mapping, hence it is a homeomorphism. Furthermore as a map from $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ it is $\mathcal{C}^\infty$ since it is just scaling. Next, note that

$$h(W) = \{(\sqrt{2}u, \sqrt{2}v) \in \mathbb{R}^n_u \times \mathbb{R}^n_v : \langle u, v\rangle = 0, |u|^2 = |v|^2 = 1\}.$$

So we have that h is a smooth bijective mapping from W to V. Finally it remains to show that it is a diffeomorphism. By question 6, f is a diffeomorphism if and only for any coordinate chart ψ, $\psi \circ f$ is a coordinate chart of V. Since f gives a bijection between the open sets in W and V, for any ψ, $\psi \circ f$ is a coordinate chart on W. Similarly, if $\psi \circ f$ is a coordinate chart on W, then $\psi \circ f \circ f^{-1} = \psi$ is a coordinate chart on V.