Q3i: Any vector in V is of the form $v = \lambda(e_1 + \cdots + e_n)$ for some $\lambda \in \mathbb{C}$. Applying any permutation $\sigma \in S_n$ we have that

$$\sigma \cdot v = \lambda(e_{\sigma(1)} + \cdots + e_{\sigma(n)}) = v.$$

Therefore V is invariant under $\mathbb{C}[S_n]$ if we extend by linearity over $\mathbb{C}$.

Q3ii: Suppose $w \in W$. Then if for any permutation $\sigma \in S_n$, we have that

$$\sigma \cdot w = \sum_i^n w_i e_{\sigma(i)}.$$

So $\sigma \cdot w \in W$ since this only relabels the basis vectors but does not adjust the coefficients. When we extend by linearity since W is a subspace we still remain in W.

Q3iii: We claim that $\mathbb{C}^n = V \oplus W$. Observe that if $v \in W \cap V$, then $v = \lambda(e_1 + \ldots e_n)$ and $n\lambda = 0$ i.e. $\lambda = 0$. We now claim that any $u \in V$ can be written in the form $u = v + w$ for $v \in V$, $w \in W$. In the basis $\{e_1, \ldots, e_n\}$ we write $u = u_1e_1 + \cdots + u_ne_n$. We see that by taking $\lambda = \frac{1}{n}\sum_i^n u_i$, and $w_i = u_i - \lambda$, we see

$$u = \lambda(e_1 + \cdots + e_n) + (u_1 - \lambda)e_1 + \cdots + (u_n - \lambda)e_n.$$

The coefficients on our choice of W work since

$$\sum_{i=1}^n (u_i - \lambda) = \sum_{i=1}^n u_i - n\lambda = 0.$$