

Q3a: If we let $A_k = \bigcap_{n \geq k} E_n$ it is clear that $A_1 \subset A_2 \dots$. Hence by measure continuity, we get that

$$\mu\left(\bigcup_{k \geq 1} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k) = \lim_{k \rightarrow \infty} \mu\left(\bigcap_{n \geq k} E_n\right)$$

However from the properties of the measure, namely $A \subset B$ implies that $\mu(A) \leq \mu(B)$, we can deduce that for any k ,

$$\mu\left(\bigcap_{n \geq k} E_n\right) \leq \inf_{n \geq k} \mu(E_n)$$

Since measure continuity holds, by applying limits we see that

$$\liminf_n E_n = \mu\left(\bigcap_{k \geq 1} \bigcup_{n \geq k} E_n\right) = \lim_{k \rightarrow \infty} \mu\left(\bigcap_{n \geq k} E_n\right) \leq \lim_{k \rightarrow \infty} \inf_{n \geq k} \mu(E_n) = \liminf_n \mu(E_n) \quad \blacksquare$$

Q3b: If we define $A_k = \bigcup_{n \geq k} E_k$ we see that $A_1 \supset A_2 \dots$, and $\mu(A_1) < \infty$ as given. Hence we can apply measure continuity to get that

$$\mu\left(\bigcap_{k \geq 1} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k) = \lim_{k \rightarrow \infty} \mu\left(\bigcup_{n \geq k} E_n\right)$$

Similarly to 3a, we can reason that

$$\mu\left(\bigcup_{n \geq k} E_n\right) \geq \sup_{n \geq k} \mu(E_n)$$

Since measure continuity holds we can apply limits and conclude that

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{k \geq 1} \bigcup_{n \geq k} E_n\right) = \lim_{k \rightarrow \infty} \mu\left(\bigcup_{n \geq k} E_n\right) \geq \lim_{k \rightarrow \infty} \sup_{n \geq k} \mu(E_n) = \limsup_n \mu(E_n) \quad \blacksquare$$

If we did not have the hypothesis that $\mu(\bigcup_{n=1}^{\infty} E_n)$ is finite this result would not hold. Consider the collection $\{E_n\}$ with $E_n = (-n, n]$. We see that $\mu(\limsup_n E_n) = \mu(E_1) = 2$ but $\limsup_n \mu(E_n) = \infty$. It is certainly false that $2 \geq \infty$.