Q1a: To show that α is a ring isomorphism, we will show that it is a group homomorphism with respect to polynomial addition, then show it maps the identity to the identity, and that it respects multiplication in the ring. Observe

$$\begin{aligned}
\alpha([a(x)]_{f(x)g(x)} + [b(x)]_{f(x)g(x)}) &= \alpha([a(x) + b(x)]_{f(x)g(x)}) \\
&= ([a(x) + b(x)]_{f(x)}, [a(x) + b(x)]_{g(x)}) \\
&= ([a(x)]_{f(x)} + [b(x)]_{f(x)}, [a(x)]_{g(x)} + [b(x)]_{g(x)}) \\
&= ([a(x)]_{f(x)}, [a(x)]_{g(x)}) + ([b(x)]_{f(x)}, [b(x)]_{g(x)}) \\
&= \alpha([a(x)]_{f(x)g(x)}) + \alpha([b(x)]_{f(x)g(x)})
\end{aligned}$$

We now show it respects the identity element;

$$\alpha([1]_{f(x)g(x)}) = ([1]_{f(x)}, 1_{g(x)})$$

Which is the identity element in the product ring. Finally we show that the mapping α respects products.

$$\begin{aligned}
\alpha([a(x)]_{f(x)g(x)} \cdot [b(x)]_{f(x)g(x)}) &= \alpha([a(x) \cdot b(x)]_{f(x)g(x)}) \\
&= ([a(x) \cdot b(x)]_{f(x)}, [a(x) \cdot b(x)]_{g(x)}) \\
&= ([a(x)]_{f(x)} \cdot [b(x)]_{f(x)}, [a(x)]_{g(x)} \cdot [b(x)]_{g(x)}) \\
&= ([a(x)]_{f(x)}, [a(x)]_{g(x)}) \cdot ([b(x)]_{f(x)}, [b(x)]_{g(x)})]) \\
&= \alpha([a(x)]_{f(x)g(x)}) \cdot \alpha([b(x)]_{f(x)g(x)})
\end{aligned}$$

As desired. Finally we will show that it is a bijection. We claim that the domain and codomain have the same carinality. Indeed,

$$\begin{aligned}
|\,\mathbb{F}_p(x)/f(x)g(x)\,\mathbb{F}_p(x)| &= p^{deg(f(x)g(x))} && \text{(A4 Q3b)} \\
&= p^{deg(f(x))+deg(g(x))} && \text{(by properties of polynomials)} \\
&= p^{deg(f(x))} \cdot p^{deg(g(x))} \\
&= |\,\mathbb{F}_p(x)/f(x)\,\mathbb{F}_p(x)| \cdot |\,\mathbb{F}_p(x)/g(x)\,\mathbb{F}_p(x)|
\end{aligned}$$

Thus it suffices to show that α is a ring injection. Suppose that $\alpha([a(x)]_{f(x)g(x)}) = \alpha([b(x)]_{f(x)g(x)})$. This implies that $[a(x)_{f(x)}] = [b(x)_{f(x)}]$ and $[a(x)_{g(x)}] = [b(x)_{g(x)}]$ Therefore, $f(x)|a(x)-b(x)$ and $g(x)|a(x)-b(x)$. Since $f(x)$ and $g(x)$ are coprime, we have that $f(x)g(x)|a(x)-b(x)$. Therefore, $[a(x)]_{f(x)g(x)} = [b(x)]_{f(x)g(x)}$, and we conclude that α is an injection.

Q1b: Since the polynomaisl $f(x), g(x)$ are coprime, there exists $z(x), y(x)$ such that $z(x)f(x)+y(x)g(x) = 1$. We see that $y(x)g(x) = 1 - z(x)f(x)$ or equivalently, $[y(x)g(x)]_{f(x)} = [1]_{f(x)}$. By almost exactly the same argument we have that $[z(x)f(x)]_g(x) = [1]_{g(x)}$

Q1c: Let $c(x) = a(x)y(x)g(x) + b(x)z(x)f(x)$. We can verify that

$$\begin{aligned}
[c(x)]_{f(x)} &= [a(x)y(x)g(x) + b(x)z(x)f(x)]_{f(x)} \\
&= [a(x)]_f(x) \cdot [y(x)g(x)]_f(x) + [b(x)z(x)f(x)]_{f(x)} \\
&= [a(x)]_{f(x)}
\end{aligned}$$

By almost the exact same computation we can verify that $[c(x)]_{g(x)} = [b(x)]_{g(x)}$