Q6:

Let $A \subset \mathbb{R}^n$, not closed. Let x be a point such that $x \in \mathbb{R}^n \setminus A$ and $x \notin \operatorname{int} \mathbb{R}^n \setminus A$. Consider $f(y) = \frac{1}{\|y-x\|}$. We want to show that this function is unbounded and continuous. We being by showing that f is continuous. Notice that if $g(y) = \|x - y\|$ and $h(z) = \frac{1}{z}$, then $f = h \circ g$. It is sufficient to show that g is continous and never 0. We will use the following fact to prove that g is continous
Claim: $| \, \|x\| - \|y\| \, | \leq \|x - y\|$
pf: since both quantities are positive we can square them

$$\begin{aligned}
&(\|x\| - \|y\|)^2 \leq \|x - y\|^2 \\
&\iff \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \leq \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle \\
&\iff -2\|x\|\|y\| \leq -2\langle x, y\rangle \\
&\iff \langle x, y\rangle \leq \|x\|\|y\| \text{ which is true by cauchy- shwartz} \quad \blacksquare
\end{aligned}$$

Now we show that g is continous. Let $\epsilon > 0$. Let $\delta = \epsilon$. Take $y, z \in A$. Then,

$$\begin{aligned}
&\|z - y\| < \epsilon \\
&\implies \|z - x - y + x\| < \epsilon \\
&\implies \|(z - x) - (y - x)\| < \epsilon \\
&\implies | \, \|z - x\| - \|y - x\| \, | \leq \|(z - x) - (y - x)\| < \epsilon \text{ (by claim)}
\end{aligned}$$

Therefore g is continous. Now h will be continous since g will never be 0, since $g(y) = 0 \iff \|y - x\| = 0 \iff y = x$, but x is not in A. As the composition of two continous functions, f is continous as well. Now we show that f is unbounded. First, note that the point x must be in the boundary of A, since it is not in the exteriour of A, and is not in A so it can not be in the interiour. Therefore, for all $\epsilon > 0$, $B_\epsilon(x)$ will contain at least one point $z \in A$. Choose $\epsilon > 0$. Suppose that this f is bounded. There must exist some M with $f(y) \leq |M|$ for all $y \in A$. So we see that

$$\begin{aligned}
&f(y) \leq |M| \\
&\iff \frac{1}{\|y - x\|} \leq |M| \\
&\iff \frac{1}{|M|} \leq \|y - x\|, \text{ for all } y \in A
\end{aligned}$$

However, we can choose any $\epsilon > 0$ and find a $y \in A$ where $\|y - x\| < \epsilon$. Choosing $\epsilon = \frac{2}{|M|}$ implies that $\frac{1}{|M|} \leq \|y - x\| < \frac{2}{|M|}$. This is a contradiction, since no such positive number M exists where $\frac{1}{|M|} < \frac{2}{|M|}$. Thus f is not bounded on A. $\blacksquare$