Problem 1.

(a) By proposition 3.6 we have that

$$\langle J(s), \gamma'(s)\rangle = \langle J'(0), \gamma'(0)\rangle s + \langle J(0), \gamma'(0)\rangle.$$

Since $\langle J'(0), \gamma'(0)\rangle = 0$, and $J'(0) = 0$ we have that $\langle J(s), \gamma'(s)\rangle = 0$.

(b) We first claim that any geodesic γ through $p = 0$ must be a meridian. Note that meridians are preserved under rotation, which is an isometry. Since isometries preserve geodesics, any geodesic with a tangent vector tangent to a meridian must be a meridian. Since meridians gro through 0 we have that the geodesics that travel through 0 are meridians. Suppose that p is conjugate to 0. There exists a geodesic γ attaining p at t_0 and a Jacobi vector field J so that $J(0) = J(t_0) = 0$. By the proof of proposition 2.4, we can write

$$J(t) = \frac{\partial f}{\partial s}(t, 0)$$

where $f(s,t) = (t\cos\theta(s), t\sin\theta(s), t^2)$, for some smooth $\theta(s)$. We have that $J(t_0) = (0,0,0)$, so

$$(0,0,0) = J(t_0) = \frac{\partial f}{\partial s}(t_0, 0) = (-t_0 \cdot \theta'(0) \cdot \sin\theta(0), t_0 \cdot \theta'(0)\cos\theta(0), 0).$$

Since sin and cos cannot vanish at the same time we have that $\theta'(0) = 0$. However this equation defines $J(t)$ for all choices of t, so we must have that $J(t) \equiv 0$.

Problem 2.

By corollary 2.5, we can write
$$J(t) = \left(d\exp_p\right)_{t\gamma'(0)}(tJ'(0))$$
for any jacobian field J. Furthermore, since M has 0 scalar curvature, we have that
$$J(t) = tw(t)$$
where $J'(0) = w(0)$, and w is a parallel unit vector field along γ. Take a normal ball $B_\varepsilon(0)$ of p. For any vectors v, w so that $J_1'(0) = v$, $J_2'(0) = w$, using the cor. and lemma, we write
$$\left\langle \left(d\exp_p\right)_{t\gamma'(0)}(tJ_1'(0)), \left(d\exp_p\right)_{s\gamma'(0)}(sJ_2'(0))\right\rangle = \langle J_1(t), J_2(s)\rangle = \langle tw_1(t), sw_2(s)\rangle.$$
We use linearity of the differential and divide out $t \cdot s$ from both sides to get that:
$$\left\langle \left(d\exp_p\right)_{t\gamma'(0)}(J_1'(0)), \left(d\exp_p\right)_{s\gamma'(0)}(J_2'(0))\right\rangle = \langle w_1(t), w_2(s)\rangle.$$
At $t = s = 0$, we have that
$$\left\langle \left(d\exp_p\right)(v), \left(d\exp_p\right)(w)\right\rangle = \langle w_1(0), w_2(0)\rangle = \langle v, w\rangle.$$

Problem 3.

Suppose that $N \subset K \subset M$, with N totally geodesic in K, and K totally geodesic in M. By prop. 9 it is sufficient to show that a geodesic in N is also a geodesic in M. Let γ be a geodesic at $p \in N$. Then since N is totally geodesic in K we have that γ is also a geodesic at p in K. Since K is totally geodesic in M, we have that γ is a geodesic in M at p. Our choice of p and γ was arbitrary so we have that N is totally geodesic in M.

Problem 4.

(a) We first compute the differential of x. We get that

$$dx(\theta,\phi)=\frac{1}{\sqrt{2}}\begin{bmatrix}-\sin\theta & 0\\ \cos\theta & 0\\ 0 & -\sin\phi\\ 0 & \cos\phi\end{bmatrix}.$$

We see that $dx(1,0)=\frac{1}{\sqrt{2}}e_1, dx(0,1)=\frac{1}{\sqrt{2}}e_2$. Therefore e_1, e_2 belong to the tangent space. We see that they are orthonormal since

$$\langle e_i, e_j\rangle=\delta_{ij}.$$

It remains to check that n_1, n_2 form am orthonormal basis for the normal space. We see that

$$\langle n_i, n_j\rangle=\delta_{ij},$$

as well as

$$\langle n_i, e_j\rangle=0.$$

(b) By prop. 2.3, we can write

$$\langle S_{n_k}(e_i), e_j\rangle=-\langle\overline{\nabla}_{e_i}n_k, e_j\rangle=\langle\overline{\nabla}_{e_i}e_j, n_k\rangle.$$

For a fixed choice of n_k computing the above for each e_i, e_j will give us the entries of the matrix of S in the basis of $\{e_1, e_2\}$. Since in $\mathbb{R}^n$ the covariant derivative agrees exactly with the usual derivative, we have that

$$\overline{\nabla}_{e_1}e_1=(-\cos\theta,-\sin,0,0),\overline{\nabla}_{e_2}e_2=(0,0,-\cos\phi,-\sin\phi).$$

For n_1, we compute that

$$\langle\overline{\nabla}_{e_1}e_1, n_1\rangle=\frac{\sqrt{2}}{\sqrt{2}}\cdot-1=-1=\langle\overline{\nabla}_{e_2}e_2, n_1\rangle,$$

and

$$\langle\overline{\nabla}_{e_1}e_2, n_1\rangle=0=\langle\overline{\nabla}_{e_2}e_1, n_1\rangle.$$

Therefore

$$S_{n_1}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}.$$

For n_2, we compute that

$$\langle\overline{\nabla}_{e_1}e_2, n_2\rangle=0=\langle\overline{\nabla}_{e_2}e_1, n_2\rangle,$$

and

$$\langle\overline{\nabla}_{e_1}e_1, n_2\rangle=1,\langle\overline{\nabla}_{e_2}e_2, n_2\rangle=-1.$$

Therefore

$$S_{n_2}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

(c) We claim that n_1 is in the tangent space of S^3. Since $x(\theta,\phi)=n_1\in S^3$, we have that $n_1\in T_pS^3$. Therefore n_2 spans $(T_pS^3)^\perp$. Any $\eta\in(T_pS^3)^\perp$ must satisfy $\eta=\alpha n_2$. Therefore by construction of S_η by prop. 2.3, $S_\eta=\alpha S_{n_2}$, and since S_{n_2} is traceless so must be S_η.

Problem 5.

(a) Consider $S^1 \hookrightarrow \mathbb{R}^2$. We have that the geodesics from x to y on S^1 are given by travelling along the arc between x, y. So $d_M(x,y) = \min(\arg(x) - \arg(y), \arg(y) - \arg(x))$. However, $d_N(x,y) = |x - y|$. We have that $d_M > d_N$, even though $\iota : S^1 \to \mathbb{R}^2$ is an isometric immersion.

(b) We first give the following riemannian structure to $\widetilde{M}$. For $\tilde{p} \in \widetilde{M}$, we define

$$\langle v, w \rangle_{\tilde{p}} = \langle d\pi_p(v), d\pi_p(w) \rangle_{\pi(\tilde{p})}.$$

This will be a local isometry exactly when π is a diffeomorphism, so as long as we take a sufficiently small open neighbourhood of $\tilde{p}$. We now claim that $\widetilde{M}$ is complete in this metric if and only if M is complete. Suppose that M is complete. Let $\gamma : [0,1] \to M$ be a geodesic. We know from topology that we can lift γ to a path $\tilde{\gamma} : [0,1] \to \widetilde{M}$ satisfying $\pi \circ \tilde{\gamma} = \gamma$. Since γ is locally length minimizing, and π is locally an isometry we have that $\tilde{\gamma}$ is also locally length minimizing and hence a geodesic. We can extend γ for all $t \in \mathbb{R}$, corresponding to extending $\tilde{\gamma}$ while still remaining a geodesic. Thus $\widetilde{M}$ is complete. Conversely suppose that $\widetilde{M}$ is complete with respect to its metric. Let $\tilde{\gamma}$ be a geodesic defined for all $t \in \mathbb{R}$. Then the path $\pi \circ \tilde{\gamma} = \gamma$ is defined for all t. We claim that γ is a geodesic. Since π locally preserves distances, and $\tilde{\gamma}$ locally minimizes lengths, we have that γ locally minimizes lengths. Therefore γ is a geodesic and M is complete.

(c) Let M_1 be the open upper hemisphere of S^2, $M_2 = S^2$. Let $f = \iota$ the inclusion map of M_1 in M_2. If we endow M_1 with the restriction of any metric on M_2, then f is an isometry. Since M_2 is compact it must be complete by corr. 2.9. Note however that M_1 is extendable, in particular it can be extended to M_2. By the contrapositive of prop. 2.3, M_1 is not complete.