

Q1: By monotonicity, it is sufficient to show that

$$m^*(E) \geq \inf \left\{ \sum_{n=1}^{\infty} m^*(B_i) : B_i \text{ ball, } E \subset \bigcup_{i=1}^{\infty} B_i \right\}.$$

Furthermore, it is sufficient to show that this inequality holds on open sets by Folland Theorem 2.40. Let U be some open set with finite measure. Take an open covering $\{B_i\}$ of U by balls such that $B_i \subset U$ for each i . Define the collection of balls \mathcal{C} to be the set of all balls that are contained in some B_i . This is going to be a Vitali covering. By Vitali's Covering theorem, for all $\varepsilon > 0$ there exists a finite collection $\{B_j\}_{j=1}^N$ with

$$m(U \setminus \bigcup_{j=1}^N B_j) < \varepsilon.$$

We now claim that $U \subset \bigcup_{j=1}^N \overline{B_j}$. If not, then there is some $x \in U \setminus \bigcup_{j=1}^N \overline{B_j}$. Elementary topology tells us that $U \setminus \bigcup_{j=1}^N \overline{B_j}$ is open, hence we can find some ball $B_x \ni x$. This is a contradiction to the Vitali covering theorem, since B_x can not have positive measure. The last claim we make is that for any open ball $m(B) = m(\overline{B})$. Since any \overline{B} is the intersection of open balls containing it, downward measure continuity implies that the measures are equal.