Q5a:
We first begin by finding an upper bound for $\frac{\|f(h,k)\|}{\|h,k\|}$

$$\begin{aligned}
&\frac{\|f(h,k)\|}{\|(h,k)\|}\\
&= \frac{\|f(\sum_{i=1}^n h_i e_i, k)\|}{\|(h,k)\|} \text{ (expressing h as the sum of its componoents)}\\
&= \frac{\|\sum_{i=1}^n h_i f(e_i, k)\|}{\|(h,k)\|} \text{ (by linearity in the first slot)}\\
&\leq \frac{\|\sum_{i=1}^n h_i f(e_i, k)\|}{\|h\|} \text{ (since norm of (h,k) is at least norm of h)}\\
&\leq \frac{\sum_{i=1}^n \|h_i f(e_i, k)\|}{\|h\|} \text{ (by triangle inequality)}\\
&= \frac{\sum_{i=1}^n |h_i| \, \|f(e_i, k)\|}{\|h\|}\\
&\leq \frac{\sum_{i=1}^n \|h\| \, f(e_i, k)}{\|h\|} \text{ (since } |x_i| \leq \|x\| \text{)}\\
&= \sum_{i=1}^n \|f(e_i, k)\|
\end{aligned}$$

So we have that

$$\frac{\|f(h,k)\|}{\|h,k\|} \leq \sum_{i=1}^n f(e_i, k)$$

Applying the limit as $(h,k) \to 0$ to both sides we get that $\lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} = 0$. ■

5b:
We want to check if in fact $Df(a,b)(x,y) = f(a,y) + f(x,b)$ is the differential of f at (a,b). We can check using Spivak's definition of differntiablilty. That is if $\lim_{h\to 0} \frac{\|f(a+h)-f(a)-Dfa(h)\|}{\|h\|} = 0$ then f will be differentiable.

$$\begin{aligned}
&\lim_{(x,y)\to 0} \frac{\|f(a+x, b+y) - f(a,b) - f(a,y) - f(x,b)\|}{\|(x,y)\|}\\
&= \lim_{(x,y)\to 0} \frac{\|f(a,b) + f(a,y) + f(x,b) + f(x,y) - f(a,b) - f(a,y) - f(x,b)\|}{\|(x,y)\|} \text{ (by bilinearity of f)}\\
&= \lim_{(x,y)\to 0} \frac{\|f(x,y)\|}{\|(x,y)\|}\\
&= 0 \text{ (by 5a)}
\end{aligned}$$

Thus, f is differentiable with $Df(a,b)(x,y) = f(a,y) + f(x,b)$
5c: Let $p(x,y) = xy$. This is billinear, by the properties of multiplication of real numbers. According to the result from $5b$, $Dp(a,b)(x,y) = p(a,y) + p(x,b) = ay + bx$, which is exactly what we proved in class.