

Q3: We claim that for any isomorphism φ between two groups G, H , for all $g \in G$, $|\phi(g)| = |g|$. Let $|g| = n$. It is clear that

$$\varphi(x)^n = \varphi(x^n) = \varphi(e) = e$$

Suppose now that for some $m < n$, we have that $\varphi(x)^m = e$. Since φ is injective this implies that $x^m = e$. Contradicting the order of x . Now suppose that $n > m$. Without loss of generality, assume that m is not an integer multiple of n . We can write $m = kn + r$ for some nonzero r . We get that

$$e = \varphi(x)^m = \varphi(x^m) = \varphi(x^{kn+r}) = \varphi(x^k n) \varphi(x^r) = \varphi(x^r)$$

Once again this contradicts the minimality of n . Now if there were an isomorphism between $C_2 \times C_2$ and C_4 , it would have to preserve the order of every element in $C_2 \times C_2$. However, $C_2 \times C_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Notice that every element but e has an order of 2. However in $C_4 = \{0, 1, 2, 3\}$ we have two elements, 1, 3 which have order 4. Hence no isomorphism can exist between them.