

Q2a:

Define $K(x, y, u) = (G(x, y, u), H(x, y, u))$. We compute the derivative of K as

$$DK = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial G}{\partial u} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial u} \end{bmatrix}$$

At the point $(2, -1, 1)$, we evaluate DK as

$$DK_{(2,-1,1)} = \begin{bmatrix} \frac{\partial f(2,-1)}{\partial x} & \frac{\partial f(2,-1)}{\partial y} & 2 \\ 1 & 9 & 5 \end{bmatrix}$$

From the implicit function theorem we can find functions $g(y) = x$ and $h(y) = u$ which satisfy $K(x, y, u) = 0$ at the point $(2, -1, 1)$ when $\frac{\partial K}{\partial(x,u)}$ is invertible. That is when $\begin{bmatrix} \frac{\partial f(2,-1)}{\partial x} & 2 \\ 1 & 5 \end{bmatrix}$ is invertible. This will happen if and only if it has nonzero determinant. We have that $\text{Det}(\frac{\partial K}{\partial(x,u)}) = 5\frac{\partial f(2,-1)}{\partial x} - 2$. So long as $\frac{\partial f}{\partial x} \neq \frac{2}{5}$ this matrix will be invertible and we can find such a h and g with $g(-1) = 2$ and $g(-1) = 1$.

Q2b

Since $f'(2, 1) = (1 \quad -3)$ we have that $DK = \begin{bmatrix} 1 & -3 & 2 \\ 1 & 9 & 5 \end{bmatrix}$. Since the matrix given by $\frac{\partial K}{\partial(x,u)}$ is invertible, we can find functions h, g which satisfy $K(g(y), y, h(y)) = 0$. By the Implicit Function Theorem, we have that

$$(g, h)' = - \left[\frac{\partial K}{\partial(x, u)} \right]^{-1} \cdot \frac{\partial K}{\partial y}$$

We compute $\left[\frac{\partial K}{\partial(x, u)} \right]^{-1} = \begin{bmatrix} \frac{5}{3} & \frac{-2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix}$. From DK we see that $\frac{\partial K}{\partial y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$.

Thus

$$(g, h)' = - \begin{bmatrix} \frac{5}{3} & \frac{-2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}$$

Therefore, $g'(-1) = 11$ and $h'(-1) = -4$