Q3: We will show that any such function f which preserves norms and maps 0 to 0 is a rotation or a rotation and conjugation. By properties of f, we have that

$$|f(z) - f(1)|^2 = |z - 1|^2$$

Using the properties of the norm, we know that

$$[\overline{f(z)} - \overline{f(1)}][f(z) - f(1)] = [\overline{z} - 1][z - 1]$$

Expanding, we see that

$$|f(z)|^2 - \overline{f(z)}f(1) - \overline{f(1)}f(z) + |f(1)|^2 = |z|^2 - \overline{z} - z + 1$$

Using the distance preserving properties, we get that

$$\overline{f(z)}f(1) + \overline{f(1)}f(z) = \overline{z} + z = 2Re(z)$$

Since $|f(1)| = 1$, we can write $f(1) = e^{i\theta}$ for some θ. We get that

$$\overline{f(z)}e^{i\theta} + f(z)e^{-i\theta} = \overline{f(z)e^{-i\theta}} + f(z)e^{-i\theta} = 2Re(z)$$

If we let $z = a + ib$ We get that

$$2Re(f(z)e^{-i\theta}) = 2Re(z) = 2a$$

And so,

$$Re(f(z)e^{-i\theta}) = a = Re(z)$$

By the norm preserving property, we see that

$$Re(f(z)e^{-i\theta})^2 + Im(f(z)e^{-i\theta})^2 = a^2 + b^2$$

And thus

$$Im(f(z)e^{-i\theta}) = \pm b$$

We see that if it is the case that $Im(f(z)e^{-i\theta}) = b$, then $f(z) = e^{i\theta}(a + ib)$ and so f is a rotation by θ. If $Im(f(z)e^{-i\theta}) = -b$ then we have that $f(z) = e^{i\theta}(a - ib)$, which is exactly a rotation by θ and complex conjugation. Note that it is not the case that for some z_1 that $f(z_1) = \overline{e^{i\theta}z_1}$ but for all other z, $f(z) = e^{i\theta}z$ since f would no longer be continuous, since we can find a neighbourhood of z_1 that does not get carried to a neighbourhood of $f(z_1)$.