Q4a: We proceed by strong induction. For the case when $n = 1$ we have that

$$x - 1 = \Phi_1(x) = \Phi_1(x) \cdot \Pi_{d1,a \leq d < 1} = \Phi_1(x)$$

Now suppose the statement is true for all $k < n$. Since

$$x^k - 1 = \Phi_k(x)\Pi_{d|k,1\leq d<k}\Phi_d(x)$$

It is clear that $\Phi_k(x)|x^k - 1$. By the given fact, for each $d \in \mathbb{N}$ where $d|n$, we know that $\gcd(x^n - 1, x^d - 1) = x^d - 1$ which implies that $x^d - 1|x^n - 1$. Hence $\Phi_d(x)|x^n - 1$. We know that $\Phi_{d_1}(x)$ is coprime to $\Phi_{d_2}(x)$ for all $d_1, d_2|n$, we have that $\Pi_{d|n,1\leq d<n}\Phi_d(x)|x^n - 1$. We now define $\Phi_n(x) = \frac{x^n-1}{\Pi_{d|n,1\leq d>n}}$. We will verify that indeed $\Phi_n(x)$ is monic, primitive and $deg(\Phi_n) = \phi(n)$. We see that

$$\begin{aligned}
n = deg(x^n - 1) &= deg(\Phi_n(x)) + \sum_{d|n,1\leq d<n} deg(\Phi_d(x)) \\
&= deg(\Phi_n(x)) + \sum_{d|n,1\leq d<n} \phi(d) \\
&= deg(\Phi_n(x)) + \sum_{d|n} \phi_d(x) - \phi(n) \\
&= deg(\Phi_n(x)) + n - \phi(n)
\end{aligned}$$

Which implies that $deg(\Phi_n) = \phi(n)$. Next note that for some $a_{\phi(n)}$ we can write

$$x^n - 1 = \Phi_n(x) \cdot \Pi_{d|n,1\leq d<n}\Phi_d(x) = (a_{\phi(n)}x^{\phi(n)} + \dots) \cdot (x^{n-\phi(n)} + \dots)$$

We see that the coffiecient on x^n must be 1. Hence $\Phi_n(x)$ is monic. Finally, we see that $\Phi_n(x)$ is primitive since it has at least one coefficient equal to 1, hence the gcd over all of its coefficients is 1.

Q4b: Using the definition of $\Phi_n(x)$

$$\Phi_3(x) = x^2 + x + 1$$

$$\Phi_4(x) = \frac{x^4 - 1}{(x-1)(x+1)} = x^2 + 1$$

$$\Phi_6(x) = \frac{x^6 - 1}{(x-1)(x+1)(x^2+x+1)} = x^2 - x + 1$$

$$\Phi_8(x) = \frac{x^8 - 1}{(x-1)(x+1)(x^1+1)} = x^4 + 1$$

$$\Phi_10(x) = \frac{x^{10} - 1}{(x-1)(x+1)(x^4+x^3+x^2+x+1)} = x^4 - x^3 + x^2 - x + 1$$

4c: We see that the roots to $\Phi_3(x)$ take the form of $x = \frac{-1\pm\sqrt{1-4}}{2}$. This is not in $\mathbb{Z}$. Similarly, for $\Phi_4(x)$ the roots must be of the form $x = \frac{\pm\sqrt{-4}}{2} \notin \mathbb{Z}$. Finally the roots of $\Phi_6(x)$ must be of the form $\frac{1\pm\sqrt{1-4}}{2}$. Once again this is not in $\mathbb{Z}$. We conclude that these polynomials are irreducible.