Q6a: Let $f(z) = \frac{az-b}{cz-d}$ be a fractional linear transformation which maps the upper half plane H to the unit disk. Since f is a homeomorphism, we have that $f(\infty) = \frac{a}{c} \in S^1$, since $\infty \in \mathbb{R}$ when we consider the riemann sphere, and homeomorphisms preserve boundaries. Hence we have that

$$\left|\frac{a}{c}\right| = 1$$

which implies that

$$|a| = |c|$$

Letting $\frac{a}{c} = \eta$, we can rewrite f as

$$f(z) = \eta \frac{z - \frac{b}{a}}{z - \frac{d}{c}}$$

Let $\frac{b}{a} = w$. We claim that $w \in H$. Note that $f : \mathbb{C} \to \mathbb{C}$ is a bijection, no point below the real line can map to the disk. Therefore the unique z' satisfying $0 = f(z')$ must belong to H. By inspecting f we see that $z' = w$. Thus we have that

$$f(z) = \eta \frac{z - w}{z - \frac{d}{c}}$$

. We finally claim that $\overline{w} = \frac{d}{c}$. We define $\frac{d}{c} = u$. Hence we have that

$$f(z) = \eta \frac{z - w}{z - u}$$

Since f is a homeomorphism between H and D, the boundary of H gets mapped to S^1. Hence we have that for all $z \in \mathbb{R}$, $|f(z)| = 1$. We get that

$$1 = |f(z)| = |\eta| \left|\frac{z - w}{z - u}\right| = \frac{|z - w|}{|z - u|}$$

Using the properties of the norm, we get that

$$1 = \frac{(z - w)(\overline{z} - \overline{w})}{(z - u)(\overline{z} - \overline{u})} \implies \frac{z\overline{z} - w\overline{z} - \overline{w}z + w\overline{w}}{\overline{z}z - \overline{u}z - u\overline{z} + u\overline{u}} \implies w\overline{w} - u\overline{u} = z(\overline{w} - \overline{u}) + \overline{z}(w - u)$$

Since this is true for all $z \in \mathbb{R}$, taking $z = 0$ implies that $|w| = |u|$. Next taking $z = 1$, we get that

$$0 = (\overline{w} - \overline{u} + w - u) \implies w + \overline{w} = u + \overline{u}$$

Hence we have that $Re(w) = Re(u)$. Since their norms are equal, we get that $Im(w) = \pm Im(u)$ We can not have that $Im(w) = Im(u)$ since this function would be constant, thus we conclude that $Im(w) = -Im(u)$. Therefore, $u = \overline{w}$. Hence f can be written as

$$f(z) = \eta \frac{z - w}{z - \overline{w}}$$

Conversely, suppose that $f(z) = \eta \frac{z-a}{z-\overline{a}}$ with $|\eta| = 1$, and $Im(a) > 0$. We wish to show that on the upper half plane H, $f(z) \leq 1$. Suppose for the sake of contradiction that for some $z \in H$, $|f(z)| \geq 1$. We see that

$$1 \leq |f(z)| = |\eta| \left|\frac{z - a}{z - \overline{a}}\right| \implies |z - \overline{a}| \leq |z - a|$$

Using the properties of the modulus of a complex number, we get that

$$(z - \overline{a})(\overline{z} - a) \leq (z - a)(\overline{z} - \overline{a}) \implies z\overline{z} - \overline{z}a - \overline{za} + \overline{a}a \leq \overline{z}z - z\overline{a} - \overline{z}a + a\overline{a}$$

Cleaning up this expression, we see that

$$0 \leq \overline{z}(a - a) + z(a - \overline{a}) \implies 0 \leq z \cdot 2iIm(a)$$

However, we note that from algebraic proprties of $\mathbb{C}$, that this inequality is only met when $Re(z) = 0$ and $Im(z) \leq 0$. We obtain a contradiction.

Q6b: Note that since f is a bijection on $\mathbb{C}$, and it maps the upper half plane onto itself we can deduce that it is a bijection on the upper half plane. Similarly by conjugating, we get the equivalent result on the lower half plane. Therefore f is a bijection on the upper half plane, the lower half plane, and on $\mathbb{C}$. Therefore it is a bijection on $\mathbb{R}$. Hence we apply our result from Q5