5.5.66a: Suppose $\{f_n\}$ is a sequence in M so that $f_n \to f$. Furthermore let T be the map which includes M into $C([0,1])$. Furthermore suppose that $Tf_n \to g$. We claim that $g = Tf$. Since M is a closed subspace, we have that $f_n \to f$ in M. We see that

$$|Tf - g| \leq |Tf - Tf_n| + |Tf_n - g|.$$

We know $Tf_n \to g$, and if $f_n \to f$ in L^2 then $f_n \to f$ uniformly. Therefore by the close graph theorem, T is continous hence bounded, and we have that $|f|_u \leq C|f|_2$ for some C.

5.5.66b: Note that the assignment map $\hat{x}(f) : f \mapsto f(x)$. We have that

$$|\hat{x}| \leq |f|_u \leq C|f|_2.$$

So $\hat{x}(f) = \langle f, g_x \rangle$ for some g_x. Furthermore we see that

$$g_x(x) = \langle g_x, g_x \rangle \leq |g_x|_u \leq C|g|_x,$$

So we must have that $|g_x|_2 \leq C$.

5.5.66c: Let $\{f_i\}$ be an orthonormal sequence of vectors in M. Therefore,

$$\sum_i |f_i(x)|^2 = \sum_i |\langle f_i, g_x \rangle|^2 = \sum_i |\langle g_x, f_i \rangle| \leq |g_x|^2 \leq C^2.$$

But also, since $|f_i| = 1$ there is at most C^2 linearly independant $f_i's$.