Q2i: First suppose that I is principal. Then for some $a+b\sqrt{-5}$, we have $I=(a+b\sqrt{-5})$. Since $2\in I$ for some $\alpha\in\mathcal{R}$, $\alpha(a+b\sqrt{-5})=2$. Taking the usual norm we get

$$4=N(\alpha)(a^2+5b^2).$$

Since both sides are strictly positive we must have that $a^2+5b^2=1$ or 2. If $a^2+5b^2=1$ then the only solution over $\mathbb{Z}$ is $(a,b)=(1,0)$. Therefore $I=(1)$ which is clearly absurd since this implies that $\mathcal{R}=I$ which is not true. The equation $a^2+5b^2=2$ has no solutions over $\mathbb{Z}$. Thus I is not principle. Similarly, if $J=(a+b\sqrt{-5})$ for some $a+b\sqrt{-5}$. Then for some $\beta\in\mathcal{R}$, we have that $3=\beta(a+b\sqrt{-5})$ and so taking norms get

$$9=N(\beta)(a^2+5b^2).$$

Similarly to I, a^2+5b^2 must be $1,3$ or 9. If it is equal to 1, we have that $(a,b)=(1,0)$ which can not be for the same reason as I. If we have $a^2+5b^2=3$, we see that this has no solutions over the naturals. Finally if $a^2+5b^2=9$, possible solutions for (a,b) will be $(3,0)$ and $(2,1)$. We have that $(3,0)$ does not generate the ideal for a similar reason as 1 does not. Finally we show that $I\neq(2+\sqrt{-5})$. Since $3\in I$, there must be some $c+d\sqrt{-5}$ with $(2+\sqrt{-5})(c+d\sqrt{-5})=3$. We can solve this to get that $2c-5d=3, 2d+c=0$. This has no solutions over $\mathbb{Z}$. Thus this ideal is not principal.

Q2ii: We have that $I^2=(4,2-2\sqrt{-5},-4-2\sqrt{-5})$. We wish to find some $r\in I^2$ so that $I^2=(r)$. Note that any element i in I^2 must be written of the form

$$i=4(a_1+b_1\sqrt{-5})+(2-2\sqrt{-5})(a_2+b_2\sqrt{-5})+(-4-2\sqrt{-5})(a_3+b_3\sqrt{-5}).$$

Expanding this out we get

$$i=2(2a_1+2b_2\sqrt{-5}+a_2+b_2\sqrt{-5}-a_2\sqrt{-5}+5b_2+-2a_3-2b_3\sqrt{-5}-a_3\sqrt{-5}+5b_3).$$

We can write any element of I as 2 times an arbitrary element of $\mathcal{R}$. Therefore, $I=(2)$ and so it is principal.

Q2iii: Note that $IJ=(6,3-3\sqrt{-5},2-2\sqrt{-5},(1-\sqrt{-5})^2)$. We can write

$$6=1^2+5^2=(1-\sqrt{-5})(1+\sqrt{-5}).$$

Therefore every element that generates IJ is a multiple of $1-\sqrt{-5}$. Therefore $IJ=(1-\sqrt{-5})$