Q6: We claim that if $f:\mathbb{R}^4 \to \mathbb{R}$ is the determinant on 2 by 2 matrices, then $Df(a,b,c,d)(x,y,z,w) = dx - cy - bz + aw$. We proceed by showing that $\lim_{h\to 0}\frac{\|f(x+h)-f(x)-Df(x)(h)\|}{\|h\|} = 0$

$$\begin{aligned}
&\lim_{(x,y,z,w)\to 0} \frac{\|det(a+x,b+y,c+z,d+w) - det(a,b,c,d) - Df(a,b,c,d)(x,y,z,w)\|}{\|x,y,z,w\|}\\
&= \lim_{(x,y,z,w)\to 0} \frac{\|(a+x)(d+w)-(b+y)(c+z)-(ad-bc)-(dx-cy-bz+aw)\|}{\|(x,y,z,w)\|}\\
&= \lim_{(x,y,z,w)\to 0} \frac{\|ad+aw+dx+xw-bc-bz-cy-yz-ad+bc-dx+cy+bz-aw\|}{\|(x,y,z,w)\|}\\
&= \lim_{(x,y,z,w)\to 0} \frac{\|xw-yz\|}{\|(x,y,z,w)\|}
\end{aligned}$$

.Note that $xw-yz = Det\begin{pmatrix} x & y \\ z & w\end{pmatrix}$ Since the determinant is bilinear in each column it follows that the above limit is equal to 0 by 5a. The claim follows. ■