Q7a: Suppose f is an entire function with a pole at ∞. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion. Consider the substitution $z = \frac{1}{\zeta}$. We have that

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \zeta = 0 \implies f(0) = \infty.$$

Therefore we have that 0 is a pole of $f(\zeta)$ which by definition means that the power series of $f(\zeta)$ is finite. i.e. f is a polynomial.

Q7b: Let

$$f(z) = \frac{g(z)}{h(z)} = \frac{\sum_{n=0}^{\infty} a_n z^n}{\sum_{m=0}^{\infty} b_m z^m}.$$

Let $\{z_k\}$ be the set of all poles of $f(z)$. Consider the change of variables

$$h\Big(\frac{1}{z - z_i}\Big) = \sum_{m=0}^{\infty} b_m \Big(\frac{1}{z - z_i}\Big)^m$$

We have that $h(\infty) = \infty$. Hence it must be a polynomial by $7a$. Similarly $g(z)$ must be a polynomial, after we repead this process with $\frac{1}{f(z)}$. Hence if f is meromorphic on S^2 it must be rational.