Q4: We define $\underline{n}_s^k = \{(i_1, \dots i_k) : 1 \leq i_1 \leq \dots \geq i_k \leq n\}$. Define $\sigma_I = \sum_{\sigma \in S_k} \varphi_I \circ \sigma^*$. We make the claim that σ_I is a basis for $S^k(V)$. We will first show that indeed $\sigma_I \in S^k(V)$. It will definitely be k-linear, since it is the sum of k-linear maps. It is enough to show that it is symmetric on some list of vectors $u_1 \dots u_k$. We let $\tau \in S_k$. We evaluate:

$$\begin{aligned}
\sigma_I \circ \tau(u_1, \dots, u_k) &= \sum_{\sigma \in S_k} \varphi_I \circ \sigma^*(u_{\tau(1)}, \dots, u_{\tau(k)}) \\
&= \sum_{\sigma \in S_k} \varphi_I(u_{\sigma(\tau(1))}, \dots u_{\sigma(\tau(k))}) \\
&= \sum_{\lambda \in S_k} \varphi_I \circ \lambda^*(u_1, \dots, u_k) && \text{(since for fixed } \tau, \sigma \circ \tau \text{ ranges over } S_k) \\
&= \sigma_I(u_1, \dots u_k)
\end{aligned}$$