Problem 1.

Since f is a polynomial of degree 2, it is irreducible if and only if it does not have any roots. Any root $x = \frac{a}{b}$ must satisfy $a, b|1$. Therefore if any roots of f exist they must be $x = \pm 1$. However $f(1) = 3$ and $f(-1) = 1$. Thus f is not reducible and $\mathbb{Q}[x]/(f(x))$ is a field. We claim that $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}(\sqrt{-3})$, and $\overline{x}$ can be identified with $-\frac{1}{2} + \frac{\sqrt{-3}}{2}$. Under this identification, we have for $g \in \mathbb{Q}[x]/(f(x))$,

$$g = a + b\overline{x} = a + b\left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) = (a - \frac{b}{2}) + \frac{b}{2}(\sqrt{-3}) \in \mathbb{Q}(\sqrt{-3}).$$

Similarly for $z = a + b\sqrt{-3}$,

$$z = a + b\sqrt{-3} = a + b(2\overline{x} + 1) = (b + a) + 2b(\overline{x}).$$

We conclude $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}(\sqrt{-3})$.

Problem 2.

Suppose that $K(\sqrt{a}) = K(\sqrt{b})$. Without loss of generality, assume that $\sqrt{a}, \sqrt{b} \notin K$. Then certainly there exists some $c, d \in K$ with $\sqrt{a} = c + d\sqrt{b}$. We claim that $c = 0$. If not, then we have that

$$a = c^2 + 2cd\sqrt{b} + d^2 b \implies \sqrt{b} \in K.$$

A similar argument for $\sqrt{b} = c' + d'\sqrt{a}$ will yield the same contradiction. Therefore $\sqrt{a} = d\sqrt{b}$ for some $d \in K$. Therefore

$$a = y^2 b \implies ab = y^2 b^2.$$

So ab is a square. Conversely suppose that $ab = c^2$ for some c. Then, $c = \sqrt{a}\sqrt{b}$ since $b^2 a = cb$. Thus we have that $\sqrt{a} = \frac{c}{b}\sqrt{b}$. Any expression of the form $x + y\sqrt{a} = x + y\frac{c}{b}\sqrt{b}$ holds. Similarly we have $x + y\sqrt{b} = x + \frac{b}{c}\sqrt{a}$.

Problem 3.

Let $\alpha = \sqrt{2} + \sqrt{3}$. We have that

$$\alpha^2 = 5 + 2\sqrt{6} \implies \alpha^2 - 5 = 2\sqrt{6} \implies (\alpha^2 - 5)^2 - 24 = 0.$$

Take $f(x) = (x^2 - 5)^2 - 24$. By construction f will satisfy $f(\alpha) = 0$.

Problem 4.

We compute the cubes of elements in $\mathbb{F}_7$:

$$\begin{aligned}
0^3 &= 0 \\
1^3 &= 1 \\
2^3 &= 1 \\
3^3 &= 6 \\
4^3 &= 1 \\
5^3 &= 6 \\
6^3 &= 6
\end{aligned}$$

The polynomial $x^3 + 2$ is degree 3 so it is irreducible if it has a root. No such roots exist in $\mathbb{F}_7$ since a root β must satisfy $\beta^3 = 5$, which cannot happen by our above computation. Suppose that α is a root in $\mathbb{F}_7[x]/(q(x))$. Then, 2α and 4α will also be solutions to $x^3 + 2 = 0$ since $2^3 = 4^3 = 1$.