Q3:

We first show that T is a linear mapping.

$$
\begin{aligned}
&\phi_{\alpha x+y} \\
&= T(\alpha x + y) \\
&= \langle \alpha x + y, z\rangle \\
&= \alpha\langle x, z\rangle + \langle y, z\rangle \\
&= \alpha\phi_x + \phi_y
\end{aligned}
$$

Therefore T is a linear map from $\mathbb{R}^n$ to $(\mathbb{R}^n)^*$

We now claim that T is injective. Suppose that $\phi_x = \phi_y$.

$$
\begin{aligned}
&\implies T(x) = T(y) \\
&\implies \langle x, z\rangle = \langle y, z\rangle \quad \forall z \in \mathbb{R}^n \\
&\implies \langle x, z\rangle - \langle y, z\rangle = 0 \quad \forall z \in \mathbb{R}^n \\
&\implies \langle x - y, z\rangle = 0 \quad \forall z \in \mathbb{R}^n \\
&\implies x - y = 0 \\
&\implies x = y
\end{aligned}
$$

Thus $T(y) = T(x) \implies y = x$. We can conclude that T is an injective mapping and so the dimension of the null space of T is 0. The dual space is $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ it will have a dimension of n. It follows from the rank-nullity theorem that the image of T is also n dimensional. Equivalently, T is a surjective mapping.Therefore T is a bijection between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. So for each $\phi \in (\mathbb{R}^n)^*$ there exists a unique $x \in \mathbb{R}^n$ such that $T(x) = \phi_x = \phi$