Q3a: Suppose that for $f(x) \in \mathbb{Z}[x]$ there exists some $g(x)$ where $f(x)g(x) = 1$. Taking the degrees of both sides we have that the degree of f and g must both be 0, and hence f and g are constant. Furthermore, if the product of two integers are 1, then they must both be 1 or -1. Therefore, $f(x) = \pm 1$

Q3bi: Consider the polynomial $f(x) = (x-3)(x-4) = x^2 - 7x + 12$. This is clearly primitive since $\gcd(1, -7, 12) = 1$ but it can be written as the product of two linear polynomials.

Q3bii: Consider the polynomial $f(x) = 2x^2 + 6$. This polynomial is irreducible over $\mathbb{Q}$ since it does not split and hence can not be prime. It is clearly not primitive, since $\gcd(2, 6) = 2 \neq 1$. f is not irreducible in $\mathbb{Z}[x]$, since $f(x) = 2(x^2 + 3)$

Q3c: Since $f(r) = 0$, for some g we can write $f(x) = (x - \frac{a}{b})g(x)$. Let $g(x) = \sum_{i=0}^{n} c_i x^i$. We claim that $b|c_i$ for each i. Indeed, we can verify that by expanding the product,

$$f(x) = (x - \frac{a}{c})g(x) = \sum_{i=0}^{n} c_i x^{i+1} - \sum_{i=0}^{n} \frac{a}{b} c_i x^i$$

Since the coefficients of f are in $\mathbb{Z}$, we have that $c_i \cdot \frac{a}{b} \in \mathbb{Z}$ and so we conclude that $b|c_i$. Therefore we can write $c_i = b \cdot d_i$. Therefore

$$f(x) = (x - \frac{a}{b}) \sum_{i=0}^{n} c_i x^i = (x - \frac{a}{b}) b \sum_{i=0}^{n} d_i x^i = (bx - a) \sum_{i=0}^{n} d_i x^i$$

As desired.

Q3d: We know that when we multiple two polynomials, $c_{j_0+i_0} = \sum_{k=0}^{i_0+j_0} a_k b_{i_0+j_0-k}$. Assume that $j_0 > i_0$ and we see that p will divide the first i_0 terms, but at the $j_0'th$ term $p \nmid a_{j_0} \cdot b_{i_0}$