Q4a: We claim that $U$ is a contraction with constant $\frac{L}{M}$. We first observe the following:

$$\frac{d}{dt}e^{-Mt}\int_0^t |\gamma_1(s)-\gamma_2(s)|ds = -Me^{-Mt}\int_0^t |\gamma_1(s)-\gamma_2(s)| + e^{-Mt}|\gamma_1(t)-\gamma_2(t)|$$

Hence at some $t_0$, we have a maximum and so

$$\frac{1}{M}|\gamma_1(t_0)-\gamma_2(t_0)| = \int_0^{t_0} |\gamma_1(s)-\gamma_2(s)|ds$$

We now will show that $U$ is indeed a contraction. Observe:

$$\begin{aligned}
\|U(\gamma_1)-U(\gamma_2)\|_M &= \left\|\int_0^t f(\gamma_1(s))-f(\gamma_2(s))ds\right\|_M \\
&= \sup_{t\geq 0} e^{-Mt}\left|\int_0^t f(\gamma_1(s)-f(\gamma_2(s))ds\right| \\
&\leq \sup_{t\geq 0} e^{-Mt}\int_0^t \left|f(\gamma_1(s))-f(\gamma_2(s))ds\right| \\
&\leq \sup_{t\geq 0} Le^{-Mt}\int_0^t |\gamma_1(s)-\gamma_2(s)|ds && \text{(by Lipschitz)} \\
&\leq \frac{L}{M}e^{-Mt_0}|\gamma_1(t_0)-\gamma_2(t_0)| \\
&\leq \frac{L}{M}\sup_{t\geq 0} e^{-Mt}|\gamma_1(t)-\gamma_2(t)| \\
&= \frac{L}{M}\,\|\gamma_1-\gamma_2\|_M
\end{aligned}$$

If it is the case that $M > L$, then this map will be a contraction

Q4b: Since $U$ is a contraction if the condition from 4a is met, we have that there exists a unique fixed point $y$ where $y(t) = U(y(t))$ or equivalently $y' = f(y)$, with $y(0) = v$

Q4c: By 4b, $y(t)$ exists on all $t \geq 0$. Since $y \in C_M$ we have that

$$\sup_{t\geq 0} e^{-Mt}|y(t)| \leq K \implies \exists t_0 : |y(t_0)| \leq Ke^{Mt} \implies |y(t)| \leq Le^{-Mt}$$