Problem 1.

We compute that

$$P \circ h = \left(z + \frac{1}{z}\right)^4 - 4\left(z + \frac{1}{z}\right)^2 + 2 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} - 4z^2 - 8 - \frac{4}{z^2} + 2 = z^4 + \frac{1}{z^4} = h \circ Q.$$

Now, notice that

$$P^n \circ h = \overbrace{P \circ \cdots \circ P}^{\text{n times}} \circ h = h \circ Q^n = z^{4n} + \frac{1}{z^{4n}}.$$

Furthermore, from MAT354 we know that $h(z)$ is a conformal mapping of D to $\mathbb{C} \setminus [-2, 2]$. Thus $\{P^n|_U\}$ is a normal family on some U, a neighbourhood of some $a \in [-2, 2]$ if and only if $\left\{(P^n \circ h)|_{h(U)}\right\}$ is normal. Our previous computation verifies that $\left\{(P^n \circ h)|_{h(U)}\right\} = \left\{(h \circ Q^n)|_{h(U)}\right\}$. For any choice of U,the family $\left\{(h \circ Q^n)|_{h(U)}\right\}$ becomes pointwise unbounded as $n \to \infty$. Therefore not a normal family.

Problem 2.

(a) If $f_n \to \infty$ we are done and $d(f_n, \infty) = \frac{2}{\sqrt{1+|f_n|^2}} \rightrightarrows 0$. Suppose that $f_n \to f$ uniformly for f not identically ∞ on compact subsets. Let z be a point so that $f_n(z) \to f(z)$ with $f(z) \neq \infty$. By marty's theorem, it is sufficient to show that $\{f_n^\sharp\}$ is locally bounded in a neighbourhood of z, then the limit of this sequence will be holomorphic. Notice that $f_n^\sharp(z) = \frac{2|f_n'(z)|}{1+|f_n(z)|^2}$. If we had that $f_n^\sharp(z) \to \infty$ then it must be that $f_n'(z) \to \infty$. However, this would contradict Montel's Little theorem, since $f_n(z)$ is bounded by $|f(z)| + c$ for some constant c. Therefore f must have no poles i.e. it is holomorphic.

(b) If $f_n \to f$ with f holomorphic, then on any compact subset, f_n and f are both bounded uniformly, say by M. So convergence in the chordal metric implies that

$$d(f_n, f) = \frac{2|f_n - f|}{\sqrt{(1+|f|^2)(1+|f_n|^2)}} \leqslant \frac{2|f_n - f|}{1+M} \rightrightarrows 0 \implies |f - f_n| \rightrightarrows 0.$$

Problem 3.

Let f be a periodic holomorphic function with period c. Suppose f has no fixed points, that is $f(z)-z \neq 0$. Define the holomorphic function $g(z) = f(z) - z$. We have that $g(z) \neq 0$. Furthermore, $g(z-c) = f(z-c)-z+c = g(z)+c$. Since $g(z)$ omits 0, it must also omit c. This contradicts Picards little theorem.

Problem 4.

(a) Consider the function $f(z) = e^z + z$. This has a fixed point if there is some z so that $z = f(z) = e^z + z$. Note that no such z can exist however, since such z must also satisfy $e^z = 0$.

(b) First suppose that $f \circ f$ has no fixed points. Then it follows that f has no fixed points, since any fixed point of f will be a fixed point of $f \circ f$. Consider the function

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}.$$

Note that g is entire since f is, and g omits 0. By Picards theorem, we have that at some w, $g(w) = 1$. Therefore $f(f(w)) = f(w)$. Which contradicts the assumption that $f \circ f$ has no fixed points.

Problem 5.

We write

$$w = z - \sqrt{z^2 - 1} \implies \sqrt{z^2 - 1} = w - z \implies w^2 - 2zw + 1 = 0.$$

This defines a Riemann surface with branch points at $z = \pm 1$. So it looks like two copies of $\mathbb{C}$ identified along the interval $[-1, 1]$. If we have the holomorphic differential form

$$\omega = \frac{dz}{\sqrt{z^2 - 1}},$$

we can write it as

$$\omega = \frac{dz}{z - w}$$

in a neighbourhood of $(z_0, w_0) \in X$ away from $w = z$. In a neighbourhood of $z = w$, we have

$$d(w^2 - 2zw + 1) = 0 \implies w dz = (w - z) dw.$$

So $\omega = \frac{w-z}{w\sqrt{z^2-1}} dw$. Homogenizing, our equation becomes

$$\left(\frac{w}{t}\right)^2 - 2\left(\frac{w}{t} \cdot \frac{z}{t}\right) + 1 = 0 \implies w^2 - 2zw + t^2 = 0.$$

Setting $t = 0$, we determine that the coordinates at ∞ are $[0, 1, 0]$ and $[\frac{1}{2}, 1, 0]$ By finding the values of z, w when w, z are local coordinates respectively at $t = 0$. We compute the residues of ω at ∞ as:

$$res(\omega, [0, 1, 0]) = 1, res(\omega, [\frac{1}{2}, 1, 0]) = -1.$$