

Q1a:
 \Rightarrow

Suppose that T is inner product preserving. Then,

$$\langle Tx, Tx \rangle = \langle x, x \rangle = \|Tx\|^2 = \|x\|^2$$

Since norms are positive, this implies that $\|Tx\| = \|x\|$

\Leftarrow

Suppose that T is norm preserving, then by the polarization identity,

$$\begin{aligned} \langle Tx, Ty \rangle &= \frac{\|T(x+y)\|^2 - \|T(x-y)\|^2}{4} \\ &= \frac{\|x+y\|^2 - \|x-y\|^2}{4} \\ &= \langle x, y \rangle \end{aligned}$$

■

1b:

Suppose that T is a norm preserving linear map from $\mathbb{R}^n \rightarrow \mathbb{R}^n$. Consider the case when $\|T(x)\| = 0$. By assumption, it must be that $\|x\| = 0$. By the properties of the norm, this is equivalent to $x = 0$. Therefore, T is an injective mapping. From the rank-nullity theorem, it follows that the dimension of the range of T is n , so T must also be a surjective linear map. Hence T is a bijective mapping. Therefore the linear map T^{-1} must exist. By assumption T is norm preserving so $\|T^{-1}(T(x))\| = \|x\| = \|T(x)\|$. By part a, it follows that T^{-1} is also inner product preserving. ■