Q1a: We compute that

$$|Ax|^2 = \langle Ax, Ax\rangle = \langle A^\intercal Ax, x\rangle = \langle x, x\rangle = |x|^2$$

Hence A preserves norms, and so the image of $\mathcal{S}^2$ under A is $\mathcal{S}^2$

Q1b: Letting $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, we can verify through a very long and strenuous computation that

$$\begin{aligned}
A^*\omega &= A^*(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \\
&= (x \circ A)d(y \circ A) \wedge d(z \circ A) + (y \circ A)d(z \circ A) \wedge d(x \circ A) + (z \circ A)d(x \circ A) \wedge d(y \circ A) \\
&= (xA)[da_2x + db_2y + dc_2z] \wedge [da_3x + db_3y + dc_3z] + yA[da_3x + db_3y + dc_3z] \wedge [da_1x + db_1y + dc_2z] \\
&+ (zA)[da_1x + db_1y + dc_1z] \wedge [da_2x + db_2y + dc_2z] \\
&= [xA(b_2c_3 - b_3c_2) + yA(b_3c_1 - b_1c_3) + zA(b_1c_2 - b_2c_1)](dy \wedge dz) \\
&+ [xA(a_3c_2 - a_2c_3) + yA(a_1c_3 - a_3c_1) + zA(a_2c_1 - a_1c_2)](dz \wedge dx) \\
&+ [xA(a_2b_3 - a_3b_2) + yA(a_3b_1 - a_1b_3) + zA(a_1b_2 - a_2b_1)](dx \wedge dy)
\end{aligned}$$

For the sake of simplicity and readibility, we will manage only the term containing $dy \wedge dz$. The other terms will admit almost identical manipulations and hence will be omitted.

$$\begin{aligned}
&= [xA(b_2c_3 - b_3c_2) + yA(b_3c_1 - b_1c_3) + zA(b_1c_2 - b_2c_1)](dy \wedge dz) \\
&= [(a_1x + b_1y + c_1z)(b_2c_3 - b_3c_2) + (a_2x + b_2y + c_2z)(b_3c_1 - b_1c_2) + (a_3x + b_3y + c_3z)(b_1c_2 - b_2c_1)](dy \wedge dz) \\
&= (a_1b_2c_3zx - a_1b_3c_2x + a_2b_3c_1x - a_2b_1c_3x + a_3b_1c_2x - a_3b_2c_1x)(dy \wedge dz) \\
&= \det(A)xdy \wedge dz
\end{aligned}$$

Therefore,

$$A^*\omega = \det(A)xdy \wedge dz + \det(A)ydz \wedge dy + \det(A)zdx \wedge dy = \det(A)\omega$$

Q1c: Since $A \in O(3)$, we have that $1 = \det(I) = \det(A)\det(A^\intercal) = \det(A)^2$ so $\det(A) = \pm 1$. Hence $SO(3) \subset O(3)$ Thus by 1b, for any $B \in SO(3)$, we have that

$$B^*\omega = \omega$$

Since ω is a nowhere vanishing top form on $\mathcal{S}^2$, and pulling back by B amounts to scaling by a positive number, we conclude that B is an orientation preserving map of $\mathcal{S}^2$.