MAT458 Solution Set

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Problem 1. *Folland 5.3.42*

(a) Let $f \in E_n$. That is for some $x_0 \in [0,1]$ we have that $|f(x_0) - f(x)| \leqslant n|x_0 - x|$. By Stone-Weierstrass, we can uniformly approximate f with some piecewise linear function h with slope $\pm 2n$. Thus for any $\varepsilon > 0$ we have an h so that $\|f - h\|_u < \varepsilon$. We claim that $h \notin E_n$. For any $x \in [0,1]$,

$$\frac{|h(x_0) - h(x)|}{|x - x_0|} \geqslant 2n \implies |h(x_0) - h(x)| \geqslant 2n|x - n_0| > n|x - x_0|.$$

Therefore h is not in E_n, and so E_n is nowhere dense in C^0.

(b) The countable union $E = \bigcup_{n=1}^{\infty} E_n$ is nowhere dense in C^0. It follows that the set $C^0 \setminus E$ is residual, and nonempty. Since E is the set of all somewhere differentiable functions, the compliment must be set of nowhere differentiable functions.

Problem 2. *Folland 5.3.27*

Let x_n be an enumeration of the rationals. Define:

$$E_n = \bigcup_{k=1}^{\infty} \left(x_k - \frac{1}{2^{k-1}n}, x_k - \frac{1}{2^{k-1}n} \right).$$

We have that $m(E_n) = \frac{1}{n}$. It follows from measure continuity that $m\left(\cap_{n=1}^{\infty} E_n\right) = 0$. Since each E_n is dense in $\mathbb{R}$, so is E by Baire Category theorem. The compliment is nowhere dense. Take the compliement of E as our desired set.

Problem 3. *Extra Credit:*

Problem 4. *Folland 5.3.32*

Consider the identity mapping $I : (\mathfrak{X}, \|\cdot\|_1) \to (\mathfrak{X}, \|\cdot\|_2)$. I is bijective, and continuous by assumption. It follows that the inverse is bounded by Folland Cor 5.11. So there exists some constant C so that $\|\cdot\|_2 \leqslant C\|\cdot\|_1$.

Problem 5. *(Extra Credit) Folland 5.3.33*

Suppose that there exists a sequence $\{a_n\}$, $a_i \geqslant 0$ so that $\sum_n a_n|c_n| < \infty$ if and only if $\{c_n\}$ is bounded. Define $T : B(\mathbb{N}) \to L^1(\mu)$ as $Tf(n) = a_n f(n)$. We first claim that $\{g_n\}$ so that g_n is nonzero for finitely many n is dense in $L^1(\mu)$. Given some $h(n) \in L^1(\mu)$, for any $\varepsilon > 0$ there is some N so that $\sum_{n \geqslant N} |h(n)| < \varepsilon$. Define

$$g = \begin{cases} h(n) & n < N \\ 0 & n \geqslant N \end{cases}.$$

Then, $\sum_{n \in \mathbb{N}} |h(n) - g(n)| = \sum_{n \geqslant N} |h(n)| < \varepsilon$. Note however this family of functions is not dense in $B(\mathbb{N})$, since if we take a constant sequence of 1, then $|f(n) - 1|_1 = \infty$. By the uniform boundedness principle, we have that $\|Tf(n)\| < \infty$ for all n, so $\sup_n \|Tf(n)\| < \infty$. There exists some c so that $a_n f(n) = Tf(n) \leqslant C$. Clearly, we can modify $f(n)$ so that this inequality breaks however.

Problem 6. *Folland 5.3.37*

Let $\{x_n\}$ be a sequence converging to x. Let $\lim_{n\to\infty} Tx_n = y$. By continuity of linear functionals, we have that $f(Tx_n) \to f(Tx)$. We also have that $f(Tx_n) \to f(y)$. Since continuous linear functionals seperate points, we have $y = Tx$. By the closed graph theorem T is bounded.

Problem 7. *Folland 5.3.38*

Note that T is linear. It remains to show that it is continuous. By the uniform boundedness principle, we there exists some constant C so that $\sup_n \|T_n\| \leqslant C$. Note that the following chain of inequalities holds:

$$\|Tx\| \leqslant \sup_n \|T_n x\| \leqslant C\|x\|.$$

Therefore T is bounded and thus continuous.

Problem 8. *Folland 5.3.39*

Let $B : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z}$ be a separately continuous linear map. By bilinearity, it is enough to show that there exists a constant C so that $\|B(x,y)\| \leqslant C\|x\| \cdot \|y\|$. Let $B_x(y) = B(x,y), B_y(x) = B(x,y)$. Since B is seperately continuous, we have that there exists some C_x, D_y so that $\|B_x(y)\| \leqslant C_x\|y\|, \|B_y(x)\| \leqslant D_y\|x\|$. By uniform boundedness principle, there exist C, D so that $\|B_y(x)\| \leqslant C\|x\|$ and $\|B_x(y)\| \leqslant D\|y\|$ for all x, y respectively. Thus we have that $\sup_{\|x\|,\|y\|=1} \|B(x,y)\| \leqslant \max(C,D)$. Thus B is continuous.

Problem 9. *Folland 5.3.40 (Principle of Condensation of Singularities)*

Suppose not. Then $\sup_k\{\|T_{jk}x\| : j \in \mathbb{N}\} < \infty$ for all $x \in \mathfrak{X}$. By uniform boundedness we have that $\sup_k\{\|T_{jk}\| : j \in \mathbb{N}\} < \infty$. This contradicts the assumption.

Problem 10. *Folland 5.4.47*

(a) Suppose that $T_n \to T$ weakly. That is for all $f \in \mathfrak{Y}^*$, we have $fT_n x \to fTx$. Therefore $\sup_n \|fT_n x\| < \infty$ for all f. By hahn banach, take g so that $|g| = 1$, so

$$\|T_n x\| = |g(T_n x)| \leqslant \sup_{f \in \mathfrak{Y}^*, \|f\|=1} |f(T_n x)| = \sup_{f \in \mathfrak{Y}^*, \|f\|=1} |\hat{x} \circ (T_n^* \circ f)| \leqslant \sup_{f \in \mathfrak{Y}^*, \|f\|=1} |\hat{x} \circ T_n^*| \leqslant \|\hat{x} \circ T_n^*\| < \infty.$$

By uniform boundedness princple, we have that $\sup_n \|\hat{x} \circ T_n^*\| < \infty$. Since if $T_n \to T$ strongly implies weakly, we are done.

(b) Let $\{x_n\}$ be a weakly convergent sequence in $\mathfrak{X}$. That is for all $f \in \mathfrak{X}^*$, $f(x_n) \to f(x)$. Therefore $\|f(x_n)\| \to \|f(x)\|$. We also have that $\hat{x}_n(f) \to \hat{x}(f)$, with norms converging to $\|f(x)\|$. Thus we have that $\sup_n \|\hat{x}_n(f)\| < \infty$. Therefore $\|\hat{x}\| = \|x\| < \infty$. Now let $\{f_n\}$ be a sequence in $\mathfrak{X}$ converging to f in the weark star topology i.e. $f_n(x) \to f(x)$ for all x. Thus we have that

$$\sup_n \|f_n(x)\| = \sup_n \|\hat{x}(f_n)\| < \infty,$$

by convergence. Therefore taking $\|\hat{x}\| = 1$ we have that f_n is bounded.

Problem 11. *Folland 5.4.48*

(a) Let $\{x_n\}$ be a sequence in B converging to some $x \in B$. For any $f \in \mathfrak{X}$, $\|f(x_n)\| = \|\hat{x}_n(f)\| \leqslant \|\hat{x}_n\| \leqslant 1$ by theorem 5.8d. Therefore $\|\hat{x}(f)\| = \|f(x)\| \leqslant 1$.

(b) Let E be a bounded set in $\mathfrak{X}$. Let $\langle x_\alpha \rangle$ be a net in E converging to x , and for $f \in \mathfrak{X}$, $f(x_\alpha) \to f(x)$. We have that

$$\sup_\alpha \|f(x_\alpha)\| = \sup_\alpha \|\hat{x}_\alpha(f)\| = \sup_\alpha \|\hat{x}_\alpha\| = \sup_\alpha \|x_\alpha\| < \infty.$$

(c) Let F be a bounded subset of $\mathfrak{X}^*$. Let $\langle f_\alpha \rangle$ be a net in F converging to f. Then for all $\|x\| = 1$ we have

$$\sup_\alpha \|f_\alpha(x)\| < C \implies \lim_{\alpha \to \infty} \|f_\alpha\| < \infty.$$

(d) Let $\langle f_\alpha \rangle$ be a net so that $\langle f_i - f_j \rangle \to 0$. Then for sufficiently large n, m we have $\|f_n(x) - f_m(x)\| \to 0$. So $\{f_n(x)\}$ is a cauchy sequence. It converges to some f by 5.3.38.

Problem 12. *Folland 5.4.49*

(a) It is sufficient to show that any element of the basis is unbounded. Elements of the basis take the form

$$U_{f,\varepsilon}(x) = \{y \in \mathfrak{X} : |f(x) - f(y)| < \varepsilon\}.$$

Taking any $v \in f^{-1}(0)$, we have that $x + \alpha v \in U_{f,\varepsilon}(x)$ for all scalars α. Thus this set is unbounded. For the weak $*$ topology, the basis elements take the form

$$V_{f,\varepsilon} = \{g \in \mathfrak{X} : \|f - g\| < \varepsilon\}.$$

It is sufficient to show that these sets are unbounded. For all $f \in V_{f,\varepsilon}$,

$$\sup_{\|x\|=1} \|f(x) - g(x)\| = \sup_{\|x\|=1} \hat{x}(f - g) < \varepsilon.$$

Takeing any l so that $\hat{x}(l) = 0$ we have that $f + \alpha l \in V_{f,\varepsilon}$ for all α. Thus this set is unbounded.

(b) If E is a bounded subset of $\mathfrak{X}$, then so is its weak closure by 5.4.48b. By part a we have that the interiour must be empty. The same result follows for $F \subset \mathfrak{X}^*$ bounded by 5.4.48c and a.

(c) Let $E_n = \{x \in \mathfrak{X} : \|x\| \leqslant n\}$. Each E_n is nowhere dense in weak topology, and $\mathfrak{X} = \bigcup_n E_n$. So $\mathfrak{X}$ is meager in the weak topology. The result for $\mathfrak{X}^*$ is obtained in the exact same way.

(d) IDK ask rob

Problem 13. *Folland 5.4.50*

Let $\{x_n\}, \{q_n\}$ be enumerations of the dense subsets of $B, \mathbb{Q}$ respectively. Take $f_n \in \mathfrak{X}$ so that $f_x(x_n) = q_n$. Let $\varepsilon > 0$. Take $V_{f_n,\varepsilon}$ as defined earlier. We claim that $\{V_{f_n,\varepsilon}\}$ is a covering of B^*. Let $\|g\| \leqslant 1$. Then at some $x \in B$, g attains a maximum since g bounded. Then,

$$\|g(x) - f(x)\| \leqslant \|g(x_n) - g(x)\| + \|f(x_n) - f(x)\| + \|f(x_m) - f(x_n)\| + \|f(x_m) - f(x)\|.$$

Since the norms of all the operators are 1, taking m, n sufficiently large we can make each term less than $\frac{\varepsilon}{4}$. Therefore we have a countable covering of B^* by basis elements. Hence it is second countable. Since it is compact and Hausdorff, it must be metrizable by topology results.

Problem 14. *Folland 5.4.51*

Let $\mathfrak{Y} \subset \mathfrak{X}$ be a vector subspace. Let $\{x_n\}$ be a sequence in $\mathfrak{Y}$ so that $\|x_n - x\| \to 0$ implies $x \in \mathfrak{Y}$. Let $f \in \mathfrak{X}$. Then we have that $\|f(x_n) - f(x)\| \leqslant C_f \|x_n - x\| \to 0$. Conversely suppose that for all $f \in \mathfrak{X}^*$, $f(x_n) \to f(x)$, for $\{x_n\} \subset \mathfrak{Y}$ and $\|x_n - x\| \to 0$. We claim that $x \in \mathfrak{Y}$. By theorem 5.8 we can take f so that $f|_{\mathfrak{Y}} = 0$. We have that $f(x) = 0$ and so we are done.

Problem 15. *Folland 5.4.52*

(a)

Problem 16. *Folland 5.5.56*

The smallest closed subspace that contains E is by definition $\overline{span(E)}$. We claim that $\overline{span(E)} = E^{\perp\perp}$. First suppose that $v \in \overline{span(E)}$. Then there exists some sequence $\{v_n\}$ converging to v. For all $u \in E^{\perp}$, we have that

$$\langle v_n, u\rangle = 0,$$

so $v_n \in E^{\perp\perp}$, and by continuity of the inner product, $v \in E^{\perp\perp}$. Now suppose that $\{v_n\}$ is a sequence in $E^{\perp\perp}$ converging to some v. Then for all $u \in \overline{span(E)}$ we have that

$$\langle u, v_n\rangle = 0$$

and so $\{v_n\} \subset \overline{span(E)}$. By contiuity we have that $v \in \overline{span(E)}$.

Problem 17. *Folland 5.5.57*

(a) We claim that $T^* = V^{-1}T^{\dagger}V$. We see that it satisfies

$$\langle x, T^*y\rangle = \langle x, V^{-1}T^{\dagger}Vy\rangle = \left(VV^{-1}T^{\dagger}Vy\right)(x) = (T^{\dagger}Vy)(x) = (Vy)(Tx) = \langle Tx, y\rangle.$$

We now claim uniqueness holds. If S^*, T^* both satisfy the equality, then we have that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, S^*y\rangle \implies \langle x, T^* - S^*y\rangle = 0, \forall x, y \implies T^* = S^*.$$

(b) We first claim that $T^{**} = T$. Notice that if T^* satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$, then we must have that

$$\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle.$$

We now claim that

$$\|T^*\| = \|T\|.$$

Observe that for any x,

$$\|T^*\| = \sup_{\|y\|=1} \|T^*y\| = \sup_{\|y\|=\|x\|=1} |\langle x, T^*y\rangle| = \sup_{\|y\|=\|x\|=1} |\langle Tx, y\rangle| = \|T\|.$$

Next, we have that

$$\langle (aS + bT)x, y\rangle = \langle aSx, y\rangle + \langle bTx, y\rangle = \langle x, \overline{a}S^*y\rangle + \langle x, \overline{b}T^*y\rangle = \langle x, \overline{a}S^* + \overline{b}T^*y\rangle.$$

Finally,

$$\langle STx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle.$$

(c) We first show that $R(T)^{\perp} = N(T^*)$. Let $y \in R(T)^{\perp}$. Then for all $x \in \mathcal{H}$, we have

$$0 = \langle Tx, y\rangle = \langle x, T^*y\rangle \implies T^*y = 0.$$

Conversely, if $y \in N(T^*)$, we have that for all $x \in \mathcal{H}$,

$$0 = \langle x, T^*y\rangle = \langle Tx, y\rangle \implies y \in R(T)^{\perp}.$$

Next we claim that $N(T)^{\perp} = \overline{R(T^*)}$. Suppose for some $x \in N(T)$, and for all $y \in \mathcal{H}$, we have that

$$0 = \langle Tx, y\rangle = \langle x, T^*y\rangle \implies \overline{R(T^*)} \subset N(T)^{\perp}.$$

Conversely, let $x \in N(T)^{\perp}$. Then we have that

$$0 = \langle Tx, y\rangle = \langle x, T^*y\rangle \implies x \in \overline{R(T)}.$$

(d) Suppose that T is unitary. Then it must also be invertible. We claim that $T^* = T^{-1}$. We have that
$$\langle x, y\rangle = \langle Tx, Ty\rangle = \langle x, T^*Ty\rangle \implies 0 = \langle x, (T^*T - I)y\rangle \implies T^*T = I.$$
Therefore we have that $T^{-1} = T^*$. Conversely suppose that T is invertible with $T^{-1} = T^*$. Then we have that
$$\langle x, x\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle.$$
By the polarization identity we have that
$$\langle Tx, Ty\rangle = \langle x, y\rangle.$$

Problem 18. *Folland 5.5.58*

(a) First note that by definition of P, we have that $\langle Px - x, Px\rangle = 0$. This implies that $\langle Px, Px\rangle = \langle x, Px\rangle$. By Cauchy-Schwartz's inequality, we have that
$$\|Px\|^2 = \langle x, Px\rangle \leqslant \|x\| \cdot \|Px\| \implies \|Px\| \leqslant \|x\|.$$
Therefore $\|P\| \leqslant 1$, so $P \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. We now claim that $P^* = P$. Note that $\langle Px - x, Px\rangle = 0 = \langle Px, Px - x\rangle$ which implies that $\langle Px, x\rangle = \langle x, Px\rangle$. Therefore we have
$$\langle P^*x - Px, x\rangle = \langle P^*x, x\rangle - \langle Px, x\rangle = \langle x, Px\rangle - \langle Px, x\rangle = 0.$$
Since this holds for all x we have that $P^* = P$. Finally, we have that
$$\langle Px, Px\rangle = \langle x, P^2x\rangle = \langle x, Px\rangle \implies P^2 = P.$$
We next claim that $N(P) = M^\perp$ and $R(P) = M$. Note that by definition, $Px \in M$. Now if $y \in M$, since $y - Py \in M^\perp \cap M$ we have that $Py = y$. Therefore $R(P) = M$ and so by 57c) we must have $N(P) = M^\perp$.

(b) First suppose that $\{x_n\} \subset R(P)$ with limit x and that P satisfies the definition of a projection. We will have that $x \in R(P)$ if $Px = x$. We have that
$$\lim_{n\to\infty} \langle Px_n - x_n, x_n\rangle = 0, \forall n \implies \langle Px - x, x\rangle = 0$$
Therefore $R(P)$ is closed. We now claim that for all x, $\langle Px - x, Px\rangle = 0$. Using the properties of P, we see that
$$\langle Px - x, Px\rangle = \langle P^2x - Px, x\rangle = 0 \forall x.$$
Therefore such P must be a projection.

(c) We claim that $Px = \sum_\alpha \langle x, u_\alpha\rangle u_\alpha$ satisfies $\langle Px - x, Px\rangle = 0$. This is clearly a continuous operator, so by the previous result we can conclude that P is indeed a projection.
$$\begin{aligned}
\langle Px - x, Px\rangle &= \langle \sum_\alpha \langle x, u_\alpha\rangle u_\alpha - x, \sum_\alpha \langle x, u_\alpha\rangle u_\alpha\rangle \\
&= \langle \sum_\alpha \langle x, u_\alpha\rangle u_\alpha, \sum_\alpha \langle x, u_\alpha\rangle u_\alpha\rangle - \langle x, \sum_\alpha \langle x, u_\alpha\rangle u_\alpha\rangle \\
&= \sum_\alpha \langle x, u_\alpha\rangle^2 - \sum_\alpha \langle x, u_\alpha\rangle^2 \\
&= 0
\end{aligned}$$
As desired.

Problem 19. *Folland 5.5.59*

Assume that $0 \notin K$. Let $\delta = \inf_{v\in K} \|v\|$. Let $\{v_n\}$ be a sequence so that $\|v_n\| \to \delta$. By closedness of K, we have that the limit $v \in K$. We now claim uniqueness of v. Suppose that u is another vector that attains minimal norm. Test.