

Q2: Since *NBV* functions are the difference of increasing functions, it is sufficient to show that the inequality holds for bounded increasing functions. Writing F, F_t as the difference of increasing functions we observe:

$$\int |F - F_t| dm = \int |(F^1 - F^2) - (F_t^1 - F_t^2)| \leq \int |F^1 - F_t^1| + \int |F^2 - F_t^2|.$$

Hence we can seek an upper bound for each of these terms. Furthermore we can normalize F so that $F(-\infty) = 0$. We define $C = \sup_{x \in \mathbb{R}} (F(x))$. Let $E_n = (-nt, nt]$. Let $g_n(x) = \chi_{E_n}[F(x) - F(x - t)]$. Since $F(x)$ increasing, we have that $g_n \nearrow F(x) - F(x - t)$ on \mathbb{R} . First for $t \geq 0$, we compute that

$$\begin{aligned} \int_{\mathbb{R}} g_n dm &= \int_{(-nt, nt]} F(x) - F(x - t) \\ &= \int_{(-nt, nt]} F(x) dm - \int_{(nt, nt]} F(x - t) dm \\ &= \int_{(-nt, nt]} F(x) dm - \int_{(t - nt, t + nt]} F(x) dm && \text{(by change of variables)} \\ &\leq \int_{(-nt, nt]} F(x) dm - \int_{(-nt, t(n+1)]} F(x) dm && \text{(since } F(x) \text{ increasing)} \\ &= \int_{(nt, n(t+1)]} F(x) dm \\ &\leq C|t|. && \text{(since } F(x) \text{ bounded)} \end{aligned}$$

The argument is almost identical for $t < 0$, we simply change the direction of the shifting of intervals. Hence we have that

$$\int_{\mathbb{R}} g_n < C|t|.$$

The monotone convergence theorem implies that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} g_n dm = \int_{\mathbb{R}} \lim_{n \rightarrow \infty} g_n dm = \int_{\mathbb{R}} F(x) - F(x - t) dm < C|t|.$$

Thus we are done.