Q5a: We claim that Picard Iteration is monotone. We claim this is true by strong induction. For $n = 0$, we have that by assumption

$$x = x_0 \leq a + b\int_0^t x = x_1$$

Now suppose that for all $i < k$, we have that $x_i \leq x_{i+1}$. We wish to show that $x_k \leq x_{k+1}$. Observe that

$$x_{k-1} \leq x_k \implies a + b\int_0^t x_{k-1} \leq a + b\int_0^t x_k \implies x_k \leq x_{k+1}$$

Where the first implication follows by assumption, that $0 \leq x_{k-1}, x_k$.

Q5b: We claim that $\{x_k\}$ is a cauchy sequence. Let $\sup_{t\in[0,T_1]|} |x(t)| = M$. We see that

$$|x_1 - x_0| = |a + b\int_0^t x - x| \leq a + bT_1M + M$$

For $|x_2 - x_1|$, we see

$$|x_2 - x_1| \leq b\int_0^t |x_1 - x_0| \leq bT_1M$$

For $|x_3 - x_2|$, we see that

$$|x_3 - x_2| \leq (bT_1)^2 M$$

Extending this we see $|x_{k+1} - x_k| \leq (bT_1)^k M$. Now we compute that

$$\begin{aligned}
|x_n - x_m| &\leq \sum_{k=m}^{n} |x_k - x_{k-1}| \\
&\leq \sum_{k=m}^{n} (bT_1)^k M \\
&< \varepsilon && \text{(since } bT_1 < 1 \text{ can be made sufficiently small)}
\end{aligned}$$

Hence this sequence is Cauchy and thus converges. By 5a, it converges upwards to some $y(t)$. Since the solution to our system is $y(t) = ae^{bt}$ we have that $0 \leq x(t) \leq ae^{bt}$.

Q5c: By partitioning $[0, T]$ into intervals of length less than T_1 we apply the same argument form 5b, and we get that $x(t)$ exists on $[0, T]$ and $x(t) \leq ae^{tb}$