We claim that ρ is a metric on measurable functions on X. First, note that

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}d\mu = \int \frac{|g-f|}{1+|g-f|}d\mu = \rho(g,f)$$

We now claim positive definiteness. First note that the function

$$0 \leq \frac{|f-g|}{1+|f-g|} \leq 1$$

for all f, g. Therefore we have that for all f, g,

$$0 \leq \int_X \frac{|f-g|}{1+|f-g|} \leq \int_X 1 = \mu(X) < \infty$$

We claim that $\rho(f,g) = 0$ if and only if $f = g$. First suppose that $\rho(f,g) = 0$ i.e.

$$\int_X \frac{|f-g|}{1+|f-g|} = 0$$

We know from properties of the integral that if the integral of a positive function is 0, then the function must be 0 almost everywhere. Therefore

$$\frac{|f-g|}{1+|f-g|} = 0$$

almost everywhere. This is only satisfied when $|f-g| = 0$ a.e., which is the same as $f = g$ a.e. If $f = g$ a.e. We denote E to be the measure 0 set where they differ. Then

$$\rho(f,g) = \int_X \frac{|f-g|}{1+|f-g|} = \int_E \frac{|f-g|}{1+|f-g|} + \int_{E^c} \frac{|f-g|}{1+|f-g|} = \int_E \frac{|f-g|}{1+|f-g|} = 0$$

Where we use the fact that the integral over a measure 0 set will be 0. We now claim that the triangle inequality holds. Let f, g, h be measurable functions. It is sufficient to show that

$$\rho(g,h) + \rho(h,f) - \rho(f,g) \geq 0$$

We can compute that

$$\begin{aligned}
\rho(g,h) + \rho(h,f) - \rho(f,g) &= \int \frac{|g-h|}{1+|g-h|} + \frac{|h-f|}{1+|h-f|} - \frac{|f-g|}{1+|f-g|}d\mu \\
&= \int \frac{2(|h-f|\cdot|g-h|) + (|h-f|\cdot|g-h|\cdot|f-g|) + |h-f| + |g-h| - |f-g|}{(1+|g-h|)(1+|h-f|)(1+|f-g|)} \\
&\geq \int \frac{2(|h-f|\cdot|g-h|) + (|h-f|\cdot|g-h|\cdot|f-g|)}{(1+|g-h|)(1+|h-f|)(1+|f-g|)} && \text{(by triangle inequality)} \\
&\geq 0 && \text{(since this function is positive)}
\end{aligned}$$

Hence we conclude that ρ is a metric. We now claim that $f_n \to f$ in measure if and only if $f_n \to f$ in the ρ metric. We first claim that for any f_n, $\frac{|f_n-f|}{1+|f_n-f|} \in L^1$. It is clear, since

$$\frac{|f_n-f|}{1+|f_n-f|} \leq 1 \implies \int_X \frac{|f_n-f|}{1+|f_n-f|}d\mu \leq \int_X 1 d\mu = \mu(X) < \infty$$

This function is therefore in L^1. Now we let $\varepsilon > 0$, it has been proven in class that there exists a $\delta > 0$ such that $\int_E |\frac{|f_n-f|}{1+|f_n-f|}| < \varepsilon$ for any E with $\mu(E) < \delta$. Take $E_n = \{x : |f_n - f| \geq \varepsilon\}$. By convergence in measure, take n sufficiently large so that $\mu(E_n) < \delta$. We see that

$$\begin{aligned}\rho(f_n, f) &= \int_X \frac{|f_n - f|}{1 + |f_n - f|} d\mu \\ &= \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|} d\mu + \int_{E_n^c} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \\ &\leq \varepsilon + \int_{E_n^c} \frac{\varepsilon}{1 + |f_n - f|} d\mu \\ &\leq \varepsilon + \varepsilon \int_X 1 d\mu \\ &\leq \varepsilon + \varepsilon(\mu(X))\end{aligned}$$

Since ε is arbitrary, we have that $\rho(f_n, f) \to 0$ as $n \to \infty$. Now suppose that $\rho(f_n, f) \to 0$ as $n \to \infty$. Let $\varepsilon, \delta > 0$. Choose n sufficiently large so that $\rho(f_n, f) < \delta$. Define $E_{\varepsilon,n} = \{x : |f_n - f| \geq \varepsilon\}$. We have that

$$\rho(f_n, f) < \delta \implies \int_{E_{\varepsilon,n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{(E_{\varepsilon,n})^c} \frac{|f_n - f|}{1 + |f_n - f|} < \delta$$

Positiveness of $\frac{|f_n - f|}{1+|f_n - f|}$ implies that

$$\int_{E_{\varepsilon,n}} \frac{|f_n - f|}{1 + |f_n - f|} < \delta$$

Since for positive F, the mapping $E \mapsto \int_E F d\mu$ is a measure, this implies that $\mu(E_{\varepsilon,n}) < \delta$ for arbitrarily large n, as desired.