Q1a: Consider $M = \mathbb{R}$ and $\omega = x$. We have that M is clearly not compact and $\partial M = \emptyset$, and

$$\int_M d\omega \int_{\mathbb{R}} 1dx = \infty \neq \int_{\partial M} \omega = \int_{\emptyset} x = 0$$

Now suppose that M is a noncompact manifold and ω is a differential $k-1$ form which is compactly supported on some subset $S \subset M$. By linearity of integration, it is sufficient to assume that S is connected. Let c be some k orientation preserving chain such that its image is S. We will check 2 cases, if $c(I^k) \cap \partial M = \emptyset$ or if $c(I^K) \cap \partial M \neq \emptyset$. For the first case evaluate that

$$\int_M d\omega = \int_c d\omega = \int_{I^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c} \omega$$

Where the equalities follow from Stokes' Theorem on chains. Since $\omega = 0$ on the boundary of c, we have that

$$0 = \int_{\partial M} \omega = \int_M d\omega$$

Now we check the second case. Suppose that there is a singular orientation preserving k cube with $c_{(k,0)}$ being the only face in ∂M, with $\omega = 0$ outside of $c(I^k)$. Then by Stokes' Theorem on chains we have that

$$\int_M d\omega = \int_c d\omega = \int_{\partial c} \omega = \int_{\partial M} \omega$$

As desired.

Q1b: Let $\omega = d\eta$. Let M be a compact oriented manifold with no boundary. Then we can evaluate that

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0$$

Where the second equality follows from Stokes' Theorem. Use the counterexample from $1a$ as a non-compact manifold where the integral does not vanish.