Q4: This result is false. Consider $D_8 = G$, with $H = Z(G) = \langle \rho^2 \rangle$ and $H' = \{e, \sigma\}$. We have that $\varphi : H \to H'$ is an isomorphism when we define $\varphi(\rho^2) = \sigma$. Furthermore, we have that $H = Z(G) \triangleleft G$. However, we have that

$$G/H = \{\overline{e}, \overline{\rho}, \overline{\sigma}, \overline{\sigma\rho}\},$$

and

$$G/H' = \{\overline{e}, \overline{\rho}, \overline{\rho^2}, \overline{\rho^3}\}.$$

These groups are certainly not isomorphic, since every element in G/H has an order of 2 or 1, while $\overline{\rho^3} \in G/H'$ has an order of 3. By the proof of A4Q3 we know that isomorphisms must preserve the order of elements. Hence no isomorphism between these groups can exist.