Q1a: Writing $u + iv = w = cos(z)$, we compute that

$$\begin{aligned}
u + iv &= cos(z) \\
&= \frac{e^{iz} - e^{-iz}}{2} \\
&= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
&= \frac{e^{ix} \cdot e^{-y} + e^{i(-x)} \cdot e^{y}}{2} \\
&= \frac{(\cos(x) + i\sin(x))e^{-y} + (\cos(x) - i\sin(x))e^{y}}{2} \\
&= \frac{\cos(x)(e^{y} + e^{-y})}{2} + i\frac{-\sin(x)(e^{y} - e^{-y})}{2} \\
&= \cos(x)\cosh(y) - i\sin(x)\sinh(y).
\end{aligned}$$

We conclude that

$$u = \cos(x)\cosh(x), v = -\sin(x)\sinh(y)$$

Q1b: Fix some $y \in \mathbb{R}$, and we let $\alpha = \cosh(y), \beta = \sinh(y)$. We have that

$$(u(x), v(x)) = (\alpha\cos(x), \beta\sin(x)).$$

Note that this satisfies the elipse equation

$$\frac{u(x)^2}{\alpha^2} + \frac{v(x)^2}{\beta^2} = 1.$$

We claim that for $x \in (0, 2\pi]$, $(u(x), v(x))$ is injective. Suppose that for some $x, y \in (0, 2\pi]$ we have that

$$(u(x), v(x)) = (u(y), v(y)).$$

This would imply that

$$\cos(x) = \cos(y), \sin(x) = \sin(y).$$

The first equation implies that either $x = y$ or $x = 2\pi - y$. Suppose that $x = 2\pi - y$. This yields

$$\sin(x) = \sin(2\pi - x) = -\sin(x)$$

Which implies that $x = \pi$ or $x = 2\pi$. If $x = \pi$, $y = \pi$. If $x = 2\pi$ then $y = 0$, which can not happen. Thus we conclude that $x = y$ and hence this chain is injective.

Q1c: For fixed x, define $\alpha = \cos(x), \beta = -\sin(x)$. We have that

$$(u(y), v(y)) = (\alpha\cosh(y), \beta\sinh(y)).$$

Note that this satisfies the hyperbola equation

$$\frac{u(x)^2}{\alpha^2} - \frac{v(x)^2}{\beta^2} = 1.$$

Since $\cosh(y) > 0$ for all $y \in \mathbb{R}$, if $\alpha > 0$ this will define the right branch and if $\alpha < 0$ this will define the left branch.