Q1i: We verify that $\|x+\mathcal{M}\|$ is indeed a norm on $\mathfrak{X}/\mathcal{M}$. We verify semilinearity first:

$$\|\alpha x+\mathcal{M}\| = \inf_{y\in\mathcal{M}} \|\alpha x+y\| = \inf_{\alpha y\in\mathcal{M}} \|\alpha x+\alpha y\| = |\alpha| \inf_{y\in\mathcal{M}} \|x+y\| = |\alpha|\cdot\|x+\mathcal{M}\|.$$

Similarly for sublinearity, we check that

$$\|(x+w)+\mathcal{M}\| = \inf_{y\in\mathcal{M}} \|x+w+y\| = \inf_{2y\in\mathcal{M}} \|x+w+y+y\| \leq \inf_{y\in\mathcal{M}} \|x+y\|+\inf_{y\in\mathcal{M}} \|w+y\| = \|x+\mathcal{M}\|+\|w+\mathcal{M}\|.$$

Finally we verify that $\|x+\mathcal{M}\|=0$ if and only if $x\in\mathcal{M}$. First if $x\in\mathcal{M}$, we have that

$$\|x+\mathcal{M}\| = \inf_{y\in\mathcal{M}} \|x+y\| \leq \|x+(-x)\| = 0.$$

Now suppose we have that $\inf_{y\in\mathcal{M}}\|x+y\| = \|x+\mathcal{M}\| = 0$. If this infimum is never attained, then we have that for any convergent sequence $\{y_n\}$ such that $\|x+y_n\|$ is decreasing, this will never be 0. However since $\mathcal{M}$ is closed this sequence must converge to a point $y\in\mathcal{M}$ and so for some $y, \|x+y\|=0$. This only happens if $y=-x$. Hence $x\in\mathcal{M}$.

Q1ii: Note that since $\mathcal{M}$ is a closed linear subspace, the projection onto $\mathfrak{X}/\mathcal{M}$ is linear and well defined. We compute that

$$\|\pi(x)\| = \sup_{\|x\|=1} \|x+\mathcal{M}\| = \sup_{\|x\|=1}\inf_{y\in\mathcal{M}} \|x+y\| \leq \sup_{\|x\|=1} \|x\| + \inf_{y\in\mathcal{M}} \|y\| = \sup_{\|x\|=1} \|x\| = 1.$$

Now by Riesz' lemma, we have for all $\alpha\in(0,1)$, there is some $\|x\|=1$ such that

$$\inf_{y\in\mathcal{M}} \|x-y\| \geq \alpha \implies \sup_{\|x\|=1}\inf_{y\in\mathcal{M}} \|x-y\| \geq \sup_{\|x\|=1} \alpha \geq 1.$$

Thus we have that $\|\pi\|=1$.

Q1iii: It is sufficient to show that any absolutely convergent sequence also converges. Let $\{x_n+\mathcal{M}\}$ be an absoutely convergent series. Furthermore suppose that $\sum_n^\infty x_n \to x$. We have that

$$\left\|\sum_{n=1}^N x_n + \mathcal{M} - x + \mathcal{M}\right\| \leq \left\|\sum_{n=1}^N x_n - x\right\|,$$

which goes to 0 as $N\to\infty$ by completeness of $\mathfrak{X}$.