Q1:

$(a) \implies (b)$

Suppose a holds. Take $a \in M$, with open neighbourhood U and function f which satisfies the assumption. We can regard our function f as a map from $U \cap (\mathbb{R}^k_y \times \mathbb{R}^{n-k}_z)$ to $\mathbb{R}^{n-k}$ with coordinates permuted so that $\left[\frac{\partial f}{\partial z}\right]$ has rank $n-k$. By the implicit function theorem there is a C^r function from $g : U' \subset \mathbb{R}^k_y \to V' \subset \mathbb{R}^{n-k}_z$ which satisfies $f(x, g(x)) = 0$ and $(a_{k+1}, \ldots, a_n) = g(a_1, \ldots, a_k)$. This is the desired graph, and it will parametrize some neighbourhood of the point a on M since $f = 0$ exactly on the manifold.

$(b) \implies (a)$

Now suppose that condition b holds. Let U be a neighbourhood of M such that M is the graph of function g. We define $F(y, z) : U \to \mathbb{R}^{n-k}$ by $F(y, z) = g(y) - z$. We have that $F \in C^r$ and $F = 0$ if and only if $(y, z) \in M \cap U$. Furthermore, we have that

$$DF = \left[Dg \middle| -I\right]$$

which will have a rank of $n-k$ on $M \cap U$ since the identity is of maximal rank.

$(a) \implies (c)$

Suppose condition c holds. Pick coordinates (x, y) such that when we regard f as a map from $U \cap (\mathbb{R}^k_x \times \mathbb{R}^{n-k}_y)$ we have that $\left[\frac{\partial f}{\partial y}\right]$ is non singular. Define our function h as $h(x, y) = (x, f(x, y))$. We have that

$$Dh = \begin{bmatrix} I & 0 \\ * & \frac{\partial f}{\partial y} \end{bmatrix}$$

This is invertible by assumption, hence h is a C^r diffeomorphism by the inverse function theorem. Furthermore, if $(x, y) \in M$ then

$$h(x, y) = (x, f(x, y)) = (x, 0).$$

The point $x \in \mathbb{R}^k$ so h is of desired form.

$(c) \implies (a)$

We now show that c implies a. Given $h : U \to V$, a diffeomorphism satisfying the hypothesis of c, define our function f as $\pi_{n-k} \circ h$, where π_{n-k} is the projection onto the last $n-k$ coordinates. If $x \in M \cap U$, then

$$\pi_{n-k} \circ h(x) = \pi_{n-k}(y_1 \ldots y_k, 0 \ldots 0) = 0.$$

Note that since h has full rank as it is a diffeomorphism from a subsets of $\mathbb{R}^n$ to $\mathbb{R}^n$, it must have full rank. Composing it with π_{n-k} leaves the composition with rank $n-k$.

$(c) \implies (d)$

Given a diffeomorphism $h : M \to \mathbb{R}^n$ Next we show that condition c implies d. Let h, U, V be as given by c. Define the set W as the projection onto the first k coordinates of the set V. We take our coordinate patch $\varphi := h^{-1} \circ \iota_k$ where ι_k is the injection from $\mathbb{R}^k$ into $\mathbb{R}^n$. Note that φ is injective, since it is the composition of two injective functions. We also have that

$$D\varphi = D(h^{-1} \circ \iota_k) \cdot D\iota_k.$$

The rank of $D(h' \circ \iota_k)$ is n since h^{-1} is a diffeomorphism, and the rank of $D\iota_k$ is k. Hence their product will be rank k as well. Furthermore, by definition of φ we have that

$$\varphi(W) = h^{-1}(V \times \{0\}) = M \cap U.$$

It remains to show that φ is continuous with respect to the subspace topology. Take $\Omega \subset W$ to be any open set. Then,

$$\varphi(\Omega) = h^{-1}(\iota(\Omega)).$$

The inclusion is continuous with respect to the subspace toplogy, and since h^{-1} is a continuous bijection it is as well. Hence their composition must be as well.

$(d) \implies (c)$

Finally we show that d implies c. On $W \times \mathbb{R}^{n-k}_y$ define $l(x,y) = \varphi(x) + (0,y)$. This is a map into a manifold, and we have that

$$l'(x,y) = \begin{bmatrix} \frac{\partial \varphi}{\partial x} & 0 \\ 0 & I \end{bmatrix}$$

Since $\frac{\partial \varphi}{\partial x}$ has rank k, $l'(x,y)$ is invertible and so $l(x,y)$ is invertible on some neighbourhood $V \subset W \times \mathbb{R}^{n-k}$. Set $U = l(V)$. It remains to show that if $V' \subset V$ open, then $l|_{V' \cap \mathbb{R}^k}$ is contained in some $U' \cap M \subset U \cap M$ open. Since on $V \cap \mathbb{R}^k$, l agrees with φ, it is enough to show that $\varphi(V' \cap \mathbb{R}^k)$ is open in M. Indeed since φ has a continuous inverse and is 1-1, it must carry open sets to open sets. So $\varphi(V' \cap \mathbb{R}^k)$ must be open in $M \cap U$. There exists some U'' so that $\varphi(V' \cap \mathbb{R}^k) = U'' \cap M$. Set $V'' = l^{-1}(U'')$