Problem 1. *Griffiths 3.44*

First note that $\rho = 0$ everywhere except for $(0, 0, z)$ for $z \in [-a, a]$. The formula for potential is given by

$$V(r) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n \cos\theta \rho(r') d\tau'.$$

Substituting $r' = z\cos\theta$, and evaluating when $\theta = 0$ and $\theta = \frac{\pi}{2}$, we see:

$$\begin{aligned}
V(r) &= \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \frac{Q}{2a} \left[\int_0^a (r')^n P_n \cos\alpha \Big|_{\theta=0} dr' + \int_0^a (r')^n P_n \cos\alpha \Big|_{\theta=\pi} dr' \right] \\
&= \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \frac{Q}{2a} \left[\int_0^a z^n P_n \cos\theta dz + \int_0^a z^n P_n \cos(\theta - \pi) dz \right] && \text{(making the substitution)} \\
&= \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \frac{Q}{2a} [P_n \cos\theta + P_n \cos(\theta - \pi)] \int_0^a z^n dz \\
&= \frac{Q}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} P_n \cos\theta && \text{(Since when n is odd } P_n \cos\theta + P_n \cos(\theta - \pi) = 0\text{)}
\end{aligned}$$

As Desired.

Problem 2. *Griffiths 3.47*

(a) We compute the average electric field as follows:

$$\mathsf{E}_{avg} = \frac{1}{4/3\pi R^3}\int \mathsf{E}(\mathfrak{r})d\tau' = \frac{1}{4/3\pi R^3}\int \frac{-q}{4\pi\varepsilon_0}\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}d\tau' = \frac{1}{4/3\pi R^3}\cdot\frac{-q}{4\pi\varepsilon_0}\int\frac{1}{\mathfrak{r}^2}\hat{\mathfrak{r}}d\tau'.$$

Using formula 2.15 we compute the field of a sphere at as:

$$\mathsf{E}(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\rho(r')d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{1}{4/3\pi R^3}\int\frac{-q}{\mathfrak{r}^2}\hat{\mathfrak{r}}d\tau'.$$

(b) Using Gauss' Law, we compute:

$$\oint \mathsf{E}\cdot da = \frac{q_{enc}}{\varepsilon_0} \implies |\mathsf{E}|\cdot 4\pi r^2 = \frac{-q\cdot(4/3)\pi r^3}{\varepsilon_0\cdot 4/3\pi R^3} \implies \mathsf{E} = \frac{-qr}{4\pi\varepsilon_0}\hat{r} = \frac{-\mathsf{P}}{4\pi\varepsilon_0}.$$

(c) We can write an arbitrary charge distribution as $\mathsf{P} = \sum q_i r_i'$. Since integration is linear by superposition the formula above should hold as well.

(d) If we place a charge q outside of the sphere, the average E will be :

$$\mathsf{E}_{avg} = \frac{1}{4/3\pi R^3}\int \mathsf{E}da = \frac{1}{4/3\pi R^3}\frac{-q}{r^2}\cdot\frac{4/3\pi R^3}{4\pi\varepsilon_0} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}.$$

We can use superposition to extend to any distribution of charges.

Problem 3. *Griffiths 3.56*

First note that $F_{dip}(r,\theta) = \frac{q\mathbf{p}}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{r} + sin\theta\hat{\theta})$. We assume that $\phi = 0$, and r is fixed. In spherical coordinates, we have that acceleration can be written as:

$$(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}\sin^2\theta)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\phi} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} + (r\ddot{\phi} + 2\dot{r}\dot{\phi}\cos\theta)\hat{\phi}.$$

Equating this to Force, and approximating $\sin\theta$ as θ, we get that the last term vanishes, and

$$\ddot{\theta} = \frac{q \cdot p}{4\pi\varepsilon_0 r^4 m}\theta.$$

This is solved by a θ which is periodic in t.
Since r is fixed and ϕ is 0, we have that the particle traces out an arc.

Problem 4. *Griffiths 4.2*

We first compute the electric field due to the electron cloud. At a radius of r, the enclosed charge is:

$$Q_{enc} = \int_0^r \rho d\tau = \frac{4q}{3a^3}\int_0^r (r')^2 e^{-\frac{2r'}{a}} dr' = q\left(1 - e^{-\frac{2r}{a}}(1 + \frac{2r}{a} + \frac{2r^2}{a^2})\right).$$

Therefore by Gauss' Law, we have that

$$E_{in} = \frac{1}{4\pi\varepsilon_0 r^2} q\left(1 - e^{-\frac{2a}{r}}(1 + \frac{2r}{a} + \frac{2r^2}{a})\right).$$

If we apply some small electric field E_{ext}, the nucleus will get shifted by some small quantity d so that $E_{ext} = E_{in}$. We taylor expand $e^{-\frac{2r}{a}}$ as:

$$e^{-\frac{2r}{a}} = 1 - \frac{2d}{a} + \frac{2d^2}{a^2} - \frac{4d^3}{3a^3} + \dots$$

And so

$$1 - e^{-\frac{2a}{d}}\left(1 + \frac{2d}{a} + \frac{2d^2}{a^2}\right) = 1 - 1 - \frac{2d}{a} - \frac{2d^2}{a^2} + \frac{2d}{a} + \frac{4d^2}{a^2} - \frac{2d^2}{a^2} + \frac{4d^3}{3a^3} + \dots = \frac{4}{3}\frac{d^3}{a^3} + O\left(\frac{d^5}{a^5}\right).$$

Thus at d we have $E_{in} = E_{out}$ and so

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{d^2} \cdot \frac{4d^3}{3a^3} = \frac{qd}{3a^3\pi\varepsilon_0} = \frac{P}{3a^3\pi\varepsilon_0}.$$

Therefore $\alpha = 3a^3\pi\varepsilon_0$.

Problem 5. *Griffiths 4.4*

We have that the electric field of q at distance r is

$$|E| = \frac{q}{4\pi\varepsilon_0 r^2}.$$

We can place the charge on the x axis so that $\vec{p} = \alpha E = \frac{\alpha q}{4\pi\varepsilon_0 r^2}\hat{x}$. We know that $E_{dip}(r) = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(p \cdot \hat{r}) - p\right]$. Since $\hat{p} = \hat{x}$, we have that

$$3p \cdot \hat{r} - p = \frac{3\alpha q}{4\pi\varepsilon_0 r^3}\hat{x} - \frac{\alpha q}{4\pi\varepsilon_0 r^3}\hat{x} = \frac{2\alpha q}{4\pi\varepsilon_0 r^3}\hat{x}.$$

Therefore $F = qE = \frac{2\alpha q^2}{16\pi^2\varepsilon_0^2 r^3}\hat{x}$.

Problem 6. *Griffiths 4.6*

We first place an dipole at $-z$. The boundary condtions are the same, so we can instead compute the torque this dipole exerts on p. We have that $\mathrm{E}_{\mathrm{dip}}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$. At a distance of $2z$ this becomes:

$$\mathrm{E}_{\mathrm{dip}} = \frac{p}{4\pi\varepsilon_0(2z)^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}).$$

We can also write the dipole as $p = p\cos\theta\hat{r} + p\sin\theta\hat{\theta}$. We compute the torque as

$$\mathrm{N} = p \times \mathrm{E} = \frac{p^2}{4\pi\varepsilon_0(2z)^3}\left[(\cos\theta\hat{r} + \sin\theta\hat{\theta}) \times (2\cos\theta\hat{r} + \sin\theta\hat{\theta})\right] = \frac{-p^2\sin\theta\cos\theta}{4\pi\varepsilon_0(2z^3)}\hat{\phi}.$$

If we allow the dipole to rotate, it will come to a rest when the torque is at a minimum. so at $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$.