5.5.58a: We have as given that

$$\langle Px - x, Px\rangle = \langle Px, Px - x\rangle = 0.$$

Together with linearity this implies that for all x,

$$\langle Px, x\rangle = \langle x, Px\rangle.$$

Thus we have that $P^* = P$. Furthermore, linearity implies that

$$\langle Px, Px\rangle = \langle P^2 x, x\rangle = \langle Px, x\rangle.$$

Therefore $P^2 = P$.

5.5.58b: We first claim that $R(P)$ is closed. Let $\{x_n\}$ be a sequence in $R(P)$, that converges to some x. Then we have that $Px_n = x_n$, and so

$$\langle Px_n - x_n, x_n\rangle = 0.$$

Inner product continuity implies that

$$\langle Px - x, x\rangle = 0.$$

Therefore $Px = x$ and so $x \in R(P)$. Thus $R(P)$ is closed. We now verify that $\langle Px - x, Px\rangle = 0$. Since $P^2 = P = P^*$.

$$\langle Px - x, Px\rangle = \langle Px - x, P^2 x\rangle = \langle P^2 x - Px, Px\rangle = 0.$$

5.5.58c: It is enough to check that $Px = \sum\langle x, u_\alpha\rangle u_\alpha$ satisfies $\langle Px - x, Px\rangle = 0$.

$$\begin{aligned}
\langle \sum\langle x, u_\alpha\rangle u_\alpha - x, \sum\langle x, u_\alpha\rangle u_\alpha\rangle &= \langle \sum\langle x, u_\alpha\rangle u_\alpha, \sum\langle x, u_\alpha\rangle u_\alpha\rangle - \langle x, \sum\langle x, u_\alpha\rangle u_\alpha\rangle \\
&= \langle \sum\langle x, u_\alpha\rangle u_\alpha, \sum\langle x, u_\alpha\rangle u_\alpha\rangle - \sum\langle x, u_\alpha\rangle \cdot \sum\langle x, u_\alpha\rangle u_\alpha \\
&= 0
\end{aligned}$$