Problem 1.

(a) We claim that $P(t)$ is conserved. We compute that:

$$\begin{aligned}\dot{P}(t) &= \int \frac{d}{dt} u_t u_x \, dx \\ &= \int u_{tt}u_x + u_{tx}u_t \, dx \\ &= \int c^2 u_{xx}u_x + u_{tx}u_x \, dx\end{aligned}$$

Computing each summand seperately, we have that

$$\int u_{tx}u_t \, dx = u_t \cdot u_t|_{\mathbb{R}} - \int u_{tx}u_t \, dx = -\int u_{tx}u_t \, dx \implies \int u_{tx}u_t \, dx = 0,$$

assuming that u_t vanishes at ∞. Similarly if we assume that u_x vanishes at ∞. we have that

$$\int u_{xx}u_x \, dx = -\int u_x x u_x \, dx \implies \int u_x x u_x \, dx = 0.$$

(b) We compute:

$$e_t = u_t \cdot u_{tt} + c^2 u_x \cdot u_{xt} = c^2 \left(u_{tx} \cdot u_x + \frac{1}{c^2} u_t \cdot u_{tt} \right) = c^2 p_x.$$

Similarly, we compute that

$$p_t = u_{tt} \cdot u_t + u_t \cdot u_{xt} = e_x.$$

Thus we see that

$$e_{tt} = c^2 p_{xt}, e_{xx} = p_{xt} \implies e_{tt} - c^2 e_{xx} = 0.$$

For p we get

$$p_{tt} = e_{xt}, c^2 p_{xx} = e_{xt} \implies p_{xx} - \frac{1}{c^2} p_{tt}.$$

(c) Since we have that E, P are both conserved. Therefore they are independant of the choice of t. So we can write

$$E(t) = E(0) = \int \frac{1}{2} \phi_0^2(x) + c^2 \phi_1^2(x) \, dx,$$

and

$$P(t) = \int \phi_0(x)\phi_1(x) \, dx$$

Problem 2.

(a) As a reminder we write

$$E(t) = \int \frac{1}{2}u_t^2 + c^2u_x^2\,dx.$$

Taking the time derivative, we see

$$\frac{d}{dt}E(t) = \int u_t u_{tt} + c^2 u_x u_{xt}\,dx = \int u_t(u_{tt} - c^2u_{xx})\,dx = -r\int u_t^2\,dx.$$

Where we use integration by parts, and the PDE condition. Note that $E(0)$ is positive, and the derivative is decreasing strictly. Therefore as $t \to \infty$ then $E(t) \to 0$.

(b) Yes solutions are unique. Suppose u_1, u_2 are both solutions with the same initial datum $u^i(x,0) = \phi_0(x)$, $u_t^i(x,0) = \phi_1(x)$. Write $v = u_1 - u_2$. Note that v has 0 boundary conditions. Note that $E(0) = 0$, but is also always at least 0. Since E is decreasing we have that $E(t) = 0$ for all time. Therefore $v \equiv 0$ and $u_1 = u_2$.

(c) If $r < 0$ then

$$\dot{E}(t) = -r\int u_t^2\,dx > 0.$$

Thus the energy must go off to infinity as time goes to infinity. Note that we can write

$$\dot{E}(t) = -2rE(t) - r\int c^2u_{xx}\,dx \implies \dot{E}(t) \leqslant -3rE(t).$$

By Gronwalls inequality, we have that $E(t) = E(0)e^{-3rt}$. Therefore if we take two initial value problems as we did in b), and apply energy to their difference we have that $E(0) = 0$. By gronwalls we have $E(t) = 0$ for all t. Thus $v \equiv 0$ and so uniqueness also holds.

Problem 3.

(a) If L lorentz, we have
$$L^{-1} = \Gamma L^{T} \Gamma.$$
Since $\Gamma^{-1} = \Gamma$ taking the inverse of both sides we see that
$$L = \Gamma {L^{-1}}^{T} \Gamma.$$
Therefore L^{-1} is lorentz. Now suppose that L, M are both lorzentz. Then,
$$(LM)^{-1} = M^{-1}L^{-1} = \Gamma M^{T} \Gamma \Gamma L^{T} \Gamma = \Gamma (LM)^{T} \Gamma.$$

(b) Note that $m(w) = w^{T} \Gamma w$. First suppose that L is lorentz. Then,
$$m(Lw) = (Lw)^{T} \Gamma L w = w^{T} \Gamma \Gamma L^{T} \Gamma L w = w^{T} \Gamma L^{-1} L L w = w \Gamma w.$$
Suppose the converse. Then we have that
$$w^{T} L^{T} \Gamma L w = w^{T} \Gamma w.$$
Since this is true for all w, we have that $L^{T} \Gamma L = \Gamma$, and so L is invertible. Thus we have that
$$L^{T} \Gamma = \Gamma L^{-1} \implies L^{-1} = \Gamma L^{T} \Gamma.$$

(c) We write the following vector $v = \begin{bmatrix} \partial_x & \partial_y & \partial_z & \partial_t \end{bmatrix}$. We have that $m(v)u = 0$. Therefore $m(Lv)u = 0$ or
$$(vL^{T} \Gamma L v)u = u(L(x, y, z, t))$$

(d) We write the matrix of L as follows:
$$L = \begin{bmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{bmatrix}.$$
We compute the inverse as
$$L^{-1} = \begin{bmatrix} -\frac{-1}{\gamma(v+1)(v-1)} & 0 & 0 & \frac{v}{\gamma - \gamma v^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v}{\gamma - \gamma v^2} & 0 & 0 & \frac{1}{\gamma - \gamma v^2} \end{bmatrix}.$$
Using the fact that $L^{-1} = \Gamma L^{T} \Gamma$, we have that
$$\begin{bmatrix} -\frac{-1}{\gamma(v+1)(v-1)} & 0 & 0 & \frac{v}{\gamma - \gamma v^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v}{\gamma - \gamma v^2} & 0 & 0 & \frac{1}{\gamma - \gamma v^2} \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{bmatrix}.$$
For this to hold i.e. L be lorenztiam we must have that $v \in (-1, 1), \gamma > 1$ and
$$\gamma^2 = \frac{1}{1 - v^2}$$

Problem 4.

(a) Let $w = v - u$. Note that $w \geqslant 0$ on the boundary. If at some point a, b we had that $u - v(a, b) > 0$, then $w(a, b) < 0$. This can not happen by the minimum principle.

(b) We first claim that if u solves $u_t = u_{xx} + f$, then $\min_{int} u \geqslant \min_{bd} u + l^2 \min_{bd} f$. Consider the function $v = u + cx^2 - l^2 c$, where we choose $c = \frac{\min f}{2}$. We have that $v_t = u_t$ and $v_{xx} = u_{xx} + \frac{\min f}{2}$. We have that $v_t \geqslant v_{xx}$, Therefore v attains its min when $x = 0$ or $x = l$ or $t = 0$ by the minimum principle. Therefore we have $\min_{int} u \geqslant \min_{bd} u + l^2 \min_{bd} f$. We now apply this to $w = v - u$. We must have that $\min_{int} w \geqslant \min_{bd} w + l^2 \min_{bd} l^2(g - f)$. The lefthand side ofe the equality must always be positive since the righthand side is always positive. Therefore $v \geqslant u$.

(c) Take $u(t, x) = (1 - e^{-t}) \sin x$. We see that $u_t - u_{xx} = \sin x$. As well, we have $u(0, x) = \sin x$, $u(t, 0) = 0 = u(t, \pi)$. Since we have that $v \geqslant u$ on the boundary, then by part b it must also be true on the interiour i.e. , $v(x, t) \geqslant (1 - e^{-t}) \sin x$.

Problem 5.

(a) We define the function $v(x,t) = u(x,t) - \varepsilon\left(t + \frac{1}{2}|x-y|^2\right)$ for arbitrary y. Set $\rho = |x-y|$. Define

$$\Omega = \{x, t : |x-y| < \rho, 0 < t < T\}.$$

By the proof of the weak max principle, have that v must satisfy the weak max principle, so therefore

$$v(x,t) \leqslant \max\left\{f(x) - \frac{\varepsilon}{2}\rho^2, u(y-\rho,t) - \varepsilon t + \frac{\varepsilon}{2}\rho^2, u(y+\rho,t) - \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\rho^2\right\}.$$

If we take $x = y$, then for all y, ρ,

$$v(y,t) \leqslant \sup_{x\in\mathbb{R}} f(x),$$

since we can range x over all $\mathbb{R}$. Since this is true for all ε, we have that $u(y,t) \leqslant \sup_{x\in\mathbb{R}} f(x)$. Since this system is linear if we have two solutions u_1, u_2 for the initial value $f(x)$, their difference will be a solution to the same equation with initial value 0. Since the solution is bounded above by 0 is must be identically 0.

(b) We have solved the heat equation on the real line. We have that

$$u(x,t) = \frac{1}{2\sqrt{kt}}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} f(y)\,dy.$$

Therefore,

$$|u(x,t)| = \left|\frac{1}{2\sqrt{kt}}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} f(y)dy\right| \leqslant Ct^{-1/2}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4kt}} dy \cdot \int_{\mathbb{R}} |f(y)|dy \leqslant Dt^{-1/2},$$

Where we use the fact that $\int e^{\frac{-(x-y)^2}{4kt}} dy$ is a constant independant of t, and $f \in L^1$.