

Q3: Suppose that there is some S so that $S \circ T = id_E$. First we claim that $R(T)$ is closed. Suppose that $x_n \rightarrow x$, $Tx_n \rightarrow y$. Then we have that $Tz = y$ for some z . By uniqueness of limits and injectivity we have that $Tx_n \rightarrow Tx$. Now let S be the left inverse of T . We claim that $\ker S$ is the complement of $R(T)$. If $v \in \ker S \cap R(T)$, then $Sv = 0$ but for some w , $Tw = v$ so $0 = STw = id_E w$ so $w = 0$ and so $v = 0$. Now if $v \in F$, $TSv - v = u \in \ker S$ so $v = TSv - u$. Suppose that $R(T)$ is closed and admits a complement G . Every $v \in F$ can be written as $v = x + y$, $x \in R(T)$, $y \in G$. Since $x \in R(T)$ we can write $x = Tu$. Define $Sv = x$. This will be our left inverse since $STx = x$.