

THE GEOMETRY OF DIMER MODELS

DAVID CIMASONI

ABSTRACT. This is an expanded version of a three-hour minicourse given at the winterschool *Winterbraids IV* held in Dijon in February 2014. The aim of these lectures was to present some aspects of the dimer model to a geometrically minded audience. We spoke neither of braids nor of knots, but tried to show how several geometrical tools that we know and love (e.g. (co)homology, spin structures, real algebraic curves) can be applied to very natural problems in combinatorics and statistical physics. These lecture notes do not contain any new results, but give a (relatively original) account of the works of Kasteleyn [14], Cimasoni-Reshetikhin [4] and Kenyon-Okounkov-Sheffield [16].

CONTENTS

Foreword	1
1. Introduction	1
2. Dimers and Pfaffians	2
3. Kasteleyn's theorem	4
4. Homology, quadratic forms and spin structures	7
5. The partition function for general graphs	8
6. Special Harnack curves	11
7. Bipartite graphs on the torus	12
References	15

FOREWORD

These lecture notes were originally not intended to be published, and the lectures were definitely not prepared with this aim in mind. In particular, I would like to stress the fact that they do not contain any new results, but only an exposition of well-known results in the field. Also, I do not claim this treatement of *the geometry of dimer models* to be complete in any way. The reader should rather take these notes as a personal account by the author of some selected chapters where the words *geometry* and *dimer models* are not completely irrelevant, chapters chosen and organized in order for the resulting story to be almost self-contained, to have a natural beginning, and a happy ending.

1. INTRODUCTION

Let Γ be a finite connected graph, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A *perfect matching* on Γ is a choice of edges of Γ such that each vertex of Γ is adjacent to exactly one of these edges. In statistical mechanics, a perfect matching on Γ is also known as a *dimer configuration* on Γ , and the edges of the perfect matching

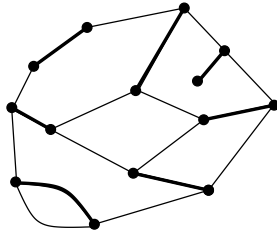


FIGURE 1. A dimer configuration on a graph.

are called *dimers*. We shall denote by $\mathcal{D}(\Gamma)$ the set of dimer configurations on Γ . An example is given in Figure 1.

The first natural question to address is whether a given finite graph Γ admits a perfect matching at all. Clearly, Γ needs to have an even number of vertices, but this condition is not sufficient: the γ -shaped graph has 4 vertices but no perfect matching. There is an efficient algorithm to answer this question: it is the so-called *Hungarian method* in the case of bipartite graphs [17], that was extended by Edmonds [7] to the general case. We refer to these sources for purely combinatorial aspects of matching theory. Unless otherwise stated, we will only consider finite graphs which admit perfect matchings. In particular, all the graphs will have an even number of vertices.

The second question is: given a fixed finite graph Γ , can one count the number $\#\mathcal{D}(\Gamma)$ of perfect matchings on Γ ? The answer to this question turns out to be much more subtle. As proved by Valient [22], there is no efficient algorithm to compute this number for an arbitrary finite graph. However, in the case of planar graphs (and more generally, of graphs of a fixed genus), there is a polynomial time algorithm to compute the number of perfect matchings. This will be the subject of a good part of these notes.

In fact, it is possible to compute not only the number of dimer configurations, but also the weighted sum of these configurations, that now we define. An *edge-weight system* on Γ is a positive real-valued function ν on the set of edges of Γ . Such a weight system ν defines a probability measure on the set of dimer configurations:

$$\mathbb{P}(D) = \frac{\nu(D)}{Z(\Gamma; \nu)},$$

where $\nu(D) = \prod_{e \in D} \nu(e)$ and

$$Z(\Gamma; \nu) = \sum_{D \in \mathcal{D}(\Gamma)} \nu(D).$$

The normalization constant $Z(\Gamma; \nu)$ is called the *partition function* for the *dimer model* on the graph Γ with weight system ν . Note that if the weight system is constant equal to 1, then $Z(\Gamma; \nu)$ is nothing but the number $\#\mathcal{D}(\Gamma)$ of dimer configurations on Γ .

2. DIMERS AND PFAFFIANS

Recall that the determinant of a skew-symmetric matrix $A = (a_{ij})$ of size $2n$ is the square of a polynomial in the a_{ij} 's. This square root, called the *Pfaffian* of A ,

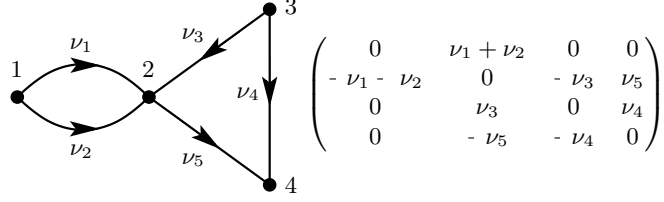


FIGURE 2. An oriented weighted graph, and the corresponding weighted skew-adjacency matrix.

is given by

$$\text{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)},$$

where the sum is over all permutations of $\{1, \dots, 2n\}$ and $\text{sgn}(\sigma) \in \{\pm 1\}$ denotes the signature of σ . Since A is skew-symmetric, the monomial corresponding to $\sigma \in S_{2n}$ only depends on the matching of $\{1, \dots, 2n\}$ into unordered pairs $\{\{\sigma(1), \sigma(2)\}, \dots, \{\sigma(2n-1), \sigma(2n)\}\}$. As there are $2^n n!$ different permutations defining the same matching, we get

$$\text{Pf}(A) = \sum_{[\sigma] \in \Pi} \text{sgn}(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)},$$

where the sum is on the set Π of matchings of $\{1, \dots, 2n\}$. Note that there exists a skew-symmetric version of the Gauss elimination algorithm, based on the equality $\text{Pf}(BAB^T) = \det(B)\text{Pf}(A)$. This allows us to compute the Pfaffian of a matrix of size $2n$ in $\mathcal{O}(n^3)$ time.

The celebrated *FKT algorithm* – named after Fisher, Kasteleyn and Temperley [8, 12, 20] – allows us to express the dimer partition function of a certain class of graphs as a Pfaffian. It is based on the following beautifully simple computation.

Enumerate the vertices of Γ by $1, 2, \dots, 2n$, and fix an arbitrary orientation K of the edges of Γ . Let $A^K = (a_{ij}^K)$ denote the associated weighted skew-adjacency matrix; this is the $2n \times 2n$ skew-symmetric matrix whose coefficients are given by

$$a_{ij}^K = \sum_e \varepsilon_{ij}^K(e) \nu(e),$$

where the sum is on all edges e in Γ between the vertices i and j , and

$$\varepsilon_{ij}^K(e) = \begin{cases} 1 & \text{if } e \text{ is oriented by } K \text{ from } i \text{ to } j; \\ -1 & \text{otherwise.} \end{cases}$$

An example is given in Figure 2.

Now, let us consider the Pfaffian of this matrix. A matching of $\{1, \dots, 2n\}$ contributes to $\text{Pf}(A^K)$ if and only if it is realized by a dimer configuration on Γ , and this contribution is $\pm \nu(D)$. More precisely,

$$(1) \quad \text{Pf}(A^K) = \sum_{D \in \mathcal{D}(\Gamma)} \varepsilon^K(D) \nu(D),$$

where the sign $\varepsilon^K(D)$ can be computed as follows: if the dimer configuration D is given by edges e_1, \dots, e_n matching vertices i_ℓ and j_ℓ for $\ell = 1, \dots, n$, let σ denote

the permutation sending $(1, \dots, 2n)$ to $(i_1, j_1, \dots, i_n, j_n)$; the sign is given by

$$\varepsilon^K(D) = \text{sgn}(\sigma) \prod_{\ell=1}^n \varepsilon_{i_\ell j_\ell}^K(e_\ell).$$

The problem of expressing $Z(\Gamma; \nu)$ as a Pfaffian now boils down to finding an orientation K of the edges of Γ such that $\varepsilon^K(D)$ does not depend on D .

Let us fix $D, D' \in \mathcal{D}(\Gamma)$, and try to compute the product $\varepsilon^K(D)\varepsilon^K(D')$. Note that the symmetric difference $D \Delta D'$ consists of a disjoint union of simple closed curves C_1, \dots, C_m of even length in Γ . Since the matchings D and D' alternate along these cycles, one can choose permutations σ (resp. σ') representing D (resp. D') such that $\tau := \sigma' \circ \sigma^{-1} \in S_{2n}$ is the product of the cyclic permutations defined by the cycles C_1, \dots, C_m . Using this particular choice of representatives, and the fact that $\text{sgn}(\tau) = (-1)^m$, we find

$$(2) \quad \varepsilon^K(D)\varepsilon^K(D') = \prod_{i=1}^m (-1)^{n^K(C_i)+1},$$

where $n^K(C_i)$ denotes the number of edges of C_i where a fixed orientation of C_i differs from K . Since C_i has even length, the parity of this number is independent of the chosen orientation of C_i .

Therefore, we are now left with the problem of finding an orientation K of Γ with the following property: for any cycle C of even length such that $\Gamma \setminus C$ admits a dimer configuration, $n^K(C)$ is odd. Such an orientation was called *admissible* by Kasteleyn [14]; nowadays, the term of *Pfaffian orientation* is commonly used. By the discussion above, if K is a Pfaffian orientation, then $Z(\Gamma; \nu) = |\text{Pf}(A^K)| = \sqrt{|\det(A^K)|}$.

3. KASTELEYN'S THEOREM

Kasteleyn's celebrated theorem asserts that every planar graph admits a Pfaffian orientation. More precisely, let Γ be a graph embedded in the plane. Each face f of $\Gamma \subset \mathbb{R}^2$ inherits the (say, counterclockwise) orientation of \mathbb{R}^2 , so ∂f can be oriented as the boundary of the oriented face f .

Theorem 3.1 (Kasteleyn [13, 14]). *Given $\Gamma \subset \mathbb{R}^2$, there exists an orientation K of Γ such that, for each face f of $\Gamma \subset \mathbb{R}^2$, $n^K(\partial f)$ is odd. Furthermore, such an orientation is Pfaffian, so $Z(\Gamma; \nu) = |\text{Pf}(A^K)|$.*

A striking consequence of this theorem is that it enables to compute the dimer partition function for planar graphs in polynomial time.

Following Kasteleyn, we shall break the proof into three lemmas. Also, we shall say that an orientation K satisfies the *Kasteleyn condition* around a face f if $n^K(\partial f)$ is odd. An orientation satisfying the Kasteleyn condition around each face is a *Kasteleyn orientation*, and the corresponding skew-adjacency matrix a *Kasteleyn matrix*.

Lemma 3.2. *There exists a Kasteleyn orientation on any planar graph $\Gamma \subset \mathbb{R}^2$.*

Proof. Fix a spanning tree T in $\Gamma^* \subset S^2$, the graph dual to the graph $\Gamma \subset S^2$, rooted at the vertex corresponding to the outer face of $\Gamma \subset S^2$. Orient arbitrarily the edges of Γ which do not intersect T . Moving down the tree from the leaves to the root, remove the edges of T one by one, and orient the corresponding edge of Γ

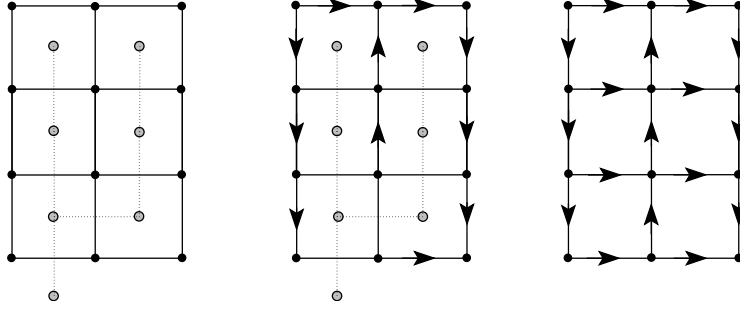


FIGURE 3. Explicit construction of a Kasteleyn orientation.

in the unique way such that the Kasteleyn condition is satisfied around the newly oriented face. When all the edges of T are removed, we are left with an orientation K of Γ satisfying the Kasteleyn condition around each face of $\Gamma \subset \mathbb{R}^2$. This is illustrated in Figure 3. \square

One might wonder what happens with the outer face of $\Gamma \subset \mathbb{R}^2$. It turns out that the orientation K of $\Gamma \subset \mathbb{R}^2$ also satisfies the Kasteleyn condition around the outer face if and only if Γ has an even number of vertices. This will follow from a more general statement in Section 5.

Lemma 3.3. *Let K be a Kasteleyn orientation on $\Gamma \subset \mathbb{R}^2$. Then, given any counterclockwise oriented simple closed curve C in Γ , $n^K(C)$ and the number of vertices of Γ enclosed by C have opposite parity.*

Proof. Let us check this result by induction on the number $k \geq 1$ of faces enclosed by C . The case $k = 1$ is exactly the Kasteleyn condition, so let us assume the statement true for any simple closed curve enclosing up to $k \geq 1$ faces, and let $C = \partial S$ be a simple closed curve enclosing $k + 1$ faces and m vertices. Pick a face f in S such that the closure of f intersects C in an interval. (This always exists, except in the case where a unique face f meets C in more than a vertex; this degenerate case can be treated in a similar way.) Then, $S' := S \setminus f$ contains k faces, and $C' := \partial S'$ is still a simple closed curve. Also, f meets S' in an interval made of a certain number – say $\ell \geq 1$ – of edges of Γ . It follows that S' contains $m - \ell + 1$ vertices. Using the induction hypothesis and the Kasteleyn condition around f , we get the modulo 2 equality

$$n^K(C) = n^K(C') + n^K(\partial f) - \ell \equiv (m - \ell) + 1 - \ell \equiv m + 1,$$

and the lemma is proved. \square

Lemma 3.4. *Any Kasteleyn orientation on $\Gamma \subset \mathbb{R}^2$ is a Pfaffian orientation.*

Proof. Let C be a cycle of even length in Γ such that $\Gamma \setminus C$ admits a dimer configuration D . This implies that C encloses an even number of vertices, as these vertices are matched by D . By Lemma 3.3, $n^K(C)$ is odd, so K is a Pfaffian orientation. \square

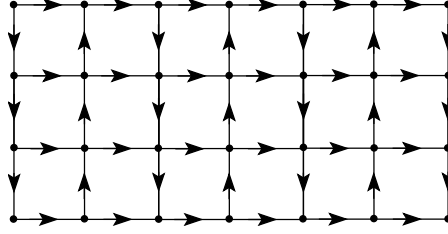


FIGURE 4. A Kasteleyn orientation on a square lattice.

Example 3.1. The orientation given in Figure 2 is Kasteleyn. The corresponding Pfaffian is equal to

$$\text{Pf}(A^K) = \text{Pf} \begin{pmatrix} 0 & \nu_1 + \nu_2 & 0 & 0 \\ -\nu_1 - \nu_2 & 0 & -\nu_3 & \nu_5 \\ 0 & \nu_3 & 0 & \nu_4 \\ 0 & -\nu_5 & -\nu_4 & 0 \end{pmatrix} = \nu_1 \nu_4 + \nu_2 \nu_4.$$

This is clearly equal to the dimer partition function $Z(\Gamma; \nu)$.

As a less trivial example, let us now consider the case of the square lattice. Note that this example is a very special case of Theorem 7.3 below.

Example 3.2. Let Γ_{mn} be the $m \times n$ square lattice, and let ν be the edge weight system assigning to each horizontal (resp. vertical) edge the weight x (resp. y). If neither m nor n are even, then Γ_{mn} has an odd number of vertices and the partition function $Z(\Gamma_{mn}; \nu)$ vanishes. Let us therefore assume that m is even. Let K be the orientation of $\Gamma_{mn} \subset \mathbb{R}^2$ illustrated in Figure 4. This orientation is obviously Kasteleyn, so $Z(\Gamma_{mn}; \nu) = |\text{Pf}(A^K)| = \sqrt{\det(A^K)}$, where A_K is the $mn \times mn$ weighted skew-adjacency matrix associated to K . The determinant of A^K can be computed explicitly, leading to the following formula (see [12]):

$$Z(\Gamma_{mn}; x, y) = \prod_{k=1}^{\frac{1}{2}m} \prod_{\ell=1}^n 2 \left(x^2 \cos^2 \frac{k\pi}{m+1} + y^2 \cos^2 \frac{\ell\pi}{n+1} \right)^{1/2}.$$

Using this equation, Kasteleyn computed the following limit:

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z(\Gamma_{nn}; x, y) = \frac{1}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \log(4(x^2 \cos^2 \varphi + y^2 \cos^2 \psi)) d\varphi d\psi.$$

In particular, setting $x = y = 1$ in the first equation above, we find that the number of dimer configurations on the $m \times n$ square lattice is equal to

$$\#\mathcal{D}(\Gamma_{mn}) = \prod_{k=1}^{\frac{1}{2}m} \prod_{\ell=1}^n 2 \left(\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{\ell\pi}{n+1} \right)^{1/2}.$$

For example, the 4×3 lattice illustrated in Figure 3 admits 11 dimer configurations. For $n \times n$ lattices with $n = 2, 4, 6, 8$, this number is equal to 2, 36, 6728, 12988816, respectively. Finally, setting $x = y = 1$ in the second equation displayed above, Kasteleyn established that this number grows as

$$\#\mathcal{D}(\Gamma_{nn}) \sim e^{\frac{G}{\pi} n^2},$$

where $G = 1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \dots = 0.915965\dots$ is Catalan's constant.

Remark 3.5. If a finite graph Γ is bipartite (with B black and W white vertices), then any associated skew-adjacency matrix will be of the form

$$A^K = \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix},$$

with M of size $B \times W$. If Γ admits a dimer configuration, then this matrix is square (say, of size k), and

$$\text{Pf}(A^K) = (-1)^{\frac{k(k-1)}{2}} \det(M).$$

Therefore, if Γ is a planar graph and K is a Kasteleyn orientation, then

$$Z(\Gamma; \nu) = |\det(M)|.$$

What about non-planar graphs? There is no hope to extend Theorem 3.1 *verbatim* to the general case, as some graphs do not admit a Pfaffian orientation. (The complete bipartite graph $K_{3,3}$ is the simplest example). More generally, enumerating the dimer configurations on a graph is a $\#P$ -complete problem [22].

However, Kasteleyn [12] was able to compute the dimer partition function for the square lattice on the torus using 4 Pfaffians (see Example 5.1 below). He also stated without further detail that the partition function for a graph of genus g requires 2^{2g} Pfaffians [13]. Such a formula was found much later by Tesler [21] and Galluccio-Loebl [9], independently. (See also [6].) These authors generalize by brute force, so to speak, the combinatorial proof given by Kasteleyn in the planar case and in the case of the biperiodic square lattice.

The aim of the following two sections is to explain an alternative, geometric version of this Pfaffian formula [4, 5, 3]. This approach relies on some geometric tools that we very briefly recall in the following (somewhat dry) section.

4. HOMOLOGY, QUADRATIC FORMS AND SPIN STRUCTURES

The planarity of Γ was used in a crucial way in the proof of Kasteleyn's theorem: we used the fact that a cycle in a planar graph bounds a collection of faces. In general, any finite graph can be embedded in a closed orientable surface Σ of some genus g , but if g is non-zero, then some cycles might not bound faces.

There is a standard tool in algebraic topology to measure how badly the fact above does not hold: it is called *homology*, and we now briefly recall its definition in our context.

Given a graph $\Gamma \subset \Sigma$ whose complement consists of topological discs, let C_0 (resp. C_1 , C_2) denote the \mathbb{Z}_2 -vector space with basis the set of vertices (resp. edges, faces) of $\Gamma \subset \Sigma$. Also, let $\partial_2: C_2 \rightarrow C_1$ and $\partial_1: C_1 \rightarrow C_0$ denote the *boundary operators* defined in the obvious way. Since $\partial_1 \circ \partial_2$ vanishes, the space of 1-cycles $\ker(\partial_1)$ contains the space $\partial_2(C_2)$ of 1-boundaries. The *first homology space* $H_1(\Sigma; \mathbb{Z}_2) := \ker(\partial_1) / \partial_2(C_2)$ turns out not to depend on Γ , but only on Σ : it has dimension $2g$, where g is the genus of the closed connected orientable surface Σ . Note that the intersection of curves defines a non-degenerate bilinear form on $H_1(\Sigma; \mathbb{Z}_2)$, that will be denoted by $(\alpha, \beta) \mapsto \alpha \cdot \beta$.

We now turn to quadratic forms. Let H be a \mathbb{Z}_2 -vector space endowed with a non-degenerate bilinear form $(\alpha, \beta) \mapsto \alpha \cdot \beta$. A *quadratic form* on (H, \cdot) is a map $q: H \rightarrow \mathbb{Z}_2$ such that $q(\alpha + \beta) = q(\alpha) + q(\beta) + \alpha \cdot \beta$ for all $\alpha, \beta \in H$. Note that there are exactly $|H|$ quadratic forms on (H, \cdot) , as the set of such forms is an affine

space over $\text{Hom}(H, \mathbb{Z}_2)$. Furthermore, it can be showed [1] that these forms are classified by their *Arf invariant* $\text{Arf}(q) \in \mathbb{Z}_2$. We shall need a single property of this invariant, namely that it satisfies the equality

$$(3) \quad \frac{1}{\sqrt{|H|}} \sum_q (-1)^{\text{Arf}(q) + q(\alpha)} = 1$$

for any $\alpha \in H$, where the sum is over all quadratic forms on (H, \cdot) .

Quadratic forms are the algebraic avatar of a geometric object called a *spin structure*. We shall not go into the trouble of giving the formal definition (see e.g. [2]); let us simply recall that any spin structure on a closed orientable surface is given by a vector field with zeroes of even index.

The relationship between spin structures and quadratic forms is given by the following classical result of Johnson [11]. Consider a spin structure λ on Σ represented by a vector field Y on Σ with zeroes of even index. Given a piecewise smooth closed curve γ in Σ avoiding the zeroes of Y , let $\text{rot}_\lambda(\gamma) \in 2\pi\mathbb{Z}$ denote the rotation angle of the velocity vector of γ with respect to Y . Then, given a homology class $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$ represented by the disjoint union of oriented simple closed curves γ_j , the equality $(-1)^{q_\lambda(\alpha)} = \prod_j -\exp(\frac{i}{2}\text{rot}_\lambda(\gamma_j))$ gives a well-defined quadratic form on $(H_1(\Sigma; \mathbb{Z}_2), \cdot)$. Furthermore, Johnson's theorem asserts that the mapping $\lambda \mapsto q_\lambda$ defines a bijection between the set of spin structures on Σ and the set of quadratic forms on $(H_1(\Sigma; \mathbb{Z}_2), \cdot)$.

5. THE PARTITION FUNCTION FOR GENERAL GRAPHS

Let $\Gamma \subset \Sigma$ be a fixed *surface graph*, i.e. a graph embedded in an orientable closed surface Σ such that $\Sigma \setminus \Gamma$ consists of topological disks. Throughout this section, we shall use the same notation X for the surface graph and the induced cellular decomposition of Σ .

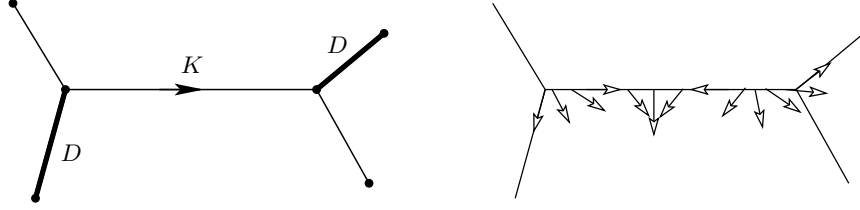
We will now try to encode combinatorially a spin structure on a surface Σ , or equivalently, a vector field on Σ with isolated zeroes of even index. This will lead us to a very natural “re-discovery” of the notion of a Kasteleyn orientation. Let X be a fixed cellular decomposition of Σ .

- To construct a (unit length) vector field along the 0-skeleton X^0 , we just need to specify one tangent direction at each vertex of X . Such an information is given by a dimer configuration D on X^1 : at each vertex, point in the direction of the adjacent dimer.

- This vector field along X^0 extends to a unit vector field on X^1 , but not uniquely. Roughly speaking, it extends in two different natural ways along each edge of X^1 , depending on the sense of rotation of the resulting vector field. We shall encode this choice by an orientation K of the edges of X^1 , together with the following convention: moving along an oriented edge, the tangent vector first rotates counterclockwise until it points in the direction of the edge, then rotates clockwise until it points backwards, and finally rotates counterclockwise until it coincides with the tangent vector at the end vertex. This is illustrated in Figure 5.

- Each face of X being homeomorphic to a 2-disc, the unit vector field defined along X^1 naturally extends to a vector field Y_D^K on X , with one isolated zero in the interior of each face.

One easily checks that for each face f of X , the index of the zero of Y_D^K in f has the parity of $n^K(\partial f) + 1$. Therefore, the vector field Y_D^K defines a spin structure

FIGURE 5. Construction of the vector field along the 1-skeleton of X .

if and only if K satisfies the Kasteleyn condition around each face. Hence, we shall say that an orientation K of the 1-cells of a surface graph X is a *Kasteleyn orientation on X* if, for each face f of X , the number $n^K(\partial f)$ is odd.

Given a Kasteleyn orientation on X , there is an obvious way to obtain another one: pick a vertex of X and flip the orientation of all the edges adjacent to it. Two Kasteleyn orientations are said to be *equivalent* if they can be related by such moves. Let us denote by $\mathcal{K}(X)$ the set of equivalence classes of Kasteleyn orientations on X .

Using standard cohomological techniques (see [4]), it can be showed that a surface graph X admits a Kasteleyn orientation if and only if X has an even number of vertices. In this case, the set $\mathcal{K}(X)$ admits a freely transitive action of the group $H^1(\Sigma; \mathbb{Z}_2) = \text{Hom}(H_1(\Sigma; \mathbb{Z}_2), \mathbb{Z}_2)$.

Summarizing this section so far, a dimer configuration D on X^1 and a Kasteleyn orientation K on X determine a spin structure on Σ , i.e: a quadratic form $q_D^K: H_1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$. Furthermore, this quadratic form can be computed explicitly. If C is an oriented simple closed curve in Γ , then

$$(4) \quad q_D^K([C]) = n^K(C) + \ell_D(C) + 1,$$

where $\ell_D(C)$ denotes the number of vertices of C where the adjacent dimer of D sticks out to the left of C . With this formula, one immediately sees that the quadratic form q_D^K only depends on the equivalence class $[K]$ of the Kasteleyn orientation K . With some more work, one can show that the map $[K] \mapsto q_D^K$ defines an $H^1(\Sigma; \mathbb{Z}_2)$ -equivariant bijection from the set of equivalence classes of Kasteleyn orientations on X to the set of spin structures on Σ . Therefore, Equation (3) translates into

$$(5) \quad \frac{1}{2g} \sum_{[K] \in \mathcal{K}(X)} (-1)^{\text{Arf}(q_D^K) + q_D^K(\alpha)} = 1$$

for all $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$.

However, the most important outcome of this discussion is that we now know “for free” that the formula displayed above gives a well-defined quadratic form on $H_1(\Sigma; \mathbb{Z}_2)$. Let us use this fact to solve our original problem, i.e. the computation of the dimer partition function.

Let Γ be a finite connected graph endowed with an edge weight system ν . If Γ does not admit any dimer configuration, then the partition function $Z(\Gamma; \nu)$ is obviously zero, so let us assume that Γ admits a dimer configuration D_0 . Enumerate the vertices of Γ by $1, 2, \dots, 2n$ and embed Γ in a closed orientable surface Σ of genus g as the 1-skeleton of a cellular decomposition X of Σ .

Since Γ has an even number of vertices, it admits a Kasteleyn orientation K . Replacing it with an equivalent orientation, we can assume that $\varepsilon^K(D_0) = 1$. The Pfaffian of the associated Kasteleyn matrix then satisfies

$$\begin{aligned} \text{Pf}(A^K) &\stackrel{(1)}{=} \sum_{D \in \mathcal{D}(\Gamma)} \varepsilon^K(D_0) \varepsilon^K(D) \nu(D) \\ &\stackrel{(2)}{=} \sum_{D \in \mathcal{D}(\Gamma)} (-1)^{\sum_i (n^K(C_i) + 1)} \nu(D), \end{aligned}$$

where $D\Delta D_0 = \bigsqcup_i C_i$. Note that given any vertex of C_i , the adjacent dimer of D_0 lies on C_i , so that $\ell_{D_0}(C_i) = 0$. Since the cycles C_i are disjoint, the quadratic form is linear on them and we get

$$\sum_i (n^K(C_i) + 1) = \sum_i (n^K(C_i) + \ell_{D_0}(C_i) + 1) \stackrel{(4)}{=} \sum_i q_{D_0}^K(C_i) = q_{D_0}^K([D\Delta D_0]).$$

Therefore, for every element $[K]$ of $\mathcal{K}(X)$, we have

$$\text{Pf}(A^K) = \sum_{D \in \mathcal{D}(\Gamma)} (-1)^{q_{D_0}^K([D\Delta D_0])} \nu(D) = \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} (-1)^{q_{D_0}^K(\alpha)} \sum_{[D\Delta D_0] = \alpha} \nu(D),$$

the last sum being over all $D \in \mathcal{D}(\Gamma)$ such that the homology class of $D\Delta D_0$ is α . This leads to:

$$\begin{aligned} Z(\Gamma; \nu) &= \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} \sum_{[D\Delta D_0] = \alpha} \nu(D) \\ &\stackrel{(5)}{=} \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} \left(\frac{1}{2^g} \sum_{[K] \in \mathcal{K}(X)} (-1)^{\text{Arf}(q_{D_0}^K) + q_{D_0}^K(\alpha)} \right) \sum_{[D\Delta D_0] = \alpha} \nu(D) \\ &= \frac{1}{2^g} \sum_{[K] \in \mathcal{K}(X)} (-1)^{\text{Arf}(q_{D_0}^K)} \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} (-1)^{q_{D_0}^K(\alpha)} \sum_{[D\Delta D_0] = \alpha} \nu(D) \\ &\stackrel{(6)}{=} \frac{1}{2^g} \sum_{[K] \in \mathcal{K}(X)} (-1)^{\text{Arf}(q_{D_0}^K)} \text{Pf}(A^K). \end{aligned}$$

We have proved the following *Pfaffian formula*, first obtained in [4].

Theorem 5.1. *Let Γ be a graph embedded in a closed oriented surface Σ of genus g such that $\Sigma \setminus \Gamma$ consists of topological disks. Then, the partition function of the dimer model on Γ is given by the formula*

$$Z(\Gamma; \nu) = \frac{1}{2^g} \sum_{[K] \in \mathcal{K}(\Gamma \subset \Sigma)} (-1)^{\text{Arf}(q_{D_0}^K)} \text{Pf}(A^K).$$

Let us conclude this section with one example.

Example 5.1. Consider the $m \times n$ square lattice wrapped up around the torus. As usual, let ν be the edge weight system assigning to each horizontal (resp. vertical) edge the weight x (resp. y), and let us assume without loss of generality that m is even. Let K be the orientation of $\Gamma \subset \Sigma$ illustrated in Figure 4. We get (see [12]):

$$Z(\Gamma; x, y) = \frac{1}{2} (P_{00} + P_{10} + P_{01} - P_{11}),$$

where

$$P_{\epsilon_1 \epsilon_2} = \prod_{k=1}^{\frac{1}{2}m} \prod_{\ell=1}^n 2 \left(x^2 \sin^2 \frac{(2\ell + \epsilon_2 - 1)\pi}{n} y^2 \sin^2 \frac{(2k + \epsilon_1 - 1)\pi}{m} \right)^{1/2}.$$

The Pfaffian P_{11} vanishes (because of the factor corresponding to $k = m/2$ and $\ell = n$), so $Z(\Gamma; x, y) = (P_{00} + P_{10} + P_{01})/2$. This also shows that, unlike in the planar case, there exist graphs that admit dimer configurations but where some Pfaffian vanishes in the Pfaffian formula.

6. SPECIAL HARNACK CURVES

We shall now make a small detour into an area *a priori* totally unrelated: real algebraic geometry. We shall be rather sketchy and refer the interested reader to [18, 19] for further details.

Consider a real polynomial $P \in \mathbb{R}[z^{\pm 1}, w^{\pm 1}]$. The associated *real plane curve* is

$$\mathbb{R}A := \{(z, w) \in (\mathbb{R}^*)^2; P(z, w) = 0\} \subset (\mathbb{R}^*)^2.$$

One can also consider the associated *complex plane curve*

$$A := \{(z, w) \in (\mathbb{C}^*)^2; P(z, w) = 0\} \subset (\mathbb{C}^*)^2.$$

It is a classical result due to Harnack [10] that the number c of connected components (or *ovals*) of a real algebraic plane curve of degree d in $\mathbb{R}P^2 \supset (\mathbb{R}^*)^2$ is bounded by

$$c \leq \frac{(d-1)(d-2)}{2} + 1.$$

Which configurations of ovals can be realized is the subject of Hilbert's sixteenth problem, only solved up to degree $d = 7$.

More generally, if $P(z, w) = \sum_{(m,n) \in \mathbb{Z}^2} a_{mn} z^m w^n$, one can define:

- $\Delta := \text{conv}\{(m, n) \in \mathbb{Z}^2 \mid a_{mn} \neq 0\}$, the *Newton polygon* of P , with g interior integral points and s sides $\delta_1, \dots, \delta_s$;
- using Δ , a natural compactification $\mathbb{R}T_\Delta$ of $(\mathbb{R}^*)^2$, the associated *toric surface*, with $\mathbb{R}T_\Delta \setminus (\mathbb{R}^*)^2$ consisting of s real lines $\ell_1 \cup \dots \cup \ell_s$ intersecting cyclically;
- the closure $\mathbb{R}\bar{A}$ of $\mathbb{R}A$ in $\mathbb{R}T_\Delta$.

It turns out that the number of ovals of $\mathbb{R}\bar{A}$ is bounded above by $g + 1$, and that $\mathbb{R}\bar{A}$ meets the axis ℓ_i corresponding to the side δ_i at most d_i times, where $d_i - 1$ denotes the number of integral points in the interior of δ_i . We shall say that $\mathbb{R}A$ is a *special Harnack curve* if the number of ovals of $\mathbb{R}\bar{A}$ is $g + 1$, only one of these ovals intersects the axes $\ell_1 \cup \dots \cup \ell_s$, and it does so in the maximal and cyclic way: d_1 times ℓ_1 , then d_2 times ℓ_2 , and so on.

Note that if Δ is a triangle, then $\mathbb{R}T_\Delta = \mathbb{R}P^2 = (\mathbb{R}^*)^2 \cup \ell_1 \cup \ell_2 \cup \ell_3$, where the ℓ_i 's are the coordinate axes. In this case, $\mathbb{R}\bar{A}$ is simply given by the zeroes of the homogeneous polynomial associated to P , and the inequality $c \leq g + 1$ corresponds to Harnack's upper bound.

Example 6.1. Consider the very simple example given by $P(z, w) = a + bz + cw$, with $a, b, c \in \mathbb{R}^*$. The associated Newton polygon is the triangle with vertices $(0, 0)$,

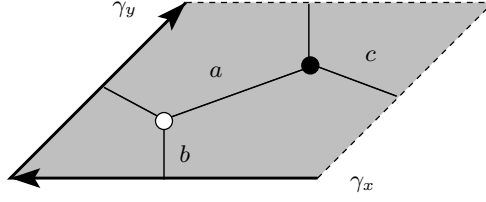


FIGURE 6. A fundamental domain for the hexagonal lattice wrapped up on the torus.

$(1, 0)$ and $(0, 1)$, so we get $g = 0$ and $d_1 = d_2 = d_3 = 1$. The corresponding curve is a special Harnack curve, since

$$\mathbb{R}\overline{A} = \{[z : w : t] \in \mathbb{R}P^2; at + bz + cw = 0\}$$

has one oval and meets each axis once.

The following result of Mikhalkin is of great importance for the application of Harnack curves to the study of dimer models.

Theorem 6.1 (Mikhalkin [18, 19]). *Fix $P \in \mathbb{R}[z^{\pm 1}, w^{\pm 1}]$ with Newton polygon of positive area. Then, the following are equivalent.*

1. $\mathbb{R}A$ is a (possibly singular) special Harnack curve.
2. $A \subset (\mathbb{C}^*)^2$ meets every torus $\{(z, w) \in (\mathbb{C}^*)^2; |z| = r_1 |w| = r_2\}$ at most twice.

7. BIPARTITE GRAPHS ON THE TORUS

After this small detour, let us come back to dimers. Consider the case of a bipartite graph Γ (with same number of black and white vertices) embedded in the torus \mathbb{T}^2 , and fix a Kasteleyn orientation on $\Gamma \subset \mathbb{T}^2$.

Recall that the corresponding Kasteleyn matrix is of the form $A^K = \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}$, so $\text{Pf}(A^K) = \pm \det(M)$. Let us modify M as follows: pick oriented simple closed curves γ_x and γ_y on the torus, transverse to Γ , representing a basis of the homology $H_1(\mathbb{T}^2)$, and multiply the coefficient corresponding to each edge e by $z^{\gamma_x \cdot e} w^{\gamma_y \cdot e}$, where \cdot denotes the intersection number and e is oriented from the white to the black vertex. Let $M(z, w)$ denote the resulting matrix. Its determinant

$$P(z, w) := \det(M(z, w)) \in \mathbb{R}[z^{\pm 1}, w^{\pm 1}]$$

is called the *characteristic polynomial* of Γ . We shall say that a weighted bipartite toric graph $\Gamma \subset \mathbb{T}^2$ is *non-degenerate* if the Newton polygon of its characteristic polynomial has positive area.

Note that the formula for the dimer partition function now reads

$$(7) \quad Z(\Gamma; \nu) = \frac{1}{2} (-P(1, 1) + P(-1, 1) + P(1, -1) + P(-1, -1))$$

for a well-chosen Kasteleyn orientation.

Example 7.1. Consider the weighted toric bipartite graph illustrated in Figure 6. Note that this is nothing but the simplest fundamental domain of the hexagonal lattice. On this example, the orientation from the white to the black vertex is Kasteleyn, and with the choice of curves γ_x and γ_y illustrated in this figure, one computes

$$P(z, w) = M(z, w) = a + bz + cw.$$

As seen in Example 6.1, it is a special Harnack curve.

This is no accident.

Theorem 7.1 (Kenyon-Okounkov-Sheffield [15, 16]). *Let $(\Gamma, \nu) \subset \mathbb{T}^2$ be a non-degenerate weighted bipartite graph embedded in the torus. Then, the real algebraic curve $\mathbb{R}A$ associated with $P(z, w)$ is a (possibly singular) special Harnack curve.*

By Theorem 6.1, we immediately get the following corollary.

Corollary 7.2. *For any non-degenerate weighted bipartite graph $(\Gamma, \nu) \subset \mathbb{T}^2$, the associated complex algebraic curve meets the unit torus*

$$S^1 \times S^1 = \{(z, w) \in (\mathbb{C}^*)^2; |z| = |w| = 1\}$$

at most twice. □

This seemingly technical result is extremely deep, and actually provides the necessary information for the rigorous study of large scale properties of dimer models on bipartite graphs. The paper [16] contains several applications of this result; we shall only explain the simplest of these, namely the computation of the *free energy*, that we now define.

For a graph Γ embedded in the torus, there is a very natural way to make the graph finer and finer: simply consider a fundamental domain for $\Gamma \subset \mathbb{T}^2$ and paste together $n \times n$ copies of these fundamental domains before wrapping up the torus again. (Topologically, this corresponds to considering a covering map of degree n^2 .) Let us denote by $\Gamma_n \subset \mathbb{T}^2$ the corresponding graph. Then, the *free energy (per fundamental domain)* of the model is defined by

$$\log Z := \lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z(\Gamma_n; \nu).$$

The existence of this limit follows from standard subadditivity arguments. Note that in particular, $\log Z$ evaluated at $\nu = 1$ measures the exponential growth of the number of dimer configurations on Γ_n .

Theorem 7.3 (Kenyon-Okounkov-Sheffield [16]). *For any weighted bipartite graph Γ on the torus, the free energy is given by*

$$\log Z = \frac{1}{(2\pi i)^2} \int_{S^1 \times S^1} \log |P(z, w)| \frac{dz}{z} \frac{dw}{w}.$$

Sketch of proof. With Equation (7) and Corollary 7.2 in hand, the demonstration of Theorem 7.3 is quite straightforward. It can be divided into three steps.

Step 1. Let us write P_n for the characteristic polynomial of Γ_n and use the notation $Z_n^{\theta\tau} := P_n((-1)^\theta, (-1)^\tau)$. We have

$$|Z_n^{\theta\tau}| \stackrel{(1)}{\leq} Z(\Gamma_n; \nu) \stackrel{(7)}{=} \frac{1}{2} (-Z_n^{00} + Z_n^{10} + Z_n^{01} + Z_n^{11}) \leq 2 \max_{(\theta, \tau)} |Z_n^{\theta\tau}|$$

for all $\theta, \tau \in \{0, 1\}$. It follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z(\Gamma_n; \nu) = \lim_{n \rightarrow \infty} \frac{1}{n^2} \log \max_{(\theta, \tau)} |Z_n^{\theta\tau}|,$$

that is to say,

$$\log Z = \lim_{n \rightarrow \infty} \frac{1}{n^2} \log \max_{(\theta, \tau)} |P_n((-1)^\theta, (-1)^\tau)|.$$

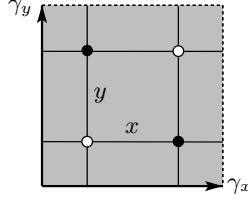


FIGURE 7. A fundamental domain for the bipartite square lattice wrapped up on the torus.

Step 2. Using the symmetry of the twisted Kasteleyn matrix for Γ_n , it can be block-diagonalized into n^2 copies of the twisted Kasteleyn matrix for $\Gamma = \Gamma_1$. Computing the determinants leads to the following formula:

$$P_n(z, w) = \prod_{u^n = z} \prod_{v^n = w} P(u, v).$$

In particular, for $\theta, \tau \in \{0, 1\}$, we have

$$\frac{1}{n^2} \log |P_n((-1)^\theta, (-1)^\tau)| = \frac{1}{n^2} \sum_{u^n = (-1)^\theta} \sum_{v^n = (-1)^\tau} \log |P(u, v)|,$$

which is nothing but a Riemann sum for the integral

$$\int_0^1 \int_0^1 \log |P(e^{2\pi i \varphi}, e^{2\pi i \psi})| d\varphi d\psi = \frac{1}{(2\pi i)^2} \int_{S^1 \times S^1} \log |P(z, w)| \frac{dz}{z} \frac{dw}{w}.$$

Step 3. By step 1, we only need to check that this sum converges for at least one choice of (θ, τ) . By Corollary 7.2, $P(z, w)$ has at most two (conjugate) zeroes on the unit torus $S^1 \times S^1$. Since the four Riemann sums above are on four staggered lattices, at least three of them converge. \square

We conclude these notes with two examples.

Example 7.2. By Example 7.1 and Theorem 7.3, the free energy for the dimer model on the hexagonal lattice (with fundamental domain as in Figure 6) is given by

$$\log Z = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |a + be^{i\varphi} + ce^{i\psi}| d\varphi d\psi.$$

Example 7.3. Consider the weighted toric bipartite graph illustrated in Figure 7, which is nothing but the simplest fundamental domain for the *bipartite* square lattice on the torus. One easily computes its characteristic polynomial

$$P(z, w) = y^2(2 + (z + z^{-1})) + x^2(2 + (w + w^{-1})).$$

By Theorem 7.3, the corresponding free energy is given by

$$\begin{aligned} \log Z &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log(2(x^2 + y^2 + x^2 \cos \varphi + y^2 \cos \psi)) d\varphi d\psi \\ &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \log(4(x^2 \cos^2 \varphi + y^2 \cos^2 \psi)) d\varphi d\psi. \end{aligned}$$

This is four times the value obtained by Kasteleyn (recall Example 3.2), as it should, the fundamental domain for the bipartite square lattice being of size two by two.

REFERENCES

- [1] Cahit Arf. Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. I. *J. Reine Angew. Math.*, 183:148–167, 1941.
- [2] Michael F. Atiyah. Riemann surfaces and spin structures. *Ann. Sci. École Norm. Sup. (4)*, 4:47–62, 1971.
- [3] David Cimasoni. Dimers on graphs in non-orientable surfaces. *Lett. Math. Phys.*, 87(1-2):149–179, 2009.
- [4] David Cimasoni and Nicolai Reshetikhin. Dimers on surface graphs and spin structures. I. *Comm. Math. Phys.*, 275(1):187–208, 2007.
- [5] David Cimasoni and Nicolai Reshetikhin. Dimers on surface graphs and spin structures. II. *Comm. Math. Phys.*, 281(2):445–468, 2008.
- [6] N. P. Dolbilin, Yu. M. Zinov'ev, A. S. Mishchenko, M. A. Shtan'ko, and M. I. Shtogrin. Homological properties of two-dimensional coverings of lattices on surfaces. *Funktsional. Anal. i Prilozhen.*, 30(3):19–33, 95, 1996.
- [7] Jack Edmonds. Optimum branchings. *J. Res. Nat. Bur. Standards Sect. B*, 71B:233–240, 1967.
- [8] Michael E. Fisher. Statistical mechanics of dimers on a plane lattice. *Phys. Rev.*, 124(6):1664–1672, Dec 1961.
- [9] Anna Galluccio and Martin Loebl. On the theory of Pfaffian orientations. I. Perfect matchings and permanents. *Electron. J. Combin.*, 6:Research Paper 6, 18 pp. (electronic), 1999.
- [10] Axel Harnack. Ueber die Vieltheiligkeit der ebenen algebraischen Curven. *Math. Ann.*, 10(2):189–198, 1876.
- [11] Dennis Johnson. Spin structures and quadratic forms on surfaces. *J. London Math. Soc. (2)*, 22(2):365–373, 1980.
- [12] P. W. Kasteleyn. The statistics of dimers on a lattice. *Physica*, 27:1209–1225, 1961.
- [13] P. W. Kasteleyn. Dimer statistics and phase transitions. *J. Mathematical Phys.*, 4:287–293, 1963.
- [14] P. W. Kasteleyn. Graph theory and crystal physics. In *Graph Theory and Theoretical Physics*, pages 43–110. Academic Press, London, 1967.
- [15] Richard Kenyon and Andrei Okounkov. Planar dimers and Harnack curves. *Duke Math. J.*, 131(3):499–524, 2006.
- [16] Richard Kenyon, Andrei Okounkov, and Scott Sheffield. Dimers and amoebae. *Ann. of Math. (2)*, 163(3):1019–1056, 2006.
- [17] L. Lovász and M. D. Plummer. *Matching theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Annals of Discrete Mathematics, 29.
- [18] G. Mikhalkin. Real algebraic curves, the moment map and amoebas. *Ann. of Math. (2)*, 151(1):309–326, 2000.
- [19] Grigory Mikhalkin and Hans Rullgård. Amoebas of maximal area. *Internat. Math. Res. Notices*, (9):441–451, 2001.
- [20] H. N. V. Temperley and Michael E. Fisher. Dimer problem in statistical mechanics—an exact result. *Philos. Mag. (8)*, 6:1061–1063, 1961.
- [21] Glenn Tesler. Matchings in graphs on non-orientable surfaces. *J. Combin. Theory Ser. B*, 78(2):198–231, 2000.
- [22] L. G. Valiant. The complexity of computing the permanent. *Theoret. Comput. Sci.*, 8(2):189–201, 1979.