

Q3:

We first claim that $f(0) = 0$.

$$\begin{aligned}\|f(0)\| &\leq \|0\|^2 \\ \implies \|f(0)\| &\leq 0 \\ \implies \|f(0)\| &= 0 \\ \implies f(0) &= 0\end{aligned}$$

We now show that $f \in o(h)$. Now consider $f(h)$ for some $h \in \mathbb{R}^n$.

$$\begin{aligned}\|f(h)\| &\leq \|h\|^2 \\ \implies \frac{\|f(h)\|}{\|h\|} &\leq \|h\| \\ \implies \lim_{h \rightarrow 0} \frac{\|f(h)\|}{\|h\|} &\leq \lim_{h \rightarrow 0} \|h\| \\ \implies \lim_{h \rightarrow 0} \frac{\|f(h)\|}{\|h\|} &= 0\end{aligned}$$

Thus we can write $f(h) = f(0) + Lh + o(h)$ for some linear mapping L . Since $f \in o(h)$, this implies that $L \in o(h)$. By the lemma from class any linear mapping in $o(h)$ is 0. And so f is differentiable with $Df(0) = 0$.