Q4a: If f is open, then it is not necessarily continuous. Consider the following function, $\sigma(x) : \mathbb{R} \to \{-1, 0, 1\}$ which maps $x > 0$ to 1, $x < 0$ to -1, and 0 to 0, with the codomain equipped with the discrete topology. This is an open map, since the image of any open set will be either -1,0,1 or some union of them. However, $f^{pre}(\{0\}) = 0$ which is not open in $\mathbb{R}$.

4b We claim any homeomorphism, $f : M \to N$, is an open map. Let U be some open set in M. Then, ${f^{-1}}^{pre}(U)$ is open. Since f is a bijection, ${f^{-1}}^{pre}(U) = f(U)$. Thus f is an open map.

4c: We claim an open, continuous bijection is a homeomorphism. Let U be an open set in N. Then $f^{pre}(U)$ is open, and by bijecivity, $f(f^{pre}(U)) = U$, and so $f^{pre}(U) = f^{-1}(U)$ is open. Thus f^-1 is a homeomorphism

4d: Consider the function $f(x) = x^3 - 4x$. This is clearly surjective and continuous. Consider the open set $U = (0, 1.5)$. We evaluate $f(U) = [-3.079, 0)$, which is not an open set in $\mathbb{R}$

4e: We claim that and $f : \mathbb{R} \to \mathbb{R}$, which is a continuous, surjective and open, is a homeomorphism. It suffices to show injectivity, then we can apply the result from c and conclude it is a homeomorphism. Suppose that f is not injective. Then for some $x, y \in \mathbb{R}$, $x < y$, we have $f(x) = f(y)$. By continuity, f attains a maximum and minimum on the set $[x, y]$. Let a correspond to the point at which f attains a minimum, b be the point where f attains its maximum. First consider the case where $a, b \in \{x, y\}$. This would imply that f is constant on $[x, y]$, violating openness, since the image of any open set contained in $[x, y]$ will be a point, which is closed. Suppose that $a \in \{x, y\}$, and $b \in (x, y)$. Then the image of (x, y) under f should be open, but since it attains its minimum on the interiour, $f((x, y)) = [f(a), f(b))$. Similarly, if $b \in \{x, y\}$ and $a \in (x, y)$ then $f((x, y)) = (f(a), f(b)]$. Either case contradicts openness. Now consider the case $a, b \in (x, y)$. We see that $f((x, y)) = [f(a), f(b)]$. We contradict openness again. Hence f must be injective, and we conclude it is a homeomorphism.

4f: Consider the map $f : S^1 \to S^1$ which maps $z \mapsto z^2$ in the complex plane. This is a continuous, open surjection, which doubles the argument of any $x \in S^1$. We notice that it is not injective since $f((1, 0)) = f((-1, 0)) = 0$