Problem 1.

(a) First note that on the domain $(0,\infty)$ we have that $\varphi_t(x)$ is smooth. Furthermore, we have that

$$\frac{d}{dx}\varphi_t(x) = (\sqrt{x}+t)\cdot\frac{1}{\sqrt{x}}.$$

By the inverse function theorem we have that $\varphi_t(x)$ is a diffeomorphism. Finally we show that $\varphi_t(x)$ defines a 1 parameter group in t.

$$\varphi_t \circ \varphi_s(t) = \varphi_t\left((\sqrt{x}+s)^2\right) = \left(\sqrt{(\sqrt{x}+s)^2}+t\right)^2 = \left(\sqrt{x}+s+t\right)^2 = \varphi_{s+t}(x).$$

Therefore $\varphi_t(x)$ defines the flow of a vector field.

(b) To find an X which $\varphi_t(x)$ generates, we compute

$$\left.\frac{d}{dt}\right|_{t=0}\varphi_t(x) = 2\sqrt{x}\frac{d}{dt}.$$

Problem 2.

(a) By a long computation we verify the Jacobi identity holds for Lie Bracket:

$$\begin{aligned}
[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]] &= [X, YZ-ZY]+[Z, XY-YX]+[Y, ZX-XZ] \\
&= XYZ - XZY - YZX + ZYX + ZXY - ZYX \\
&- XYZ + YXZ + YZX - YXZ - ZXY + XZY \\
&= 0
\end{aligned}$$

(b) i) Using the Jacobi identity for the lie bracket, along witg the fact that $L_X Y = [X, Y]$ we see that

$$0 = [X,[Y,Z]]+[Z,[X,Y]]-[Y,[X,Z]] = L_X[Y,Z]-[L_X Y, Z]-[Y, L_X Z] \implies L_X[Y,Z] = [L_X Y, Z]-[Y, L_X Z]$$

ii) Since the Lie bracket is bilinear, we get that

$$L_{[X,Y]}f = L_{XY}f - L_{YX}f = XYf - YXf = L_x \circ L_y f - L_y \circ L_x f$$

Problem 3.

(a) Suppose ϕ_h and ψ_h are the flows associated to X, Y respectively. Furthermore suppose that $f_*X = Yf$. We first claim that $f \circ \psi_h = \phi_h \circ f$. We see that

$$\frac{d}{dt}\bigg|_{t=0} (f \circ \psi_h(q)) = \frac{d}{dt}\bigg|_{t=0} (f(\psi_t(q)) \circ \frac{d}{dt}\bigg|_{t=0} \psi_t(q) = f_*X_q = Y_{f(q)}f = \frac{d}{dt}\bigg|_{t=0} \phi_t(f(q)).$$

This computation along with the initial condition $f_*X = Yf$ implies that $f \circ \psi_h = \phi_h \circ f$. It follows that $(\phi_h \circ f)_* = (f \circ \psi_h)_*$. Therefore we compute that

$$\begin{aligned}
[Y_1, Y_2] &= L_{Y_1}Y_2 \\
&= \lim_{h\to 0} \frac{1}{h}\left[Y_{2f(p)} - \psi_{h*}Y_{2f(p)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[f_*X_{2p} - \psi_{h*}f_*X_{2p}\right] && \text{(since } X \text{ is a lifting of } Y\text{)} \\
&= \lim_{h\to 0} \frac{1}{h}\left[f_*X_{2p} - (\psi_h \circ f)_{*p}X_{2p}\right] && \text{(by chain rule)} \\
&= \lim_{h\to 0} \frac{1}{h}\left[f_*X_{2p} - (f \circ \phi_h)_{*p}X_{2p}\right] && \text{(by claim)} \\
&= \lim_{h\to 0} \frac{1}{h}\left[f_*X_{2p} - f_* \circ \phi_{h*p}X_{2p}\right] && \text{(by chain rule)} \\
&= f_*\left(\lim_{h\to 0} \frac{1}{h}\left[X_{2p} - \phi_{h*}X_{2p}\right]\right) && \text{(since } f_* \text{ is continous)} \\
&= f_*L_{X_1}X_2 \\
&= f_*[X_1, X_2]
\end{aligned}$$

Therefore $[X_1, X_2]$ is a lifting of $[Y_1, Y_2]$.

(b) Suppose that $[X_1, X_2]$ is tangent to $f^{-1}(q)$ for all $q \in N$. First, note that since f is a surjective submersion, we have that $f^{-1}(q)$ is a submanifold of M. Since f is constant on $f^{-1}(q)$, and for $p \in f^{-1}(q)$ we have $[X_1, X_2]_p \in T_pf^{-1}(q)$ and

$$f_*[X_1, X_2]_p = [Y_1, Y_2]_q = 0.$$

Conversely, suppose that $[Y_1, Y_2] = 0$. Then we have that $f_*[X_1, X_2] = 0$ by $3a$. Therefore on every fiber $f^{-1}(q)$ we have that $f_*[X_1, X_2]_{f^{-1}(q)} = 0$, so $[X_1, X_2]$ is tangent to every fiber.

Problem 4.

(a) Note that $\mathfrak{X}(G)^L$ is a vector space, since it is the image of a vector space, namely TM_h under the mapping $(\mu_g)_*$. We claim that the evaluation map $X \mapsto X_e$ is a linear mapping. Indeed,

$$(Y + \alpha X)_e = Y_e + \alpha X_e.$$

We now claim that is a linear isomorphism. We show that it is injective and surjective. First suppose that for some left invariant vector fields X, Y we have that $X_e = Y_e$. By left invariance, we have that

$$0 = (\mu_g)_*(Y_e - X_e) = Y_g - X_g.$$

We have $X_g = Y_g$ for all g so $X = Y$. Now suppose that $v \in TG_e$. Define the vector field $X_g = (\mu_g)_* v$. Observe that this is a smooth mapping into the tangent space, so X_g is a vector field, and $X_e = v$. We quickly verify that X is left invariant:

$$(\mu_h)_* X_g = (\mu_h)_* \circ (\mu_g)_* v = (\mu_{hg})_* v = X_{hg}.$$

Therefore the evaluation map is a isomorphism of vector spaces.

(b) As proven in a previous assignment, $(\mu_g)_*$ is a linear isomorphism since μ_g is a diffeomorphism. It follows that $(\mu_g)_*$ is a surjective submersion. Furthermore, note that X, Y are left invariant if and only if they are liftings of themselves. By $3a$, we have

$$(\mu_g)[X, Y]_h = [X, Y]_{\mu(g,h)}.$$

So the left invariant vector fields form a Lie Algebra.

Problem 5.

We can rewrite the vector fields X, Y as $X_{(x,y,z)} = (y - z, 0, 0), Y_{(x,y,z)} = (0, 1, 1)$. Define the function $p(t, s)$ so that p satisfies X, Y and $p(0, 0) = a$. To satisfy Y we must have

$$\frac{\partial p}{\partial s}(s, t) = (0, 1, 1) \implies p(s, t) = (f(t), s + c, s + d).$$

The condition X gives us that

$$\frac{\partial p}{\partial t}(s, t) = c - d \implies p(s, t) = (e + (c - d)t, s + c, s + d).$$

Initial conditions imply that $(e, c, d) = a$, so the solution surface is given as

$$p(s, t) = a + ((a_2 - a_3)t, s, s).$$

Problem 6.

(a) Let $\{U_\alpha\}$ be a finite covering of $f^{-1}(\{0\})$ by coordinate charts so that we can apply the submersion theorem on each U_α. Take $\{\psi_i\}$ to be a partition of unity subordinate to this cover. On each U_i define the vector field $X^i_y = \frac{\partial}{\partial x_1}\Big|_y$. Let $\varphi_{it}(x)$ be the associated flow of X^i defined for $t \in (-\varepsilon_i, \varepsilon_i)$. We define the vector field $X = \sum_i \psi_i X^i$. By the submersion theorem, we have that

$$f_* X = \sum_i \psi_i f_* X^i = \frac{d}{dt}.$$

We now define the mapping $\phi_t(x)$ to be $\phi_{t_i}(x)$ for x belonging to U_i, and $t \in \bigcap_i^n (-\varepsilon_i, \varepsilon_i) = (-\varepsilon, \varepsilon)$. We claim that this is a diffeomorphism, and thus will be the flow of X. Note this function is smooth since the $\phi_{it}(x)'s$ agree on the intersection of the $U_i's$ by uniqueness. First we have that $\frac{\partial}{\partial t}\phi_t(x)$ is nonsingular, since for each i, $\frac{\partial}{\partial t}\phi_{it}(x)$ is non singular. Furthermore, $\frac{\partial}{\partial x}\phi_t(x)$ will also be nonsingular since each $\frac{\partial}{\partial x}\phi_{it}(x)$ is nonsingular. Hence $\phi_t(x)$ is a diffeomorphism. By uniqueness it corresponds to the flow of X. It follows that $f(\phi(t,x)) = t$ since $f_*(X) = \frac{d}{dt}$ and $f(f^{-1}(0)) = 0$. Therefore $\phi_t(x)$ is a diffeomorphism from $f^{-1}(0) \times (-\varepsilon, \varepsilon)$ to $f^{-1}(-\varepsilon, \varepsilon)$.

(b) First assume that $N = \mathbb{R}^m$ and $b = 0$. Since 0 is a regular point of f, by the submersion theorem we can take a covering of $f^{-1}(\{0\})$ by charts $\{U_i, x^i\}$ so that f is a projection i.e. $f_i(x^j) = x^j_i$. By part a) we can find a diffeomorphism $\phi^j_i(y,t)$ defined on $f_i^{-1}(0) \cap U_j \times (-\varepsilon_i, \varepsilon_i) \to f^{-1}((-\varepsilon, \varepsilon))$ so that $f_i(\phi^j_i(y,t)) = t$. By the same process as a), we can extend ϕ^j_i to a diffeomorphism ϕ_i, which satisfies $f_i(\phi_i(x,t)) = t$. Let $(-\varepsilon_1, \varepsilon_1) \times \cdots \times (-\varepsilon_m, \varepsilon_m) = I$ and define the mapping $\phi(x,t) : f^{-1}(0) \times I \to f^{-1}(I)$ as

$$\phi(x, t_1, \dots, t_m) = (\phi_1(x, t_1), \dots, \phi_m(x, t_m)).$$

First, notice that by the submersion theorem we have that $f^{-1}(0) = \prod_i f_i^{-1}(0)$ and $f^{-1}(I) = \prod_i f^{-1}((-\varepsilon_i, \varepsilon_i))$, so $\phi(x,t)$ is actually defined from $\prod_i f^{-1}(0) \times I$ with values into $\prod_i f^{-1}((-\varepsilon_i, \varepsilon_i))$. By the submersion theorem we have that $f(\phi(x, t_1, \dots, t_m)) = (t_1, \dots, t_m)$. Finally, $\phi(x,t)$ is a diffeomorphisms since it is a product of diffeomorphisms with images into a product of manifolds. If N is a manifold, we can take a sufficiently small neighbourhood U of b and coordinates so that U is an open set in $\mathbb{R}^m$ and $b = 0$.

(c) The claim is false. Consider the Hopf Fibration:

$$S^1 \hookrightarrow S^3 \xrightarrow{p} S^2 .$$

It has been shown that for any $\lambda \in S^2$, $p^{-1}(\lambda) = S^1$. If it were the case that $S^1 \times S^2 \cong S^3$ that would imply that $S^1 \times S^2$ has trivial tangent bundle, since S^3 is a lie group after identification with $SU(2)$. However a product of manifolds has trivial tangent bundle if and only if each factor has trivial tangent bundle. S^2 does not have a trivial tangent bundle, a contradiction.