Problem 1.

Endow $\mathbb{RP}^2$ with the smooth structure given by quotienting S^2 in the usual way. We compute the local derivative of f as

$$Df(\varphi_1^{-1})(\varphi([x,y,z])) = D(zy, z\sqrt{1-y^2-z^2}, y\sqrt{1-y^2-z^2}) = \begin{bmatrix} z & y \\ \frac{-zy}{\sqrt{1-y^2-z^2}} & \frac{1-y^2-2z^2}{\sqrt{1-y^2-z^2}} \\ \frac{1-2y^2-z^2}{\sqrt{1-y^2-z^2}} & \frac{-zy}{\sqrt{1-y^2-z^2}}. \end{bmatrix}$$

The mapping f will fail to be an immersion at points (y,z) where the columns are linearly dependant i.e. their cross product is 0. For notation set $x = \sqrt{1-y^2-z^2}$. We aim to solve

$$\left(z^2+y^2-x^2, \frac{y(x^2-y^2)+z^2y}{x}, \frac{z(x^2-z^2)+zy^2}{x}\right) = 0.$$

The constraint on the first coordinate implies that

$$y^2+z^2 = \frac{1}{2}$$

and the second and third coordinate constraints imply that y or z is 0 but not both. Substituting back into φ^{-1} we get four points where f fails to be an immersion in this chart,

$$\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right], \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right], \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right].$$

We repeat this process for φ_2, seeing that

$$Df(\varphi_2^{-1})(\varphi_2([x,y,z])) = \begin{bmatrix} \frac{-xz}{\sqrt{1-x^2-z^2}} & \frac{1-x^2-2z^2}{\sqrt{1-x^2-z^2}} \\ z & x \\ \frac{1-2x^2-z^2}{\sqrt{1-x^2-z^2}} & \frac{-xz}{\sqrt{1-x^2-z^2}} \end{bmatrix}.$$

A similar computation reveals that this matrix has rank less than 2 at

$$\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right], \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right], \left[0\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \left[0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right].$$

Finally, for φ_3 we have that

$$Df(\varphi_3^{-1})(\varphi([x,y,z])) = \begin{bmatrix} \frac{-xy}{\sqrt{1-x^2-y^2}} & \frac{1-x^2-2y^2}{\sqrt{1-x^2-y^2}} \\ \frac{1-2x^2-y^2}{\sqrt{1-x^2-y^2}} & \frac{-xy}{\sqrt{1-x^2-y^2}} \\ y & x \end{bmatrix}.$$

This will fail to be an immersion at the points

$$\left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \left[0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], \left[\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right].$$

We conclude that the mapping f will not be an immersion at

$$\left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \left[0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], \left[\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right], \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right].$$

The images of these points will be

$$\left(\pm\frac{1}{2}, 0, 0\right), \left(0, \pm\frac{1}{2}, 0\right), \left(0, 0, \pm\frac{1}{2}\right).$$

Problem 2.

(a) Let f be the map given. We show that the following hold to conclude that it is indeed an imbedding.

i) f is injective

ii) f is an immersion

iii) f is a homeomorphism onto its image.

First we show that f is injective. If

$$[y, 0] = [x, 0]$$

then $(y, 0)$ and $(x, 0)$ are either equal or andipodal. Clearly we must have that $x \sim y$ and $[x] = [y]$. Hence f is injective. Now we claim that f is an immersion. Endow $\mathbb{R}P^n$ and $\mathbb{R}P^{n+1}$ with the smooth structure induced by quotienting the sphere by antipodal points. Then if ψ_j and φ_i are the standard charts on $\mathbb{R}P^{n+1}$ and $\mathbb{R}P^n$, for $j \neq i$ we have that

$$(\psi_j \circ f \circ \varphi_i^{-1})(x_1, \dots, x_n) = (x_1, \dots, \hat{x}_j, \dots, \sqrt{1 - x_1^2 - \dots x_n^2}, \dots, x_n, 0),$$

which will evaluate as

$$D(\psi_j \circ f \circ \varphi_i^{-1}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-2x_1}{\sqrt{1-x_1^2-\dots x_n^2}} & \cdots & \frac{-2x_{n-1}}{\sqrt{1-x_1^2-\dots x_n^2}} & \frac{-2x_n}{\sqrt{1-x_1^2-\dots x_n^2}} & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix}.$$

This will be of rank n. If $i = j$ then the Jacobian matrix will be

$$D(\psi_j \circ f \circ \varphi_j^{-1}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where I is the $n \times n$ identity matrix. Therefore f is an immersion. It remains to show that it is a homeomorphism onto its image. First, note that f must be continuous since f is smooth, and ψ_j, φ_i cover the manifold, the gluing lemma gives us the desired result. Furthermore since f is injective, we have that f is a bijection onto $f(\mathbb{R}P^n)$. Since $\mathbb{R}P^n$ is compact and hausdorff, it follows from topology that f is a homeomorphism onto its image. Therefore f defines an imbedding of $\mathbb{R}P^n$ in $\mathbb{R}P^{n+1}$.

(b) We first check that the Segre imbedding is in fact an imbedding. First we show that it is an immersion. Regard $\mathbb{C}P^1$ and $\mathbb{C}P^3$ as quotients of complex spheres of same dimension. Let $\{\varphi_i, U_i\}, \{\psi_j, V_j\}$ be atlases on $\mathbb{C}P^1, \mathbb{C}P^3$ with coordinates given by projection. We compute that S looks like

$$\psi_1 \circ S \circ (\varphi_1^{-1}, \varphi_1^{-1})(z, w) = (w\sqrt{1 - z^2}, z\sqrt{1 - w^2}, \sqrt{1 - w^2}\sqrt{1 - z^2}),$$

and the differential will be

$$D(\psi_1 \circ S \circ (\varphi_1^{-1}, \varphi_1^{-1}))(z, w) = \begin{bmatrix} \frac{-zw}{\sqrt{1-z^2}} & \sqrt{1 - z^2} \\ \sqrt{1 - w^2} & \frac{-zw}{\sqrt{1-w^2}} \\ \frac{-z\sqrt{1-w^2}}{\sqrt{1-z^2}} & \frac{-w\sqrt{1-z^2}}{\sqrt{1-w^2}} \end{bmatrix}.$$

This will have a complex rank 2. A similar computation for different choices of ψ_i, φ_j will yield the same result and so we conclude that S is an immersion. We now claim that S is a homeomorphism onto its image. First we show that S is injective. Suppose that

$$S([z_0, z_1], [w_0, w_1]) = S([u_0, u_1], [v_0, v_1]).$$

This gives us that

$$[z_0w_0, z_1w_0, z_0w_1, z_1w_1] = [u_0v_0, u_1v_0, u_0v_1, u_1v_1].$$

By the equivalence relation we have that

$$\begin{aligned} z_0w_0 &= \pm u_0v_0 \\ z_1w_0 &= \pm u_1v_0 \\ z_0w_1 &= \pm u_0v_1 \\ z_1w_1 &= \pm u_1v_1 \end{aligned}$$

which implies that $[z_0, z_1] = [u_0, u_1]$ and $[w_0, w_1] = [v_0, v_1]$. Furthermore since S is smooth, and since ψ_j, φ_i cover our manifolds, the gluing lemma implies that S is continuous. Since S is a bijection onto its image, and $\mathbb{C}P^1 \times \mathbb{C}P^1$ is compact and hausdorff, and $\mathbb{C}P^3$ is hausdorff we have that S must be a homeomorphism onto its image. Define the generalized Segre imbedding as $S : [x_0, \ldots, x_j] \times [y_0, \ldots, y_k] \mapsto [x_iy_l]$ where x_iy_l is the vector given with entries ranging over all possible products of x_i with y_l. We claim that S is an imbedding. Let φ be a chart of $\mathbb{C}P^j$, ψ be a chart of CP^k and λ be a chart of $\mathbb{C}P^{(j+1)(k+1)-1}$. We have that

$$\lambda \circ S \circ (\varphi^{-1}, \psi^{-1})(z, y) = (z_0y_0, \ldots z_0\sqrt{1 - y_0^2 - \ldots}, \ldots \widehat{z_hy_k}, \ldots z_jy_k),$$

and will have a differential of

$$D(\lambda \circ S \circ (\varphi^{-1}, \psi^{-1}))(z, y) = \begin{bmatrix} y_0 & 0 & \cdots & \cdots & z_0 & 0 & \cdots & \cdots & 0 \\ y_1 & 0 & \cdots & \cdots & 0 & z_0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \sqrt{(1 - y_0^2 - \ldots)} & \cdots & \frac{-z_0y_1}{\sqrt{1-y_0^2-\ldots}} & \cdots & \cdots & \cdots & \frac{-z_0y_k}{\sqrt{1-y_0^2-\ldots}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & y_{n+1} & z_0 & 0 & \cdots & \cdots & 0 \\ 0 & y_0 & 0 & \cdots & z_1 & 0 & \cdots & 0 & \end{bmatrix}.$$

One can verify that this matrix has a complex rank of $j + k$. Hence S is an immersion. By a similar argument as before, it is an imbedding.

Problem 3.

Let U be an open set around B thats disjoint from A. We have that U is a submanifold, and A^c is an open covering of it. There exists a partition of unity $\{\psi_i\}$ subordinate to A^c. We have that $supp(\psi_i) \subset A^c$. Therefore the function

$$f(p) = \begin{cases} \sum_i \psi_i(p) & p \notin A \\ 0 & p \in A \end{cases}$$

will be smooth and satisfies our requirements.

Problem 4.

Let C be a closed subset of $\mathbb{R}^n$. Cover $\mathbb{R}^n \setminus C$ with a countable covering of open balls $\{B_n\}$. Each ball B_i contains some compact set C_i, and we can take some smooth functions f_i so that $f_i|_{C_i} = 1$ and $f_i = 0$ outside of B_i. Define

$$f = \sum_i \frac{f_i}{2^i M_i}$$

where M_i is the supremum of the absolute value of all mixed partials of orders less than or equal to i, of f_i. We claim that f as defined is 0 exactly on C. Notice that if $x \in C$, then each f_i is 0 so $f(x) = 0$. If $x \in C^c$, then it belongs to some B_i and so $f(x) \geqslant \frac{f_i(x)}{2^i M_i} > 0$. It remains to show that f is smooth. By comparison test, we have that

$$\sum_i \frac{f_i}{2^i M_i}$$

is an absolutely convergent series, hence f is differentiable since each f_i is. We claim that f is smooth. Note that if we apply any mixed order partial derivative operator, we get that

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} f\right| = \left|\sum_i \frac{1}{2^i M_i} \frac{\partial^\alpha}{\partial x^\alpha} f_i\right| \leqslant M_i \sum_i \frac{1}{2^i}.$$

Therefore all mixed partials of f exist hence f is smooth. Now suppose that C is a closed subset of a manifold. By the Whitney Imbedding theorem, there exists an imbedding $\psi : M \to \mathbb{R}^M$ for sufficiently large M. Since ψ is a homeomorphism onto its image, we have that $\psi(C)$ is a closed subset of $\mathbb{R}^M$. Choose f as per above defined on $\psi(M)$ so that $f^{-1}\{(0)\} = \psi(C)$. Then the smooth function $f \circ \psi : M \to \mathbb{R}$ will suffice.

Problem 5.

(a) Let $X_0 \in M(m, n; k)$. Let $v_{i_1}, \dots, v_{i_k}$ be the k linearly independant columns. Choose a column permutation matrix Q that sends $v_{i_1} \dots v_{i_k}$ to the first k columns. Now let $u_{j_1}, \dots, u_{j_k}$ be the k linearly independant rows of $X_0 Q$. Take P to be a permutation matrix which sends $u_{j_1}, \dots, u_{j_k}$ to the first k rows. Our matrix PX_0Q will be of the form

$$PX_0Q = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is an invertible k by k matrix.

(b) Since $\det(A)$ is a smooth polynomial in the entries of A, if $\det(A) \neq 0$ we can find a sufficiently small ε so that $\det(A_0) \neq 0$ when the entries of $A - A_0$ are less than ε.

(c) Suppose that

$$Y = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

for A $k \times k$ and nonsingular. Suppose that Y is rank k. Then for some matrix

$$X = \begin{bmatrix} I & 0 \\ Z & 0 \end{bmatrix},$$

we have that

$$XY = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}.$$

Computing the matrix multiplication, we see that

$$XY = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ ZA + C & ZB + D \end{bmatrix}.$$

This implies that $Z = -CA^{-1}$, and so $-CA^{-1}B + D = 0$ as desired. Now suppose that $D = CA^{-1}B$. Then we have that

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}.$$

Since A has rank k, Y must as well.

(d) Define the map $f : \mathbb{R}^{nm} \to \mathbb{R}^{(m-k)(n-k)}$ by

$$f\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = D - CA^{-1}B.$$

We can always take a matrix of rank k to be in this form, and in some neighbourhood of A this matrix will be of the same form by a, b, c. Evidently by c) this will vanish exactly when $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ has rank k. The zero set of this function will be a neighbourhood of $M(n, m; k)$, and f will have full rank since it consists of linear terms. Therefore the dimension of this manifold will be

$$nm - (m - k)(n - k) = k(m + n - k)$$

Problem 6.

(a) Matrix multiplication is an algebraic operation, hence smooth. Similarly, the inverse of a matrix is a polynomial in its entries, so it is smooth as well. Since $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ we have that $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and hence has dimension of n^2.

(b) $O(n)$ is the set of all matrices satisfying $A^{\perp} = A^{-1}$ or equivalently $A^{\perp}A = I$. Notice that this is a lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. Consider the mapping $f : \mathrm{GL}_n(\mathbb{R}) \to Sym_n(\mathbb{R})$ defined by

$$f(A) = A^{\perp}A.$$

We claim that I is a regular value of f, which would imply that $O(n)$ is a manifold since $f^{-1}(I) = O(n)$. By the computation done in tutorial, we have that $Df_A(X) = A^{\perp}X + X^{\perp}A$. We claim that this is surjective for $A \in f^{-1}(I)$. Let $Y \in Sym_n(\mathbb{R})$. Then taking $X = \frac{1}{2}AY$ will solve the equation. So I is a regular value and so $O(n)$ is a manifold of dimension

$$\dim(\mathrm{GL}_n(\mathbb{R})) - \dim(Sym_n(\mathbb{R})) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$$

Problem 7.

(a) Consider the mapping $f : \mathbb{R} \to S^1$ defined by

$$x \mapsto e^{2\pi i x}.$$

We have that $f' = 2\pi i f(x)$. This is nonzero so f is a submersion. Clearly this is not a diffeomorphism since it is periodic on $\mathbb{R}$, yet S^1 and $\mathbb{R}$ are both 1−manifolds.

(b) Let $a \in M$. Taking a suitable chart (φ, U) around a, and a chart (ψ, V) around $f(a)$ consider the following commutative diagram:

$$\begin{tikzcd} TM_a \arrow[r, "f_{*a}"'] \arrow[d, "\varphi_{*a}"'] & TN_{f(a)} \arrow[d, "\psi_{*f(a)}"] \\ T\mathbb{R}^n_{\varphi(a)} \arrow[r, "\psi_{*f(a)} \circ f_{*a} \circ \varphi^{-1}_{*\varphi(a)}"'] & T\mathbb{R}^m_{\psi(f(a))} \end{tikzcd}$$

This diagram commutes, and since ψ_{*a} and φ_{*a} are isomorphisms, we have that $n = m$. Furthermore, We have that the mapping

$$\psi_{*f(a)} \circ f_{*a} \circ \varphi^{-1}_{*\varphi(a)} = (\psi \circ f \circ \varphi^{-1})_{*a}$$

is an isomorphism. So $(\psi \circ f \circ \varphi^{-1})$ is a diffeomorphism by the inverse function theorem. So f must be a diffeomorphism.

(c) First note that f is injective and continuous. Hence it is an open mapping. Therefore $f(M)$ is open in N. Since M is compact then so is $f(M)$. Therefore $f(M)$ is closed and open and nonempty. So $f(M) = N$. We have that $f : M \to N$ is a bijection. By $b)$ f must be a diffeomorphism.

Problem 8.

(a) For $X \in M(n, \mathbb{R})$, let $X_{\mathbb{C}} = X \otimes_{\mathbb{C}} 1$ be the complexification of the matrix X. We have that by linear algebra,

$$\det(I + tX_{\mathbb{C}}) = t^n \det(t^{-1}I - (-X_{\mathbb{C}})) = t^n \left(t^{-n} + (\mathrm{Tr}(X_{\mathbb{C}}))t^{-n+1} + \dots\right) = 1 + (\mathrm{Tr}(X_{\mathbb{C}}))t + \dots.$$

This is a polynomial in t, so differentiating at $t = 0$ gives us that

$$\frac{d}{dt} \det(I + tX_{\mathbb{C}}) = \mathrm{tr}(X_{\mathbb{C}}).$$

Since $\mathrm{tr}(X) = \mathrm{tr}(X_{\mathbb{C}})$ we obtain the desired result.

(b) We have that

$$f(A + tX) = \det(A + tX) = \det(A)\det(I + A^{-1}X).$$

By part a) we have that $Df(A)X = \det(A)\mathrm{tr}(A^{-1}X)$. This is a linear map in X. Thus we are done.

(c) We claim that f is a submersion. It is sufficient to show that $Df(A)X$ is a surjective mapping onto $\mathbb{R}$. Given $A \in GL_n(\mathbb{R})$ and $c \in \mathbb{R}$ we wish to find an X so that

$$\det(A)\mathrm{tr}(A^{-1}X) = c.$$

Taking $X = \frac{c}{n \det(A)} A$ gives us

$$\det(A)\mathrm{tr}\left(A^{-1} \frac{c}{n \det(A)} A\right) = \frac{c}{n} \mathrm{tr}(I) = c.$$

Therefore f is a submersion.

(d) By results from A1Q2, we have that the tangent space to I is given by the kernel of $Df(I)$. So

$$X \in TM_I \iff Df(I)X = 0 \iff \det(I)\mathrm{tr}(I^{-1}X) = 0 \iff \mathrm{tr}(X) = 0.$$