Riemannian Geometry Solution Set

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Problem 1. *1.1*

The antipodal mapping is linear, so its tangent mapping df is itself. At any $p \in S^n$, we compute that

$$\langle u, v\rangle_p = \langle -u, -v\rangle_p = \langle dA_p(u), dA_p(v)\rangle_p.$$

On $\mathbb{R}P^p$ we define a riemannian metric by:

$$\langle u, v\rangle_{\pi(p)} = \langle u, v\rangle_p.$$

By above, this is well defined. We now claim that the natural projection is an isometry. Let $p \in S^n$, take $p \ni U_i = \{x \in S_n : x_i \neq 0\}$. Then $A(U_i)$ is a coordinate chart of $\mathbb{R}P^n$. Therefore, under the natural projection, on U_i we compute that

$$\langle d\pi_p(u), d\pi_p(v)\rangle_{[p]} = \langle \pm u, \pm v\rangle_{[p]} = \langle u, v\rangle_p.$$

Problem 2. *1.2*

Define the riemannian metric on $\mathbb{T}^n$ as:

$$\langle u, v\rangle_{\pi(x)} = -\langle u, v\rangle_x.$$

The following computation verifies that π is locally an isometry:

$$\langle d\pi(u), d\pi(v)\rangle_{\pi(x)} = \langle iu, iv\rangle_{\pi(x)} = -\langle u, v\rangle_{\pi(x)} = \langle u, v\rangle_x$$

We now claim that $\mathbb{T}^n$ is isometric with the flat torus i.e. $\mathbb{R}^n/\Gamma$, where Γ is a full rank lattice, generated by $2\pi e_i$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$. The riemannian metric on the flat torus is given by the product metric on $S^1 \times \cdots \times S^1$, so for any vectors $u, v \in T_x\mathbb{R}^n/\Gamma$, we have that

$$\langle u, v\rangle_x = \sum_{i=1}^{n} \langle u_i, v_i\rangle = \sum_{i=1}^{n} u_i\overline{v_i} = \langle u, v\rangle_{\pi(x)}.$$

Problem 3. *1.3*

Let $f : \mathbb{T}^n \to \mathbb{R}^{2n}$ be defined by $f(u_1 + iv_1, \ldots, u_n + iv_n) = (u_1, v_1, \ldots, u_n, v_n)$. We claim that this is an isometric immersion of $\mathbb{T}^n$ into $\mathbb{R}^{2n}$. This map is clearly a diffeomorphism, we will show that it is an isometry of manifolds. We compute that

$$\langle u, v\rangle_x = \sum_{i=1}^{2n} \langle u_i, v_i\rangle = \langle df_x(u), df_x(v)\rangle_{f(x)}$$

Problem 4. *1.4*

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