Q2a:

Suppose not. That is, assume that for all $d > 0$ and $x \in A^c$ and $y \in A$, $\|x - y\| < d$. Choose $B = B_d(x)$. We notice that R will always contain a point in A, namely y, since it will be within d of x. x is not part of A yet every open ball centered at it will contain at least y and itself. Therefore, $x \in \text{bd}A$. Since A is closed, it must be that $\text{bd}A \subset A$. Therefore, $x \in A$. We obtain a contradiction since we supposed $x \in A^c$.

2b:

By compactness of B, for all open covers O, there exists a finite subcover $U_1 \ldots U_n$. From 2a, we know for each point,x in A^c, there exists some d_x such that $\|x - a\| \geq d_x \forall a \in A$. For each $x \in U_i \cap B$ we define d_i as follows: $d_i = inf\{d_x : x \in B \cap U_i\}$. This is well defined and exists, since each U_i was chosen to contain at least 1 point in B. This set is also bounded from below, since each $d_x > 0$.We choose $d = min\{d_1 \ldots d_n\}$. From our choice of d, we get that for all $x \in A, y \in B, \|x - y\| \geq d$.

2C:

Consider $A = \{(x, y) \in R^2 : x \in \mathbb{R}, e^x \leq y\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \leq -e^x\}$. Both of these sets are closed, yet the distance between them can be made arbitrarily small. Take for example at some point x_0. The smallest distance between the two sets will be $\|e^{x_0} - (-e^{x_0})\| = 2e^{x_0}$, since as x_0 approaches $-\infty$, $2e^{x_0}$ approaches 0.