Q4a: For $I \in \underline{n}_a^k$ define $I^c \in \underline{n}_a^{n-k}$ to be the ascending list of $n-k$ elements such that $I^c \cap I = \emptyset$. It is sufficient to define $\star$ on ω_I and define it to be linear. Let $\omega_I \in \Lambda^k(\mathbb{R}^n)$; we define $\star\omega_I = (-1)^{\sigma(I \cup I^c)}\omega_{I^c}$, with $\star(\alpha\lambda + \eta) = \alpha \star \lambda + \star\eta$, for all $\alpha \in \mathbb{R}$. We can verify that indeed, for some $\omega_I, \omega_J \in \underline{n}_a^k$;

$$\omega_I \wedge \star\omega_J = \omega_I \wedge (-1)^{\sigma(J \cup J^c)}\omega_{J^c} = \delta_{IJ}\omega_n = \langle \omega_I, \omega_J \rangle \omega_n$$

Where the second equality holding because when $I = J$, $I \cup J^c = \{1, 2 \dots n\}$. The $(-1)^{\sigma(J \cup J^c)}$ term takes care of sign swaps occuring when we rearrange each $\varphi_{i_k}, \varphi_{j_k}$ used to construct ω_I and ω_J. Additionally, take note that if $I \neq J$, then $I \cap J^c \neq \emptyset$ and the following happens. Assume that $I = \{i_1, \dots i_k\}$ and $J^c = \{j_1, \dots j_{n-k}\}$. At some indices, $i_\alpha = j_\beta$ and so

$$\begin{aligned}
\omega_I \wedge (-1)^{\sigma(J \cup J^c)}\omega_{J^c} &= (-1)^{\sigma(J \cup J^c)}\varphi_{i_1} \wedge \cdots \varphi_{i_\alpha} \cdots \wedge \varphi_{i_k} \wedge \varphi_{j_i} \wedge \cdots \varphi_{j_\beta} \cdots \wedge \varphi_{j_{n-k}} \\
&= (-1)^{\sigma(J \cup J^c)+1}\varphi_{i_1} \wedge \cdots \varphi_{j_\beta} \cdots \wedge \varphi_{i_k} \wedge \varphi_{j_i} \wedge \cdots \varphi_{i_\alpha} \cdots \wedge \varphi_{j_{n-k}} \\
&\qquad \text{(swapping the equal } \varphi) \\
&= 0 \qquad \text{(since sign changes but the value does not)}
\end{aligned}$$

We now claim uniqueness of $\star$. Suppose there is $\star_1$, $\star_2$ which both satisfy $\lambda \wedge \star\eta = \langle \lambda\eta \rangle \omega_n$. Then we have that for any $\lambda \in \Lambda^k(\mathbb{R}^n)$

$$\lambda \wedge \star_1\eta - \star_2\eta = \lambda \wedge \star_1\eta - \lambda \wedge \star_2\eta = \langle \lambda, \eta \rangle \omega_n - \langle \lambda, \eta \rangle \omega_n = 0$$

Taking $\lambda = \star_1(\star_1\eta - \star_2\eta)$

$$0 = \star_1(\star_1\eta - \star_2\eta) \wedge (\star_1\eta - \star_2\eta) = (-1)^{(n-k)^2}(\star_1\eta - \star_2\eta) \wedge \star_1(\star_1\eta - \star_2\eta) = \langle (\star_1\eta - \star_2\eta), (\star_1\eta - \star_2\eta) \rangle \omega_n$$

By the properties of the inner product, $(\star_1\eta - \star_2\eta) = 0$ or equivalently $\star_1\eta = \star_2\eta$. Hence the $\star$ operation is unique.

Q4b: using the formula for $\star\omega_I$ in Q4a, for $\omega_I \in \Lambda^1(\mathbb{R}^3)$, we compute the following.

$$\star\omega_1 = \omega_2 \wedge \omega_3, \star\omega_2 = -\omega_1 \wedge \omega_3, \star\omega_3 = \omega_1 \wedge \omega_2$$

Similarly, when $n = 4$ and $k = 2$, using our definition of $\star$,

$$\star\omega_{12} = \omega_3 \wedge \omega_4, \star\omega_{13} = -\omega_2 \wedge \omega_4, \star\omega_{14} = \omega_2 \wedge \omega_3, \star\omega_{23} = \omega_1 \wedge \omega_4, \star\omega_{24} = -\omega_1 \wedge \omega_3, \star\omega_{34} = \omega_1 \wedge \omega_2$$

Q4c: It is sufficient to show that $\star \circ \star$ applied to some basis element of $\Lambda^k(V)$ is scaled by the desired constant. Let $I \in \underline{n}_a^k$, $I = \{i_1, \dots, i_k\}$. Then we see that

$$\star \circ \star(\omega_I) = \star(-1)^{\sigma(I \cup I^c)}\omega_{I^c} = (-1)^{\sigma(I^c \cup I)} \cdot (-1)^{\sigma(I \cup I^c)}\omega_I = (-1)^{(k)(n-k)}\omega_I$$

Where the last equality holds since by applying the $\star$ operation twice, we make $k(n-k)$ swaps of the constituent ω_i.