Q2a: We first claim that $I(P)$ converges if $f \in L^1(\mu)$. First, note that given any partition we can without loss of generality assume that $y_0 = 0$, since we can always partition the interval containing 0 into two smaller intervals with lengths less than $\delta(P)$ by the fact that $\delta(P) < \infty$. We know that

$$I(P) = \sum_{i=-\infty}^{\infty} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\}) = \sum_{i=0}^{\infty} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\}) + \sum_{i=-\infty}^{0} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\})$$

It is sufficient to check that these sums do not diverge, so we can conclude that their sum will not diverge either. Suppose for the sake of contradiction that

$$\sum_{i=0}^{\infty} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\}) = \infty$$

Then for any simple functions $\{\phi_n\}$, we would have that

$$\sum_{i=1}^{n} a_i \phi_i \leq \sum_{i=0}^{\infty} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\})$$

We can make this true for arbitrarily large a_i, hence we have that $\int f^+$ diverges. This contradicts the integrability of f. It remains to show that $\sum_{i=-\infty}^{0} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\})$ converges. Suppose not. Then we have that

$$|\sum_{i=-\infty}^{0} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\})| = \infty$$

For any simple functions $\{\phi_n\}$, we can take a_i such that $(a_i - \delta(P))\phi_i \leq f^-$, which we can do by integrability of f^-. Since $|\sum_{i=-\infty}^{0} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\})|$ is unbounded, and must always be greater than or equal to $\sum_{i=1}^{n} a_i \phi_i$, we can choose a_i arbitrarily large. Once again this contradicts the integrability of f^-. Hence we have that $I(P)$ is well defined.

We now claim that

$$\int f = \lim_{\delta(P) \to 0} I(P)$$

First we will show that $\lim_{\delta(P)\to 0} I(P) \geq \int f$. First note that for any interval, $(y_i, y_{i+1}]$ we have that $\delta(P) \geq y_{i+1} - y_i$, and for $x \in f^{-1}\{(y_i, y_{i+1})\}$ we get that $f(x) \leq y_{i+1} - (y_i - y_{i+1})$ and so we get that $y_i \geq f(x) - \delta(P)$.

$$\begin{aligned}
I(P) &= \sum_{i=-\infty}^{\infty} \int_{f^{-1}\{(y_i, y_{i+1})\}} y_i \\
&\geq \sum_{i=-\infty}^{\infty} \int_{f^{-1}\{(y_i, y_{i+1})\}} f(x) - \delta(P) \\
&= \int_{\bigcup_{i=-\infty}^{\infty} (y_i, y_{i+1}]} f(x) - \delta(P) \\
&= \int_X f d\mu - \int_X \delta(P) \\
&= \int_X f d\mu - \mu(X)\delta(P)
\end{aligned}$$

Taking the limit as $\delta(P) \to 0$ we get that $\lim_{\delta(P)\to 0} I(P) \geq \int f$. We will now show that $\lim_{\delta(P)\to 0} I(P) \leq \int f$. Since $f(x) \geq y_i$, we have that

$$\int f = \int f^+ - \int f^- \geq \sum_{i=0}^{\infty} y_i \mu(\{x : f(x) \in (y_i, y_{i+1}]\}) - \sum_{i=-\infty}^{0} \mu(\{x : f(x) \in (y_i, y_{i+1}]\}) = I(P)$$

Taking the limit as $\delta(P) \to 0$ we get the desired inequality and conclude that $\int f = \lim_{\delta(P)\to 0} I(P)$

Q2b: We claim that $\int |f|$ is finite. We have that

$$\int |f| \leq \int_X \delta(P) d\mu = \mu(X)\delta(P) < \infty$$

Since the value of the integral will be at most δ