Problem 1. *Folland 8.1.3*

(a) We proceed by induction on k. For $k = 1$ we have

$$\eta^1(t) = \frac{1}{t^2}e^{-\frac{1}{t}} = P_1\left(1/t\right)\eta(t).$$

Suppose the result holds for k. Then,

$$\eta^{(k+1)}(t) = \left(P_k\left(1/t\right)\eta(t)\right)^{(1)} = \left(P_k(1/t)\right)'\eta(t) + P_k(1/t)P_1(1/t)\eta(t) = \eta(t)\left(P_k(1/t)' + P_k(1/t)P_1(1/t)\right).$$

This is what we wanted to show.

(b) First we claim that $\lim_{t\to 0}\eta^{(1)}(t) = 0$.

$$\lim_{t\to 0}\eta^1(t) = \lim_{t\to 0}\frac{1}{t^2}e^{-1/t} = \lim_{y\to\infty} y^2 e^{-y} = 0$$

by L'Hopitals Rule. This is true for all k by induction and L'Hopitals rule.

Problem 2. *Folland 8.1.4*

First note that such f must belong to L^1_{loc}, since on any compact set K we have that

$$\int_K |f| dx \leqslant \|f\|_\infty m(K) < \infty.$$

Define $A_r f(x)$ as

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) dy.$$

We claim that $\lim_{r \to 0} A_r f(x) = g(x)$ is uniformly continuous. we have that:

$$\begin{aligned} |g(x) - g(y)| &= \lim_{r \to 0} \frac{1}{B_x(r)} \left| \int_{B_x(r)} f(z) dz - \int_{B_y(r)} f(z) dz \right| \\ &= \lim_{r \to 0} \frac{1}{B_x(r)} \int_{B_x(r)} \tau_y f(z) - f(z) dz \\ &\leqslant \|\tau_y f(x) - f(x)\|_u \end{aligned}$$

Which can be made arbitrarily small. As $r \to 0$, $A_r f(x) \to f(x)$. We have

$$\|A_r f - f\|_\infty \leqslant \|A_r f - \tau_y f\|_\infty + \|\tau_y f - f\|_\infty,$$

which can be made arbitrarily small since $\|A_r f - \tau_y f\|_\infty \to 0$ uniformly as $y \to 0, r \to 0$. Therefore f agrees with $\lim_{r \to 0} A_r f(x)$ except on a set of measure 0.

Problem 3. *Folland 8.2.6*

The following chain of inequalities holds:

$$\begin{aligned}
|f * g(x)|^r &= \left|\int f(y) g(x-y)\, dy\right|^r \\
&\leqslant \left(\int |f(y)|\, |g(x-y)|\right)^r \\
&= \left(\int |f(x)|^{1+p/q-p/q}\, |g(x-y)|^{1+q/p-q/p}\, dy\right)^r \\
&= \left(\int |f(y)|^{p/r}\, |g(x-y)|^{q/r}\, |f(y)|^{(r-p)/r}\, |g(x-y)|^{(r-q)/r}\, dy\right)^r \\
&= \left(\int \left(|f(y)|^p\, |g(x-y)|^q\right)^{1/r}\, |f(y)|^{(r-p)/r}\, |g(x-y)|^{(r-q)/r}\, dy\right)^r \\
&\leqslant \|f(y)^p g(x-y)^p\|_r^r \cdot \left\|f(y)^{\frac{r-p}{p}}\right\|_{\frac{pr}{r-p}}^r \cdot \left\|g(x-y)^{\frac{r-q}{q}}\right\|_{\frac{qr}{r-q}}^r \qquad \text{(by Generalized Holders Inequality)} \\
&\leqslant \|f\|_p^{r-p}\, \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q\, dy
\end{aligned}$$

Therefore

$$\begin{aligned}
\int |f * g(x)|^r\, dx &\leqslant \|f\|_p^{r-p}\, \|g\|_q^{r-q} \int\int |f(y)|^p |g(x-y)|^q\, dy\, dx \\
&= \|f\|_p^{r-p}\, \|g\|_q^{r-q} \int |f(y)|^p\, dy \int |g(x-y)|^q\, dx \qquad \text{(by Fubini-Tonelli)} \\
&= \|f\|_p^r\, \|g\|_q^r
\end{aligned}$$

As desired.

Problem 4. *Folland 8.2.7*

Since g has compact support, then for every multi index $\alpha, |\alpha| \leqslant k$ we have $\partial^\alpha g \in C_c(\mathbb{R}^n)$. Thus

$$\partial^\alpha f * g(x) = \int_{\mathbb{R}^n} \partial^\alpha g(x-y) f(y) dy$$

will exist, since $f \in L^1_{loc}$ and $\partial^\alpha g$ is compactly supported.

Problem 5. *Folland 8.2.8*

By the fundamental theorem of calculus, $\partial_j(f * g)$ exists in the regular sense. We now claim that it equals $(\partial_j f) * g$. We compute that

$$\begin{aligned}
&\lim_{y\to 0} \left\|y^{-1}\left(\tau_{-y} f * g - f * g\right) - (\partial_i f) * g\right\| \\
&= \lim_{y\to 0} \left\|y^{-1}\left[\int f(x + ye_j - z)g(z)dz - \int f(x-z)g(z)dz\right] - \int \partial_j f(x-z)g(z)dz\right\| \\
&= \lim_{y\to 0} \left\|y^{-1}\int \left[\left|f(x+ye_j - z) - f(x-z) - y\partial_j f(x-z)\right|\right] |g(z)|\, dz\right\| \\
&\leqslant \lim_{y\to 0}\left[\left\|y^{-1}(\tau_{-y} f - f) - \partial_j f\right\|_p \|g\|_q\right] && \text{(By Holders Inequality)} \\
&= 0 && \text{(since } f \text{ is strong } L^p \text{ differentiable)}
\end{aligned}$$

Problem 6. *Folland 8.2.9*

First suppose that f' exists almost everywhere.Then, by taking any $g \in C_c$ with $\int g = 1$ we have by Folland 8.2.8 $f * g$ is L^p differentiable in the usual sense. We also have that $(f * g_t)' = f' * g_t$. By Folland Theorem 8.14 we have $f' * g_t \to f'$ as $t \to 0$. Therefore we have that f is absolutely continuous by the Fundamental Theorem of Lebesgue integrals. Conversely suppose that f is absolutely continuous on bounded intervals. We can write using the fundamental theorem of Calculus,

$$\frac{f(x+y) - f(x)}{y} - f'(x) = \frac{1}{y}\int_0^y [f'(x+t) - f'(x)]\, dx.$$

Taking the L^p norm we have

$$\left\| \frac{1}{y}\int_0^y [f'(x+t) - f'(x)]\, dx \right\|_p \leqslant \left\| \frac{1}{y}\int_0^y f'(x)dt \right\|_p + \left\| \frac{1}{y}\int_0^y f'(x+t)dt \right\| \to 0 \text{ as } y \to 0$$

Problem 7. *Folland 8.2.10*

We can write our integral as

$$|f * \phi_t(x)| \leqslant \int |f(x-y)|\,|\phi_t(y)|\,dy = \int_{|x|\leqslant t} |f(x-y)|\,|\phi_t(y)|\,dy + \sum_{k=0}^{\infty} \int_{2^k t \leqslant |x| \leqslant 2^{k+1} t} |f(x-y)|\,|\phi_t(y)|\,dy.$$

We bound each term in the following way:

$$\begin{aligned}
\int_{|x|\leqslant t} |f(x-y)|\,|\phi_t(y)|\,dy &= t^{-n} \int_{|x|\leqslant t} |f(x-y)|\,\left|\phi(t^{-1}y)\right| dy \\
&\leqslant Ct^{-n} \int_{|x|\leqslant t} |f(x-y)| dy \\
&\leqslant C \frac{m(B_x(1))}{m(B_x(t))} \int_{B_x(t)} |f(y)|\,dy \\
&= Cm(B_x(1))Hf(x).
\end{aligned}$$

For The second summand, we estimate that

$$\begin{aligned}
\sum_{k=0}^{\infty} \int_{2^k t \leqslant |x| \leqslant 2^{k+1} t} |f(x-y)|\,|\phi_t(y)|\,dy &\leqslant \sum_{k=0}^{\infty} Ct^{-n} \int_{2^k t \leqslant |x| \leqslant 2^{k+1} t} |f(x-y)| \left(1 + |t^{-1}y|\right)^{-n-\varepsilon} dy \\
&\leqslant \sum_{k=0}^{\infty} Cm(B_x(1)) \left(2^k\right)^{-n-\varepsilon} Hf(x) \\
&\leqslant 2Cm(B_x(1))Hf(x).
\end{aligned}$$

Therefore $M_\phi(f) \leqslant C \cdot Hf(x)$.

Problem 8. *Folland 8.2.11*

(a) Let $\mathcal{J}$ be an ideal in L^1. Let $\overline{\mathcal{J}}$ be its closure. Take any sequence $\{f_n\}$ in $\mathcal{J}$ with limit f. Then by youngs inequality with $p = 1$, we have for any $g \in L^1$,

$$\|f_n * g\|_1 \leqslant \|f_n\|_1 \|g\|_1 \implies \|f * g\|_1 \leqslant \|f\|_1 \|g\|_1 .$$

(b) Let U be the subspace. First, suppose that $g \in C_c$. Then, for a finite partition of the support of g we estimate $f * g$ by

$$\sum_{i=1}^{n} f(x - y_j)g(y_i)(y_{i+1} - y_i) = \sum_{i=1}^{n} (y_{i+1} - y_i)\tau_{y_i} f(x) \leqslant \sum_{i}^{n} \int_{(y_i, y_{i+1}]} |f(x - y)| dy < \infty.$$

Conversely, consider the sequence $\{\phi_{1/n}\}$ with $\int \phi = 1$, then by prop 8.6, theorem 8.14

$$f * \tau_y \phi_{1/n} \to \tau_y f.$$

Since I is closed we have that this is in I.

Problem 9. *60 [Extra Credit]*

Take $\{U_i\}$ to be a countable covering of $\mathbb{R}^n \setminus E$. Take f_i to be smooth, > 0 on a compact set contained inside U_i, and 0 outside of U_i. Define

$$f = \sum_{i=1}^{\infty} \frac{f_i}{2^i M_i},$$

where $M_i = \sup_{\alpha \leqslant i} |\partial^\alpha f_i|$. This sequence absolutely and uniformly converges. The partial derivatives of all orders of f are bounded, so $f \in C^\infty$. Furthermore $f(x) = 0$ if and only if $f_i(x) = 0$ for all i i.e. $x \in E$. Thus we are done.