Q2: This is not true. Consider the function algebra generated by $\mathcal{A} = \{1, x^2, \dots x^n\}$, where we allow finite sums, products and scaling with real numbers between elements of $\mathcal{A}$ on $[0,1]$. Note that this is indeed a function algebra, since scaling a polynomial with no linear term will yield another polynomial with no linear term, and similarly for addition and multiplication. This will split points since x^2 is injective on $[0,1]$. It also vanishes nowhere since the constant function is a part of this algebra. Hence by the Stone Weierstrass Theorem, for each $\varepsilon > 0$ there is a $g = b_0 + b_2x^2 + \dots b_nx^n$ with $|f - g| < \varepsilon$. Then observe:

$$\begin{aligned}\int_0^1 f^2 dx &= |\int_0^1 fg dx + \int_0^1 f(f-g)dx| \\ &\leq |\int_0^1 fg dx| + \int_0^1 |f||g-f|dx \\ &= |\int_0^1 f(x)(b_0 + b_2x^2 + \cdots + b_nx^n dx| + \int_0^1 |f||f-g|dx \\ &\leq 0 + sup|f|\,\varepsilon\end{aligned}$$

Since ε can be made arbitrarily small, we have that $\int_0^1 f^2 dx = 0$. Thus it is not possible that $\int_0^1 x \cdot f dx = 1$.