5.5.61: We show that the set $\{h_{nm}\}$ satisfies mutual orthongonality and is normalized to 1. Observe that:

$$\begin{aligned}
\langle h_{nm}, h_{pq}\rangle &= \int_{X\times Y} \overline{h_{nm}} h_{pq} d(\mu\times\nu) \\
&= \int_{X\times Y} \overline{f_n(x)g_m(y)} f_p(x)g_p(y) d(\mu\times\nu) \\
&= \int_X \int_Y \overline{f_n(x)} f_p(x) \overline{g_m(y)} g_q(y) d\nu d\mu && \text{(by Fubini-Tonelli's Theorem)} \\
&= \Big[\int_X \overline{f_n(x)} f_p(x) d\mu\Big] \cdot \Big[\int_Y \overline{g_m(y)} g_q(y) d\nu\Big] \\
&= \delta_{np}\cdot\delta_{mq} && \text{(Since } f_i, g_j \text{ form orthonormal basis)}
\end{aligned}$$

This is exactly what we wanted to show. Now we claim that the set $\{h_{nm}\}$ spans $L^2(\mu\times\nu)$. We check that this satisfies the properties $(a)-(c)$ of theorem 5.27 of folland. First suppose $k \in L^2(\mu\times\nu)$ such that $\langle k, h_{nm}\rangle = 0$ for all (n,m). We have that

$$0 = \int_{X\times Y} \overline{k(x,y)} h_{nm} d(\mu\times\nu) = \int_X f_n(x)\Big[\int_Y \overline{k(x,y)} g_m(y) d\nu\Big] d\mu = \int_Y g_m(y)\Big[\int_X \overline{k(x,y)} f_n(x) d\mu\Big] d\nu$$

Which implies that for all m, n,

$$0 = \int_Y \overline{k(x,y)} g_m(y) d\nu = \int_X \overline{k(x,y)} f_n(x) d\mu.$$

Letting $k_x(y) = k(x,y)$, and $k_y(x) = k(x,y)$. Using orthonormality we get that $0 = k_x(y) = k_y(x) = k(x,y)$. We now show that Parseval's Identity holds. By Bessels Identity, it is sufficient to show that $|k|^2 \leq \sum_{n,m} |\langle k, h_{nm}\rangle|^2$ Let $k \in L^2(\mu\times\nu)$. Then, we write

$$\sum_{(n,m)} \Big|\langle k, h_{nm}\rangle\Big|^2 = \sum_{(n,m)} \Big|\int_X f_n(x)\Big[\int_Y \overline{k(x,y)} g_m(y) d\nu\Big] d\mu\Big|^2 = \sum_{(n,m)} \Big|\int_Y g_m(y)\Big[\int_X \overline{k(x,y)} f_n(x) d\mu\Big] d\nu\Big|$$

Using orthonormality, we get that

$$\Big|\int \overline{k(x,y)} g_m(y)\Big|^2 = \Big|\int \overline{k(x,y)} f_n(x)\Big|^2.$$

This implies that

$$|k|^2 = \int_{X\times Y} \overline{k(x,y)} k(x,y) \langle h_{nm}, h_{nm}\rangle d(\mu\times\nu) \leq \sum_{(n,m)} \Big|\langle k(x,y), f_n(x)g_m(x)\rangle\Big|^2$$

From orthonormality of f, g.