Problem 1.

(a) Take $\{(\varphi_n, U_n)\}, \{(\psi_m, V_m)\}$ maximal atlases for M, N. For any choice of charts $\varphi_i, \varphi_j, \psi_k$ we have that

$$\varphi_i \circ \pi_1 \circ (\varphi_j^{-1}, \psi_k^{-1}) = \varphi_i \circ \varphi_j^{-1} \in C^\infty.$$

A similar computation for π_2 yields

$$\psi_j \circ \pi_2(\varphi_k^{-1}, \psi_i^{-1}) = \psi_j \circ \psi_i^{-1} \in C^\infty.$$

(b) First we notice that $\dim(M^m \times N^n) = \dim(M^m) + \dim(N^n) = m + n$ so

$$\dim\left(T(M \times N)_{(x,y)}\right) = \dim\left(TM_x\right) + \dim(TN_y) = \dim(TM_x \oplus TN_y).$$

It is sufficient to show that the mapping $f(v)$ defined by

$$v \mapsto \left((\pi_1)_{*(x,y)} v, (\pi_2)_{*(x,y)} v\right)$$

has trivial kernel. Suppose that for some v we have that $f(v) = 0$. Notably we have that $(\pi_1)_{*(x,y)} v = 0$. Viewed from the point of view of charts, we have that

$$0 = D\left(\phi_i \circ \pi_1\left(\varphi_j^{-1}, \psi_k^{-1}\right)\right) v = \left[D(\varphi_i \circ \varphi_j^{-1}) \;\mid\; 0\right] v.$$

Since $D(\varphi_i \circ \varphi_j^{-1})$ is an isomorphism, we have that

$$D(\varphi_i \circ \varphi_j^{-1}) \pi_x(v) = 0.$$

So the first m coordinates must be 0. A similar argument with $(\pi_2)_{*(x,y)}$ shows that the last n coordinates are 0. Therefore $v = 0$.

(c) By part b, we have the mapping $(i_y)_{*x} : TM_x \to TM_x \oplus TN_y$ given by

$$v \mapsto \left((\pi_1)_{*(x,y)} \circ (i_y)_{*(x)} v, (\pi_2)_{*(x,y)} \circ (i_y)_{*(x)} v\right)$$

This simplifies to

$$\left((\pi_1 \circ i_y)_{*x} v, (\pi_2 \circ i_y)_{*x} v\right) = (v, 0),$$

since the first coordinate is the identity, and the second is constant.

(d) First note that $(f \times g)(x, y)$ is smooth since for any choice of charts on M, N, P, Q the function

$$(\lambda, \eta) \circ (f, g) \circ (\varphi^{-1}, \psi^{-1}) = (\lambda \circ f \circ \varphi^{-1}, \eta \circ g \circ \psi^{-1})$$

is smooth in both components. First, notice that the identity mapping on $T(M \times N)$ is of the form

$$id_{T(P\times Q)} = (i_{g(y)})_{*f(x)} \circ (\pi_1)_{*(f(x),g(y))} + (i_{f(x)})_{*g(y)} \circ (\pi_2)_{*(f(x),g(y))}.$$

If we compose with $(f \times g)_{*(x,y)}$, using the chain rule we get

$$\begin{aligned}
(f \times g)_{*(x,y)} &= id_{T(P\times Q)} \circ (f \times g)_{*(x,y)} \\
&= \left[(i_{g(y)})_{*f(x)} \circ (\pi_1)_{*(f(x),g(y))} + (i_{f(x)})_{*g(y)} \circ (\pi_2)_{*(f(x),g(y))}\right] \circ (f \times g)_{*(x,y)} \\
&= \left[(i_{g(y)})_{*f(x)} \circ (\pi_1)_{*(f(x),g(y))} \circ (f \times g)_{*(x,y)}\right] + \left[(i_{f(x)})_{*g(y)} \circ (\pi_2)_{*(f(x),g(y))} \circ (f \times g)_{*(x,y)}\right] \\
&= \left[(i_{g(y)} \circ \pi_1 \circ (f(x), g(y)))_{*(x,y)}\right] + \left[(i_{f(x)} \circ \pi_2 \circ (f(x), g(y)))_{*(x,y)}\right] \\
&= (f_{*x}, 0) + (0, g_{*y}) \\
&= (f_{*x}, g_{*y})
\end{aligned}$$

Problem 2.

(a) When we restrict μ to $G \times \{e\}, \{e\} \times G$ we see that

$$\mu(e, g) = \mu(g, e) = g.$$

The multiplication map behaves like projection when restricted to these groups.

(b) For $u + v \in TG_e \oplus TG_e$ the inverse of the mapping given in 1b is

$$v + w \mapsto (i_e)_{*e}v + (i_e)_{*e}w.$$

Therefore by the chain rule,

$$\begin{aligned}
\mu_{*(e,e)}(v, w) &= \mu_{*(e,e)} \circ [(i_e)_{*e}v + (i_e)_{*e}w] \\
&= \mu_{*(e,e)} \circ (i_e)_{*e}v + \mu_{*(e,e)} \circ (i_e)_{*e}w \\
&= (\mu \circ i_e)_{*e}\, v + (\mu \circ i_e)_{*e}\, w \\
&= \mu_{*(e\times G)}v + \mu_{*(G\times e)}w \\
&= v + w
\end{aligned}$$

(c) We compute the composition $\mu \circ (\mathrm{id} \times \iota) \circ \Delta$ as

$$\mu \circ (\mathrm{id} \times \iota) \circ \Delta(g) = \mu \circ (\mathrm{id} \times \iota)(g, g) = \mu(g, g^{-1}) = e.$$

(d) Since the mapping $\mu \circ (\mathrm{id} \times \iota) \circ \Delta$ is constant, we have that

$$(\mu \circ (\mathrm{id} \times \iota) \circ \Delta)_{*e} = 0.$$

Since $\Delta = \mathrm{id} \times \mathrm{id}$ we know that $\Delta_{*e}v = (\mathrm{id}_{*e}, \mathrm{id}_{*e})v = (v, v)$. Thus by the chain rule and 2b,

$$0 = (\mu \circ (\mathrm{id} \times \iota) \circ \Delta)_{*e}\, v = \mu_{*(e,e)} \circ (\mathrm{id}_{*e}, \iota_{*e}) \circ \Delta_{*(e)}v = \mu_{*(e,e)} \circ (\mathrm{id}_{*e}v, \iota_{*e}v) = v + \iota_{*v} \implies \iota_{*e}v = -v$$

Problem 3.

(a) Since $\mu : G \times G \to G$ is a C^∞ mapping, the restriction mapping $\mu_g : G \to G$ is also C^∞. Furthermore, μ_g is a diffeomorphism, since it is smooth and bijective, and has smooth inverse $\mu_{g^{-1}}$. Therefore the tangent mapping $(\mu_g)_{*e}$ is an isomorphism of TG_e into TG_g.

(b) Consider the mapping $f : TG \to G \times \mathbb{R}^n$ defined by $[g, v] \mapsto (g, (\mu_g)^{-1}_{*e} v)$. We claim that this is an isomorphism. Since $(\mu_g)^{-1}_{*e}$ is a linear isomorphism, this is a bijection. Furthermore, the following diagram commutes

$$\begin{tikzcd} TG \arrow[rr, "f"] \arrow[dr, "\pi"'] & & G \times \mathbb{R}^n \arrow[dl, "\pi'"] \\ & G & \end{tikzcd}$$

Thus the tangent bundle of G must be trivial.

Problem 4.

Let $\mathscr{A} = \{(\phi_i, U_i)\}$ be a maximal atlas on M. Then the charts on TM are of the form

$$\begin{aligned} f_i : U_i \times \mathbb{R}^n &\to \varphi_i(U_i) \times \mathbb{R}^n \\ (v, p) &\mapsto (\varphi_i(p), (\varphi_i)_{*p} v). \end{aligned}$$

On a suitable domain, the transition maps are of the form $(\varphi_j \circ \varphi_i^{-1}(x), (\varphi_i \circ \varphi_j^{-1})_{*x} v)$, and the jacobian will be

$$D(\varphi_j \circ \varphi_i^{-1}(x), (\varphi_i \circ \varphi_j^{-1})_{*x} v) = \begin{bmatrix} (\varphi_j \circ \varphi_i^{-1})'(x) & 0 \\ * & (\varphi_j \circ \varphi_i^{-1})_{*x} \end{bmatrix}.$$

Since $(\varphi_j \circ \varphi_i^{-1})'(x) = (\varphi_j \circ \varphi_i^{-1})_{*x}$ as linear maps, and are both nonsingular by the inverse function theorem,

$$\mathrm{Det}(D(\varphi_j \circ \varphi_i^{-1}(x), (\varphi_i \circ \varphi_j^{-1})_{*x} v)) = \mathrm{Det}(\varphi_j \circ \varphi_i^{-1}(x))^2 > 0.$$

The transition mappings on TM have positive determinants hence TM is orientable.

Problem 5.

Let M be a nonorientable manifold. Suppose $TM \cong M \times \mathbb{R}^n$. By problem 4 we have that TM is orientable. Since some isomorphism f maps $TM \to M \times \mathbb{R}^n$, the orientation on TM gets pushed to an orientation of $M \times \mathbb{R}^n$. Then we must have that $M \times \mathbb{R}^n$ is orientable. Note however that charts on $M \times \mathbb{R}^n$ are of the form $\varphi_i \times \mathrm{id}$, where φ_i is a chart on M and id is the identity on $\mathbb{R}^n$. The transition maps on $M \times \mathbb{R}^n$ will be the product of transition maps on M with the identity. For some choice of charts φ_i, φ_j the tangent mapping $D(\varphi_i \circ \varphi_j^{-1}, \mathrm{id})$ will have negative determinant. Thus $M \times \mathbb{R}^n$ is not orientable, a contradiction.

Problem 6.

Given that the transition mappings of E are of the form:

$$\begin{aligned} T\varphi_i(U_i \cap U_j) &\to T\varphi_j(U_i \cap U_j) \\ (x, v) &\mapsto (\varphi_j \circ \varphi_i^{-1}(x), \Lambda_{ij}(x)v) \end{aligned}$$

For a smooth mapping $\Lambda_{ij} : \varphi_i(U_i \cap U_j) \to GL(k, \mathbb{R})$. Note that $\Lambda_{ij}(x)$ induces a map $\Lambda_{ij}^*(x) : (\mathbb{R}^k)^* \to (\mathbb{R}^k)^*$ by pullback i.e. $\Lambda_{ij}^*(\eta)(v) = \eta(\Lambda_{ij}(x)(v))$. So we have bundle charts with transition maps on E^* given by

$$\begin{aligned} T^*\varphi_j(U_i \cap U_j) &\to T^*\varphi_i(U_i \cap U_j) \\ (y, \eta) &\mapsto (\varphi_i \circ \varphi_j^{-1}(y), \Lambda_{ij}^*((\varphi_i \circ \varphi_j^{-1}(y))\eta) \end{aligned}$$

The transition maps induce an equivalence relation $\sim$ on $\bigsqcup T^*\varphi_i(U_i)$ where $(x, \lambda) \sim (y, \eta)$ if for some i, j,

$$(x, \lambda) = (\varphi_i \circ \varphi_j^{-1}(y), \Lambda_{ij}^*((\varphi_i \circ \varphi_j^{-1}(y))\eta).$$

By $A2Q7$ this gives a manifold structure to E^*. We have constructed a dual bundle (E^*, M, π).

Problem 7.

(a) Recall from basic algebra, a root b of $p(z)$ is of multiplicity greater than 1 if and only if it is a root of $p'(z)$. Take $a_0, \dots, a_n$ so that $p(z) = a_0 + \cdots + a_{n-1}z^{n-1} + z^n$ has n distinct roots $\{b_1, \dots, b_n\}$. We have that $p(b_i) - p'(b_i)$ is nonzero. Since $p(z) - p'(z)$ is continous when viewed as a function of $z, a_0, \dots, a_n$, there exists some ε ball B_ε around $(a_0, \dots, a_n) \in \mathbb{C}^n$ so that $p'(b_i) - p(b_i)$ does not vanish for all $a \in B_\varepsilon$. Since the roots b_i smoothly vary with $(a_0, \dots, a_{n-1})$, M_n is an open set in $\mathbb{C}^n$, so it must be an $n-$manifold.

(b) This is not a smooth manifold. Consider when $n = 2$. Let $A \subset \mathbb{C}^2 \cong \mathbb{R}^4$ be the space of polynomials with complex coefficients with at least one complex root. If we notate coordinates in $\mathbb{C}^2$ as (b_1, b_2, c_1, c_2) then A will be the union of subsets parametrized by $A_1 = (b_1, 0, c_1, c_2)$ and $A_2 = (b_1, b_2, c_1, 0)$. Open neighbourhoods of this subset will look like $(b_1 \pm \varepsilon, 0, c_1 \pm \varepsilon, c_2 \pm \varepsilon)$ and $(b_1 \pm \varepsilon, b_2 \pm \varepsilon, c_1 \pm \varepsilon, 0)$. However on $A_1 \cap A_2$, the open neighborhoods will look like $(b_1 \pm \varepsilon, 0, c_1 \pm \varepsilon, 0)$. These open sets are not of the same dimension hence this can not be a smooth manifold.