Q7: We wish to minimize the integral

$$\int_{[-1,1]} |x^3 - a - bx - cx^2|^2.$$

Equivalently, we minimize

$$\langle x^3 - a - bx - cx^2, x^3 - a - bx - cx^2 \rangle.$$

If we choose an orthonormal basis of polynomials, we get that the minimum is given by

$$\langle x^3 - a - bx - cx^2, x^3 - a - bx - cx^2 \rangle = \langle c_1 x^3 + c_2 x^2 + c_3 x + c_4, c_1 x^3 + c_2 x^2 + c_3 x + c_4 \rangle = c_1^2 + c_2^2 + c_3^2 + c_4^2.$$

We see that this quantity will be minimzed when $c_1 = 1, c_2 = c_3 = c_4 = 0$. i.e. the minimum is $\int_{-1}^{1} x^6 = \frac{2}{7}$.

By cauchy Swartz Inequality, we have that

$$\int_{[-1,1]} x^3 g(x) \leq \Big[\int_{[-1,1]} x^6 \Big] dx \cdot \int_{[-1,1]} |g(x)|^2 = \frac{\sqrt{2}}{\sqrt{7}}.$$