Discrete Riemann surfaces:
Linear discretization and its convergence

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Abstract. We develop linear discretization of complex analysis, originally introduced by R. Isaacs, J. Ferrand, R. Duffin, and C. Mercat. We prove convergence of discrete period matrices and discrete Abelian integrals to their continuous counterparts. We also prove a discrete counterpart of the Riemann–Roch theorem. The proofs use energy estimates inspired by electrical networks.

1. Introduction

The history of linear discretizations of harmonic and holomorphic functions can be traced back to the early finite-element literature [11,12]. The discretization of the Dirichlet variational principle leads to linear discretizations of the Laplace equation and of the Cauchy–Riemann equations on a square grid. Lelong-Ferrand developed the theory of discrete harmonic and holomorphic functions on a square grid to a level that allowed her to prove the Riemann mapping theorem [15,19]. Duffin took first steps to extend the theory beyond the square grid. He considered arbitrary triangulations of planar domains and discovered the now famous cotanweights [13]. The cotan-Laplace-operator turned out to be important in discrete differential geometry of surfaces [26] and in computer graphics [24].

A striking feature of some 2-dimensional discrete models in statistical physics is that they exhibit conformally invariant properties in the scaling limit: for site percolation on a triangular grid [33], for the random cluster model [34], for the Ising model [9], for domino tiling [18]. In all cases, the linear theory of discrete analytic functions on regular grids was instrumental.

Mercat generalized the linear theory from planar domains to discrete Riemann surfaces [20] and in particular introduced discrete period matrices [23]. Numerical experiments to compute period matrices for triangulated surfaces and compare them with known period matrices

for the smooth surfaces were performed in [4]. First attempts to prove the convergence were made in [21] but it remained an open problem; see Remark 7.6 below.

A very different "nonlocal" linear theory for discrete holomorphic functions on triangulated surfaces was introduced by Wilson [39]. For this discretization the convergence of period matrices to their continuous counterparts is also discussed in [39]. Yet another linear theory for discrete holomorphic functions on regular triangle lattices, due to Dynnikov–Novikov [14], cf. [7, 17], was motivated by the theory of integrable systems. A discretization of the Riemann–Roch theorem for graphs was introduced by Baker–Norine [1].

Important nonlinear discrete models of complex analysis and Riemann surfaces involve circle packings or, more generally, patterns of intersecting or disjoint circles [28, 36, 37]. The linear theory of discrete holomorphic functions on a square grid can be obtained by linearization from the nonlinear circle pattern theory. Discrete holomorphic functions describe infinitesimal deformations of circle patterns [5]. A strikingly simple concept of discrete conformal equivalence for triangle meshes leads to a new nonlinear model of discrete Riemann surfaces [6, 35].

The problem of convergence of discrete analytic functions to their continuous counterparts is certainly one of the most important issues. Convergence for the linear theory was proved by Lusternik and Courant–Friedrichs–Lewy [11] for square lattices, by Chelkak–Smirnov [10] for rhombic lattices, and by one of the authors [30] for orthogonal quadrilateral lattices. Although there are some striking convergence results [16, 28, 29], see also [8], the situation is far from satisfying.

In this paper we prove convergence of discrete period matrices (Theorem 2.5) and discrete Abelian integrals (Theorem 2.6) to their continuous counterparts. In particular, we obtain an approximation error estimate (2.3) for period matrices. We also prove a discrete analogue of the Riemann–Roch theorem (Theorem 6.6).

The proofs use energy estimates inspired by electrical networks, which is a highly nontrivial extension of [30] to the case when the curvature is present. Energy estimates allow to prove convergence of discrete period matrices directly, without proving convergence of Abelian integrals first (as in previous approaches). An elementary introduction to our method can be found in [25, 31, 32].

The main results of the paper are stated in Section 2 and proved in Sections 3–5. The discrete Riemann–Roch theorem is stated and proved in Section 6. A generalization of our setup, results of numerical experiments, and some open problems are given in Section 7.

2. Main ideas

2.1. Discrete harmonic and discrete analytic functions. Let S be a *polyhedral surface*, i.e., an oriented 2-dimensional manifold without boundary equipped with a piecewise flat metric having isolated conical singularities. An example of a polyhedral surface is the surface of a polyhedron in 3-dimensional space. Let T be a geodesic triangulation of the polyhedral surface S such that all faces are flat triangles; in particular, all the singular points of the metric are vertices of T. Denote by T^0, T^1, $\vec{T}^1$, T^2 the sets of vertices, edges, oriented edges, faces, respectively. Introduce *cotan edge weights* by the formula

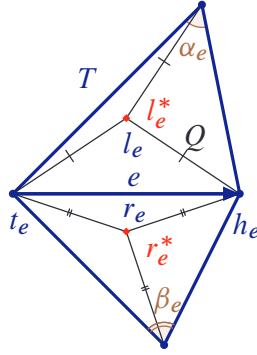$$c(e) := \frac{1}{2}\cot\alpha_e + \frac{1}{2}\cot\beta_e,$$

Figure 1. Notation associated with an edge $e \in T^1$; see Sections 2.1 and 7.1 for explanation.

where α_e and β_e are the angles opposite to an edge $e \in T^1$ in the two triangles sharing e; see Figure 1. The weight $c(e) < 0$, if $\alpha_e + \beta_e > \pi$.

A function $u\colon T^0 \to \mathbb{R}$ is *discrete harmonic*, if for each vertex $z \in T^0$ we have

$$(2.1) \qquad \sum_{xy \in T^1 : x = z} c(xy)(u(x) - u(y)) = 0.$$

Example 2.1 (see Figure 2 to the left). Let $\tilde{T}$ be an infinite square grid in the plane with the SW–NE diagonal drawn in each square. Then the sides of the squares have unit weights and the weights of diagonals vanish. Thus a function on the vertices of $\tilde{T}$ is discrete harmonic if and only if the value at a vertex equals the average of the values at the neighbors (diagonal neighbors are not taken into account).

For an oriented edge $e \in \vec{T}^1$ denote by $h_e \in T^0, t_e \in T^0, l_e \in T^2, r_e \in T^2$ the *head*, the *tail*, the *left shore*, the *right shore* of e, respectively; see Figure 1. Two functions $u\colon T^0 \to \mathbb{R}$ and $v\colon T^2 \to \mathbb{R}$ are *conjugate*, if for each oriented edge $e \in \vec{T}^1$ we have

$$(2.2) \qquad v(l_e) - v(r_e) = c(e)(u(h_e) - u(t_e)).$$

The pair $f = (u\colon T^0 \to \mathbb{R}, v\colon T^2 \to \mathbb{R})$ of two conjugate functions is called a *discrete analytic function*; for an example see Figure 2 to the left. We write $f = \text{const}$, if both u and v are constant functions, not necessarily equal to each other. We write $\operatorname{Re} f := u$ and $\operatorname{Im} f := v$. So the real and the imaginary part of a discrete analytic function are defined on different sets; such folklore approach to discretization is most convenient for us, cf. the works by Dynnikov–Novikov [14] and Mercat [20]. A direct checking shows that on simply-connected surfaces S discrete harmonic functions are precisely the real parts of discrete analytic functions.

2.2. Discrete Abelian integrals of the first kind. We are going to consider multi-valued functions on the vertices and the faces of the triangulation T. Informally, a multi-valued function is the one whose value changes after performing a nontrivial loop on the surface S. For formal definition we need the following notions.

In what follows assume that the surface S is closed and has genus $g \geq 1$. Denote by $p\colon \tilde{S} \to S$ the universal covering of S and by $p\colon \tilde{T} \to T$ the induced universal covering of T. Fix a base point $z_0 \in \tilde{S}$ and closed paths $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\colon [0, 1] \to S$ forming
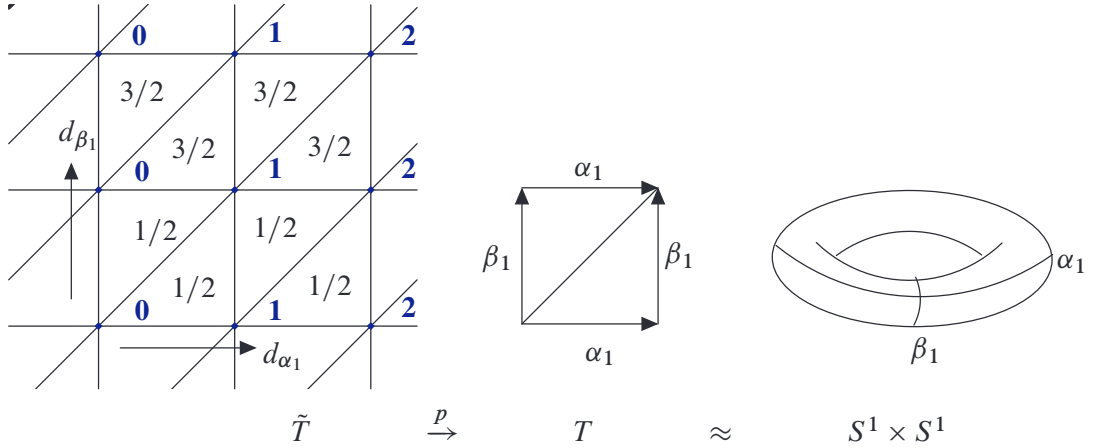
Figure 2. The plane (left) is the universal covering of a torus (right). The triangulation $\tilde{T}$ covers the "triangulation" T of the torus. The deck transformations of the covering are compositions of translations along the vectors d_{α_1} and d_{β_1}. Numbers represent the values of a discrete analytic function, namely, the discrete Abelian integral ϕ_T^1 of the first kind. Bold numbers near the vertices are the values of the real part $\operatorname{Re} \phi_T^1$. It is a discrete harmonic function. Thin numbers inside the faces are the values of the imaginary part $\operatorname{Im} \phi_T^1$. See Examples 2.1 and 2.2 for the details.

a standard basis of the fundamental group $\pi_1(S, pz_0)$ so that $\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \dots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}$ is null-homotopic. Each closed path $\alpha \colon [0,1] \to S$ with $\alpha(0) = \alpha(1) = pz_0$ determines the *deck transformation* $d_\alpha \colon \tilde{S} \to \tilde{S}$, i.e., the homeomorphism such that $p \circ d_\alpha = p$ and $d_\alpha(z_0) = \tilde{\alpha}(1)$, where $\tilde{\alpha} \colon \mathbb{R} \to \tilde{S}$ is the lift of $\alpha \colon [0,1] \to S$ such that $\tilde{\alpha}(0) = z_0$. The induced deck transformation of $\tilde{T}$ is also denoted by $d_\alpha \colon \tilde{T} \to \tilde{T}$.

A *multi-valued function with periods* $A_1, \dots, A_g, B_1, \dots, B_g \in \mathbb{C}$ is a pair of functions $f = (\operatorname{Re} f \colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} f \colon \tilde{T}^2 \to \mathbb{R})$ such that for each $k = 1, \dots, g$ and each $z \in \tilde{T}^0$, $w \in \tilde{T}^2$ we have

$$[\operatorname{Re} f](d_{\alpha_k} z) - [\operatorname{Re} f](z) = \operatorname{Re} A_k, \qquad [\operatorname{Re} f](d_{\beta_k} z) - [\operatorname{Re} f](z) = \operatorname{Re} B_k,$$
$$[\operatorname{Im} f](d_{\alpha_k} w) - [\operatorname{Im} f](w) = \operatorname{Im} A_k, \qquad [\operatorname{Im} f](d_{\beta_k} w) - [\operatorname{Im} f](w) = \operatorname{Im} B_k.$$

The numbers $A_1, \dots, A_g$ and $B_1, \dots, B_g$ are called the *A-periods* and the *B-periods* of the multi-valued function f, respectively. A multi-valued discrete analytic function is called a *discrete Abelian integral of the first kind* or *discrete holomorphic integral*.

Example 2.2 (see Figure 2). Let S be the surface obtained by gluing opposite sides of a unit square. Let T be the "triangulation" obtained by drawing the SE–NW diagonal of the square. Then $\tilde{S}$ is the complex plane $\mathbb{C}$ and $\tilde{T}$ is the triangulation from Example 2.1. The deck transformations $d_{\alpha_1}, d_{\beta_1} \colon \tilde{S} \to \tilde{S}$ are the translations along the sides of the squares. For a face $w \in \tilde{T}^2$ denote by w^* the circumcenter of w. Then a discrete Abelian integral of the first kind with the period $A_1 \in \mathbb{C}$ is given by the formula

$$[\operatorname{Re} f](z) := \operatorname{Re}(A_1 z) \qquad \text{for each vertex } z \in \tilde{T}^0,$$
$$[\operatorname{Im} f](w) := \operatorname{Im}(A_1 w^*) \quad \text{for each face } w \in \tilde{T}^2.$$

Actually this formula produces discrete Abelian integrals of the first kind on any triangulated flat torus $S = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\eta)$, where $\eta \in \mathbb{C} - \mathbb{R}$.

2.3. Discrete period matrices.   To proceed we need the following theorem essentially due to Mercat. It is proved in Section 3.3.

Theorem 2.3 (First existence and uniqueness theorem). *For arbitrary numbers $A_1, \ldots, A_g \in \mathbb{C}$ there exists a discrete Abelian integral of the first kind with A-periods $A_1, \ldots, A_g$. It is unique up to constant.*

For each $l = 1, \ldots, g$ denote by $\phi_T^l = (\operatorname{Re} \phi_T^l \colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} \phi_T^l \colon \tilde{T}^2 \to \mathbb{R})$ the unique (up to constant) discrete Abelian integral of the first kind with the A-periods given by the formula $A_k = \delta_{kl}$, where $k = 1, \ldots, g$. The $g \times g$ matrix Π_T whose l-th column is formed by the B-periods of ϕ_T^l, where $l = 1, \ldots, g$, is called the *period matrix* of the triangulation T.

Example 2.4.   Let T be an arbitrary triangulation of a flat torus $S = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\eta)$ with $\eta \in \mathbb{C} - \mathbb{R}$. Then $\Pi_T = \eta = \Pi_S$ by the explicit construction of discrete Abelian integrals in Example 2.2.

2.4. Convergence of discrete period matrices.   The polyhedral surface S has a natural complex structure (see Section 4.2). Indeed, identify each face $w \in \tilde{T}^2$ with a triangle in the complex plane $\mathbb{C}$ by an orientation-preserving isometry. A function $f \colon \tilde{S} \to \mathbb{C}$ is *analytic*, if it is continuous and its restriction to the interior of each face is analytic. Given the notion of an analytic function, a *basis of Abelian integrals* $\phi_S^l \colon \tilde{S} \to \mathbb{C}$ of the first kind and the *period matrix* Π_S of the surface S are defined analogously to the discrete case above. The *aperture* of a vertex $z \in T^0$ is the sum of all the face angles meeting at the vertex. Denote by γ_z the value 2π divided by the aperture. Denote $\gamma_S := \min\{1, \min_{z \in T^0} \gamma_z\}$. Clearly, the value γ_S depends only on the metric of S. For a $g \times g$ matrix Π denote

$$\|\Pi\| := \sqrt{\sum_{1 \leq k, l \leq g} |\Pi_{kl}|^2}.$$

Theorem 2.5 (Convergence theorem for period matrices).   *For each $\delta > 0$ there are two constants $\operatorname{Const}_{\delta, S}, \operatorname{const}_{\delta, S} > 0$ depending only on δ and the metric of the surface S such that for any triangulation T of S with the maximal edge length $h < \operatorname{const}_{\delta, S}$ and with the minimal face angle $> \delta$ we have*

$$(2.3) \qquad \|\Pi_T - \Pi_S\| \leq \operatorname{Const}_{\delta, S} \cdot \begin{cases} h & \text{if } \gamma_S > 1/2, \\ h|\log h| & \text{if } \gamma_S = 1/2, \\ h^{2\gamma_S} & \text{if } \gamma_S < 1/2. \end{cases}$$

In particular, for a sequence of triangulations of the surface S with the maximal edge length tending to zero and with face angles bounded from zero the discrete period matrices converge to the period matrix Π_S. The approximation order in (2.3) agrees with numerical experiments; see Section 7.3. Theorem 2.5 is proved in Section 4.4.

2.5. Convergence of discrete Abelian integrals of the first kind.   To state the next result we need the following notions. We say that a discrete Abelian integral

$$\phi_T^l = \big(\operatorname{Re} \phi_T^l \colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} \phi_T^l \colon \tilde{T}^2 \to \mathbb{R}\big)$$

of the first kind is *normalized* at a vertex $z \in \tilde{T}^0$ and a face $w \in \tilde{T}^2$, if

$$[\operatorname{Re} \phi_T^l](z) = [\operatorname{Im} \phi_T^l](w) = 0.$$

Similarly, we say that an Abelian integral $\phi_S^l \colon \tilde{S} \to \mathbb{C}$ is *normalized* at a point $z \in \tilde{S}$, if $\phi_S^l(z) = 0$.

A triangulation T is *Delaunay*, if for each edge $e \in T^1$ we have $\alpha_e + \beta_e \leq \pi$. Any polyhedral surface has an (essentially unique) Delaunay triangulation [2, 27]. Let $\{T_n\}$ be a sequence of triangulations of the surface S such that all the faces are flat triangles (the piecewise-flat metric on the surface S is fixed and does not depend on n). The sequence of triangulations $\{T_n\}$ is *nondegenerate uniform*, if there is a constant Const (not depending on n) such that for each member of the sequence:

 (A) the angles of each face are greater than $1/\text{Const}$;

 (D) for each edge the sum of opposite angles in the two triangles containing the edge is less than $\pi - 1/\text{Const}$ (in particular, the triangulation is Delaunay);

 (U) the number of vertices in an arbitrary intrinsic disk of radius equal to the maximal edge length is less than Const.

A sequence of pairs of functions $f_n = (\operatorname{Re} f_n \colon \tilde{T}_n^0 \to \mathbb{R}, \operatorname{Im} f_n \colon \tilde{T}_n^2 \to \mathbb{R})$ *converges* to a function $f \colon \tilde{S} \to \mathbb{C}$ *uniformly on each compact subset*, if for each compact set $K \subset \tilde{S}$ as $n \to \infty$ we have

$$\max_{z \in K \cap \tilde{T}_n^0} \left| [\operatorname{Re} f_n](z) - \operatorname{Re} f(z) \right| \to 0$$

and

$$\max_{xyz \in \tilde{T}_n^2 : K \cap xyz \neq \emptyset} \left| [\operatorname{Im} f_n](xyz) - \operatorname{Im} f(z) \right| \to 0.$$

Theorem 2.6 (Convergence theorem for Abelian integrals). *Let $\{T_n\}$ be a nondegenerate uniform sequence of Delaunay triangulations of S with maximal edge length approaching zero as $n \to \infty$. Let $z_n \in \tilde{T}_n^0$ be a sequence of vertices converging to a point $z_0 \in \tilde{S}$. Let $w_n \in \tilde{T}_n^2$ be a sequence of faces with the vertices converging to z_0. Then for each $1 \leq l \leq g$ the discrete Abelian integrals $\phi_{T_n}^l = (\operatorname{Re} \phi_{T_n}^l \colon \tilde{T}_n^0 \to \mathbb{R}, \operatorname{Im} \phi_{T_n}^l \colon \tilde{T}_n^2 \to \mathbb{R})$ of the first kind normalized at z_n and w_n converge uniformly on each compact set to the Abelian integral $\phi_S^l \colon \tilde{S} \to \mathbb{C}$ of the first kind normalized at z_0.*

This theorem is proved in Section 5.4.

2.6. Convergence of energy.   Our results are proved by energy estimates inspired by alternating-current networks. Let us state our main lemma establishing convergence of energy.

Multi-valued functions $\tilde{T}^0 \to \mathbb{R}$, $\tilde{T}^2 \to \mathbb{R}$, and $\tilde{S} \to \mathbb{R}$ are defined analogously to the multi-valued functions from Section 2.2. For each multi-valued function $u \colon \tilde{T}^0 \to \mathbb{R}$ and each edge $xy \in T^1$ the difference square $(u(x) - u(y))^2$ is well-defined. The *energy* of the multi-valued function is

$$E_T(u) := \sum_{xy \in T^1} c(xy)(u(x) - u(y))^2.$$

For each multi-valued function $u \colon \tilde{S} \to \mathbb{R}$ and each point inside a face $w \in T^2$ the gradient norm $|\nabla u|^2$ is well-defined. The *energy* of the multi-valued function is

$$E_S(u) := \sum_{w \in T^2} \int_w |\nabla u|^2 \, dx dy.$$

Lemma 2.7 (Energy convergence lemma). *For each $\delta > 0$ and each smooth multi-valued function $u \colon \tilde{S} \to \mathbb{R}$ there are two constants $\mathrm{Const}_{u,\delta,S}, \mathrm{const}_{u,\delta,S} > 0$ such that for any triangulation T of S with the maximal edge length $h < \mathrm{const}_{u,\delta,S}$ and with the minimal face angle $> \delta$ we have*

$$\left| E_T(u|_{\tilde{T}^0}) - E_S(u) \right| \leq \mathrm{Const}_{u,\delta,S} \cdot \begin{cases} h & \text{if } \gamma_S > 1/2, \\ h|\log h| & \text{if } \gamma_S = 1/2, \\ h^{2\gamma_S} & \text{if } \gamma_S < 1/2. \end{cases}$$

Such approximation order is informally explained as follows. There are two competing main sources of energy approximation error. In the flat regions apart from the conical singularities there is a nice approximation with the error of order h. In the h-neighborhood of conical singularities there is essentially no approximation, and the error has order of the energy itself, i.e., $h^{2\gamma_S}$. Depending on whether $\gamma_S > 1/2$ or $\gamma_S < 1/2$, one of the error sources dominates.

The assumption on the minimal face angle in the lemma cannot be dropped; see Example 4.14. The lemma is proved in Section 4.3.

3. Discrete period matrices

In this section we prove basic properties of discrete period matrices.

3.1. Riemann bilinear identity. We start with the following result. In what follows fix an arbitrary orientation of each edge of T^1.

Lemma 3.1 (Riemann bilinear identity). *Let $u \colon \tilde{T}^0 \to \mathbb{R}$ and $u' \colon \tilde{T}^2 \to \mathbb{R}$ be two multi-valued functions with periods $A_1, \ldots, A_g, B_1, \ldots, B_g$ and $A'_1, \ldots, A'_g, B'_1, \ldots, B'_g$, respectively. Then*

$$\sum_{e \in T^1} (u'(l_e) - u'(r_e))(u(h_e) - u(t_e)) = \sum_{k=1}^{g} (A_k B'_k - B_k A'_k).$$

The left-hand side of this identity makes sense for an arbitrary cell decomposition T, not necessarily a triangulation and not necessarily equipped with a metric. Such generalization appears in the following example.

Example 3.2. A *canonical cell decomposition* T of a surface S of genus g is obtained from a $4g$-gon with the sides $e_1 \ldots e_{4g}$ oriented counterclockwise by gluing together the pairs of sides e_{4k-3}, e_{4k-1} and e_{4k-2}, e_{4k} for each $k = 1, \ldots, g$ with orientation reversal. The $4g$-gon itself is considered as a face of the universal cover $\tilde{T}$; see Figure 3. The paths
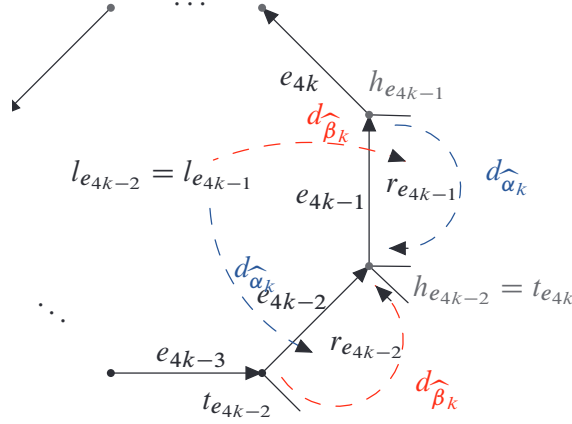
Figure 3.   Action of deck transformations on the universal covering of the canonical cell decomposition; see Example 3.2 for the details.

$\alpha_k := pe_{4k-3}$ and $\beta_k := pe_{4k-2}$ for $k = 1, \ldots, g$ (called the paths *going along the sides of the 4g-gon*) form a standard basis of the fundamental group of S. It is easy to see that there are deck transformations $d_{\hat{\alpha}_k}, d_{\hat{\beta}_k} : \tilde{T} \to \tilde{T}$ such that

$$d_{\hat{\alpha}_k} h_{e_{4k-1}} = t_{e_{4k-1}}, \qquad d_{\hat{\alpha}_k} l_{e_{4k-2}} = r_{e_{4k-2}},$$
$$d_{\hat{\beta}_k} t_{e_{4k-2}} = h_{e_{4k-2}}, \qquad d_{\hat{\beta}_k} l_{e_{4k-1}} = r_{e_{4k-1}},$$

where $\hat{\alpha}_k$ and $\hat{\beta}_k$ are appropriate elements of $\pi_1(S, pz_0)$ conjugate to α_k and β_k, respectively. Indeed, if the base point $z_0 = h_{e_{4k-1}}$ then we can just take $\hat{\alpha}_k = \alpha_k$, and change of the base point leads to conjugation in $\pi_1(S, pz_0)$. This implies that

$$u(t_{e_{4k-1}}) - u(h_{e_{4k-1}}) = A_k, \qquad u'(r_{e_{4k-2}}) - u'(l_{e_{4k-2}}) = A'_k,$$
$$u(h_{e_{4k-2}}) - u(t_{e_{4k-2}}) = B_k, \qquad u'(r_{e_{4k-1}}) - u'(l_{e_{4k-1}}) = B'_k.$$

Thus, for the canonical cell decomposition, the Riemann bilinear identity in Lemma 3.1 is checked directly:

$$\sum_{e \in T^1} \big(u'(l_e) - u'(r_e)\big)\big(u(h_e) - u(t_e)\big)$$

$$= \sum_{k=1}^{g} \big(u'(l_{e_{4k-1}}) - u'(r_{e_{4k-1}})\big)\big(u(h_{e_{4k-1}}) - u(t_{e_{4k-1}})\big)$$

$$+ \sum_{k=1}^{g} \big(u'(l_{e_{4k-2}}) - u'(r_{e_{4k-2}})\big)\big(u(h_{e_{4k-2}}) - u(t_{e_{4k-2}})\big)$$

$$= \sum_{k=1}^{g} (A_k B'_k - B_k A'_k).$$

Now we proceed to general triangulations. Note that there are triangulations which are not subdivisions of the canonical cell decomposition, e.g., a triangulation of a surface of genus $g > 3$ with all vertex degrees $< 2g$. To prove the identity for general triangulations we need the following auxiliary assertion.

Lemma 3.3 (Stokes' formula). *Let $\hat{T} \subset \tilde{T}$ be a triangulated polygon. Then*

$$\sum_{e \in \hat{T}^1} \big(u'(l_e) - u'(r_e)\big)\big(u(h_e) - u(t_e)\big) = \sum_{e \in (\partial \hat{T})^1} u'(r_e)\big(u(t_e) - u(h_e)\big),$$

where each edge $e \in (\partial \hat{T})^1$ is oriented so that $l_e \subset \hat{T}$.

Proof. In the case when the polygon $\hat{T}$ is a single face of the triangulation $\tilde{T}$ Stokes' formula is checked directly. Sum such Stokes' formulas over all the faces of the triangulated polygon $\hat{T}$. Each interior edge e appears two times with each of the two possible orientations. Hence it contributes $2(v(l_e) - v(r_e))(u(h_e) - u(t_e))$ to the left-hand side of the sum and $(v(r_e) - v(l_e))(u(t_e) - u(h_e))$ to the right-hand side. Canceling the repeating terms, we get the required formula.   □

Proof of Lemma 3.1. We are going to transform the triangulation T without changing the left-hand side of the required identity and this way reduce the identity to Example 3.2. An *edge subdivision* is replacement of two adjacent faces xyz and xyw of the triangulation by four new faces xzv, zvy, yvw, wvx sharing a common new vertex v. It is well known that any two triangulations of one surface can be transformed to each other by edge subdivisions and their inversions.

Let us check that the left-hand side of the required identity is invariant under an edge subdivision and arbitrary extension of the functions u and u' to the obtained new vertices and faces. Indeed, the contributions of all the edges not contained in the quadrilateral $xzyw$ to the left-hand side are not affected by the edge subdivision. Stokes' formula (Lemma 3.3) immediately implies that the total contributions of the edges contained in the quadrilateral $xzyw$ are the same before and after the edge subdivision. So the left-hand side is invariant.

Thus without loss of generality we may assume that the triangulation T is a subdivision of a canonical cell decomposition with the given paths $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ going along the sides of the $4g$-gon. It follows by Stokes' formula (Lemma 3.3) that the left-hand side of the required identity is invariant under adjustment of the value of the function u' at one particular face and its images under all possible deck transformations. Thus without loss of generality we may assume that u' assigns the same values at all the faces inside the $4g$-gon of Example 3.2. Then neither of the edges lying inside the $4g$-gon contributes to the left-hand side of the required identity. This means that the left-hand side equals to that of the canonical cell decomposition. By Example 3.2 the identity follows.   □

3.2. Energy principles. We are going to use the following basic properties of energy; cf. [30, Principles 2.1, 2.2, 5.6].

Lemma 3.4 (Energy conservation principle). *Let $f = (\operatorname{Re} f \colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} f \colon \tilde{T}^2 \to \mathbb{R})$ be a discrete Abelian integral of the first kind with periods $A_1, \ldots, A_g, B_1, \ldots, B_g$. Then*

$$E_T(\operatorname{Re} f) = -\operatorname{Im} \sum_{k=1}^{g} A_k \bar{B}_k.$$

Proof. By equation (2.2) and the Riemann bilinear identity (Lemma 3.1) we have

$$
\begin{aligned}
E_T(\operatorname{Re} f) &= \sum_{e \in T^1} c(e)\big(\operatorname{Re} f(h_e) - \operatorname{Re} f(t_e)\big)^2 \\
&= \sum_{e \in T^1} \big(\operatorname{Im} f(l_e) - \operatorname{Im} f(r_e)\big)\big(\operatorname{Re} f(h_e) - \operatorname{Re} f(t_e)\big) \\
&= \sum_{k=1}^{g} (\operatorname{Re} A_k \operatorname{Im} B_k - \operatorname{Re} B_k \operatorname{Im} A_k) \\
&= -\operatorname{Im} \sum_{k=1}^{g} A_k \bar{B}_k, \qquad \qquad \qquad \square
\end{aligned}
$$

Lemma 3.5 (Energy positivity principle).   *For each nonconstant multi-valued function* $u: \tilde{T}^0 \to \mathbb{R}$ *we have* $E_T(u) > 0.$

Proof.   For $u \neq \text{const}$ using the formula

$$
\cot zxy = \frac{1 - \cot xyz \cot yzx}{\cot xyz + \cot yzx},
$$

we get

$$
\begin{aligned}
E_T(u) &= \sum_{xy \in T^1} c(xy)(u(x) - u(y))^2 \\
&= \frac{1}{2} \sum_{xyz \in T^2} \big(\cot xyz\,(u(x) - u(z))^2 + \cot yzx\,(u(y) - u(x))^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \cot zxy\,(u(z) - u(y))^2\big) \\
&= \frac{1}{2} \sum_{xyz \in T^2} \frac{\big(\cot xyz\,(u(x) - u(z)) + \cot yzx\,(u(x) - u(y))\big)^2 + (u(z) - u(y))^2}{\cot xyz + \cot yzx} \\
&> 0.
\end{aligned}
$$

Another proof is by application of the interpolation lemma (Lemma 4.1 below). $\square$

Lemma 3.6 (Variational principle).   *A multi-valued discrete harmonic function has minimal energy among all multi-valued functions with the same periods.*

Proof.   Consider the finite-dimensional affine space of multi-valued functions $\tilde{T}^0 \to \mathbb{R}$ with the same periods as a given multi-valued discrete harmonic function $u: \tilde{T}^0 \to \mathbb{R}$. By the energy positivity principle (Lemma 3.5) it follows that the energy $E_T(\cdot)$ is a (nonstrictly) convex functional on this space. Since $u: \tilde{T}^0 \to \mathbb{R}$ is discrete harmonic, it follows by differentiation that u is a critical point of the functional $E_T(\cdot)$. Thus u is a global minimum. $\square$

If $c(xy) \neq 0$ for each edge $xy \in T^1$ then the *energy* of a multi-valued function $v: \tilde{T}^2 \to \mathbb{R}$ is defined by the formula $E_T(v) := \sum_{xy \in T^1} (v(x) - v(y))^2 / c(xy)$. Identity (2.2) immediately implies the following assertion.

Lemma 3.7 (Conjugate functions principle).   *Let* $f = (\operatorname{Re} f\colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} f\colon \tilde{T}^2 \to \mathbb{R})$ *be a discrete Abelian integral of the first kind. Then* $E_T(\operatorname{Re} f) = E_T(\operatorname{Im} f)$.

3.3. Existence and uniqueness theorem.

Corollary 3.8.   *Let* $f = (\operatorname{Re} f\colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} f\colon \tilde{T}^2 \to \mathbb{R})$ *be a discrete Abelian integral of the first kind with the periods* $A_1, \ldots, A_g, B_1, \ldots, B_g$. *Then the following three conditions are equivalent:*

(1) $f = \text{const}$;

(2) $A_1 = \cdots = A_g = 0$;

(3) $\operatorname{Re} A_1 = \cdots = \operatorname{Re} A_g = \operatorname{Re} B_1 = \cdots = \operatorname{Re} B_g = 0$.

Proof.   The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious. Let us prove that (2) $\Rightarrow$ (1); the implication (3) $\Rightarrow$ (1) is proved analogously. By the energy conservation principle (Lemma 3.4) we have

$$E_T(\operatorname{Re} f) = \sum_{k=1}^{g}(\operatorname{Re} A_k \operatorname{Im} B_k - \operatorname{Re} B_k \operatorname{Im} A_k) = 0.$$

Hence by the energy positivity principle (Lemma 3.5) we get $\operatorname{Re} f = \text{const}$. By (2.2) we get also $\operatorname{Im} f = \text{const}$. Hence $f = \text{const}$.   □

Proof of Theorem 2.3.   *Uniqueness*. Let f, f' be two discrete Abelian integrals of the first kind as required. Then $f - f'$ is a discrete Abelian integral of the first kind with vanishing A-periods. By the implication (2) $\Rightarrow$ (1) of Corollary 3.8 it follows that $f - f' = \text{const}$.

Existence. Denote by $f = (\operatorname{Re} f\colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} f\colon \tilde{T}^2 \to \mathbb{R})$ an unknown multi-valued function. The real and imaginary parts of the A-periods $A_1, \ldots, A_g$ are considered as parameters.

Consider the following system of linear equations. Identify T^0, T^1, T^2 with some subsets of $\tilde{T}^0, \tilde{T}^1, \tilde{T}^2$ which project bijectively to T^0, T^1, T^2, respectively. Let the variables be the values of the function $f = (\operatorname{Re} f\colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} f\colon \tilde{T}^2 \to \mathbb{R})$ at the elements of the sets $T^0 \subset \tilde{T}^0, T^2 \subset \tilde{T}^2$, and also unknown real and imaginary parts of the B-periods $B_1, \ldots, B_g$ of this function. Express the values of the function $\operatorname{Re} f\colon \tilde{T}^0 \to \mathbb{R}$ at the vertices not belonging to $T^0 \subset \tilde{T}^0$ through the values at T^0 and the periods $A_1, \ldots, A_g, B_1, \ldots, B_g$. Similarly, express the values of the function $\operatorname{Im} f\colon \tilde{T}^2 \to \mathbb{R}$ at the faces not belonging to $T^2 \subset \tilde{T}^2$ through the values at T^2 and the periods $A_1, \ldots, A_g, B_1, \ldots, B_g$. For each edge $e \in T^1$ choose an orientation and write one linear equation (2.2). Also pick up a vertex $z \in \tilde{T}^0$, a face $w \in \tilde{T}^2$, and write two equations $\operatorname{Re} f(z) = 0$, $\operatorname{Im} f(w) = 0$.

We have written a system of $|T^1|+2$ linear equations in $|T^0|+|T^2|+2g$ variables, where $|F|$ denotes the number of elements in a finite set F. By the Euler formula for surface triangulations the number of equations in the system equals the number of variables. The free terms of these equations are linear combinations of the parameters $\operatorname{Re} A_1, \operatorname{Im} A_1, \ldots, \operatorname{Re} A_g, \operatorname{Im} A_g$. By the uniqueness part of the theorem it follows that the homogeneous system obtained for $A_1 = \cdots = A_g = 0$ has only the trivial solution. Then by the finite-dimensional Fredholm alternative it follows that for arbitrary $A_1, \ldots, A_g \in \mathbb{C}$ the inhomogeneous system has at least one solution. Thus there exists a discrete Abelian integral of the first kind as required.   □

The following theorem is proved analogously using $(3) \Rightarrow (1)$ of Corollary 3.8.

Theorem 3.9 (Second existence and uniqueness theorem). *For each real vector* $P = (A_1, \ldots, A_g, B_1, \ldots, B_g) \in \mathbb{R}^{2g}$ *there is a unique (up to constant) discrete Abelian integral* $\phi_{T,P} = (\mathrm{Re}\,\phi_{T,P} \colon \tilde{T}^0 \to \mathbb{R}, \mathrm{Im}\,\phi_{T,P} \colon \tilde{T}^2 \to \mathbb{R})$ *of the first kind whose periods have real parts* $A_1, \ldots, A_g, B_1, \ldots, B_g$.

3.4. Discrete period matrices. For each $l = 1, \ldots, g$ denote by

$$\phi_{T*}^l = \left(\mathrm{Re}\,\phi_{T*}^l \colon \tilde{T}^0 \to \mathbb{R}, \mathrm{Im}\,\phi_{T*}^l \colon \tilde{T}^2 \to \mathbb{R} \right)$$

the unique (up to constant) Abelian integral of the first kind with A-periods given by the formula $A_k = i\delta_{kl}$, where $k = 1, \ldots, g$. The $g \times g$ matrix Π_{T*} whose l-th column is the B-periods of ϕ_{T*}^l divided by i is called the *dual period matrix* of the triangulation T.

Example 3.10. The map taking the vector of A-periods of a discrete Abelian integral of the first kind to the vector of its B-periods is not necessarily $\mathbb{C}$-linear, and thus $\Pi_{T*} \neq \Pi_T$ in general. For instance, if T is obtained from the side surface of a regular square pyramid by identifying the opposite sides of the base with orientation reversal then $\Pi_T = 2i/\sqrt{3}$ whereas $\Pi_{T*} = i\sqrt{3}/2$.

Lemma 3.11 (Period matrix lemma). *The matrices* $\mathrm{Im}\,\Pi_T$ *and* $\mathrm{Im}\,\Pi_{T*}$ *are symmetric positively definite, and* $\mathrm{Re}\,\Pi_{T*} = (\mathrm{Re}\,\Pi_T)^{\mathrm{T}}$.

Remark 3.12. The *discrete period matrix* by Mercat [21] is $\Pi_Q := (\Pi_T + \Pi_{T*})/2$. The matrix Π_Q is symmetric with positively definite imaginary part. Our proof of convergence of period matrices (Theorem 2.5) implies an analogue of error estimate (2.3) for this matrix as well:

$$\|\Pi_Q - \Pi_S\| \leq \mathrm{Const}_{\delta, S} \cdot \begin{cases} h & \text{if } \gamma_S > 1/2, \\ h|\log h| & \text{if } \gamma_S = 1/2, \\ h^{2\gamma_S} & \text{if } \gamma_S < 1/2. \end{cases}$$

For the proof of the lemma we need the following auxiliary assertion.

Lemma 3.13. *Let*

$$f = \left(\mathrm{Re}\,f \colon \tilde{T}^0 \to \mathbb{R}, \mathrm{Im}\,f \colon \tilde{T}^2 \to \mathbb{R} \right), \quad f' = \left(\mathrm{Re}\,f' \colon \tilde{T}^0 \to \mathbb{R}, \mathrm{Im}\,f' \colon \tilde{T}^2 \to \mathbb{R} \right)$$

be two discrete Abelian integrals of the first kind with the periods $A_1, \ldots, A_g, B_1, \ldots, B_g$ *and* $A_1', \ldots, A_g', B_1', \ldots, B_g'$, *respectively. Then*

$$\mathrm{Im} \sum_{k=1}^{g} (A_k B_k' - B_k A_k') = 0.$$

Proof. Using the Riemann bilinear identity (Lemma 3.1) and then formula (2.2) to cancel repeating terms, we get

$$\text{Im} \sum_{k=1}^{g} (A_k B_k' - B_k A_k')$$

$$= \sum_{k=1}^{g} (\text{Re}\, A_k \, \text{Im}\, B_k' - \text{Re}\, B_k \, \text{Im}\, A_k' - \text{Re}\, A_k' \, \text{Im}\, B_k + \text{Re}\, B_k' \, \text{Im}\, A_k)$$

$$= \sum_{e \in T^1} \big(\text{Im}\, f'(l_e) - \text{Im}\, f'(r_e)\big)\big(\text{Re}\, f(h_e) - \text{Re}\, f(t_e)\big)$$

$$- \sum_{e \in T^1} \big(\text{Im}\, f(l_e) - \text{Im}\, f(r_e)\big)\big(\text{Re}\, f'(h_e) - \text{Re}\, f'(t_e)\big) = 0. \qquad \square$$

Proof of Lemma 3.11. Applying Lemma 3.13 for $f := \phi_T^i$ and $f' := \phi_T^j$, we get $(\text{Im}\, \Pi_T)_{ij} - (\text{Im}\, \Pi_T)_{ji} = 0$, that is, $\text{Im}\, \Pi_T$ is symmetric. Applying Lemma 3.13 for $f := \phi_T^i$, $f' := \phi_{T*}^j$, we get $(\text{Re}\, \Pi_{T*})_{ij} - (\text{Re}\, \Pi_T)_{ji} = 0$, that is, $\text{Re}\, \Pi_{T*} = (\text{Re}\, \Pi_T)^{\text{T}}$.

Finally, let f be a discrete Abelian integral of the first kind whose A-periods $A_1, \ldots, A_g$ are arbitrary *real* numbers not vanishing simultaneously. Then the B-periods of this integral are equal to $B_l = \sum_{k=1}^{g} (\Pi_T)_{kl} A_k$. Hence by the energy conservation and energy positivity principles (Lemmas 3.4 and 3.5) it follows that

$$\sum_{1 \le k,l \le g} (\text{Im}\, \Pi_T)_{kl} A_k A_l = -\sum_{k=1}^{g} \text{Im}(A_k \bar{B}_k) = E_T(\text{Re}\, f) > 0.$$

Thus $\text{Im}\, \Pi_T$ is positively definite. Analogously, $\text{Im}\, \Pi_{T*}$ is symmetric and positively definite.

$\square$

Denote $u_{T,P} := \text{Re}\, \phi_{T,P}$, where $\phi_{T,P}$ is the discrete Abelian integral of the first kind defined in the second existence and uniqueness theorem (Theorem 3.9) for each vector $P \in \mathbb{R}^{2g}$.

Lemma 3.14 (Energy matrix lemma). *The energy $E_T(u_{T,P})$ is a quadratic form in the vector $P \in \mathbb{R}^{2g}$ with the block matrix*

$$E_T := \begin{pmatrix} \text{Re}\, \Pi_{T*}(\text{Im}\, \Pi_{T*})^{-1} \text{Re}\, \Pi_T + \text{Im}\, \Pi_T & -(\text{Im}\, \Pi_{T*})^{-1} \text{Re}\, \Pi_T \\ -\text{Re}\, \Pi_{T*}(\text{Im}\, \Pi_{T*})^{-1} & (\text{Im}\, \Pi_{T*})^{-1} \end{pmatrix}.$$

Proof. Denote by $A_1, \ldots, A_g, B_1, \ldots, B_g \in \mathbb{C}$ the periods of the discrete Abelian integral $\phi_{T,P}$ of the first kind. Denote $A := (A_1, \ldots, A_g)$ and $B := (B_1, \ldots, B_g)$. Then $P = (\text{Re}\, A, \text{Re}\, B) = (\text{Re}\, A_1, \ldots, \text{Re}\, A_g, \text{Re}\, B_1, \ldots, \text{Re}\, B_g)$. By the first existence and uniqueness theorem (Theorem 2.3) we get

$$\phi_{T,P} = \text{Re}\, A_1 \cdot \phi_T^1 + \cdots + \text{Re}\, A_g \cdot \phi_T^g + \text{Im}\, A_1 \cdot \phi_{T*}^1 + \cdots + \text{Im}\, A_g \cdot \phi_{T*}^g + \text{const}.$$

Thus

$$\text{Re}\, B = \text{Re}\, \Pi_T \, \text{Re}\, A - \text{Im}\, \Pi_{T*} \, \text{Im}\, A, \quad \text{Im}\, B = \text{Im}\, \Pi_T \, \text{Re}\, A + \text{Re}\, \Pi_{T*} \, \text{Im}\, A.$$

The matrix $\text{Im}\, \Pi_{T*}$ is invertible by Lemma 3.11. Solving in $\text{Im}\, A$ and $\text{Im}\, B$ we get

$$\text{Im}\, A = (\text{Im}\, \Pi_{T*})^{-1} \text{Re}\, \Pi_T \, \text{Re}\, A - (\text{Im}\, \Pi_{T*})^{-1} \text{Re}\, B;$$

$$\text{Im}\, B = \big(\text{Im}\, \Pi_T + \text{Re}\, \Pi_{T*}(\text{Im}\, \Pi_{T*})^{-1} \text{Re}\, \Pi_T\big) \text{Re}\, A - \text{Re}\, \Pi_{T*}(\text{Im}\, \Pi_{T*})^{-1} \text{Re}\, B.$$

Applying the energy conservation principle (Lemma 3.4) for $f := \phi_{T,P}$, we get the required expression for the matrix of $E_T(u_{T,P})$. □

Let us conclude the section by a continuous counterpart of the energy matrix lemma (Lemma 3.14), which is proved analogously. For $P = (A_1, \ldots, A_g, B_1, \ldots, B_g) \in \mathbb{R}^{2g}$ let $\phi_{S,P} \colon \tilde{S} \to \mathbb{C}$ be an Abelian integral of the first kind whose periods have *real parts* $A_1, \ldots, A_g, B_1, \ldots, B_g$. Denote $u_{S,P} := \operatorname{Re} \phi_{S,P}$.

Lemma 3.15. *The energy $E_S(u_{S,P})$ is a quadratic form in the vector $P \in \mathbb{R}^{2g}$ with the block matrix*

$$E_S := \begin{pmatrix} \operatorname{Re} \Pi_S (\operatorname{Im} \Pi_S)^{-1} \operatorname{Re} \Pi_S + \operatorname{Im} \Pi_S & -(\operatorname{Im} \Pi_S)^{-1} \operatorname{Re} \Pi_S \\ -\operatorname{Re} \Pi_S (\operatorname{Im} \Pi_S)^{-1} & (\operatorname{Im} \Pi_S)^{-1} \end{pmatrix}.$$

The last two lemmas reduce the proof of the convergence of discrete period matrices to convergence of the energy form. We are going to establish the latter in the next section.

4. Convergence of energy and discrete period matrices

In this section we prove convergence of discrete energy and discrete period matrices to their continuous counterparts. In what follows we use the following notation:

- Const is a positive constant (like 2π or 10^{100}) that does not depend on any parameters of a configuration under consideration (e.g., h, γ_S, the shape of T); Const may denote distinct constants at distinct places of the text (e.g., in the sides of a formula like $2 \cdot \text{Const} \leq \text{Const}$);

- $\text{Const}_{x,y,z}$ is a positive constant depending only on the parameters x, y, z;

- $\|D^k u(z)\| := \max_{0 \leq j \leq k} |\frac{\partial^k u}{\partial^j x \, \partial^{k-j} y}(z)|$.

4.1. Convergence of energy in a triangle. We start with several well-known lemmas; cf. [30, Lemmas 4.2, 4.5, 2.3]. Throughout this subsection Δ is a triangle in the plane, x, y, z are its vertices, and $u \colon \Delta \to \mathbb{R}$ is a smooth function which smoothly extends to a neighborhood of Δ. The *energy* and the *discrete energy* of the function $u \colon \Delta \to \mathbb{R}$ are $E_{S_\Delta}(u) := \int_\Delta |\nabla u|^2 \, dx \, dy$ and

$$E_{T_\Delta}(u) := \frac{1}{2} \cot xyz \cdot (u(z) - u(x))^2$$
$$+ \frac{1}{2} \cot yzx \cdot (u(x) - u(y))^2 + \frac{1}{2} \cot zxy \cdot (u(y) - u(z))^2,$$

respectively. The *interpolation* of the function $u \colon \Delta \to \mathbb{R}$ in the triangle Δ is the linear function $I_\Delta u \colon \Delta \to \mathbb{R}$ which coincides with u at the vertices of the triangle Δ. The following well-known lemma is the main motivation for using cotangent weights.

Lemma 4.1 (Interpolation lemma, see [13, §4]). $E_{T_\Delta}(u) = E_{S_\Delta}(I_\Delta u)$.

Denote by h and δ the maximal side length and the minimal angle of the triangle Δ, respectively.

Lemma 4.2 (Projection lemma).   *For any vector $\vec{v} \in \mathbb{R}^2$ we have*

$$|\vec{v}| \leq 2 \csc \delta \cdot \left(\frac{|\vec{v} \cdot \overrightarrow{xy}|}{|xy|} + \frac{|\vec{v} \cdot \overrightarrow{yz}|}{|yz|} + \frac{|\vec{v} \cdot \overrightarrow{zx}|}{|zx|} \right).$$

Proof.   Without loss of generality assume that $\angle(\overrightarrow{xy}, \overrightarrow{xz}) = \delta$. Since

$$\angle(\overrightarrow{xy}, \overrightarrow{xz}) = \pm\angle(\vec{v}, \overrightarrow{xy}) \pm \angle(\vec{v}, \overrightarrow{xz}),$$

it follows that at least one of the angles in the right-hand side, say, the first one, does not belong to the interval $(\frac{\pi}{2} - \frac{\delta}{2}, \frac{\pi}{2} + \frac{\delta}{2})$. Then

$$|\vec{v}| \leq \csc \frac{\delta}{2} \cdot \frac{|\vec{v} \cdot \overrightarrow{xy}|}{|xy|} \leq 2 \csc \delta \cdot \frac{|\vec{v} \cdot \overrightarrow{xy}|}{|xy|}. \qquad \square$$

Lemma 4.3 (Gradient approximation lemma).   *We have*

$$\max_{w \in \Delta} |\nabla u(w) - \nabla I_\Delta u(w)| \leq \mathrm{Const}_\delta \cdot h \cdot \max_{w \in \Delta} \|D^2 u(w)\|.$$

Proof.   Denote $\vec{v}(w) := \nabla u(w) - \nabla I_\Delta u(w)$. By the Rolle theorem there is a point $w \in xy$ such that $\vec{v}(w) \cdot \overrightarrow{xy}/|\overrightarrow{xy}| = 0$. Thus $|\vec{v} \cdot \overrightarrow{xy}|/|\overrightarrow{xy}| \leq \mathrm{Const} \cdot h \cdot \max_\Delta \|D^2 u\|$ in the triangle Δ because $\nabla I_\Delta u$ is a constant vector. The same inequality holds with xy replaced by yz or zx. Thus the lemma follows from the projection lemma (Lemma 4.2). $\square$

Lemma 4.4 (Energy approximation lemma).   *We have*

$$|E_{T_\Delta}(u) - E_{S_\Delta}(u)|$$
$$\leq \mathrm{Const}_\delta \cdot \left(\max_{w \in \Delta} \|D^1 u(w)\| + h \max_{w \in \Delta} \|D^2 u(w)\| \right) \cdot h \max_{w \in \Delta} \|D^2 u(w)\| \cdot \mathrm{Area}(\Delta).$$

Proof.   By the interpolation lemma (Lemma 4.1) we get

$$E_{T_\Delta}(u) - E_{S_\Delta}(u) = \int_\Delta \left(|\nabla u|^2 - |\nabla I_\Delta u|^2 \right) dx dy$$
$$= \int_\Delta \left(2\nabla u + \nabla I_\Delta u - \nabla u \right)(\nabla u - \nabla I_\Delta u) \, dx dy.$$

Using the gradient approximation lemma (Lemma 4.3) completes the proof. $\square$

4.2. Convergence of energy in a cone.   Now we are going to estimate the energy near a conical singularity O of S. Let S_O be a neighborhood of the point O not containing other conical singularities bounded by a piecewise-geodesic closed broken line. Metrically S_O is a part of a cone of aperture $2\pi/\gamma_O$ with the vertex O. Introduce "polar coordinates" (ρ, ϕ) on S_O with the origin at the vertex O.

Let us give the proposed construction of a natural complex structure on the polyhedral surface S (see, e.g., [3, Section 1.1.3]). Define a chart $g_O : S_O \to \mathbb{C}$ by the formula

$g_O(\rho,\phi) := \rho^{\gamma_O} \exp(i\gamma_O\phi)$. For a point $w \in S_O$, $w \neq O$, define a chart $g_w: N_w \to \mathbb{C}$ to be an orientation-preserving and distance-preserving map of a neighborhood $N_w \not\ni O$ of the point w. For instance, for w not belonging to the ray $\phi = 2\pi/\gamma_O$ this map can be given by the formula $g_w(\rho,\phi) := \rho \exp(i\phi)$. The constructed charts form a complex analytic atlas because all transition functions are compositions of maps of the form $w \mapsto aw + b$ and $w \mapsto w^{\gamma_z}$ (away from the origin). The definition of an analytic function given in Section 2.4 is compatible with the constructed complex structure by the singularity removal theorem. In fact any complex structure on a closed surface can be realized in this way by a piecewise flat metric [38].

For a function $u: S_O \to \mathbb{R}$ denote $u_O := u \circ g_O^{-1}$ and $u_w := u \circ g_w^{-1}$. Throughout this subsection $u: \operatorname{Int} S_O \to \mathbb{R}$ is a function such that $u_O: \operatorname{Int} g_O S_O \to \mathbb{R}$ is smooth and smoothly extends to a neighborhood of $g_O S_O$. Denote by $|xy|$ the length of a geodesic segment $xy \subset S_O$.

Let T_O be a geodesic triangulation of S_O with $O \in T_O^0$, the maximal edge length $\leq h$ and the minimal face angle $> \delta$. Let $S_{O,h}$ be the union of faces of T_O intersecting the h-neighborhood of the vertex O, and let $T_{O,h}$ the restriction of the triangulation T_O to $S_{O,h}$. We are going to estimate the difference between

$$E_{T_O}(u) := \sum_{\Delta \in T_O^2} E_{T_\Delta}(u) \quad \text{and} \quad E_{S_O}(u) := \sum_{\Delta \in T_O^2} E_{S_\Delta}(u).$$

The main difficulty in our estimates is that the partial derivatives of u_w are not necessarily bounded near the vertex O.

Lemma 4.5 (Derivative estimation lemma). *For each $w = (\rho,\phi) \in S_O$ such that $w \neq O$ we have*

$$\left\| D^1 u_w(z)|_{z=g_w(w)} \right\| \leq \operatorname{Const}_{u,S_O} \cdot \rho^{\gamma_O-1},$$
$$\left\| D^2 u_w(z)|_{z=g_w(w)} \right\| \leq \operatorname{Const}_{u,S_O} \cdot \rho^{\gamma_O-2}.$$

Proof. By the chain rule and the Leibnitz rule we have

$$\left\| D^1 u_w(z)|_{z=g_w(w)} \right\| \leq \operatorname{Const} \cdot \left\| D^1(g_O \circ g_w^{-1})(z)|_{z=g_w(w)} \right\| \cdot \left\| D^1 u_O(z)|_{z=g_O(w)} \right\|$$

and

$$\left\| D^2 u_w(z)|_{z=g_w(w)} \right\| \leq \operatorname{Const} \cdot \left\| D^2(g_O \circ g_w^{-1})(z)|_{z=g_w(w)} \right\| \cdot \left\| D^1 u_O(z)|_{z=g_O(w)} \right\|$$
$$+ \operatorname{Const} \cdot \left\| D^1(g_O \circ g_w^{-1})(z)|_{z=g_w(w)} \right\|^2 \cdot \left\| D^2 u_O(z)|_{z=g_O(w)} \right\|.$$

The map $g_O \circ g_w^{-1}: g_w N_w \to \mathbb{C}$ has the form $z \mapsto Az^{\gamma_O}$, where $|A| = 1$. A direct computation shows that $\|D^k(g_O \circ g_w^{-1})\| \leq \operatorname{Const}_{\gamma_O} \rho^{\gamma_O-k}$ for each $k = 1, 2$. In particular,

$$\left\| D^1(g_O \circ g_w^{-1}) \right\|^2 \leq \operatorname{Const}_{\gamma_O} \rho^{2\gamma_O-2} \leq \operatorname{Const}_{S_O} \rho^{\gamma_O-2}.$$

Since $u_O: \operatorname{Int} g_O S_O \to \mathbb{R}$ is smooth and smoothly extends to a neighborhood of $g_O S_O$, it follows that $\|D^1 u_O\|, \|D^2 u_O\| \leq \operatorname{Const}_{u,S_O}$ on the compact set $g_O S_O$. The required estimate follows. $\square$

Lemma 4.6. *For each $\Delta \in T_O^2 - T_{O,h}^2$ we have*

$$|E_{T_\Delta}(u) - E_{S_\Delta}(u)| \le \mathrm{Const}_{u,\delta,S_O} \cdot h \cdot \int_\Delta \rho^{2\gamma_O - 2} \, d\rho d\phi.$$

Proof. Let $z \in \Delta$ be the vertex closest to O. Since $\Delta \in T_O^2 - T_{O,h}^2$, it follows that for each point $(\rho, \phi) \in \Delta$ we have $h \le |Oz| \le \rho \le |Oz| + h \le 2|Oz|$. Thus by the derivative estimation lemma (Lemma 4.5) and the energy approximation lemma (Lemma 4.4) we get

$$|E_{T_\Delta}(u) - E_{S_\Delta}(u)| \le \mathrm{Const}_{u,\delta,S_O}\left(|Oz|^{\gamma_O - 1} + h|Oz|^{\gamma_O - 2}\right) \cdot h|Oz|^{\gamma_O - 2} \, \mathrm{Area}(\Delta)$$

$$\le \mathrm{Const}_{u,\delta,S_O} \cdot h \cdot \int_\Delta \rho^{2\gamma_O - 2} \, d\rho d\phi. \qquad \square$$

Lemma 4.7. *We have*

$$|E_{T_O - T_{O,h}}(u) - E_{S_O - S_{O,h}}(u)| \le \mathrm{Const}_{u,\delta,S_O} \cdot \begin{cases} h & \text{if } \gamma_O > 1/2, \\ h|\log \frac{h}{\mathrm{Diam}(S_O)}| & \text{if } \gamma_O = 1/2, \\ h^{2\gamma_O} & \text{if } \gamma_O < 1/2. \end{cases}$$

Proof. Summing the inequalities of Lemma 4.6, enlarging the integration domain, and evaluating the integral, we get

$$|E_{T_O - T_{O,h}}(u) - E_{S_O - S_{O,h}}(u)| \le \sum_{\Delta \in T_O^2 - T_{O,h}} |E_{T_\Delta}(u) - E_{S_\Delta}(u)|$$

$$\le \mathrm{Const}_{u,\delta,S_O} \cdot h \cdot \int_{S_O - S_{O,h}} \rho^{2\gamma_O - 2} \, d\rho d\phi$$

$$\le \mathrm{Const}_{u,\delta,S_O} \cdot h \cdot \int_{h \le \rho \le \mathrm{Diam}(S_O), \, 0 \le \phi \le 2\pi/\gamma_O} \rho^{2\gamma_O - 2} \, d\rho d\phi$$

$$\le \mathrm{Const}_{u,\delta,S_O} \cdot \begin{cases} h & \text{if } \gamma_O > 1/2, \\ h|\log \frac{h}{\mathrm{Diam}(S_O)}| & \text{if } \gamma_O = 1/2, \\ h^{2\gamma_O} & \text{if } \gamma_O < 1/2. \end{cases} \qquad \square$$

Remark 4.8. Lemma 4.7 and its proof remain true, if we replace S_O and T_O in the left-hand side by a triangulated subset $S_O' \subset S_O$ with a triangulation T_O' of the maximal edge length $\le h$ and the minimal face angle $> \delta$ (the constant in the right-hand side still depends on S_O but not S_O').

Lemma 4.9. *We have $E_{S_{O,h}}(u) \le \mathrm{Const}_{u,S_O} \cdot h^{2\gamma_O}$.*

Proof. By the derivative estimation lemma (Lemma 4.5) we have

$$E_{S_{O,h}}(u) = \sum_{\Delta \in T_{O,h}^2} \int_\Delta |\nabla u|^2 dx dy$$

$$\le \mathrm{Const}_{u,S_O} \int_{0 \le \rho \le 2h, \, 0 \le \phi \le 2\pi/\gamma_O} \rho^{2\gamma_O - 1} \, d\rho d\phi$$

$$\le \mathrm{Const}_{u,S_O} \cdot h^{2\gamma_O}. \qquad \square$$

For an analogous estimate of the discrete energy we need the following auxiliary assertions.

Lemma 4.10. *For each edge* $xy \in T_O^1$ *we have*

$$|u(x) - u(y)| \leq \text{Const}_{u, S_O} \cdot |xy| \cdot \max\{|Ox|, |Oy|\}^{\gamma_O - 1}.$$

Proof. Assume $|Ox| \leq |Oy|$ without loss of generality. Let a point $w = (\rho, \phi)$ move along the segment xy. Denote by Ov the perpendicular from the point O to the geodesic containing xy. By the derivative estimation lemma (Lemma 4.5), the Bernoulli inequality, the inequality $\phi \leq 2 \sin \phi$ for $0 \leq \phi \leq \pi/2$, and the sine theorem for the triangle Oxy (in the development of the cone S_O into the plane) we have

$$|u(x) - u(y)| \leq \int_{xy} \left\| D^1 u_w(z)|_{z = g_w(w)} \right\| \cdot (|d\rho| + \rho|d\phi|)$$

$$\leq \text{Const}_{u, S_O} \int_{xy} \rho^{\gamma_O - 1}(|d\rho| + \rho|d\phi|)$$

$$\leq \text{Const}_{u, S_O} \begin{cases} \int_{xy} \rho^{\gamma_O - 1} d\rho + |\int_{xy} \rho^{\gamma_O} d\phi| & \text{if } x \text{ is between } y \text{ and } v \\ \int_{vx} \rho^{\gamma_O - 1} d\rho + \int_{vy} \rho^{\gamma_O - 1} d\rho + |\int_{xy} \rho^{\gamma_O} d\phi| & \text{if } v \text{ is between } x \text{ and } y \end{cases}$$

$$\leq \text{Const}_{u, S_O} \cdot \left(2|Oy|^{\gamma_O} - 2\max\{0, |Oy| - |xy|\}^{\gamma_O} + |Oy|^{\gamma_O} \cdot \angle xOy\right)$$

$$\leq \text{Const}_{u, S_O} \cdot \begin{cases} 2|Oy|^{\gamma_O} + |Oy|^{\gamma_O} \cdot \pi & \text{if } |xy| \geq |Oy| \\ 2\max\{1, \gamma_O\} \cdot |xy| \cdot |Oy|^{\gamma_O - 1} + 2|Oy|^{\gamma_O} \sin xOy & \text{if } |xy| < |Oy| \end{cases}$$

$$\leq \text{Const}_{u, S_O} \cdot |xy| \cdot |Oy|^{\gamma_O - 1}$$

Here we set $\angle xOy = 0$, if the segment Oy contains x. □

Lemma 4.11. *For each* $xyz \in T_O^2$ *we have*

$$|xy|^2 \leq \text{Const}_\delta \cdot \text{Area}(xyz),$$
$$\max\{|Ox|, |Oy|, |Oz|\} \leq \text{Const}_\delta \cdot \max\{|Ox|, |Oy|\}.$$

Proof. Take the point w such that $\angle xyw = \angle yxw = \delta$ and the triangles xyz and xyw have the same orientation. Since $\angle xyz, \angle yxz > \delta$, it follows that $xyw \subset yxz$. Thus $\text{Area}(xyz) > \text{Area}(xyw) = \tan \delta \cdot |xy|^2/4$ and the first inequality follows. Assume that $|Ox| \leq |Oy| \leq |Oz|$ without loss of generality. By the triangle inequality and the sine theorem we get the second inequality

$$|Oz| - |Oy| \leq |yz| \leq \csc \delta \cdot |xy| \leq \csc \delta \cdot (|Ox| + |Oy|) \leq 2 \csc \delta \cdot |Oy|. \quad \square$$

Lemma 4.12. *For each* $\Delta \in T_{O,h}^2$ *we have*

$$E_{T_\Delta}(u) \leq \text{Const}_{u, \delta, S_O} \cdot \int_\Delta \rho^{2\gamma_O - 1} d\rho d\phi.$$

Proof. Denote by x, y, z the vertices of Δ so that $|Ox| \leq |Oy| \leq |Oz|$. Since the minimal face angle is greater than δ, using Lemmas 4.10 and 4.11 and estimating the sum via

the integral, we get

$$E_{T_\Delta}(u) = \frac{1}{2}\cot xyz \cdot (u(z) - u(x))^2 + \frac{1}{2}\cot yzx \cdot (u(x) - u(y))^2$$
$$+ \frac{1}{2}\cot zxy \cdot (u(y) - u(z))^2$$
$$\le \mathrm{Const}_\delta \left((u(z) - u(x))^2 + (u(x) - u(y))^2 + (u(y) - u(z))^2\right)$$
$$\le \mathrm{Const}_{u,\delta,S_O}\left(|Oz|^{2\gamma_O-2}\cdot|zx|^2 + |Oy|^{2\gamma_O-2}\cdot|xy|^2 + |Oz|^{2\gamma_O-2}\cdot|yz|^2\right)$$
$$\le \mathrm{Const}_{u,\delta,S_O}\cdot|Oz|^{2\gamma_O-2}\cdot\mathrm{Area}(\Delta)$$
$$\le \mathrm{Const}_{u,\delta,S_O}\cdot\int_\Delta \rho^{2\gamma_O-1}\,d\rho d\phi.$$

The latter inequality is straightforward for $\gamma_O \le 1$ and requires the following estimate for $\gamma_O > 1$:

$$|Oz|^{2\gamma_O-2}\cdot\mathrm{Area}(\Delta) = 16|Oz|^{2\gamma_O-2}\cdot\mathrm{Area}\left(\frac{1}{4}\Delta\right)$$
$$\le 16\int_{\frac{1}{4}\Delta}(2\rho)^{2\gamma_O-2}\rho\,d\rho d\phi \le 2^{2\gamma_O+2}\int_\Delta \rho^{2\gamma_O-1}\,d\rho d\phi,$$

where $\frac{1}{4}\Delta \subset \Delta$ is obtained from Δ by the homothety with the center z and the coefficient $\frac{1}{4}$.
$\square$

Lemma 4.13. *We have $E_{T_{O,h}}(u) \le \mathrm{Const}_{u,\delta,S_O}\cdot h^{2\gamma_O}$.*

Proof. Summing the inequalities of Lemma 4.12, enlarging the integration domain, and evaluating the integral, we get

$$E_{T_{O,h}}(u) = \sum_{\Delta\in T_{O,h}^2} E_{T_\Delta}(u)$$
$$\le \mathrm{Const}_{u,\delta,S_O}\cdot\int_{S_{O,h}}\rho^{2\gamma_O-1}\,d\rho d\phi$$
$$\le \mathrm{Const}_{u,\delta,S_O}\cdot\int_{0\le\rho\le2h,\,0\le\phi\le2\pi/\gamma_O}\rho^{2\gamma_O-1}\,d\rho d\phi$$
$$\le \mathrm{Const}_{u,\delta,S_O}\cdot h^{2\gamma_O}. \qquad \square$$

Example 4.14. The assumption that the minimal face angle is greater than δ in Lemmas 4.12 and 2.7 cannot be dropped. Indeed, take $0 < \epsilon < 1/2$, $\gamma_O := 1/2 - \epsilon$, and $u(\rho,\phi) := \rho^{\gamma_O}\cos(\gamma_O\phi)$. Consider the triangle Δ with the vertices O, $x := (1/n, 0)$, and $y := (1/n^\epsilon, \pi/2)$. We have $\angle Oyx \to 0$ and

$$E_{T_\Delta}(u) \ge \cot Oyx \cdot \frac{1}{2}(u(x) - u(O))^2 = \frac{1}{2}n^{1-\epsilon-2\gamma_O} \to \infty$$

as $n \to \infty$. Let T_O be an arbitrary triangulation of S_O containing the triangle Δ and having maximal edge length $< 2/n^\epsilon$. Then the inequalities of Lemmas 4.12 and 2.7 cannot hold.

4.3. Convergence of energy in polyhedral surfaces.   Denote

$$
\lambda_S(h) := \begin{cases} h & \text{if } \gamma_S > 1/2, \\ h|\log h| & \text{if } \gamma_S = 1/2, \\ h^{2\gamma_S} & \text{if } \gamma_S < 1/2. \end{cases}
$$

Proof of Lemma 2.7.   First take an auxiliary triangulation T_1 of the surface S and set S_z to be the union of faces containing a vertex $z \in T_1^0$. Denote by $tS_z \subset S_z$ the set obtained from the cone S_z by the homothety with the center $z \in T_1^0$ and the coefficient $0 < t < 1$. Let $\mathrm{const}_{u,\delta,S}$ be $1/10$ of the minimal face height of T_1 (actually it depends only on S). Now take a triangulation T of maximal edge length $h < \mathrm{const}_{u,\delta,S}$. Let $S_z' \subset S_z$ be the union of faces of T intersecting the set $\frac{8}{10}S_z$ and T_z' be the restriction of T to S_z'. Then $S_z' \subset \frac{9}{10}S_z$ and $u_z \colon g_z(\frac{9}{10}S_z) \to \mathbb{R}$ smoothly extends to a neighborhood of $g_z(\frac{9}{10}S_z)$. On the other hand, $S_z' \supset \frac{8}{10}S_z$ and thus $\bigcup_{z \in T_1^0} S_z' \supset S$. Replace the sets S_z' and T_z' by their appropriate subsets forming *decompositions* of S and T, respectively. Estimating the energy separately in the h-neighborhoods of the vertices of T_1 and outside them by Lemmas 4.9, 4.13, 4.7, Remark 4.8, and the inequality $\gamma_S \leq 1$, we get

$$
\left| E_T(u|_{\tilde{T}^0}) - E_S(u) \right| \leq \sum_{z \in T_0^1} \left(E_{T_{z,h}}(u) + E_{S_{z,h}}(u) + |E_{T_z'-T_{z,h}}(u) - E_{S_z'-S_{z,h}}(u)| \right)
$$

$$
\leq \sum_{z \in T_1^0} \mathrm{Const}_{u,\delta,S_z} \cdot \begin{cases} h & \text{if } \gamma_z > 1/2, \\ h|\log h| & \text{if } \gamma_z = 1/2, \\ h^{2\gamma_z} & \text{if } \gamma_z < 1/2. \end{cases}
$$

$$
\leq \mathrm{Const}_{u,\delta,S} \cdot \lambda_S(h). \qquad \square
$$

Lemma 4.15.   *For any $P \in \mathbb{R}^{2g}$ we have $E_T(u_{T,P}) \geq E_S(u_{S,P})$.*

Proof.   Let $I_T u_{T,P} \colon \tilde{S} \to \mathbb{R}$ be the linear interpolation of $u_{T,P} \colon \tilde{T}^0 \to \mathbb{R}$ in each face of $\tilde{T}$. By the interpolation lemma (Lemma 4.1) we get $E_T(u_{T,P}) = E_S(I_T u_{T,P})$ because the metric of the surface $\tilde{S}$ may have singularities only at the vertices of $\tilde{T}$. Since the function $I_T u_{T,P} \colon \tilde{S} \to \mathbb{R}$ is continuous by the continuous counterpart of the variational principle (Lemma 3.6), the result follows. $\qquad \square$

Lemma 4.16 (Energy form convergence lemma).   *For each $\delta > 0$ and each vector $P \in \mathbb{R}^{2g}$ there are constants $\mathrm{Const}_{P,\delta,S}, \mathrm{const}_{P,\delta,S} > 0$ such that for any triangulation T of S with the maximal edge length $h < \mathrm{const}_{P,\delta,S}$ and the minimal face angle $> \delta$ we have*

$$
|E_T(u_{T,P}) - E_S(u_{S,P})| \leq \mathrm{Const}_{P,\delta,S} \cdot \lambda_S(h).
$$

Proof.   The lemma follows from the following sequence of estimates provided by Lemmas 4.15, 3.6, and 2.7:

$$
0 \leq E_T(u_{T,P}) - E_S(u_{S,P}) \leq E_T(u_{S,P}|_{\tilde{T}^0}) - E_S(u_{S,P}) \leq \mathrm{Const}_{P,\delta,S} \cdot \lambda_S(h). \qquad \square
$$

Corollary 4.17. *Let $\{T_n\}$ be a nondegenerate uniform sequence of Delaunay triangulations of S with maximal edge length approaching zero as $n \to \infty$. Let $P_n \in \mathbb{R}^{2g}$ be a sequence of $2g$-dimensional real vectors converging to a vector $P \in \mathbb{R}^{2g}$. Then*

$$E_{T_n}(u_{T_n,P_n}) \to E_S(u_{S,P}) \quad as \; n \to \infty.$$

Proof. Both $E_{T_n}(u_{T_n,P})$ and $E_S(u_{S,P})$ are quadratic forms in $P \in \mathbb{R}^{2g}$. Thus the result follows by Lemma 4.16. □

4.4. Convergence of period matrices.

Proof of Theorem 2.5. By Lemmas 3.14 and 3.15 the energies $E_T(u_{T,P})$ and $E_S(u_{S,P})$ are quadratic forms in $P \in \mathbb{R}^{2g}$ with the block matrices E_T and E_S, respectively. Thus by the energy form convergence lemma (Lemma 4.16) for each $\delta > 0$ there exist two constants $\mathrm{Const}_{\delta,S}, \mathrm{const}_{\delta,S} > 0$ such that for any triangulation T of S with maximal edge length $h < \mathrm{const}_{\delta,S}$ and the minimal face angle $> \delta$ we have $\|E_T - E_S\| \leq \mathrm{Const}_{\delta,S} \cdot \lambda_S(h)$. In particular, $\|(\mathrm{Im}\,\Pi_{T^*})^{-1} - (\mathrm{Im}\,\Pi_S)^{-1}\| \leq \mathrm{Const}_{\delta,S} \cdot \lambda_S(h)$ for $h < \mathrm{const}_{\delta,S}$. Hence there exist two new constants $\mathrm{Const}'_{\delta,S}, \mathrm{const}'_{\delta,S} > 0$ such that $\|(\mathrm{Im}\,\Pi_{T^*})\| \leq \mathrm{Const}'_{\delta,S}$ for $h < \mathrm{const}'_{\delta,S}$. Assume further that $h < \min\{\mathrm{const}'_{\delta,S}, \mathrm{const}_{\delta,S}\}$.

By the inequality $\|E_T - E_S\| \leq \mathrm{Const}_{\delta,S} \cdot \lambda_S(h)$ and the properties of the norm we have

$$\begin{aligned}
\mathrm{Const}_{\delta,S} \cdot \lambda_S(h) &\geq \|(\mathrm{Im}\,\Pi_{T^*})^{-1}\,\mathrm{Re}\,\Pi_T - (\mathrm{Im}\,\Pi_S)^{-1}\,\mathrm{Re}\,\Pi_S\| \\
&= \|(\mathrm{Im}\,\Pi_{T^*})^{-1}(\mathrm{Re}\,\Pi_T - \mathrm{Re}\,\Pi_S) - ((\mathrm{Im}\,\Pi_S)^{-1} - (\mathrm{Im}\,\Pi_{T^*})^{-1})\,\mathrm{Re}\,\Pi_S\| \\
&\geq \|(\mathrm{Im}\,\Pi_{T^*})\|^{-1} \cdot \|\mathrm{Re}\,\Pi_T - \mathrm{Re}\,\Pi_S\| - \|(\mathrm{Im}\,\Pi_S)^{-1} - (\mathrm{Im}\,\Pi_{T^*})^{-1}\| \cdot \|\mathrm{Re}\,\Pi_S\| \\
&\geq (\mathrm{Const}'_{\delta,S})^{-1} \cdot \|\mathrm{Re}\,\Pi_T - \mathrm{Re}\,\Pi_S\| - \mathrm{Const}_{\delta,S} \cdot \lambda_S(h) \cdot \|\mathrm{Re}\,\Pi_S\|.
\end{aligned}$$

Thus

$$\|\mathrm{Re}\,\Pi_T - \mathrm{Re}\,\Pi_S\| \leq \mathrm{Const}''_{\delta,S} \cdot \lambda_S(h),$$

where

$$\mathrm{Const}''_{\delta,S} := \mathrm{Const}'_{\delta,S} \cdot \mathrm{Const}_{\delta,S} \cdot (1 + \|\mathrm{Re}\,\Pi_S\|).$$

Finally, since $\|E_T - E_S\| \leq \mathrm{Const}_{\delta,S} \cdot \lambda_S(h)$, it follows that

$$\begin{aligned}
&\|\mathrm{Re}\,\Pi_{T^*}(\mathrm{Im}\,\Pi_{T^*})^{-1}\,\mathrm{Re}\,\Pi_T + \mathrm{Im}\,\Pi_T - \mathrm{Re}\,\Pi_S(\mathrm{Im}\,\Pi_S)^{-1}\,\mathrm{Re}\,\Pi_S - \mathrm{Im}\,\Pi_S\| \\
&\qquad \leq \mathrm{Const}_{\delta,S} \cdot \lambda_S(h).
\end{aligned}$$

Using the result of the previous paragraph and similar estimates, we conclude that there exist a new constant $\mathrm{Const}'''_{\delta,S}$ such that

$$\|\mathrm{Re}\,\Pi_{T^*}(\mathrm{Im}\,\Pi_{T^*})^{-1}\,\mathrm{Re}\,\Pi_T - \mathrm{Re}\,\Pi_S(\mathrm{Im}\,\Pi_S)^{-1}\,\mathrm{Re}\,\Pi_S\| \leq \mathrm{Const}'''_{\delta,S} \cdot \lambda_S(h).$$

Thus $\|\mathrm{Im}\,\Pi_T - \mathrm{Im}\,\Pi_S\| \leq (\mathrm{Const}'''_{\delta,S} + \mathrm{Const}_{\delta,S}) \cdot \lambda_S(h)$, which completes the proof of the theorem. □

5. Convergence of discrete Abelian integrals

In this section we prove convergence of Abelian integrals (Theorem 2.6). The proof uses lemmas stated in the next two subsections.

5.1. Equicontinuity.   Consider a Delaunay triangulation T of a polyhedral surface with boundary such that all faces are flat triangles. A function $u\colon T^0 \to \mathbb{R}$ is *discrete harmonic*, if it satisfies equation (2.1) at each nonboundary vertex z. Denote

$$E'_T(u) := \sum_{xy \in T^1,\, xy \not\subset \partial T} c(xy)(u(x) - u(y))^2,$$

where the sum is over nonboundary edges. The *eccentricity* of the triangulation is the infimum of the numbers Const such that the triangulation satisfies conditions (D) (for each nonboundary edge) and (U) from Section 2.5. Denote by e and h' the eccentricity and twice the maximal circumradius of faces of the triangulation, respectively. To make the proofs below simple we need the following technical notion. We say that the triangulation has *regular boundary*, if it extends to a Delaunay triangulation of a closed polyhedral surface.

Lemma 5.1 (Equicontinuity lemma).   *Let S_O be the part of a cone with the vertex O and the aperture $2\pi/\gamma_O$ bounded by a piecewise-geodesic broken line. Let T_O be a Delaunay triangulation of S_O with regular boundary and with flat faces. Let $u\colon T^0_O \to \mathbb{R}$ be a discrete harmonic function. Let $z, w \in T^0_O$ be at the distance $|zw| \geq h'$. Denote*

$$r := \mathrm{Dist}(O, \partial S_O) - \max\{|Oz|, |Ow|\} - h', \quad \gamma'_O := \max\{\gamma^2_O, 1/\gamma^2_O\}.$$

Assume that $r > 3\gamma'_O|zw|$. Then there is a constant $\mathrm{Const}_{e,\gamma_O}$ such that

$$(5.1) \qquad |u(z) - u(w)| \leq \mathrm{Const}_{e,\gamma_O} \cdot E'_{T_O}(u)^{1/2} \cdot \log^{-1/2} \frac{r}{3\gamma'_O|zw|}.$$

For $|zw| < h' < r/3\gamma'_O$ the same inequality holds with $|zw|$ replaced by h'.

Proof.   The lemma is proved analogously to a similar estimate for *quadrilateral lattices* in the plane [30, Equicontinuity Lemma 2.4]. Actually one can derive the lemma from that estimate as follows. The used notions and results can be found in [30, §1, Remarks 3.4, 4.8] and Section 7.2 below.

For each face of T_O at distance $> h'/2$ from the boundary ∂S_O, draw three circumradii from the circumcenter to the vertices; see Figure 1. Since the triangulation T_O has regular boundary, by Theorem 7.4 it follows that the drawn segments do not have common interior points. Erase hanging edges from the obtained graph. We get an orthogonal quadrilateral lattice Q in S_O. Its twice maximal edge length is at most h' and its eccentricity in the relaxed sense of [30, Remark 4.8] is Const_e. Extend the function $u\colon T^0_O \to \mathbb{R}$ to Q^0 by identical zero at $Q^0 - T^0_O$. The obtained function $Q^0 \to \mathbb{R}$ is discrete harmonic and has the energy at most $E'_{T_O}(u)$.

Let the map $q\colon S_O \to \mathbb{R}^2$ be given in polar coordinates by the formula

$$q\colon (\rho, \phi) \mapsto \left(\sqrt{\gamma'_O}\rho, \gamma_O\phi\right).$$

Identify each face of Q (respectively, its image under q) with a quadrilateral $z_1 z_2 z_3 z_4 \subset \mathbb{C}$ by an orientation-preserving isometry (respectively, composed with q^{-1}). This makes $q(Q)$ a quad-surface and $u \circ q^{-1}\colon q(Q^0) \to \mathbb{R}$ a discrete harmonic function. Since the map $q\colon S_O \to \mathbb{R}^2$

increases distances by a factor at most γ'_O, it follows that $q(Q)$ has twice the maximal edge length at most $\gamma'_O h'$ and the eccentricity at most $\mathrm{Const}_{\gamma_O,e}$.

Draw the square R of side length r with the center at the midpoint x of the segment $q(z)q(w)$ and the sides parallel and orthogonal to $q(z)q(w)$. Assume $|Oz| \geq |Ow|$. Since

$$r = \mathrm{Dist}(O, \partial S_O) - |Oz| - h' \leq \mathrm{Dist}(O, \partial Q) - |Oz|$$
$$\leq \mathrm{Dist}(q^{-1}(x), \partial Q) \leq \mathrm{Dist}(x, q(\partial Q)),$$

it follows that $R \cap q(\partial Q) = \emptyset$. Since $r > 3\gamma'_O|zw| \geq 3|q(z) - q(w)|$, by [30, Lemma 2.4, Remarks 4.8, 4.9] the lemma follows. $\square$

5.2. Harmonicity of a uniform limit. A sequence of triangulated polygons $\{T_n\}$ *approximates* a domain $\Omega \subset \mathbb{C}$, if for $n \to \infty$:

- the maximal distance from a point of ∂T_n to the set $\partial\Omega$ tends to zero;
- the maximal distance from a point of $\partial\Omega$ to the set ∂T_n tends to zero;
- the maximal edge length of the triangulation T_n tends to zero.

Lemma 5.2. *Let $\{T_n\}$ be a nondegenerate uniform sequence of Delaunay triangulations of polygons with regular boundary approximating a domain $\Omega \subset \mathbb{C}$. Let $u_n: T_n^0 \to \mathbb{R}$ be a sequence of discrete harmonic functions uniformly converging to a continuous function $u: \Omega \to \mathbb{R}$. Then the function $u: \Omega \to \mathbb{R}$ is harmonic.*

This lemma is proved analogously to a similar result for quadrilateral lattices (see [30, Lemma 4.13, remark at the end of §4.5]). Actually it immediately reduces to that result analogously to the second paragraph of the proof of Lemma 5.1 above.

5.3. Convergence of multi-valued discrete harmonic functions. We say that the function $u_{T,P}: \tilde{T}^0 \to \mathbb{R}$ (respectively, $u_{S,P}: \tilde{S} \to \mathbb{R}$) is *normalized* at a point z if $u_{T,P}(z) = 0$ (respectively, $u_{S,P}(z) = 0$).

Theorem 5.3 (Convergence theorem for multi-valued discrete harmonic functions). *Let $\{T_n\}$ be a nondegenerate uniform sequence of Delaunay triangulations of S with maximal edge length approaching zero as $n \to \infty$. Let $z_n \in \tilde{T}_n^0$ be a sequence of vertices converging to a point $z_0 \in \tilde{S}$. Let $P_n \in \mathbb{R}^{2g}$ be a sequence of vectors converging to a vector $P \in \mathbb{R}^{2g}$. Then the function $u_{T_n,P_n}: \tilde{T}_n^0 \to \mathbb{R}$ normalized at z_n converges to $u_{S,P}: \tilde{S} \to \mathbb{R}$ normalized at z_0 uniformly on each compact subset.*

The proof goes along the lines of the proof of [30, Convergence Theorem 1.2] with several improvements required by the presence of curvature. We need an auxiliary lemma.

Lemma 5.4. *Let T be a triangulation of a polyhedral surface with boundary such that all face angles are $> \delta$. Then for each $u: T^0 \to \mathbb{R}$ we have $E'_T(u) \leq \mathrm{Const}_\delta \cdot E_T(u)$.*

Proof. Take a face $\Delta = xyz \in T^2$ such that $xz \subset \partial T$ and $\angle xyz > \pi/2$. By Lemmas 4.1 and 4.11 we get

$$\frac{1}{2}|\cot xyz| \cdot (u(x) - u(z))^2 \leq \mathrm{Const}_8 \cdot |\nabla I_\Delta u|^2 \cdot |xz|^2 \leq \mathrm{Const}_8 \cdot E_{T_\Delta}(u).$$

Summing these inequalities over all such faces Δ, we get $E'_T(u) - E_T(u) \leq \mathrm{Const}_8 \cdot E_T(u)$.

$\square$

Proof of Theorem 5.3. Take an arbitrary subsequence $\{\tilde{T}_{n_k}\}$ of the given sequence $\{\tilde{T}_n\}$. For brevity denote $\tilde{T}_k := \tilde{T}_{n_k}$. Take a sequence of compact sets $K_1 \subset K_2 \subset \cdots \subset \tilde{S}$ such that $\tilde{S} = \bigcup_{j=1}^\infty K_j$. Assume that K_1 contains all the points of the converging sequence $\{z_k\}$. Since the sequence $\{T_n\}$ is nondegenerate uniform, it follows that the face angles are greater than some constant $\delta > 0$.

We start with individual estimates near each point $v \in \tilde{S}$. Let $10r'$ be the minimal distance between the conical singularities of $\tilde{S}$ (if there are no singularities then $10r'$ is defined to be the diameter of S). Let $S'_v \subset S$ be the set of points within distance at most $3r'$ from a point $v \in \tilde{S}$. Assume further that v is either a conical singularity or v is at distance at least $5r'$ from the nearest conical singularity. Take k_1 such that for each $k > k_1$ the maximal edge length of $\tilde{T}_k$ is less than $r' \sin \delta/12\gamma'_v$. Take $k > k_1$. Let S_v be the union of all the faces of the triangulation $\tilde{T}_k$ within distance at most $4r'$ from the vertex v. By construction S_v is a part of a cone with regular boundary. Let T_v be the restriction of the triangulation $\tilde{T}_k$ to S_v. Let us estimate the right-hand side of inequality (5.1) from the equicontinuity lemma (Lemma 5.1) for $u := u_{T_k, P_k}|_{(T_v)^0}$ and $z, w \in S'_v$ at the distance $|zw| < r'/12\gamma'_v$. Since

$$r = \mathrm{Dist}(v, \partial S_v) - \max\{|vz|, |vw|\} - h' > \frac{1}{2}r',$$

it follows that $r/3\gamma'_v|zw| > 2$. By Lemma 5.4 and Corollary 4.17 the sequence of energies $E'_{T_v}(u_{T_k, P_k})$ is bounded. Thus by Lemma 5.1 it follows that the function $u_{T_k, P_k}|_{S'_v \cap \tilde{T}^0_k}$ has *uniformly bounded differences*, i.e., there is a constant $\mathrm{Const}_{S, \{T_n\}, \{P_n\}}$ not depending on k, z, w such that for each $k > k_1$ and $z, w \in S'_v \cap \tilde{T}^0_k$ we have

$$|u_{T_k, P_k}(z) - u_{T_k, P_k}(w)| < \mathrm{Const}_{S, \{T_n\}, \{P_n\}}.$$

By the same lemma the sequence $\{u_{T_k, P_k}|_{S'_v \cap \tilde{T}^0_k}\}$ is *equicontinuous*, i.e., there is a positive function $\delta(\epsilon)$ not depending on k, z, w such that for each $k > k_1$ and any points $z, w \in S'_v \cap \tilde{T}^0_k$ at the distance $|zw| < \delta(\epsilon)$ we have

$$|u_{T_k, P_k}(z) - u_{T_k, P_k}(w)| < \epsilon.$$

Let us combine the obtained estimates. Cover the compact set K_1 by finitely many sets of the form S'_v, where v is either a conical singularity or at the distance at least $5r'$ from the nearest conical singularity, so that sufficiently many pairs of these sets have intersection containing an intrinsic disk of radius $> r'/4$. We obtain that the sequence $\{u_{T_k, P_k}|_{K_1 \cap \tilde{T}^0_k}\}$ also has uniformly bounded differences (with the bound depending also on K_1) and is equicontinuous. Moreover, the sequence is uniformly bounded, because all $z_k \in K_1$ and $u_{T_k, P_k}(z_k) = 0$. Then by the Arzelà–Ascoli theorem it follows that there is a continuous function $u_1: K_1 \to \mathbb{R}$ and

a subsequence $\{l_k\}$ of the sequence $1, 2, \ldots$ such that $l_1 = k_1$ and u_{T_l, P_l} converges to u_1 uniformly in K_1.

Proceed to the next compact set K_2. Analogously, there are a continuous function $u_2 \colon K_2 \to \mathbb{R}$ and a subsequence $\{m_k\}$ of the sequence $\{l_k\}$ such that $m_1 = l_1$, $m_2 = l_2$, and u_{T_m, P_m} converges to u_2 uniformly on K_2. Clearly, $u_1 = u_2$ on K_1. Thus the extension can be continued, and eventually we get a continuous function $u \colon \tilde{S} \to \mathbb{R}$ and a subsequence $\{p_k\}$ of the sequence $1, 2, \ldots$ such that u_{T_p, P_p} converges to u uniformly on each compact subset of $\tilde{S}$.

Clearly, $u \colon \tilde{S} \to \mathbb{R}$ has the same periods P as $u_{S,P}$, and $u(z_0) = 0$. For each edge $e \in \tilde{T}_1^1$ denote by $S_e \subset \tilde{S}$ the union of the edge e (without the endpoints) and the interiors of the two faces of $\tilde{T}_1$ containing the edge e. Then each set S_e is isometric to a planar domain. By Lemma 5.2 applied to each domain S_e it follows that the limit function $u \colon \tilde{S} \to \mathbb{R}$ is harmonic in $\tilde{S}$ possibly except the vertices of the triangulation $\tilde{T}_1$ (where the metric of $\tilde{S}$ may have conical singularities). By the singularity removal theorem it follows that the continuous function $u \colon \tilde{S} \to \mathbb{R}$ is harmonic in the whole surface $\tilde{S}$. Thus $u = u_{S,P}$ normalized at z_0.

Since the limit function u is unique, it follows that the initial sequence $u_{T_n, P_n} \colon \tilde{T}_n^0 \to \mathbb{R}$, not just the subsequence u_{T_k, P_k}, converges to $u_{S,P}$ uniformly on each compact subset. □

5.4. Convergence of discrete Abelian integrals of the first kind.

Proof of Theorem 2.6. Let P_n, $P \in \mathbb{R}^{2g}$ be the periods of the real parts $\operatorname{Re} \phi_{T_n}^l \colon \tilde{T}_n^0 \to \mathbb{R}$ and $\operatorname{Re} \phi_S^l \colon \tilde{S} \to \mathbb{R}$, respectively. Then by the second existence and uniqueness theorem (Theorem 3.9) it follows that $\operatorname{Re} \phi_{T_n}^l = u_{T_n, P_n}$ and $\operatorname{Re} \phi_S^l = u_{S,P}$. By convergence of period matrices (Theorem 2.5) we have $P_n \to P$ as $n \to \infty$. Thus by Theorem 5.3 the real parts $\operatorname{Re} \phi_{T_n}^l \colon \tilde{T}_n^0 \to \mathbb{R}$ converge to $\operatorname{Re} \phi_S^l \colon \tilde{S} \to \mathbb{R}$ uniformly on each compact subset.

Convergence of the imaginary parts is proved analogously using the conjugate functions principle (Lemma 3.7). □

6. Discrete Riemann–Roch theorem

6.1. Discrete Abelian integrals of the second kind.
A *discrete Abelian integral of the second kind* is an arbitrary multi-valued function $f = (\operatorname{Re} f \colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} f \colon \tilde{T}^2 \to \mathbb{R})$. If it does not satisfy equation (2.2) for an edge $e \in \vec{T}^1$ then we say that f has a *simple pole* at the edge e. The value

$$\operatorname{res}_e f := \operatorname{Im} f(r_e) - \operatorname{Im} f(l_e) + c(e) \operatorname{Re} f(h_e) - c(e) \operatorname{Re} f(t_e)$$

is called the *residue* of f. In the case when all the periods vanish then the discrete Abelian integral of the second kind is called a *discrete meromorphic function*; it can be viewed as a pair of single-valued functions $(\operatorname{Re} f \colon T^0 \to \mathbb{R}, \operatorname{Im} f \colon T^2 \to \mathbb{R})$. In questions concerning continuous limit one should assume that under triangulation refinement the number of poles of discrete meromorphic functions is fixed.

Theorem 6.1 (Third existence and uniqueness theorem). *For each edge $e \in \vec{T}^1$ there is a unique (up to constant) discrete Abelian integral $\phi_e = (\operatorname{Re} \phi_e \colon \tilde{T}^0 \to \mathbb{R}, \operatorname{Im} \phi_e \colon \tilde{T}^2 \to \mathbb{R})$ of the second kind with vanishing A-periods, with the only pole at the edge e, and with $\operatorname{res}_e \phi_e = 1$.*

This theorem is proved analogously to the first existence and uniqueness theorem (Theorem 2.3). The only difference is that the value $\mathrm{res}_e\,\phi_e$ should be considered as a parameter instead of the values of the A-periods.

Lemma 6.2. *The B-periods of* $\phi_e = (\mathrm{Re}\,\phi_e\colon \tilde{T}^0 \to \mathbb{R}, \mathrm{Im}\,\phi_e\colon \tilde{T}^2 \to \mathbb{R})$ *are*

$$B_l = \mathrm{Re}\,\phi^l_{T*}(t_e) - \mathrm{Re}\,\phi^l_{T*}(h_e) + i\,\mathrm{Re}\,\phi^l_T(t_e) - i\,\mathrm{Re}\,\phi^l_T(h_e),$$

where $l = 1, \dots, g.$

Proof. Let $A_1, \dots, A_g, B_1, \dots, B_g$ and $A'_1, \dots, A'_g, B'_1, \dots, B'_g$ be the periods of ϕ_e and ϕ^l_T, respectively. All the A-periods $A_1, \dots, A_g, A'_1, \dots, A'_g$ vanish except for $A'_l = 1$. Using the Riemann bilinear identity (Lemma 3.1) and formula (2.2), we get

$$\begin{aligned}
\mathrm{Im}\,B_l &= \sum_{k=1}^{g} (\mathrm{Re}\,A'_k\,\mathrm{Im}\,B_k - \mathrm{Re}\,B'_k\,\mathrm{Im}\,A_k - \mathrm{Re}\,A_k\,\mathrm{Im}\,B'_k + \mathrm{Re}\,B_k\,\mathrm{Im}\,A'_k) \\
&= \sum_{f\in T^1} \big(\mathrm{Im}\,\phi_e(l_f) - \mathrm{Im}\,\phi_e(r_f)\big)\big(\mathrm{Re}\,\phi^l_T(h_f) - \mathrm{Re}\,\phi^l_T(t_f)\big) \\
&\quad - \sum_{f\in T^1} \big(\mathrm{Im}\,\phi^l_T(l_f) - \mathrm{Im}\,\phi^l_T(r_f)\big)\big(\mathrm{Re}\,\phi_e(h_f) - \mathrm{Re}\,\phi_e(t_f)\big) \\
&= \sum_{f\in T^1} \mathrm{res}_f\,\phi_e \cdot \big(\mathrm{Re}\,\phi^l_T(t_f) - \mathrm{Re}\,\phi^l_T(h_f)\big) \\
&= \mathrm{Re}\,\phi^l_T(t_e) - \mathrm{Re}\,\phi^l_T(h_e),
\end{aligned}$$

Analogously, $\mathrm{Re}\,B_l = \mathrm{Re}\,\phi^l_{T*}(t_e) - \mathrm{Re}\,\phi^l_{T*}(h_e)$. $\qquad\square$

6.2. Discrete Abelian differentials. The *differential* of a discrete Abelian integral $f = (\mathrm{Re}\,f\colon \tilde{T}^0 \to \mathbb{R}, \mathrm{Im}\,f\colon \tilde{T}^2 \to \mathbb{R})$ of the first kind is the function $df\colon \vec{T}^1 \to \mathbb{R}$ given by the formula

$$df(e) := \mathrm{Re}\,f(h_e) - \mathrm{Re}\,f(t_e)$$

for each edge $e \in \vec{T}^1$. The functions $\omega\colon \vec{T}^1 \to \mathbb{R}$ which are differentials of discrete Abelian integrals of the first kind are called *discrete Abelian differentials of the first kind.*

Abelian differentials of the second kind have no discrete counterparts in our setup.

A *discrete Abelian differential of the third kind* is an arbitrary map $\omega\colon \vec{T}^1 \to \mathbb{R}$ satisfying the condition $\omega(-e) = -\omega(e)$. Its *integral* $\int_\gamma \omega$ along an oriented path γ in the graph T^1 is the sum of values on the oriented edges of the path γ. The *residue* $\mathrm{res}_w\,\omega$ of ω at a face $w \in T^2$ is the counterclockwise integral along the boundary of the face. Given a collection of closed paths $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\colon [0;1] \to T^1$ in the graph T^1 forming a standard basis of the fundamental group, define the *real periods* of ω as the integrals along these paths. The real periods of ω may depend on the choice of the paths themselves and not only their homotopy classes.

Similarly, a discrete Abelian differential $\omega\colon \vec{T}^1 \to \mathbb{R}$ of the third kind can be integrated along any chain of oriented triangles such that each two consecutive triangles in the chain induce opposite orientations on their unique common side. For example, if the chain consists of two oriented triangles xyz and zyw then the integral of ω is by definition $c(yz)\omega(yz)$. The

details of the definition for a general chain are left to the reader. The *residue* of ω at a vertex $z \in T^0$ is the number

$$\mathrm{res}_z \, \omega := i \sum_{e \in \vec{T}^1 : h_e = z} c(e) \omega(e).$$

The *imaginary periods* of ω are defined analogously to the real ones as the integrals along certain fixed chains of triangles.

A direct checking shows that $\omega \colon \vec{T}^1 \to \mathbb{R}$ is a discrete Abelian differential of the first kind if and only if its residues at all faces and all vertices vanish. It is also easy to see that for a discrete Abelian differential of the third kind the sum of the residues at all the vertices and all the faces vanishes.

Theorem 6.3 (Fourth existence and uniqueness theorem). *For any two distinct vertices* $z, w \in T^0$ *there is a unique discrete Abelian differential* $d\phi_{z,w} \colon \vec{T}^1 \to \mathbb{R}$ *of the third kind with* $\mathrm{res}_z \, d\phi_{z,w} = -\mathrm{res}_w \, d\phi_{z,w} = i$ *and with all the other residues and both real and imaginary A-periods vanishing.*

Proof. *Uniqueness.* Let ω, ω' be two discrete Abelian differentials of the third kind as required. Then all residues of $\omega - \omega'$ vanish. Hence $\omega - \omega'$ is a discrete Abelian differential of the first kind. Since all A-periods of $\omega - \omega'$ vanish by Corollary 3.8 (2) $\Rightarrow$ (1), it follows that $\omega = \omega'$.

Existence. Let $\omega \colon \vec{T}^1 \to \mathbb{R}$ be an unknown function satisfying $\omega(-e) = -\omega(e)$. Consider its values at all the edges as variables, and the residue ir, where $r \in \mathbb{R}$, at the vertex z as a parameter. For each vertex $y \in T^0$ distinct from w and z write one linear equation $\mathrm{res}_y \, \omega = 0$. Write the linear equation $\mathrm{res}_z \, \omega = ir$. (The equality $\mathrm{res}_w \, \omega = -ir$ is obtained by summation of the written ones and therefore is not written.) For each face $x \in T^2$ but one write one linear equation $\mathrm{res}_x \, \omega = 0$. (A similar equation for the remaining face is obtained by summation of the written ones.) For each $k = 1, \dots, g$ write one linear equation $\int_{\alpha_k} \omega = 0$ meaning that the real A-periods of ω vanish. Write g analogous linear equations meaning that the imaginary A-periods of ω vanish. We get a system of $|T^0| + |T^2| + 2g - 2$ equations in $|T^1|$ variables. By the Euler formula for triangulated surfaces it follows that the number of equations equals the number of variables. By the uniqueness part of the theorem it follows that for $r = 0$ the system has only trivial solution. Thus by the finite-dimensional Fredholm alternative it follows that for $r = 1$ the system has a solution. This solution is the required discrete Abelian differential of the third kind. $\qquad\square$

Lemma 6.4. *For any two distinct vertices* $z, w \in T^0$ *and any discrete meromorphic function* $f = (\mathrm{Re} \, f \colon \tilde{T}^0 \to \mathbb{R}, \mathrm{Im} \, f \colon \tilde{T}^2 \to \mathbb{R})$ *we have*

$$\mathrm{Re} \, f(z) - \mathrm{Re} \, f(w) = \sum_{e \in T^1} d\phi_{z,w}(e) \cdot \mathrm{res}_e \, f.$$

Proof. The lemma follows from the sequence of equalities

$$\sum_{e \in T^1} d\phi_{z,w}(e) \cdot \mathrm{res}_e \, f$$

$$= \sum_{e \in T^1} d\phi_{z,w}(e) \cdot \big(\mathrm{Im} \, f(r_e) - \mathrm{Im} \, f(l_e) + c(e) \, \mathrm{Re} \, f(h_e) - c(e) \, \mathrm{Re} \, f(t_e) \big)$$

$$= \sum_{x \in T^2} \operatorname{Im} f(x) \sum_{e \in \vec{T}^1 : r_e = x} d\phi_{z,w}(e) + \sum_{y \in T^0} \operatorname{Re} f(y) \sum_{e \in \vec{T}^1 : h_e = y} c(e) d\phi_{z,w}(e)$$

$$= - \sum_{x \in T^2} \operatorname{Im} f(x) \cdot \operatorname{res}_x d\phi_{z,w} - i \sum_{y \in T^0} \operatorname{Re} f(y) \cdot \operatorname{res}_y d\phi_{z,w}$$

$$= \operatorname{Re} f(z) - \operatorname{Re} f(w). \qquad \qquad \square$$

Similar results hold, if one takes two distinct *faces* $z, w \in T^2$ instead of two distinct *vertices*.

6.3. Discrete Riemann–Roch theorem. An arbitrary map $T^0 \sqcup T^1 \sqcup T^2 \to \{0, \pm 1\}$ is a *divisor* on T. The *divisor*

$$(f): T^0 \sqcup T^1 \sqcup T^2 \to \{0, \pm 1\}$$

of a discrete meromorphic function $f = (\operatorname{Re} f : T^0 \to \mathbb{R}, \operatorname{Im} f : T^2 \to \mathbb{R})$ is defined by the formulae

$$(f)(z) = \begin{cases} +1 & \text{if } \operatorname{Re} f(z) = 0, \\ 0 & \text{if } \operatorname{Re} f(z) \neq 0, \end{cases} \qquad (f)(e) = \begin{cases} -1 & \text{if } \operatorname{res}_e f \neq 0, \\ 0 & \text{if } \operatorname{res}_e f = 0, \end{cases}$$

$$(f)(w) = \begin{cases} +1 & \text{if } \operatorname{Im} f(w) = 0, \\ 0 & \text{if } \operatorname{Im} f(w) \neq 0, \end{cases}$$

for each $z \in T^0$, $e \in T^1$, $w \in T^2$. The *divisor*

$$(\omega): T^0 \sqcup T^1 \sqcup T^2 \to \{0, \pm 1\}$$

of a discrete Abelian differential $\omega : \vec{T}^1 \to \mathbb{R}$ of the third kind is defined by the formulae

$$(\omega)(z) = \begin{cases} -1 & \text{if } \operatorname{res}_z \omega \neq 0, \\ 0 & \text{if } \operatorname{res}_z \omega = 0, \end{cases} \qquad (\omega)(e) = \begin{cases} +1 & \text{if } \omega(e) = 0, \\ 0 & \text{if } \omega(e) \neq 0, \end{cases}$$

$$(\omega)(w) = \begin{cases} -1 & \text{if } \operatorname{res}_w \omega \neq 0, \\ 0 & \text{if } \operatorname{res}_w \omega = 0. \end{cases}$$

Remark 6.5. As opposed to the continuous theory, on a discrete Riemann surface divisors do *not* form an Abelian group. This reflects the fact that the product of discrete analytic functions is not necessarily a discrete analytic function. Also the notions of multiple zeroes and multiple poles have no discrete counterparts in our setup.

The *degree* $\deg D$ of a divisor $D: T^0 \sqcup T^1 \sqcup T^2 \to \{0, \pm 1\}$ is the sum of all its values. The divisor $D: T^0 \sqcup T^1 \sqcup T^2 \to \{0, \pm 1\}$ is *greater or equal* to a divisor $D': T^0 \sqcup T^1 \sqcup T^2 \to \{0, \pm 1\}$, if for each $z \in T^0 \sqcup T^1 \sqcup T^2$ we have $D(z) \geq D'(z)$. Denote by $l(D)$ the dimension of the space of discrete meromorphic functions on T with divisors greater or equal to the divisor D. Denote by $i(D)$ the dimension of the space of Abelian differentials of the third kind on T with divisors greater or equal to the divisor D. A divisor D is *admissible*, if $D(z) \leq 0$ for each vertex $z \in T^0$, $D(e) \geq 0$ for each edge $e \in T^1$, and $D(w) \leq 0$ for each face $w \in T^2$.

Theorem 6.6 (Riemann–Roch theorem).   *For any admissible divisor D on a triangulated surface of genus g we have*

$$(6.1) \qquad\qquad l(-D) = \deg D - 2g + 2 + i(D).$$

Remark 6.7.   This is a real version of the classical Riemann–Roch theorem:

$$l_S(-D_S) = \deg D_S - g + 1 + i(D_S),$$

where D_S is a divisor on a Riemann surface S and $l_S(-D_S)$, $i_S(D_S)$ are the *complex* dimensions of continuous counterparts of the spaces from the definition of $l(-D)$, $i(D)$. The discrete counterpart D of a divisor D_S has twice larger degree. An informal explanation is that a counterpart of one complex equation describing vanishing of a meromorphic function at a point is two real equations describing vanishing of the real and imaginary parts of a discrete meromorphic function.

Proof of Theorem 6.6.   Let $D: T^0 \sqcup T^1 \sqcup T^2 \to \{0, \pm 1\}$ be an admissible divisor. The *support* of D is the set $\operatorname{supp} D := \{z \in T^0 \sqcup T^1 \sqcup T^2 : D(z) \neq 0\}$. Denote $D_\infty := D|_{T^1}$ and $D_0 := D|_{T^0 \sqcup T^2}$.

Consider a discrete Abelian integral $f = (\operatorname{Re} f: T^0 \to \mathbb{R}, \operatorname{Im} f: T^2 \to \mathbb{R})$ of the second kind with vanishing A-periods whose poles belong to $\operatorname{supp} D_\infty$. There are unique $\lambda_e \in \mathbb{R}$ such that

$$(6.2) \qquad\qquad f = \sum_{e \in \operatorname{supp} D_\infty} \lambda_e \phi_e + \text{const.}$$

Indeed, by Corollary 3.8 for $\lambda_e = \operatorname{res}_e f$ equality (6.2) holds. Conversely, if (6.2) holds then, taking the residues of both sides at each $e \in \operatorname{supp} D_\infty$, we get $\lambda_e = \operatorname{res}_e f$.

By Lemma 6.2 the B-periods of f vanish if and only if for each $k = 1, \ldots, g$ we have

$$(6.3) \qquad \sum_{e \in \operatorname{supp} D_\infty} \lambda_e d\phi_T^k(e) = 0 \quad \text{and} \quad \sum_{e \in \operatorname{supp} D_\infty} \lambda_e d\phi_{T*}^k(e) = 0.$$

Assume that equalities (6.3) hold. Assume that $T^0 \cap \operatorname{supp} D \neq \varnothing$ and $T^2 \cap \operatorname{supp} D \neq \varnothing$ (the opposite case is analogous). Take $w \in T^0 \cap \operatorname{supp} D$ and $y \in T^2 \cap \operatorname{supp} D$. Choose constant in (6.2) so that $f(w) = f(y) = 0$. By Lemma 6.4, $f(\operatorname{supp} D_0) = 0$ if and only if for each $z \in T^0 \cap \operatorname{supp} D_0 - \{w\}$ and each $x \in T^2 \cap \operatorname{supp} D_0 - \{y\}$ we have

$$(6.4) \qquad \sum_{e \in \operatorname{supp} D_\infty} \lambda_e d\phi_{z,w}(e) = 0 \quad \text{and} \quad \sum_{e \in \operatorname{supp} D_\infty} \lambda_e d\phi_{x,y}(e) = 0.$$

Consider (6.3)–(6.4) as a system of linear equations in variables λ_e for all $e \in \operatorname{supp} D_\infty$. By the previous part of the proof the dimension of the space of solutions of this system equals $l(-D)$. Thus the rank of the system (6.3)–(6.4) equals $\deg D_\infty - l(-D)$.

Transposing the matrix of the system (6.3)–(6.4), we get a new system of linear equations. Similarly to the previous part of the proof the dimension of the space of solutions of the new system equals $i(D)$. Thus the rank of the new system equals $\deg D_0 + 2g - 2 - i(D)$. Since transposition does not change the rank, we obtain the required identity

$$\deg D_\infty - l(-D) = \deg D_0 + 2g - 2 - i(D). \qquad\qquad \square$$

Example 6.8. In the particular case of a triangulated flat torus and a divisor supported by one or two edges, the Riemann–Roch theorem (Theorem 6.6) implies the following:

- there are no discrete meromorphic functions with exactly one simple pole;

- there exists a discrete meromorphic function with exactly two simple poles at two given edges if and only if the edges are parallel to each other.

This is proved by a direct computation of the right-hand side of (6.1) using explicit construction of discrete Abelian integrals in Example 2.2.

7. Generalizations

7.1. Quad-surfaces. Let us discuss a generalization of discrete Riemann surfaces considered in this paper above. This generalization has been introduced by Mercat [23]. It has a physical interpretation in terms of alternating-current networks; cf. [30, Section 5.2].

Let Q be a cell decomposition of the surface S with quadrilateral faces. Assume that each vertex of Q is paint either black or white so that the endpoints of each edge have different colors. Assume also that the intersection of any two faces is either empty, or a single vertex, or a single edge. A *chart* on Q is an orientation-preserving homeomorphism of a face and a (not necessarily convex) quadrilateral $z_1 z_2 z_3 z_4$ in the complex plane $\mathbb{C}$. An *atlas* on Q is a collection of charts, one for each face (no agreement of charts for distinct faces is assumed). A cell decomposition Q with a fixed atlas is called a *quad-surface*.

A function $f \colon Q^0 \to \mathbb{C}$ is *discrete analytic on Q*, if the difference quotients along the two diagonals of each face are equal, i.e., for each face $z_1 z_2 z_3 z_4 \in Q^2$ we have

$$(7.1) \qquad \frac{f(z_1) - f(z_3)}{z_1 - z_3} = \frac{f(z_2) - f(z_4)}{z_2 - z_4}.$$

A *multi-valued function on Q with periods*

$$\mathbf{A}_1, \ldots, \mathbf{A}_g, \mathbf{B}_1, \ldots, \mathbf{B}_g, \mathbb{A}_1, \ldots, \mathbb{A}_g, \mathbb{B}_1, \ldots, \mathbb{B}_g \in \mathbb{C}$$

is a function $f \colon \tilde{Q}^0 \to \mathbb{C}$ such that for each $k = 1, \ldots, g$, each black vertex $z \in \tilde{Q}^0$, and each white vertex $w \in \tilde{Q}^0$ we have

$$f(d_{\alpha_k} z) - f(z) = \mathbf{A}_k, \qquad f(d_{\beta_k} z) - f(z) = \mathbf{B}_k,$$
$$f(d_{\alpha_k} w) - f(w) = \mathbb{A}_k, \qquad f(d_{\beta_k} w) - f(w) = \mathbb{B}_k.$$

The numbers $\mathbf{A}_1, \ldots, \mathbf{A}_g, \mathbb{A}_1, \ldots, \mathbb{A}_g$ are called the *A-periods* of the multi-valued function f. A multi-valued discrete analytic function is called a *discrete Abelian integral of the first kind* on Q.

The following theorem is proved analogously to Lemma 3.1.

Theorem 7.1. *For any two multi-valued functions $f, f' \colon \tilde{Q}^0 \to \mathbb{C}$ with periods*

$$\mathbf{A}_1, \ldots, \mathbf{A}_g, \mathbf{B}_1, \ldots, \mathbf{B}_g, \mathbb{A}_1, \ldots, \mathbb{A}_g, \mathbb{B}_1, \ldots, \mathbb{B}_g,$$
$$\mathbf{A}_1', \ldots, \mathbf{A}_g', \mathbf{B}_1', \ldots, \mathbf{B}_g', \mathbb{A}_1', \ldots, \mathbb{A}_g', \mathbb{B}_1', \ldots, \mathbb{B}_g',$$

respectively, we have

$$\sum_{z_1 z_2 z_3 z_4 \in Q^2} \left((f(z_1) - f(z_3))(f'(z_2) - f'(z_4)) - (f(z_2) - f(z_4))(f'(z_1) - f'(z_3)) \right)$$

$$= \sum_{k=1}^{g} (\mathbf{A}_k \mathbb{B}'_k - \mathbb{B}_k \mathbf{A}'_k + \mathbb{A}_k \mathbf{B}'_k - \mathbf{B}_k \mathbb{A}'_k).$$

Analogously to the first existence and uniqueness theorem (Theorem 2.3) one can prove that there is a unique (up to constant) Abelian integral $\phi^l_Q \colon \tilde{Q}^0 \to \mathbb{C}$ of the first kind whose A-periods are given by the formulae $\mathbf{A}_k = \delta_{kl}$ and $\mathbb{A}_k = \delta_{kl}$, where $k = 1, \dots, g$. Let $\mathbf{B}^l_1, \dots, \mathbf{B}^l_g, \mathbb{B}^l_1, \dots, \mathbb{B}^l_g$ be the B-periods of $\phi^l_Q \colon \tilde{Q}^0 \to \mathbb{C}$. The matrix $(\Pi_Q)_{kl} := \frac{1}{2}(\mathbf{B}^l_k + \mathbb{B}^l_k)$ is the *period matrix* of the quad-surface Q.

Problem 7.2. Generalize Theorem 2.5, the convergence theorem for period matrices, to quad-surfaces.

Convergence of period matrices for quad-surfaces cannot be proved similarly to triangulated surfaces. Difficulties appear in the proof of a counterpart of Lemma 4.15 because for a quad-surface the natural "interpolation" $I_Q u$ from [30, Section 5.1] is not continuous.

Problem 7.3. Which complex $g \times g$ matrices arise as period matrices of quad-surfaces of genus g?

This discrete counterpart of the Shottky problem is related to an inverse problem for alternating-current networks and seems to be very difficult.

7.2. Delaunay–Voronoi quadrangulation. Let us show that the discretization of Riemann surfaces studied in Sections 2–6 is a particular case of the one from Section 7.1 in a natural sense. By the *circumcenter* w^* of a face w of a Delaunay triangulation we mean the center of the immersed flat intrinsic disk containing the face w in the interior and the vertices of w in the boundary (such disk exists by the results of [2]). By the *circumradius* we mean a geodesic segment lying in the disk and joining the circumcenter with a point in the boundary.

Theorem 7.4. *Let T be a Delaunay triangulation of a closed polyhedral surface S satisfying condition* (D) *from Section 2.5. For each face of T, draw three circumradii from the circumcenter to the vertices; see Figure* 1. *Then the drawn segments do not have common interior points and thus form a quad-surface Q, in which each face is identified with a planar quadrilateral by an orientation-preserving isometry.*

Proof. Take an oriented edge $e \in \vec{T}^1$. Let $w \in T^2$ be a face containing e. By the results of [2] there is an orientation-preserving and locally isometric immersion $g_w \colon D_w \to S$ of a disk $D_w \subset \mathbb{C}$ such that the face w and the vertices of w are in the interior and the boundary of $g_w D_w$, respectively. Without loss of generality assume that

$$g_{r_e}^{-1} t e = g_{l_e}^{-1} t e \quad \text{and} \quad g_{r_e}^{-1} h e = g_{l_e}^{-1} h e.$$

Denote $z_1 := g_{r_e}^{-1} t_e$ and $z_3 := g_{r_e}^{-1} h_e$. Let $z_2 := g_{l_e}^{-1} l_e^*$ and $z_4 := g_{r_e}^{-1} r_e^*$ be the centers of D_{l_e} and D_{r_e}, respectively. Define a locally isometric immersion $g_e : z_1 z_2 z_3 z_4 \to S$ to be equal to g_{l_e} in the triangle $z_1 z_2 z_3$ and to g_{r_e} in the triangle $z_1 z_3 z_4$. The images $g_e(z_1 z_2 z_3 z_4)$ are the faces of the future quad-surface Q.

Compare the sum of the areas of the quadrilaterals $z_1 z_2 z_3 z_4$ with the area of the union of their images. Clearly, $\mathrm{Area}(z_1 z_2 z_3 z_4) = |h_e t_e|^2 (\cot \alpha_e + \cot \beta_e)/4$ for each edge $e \in T^1$. Since T is Delaunay and satisfies condition (D), it follows that these oriented areas are all positive. Expressing the area of each face of T as an algebraic sum of three terms of the form $\mathrm{Area}(z_1 z_2 z_3)$, we get $\sum_{e \in T^1} \mathrm{Area}(z_1 z_2 z_3 z_4) = \mathrm{Area}(S)$. Finally, $\bigcup_{e \in T^1} g_e(z_1 z_2 z_3 z_4) \supset S$ because $\partial(\bigcup_{e \in T^1} g_e(z_1 z_2 z_3 z_4)) = \emptyset$. This implies that the maps $g_e : z_1 z_2 z_3 z_4 \to S$ neither self-intersect nor overlap with each other, hence the drawn segments do not have common interior points. □

Now it is easy to construct an "isomorphism" between the discretizations on T and Q. For two arbitrary functions $u : T^0 \to \mathbb{R}$ and $v : T^2 \to \mathbb{R}$ define a function $f : Q^0 \to \mathbb{R}$ by the formula $f(z) := u(z)$ for each $z \in T^0$ and $f(w^*) := i v(w)$ for each $w \in T^2$. Then for each face $z_1 z_2 z_3 z_4 \in Q^2$ with the diagonal $e := z_1 z_3 \in T^1$ equations (2.2) and (7.1) are equivalent.

Problem 7.5. Find a direct proof of the existence of the "Delaunay–Voronoi quadrangulation" Q of a polyhedral surface as in Theorem 7.4 not using the existence of a Delaunay triangulation.

7.3. Numerical experiments. Let us present numerical analysis of the above convergence results using a software by S. Tikhomirov. Take the surface S obtained by gluing three unit squares as shown in Figure 4 to the left. This surface has genus $g = 2$, the parameter $\gamma_S = 1/3$, and the period matrix

$$\Pi_S = \frac{i}{3} \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix};$$

see [4, p. 220]. Take the triangulation T_n of the surface S shown in Figure 4 to the right. Computing the discrete harmonic functions $u_{T_n, P} : \tilde{T}_n^0 \to \mathbb{R}$ for all basis vectors $P \in \mathbb{R}^{2g}$ (using classical method of relaxations), then the energy matrix E_{T_n}, and then the discrete period matrix Π_{T_n} (using Lemma 3.14), we get the values in Table 1. The values in the last column are nearly equal, which shows that the theoretical approximation order in Theorem 2.5 agrees with the experimental one. Table 1 corrects wrong values in [4, p. 220, Table, top part].

7.4. Underwater reefs. Let us warn the reader on possible dangers arising on the way of proving convergence results.

Remark 7.6. The proof of convergence of discrete period matrices suggested in the proof of [21, p. 14, Theorem 3.1] is erroneous. It essentially relies on the assumption that the approximation of discrete analytic functions by discrete polynomials constructed in [22] is *uniform* while the approximation is in fact *pointwise* and *not* uniform by [22, Section D].
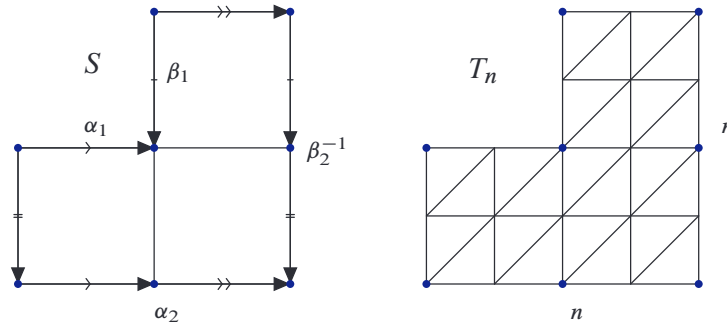
Figure 4. The surface S and its "triangulation" T_n for $n = 4$; see Section 7.3 for details. The paths $\alpha_1, \beta_1, \alpha_2, \beta_2 \colon [0, 1] \to S$ form a standard basis of the first homology (but not homotopy) group.

n	$\|\Pi_{T_n} - \Pi_S\|$	$\|\Pi_{T_n} - \Pi_S\| \cdot h^{-2\gamma_S}$
8	0.611	1.22
16	0.363	1.15
32	0.220	1.11
64	0.136	1.08
128	0.084	1.07
256	0.053	1.06

Table 1. Numerical experiments on approximation of period matrices by their discrete counterparts.

References

[1] *M. Baker* and *S. Norine*, Riemann–Roch and Abel–Jacobi theory on a finite graph, Adv. Math. **215** (2007), no. 2, 766–788.

[2] *A. Bobenko* and *B. Springborn*, A discrete Laplace–Beltrami operator for simplicial surfaces, Discrete Comput. Geom. **38** (2007), 740–756.

[3] *A. I. Bobenko*, Introduction to compact Riemann surfaces, in: Computational approach to Riemann surfaces, Lecture Notes in Math. **2013**, Springer, Berlin (2011), 3–64.

[4] *A. I. Bobenko*, *C. Mercat* and *M. Schmies*, Period matrices of polyhedral surfaces, in: Computational approach to Riemann surfaces, Lecture Notes in Math. **2013**, Springer, Berlin (2011), 213–226.

[5] *A. I. Bobenko*, *C. Mercat* and *Y. B. Suris*, Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green's function, J. reine angew. Math. **583** (2005), 117–161.

[6] *A. I. Bobenko*, *U. Pinkall* and *B. Springborn*, Discrete conformal maps and ideal hyperbolic polyhedra, preprint 2010, http://arxiv.org/abs/1005.2698v1.

[7] *C. Bohle*, *F. Pedit* and *U. Pinkall*, Discrete holomorphic geometry I. Darboux transformations and spectral curves, J. reine angew. Math. **637** (2009), 99–139.

[8] *U. Bücking*, Approximation of conformal mappings by circle patterns, Geom. Dedicata **137** (2008), 163–197.

[9] *D. Chelkak* and *S. Smirnov*, Universality in the 2D Ising model and conformal invariance of fermionic observables, Invent. Math. **189** (2012), 515–580.

[10] *D. Chelkak* and *S. Smirnov*, Discrete complex analysis on isoradial graphs, Adv. Math. **228** (2011), no. 3, 1590–1630.

[11] *R. Courant*, *K. Friedrichs* and *H. Lewy*, Über die partiellen Differentialgleichungen der mathematischen Physik, Math. Ann. **100** (1928), 32–74.

[12] *R. J. Duffin*, Discrete potential theory, Duke Math. J. **20** (1953), 233–251.

[13] *R. J. Duffin*, Distributed and lumped networks, J. Math. Mech. **8** (1959), 793–826.

[14] *I. A. Dynnikov* and *S. P. Novikov*, Geometry of the triangle equation on two-manifolds, Mosc. Math. J. **3** (2003), no. 2, 419–438.

[15] *J. Ferrand*, Fonctions préharmoniques et fonctions préholomorphes, Bull. Sci. Math. (2) **68** (1944), 152–180.

[16] *Z.-X. He* and *O. Schramm*, On the convergence of circle packings to the Riemann map, Invent. Math. **125** (1996), no. 2, 285–305.

[17] *R. Isaacs*, A finite difference function theory, Univ. Nac. Tucuman. Revista A. **2** (1941), 177–201.

[18] *R. Kenyon*, Conformal invariance of domino tiling, Ann. Probab. **28** (2000), no. 2, 759–795.

[19] *J. Lelong-Ferrand*, Représentation conforme et transformations à intégrale de Dirichlet bornée, Gauthier-Villars, Paris 1955.

[20] *C. Mercat*, Discrete Riemann surfaces and the Ising model, Comm. Math. Phys. **218** (2001), no. 1, 177–216.

[21] *C. Mercat*, Discrete period matrices and related topics, preprint 2002, `http://arxiv.org/abs/math-ph/0111043`.

[22] *C. Mercat*, Discrete polynomials and discrete holomorphic approximation, preprint 2002, `http://arxiv.org/abs/math-ph/0206041`.

[23] *C. Mercat*, Discrete Riemann surfaces, in: Handbook of Teichmüller theory, vol. I, IRMA Lect. Math. Theor. Phys. **11**, European Mathematical Society, Zürich (2007), 541–575.

[24] *M. Meyer*, *M. Desbrun*, *P. Schröder* and *A. H. Barr*, Discrete differential-geometry operators for triangulated 2-manifolds, in: Visualization and mathematics III, Springer, Berlin (2003), 35–57.

[25] *A. Pakharev*, *M. Skopenkov* and *A. Ustinov*, Through the resisting net, Mat. Prosv. **18** (2014), 33–65.

[26] *U. Pinkall* and *K. Polthier*, Computing discrete minimal surfaces and their conjugates, Experiment. Math. **2** (1993), no. 1, 15–36.

[27] *I. Rivin*, Euclidean structures on simplicial surfaces and hyperbolic volume, Ann. of Math. (2) **139** (1994), no. 3, 553–580.

[28] *B. Rodin* and *D. Sullivan*, The convergence of circle packings to the Riemann mapping, J. Differential Geom. **26** (1987), no. 2, 349–360.

[29] *O. Schramm*, Circle patterns with the combinatorics of the square grid, Duke Math. J. **86** (1997), 347–389.

[30] *M. Skopenkov*, The boundary value problem for discrete analytic functions, Adv. Math. **240** (2013), 61–87.

[31] *M. Skopenkov*, *M. Prasolov* and *S. Dorichenko*, Dissections of a metal rectangle, Kvant **3** (2011), 10–16.

[32] *M. Skopenkov*, *V. Smykalov* and *A. Ustinov*, Random walks and electric networks, Mat. Prosv. **16** (2012), 25–47.

[33] *S. Smirnov*, Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits, C.R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 3, 239–244.

[34] *S. Smirnov*, Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model, Ann. of Math. (2) **172** (2010), no. 2, 1435–1467.

[35] *B. Springborn*, *P. Schröder* and *U. Pinkall*, Conformal equivalence of triangle meshes, in: ACM SIGGRAPH 2008, ACM, New York (2008), article no. 77.

[36] *K. Stephenson*, Introduction to circle packing. The theory of discrete analytic functions, Cambridge University Press, Cambridge 2005.

[37] *W. P. Thurston*, The geometry and topology of three-manifolds, lecture notes 2002, `http://library.msri.org/books/gt3m/`.

[38] *M. Troyanov*, Les surface euclidiennes à singularités coniques, Enseign. Math. **32** (1986), 79–94.

[39] *S. Wilson*, Conformal cochains, Trans. Amer. Math. Soc. **360** (2008), no. 10, 5247–5264.

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