Problem 1. *Folland 8.6.39*

First assume that μ is not the finite sum of point masses. Then we have that

$$\hat{\mu}(k) = \int_{\mathbb{T}} e^{-2\pi i k x} d\mu(x) \leqslant \frac{1}{k} \int_{\mathbb{T}} e^{2\pi i y} d\mu(y) = \frac{1}{k} < 1.$$

If μ is a linear combination of point masses with the given properties, we have that

$$\hat{\mu}(jm) = \int e^{-2\pi i j m x} d\mu = \sum_{k=1}^{n} a_k \int e^{-2\pi i j m x} d\delta_{\frac{\alpha+(k-1)}{m}} = \sum_{k=1}^{n} a_k e^{-2\pi i j(\alpha+k-1)} = \sum_{k=1}^{n} e^{-2\pi i j \alpha} = e^{-2\pi i j \alpha}$$

Problem 2. *Folland 8.6.40*

Let $\mu \in M(\mathbb{R}^n)$. Let $\{\phi_t\}$ be an approximate identity. Then we have that $\phi_t * \mu \in L^1$ by prop 8.49. Thus for every g, we have that

$$\int g d(\phi_t * \mu) \to \int g d\mu,$$

so for sufficiently small t we can take $|\phi_t * \mu - \mu| < \varepsilon$ in the weak $*$ topology.

Problem 3. *Folland 8.6.41*

It is sufficient to show that Δ is vaguely dense in L^1 since L^1 is vaguely dense in $M(\mathbb{R}^n)$ by 8.6.40. Take $f \in C_c(\mathbb{R}^n)$. Then for any $g \in C_0(\mathbb{R}^n)$, we have that fg is riemann integrable. Therefore for $\varepsilon > 0$, choose a partition $\{R_i\}$ so that

$$\left| \int fg - \sum_{i=1}^{n} \mathrm{Vol}(R_i) \sup_{R_i}(fg) \right| < \varepsilon.$$

Therefore we can take $\mu \in \Delta$ as $\mu = \sum_{i=1}^{n} \mathrm{Vol}(R_i) \sup_{R_i}(fg)\delta_{y_i}$ for some $y_i \in R_i$.

Problem 4. *Folland 8.7.43*

We rewrite our PDE as $(1-\partial^2)u = f$. Applying the fourier transformation and inverting, we get the condition that

$$\hat{u} = \frac{1}{1-\xi^2}\hat{f}.$$

We also verify that

$$\int \frac{1}{2}e^{-|x|} \cdot e^{2\pi i\xi x}dx = \frac{1}{1-\xi^2}.$$

It follows that the solution to the PDE will be given as $f * \phi$. We can verify this by a straightforward computation to see that

$$u - u'' = f * (\phi - \phi'') = f * \delta = f.$$

As long as $f \in L^1$ this solution will make sense, since Fourier inversion is defined.

Problem 5. *Folland 8.7.44*

First we show that $u(x,t) = f * G_t(x)$ is well defined. Taking ε sufficiently small, we have that

$$|u(x,t)| = |f * G_t(x)| = \left|\int f(y)G_t(x-y)dy\right| \leqslant \int |f(y)|G_t(x-y)dy \leqslant \int C_\varepsilon e^{\varepsilon|x^2|}G_t(x-y)dy < \infty.$$

Next we claim that $\lim_{t\to 0} u(x,t) = f(x)$ a.e. Take V as a bounded open set. By Urysohns lemma, take ϕ which is 1 on V. We write $f = \phi f + (1-\phi)f$. Since $G_t(x)$ is an approximate identity, we have that $(1-\phi)f * G_t(x) \to 0$ on V, and $\phi f * G_t(x) \to \phi f(x) = f(x)$ on V. Therefore $\lim_{t\to 0} u(x,t) = f(x)$ a.e. We now check that is satisfies the PDE. We compute that:

$$\begin{aligned}
(\partial_t - \Delta)(f * G_t(x)) &= (\partial_t - \Delta)\left(\int f(y)G_t(x-y)dy\right) \\
&= \int f(y)\left[\partial_t(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}} - \Delta(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x-y|}{4t}}\right]dy \\
&= \int f(y)\left[-\frac{n}{2}4\pi e^{-\frac{-|x-y|}{4t}} + (4\pi t)^{-\frac{n}{2}}\frac{|x-y|^2}{4t^2}e^{-\frac{|x-y|^2}{4t}} + (4\pi t)^{-\frac{n}{2}}\right] \\
&+ \left[-(4\pi t)^{-\frac{n}{2}}\frac{|x-y|^2}{4t^2}e^{-\frac{|x-y|^2}{4t}}(4\pi t)^{-\frac{n}{2}}\frac{n}{2t}e^{-\frac{|x-y|^2}{4t}}\right]dy \\
&= 0.
\end{aligned}$$

Problem 6. *Folland 8.7.45*

Using 8.55, we can write the solution as

$$u(x,t) = \partial_t(f * W_t(x)) + g * W_t(x),$$

where $W_t = \left[\frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}\right]^\vee$. By exercise 15a we know that $W_t = \frac{1}{2}\chi_{[-t,t]}$. So we compute that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\partial_t \int f(x-s)\chi_{[-t,t]}ds\right) + \frac{1}{2}\left(\int g(x-s)\chi_{[-t,t]}ds\right) \\ &= \frac{1}{2}\partial_t \int_{x-t}^{x+t} f(s)ds + \frac{1}{2}\int_{x-t}^{x-t} g(s)ds \\ &= \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds && \text{(by FTC)} \end{aligned}$$

Problem 7. *Folland 9.1.1*

(a) If $f_n \to f$ in L^p norm we have that $\langle f_n, \phi \rangle \to \langle f, \phi \rangle$ for any $\phi \in L^q$. If $f_n \to f$ weakly then for $\phi \in L^q$,

$$\langle g, \tau_x(f_n - f) \rangle = |g * f_n - f| \leqslant \|g\|_1 \|f_n - f\| \to 0.$$

(b) Take $h \in C_c^\infty$ Then, we have that

$$\int |f_n||h| \leqslant \int |g|||h|.$$

Therefore by the DCT, we have that $\int f_n h \to \int fh$ for all $h \in C_c^\infty$

(c) Consider $\{f_n\}$, the growing steeples. Then $f_n \to 0$ pointwise, but if we take $g = 1$ on $[0, 1]$ and 0 outside of some open interval, we have that $\int f_n g > 0$ for all n but $\int fg = 0$.

Problem 8. *Folland 9.1.5*

We verify that f' satisfies $\langle f', \phi \rangle = -\langle f, \phi' \rangle$.

$$\begin{aligned}
\langle f', \phi \rangle &= \int \left(\frac{df}{dx} + \sum^{m} \left(f(x_j+) - f(x_j-) \right) \tau_{x_j} \delta \right) \phi dx \\
&= -\int_{\mathbb{R} \setminus x_1, \ldots x_m} f\phi' dx + \int_{\mathbb{R}} \sum^{m} \left(f(x_j+) - f(x_j-) \right) \tau_{x_j} \delta dx \\
&= -\int_{\mathbb{R} \setminus x_1, \ldots x_m} f\phi' dx - \int \sum^{m} \left(f(x_j+) - f(x_j-) \right) \phi'(x_j) dx \\
&= -\int f\phi' dx
\end{aligned}$$