Q5: We will reason contrapositively. Assume that M is not a compact metric space. Thus there exists a sequence (p_n) with no convergent subsequence. WLOG assume that each p_i is distinct. For each n, find a $r_n > 0$ such that the neighbourhoods $M_{r_n}(p_n)$ are disjoint from one another and no sequence $q_n \in M_{r_n}(p_n)$ converges. We define $f_n(x)$ as:

$$f_n(x) = \frac{r_n - d(x, p_n)}{\frac{r_n}{n} + d(x, p_n)}$$

We set $f(x) = f_n(x)$ if $x \in M_{r_n}(p_n)$ and $f(x) = 0$ else. Observe that f is a composition of continuous maps, hence is continuous. We see that $f(p_n) = \frac{r_n - d(p_n, p_n)}{\frac{r_n}{n} - d(p_n, p_n)} = \frac{r_n}{\frac{r_n}{n}} = n$. We see that $f(p_n)$ is unbounded. Thus we have proved the claim.