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ON THE RANDOM-CLUSTER MODEL

I. INTRODUCTION AND RELATION TO OTHER MODELS

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Synopsis

The random-cluster model is defined as a model for phase transitions and other phenomena in lattice systems, or more generally in systems with a graph structure. The model is characterized by a (probability) measure on a graph and a real parameter κ. By specifying the value of κ to 1, 2, 3, 4, ... it is shown that the model covers the percolation model, the Ising model, the Ashkin–Teller–Potts model with 3, 4, ... states per atom, respectively, and thereby, contains information on graph-colouring problems; in the limit $\kappa \downarrow 0$ it describes linear resistance networks. It is shown that the function which for the random-cluster model plays the role of a partition function, is a generalization of the dichromatic polynomial earlier introduced by Tutte, and related polynomials.

1. *Introduction.* This paper is the first of a sequence of papers devoted to a model for phase transitions which was recently introduced by the authors [1]†. This model, to be called the random-cluster model, is actually a one-parameter family of systems, which includes among its members the spin-$\frac{1}{2}$ Ising model and the percolation model, but also systems representing graph colourings and certain electrical networks.

The member of the family which hitherto has been most thoroughly investigated is the Ising model, introduced by Ising in 1925 as a model for ferromagnetism upon a suggestion by Lenz [2]), and later on also applied to antiferromagnetism, ordering in binary alloys, condensation of a lattice gas and many other phenomena. In 1943 Ashkin and Teller introduced a lattice model in which each atom can be in four states, which was a direct analogue of the two-state Ising model [3]). In 1952 Potts generalized both models to one with an arbitrary number of states per atom [4]).

A less-known member of the family is the percolation model (connectivity model) which was introduced in 1957 by Broadbent and Hammersley [5]) as a

† A preliminary account of this work was given at the Summer School and Seminar on Critical Phenomena at Banff (August 1968).

model for the percolation of a liquid through a porous medium, the spread of a disease through a community and similar phenomena. Its resemblance to the Ising model was first recognized by Hammersley[6]), and various methods developed for the Ising model were translated and applied to it by Sykes and Essam[7]), but a precise relation between the models was not established until 1968 (see ref. 1).

The problem of finding the number of ways in which the vertices of a given graph can be coloured with not more than a given number of colours n so that adjacent vertices have different colours (n-colourings) has a longer tradition than the models mentioned above; in the form of the four-colour conjecture it has a history which goes back to the middle of the 19th century. In his research on the colouring problem, Birkhoff introduced in 1912 the chromatic polynomial, which is an extension of the number of n-colourings from integral values to arbitrary real values of n[8]). It is easy to see that the number of n-colourings of a graph is equal to the degeneracy of the ground state of the "antiferromagnetic" Ashkin–Teller–Potts model. This establishes the relation of the colouring problem to the models discussed above.

Finally, the oldest member of the family is the linear electrical network, investigated since the beginning of the 19th century. It was Kirchhoff who showed in 1847 that a central role in the systematic analysis of these networks is played by what nowadays is called the generating function for spanning trees[9]). In 1954 an important relation between this generating function and the chromatic polynomial was discovered by Tutte[10]). He showed that for a given graph both functions are special cases of a two-variable polynomial which he called the dichromate of the graph and which is now generally referred to as the Tutte polynomial. Another two-variable polynomial, which later on was called the dichromatic polynomial and was shown to be identical, apart from a certain factor and a shift of variables, with the Tutte polynomial, had been introduced by Tutte in 1947[11]). This polynomial was introduced in a different way by Zykov in a study of recursive functions on graphs[12]). The generating function for spanning trees also served as a model for the partition function of a branched polymer without rings[13]).

As a final step in establishing the relations between these models and problems the random-cluster model was introduced[1]), which, as we shall show in detail in this paper, embodies the entire family. The model is defined for an arbitrary graph, and associates with each edge e of the graph a real parameter p_e; if $0 \leq p_e \leq 1$ for all edges e, the model has a probabilistic interpretation. In addition, one real positive parameter κ occurs in the description of the model; it represents in a way the complexity of the model. Different values of κ describe systems with different properties, the various systems discussed above appearing as the cases where κ is an inte-

ger ≥ 0, sometimes combined with special limiting values of the p_e; $\kappa = 0$ corresponds to the electrical network, $\kappa = 1$ to the percolation model, $\kappa = 2$ to the Ising model, $\kappa = n \geq 2$ to the Ashkin–Teller–Potts model with n states per atom and to the n-colouring problem.

After having introduced the random-cluster model we observed that if all parameters p_e are given equal values, the function which plays the central role in the theory of the model reduces, after a simple change of variables, to the dichromatic polynomial. Temperley independently made a similar observation[14]), and, together with Lieb, developed a transfer-matrix approach for the case of a quadratic lattice[15]). Essam also investigated the relation between the aforesaid problems, paying particular attention to cluster expansions[16]).

The aim of the present paper is the precise definition of the random-cluster model for an arbitrary countable graph. After an introductory section 2 on graph-theoretic notions we first define in section 3 the percolation model and a number of characteristic quantities related with it. We then derive a recursion relation for these quantities and a differentiation relation connecting some of them. In section 4 we show that the Ising model, and more generally, the Ashkin–Teller–Potts model, can be formulated entirely in terms of the percolation model. The same procedure is applied to the chromatic polynomial in section 5, and to a certain class of electrical networks in section 6. In all these cases the recursion relation derived in section 3 plays an essential role. It is a special case of a more general recursion relation studied by Zykov[12]). Therefore, the characteristic functions of the various models are recursive functions on graphs in the sense of Zykov. Their interrelation is discussed in section 7, in which the random-cluster model is defined and some of its properties are discussed. Finally, in section 8, the position of the random-cluster model with respect to the above-mentioned systems and problems, and with respect to the branch of combinatorial mathematics to which it belongs is briefly sketched.

2. *Graphs and clusters.* A *graph* G is defined by a set V of *vertices*, a set E of *edges* and an *incidence relation* i between edges and vertices, associating with each edge $e \in E$ an unordered pair $i(e)$ of vertices $v, v' \in V$, the *ends* of e; if $v = v'$ the edge is called a *loop*. The edge e is said to be *incident* with the vertices in $i(e)$ and *vice versa*. If G is the graph thus defined, we write $G = (V, E, i)$. If more than one graph is considered, the vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively.

We shall frequently encounter products of commuting quantities Q_a over all elements a of a set A; we shall denote them by $Q^A \equiv \prod_{a \in A} Q_a$. For convenience we put $Q^\phi \equiv 1$.

The (cardinal) number (of elements) of a set A will be denoted by $|A|$. So the number of vertices is $|V|$, the number of edges $|E|$. A graph G is *finite*

if both $|V(G)|$ and $|E(G)|$ are finite, and *infinite* otherwise. If both $|V(G)|$ and $|E(G)|$ are finite or infinite countable the graph G is *countable*. If the number of edges incident with a vertex v is finite for all $v \in V(G)$ the graph G is *locally finite*.

A *subgraph* of a graph $G = (V, E, i)$ is a graph $G' = (V', E', i')$ such that $V' \subseteq V$, $E' \subseteq E$ and $i'(e) = i(e)$ for all $e \in E'$. Since i' is the restriction of i to the domain E' we shall denote it simply by i. If $G' = (V', E', i)$ and $G'' = (V'', E'', i)$ are subgraphs of a graph $G = (V, E, i)$ then $(V' \cup V'', E' \cup E'', i)$ and $(V' \cap V'', E' \cap E'', i)$ are subgraphs of G, to be called the *union graph* and the *intersection graph* of G' and G'' in G, and to be denoted by $G' \cup G''$ and $G' \cap G''$. If $V' \subseteq V''$ and $E' \subseteq E''$, then G' is a subgraph of G'', and we write $G' \subseteq G''$. In particular for subgraphs G' of G we have $G' \subseteq G$. A *spanning subgraph* or *partial graph* of G is a subgraph with $V' = V$. The spanning subgraph G' with the set of edges E' will be denoted by $G_{E'}$.

A *path* between two vertices v and v' in a graph G is a finite sequence of alternatingly vertices and edges of G: $v = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = v'$ such that $i(e_k) = \{v_{k-1}, v_k\}$ for $k = 1, 2, \ldots n$; it is often represented only by the edges which it contains. Two vertices v and $v' \in V(G)$ are *connected in* G if there is a path in G between them; if not, they are *disconnected*. The relation of connection between vertices is an equivalence relation. A graph G is *connected* if any two of its vertices are connected. A *cluster* (or connected component) of a graph is a maximal non-empty connected subgraph. The smallest cluster consists of one vertex and no edges (*isolated* vertex). The number of clusters of a graph is easily seen to equal the number of equivalence classes under the relation of connection.

A *polygon* in a graph G is the subgraph consisting of the vertices and edges of a path in G, containing at least one edge, between two coinciding vertices and with all vertices distinct except the first and last vertex. The smallest polygon is a vertex with a loop. A set of polygons is called *dependent* if the sum modulo 2 (symmetric difference) of the edge sets of a finite subset of polygons is the empty set. The number of independent polygons of a graph is called the *cyclomatic number* of that graph. A *tree* is a connected non-empty graph having no polygon as a subgraph. The smallest tree consists of one isolated vertex. A *tree in a graph* is a subgraph which is a tree. A *spanning tree* in a graph is a spanning subgraph which is a tree. A *forest* is a non-empty graph having no polygon as a subgraph. A *forest in a graph* is a subgraph which is a forest. A *maximal spanning forest* in a graph is a maximal spanning subgraph which is a forest.

We shall find it convenient to have defined the operations of deleting and contracting edges from a graph. Let $G = (V, E, i)$ be a graph and $E' \subseteq E$ a subset of edges. Then we shall say that the spanning subgraph $G_{E-E'} = (V, E - E', i)$ is obtained from G by *deleting* the edges of E' from G, and we denote it by $\mathcal{D}^{E'}G \equiv (V, E - E', i)$. Let further $\overline{V}$ be the set of

equivalence classes of the set of vertices V under the relation of connection in $G_{E'}$ (in other words, let the vertices of G which are connected in $G_{E'}$ be identified), and let $\bar{\imath}$ be a relation on the edges of $E - E'$ such that if the edge $e \in E - E'$ is incident in G with $i(e) = \{v, v'\}$, then $\bar{\imath}(e) \equiv \{\bar{v}, \bar{v}'\}$, where $\bar{v} \in \bar{V}$ is the class of V containing v. We shall say that the edges of E' are *contracted from* G in order to obtain the graph $\mathscr{C}^{E'}G \equiv (\bar{V}, E - E', \bar{\imath})$.

Obviously, if E', E'' are disjoint subsets of E we have $\mathscr{D}^{E'}(\mathscr{D}^{E''}G) = \mathscr{D}^{E' \cup E''}G = \mathscr{D}^{E''}(\mathscr{D}^{E'}G)$ and $\mathscr{C}^{E'}(\mathscr{D}^{E''}G) = \mathscr{D}^{E''}(\mathscr{C}^{E'}G)$. Moreover we can prove that $\mathscr{C}^{E'}(\mathscr{C}^{E''}G) \sim \mathscr{C}^{E' \cup E''}G \sim \mathscr{C}^{E''}(\mathscr{C}^{E'}G)$ (see lemma 1 in section 3.2), where $G \sim G'$ denotes that the graphs G and G' are isomorphic; we shall write $=$ instead of $\sim$. It follows that we may define $\prod_{e \in E'} \mathscr{D}_e \equiv \mathscr{D}^{E'}$ with $\mathscr{D}_e \equiv \mathscr{D}^{\{e\}}$ and $\prod_{e \in E'} \mathscr{C}_e \equiv \mathscr{C}^{E'}$ with $\mathscr{C}_e \equiv \mathscr{C}^{\{e\}}$, where $\mathscr{D}_e$ is the operation of deleting the edge e from the graph, and $\mathscr{C}_e$ the operation of contracting the edge e from the graph. Observe that by definition $\mathscr{D}^{\phi}G = G$ and $\mathscr{C}^{\phi}G = G$. A graph $\mathscr{C}^{E'}\mathscr{D}^{E''}G$ with $E', E'' \subseteq E(G)$ disjoint is called a *descendant* of G. The vertex $\bar{v} \in \bar{V} = V(\mathscr{C}^{E'}\mathscr{D}^{E''}G)$ is called the vertex of the descendant *associated* with the vertex v of the graph G.

3. *The percolation model.* 3.1. Description of the model. With an arbitrary graph we can associate various mathematical systems which serve as models for certain physical systems. One of these is the (bond) percolation model, introduced in 1957 by Broadbent and Hammersley[5]) as a model for a medium with randomly distributed pores through which a liquid percolates. In this section we shall discuss this model for the case of a graph with non-directed edges. We shall successively introduce events on a graph (*cf.* Rényi[17]) ch. I), local events, random events, probabilities of random events (*cf.* Rényi[17]) ch. II), random variables and expectation values (*cf.* Zaanen[18]) ch. 3). Special care is to be taken in the definition of probabilities and expectation values on infinite graphs because of the occurrence of infinite products and sums and of the values $\pm\infty$ of certain expectation values.

Let $G = (V, E, i)$ be an arbitrary graph, each edge of which can be in two different states, to be denoted by c and d. For each $e \in E$ we consider the two events: "e is in the state c" $\equiv$ "e is a c edge" and "e is in the state d" $\equiv$ "e is a d edge". These events are considered to be each other's negation; we denote them by c_e and d_e, respectively. In the literature on the percolation model the basic elements such as vertices, c edges, d edges, occur under various names. For convenience we give a short translation list:

vertex	edge	c(onstituting)	d(ummy)	this paper
atom	bond	undammed	dammed	Broadbent and Hammersley[5])
site	bond	black	white	Sykes and Essam[7])
vertex	link	active	passive	Harris[19])
atom	bond	occupied	vacant	Fisher[20])

From these events, to be called *edge events*, we construct more detailed events by taking (logical) products; we call these events *product events* and denote them as algebraic products. Thus $c_e d_{e'}$ is the event "e is a c edge and e' is a d edge". Using the symbolic power introduced in section 2 we can denote the general product event by $c^{E'}d^{E''}$ where $E', E'' \subseteq E$; $c_e d_e = 0 \equiv$ "the false event". For completeness we write $c^{\phi} = d^{\phi} = 1 =$ "the true event". The most detailed (smallest) product events are those of the form $c^C d^D$ with $C \cup D = E$ and $C \cap D = \phi$. We call them *elementary events*. The set of all elementary events is called the *event space*, denoted by Ω. The (logical) sum of two events a and a' is denoted $a + a'$. Obviously $c_e + d_e = 1$. From now on we shall assume complete distributivity for logical sums and products. Two events a and a', say, are called *incompatible*, or *disjoint*, if $aa' = 0$.

The events formed by finite sums of finite product events (that is, the events obtained by closing the collection of edge events under finite sums and finite products) are called *local events*. The events formed by closing the collection of local events under countable sums and countable products are called *random events*. The most general events are obtained by closing the collection of random events under arbitrary sums and products. By the assumption of complete distributivity each event can be written uniquely as a sum of elementary events, so that there is a one- to-one correspondence between events and subsets of the event space. This correspondence will be used extensively in the following.

We next define the *probability* $P(a)$ of local events a. Firstly $P(0) = 0$ and $P(1) = 1$, secondly for the edge events $P(c_e) = p_e$ and $P(d_e) = q_e = 1 - p_e$, where $0 \leq p_e \leq 1$. For finite product events $c^{E'}d^{E''}$ with $E' \cap E'' = \phi$ we define $P(c^{E'}d^{E''}) = p^{E'}q^{E''}$, *i.e.* the edge events are considered to be independent. For finite sums of disjoint finite product events we have $P(\sum_{i=1}^{n} a_i) = \sum_{i=1}^{n} P(a_i)$. Using the above-mentioned correspondence between events and subsets of the event space, we see that the probability on local events corresponds to a normed measure on the algebra of the cylinder sets corresponding to the local events.

A *local variable* will be a real function f on the event space Ω which assumes only a finite number of different values f_i such that for each i the sum of all elementary events with $f(c^C d^D; G) = f_i$ is a local event a_i. For brevity we shall often write $f(c^C d^D; G) = f(c^C d^D) = f(C; G) = f(C)$. The *expectation value with respect to* P of a local variable f is defined to be

$$\langle f \rangle = \sum_{i=1}^{n} f_i P(a_i) = \langle f; G, P \rangle.$$

The local variables correspond to the simple functions with respect to the algebra of cylinder sets, the expectation value corresponds to the integral with respect to P of a simple function. The functions obtained by closing

the collection of non-negative local variables under the suprema and infima of countable collections (admitting the value $+\infty$) are called *non-negative random variables*. The difference between two non-negative random variables, not both assuming a value $\neq 0$ at the same time, is called a *random variable*; the non-negative random variables are called its positive and negative part.

Using the extension procedure of measures on semirings together with the Daniell integral scheme, we can, given a probability P on local events with the corresponding expectation value $\langle f \rangle$, extend these uniquely to a probability on random events and an expectation value of random variables (Zaanen[18]) ch. 2, 3), for which we use again the notation $P(a)$ and $\langle f \rangle$. If the expectation value of a random variable is finite, the random variable is said to be *summable*. If not both the expectation values of the positive and negative part of a random variable are $+\infty$, the random variable is said to be *integrable*. In the special case that the graph is finite, the expectation value of a random variable reduces to a sum: $\langle f \rangle = \sum_{C \subseteq E} f(C)\, p^C q^D$; here, and in the following we understand by D the set $E - C$. In general we write $\langle f \rangle = \int_{C \subseteq E} \mathrm{d}P(C)\, f(C)$.

A particular class of (non-negative) random variables is formed by the indicators of random events; the *indicator* of an event a is the function which takes the value 1 if a occurs and the value 0 if a does not occur. For convenience we use the same symbol for the event and for the indicator of that event. So c_e will represent both the edge event c_e and the indicator (of the event) that e is a c edge. We have $\langle a \rangle = P(a)$ for random events a.

A countable graph G together with a probability P as described above we call a *percolation model*, to be denoted (G, P). Notice that the absence of correlations constitutes a most essential feature of the model. The probability P is completely characterized by the mapping p from E into the real interval $[0, 1]$ such that $p(e) = p_e = P(c_e)$. We shall say that the measure P is *generated* by the mapping p.

We shall say that two vertices v and v' of G are *c-connected* in G, if there is a path in G between v and v' such that all edges in that path are c edges. If G_C is the spanning subgraph of G with $E(G_C) = C \equiv$ the set of all c edges, we may equivalently say that v and v' are c-connected in G if they are connected in G_C. Analogously we define a *c cluster*, a *c polygon*, the *c-cyclomatic number etc.*

Finally, we shall list some functions which will be considered in this or in subsequent papers. Most of these are indicators; for brevity we shall omit from their definition the words "the indicator of the event that":

γ $\equiv$ the number of c clusters;

ω $\equiv$ the c-cyclomatic number;

$\gamma_{G'}$ $\equiv$ G' is a c cluster;

$\gamma_{G';v} \equiv G'$ is a c cluster containing the vertex v;

$\gamma^{\mathrm{f}}_{G';v} \equiv G'$ is a finite c cluster containing the vertex v;

$\gamma^{\infty}_{G';v} \equiv G'$ is an infinite c cluster containing the vertex v;

$\gamma^{\mathrm{f}}_{v} \equiv$ there is a finite c cluster containing the vertex v;

$\gamma^{\infty}_{v} \equiv$ there is an infinite c cluster containing the vertex v; obviously $\gamma^{\infty}_{v} = 1 - \gamma^{\mathrm{f}}_{v}$;

$\gamma_{vv'} \equiv$ the vertices v and v' are c-connected $=$ v and v' belong to the same c cluster;

$\gamma_{e} \equiv$ the ends of the edge e are c-connected;

$\gamma_{vV'} \equiv$ the vertex v is c-connected with at least one vertex v' of the set of vertices V'.

For negations of γ indicators we use the symbol δ:

$\delta_{vv'} \equiv$ the vertices v and v' are not c-connected (are c-disconnected); $\delta_{vv'} = 1 - \gamma_{vv'}$;

$\delta_{e} \equiv$ the ends of the edge e are not c-connected (are c-disconnected); $\delta_{e} = 1 - \gamma_{e}$;

$\delta_{vV'} \equiv$ the vertex v is not c-connected with any of the vertices of V'; $\delta_{vV'} = 1 - \gamma_{vV'}$;

$\delta^{\mathrm{f}}_{v} = \gamma^{\infty}_{v}$;

$\delta^{\infty}_{v} = \gamma^{\mathrm{f}}_{v}$.

Obviously, we have for finite graphs

$$\gamma = \sum_{G' \subseteq G} \gamma_{G'}.$$

For finite graphs the above-mentioned functions are random variables. We shall prove in a subsequent paper that for countable graphs they are also random variables.

3.2. Associated random variables on descendants. Recursion theorem. Let f be a random variable defined on the event space of a graph G. Let E' and E'' be disjoint subsets of $E(G)$, then $\mathscr{C}^{E'}\mathscr{D}^{E''}G$ is a descendant of G. Now we associate with f a function $\bar{f}$ on the event space of $\mathscr{C}^{E'}\mathscr{D}^{E''}G$ by the definition (for disjoint sets E' and E'' the union $E' \cup E''$ is alternatively denoted $E' + E''$):

$$\bar{f}(C; \mathscr{C}^{E'}\mathscr{D}^{E''}G) \equiv f(C + E'; G) \quad \text{for all} \quad C \subseteq E(\mathscr{C}^{E'}\mathscr{D}^{E''}G) = E(G) - E' - E''. \tag{3.1}$$

This function $\bar{f}$ is nothing but the section of f determined by the product events $c^{E'}d^{E''}$ (*cf.* Halmos[21]) section 34), and it follows that if f is a random

variable, $\bar{f}$ is a random variable. Moreover, if f is summable, $\bar{f}$ is summable. This procedure uniquely defines $\bar{f}$ on all descendants of G. In particular $\bar{f}(C; G) = f(C; G)$ and $\bar{f}(\phi; \mathscr{C}^C\mathscr{D}^D G) = f(C; G)$.

The state of an edge is a property of the edge alone. Therefore, if the state of all edges of G is given, the relation of c connection is determined on all descendants of G, and also on all subgraphs of G. But this relation of c connection on descendants is ultimately connected with the relation of c connection in G, as is shown by the next lemma.

Lemma 1: Let G be a graph, let E', E'' be disjoint subsets of $E(G)$ and v, v' be vertices of $V(G)$. Let $\mathscr{C}^{E'}G$ be the graph obtained by contracting the edges of E' from G, and $\bar{v}$, $\overline{v'}$ the vertices of $V(\mathscr{C}^{E'}G)$ associated with v, v'. Then v and v' are connected in $G_{E'\cup E''}$ if and only if $\bar{v}$ and $\overline{v'}$ are connected in $(\mathscr{C}^{E'}G)_{E''}$.

Proof. Preliminary remarks: (a) if v and v' are connected in a graph G, there is, by definition, a path in G connecting them, and we may even say that there is a *vertex-disjoint* path in G connecting them, *i.e.* a path in which each vertex occurs only once; (b) all edges of $(\mathscr{C}^{E'}G)_{E''}$ and $G_{E'}$ are edges of $G_{E'\cup E''}$.

If there is a vertex-disjoint path in $G_{E'\cup E''}$ between v and v' we construct from it a path in $(\mathscr{C}^{E'}G)_{E''}$ between $\bar{v}$ and $\overline{v'}$ in the following way: if any edge of E' occurs in the path, we remove it together with the preceding and succeeding vertex and we replace it by the vertex of $\mathscr{C}^{E'}G$ associated with the preceding vertex.

If there is a vertex-disjoint path in $(\mathscr{C}^{E'}G)_{E''}$ between $\bar{v}$ and $\overline{v'}$, we construct from it a path in $G_{E'\cup E''}$ between v and v' in the following way. First, if v is incident in G with the first edge e_1 of the path, let v be the first vertex of the path in $G_{E'\cup E''}$ to be constructed. If v is not incident in G with e_1, there is a vertex v'' of $V(G)$ incident with e_1 in G such that $\overline{v''} = \bar{v}$, so there is, by definition of contraction, a path in $G_{E'}$ between v and v'' and this path will be taken as the first part of the path in $G_{E'\cup E''}$ from v to v' to be constructed. Secondly, e_1 is also incident in G with a vertex $v''' \neq v, v''$, so the second vertex of the path in $(\mathscr{C}^{E'}G)_{E''}$ must be $\overline{v'''}$, and we can repeat the procedure on the second vertex and the second edge of the latter path. By the finiteness of the path in $(\mathscr{C}^{E'}G)_{E''}$ we obtain in this way a path in $G_{E'\cup E''}$ between v and v'. ∎

Corollary : For all descendants of a graph G:

$$\bar{\gamma} = \gamma, \quad \overline{\gamma_{vv'}} = \gamma_{\bar{v}\bar{v}'}, \quad \overline{\delta_{vv'}} = \delta_{\bar{v}\bar{v}'}, \quad \overline{\gamma_{vV'}} = \gamma_{\bar{v}\overline{V'}}, \quad \overline{\delta_{vV}} = \delta_{\bar{v}\overline{V}}.$$

Functions having this property are called $\mathscr{C}\mathscr{D}$-*invariant*. The $\mathscr{C}\mathscr{D}$-invariance property does not hold for all random variables; *e.g.* $\bar{\omega} \neq \omega$, $\overline{\gamma_{G'}} \neq \gamma_{\overline{G'}}$,

$\overline{\gamma_v^f} \neq \gamma_{\bar{v}}^{\bar{f}}$ in general. However, if the number of contractions is finite, then $\overline{\gamma_v^f} = \gamma_{\bar{v}}^{\bar{f}}$, $\overline{\gamma_v^\infty} = \gamma_{\bar{v}}^\infty$. For convenience we shall sometimes drop the association bar over functions and vertices, so we shall write f instead of $\bar{f}$, and v instead of $\bar{v}$, where no confusion can arise.

Using the extension of f to all descendants of G, we shall prove a recursion theorem in which the expectation value $\langle f; G\rangle$ is expressed in terms of the expectation values $\langle \bar{f}; \mathscr{C}_e G\rangle$ and $\langle \bar{f}; \mathscr{D}_e G\rangle$ defined on smaller graphs. This property is especially useful in the case of $\mathscr{C}\mathscr{D}$-invariant random variables.

Theorem 1 (*Recursion theorem*): Let (G, P) be a percolation model and f an integrable random variable. Then for all edges $e \in E(G)$

$$\langle f; G\rangle = p_e \langle \bar{f}; \mathscr{C}_e G\rangle + q_e \langle \bar{f}; \mathscr{D}_e G\rangle. \tag{3.2}$$

Proof. By definition $\langle f\rangle = \int \mathrm{d}P(C)\, f(C)$. By construction P can be regarded as a product measure *i.e.* $P = P^E = P^e \times P^{E-e}$, where the upper index specifies the domain of P. If f is summable we can apply Fubini's theorem, if f is non-negative it is the limit of a non-decreasing sequence of summable random variables and we may again apply Fubini's theorem:

$$\begin{aligned}
&\int_{C\subseteq E} \mathrm{d}P^{E-e}(C)\, f(C) \\
&\quad = \int_{C'\subseteq\{e\}} \mathrm{d}P^e(C') \int_{C''\subseteq E-e} \mathrm{d}P^{E-e}(C'')\, f(C'+C''; G) \\
&\quad = p_e \int_{C\subseteq E-e} \mathrm{d}P^{E-e}(C)\, f(C+e; G) \\
&\quad + q_e \int_{C\subseteq E-e} \mathrm{d}P^{E-e}(C)\, f(C; G).
\end{aligned} \tag{3.3}$$

By the definition of the extension of f to the descendants $\mathscr{C}_e G$ and $\mathscr{D}_e G$ this is equal to

$$\begin{aligned}
&p_e \int_{C\subseteq E-e} \mathrm{d}P(C)\, \bar{f}(C; \mathscr{C}_e G) + q_e \int_{C\subseteq E-e} \mathrm{d}P(C)\, \bar{f}(C; \mathscr{D}_e G) \\
&\quad = p_e \langle \bar{f}; \mathscr{C}_e G\rangle + q_e \langle \bar{f}; \mathscr{D}_e G\rangle.
\end{aligned} \tag{3.4}$$

Finally, if f is integrable, but not necessarily summable or non-negative, then either the positive part f^+ of f or the negative part f^- of f is summable, say f^-, without loss of generality. We may use Fubini's theorem on the positive and negative part of f and collect the terms with p_e and q_e:

$$\begin{aligned}
\langle f; G\rangle &\equiv \langle f^+; G\rangle - \langle f^-; G\rangle \\
&= \{p_e \langle \overline{f^+}; \mathscr{C}_e G\rangle + q_e \langle \overline{f^+}; \mathscr{D}_e G\rangle\} - \{p_e \langle \overline{f^-}; \mathscr{C}_e G\rangle + q_e \langle \overline{f^-}; \mathscr{D}_e G\rangle\} \\
&= p_e\{\langle \overline{f^+}; \mathscr{C}_e G\rangle - \langle \overline{f^-}; \mathscr{C}_e G\rangle\} + q_e\{\langle \overline{f^+}; \mathscr{D}_e G\rangle - \langle \overline{f^-}; \mathscr{D}_e G\rangle\}.
\end{aligned} \tag{3.5}$$

Because $\bar{f}^+ = \bar{f}^+$ and $\bar{f}^- = \bar{f}^-$ this equals

$$p_e\{\langle \bar{f}^+; \mathscr{C}_e G\rangle - \langle \bar{f}^-; \mathscr{C}_e G\rangle\} + q_e\{\langle \bar{f}^+; \mathscr{D}_e G\rangle - \langle \bar{f}^-; \mathscr{D}_e G\rangle\}$$
$$\equiv p_e \langle \bar{f}; \mathscr{C}_e G\rangle + q_e \langle \bar{f}; \mathscr{D}_e G\rangle. \quad \blacksquare \tag{3.6}$$

Since the summability of a function implies the summability of the sections we have the following corollary.

Corollary: The expectation value of a summable random variable f is a linear function of p_e with the finite boundary values:

$$\langle f; G, p_e = 0\rangle = \langle \bar{f}; \mathscr{D}_e G\rangle \qquad \text{and} \qquad \langle f; G, p_e = 1\rangle = \langle \bar{f}; \mathscr{C}_e G\rangle. \tag{3.7}$$

The recursion theorem may be generalized using the extension of functions to general descendants in the case of summable or non-negative random variables:

$$\langle f; G\rangle = \int_{C' \subseteq E'} \mathrm{d}P^{E'}(C') \langle \bar{f}; \mathscr{C}^{C'} \mathscr{D}^{D'} G\rangle,$$
$$\text{for all} \qquad E' \subseteq E(G), \; D' = E' - C'. \tag{3.8}$$

In particular we obtain for $E' = E$:

$$\langle f; G\rangle = \int_{C \subseteq E(G)} \mathrm{d}P^{E}(C) \langle \bar{f}; \mathscr{C}^{C} \mathscr{D}^{D} G\rangle = \int_{C \subseteq E(G)} \mathrm{d}P(C)\, f(C; G). \tag{3.9}$$

3.3. A relation between γ and δ_e. An application of the recursion theorem is the proof of a proposition relating the expectation value of the number of c clusters, γ, with the expectation value of the indicator δ_e of the event that the ends of the edge e are in different c clusters. To prove this we need a lemma which contains the essential feature of the relationship (*cf.* Berge[22] ch. 4, Th. 1). We recall that the number of clusters of a graph G is the number of equivalence classes of the vertices of G under the relation of connection in G: if G' is a cluster of G, $V(G')$ is an equivalence class of $V(G)$ under connection in G.

Lemma 2: Let G be an arbitrary graph. Then for all edges $e \in E(G)$ and all subsets $C \subseteq E(G) - e$

$$\gamma(C; G) = \gamma(C + e; G) + \delta_e(C; G). \tag{3.10}$$

Proof. We recall that $\gamma(C; G)$ is the number of clusters of G_C, $\gamma(C + e; G)$ is the number of clusters of G_{C+e}, and $\delta_e(C; G)$ is the indicator of the event that the ends of e are disconnected in G_C. Evidently, a path in G_C between two vertices is a path in G_{C+e} between the same vertices. Let $v, v' \in V(G)$ belong to a cluster of G_{C+e}, then there is a path in G_{C+e} between them. If the cluster of G_{C+e} containing v, v' does not contain e, this path is a path in G_C, too. Therefore, a cluster of G_{C+e} which does not contain e is a cluster of G_C which does not contain an end of e. There is just one cluster of G_{C+e}

containing e, with vertex set V', say, and the vertices of V' may belong to several clusters of G_C. We shall prove that "several" can be only 1 or 2. Either $\delta_e(C; G) = 0$ or $\delta_e(C; G) = 1$. First, let $\delta_e(C; G) = 0$; then there is a path in G_C between the ends of e. Consequently, a path in G_C between any two vertices of V' can be obtained as follows: by definition, there is a path connecting them in G_{C+e}; if this path contains e, replace e by the path in G_C between the ends of e. Hence, there is just one cluster of G_C containing the vertices V', and so $\gamma(C; G) = \gamma(C + e; G)$. Secondly, let $\delta_e(C; G) = 1$, then there is no path in G_C between the ends of e. In this case, each vertex v of V' is connected in G_C either with one end or with the other end of e. For, there is a path in G_{C+e} to a given end of e; the part from v to the first end of e occurring in it (which may be the given end) is a path in G_C from v to that end, not containing e. If there was also a path in G_C to the other end of e, we could construct a path in G_C between the ends of e, contrary to the hypothesis. Hence, there are just two clusters of G_C, containing together the vertices of V', and so $\gamma(C; G) = \gamma(C + e; G) + 1$. ∎

Proposition 1 (*Differentiation relation*): Let (G, P) be a percolation model such that γ is summable.

Then for all edges $e \in E(G)$, with $p_e = 1 - q_e$:

$$q_e \frac{\partial}{\partial q_e} \langle \gamma \rangle = \langle \delta_e \rangle. \tag{3.11}$$

Proof. By the preceding lemma $\gamma(C; G) = \gamma(C + e; G) + \delta_e(C; G)$ for each $C \subseteq E(G) - e = E(\mathscr{D}_e G) = E(\mathscr{C}_e G)$. By the $\mathscr{C}\mathscr{D}$-invariance of γ and of δ_e this is equivalent to $\gamma(C; \mathscr{D}_e G) = \gamma(C; \mathscr{C}_e G) + \delta_e(C; \mathscr{D}_e G)$, and integrating we get

$$\langle \gamma; \mathscr{D}_e G \rangle = \langle \gamma; \mathscr{C}_e G \rangle + \langle \delta_e; \mathscr{D}_e G \rangle. \tag{3.12}$$

By the recursion theorem $\langle \gamma; G \rangle = p_e \langle \gamma; \mathscr{C}_e G \rangle + q_e \langle \gamma; \mathscr{D}_e G \rangle$, from which it follows that

$$\langle \gamma; G \rangle = \langle \gamma; \mathscr{C}_e G \rangle + q_e \langle \delta_e; \mathscr{D}_e G \rangle. \tag{3.13}$$

By the summability of γ this can be differentiated, giving:

$$q_e \frac{\partial}{\partial q_e} \langle \gamma; G \rangle = q_e \langle \delta_e; \mathscr{D}_e G \rangle. \tag{3.14}$$

The proposition follows because $\langle \delta_e; G \rangle = p_e \langle \delta_e; \mathscr{C}_e G \rangle + q_e \langle \delta_e; \mathscr{D}_e G \rangle$, again by the recursion theorem, and $\langle \delta_e; \mathscr{C}_e G \rangle = 0$ because in $\mathscr{C}_e G$ the ends of the edge e coincide so that $\delta_e = 0$. ∎

Note: obviously, for a finite graph γ is summable for any P.

4. *The Ising model of ferromagnetism.* 4.1. Reformulation of the partition function. We consider a finite spin-$\frac{1}{2}$ Ising system and we represent this system by a finite graph G, so that to each spin there corresponds a vertex of the graph, and to each interaction between a pair of spins there corresponds an edge incident with the corresponding vertices. With each vertex v we associate a spin variable σ_v, which can take the values $+1$ or -1. With each edge e we associate a coupling constant J_e and an edge variable $\sigma_e = \sigma_v\sigma_{v'}$, where v and v' are the ends of e, *i.e.* $\{v, v'\} = i(e)$. The hamiltonian of the system is taken to be

$$H = -\sum_{e\in E(G)} J_e(\sigma_e - 1), \tag{4.1}$$

where $E(G) = E$ is the set of edges of G, and the energy of the ferromagnetic ground state has been chosen to be zero.

Let σ represent a sequence of values of σ_v for all $v \in V(G)$ and $\sum_\sigma$ denote the summation over all possible sequences σ. The canonical partition function Z of the model is defined as

$$Z \equiv \sum_\sigma \exp\{-\beta H(\sigma)\}. \tag{4.2}$$

We shall formulate the partition function in terms of the percolation model[1]). To this end we shall first show that the partition function $Z(G)$ of a system with finite graph G satisfies a recursion relation. For any edge $e \in E(G)$

$$Z(G) = p_e Z(\mathscr{C}_e G) + q_e Z(\mathscr{D}_e G), \tag{4.3}$$

where

$$q_e \equiv \exp(-2\beta J_e), \qquad p_e \equiv 1 - q_e. \tag{4.4}$$

To prove this relation one notices that the sum over all states of G can be split up into a sum over all states with $\sigma_e = +1$, and a sum over all states with $\sigma_e = -1$:

$$\begin{aligned} Z(G) &= \sum_\sigma \exp\{\beta \sum_{e'\in E(G)} J_{e'}(\sigma_{e'} - 1)\} \\ &= \Big(\sum_{\sigma:\,\sigma_e=+1} + \sum_{\sigma:\,\sigma_e=-1}\Big) \exp\{\sum_{e'\in E} \beta J_{e'}(\sigma_{e'} - 1)\} \\ &= 1\cdot \sum_{\sigma:\,\sigma_e=+1} \exp\{\sum_{e'\in E-e} \beta J_{e'}(\sigma_{e'} - 1)\} \\ &\quad + q_e \cdot \sum_{\sigma:\,\sigma_e=-1} \exp\{\sum_{e'\in E-e} \beta J_{e'}(\sigma_{e'} - 1)\}. \end{aligned} \tag{4.5}$$

The spin states of the graph $\mathscr{C}_e G$, with the edge e contracted, are in one-to-one correspondence with the spin states of the graph G which have $\sigma_e = +1$, and $E(\mathscr{C}_e G) = E(G) - e$, so

$$Z(\mathscr{C}_e G) = \sum_{\sigma:\,\sigma_e=+1} \exp\{\sum_{e'\in E-e} \beta J_{e'}(\sigma_{e'} - 1)\}. \tag{4.6}$$

Because $V(\mathscr{D}_e G) = V(G)$, the spin states of the graph $\mathscr{D}_e G$ are just the same as those of the graph G, and the summation over all states may be split up as before. Using $E(\mathscr{D}_e G) = E(G) - e$, we obtain

$$Z(\mathscr{D}_e G) = \sum_{\sigma:\, \sigma_e = +1} \exp\{\sum_{e' \in E-e} \beta J_{e'}(\sigma_{e'} - 1)\} + \sum_{\sigma:\, \sigma_e = -1} \exp\{\sum_{e' \in E-e} \beta J_{e'}(\sigma_{e'} - 1)\}. \tag{4.7}$$

The above-mentioned recursion relation for $Z(G)$ follows by elimination of the partial sums from $Z(G)$, $Z(\mathscr{C}_e G)$ and $Z(\mathscr{D}_e G)$ in eqs. (4.5), (4.6) and (4.7).

The recursion relation for $Z(G)$ implies an interpretation for $Z(\mathscr{C}_e G)$ and $Z(\mathscr{D}_e G)$:

$$Z(\mathscr{C}_e G) = \lim_{J_e \to +\infty} Z(G), \qquad Z(\mathscr{D}_e G) = \lim_{J_e \to 0} Z(G); \tag{4.8}$$

so taking the limit of strong ferromagnetic coupling is equivalent with contracting edges, and taking the limit of weak coupling is equivalent with deleting edges as one would expect.

If we iterate the recursion relation with respect to all edges we finally get

$$Z(G) = \sum_{C \subseteq E} p^C q^D Z(\mathscr{C}^C \mathscr{D}^D G), \tag{4.9}$$

which expresses the partition function of G in terms of the partition function of systems without interaction. Evidently $Z(\mathscr{C}^C \mathscr{D}^D G) = 2^{|V(\mathscr{C}^C \mathscr{D}^D G)|}$. Because the number of vertices of such a graph equals the number of clusters, and the latter, unlike the former, is a $\mathscr{C}\mathscr{D}$-invariant random variable, we can write $|V(\mathscr{C}^C \mathscr{D}^D G)| = \gamma(\phi; \mathscr{C}^C \mathscr{D}^D G) = \gamma(C; G)$, and therefore:

$$Z(G) = \sum_{C \subseteq E} p^C q^D 2^{\gamma(C;G)} = \langle 2^\nu; G, P \rangle, \tag{4.10}$$

in the terminology of the percolation model, with the probability measure P generated by (4.4). The term probability is justified only in the ferromagnetic case, *i.e.* when $J_e \geq 0$ for all edges $e \in E(G)$, because only then $0 \leq q_e \leq 1$. By eq. (4.10) we have rewritten the partition function of an Ising system, which by definition is a sum over spin states, as a percolation-model average, *i.e.* as a sum over edge states.

4.2. Generalized reformulation of the Ising model. In the theory of the Ising model one is not only interested in the partition function but also in canonical averages of spin functions $f(\sigma)$:

$$\langle f \rangle_{\text{can}} \equiv \sum_\sigma f(\sigma) \exp\{-\beta H(\sigma)\} / \sum_\sigma \exp\{-\beta H(\sigma)\}. \tag{4.11}$$

The denominator in the right-hand side of this definition is the partition function Z, which in the previous section has been rewritten as a percolation-model average. The numerator of the right-hand side can also be

rewritten as a percolation-model average, but the method used in the previous section cannot be applied. Instead we shall use an alternative method, which, of course, also applies to Z.

We start by writing $1 = \frac{1}{2}(1 + \sigma_e) + \frac{1}{2}(1 - \sigma_e)$, and thereby

$$\begin{aligned}
&\sum_\sigma f(\sigma) \exp\{-\beta H(\sigma)\} \\
&\quad = \sum_\sigma \tfrac{1}{2}(1 + \sigma_e) f(\sigma) \exp\{-\beta H(\sigma; G)\} \\
&\quad + \sum_\sigma \tfrac{1}{2}(1 - \sigma_e) f(\sigma) \exp\{-\beta H(\sigma; G)\} \\
&\quad = \sum_\sigma \tfrac{1}{2}(1 + \sigma_e) f(\sigma) \exp\{-\beta H(\sigma; \mathscr{D}_e G)\} \\
&\quad + q_e \sum_\sigma \tfrac{1}{2}(1 - \sigma_e) f(\sigma) \exp\{-\beta H(\sigma; \mathscr{D}_e G)\},
\end{aligned} \tag{4.12}$$

by the same argument that was used in section 4.1. Notice however that both terms in the last member are sums defined for the same graph $\mathscr{D}_e G$. Multiplying the first sum by $1 = p_e + q_e$ and collecting the terms with p_e and those with q_e we obtain:

$$\begin{aligned}
&q_e \sum_\sigma f(\sigma) \exp\{-\beta H(\sigma; \mathscr{D}_e G)\} \\
&\quad + p_e \sum_\sigma \tfrac{1}{2}(1 + \sigma_e) f(\sigma) \exp\{-\beta H(\sigma; \mathscr{D}_e G)\}.
\end{aligned} \tag{4.13}$$

Iterating this with respect to all edges we get the expansion

$$\begin{aligned}
&\sum_\sigma f(\sigma) \exp\{-\beta H(\sigma; G)\} \\
&\quad = \sum_{C \subseteq E} p^C q^{E-C} \sum_\sigma \{\tfrac{1}{2}(1 + \sigma)\}^C f(\sigma) \exp\{-\beta H(\sigma; \mathscr{D}^E G)\} \\
&\quad = \sum_{C \subseteq E} p^C q^{E-C} \sum_\sigma \{\tfrac{1}{2}(1 + \sigma)\}^C f(\sigma),
\end{aligned} \tag{4.14}$$

because the hamiltonian of a graph without interactions is zero. If we define a function on the event space of G, thus for every $C \subseteq E(G)$, by

$$\tilde{f}(C; G) \equiv \sum_\sigma \{\tfrac{1}{2}(1 + \sigma)\}^C f(\sigma; G), \tag{4.15}$$

then by the definition of canonical averages we have

$$\langle f \rangle_{\text{can}} = \langle \tilde{f} \rangle / \langle \tilde{1} \rangle. \tag{4.16}$$

In this way we have arrived at a description of canonical averages in terms of percolation-model averages. The factor $\{\frac{1}{2}(1 + \sigma)\}^C$ can be interpreted as the restriction that all spins connected by c edges must be parallel in order to give a non-zero contribution to the summation. So each c-connected component of G acts as one spin. Since the operation $\tilde{}$ is linear and all spin functions $f(\sigma)$ can be linearly expressed in terms of 1, σ_v, $\sigma_v \sigma_{v'}$,

etc., it is sufficient to consider the random variables related with these special functions:

$$\widetilde{1}(C) = \sum_{\sigma} \{\tfrac{1}{2}(1+\sigma)\}^{C} \cdot 1 = 2^{\nu(C)},$$

$$\widetilde{\sigma_v}(C) = \sum_{\sigma} \{\tfrac{1}{2}(1+\sigma)\}^{C} \cdot \sigma_v = 0,$$

$$\widetilde{\sigma_v \sigma_{v'}}(C) = \sum_{\sigma} \{\tfrac{1}{2}(1+\sigma)\}^{C} \cdot \sigma_v \sigma_{v'} = \gamma_{vv'}(C)\, 2^{\nu(C)},$$

and in general for $V' \subseteq V(G)$:

$$\widetilde{\sigma^{V'}}(C) = \sum_{\sigma} \{\tfrac{1}{2}(1+\sigma)\}^{C} \sigma^{V'} = \varepsilon_{V'}(C)\, 2^{\nu(C)}, \tag{4.17}$$

where $\varepsilon_{V'}$ is the indicator that each c cluster contains an even number of the vertices of V'. The random variables (4.17) are all $\mathscr{CG}$-invariant and obviously satisfy the recursion relation. From eqs. (4.15)–(4.17) we find:

$$\begin{aligned} Z(G) &= \langle 2^{\nu}; G\rangle, \\ \langle \sigma_v \rangle_{\mathrm{can}} &= 0, \\ \langle \sigma_v \sigma_{v'} \rangle_{\mathrm{can}} &= \langle \gamma_{vv'} 2^{\nu} \rangle / \langle 2^{\nu} \rangle, \\ \langle \sigma^{V'} \rangle_{\mathrm{can}} &= \langle \varepsilon_{V'} 2^{\nu} \rangle / \langle 2^{\nu} \rangle. \end{aligned} \tag{4.18}$$

If in (4.18) v and v' are the ends of the edge $e \in E(G)$ we have in particular $\langle \sigma_e \rangle_{\mathrm{can}} = \langle \gamma_e 2^{\nu} \rangle / \langle 2^{\nu} \rangle$. On the other hand, one easily sees by differentiating the free energy $F \equiv -\beta^{-1} \ln Z$ with respect to J_e:

$$\begin{aligned} \frac{\partial F}{\partial J_e} &= Z^{-1} \sum_{\sigma} \frac{\partial H(\sigma)}{\partial J_e} \exp\{-\beta H(\sigma)\} \\ &= Z^{-1} \sum_{\sigma} (1 - \sigma_e) \exp\{-\beta H(\sigma)\} = 1 - \langle \sigma_e \rangle_{\mathrm{can}}. \end{aligned} \tag{4.19}$$

Using eqs. (4.4), (4.10) and (4.18) we obtain

$$q_e \frac{\partial}{\partial q_e} 2 \ln \langle 2^{\nu} \rangle = \langle \delta_e 2^{\nu} \rangle / \langle 2^{\nu} \rangle. \tag{4.20}$$

This equation (4.20) is analogous to eq. (3.11), and shows that in the percolation model the function $\langle \gamma \rangle$ plays the same role as the free energy in the Ising model.

Up to now we have chosen the ground-state energy for ferromagnetic interaction zero in order to normalize the measure P: $P(1) = 1$, and to make it possible to interpret P as a probability: $0 \leq P(d_e) = q_e \leq 1$. In the antiferromagnetic case the procedure given above leads by eq. (4.4) to values $q_e > 1$ and $p_e < 0$. It is possible to retain a probabilistic interpretation of the p's and q's by replacing for "antiferromagnetic edges" the factor $(\sigma_e - 1)$ in eq. (4.1) by $(\sigma_e + 1)$ and the factors $\frac{1}{2}(1 + \sigma_e)$ in

eqs. (4.12) *ff.* by $\frac{1}{2}(1 - \sigma_e)$, and *vice versa*, but in that case the function 2^ν in eq. (4.10) is replaced by a more complicated one.

4.3. Ising model in magnetic field. In section 4.1 and section 4.2 we have considered the spin-$\frac{1}{2}$ Ising system without an external magnetic field. In the case where there is a magnetic field, which has the value B_v at the position of the spin v with magnetic moment m_v, the hamiltonian of the system is

$$H = - \sum_{e \in E(G)} J_e(\sigma_e - 1) - \sum_{v \in V(G)} m_v B_v(\sigma_v - 1). \tag{4.21}$$

It is well known (*cf.* Griffiths[23]), Suzuki[24])) that such an external magnetic field can be replaced by one supplementary "ghost spin" which interacts with any spin v with a coupling constant $m_v B_v$. We may therefore replace the hamiltonian of the system with graph G in an external magnetic field, given in (4.21), by the hamiltonian of the system with graph G^o, obtained from G by adding one vertex o and for each vertex v of G an edge incident with v and o:

$$H^o = - \sum_{e \in E(G^o)} J_e(\sigma_e - 1). \tag{4.22}$$

The partition function Z^o calculated from this hamiltonian H^o is twice the partition function Z calculated from H, which has, of course, no influence on the expectation values of spin correlations.

4.4. Ashkin–Teller–Potts model. There is a straightforward generalization of the Ising model in which each atom can be in n different states, where n is an arbitrary number ≥ 2 [3, 4]). In this so-called Ashkin–Teller–Potts model the energy between two interacting spins is taken to be zero if the atoms are in the same state, and equal to a constant if they are in different states. If the system is represented by a graph, just as in the Ising model, and if the above-mentioned constant is denoted by $2J_e$, the hamiltonian can again be written in the form (4.1), where the edge variables σ_e have the values $+1$ and -1, accordingly as the atoms at the ends of e are in the same state or not. Although we cannot introduce simple spin variables for the states of the atoms (*i.e.* σ_e cannot be written as a simple product of two spin variables), we shall still denote the states of the system by σ. We can then apply the same procedure as was used in sections 4.1 and 4.2. We thus get a recursion relation for the partition function $Z = Z_n(G, \beta, J)$ just as in the case of two states per atom [see eq. (4.3)].

$$Z(G) = p_e Z(\mathscr{C}_e G) + q_e Z(\mathscr{D}_e G), \tag{4.3}'$$

where p_e and q_e are again defined by eq. (4.4). But now, after iterating eq. (4.3), we have to substitute in eq. (4.9) $Z(\mathscr{C}^C \mathscr{D}^D G) = n^{\gamma(C;G)}$, because in the graph $\mathscr{C}^C \mathscr{D}^D G$ each of the "atoms" can be independently in n states.

So eq. (4.10) is generalized to

$$Z(G) = \sum_{C \subseteq E} p^C q^D n^{\gamma(C;G)} = \langle n^\gamma; G, P \rangle. \tag{4.23}$$

In case the interaction energy $2J_e$ is positive, we have again $0 \leq P \leq 1$, *i.e.* P is a probability measure.

5. *Graph colourings.* 5.1. Formulation of the problem. An old and well-known problem in graph theory is the following. Let $G = (V, E, i)$ be a finite graph and Q_n a set of n elements called "colours". Each mapping f of the set of vertices V into the set of colours Q_n is called a (vertex)colouring of the graph with at most n colours; colourings with the property that for each edge the ends have different colours are called *n-colourings*. The problem is to study the total number of n-colourings of the graph, which is denoted by $P(n; G)$, as a function of n.

In the special case that G is planar, *i.e.* if there exists a faithful representation of the graph as a map in a plane such that lines representing edges do not cross, the number of n-colourings of G is equal to the number of country colourings of the dual map such that neighbouring countries have different colours.

5.2. Recursion relation. As found by R. M. Foster (unpublished; see note ref. 25) there exists a recursion relation for the total number of n-colourings $P(n; G)$. It is derived by dividing the n-colourings of $\mathscr{D}_e G$ for a given edge e into those which have the property $f(v) \neq f(v')$ where v and v' are the ends of the edge e, and those with $f(v) = f(v')$ (possible because e is not an edge of $\mathscr{D}_e G$). The former ones are just the n-colourings of G, the latter ones just the n-colourings of $\mathscr{C}_e G$, because in the latter graph $v = v'$. So

$$P(n; \mathscr{D}_e G) = P(n; G) + P(n; \mathscr{C}_e G), \tag{5.1}$$

and we arrive at the recursion relation:

$$P(n; G) = P(n; \mathscr{D}_e G) - P(n; \mathscr{C}_e G). \tag{5.2}$$

This may be compared with eq. (3.2) and eq. (4.3). Iterating eq. (5.2) with respect to all edges we get Birkhoff's formula[8])

$$P(n; G) = \sum_{C \subseteq E} (-)^C P(n; \mathscr{C}^C \mathscr{D}^D G) = \sum_{C \subseteq E} (-)^C n^{\gamma(C;G)}, \tag{5.3}$$

because the graph $\mathscr{C}^C \mathscr{D}^D G$ consists of just $\gamma(C; G)$ isolated vertices which can be coloured each one independently with n colours.

We can write, with $p = 1 - q$, $D = E - C$:

$$\sum_{C \subseteq E} (-)^C n^{\gamma(C)} = \lim_{q \to \infty} q^{-E} \sum_{C \subseteq E} p^C q^D n^{\gamma(C)}. \tag{5.4}$$

If we allow also negative values of measures in the percolation model, as we did in the case of the antiferromagnetic Ising model, we can write eq. (5.3), with the aid of (5.4), as

$$P(n; G) = \lim_{q \to \infty} q^{-E(G)} \langle n^{\nu}; G, P \rangle, \qquad p = 1 - q. \tag{5.5}$$

This should be compared with eq. (4.10) and eq. (4.23). For the antiferromagnetic case of the Ashkin–Teller–Potts model the probability $q = \infty$ corresponds to temperature zero, so the number of n-colourings is equal to the degeneracy of the ground state of the antiferromagnetic Ashkin–Teller–Potts model with n states per atom.

6. *Linear resistance networks.* 6.1. Formulation of Kirchhoff's problem. In this section we shall consider finite electrical networks consisting of linear resistors and generators of electromotive force; the electrical character of the network is in no way essential to what follows. We shall represent such an electrical-resistance network by a finite connected graph $G = (V, E, i)$ where V is the set of nodes of the network, E the set of branches of the network (resistors or generators) and i the incidence relation. Kirchhoff[9]) solved in 1847 the problem of finding the currents through the branches of a finite network each of which has a resistance and an electromotive force. By virtue of the superposition principle, however, it is sufficient to solve the case where only one edge e has an electromotive force U_e, say, while every other edge $e' \neq e$ has a resistance $R_{e'}$. Moreover we shall concentrate on Kirchhoff's solution for the electric current I_e through the edge e. This solution can in our notation be written as follows. If U_e and I_e are measured in the same sense, then

$$I_e = -U_e \sum_{T \in \mathcal{T}(\mathscr{D}_e G)} R^{E-T-e} \Big/ \sum_{T \in \mathcal{T}(\mathscr{C}_e G)} R^{E-T-e} \tag{6.1}$$

where for any graph G, $\mathcal{T}(G)$ is the collection of edge sets of all spanning trees in G. To get this in a more usual form we multiply the numerator and denominator of (6.1) by the product S^{E-e} of all conductances $S_{e'} \equiv R_{e'}^{-1}$:

$$I_e = -U_e \sum_{T \in \mathcal{T}(\mathscr{D}_e G)} S^T \Big/ \sum_{T \in \mathcal{T}(\mathscr{C}_e G)} S^T. \tag{6.2}$$

Eq. (6.2) expresses the current I_e as a quotient of the generating functions of spanning trees of the graphs $\mathscr{D}_e G$ and $\mathscr{C}_e G$. The effective resistance $R_e^{\text{eff}} \equiv -U_e/I_e$ "seen by" the electromotive force is, in the special case when e is parallel to some resistance, say e',

$$R_e^{\text{eff}} = \sum_{T \in \mathcal{T}(\mathscr{C}_e G)} S^T \Big/ \sum_{T \in \mathcal{T}(\mathscr{D}_e G)} S^T = \frac{\partial}{\partial S_{e'}} \ln \sum_{T \in \mathcal{T}(\mathscr{D}_e G)} S^T; \; i(e) = i(e'). \tag{6.3}$$

Indeed, the spanning trees of $\mathscr{C}_e G$ can be made to spanning trees of $\mathscr{D}_e G$ by

undoing the identification of the ends of e (which are the ends of e' too) and adding the edge e'. These spanning trees are just the spanning trees of $\mathscr{D}_e G$ containing e'; hence

$$\frac{\partial}{\partial S_{e'}} \sum_{T \in \mathscr{T}(\mathscr{D}_e G)} S^T = \sum_{T \in \mathscr{T}(\mathscr{C}_e G)} S^T \qquad i(e) = i(e'), \tag{6.4}$$

which proves (6.3). A comparison of eq. (6.3) with (4.20) shows that the generating function of spanning trees $\sum_{T \in \mathscr{T}(G)} S^T$ plays a role similar to that of the partition function of the Ising model.

6.2. Reformulation. We may observe that the generating function of spanning trees $Z_T(G, S) = Z_T(G) \equiv \sum_{T \in \mathscr{T}(G)} S^T$ obeys a recursion relation:

$$Z_T(G) = Z_T(\mathscr{D}_e G) + S_e Z_T(\mathscr{C}_e G) \qquad \text{if } e \text{ is not a loop in } G, \tag{6.5}$$

because we can divide the spanning trees of G into two classes according to the occurrence or non-occurrence of the edge e: if a spanning tree of G does not contain e, it is also a spanning tree of $\mathscr{D}_e G$; if it does contain e, $\mathscr{C}_e G_T$ is just a spanning tree of $\mathscr{C}_e G$. This recursion relation is to be compared with those in eqs. (3.2), (4.3) or (5.1).

We can derive an expression for the generating function of spanning trees which closely resembles expression (4.10) for the partition function of the Ising model. To that end we observe that we may characterize the spanning trees G_T of a finite connected graph G by the property $\omega(T; G) = 0$, $\gamma(T; G) = 1$, or equivalently by

$$\omega(T; G) + \gamma(T; G) = 1 = \inf_{C \subseteq E(G)} \{\omega(C; G) + \gamma(C; G)\}. \tag{6.6}$$

The last equality follows from the inequalities $\omega \geq 0$ and $\gamma \geq 1$. This characterization can be used to generate all spanning trees of a given finite connected graph G by a polynomial in x:

$$\begin{aligned} \sum_{T \in \mathscr{T}(G)} S^T &= \sum_{C \subseteq E} S^C \lim_{x \downarrow 0} x^{\{\omega(C; G) + \gamma(C; G) - 1\}} \\ &= \lim_{x \downarrow 0} x^{-1} \sum_{C \subseteq E} S^C x^{\{\omega(C) + \gamma(C)\}}. \end{aligned} \tag{6.7}$$

Now putting $x = \kappa^{\frac{1}{2}}$ for positive κ and using Euler's formula (*cf.* ref. 22 ch. 4 th. 2)

$$\omega(C; G) = |C| - |V(G)| + \gamma(C; G), \tag{6.8}$$

in eq. (6.7) we get, with

$$p \equiv \kappa^{\frac{1}{2}} S (1 + \kappa^{\frac{1}{2}} S)^{-1}, \qquad q \equiv (1 + \kappa^{\frac{1}{2}} S)^{-1}, \tag{6.9}$$

the equality

$$\sum_{T\in\mathcal{T}(G)} S^T = \lim_{\kappa\downarrow 0} \kappa^{-\frac{1}{2}} \sum_{C\subseteq E} S^C \kappa^{\frac{1}{2}\{|C|-|V|+2\gamma(C)\}}$$
$$= \lim_{\kappa\downarrow 0} \kappa^{-\frac{1}{2}\{|V|+1\}} q^{-E} \sum_{C\subseteq E} \kappa^{\gamma(C)} p^C q^D. \tag{6.10}$$

Notice that $p + q = 1$ and $0 \leq p \leq 1$ for $S \geq 0$, so that we may write the generating function of spanning trees in terms of the percolation model as:

$$Z_T(G; S) = \lim_{\kappa\downarrow 0} \kappa^{-\frac{1}{2}\{|V|+1\}} \langle \kappa^\gamma; G, P\rangle. \tag{6.11}$$

Eq. (6.11) is to be compared with eqs. (4.10) and (5.5).

7. *Random-cluster model.* 7.1. Description of the model. After having shown, in the preceding sections, that for a number of models and problems the functions which play a key role in the calculations can be expressed in a uniform way in terms of the percolation model, we shall in this section introduce a new model with a "key function" which includes the above-mentioned key functions as special cases.

Let first $G = (V, E, i)$ be a finite graph and P a normed measure $[P(1)=1]$ on the event space of G, generated by a function p on E. We shall find it convenient to allow negative values for p, *i.e.* we consider P to be a *signed measure*. Let κ be a real number and let γ denote the number of c clusters. We define the *cluster (generating) function* of G by

$$Z(G; p, \kappa) \equiv \sum_{C\subseteq E} p^C q^{E-C} \kappa^{\gamma(C;G)}, \tag{7.1}$$

and a normed signed *measure* by

$$\mu(C) = \mu(C; G, p, \kappa) = p^C q^{E-C} \kappa^{\gamma(C;G)} Z^{-1}(G, p, \kappa), \tag{7.2}$$

for all subsets $C \subseteq E(G)$ and for all (G, p, κ) such that $Z(G, p, \kappa) \neq 0$.

Next, let G be an infinite countable graph, and let G_n be an increasing sequence of finite subgraphs of G such that $\bigcup_{n=1}^{\infty} G_n = G$ and $Z(G_n, p, \kappa) \neq 0$ for almost all n. For any local event a on G there is an $n(a)$ such that for $n \geq n(a)$, a is a local event on G_n. The normal signed measure $\mu(a)$ of a local event on G will be defined by

$$\mu(a) = \lim_{n\to\infty} \mu_n(a), \qquad n \geq n(a), \tag{7.3}$$

where μ_n is the signed measure defined on G_n. A necessary and sufficient condition for this limit to exist is that $\mu(c^C)$ exists for all finite subsets $C \subseteq E(G)$. This signed measure μ may be extended to random events by the procedure mentioned in section 3.1, and the corresponding expectation value to random variables f, to be denoted by $\langle f; G, \mu\rangle = \langle f; G, p, \kappa\rangle = \langle f\rangle$. Notice that, unlike P, the measure μ is not a product measure. The influ-

ence of the c clusters makes the edge events dependent on each other, and thus introduces a global effect in the measure.

A countable graph G together with a normed signed measure μ as described above we call a *random-cluster model*, to be denoted (G, μ) or (G, p, κ).

In order to have in the random-cluster model an analogue of the magnetic field in the Ising model, we shall occasionally add to the graph G a *supplementary vertex* o and to each vertex v of $V(G)$ one *supplementary edge* incident with v and o. The graph thus obtained will be called *supplemented* and denoted $G^o = (V^o, E^o, i)$, with $V^o = V(G) \cup \{o\}$, $E^o = E(G) \cup E_o$, where E_o is the set of supplementary edges. The probability for the edge incident with o and v to be a c edge will be p_{ov}. Furthermore, $1 - p_{ov} = q_{ov}$. The measures generated by $p^o \equiv p \cup p_o$ are denoted P^o and μ^o.

7.2. Some properties of the random-cluster model. In the preceding sections we have shown that in the various systems considered the "key function" obeys a recursion relation and a differentiation relation. We shall now show that in the random-cluster model the cluster function Z obeys a recursion relation and a differentiation relation.

The cluster function Z is defined for finite graphs G, so the number of c clusters lies between 1 and $|V(G)|$. Consequently, Z is finite for any finite graph. Let G be a finite graph and $e \in E(G) \equiv E$; then with $E - e \equiv E'$, $E - C \equiv D$, $E' - C' \equiv D'$ for $C' \subseteq E'$,

$$\begin{aligned} Z(G) &= \sum_{C \subseteq E} p^C q^D \kappa^{\gamma(C;G)} \\ &= \sum_{C' \subseteq E'} p^{C'} q^{D'} \{p_e \kappa^{\gamma(C'+e;G)} + q_e \kappa^{\gamma(C';G)}\} \\ &= \sum_{C' \subseteq E'} p^{C'} q^{D'} \{p_e \kappa^{\gamma(C';\mathscr{C}_e G)} + q_e \kappa^{\gamma(C';\mathscr{D}_e G)}\}, \end{aligned} \tag{7.5}$$

because γ is $\mathscr{C}\mathscr{D}$-invariant, *i.e.* $\gamma(C'; \mathscr{C}_e G) = \gamma(C' + e; G)$ and $\gamma(C'; \mathscr{D}_e G) = \gamma(C'; G)$ for all e and all $C' \subseteq E'$. From eq. (7.5) we obtain the following:

Proposition 2 (*Recursion relation*): Let (G, p, κ) be a finite random-cluster model. Then for all $e \in E(G)$

$$Z(G) = p_e Z(\mathscr{C}_e G) + q_e Z(\mathscr{D}_e G). \tag{7.6}$$

Notice that for $0 \leq p \leq 1$, eq. (7.6) is a special case of theorem 1.

In order to obtain a differentiation relation for Z, we make use of lemma 2 and the same type of argument as was used in the proof of proposition 1. Thus we easily obtain the following:

Proposition 3 (*Differentiation relation*): Let (G, p, κ) be a finite random-cluster model with $p = 1 - q$ and $Z \neq 0$. Then for all edges $e \in E(G)$:

$$q_e \frac{\partial}{\partial q_e} \ln Z(G, p, \kappa) = (1 - \kappa^{-1}) \langle \delta_e; G, p, \kappa \rangle. \tag{7.7}$$

Finally, we mention the following, almost trivial, property of the cluster function.

Proposition 4 (*Product relation*): Let (G, p, κ) be a finite random-cluster model and G', G'' disjoint subgraphs of G. Then

$$Z(G' \cup G'') = Z(G')\, Z(G''). \tag{7.8}$$

Indeed, if $C \subseteq E(G') \cup E(G'')$, $C' \equiv C \cap E'$, $C'' \equiv C \cap E''$, $D' \equiv D \cap E'$ and $D'' \equiv D \cap E''$, then $C = C' \cup C''$, $D = D' \cup D''$, $\gamma(C; G) = \gamma(C'; G') + \gamma(C''; G'')$, and therefore

$$p^C q^D \kappa^{\gamma(C; G' \cup G'')} = p^{C'} q^{D'} \kappa^{\gamma(C'; G')} \cdot p^{C''} q^{D''} \kappa^{\gamma(C''; G'')}. \tag{7.9}$$

Since summation over all $C \subseteq E' \cup E''$ is equivalent to a repeated summation over all $C' \subseteq E'$ and all $C'' \subseteq E''$, eq. (7.8) follows by summing (7.9).

7.3. Special cases of the random-cluster model. In this section we regard only finite graphs. We show that the random-cluster model generalizes the systems discussed in previous sections. The percolation problem is regained by putting $\kappa = 1$:

$$Z(G, p, 1) = 1 \qquad \text{and} \qquad \mu(C; G, p, 1) = p^C q^D, \tag{7.10}$$

i.e. μ reduces to the original measure P.

As shown by eqs. (4.10) and (4.23), together with (4.4), expressing the partition function of the Ising and Ashkin–Teller–Potts model in terms of the percolation model, we have for $\kappa = n \geq 2$:

$$Z_n(G, \beta, J) = Z(G, 1 - \exp(-2\beta J), n). \tag{7.11}$$

As shown by eq. (5.5) for the chromatic polynomial we have further

$$P(n; G) = \lim_{q \to \infty} q^{-E(G)} Z(G, 1 - q, n). \tag{7.12}$$

Finally, the generating function of spanning trees in connected graphs, occurring in the theory of linear resistance networks, can, by eq. (6.9) and (6.11), be written as

$$Z_T(G, S) = \lim_{\kappa \downarrow 0} \kappa^{-\frac{1}{2}(|V(G)|+1)} Z(G, \kappa^{\frac{1}{2}} S/(1 + \kappa^{\frac{1}{2}} S), \kappa). \tag{7.13}$$

The differentiation relations for the various systems follow from the single differentiation relation (7.7) for the random-cluster model. Eqs. (3.11), restricted to finite graphs, (4.20) and (6.3) are obtained from (7.7) using the same procedure as for the cluster function.

To begin with, for $\kappa = 2$ and probabilities (4.4), eq. (7.7) reduces to eq. (4.20). Next we observe that for $\kappa = 1$ both sides of (7.7) vanish. In order to obtain eq. (3.11) we first divide both sides of (7.7) by $(\kappa - 1)$ for $\kappa \neq 1$, and then take the limit $\kappa \to 1$. From the left-hand side of (7.7) we get,

after having interchanged limit and derivative,

$$q_e \frac{\partial}{\partial q_e} \lim_{\kappa \to 1} \frac{\ln Z}{\kappa - 1} = q_e \frac{\partial}{\partial q_e} \lim_{\kappa \to 1} \left(\frac{\partial}{\partial \kappa} \ln Z \Big/ \frac{\partial}{\partial \kappa} (\kappa - 1) \right),$$

by l'Hôpital's rule; the above expression is equal to

$$q_e \frac{\partial}{\partial q_e} \lim_{\kappa \to 1} Z^{-1} \langle \gamma \kappa^{\gamma - 1}; G, P \rangle = q_e \frac{\partial}{\partial q_e} \langle \gamma; G, P \rangle. \tag{7.14}$$

From the right-hand side of (7.7) we get

$$\lim_{\kappa \to 1} \kappa^{-1} \langle \delta_e; G, p, \kappa \rangle = \langle \delta_e; G, p, 1 \rangle = \langle \delta_e; G, P \rangle. \tag{7.15}$$

So eqs. (7.14) and (7.15), together with (7.7), give eq. (3.11).

Finally, we have seen that at least one quantity of linear resistance networks is also obtained in an asymptotic way, namely by putting $p = \kappa^{\frac{1}{2}} S/(1 + \kappa^{\frac{1}{2}} S)$, [see eq. (6.9)], and taking the limit $\kappa \downarrow 0$. Because for $\kappa \downarrow 0$ both sides of (7.7) tend to $-\infty$, they must first be multiplied by $-\kappa^{\frac{1}{2}}$, as we shall show, in order to obtain eq. (6.3). The procedure used to derive eq. (6.11) is now applied in the reversed direction to $Z = \langle \kappa^{\gamma}; G, P \rangle$ and $\langle \delta_e; G, p, \kappa \rangle = \langle \delta_e \kappa^{\gamma}; G, P \rangle / Z$. We then obtain

$$\langle \kappa^{\gamma}; G, P(\kappa, S) \rangle = \kappa^{\frac{1}{2}(|V|+1)} (1 + \kappa^{\frac{1}{2}} S)^{-E} \sum_{C \subseteq E} S^C \kappa^{\frac{1}{2}\{\omega(C) + \gamma(C) - 1\}}, \tag{7.16}$$

$$\langle \delta_e \kappa^{\gamma}; G, P(\kappa, S) \rangle = \kappa^{\frac{1}{2}(|V|+2)} (1 + \kappa^{\frac{1}{2}} S)^{-E} \sum_{C \subseteq E} S^C \delta_e(C)\, \kappa^{\frac{1}{2}\{\omega(C) + \gamma(C) - 2\}}. \tag{7.17}$$

The reason for splitting off the factor $\kappa^{\frac{1}{2}(|V|+2)}$ in eq. (7.17) is that we want the power of κ in the summation to be non-negative. Under the constraint $\delta_e(C) = 1$, which gives the non-vanishing terms, the ends of e are in different c clusters, so there are at least two c clusters. Since $\omega \geq 0$, it follows that $\omega(C) + \gamma(C) \geq 2$ in this case. The infimum is reached for those sets $T \subseteq E$ for which $\omega(T) = 0$ and $\gamma(T) = 2$. Evidently, $T + e$ is just the edge set of a spanning tree of G containing e. By the considerations leading to (6.5) we may equivalently say that $T \in \mathsf{T}(\mathscr{C}_e G)$, *i.e.*, on the analogy of eq. (6.6), we may define a spanning tree of $\mathscr{C}_e G$ with edge set T by

$$\omega(T; G) + \gamma(T, G) = 2 = \inf_{C:\, \delta_e(C) = 1} \{\omega(C) + \gamma(C)\}, \qquad T \in \mathsf{T}(\mathscr{C}_e G). \tag{7.18}$$

For the left-hand side of (7.7) we get, after multiplying by $-\kappa^{\frac{1}{2}}$ and changing the differentiation variable to S_e [eq. (6.9)],

$$\lim_{\kappa \downarrow 0} -\kappa^{\frac{1}{2}} \frac{(1 + \kappa^{\frac{1}{2}} S_e)}{-\kappa^{\frac{1}{2}}} \frac{\partial}{\partial S_e} \ln Z = \lim_{\kappa \downarrow 0} \frac{\partial}{\partial S_e} \ln Z$$

$$= \lim_{\kappa \downarrow 0} \frac{-\kappa^{\frac{1}{2}}}{(1 + \kappa^{\frac{1}{2}} S_e)}$$

$$+ \lim_{\kappa \downarrow 0} \frac{\partial}{\partial S_e} \ln \sum_{C \subseteq E} S^C \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}}, \qquad \text{by (7.16)},$$

$$= \frac{\partial}{\partial S_e} \ln \sum_{C \subseteq E} S^C \lim_{\kappa \downarrow 0} \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}}$$

$$= \frac{\partial}{\partial S_e} \ln Z_T(G, S), \qquad \text{by (6.6)}. \tag{7.19}$$

For the right-hand side of (7.7) we get after multiplying by $-\kappa^{\frac{1}{2}}$ and substituting (7.16) and (7.17)

$$\lim_{\kappa \downarrow 0} -\kappa^{\frac{1}{2}}(1-\kappa^{-1})\,\kappa^{\frac{1}{2}} \sum_{C \subseteq E} S^C \delta_e(C)\, \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-2\}} \Big/ \sum_{C \subseteq E} S^C \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}}$$

$$= \sum_{C \subseteq E} S^C \lim_{\kappa \downarrow 0} \delta_e(C)\, \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-2\}} \Big/ \sum_{C \subseteq E} S^C \lim_{\kappa \downarrow 0} \kappa^{\frac{1}{2}\{\omega(C)+\gamma(C)-1\}}$$

$$= Z_T(\mathscr{C}_e G, S)/Z_T(G, S), \tag{7.20}$$

by (6.6) and (7.18). From (7.7), (7.19) and (7.20) we obtain

$$\frac{\partial}{\partial S_e} \ln Z_T(G, S) = Z_T(\mathscr{C}_e G, S)/Z_T(G, S). \tag{7.21}$$

In order to obtain eq. (6.3), we have to apply (7.21) to an edge e' parallel to the given edge e, and to the graph $\mathscr{D}_e G$. Notice that $\mathscr{C}_{e'}\mathscr{D}_e G$ and $\mathscr{C}_e G$ differ only by a loop.

7.4. The cluster-generating function and other graph polynomials. In this section we shall derive relations between the cluster-generating function and other polynomials which in the course of time have been introduced by several authors, and which will be defined explicitly below. First we mention the two-variable polynomial Q, introduced for arbitrary graphs by Tutte in 1947[11]). In 1954, Tutte[10]), in a study of graph-colouring problems, introduced another two-variable polynomial for finite graphs, the dichromate χ. It was not until 1967 that it was explicitly stated, again by Tutte[27]), that the polynomials Q and χ are identical apart from a factor and a shift of variables. A somewhat different line of research was pursued by Zykov[12]), who in 1962, in a study of recursive functions on graphs, introduced a two-variable polynomial ψ, and showed that the four-variable polynomial ψ', also introduced by him, the two-variable polynomial P, which is identical, up to a factor, to Tutte's Q, and the dichromate χ are all particular cases of the polynomial ψ apart from factors and changes of variables. Finally, we mention the two-variable polynomial ρ, introduced by Crapo[28]) for finite pregeometries (matroids), which he showed to be identical in the above sense to the generalization of the dichromate to matroids. We shall show that the cluster-generating

function Z, which is a $(|E| + 1)$-variable polynomial, is a generalization of the above-mentioned polynomials, in the sense that different edges can have different "weights".

Before showing the connection between Z and the polynomials Q, χ, ψ, ψ' and ρ, we shall introduce a slightly generalized polynomial Z' and a corresponding measure μ'. Let $G = (V, E, i)$ be a finite graph, x and y be two mappings of E into the set of real numbers and let ξ, η be two real numbers. Then we define $(D \equiv E - C)$

$$Z'(G, x, y, \xi, \eta) \equiv \sum_{C \subseteq E} x^C y^D \xi^{\gamma(C;G)-\gamma(E;G)} \eta^{\omega(C;G)}, \tag{7.22}$$

$$\mu'(C; G, x, y, \xi, \eta) \equiv x^C y^D \xi^{\gamma(C;G)-\gamma(E;G)} \eta^{\omega(C;G)} / Z'(G, x, y, \xi, \eta). \tag{7.23}$$

This polynomial Z' and measure μ' are related to Z and μ through Euler's formula

$$|V(G)| + \omega(C; G) = |C| + \gamma(C; G). \tag{7.24}$$

Eliminating ω from Z' and μ' by (7.24), we deduce

$$Z'(G, x, y, \xi, \eta) = (x\eta + y)^E \xi^{-\gamma(E;G)} \eta^{-|V(G)|} Z(G, x\eta/(x\eta + y), \xi\eta), \tag{7.25}$$

$$\mu'(C; G, x, y, \xi, \eta) = \mu(C; G, x\eta/(x\eta + y), \xi\eta). \tag{7.26}$$

One observes that apart from factors there is no loss of generality in going from the $2(|E| + 1)$-variable polynomial Z' to the $(|E| + 1)$-variable polynomial Z.

The polynomial Q, now called the *dichromatic polynomial*, is defined for finite graphs by $Q(G, t, z) \equiv \sum_{C \subseteq E} t^{\gamma(C;G)} z^{\omega(C;G)}$. We have immediately:

$$Q(G, t, z) = (z + 1)^{|E(G)|} z^{-|V(G)|} Z(G, z/(z + 1), tz). \tag{7.27}$$

The original definition of the dichromate χ, now called the *Tutte polynomial*, is rather complicated and will be omitted here. As Zykov has shown[12]), the Tutte polynomial is uniquely determined by the following properties, which were deduced by Tutte. If all edges of G are loops or isthmi, $\chi(G, x, y) = x^{|E|-\omega(E;G)} y^{\omega(E;G)}$. If G has an edge, e say, which is neither a loop nor an isthmus, χ satisfies the recursion relation $\chi(G) = \chi(\mathscr{C}_e G) + \chi(\mathscr{D}_e G)$. One readily verifies that $Z'(G, 1, 1, x - 1, y - 1)$ obeys these conditions, so

$$\chi(G, x, y) = y^{|E(G)|}(x - 1)^{-\gamma(E;G)} \times (y - 1)^{-|V(G)|} Z(G, 1 - y^{-1}, (x - 1)(y - 1)). \tag{7.28}$$

The polynomials ψ and ψ' introduced by Zykov are defined in the following way for finite graphs. If all edges of G are loops, $\psi(G, \alpha, \beta) = 1$, $\psi'(G, \alpha, \beta, u, v) = u^{|V(G)|} v^{|E(G)|}$. If G has an edge, e say, which is not a loop, ψ and ψ' are defined recursively by the recursion relations $\psi(G) =$

$= \alpha\psi(\mathscr{D}_e G) + \beta\psi(\mathscr{C}_e G)$, $\psi'(G) = \alpha\psi'(\mathscr{D}_e G) + \beta\psi'(\mathscr{C}_e G)$. One readily verifies that the functions $\{\beta/(1-\alpha)\}^{|V(G)|} Z(G, 1-\alpha, (1-\alpha)/\beta)$ and $u^{\gamma(E;G)} \times$ $\times Z'(G, \beta, \alpha, u, (v-\alpha)/\beta)$ obey these conditions for ψ and ψ', respectively, so

$$\psi(G, \alpha, \beta) = \left(\frac{\beta}{1-\alpha}\right)^{|V(G)|} Z\left(G, 1-\alpha, \frac{1-\alpha}{\beta}\right) \tag{7.29}$$

$$\psi'(G, \alpha, \beta, u, v) = \left(\frac{\beta}{v-\alpha}\right)^{|V(G)|} v^{|E(G)|} Z\left(G, \frac{v-\alpha}{v}, \frac{u(v-\alpha)}{\beta}\right). \tag{7.30}$$

The rank generating function ρ was defined for matroids by Crapo. A *matroid*, or (*combinatorial*) *pregeometry*, is the Boolean algebra of all subsets of a finite set X together with an integral-valued rank function r on this algebra, satisfying the following relations: $r(\phi) = 0$; for all $X' \subseteq X$ and all $x, x' \in X$ $r(X' \cup x \cup x') - r(X' \cup x) - r(X' \cup x') + r(X') = 0$ or -1. If X is the edge set $E(G)$ of a finite graph G, and if the rank function is the function $|C| - \omega(C; G) = |V(G)| - \gamma(C; G)$, ρ has the following form:

$$\rho(G, \xi, \eta) \equiv \sum_{C \subseteq E} \xi^{\gamma(C;G) - \gamma(E;G)} \eta^{\omega(C;G)}.$$

So,

$$\rho(G, \xi, \eta) = (\eta + 1)^{|E(G)|} \xi^{-\gamma(E;G)} \eta^{-|V(G)|} Z(G, \eta/(\eta+1), \xi\eta). \tag{7.31}$$

8. *Discussion.* From the foregoing analysis one can draw two main conclusions.

a) A number of seemingly unrelated physical systems, such as the linear resistance network, the percolation model and the Ising model, can be considered as special cases of one single model, the random-cluster model. This model has the advantage over the Ashkin–Teller–Potts model, which constitutes another generalization of the Ising model, that the parameter κ characterizing the various special cases can take all real values, including the remaining non-negative integral values 0 and 1. This fact enables one to study the properties of the model as a function of a continuously varying additional parameter. If, *e.g.*, the system exhibits a phase transition in the thermodynamic limit, one can investigate how its critical behaviour changes with κ.

In this connection it might be of interest to study those quantities which form the generalization of the thermodynamic quantities and spin correlations of the Ising model. As such we mention the generalized free energy

$$F(G^o, p^o, \kappa) \equiv \ln Z(G^o, p^o, \kappa), \tag{8.1}$$

and its derivatives, $q_e(\partial/\partial q_e)\, F(G^o)$, $q_{e'}(\partial/\partial q_{e'})\, q_e(\partial/\partial q_e)\, F(G^o)$ *etc.*, the first

of which can, by proposition 2, be written as

$$q_e \frac{\partial}{\partial q_e} F(G^0) = (1 - \kappa^{-1}) \langle \delta_e; G^0, \mu^0 \rangle, \tag{8.2}$$

and in addition the quantities $\langle \gamma_{vv'} \rangle$, $\langle \gamma_{vv'v''} \rangle$, *etc.*

Of particular interest are the (generalized) local magnetization M and local susceptibility χ:

$$M(G^0, v) \equiv (1 - \kappa^{-1}) - q_{ov} \frac{\partial}{\partial q_{ov}} F(G^0) = (1 - \kappa^{-1}) \langle \gamma_{ov} \rangle, \tag{8.3}$$

$$\chi(G^0, v, v') \equiv q_{ov} \frac{\partial}{\partial q_{ov}} q_{ov'} \frac{\partial}{\partial q_{ov'}} F(G^0), \tag{8.4}$$

and the corresponding "global" quantities, obtained by summing over all vertices and vertex pairs, respectively.

b) The cluster-generating function $Z(G, p, \kappa)$ which takes a central place in the theory of the random-cluster model, is a straightforward generalization of a polynomial in two variables, the dichromatic polynomial, which is playing a more and more important role in the theory of graphs and its extension, the theory of matroids. The dichromatic polynomial of a given graph G is the generating function for the *number* of spanning graphs of G with a given number of clusters and a given cyclomatic number; the cluster-generating function generates all *individual* spanning subgraphs in such a form that the number of clusters and the cyclomatic number can be read off immediately.

The dichromatic polynomial has recently been put in a wider mathematical perspective by Brylawski[29]) in an interesting study on what he calls the Tutte–Grothendieck ring. The main idea of Brylawski's work goes back to Tutte's paper[11]) in which he introduced the polynomial Q. One might expect that a combination of the ideas developed in this branch of mathematics, which deals almost exclusively with finite sets, with those developed in the theoretical treatment of the thermodynamic limit in translationally invariant systems will lead to a deeper understanding of phase transitions.

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