

Q5a:

Suppose that A is a unbounded set of content 0. Then for any $\varepsilon > 0$ we can find a finite collection of closed rectangles $U_1 \dots U_n$ such that $\bigcup_{i=1}^n U_i \supset A$ and $\sum_{i=1}^n \text{vol}(U_i) < \varepsilon$. Since we are dealing with a finite collection of closed rectangles, we can choose a closed rectangle W such that each $U_i \subset W$. This is the same as saying $A \subset \bigcup_{i=1}^n U_i \subset W$. Therefore A is bounded, a contradiction.

5b:

Consider the set of all integers as a subset of \mathbb{R} . This is closed, since $\mathbb{R} \setminus \mathbb{Z} = \bigcup_{i \in \mathbb{Z}} (i, i+1)$ which is the arbitrary union of open sets hence is open. Since \mathbb{Z} is countable, we can rewrite \mathbb{Z} as the countable union of each of its elements. Since a point is of measure 0, Spivak Theorem 3-4 says a countable union of measure 0 sets is of measure 0 as well. Since \mathbb{Z} is unbounded, we have by 5a that it can not be of content 0. Thus \mathbb{Z} viewed as a subset of \mathbb{R} is closed and of measure 0 but not content 0.