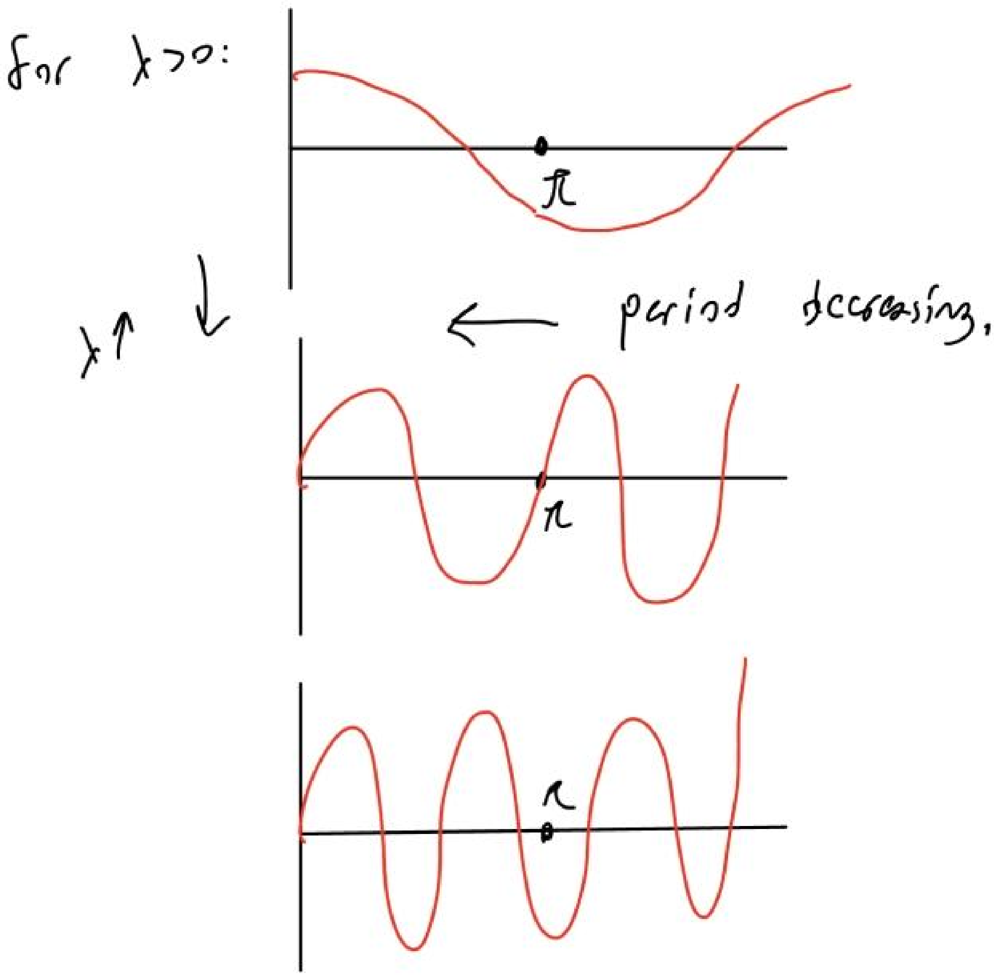
Problem 1.

(a) If $\lambda = 0$ the PDE is solved by $X(x) = ax + b$. $X(0) = 0$ implies that $b = 0$, and the robin condition tells us that $a + a\pi = 0$ so $a = 0$ Therefore the only function satisfying this condition is $X(x) = 0$. So $\lambda = 0$ is not an eigenvalue.

(b) Consider the following image:



Observe that as λ increases, the period decreases and the graph will always attain 0 at π for infinitely many λ's.

(c) We first determine the form of the solution to this PDE. We have that

$$X(x) = a \sin \sqrt{\lambda}x + b \cos \sqrt{\lambda}x.$$

Since $X(0) = 0$, $b = 0$. Furthermore the robin conditions tells us that

$$\sqrt{\lambda} \cos \sqrt{\lambda}\pi + \sin \sqrt{\lambda}\pi = 0.$$

We wish to solve for a nonzero lower bound on λ. Consider the function

$$f(x) = x \cos \pi x + \sin \pi x.$$

Verifying with graphing tools, we have that for $a_1 \approx 0.405$ $f'(a_1) = 0$ and $f(a_1) > 0$. Furthermore at $a_2 \approx 1.258$, $f'(a_2) = 0$ and $f(a_2) < 0$. So at some $a^* \in (a_1, a_2)$, $f(a^*) = 0$. Furthermore we have that $f(0) = 0$ and f' is increasing on $(0, a_1)$. Therefore a_1^2 is the lower bound on λ.

(d) Suppose there was an eigenfunction for $\lambda < 0$. The solution takes the form of $X(x) = a \sinh \sqrt{-\lambda} x$. The robin boundary condition tells us that $\sqrt{-\lambda} \cosh \sqrt{-\lambda}\pi + \sin \sqrt{-\lambda}\pi = 0$. However this is only 0 at $\lambda = 0$ since this is an increasing function. Thus no eigenvalues for $\lambda < 0$ exist.

Problem 2.

We write $u(t,x) = X(x)T(t)$. The PDE gives us that

$$X(x)T''(t) = c^2X''(x)T(t) - rX(x)T'(t).$$

Rearranging and dividing by $c^2u(x,t)$, we get:

$$\frac{T''(t) + rT'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}.$$

This must be equal to a constant, say $-\lambda$, since one side is independent of t and the other is independent of x. We first solve for $X(x)$. The general solution of $X''(x) = -\lambda X(x)$ is $X(x) = a\sin\sqrt{\lambda}x + b\cos\sqrt{\lambda}x$. The boundary condition at $x = 0$ tells us that $b = 0$, and at $x = l$ we have that

$$0 = \sin\sqrt{\lambda}l = 0 \implies \lambda = \frac{n^2\pi^2}{l}.$$

Therefore $X(x) = a_n \sin\frac{n\pi}{l}x$. We now solve for $T(t)$. We solve the following ODE:

$$T'' + rT' + \lambda c^2 T = 0.$$

We guess a solution is of the form $T(t) = e^{kt}$. Applying the ODE to this, we see that it must satisfy

$$e^{kt}(k^2 + rk + \lambda c^2) = 0.$$

Solving for k, we get that

$$k = \frac{r}{2} \pm ik_n,$$

where $k_n = \frac{1}{2}\sqrt{\frac{4c^2n^2\pi^2}{l^2} - r^2}$. Note that $k_n \in \mathbb{R}$ since $\frac{4c^2n^2\pi^2}{l^2} - r^2 = \left(\frac{2cn\pi}{l} - r\right)\left(\frac{2cn\pi}{l} + r\right) > 0$. Thus the general solution of T is

$$T(t) = c_n e^{t\left(\frac{r}{2}+ik_n\right)} + d_n e^{t\left(\frac{r}{2}-ik_n\right)}.$$

Therefore the series expansion of u is:

$$u(x,t) = \sum_{n=1}^{\infty}\left(c_n e^{t\left(\frac{r}{2}+ik_n\right)} + d_n e^{t\left(\frac{r}{2}-ik_n\right)}\right)\sin\frac{n\pi}{l}x$$

Problem 3.

Write $u(x,t) = X(x)T(t)$. The PDE gives us that

$$X(x)T''(t) = c^2X''(x)T(t).$$

Dividing by c^2u gets us:

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}.$$

Since both sides are equal for all t, x we have that they must be equal to some constant $-\lambda$. Therefore we can write $X(x) = a\sin\sqrt{\lambda}x + b\cos\sqrt{\lambda}x$. We now use the mixed boundary conditions. Since $X'(0) = 0$, we have that $a = 0$, so $X(x) = b\cos\sqrt{\lambda}x$. Since $X(l) = 0$, $\sqrt{\lambda}l = n + \frac{\pi}{2}$, so set $\lambda_n = \frac{(2n+\pi)^2}{4l^2}$. We have that $X(x) = a_n\cos\sqrt{\lambda_n}x$. Similarly, we have that $T(t) = c_n\sin\frac{\sqrt{\lambda_n}}{c}t + d_n\cos\frac{\sqrt{\lambda_n}}{c}t$. X, T are our desired eigenfunctions, and u has power expansion of the form

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\sin\frac{\sqrt{\lambda_n}}{c}t + b_n\cos\frac{\sqrt{\lambda_n}}{c}t\right)\cos\sqrt{\lambda_n}x$$

Problem 4.

(a) If f is real valued $\overline{f} = f$, then:

$$\overline{c_{-n}} = \overline{\frac{1}{2\pi}\int_0^{2\pi} e^{ixn} f(x)dx} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ixn} f(x)dx = c_n,$$

(b) First suppose that f is even. Then we compute c_n as:

$$c_n = \frac{1}{2\pi}\left(\int_0^{2\pi} f(x)\cos nx dx - i\int_0^{2\pi} f(x)\sin nx\right).$$

Note that $f(x)\sin(nx)$ is odd and 2π periodic, so the following holds:

$$\begin{aligned}
0 &= \int_{-2\pi}^{2\pi} f(x)\sin nx dx \\
&= \int_0^{2\pi} f(x)\sin nx dx + \int_{-2\pi}^{0} f(x)\sin nx dx \\
&= \int_0^{2\pi} f(x)\sin nx dx - \int_0^{-2\pi} f(x)\sin nx dx \\
&= \int_0^{2\pi} f(x)\sin nx dx + \int_0^{2\pi} f(x)\sin nx dx && \text{(changing variables, f even,)}
\end{aligned}$$

Therefore $\int_0^{2\pi} f(x)\sin nx dx = 0$, so $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos nx dx$. Similarly, if f is odd then $\int_0^{2\pi} f(x)\cos nx dx$ vanishes by the same reasoning as above and so $c_n = \frac{-ii}{2\pi}\int_0^{2\pi} f(x)\sin nx dx$.

(c) We compute $c_n(f'')$:

$$c_n(f'') = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f'' dx = \frac{1}{2\pi}\left[e^{-inx} f'(x)\right]\Big|_0^{2\pi} + \frac{-in}{2\pi}\int_0^{2\pi} e^{-inx} f'(x)dx = -inc_n(f').$$

The exact same computation where we use f' instead of f'' tells us that $c_n(f') = -inc_n(f)$. Therefore $c_n(f) = \frac{i}{n}c_n(f') = \frac{-1}{n^2}c_n(f)$.

(d) We compute the Fourier coefficients for $f = 1$.

$$c_n(1) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} dx = \frac{1}{2\pi}\cdot\frac{1}{-in}e^{-inx}\Big|_0 2\pi = 0.$$

Except for $n = 0$, when $c_1(1) = 1$, clearly. If $f = \sin 2x$, then f is odd so we apply the result from 4c).

$$c_n = \frac{-i}{2\pi}\int_0^{2\pi} \sin 2x \cdot \sin nx dx = \frac{-i}{2}\delta_{2,n}.$$

By orthogonality. Finally when $f = x$, we compute

$$c_n(x) = \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx} dx = \frac{1}{2\pi}\left[\frac{i}{n}xe^{-inx}\right]\Big|_0^{2\pi} - \int_0^{2\pi} e^{-inx} dx = \frac{i}{n}.$$