Problem 1. *Q1 Pg. 45*

The antipodal mapping is linear, hence $dA_p(v) = Av = -v$. Therefore we compute that

$$\langle dA_p(v), dA_p(u)\rangle_p = \langle -v, -u\rangle_p = \langle u, v\rangle.$$

This is true for all points in $\mathbb{R}^n$, so it must be true on S^{n-1} since it inherits the riemannian metric of $\mathbb{R}^n$. Thus the antipodal mapping A is an isometry. Now, given the projection $\pi : S^n \to \mathbb{RP}^n$ which identifies $p, -p$, we define the riemannian metric on $\mathbb{RP}^n$ as

$$\langle u, v\rangle_{[p]} = \langle (d\pi_p)^{-1}u, (d\pi_p)^{-1}v\rangle_p.$$

This is well defined since

$$\begin{aligned}
\langle d(\pi_p)^{-1}v, (d\pi_p)^{-1}u\rangle_p &= \langle d(\pi \circ A_{-p})^{-1}u, d(\pi \circ A_{-p})^{-1}v\rangle_p \\
&= \langle A^{-1}(d\pi_{-p})u, A^{-1}d(\pi_{-p})v\rangle_{-p} && \text{(chain rule)} \\
&= \langle (d\pi_{-p})u, (d\pi_{-p})v\rangle_{-p} && \text{(since } A^{-1} = A, A \text{ isometry)}
\end{aligned}$$

We now claim that π is a local isometry. For any $p \in S^n$, take open $U \ni p$ so that $A(U) \cap U = \emptyset$. We have that $\pi : U \to \pi(U)$ is a smooth bijection, and hence is a diffeomorphism. It remains to show that is an isometry on U. We evaluate:

$$\langle d\pi_p v, d\pi_p u\rangle_{\pi(p)} = \langle u, v\rangle_p,$$

since π is the identity on our choice of U.

Problem 2. *Q2 Pg. 46*

Note that $(d\pi_x)v = \sum_{j=1}^n ie^{ix_j}v_j$. We can define $\langle ., . \rangle_{\pi(x)}$ as follows:

$$\langle w, z \rangle_{\pi(x)} := \langle w, z \rangle.$$

With this choice of riemannian metric, π will be a local isometry exactly when π is injective, i.e. for any $x \in \mathbb{R}^n$, take open U satisfying $x \in U \subset B$ where B is a box of width 2π centered about x.

We now show that with this choice of metric $\mathbb{T}^n$ is isometric with the flat Torus, T. It is clear that $\mathbb{T}^n$ and T are diffeomorphic as manifolds. Let f be the identity map between the sets. We check that f is an isometry at p.

$$\begin{aligned}
\langle (df)u, (df)v \rangle_{f(p)} &= \sum_{j=1}^n \langle e^{ix_j}u_j, e^{ix_j}v_j \rangle \\
&= \sum_{j=1}^n \langle u_j, v_j \rangle \\
&= \langle u, v \rangle
\end{aligned}$$

Problem 3.

(a) Given the parametrization of the cylinder $\phi(\theta, t) = (r\cos\theta, r\sin\theta, t)$, we compute its differential as:

$$d\phi = \begin{bmatrix} -r\sin\theta & 0 \\ r\cos\theta & 0 \\ 0 & 1 \end{bmatrix}$$

We have that $g_{11} = \langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \rangle = r^2, g_{12} = g_{21} = \langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial t} \rangle = 0, g_{22} = 1$ Using the formula for the volume of a manifold we compute:

$$vol(R) = \int_R \sqrt{g_{11}g_{22}}\, dt\, d\theta = \int_0^{2\pi}\int_0^h r^2\, dt\, d\theta = 2\pi r^2 h$$

(b) Since $\phi(U)$ is the graph of a function it must be a manifold. We wish to compute the differential $d\phi$, then compute the associated g_{ij}. We compute that

$$d\phi = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ \frac{\partial g}{\partial u_1} & \frac{\partial g}{\partial u_2} & \dots & \frac{\partial g}{\partial u_n} \end{bmatrix} = \begin{bmatrix} I \\ \nabla g \end{bmatrix}.$$

Therefore we have

$$g_{ij} = \begin{cases} 1 + \left(\frac{\partial g}{\partial u_i}\right)^2 & i = j \\ \left(\frac{\partial g}{\partial u_i}\right) \cdot \left(\frac{\partial g}{\partial u_j}\right) & i \neq j \end{cases}.$$

We can compute that $\det(g_{ij})$ as

$$\begin{aligned}
\det(g_{ij}) &= \sum_{\sigma\in S_n}\left((-1)^\sigma \prod_{i=1}^n g_{i\sigma(i)}\right) \\
&= \prod_{i=1}^n \left(1 + \frac{\partial g}{\partial u_i}^2\right) + \sum_{\sigma\neq id}(-1)^\sigma \prod_{i=1}^n g_{i\sigma(i)} \\
&= 1 + \sum_{i=1}^n \frac{\partial g}{\partial u_i}^2 - \sum_{\sigma\neq id}(-1)^\sigma \prod_{i=1}^n g_{i\sigma(i)} + \sum_{\sigma\neq id}(-1)^\sigma \prod_{i=1}^n g_{i\sigma(i)} && \text{(rewriting the product)} \\
&= \left[1 + \sum_{i=1}^n \frac{\partial g}{\partial u_i}^2\right] && \text{(Since the cross terms cancel with eachother)}
\end{aligned}$$

Therefore volume of $\phi(U)$ must be

$$\int_U \left(1 + \sum_{i=1} \frac{\partial g}{\partial u_i}^2\right)^{\frac{1}{2}} du_1 \dots du_n$$

Problem 4. *Problems 1,2 Page 56 Do Carmo*

1) We first claim that $P_{c,t_0,t}$ is a linear mapping. First let v be a vector, $\alpha \in \mathbb{R}$. There exists a unique parrellel vector field $X(t)$ so that $X(t_0) = \alpha v$ and $X(t) = P_{c,t_0,t}\alpha v$. We must also have that $\frac{1}{\alpha}X(t_0) = v = \frac{1}{\alpha}P_{c,t_0,t}\alpha v$, and therefore $\frac{1}{\alpha}X(t_0) = P_{c,t_0,t}v$ and thus $P_{c,t_0,t}\alpha v = \alpha P_{c,t_0,t}v$. Similarly, if $X(t_0) = v, Y(t_0) = u$ with parrellel transports $X(t), Y(t)$. If we define $W(t) = X(t) + Y(t)$, we see that

$$P_{c,t_0,t}(u+v) = W(t) = X(t) + Y(t) = P_{c,t_0,t}u + P_{c,t_0,t}v$$

by uniqueness. We now claim that $P_{c,t_0,t}$ is an isometry. We compute that:

$$\frac{d}{dt}\langle P_{c,t_0,t}u, P_{c,t_0,t}v\rangle = \frac{d}{dt}\langle X(t), Y(t)\rangle = \langle \frac{DX}{dt}, Y(t)\rangle + \langle X(t), \frac{DY}{dt}\rangle = 0,$$

where we use compatability with the metric in the second equality, and the fact that X, Y are parrellel vector fields in the last equality. Therefore $\langle P_{c,t_0,t}u, P_{c,t_0,t}v\rangle$ is constant. Furthermore at $t = t_0$ we have that $\langle P_{c,t_0,t}u, P_{c,t_0,t}v\rangle = \langle u, v\rangle$. Therfore $P_{c,t_0,t}$ is an isometry. Further suppose that M is oriented. We first assume that the image of $c(t)$ is contained in one coordinate chart. Since $P_{c,t_0,t}$ is an isometry we have $\det P_{c,t_0,t} = \pm 1$. By continuity, it must be constant. taking $t = t_0$ we have that $P_{c,t_0,t_0} = id$, and so the determinant must always be 1. Thus $P_{c,t_0,t}$ pushes any ordered n frame into another oriented n frame of the same orientation. If $c(t)$ is covered by two coordinate charts, say ϕ, ψ, the differential along the curve will be

$$d(\phi \circ \psi^{-1}) \circ d(P_{c,t_0,t}).$$

Since M is orientable, the determinant of this will be positive. Hence $P_{c,t_0,t}$ always preserves orientation.

2) Since $\nabla_X Y(p)$ is defined locally, we can assume that $c(t)$ is inside a coordinate chart. Take an orthonormal basis at $T_{c(t_0)}M$ of $\left\{\frac{\partial}{\partial x_i}\right\} = \{X_i\}$, and parallel transport this using P to a basis $\{P_i(t)\}$. In the coordinate chart we represent $Y(c(t)) = \sum_i a_i(t)P_i(t)$, and let $V(t) = P^{-1}_{c,t_0,t}Y(c(t))$. We compute $\frac{d}{dt}V$ as:

$$\begin{aligned}
\frac{d}{dt}V(t)\Big|_{t=t_0} &= \frac{d}{dt}\sum_i a_i P^{-1}P_i(t) \\
&= \sum_i a_i' X_i + \sum_i a_i (P^{-1} \circ P_i(t))' \\
&= \sum_i a_i' X_i + \sum_i a_i \frac{DX_i}{dt} && \text{(since } P^{-1} \circ P_i = c_i(t) = X_i) \\
&= \frac{DV}{dt} && \text{(by proof of prop. 2.2 Do Carmo)} \\
&= \nabla_X Y(p) && \text{(by prop 2.2)}
\end{aligned}$$

Problem 5.

We first parametrize the sphere using spherical coordinates:

$$f : [0, \pi] \times [0, 2\pi] \to S^2 : f(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

The differential of this is

$$Df = \begin{bmatrix} \cos\varphi\cos\theta & -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta & -\sin\varphi\cos\theta \\ -\sin\varphi & 0 \end{bmatrix},$$

and so the matrix g is equal to

$$\begin{bmatrix} 1 & 0 \\ 0 & \sin^2\varphi \end{bmatrix}.$$

It's inverse is given by

$$g^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\varphi} \end{bmatrix}.$$

We parametrize the latitude in spherical coordinates as $c(t) = (\varphi_0, t)$ for $t \in [0, 2\pi]$, and define the vector field $V(t) = (a(t, b(t))$ along the curve $c(t)$ so that $V(0) = (0, 1)$. To find $V(t)$, we set $\frac{DV}{dt} = 0$ to get:

$$\begin{cases} a' + a\left(\Gamma^1_{11}u' + \Gamma^1_{12}v'\right) + b\left(\Gamma^1_{21}u' + \Gamma^1_{22}v'\right) = 0 \\ b' + a\left(\Gamma^2_{11}u' + \Gamma^2_{12}v'\right) + b\left(\Gamma^2_{21}u' + \Gamma^2_{22}v'\right) = 0 \end{cases}.$$

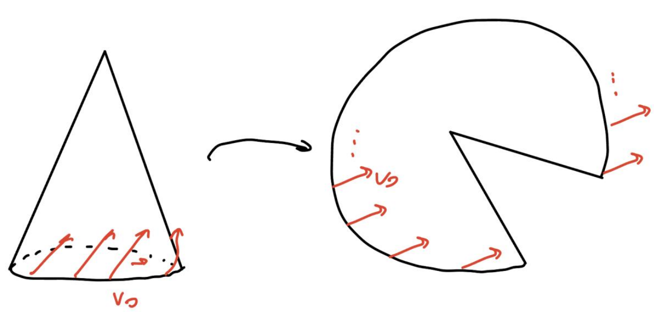
Computing the Christoffel symbols with g, g^{-1}, and noting that $u' = 0$, we get the initial value problem:

$$\begin{cases} a' = \sin\varphi_0\cos\varphi_0 b(t) & a(0) = 0 \\ b' = -\cot\varphi_0 a(t) & b(0) = 1 \end{cases}$$

We know from ODE theory that this is solved by

$$V(t) = \left(\sin\varphi_0\sin\left(\cos\varphi_0 t\right), \cos\left(\cos\varphi_0 t\right)\right).$$

Now by taking a cone tangent to $c(t)$, we have that the tangent space along $c(t)$ is equal for both manifolds, hence parallel transport of the same initial value is equivalent. This transport looks like fixing the direction of the vector and moving it along the cone.



Problem 6.

(a) We claim that a Pseudo Levi-Civita connection exists on any pseudo riemannian manifold. Note that the proof of the existence of a Levi-Civita connection does not depend on positive definiteness of the metric, so we can define a Pseudo riemannian connection in the same way,

$$\langle Z, \nabla_Y X\rangle = \frac{1}{2}\left[X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle\right].$$

This is well defined and satisfies all the properties that we desire since the riemmanian conection does as well.

(b) Given the quadratic form $Q(x) = -x_0^2 + x_1^2 + \cdots + x_n^2$, we note that it can be written in the form:

$$Q(x) = x^{\perp}Ax = x^{T}\begin{bmatrix} -1 & 0 \\ 0 & I_n \end{bmatrix} x.$$

Therefore the pseudo-riemannian metric is of the form :

$$g(x, y) = x^{T}\begin{bmatrix} -1 & 0 \\ 0 & I_n \end{bmatrix} y.$$

The matrix that corresponds to the metric has constant entries, so we know that the christoffel symbols $\Gamma_{ij}^{k} = 0$ for all i, j, k. Since the Christoffel symbols are the same as in $\mathbb{R}^{n+1}$ we have that the parrellel transports of any vector along a given curve must be the same in both metrics, since they satisfy the same ODE.