Q2: Let $|G| = p^\alpha$. We first claim that if for some i, $Z_i(G) = Z_{i+1}(G)$ then $Z_i(G) = Z_{i+1}(G) = G$. By assumption, $|Z_i(G)/Z_{i+1}(G)| = 1$. Therefore we have that $|Z(G/Z_i(G))| = 1$. This implies that $Z_i(G) = G$ by Lagranges Theorem. Hence $Z_i(G) = Z_{i+1}(G) = G$. We now claim that the sequence $\{|Z_i(G)|\}$ is strictly increasing up until some k from which point on we have equality. We will proceed by induction. First consider $|Z_2(G)/Z_1(G)|$. By definition the equality

$$|Z(G/Z_1(G))| = |Z_2(G)/Z_1(G)|$$

must hold. If we let $|Z_1(G)| = p^\beta$ for $\beta < \alpha$, we have that for some $\gamma \leq \alpha - \beta$,

$$p^\gamma = |Z_2(G)/Z_1(G)| = |Z_2(G)| \cdot p^{-\alpha}$$

Which is equivalent to $|Z_2(G)| = p^{\gamma+\alpha}$. Since $|Z_2(G)|$ must divide p^α it must also be a power of p. Now if $\gamma = \alpha - \beta$ we get that $|Z_2(G)| = p^\alpha$ and we conclude that $Z_2(G) = G$. If not, then we have that $|Z_2(G)| > |Z(G)|$. Now suppose that this holds for up until some i. We will show that $|Z_i(G)| \leq |Z_{i+1}(G)|$, with equality signaling that $Z_{i+1}(G) = G$. We let $|Z_i(G)| = p^{\beta_i}$, and note that $p^{\gamma_i} = |Z(G/Z_i(G))| \geq p^{\alpha-\beta_i}$ But also that

$$|Z_{i+1}(G)| = p^{\gamma_i+\beta_i}$$

. When $\gamma = \alpha - \beta_i$ we have equality and conclude $|Z_{i+1}(G)| = p^\alpha$. If the inequality is strict, then we get that $p^{\gamma_i+\beta_i} = |Z_{i+1}(G)| > p^{\beta_i} = |Z_i(G)|$. Hence the sequence $\{|Z_i(G)|\}$ is strictly increasing, and since it is bounded above by p^α is must attain p^α eventually, from then on it will be constant.