Problem 1. *Folland 8.4.30*

Take any x in the Lebesgue set of f. Then by fatou's lemma, we have

$$\lim_{t\to 0} f^t(x) \geqslant \int \liminf_{t\to 0} \hat{f}(\xi)\Phi(t\xi)e^{2\pi i \xi\cdot x} d\xi = \int \hat{f} d\xi$$

(Done with Robbert Liu)

Problem 2. *Folland 8.4.26*

(a) Taking $\varphi(x)$ as in 8.37, then by the inverse fourier transform we get that

$$e^{-\beta} = \int \frac{1}{\pi(1+t^2)} e^{-i\beta t} dt$$

(b) We verify that the equality holds by the following computation:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+t^2} e^{-i\beta t} &= \int_{-\infty}^{\infty} \frac{e^{-i\beta t}}{\pi} \int_0^{\infty} e^{-(1+t^2)s} ds dt && \text{(by hint)} \\
&= \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi} e^{-i\beta t} e^{-(1+t^2)s} dt ds && \text{(By Fubini-Tonelli)} \\
&= \int_0^{\infty} \int_{-\infty}^{\infty} \sqrt{\pi} e^{-s} e^{-i\beta\sqrt{\pi}x} e^{-\pi x^2 s} dx ds && \text{(By substituting } x = \tfrac{t}{\sqrt{\pi}}) \\
&= \int_0^{\infty} \sqrt{\pi} e^{-s} \int_{-\infty}^{\infty} e^{\frac{-i\beta x 2\pi}{2\sqrt{\pi}}} e^{-\pi x^2 s} \\
&= \int_0^{\infty} \sqrt{\pi} e^{-s} \int_{-\infty}^{\infty} e^{-2\pi i \xi} e^{-\pi x^2 s} dx ds \\
&= \int_0^{\infty} (\pi s)^{-1/2} e^{-s} e^{\frac{-\beta^2}{4s}} ds
\end{aligned}$$

(c) By the previous results,

$$\begin{aligned}
e^{-2\pi|\xi|} &= \int_0^{\infty} (\pi s)^{-1/2} e^{-s} e^{-\frac{\pi^2|\xi|^2}{s}} ds \\
&= \int_0^{\infty} (\pi s)^{-1/2} e^{-s} \int_{\mathbb{R}} \left(\frac{s}{\pi}\right)^{n/2} e^{-s|x|^2} e^{-2\pi i \xi \cdot x} dx ds \\
&= \int_{\mathbb{R}} \int_0^{\infty} (\pi s)^{-1/2} e^{-s} \left(\frac{s}{\pi}\right)^{n/2} e^{-s|x|^2} e^{-2\pi i \xi \cdot x} ds dx \\
&= \int_{\mathbb{R}} e^{-2\pi i \xi \cdot x} \left(\int_0^{\infty} (s)^{\frac{n-1}{2}} e^{-\pi s} e^{-\pi s|x|^2} ds \right) dx \\
&= \int_{\mathbb{R}} e^{-2\pi i \xi \cdot x} \frac{\Gamma\left(\frac{1}{2}(n+1)\right)}{\pi^{(n+1)/2}} (1+|x|)^{-\frac{n+1}{2}} dx
\end{aligned}$$

Which is exactly what we wanted to show. (Done with Robbert Liu)

Problem 3. *Folland 8.4.28*

(a) The following computation verifies what we wish to show:

$$\begin{aligned} f * P_r(x) &= P_r * f(x) \\ &= \int f(y) P_r(x-y)\,dy \\ &= \int f(y) \sum_{k\in\mathbb{Z}} r^{|k|} e^{2\pi i(x-y)}\,dy \\ &= \sum_{k\in\mathbb{Z}} r^{|k|} e^{2\pi i x} \int_{\mathbb{R}} f(y) e^{-2\pi i y}\,dy \\ &= \sum_{k\in\mathbb{Z}} r^{|k|} e^{-2\pi i k x} \hat{f}(k) \end{aligned}$$

(b) We compute $P_r(x)$ as

$$\begin{aligned} P_r(x) &= \sum_{-\infty}^{\infty} r^{|k|} e^{2\pi i k x} \\ &= 1 + \sum_{1}^{\infty} r^{|k|} e^{2\pi i k x} + \sum_{-\infty}^{-1} r^{|k|} e^{2\pi i k x} \\ &= 1 + \sum_{1}^{\infty} \left(re^{2\pi i x}\right)^k + \sum_{1}^{\infty} \left(re^{-2\pi i x}\right)^k \\ &= 1 + \left(-1 + \frac{1}{1 - re^{2\pi i x}}\right) + \left(-1 + \frac{1}{1 - re^{-2\pi i x}}\right) \\ &= -1 + \frac{1}{1 - re^{2\pi i x}} + \frac{1}{1 - re^{2\pi i x}} \\ &= \frac{-\left(1 - re^{2\pi i x}\right)\left(1 - re^{-2\pi i x}\right) + 1 - re^{2\pi i x} + 1 - re^{2\pi i x}}{(1 - re^{2\pi i x})(1 - re^{-2\pi i x})} \\ &= \frac{1 - r^2}{1 + r^2 - 2r\cos(2\pi x)} \end{aligned}$$

Problem 4. *Folland 8.4.31*

Using 8.37, and the dilation formula, we compute that

$$\frac{\pi}{a} \cdot \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} = \frac{\pi}{a}\left(\sum_{1}^{\infty} e^{-2\pi a k} + \sum_{0}^{\infty} e^{-2\pi a k}\right) = \frac{\pi}{a}\sum_{-\infty}^{\infty} e^{-2\pi a|k|} = \sum_{-\infty}^{\infty} \frac{1}{(k^2+a^2)}.$$

Therefore the following series of equalities hold:

$$\begin{aligned} 2\sum_{k=1}^{\infty} \frac{1}{k^2+a^2} &= \frac{\pi}{a} \cdot \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} - \frac{1}{a^2} \\ &= \frac{a\pi\left(1+e^{-2\pi a}\right) - \left(1-e^{2\pi a}\right)}{a^2\left(1-e^{-2\pi a}\right)} \end{aligned}$$

To take the limit as $a \to 0$, we apply L'hoptals rule 3 times. The third derivative of the numerator with respect to a is:

$$4\pi^3 e^{-2\pi a} + 4\pi^3 e^{-2\pi a} + 4\pi^3 e^{-2\pi a} + 8\pi^4 a e^{-2\pi a} - 8\pi^3 e^{-2\pi a},$$

and the third derivative of the denominator is

$$4\pi e^{-2\pi a}(2\pi^2 a^2 - 6\pi a + 3).$$

At $a = 0$ we have the quotient equal to $\frac{\pi^2}{3}$ and we conclude that

$$2\sum_{1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{3}.$$

(Done with Charles Swaney)

Problem 5. *Folland 8.5.33*

(a) We compute the convolution as:

$$\begin{aligned}
f * F_m(x) &= \int f(y) F_m(x-y)\, dy \\
&= \int f(y) \frac{1}{m+1} \sum_{0}^{m} \sum_{-k}^{k} e^{2\pi i k(x-y)}\, dy \\
&= \sum_{-k}^{k} \int f(y) \frac{m+1-|k|}{m+1} e^{-2\pi i k y} e^{2\pi i k x}\, dy && \text{(by counting like terms)} \\
&= \sum_{-k}^{k} e^{2\pi i k x} \frac{m+1-|k|}{m+1} \hat{f}(y)
\end{aligned}$$

(b) We compute F_m as

$$\begin{aligned}
F_m &= \frac{1}{m+1} \sum_{k=0}^{m} D_k \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \frac{e^{(2k+1)\pi i x} - e^{-(2k+1)\pi i x}}{2i} \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \frac{e^{(2k+1)\pi i x} - e^{-(2k+1)\pi i x}}{2i} \cdot \frac{e^{(2k+1)\pi i x} + e^{-(2k+1)\pi i x}}{e^{(2k+1)\pi i x} + e^{-(2k+1)\pi i x}} \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \frac{e^{(2k+1)\pi i x^2} - e^{-(2k+1)\pi i x^2}}{2i\left(e^{(2k+1)\pi i x} + e^{-(2k+1)\pi i x}\right)} \\
&= \frac{\sin^2([m+1]\pi x)}{(m+1)\sin^2(\pi x)} && \text{(by telescopic summations)}
\end{aligned}$$

Problem 6. *Folland 8.5.34*

Using the closed form for the Dirichlet kernel, we compute the limit as $m \to \infty$ of $\|D_m\|_1$ as:

$$\begin{aligned}
\lim_{m\to\infty} \|D_m\|_1 &= \lim_{m\to\infty} \int_{\mathbb{R}} \left| \frac{\sin([2m+1]\pi x)}{\sin(\pi x)} \right| dx \\
&= \lim_{m\to\infty} \int \left| \frac{\sin(y)}{\sin(y/\pi(2m+1))} \right| \frac{1}{\pi(2m+1)} dy && \text{(substituting } y = \pi(2m+1)x\text{)} \\
&= \int \left| \frac{\sin(y)}{y} \right| dy && \text{(by Monotone Convergence Theorem)} \\
&= \infty && \text{(Mat157)}
\end{aligned}$$

Problem 7. *Folland 8.5.35*

(a) First we show that ϕ_m is linear. We compute that

$$\phi_m(f+g) = S_m(f+g)(0) = \sum_{-m}^{m} \hat{f} + \hat{g}(k) = \sum_{-m}^{m} \hat{f}(k) + \sum_{-m}^{m} \hat{g}(k) = \phi_m(f) + \phi_m(g),$$

and

$$\phi_m(\alpha f) = S_m(\alpha f(0)) = \alpha \sum_{-m}^{m} \hat{f}(k) = \alpha \phi_m(f).$$

We now claim that ϕ_m is continuous. For $|f|_u < \varepsilon$, we have

$$\|\phi_m(f)\| = \left\| \sum_{-m}^{m} \hat{f}(k) \right\| \leqslant \sum_{-m}^{m} \left\| \hat{f} \right\| = 2m \left\| \hat{f} \right\| = 2m \int_{\mathbb{T}} \left| f(x) e^{-2\pi i k x} \right| dx \leqslant 2m\varepsilon m^*(\mathbb{T}).$$

Therefore $\phi_m \in C(\mathbb{T})^*$. We also have that by Young's Inequality that $\|\phi_m\| \leqslant \|D_m\|$, and that this equality will be attained when we take $f = 1$. Thus $\|\phi_m\| = \|D_m\|$.

(b) Suppose that the set of all such f is not meager. Then by Uniform Boundedness Principle we have that the sequence $\sup_f \{S_m f(0)\} < \infty$. This contradicts part 8.5.34 since as $m \to \infty$, $\|D_m\| \to \infty$ and by part $a)$ so does $\|\phi_m\|$.

(c) First note that the result from $b)$ holds for any $x \in T$, since if we replace 0 with any point x we still have that $\left|e^{2\pi i x k}\right| = 1$. Then, by the Principle of Condensation of Singularities (Folland 5.3.40) there is a residual subset of $C(\mathbb{T})$ so that $\{S_m f(x)\}$ diverges on a dense subset of $\mathbb{T}$.

(Done with Petar Jovasevic)