Q7: We first show that $\sim$ as defined is an equivalence relation. First $x \sim x$ since $x = \varphi \circ \varphi^{-1}(x)$ for any φ. If $x \sim y$, then $y = \varphi_i \circ \varphi_j^{-1}(x)$. Applying $\varphi_j \circ \varphi_i^{-1}$, we see that $x = \varphi_j \circ \varphi_i^{-1}(y)$. So $y \sim x$. Finally suppose that $x \sim y$ and $y \sim z$. We can write $y = \varphi_j \circ \varphi_i^{-1}(x)$ and $z = \varphi_l \circ \varphi_j^{-1}(y)$. Composing we see that

$$z = \varphi_l \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1}(x) = \varphi_l \circ \varphi_i^{-1}(x).$$

As desired. We now claim there is a bijection between X and $\bigsqcup V_i / \sim$. Define $f : X \to \bigsqcup V_i / \sim$ by $x \mapsto [\varphi_i(x)]$. We claim that such f is a bijection. Let $y \in \bigsqcup V_i$ belonging to the class $[\varphi_i(x)]$. For some φ_j, we have that $\varphi_j^{-1}(y) = x$ and so $\varphi_j(x) = y$ so $f(x) = y$. Now suppose that $[\varphi_i(x)] = [\varphi_j(y)]$. By the equivalence relation we have that $\varphi_i(x) = \varphi_i \circ \varphi_j^{-1} \circ \varphi_j(y) = \varphi_i(y)$. Since φ_i is injective we have that $x = y$. Therefore f is a bijection. Finally we show that X is a smooth manifold and the topology on X is induced by the quotient topology of $\bigsqcup V_i / \sim$. For each φ_i, using the same f as above, we define the coordinate charts on $\bigsqcup V_i$ as $\psi_i : f^{-1}(U_i) \to V_i$ with $\psi_i = \varphi_i \circ f^{-1}$. Each ψ_i is a composition of injective and continuous maps, hence they are injective and continuous as well. We also see that they are C^∞ related since

$$\psi_i \circ \psi_j^{-1} = (\varphi_i \circ f^{-1}) \circ (\varphi_j \circ f^{-1})^{-1} = \varphi_i \circ f^{-1} \circ f \circ \varphi_j^{-1} = \varphi_j \circ \varphi_i^{-1}.$$

Therefore $\bigsqcup V_i / \sim$ is a smooth manifold, and the identification with X makes X a smooth manifold as well.