Q4: Suppose that $f_n \to f$ in L^1. Let $E_k = \{x : |f_n - f| \geq \frac{1}{k}\}$. By markovs inequality, we have that

$$\mu(E_k) \leq k \int_X |f_n - f|$$

As n get sufficiently large the integral goes to 0 and we conclude that $\lim_{n\to\infty} \mu(E_k) = 0$. Hence $f_n \to f$ in measure. We now claim that $\{f_n\}$ is uniformly absolutely continuous. Let $\varepsilon > 0$. We take N sufficiently large so that $\int_X |f_n - f| < \frac{\varepsilon}{2}$ for $n \geq N$. By a result from class, there must exist some $\delta^* > 0$ so that for all E with $\mu(E) < \delta^*$, $\int_E |f| < \frac{\varepsilon}{2}$. We therefore have that

$$\int_E |f_n| \leq \int_E |f_n - f| + |f| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Now if $1 \leq n < N$, by the result from class for the same ε there exists δ_i so that

$$\int_E |f_n| \leq \varepsilon$$

for $\mu(E) < \delta_i$. We take δ to be the minimum of the δ_i's and δ^*. Hence for all E, $\mu(E) < \delta$ we have that $\int_E |f_n| < \varepsilon$ Now suppose the converse. Let $\varepsilon > 0$. By convergence in measure choose n sufficiently large so that if $E = \{x : |f_n - f| > \varepsilon\}$ then $\mu(E) < \frac{\varepsilon}{2}$. Then we have that

$$\begin{aligned}
\int_X |f_n - f| &= \int_E |f_n - f| + \int_{E^c} |f_n - f| \\
&\leq \int_E |f| + \int_E |f| + \int_{E^c} |f_n - f| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \int_{E^c} \varepsilon \\
&< \varepsilon(1 + \mu(X))
\end{aligned}$$

Since ε is arbitrary, we conclude that $\int |f_n - f| \to 0$ and so $f_n \to f$.