Q4: We begin by defining $\underline{n}_s^k = \{(i_1, \dots i_k) : 1 \leq i_1 \leq \cdots \leq i_k\}$. We claim that $\sigma_I = \sum_{\sigma \in S_k} \varphi_I \circ \sigma^*$ is a basis of $S^k(V)$. We will first show that $\sigma_I \in S^k(V)$. Let $u_1, \dots u_k$ be some vectors in V. Let $\tau \in S_k$. Then we evaluate that

$$\begin{aligned}
\sigma_I \circ \tau^*(u_1, \dots, u_k) &= \sum_{\sigma \in S_k} \varphi_I \circ \sigma^*(u_{\tau(1)}, \dots, u_{\tau(k)}) \\
&= \sum_{\sigma \in S_k} \varphi_I(u_{\sigma(\tau(1))}, \dots u_{\sigma(\tau(k))}) \\
&= \sum_{\lambda \in S_k} \varphi_I(u_{\lambda(1)}, \dots, u_{\lambda(k)}) && \text{(since for fixed } \tau,\ \sigma \circ \tau \text{ is } S_k \text{ for } \sigma \in S_k) \\
&= \sigma_I(u_1, \dots, u_k)
\end{aligned}$$

Therefore σ_I is in $S^k(V)$. We now claim that $\{\sigma_I : I \in \underline{n}_s^k\}$ forms a basis for $S^k(V)$. We will show this in several steps. Let $v_1 \dots v_n$ be a basis for V. The first claim we make is that given $I, J \in \underline{n}_s^k, \sigma_I(v_J) = \delta_{IJ}$. We can check indeed that

$$\sigma_I(v_J) = \sum_{\sigma \in S_k} \varphi_I \circ \sigma^*(v_1, \dots, v_k) = \sum_{\sigma \in S_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \sum_{\sigma \in S_k} \varphi_{i_k}(v_{\sigma(1)}) \cdot \ldots \varphi_{i_k}(v_{\sigma(k)}) = \delta_{IJ}$$

Where the last equality holds because the sum will be 0 unless under some permutation, $\sigma^*(J) = I$. Since each σ_I is a k-tensor, it follows that $S_1 = S_2$ if and only if $S_1(v_I) = S_2(v_I)$ for all $I \in \underline{n}_s^k$. We now claim that all the σ_I span $S^k(V)$. Let $S \in S^k(V)$. We want to find a_I such that $S = \sum a_I \sigma_I$. Take $a_I = S(v_I)$. Then it is enough to show that $S(v_J) = \sum a_I \sigma_I(v_J)$.

$$S(v_J) = \sum a_I \delta_{IJ} = a_J$$

Finally, we claim that $\{\sigma_I\}$ is a linearly independant set. Suppose for some b_I, $\sum b_I \omega_I = 0$. Evaluating this on v_J gives us that

$$b_J = \sum b_I \sigma_I(v_J) = 0(b_J) = 0$$

Therefore each b_I is 0, and thus this set is linearly independant and spans $S^k(V)$. So it is a basis. By our discussion in lecture 47, $|\underline{n}_s^k| = \binom{n+k-1}{k}$, and so $dim(S^k)V = |\underline{n}_s^k|$ thus we are done.