Q6a: Writing $f(x+iz) = w(x,y) + iv(x,y)$, we get that

$$F = w^2 + v^2 = u^2.$$

Using the chain rule, we compute that

$$\frac{\partial F}{\partial x} = 2u \cdot u_x,$$

and so

$$u_x = \frac{\partial F}{\partial x} \cdot \frac{1}{2u}.$$

Using this formula, we get that

$$u_x = \frac{1}{2|f|} \cdot \Big[2w \cdot w_x + 2v \cdot v_x\Big] = \frac{1}{|f|} Re\Big(w \cdot w_x + iu \cdot v_x - iv \cdot u_x + v \cdot v_x\Big) = \frac{1}{|f|} Re(\overline{f} f').$$

Similarly we have that

$$u_y = \frac{1}{2u} \frac{\partial F}{\partial y}.$$

We run through an almost identical computation to see that

$$u_y = \frac{1}{2|f|} \Big[2w \cdot w_y + 2v \cdot v_y\Big] = -\frac{1}{|f|} Im\Big(w \cdot v_y - iw \cdot w_y - iv \cdot v_y - v \cdot w_y\Big) = -\frac{1}{|f|} Im(\overline{f} f').$$

We now wish to compute

$$F_{xx} + F_{yy}.$$

Using simple calculus, we can write this as

$$F_{xx} + F_{yy} = 2u_x^2 + 2uu_{xx} + 2u_y^2 + 2u_{yy} = 2[u_x^2 + u_y^2]^2 + 2u[u_{xx} + u_{yy}].$$

We compute that

$$\begin{aligned}
F_{xx} + F_{yy} &= 2[u_x^2 + u_y^2]^2 + 2u[u_{xx} + u_{yy}] \\
&= \frac{2}{u^2}\Big[Re(\overline{f} f')^2 + Im(\overline{f} f')^2\Big] + 2u\Big[\Big(\frac{ww_x + vv_x}{u}\Big)_x + \Big(\frac{ww_y + vv_y}{u}\Big)_y\Big] \\
&= \frac{2}{u^2}\Big[(v^2 + w^2)(w_x^2 + v_x^2)\Big] + 2u\Big[\frac{u(w_x^2 + v_2^2 + w_y^2 + v_y^2)}{u^2} - u\frac{(w_x^2 + v_x^2)}{u^3}\Big] \qquad \text{(simplifying above)} \\
&= 4(w_x^2 + v_x^2) \\
&= 4|f'(z)|
\end{aligned}$$

Q6b: Since g is holomorphic, its real and imaginary parts are both holomorphic and thus both harmonic. Thus using the previous result, we get

$$0 = Re(g(z))_{xx}^2 + Re(g(z))^2 + yy = |f(z)|_{xx}^2 + |f(z)|_{yy}^2 = 4|f'(z)|^2.$$

Thus we have that for all z, $|f'(z)| = 0$ and so $f'(z) = 0$. It has been shown that any such holomorphic f is necessarily constant. Furthermore, once again by previous results we have that

$$Re(g(z))_x = Re(g(z))_y = 0.$$

Holomorphicity of g implies that g is constant, from the Cauchy-Riemann Equations.