Q1a: From previous results, we know that $\Phi_8(x) = x^4 + 1$. By assumption $s(x)$ is a factor of $\Phi_8(x)$, and ξ is a root of $s(x)$ it must also be a root of $\Phi_8(x)$. Hence $\xi^4 + 1 = 0$. Therefore $\xi^4 = -1$, and so $\xi^8 = 1$. Therefore ξ is nonzero. Hence by Lagranges theorem, the order of the subgroup generated by ξ must must divide 8. Suppose the order is not 8. Then we have that if $m = ord(\xi)$, then

$$\xi^k = \xi^{k^{\frac{4}{k}}} + 1 = 1 + 1 = 0$$

Hence $2 = 0 \mod p$, and so p is 2. This is a contradiction, since we assume that p is odd. Therefore, the order of ξ is 8.

1b: Let $\tau = \xi + \xi^{-1}$. Since $\xi^4 + 1 = 0$, we have that $\xi^{-2}(\xi^4 + 1) = 0$ and so $\xi^2 + \xi^{-2} = 0$. Therefore,

$$(\xi + \xi^{-1}) - 2(\xi \cdot \xi^{-1}) = 0$$

Which implies that $\tau^2 = 2 \cdot 1 = 2$

1c: By Eulers Criterion, we have that

$$(\frac{2}{p}) = 2^{\frac{p-1}{2}} = \tau^{2^{\frac{p-1}{2}}} = \tau^{p-1}$$

Where the second equality follows from $1b$.

1d: We will use the result that if $p = \pm 1 \mod 8$, $\tau^p = \tau$, but that we can discard the case when $p = \pm 3 \mod 8$ in our proof. First if $p = \pm 1 \mod 8$, then

$$\tau^p = \xi^{8^k}\xi^{\pm 1} + \xi^{8^k}\xi^{\pm 1} = \xi^{\pm 1} + \xi^{\pm 1} = \tau$$

Now if $p = \pm 3 \mod 8$, then

$$\tau^p = \xi^{8^k}\xi^{\pm 3} + \xi^{8^{-k}}\xi^{\pm 3} = \xi^{\pm 3} + \xi^{\pm 3} = \xi^3 + \xi^{-3} = \xi = \xi^{-1}$$

Thus if $\tau^p = \tau$,

$$\xi^6 + 1 = \xi^4 + \xi^2 \implies \xi^4(\xi^2) + 1 = -1 + \xi^2$$

Which implies that $2\xi^2 = 2$, which can not happen since the order is 8. We now prove the result.

$\implies$

Suppose that $p = \pm 1 \mod 8$. Then we have that $p = 8k + 1$, and so $p^2 - 1 = (8k + 1)^2 - 1 = 64k^2 + 16k$. It is clear that $\frac{p^2-1}{8} = 8k^2 + 2k$ which is 0 mod 2.

$\impliedby$

Suppose that $\frac{p^2-1}{8} = 0 \mod 2$. Therefore $16|p^2 - 1 = (p - 1)(p + 1)$.If we have that $4|p - 1$ and $4|p + 1$. Hence $p \equiv 3 \equiv 1 \mod 4$. Which can not be the case. Therefore $8|p + 1$ or $8|p - 1$, because $2|(p + 1)$ and so $p = \pm 1 \mod 4$.

Q1e: By c, d, we have that $(\frac{2}{p}) = 1 \iff \frac{p^2-1}{8} \equiv 0 \mod 2$, or $(\frac{2}{p}) \equiv -1 \mod 2 \equiv \frac{p^2-1}{8} \equiv 1 \mod 2$. Thus, $(\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}}$