Q3a: Consider the function $f(x) = \frac{1}{2}x^2$ on $(-1, 1)$. We claim that this is a weak contraction, yet not a contraction. Since $x, y \in (-1, 1)$, note that for distinct x, y

$$\frac{1}{2}|(x+y)| < 1 \iff \frac{1}{2}|x+y| \cdot |x-y| < |x-y| \iff \frac{1}{2}|x^2 - y^2| < |x-y|$$

Therefore $d(fx, fy) < d(x, y)$. We claim that this function is not a contraction. Let $0 < k < 1$. Choosing x, y such that $\frac{1}{2}|x+y| > k$ we have that

$$\frac{1}{2}|x+y| > k \iff \frac{1}{2}|x-y| \cdot |x+y| > k|x-y|$$

And so $d(fx, fy) > k \cdot d(x, y)$

Q3b: Use the same $f(x)$ as defined in $3a$, except we define it on $[-1, 1]$. Using the exact same proof as in $3a$, we have that this function is a weak contraction yet not a contraction.

Q3c: Define $g : M \to M$ by $g(x) = d(fx, x)$. This is a continuous map and so $g(M) = [a, b]$, for some $0 < a < b$. We claim that $a = 0$. Suppose not, that it assume that g attains its minimum at some $y \in M$, and $g(y) > 0$. Consider however $g(fy)$. We have that $g(fy) = d(f^2y, fy) < d(fy, y) = g(y)$. This is a contradiction and hence g attains a minimum of 0 at some point x_0. Therefore, $0 = x_0 = d(fx_0, x_0)$ and so $fx_0 = x_0$. We now claim uniqueness. Suppose that x, y are two distinct fixed points. We therefore have that $d(x, y) = d(fx, fy) < d(x, y)$. This is a contradiction. Hence the fixed point of f is unique.