Q1: Since $\lim_{n\to\infty} \nu_n(E)$ exists and is finite for each set we have that

$$\nu(\emptyset) = \lim_{n\to\infty} \nu_n(\emptyset) = \lim_{n\to\infty} 0 = 0.$$

Now suppose that $\{E_m\}_{m=1}^{\infty}$ is a disjoint sequence of sets in $\mathcal{M}$. We first claim that for any finite union of disjoint sets, $\nu(\bigcup_{m=1}^{M} E_m) = \sum_{m=1}^{M} \nu(E_m)$. We have that

$$\nu(\bigcup_{m=1}^{N} E_m) = \lim_{n\to\infty} \nu_n(\bigcup_{m=1}^{M} E_m) = \lim_{n\to\infty} \sum_{m=1}^{M} \nu_n(E_m) = \sum_{m=1}^{M} \lim_{n\to\infty} \nu_n(E_m) = \sum_{m=1}^{M} \nu(E_m),$$

using the fact that limits commute with finite sums. We now will prove that countable additivity holds. Let $\varepsilon > 0$ and M sufficiently large so that

$$\nu(\bigcup_{m=1}^{\infty} E_m) = \sum_{m=1}^{M} \nu(E_m) + \nu(\bigcup_{m=M+1}^{\infty} E_m) \leq \sum_{m=1}^{M} \nu(E_m) + \varepsilon$$

by setwise convergence. Furthermore, for all n, we have that

$$\nu_n(\bigcup_{m=1}^{M} E_m) \leq \nu_n(\bigcup_{m=1}^{\infty} E_m)$$

Taking the limits and applying our first claim we have that

$$\sum_{m=1}^{M} \nu(E_m) \leq \nu(\bigcup_{m=1}^{\infty} E_m)$$

Hence we have the inequality

$$\sum_{m=1}^{M} \nu(E_m) \leq \nu(\bigcup_{m=1}^{\infty} E_m) \leq \sum_{m=1}^{M} \nu(E_m) + \nu(\bigcup_{m=M+1}^{\infty} E_m) \quad (1)$$

We claim that as $M \to \infty$, $\nu(\bigcup_{m=M+1}^{\infty} E_m) \to 0$. Since each ν_n is bounded, it is sufficient to show that $\nu_n(\bigcup_{m=M+1}^{\infty} E_m) \to 0$ as $M \to \infty$. Note that (1) holds for ν_n as well since it is a measure, so we can take $\varepsilon > 0$ and choose M large enough so that $\sum_{m=M+1}^{\infty} \nu_n(E_m) = \nu_n(\bigcup_{m=M+1}^{\infty} E_m) < \varepsilon$/ By uniform absolute measure continuity, we have that there is some $\delta > 0$ with $\mu(\bigcup_{m=M+1}^{\infty} E_m) = \sum_{m=M+1} \mu(E_m) < \delta$. Since $\mu(X) < \infty$, we have that $\lim_{M\to\infty} \sum_{m=M+1} \mu(E_m) \to 0$, since it is convergent and so tails must converge. Therefore we have that $\sum_{m=M+1}^{\infty} \nu_n(E_m) \to 0$ as $M \to \infty$, and so $\nu(\bigcup_{m=M+1}^{\infty} E_m) \to 0$ as $M \to \infty$. We apply the limit of $M \to \infty$ to (1) and conclude that

$$\sum_{m=1}^{\infty} \nu(E_m) = \nu(\bigcup_{m=1}^{M} E_m)$$