5.5.57a: We claim that $V^{-1}T^tV$ satisfies the adjoint condition. We compute that

$$\langle x, V^{-1}T^tVy\rangle = \langle y, V^{-1}Tx\rangle = \langle Tx, y\rangle.$$

We now claim uniqueness. Let T_1^*, T_2^* be adjoints of T. For all $x, y \in \mathcal{H}$ we have that

$$\langle x, T_1^*y\rangle = \langle x, T_2^*y\rangle \implies \langle x, T_1^* - T_2^*y\rangle = 0.$$

Non degeneracy of the inner product implies that $T_1^* = T_2^*$.

5.5.57b: We first claim that $T^{**} = T$. We have that

$$\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle.$$

As desired. We now show that $\|T^*\| = \|T\|$. We have that

$$\|T^*x\| = \|V_y(Tx)\| = \langle Tx, y\rangle \leq \|T\|\,\|y\|\,.$$

Symmetry implies that $\|T\| \leq \|T^*\|$. Next , we have that

$$\|T^*T\| = \sup_{\|x\|=1} \langle T^*Tx, x\rangle = \sup_{\|x\|=1} \langle Tx, Tx\rangle = \sup_{\|x\|=1} \|Tx\|^2 = \|T\|^2\,.$$

Furthermore, for any linear operators T, s and scalars a, b ,

$$\langle (aS + bT)x, y\rangle = \overline{a}\langle Sx, y\rangle + \overline{b}\langle Tx, y\rangle = \overline{a}\langle x, S^*y\rangle + \overline{b}\langle x, T^*y\rangle = \langle x, \overline{a}S^* + \overline{b}T^*y\rangle.$$

5.5.57c: Suppose that $y \in R(T)^\perp$. Then for all $x \in \mathcal{H}$,

$$0 = \langle Tx, y\rangle \iff 0 = \langle x, T^*y\rangle \iff y \in N(T^*),$$

where the last implication holds by non degeneracy of the inner product. Now if $y \in \overline{R(T^*)}$, for some $u, \overline{T^*u} = y$. Therefore, if $x \in N(T)$ then

$$\langle u, Tx\rangle = 0 \iff \langle T^*(u), x\rangle = 0 \iff \langle x, y\rangle = 0.$$

As desired.

5.5.57d: Suppose that T is unitary. Then

$$\langle Tx, Tx\rangle = \langle x, x\rangle = 0 \iff x = 0.$$

Thus T is invertible. We also have that for all $x, y \in \mathcal{H}$,

$$\langle x, T^*Ty\rangle = \langle x, y\rangle.$$

Thus $T^*T = I$. Therefore by uniqueness of inverses, we have that $T^* = T^{-1}$. Now suppose that T invertible with $T^* = T^{-1}$. Then

$$\langle x, y\rangle = \langle x, T^*Ty\rangle = \langle Tx, Ty\rangle.$$

As desired.