Discrete Riemann Surfaces and Integrable Systems

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Zusammenfassung

Wir entwickeln eine lineare Theorie der diskreten Funktionentheorie auf allgemeinen Quad-Graphen und führen bisherige Arbeiten von Duffin, Mercat, Kenyon, Chelkak und Smirnov auf rhombischen Quad-Graphen fort. Unser Ansatz, der auf dem Kantenmittelgraphen basiert, liefert nicht nur instruktivere Beweise von diskreten Analoga klassischer Sätze, sondern auch neue Resultate. Wir geben diskrete Entsprechungen zu fundamentalen Konzepten der Funktionentheorie wie holomorphen Funktionen, Ableitungen, dem Laplace-Operator und Differentialformen an. Zudem diskutieren wir diskrete Varianten wichtiger klassischer Sätze wie den Greenschen Formeln und den Cauchyschen Integralformeln. Zum ersten Mal überhaupt werden die erste Greensche Formel und die Cauchysche Integralformel für die Ableitung einer holomorphen Funktion diskretisiert.

Für planare Parallelogrammgraphen mit beschränkten Innenwinkeln und Seitenverhältnissen konstruieren wir Diskretisierungen von Greenschen Funktionen und Cauchy-Kernen, deren asymptotisches Verhalten dem glatten Fall nahe kommt. Auf zweidimensionalen Gittern diskretisieren wir Cauchysche Integralformeln höherer Ableitungen.

Unsere Arbeit zur diskreten Funktionentheorie wird mit der Untersuchung von diskreten Riemannschen Flächen fortgesetzt, die wir als Zellzerlegungen von Riemannschen Flächen in Vierecke zusammen mit Linearisierungen ihrer komplexen Strukturen betrachten. Wir verallgemeinern die Resultate von Mercat, Bobenko und Skopenkov auf Zerlegungen in allgemeine Vierecke und gelangen zu neuen Erkenntnissen. Unter anderem formulieren wir eine diskrete Riemann-Hurwitz-Formel, beweisen einen diskreten Satz von Riemann-Roch auf einer größeren Klasse von Divisoren und beschreiben diskrete Abel-Jacobi-Abbildungen.

Zuletzt behandeln wir die variationelle Formulierung diskreter Laplace-Gleichungen, die durch diskrete integrable Systeme motiviert sind. Wir erklären, warum es nicht mehr als die untersuchten Realitätsbedingungen geben sollte, und leiten hinreichende Bedingungen für die Parameter her, die oft auch notwendig sind, unter denen das entsprechende Wirkungsfunktional konvex ist. Konvexität hilft uns, die Existenz und Eindeutigkeit von Lösungen Dirichletscher Randwertprobleme zu diskutieren. Außerdem analysieren wir, welche kombinatorischen Daten konvexe Wirkungsfunktionale von diskreten Laplace-Gleichungen zulassen, die gerade von diskreten integrablen Vierecksgleichungen induziert werden, und erläutern, wie die Gleichungen und Funktionale zu (Q3) mit Kreismustern zusammenhängen.

Summary

We develop a linear theory of discrete complex analysis on general quad-graphs, continuing and extending previous work of Duffin, Mercat, Kenyon, Chelkak, and Smirnov on discrete complex analysis on rhombic quad-graphs. Our approach based on the medial graph yields more instructive proofs of discrete analogs of several classical theorems, and even new results. We provide discrete counterparts of fundamental concepts in complex analysis such as holomorphic functions, derivatives, the Laplacian, and exterior calculus. Also, we discuss discrete versions of important basic theorems such as Green's identities and Cauchy's integral formulae. For the first time, we discretize Green's first identity and Cauchy's integral formula for the derivative of a holomorphic function.

In the case of planar parallelogram-graphs with bounded interior angles and bounded ratio of side lengths, we construct a discrete Green's function and discrete Cauchy's kernels with asymptotics comparable to the smooth case. Further restricting to the integer lattice of a two-dimensional skew coordinate system yields appropriate discrete Cauchy's integral formulae for higher order derivatives.

We continue our work on discrete complex analysis by investigating discrete Riemann surfaces, seen as quadrilateral cellular decompositions of Riemann surfaces together with linearizations of their complex structures. We generalize the results of Mercat, Bobenko, and Skopenkov to decompositions into general quadrilaterals, and extend the known theory. Inter alia, we give a discrete Riemann-Hurwitz formula, we prove a discrete Riemann-Roch theorem on a larger class of divisors, and we discuss discrete Abel-Jacobi maps.

Finally, we investigate the variational structure of discrete Laplace-type equations that are motivated by discrete integrable quad-equations. We explain why the reality conditions we consider should be all that are reasonable, and derive sufficient conditions (that are often necessary) on the labeling of the edges under which the corresponding generalized discrete action functional is convex. Convexity is an essential tool to discuss existence and uniqueness of solutions to Dirichlet boundary value problems. Furthermore, we study which combinatorial data allow convex action functionals of discrete Laplace-type equations that are actually induced by discrete integrable quad-equations, and present how the equations and functionals corresponding to (Q3) are related to circle patterns.

Contents

Chapter 1

Introduction

The introduction is divided into several sections corresponding to the individual chapters of this thesis. We start in Section 1.1 with a short history of discrete complex analysis, and sketch the content of Chapters 2 and 4. In the latter, we investigate the asymptotics of discretizations of certain functions on planar parallelogram-graphs. Some combinatorial properties of planar parallelogram-graphs are postponed to Appendix A. Then, we come to discrete Riemann surfaces in Section 1.2 and Chapter 3. In Section 1.3, we discuss discrete variational systems, and introduce Chapter 5 and Appendix B. We conclude the introduction with some open questions in Section 1.4, and the acknowledgments in Section 1.5.

1.1 Discrete complex analysis

Our aim is to give just a very rough overview of the linear discrete theories of complex analysis we are enhancing, and to name connections to statistical physics and the nonlinear discrete theory of complex analysis based on circle patterns. For a more detailed discussion of these topics, we refer to the survey of Smirnov [Smi00], on which our introduction is based and that also discusses applications to probability theory and mathematical physics, and the book [BS08] of Bobenko and Suris that not only investigates the influence of discrete integrable systems to discrete theories of complex analysis, but also gives a good overview of discrete differential geometry in general.

Linear theories of discrete complex analysis look back on a long and varied history. Already Kirchhoff's circuit laws describe a discrete harmonicity condition for the potential function whose gradient describes the current flowing through the electric network. A notable application of Kirchhoff's laws in geometry was the article [BSST40] of Brooks, Smith, Stone, and Tutte, who used coupled discrete harmonic functions (in fact, discrete holomorphic functions) to construct tilings of rectangles into squares with different integral side lengths.

Discrete harmonic functions on the square lattice were studied by a number of authors in the 1920s, including Courant, Friedrichs, and Lewy, who showed convergence of solutions of the Dirichlet boundary value problem to their corresponding continuous counterpart [CFL28]. Recently, Skopenkov studied Dirichlet boundary value problems of discrete analytic functions on general quad-graphs in [Sko13], and he showed a convergence result in the case of quad-graphs where the diagonals of quadrilaterals intersect orthogonally.

Discrete holomorphic functions on the square lattice were studied by Isaacs [Isa41]. He proposed two different definitions for holomorphicity. The first one said that f is *monodiffric of the first kind*, if

$$f(z+i\varepsilon)-f(z)=i(f(z+\varepsilon)-f(z)),$$

where ε denotes the side length of the squares. This definition is not symmetric on the square lattice, but it becomes symmetric on the triangular lattice obtained by inserting the diagonals parallel to $1-i$. Dynnikov and Novikov studied this notion in [DN03].

Isaac's second definition was given by

$$f(z+i\varepsilon)-f(z+\varepsilon)=i(f(z+(1+i)\varepsilon)-f(z)).$$

In this context, it is natural to consider the real part of a discrete holomorphic function as being defined on one type of vertices, say black, and the imaginary part on the other type of vertices, say white, corresponding to a bipartite decomposition of the square lattice. Lelong-Ferrand reintroduced this notion in [Fer44], and developed the theory to a level that allowed her to prove the Riemann mapping theorem using discrete methods [LF55]. Duffin also studied discrete complex analysis on the square grid [Duf56], and he was the first who extended the theory to rhombic lattices [Duf68]. Mercat [Mer01b], Kenyon [Ken02b], and Chelkak and Smirnov [CS11] resumed the investigation of discrete complex analysis on rhombic lattices, or, equivalently, isoradial graphs.

Some two-dimensional discrete models in statistical physics exhibit conformally invariant properties in the thermodynamical limit. Such conformally invariant properties were established by Smirnov for site percolation on a triangular grid [Smi01] and for the random cluster model [Smi10], by Chelkak and Smirnov for the Ising model [CS12], and by Kenyon for the dimer model on a square grid (domino tiling) [Ken02a]. In all cases, linear theories of discrete analytic functions on regular grids were highly important. Kenyon, Chelkak, and Smirnov obtained important analytic results [Ken02b, CS11], which were instrumental in the proof that the critical Ising model is universal, i.e., that the scaling limit is independent of the shape of the lattice [CS12]. Already Mercat related the theory of discrete complex analysis to the Ising model [Mer01b], and investigated criticality.

Important non-linear discrete theories of complex analysis involve circle packings, or, more generally, circle patterns. Rodin and Sullivan first proved that the Riemann mapping of a complex domain to the unit disk can be approximated by circle packings [RS87]. A similar result for isoradial circle patterns, even with irregular combinatorics, is due to Bücking [Bü08]. In their paper [BMS05], Bobenko, Mercat, and Suris showed how the linear theory of discrete holomorphic functions on quad-graphs can be obtained by linearizing the theory on circle patterns: Discrete holomorphic functions describe infinitesimal deformations of circle patterns. In the case of parallelogram-graphs, they embedded the quad-graph in $\mathbb{Z}^n$, and introduced a discrete exponential function and a discrete logarithm, generalizing Kenyon's discrete exponential and discrete Green's function [Ken02b].

Our setup in Chapter 2 is a strongly regular cellular decomposition of the complex plane into quadrilaterals, called quad-graph, that we assume to be bipartite. Of crucial importance for our work is the medial graph of a quad-graph. It provides the connection between the notions of discrete derivatives of Kenyon [Ken02b], Mercat [Mer07], and Chelkak and Smirnov [CS11], extended from rhombic to general quad-graphs, and discrete differential forms and discrete exterior calculus as suggested by Mercat [Mer01b, Mer08]. Concerning discrete differential forms, we get essentially the same definitions as Mercat proposed in [Mer08]. However, our notation of discrete exterior calculus is slightly more general, and shows its power when considering integral formulae. Discrete Stokes' Theorem 2.9 will be a consequence and not part of the definition of the discrete exterior derivate as in the work of Mercat [Mer01b, Mer08], and in Theorem 2.16 we prove that the discrete exterior derivative is a derivation of the discrete wedge-product. These two theorems are the most powerful ones in our setting. In particular, the proof of discrete Green's identities in Theorem 2.23 is an immediate corollary. Here, a discrete version of Green's first identity is provided for the first time.

According to the paper [CS11] of Chelkak and Smirnov, one of the unpleasant facts of all discrete theories of complex analysis is that (pointwise) multiplication of discrete holomorphic functions does not yield a discrete holomorphic function in general. We suggest at least a partial solution in Corollary 2.13, where we describe how the medial graph allows to (kind of pointwise) multiply discrete holomorphic functions to a function that is not defined on the vertex sets of the original graphs anymore, but discrete holomorphic in a certain sense.

Based on Skopenkov's results on the existence and uniqueness of solutions to the discrete Dirichlet boundary value problem [Sko13], we prove surjectivity of the discrete differentials and the discrete Laplacian seen as linear operators in Theorem 2.30. In particular, discrete Green's functions and discrete Cauchy's kernels z^{-1} exist. As a consequence, we can formulate discrete Cauchy's integral formulae for discrete holomorphic functions in Theorem 2.35, and for the discrete derivative of a discrete holomorphic function on the vertices of the quad-graph in Theorem 2.36. Unfortunately, we cannot require any certain asymptotic behavior of them in the general setup so far. But at least, we show in Theorem 2.31 that any discrete harmonic function with asymptotics $o(v^{-1/2})$ is constant, provided that all interior angles and side lengths of the quadrilaterals are bounded.

Chapter 4 is devoted to discrete complex analysis on planar parallelogram graphs. There, we construct discrete Green's functions and discrete Cauchy's kernels with asymptotics similar to the functions in the rhombic case [Ken02b, Bü08, CS11], and close to the smooth case. This will be proven in Theorems 4.3, 4.4, and 4.6, assuming that the interior angles and the ratio of side lengths of all parallelograms is bounded. The construction of these functions is based on the discrete exponential introduced by Kenyon on quasicrystallic rhombic quad-graphs [Ken02b], and by Bobenko, Mercat, and Suris on quasicrystallic parallelogram graphs [BMS05]. In the end, we close with the very special case of the integer lattice of a skew coordinate system in the complex plane. In this case, we show in Theorem 4.7 that discrete Cauchy's integral formulae for higher order discrete derivatives of a discrete holomorphic function exist, and that the asymptotics of these formulae match the expectations from the previous results.

Finally, in Appendix A we give the proofs of some statements concerning the combinatorics of parallelogram-graphs that we use for determining the asymptotic behavior of certain discrete functions.

1.2 Discrete Riemann surfaces

Mercat extended the theory from domains in the complex plane to discrete Riemann surfaces, first considering cellular decompositions into rhombi [Mer01b], and later generalizing the notions to general quadrilaterals [Mer08]. In the case of cellular decompositions into rhombi, he introduced discrete period matrices [Mer01a, Mer07]. In these papers, Mercat made first attempts to prove the convergence; Bobenko and Skopenkov eventually showed convergence of discrete period matrices and discrete Abelian integrals to their continuous counterparts [BS12]. The motivation for this theory of discrete Riemann surfaces is derived from statistical physics, in particular, the Ising model. Mercat defined a discrete Dirac operator and discrete spin structures, and he identified criticality in the Ising model with rhombic quad-graphs. Recently, Bobenko, Pinkall, and Springborn developed a nonlinear theory of discrete conformal mappings that discretizes the uniformization of Riemann surfaces, and metrics with constant curvature.

In Chapter 3, we follow the ideas of Mercat in [Mer01a, Mer01b, Mer07, Mer08], but differ in some aspects. One difference is certainly that we do not focus on discrete complex structures given by positive real numbers ρ_Q. Here, the discrete complex structure defines discrete holomorphic functions by demanding that

$$f(w_+) - f(w_-) = i\rho_Q \left(f(b_+) - f(b_-) \right)$$

holds on any quadrilateral Q with black vertices b_-, b_+, and white vertices w_-, w_+. But the main difference lies in our approach using the medial graph that includes local representations of discrete differentials, leading to more intuitive notions, more instructive proofs, and even new results. Nevertheless, we recover the definitions of Mercat in our setting.

Note that our method requires in addition to a discrete complex structure local charts around the vertices of the quad-graph. However, we will show in Proposition 3.1 that such charts exist for any discrete complex structure, and it turns out that these charts simplify our discussion of discrete exterior calculus, but the relevant definitions and the important theorems actually do not depend on these charts. Still, it makes sense to include these charts in the definition of discrete holomorphic mappings between discrete Riemann surfaces, since otherwise non-trivial mappings that are locally (essentially) constant exist. But even if we do not take charts around vertices into account, Theorem 3.10, a discretization of the Riemann-Hurwitz formula, holds true.

Since the notion of discrete holomorphic mappings is too rigid to go further, we concentrate on discrete meromorphic functions, and discrete one-forms. Again, the most powerful theorems are Theorems 2.9 and 2.16 from Chapter 2 that are valid on discrete Riemann surfaces as well. We will use them to prove the discrete Riemann Bilinear Identity 3.12 in a way very close to the proof of the classical Riemann bilinear identity. Concerning discrete periods, the medial graph approach has one disadvantage: For some operations such as the discrete wedge product we have to restrict to certain classes of discrete differentials. In the end, we recover the discrete period matrices of Mercat [Mer01a, Mer07]. Then, we follow the classical approach to compute the dimension of the space of discrete holomorphic differentials in Corollary 3.18 by investigating discrete harmonic differentials.

Furthermore, we recover not only the discrete Abelian integrals of the first and the third kind of Bobenko and Skopenkov [BS12] in our general setup; also, we define discrete Abelian integrals of the second kind. This leads to the discrete Riemann-Roch Theorem 3.30 that generalizes the result of Bobenko and Skopenkov [BS12] to general quad-graphs, and allows double zeroes of discrete meromorphic functions or double poles of discrete Abelian differentials. In the end, we shortly discuss discrete Abel-Jacobi maps, and discuss analogies to the classical theory.

1.3 Discrete variational systems

Discrete integrable systems on quad-graphs serve as discretizations of integrable partial differential equations with a two-dimensional space-time, as suggested by Bobenko and Suris in [BS02]. They identified integrability of such systems with their multi-dimensional consistency, as did also Nijhoff in [Nij02]. This property was used by Adler, Bobenko, and Suris in [ABS03] to classify integrable systems on quad-graphs, resulting in the now famous ABS list of discrete integrable equations.

Multi-dimensional consistent quad-equations can be consistently imposed on all two-dimensional sublattice of $\mathbb{Z}^d$, i.e., the corresponding multidimensional system possesses solutions, such that its restrictions to two-dimensional quad-surfaces are solutions of the original two-dimensional system. In [ABS03], Adler, Bobenko, and Suris described the variational (Lagrangian) formulation of such systems. Being more precise, they showed that solutions of some discrete integrable quad-equations of the ABS list on a quad-surface Σ are critical points of a certain discrete action functional $S = \int_\Sigma \mathcal{L}$, where $\mathcal{L}$ is a suitable discrete two-form on Σ, called the Lagrangian. Lobb and Nijhoff found out in [LN09] that the discrete action functional takes the same value if the underlying surface is changed only locally, meaning that $\mathcal{L}$ is closed. Bobenko and Suris extended these results to all equations from the ABS list, and gave a more conceptual proof in [BS10b]. Variational principles are a very powerful tool for numerical simulations of problems in classical mechanics. For a presentation of discrete mechanics along with a discrete variational principle and numerical applications, we refer to the paper [MW97] of Marsden and Wendlandt.

Following the papers [BS14] of Bobenko and Suris, and [BPS14] of Boll, Petrera, and Suris, the idea of Lobb and Nijhoff can be equivalently stated as follows: The solutions of discrete integrable systems give critical points simultaneously for discrete action functionals along all possible two-dimensional quad-surfaces of $\mathbb{Z}^d$, and the Lagrangian is closed on solutions. This interpretation corresponds to the classical theory of pluriharmonic functions. A pluriharmonic function is defined as a real-valued function of several complex variables that minimizes the Dirichlet energy along any holomorphic curve in its domain. The relation to pluriharmonic functions motivated Bobenko and Suris to introduce *pluri-Lagrangian problems*: Given a k-dimensional Lagrangian $\mathcal{L}$ in d-dimensional space that depends on a sought-after function u of k variables, one looks for functions u that deliver critical points to $S = \int_\Sigma \mathcal{L}$ on a k-dimensional surface Σ. They claimed that integrability of variational systems should be understood as the existence of the pluri-Lagrangian structure.

A general theory of one-dimensional pluri-Lagrangian systems was developed by Suris in [Sur13]; two-dimensional systems were discussed by Boll, Petrera, and Suris in [BPS14]. They identified the main building blocks of the discrete Euler-Lagrange equations of the pluri-Lagrangian systems as the corresponding equations at the quad-surface consisting of three elementary squares adjacent to a vertex of valence three, and called them 3D-corner equations. They discussed the notion of consistency of the system of 3D-corner equations, and analyzed it for a special class of three-point two-forms motivated by discrete integrable quad-equations of the ABS list.

Motivated by the connection to pluriharmonic functions, Bobenko and Suris studied in [BS14] the case of quadratic Lagrangian two-forms (corresponding to the Dirichlet energy in the setting of pluriharmonic functions), leading to linear 3D-corner equations. In particular, they introduced the notion of *discrete pluriharmonic functions*. These are exactly solutions of the pluri-Lagrangian problem for a Lagrangian $\mathcal{L}$ quadratic in its arguments, the corresponding discrete action functional was called the *Dirichlet energy*. As differential equations governing pluriharmonic functions are, the system of 3D-corner equations is overdetermined.

In [BS14], Bobenko and Suris also applied the setting of pluri-Lagrangian problems to discrete complex analysis. As for discrete Riemann surfaces introduced in Section 1.2 above, discrete holomorphicity is defined by a discrete complex structure given by complex numbers ρ_Q with positive real part on each quadrilateral. In Section 2.4 of Chapter 2, we describe how the discrete Laplacian and the discrete Dirichlet energy can be defined in terms of the ρ_Q. Also, Lemma 2.25 in fact shows that the discrete harmonic conjugate of a discrete harmonic function can be easily constructed knowing the ρ_Q. Defining complex numbers on all elementary squares of $\mathbb{Z}^d$ gives rise to notions of discrete holomorphic functions and discrete Dirichlet energies on arbitrary quad-surfaces in $\mathbb{Z}^d$. Functions that are discrete harmonic on every quad-surface, i.e., that are critical points of the corresponding discrete Dirichlet energy, are called *discrete pluriharmonic* in [BS14]. Now, existence of discrete pluriharmonic functions is a condition on the complex weights ρ_Q. Locally, it can be reformulated as a star-triangle relation.

Note that quantization of discrete integrable systems on quad-graphs yields solvable lattice models. Here, the consistency principle corresponds to the quantum Yang-Baxter equation. Classical discrete integrable systems on quad-graphs are then recovered in the quasi-classical limit. The corresponding action functional is derived as a quasi-classical limit of the partition function of the corresponding integrable quantum model (the Lagrangians being the quasi-classical limit of the Boltzmann weights). A quantization of circle patterns, with the corresponding quantization of the action functional introduced by Bobenko and Springborn in [BS04], was carried out by Bazhanov, Mangazeev, and Sergeev in [BMS07]. The quantum "master solution" that serves as a quantization of the discrete Laplace-type equation corresponding to the (Q4)-system was recently introduced by Bazhanov and Sergeev in [BS10a, BS11]. Functionals obtained in the quasi-classical limit are related to the variational problems considered in Chapter 5.

Our setting in Chapter 5 is again a bipartite quad-graph, and we discuss discrete Laplace-type equations that are motivated from discrete integrable quad-equations of the ABS list. We explain why the reality conditions, i.e., restrictions of the variables and parameters to certain one-dimensional complex subspaces, should be actually all that are reasonable, and compute the corresponding discrete action functionals. Our variational description turns out to be slightly more general than the previous formulation of Bobenko and Suris in [BS10b], since we include boundary terms yielding a nonintegrable generalization. Then, we derive sufficient conditions on the parameters such that the corresponding discrete generalized action functionals are strictly concave or strictly convex. In all cases except the ones corresponding to (Q4), these conditions are even necessary. Our results are summarized in Theorem 5.1.

Strict convexity of the functional implies the uniqueness of solutions to Dirichlet boundary value problems, and helps to investigate their existence. Moreover, minimization of the corresponding functional is an effective tool to construct the corresponding solution numerically. For example, Stephenson's program `circlepack` constructs circle packings by a method of Thurston that can be interpreted as a particular method to minimize the action functional of circle patterns that was derived by Bobenko and Springborn in [BS04]. By relating the discrete generalized action functionals corresponding to discrete Laplace-type equations of type (Q3) to the circle pattern functionals of Bobenko and Springborn, we can adapt their existence results to our setting. In addition, we prove that the Dirichlet boundary value problems in the case of (Q4) with rectangular or rhombic lattices can be uniquely solved.

Furthermore, we discuss some conditions on the combinatorics of a quad-graph to be able to support discrete action functionals coming from discrete integrable Laplace-type equations, i.e., the discrete Laplace equations of discrete integrable quad-equations. Finally, we continue the geometric interpretation of the discrete Laplace-type equations of type (Q3) as Euclidean and hyperbolic circle patterns with conical singularities. Our geometric interpretation of (Q3) turns out to be closely connected to the paper [BMS05] of Bobenko, Mercat, and Suris. They compared the linear theory of discrete complex analysis on rhombic quad-graphs with a nonlinear theory on circle patterns, and showed how the first can be obtained as a linearization of the second. Moreover, they discussed integrable circle patterns, and discrete integrability of a system of cross-ratio equations. Their notions of integrability fit well to ours in terms of the parameters of the discrete Laplace-type equations.

1.4 Open questions

Here, we want to just mention some questions that arise from this thesis. The first one is posed by Smirnov, asking how general bipartite quad-graphs should be embedded into the complex plane. It follows from Kenyon's and Schlenker's characterization, Proposition A.1, that not every bipartite quad-graph can be embedded as a planar rhombic quad-graph. Also, it is not hard to show that spider-graphs as discussed in Section 5.4 with a central square cannot be embedded as planar quad-graphs consisting only of convex quadrilaterals whose diagonals intersect orthogonally. But is it generally possible to use not necessarily convex quadrilaterals whose diagonals are orthogonal to each other that are in addition even of the same length? Another way to interpret quad-graphs geometrically is the paper [BSST40] of Brooks, Smith, Stone, and Tutte, investigating tilings of rectangles into squares, or, more generally, into rectangles. This corresponds to a discrete complex structure given by real ρ_Q, but what about complex weights?

Clearly, there is much work left regarding convergence results. Let us just focus on some aspects of the asymptotic behavior of certain functions. In Theorem 2.31, we have proven that under reasonable geometric conditions on the quadrilaterals, any discrete harmonic function with asymptotics $o(v^{-1/2})$ is essentially constant. Can we extend this results to asymptotics $o(1)$, yielding a discrete Liouville's theorem? If not, what is the best possible bound?

In addition, we have constructed discrete Green's functions and discrete Cauchy's kernels with appropriate asymptotics in the setting of planar parallelogram-graphs in Chapter 4. Existence of these functions without requiring certain asymptotics is a corollary of Theorem 2.30. Is it possible to explicitly construct discrete Green's functions and discrete Cauchy's kernels in the general setup? If yes, what are their asymptotics? Figure 3.4 in Proposition 3.32 suggests that some assumptions on the combinatorics might be necessary. A way to tackle this problem could be the concept of discrete pluriharmonic functions suggested by Bobenko and Suris in [BS14].

A less important question that comes into mind is a classification of quad-graphs that support convex integrable labelings, i.e., labelings of edges, such that the discrete generalized action functional of the corresponding discrete Laplace-type equation of type (Q3) or (Q4) is convex. In Section 5.4, we have proven that a large class of rhombic-embeddable quad-graphs allows convex integrable labelings, but not all rhombic quad-graphs do.

Finally, we discussed a relation between the discrete integrable systems (Q3) and circle patterns in Section 5.5, and there is a nonlinear theory of discrete complex analysis on circle patterns. Is there a geometric interpretation of (Q4), and does it support a nonlinear version of discrete complex analysis?

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Chapter 2

Discrete complex analysis on planar quad-graphs

Our setup in this chapter is a strongly regular cellular decomposition of the complex plane into quadrilaterals, called quad-graph, that we assume to be bipartite. Although we focus on planar quad-graphs here, many of our notions and theorems generalize to discrete Riemann surfaces. A corresponding linear theory of discrete Riemann surfaces will be discussed in Chapter 3.

Basic notations for quad-graphs used in this and the following chapter are introduced in Section 2.1. In particular, we introduce the medial graph of a quad-graph, what allows us to nicely define discrete derivatives and discrete differential forms in Section 2.2. We continue with investigating the discrete Laplacian in Section 2.4. Main theorems of this section are the discrete Green's Identities 2.23 and Theorem 2.30 that essentially guarantees the existence of discrete Green's functions and discrete Cauchy's kernels that we discuss in Sections 2.5 and Section 2.6. As a consequence, we can formulate discrete Cauchy's integral formulae for discrete holomorphic functions (Theorem 2.35) and the discrete derivative of a discrete holomorphic function on the vertices of the quad-graph (Theorem 2.36).

Note that the only result regarding asymptotics of functions is Theorem 2.31. It states that any discrete harmonic function with asymptotics $o(v^{-1/2})$ is zero everywhere, provided that all interior angles and side lengths of the quadrilaterals are bounded. But in Chapter 4, we will derive results concerning the asymptotics of discrete Green's functions and discrete Cauchy's kernels in the case of planar parallelogram-graphs with bounded interior angles and bounded ratios of side lengths of the parallelograms.

2.1 Basic definitions and notation

The aim of this section is to introduce first bipartite quad-graphs and some basic notation in Section 2.1.1, and then to discuss the medial graph in Section 2.1.2.

2.1.1 Bipartite quad-graphs

We consider a strongly regular cellular decomposition of the complex plane $\mathbb{C}$ described by an embedded bipartite *quad-graph* Λ, such that 0-cells correspond to vertices $V(\Lambda)$, 1-cells to edges $E(\Lambda)$, and 2-cells to quadrilateral faces $F(\Lambda)$. We refer to the maximal independent sets of vertices of Λ as *black* and *white* vertices. Furthermore, we restrict to locally finite cellular decompositions, i.e., a compact subset of $\mathbb{C}$ contains only finitely many quadrilaterals.

The assumption of strong regularity asserts that the boundary of a quadrilateral contains a particular vertex or a particular edge at most once, and two different edges or faces have at most one vertex or edge in common, respectively. As a consequence, if a line segment connecting two vertices of Λ is the (possibly outer) diagonal of a quadrilateral, there is just one such face of Λ. Let Γ and Γ^* be the graphs defined on the black and white vertices where the edges are exactly corresponding diagonals of faces of Λ. If the diagonal lies outside the face, it is more convenient to consider the corresponding edge of Γ or Γ^* not as the straight line segment connecting its two endpoints, but as a curve lying inside the face. Then, the *duality* between Γ and Γ^* becomes obvious: Black and white vertices are in one-to-one correspondence to the white and black faces they are contained in, and black and white edges dual to each other are exactly these who cross each other. For simplicity, we identify vertices of Λ, or of Γ and Γ^*, with their corresponding complex values, and to oriented edges of $\Lambda, \Gamma, \Gamma^*$ we assign the complex numbers determined by the difference of their two endpoints.

To Λ we associate its *dual* $\diamond := \Lambda^*$. In general, we look at $\diamond$ in an abstract way, identifying vertices or faces of $\diamond$ with corresponding faces or vertices of Λ, respectively. However, in the particular case that all quadrilaterals are parallelograms, it makes sense to place the vertices of $\diamond$ at the centers of the parallelograms. Here, the center of a parallelogram is the point of intersection of its two diagonals. Further details will be given in Sections 2.2.1 and 2.2.3.

If a vertex $v \in V(\Lambda)$ is a vertex of a quadrilateral $Q \in F(\Lambda) \cong V(\diamond)$, we write $Q \sim v$ or $v \sim Q$, and say that v and Q are *incident* to each other. The vertices of Q are denoted by b_-, w_-, b_+, w_+ in counterclockwise order, where $b_\pm \in V(\Gamma)$ and $w_\pm \in V(\Gamma^*)$.

Definition. For a quadrilateral $Q \in V(\diamond)$ we define

$$\rho_Q := -i\frac{w_+ - w_-}{b_+ - b_-}.$$

Moreover, let

$$\varphi_Q := \arccos\left(\operatorname{Re}\left(i\frac{\rho}{|\rho|}\right)\right) = \arccos\left(\operatorname{Re}\left(\frac{(b_+ - b_-)\overline{(w_+ - w_-)}}{|b_+ - b_-||w_+ - w_-|}\right)\right)$$

be the angle under which the diagonal lines of Q intersect.

Note that $0 < \varphi_Q < \pi$. Figure 2.1 shows a finite bipartite quad-graph together with the notations we have introduced for a single quadrilateral Q, and the notations we are using later for the *star of a vertex* v, i.e., the set of all faces incident to v.

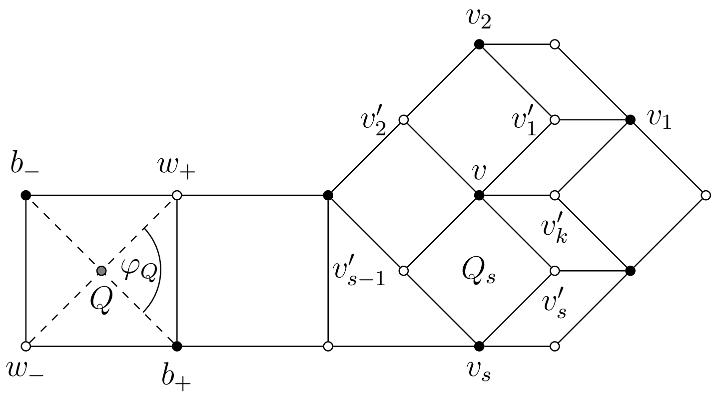


Figure 2.1: Bipartite quad-graph with notations

In addition, we denote by $\diamond_0$ always a connected subset of $\diamond$. It is said to be *simply-connected* if the corresponding set of cells in $\mathbb{C}$ is simply-connected. Its vertices induce a subgraph Λ_0 of Λ, together with subgraphs Γ_0 of Γ and Γ_0^* of Γ^*. For simplicity, we assume that the induced subgraphs are connected as well. By $\partial\Lambda_0$ we denote the subgraph of Λ_0 that consists of boundary vertices and edges.

2.1.2 Medial graph

Definition. The *medial graph* X of Λ is defined as follows. Its vertex set is given by all the midpoints of edges of Λ, and two vertices are adjacent if and only if the corresponding edges belong to the same face and have a vertex in common. The set of faces of X is in bijection with $V(\Lambda) \cup V(\diamond)$: A face F_v corresponding to $v \in V(\Lambda)$ consists of the edges of Λ incident to v, and a face F_Q corresponding to $Q \in V(\diamond)$ consists of the four edges of Λ belonging to Q.

Any edge e of X is the common edge of two faces F_Q and F_v for $Q \sim v$, denoted by $[Q, v]$.

Let $Q \in V(\diamond)$ and $v_0 \sim Q$. Due to Varignon's theorem, F_Q is a parallelogram, and the complex number assigned to the edge $e = [Q, v_0]$ connecting the midpoints of edges $v_0v'_-$ and $v_0v'_+$ of Λ is just half of $e = v'_+ - v'_-$. Note that if Q is nonconvex, some part of F_Q lies outside Q, and it may happen that v lies not inside F_v. These situations do not occur if all quadrilaterals are convex.

In Figure 2.2, showing Λ with its medial graph, the vertices of F_Q and F_v are colored gray.

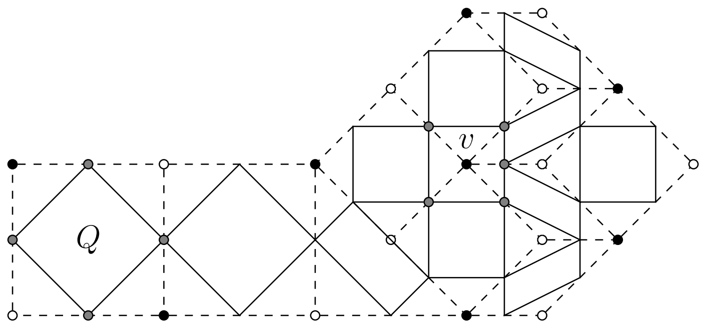


Figure 2.2: Bipartite quad-graph (dashed) with medial graph (solid)

For a subgraph $\diamond_0 \subseteq \diamond$, we denote by $X_0 \subseteq X$ the subgraph of X consisting of all edges $[Q, v]$ where $Q \in V(\diamond_0)$ and $v \sim Q$. In the case of convex quadrilaterals, this means that we take all edges of X lying inside $\diamond_0$. The medial graph X corresponds to a (strongly regular and locally finite) cell decomposition of $\mathbb{C}$ in a canonical way. In particular, we can talk about a topological disk in $F(X)$, and about a (counterclockwise oriented) boundary ∂X_0.

Definition. For $v \in V(\Lambda)$ and $Q \in V(\diamond)$, let P_v and P_Q be the closed paths on

X connecting the midpoints of edges of Λ incident to v and Q, respectively, in counterclockwise direction. In Figure 2.2, their vertices are colored gray. We say that P_v and P_Q are *discrete elementary cycles.*

2.2 Discrete holomorphicity

In the classical theory, a real differentiable function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if the Cauchy-Riemann equation $\partial f/\partial x = -i\partial f/\partial y$ is satisfied in all points of U. Moreover, holomorphic functions with nowhere-vanishing derivative preserve angles; and at a single point, lengths are uniformly scaled.

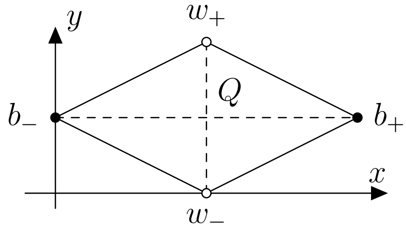


Figure 2.3: Discretization of Cauchy-Riemann equation

Now, let us imagine a rhombus Q in $\mathbb{C}$ with vertices b_-, w_-, b_+, w_+ and diagonals b_-b_+, w_-w_+ aligned to the real x- and the imaginary y-axis, see Figure 2.3. Then,

$$\begin{aligned} \frac{f(b_+) - f(b_-)}{b_+ - b_-} &\quad \text{discretizes} \quad \frac{\partial f}{\partial x}(Q), \\ \frac{f(w_+) - f(w_-)}{w_+ - w_-} &\quad \text{discretizes} \quad -i\,\frac{\partial f}{\partial y}(Q). \end{aligned}$$

This motivates the following definition of discrete holomorphicity due to Mercat [Mer08] that was also used previously in the rhombic setting by Duffin [Duf68] and others.

Definition. Let $Q \in V(\diamond)$ and f be a complex function on b_-, w_-, b_+, w_+. f is said to be *discrete holomorphic* at Q if the *discrete Cauchy-Riemann equation* is satisfied:

$$\frac{f(b_+) - f(b_-)}{b_+ - b_-} = \frac{f(w_+) - f(w_-)}{w_+ - w_-}.$$

If Q is not a rhombus, we do not have an immediate interpretation of the discrete Cauchy-Riemann equation as before. But if a discrete holomorphic f does not have the same value on both black vertices b_-, b_+, it preserves the angle φ_Q, and f uniformly scales the lengths of the diagonals of Q. However, the image of Q under f might be a degenerate quadrilateral.

We immediately see that for discrete holomorphicity, only the differences at black and at white vertices matter. Hence, we should not consider constants on $V(\Lambda)$, but biconstants [Mer07] determined by each a value on $V(\Gamma)$ and $V(\Gamma^*)$. A function that is constant on $V(\Gamma)$ and constant on $V(\Gamma^*)$ is said to be *essentially constant*.

In the following, we will define discrete analogs of $\partial, \bar{\partial}$, first of functions on $V(\Lambda)$ in Section 2.2.1, and later of functions on $V(\diamond)$ in Section 2.2.3. Before, we introduce discrete differential forms in Section 2.2.2.

2.2.1 Discrete derivatives of functions on $V(\Lambda)$

Remember that the derivatives $\partial := \partial/\partial z$ and $\bar{\partial} := \partial/\partial \bar{z}$ in complex analysis are defined through

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right);$$

the coefficients in front of the partial derivatives in $\bar{\partial}$ are complex conjugate to the coefficients appearing in ∂. Using the interpretation

$$\begin{aligned}\frac{f(b_+) - f(b_-)}{b_+ - b_-} &\cong \frac{\partial f}{\partial x},\\ \frac{f(w_+) - f(w_-)}{w_+ - w_-} &\cong -i\frac{\partial f}{\partial y}\end{aligned}$$

for rhombi Q as above, the definition of discrete derivatives in rhombic quad-graphs used by Chelkak and Smirnov in [CS11],

$$\begin{aligned}\partial_\Lambda f(Q) &= \frac{1}{2}\left(\frac{f(b_+) - f(b_-)}{b_+ - b_-} + \frac{f(w_+) - f(w_-)}{w_+ - w_-}\right),\\ \bar{\partial}_\Lambda f(Q) &= \frac{1}{2}\left(\frac{f(b_+) - f(b_-)}{\overline{b_+ - b_-}} + \frac{f(w_+) - f(w_-)}{\overline{w_+ - w_-}}\right),\end{aligned}$$

becomes plausible.

In a general quadrilateral Q, the diagonals usually do not intersect orthogonally, and their quotient is not necessarily purely imaginary. Therefore, we have to choose different factors in front of the difference quotients, taking the deviation $\varphi_Q - \pi/2$ from orthogonality into account.

Definition. Let $Q \in V(\diamondsuit)$, and let f be a complex function on b_-, w_-, b_+, w_+. The *discrete derivatives* $\partial_\Lambda f$, $\bar{\partial}_\Lambda f$ are defined by

$$\partial_\Lambda f(Q) := \lambda_Q \frac{f(b_+) - f(b_-)}{b_+ - b_-} + \bar{\lambda}_Q \frac{f(w_+) - f(w_-)}{w_+ - w_-},$$

$$\bar{\partial}_\Lambda f(Q) := \bar{\lambda}_Q \frac{f(b_+) - f(b_-)}{\overline{b_+ - b_-}} + \lambda_Q \frac{f(w_+) - f(w_-)}{\overline{w_+ - w_-}},$$

where $2\lambda_Q := \exp\left(-i\left(\varphi_Q - \frac{\pi}{2}\right)\right) / \sin(\varphi_Q)$.

Obviously, essentially constant functions have vanishing discrete derivatives. If the quadrilateral Q is a rhombus, $\varphi_Q = \pi/2$, and $\lambda_Q = 1/2$. Thus, the definition above then reduces to the previous one in [CS11]. The definition of discrete derivatives also matches the notion of discrete holomorphicity; and the discrete derivatives approximate their smooth counterparts correctly up to order one for general quad-graphs, and up to order two for parallelogram-graphs:

Proposition 2.1. *Let $Q \in V(\diamondsuit)$ and f be a complex function on b_-, w_-, b_+, w_+.*

(i) f is discrete holomorphic at Q if and only if $\bar{\partial}_\Lambda f(Q) = 0$.

(ii) If $f(v) = v$, $\bar{\partial}_\Lambda f(Q) = 0$ and $\partial_\Lambda f(Q) = 1$.

(iii) If Q is a parallelogram and $f(v) = v^2$, $\bar{\partial}_\Lambda f(Q) = 0$ and $\partial_\Lambda f(Q) = 2Q$, where Q as a vertex is placed at the center of the parallelogram.

(iv) If Q is a parallelogram and $f(v) = |v|^2$, $\bar{\partial}_\Lambda f(Q) = \overline{\partial_\Lambda f(Q)} = Q$, where Q as a vertex is placed at the center of the parallelogram.

Proof. (i) We observe that

$$\frac{2\sin(\varphi_Q)\bar{\lambda}_Q}{\overline{b_+ - b_-}} = \frac{\exp\left(i\left(\varphi_Q - \frac{\pi}{2}\right)\right)}{\overline{b_+ - b_-}} = -i\frac{w_+ - w_-}{|w_+ - w_-||b_+ - b_-|},$$

$$\frac{2\sin(\varphi_Q)\lambda_Q}{\overline{w_+ - w_-}} = \frac{\exp\left(-i\left(\varphi_Q - \frac{\pi}{2}\right)\right)}{\overline{w_+ - w_-}} = i\frac{b_+ - b_-}{|w_+ - w_-||b_+ - b_-|}.$$

So if we multiply $\bar{\partial}_\Lambda f(Q)$ by $2i|w_+ - w_-||b_+ - b_-|\sin(\varphi_Q) \neq 0$, we obtain

$$(w_+ - w_-)\left(f(b_+) - f(b_-)\right) - (b_+ - b_-)\left(f(w_+) - f(w_-)\right).$$

The last expression vanishes if and only if the discrete Cauchy-Riemann equation is satisfied.

(ii) Clearly, $f(v) = v$ satisfies the discrete Cauchy-Riemann equation. By the first part, $\bar{\partial}_\Lambda f(Q) = 0$. Due to $2\sin(\varphi_Q) = \exp\left(-i\left(\varphi_Q - \frac{\pi}{2}\right)\right) + \exp\left(i\left(\varphi_Q - \frac{\pi}{2}\right)\right)$, $\partial_\Lambda f(Q)$ simplifies to $\lambda_Q + \bar{\lambda}_Q = 1$.

(iii) For the function $f(v) = v^2$, the discrete Cauchy-Riemann equation is equivalent to $b_+ + b_- = w_+ + w_-$. But since Q is a parallelogram, both $(b_+ + b_-)/2$ and $(w_+ + w_-)/2$ equal the center Q of the parallelogram. Thus, f is discrete holomorphic at Q, and

$$\partial_\Lambda f(Q) = \lambda_Q(b_+ + b_-) + \bar{\lambda}_Q(w_+ + w_-) = 2Q(\lambda_Q + \bar{\lambda}_Q) = 2Q.$$

(iv) Since f is a real function, $\bar{\partial}_\Lambda f(Q) = \overline{\partial_\Lambda f(Q)}$ follows straight from the definition. Let $z \in \mathbb{C}$ be arbitrary. If $g(v) := v\bar{z}$, $\partial_\Lambda g(Q) = \bar{z}$ and $\partial_\Lambda \bar{g}(Q) = 0$ by the second part. So if $h(v) := |v - z|^2 = |v|^2 - v\bar{z} - \bar{v}z + |z|^2$, $\overline{\partial_\Lambda h(Q)} = \overline{\partial_\Lambda f(Q)} - z$. Hence, the statement is invariant under translation, and it suffices to consider the case when the center of the parallelogram Q is the origin. Then, $b_+ = -b_-$ and $w_+ = -w_-$ since Q is a parallelogram. It follows that $f(b_-) = f(b_+)$ and $f(w_-) = f(w_+)$, so $\partial_\Lambda f(Q) = 0$, as desired. □

Our first discrete analogs of classical theorems are immediate consequences of the discrete Cauchy-Riemann equation:

Proposition 2.2. *Let $\diamond_0 \subseteq \diamond$ and $f : V(\Lambda) \to \mathbb{C}$ be discrete holomorphic.*

(i) If f is purely imaginary or purely real, f is essentially constant.

(ii) If $\partial_\Lambda f \equiv 0$, f is essentially constant.

Proof. (i) Let b_-, b_+ be two adjacent vertices of Γ, and let $b_-, w_-, b_+, w_+ \in V(\Lambda_0)$ be the vertices of the corresponding quadrilateral in $\diamond$. Let us assume that $f(b_+) \neq f(b_-)$. Due to the discrete Cauchy-Riemann equation,

$$\frac{f(w_+) - f(w_-)}{f(b_+) - f(b_-)} = \frac{w_+ - w_-}{b_+ - b_-}.$$

The left hand side is real, but the right hand side is not, contradiction. The same argumentation goes through if we start with two adjacent vertices of Γ^*.

(ii) Since f is discrete holomorphic,

$$\frac{f(b_+) - f(b_-)}{b_+ - b_-} = \frac{f(w_+) - f(w_-)}{w_+ - w_-}.$$

$\partial_\Lambda f \equiv 0$ then yields that both sides of the discrete Cauchy-Riemann equation equal zero, so f is constant on $V(\Gamma_0)$ and constant on $V(\Gamma_0^*)$. $\square$

2.2.2 Discrete differential forms

We mainly consider two type of functions, functions $f : V(\Lambda) \to \mathbb{C}$ and functions $h : V(\diamond) \to \mathbb{C}$. An example for a function on the quadrilateral faces is $\partial_\Lambda f$. To get a unified notation, we extend these functions to functions on the faces of the medial graph X. Given $f : V(\Lambda) \to \mathbb{C}$, $f : F(X) \to \mathbb{C}$ equals f on faces corresponding to vertices of Λ, and 0 on faces corresponding to vertices of $\diamond$. Similarly, functions $h : V(\diamond) \to \mathbb{C}$ are extended to functions $h : F(X) \to \mathbb{C}$.

A *discrete one-form* ω is a complex function on the oriented edges of the medial graph X, and a *discrete two-form* Ω is a complex function on the faces of X. The evaluations of ω at an oriented edge e of X and of Ω at a face F of X are denoted by $\int_e \omega$ and $\iint_F \Omega$, respectively.

If P is a directed path in X consisting of oriented edges $e_1, e_2, \ldots, e_n$, the *discrete integral* along P is defined as $\int_P \omega = \sum_{k=1}^n \int_{e_k} \omega$. For closed paths P, we write $\oint_P \omega$ instead. In the case that P is the boundary of an oriented disk in X, we say that the discrete integral is a *discrete contour integral* with *discrete contour* P. Similarly, the *discrete integral* of Ω over a set of faces of X is defined.

Definition. The discrete one-forms dz and $d\bar{z}$ are defined in such a way that $\int_e dz = e$ and $\int_e d\bar{z} = \bar{e}$ hold for any oriented edge e of X. The discrete two-form $dz \wedge d\bar{z}$ is given by $\iint_F dz \wedge d\bar{z} = -4i\text{area}(F)$.

Remark. The reason why we define $\iint_F dz \wedge d\bar{z}$ as twice the value one would expect from the smooth setting will be clearer when we discuss discrete exterior calculus. Essentially, this factor of two gives nicer formulae. We will comment on this additional factor in more detail in Section 2.3.

For later purposes, let us compute area(F). First, if F is the Varignon parallelogram corresponding to the quadrilateral $Q \in V(\diamond)$,

$$\text{area}(F) = \frac{\text{area}(Q)}{2} = \frac{1}{4}|b_+ - b_-||w_+ - w_-|\sin(\varphi_Q).$$

Second, if F is the face of X corresponding to a vertex $v \in V(\Lambda)$, its area is a quarter of the area of the polygon $v_1'v_2' \ldots v_k'$ in the star of v. So area(F) equals

$$\frac{1}{4}\sum_{Q_s \sim v} \text{area}(\triangle v v_{s-1}' v_s') = \frac{1}{8}\sum_{Q_s \sim v} \text{Im}\left((v_s' - v)\,\overline{(v_{s-1}' - v)}\right) = \frac{1}{8}\sum_{Q_s \sim v} \text{Im}\left(v_s'\bar{v}_{s-1}'\right),$$

using that $\sum_{Q_s\sim v}\left(v\bar{v}'_{s-1}+\bar{v}v'_s\right)=\sum_{Q_s\sim v}\left(v\bar{v}'_s+\bar{v}v'_s\right)$ is real.

It turns out that discrete one-forms that actually come from discrete one-forms on Γ and Γ^* are of particular interest. We say that such a discrete one-form ω is of *type* $\diamond$. It is characterized by the property that for any $Q\in V(\diamond)$ there exist complex numbers p,q such that $\omega=pdz+qd\bar{z}$ on all edges $e=[Q,v]$, $v\sim Q$.

Definition. Let $g:F(X)\to\mathbb{C}$, and let ω be a discrete one-form and Ω a discrete two-form. Then, the products $g\omega$ and $g\Omega$ are defined by

$$\int\limits_e g\omega := (g(Q)+g(v))\int\limits_e \omega,$$
$$\iint\limits_F g\Omega := g(F)\iint\limits_F \Omega$$

for any edge $e=[Q,v]$ and any face F of X.

Remark. In general, adding a factor of one half to the definition of $g\omega$ seems to be more appropriate. In some of the discrete analogs we develop in the sequel, this missing factor causes an additional factor of $2^{\pm 1}$ compared to the classical theory. However, using the factor of one half in the definition of $g\omega$ would yield multiplicative factors in some other discrete analogs. In our opinion, the definition we choose leads to more pleasant formulae than the other. Actually, it is adapted to our focus on complex functions on vertices of Λ or $\diamond$, since the corresponding functions on $F(X)$ vanish on half of the faces of X.

Note that if f is a function on $V(\Lambda)$, the discrete integrals of fdz and $fd\bar{z}$ around a discrete elementary circle P_Q, $Q\in V(\diamond)$, do not depend on the choice of the extension of f. Also, the discrete integrals of hdz and $hd\bar{z}$ around a discrete elementary circle P_v, $v\in V(\Lambda)$, do not depend on the choice of the extension of h. Exactly these discrete integrals serve as a further justification for the choice of discrete derivatives of f, or as a definition of them for h. The corresponding discrete contours are illustrated in Figure 2.4 below.

Lemma 2.3. *Let $Q\in V(\diamond)$ and f be a complex function on the vertices of Q. Let P_Q be the discrete elementary cycle around Q and F the face of X corresponding to Q. Then,*

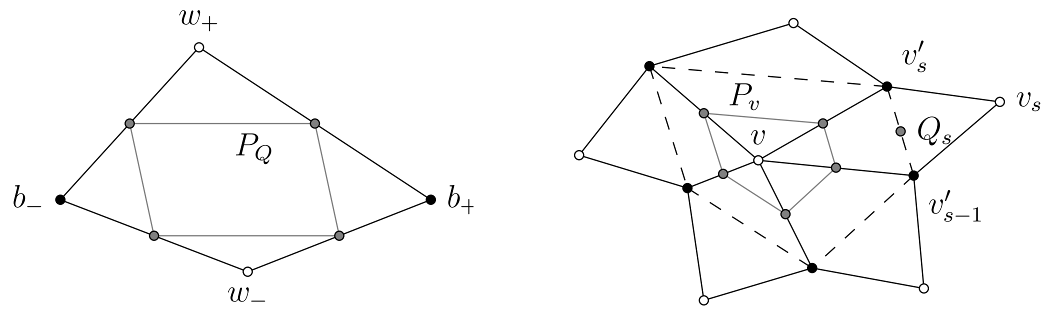
$$\partial_\Lambda f(Q)=\frac{-1}{4i\text{area}(F)}\oint\limits_{P_Q} fd\bar{z},$$

$$\bar{\partial}_\Lambda f(Q) = \frac{1}{4i\mathrm{area}(F)} \oint\limits_{P_Q} f dz.$$

Proof. Since F is a parallelogram, $f(b_+)$ and $-f(b_-)$ are multiplied by the same factor $\overline{(w_+ - w_-)}/2$ when evaluating the discrete contour integral $\oint_{P_Q} f d\bar{z}$. Therefore, the factor in front of $f(b_+) - f(b_-)$ in the right hand side of the first equation is

$$i\frac{\overline{w_+ - w_-}}{8\mathrm{area}(F)} = -\bar{i}\frac{\overline{w_+ - w_-}}{2\sin(\varphi_Q)|w_+ - w_-||b_+ - b_-|} = \frac{\exp\left(-i\left(\varphi_Q - \frac{\pi}{2}\right)\right)}{2\sin(\varphi_Q)(b_+ - b_-)} = \frac{\lambda_Q}{b_+ - b_-}$$

(compare with the proof of Proposition 2.1 (i)), which is exactly the factor appearing in $\partial_\Lambda f(Q)$. In an analogous manner, the factors in front of $f(w_+) - f(w_-)$ are equal. This shows the first equation. The second one follows from the first, noting that the factors in front of $f(b_+) - f(b_-)$ and $f(w_+) - f(w_-)$ on both sides of the second equation are just complex conjugates of the corresponding factors appearing in the first equation. □



(a) Lemma 2.3 for $\partial_\Lambda, \bar{\partial}_\Lambda$ (b) Definitions of $\partial_\diamond, \bar{\partial}_\diamond$

Figure 2.4: Integration formulae for discrete derivatives

Remark. The additional factor of 1/2 is due to the fact that in analogy to the smooth setup, we should not multiply $f(v)$ with dz (or $d\bar{z}$), but by the arithmetic mean of $f(v)$ and some intermediate value $f(Q)$ instead. Integrating fdz would then eliminate $f(Q)$.

2.2.3 Discrete derivatives of functions on $V(\diamond)$

Inspired by Lemma 2.3, we can now define the discrete derivatives for complex functions on $V(\diamond)$, see Figure 2.4 (b).

Definition. Let $v \in V(\Lambda)$ and h be a complex function defined on all quadrilaterals $Q_s \sim v$. Let P_v be the discrete elementary cycle around v and F the face of X corresponding to v. Then, the *discrete derivatives* $\partial_\diamond h$, $\bar{\partial}_\diamond h$ at v are defined by

$$\partial_\diamond h(v) := \frac{-1}{4i\text{area}(F)} \oint\limits_{P_v} h d\bar{z},$$
$$\bar{\partial}_\diamond h(v) := \frac{1}{4i\text{area}(F)} \oint\limits_{P_v} h dz.$$

h is said to be *discrete holomorphic* at v if $\bar{\partial}_\diamond h(v) = 0$.

The additional factor of $1/2$ comes for the same reason as in Lemma 2.3. Note that in the rhombic case, our definition coincides with the one used by Chelkak and Smirnov in [CS11]. As an immediate consequence of the definition, we obtain a *discrete Morera's theorem.*

Proposition 2.4. *Functions $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$ are discrete holomorphic if and only if $\oint_P f dz = 0$ and $\oint_P h dz = 0$ for all discrete contours P.*

Proof. Clearly, $\oint_{P_v} f dz = f(v) \oint_{P_v} dz = 0$ for any discrete elementary cycle P_v corresponding to a vertex $v \in V(\Lambda)$. Similarly, $\oint_{P_Q} h dz = 0$ for any $Q \in V(\diamond)$. Using Lemma 2.3 and the definition of $\bar{\partial}_\diamond$, f and h are discrete holomorphic if and only if $\oint_P f dz = 0$ and $\oint_P h dz = 0$ for all discrete elementary cycles P. To conclude the proof, we observe that any integration along a discrete contour can be decomposed into integrations along discrete elementary cycles. $\square$

The discrete derivatives of constant functions on $V(\diamond)$ vanish. As an analog of Proposition 2.1, we prove that the discrete derivatives $\partial_\diamond, \bar{\partial}_\diamond$ locally approximate their smooth counterparts correctly up to order one if the vertices Q are placed at the midpoints of black or white edges. Note that even for rhombic quad-graphs, these discrete derivatives generally do not coincide with the smooth derivatives in order two.

Proposition 2.5. *Let $v \in V(\Lambda)$, and let h be a complex function on all faces $Q_s \sim v$. Assume that the vertices Q_s are placed at the midpoints $(v'_{s-1} + v'_s)/2$ of edges of the star of v. Then, if $h(Q) = Q$, $\bar{\partial}_\diamond h(v) = 0$ and $\partial_\diamond h(v) = 1$.*

Proof.

$$4\oint\limits_{P_v} h dz = \sum_{Q_s\sim v} (v'_{s-1}+v'_s)(v'_s-v'_{s-1}) = \sum_{Q_s\sim v}\left((v'_s)^2-\left(v'_{s-1}\right)^2\right)=0,$$

$$4\oint_{P_v} h d\bar{z} = \sum_{Q_s\sim v} (v'_{s-1}+v'_s)\overline{(v'_s-v'_{s-1})} = \sum_{Q_s\sim v}\left(\left|v'_s\right|^2-\left|v'_{s-1}\right|^2-2i\operatorname{Im}\left(v'_s\bar{v}'_{s-1}\right)\right)$$
$$= -2i\sum_{Q_s\sim v}\operatorname{Im}\left(v'_s\bar{v}'_{s-1}\right) = -16i\text{area}(F_v).$$

These equations yield $\bar{\partial}_\diamond h(v)=0$ and $\partial_\diamond h(v)=1$. □

Remark. As before, parallelogram-graphs play a special role. In a parallelogram-graph, the midpoint of a black edge equals the midpoint of its dual white edge. Placing the vertices $Q\in V(\diamond)$ at the centers of the corresponding parallelograms then yields a global approximation statement.

In [CS11], Chelkak and Smirnov used averaging operators to map functions on $V(\Lambda)$ to functions on $V(\diamond)$ and vice versa. On parallelogram-graphs, the *averaging operator* $m(f)(Q) := \sum_{v\sim Q} f(v)/4$ actually maps discrete holomorphic functions $f: V(\Lambda)\to\mathbb{C}$ to discrete holomorphic functions on $V(\diamond)$. The corresponding statement for rhombic quad-graphs was shown in [CS11], our proof is similar.

Proposition 2.6. *Let Λ be a parallelogram-graph and $f: V(\Lambda)\to\mathbb{C}$ be discrete holomorphic. Then, $m(f): V(\diamond)\to\mathbb{C}$ is discrete holomorphic.*

Proof. Let us consider the star of the vertex $v\in V(\Lambda)$. Since f is discrete holomorphic, the discrete Cauchy-Riemann equation is satisfied on any $Q_s\sim v$. Therefore, we can express $f(v_s)$ in terms of $f(v)$, $f(v'_s)$ and $f(v'_{s-1})$. Plugging this in the definition of the averaging operator, we obtain

$$4m(f)(Q_s) = 2f(v) + \frac{v_s-v+v'_s-v'_{s-1}}{v'_s-v'_{s-1}}f(v'_s) + \frac{v_s-v-v'_s+v'_{s-1}}{v'_s-v'_{s-1}}f(v'_{s-1})$$
$$= 2f(v) + 2\frac{v'_s-v}{v'_s-v'_{s-1}}f(v'_s) - 2\frac{v'_{s-1}-v}{v'_s-v'_{s-1}}f(v'_{s-1}).$$

Here, we have used the properties $v_s-v'_{s-1}=v'_s-v$ and $v_s-v'_s=v'_{s-1}-v$ of the parallelogram Q_s. Therefore,

$$2\oint\limits_{P_v} m(f)dz = f(v)\oint\limits_{P_v} dz + \sum_{Q_s\sim v}(v'_s-v)f(v'_s) - \sum_{Q_s\sim v}(v'_{s-1}-v)f(v'_{s-1}) = 0,$$

and $m(f)$ is discrete holomorphic at v. $\square$

Remark. As noted by Chelkak and Smirnov, discrete holomorphic functions on $V(\diamond)$ cannot be averaged to discrete holomorphic functions on $V(\Lambda)$ in general.

Before, we extended $f : V(\Lambda) \to \mathbb{C}$ to a complex function on $F(X)$ by setting f zero on faces of X corresponding to vertices of $\diamond$. Now, it seems to be more appropriate to extend f by using its average $m(f)$ on $V(\diamond)$. The reason why we have not chosen this option is that functions on $V(\Lambda)$ and on $V(\diamond)$ behave differently. In Corollary 2.11 we will see that $\partial_\Lambda f$ is discrete holomorphic if f is, but $\partial_\diamond m(f)$ does not need to be discrete holomorphic in general. So to make sense of differentiating twice, we can only consider functions on $V(\Lambda)$.

Definition. Let g_1, g_2 be complex functions on $F(X)$. Their *discrete scalar product* is defined as

$$\langle g_1, g_2 \rangle := -\frac{1}{2i} \iint\limits_{F(X)} g_1 \bar{g}_2 dz \wedge d\bar{z},$$

whenever the right hand side converges absolutely.

In particular, we have defined discrete scalar products for complex functions g_1, g_2 on $V(\Lambda)$ or $V(\diamond)$. In this case, $\langle g_1, g_2 \rangle$ is defined by extending g_1, g_2 to complex functions on $F(X)$, setting yet undefined values to be zero. Then, on half of the faces of X, $g_1 \bar{g}_2$ is zero. The additional factor of two in $dz \wedge d\bar{z}$ compensates this.

Proposition 2.7. *$-\partial_\diamond$ and $-\bar{\partial}_\diamond$ are the formal adjoints of $\bar{\partial}_\Lambda$ and ∂_Λ, respectively. That is, if $f : V(\Lambda) \to \mathbb{C}$ or $h : V(\diamond) \to \mathbb{C}$ is compactly supported,*

$$\langle \partial_\Lambda f, h \rangle + \langle f, \bar{\partial}_\diamond h \rangle = 0 = \langle \bar{\partial}_\Lambda f, h \rangle + \langle f, \partial_\diamond h \rangle.$$

Proof. Using Lemma 2.3, and $\partial_\diamond \bar{h} = \overline{\bar{\partial}_\diamond h}$, we get

$$-2i\langle \partial_\Lambda f, h \rangle - 2i\langle f, \bar{\partial}_\diamond h \rangle = \sum_{Q \in V(\diamond)} \bar{h}(Q) \oint\limits_{P_Q} f d\bar{z} + \sum_{v \in V(\Lambda)} f(v) \oint\limits_{P_v} \bar{h} d\bar{z} = \oint\limits_{P} f \bar{h} d\bar{z} = 0,$$

where P is a large contour enclosing all the vertices of Λ and $\diamond$ where f or h does not vanish. In particular, $f\bar{h}$ vanishes in a neighborhood of P. In the same way, $\langle \bar{\partial}_\Lambda f, h \rangle + \langle f, \partial_\diamond h \rangle = 0$. $\square$

Remark. In their work on discrete complex analysis on rhombic quad-graphs, Kenyon [Ken02b] and Mercat [Mer07] defined the discrete derivatives for functions on the faces in such a way that they were the formal adjoints of the discrete derivatives for functions on the vertices of the quad-graph.

In Corollary 2.11, we will prove that $\partial_\Lambda f$ is discrete holomorphic if $f : V(\Lambda) \to \mathbb{C}$ is. Conversely, we can find discrete primitives of discrete holomorphic functions on simply-connected domains $\diamond_0$, extending the corresponding result for rhombic quad-graphs given in the paper of Chelkak and Smirnov [CS11].

Proposition 2.8. *Let $\diamond_0 \subseteq \diamond$ be simply-connected. Then, for any discrete holomorphic function h on $V(\diamond_0)$, there is a discrete primitive $f := \int h$ on $V(\Lambda_0)$, i.e., f is discrete holomorphic and $\partial_\Lambda f = h$. f is unique up to two additive constants on Γ_0 and Γ_0^*.*

Proof. Since h is discrete holomorphic, $\oint_P hdz = 0$ for any discrete contour P. Thus, hdz can be integrated to a well defined function f_X on $V(X)$ that is unique up to an additive constant. Using that hdz is a discrete one-form of type $\diamond$, we can construct a function f on $V(\Lambda)$ such that $f_X\left((v+w)/2\right) = \left(f(v)+f(w)\right)/2$ for any edge (v, w) of Λ. Given f_X, f is unique up to an additive constant.

In summary, f is unique up to two additive constants that can be chosen independently on Γ_0 and Γ_0^*. By construction, f satisfies

$$\frac{f(b_+) - f(b_-)}{b_+ - b_-} = h(Q) = \frac{f(w_+) - f(w_-)}{w_+ - w_-}$$

on any quadrilateral Q. So f is discrete holomorphic, and $\partial_\Lambda f = h$. $\square$

2.3 Discrete exterior calculus

Our notation of discrete exterior calculus is similar to the approach of Mercat in [Mer01b, Mer07, Mer08], but differs in some aspects. The main differences are due to our different notation of multiplication of functions with discrete one-forms, which allows us to define a discrete exterior derivative on a larger class of discrete one-forms in Section 2.3.1. It coincides with Mercat's discrete exterior derivative in the case of discrete one-forms of type $\diamond$ that Mercat considers. In contrast, our definitions are based on a coordinate representation, making the connection to the smooth case evident. Eventually, they lead to essentially the same definitions of a discrete wedge product in Section 2.3.2 and a discrete Hodge star in Section 2.3.3 as in [Mer08].

2.3.1 Discrete exterior derivative

Definition. Let $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$. We define the *discrete exterior derivatives* df and dh as follows:

$$\begin{aligned} df &:= \partial_\Lambda f dz + \bar{\partial}_\Lambda f d\bar{z}; \\ dh &:= \partial_\diamond h dz + \bar{\partial}_\diamond h d\bar{z}. \end{aligned}$$

Let ω be a discrete one-form. Around faces F_v and F_Q of X corresponding to vertices $v \in V(\Lambda)$ and $Q \in V(\diamond)$, respectively, we write $\omega = pdz + qd\bar{z}$ with functions p, q defined on faces $Q_s \sim v$ or vertices $b_\pm, w_\pm \sim Q$, respectively. The *discrete exterior derivative* $d\omega$ is given by

$$\begin{aligned} d\omega|_{F_v} &:= \left(\partial_\diamond q - \bar{\partial}_\diamond p\right) dz \wedge d\bar{z}, \\ d\omega|_{F_Q} &:= \left(\partial_\Lambda q - \bar{\partial}_\Lambda p\right) dz \wedge d\bar{z}. \end{aligned}$$

The reason why we add a factor of two in the definition of $d\omega$ (hidden in $dz \wedge d\bar{z}$) is the same as the factor of $1/2$ in the definition of $\partial_\diamond, \bar{\partial}_\diamond$: In the definition of the discrete exterior derivative, p and q are defined on the vertices of Λ or $\diamond$, but ω lives halfway between two incident vertices of Λ and $\diamond$. Thus, an arithmetic mean of values at v and $Q_s \sim v$, or Q and $v \sim Q$, would be more appropriate. The term involving v or Q cancels after integration, but to adjust the missing factor $1/2$, we have to multiply the result by 2.

The representation of ω as $pdz + qd\bar{z}$ (p, q defined on edges of X) we have used above may be nonunique. However, $d\omega$ is well defined by *discrete Stokes' theorem*, that also justifies our definition of df and $d\omega$. Note that Mercat defined the discrete exterior derivative by the discrete Stokes' theorem [Mer01b].

Theorem 2.9. *Let $f : V(\Lambda) \to \mathbb{C}$ and ω be a discrete one-form. Then, for any directed edge e of X starting in the midpoint of the edge vv'_- and ending in the midpoint of the edge vv'_+ of Λ, and for any face F of X with counterclockwise oriented boundary ∂F we have:*

$$\int\limits_e df = \frac{f(v'_+) - f(v'_-)}{2} = \frac{f(v) + f(v'_+)}{2} - \frac{f(v) + f(v'_-)}{2};$$

$$\iint\limits_F d\omega = \oint\limits_{\partial F} \omega.$$

Proof. Let v_- be the other vertex of the quadrilateral Q with vertices v, v'_- and v'_+. Without loss of generality, let v be white. Then, $\int_e df$ equals

$$\begin{aligned}
&\partial_\Lambda f \frac{v'_+ - v'_-}{2} + \bar{\partial}_\Lambda f \overline{\frac{v'_+ - v'_-}{2}}\\
=&\frac{1}{2}(\lambda_Q + \bar{\lambda}_Q)(f(v'_+) - f(v'_-)) + \frac{1}{2}\left(\bar{\lambda}_Q \frac{v'_+ - v'_-}{v - v_-} + \lambda_Q \overline{\frac{v'_+ - v'_-}{v - v_-}}\right)(f(v) - f(v_-))\\
=&\frac{f(v) + f(v'_+)}{2} + \operatorname{Re}\left(\bar{\lambda}_Q \frac{v'_+ - v'_-}{v - v_-}\right)(f(v) - f(v_-))\\
=&\frac{f(v'_+) - f(v'_-)}{2}.
\end{aligned}$$

Here, we have used $\lambda_Q + \bar{\lambda}_Q = 1$ and

$$\arg\left(\bar{\lambda}_Q \frac{v'_+ - v'_-}{v - v_-}\right) = \arg\left(\pm \exp\left(i\left(\varphi_Q - \frac{\pi}{2}\right)\right)\exp\left(-i\varphi_Q\right)\right) = \pm\pi/2.$$

The sign depends on the orientation of the vertices $v, v'_- v_-, v'_+$. But in any case, the expression inside the arg is purely imaginary.

Let us write $\omega = pdz + qd\bar{z}$ around F, where p, q are functions defined on $V(\Lambda)$ or $V(\diamond)$, depending on the type of F. By $\partial, \bar{\partial}$ we denote the corresponding discrete derivatives $\partial_\Lambda, \bar{\partial}_\Lambda$ or $\partial_\diamond, \bar{\partial}_\diamond$. Then, by Lemma 2.3 and the definition of the discrete derivatives $\partial_\diamond, \bar{\partial}_\diamond$, $\iint_F d\omega$ equals

$$\iint\limits_F \left(\partial q - \bar{\partial} p\right) dz \wedge d\bar{z} = -4i\text{area}(F)\left(\partial q - \bar{\partial} p\right) = \oint\limits_{\partial F} pdz + \oint\limits_{\partial F} qd\bar{z} = \oint\limits_{\partial F} d\omega.$$

□

Note that if ω is a discrete one-form of type $\diamond$, $\iint_F d\omega = 0$ for any face F corresponding to a vertex of $\diamond$. A discrete one-form ω is said to be *closed*, if $d\omega \equiv 0$. Examples for closed discrete one-forms are discrete exterior derivatives of complex functions on $V(\Lambda)$:

Proposition 2.10. *Let $f : V(\Lambda) \to \mathbb{C}$. Then, $ddf = 0$.*

Proof. By discrete Stokes' Theorem 2.9, $ddf = 0$ if $\oint_P df = 0$ for any discrete elementary cycle P. Since df is of type $\diamond$, the statement is trivially true if $P = P_Q$ for $Q \in V(\diamond)$. So let $P = P_v$ for $v \in V(\diamond)$. Using discrete Stokes' Theorem 2.9 again,

$$\oint\limits_{P_v} df = \sum_{Q_s \sim v} \frac{f(v'_s) - f(v'_{s-1})}{2} = 0.$$

□

An immediate corollary of the last proposition is the commutativity of discrete differentials, generalizing the known result for rhombic quad-graphs as provided in [CS11].

Corollary 2.11. *Let $f : V(\diamond) \to \mathbb{C}$. Then, $\partial_\diamond \bar{\partial}_\Lambda f \equiv \bar{\partial}_\diamond \partial_\Lambda f$. In particular, $\partial_\Lambda f$ is discrete holomorphic if f is discrete holomorphic.*

Proof. Due to Proposition 2.10, $0 = ddf = \left(\partial_\diamond \bar{\partial}_\Lambda f - \bar{\partial}_\diamond \partial_\Lambda f\right) dz \wedge d\bar{z}$. □

Remark. Note that even in the generic rhombic case, $\partial_\Lambda \bar{\partial}_\diamond h$ does not always equal $\bar{\partial}_\Lambda \partial_\diamond h$ for $h : V(\diamond) \to \mathbb{C}$ [CS11]. Hence, an analog of Proposition 2.10 cannot hold for such functions h in general.

Corollary 2.12. *Let $f : V(\Lambda) \to \mathbb{C}$. Then, f is discrete holomorphic if and only if $df = pdz$ for some $p : V(\diamond) \to \mathbb{C}$. In this case, p is discrete holomorphic.*

Proof. Since all quadrilaterals Q are nondegenerate, the representation of $df|_{\partial F_Q}$ as $pdz + qd\bar{z}$ is unique. Now, $df = \partial_\Lambda f dz + \bar{\partial}_\Lambda f d\bar{z}$. It follows that f is discrete holomorphic at Q if and only if $df|_{\partial F_Q} = pdz$.

Assuming that $df = pdz$ for some $p : V(\diamond) \to \mathbb{C}$, $ddf = 0$ by Proposition 2.10 yields $\bar{\partial}_\diamond p = 0$. □

Let us say that a discrete one-form ω is *discrete holomorphic* if $\omega = pdz$ for some $p : V(\diamond) \to \mathbb{C}$ and $d\omega = 0$. This notion will recur in the more general setting of discrete Riemann surfaces in Section 3.1.3 of Chapter 3. By Corollary 2.12, df is discrete holomorphic if f is, and by Proposition 2.8, any discrete holomorphic one-form ω on a simply-connected domain is the exterior derivative of a discrete holomorphic function on $V(\Lambda)$.

Due to Chelkak and Smirnov [CS11], one of the unpleasant facts of all discrete theories of complex analysis is that (pointwise) multiplication of discrete holomorphic functions does not yield a discrete holomorphic function in general. We can define a product of complex functions on $V(\Lambda)$ that is defined on $V(X)$, and a product of complex functions on $V(\Lambda)$ with functions on $V(\diamond)$ that is defined on $E(X)$. In general, the product of two discrete holomorphic functions is not discrete holomorphic according to the classical quad-based definition, but it will be discrete holomorphic in the sense that a discretization of its exterior derivative is closed and of the form pdz, $p : E(X) \to \mathbb{C}$, or that it fulfills a discrete Morera's theorem.

Corollary 2.13. *Let $f, g : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$.*

(i) $fdg + gdf$ is a closed discrete one-form.

(ii) If f and h are discrete holomorphic, $fhdz$ is a closed discrete one-form.

Proof. (i) Let $\omega := fdg + gdf$. By Proposition 2.10, df and dg are closed. Thus,

$$\oint\limits_{\partial F_v} \omega = f(v) \oint\limits_{\partial F_v} dg + g(v) \oint\limits_{\partial F_v} df = 0$$

for any face F_v corresponding to $v \in V(\Lambda)$. Using Lemma 2.3,

$$\begin{aligned} 4i\text{area}(F_Q) \oint\limits_{\partial F_Q} \omega &= 4i\text{area}(F_Q) \oint\limits_{\partial F_Q} \left(f\partial_\Lambda g dz + f\bar{\partial}_\Lambda g d\bar{z} + g\partial_\Lambda f dz + g\bar{\partial}_\Lambda f d\bar{z}\right) \\ &= \bar{\partial}_\Lambda f \partial_\Lambda g + \partial_\Lambda f \bar{\partial}_\Lambda g - \bar{\partial}_\Lambda g \partial_\Lambda f - \partial_\Lambda g \bar{\partial}_\Lambda f = 0 \end{aligned}$$

for any face F_Q corresponding to $Q \in V(\diamond)$. It follows by discrete Stokes' Theorem 2.9 that $d\omega = 0$.

(ii) By discrete Morera's Theorem 2.4, $\oint_{\partial F} fhdz = 0$ for any face F of X, using that f and h are discrete holomorphic. Therefore, $fhdz$ is closed. $\square$

Remark. In particular, a product $f \cdot g : V(X) \to \mathbb{C}$ can be defined by integration, and $f \cdot g$ is defined up to an additive constant. Furthermore, $f \cdot h : E(X) \to \mathbb{C}$ can be defined by "pointwise" multiplication. If f, g, h are discrete holomorphic, $fdg + gdf = pdz$ is closed, where $p = f \cdot \partial_\Lambda g + g \cdot \partial_\Lambda f : E(X) \to \mathbb{C}$, and so to say a discrete holomorphic one-form, meaning that $f \cdot g$ is discrete holomorphic in this sense. Similarly, $fhdz$ is closed, so $f \cdot h$ is discrete holomorphic in the sense of a discrete Morera's theorem. However, $f \cdot g$ and $f \cdot h$ are generally not discrete holomorphic everywhere according to the classical quad-based definition of discrete holomorphicity on the dual of a bipartite quad-graph. For this, we place any $Q \in V(\Diamond)$ as a vertex somewhere in the interior of the corresponding quadrilateral.

Indeed, $f \cdot g$ is a complex function on the vertices of X. The medial graph X is not a quad-graph itself, but X is the dual of a bipartite quad-graph. More precisely, X is the dual of the bipartite quad-graph with vertex set $V(\Lambda) \sqcup V(\Diamond)$, edges connecting points $Q \in V(\Diamond)$ with all incident vertices $v \in V(\Lambda)$, and faces corresponding to edges of Λ. Then, $f \cdot g$ does not need to be a discrete holomorphic function on the faces of the latter quad-graph. For example, consider $f(v) = 0$ if v is black and $f(v) = 1$ if v is white, and a discrete holomorphic g that is not essentially constant. Then, the product $f \cdot g$ is not discrete holomorphic at all $Q \in V(\Diamond)$ (seen as vertices of the quad-graph described above) where $\partial_\Lambda g(Q) \neq 0$.

Furthermore, $f \cdot h$ is a complex function on the edges of X, so it is a function on the vertices of the medial graph of X. The medial graph of the medial graph of Λ is usually not a quad-graph, but it is the dual of a bipartite quad-graph. Namely, it is the dual of the quad-graph with vertex set $(V(\Lambda) \cup V(\Diamond)) \sqcup V(X)$, edges connecting points $v \in V(\Lambda)$ or $Q \in V(\Diamond)$ with the midpoints of all incident edges, and each face having an edge of X as a diagonal. Since $fhdz$ is closed, $f \cdot h$ is discrete holomorphic at vertices of Λ or $\Diamond$ by discrete Morera's Theorem 2.4. But there is no need for $f \cdot h$ to be discrete holomorphic at vertices of X, even for constant h, and f defined by $f(v) = 0$ if v is black and $f(v) = 1$ if v is white.

In summary, we defined reasonable products $f \cdot g$ and $f \cdot h$, where $f, g : V(\Lambda) \to \mathbb{C}$ and $h : V(\Diamond) \to \mathbb{C}$ are discrete holomorphic. Somehow missing is a product $h \cdot h'$, where $h' : V(\Diamond) \to \mathbb{C}$. In the general case, we do not know an appropriate product so far. But we want to point out that Chelkak and Smirnov defined such a product for so-called *spin holomorphic* functions h, h' in [CS12]. This product satisfies $\operatorname{Re}\left(\bar{\partial}_\Diamond \left(h \cdot h'\right)\right) \equiv 0$.

2.3.2 Discrete wedge product

Whereas Mercat also defined a discrete wedge product for discrete one-forms living on the edges of Λ [Mer01b], we concentrate on discrete one-forms essentially living on the edges of Γ and Γ^*, i.e., discrete one-forms of type $\Diamond$.

Lemma 2.14. *Let ω be a discrete one-form of type $\Diamond$. Then, there is a unique representation $\omega = pdz + qd\bar{z}$ with functions $p, q : V(\Diamond) \to \mathbb{C}$. On a quadrilateral $Q \in V(\Diamond)$, p and q are given by*

$$p(Q) = \lambda_Q \frac{\int_e \omega}{e} + \bar{\lambda}_Q \frac{\int_{e^*} \omega}{e^*},$$
$$q(Q) = \bar{\lambda}_Q \frac{\int_e \omega}{\bar{e}} + \lambda_Q \frac{\int_{e^*} \omega}{\bar{e}^*}.$$

Here, e is an edge of X parallel to a black edge of Γ, and e^* corresponds to a white edge of Γ^*.

Proof. Since ω is of type $\Diamond$, a representation $\omega = pdz + qd\bar{z}$ exists for any quadrilateral Q. In fact, given ω, we have to solve a nondegenerate system of two linear equations in the variables p and q. Thus, p, q are uniquely defined on $V(\Diamond)$.

Furthermore, we can find for any quadrilateral Q a function f on the vertices of Q such that $2\int_e \omega = f(b_+) - f(b_-)$ and $2\int_{e^*} \omega = f(w_+) - f(w_-)$, where e is one of the two oriented edges of X going from the midpoint of b_- and $w_\pm$ to the midpoint of b_+ and $w_\pm$, and e^* is one of the two edges connecting the midpoint of w_- and $b_\pm$ with the midpoint of w_+ and $b_\pm$. By discrete Stokes's theorem 2.9, we locally get $\omega = df = pdz + qd\bar{z}$ with $p = \partial_\Lambda f$ and $q = \bar{\partial}_\Lambda f$. Replacing the differences of f by discrete integrals of ω yields the desired result. $\square$

Definition. Let $\omega = pdz + qd\bar{z}$ and $\omega' = p'dz + q'd\bar{z}$ be two discrete one-forms of type $\Diamond$, $p, p', q, q' : V(\Diamond) \to \mathbb{C}$ given by Lemma 2.14. Then, the *discrete wedge product* $\omega \wedge \omega'$ is defined as the discrete two-form being 0 on faces of X corresponding to vertices of Λ that equals

$$(pq' - qp')\, dz \wedge d\bar{z}$$

on faces corresponding to $V(\Diamond)$.

By definition, the discrete wedge product vanishes on faces of X corresponding to $V(\Lambda)$. Since the faces of X corresponding to $V(\Diamond)$ cover exactly half of the area of the quadrilaterals, the factor of two in the definition of $dz \wedge d\bar{z}$ incorporates the vanishing regions of the discrete wedge product.

Proposition 2.15. *Let F be a face of X corresponding to $Q \in V(\Diamond)$, and let e, e^* be oriented edges of X parallel to the black and white diagonal of Q, respectively, such that* $\operatorname{Im}(e^*/e) > 0$. *Then,*

$$\iint\limits_F \omega \wedge \omega' = 2\int\limits_e \omega \int\limits_{e^*} \omega' - 2\int\limits_{e^*} \omega \int\limits_e \omega'.$$

Proof. Both sides of the equation are bilinear and antisymmetric in ω, ω'. Hence, it suffices to check the identity for $\omega = dz$, $\omega' = d\bar{z}$. On the left hand side, we get $\iint_F \omega \wedge \omega' = -4i\text{area}(F)$. This equals the right hand side

$$2e\bar{e}^* - 2e^*\bar{e} = 4i\operatorname{Im}(e\bar{e}^*) = -i|2e||2e^*|\sin(\varphi_Q) = -4i\text{area}(F).$$

$\square$

Remark. Since the complex numbers e and e^* are just half of the oriented diagonals, we recover the definition of the discrete wedge product given by Mercat in [Mer01b, Mer07, Mer08].

The discrete exterior derivative is a derivation for the discrete wedge product if one considers functions on Λ and discrete one-forms of type $\Diamond$:

Theorem 2.16. *Let $f : V(\Lambda) \to \mathbb{C}$ and ω be a discrete one-form of type $\Diamond$. Then, $d(f\omega) = df \wedge \omega + fd\omega$.*

Proof. Let $\omega = pdz + qd\bar{z}$ with $p, q : V(\Diamond) \to \mathbb{C}$ given by Lemma 2.14. If F_v and F_Q are faces of X corresponding to a vertex v and a face Q of Λ,

$$\begin{aligned} d(f\omega)|_{F_v} &= \left(f(v)\left(\partial_\Diamond q\right)(v) - f(v)\left(\bar{\partial}_\Diamond p\right)(v)\right) dz \wedge d\bar{z} = fd\omega|_{F_v}, \\ d(f\omega)|_{F_Q} &= \left(q(Q)\left(\partial_\Lambda f\right)(Q) - p(Q)\left(\bar{\partial}_\Lambda f\right)(Q)\right) dz \wedge d\bar{z} = (df \wedge \omega))|_{F_Q}. \end{aligned}$$

But $(df \wedge \omega)|_{F_v} = 0$ and $fd\omega|_{F_Q} = 0$, so $d(f\omega) = df \wedge \omega + fd\omega$. $\square$

Remark. In [Mer01b], Mercat proved an analog of Theorem 2.16 in a setting where discrete one-forms are defined on edges of Λ. He also considered discrete differential forms as we do, however, the claim $d(f\omega) = df \wedge \omega + fd\omega$ could not be well defined in his setting. At first, our approach using the medial graph allows to make sense out of this statement. It turns out that Theorem 2.16 is a very powerful theorem leading to discretizations of Green's identities in Section 2.4.1, of a Cauchy's integral formula for the discrete derivative of a discrete holomorphic function in Section 2.6, and of Riemann's bilinear identity later in Section 3.3.2 of Chapter 3.

2.3.3 Discrete Hodge star

Definition. Let $g : F(X) \to \mathbb{C}$, and let $\omega = pdz + qd\bar{z}$ be a discrete one-form of type $\Diamond$ with $p, q : V(\Diamond) \to \mathbb{C}$ given by Lemma 2.14 and Ω a discrete two-form. The *discrete Hodge star* is defined by

$$\begin{aligned} \star g &:= -\frac{1}{2i} g dz \wedge d\bar{z}; \\ \star\omega &:= -ipdz + iqd\bar{z}; \\ \star\Omega &:= -2i\frac{\Omega}{dz \wedge d\bar{z}}. \end{aligned}$$

If ω and ω' are both discrete one-forms of type $\Diamond$, we define their *discrete scalar product*

$$\langle \omega, \omega' \rangle := \iint\limits_{F(X)} \omega \wedge \star\bar{\omega}',$$

whenever the right hand side converges absolutely. Similarly, the discrete scalar product between discrete two-forms is defined:

$$\langle \Omega, \Omega' \rangle := \iint\limits_{F(X)} (\star\bar{\Omega}')\Omega.$$

The following corollary results directly from the definitions:

Corollary 2.17. *(i)* $\star^2 = -\mathrm{Id}$ *on discrete one-forms of type* $\Diamond$.

(ii) $\star^2 = \mathrm{Id}$ *on complex functions on* $F(X)$ *and discrete two-forms.*

(iii) $f : V(\Lambda) \to \mathbb{C}$ *is discrete holomorphic if and only if* $\star df = -idf$.

(iv) $\langle g_1, g_2 \rangle = \iint_{F(X)} g_1 \overline{\star g_2}$ *for* $g_1, g_2 : F(X) \to \mathbb{C}$.

(v) $\langle \cdot, \cdot \rangle$ *is a Hermitian scalar product on discrete differential forms.*

Proposition 2.18. *Let ω be a discrete one-form of type $\diamond$, let $Q \in V(\diamond)$, and let e, e^* be oriented edges of X parallel to the black and white diagonal of Q, respectively, such that* $\operatorname{Im}(e^*/e) > 0$. *Then,*

$$\int_e \star\omega = \cot(\varphi_Q) \int_e \omega - \frac{|e|}{|e^*| \sin(\varphi_Q)} \int_{e^*} \omega,$$
$$\int_{e^*} \star\omega = \frac{|e^*|}{|e| \sin(\varphi_Q)} \int_e \omega - \cot(\varphi_Q) \int_{e^*} \omega.$$

Proof. Both sides of any of the two equations are linear and behave the same under complex conjugation. Thus, it suffices to check the statement for $\omega = dz$. Hence, it remains to show that

$$-ie = \cot(\varphi_Q)\, e - \frac{|e|}{|e^*| \sin(\varphi_Q)} e^*,$$
$$e^* = \frac{|e^*|}{|e| \sin(\varphi_Q)} e - \cot(\varphi_Q)\, e^*.$$

Now, all terms of the first equation behave the same under scaling and rotation of e and e^*, the same statement is true for the second equation. Thus, we may assume $e = 1$ and $e^* = \exp(i\varphi_Q) = \cos(\varphi_Q) + i\sin(\varphi_Q)$. Multiplying both equations by $\sin(\varphi_Q)$ gives the equivalent statements

$$-i\sin(\varphi_Q) = \cos(\varphi_Q) - (\cos(\varphi_Q) + i\sin(\varphi_Q)),$$
$$-i\sin(\varphi_Q)\exp(i\varphi_Q) = 1 - \cos(\varphi_Q)\exp(i\varphi_Q).$$

Both equations are true, noting for the second that $\exp(-i\varphi_Q)\exp(i\varphi_Q) = 1$. ☐

Remark. Proposition 2.18 shows that our definition of a discrete Hodge star on discrete one-forms coincides with Mercat's definition given in [Mer08]. But on discrete two-forms and complex functions, our definition of the discrete Hodge star includes an additional factor of the area of the corresponding face of X. As before, the additional factor of two encoded in $dz \wedge d\bar{z}$ reflects our focus on complex functions that are defined on $V(\Lambda)$ or on $V(\diamond)$ only.

Proposition 2.19. $\delta := - \star d \star$ *is the formal adjoint of the discrete exterior derivative d: Let $f : V(\Lambda) \to \mathbb{C}$, and let ω be a discrete one-form of type $\diamond$ and Ω a discrete two-form being zero on all faces corresponding to vertices of $\diamond$. Assume that all of them are compactly supported. Then,*

$$\langle df, \omega \rangle = \langle f, \delta\omega \rangle \text{ and } \langle d\omega, \Omega \rangle = \langle \omega, \delta\Omega \rangle.$$

Proof. By discrete Stokes' Theorem 2.9, Theorem 2.16, and Corollary 2.17 (ii),

$$0 = \iint\limits_{F(X)} d(f \star \bar{\omega}) = \iint\limits_{F(X)} df \wedge \star\bar{\omega} + \iint\limits_{F(X)} f d \star \bar{\omega} = \langle df, \omega \rangle + \langle f, \star d \star \omega \rangle,$$

$$0 = \iint\limits_{F(X)} d(\star\bar{\Omega}\omega) = \iint\limits_{F(X)} d \star \bar{\Omega} \wedge \omega + \iint\limits_{F(X)} \star\bar{\Omega} d\omega = \langle \omega, \star d \star \Omega \rangle + \langle d\omega, \Omega \rangle.$$

□

2.4 Discrete Laplacian

The discrete Laplacian and the discrete Dirichlet energy on general quad-graphs were first introduced by Mercat in [Mer08]. Later, Skopenkov reintroduced these definitions in [Sko13], taking the same definition in a different notation. In our discussion of the discrete Laplacian in Section 2.4.1, we follow the classical approach of Mercat (up to sign), and adapt it to our notations. A feature of our notation is that we are able to formulate a discrete analog of Green's first identity, and a more intuitive formulation of a discrete Green's second identity than the previous ones of Mercat, Chelkak and Smirnov, and Skopenkov [Mer08, CS11, Sko13].

In Section 2.4.2, we investigate the discrete Dirichlet energy. In particular, we show how uniqueness and existence of solutions to the discrete Dirichlet boundary value problem imply surjectivity of the discrete differentials and the discrete Laplacian in Theorem 2.30. We conclude this section with a result concerning the asymptotics of discrete harmonic functions.

2.4.1 Definition and basic properties

Definition. The *discrete Laplacian* on discrete differential forms is defined as the linear operator

$$\triangle := -\delta d - d\delta = \star d \star d + d \star d \star .$$

$f : V(\Lambda) \to \mathbb{C}$ is said to be *discrete harmonic* at $v \in V(\Lambda)$ if $\triangle f(v) = 0$.

The following factorization of the discrete Laplacian in terms of discrete derivatives generalizes the corresponding result given by Chelkak and Smirnov in [CS11] to general quad-graphs. The local representation of $\triangle f$ at $v \in V(\Lambda)$ is, up to a factor involving the area of the face F_v of X corresponding to v, the same as Mercat's in [Mer08].

Corollary 2.20. *Let $f : V(\Lambda) \to \mathbb{C}$. Then, $\triangle f = 4\partial_\diamond \bar{\partial}_\Lambda f = 4\bar{\partial}_\diamond \partial_\Lambda f$. At a vertex v of Λ, $\triangle f(v)$ equals*

$$\frac{1}{4\mathrm{area}(F_v)} \sum_{Q_s \sim v} \frac{1}{\mathrm{Re}\,(\rho_s)} \left(|\rho_s|^2 \left(f(v_s) - f(v) \right) + \mathrm{Im}\,(\rho_s) \left(f(v'_s) - f(v'_{s-1}) \right) \right).$$

Here, $\rho_s := \rho_{Q_s}$ if v is black, and $\rho_s := 1/\rho_{Q_s}$ if v is white.

In particular, $\mathrm{Re}(\triangle f) \equiv \triangle \mathrm{Re}(f)$ *and* $\mathrm{Im}(\triangle f) \equiv \triangle \mathrm{Im}(f)$.

Proof. The first statement follows from $\triangle f = \star d \star df = 2\partial_\diamond \bar{\partial}_\Lambda f + 2\bar{\partial}_\diamond \partial_\Lambda f$, noting that $\partial_\diamond \bar{\partial}_\Lambda f = \bar{\partial}_\diamond \partial_\Lambda f$ by Corollary 2.11.

For the second statement, let us assume without loss of generality that $v \in V(\Gamma)$. Then, we have to show that $\triangle f(v)$ equals

$$\frac{1}{4\mathrm{area}(F_v)} \sum_{Q_s \sim v} \left(\frac{|\rho_{Q_s}|}{\sin(\varphi_{Q_s})} \left(f(v_s) - f(v) \right) - \cot(\varphi_{Q_s}) \left(f(v'_s) - f(v'_{s-1}) \right) \right).$$

The structure is similar to the formula of the discrete Hodge star in Proposition 2.18. Indeed, if e_s denotes an edge of X parallel to the black diagonal vv_s, and e^*_s an edge parallel to the dual diagonal, $\triangle f(v)$ equals

$$\begin{aligned}
&\frac{1}{2\mathrm{area}(F_v)} \iint\limits_{F_v} d \star df = \frac{1}{2\mathrm{area}(F_v)} \oint\limits_{\partial F_v} \star df \\
=&\frac{1}{2\mathrm{area}(F_v)} \sum_{Q_s \sim v} \left(\frac{|e^*_s|}{|e_s| \sin(\varphi_{Q_s})} \int\limits_{e_s} df - \cot(\varphi_{Q_s}) \int\limits_{e^*_s} df \right) \\
=&\frac{1}{4\mathrm{area}(F_v)} \sum_{Q_s \sim v} \left(\frac{|\rho_{Q_s}|}{\sin(\varphi_{Q_s})} \left(f(v_s) - f(v) \right) - \cot(\varphi_{Q_s}) \left(f(v'_s) - f(v'_{s-1}) \right) \right),
\end{aligned}$$

using discrete Stokes' Theorem 2.9, Proposition 2.18, and $|\rho_{Q_s}| = |e^*_s|/|e_s|$. ☐

Remark. In the case that the diagonals of the quadrilaterals are orthogonal to each other, ρ_Q is always a positive real number. In this case, the discrete Laplacian splits into two separate discrete Laplacians on Γ and Γ^*. In this case, it is known, and actually an immediate consequence of the local representation in Corollary 2.20, that a discrete maximum principle holds true, i.e., a discrete harmonic function can attain its maximum only at the boundary of a region. This is not true for general quad-graphs, see for example Skopenkov's paper [Sko13].

Corollary 2.21. *Let $f : V(\Lambda) \to \mathbb{C}$.*

(i) If f is discrete harmonic, $\partial_\Lambda f$ is discrete holomorphic.

(ii) If f is discrete holomorphic, f, $\operatorname{Re} f$, and $\operatorname{Im} f$ are discrete harmonic.

Proof. By Corollary 2.20, $\triangle f \equiv 4\bar{\partial}_\diamond \partial_\Lambda f \equiv 4\partial_\diamond \bar{\partial}_\Lambda f$. In particular, $\bar{\partial}_\diamond \partial_\Lambda f \equiv 0$ if $\triangle f \equiv 0$, which shows (i). Also, f is discrete harmonic if it is discrete holomorphic. Using $\operatorname{Re}(\triangle f) \equiv \triangle \operatorname{Re}(f)$ and $\operatorname{Im}(\triangle f) \equiv \triangle \operatorname{Im}(f)$, $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are discrete harmonic if f is. □

Similar to Proposition 2.1, the discrete Laplacian coincides with the smooth one up to order one in the general case, and up to order two for parallelogram-graphs. This was already shown by Skopenkov in [Sko13]. Since this result follows immediately from our previous ones, we give a proof here as well.

Proposition 2.22. *Let $f_\mathbb{C} : \mathbb{C} \to \mathbb{C}$ and f its restriction to $V(\Lambda)$.*

(i) If $f_\mathbb{C}(z)$ is a polynomial in $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ of degree at most one, then the smooth and the discrete Laplacian coincide on vertices: $\triangle_\mathbb{C} f_\mathbb{C}(v) = \triangle f(v)$.

(ii) Let all quadrilaterals of $\diamond$ be parallelograms. If $f_\mathbb{C}(z)$ is a polynomial in $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ of degree at most two, then the smooth and the discrete Laplacian coincide on vertices: $\triangle_\mathbb{C} f_\mathbb{C}(v) = \triangle f(v)$.

Proof. (i) It follows from Proposition 2.1 (ii) and Corollary 2.21 (ii) that constants as well as the complex function $f(v) = v$ are discrete harmonic since they are discrete holomorphic. Similarly, $f(v) = \overline{v}$ is discrete harmonic.

(ii) In the parallelogram case, let us place the vertices of $\diamond$ at the centers of the parallelograms. Then, $f(v) = v^2$ is discrete harmonic by Proposition 2.1 (iii) and Corollary 2.21 (ii). Looking at real and imaginary part separately, $\triangle f_1^2 = \triangle f_2^2$ and $\triangle (f_1 f_2) = 0$ with $f_1(v) = \operatorname{Re}(v)$, $f_2(v) = \operatorname{Im}(v)$. Finally,

$$\triangle |f|^2 = 4\partial_\diamond \bar{\partial}_\Lambda |f|^2 = 4\partial_\diamond h = 4$$

with $h(Q) = Q$, due to Propositions 2.1 (iv) and 2.5. Since any polynomial in $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ of monomials of degree two is a linear combination of $f_1^2 - f_2^2$, $f_1^2 + f_2^2$, and $f_1 f_2$, and we have shown that the discrete Laplacian $\triangle$ and the smooth Laplacian $\triangle_\mathbb{C}$ coincide on these, we are done. □

Remark. The second part of the last proposition generalizes the known result for rhombi given by Chelkak and Smirnov in [CS11]. Note that this is not true for general quadrilaterals even if one assumes that the diagonals of quadrilaterals are orthogonal to each other. For this, consider the following (finite) bipartite quad-graph of Figure 2.5: $0 \in \Gamma$ is adjacent to the white vertices ± 1 and $\pm i$ in Λ, and adjacent to the black vertices $2+2i$, $-1 \pm i$ and $1-i$ in Γ. There are no further vertices. Then, $\triangle f(0) \neq 0$ for $f(v) = v^2$. Indeed, we would get $\triangle f(0) = 0$ if we had replaced $v = 2+2i$ by $v = 1+i$, obtaining a rhombic quad-graph; but $|\rho_Q|^2/\operatorname{Re}(\rho_Q)\,(f(v) - f(0))$ scales by a factor of 2, whereas the other nonzero summands in the local representation of $\triangle f(0)$ remain invariant.

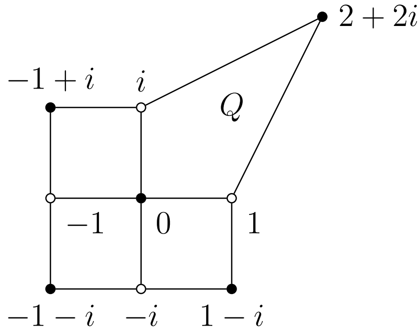


Figure 2.5: $\triangle f(0) \neq 0$ for $f(v) = v^2$

In the case of general quad-graphs, smooth functions $f_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ and restrictions f to $V(\Lambda)$, Skopenkov compared the integral of $\triangle f_{\mathbb{C}}$ over a square domain R and a sum of $\triangle f(v)$ over black vertices of Λ in R [Sko13]. Moreover, he showed that for $f(v) = |v|^2$,

$$\triangle f(v) = \frac{1}{\operatorname{area}(F_v)} \sum_{Q_s \sim v} \operatorname{area}(v v'_{s-1} Q_s v'_s)$$

when Q_s is placed at the intersection point of the middle perpendiculars to the diagonals of the corresponding quadrilateral (which is equal to the intersection point of the diagonals if the quadrilateral is a parallelogram). Note that in general, $h(Q) = Q$ is not discrete holomorphic if the vertices $Q \in V(\diamond)$ are placed on the intersection of middle perpendiculars.

For a finite subset $\diamond_0 \subset \diamond$ and two functions $f, g : V(\Lambda_0) \to \mathbb{C}$, we denote by

$$\langle f, g \rangle_{\diamond_0} := -\frac{1}{2i} \iint\limits_{F(X_0)} f \bar{g} \, dz \wedge d\bar{z}$$

the discrete scalar product of f and g restricted to $\Diamond_0$. Similarly, the restriction of the discrete scalar product of two discrete one-forms on $\Diamond_0$ is defined.

In the rhombic setup, discrete versions of Green's second identity were already stated by Mercat [Mer01b], whose integrals were not well defined separately, and Chelkak and Smirnov [CS11], whose boundary integral was an explicit sum involving boundary angles. Skopenkov formulated a discrete Green's second identity with a vanishing boundary term [Sko13]. We are able to provide a *discrete Green's first identity* for the first time, prove it completely analogously to the smooth setting, and deduce *discrete Green's second identity* out of it.

Theorem 2.23. *Let $\Diamond_0 \subset \Diamond$ be finite, and let $f, g : V(\Lambda_0) \to \mathbb{C}$.*

(i) $\langle f, \triangle g \rangle_{\Diamond_0} + \langle df, dg \rangle_{\Diamond_0} = \oint\limits_{\partial X_0} f \star d\bar{g}.$

(ii) $\langle \triangle f, g \rangle_{\Diamond_0} - \langle f, \triangle g \rangle_{\Diamond_0} = \oint\limits_{\partial X_0} \left(f \star d\bar{g} - \bar{g} \star df \right).$

Proof. (i) Since the discrete exterior derivative is a derivation for the discrete wedge product by Theorem 2.16, $d\left(f \star d\bar{g}\right) = df \wedge \star d\bar{g} + f \star (\star d \star d\bar{g})$. Now, discrete Stokes' Theorem 2.9 yields the desired result.

(ii) Just apply twice discrete Green's first identity. □

Discrete Weyl's lemma is a direct consequence of discrete Green's second identity, Theorem 2.23 (ii). A version for rhombic quad-graphs was given by Mercat in [Mer01b], proven by an explicit calculation.

Corollary 2.24. *$f : V(\Lambda) \to \mathbb{C}$ is discrete harmonic if and only if $\langle f, \triangle g \rangle = 0$ for every compactly supported $g : V(\Lambda) \to \mathbb{C}$.*

Skopenkov introduced the notion of discrete harmonic conjugates in [Sko13]. We recover his definitions in our notation, observing that his discrete gradient corresponds to our discrete exterior derivative, and his counterclockwise rotation by $\pi/2$ corresponds to our discrete Hodge star.

Definition. Let $\Diamond_0 \subseteq \Diamond$ and f be a real (discrete harmonic) function on $V(\Lambda_0)$. A real discrete harmonic function $\widetilde{f}$ on $V(\Lambda_0)$ is said to be a *discrete harmonic conjugate of f*, if $f + i\widetilde{f}$ is discrete holomorphic.

Note that the existence of a real function $\widetilde{f}$ such that $f+i\widetilde{f}$ is discrete holomorphic requires already that f is discrete harmonic due to Corollary 2.21 (ii).

Lemma 2.25. *Let $\diamondsuit_0 \subseteq \diamondsuit$ and f a real discrete harmonic function on $V(\Lambda_0)$.*

(i) The discrete harmonic conjugate $\widetilde{f}$ is unique up to two additive real constants on Γ_0 and Γ_0^*.

(ii) If $\diamondsuit_0$ is simply-connected, a discrete harmonic conjugate $\widetilde{f}$ exists.

Proof. (i) If $\widetilde{f}_1$ and $\widetilde{f}_2$ are two real discrete harmonic conjugates, their difference $\widetilde{f}_1 - \widetilde{f}_2$ is real and discrete holomorphic. So by Proposition 2.2 (ii), it is essentially constant.

(ii) Since f is harmonic, $d \star df = 0$, i.e., $\star df$ is closed and of type $\diamondsuit$. Moreover, $\star df = -i\partial_\Lambda f dz + i\bar{\partial}_\Lambda f d\bar{z} = 2\,\mathrm{Im}\,(\partial_\Lambda f dz)$ because f is real. So in the same manner as in the proof of Proposition 2.8, $\star df$ can be integrated to a real function $\widetilde{f}$ on $V(\Lambda_0)$. Finally, $f + i\tilde{f}$ is discrete holomorphic by Corollary 2.12 since $df + i \star df = 2\,\mathrm{Re}\,(\partial_\Lambda f dz) + 2i\,\mathrm{Im}\,(\partial_\Lambda f dz) = 2\partial_\Lambda f dz$. $\square$

Note that in the case of quadrilaterals with orthogonal diagonals, such that $\triangle$ splits into two discrete Laplacians on Γ and Γ^*, it follows that a discrete harmonic conjugate of a discrete harmonic function on $V(\Gamma)$ can be defined on $V(\Gamma^*)$ and vice versa, as already noted by Chelkak and Smirnov in [CS11].

Corollary 2.26. *Let $\diamondsuit_0 \subseteq \diamondsuit$ and $f : V(\Lambda_0) \to \mathbb{C}$ be discrete holomorphic. Then,* $\mathrm{Im}(f)$ *is uniquely determined by* $\mathrm{Re}(f)$ *up to two additive real constants on Γ_0 and Γ_0^*.*

2.4.2 Discrete Dirichlet energy

We follow the classical approach of discretizing the Dirichlet energy introduced by Mercat in [Mer08]. Note that Skopenkov's definition in [Sko13] is exactly the same. In particular, Skopenkov's results, including an approximation property of the Laplacian, convergence of the discrete Dirichlet energy to the smooth Dirichlet energy for nondegenerate uniform sequences of quad-graphs, and further theorems for quad-graphs with orthogonal diagonals, apply as well in our setting. We refer to his work [Sko13] for details on these results.

Definition. Let $\diamond_0 \subseteq \diamond$. For $f : V(\Lambda_0) \to \mathbb{C}$, we define the *discrete Dirichlet energy* of f on $\diamond_0$ as $E_{\diamond_0}(f) := \langle df, df \rangle_{\diamond_0} \in [0, \infty]$.

If $\diamond_0$ is finite, the *discrete Dirichlet boundary value problem* asks for a real discrete harmonic function f on $V(\Lambda_0)$ such that f agrees with a preassigned real function f_0 on the boundary $V(\partial\Lambda_0)$.

Proposition 2.27.

$$\begin{aligned} E_{\diamond_0}(f) = &\sum_{Q \in V(\diamond_0)} \frac{1}{2\operatorname{Re}(\rho_Q)} \left(|\rho_Q|^2 \, |f(b_+) - f(b_-)|^2 + |f(w_+) - f(w_-)|^2 \right) \\ &+ \sum_{Q \in V(\diamond_0)} \frac{\operatorname{Im}(\rho_Q)}{\operatorname{Re}(\rho_Q)} \operatorname{Re}\left((f(b_+) - f(b_-)) \, \overline{(f(w_+) - f(w_-))} \right). \end{aligned}$$

Proof. Let us denote the right hand side of the upper equation by $E'_{\diamond_0}(f)$. Then, $E_{\diamond_0}(f) = E_{\diamond_0}(\operatorname{Re}(f)) + E_{\diamond_0}(\operatorname{Im}(f))$ and $E'_{\diamond_0}(f) = E'_{\diamond_0}(\operatorname{Re}(f)) + E'_{\diamond_0}(\operatorname{Im}(f))$. Therefore, we can restrict to real functions f. Furthermore, it suffices to check the identity for just a singular quadrilateral Q, because both $E_{\diamond_0}(f)$ and $E_{\diamond_0}(f)$ are sums over $Q \in V(\diamond_0)$.

Also, both expressions depend only on df, so we may assume $f(b_-) = f(w_-) = 0$. In addition, both terms do not change if Q is rotated or scaled. Thus, we can assume $b_+ - b_- = 1$ as well, such that $w_+ - w_- = i\rho_Q$. Then, $E_Q(f)$ equals

$$\begin{aligned} &\iint_{F_Q} df \wedge \star df = 4\text{area}(Q)|\partial_\Lambda f|^2 = 2\operatorname{Re}(\rho_Q)|\partial_\Lambda f|^2 \\ =&2\operatorname{Re}(\rho_Q) \left| \lambda_Q f(b_+) + \frac{\bar{\lambda}_Q}{i\rho_Q} f(w_+) \right|^2 \\ =&2|\rho_Q|\sin(\varphi_Q) \left(\frac{f^2(b_+)}{4\sin^2(\varphi_Q)} + \frac{f^2(w_+)}{4\sin^2(\varphi_Q)|\rho_Q|^2} + 2f(b_+)f(w_+)\operatorname{Im}\left(\frac{\bar{\lambda}_Q^2}{\rho_Q} \right) \right) \\ =&\frac{|\rho_Q|^2}{2\operatorname{Re}(\rho_Q)} f^2(b_+) + \frac{1}{2\operatorname{Re}(\rho_Q)} f^2(w_+) + 2f(b_+)f(w_+)\frac{\operatorname{Im}(\rho_Q)}{2\operatorname{Re}(\rho_Q)} = E'_Q(f). \end{aligned}$$

Here, we used that $2\text{area}(Q) = |w_+ - w_-|\sin(\varphi_Q) = \operatorname{Re}(\rho_Q) = |\rho_Q|\sin(\varphi_Q)$, $1/|\lambda_Q| = 2\sin(\varphi_Q)$, and $\bar{\lambda}_Q = -\rho_Q/(4\text{area}(Q))$ (see the proof of Lemma 2.3). $\square$

The same formula of $E_{\diamond_0}(f)$ was given by Mercat in [Mer08].

In the case of rhombic quad-graphs, Duffin proved in [Duf68] that the discrete Dirichlet boundary value problem has a unique solution. The same argument applies for general quad-graphs with the discrete Dirichlet energy defined here. Using a different notation, Skopenkov proved existence and uniqueness of solutions of the discrete Dirichlet boundary value problem as well [Sko13].

Lemma 2.28. *Let $\diamond_0 \subset \diamond$ be finite and $f_0 : V(\partial\Lambda_0) \to \mathbb{R}$. We consider the vector space of real functions $f : V(\Lambda_0) \to \mathbb{R}$ that agree with f_0 on the boundary. Then, $E_{\diamond_0}$ is a strictly convex nonnegative quadratic functional in terms of the interior values $f(v)$. Furthermore,*

$$-\frac{\partial E_{\diamond_0}}{\partial f(v)}(f) = 4\text{area}(F_v)\triangle f(v)$$

for any $v \in V(\Lambda_0)\backslash V(\partial\Lambda_0)$. In particular, the discrete Dirichlet boundary value problem is uniquely solvable.

Proof. By construction, $E_{\diamond_0}$ is a quadratic form in the vector space of real functions $f : V(\Lambda) \to \mathbb{R}$. In particular, it is convex, nonnegative, and quadratic in terms of the values $f(v)$. Thus, global minima exist. To prove strict convexity, it suffices to check that the minimum is unique.

For an interior vertex $v_0 \in V(\Lambda_0)\backslash V(\partial\Lambda_0)$, let $\phi(v) = \delta_{vv_0}$ be the Kronecker delta function on $V(\Lambda_0)$. Then,

$$\frac{\partial E_{\diamond_0}}{\partial f(v_0)}(f) = \frac{d}{dt}E_{\diamond_0}(f+t\phi)|_{t=0} = 2\langle df, d\phi\rangle_{\diamond_0} = -2\langle \triangle f, \phi\rangle = -4\text{area}(F_{v_0})\triangle f(v_0)$$

due to Proposition 2.19. It follows that minima of $E_{\diamond_0}$ are discrete harmonic in the interior of $\diamond_0$. The difference of two minima is a discrete harmonic function vanishing on the boundary, so its energy has to be 0. But only essentially constant functions have zero energy. Thus, the difference has to vanish everywhere, i.e., minima are unique. □

In the following, we apply Lemma 2.28 to show that $\partial_\Lambda, \bar{\partial}_\Lambda, \partial_\diamond, \bar{\partial}_\diamond, \triangle$ are surjective operators. This implies immediately the existence of discrete Green's functions and discrete Cauchy's kernels, as we will see in Sections 2.5 and 2.6.

Lemma 2.29. *Let $\diamond_0 \subset \diamond$ be finite and homeomorphic to a disk. Then, the discrete derivatives $\partial_\Lambda, \bar{\partial}_\Lambda, \partial_\diamond, \bar{\partial}_\diamond$ and the discrete Laplacian $\triangle$ are surjective operators. That means, given any complex functions h_0 on $V(\diamond_0)$ and f_0 on $V(\Lambda_0\backslash\partial\Lambda_0)$, there exist functions $h_\partial, h_{\bar{\partial}}$ on $V(\diamond_0)$ and functions $f_\partial, f_{\bar{\partial}}, f_\triangle$ on $V(\Lambda_0)$ such that $\partial_\diamond h_\partial = \bar{\partial}_\diamond h_{\bar{\partial}} = \triangle f_\triangle = f_0$ and $\partial_\Lambda f_\partial = \bar{\partial}_\Lambda f_{\bar{\partial}} = h_0$. If f_0 is real-valued, $f_\triangle$ can be chosen real-valued as well.*

Proof. Let V, E, and F denote the number of vertices, edges, and faces of Λ_0, respectively. Denote by BV and BE the number of vertices and edges of $\partial\Lambda_0$, respectively. By assumption, $\partial\Lambda_0$ is a simple polygon, and $BV = BE$.

By Lemma 2.28, the space of real discrete harmonic functions on $V(\Lambda_0)$ has dimension BV. Clearly, real and imaginary part of a discrete harmonic function are itself discrete harmonic. Therefore, the complex dimension of the space of complex discrete harmonic functions is BV as well. Thus, we have shown that $\triangle : \mathbb{K}^{V(\Lambda_0)} \to \mathbb{K}^{V(\Lambda_0 \backslash \partial\Lambda_0)}$ is a surjective linear operator with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Now, $\triangle = 4\partial_\diamond\bar{\partial}_\Lambda = 4\bar{\partial}_\diamond\partial_\Lambda$ by Corollary 2.20, so $\partial_\diamond, \bar{\partial}_\diamond : \mathbb{C}^{V(\diamond_0)} \to \mathbb{C}^{V(\Lambda_0 \backslash \partial\Lambda_0)}$ are surjective as well. The kernel of $\bar{\partial}_\diamond$ consists of all discrete holomorphic functions on $V(\diamond_0)$. By Proposition 2.8, any such function has a discrete primitive, i.e., the kernel is contained in the image of ∂_Λ. Using the surjectivity of $\triangle$, it follows that $\partial_\Lambda : \mathbb{C}^{V(\Lambda_0)} \to \mathbb{C}^{V(\diamond_0)}$ is surjective. The same is true for $\bar{\partial}_\Lambda$. □

Theorem 2.30. *The discrete derivatives $\partial_\Lambda, \bar{\partial}_\Lambda, \partial_\diamond, \bar{\partial}_\diamond$ and the discrete Laplacian $\triangle$ (defined on complex or real functions) are surjective operators on the vector space of functions on $V(\Lambda)$ or $V(\diamond)$, respectively.*

Proof. Let $\diamond_0 \subset \diamond_1 \subset \diamond_2 \subset \ldots \subset \diamond$ be a sequence of simply-connected finite domains such that $\bigcup_{k=0}^{\infty} \diamond_k = \diamond$. By Λ_k we denote the subgraph of Λ whose vertices and edges are the vertices and edges of quadrilaterals in $\diamond_k$.

Let us first prove that any $h : V(\diamond) \to \mathbb{C}$ has a preimage under the discrete derivatives $\partial_\Lambda, \bar{\partial}_\Lambda$. By Lemma 2.29, the affine space $A_k^{(0)}$ of all complex functions on $V(\Lambda_k)$ that are mapped to $h|_{V(\diamond_k)}$ by ∂_Λ (or $\bar{\partial}_\Lambda$), is nonempty. Let $A_k^{(0)}\Big|_{\Lambda_j}$ denote the affine space of restrictions of these functions to $V(\Lambda_j) \subseteq V(\Lambda_k)$. Clearly,

$$A_0^{(0)} \supseteq A_1^{(0)}\Big|_{\Lambda_0} \supseteq A_2^{(0)}\Big|_{\Lambda_0} \supseteq \ldots$$

Since all affine spaces are finite-dimensional and nonempty, this chain becomes stationary at some point, giving a function f_0 on $V(\Lambda_0)$ mapped to $h|_{V(\diamond_0)}$ by ∂_Λ (or $\bar{\partial}_\Lambda$) that can be extended to a function in $A_k^{(0)}$ for any k.

Inductively, assume that $f_j : V(\Lambda_j) \to \mathbb{C}$ is mapped to $h|_{V(\diamond_j)}$ by ∂_Λ (or $\bar{\partial}_\Lambda$), and that f_j can be extended to a function in $A_k^{(j)}$ for all $k \geq j$. Let $A_k^{(j+1)}$, $k \geq j+1$, be the affine space of all complex functions on $V(\Lambda_k)$ that are mapped to $h|_{V(\diamond_k)}$ by ∂_Λ (or $\bar{\partial}_\Lambda$), and whose restriction to $V(\Lambda_j)$ is equal to f_j. By assumption, all these spaces are nonempty. In the same way as above, there is a function f_{j+1} extending f_j to $V(\Lambda_{j+1})$ that is mapped to $h|_{V(\diamond_{j+1})}$ by ∂_Λ (or $\bar{\partial}_\Lambda$), and that can be extended to a function in $A_k^{(j+1)}$ for all $k \geq j+1$.

For $v \in V(\Lambda_k)$, define $f(v) := f_k(v)$. f is a well defined complex function on $V(\Lambda)$ with $\partial_\Lambda f = h$ (or $\bar{\partial}_\Lambda f = h$). Hence, $\partial_\Lambda, \bar{\partial}_\Lambda : \mathbb{C}^{V(\Lambda)} \to \mathbb{C}^{V(\diamond)}$ are surjective.

Replacing $V(\diamond_k)$ by $V(\Lambda_k \backslash \partial\Lambda_k)$, we obtain with the same arguments that $\triangle$ is surjective, regardless whether $\triangle$ is defined on real or complex functions.

Finally, $\partial_\diamond, \bar{\partial}_\diamond : \mathbb{C}^{V(\diamond)} \to \mathbb{C}^{V(\Lambda)}$ are surjective due to $\triangle = 4\partial_\diamond\bar{\partial}_\Lambda = 4\bar{\partial}_\diamond\partial_\Lambda$ by Corollary 2.20. $\square$

In the case of rhombic quad-graphs with bounded interior angles, Kenyon proved the existence of a discrete Green's function and a discrete Cauchy's kernel with an asymptotic behavior similar to the classical setting [Ken02b]. But in the general case, it seems to be practically impossible to speak about any asymptotic behavior of certain discrete functions. For this reason, we will not require any asymptotic behavior of discrete Green's functions and discrete Cauchy's kernels in Sections 2.5 and 2.6.

Still, one can expect results concerning the asymptotics of special discrete functions if the interior angles and the side lengths of the quadrilaterals are bounded, meaning that the quadrilaterals do not degenerate at infinity. And indeed, on such quad-graphs we can show that any discrete harmonic function whose difference functions on $V(\Gamma)$ and $V(\Gamma^*)$ have asymptotics $o(v^{-1/2})$, is essentially constant. In the rhombic setting, Chelkak and Smirnov showed that a discrete Liouville's theorem holds true, i.e., any bounded discrete harmonic function on $V(\Lambda)$ vanishes.

Theorem 2.31. *Assume that there exist constants $\alpha_0 > 0$ and $\infty > E_1 > E_0 > 0$ such that $\alpha \geq \alpha_0$ and $E_1 \geq e \geq E_0$ for all interior angles α and side lengths e of quadrilaterals $Q \in V(\diamond)$.*

If $f : V(\Lambda) \to \mathbb{C}$ is discrete harmonic and $f(v_+) - f(v_-) = o(v_\pm^{-1/2})$ for any $Q \in V(\diamond)$ and two of its vertices $v_\pm \in V(\Gamma)$ or $v_\pm \in V(\Gamma^*)$, f is essentially constant.

Proof. Without loss of generality, we can restrict to real functions f. Assume that f is not essentially constant. Then, $df \wedge \star df$ is nonzero somewhere on a face F of X. In particular, the discrete Dirichlet energy of f is bounded away from zero if the domain contains F. Now, the idea of proof is to show that if the domain is large enough, the function being zero in the interior and equal to f on the boundary has a smaller discrete Dirichlet energy than f, contradicting Lemma 2.28.

Let us first bound the intersection angles and the lengths of diagonals of the quadrilaterals. Take $Q \in V(\diamondsuit)$. Then, there are two opposite interior angles that are less than π, say $\alpha_\pm$ at vertices $b_\pm$. Since all interior angles are bounded by α_0 from below, one of $\alpha_\pm$ is less than or equal to $\pi-\alpha_0$, say $\alpha_0 \leq \alpha_- \leq \pi-\alpha_0$. By triangle inequality, $|b_+ - b_-|, |w_+ - w_-| < 2E_1$. Twice the area of Q equals

$$|w_- - b_-||w_+ - b_-|\sin(\alpha_-) + |w_- - b_+||w_+ - b_+|\sin(\alpha_+) \geq E_0^2 \sin(\alpha_0).$$

On the other hand, twice the area of Q is equal to $|b_+ - b_-||w_+ - w_-|\sin(\varphi_Q)$. Combining the two inequalities yields $|b_+ - b_-|, |w_+ - w_-| > E_0' := E_0^2 \sin(\alpha_0)/(2E_1)$. Furthermore, $\sin(\varphi_Q) > E_0^2\sin(\alpha_0)/(4E_1^2) = E_0'/(2E_1)$. Thus, we can bound

$$\rho_Q = \frac{|w_+ - w_-|}{|b_+ - b_-|} \exp\left(i\left(\varphi_Q - \frac{\pi}{2}\right)\right) = \frac{|w_+ - w_-|}{|b_+ - b_-|}\left(\sin(\varphi_Q) - i\cos(\varphi_Q)\right),$$

$$\text{by } |\rho_Q| < \frac{2E_1}{E_0'} \text{ and } \operatorname{Re}(\rho_Q) > \left(\frac{E_0'}{2E_1}\right)^2.$$

For some $r > 0$, denote by $B_\diamond(0, r) \subset V(\diamondsuit)$ the set of quadrilaterals that have a nonempty intersection with the open ball around 0 and radius r. Let $R > 2E_1$, and consider the ball $B_\diamond(0, R) \subset V(\diamondsuit)$.

Since edge lengths are bounded by E_1, all faces in the boundary of $B_\diamond(0, R)$ are contained in the set $B(0, R + 2E_1)\backslash B(0, R - 2E_1) \subset \mathbb{C}$. The area of the latter is $8\pi RE_1$. Any quadrilateral has area at least $E_0^2 \sin(\alpha_0)/2$, so at most $16\pi RE_1/(E_0^2 \sin(\alpha_0))$ quadrilaterals are in the boundary of $B_\diamond(0, R)$.

Consider the real function f_R defined on the vertices of all quadrilaterals in $B_\diamond(0, R)$ that is equal to f at the boundary and equal to 0 in the interior of $B_\diamond(0, R)$. When computing the discrete Dirichlet energy, only boundary faces can give nonzero contributions. If we look at the formula of the discrete Dirichlet energy in Proposition 2.27, and use that $f(v_+) - f(v_-) = o(R^{-1/2})$ at the boundary, we see that any contribution of a boundary face has asymptotics $o(R^{-1})$. For this, we use that $|\operatorname{Re}(\rho_Q))|$ is bounded from below by a constant, and $|\mathrm{Im}\,(\rho_Q)| \leq |\rho_Q| < 2E_1/E_0'$. Using that there are only $O(R)$ faces in the boundary, the discrete Dirichlet energy $E_{B_\diamond(0,R)}(f_R)$, considered as a function of R, behaves as $o(1)$. So if R is large enough,

$$E_{B_\diamond(0,R)}(f_R) < \iint\limits_F df \wedge \star df \leq E_{B_\diamond(0,R)}(f_R),$$

contradicting that f minimizes the discrete Dirichlet energy by Lemma 2.28. □

2.5 Discrete Green's functions

In this section, we discuss discrete Green's functions.

Definition. Let $v_0 \in V(\Lambda)$. A real function $G(\cdot; v_0)$ on $V(\Lambda)$ is a *(free) discrete Green's function* for v_0 if

$$G(v_0; v_0) = 0 \text{ and } \triangle G(v; v_0) = \frac{1}{4\mathrm{area}(F_{v_0})}\delta_{vv_0} \text{ for all } v \in V(\Lambda).$$

Corollary 2.32. *A discrete Green's function exists for any $v_0 \in V(\Lambda)$.*

Proof. By Theorem 2.30, there exists a function $G : V(\Lambda) \to \mathbb{R}$ such that $\triangle G(v) = \delta_{vv_0}/\left(4\mathrm{area}(F_{v_0})\right)$. Since constant functions are discrete harmonic, we can adjust G to get $G(v_0) = 0$. □

Remark. In contrast to the smooth setting or the rhombic case investigated by Kenyon [Ken02b] and Chelkak and Smirnov [CS11], we do not require any asymptotic behavior of the discrete Green's function. But when considering planar parallelogram-graphs with bounded interior angles and bounded ratio of side lengths in Section 4.2 of Chapter 4, we will prove the existence of a discrete Green's function with asymptotics generalizing the corresponding result for rhombic quad-graphs.

Our notion of discrete Greens' functions in a discrete domain and the proof of their existence follows the presentation of Chelkak and Smirnov in [CS11].

Definition. Let $\diamond_0 \subset \diamond$ be finite. For $v_0 \in V(\Lambda_0 \backslash \partial\Lambda_0)$, a real function $G_{\Lambda_0}(\cdot; v_0)$ on $V(\Lambda_0)$ is a *discrete Green's function in* Λ_0 for v_0 if

$$\begin{aligned} G_{\Lambda_0}(v; v_0) &= 0 \text{ for all } v \in V(\partial\Lambda_0), \\ \text{and } \triangle G_{\Lambda_0}(v; v_0) &= \frac{1}{4\mathrm{area}(F_{v_0})}\delta_{vv_0} \text{ for all } v \in V(\Lambda_0 \backslash \partial\Lambda_0). \end{aligned}$$

Corollary 2.33. *Let $\diamond_0 \subset \diamond$ be finite, and $v_0 \in V(\Lambda_0 \backslash \partial\Lambda_0)$. Then, there exists a unique discrete Green's function in Λ_0 for v_0.*

Proof. Let $G_0 : V(\Lambda) \to \mathbb{C}$ be the free discrete Green's function for v_0 given by Corollary 2.32. By Lemma 2.28, there exists a real discrete harmonic function G_1 on $V(\Lambda_0)$ such that G_1 and G_0 coincide on $V(\partial\Lambda_0)$. Thus, $G_{\Lambda_0}(\cdot; v_0) := G_0 - G_1$ is a discrete Green's function in Λ_0 for v_0. Since the difference of two discrete Green's functions in Λ_0 for v_0 is discrete harmonic on $V(\Lambda_0)$ and equals zero on the boundary $V(\partial\Lambda_0)$, it has to be identically zero by Lemma 2.28. □

2.6 Discrete Cauchy's integral formulae

In this section, we first formulate discretizations of the standard Cauchy's integral formula, both for discrete holomorphic functions on $V(\Lambda)$ and $V(\diamond)$. Later, we give with Theorem 2.36 a discrete formulation of Cauchy's integral formula for the derivative of a discrete holomorphic function. We conclude this chapter with Section 2.6.1, where we relate our formulation of the discrete Cauchy's integral formula applied to a discrete holomorphic function on $V(\Lambda)$ with the notation used by Chelkak and Smirnov in [CS11]. Besides being of interest in its own right, we will encounter the decomposition into a black and a white integration again when discussing discrete period matrices in Section 3.3 of Chapter 3.

Definition. *Discrete Cauchy's kernels with respect to $Q_0 \in V(\diamond)$ and $v_0 \in V(\Lambda)$* are functions $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ and $K_{v_0} : V(\diamond) \to \mathbb{C}$, respectively, that satisfy for all $Q \in V(\diamond)$, $v \in V(\Lambda)$:

$$\bar{\partial}_\Lambda K_{Q_0}(Q) = \delta_{QQ_0} \frac{\pi}{2\text{area}(F_Q)} \text{ and } \bar{\partial}_\diamond K_{v_0}(v) = \delta_{vv_0} \frac{\pi}{2\text{area}(F_v)}.$$

Remark. As for the discrete Green's functions in Section 2.5, we do not require any asymptotic behavior of the discrete Cauchy's kernels. However, if interior angles and side lengths of quadrilaterals are bounded, it follows from Theorem 2.31 that any discrete Cauchy's kernel with respect to a vertex of $\diamond$ with asymptotics $o(v^{-1/2})$ is necessarily unique. In Section 4.3 of Chapter 4, we will construct discrete Cauchy's kernels with asymptotics similar to the smooth setting, generalizing Kenyon's result [Ken02b] on rhombic quad-graphs to parallelogram-graphs.

The existence of discrete Cauchy's kernels follows from Theorem 2.30:

Corollary 2.34. *Let $Q_0 \in V(\diamond)$ and $v_0 \in V(\Lambda)$ be arbitrary. Then, discrete Cauchy's kernels with respect to Q_0 and v_0 exist.*

Theorem 2.35. *Let f and h be discrete holomorphic functions on $V(\Lambda)$ and $V(\diamond)$, respectively. Let $v_0 \in V(\Lambda)$ and $Q_0 \in V(\diamond)$, and let $K_{v_0} : V(\diamond) \to \mathbb{C}$ and $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ be discrete Cauchy's kernels with respect to v_0 and Q_0, respectively.*

Then, for any discrete contours C_{v_0} and C_{Q_0} on X surrounding v_0 and Q_0 once in counterclockwise order, respectively, discrete Cauchy's integral formulae hold:

$$f(v_0) = \frac{1}{2\pi i} \oint\limits_{C_{v_0}} f K_{v_0} dz,$$
$$h(Q_0) = \frac{1}{2\pi i} \oint\limits_{C_{Q_0}} h K_{Q_0} dz.$$

Proof. Let P_v and P_Q be discrete elementary cycles, v being a vertex and Q a face of Λ. By Lemma 2.3 and the definition of $\bar{\partial}_\diamond$, we get:

$$\frac{1}{2\pi i} \oint\limits_{P_v} f K_{v_0} dz = \frac{2}{\pi} \text{area}(F_v) f(v) \bar{\partial}_\diamond K_{v_0}(v) = \delta_{v v_0} f(v),$$
$$\frac{1}{2\pi i} \oint\limits_{P_Q} f K_{v_0} dz = \frac{2}{\pi} \text{area}(F_Q) \bar{\partial}_\Lambda f(Q) K_{v_0}(Q) = 0.$$

By definition, the discrete contour C_{v_0} bounds a topological disk, and we can decompose the integration along C_{v_0} into a couple of integrations along discrete elementary cycles P_v and P_Q as above. Summing up, only the contribution of P_{v_0} is nonvanishing, and we get the desired result. The second formula is shown in an analog fashion. $\square$

Remark. In the case of rhombic quad-graphs, Mercat formulated a discrete Cauchy's integral formula for the average of a discrete holomorphic function on $V(\Lambda)$ along an edge of Λ. In [CS11], Chelkak and Smirnov provided a discrete Cauchy's integral formula for discrete holomorphic functions on $V(\diamond)$ using an integration along cycles on Γ and Γ^*.

Theorem 2.36. *Let f be a discrete holomorphic function on $V(\Lambda)$, $Q_0 \in V(\diamond)$, and let $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ be a discrete Cauchy's kernel with respect to Q_0.*

Then, for any discrete contour C_{Q_0} in X surrounding Q_0 once in counterclockwise order that does not contain any edge inside Q_0 (see Figure 2.6), the discrete Cauchy's integral formula is true:

$$\partial_\Lambda f(Q_0) = -\frac{1}{2\pi i}\oint\limits_{C_{Q_0}} f\partial_\Lambda K_{Q_0} dz.$$

Proof. Let D be the discrete domain in X bounded by C_{Q_0}. Since no edge of C_{Q_0} passes through Q_0, the discrete one-form $\bar{\partial}_\Lambda K_{Q_0} d\bar{z}$ vanishes on C_{Q_0}. Therefore,

$$\int\limits_{C_{Q_0}} f\partial_\Lambda K_{Q_0} dz = \int\limits_{C_{Q_0}} f dK_{Q_0} = \iint\limits_{D} d(f dK_{Q_0}) = \iint\limits_{D} df \wedge dK_{Q_0}$$

due to discrete Stokes' Theorem 2.9, Theorem 2.16, and Proposition 2.10. Now, f is discrete holomorphic, so $df \wedge dK_{Q_0} = \partial_\Lambda f \bar{\partial}_\Lambda K_{Q_0} dz \wedge d\bar{z}$. But $\bar{\partial}_\Lambda K_{Q_0}$ vanishes on all vertices of $\Diamond$ but Q_0. Finally,

$$-\frac{1}{2\pi i}\oint\limits_{C_{Q_0}} f\partial_\Lambda K_{Q_0} dz = -\frac{1}{2\pi i}\iint\limits_{F_{Q_0}} \partial_\Lambda f \bar{\partial}_\Lambda K_{Q_0} dz \wedge d\bar{z} = \partial_\Lambda f(Q_0).$$

□

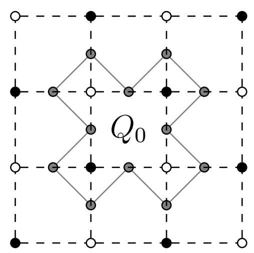


Figure 2.6: Discrete contour as in Theorem 2.36

Remark. In general, there exists no analog of Theorem 2.36 for the discrete derivative of a discrete holomorphic function on $V(\Diamond)$, because the discrete derivative itself does not need to be discrete holomorphic. However, in the special case of integer lattices, any discrete derivate of a discrete holomorphic function is itself discrete holomorphic. In Section 4.4 of Chapter 4, we will obtain discrete analogs of Cauchy's integral formulae for higher derivatives of discrete holomorphic functions on $V(\Lambda)$ or $V(\Diamond)$.

2.6.1 A different notation

Let W be a cycle on the edges of Γ^*, having (counterclockwise ordered) vertices $w_0, w_1, \ldots, w_m \in V(\Gamma^*)$, $w_m = w_0$. Then, any edge connecting two consecutive vertices w_k, w_{k+1} forms the diagonal of a face $Q(w_k, w_{k+1}) \in V(\diamond)$. We denote the set of such faces together with the induced orientation of their white diagonals by $W_\diamond$. For $Q \in W_\diamond$, we denote its white vertices by $w_-(Q), w_+(Q)$ such that the corresponding oriented diagonal goes from $w_-(Q)$ to $w_+(Q)$. Its black vertices are denoted by $b(Q), b'(Q)$ in such a way that $w_-(Q), b(Q), w_+(Q), b'(Q)$ appear in counterclockwise order. The reason why we do not choose our classical notation of Figure 2.1 is that black and white vertices now play a different role that shall be indicated by the notation.

Now, we construct a cycle B on the edges of Γ, having (counterclockwise ordered) vertices $b_0, b_1, \ldots, b_n \in V(\Gamma)$, $b_n = b_0$, in the following way. We start with $b_0 := b\left(Q\left(w_0, w_1\right)\right)$. In the star of the vertex w_1, there are two simple paths on Γ connecting b_0 and $b\left(Q\left(w_1, w_2\right)\right)$, and we choose the path that does not go through $Q(w_0, w_1)$. Note that it may happen that $b\left(Q\left(w_1, w_2\right)\right) = b_0$; in this case, we do not add any vertices to B. Also, $w_2 = w_0$ is possible, which causes adding the nondirect path connecting b_0 and $b\left(Q\left(w_1, w_2\right)\right) = b'\left(Q\left(w_0, w_1\right)\right)$.

Continuing this procedure till we have connected $b\left(Q\left(w_{m-1}, w_m\right)\right)$ with b_0, we end up with a closed path B on Γ. Without loss of generality, any two consecutive vertices in B are different. As above, any edge connecting two consecutive vertices b_k, b_{k+1} forms the diagonal of a face $Q(b_k, b_{k+1}) \in V(\diamond)$. We denote the set of such faces together with the induced orientation of their black diagonals by $B_\diamond$. For $Q \in B_\diamond$, we denote its black vertices by $b_-(Q), b_+(Q)$ such that the corresponding oriented diagonal goes from $b_-(Q)$ to $b_+(Q)$. Finally, its white vertices are denoted by $w(Q), w'(Q)$ in such a way that $b_-(Q), w'(Q), b_+(Q), w(Q)$ appear in counterclockwise order.

Definition. Let W and B be as above, and h a function defined on $W_\diamond \cup B_\diamond$. We define the *discrete integrals* along W and B by

$$\oint_W h(Q)dz := \sum_{k=0}^{m-1} f\left(Q\left(w_k, w_{k+1}\right)\right)\left(w_{k+1} - w_k\right),$$
$$\oint_W h(Q)d\bar{z} := \sum_{k=0}^{m-1} f\left(Q\left(w_k, w_{k+1}\right)\right)\overline{\left(w_{k+1} - w_k\right)};$$

$$\oint_B h(Q)dz := \sum_{k=0}^{n-1} f\left(Q\left(b_k, b_{k+1}\right)\right)\left(b_{k+1} - b_k\right),$$
$$\oint_B h(Q)d\bar{z} := \sum_{k=0}^{n-1} f\left(Q\left(b_k, b_{k+1}\right)\right)\overline{\left(b_{k+1} - b_k\right)}.$$

In between the closed paths B and W, there is a cycle P on the medial graph X that comprises exactly all edges $[Q, v]$ with $Q \in W_\diamond$ and $v \in B$ incident to Q, and all edges $[Q, v]$ with $Q \in B_\diamond$ and $v \in W$ incident to Q. The orientation of $[Q, v]$ is induced by the orientation of the corresponding parallel white or black diagonal. Figure 2.7 gives an example for this construction.

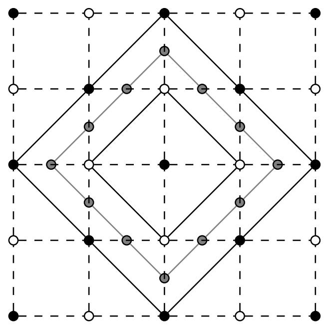


Figure 2.7: Cycles W on Γ^*, B on Γ, and closed path P on X in between

Remark. Note that a discrete contour P on X induces a white cycle W and a black cycle B in such a way that W, P, and B are related as above. We will discuss this construction in more detail in Section 3.3 of Chapter 3.

Lemma 2.37. *Let W, B, and P be as above. Let f and h be functions defined on the vertices of W and B and on $W_\diamond \cup B_\diamond$, respectively. Then,*

$$\oint_W f(b(Q))h(Q)dz + \oint_B f(w(Q))h(Q)dz = 2\oint_P fhdz.$$

Proof. Any edge $e = [Q, b(Q)]$ ($Q \in W_\diamond$) or $[Q, w(Q)]$ ($Q \in B_\diamond$) of P corresponds to either an edge $w_-(Q)w_+(Q)$ of W, or to an edge $b_-(Q)b_+(Q)$ of B, respectively, and vice versa. Since this edge of X is just half as long as the parallel edge of Γ or Γ^*, $2\int_e fh = f\left(b\left(Q\right)\right)h(Q)(w_+ - w_-)$ in the first and $2\int_e fh = f\left(w\left(Q\right)\right)h(Q)(b_+ - b_-)$ in the second case. Therefore, the discrete integral along P decomposes into one along B and one along W. $\square$

Note that the construction of B and Lemma 2.37 are also valid if W consists of a single point or of only two edges (being the same, but traversed in both directions). In any case, P will be a discrete contour.

The discrete Cauchy's integral formula in the notation of Chelkak and Smirnov in [CS11] reads as

$$\pi i h(Q_0) = \oint_W h(Q) K_{Q_0}(b(Q)) dz + \oint_B h(Q) K_{Q_0}(w(Q)) dz$$

if $Q_0 \in V(\diamond)$ is surrounded once by W, h is discrete holomorphic on $V(\diamond)$, and $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ is a discrete Cauchy's kernel with respect to Q_0. Lemma 2.37 directly relates this formulation to ours in Theorem 2.35.

Finally, we conclude this chapter with a proposition relying on the decomposition of a discrete contour into black and white cycles. In Corollary 2.13 (i), we have already seen that $f dg + g df$ is closed for functions $f, g : V(\Lambda) \to \mathbb{C}$. Actually, a slightly stronger statement is true:

Proposition 2.38. *Let W and B be as above, and let $f, g : V(\Lambda) \to \mathbb{C}$. Then,*

$$\oint_W f(b(Q)) \left(\partial_\Lambda g(Q) dz + \bar{\partial}_\Lambda g(Q) d\bar{z}\right) + \oint_B g(w(Q)) \left(\partial_\Lambda f(Q) dz + \bar{\partial}_\Lambda f(Q) d\bar{z}\right) = 0.$$

Proof. We first rewrite the discrete integral along W. By definition, we have $dg = \partial_\Lambda g dz + \bar{\partial}_\Lambda g d\bar{z}$. Using discrete Stokes' Theorem 2.9,

$$\oint_W f(b(Q)) \left(\partial_\Lambda g(Q) dz + \bar{\partial}_\Lambda g(Q) d\bar{z}\right) = \sum_{Q \in W_\diamond} f(b(Q)) \left(g\left(w_+(Q)\right) - g\left(w_-(Q)\right)\right).$$

We can change the summation along the path W into a summation along the path B by

$$\begin{aligned} &\sum_{Q \in W_\diamond} f(b(Q)) \left(g\left(w_+(Q)\right) - g\left(w_-(Q)\right)\right) \\ = &-\sum_{Q \in B_\diamond} g\left(w\left(Q\right)\right) \left(f\left(b_+(Q)\right) - f\left(b_-(Q)\right)\right). \end{aligned}$$

In an analog way as for W, we can rewrite the discrete integral along B as

$$\oint_B g(w(Q)) \left(\partial_\Lambda f(Q) dz + \bar{\partial}_\Lambda f(Q) d\bar{z}\right) = \sum_{Q \in B_\diamond} g(w(Q)) \left(f\left(b_+(Q)\right) - f\left(b_-(Q)\right)\right).$$

In summary,

$$\oint_W f(b(Q))\left(\partial_\Lambda g(Q)dz + \bar{\partial}_\Lambda g(Q)d\bar{z}\right) + \oint_B g(w(Q))\left(\partial_\Lambda f(Q)dz + \bar{\partial}_\Lambda f(Q)d\bar{z}\right) = 0.$$

$\square$

Chapter 3

Discrete Riemann surfaces

This chapter is devoted to a linear theory of discrete Riemann surfaces. Our setup is a strongly regular cellular decomposition of an oriented surface into (nonoverlapping) quadrilaterals, called quad-graph, which we assume to be bipartite. The precise definition of the complex structure will be given in Section 3.1. We also show how polyhedral surfaces can be seen as discrete Riemann surfaces. Furthermore, we comment how the discrete exterior calculus we discussed in Chapter 2 generalizes to discrete Riemann surfaces.

Although Mercat already developed such a theory in [Mer01b, Mer01a, Mer07, Mer08], and Bobenko and Skopenkov continued Mercat's work in [BS12], we want to point out that so far, one concentrated on the case of real ρ_Q, i.e., to the case that the diagonals of quadrilaterals are orthogonal to each other. In [Mer08], Mercat generalized just discrete exterior calculus to discrete Riemann surfaces; in the end of [BS12], Bobenko and Skopenkov sketched how one gets a discrete Riemann bilinear identity and discrete period matrices for general quad-graphs. But even in the case of real ρ_Q, this chapter still contains new insights and new results. In contrast to previous papers, our approach using the medial graph and local representations of discrete differentials allows more intuitive notations and proofs that often coincide with the classical theory. Still, we recover the definitions of Mercat in our setting.

In Section 3.2 we start with some elementary topological properties of discrete holomorphic mappings such as the discrete Riemann-Hurwitz Formula 3.10. This section gives an idea, how branch points of higher order can be modeled on discrete Riemann surfaces. However, the notion of discrete holomorphic mappings between discrete Riemann surfaces is too rigid to go much further.

For this reason, we continue with a theory of discrete differentials. In Section 3.3, periods of discrete differentials are introduced, and the discrete Riemann Bilinear Identity 3.12 is proven in a way very close to the classical proof.

By looking at discrete harmonic differentials in Section 3.4, we compute the dimension of the space of discrete holomorphic differentials. The way of computation is literally the same as in the continuous setup. Moreover, we recover the discrete period matrices Mercat defined in [Mer01a, Mer07].

We conclude this chapter with Section 3.5, in which we recover the discrete Abelian integrals of the first and the third kind of Bobenko and Skopenkov [BS12] in our general setup. Furthermore, we define discrete Abelian integrals of the second kind for the first time, and compute their periods. Eventually, this leads to the discrete Riemann-Roch Theorem 3.30 that generalizes the result of Bobenko and Skopenkov [BS12] to a wider class of quad-graphs, and that includes double values of discrete meromorphic functions, or double poles of discrete Abelian differentials. Finally, we shortly discuss discrete Abel-Jacobi maps.

3.1 Basic definitions

The aim of this section is to first introduce discrete Riemann surfaces in Section 3.1.1, giving polyhedral surfaces as an example in Section 3.1.2, and then to describe how the theory of discrete complex analysis on planar quad-graphs we developed in Chapter 2 applies in the more general setting of discrete Riemann surfaces in Section 3.1.3.

3.1.1 Discrete Riemann surfaces

Let Σ be a connected oriented surface without boundary together with a *strongly regular* cellular decomposition into nonoverlapping quadrilaterals described by a bipartite quad-graph Λ. Here, 0-cells correspond to vertices $V(\Lambda)$, 1-cells to edges $E(\Lambda)$, and 2-cells to quadrilateral faces $F(\Lambda)$ of Λ. The assumption of strong regularity asserts that the boundary of a quadrilateral contains a vertex or an edge at most once, and two different edges or faces have at most one vertex or edge in common, respectively. Furthermore, we restrict to *locally finite* cellular decompositions, i.e., a compact subset of Σ contains only finitely many quadrilaterals.

Exactly as in Section 2.1.1, we refer to the maximal independent sets of $V(\Lambda)$ as *black* and *white* vertices, and denote the induced subgraphs on black and white diagonals of faces of Λ by Γ and Γ^*, respectively. As before, let $\diamond := \Lambda^*$ denote the dual graph of Λ.

All faces of Λ inherit an orientation from Σ. We may assume that the orientation on Σ is chosen in such a way that the image of any orientation-preserving embedding of a two-cell Q into the complex plane is positively oriented. If such a map z_Q maps Q to a quadrilateral in $\mathbb{C}$ such that the image points of the vertices of Q are vertices of the quadrilateral, it is called a *chart* of Q. Two such charts are called *compatible* if the oriented diagonals of the image quadrilaterals are in the same complex ratio. Thus, the angle φ_Q defined in Section 2.1.1, as well as the complex number

$$\rho_Q := -i\frac{z_Q(w_+) - z_Q(w_-)}{z_Q(b_+) - z_Q(b_-)}$$

(using the notation of Figure 2.1) are the same for compatible charts.

In particular, an equivalence class of charts z_Q of a single quadrilateral Q is uniquely characterized by the complex number ρ_Q with a positive real part. An assignment of positive real numbers ρ_Q to all faces Q of Λ was the definition of a discrete complex structure Mercat used in [Mer01b]. In his subsequent work [Mer08], he proposed a generalization to complex ρ_Q with positive real part.

Note that real ρ_Q correspond to quadrilaterals Q whose diagonals are orthogonal to each other. They arise naturally if one considers a triangulated polyhedral surface Σ and places the vertices of the dual at the circumcenters of the triangles. Discrete Riemann surfaces based on this structure were investigated in the paper [BS12] of Bobenko and Skopenkov. In the end, they also suggested a generalization of discrete Riemann surfaces given by an atlas of charts z_Q whose image quadrilaterals do not necessarily have the property that their diagonals are orthogonal to each other.

It turns out that in our setting, this definition of a discrete Riemann surface allows discrete holomorphic mappings that are nonconstant, but locally constant in some neighborhoods. To overcome this obstacle, it is necessary to take charts around vertices of Λ into account. Furthermore, these charts are needed to locally define discrete derivatives, and to locally represent discrete differentials.

For a vertex $v \in V(\Lambda)$, an orientation-preserving embedding z_v of the star of v

to the star of a vertex of a planar quad-graph embedded in the complex plane that maps vertices of Λ to vertices of the embedded quad-graph, is called a *chart*. Two charts z_v and $z_{v'}$ are *compatible* if they differ only by an Euclidean transformation. If $Q \sim v$, i.e., Q is a quadrilateral incident to v, the charts z_Q and z_v are *compatible* if the induced chart of z_v on Q is compatible to z_Q.

An *atlas* of Λ is a collection z of charts z_Q of all quadrilateral faces Q, and of charts z_v of all stars of vertices that are compatible to each other.

Definition. A triple (Σ, Λ, z) of a quadrilateral cellular decomposition Λ of Σ together with an atlas z is called a *discrete Riemann surface*.

When we later speak about particular charts, we always refer to charts in the atlas of the discrete Riemann surface.

As the following proposition shows, the request of having charts around vertices is just an additional datum. To any discrete Riemann surface according to the definitions of Mercat, Bobenko and Skopenkov, i.e., to any set of compatible charts z_Q, $Q \in V(\diamond)$, we can find compatible charts z_v, $v \in V(\Lambda)$.

Proposition 3.1. *Let Λ be a quadrilateral cellular decomposition of a surface Σ, and consider any assignment of complex numbers ρ_Q with positive real parts to faces $Q \in V(\diamond)$. Then, there exists an atlas z such that the image quadrilaterals of charts z_Q are parallelograms with the oriented ratio of diagonals equal to $i\rho_Q$.*

Proof. The construction of the charts z_Q is simple: In the complex plane, the quadrilateral with black vertices ± 1 and white vertices $\pm i\rho_Q$ is a parallelogram with the desired oriented ratio of diagonals.

In contrast, the construction of charts z_v is more delicate. See Figure 3.1 for a visualization. Let us consider the star of a vertex $v \in V(\Gamma)$. If v is white, just replace ρ_Q by $1/\rho_Q$, and φ_Q by $\pi - \varphi_Q$ in the following. Let $Q_1, Q_2, \ldots, Q_k$ be the quadrilaterals incident to v.

We choose $0 < \theta < \pi$ in such a way that $\theta < \varphi_{Q_1} < \pi - \theta$. Define $\alpha(Q_1) := \pi - \theta$, and $\alpha(Q_s) := (\pi + \theta)/(k-1)$ for the other s. Then, all $\alpha(Q_s)$ sum up to 2π.

First, we construct the images of Q_s, $s \neq 1$, starting with an auxiliary construction. As in Figure 2.1, let v, v'_{s-1}, v_s, v'_s be the vertices of Q_s in counterclockwise order. Then, we map v'_{s-1} to -1 and v'_s to 1. All points x that enclose an oriented angle $\alpha(Q_s) > 0$ with ± 1 lie on a circular arc above the real axis. Since the real part of ρ_Q is positive, the ray $ti\rho_Q$, $t > 0$, intersects this arc in exactly one point. If we choose the intersection point x_v as image of v, and $x_{v_s} := x_v - i\rho_Q$ as the image of v_s, we get a quadrilateral in $\mathbb{C}$ that has the desired oriented ratio of diagonals $i\rho_{Q_s}$. The quadrilateral is convex if and only if $x_v - i\rho_Q$ has nonpositive imaginary part.

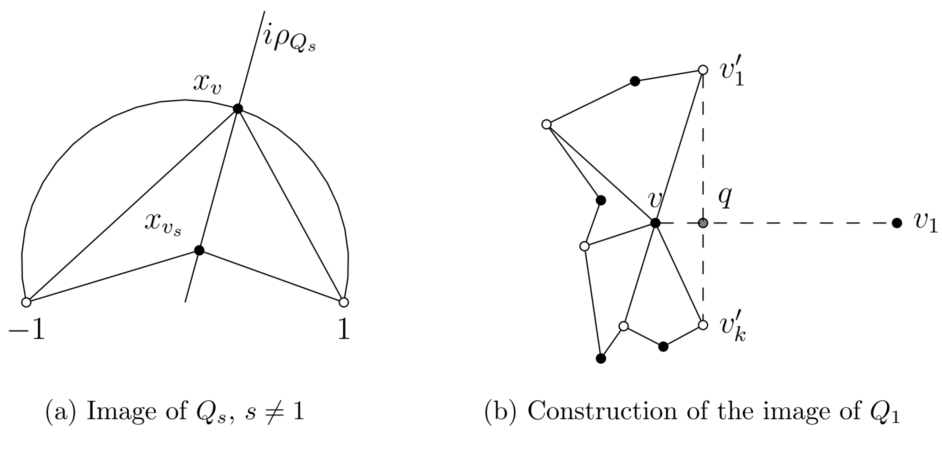
Now, we translate all the image quadrilaterals such that v is always mapped to zero. By construction, the image of Q_s is contained in a cone of angle $\alpha(Q_s)$. Thus, we can rotate and scale the images of Q_s, $s \neq 1$, in such a way that they do not overlap, and the images of edges vv'_s coincide. Since all $\alpha(Q_s)$ sum up to 2π, there is still a cone of angle $\alpha(Q_1) = \pi - \theta$ empty.

Let us identify the vertices v, v'_k, and v'_1 with their corresponding images, and choose q on the line segment $v'_k v'_1$. If q approaches v'_k, $\angle vqv'_k \to \pi - \angle v'_1 v'_k v$, and if q approaches v'_1, $\angle vqv'_k \to \angle vv'_1 v'_k$. Since

$$\angle vv'_1 v'_k < \theta < \varphi_{Q_1} < \pi - \theta < \pi - \angle v'_1 v'_k v,$$

there is a point q on the line segment such that $\angle vqv'_k = \varphi_Q$. If we take the image of v_1 on the ray tq, $t > 0$, such that its distance to the origin is $|v'_k - v'_1|/|\rho_{Q_1}|$, we obtain a quadrilateral with the oriented ratio of diagonals $i\rho_{Q_1}$.

Thus, we have constructed an orientation-preserving embedding of the star of v to the star of a vertex of a quad-graph in $\mathbb{C}$ such that the corresponding oriented ratios of diagonals coincide with $i\rho_Q$, $Q \sim v$. □



(a) Image of Q_s, $s \neq 1$

(b) Construction of the image of Q_1

Figure 3.1: Proof of Proposition 3.1

Remark. In general, it is not possible to realize the image of z_v as a star of convex quadrilaterals. For example, if the ρ_Q are very large positive real numbers, the interior angles at v are close to π if Q is convex. In particular, they would sum up to more than 2π at v.

It turns out that the particular choice of maps z_v is not that important for our consideration. For this reason, we define a discrete complex structure in the same way as Mercat did in [Mer08].

Definition. Let Λ be a quadrilateral cellular decomposition of a surface Σ. A *discrete complex structure* is an assignment of complex numbers ρ_Q with positive real part to faces $Q \in V(\diamond)$.

In Chapter 2, the medial graph of Λ played an important role, as it will do for the analysis of discrete Riemann surfaces. The definition given in Section 2.1.2 carries over to discrete Riemann surfaces on a combinatorial level, so from the quad-graph Λ we get the *medial graph* X as a cellular decomposition of Σ. Charts z_v and z_Q induce embeddings of the faces F_v and F_Q corresponding to $v \in V(\Lambda)$ and $Q \in V(\diamond)$, respectively, into the complex plane. For this, we identify vertices of X with the midpoints of the images of corresponding edges, and map the edges of X to straight line segments. As in the planar case, it follows that $z_Q(F_Q)$ is a parallelogram, even if $z_Q(Q)$ is not.

3.1.2 Polyhedral surfaces as discrete Riemann surfaces

A polyhedral surface Σ without boundary consists of Euclidean polygons glued together along common edges. Now, triangulate all faces, and take a barycentric subdivision of each triangle into three quadrilaterals. We get a strongly regular cellular decomposition into quadrilaterals that is described by a bipartite quad-graph Λ. Its black vertices correspond to the vertices and the faces of the triangulated polyhedral surface, its white vertices correspond to the midpoints of edges of triangles.

Let a choice of Euclidean embeddings of the triangles into the complex plane be given. These embeddings induce complex numbers ρ_Q on all quadrilateral faces of Q. If the orientation of Σ is chosen properly, the real parts of ρ_Q are positive. Thus, they define a discrete complex structure. To make Σ a discrete Riemann surface, we could use Proposition 3.1, but note that the interior angles at white vertices and at black vertices corresponding to faces of the triangulation already sum up to 2π in any case, so the Euclidean embeddings can be arranged properly to obtain charts around the vertices. Only at the other black vertices v, where the interior angles sum up to a cone angle Θ, we have to apply the idea of Proposition 3.1. The problem in finding a chart of the vertex star such that each quadrilateral is contained in a cone of angle $2\pi/\Theta$ times the original angle is that we have to ensure that corresponding side lengths coincide.

3.1.3 Discrete holomorphicity and discrete differentials

In this section, we consider a discrete Riemann surface (Σ, Λ, z), and will comment how we can adapt the fundamental notions and properties of discrete complex analysis discussed in Sections 2.2 and 2.3 to discrete Riemann surfaces. In contrast to the papers [Mer01b, Mer07] of Mercat, we use the medial graph for our definitions, but as we have seen in Chapter 2, we still recover his notions.

In the following, let $\diamond_0 \subseteq \diamond$ be a connected subset with vertex set $V(\Lambda_0)$, and let Γ_0 and Γ_0^* be the induced subgraphs on black and white vertices, respectively. By $X_0 \subseteq X$ we denote the subgraph of X consisting of all edges $[Q, v]$ where $Q \in V(\diamond_0)$ and $v \sim Q$.

As before, the discrete Cauchy-Riemann equation due to Mercat [Mer08] is the starting point of our analysis. Again, we use the notation of Figure 2.1.

Definition. Let $f : V(\Lambda_0) \to \mathbb{C}$. f is said to be *discrete holomorphic* if the *discrete Cauchy-Riemann equation*

$$\frac{f(b_+) - f(b_-)}{z_Q(b_+) - z_Q(b_-)} = \frac{f(w_+) - f(w_-)}{z_Q(w_+) - z_Q(w_-)}$$

is satisfied for all quadrilaterals $Q \in V(\diamond_0)$. f is *discrete antiholomorphic* if $\bar{f}$ is discrete holomorphic.

Note that the discrete Cauchy-Riemann equation in the chart z_Q is equivalent to the corresponding equation in a compatible chart z'_Q. In particular, the notion of discrete holomorphicity is well defined, i.e., it does not depend on the choice of charts. Actually, it depends on the discrete complex structure only.

Discrete differential forms on discrete Riemann surfaces are defined in the same way as in Section 2.2.2. Clearly, the discrete differential forms dz, $d\bar{z}$, and $dz \wedge d\bar{z}$ are now only defined in a chart z of the discrete Riemann surface. Still, discrete one-forms of type $\diamond$ are well defined, since they are defined by the property that the discrete integrals of the one-form along opposite edges of a face F_Q of X corresponding to $Q \in V(\diamond)$ coincide. In a chart z_Q, they can be represented as $pdz_Q + qd\bar{z}_Q$, where $p, q \in \mathbb{C}$.

Whereas the discrete derivatives of Sections 2.2.1 and 2.2.3 depend on the chosen chart, the notion of discrete holomorphic functions $h : V(\diamond) \to \mathbb{C}$ carries over. Indeed, the equation $\bar{\partial}_\diamond h(v) = 0$ does not depend on the choice of chart, because compatible charts differ by an Euclidean transformation only. Moreover, the discrete derivatives commute in a chart z_v, $v \in V(\Lambda)$, by Corollary 2.11. However, discrete holomorphicity of functions on $V(\diamond)$ is not consistently defined by the discrete complex structure. Indeed, if all ρ_Q are equal to one, any cyclic polygon defines an admissible chart z_v around a vertex v of Λ, and the equation $\bar{\partial}_\diamond h(v) = 0$ depends on the choice of the cyclic polygon.

Also, the discrete exterior derivative, the discrete wedge product, and the discrete Hodge star on discrete one-forms are well defined operators. Though we defined them using coordinates, discrete Stokes' Theorem 2.9, and Propositions 2.15 and 2.18 show that these operators depend on the discrete complex structure only. Proposition 2.10, stating that $ddf = 0$ for $f : V(\Lambda) \to \mathbb{C}$, and Theorem 2.16, stating that the discrete exterior derivative is a derivation of the discrete wedge product, generalize to discrete Riemann surfaces. The same is true for Corollary 2.12 that assures that f is discrete holomorphic in all faces incident to $v \in V(\Lambda)$ if and only if $df = pdz_v$ in a chart z_v around v, where p is a function defined on the faces incident to v. In this case, p is discrete holomorphic in v. This motivates the following definition, observing that charts z_v induce charts z_Q of quadrilaterals:

Definition. A discrete differential ω of type $\diamond$ is *discrete holomorphic* if ω is *closed*, i.e., $d\omega = 0$, and if in any chart z_Q of a quadrilateral $Q \in V(\diamond)$, $\omega = pdz_Q$. ω is *discrete antiholomorphic* if $\bar{\omega}$ is discrete holomorphic.

Remark. It suffices to check this condition for just one chart of Q, as follows from Lemma 3.2 below. In particular, discrete holomorphicity of discrete one-forms depends on the discrete complex structure only. If ω is discrete holomorphic, we can write $\omega = pdz_v$ in a chart z_v around $v \in V(\Lambda)$, where p is a function defined on the faces incident to v. In this case, p is discrete holomorphic in v. Conversely, the closeness condition can be replaced by requiring that p is discrete holomorphic.

Lemma 3.2. *A discrete differential ω of type $\diamond$ is of the form $\omega = pdz_Q$ (or $\omega = qd\bar{z}_Q$) in any chart z_Q of $Q \in V(\diamond)$, if and only if $\star\omega = -i\omega$ (or $\star\omega = i\omega$).*

Proof. Due to Lemma 2.14, there is a unique representation $\omega = pdz_Q + qd\bar{z}_Q$ in a coordinate chart z_Q of $Q \in V(\diamond)$. By definition, $\star\omega = -i\omega$ is equivalent to $q = 0$. Analogously, $\star\omega = i\omega$ is equivalent to $p = 0$. □

We have seen that the discrete exterior derivative of a discrete holomorphic function on $V(\Lambda_0)$ is discrete holomorphic. Conversely, closed discrete differentials can be integrated on simply-connected domains, and the resulting function is discrete holomorphic if the discrete differential is. This generalizes Proposition 2.8, and is proven similarly.

Proposition 3.3. *Let $\diamond_0 \subseteq \diamond$ be simply-connected and ω a closed discrete differential of type $\diamond$ on X_0. Then, there is a function $f := \int \omega : V(\Lambda_0) \to \mathbb{C}$ such that $\omega = df$. f is unique up to two additive constants on Γ_0 and Γ_0^*. If ω is discrete holomorphic, f is as well.*

Proof. Since ω is closed, $\oint_P \omega = 0$ for any discrete contour P by discrete Stokes' Theorem 2.9. Thus, ω can be integrated to a well defined function f_X on $V(X_0)$ that is unique up to an additive constant. Using that ω is of type $\diamond$, we can construct a function f on $V(\diamond_0)$ such that $f_X\left((v+w)/2\right) = \left(f(v)+f(w)\right)/2$ for any edge (v,w) of Λ. Given f_X, f is unique up to an additive constant.

In summary, f is unique up to two additive constants that can be chosen independently on black and white vertices, and by discrete Stokes' Theorem 2.9, $df = \omega$. If ω is discrete holomorphic, f satisfies

$$\frac{f(b_+) - f(b_-)}{z_Q(b_+) - z_Q(b_-)} = p = \frac{f(w_+) - f(w_-)}{z_Q(w_+) - z_Q(w_-)}$$

on any quadrilateral Q, where $\omega = pdz_Q$ in the chart z_Q. It follows that f is discrete holomorphic. □

Now, let us come to the definitions that do not generalize as immediately than the notions before.

Definition. Let Ω_Σ be a fixed nowhere vanishing two-form on the discrete Riemann surface (Σ, Λ, z). Let $g : F(X) \to \mathbb{C}$, and Ω a discrete two-form. Then, the *discrete Hodge star* is defined by

$$\star g := g\Omega_\Sigma,$$
$$\star\Omega := \frac{\Omega}{\Omega_\Sigma}.$$

As in Section 2.3.3, $\star^2 = \mathrm{Id}$ on complex functions on $F(X)$ and discrete two-forms. In the planar case, the choice of $\Omega_\Sigma = i/2dz \wedge d\bar{z}$ is the most natural one. Throughout the remainder of this chapter, Ω_Σ is a fixed positive real two-form on (Σ, Λ, z). Given a chart z, let $\Omega_{\Sigma,z}$ be the positive real number defined by $\Omega_\Sigma = \Omega_{\Sigma,z} i/2dz \wedge d\bar{z}$.

In the classical setup, there is a canonical nonvanishing two-form coming from a complete Riemannian metric of constant curvature. An interesting question is whether there exists some canonical two-form for discrete Riemann surfaces as we defined them. Note that the nonlinear theory developed by Bobenko, Pinkall, and Springborn in [BPS10] contains a uniformization of discrete Riemann surfaces, and discrete metrics with constant curvature.

Having the discrete Hodge star defined on all discrete differential forms, we can define a discrete scalar product $\langle \cdot, \cdot \rangle$ on discrete differential forms exactly as in Section 2.3.3. This scalar product gives rise to the Hilbert space of *square integrable* discrete differentials $L_2(\Sigma)$.

In addition, Proposition 2.19 holds true for discrete Riemann surfaces, assuring that $\delta := -\star d \star$ is the formal adjoint of the discrete exterior derivative d. Defining the discrete Laplacian

$$\triangle := -\delta d - d\delta = \star d \star d + d \star d \star$$

as in Section 2.4.1, Corollary 2.20 now reads as

$$\Omega_{\Sigma,z_v} \triangle f = \Omega_{\Sigma,z_v} \star d \star df = 4\partial_\diamond \bar{\partial}_\Lambda f = 4\bar{\partial}_\diamond \partial_\Lambda f$$

for a function $f : V(\Lambda) \to \mathbb{C}$ in the chart z_v around $v \in V(\Lambda)$. f is said to be *discrete harmonic* if $\triangle f \equiv 0$.

In Section 2.4.2, we defined the discrete Dirichlet energy, and in Lemma 2.28, we proved that the discrete Dirichlet boundary value problem is uniquely solvable. Although the definition of the discrete Dirichlet energy

$$E_{\diamond_0}(f) := \langle df, df \rangle_{\diamond_0} = \iint_{F(X_0)} df \wedge \star d\bar{f}$$

does not change, we want to reformulate Lemma 2.28 in the way we will use it in the sequel. The proof is still the same.

Lemma 3.4. *Let $\diamondsuit_0 \subseteq \diamondsuit$ be finite. Then, $E_{\diamondsuit_0}$ is a convex nonnegative quadratic functional in the space of functions $f : V(\Lambda_0) \to \mathbb{R}$. Furthermore,*

$$-\frac{\partial E_{\diamondsuit_0}}{\partial f(v)}(f) = 2\triangle f(v) \iint\limits_{F_v} \Omega_\Sigma$$

for any $v \in V(\Lambda_0)\backslash V(\partial\Lambda_0)$. In particular, extremal points of this functional are functions that are discrete harmonic at all interior vertices.

If all values of the function at the boundary are prescribed, there is exactly one discrete harmonic extension f to $V(\Lambda_0)$.

As a conclusion of this section, we state and prove *discrete Liouville's theorem.*

Theorem 3.5. *Let (Σ, Λ, z) be a compact discrete Riemann surface. Then, any discrete harmonic function $f : V(\Lambda) \to \mathbb{C}$ is essentially constant, i.e., constant on $V(\Gamma)$ and constant on $V(\Gamma^*)$. In particular, any discrete holomorphic function is essentially constant.*

Proof.

$$E_\diamondsuit(f) = \langle df, df \rangle = \langle f, \delta df \rangle = \langle f, \triangle f \rangle = 0$$

according to Proposition 2.19. Now, $\langle df, df \rangle \geq 0$, and equality holds only if $df = 0$, i.e., f is essentially constant. □

3.2 Discrete holomorphic mappings

Throughout this section, let (Σ, Λ, z) and (Σ', Λ', z') be discrete Riemann surfaces. Moreover, let $\diamondsuit_0 \subseteq \diamondsuit$ and $\diamondsuit'_0 \subseteq \diamondsuit'$ be connected subsets with vertex sets $V(\Lambda_0)$ and $V(\Lambda'_0)$, respectively, and let Γ_0, Γ'_0 and $\Gamma^*_0, {\Gamma'}^*_0$ be the induced subgraphs on black and white vertices.

Definition. A mapping $f : V(\Lambda) \to V(\Lambda')$ is said to be *discrete semi-holomorphic* if the following conditions are satisfied:

(i) $f(V(\Gamma)) \subseteq V(\Gamma')$ and $f(V(\Gamma^*)) \subseteq V({\Gamma'}^*)$;

(ii) for any quadrilateral $Q \in V(\diamondsuit)$, there exists a face $Q' \in V(\diamondsuit')$ such that $f(v) \sim Q'$ for all $v \sim Q$;

(iii) for any quadrilateral $Q \in V(\diamondsuit)$, the function $z'_{Q'} \circ f : V(Q) \to \mathbb{C}$ is discrete holomorphic.

If, in addition, for all black (or white) $v \in V(\Lambda)$ f either maps all adjacent white (or black) vertices of v to just one point, or maps each quadrilateral incident to v bijectively to a quadrilateral of Λ', f is said to be *discrete holomorphic*.

The first condition asserts that f respects the bipartite structures of the quad-graphs. The second one discretizes continuity, and guarantees that the third holomorphicity condition makes sense. Finally, the extra condition asserts that if the discrete holomorphic mapping f is essentially constant at one quadrilateral, it has to be globally essentially constant. This corresponds to the smooth case, where any holomorphic mapping that is locally constant somewhere is constant (provided that the source space is connected). Note that a discrete semi-holomorphic function may be essentially constant at some quadrilaterals, but not at all. However, we mainly investigate discrete semi-holomorphic mappings, and show that they almost behave as holomorphic mappings in the smooth case.

3.2.1 Simple properties and branch points

The following lemma is a discretization of the classical fact that nonconstant holomorphic mappings are open.

Lemma 3.6. *Let $f : V(\Lambda) \to V(\Lambda')$ be a discrete semi-holomorphic mapping. Then, for any $v \in V(\Lambda)$, there exists a nonnegative integer k such that the image of the star of v goes k times along the star of $f(v)$ (preserving the orientation).*

Proof. By definition of discrete semi-holomorphicity, the image of the star of v is contained in the star of $f(v)$, and the orientation is preserved. If f is essentially constant around the star of v, the statement is true with $k = 0$. So assume that f is not essentially constant. Then, at least one quadrilateral in the star is mapped to a complete quadrilateral in the star of $f(v)$. The next quadrilateral is either mapped to an edge if f is essentially constant at this quadrilateral, or to the neighboring quadrilateral. Thus, the images go along the star of $f(v)$. Since this has to close up in the end, the image goes $k > 0$ times along the star of $f(v)$. $\square$

Definition. If the number k in the lemma above is zero, we say that v is a *vanishing point*. Otherwise, v is a *regular point*. If $k > 1$, we say that v is a *branch point* of multiplicity k. In any case, we define $b_f(v) = k - 1$ as the *branch number* of f at v.

If f is essentially constant at $Q \in V(\diamond)$, we say that Q is a *branch point* of multiplicity two, with branch number $b_f(Q) = 1$. Otherwise, Q is not a branch point, and $b_f(Q) = 0$.

Note that even if f is not globally essentially constant, it may have vanishing points. The reason for saying that quadrilaterals where f is essentially constant are branch points of multiplicity two is, that if we go along the vertices b_-, w_-, b_+, w_+ of Q, its images are $f(b_-), f(w_-), f(b_-), f(w_-)$. However, in combination with vanishing points, this definition of branching might be misleading. It is more appropriate to consider a topological disk $\diamond_0$ consisting of F quadrilaterals, I interior points V_{int} (all of them vanishing points) and $B = 2(F - I + 1)$ boundary points (all of them regular points), as one single branch point of multiplicity $F - I + 1$. Indeed, black and white points alternate at the boundary, and they are always mapped to the same black or white image point, respectively. In terms of branch numbers this interpretation is fine, since

$$F - I = \sum_{Q \in V(\diamond_0)} b_f(Q) + \sum_{v \in V_{\text{int}}} b_f(v).$$

Note that this situation does not occur for discrete holomorphic mappings.

Corollary 3.7. *Let $f : V(\Lambda) \to V(\Lambda')$ be discrete semi-holomorphic and not essentially constant. Then, f is surjective. If, in addition, Σ is compact, Σ' is compact as well.*

Proof. Assume that f is not surjective. Then, there is $v' \in V(\Lambda')$ not contained in the image. Without loss of generality, v' is black. Take $v'_0 \in f(\Gamma)$ such that the length of a shortest path connecting v' and v'_0 is minimal. Since all black neighbors of a black vanishing point of f have the same image and f is not essentially constant, there is a regular point v_0 in the preimage of v'_0. By Lemma 3.6, the image of the star of v_0 equals the star of v'_0. Thus, there is an image point combinatorially nearer to v' as v'_0, contradiction.

If Σ is compact, Λ is finite. So Λ' is finite as well, and Σ' is compact. $\square$

Note the difference between the smooth and the discrete setup: In the smooth case, compactness of Σ is required to show surjectivity, in the discrete case it is the local finiteness that asserts surjectivity. From the last corollary the following discrete topological version of Liouville's theorem follows.

Corollary 3.8. *Let Σ be compact, and Σ' be homeomorphic to a plane. Then, any discrete semi-holomorphic mapping $f : V(\Lambda) \to V(\Lambda')$ is essentially constant.*

Still, discrete Liouville's Theorem 3.5 is not a consequence of Corollary 3.8, since discrete holomorphic functions are more general than discrete semi-holomorphic mappings.

Theorem 3.9. *Let $f : V(\Lambda) \to V(\Lambda')$ be a discrete semi-holomorphic mapping. Then, there exists $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that for all $v' \in V(\Lambda')$,*

$$m = \sum_{v \in f^{-1}(v')} \left(b_f(v) + 1\right).$$

Furthermore, for any $Q' \in V(\Diamond')$, m equals the number of $Q \in V(\Diamond)$ such that f maps the vertices of Q bijectively to the vertices of Q'.

Proof. If f is essentially constant, all $b_f(v)+1$ are zero, and no image of a quadrilateral in Λ is a quadrilateral itself, such that $m = 0$ fulfills the requirements.

Assume now that f is not essentially constant. By Corollary 3.7, f is surjective. Let $Q' \in V(\Diamond')$, and let v' be a vertex of Q'. We want to count the number m of $Q \in V(\Diamond)$ such that f maps the vertices of Q bijectively to the vertices of Q'. Let $v \in f^{-1}(v')$. By Lemma 3.6, exactly $b_f(v)+1$ quadrilaterals incident to v are mapped exactly to Q'. Conversely, any $Q \in V(\Diamond)$ such that f maps the vertices of Q bijectively to the vertices of Q' has exactly one vertex in the preimage of $f^{-1}(v')$. Therefore,

$$m = \sum_{v \in f^{-1}(v')} \left(b_f(v) + 1\right).$$

The same formula holds true if we replace Q' by some other face $Q'' \in V(\Diamond')$ incident to v'. Using that Λ' is connected, it follows that m does not depend on the choice of the face Q' and the incident vertex v'. $\square$

Definition. If $m > 0$ (i.e., f is not essentially constant), m is called the *degree* of f, and f is called an m-sheeted discrete semi-holomorphic covering.

If Σ is compact, $m < \infty$. The characterization of m as the number of preimage quadrilaterals nicely explains why m is called the number of sheets of f. However, each quadrilateral of Λ corresponds to one of the m sheets (and not to just two single points) only if f is discrete holomorphic.

Finally, we state and prove a discrete Riemann-Hurwitz formula.

Theorem 3.10. *Let Σ be compact and $f : V(\Lambda) \to V(\Lambda')$ be an N-sheeted discrete semi-holomorphic covering of the compact discrete Riemann surface Σ' of genus g'. Then, the genus g of Σ is equal to*

$$g = N(g'-1) + 1 + \frac{b}{2},$$

where b is the total branching number of f:

$$b = \sum_{v \in V(\Lambda)} b_f(v) + \sum_{Q \in V(\diamond)} b_f(Q).$$

Proof. Since we consider quad-graphs, the number of edges of Λ equals twice the number of faces. Thus, the Euler characteristic $2-2g$ of Σ is given by $|V(\Lambda)| - |V(\diamond)|$. By Theorem 3.9,

$$|V(\Lambda)| = N|V(\Lambda')| - \sum_{v \in V(\Lambda)} b_f(v).$$

If we count the number of faces of Λ, we have $N|V(\diamond')|$ quadrilaterals that are mapped to a complete quadrilateral of Λ' by Theorem 3.9. By definition, $\sum_{Q \in V(\diamond)} b_f(Q)$ faces are mapped to an edge of Λ'. Hence,

$$|V(\diamond)| = N|V(\diamond')| + \sum_{Q \in V(\diamond)} b_f(Q).$$

We obtain

$$\begin{aligned} 2-2g = |V(\Lambda)| - |V(\diamond)| &= N|V(\Lambda')| - \sum_{v \in V(\Lambda)} b_f(v) - N|V(\diamond')| - \sum_{Q \in V(\diamond)} b_f(Q) \\ &= N(2-2g') - b, \end{aligned}$$

from which the result follows. $\square$

3.3 Periods of discrete differentials

In this section, we define the (discrete) periods of a closed discrete differential of type $\diamond$ on a compact discrete Riemann surface (Σ, Λ, z) of genus g in Section 3.3.1, and state and prove a discrete Riemann bilinear identity in Section 3.3.2. Although we aim in being as close as possible to the smooth case in our presentation, the bipartite structure of Λ prevents us from doing so. We struggle with the same problem of white and black periods as Mercat did for discrete Riemann surfaces with real ρ_Q in [Mer07], and Bobenko and Skopenkov mentioned for general discrete Riemann surfaces in the end of [BS12]. The reason for that is that a discrete differential of type $\diamond$ corresponds to a pair of discrete differentials on each of Γ and Γ^*.

Mercat constructed out of a canonical homology basis on Λ certain canonical homology bases on Γ and Γ^*. By solving a discrete Neumann problem, he then proved the existence of dual cohomology bases on Γ and Γ^*. The discrete Riemann bilinear identity for the elements of the bases (and by linearity, for general closed discrete differentials) was a direct consequence of the construction.

On the contrary, Bobenko and Skopenkov used the notion of multi-valued functions on $V(\Lambda)$, i.e., complex functions on the universal cover. Their proof of a discrete Riemann bilinear identity follows the ideas of the smooth case, but the relation to discrete wedge products cannot be seen immediately from their notation. Although the proof is given explicitly only in a special case of quadrilaterals with orthogonal diagonals and multi-valued functions defined on either $V(\Gamma)$ or $V(\Gamma^*)$, their ideas work for more general discrete Riemann surfaces, as they noted in the end of [BS12].

We will give a full proof of the general discrete Riemann bilinear identity that follows the lines of the proof of the classical Riemann bilinear identity, using almost the same notation. The main difference to the paper [BS12] of Bobenko and Skopenkov is that we use a different refinement of the cellular decomposition to profit of a cellular decomposition of the canonical polygon with $4g$ vertices. The appearance of black and white periods indicates the analogy to Mercat's approach in [Mer07].

3.3.1 Universal cover and periods

Let $p : \tilde{\Sigma} \to \Sigma$ denote the universal covering of Σ. p gives rise to a quadrilateral cellular decomposition $\tilde{\Lambda}$ with medial graph $\tilde{X}$ and a covering $p : \tilde{\Lambda} \to \Lambda$. Now, $(\tilde{\Sigma}, \tilde{\Lambda}, p \circ z)$ is a discrete Riemann surface as well, and $p : V(\tilde{\Lambda}) \to V(\Lambda)$ is a discrete holomorphic mapping.

We fix a base vertex $\tilde{v}_0 \in V(\tilde{\Lambda})$. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be smooth loops on Σ with base point $v_0 := p(\tilde{v}_0)$ such that these loops cut out a fundamental $4g$-gon F_g. It is well known that such loops exist; the order of loops at the boundary of F_g is $\alpha_k, \beta_k, \alpha_k^{-1}, \beta_k^{-1}$, k going in order from 1 to g. Their homology classes $a_1, \ldots, a_g, b_1, \ldots, b_g$ form a canonical homology basis of $H_1(\Sigma, \mathbb{Z})$.

Using the same notation as Bobenko and Skopenkov used in [BS12], each loop α on Σ with base point v_0 induces deck transformations on $\tilde{\Sigma}$, $\tilde{\Lambda}$, and $\tilde{X}$, denoted all by d_α.

Clearly, there are homotopies between the smooth loops $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, and closed paths $\alpha'_1, \ldots, \alpha'_g, \beta'_1, \ldots, \beta'_g$ on X, all of the latter having the same base point $[Q_0, v_0]$, where $Q_0 \in V(\diamond)$ is incident to v_0. Having in mind that each edge of X corresponds to a black or white diagonal of a quadrilateral, we call an edge of X *black*, if it corresponds to a black edge of Γ, and *white*, if it corresponds to a white edge of Γ^*.

Let P be an oriented cycle on X. P induces closed paths on Γ and Γ^*, that we call BP and WP, in the following way: For an oriented edge $[Q, v]$ of P, we add the black (or white) vertex v to BP (or WP), and the corresponding white (or black) diagonal of Q to WP (or BP). The orientation of the diagonal is induced by the orientation of $[Q, v]$ in a chart z_Q. An equivalent description of BP (or WP) is that all black (or white) edges of P are replaced by the corresponding black (or white) diagonal, and all white (or black) edges of P are replaced by the nearest black (or white) vertex of Λ. Thus, BP and WP are indeed cycles on Γ and Γ^* that are homotopic to P. An example of this construction is given in Figure 3.2.

Remark. BP and WP may contain the same edge traversed in both directions, or even degenerate to a point. The construction of BP and WP is the inverse construction of P out of BP and WP we described in Section 2.6.1.

For notational simplicity, we denote the one-chains on X consisting of all the

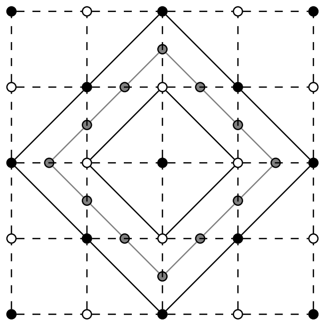


Figure 3.2: Cycles P on X, BP on Γ, and WP on Γ^*

black edges corresponding to BP, or all the white edges corresponding to WP, by BP and WP, respectively, as well.

Definition. Let ω be a closed discrete differential of type $\diamond$. We define its

$$\textit{a_k-periods } A_k := \oint_{\alpha'_k} \omega \quad \text{and} \quad \textit{b_k-periods } B_k := \oint_{\beta'_k} \omega;$$

$$\textit{black a_k-periods } A_k^B := 2\int_{B\alpha'_k} \omega \ \text{ and } \ \textit{black b_k-periods } B_k^B := 2\int_{B\beta'_k} \omega;$$

$$\textit{white a_k-periods } A_k^W := 2\int_{W\alpha'_k} \omega \ \text{ and } \ \textit{white b_k-periods } B_k^W := 2\int_{W\beta'_k} \omega.$$

Here, $1 \leq k \leq g$.

The reason for the factor of two in the definition of black and white periods is that to compute the black or white periods, we actually integrate ω on Γ or Γ^*, and not on X. Clearly, $2A_k = A_k^B + A_k^W$ and $2B_k = B_k^B + B_k^W$.

Lemma 3.11. *The periods of the closed discrete differential ω of type $\diamond$ depend only on the homology classes a_k and b_k, i.e., if α''_k, β''_k, $1 \leq k \leq g$, are loops on X that are in the homology classes a_k and b_k, respectively, then*

$$\int_{a_k} \omega := A_k = \oint_{\alpha''_k} \omega, \int_{Ba_k} \omega := A_k^B = 2\int_{B\alpha''_k} \omega, \int_{Wa_k} \omega := A_k^W = 2\int_{W\alpha''_k} \omega;$$

$$\int_{b_k} \omega := B_k = \oint_{\beta''_k} \omega, \int_{Bb_k} \omega := B_k^B = 2\int_{B\beta''_k} \omega, \int_{Wb_k} \omega := B_k^W = 2\int_{W\alpha''_k} \omega.$$

Proof. Since ω is closed, $\oint_{\partial F} \omega = 0$ for any boundary of a face F of X by discrete Stokes' Theorem 2.9. In particular, $\int_{\alpha'_k} \omega = \int_{\alpha''_k} \omega$, since α'_k and α''_k are homologically equivalent. For the same reason, the b_k-period of ω depends on b_k only. Using that ω is of type $\diamond$, ω implies discrete differentials on Γ and Γ^* in the obvious way. In the notation of Figure 2.1, they are closed in the sense that the integral along the black (or white) cycle $v'_1, v'_2, \ldots, v'_n, v'_1$ around any white (or black) vertex v of Λ vanishes. Since the paths $B\alpha'_k$ and $B\alpha''_k$ on Γ are both in the homology class a_k, $\int_{B\alpha'_k} \omega = \int_{B\alpha''_k} \omega$. Similarly, $\int_{W\alpha'_k} \omega = \int_{W\alpha''_k} \omega$. The same reasoning applies for the black and white b_k-periods. $\square$

3.3.2 Discrete Riemann bilinear identity

We repeat the notion of multi-valued functions given by Bobenko and Skopenkov in [BS12].

Definition. $f : V(\tilde{\Lambda}) \to \mathbb{C}$ is *multi-valued* with black periods $A_1^B, A_2^B, \ldots, A_g^B$, $B_1^B, B_2^B, \ldots, B_g^B$, and white periods $A_1^W, A_2^W, \ldots, A_g^W$, $B_1^W, B_2^W, \ldots, B_g^W$, if

$$\begin{aligned} f(d_{\alpha_k} b) &= f(b) + A_k^B, \\ f(d_{\alpha_k} w) &= f(w) + A_k^W, \\ f(d_{\beta_k} b) &= f(b) + B_k^B, \\ f(d_{\beta_k} w) &= f(w) + B_k^W, \end{aligned}$$

for any $1 \leq k \leq g$, each black vertex $b \in V(\Gamma)$, and each white vertex $w \in V(\Gamma^*)$.

An easy observation is that for such a function f, df defines a closed discrete differential of type $\diamond$, and df has the same black and white periods as f. Conversely, if ω is a closed discrete differential of type $\diamond$, its lift $\tilde{\omega}$ to $\tilde{X}$ can be integrated to a multi-valued function $f := \int \tilde{\omega} : V(\tilde{\Lambda}) \to \mathbb{C}$ that has the same black and white periods as ω by Proposition 3.3. f is unique up to two additive constants on $\tilde{\Gamma}$ and $\tilde{\Gamma}^*$. This shows how closed discrete differentials of type $\diamond$ and multi-valued functions are related to each other. Another consequence is that the white and black periods of ω are not determined by its periods, it may even happen that all a- and b-periods vanish, although ω does not vanish. In this case, there is at least one nontrivial white and one nontrivial black period.

In a slightly different notation, the following *discrete Riemann bilinear identity* was already discussed by Bobenko and Skopenkov in [BS12].

Theorem 3.12. *Let ω and ω' be closed discrete differentials of type $\Diamond$. Let their black periods are given by A_k^B, B_k^B and ${A'_k}^B, {B'_k}^B$, respectively, and their white periods by A_k^W, B_k^W and ${A'_k}^W, {B'_k}^W$, respectively, $k=1,\dots,g$. Then,*

$$\iint\limits_{F(X)} \omega \wedge \omega' = \frac{1}{2}\sum_{k=1}^{g}\left(A_k^B {B'_k}^W - B_k^B {A'_k}^W\right) + \frac{1}{2}\sum_{k=1}^{g}\left(A_k^W {B'_k}^B - B_k^W {A'_k}^B\right).$$

Proof. As mentioned above, we can find a multi-valued function $f : V(\tilde{\Lambda}) \to \mathbb{C}$ such that $df = \omega$. The black and white periods of f coincide with the ones of ω.

Let F_v and F_Q be faces of X corresponding to $v \in V(\Lambda)$ and $Q \in V(\Diamond)$. Consider any lifts of the star of v and of Q to $\tilde{\Lambda}$, and denote by $\tilde{F}_v$ and $\tilde{F}_Q$ the corresponding lifts of F_v and F_Q to $F(\tilde{X})$. By Theorem 2.16, $\omega \wedge \omega' = d(f\omega')$, lifting ω, ω' to $\tilde{X}$, and using that ω' is closed. So by discrete Stoke's Theorem 2.9,

$$\iint\limits_{F} \omega \wedge \omega' = \oint\limits_{\partial \tilde{F}} f\omega',$$

where F is either F_v or F_Q. Note that the independence of the right hand side of the chosen lift follows from the closeness of ω'. It follows that

$$\iint\limits_{F(X)} \omega \wedge \omega' = \int\limits_{\partial \tilde{F}(X)} f\omega',$$

where $\tilde{F}(X)$ is any collection of lifts of each a face of X to $\tilde{X}$, and $\partial \tilde{F}(X)$ is its counterclockwise oriented boundary (not necessarily connected, but being a sum of cycles).

It remains to compute $\int_{\partial \tilde{F}(X)} f\omega'$. If $g=0$, $\tilde{\Sigma} = \Sigma$, and f is a complex function on $V(\Lambda)$. Furthermore, the boundary of $F(X)$ is empty, so $\iint_{F(X)} \omega \wedge \omega' = 0$ as claimed. In what follows, let $g > 0$.

By definition, if $e = [Q, v]$ is an edge of $\tilde{X}$ ($Q \sim v \in V(\tilde{\Lambda})$), $\int_e f\omega' = f(v)\int_e \omega'$. Thus, we may consider f as a function on $E(\tilde{X})$ defined by $f([Q,v]) := f(v)$. Clearly, $f : E(\tilde{X}) \to \mathbb{C}$ fulfills

$$f(d_{\alpha_k}[Q,v]) = f([Q,v]) + A_k^B \text{ and } f(d_{\beta_k}[Q,v]) = f([Q,v]) + B_k^B \text{ if } v \in V(\Gamma),$$
$$f(d_{\alpha_k}[Q,v]) = f([Q,v]) + A_k^W \text{ and } f(d_{\beta_k}[Q,v]) = f([Q,v]) + B_k^W \text{ if } v \in V(\Gamma^*).$$

In this sense, f is multi-valued on $E(\tilde{X})$ with black periods defined on white edges, and white periods defined on black edges.

To finally compute the integral, we construct a subdivision X' of the cellular decomposition X of Σ in such a way that it allows to cut out the fundamental polygon with $4g$-vertices. Then, we extend f and ω' to the new cellular decomposition $\tilde{X}'$ of $\tilde{\Sigma}$ without changing their black and white periods.

Since f and ω' are now determined by topological data, we may forget the discrete complex structure of $\tilde{\Sigma}$, and can consider ω' and $f\omega'$ as functions on the oriented edges. Their evaluation on an edge e will still be denoted by $\int_e$. Let $\tilde{\Sigma}'$ be the polyhedral surface that is given by $\tilde{X}$, requiring that all faces are regular polygons of side length one. Similarly, Σ' is constructed. Now, p induces a covering $\tilde{\Sigma}' \to \Sigma'$ in a natural way, requiring that p on a face is an isometry. We denote this covering by p as well.

The homeomorphic images of the paths α_k, β_k are loops on Σ' with the base point being somewhere inside the face F_{v_0}. Let us choose piecewise smooth paths on Σ' with base point being the center of F_{v_0} homotopic to the previous loops such that the new paths (that will be denoted the same) still cut out a fundamental $4g$-gon.

For $v \in V(\Lambda)$, consider any subdivision of the regular polygon corresponding to F_v (being the same for all its lifts to $\tilde{\Sigma}'$) into smaller polygonal cells induced by straight lines. All new edges get the same color as the original edges of F_v had, i.e., the opposite color to the one of v. We extend f on the new edges by $f(v)$. Obviously, the new function is still multi-valued with the same periods. We define the one-form ω' on the new edges consecutively by inserting straight lines. Each time an existing oriented edge e is subdivided into two equally oriented parts e' and e'', we define $\int_{e'} \omega' = \int_{e''} \omega' := \int_e \omega'/2$. On segments of the inserted line, we define ω' by the condition that it should remain closed. Defining a black (or white) c-period of ω' on the subdivided cellular decomposition as the discrete integral over all black (or white) edges of a closed path with homology c, we see that these periods are the same as before, since all new edges in a polygon have the same color.

Now, let F_Q be the square corresponding to $Q \in V(\Diamond)$. We consider a subdivision of F_Q (and all its lifts) into smaller polygonal cells induced by straight lines parallel to the edges of the square, requiring in addition that all subdivision points on the edges of F_Q coming from the previous subdivisions of F_v, $v \sim Q$, are part of it. A new edge is black (or white) if it is parallel to an original black (or white) edge of X. Any new edge e' is of length $0 < l \leq 1$, and e' is parallel to an edge e of F_Q. Since ω' is of type $\Diamond$, it coincides on parallel edges, so we can define $\int_{e'} \omega' := l \int_e \omega'$. By construction, the new discrete one-form ω' is closed, and its black and white periods do not change. f is extended in such a way that if the new edge e' is parallel to the edges $[Q, v]$ and $[Q, v']$, having distance $0 \leq l \leq 1$ to $[Q, v]$ and distance $1 - l$ to $[Q, v']$, $f(e) := (1 - l)f([Q, v]) + lf([Q, v'])$. f is still multi-valued with the same periods.

If the subdivisions of faces F_v and F_Q are fine enough, we find cycles homotopic to α_k, β_k on the resulting cellular decomposition X' on Σ in such a way that they still cut out a fundamental polygon with $4g$ vertices. Without changing the notation, we denote these loops by α_k, β_k as well.

By construction, $\oint_{\partial F} f\omega'$ equals the sum of all discrete contour integrals of $f\omega'$ around faces of the subdivision of the face F of X. It follows that

$$\int\limits_{\partial \tilde{F}(X)} f\omega' = \int\limits_{\partial \tilde{F}(X')} f\omega'$$

for any collection $\tilde{F}(X')$ of lifts of faces of X', using that ω' is closed. Let us choose $\tilde{F}(X')$ in such a way that it builds a fundamental $4g$-gon, whose boundary consists of lifts $\tilde{\alpha}_k, \tilde{\beta}_k$ of α_k, β_k, and lifts $\tilde{\alpha}_k^{-1}, \tilde{\beta}_k^{-1}$ of its reverses. Since interior edges of the polygon are traversed twice in both directions, they do not contribute to the discrete integral, and we get

$$\int\limits_{\partial \tilde{F}(X')} f\omega' = \sum_{k=1}^{g} \left(\int\limits_{\tilde{\alpha}_k} f\omega' + \int\limits_{\tilde{\alpha}_k^{-1}} f\omega' \right) + \sum_{k=1}^{g} \left(\int\limits_{\tilde{\beta}_k} f\omega' + \int\limits_{\tilde{\beta}_k^{-1}} f\omega' \right).$$

Let e be an edge of $\tilde{\alpha}_k$, and e' the corresponding edge of $\tilde{\alpha}_k^{-1}$. Then, $d_{\beta_k} e = -e'$. Hence, ω' has opposite signs on e and e', and f differs by B_k^W on black edges, and by B_k^B on white edges. Therefore,

$$\begin{aligned} \int\limits_{\tilde{\alpha}_k} f\omega' + \int\limits_{\tilde{\alpha}_k^{-1}} f\omega' &= \int\limits_{B\tilde{\alpha}_k} \left(f\omega' - (f + B_k^W)\omega' \right) + \int\limits_{W\tilde{\alpha}_k} \left(f\omega' - (f + B_k^B)\omega' \right) \\ &= -\frac{1}{2} B_k^W {A'_k}^B - \frac{1}{2} B_k^B {A'_k}^W. \end{aligned}$$

If e is an edge of $\tilde{\beta}_k$, and e' the corresponding edge of $\tilde{\beta}_k^{-1}$, $d_{\alpha_k^{-1}}e = -e'$. Thus,

$$\int_{\tilde{\beta}_k} f\omega' + \int_{\tilde{\beta}_k^{-1}} f\omega' = \frac{1}{2}A_k^W {B'_k}^B + \frac{1}{2}A_k^B {B'_k}^W.$$

In summary,

$$\iint_{F(X)} \omega \wedge \omega' = \int_{\partial \tilde{F}(X)} f\omega' = \frac{1}{2}\sum_{k=1}^{g} \left(A_k^B {B'_k}^W - B_k^B {A'_k}^W + A_k^W {B'_k}^B - B_k^W {A'_k}^B \right).$$

□

Remark. Note that as in the classical case, the formula is true for any canonical homology basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$, not necessarily the one we started with. Indeed, if the symplectic $2g \times 2g$-matrix S describes the change from the original to the new homology basis, the black and white periods of ω and ω' change by S as well. For this, we use that the discrete integral is linear, and that the induced black and white periods Ba_k, Bb_k, Wa_k, Wb_k equal a_k, b_k. So if P_B and P_W denote the column vectors of the original black and white periods of ω, respectively, and $\tilde{P}_B$ and $\tilde{P}_W$ the new ones, $\tilde{P}_B = SP_B$ and $\tilde{P}_W = SP_W$. The same is true for the period vectors P'_B, P'_W and $\tilde{P}'_B, \tilde{P}'_W$ of ω'. Now, twice the right hand side of the discrete Riemann Bilinear Identity 3.12 can be written as

$$P_B^T J P'_W + P_W^T J P'_B,$$

where J is the $2g \times 2g$-block matrix that has the identity matrix I in the upper right block, $-I$ in the lower left block, and 0 in the other two blocks. Since S is symplectic, $SJS^T = I \Leftrightarrow S^T J S = I$, and

$$\tilde{P}_B^T J \tilde{P}'_W + \tilde{P}_W^T J \tilde{P}'_B = P_B^T S^T J S P'_W + P_W^T S^T J S P'_B = P_B^T J P'_W + P_W^T J P'_B.$$

In the case that white and black a-periods coincide, Theorem 3.12 yields the following more intuitive discretization of the Riemann bilinear identity:

Corollary 3.13. *Let ω and ω' be closed discrete differentials of type $\diamond$. Let their periods are given by A_k, B_k and A'_k, B'_k, respectively, and assume that the black a-periods of ω, ω' coincide with corresponding white a-periods. Then,*

$$\iint_{F(X)} \omega \wedge \omega' = \sum_{k=1}^{g} \left(A_k B'_k - B_k A'_k\right).$$

3.4 Discrete harmonic and discrete holomorphic differentials

The aim of this section is to investigate discrete harmonic and discrete holomorphic differentials. In Section 3.4.1, we consider a not necessarily compact discrete Riemann surface, and collect some basic properties of discrete harmonic differentials analogous to the classical case. In particular, we provide a discrete Hodge decomposition. The proof goes along the lines of the classical one. Afterwards, we prove the existence of discrete harmonic forms with prescribed black and white periods for compact discrete Riemann surfaces, and deduce the dimension of the space of discrete holomorphic differentials in Section 3.4.2. Finally, we introduce the discrete period matrix in Section 3.4.3.

Throughout this section, let (Σ, Λ, z) be a discrete Riemann surface. Starting in Section 3.4.2, we additionally assume that Σ is compact and of genus g. In this case, let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a canonical basis of $H_1(\Sigma, \mathbb{Z})$.

3.4.1 Discrete Hodge decomposition

Definition. A discrete differential ω of type $\diamond$ is *discrete harmonic* if it is closed and *co-closed*, i.e., $d\omega = 0$ and $d \star \omega = 0$ (or, equivalently, $\delta\omega = 0$).

Lemma 3.14. *Let ω be a discrete differential of type $\diamond$.*

(i) ω is discrete harmonic if and only if in any simply-connected domain $\diamond_0$, $\omega = dh$ with a discrete harmonic function $h : V(\Lambda_0) \to \mathbb{C}$.

(ii) Let Σ be compact. Then, ω is discrete harmonic if and only if $\triangle\omega = 0$.

Proof. (i) Suppose that ω is discrete harmonic. Then, it is closed, so by Proposition 3.3, $\omega = dh$ for $h : V(\Lambda_0) \to \mathbb{C}$ on a simply-connected domain $\diamond_0$. Now, $\triangle h = \delta dh = \delta\omega = 0$, so h is discrete harmonic. Conversely, if $\omega = dh$ locally, $d\omega = ddh = 0$ by Proposition 2.10, and $\delta\omega = \delta dh = \triangle h = 0$ by definition. Hence, ω is closed and co-closed.

(ii) If ω is discrete harmonic, $d\omega = \delta\omega = 0$ implies $\triangle\omega = 0$. Conversely, let $\triangle\omega = 0$. Using Proposition 2.19,

$$0 = \langle \triangle\omega, \omega \rangle = \langle d\omega, d\omega \rangle + \langle \delta\omega, \delta\omega \rangle.$$

The right hand side vanishes only for $d\omega = \delta\omega = 0$, so ω is closed and co-closed. □

The proof of the *discrete Hodge decomposition* follows the lines of the classical proof given in the book [FK80] of Farkas and Kra. The proof is even easier since we do not have to care about closures or smoothness.

Theorem 3.15. *Let E, E^* denote the sets of exact and co-exact square integrable discrete differentials of type $\diamond$, i.e., E and E^* consist of all $\omega = df$ and $\omega = \star df$, respectively, where $f : V(\Lambda) \to \mathbb{C}$, and $\langle \omega, \omega \rangle < \infty$. Let H be the set of square integrable discrete harmonic differentials. Then, $L_2(\Sigma) = E \oplus E^* \oplus H$, and this decomposition is orthogonal.*

Proof. Clearly, E and E^* are the closures of all exact and co-exact square integrable discrete differentials of type $\diamond$ of compact support. Let $E^\perp$ and $E^{*\perp}$ denote the orthogonal complements of E and E^* in $L_2(\Sigma)$. Then, $\omega \in E^\perp$ if and only if $\langle \omega, df \rangle = 0$ for all $f : V(\Lambda) \to \mathbb{C}$ of compact support. To compute the scalar product, we may restrict ω to a finite neighborhood of the support of f, so we can use Proposition 2.19 to show that $\langle \omega, df \rangle = \langle \delta\omega, f \rangle = 0$. Since this is true for any f of compact support, $\delta\omega = 0$, i.e., ω is co-closed. Thus, $E^\perp$ consists of all co-closed discrete differentials of type $\diamond$. Similarly, $E^{*\perp}$ is the space of all closed discrete differentials of type $\diamond$. By Proposition 2.10, any (co-)exact discrete differential $\diamond$ is (co)-closed, so we get an orthogonal decomposition $L_2(\Sigma) = E \oplus E^* \oplus H$, $H = E^\perp \cap E^{*\perp}$ being the set of all closed and co-closed discrete differentials of type $\diamond$, i.e., all discrete harmonic ones. □

3.4.2 Existence of certain discrete differentials

First, we want to show that to any set of black and white periods, there is a discrete harmonic differential with these periods. In [Mer07], Mercat proved this statement by referring to a (discrete) Neumann problem. The proof given by Bobenko and Skopenkov in [BS12] used the finite-dimensional Fredholm alternative. Here, we give a proof based on the (discrete) Dirichlet energy that resembles Skopenkov's proof of the existence of discrete harmonic functions in [Sko13].

Theorem 3.16. *Let $A_k^B, B_k^B, A_k^W, B_k^W$, $1 \leq k \leq g$, be $4g$ given complex numbers. Then, there exists a unique discrete harmonic differential ω with these black and white periods.*

Proof. Since periods are linear in the discrete differentials, it suffices to prove the statement for real periods. Let us consider the vector space of all multi-valued functions $f : V(\tilde{\Lambda}) \to \mathbb{R}$ having these black and white periods. For such a function f, df is well defined on X, as is the discrete Dirichlet energy $E_\diamond(f) = \langle df, df \rangle$. By Lemma 2.28, the critical points of this functional are discrete harmonic functions, noting that $\triangle f$ is a function on $V(\Lambda)$. Since the discrete Dirichlet energy is convex, quadratic, and nonnegative, a minimum has to exist. This yields a discrete harmonic function $f : V(\tilde{\Lambda}) \to \mathbb{R}$ with periods $A_k^B, B_k^B, A_k^W, B_k^W$. By Lemma 3.14 (i), $\omega := df$ is discrete harmonic and has the required periods.

Suppose that ω and ω' are two discrete harmonic differentials with the same black and white periods. Since $\omega - \omega'$ is closed, it can be integrated to a function $f : V(\tilde{\Lambda}) \to \mathbb{C}$ by Proposition 3.3. The black and white periods of f vanish, so f is a well defined discrete harmonic function on $V(\Lambda)$. By discrete Liouville's Theorem 3.5, f is essentially constant, so $\omega - \omega' = df = 0$. $\square$

The following relation between discrete harmonic and discrete holomorphic differentials is the same as for its classical counterparts, and will be used to compute the dimension of the space of discrete holomorphic differentials.

Lemma 3.17. *Let ω be a discrete differential of type $\diamond$.*

(i) ω is discrete harmonic if and only if it can be decomposed as $\omega = \omega_1 + \bar{\omega}_2$, where ω_1, ω_2 are discrete holomorphic differentials.

(ii) ω is discrete holomorphic if and only if it can be decomposed as $\omega = \alpha + i\star\alpha$, where α is a discrete harmonic differential.

Proof. (i) Suppose that $\omega = \omega_1 + \bar{\omega}_2$, where ω_1, ω_2 are discrete holomorphic. Then, ω is closed, since ω_1, ω_2 are, and it is co-closed, since $d \star \omega_k = -id\omega_k = 0$ by Lemma 3.2. Thus, ω is discrete harmonic.

Conversely, let ω be discrete harmonic. Then, we can write $\omega = pdz_v + qd\bar{z}_v$ in a chart z_v around $v \in V(\Lambda)$, where p, q are complex functions on the faces incident to v. Define $\omega_1 := pdz_v$ and $\omega_2 := \bar{q}dz_v$ in the chart z_v. By Lemma 3.2, ω_1, ω_2 are well defined on the whole discrete Riemann surface as the projections of ω onto the $\pm i$-eigenspaces of $\star$. Since ω is closed, $0 = d\omega|_{F_v} = \left(\partial_\diamond q(v) - \bar{\partial}_\diamond p(v)\right) dz_v \wedge d\bar{z}_v$, so $\partial_\diamond q(v) = \bar{\partial}_\diamond p(v)$. Similarly, $\partial_\diamond q(v) = -\bar{\partial}_\diamond p(v)$ follows from $d \star \omega = 0$. Thus,

$\bar{\partial}_\diamond p(v) = 0 = \bar{\partial}_\diamond q(v)$, i.e., p, q are discrete holomorphic in v. It follows that ω_1, ω_2 are discrete holomorphic.

(ii) Suppose that $\omega = \alpha + i \star \alpha$. Then, $d\omega = 0$, because α is closed and co-closed. In addition, $\star\omega = \star\alpha - i\alpha = -i\omega$. By Lemma 3.2, ω is discrete holomorphic.

Conversely, let ω be discrete harmonic, and define $\alpha := (\omega + \bar{\omega})/2$. By (i), α is discrete harmonic, and by construction, $\omega = \alpha + i \star \alpha$. □

Corollary 3.18. *The complex vector space $\mathcal{H}$ of discrete holomorphic differentials has dimension $2g$.*

Proof. Using that $\omega_1 \wedge \star\bar{\omega}_2 = 0$ for discrete holomorphic differentials ω_1, ω_2, Lemma 3.17 implies that the space of discrete harmonic differentials H is a direct orthogonal sum of the spaces of discrete holomorphic and discrete antiholomorphic one-forms, $\mathcal{H}$ and $\bar{\mathcal{H}}$. Due to Theorem 3.16, dim $H = 4g$. Since $\mathcal{H}$ and $\bar{\mathcal{H}}$ are isomorphic, dim $\mathcal{H} = 2g$. □

As for the space of discrete harmonic differentials, the dimension of $\mathcal{H}$ is twice as high as its classical counterpart, due to the splitting of periods into black and white periods. To finally prove the existence of discrete holomorphic differentials with prescribed black and white a-periods, we follow the approach in the classical theory.

Lemma 3.19. *Let $\omega \neq 0$ be a discrete holomorphic differential whose black and white periods are given by A_k^B, B_k^B and A_k^W, B_k^W, $1 \leq k \leq g$. Then,*

$$\mathrm{Im}\left(\sum_{k=1}^{g}\left(A_k^B \bar{B}_k^W + A_k^W \bar{B}_k^B\right)\right) < 0.$$

Proof. Since ω is discrete holomorphic, ω and $\bar{\omega}$ are closed. Thus, we can apply the discrete Riemann Bilinear Identity 3.12 to them:

$$\begin{aligned}\iint_{F(X)} \omega \wedge \bar{\omega} &= \frac{1}{2}\sum_{k=1}^{g}\left(A_k^B \bar{B}_k^W - B_k^B \bar{A}_k^W\right) + \frac{1}{2}\sum_{k=1}^{g}\left(A_k^W \bar{B}_k^B - B_k^W \bar{A}_k^B\right) \\ &= i\,\mathrm{Im}\left(\sum_{k=1}^{g}\left(A_k^B \bar{B}_k^W + A_k^W \bar{B}_k^B\right)\right).\end{aligned}$$

On the other hand, $\omega \wedge \bar{\omega}$ vanishes on faces F_v of X corresponding to vertices $v \in V(\Lambda)$, and in a chart z_Q of $Q \in V(\diamondsuit)$, $\omega \wedge \bar{\omega} = |p|^2 dz_Q \wedge d\bar{z}_Q$ if $\omega|_{\partial F_Q} = p dz_Q$. As a consequence,

$$\operatorname{Im}\left(\sum_{k=1}^{g}\left(A_k^B \bar{B}_k^W + A_k^W \bar{B}_k^B\right)\right) = \operatorname{Im}\left(\iint\limits_{F(X)} \omega \wedge \bar{\omega}\right) \leq 0,$$

where equality holds only if $\omega = 0$. □

Corollary 3.20. *Let ω be a discrete holomorphic differential.*

(i) If all black and white a-periods of ω vanish, then $\omega = 0$.

(ii) If all black and white periods of ω are real, then $\omega = 0$.

Proof. If all black and white a-periods vanish, or all black and white periods of ω are real, then

$$\operatorname{Im}\left(\sum_{k=1}^{g}\left(A_k^B \bar{B}_k^W + A_k^W \bar{B}_k^B\right)\right) = 0.$$

In particular, $\omega = 0$ by Lemma 3.19. □

Theorem 3.21. *Let (Σ, Λ, z) be a compact discrete Riemann surface of genus g.*

(i) For any $2g$ complex numbers A_k^B, A_k^W, $1 \leq k \leq g$, there exists exactly one discrete holomorphic differential ω with these black and white a-periods.

(ii) For any $4g$ real numbers $\operatorname{Re}\left(A_k^B\right), \operatorname{Re}\left(B_k^B\right), \operatorname{Re}\left(A_k^W\right), \operatorname{Re}\left(B_k^W\right)$, k an integer between 1 and g, there exists exactly one discrete holomorphic differential ω such that its black and white periods have these real parts.

Proof. Consider the complex-linear map $P_1 : \mathcal{H} \to \mathbb{C}^{2g}$ that assigns to each discrete holomorphic differential its black and white a-periods, and the real-linear map $P_2 : \mathcal{H} \to \mathbb{R}^{4g}$ that assigns to each discrete holomorphic differential the real parts of its black and white periods. By Corollary 3.20, P_1 and P_2 are injective. But by Corollary 3.18, the complex dimension of $\mathcal{H}$ is $2g$, so its real dimension is $4g$. Hence, P_1 and P_2 have to be surjective. □

3.4.3 Discrete period matrices

Discrete period matrices in the case of a cellular decomposition whose quadrilaterals have orthogonal diagonals were already studied by Mercat in [Mer01a, Mer07], and by Bobenko and Skopenkov in [BS12], who also sketched the generalization to general quadrilaterals. The first correct proof of convergence of discrete period matrices to their continuous counterparts in the case of a Delaunay-Voronoi quadrangulation of a Riemann surface was given by Bobenko and Skopenkov in [BS12]. Here, a Delaunay-Voronoi quadrangulation is the quadrangulation one gets if one starts with a Delaunay triangulation of a polyhedral surface as black subgraph, and adds the circumcenters of triangles as white vertices.

By Theorem 3.21, there exists exactly one discrete holomorphic differential with prescribed black and white a-periods. Having a limit of finer and finer quadrangulations of a Riemann surface in mind, it is natural to demand that black and white a-periods coincide.

Definition. The unique set of g holomorphic differentials ω_k, $k = 1, \ldots, g$, that satisfies

$$2\int_{Ba_j} \omega_k = 2\int_{Wa_j} \omega_k = \delta_{jk}$$

for all $1 \leq j, k \leq g$, is called *canonical.*

The $g \times g$-matrix $(B_{jk})_{j,k=1}^{g}$ with entries

$$B_{jk} := \int\limits_{b_j} \omega_k$$

is the *discrete period matrix* of the discrete Riemann surface (Σ, Λ, z).

The definition of the discrete period matrix as the arithmetic mean of black and white periods was already given by Bobenko and Skopenkov in [BS12], adapting Mercat's definition in [Mer01a, Mer07]. In our notation with discrete differentials defined on the medial graph, it becomes clear why this is a natural choice.

Example. As Bobenko and Skopenkov did in [BS12], let us consider the example of a bipartitely quadrangulated flat torus $\Sigma = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ of modulus $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$. Then, the classical period of the Riemann surface Σ is exactly τ. In the discrete setup, dz is globally defined on the medial graph of the quadrangulation. Clearly, dz is discrete holomorphic. Thus, $z = \int dz$ is discrete holomorphic on the vertices of the quadrangulation of the universal covering $\mathbb{C}$. Its black and white periods coincide, so do the black and white periods of dz. Furthermore, the a-period of dz is 1, where $\{a, b\}$ is the standard canonical homology basis coming from the two spanning vectors $1, \tau$. Therefore, the discrete period B equals the b-period of dz, that is τ. Thus, discrete and classical period coincide.

Remark. Although the black and white a-periods of a canonical set of discrete holomorphic differentials coincide by definition, the black and white b-periods must not in general. Bobenko and Skopenkov gave a counterexample in [BS12], namely the triangulated torus obtained by identifying opposite sides of the base of the side surface of a regular square pyramid, together with its dual. Let us repeat their example in our notation.

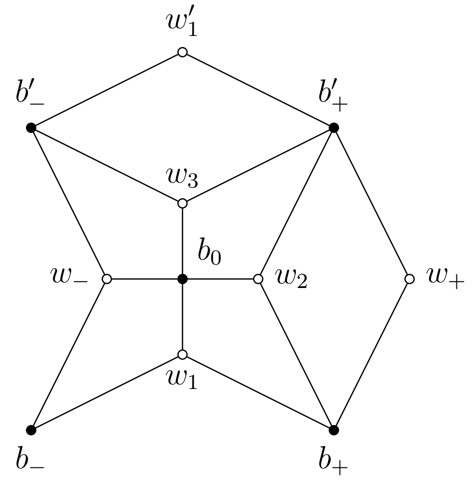


Figure 3.3: Quadrangulated torus

Figure 3.3 shows a quadrangulated torus. Here, the black vertices b_-, b_-', b_+, b_+' correspond to just one vertex of the torus; the white vertices w_- and w_+ are identified, as well as w_1 and w_1'. To each of the six quadrilaterals Q we assign $\rho_Q := 1/\sqrt{3}$. This defines a discrete complex structure on the torus, and by Proposition 3.1, the quadrangulated torus becomes a discrete Riemann surface.

As one can easily check, the multi-valued function f on the vertices of the induced quadrangulation of the complex plane that is determined by the values $f(b_-) = 0$, $f(b_+) = 1$, $f(b_0) = 1/2 + i\sqrt{3}/4$, $f(b_-') = i\sqrt{3}/2$, $f(b_+') = 1 + i\sqrt{3}/2$ on black vertices, and $f(w_1) = 0$, $f(w_1') = i2/\sqrt{3}$, $f(w_2) = 1/4 + i/(2\sqrt{3})$, $f(w_3) = i/\sqrt{3}$,

$f(w_-) = -1/4 + i/(2\sqrt{3})$, $f(w_+) = 3/4 + i/(2\sqrt{3})$ on white vertices, is discrete holomorphic.

If a is the homology class of the path $b_-w_1b_+$, and b the homology class of $b_-w_-b'_-$, $\{a, b\}$ is a canonical basis of the homology of the torus. The black and the white a-period of f equal one. Hence, $\{df\}$ is a canonical set of discrete holomorphic differentials. However, its black b-period is $i\sqrt{3}/2$, whereas its white a-period equals $i2/\sqrt{3}$.

Theorem 3.22. *The discrete period matrix B is symmetric, and its imaginary part is positive definite.*

Proof. Let $\{\omega_1, \ldots, \omega_g\}$ be the canonical set of discrete holomorphic differentials used to compute B. By looking at the coordinate representations, one immediately sees that $\omega_j \wedge \omega_k = 0$ for all j, k. Plugging this into the discrete Riemann Bilinear Identity 3.12, the periods of $\omega := \omega_j$ and $\omega' := \omega_k$ satisfy

$$\begin{aligned} 0 &= \sum_{l=1}^{g} \left(A_l^B B_l'^W - B_l^B A_l'^W \right) + \sum_{l=1}^{g} \left(A_l^W B_l'^B - B_l^W A_l'^B \right) \\ &= B_j'^W - B_k^B + B_j'^B - B_k^W = 2B_{jk} - 2B_{kj}. \end{aligned}$$

Thus, B is symmetric.

Let $\alpha = (\alpha_1, \ldots, \alpha_g)^T$ be a nonzero real column vector. Applying Lemma 3.19 to the discrete holomorphic differential $\omega := \sum_{k=1}^{g} \alpha_k \omega_k$ yields

$$0 > \mathrm{Im} \left(\sum_{k=1}^{g} \left(\alpha_k \sum_{j=1}^{g} \alpha_j 2\bar{B}_{kj} \right) \right) = -2\, \mathrm{Im} \left(\alpha^T B \alpha \right).$$

Hence, $\mathrm{Im}(B)$ is positive definite. □

3.5 Discrete theory of Abelian differentials

After introducing discrete Abelian differentials in Section 3.5.1, and discussing several properties of them, the aim of Section 3.5.2 is to state and prove the discrete Riemann-Roch Theorem 3.30. We conclude this chapter by discussing discrete Abel-Jacobi maps in Section 3.5.3.

Again, we consider a compact discrete Riemann surface (Σ, Λ, z) of genus g. Let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a canonical basis of its homology, and $\{\omega_1, \ldots, \omega_g\}$ a canonical set of discrete holomorphic differentials.

3.5.1 Discrete Abelian differentials

In the case of Delaunay-Voronoi quadrangulations of Riemann surfaces, Bobenko and Skopenkov investigated discrete Abelian integrals of any kind in [BS12]. Though they introduced a notion of discrete Abelian integrals of the second kind, discrete Abelian differentials of the second kind remained undefined. For general quad-surfaces, they provided a notion of discrete Abelian differentials of the first kind. It turns out that in the special case of a Delaunay-Voronoi quadrangulation, the discrete Abelian differentials of the second kind we define are the discrete exterior derivatives of the corresponding discrete Abelian integrals of Bobenko and Skopenkov in [BS12].

Definition. A discrete differential ω of type $\diamond$ is said to be a *discrete Abelian differential.* For a vertex $v \in V(\Lambda)$, the *residue* of ω at v is defined as

$$\text{res}_v(\omega) := \frac{1}{2\pi i} \oint\limits_{\partial F_v} \omega,$$

where F_v is the face of the medial graph X corresponding to v.

Remark. By definition, the discrete integral of a discrete differential of type $\diamond$ around a face F_Q corresponding to $Q \in V(\diamond)$ is always zero. For this reason, a residue at faces $Q \in V(\diamond)$ is not defined.

At this point, it is not clear why it is reasonable to consider all discrete differentials of type $\diamond$ as discrete Abelian differentials, and not only those that can be represented in the form pdz_Q in charts z_Q of quadrilaterals $Q \in V(\diamond)$. As we will see soon, a quadrilateral Q such that ω cannot be written as pdz_Q corresponds to a pole of ω of order two.

Using the finiteness of the quadrangulation Λ, the *discrete residue theorem* can be proven even simpler as in the smooth case. Still, the fact that Λ is bipartite comes into play.

Proposition 3.23. *Let ω be a discrete Abelian differential. Then, the sum of all residues of ω at black vertices vanishes, as well as the sum of all residues of ω at white vertices:*

$$\sum_{b \in V(\Gamma)} \text{res}_b(\omega) = 0 = \sum_{w \in V(\Gamma^*)} \text{res}_w(\omega).$$

Proof. Since ω is of type $\diamond$, $\int_{[Q,b_-]} \omega = -\int_{[Q,b_+]} \omega$ if b_-, b_+ are two black vertices incident to a quadrilateral $Q \in V(\diamond)$, and $[Q, b_-]$ and $[Q, b_+]$ are oriented in such a way that they go clockwise around F_Q. Equivalently, they are oriented in such a way that they go counterclockwise around F_{b_-} and F_{b_+}, respectively.

Now, the sum of all residues of ω at black vertices equals the sum of all discrete integrals of ω along all oriented edges $[Q, v]$, $Q \in V(\diamond)$, $Q \sim v \in V(\Gamma)$. These edges can be paired in oriented edges $[Q, b_-]$ and $[Q, b_+]$ as above. Since the sum of the two discrete integrals along these edges is zero, the sum of all residues is zero. Similarly, $\sum_{w \in V(\Gamma^*)} \mathrm{res}_w(\omega) = 0$. □

Let ω be a discrete Abelian differential, and z_v a chart around $v \in V(\Lambda)$. In this chart, we may represent ω in the form $p dz_v + q d\bar{z}_v$, where p and q are complex functions on the quadrilaterals incident to v. If $q = 0$ and p is discrete holomorphic in v, the restriction of ω to the medial edges in the star of v is discrete holomorphic. If $q = 0$, but p is not discrete holomorphic in v, ω has a residue at v. In this case, we say that v is a *simple pole* or *pole of order one* of ω. Finally, each quadrilateral Q where $q(Q) \neq 0$ is said to be a *double pole* or *pole of order two* of ω.

To say that quadrilaterals where $q(Q) \neq 0$ are double poles is well motivated. In Section 2.6 of Chapter 2, we have defined discrete Cauchy's kernels on the plane, and have proven that they exist. As a reminder, discrete Cauchy's kernels with respect to $Q \in V(\diamond)$ are functions on $V(\Lambda)$ that are discrete holomorphic at all but one face Q. They appear in the discrete Cauchy's integral formulae in Section 2.6, and model z^{-1}. There, we have seen that the discrete derivative of a discrete Cauchy's kernel models $-z^{-2}$. Now, the discrete exterior derivative of such a discrete Cauchy's kernel should model $-z^{-2}dz$, and in particular, a double pole at Q. By construction, the discrete exterior derivative of a discrete Cauchy's kernel is a discrete Abelian differential that is of the form $p dz_{Q'}$ in any chart $z_{Q'}$ around a quadrilateral $Q' \neq Q$. But in a chart z_Q, it is of the form $p dz_Q + q d\bar{z}_Q$ with $q \neq 0$. This motivates our notion of double poles.

Definition. Let ω be a discrete Abelian differential. If ω is discrete holomorphic, we say that ω is a *discrete Abelian differential of the first kind.* If ω is not discrete holomorphic, but all its residues vanish, it is a *discrete Abelian differential of the second kind.* A discrete Abelian differential whose residues do not vanish identically is said to be a *discrete Abelian differential of the third kind.*

As in the classical setup, there exists a set of normalized discrete Abelian differentials with certain prescribed poles and residues that can be normalized such that their a-periods vanish. Their b-periods can be similarly computed. We remark that in the case of a Delaunay-Voronoi quadrangulation, the existence of corresponding normalized discrete Abelian integrals of the second kind, and of normalized discrete Abelian differentials of the third kind was shown by Bobenko and Skopenkov in [BS12]. Our proofs will be similar, but in addition, we obtain the existence of certain discrete Abelian differentials of the second kind as a corollary. Bobenko and Skopenkov computed also the b-periods of the normalized discrete Abelian integrals of the second kind in [BS12]. However, our notation makes the connection to the smooth case much clearer, and the calculation of the b-periods of the normalized discrete Abelian integrals of the third kind is new.

Proposition 3.24. *Let $v, v' \in V(\Gamma)$, or $v, v' \in V(\Gamma^*)$, be two given black or white vertices. Then, there exists a discrete Abelian differential of the third kind ω, whose only poles are at v and v', and whose residues at this points are given by $\mathrm{res}_\omega(v) = -\mathrm{res}_\omega(v') = 1$. Any two such discrete differentials differ just by a discrete holomorphic differential.*

Proof. Clearly, the difference of two discrete Abelian differentials of the third kind with equal residues and no double poles has no poles at all, so it is discrete holomorphic.

Let V be the vector space of all discrete Abelian differentials that have no double poles. For any $Q \in V(\diamondsuit) \cong F(\Lambda)$, we choose one chart z_Q. By definition, each $\omega \in V$ is of the form pdz_Q at Q. Conversely, any function $p : V(\diamondsuit) \to \mathbb{C}$ defines by pdz_Q a discrete Abelian differential that has no double poles. Thus, the complex dimension of V equals $|F(\Lambda)|$.

Now, let W be the image in $\mathbb{C}^{|V(\Lambda)|}$ of the linear map res that assigns to each $\omega \in V$ all its residues at vertices of Λ. By Proposition 3.23, the residues at all black points sum up to zero, as well as all residues at white vertices. Thus, the complex dimension of W is at most $|V(\Lambda)| - 2$. Since Λ is a quad-graph, $2|E(\Lambda)| = 4|F(\Lambda)|$. By the Euler formula, it follows that

$$2 - 2g = |V(\Lambda)| - |E(\Lambda)| + |F(\Lambda)| = |V(\Lambda)| - |F(\Lambda)|,$$

so $|V(\Lambda)| - 2 = |F(\Lambda)| - 2g$. Therefore, the dimension of W is at most $|F(\Lambda)| - 2g$.

On the other hand, the dimension of W equals $|F(\Lambda)|$ minus the dimension of the kernel of the map res. But if $\omega \in V$ has vanishing residues, it is discrete holomorphic. Due to Corollary 3.18, the space of discrete holomorphic differentials is $2g$-dimensional. Hence, dim W even equals $|F(\Lambda)| - 2g = |V(\Lambda)| - 2$. In particular, we can find a discrete Abelian differential without double poles for any prescribed residues that sum up to zero at all black and at all white vertices. $\Box$

Corollary 3.25. *Given a quadrilateral $Q \in V(\diamond)$ and a chart z_Q, there exists a unique discrete Abelian differential of the second kind that is of the form*

$$pdz_Q - \frac{\pi}{2\text{area}(F_Q)}d\bar{z}_Q$$

in the chart z_Q, that has no other poles, and whose black and white a-periods vanish. This discrete differential is denoted by ω_Q. Here, area(F_Q) *denotes the Euclidean area of the parallelogram $z_Q(F_Q)$.*

Proof. Consider the discrete Abelian differential of the third kind ω that is given by the local representation $-\pi/\left(2\text{area}(F_Q)\right) d\bar{z}_Q$ at the four edges of X in Q, and zero everywhere else. Its only poles other than Q are at the four vertices incident to Q. Using Proposition 3.24, we can find a discrete Abelian differential ω' that has no double poles, and whose residues equal the residues of ω. Now, $\omega - \omega'$ fulfills almost all the requirements, only its black and white a-periods may be nontrivial. Using Theorem 3.21, we can add a discrete holomorphic differential ω'' such that $\omega_Q := \omega - \omega' + \omega''$ has vanishing black and white a-periods. ω_Q is the discrete Abelian differential of the second kind we are looking for, and if there were two such discrete differentials, their difference would be a discrete holomorphic differential with vanishing a-periods, thus zero by Corollary 3.20. $\Box$

Remark. As in the classical case, ω_Q depends on the choice of the chart z_Q. In our setting, the coefficient of $d\bar{z}_Q$ of ω_Q equals $-\bar{\partial}_\Lambda K_Q(Q) = -\pi/\left(2\text{area}(F_Q)\right)$, where K_Q is a discrete Cauchy's kernel with respect to Q. Having in mind that $-K_Q$ discretizes $-z^{-1}$, its discrete exterior derivative discretizes $z^{-2}dz$.

Lemma 3.26. *Let $Q \neq Q' \in V(\diamond)$, and let $\omega_Q, \omega_{Q'}$ be the discrete Abelian differentials of the second kind corresponding to the charts $z_Q, z_{Q'}$. Define the complex numbers α, β in such a way that $\omega_Q = \alpha dz_{Q'}$ on the four edges of $F_{Q'}$, and $\omega_{Q'} = \beta dz_Q$ on the four edges of F_Q. Then, $\alpha = \beta$.*

Proof. By definition, ω_Q and $\omega_{Q'}$ are closed discrete differentials whose black and white a-periods vanish. So by the discrete Riemann Bilinear Identity 3.12, $\iint_{F(X)} \omega_Q \wedge \omega_{Q'} = 0$. Since ω_Q and $\omega_{Q'}$ have no pole at a face of X corresponding to a quadrilateral $Q'' \neq Q, Q'$, $(\omega_Q \wedge \omega_{Q'})|_{F_{Q''}} = 0$. It follows that

$$\begin{aligned} 0 = \iint\limits_{F(X)} \omega_Q \wedge \omega_{Q'} &= -\iint\limits_{F_Q} \omega_{Q'} \wedge \omega_Q + \iint\limits_{F_{Q'}} \omega_Q \wedge \omega_{Q'} \\ &= \frac{\beta\pi}{2\text{area}(F_Q)} \iint\limits_{F_Q} dz_Q \wedge d\bar{z}_Q - \frac{\alpha\pi}{2\text{area}(F_{Q'})} \iint\limits_{F_{Q'}} dz_{Q'} \wedge d\bar{z}_{Q'} \\ &= -2\pi i(\beta - \alpha). \end{aligned}$$

Thus, $\alpha = \beta$. □

Proposition 3.27. *Let $Q \in V(\diamond)$, and let ω_Q be the discrete Abelian differential of the second kind corresponding to the chart z_Q. Suppose that $\omega_k|_{\partial F_Q} = \alpha_k dz_Q$ for $k = 1, \ldots, g$. Then, $\int_{b_k} \omega_Q = 2\pi i \alpha_k$.*

Proof. Let us apply the discrete Riemann Bilinear Identity 3.12 to the closed discrete differentials ω_k and ω_Q. Then,

$$\iint\limits_{F(X)} \omega_k \wedge \omega_Q = \int\limits_{Wb_k} \omega_Q + \int\limits_{Bb_k} \omega_Q = \int\limits_{b_k} \omega_Q,$$

using that all black and white a-periods of ω_Q and ω_k vanish but the black and white a_k-periods of the canonical discrete holomorphic differentials ω_k. By definition, the discrete wedge product vanishes at faces of X corresponding to vertices of Λ. In a chart $z_{Q'}$ of a face $Q' \neq Q$, ω_k and ω_Q are both of the form $p dz_{Q'}$, so $\omega_k \wedge \omega_Q$ vanishes at $F_{Q'}$. Therefore,

$$\int\limits_{b_k} \omega_Q = \iint\limits_{F(X)} \omega_k \wedge \omega_Q = \frac{-\alpha_k \pi}{2\text{area}(F_Q)} \iint\limits_{F_Q} dz_Q \wedge d\bar{z}_Q = 2\pi i \alpha_k.$$

□

Let us come back to discrete Abelian differentials of the third kind. Since they have residues, periods are not well defined. However, periods of the discrete Abelian differentials constructed in Proposition 3.24 are defined modulo $2\pi i$. To normalize them similarly to the discrete Abelian differentials of the second kind

ω_Q, we now think of a_k, b_k as given closed curves α'_k, β'_k on X. Then, for any given $v, v' \in V(\Gamma)$, or $v, v' \in V(\Gamma^*)$, there exists a unique discrete Abelian differential $\omega_{vv'}$ with vanishing black and white a-periods, whose only poles are at v and v', and whose residues at this points are given by $\text{res}_\omega(v) = -\text{res}_\omega(v') = 1$.

Definition. Let R be an oriented path on Γ or Γ^* and ω a discrete Abelian differential. To an edge in R we assign one of the corresponding parallel edges of X. By R_X, we denote the resulting one-chain on X. Then, $\int_R \omega := 2\int_{R_X} \omega$.

Up to now, we integrated discrete differentials mainly on the medial graph X. But already black and white periods were actually discrete integrals on Γ or Γ^*. For the calculation of the b-periods of the discrete Abelian differentials $\omega_{vv'}$, it will be necessary to integrate a discrete holomorphic differential between the two poles of $\omega_{vv'}$. Thus, discrete integration along paths on Γ or Γ^* reappears.

Proposition 3.28. *Let $v, v' \in V(\Gamma)$, or $v, v' \in V(\Gamma^*)$, be given. Suppose that the closed paths α'_k, β'_k are homotopic to closed paths α_k, β_k cutting out a fundamental polygon with $4g$ vertices on the surface $\Sigma \backslash \{v, v'\}$. In addition, let R be an oriented path on Γ, or Γ^*, from v' to v that does not intersect any of the curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$. Then,*

$$\int\limits_{b_k} \omega_{vv'} = 2\pi i \int\limits_R \omega_k.$$

Proof. Let us consider the expression $\iint_{F(X)} \omega_k \wedge \omega_{vv'}$. On the one hand, $\omega_k \wedge \omega_{vv'}$ vanishes identically since both discrete Abelian differentials are of the form $p dz_Q$ in any chart z_Q.

On the other hand, we can find a discrete holomorphic multi-valued function $f : V(\tilde{\Lambda}) \to \mathbb{C}$ such that $df = \omega_k$ (see Section 3.3.2). Its black and white periods coincide with the ones of ω_k. To some vertex $v'' \in V(\Lambda)$ or some face $Q \in V(\Diamond)$, consider a lift of $F_{v''}$ or F_Q to $F(\tilde{X})$, denoted by $\tilde{F}_{v''}$ and $\tilde{F}_Q$, respectively. By Theorem 2.16,

$$0 = \omega_k \wedge \omega_{vv'} = d(f\omega_{vv'}) - f d\omega_{vv'}$$

is true if $\omega_k, \omega_{vv'}$ are lifted to $\tilde{X}$.

Now, choose a collection $\tilde{F}(X)$ of lifts of each face of X to $\tilde{X}$ such that the corresponding lifts $\tilde{v}$ and $\tilde{v}'$ of v and v' are connected by a lift of R in $\tilde{\Gamma}$ or $\tilde{\Gamma}^*$.

It is not necessary that all faces of $\tilde{X}$ intersecting the lift of R are contained in $\tilde{F}(X)$. By what we have shown above,

$$\iint\limits_{\tilde{F}(X)} d(f\omega_{vv'}) = \iint\limits_{\tilde{F}(X)} f d\omega_{vv'}.$$

Due to discrete Stoke's Theorem 2.9, $d\omega_{vv'} = 0$ on all lifts $\tilde{F}_Q$ of a face F_Q corresponding to $Q \in V(\diamond)$, and $d\omega_{vv'}$ vanishes on all lifts $\tilde{F}_{v''}$ of a face $F_{v''}$ corresponding to a vertex $v'' \neq v, v'$, since all residues of $\omega_{vv'}$ vanish but the ones at v and v'. Using that their residues are given by ± 1,

$$\iint\limits_{\tilde{F}(X)} f d\omega_{vv'} = 2\pi i \left(f(\tilde{v}) - f(\tilde{v}') \right) = 2\pi i \int\limits_R \omega_k.$$

By discrete Stoke's Theorem 2.9,

$$\iint\limits_{\tilde{F}(X)} d(f\omega_{vv'}) = \int\limits_{\partial \tilde{F}(X)} f\omega_{vv'}.$$

The right hand side can be calculated in exactly the same way as we did in the proof of the discrete Riemann Bilinear Identity 3.12. The only essential difference is that when we extend $\omega_{vv'}$ to a subdivision of $\tilde{F}_v$ or $\tilde{F}_{v'}$, the extended one-form shall have zero residues at all new faces but one containing v or v', where it should remain 1 or -1. As a result, we obtain $\int_{b_k} \omega_{vv'}$, observing that almost all black and white a-periods of f and $\omega_{vv'}$ vanish. In summary,

$$\int_{b_k} \omega_{vv'} = \int\limits_{\partial \tilde{F}(X)} f\omega_{vv'} = \iint\limits_{\tilde{F}(X)} d(f\omega_{vv'}) = \iint\limits_{\tilde{F}(X)} f d\omega_{vv'} = 2\pi i \int\limits_R \omega_k.$$

$\square$

Remark. Analog results are true for the black and white b-periods of ω_Q and $\omega_{vv'}$, replacing ω_k by the unique discrete holomorphic differential of which all black and white a-periods vanish but the black (or white) a_k-period that equals 2. Let us denote these differentials by ω_k^B and ω_k^W, respectively.

As in the smooth case, the normalized discrete Abelian differentials form a basis of the space of discrete Abelian differentials.

Proposition 3.29. *Let a chart z_Q to each $Q \in V(\diamondsuit)$ be given. Fix $b \in V(\Gamma)$ and $w \in V(\Gamma^*)$. Then, the normalized discrete Abelian differentials of the first kind ω_k^B and ω_k^W, $k = 1, \ldots, g$, of the second kind ω_Q, $Q \in V(\diamondsuit)$, and of the third kind $\omega_{bb'}$ and $\omega_{ww'}$, $b' \neq b$ a black and $w' \neq w$ a white vertex, form a basis of the space of discrete Abelian differentials.*

Proof. Linear independence is clear. Given any discrete Abelian differential, we can first use the ω_Q to get rid of all double poles. Then, we can find linear combinations of $\omega_{bb'}$ and $\omega_{ww'}$ that have the same residues at black and white vertices, respectively. We end up with a discrete holomorphic differential, that can be represented by the ω_k^B and ω_k^W, since the space of discrete holomorphic differentials is $2g$-dimensional by Corollary 3.18. □

3.5.2 Divisors and the discrete Riemann-Roch Theorem

Concerning the notion of divisors on a compact discrete Riemann surface (Σ, Λ, z) of genus g we generalize the approach of Bobenko and Skopenkov in [BS12] to general quadrilateral cell decompositions. Furthermore, we will allow double poles of discrete Abelian differentials, and double values of *discrete meromorphic functions*, i.e., functions $f : V(\Lambda) \to \mathbb{C}$.

Definition. A *divisor* D is a formal linear combination

$$D = \sum_{j=1}^{M} m_j v_j + \sum_{k=1}^{N} n_k Q_k,$$

where $m_j \in \{-1, 0, 1\}$, $v_j \in V(\Lambda)$, $n_k \in \{-2, -1, 0, 1, 2\}$, and $Q_k \in V(\diamondsuit)$.

D is *admissible* if even $m_j \in \{-1, 0\}$, and $n_k \in \{-2, 0, 1\}$.

Its *degree* is defined as

$$\deg D := \sum_{j=1}^{M} m_j + \sum_{k=1}^{N} \operatorname{sign}(n_k).$$

D is *greater than or equal to a divisor* D', $D \geq D'$, if the formal sum $D - D'$ is a divisor whose coefficients are all nonnegative.

Remark. The definition of the degree seems to be odd at first glance, since double points just count once. However, it turns out that these points correspond to double values of a discrete meromorphic function, and not to a double zero. With respect to discrete Abelian differentials, a double pole does not include a simple pole. This explains why double points just count once in the degree.

As noted by Bobenko and Skopenkov in [BS12], divisors on a discrete Riemann surface do not form an Abelian group. One of the reasons is certainly that the pointwise product of discrete holomorphic functions does not need to be discrete holomorphic itself, but another reason is the asymmetry of point spaces. Whereas discrete meromorphic functions are defined on $V(\Lambda)$, discrete Abelian differentials are essentially defined by complex functions on $V(\Diamond)$, supposed that a chart for each quadrilateral is fixed. This reflects the fact that in the planar case discussed in Chapter 2, discrete derivatives map functions on $V(\Lambda)$ to functions on $V(\Diamond)$, and vice versa.

Definition. Let $f : V(\Lambda) \to \mathbb{C}$, $v \in V(\Lambda)$, and $Q \in V(\Diamond)$.

- f has a *zero* at v if $f(v) = 0$.
- f has a *simple pole* at Q if $\bar{\partial}_\Lambda f(Q) \neq 0$ in any chart z_Q.
- f has a *double value* at Q if $\bar{\partial}_\Lambda f(Q) = \partial_\Lambda f(Q) = 0$ in any chart z_Q.

Now, if f has zeroes $v_1, \ldots, v_M \in V(\Lambda)$, double values $Q_1, \ldots, Q_N \in V(\Diamond)$, and poles at $Q'_1, \ldots, Q'_{N'} \in V(\Diamond)$, its divisor is defined as

$$(f) := \sum_{j=1}^{M} v_j + \sum_{k=1}^{N} 2Q_k - \sum_{k'=1}^{N'} Q'_{k'}.$$

Remark. Note that a double value at Q discretizes a smooth function f such that $f - c$ has a double zero for some constant c. Indeed, a double value implies that the values of the discrete function f at both black vertices coincide, as well as at the two white vertices. In this sense, double values are separated from the points where the function is evaluated, in contrast to the classical case. In the spirit of the discrete Riemann-Roch Theorem 3.30 that we will discuss later, a double value corresponds to allowing a double pole of a discrete Abelian differential, but simple and double poles occur on different vertex sets. In a limit of finer and finer discretizations, double values eventually merge with the points where the function is defined, giving double zeroes.

In Corollary 2.11 of Chapter 2, we have seen that the discrete derivative of a discrete holomorphic function is discrete holomorphic, but the analog statement for functions on $V(\diamond)$ is not true. Thus, a double pole of a discrete meromorphic function cannot be motivated by differentiating a discrete Cauchy's kernel. Furthermore, we cannot detect more than a simple zero at a vertex and a double value at a single quadrilateral. Though, in Section 3.2.1 we were able to merge several branch points to define one branch point of higher order. Unfortunately, the image of a discrete meromorphic function generally does not define a discrete Riemann surface, so the approach of merging poles to a pole of higher order does not carry over. Still, there might be a way to define poles and zeroes of higher order by investigating larger neighborhoods of vertices.

Definition. Let ω be a discrete Abelian differential. If $\omega = 0$ in a chart z_Q, $Q \in V(\diamond)$, we say that ω has a *zero* at Q.

Now, if ω has zeroes $Q_1, \ldots, Q_N \in V(\diamond)$, double poles at $Q'_1, \ldots, Q'_{N'} \in V(\diamond)$, and simple poles at $v_1, \ldots, v_M \in V(\Lambda)$, its divisor is defined as

$$(\omega) := \sum_{k=1}^{N} Q_k - \sum_{k'=1}^{N'} 2Q'_{k'} - \sum_{j=1}^{M} v_j.$$

As for discrete meromorphic functions, one would have to consider larger neighborhoods of points to define poles and zeroes of higher order .

Definition. Let D be a divisor. By $L(D)$ we denote the complex vector space of discrete meromorphic functions f that vanish identically, or whose divisor satisfies $(f) \geq D$. Similarly, $H(D)$ denotes the complex vector space of discrete Abelian differentials ω such that $\omega \equiv 0$, or $(\omega) \geq D$. The dimensions of these spaces are denoted by $l(D)$ and $i(D)$, respectively.

We are now able to formulate and prove the *discrete Riemann-Roch theorem.* Bobenko and Skopenkov already provided such a theorem for the Delaunay-Voronoi quadrangulation of a polyhedral surface in [BS12], but without considering double poles or double values as we do.

Theorem 3.30. *Let D be an admissible divisor on the compact discrete Riemann surface (Σ, Λ, z) of genus g. Then,*

$$l(-D) = \deg D - 2g + 2 + i(D).$$

Proof. We write $D = D_0 - D_\infty$, where $D_0, D_\infty \geq 0$. Since D is admissible, D_0 is a sum of elements of $V(\diamondsuit)$, all coefficients being one. Let V_0 denote the set of $Q \in V(\diamondsuit)$ such that $D_0 \geq Q$.

For each $Q \in V(\diamondsuit)$, we fix a chart z_Q. As in Proposition 3.29, we denote the normalized Abelian differentials of the first kind by ω_k^B and ω_k^W, $k = 1, \ldots, g$, these of the second kind by ω_Q, $Q \in V(\diamondsuit)$, and these of the third kind by $\omega_{bb'}$ and $\omega_{ww'}$, $b' \neq b$ being black and $w' \neq w$ being white vertices.

In the following, we investigate the image H of the discrete exterior derivative d on functions in $L(-D)$. Clearly, the space H consists of discrete Abelian differentials, and only essentially constant functions are in the kernel of the map.

Let $f \in L(-D)$. Then, df is a discrete Abelian differential that might have double poles at the points of D_0. In addition, all the residues and periods of df vanish. So due to Proposition 3.29,

$$df = \sum_{Q \in V_0} f_Q \omega_Q$$

for some complex numbers f_Q.

Now, all black and white b-periods of df vanish. Using Proposition 3.27 (and the remark at the end of Section 3.5.1),

$$\sum_{Q \in V_0} f_Q \alpha_k^B(Q) = 0 = \sum_{Q \in V_0} f_Q \alpha_k^W(Q),$$

where $\omega_k^B|_{\partial F_Q} = \alpha_k^B(Q) dz_Q$ and $\omega_k^W|_{\partial F_Q} = \alpha_k^W(Q) dz_Q$ for $k = 1, \ldots, g$.

In the chart $z_{Q'}$ of a quadrilateral $Q' \neq Q$, ω_Q can be written as $\beta_Q(Q') dz_{Q'}$. So if $D_\infty \geq 2P$, $P \in V(\diamondsuit)$, f has a double value at P, and $df = 0$ at the edges of F_P. Thus,

$$0 = \sum_{Q \in V_0} f_Q \beta_Q(P) = \sum_{Q \in V_0} f_Q \beta_P(Q)$$

using Lemma 3.26.

Suppose that $D_\infty \geq v + v'$, where $v, v' \in V(\Gamma)$, or $v, v' \in V(\Gamma^*)$. By definition, f has zeroes at v, v', so $f(v) = f(v')$. The last equality remains true when an essentially constant function is added. In particular, this yields an additional restriction to the image H. Now, using discrete Stokes' Theorem 2.9, $d\omega_{vv'}$ equals $2\pi i$ when integrated over F_v, $-2\pi i$ when integrated over $F_{v'}$, and zero everywhere else. So by Theorem 2.16,

$$2\pi i\left(f(v)-f(v')\right)=\iint\limits_{F(X)} f d\omega_{vv'}=\iint\limits_{F(X)} d\left(f\omega_{vv'}\right)-\iint\limits_{F(X)} df\wedge\omega_{vv'}.$$

But again by discrete Stokes' Theorem 2.9,

$$\iint\limits_{F(X)} d\left(f\omega_{vv'}\right)=\sum_{F\in F(X)}\oint\limits_{\partial F} f\omega_{vv'}=0,$$

since any edge of X is traversed twice. Writing $\omega_{vv'}|_{\partial F_Q}=\gamma_{vv'}(Q)dz_Q$,

$$\left(df\wedge\omega_{vv'}\right)|_{F_Q}=f_Q\gamma_{vv'}(Q)\frac{\pi}{2\mathrm{area}(F_Q)}dz_Q\wedge d\bar{z}_Q$$

for any $Q\in V(\diamondsuit)$ such that $D_0\geq Q$, and $df\wedge\omega_{vv'}=0$ everywhere else. Thus,

$$0=f(v)-f(v')=\sum_{Q\in V_0} f_Q\gamma_{vv'}(Q).$$

In the case that there are more than two black (or two white) vertices v that satisfy $D_\infty\geq v$, we fix one such black vertex b (or white vertex w). The set of such black and white vertices is denoted by B_0 and W_0, respectively. Then, f is constant on B_0 and W_0, if and only if

$$\sum_{Q\in V_0} f_Q\gamma_{bb'}(Q)=0=\sum_{Q\in V_0} f_Q\gamma_{ww'}(Q),$$

for any $b'\in B_0$, $b'\neq b$, and $w'\in W_0$, $w'\neq w$.

Consider the matrix M whose k-th column is the column vector $\left(\alpha_k^B(Q)\right)_{Q\in V_0}$, whose $(g+k)$-th column is the column vector $\left(\alpha_k^W(Q)\right)_{Q\in V_0}$, and whose next columns are the column vectors $\left(\beta_P(Q)\right)_{Q\in V_0}$, $P\in V(\diamondsuit)$ such that $D_\infty\geq 2P$. In the case that $|B_0|\geq 2$, we add the column vectors $\left(\gamma_{bb'}(Q)\right)_{Q\in V_0}$, $b'\in B_0$ different from b, to M, and if $|W_0|\geq 2$, we additionally add the column vectors $\left(\gamma_{ww'}(Q)\right)_{Q\in V_0}$, $w'\in W_0$ different from w.

By our consideration above, $\sum_{Q\in V_0} f_Q\omega_Q$ is in H only if the column vector $(f_Q)_{Q\in V_0}$ is in the kernel of M^T. Conversely, any element of the kernel is a closed discrete differential with vanishing periods, so it can be integrated to a discrete meromorphic function f using Proposition 3.3 (the property of vanishing periods assures that the discrete integral of the discrete differential that is a priori defined on the universal covering, actually is a function on $V(\Lambda)$). f can have poles only at $Q \in V_0$, it is essentially constant on $B_0 \cup W_0$, and it has double values at all $P \in V(\diamondsuit)$ such that $D_\infty \geq 2P$. Hence, H is isomorphic to the kernel of M^T. We obtain

$$\dim H = \dim \ker M^T = |V_0| - \text{rank } M^T = \deg D_0 - \text{rank } M.$$

Let us first suppose that B_0 and W_0 are both nonempty. This means that at least one zero of f at a black and one at a white vertex is fixed. Therefore, $d : L(-D) \to H$ has a trivial kernel, and $l(-D) = \dim H$. In addition,

$$\text{rank } M = 2g + \deg D_\infty - 2 - \dim \ker M.$$

But the kernel of M consists of discrete Abelian differentials

$$\omega = \sum_{k=1}^{g} \left(\lambda_k \omega_k^B + \lambda_{k+g}\omega_k^W\right) + \sum_{P:D_\infty \geq 2P} \lambda_P \omega_P + \sum_{b' \in B_0, b' \neq b} \lambda_{b'} \omega_{bb'} + \sum_{w' \in W_0, w' \neq w} \lambda_{w'} \omega_{ww'},$$

with complex numbers λ_j such that $\omega|_{\partial F_Q} = 0$ for any $Q \in V_0$. Note that if B_0 or W_0 just contains one point, there are no $\omega_{bb'}$ or $\omega_{ww'}$, respectively, in the sum above. In any case, the kernel is exactly $H(D)$. It follows that

$$\begin{aligned} l(-D) = \dim H &= \deg D_0 - \text{rank } M \\ &= \deg D_0 - 2g - \deg D_\infty + 2 + \dim \ker M \\ &= \deg D - 2g + 2 + i(D). \end{aligned}$$

If B_0 or W_0 is empty, we can add an additive constant to all values of f at black or white vertices, respectively, still getting an element of $L(-D)$. As a consequence, the dimension of the kernel of $d : L(-D) \to H$ is one, or even two, when both B_0 and W_0 are empty. On the other hand, $2g + \deg D_\infty - \dim \ker M - \text{rank } M$ is not two anymore, but one, or even zero. With the same arguments as above,

$$l(-D) = \deg D - 2g + 2 + i(D).$$

$\square$

Remark. In the smooth theory, the standard Riemann-Roch theorem reads as $l(-D) = \deg D - g + 1 + i(D)$. The difference between the classical and the discrete theorem is explained by the fact that Λ is bipartite: The space of constant functions is no longer one-, but two-dimensional; instead of just g a-periods of Abelian differentials we now have $2g$.

In the case of a quadrangulated flat torus, we obtain the same corollaries as Bobenko and Skopenkov got in [BS12].

Corollary 3.31. *Consider a bipartitely quadrangulated flat torus* $\Sigma = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ *of modulus* $\tau \in \mathbb{C}$ *with* $\operatorname{Im}\tau > 0$.

(i) There exists no discrete meromorphic function with exactly one simple pole.

(ii) Suppose that in addition, the diagonals of any quadrilateral of the cellular decomposition are orthogonal to each other, i.e., the complex numbers ρ_Q *that define the discrete complex structure of* Σ *are all positive real numbers. Then, there exists a discrete meromorphic function with exactly two simple poles at* $Q, Q' \in V(\diamond)$ *if and only if the black diagonals of* Q, Q' *are parallel to each other.*

Proof. (i) Let $Q \in V(\diamond)$. By the discrete Riemann-Roch Theorem 3.30, we have $l(-Q) = 1 + i(Q)$. We have to show $l(-Q) \leq 2$ or, equivalently, $i(Q) \leq 1$. Since the space of discrete holomorphic differentials is two by Corollary 3.18, $i(Q) \geq 2$ would imply that all discrete holomorphic differentials vanish at Q. But in the example in Section 3.4.3 we have seen that dz is a discrete holomorphic differential. Clearly, dz has no zeroes at all, so indeed $l(Q) \leq 1$. Actually, $i(Q) = 1$, because the essentially constant functions are in $L(-Q)$.

(ii) Since essentially constant functions are in $L(-Q-Q')$, the existence of a nontrivial one is equivalent to $l(-Q-Q') \geq 3$. Again by the discrete Riemann-Roch Theorem 3.30, we have to investigate when $i(Q+Q') \geq 1$ and when not.

We already know that dz is a discrete holomorphic differential, so due to $i(0) = 2$ we just have to find another one. Using that all the diagonals of quadrilaterals are orthogonal to each other, the function $f : V(\tilde{\Lambda}) \to \mathbb{C}$ defined by $f(v) = \operatorname{Re}(v)$ if $v \in V(\tilde{\Gamma})$, and $f(v) = i \operatorname{Im}(v)$ if $v \in V(\tilde{\Gamma}^*)$, is a multi-valued discrete holomorphic function. Its differential df is discrete holomorphic, its black periods are real, and its white periods are purely imaginary. In particular, df is not a multiple of dz, and any discrete holomorphic differential ω is a linear combination of df and dz.

Let us color the edges of X in the same color as the corresponding parallel diagonals. Now, both the discrete integrals of df and dz along an edge e are determined by their direction and their length in $\mathbb{C}$. In the case of df, the color of the edge is also important. As the length of an edge scales by λ, the discrete integrals of df and dz scale by λ as well. In particular, if ω vanishes at one black edge, it vanishes at all parallel black edges. Conversely, if ω vanishes at two nonparallel black edges e, e', $\omega \equiv 0$. Indeed, df just looks at the real parts of e, e', whereas dz depends linearly on e, e'.

Therefore, ω vanishes in both Q and Q' if only if their black diagonals are parallel to each other, and $\omega \in H(Q)$. But from the first part, we know already that $i(Q) = 1$. $\square$

Note that in the case that there are no restrictions on the quadrilaterals in the cellular decomposition, the calculation of the space of discrete holomorphic differentials is generally harder, since the decomposition into real and imaginary part of a function to black and white vertices does not work. Furthermore, in contrast to the classical setup, the first part of Corollary 3.31 does not remain true if we consider general discrete Riemann surfaces.

Proposition 3.32. *For any $g \geq 0$, there exists a compact discrete Riemann surface (Σ, Λ, z) of genus g such that there exists a discrete meromorphic function $f : V(\Lambda) \to \mathbb{C}$ that has exactly one simple pole.*

Proof. We start with any compact discrete Riemann surface (Σ', Λ', z') of genus g and pick one quadrilateral $Q' \in V(\diamond')$. Now, Q' is combinatorially replaced by the five quadrilaterals of Figure 3.4. We define the discrete complex structure of the central quadrilateral Q by the complex number ϱ_1, and the discrete complex structure of the four neighboring quadrilaterals Q_k by ϱ_2, both ϱ_1 and ϱ_2 having positive real part. By Proposition 3.1, this construction yields a new compact discrete Riemann surface (Σ, Λ, z) of genus g.

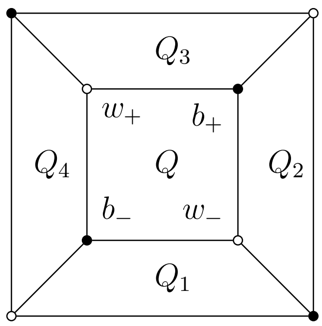


Figure 3.4: Replacement of chosen quadrilateral by five new quadrilaterals

Now, for a complex number $x \neq 0$, consider the function $f : V(\Lambda) \to \mathbb{C}$ that fulfills $f(b_-) = x = -f(b_+)$, $f(w_+) = i\varrho_2 x = -f(w_-)$, and $f(v) = 0$ for all other vertices. Clearly, f is discrete holomorphic outside Q'. The ratio of the differences at the white and black diagonals is exactly $i\varrho_2$ at all Q_k, so f is discrete holomorphic outside Q. But in the central quadrilateral, the ratio is given by $-i\varrho_1$. In particular, it cannot be equal to $i\varrho_1$ since ϱ_1, ϱ_2 have positive real parts. Thus, f is a discrete meromorphic function that has exactly one simple pole, namely at Q. □

3.5.3 Discrete Abel-Jacobi maps

In this section, we discuss discrete Abel-Jacobi maps. Due to the fact that black and white periods of discrete holomorphic one-forms do not have to coincide, we cannot define a discrete Abel-Jacobi map on all of $V(\Lambda)$ and $V(\diamond)$. However, by either restricting to black vertices (and faces) or white vertices (and faces), or considering the universal covering, we get reasonable discretizations of the Abel-Jacobi map.

Let ω denote the column vector with entries ω_k, $\{\omega_1, \ldots, \omega_g\}$ being a canonical set of discrete holomorphic differentials. The corresponding discrete period matrix is denoted by B. We also define the $g \times g$-matrices $\left(B^B_{jk}\right)^g_{j,k=1}$ and $\left(B^W_{jk}\right)^g_{j,k=1}$ with entries $B^B_{jk} := 2\int_{Bb_j} \omega_k$ and $B^W_{jk} := 2\int_{Wb_j} \omega_k$ as the *black* and *white period matrix*, respectively. In analogy to the classical case, let us denote by L the lattice $L := \{Im + Bn | m, n \in \mathbb{Z}^g\}$, where I is the $g \times g$-identity matrix. Similarly, the lattices L^B and L^W are defined.

Then, the complex tori $\mathcal{J} := \mathbb{C}^g/L$, $\mathcal{J}^B := \mathbb{C}^g/L^B$, and $\mathcal{J}^W := \mathbb{C}^g/L^W$ are the *discrete*, the *black*, and the *white Jacobian variety*, respectively.

As a base point, we fix $Q := p(\tilde{Q})$, where $\tilde{Q} \in F(\tilde{\Lambda})$. Having Proposition 3.28 in mind, an integration along $\tilde{\Gamma}$ might be the best option to reach a black vertex v. Thinking of $\tilde{Q}$ as placed at the midpoint of its black diagonal, we choose a path R starting by half the black diagonal of $\tilde{Q}$, and then connecting its black endpoint b incident to $\tilde{Q}$ with v. The discrete integral of a discrete holomorphic differential along R is then defined in essentially the same way as in Section 3.5.1. Similarly, discrete integrals are defined for paths connecting $\tilde{Q}$ with white vertices, starting with half a white diagonal, and then using edges of $\tilde{\Gamma}^*$ only.

In this sense, $\tilde{\mathcal{A}}_{\tilde{Q}}(v) := \tilde{\mathcal{A}}^B_{\tilde{Q}}(v) := \int_{\tilde{Q}}^v \omega$ and $\tilde{\mathcal{A}}_{\tilde{Q}}(v') := \tilde{\mathcal{A}}^W_{\tilde{Q}}(v') := \int_{\tilde{Q}}^{v'} \omega$ are well defined for $v \in V(\tilde{\Gamma})$ and $v' \in V(\tilde{\Gamma}^*)$, using that any discrete holomorphic differential ω can be lifted to a closed discrete holomorphic differential on the universal covering of the discrete Riemann surface (Σ, Λ, z).

To connect $\tilde{Q}$ with another $\tilde{Q}' \in F(\tilde{\Lambda})$, one could consider them as midpoints of their black diagonals, and use a path on $\tilde{\Gamma}$ similar to above. Let us denote by $\tilde{\mathcal{A}}^B_{\tilde{Q}}(\tilde{Q}')$ the corresponding discrete integral of ω from $\tilde{Q}$ to $\tilde{Q}'$, and analogously $\tilde{\mathcal{A}}^W_{\tilde{Q}}(\tilde{Q}')$ the one if $\tilde{\Gamma}$ is replaced by $\tilde{\Gamma}^*$.

Then, $\tilde{\mathcal{A}}^B_{\tilde{Q}} : V(\tilde{\Gamma}) \cup F(\tilde{\Lambda}) \to \mathbb{C}^g$ and $\mathcal{A}^W_{\tilde{Q}} : V(\tilde{\Gamma}^*) \cup F(\tilde{\Lambda}) \to \mathbb{C}^g$ actually project to well defined maps $\mathcal{A}^B_Q : V(\Gamma) \cup V(\diamond) \to \mathcal{J}^B$ and $\mathcal{A}^W_Q : V(\Gamma^*) \cup V(\diamond) \to \mathcal{J}^W$. These *black* and *white Abel-Jacobi maps* discretize the classical Abel-Jacobi map at least for divisors that do not include white or black vertices, respectively. Clearly, they do not depend on the base point Q for divisors of degree 0.

$\tilde{Q}$ can be connected with another $\tilde{Q}' \in F(\tilde{\Lambda})$ in a more symmetric way that does not depend on a choice of either black or white, using the medial graph. By Proposition 3.1, there is a chart $z_{\tilde{Q}}$ that maps $\tilde{Q}$ to a parallelogram. In such a chart, we can place $\tilde{Q}$ on its center. Then, if e and e' are the vectors corresponding to half of each of the oriented diagonals, thus corresponding to oriented edges of $\tilde{X}$, $\tilde{Q} + e/2 + e'/2$ is a vertex x of $\tilde{X}$. Defining $\int_{\tilde{Q}}^x \omega := \int_e \omega/2 + \int_{e'} \omega/2$,

$$\tilde{\mathcal{A}}_{\tilde{Q}}(\tilde{Q}') := \int\limits_{\tilde{Q}}^{\tilde{Q}'} \omega := \int\limits_{\tilde{Q}}^{x} \omega + \int\limits_{x}^{x'} \omega - \int\limits_{\tilde{Q}'}^{x'} \omega$$

is well defined if x' is a vertex incident to $\tilde{Q}'$.

In Section 3.3.1, we described how a closed path on the medial graph induces closed paths on the black and white subgraph, see Figure 3.2. In a similar way, a path connecting $\tilde{Q}$ with $\tilde{Q}'$ as above induces two other paths connecting both faces, a black path just using (half)edges of $\tilde{\Gamma}$, and a white path just using (half)edges of $\tilde{\Gamma}^*$. This construction shows that $2\tilde{\mathcal{A}}_{\tilde{Q}}(\tilde{Q}') = \tilde{\mathcal{A}}^B_{\tilde{Q}}(\tilde{Q}') + \tilde{\mathcal{A}}^W_{\tilde{Q}}(\tilde{Q}')$.

Thus, $\tilde{\mathcal{A}}_{\tilde{Q}}$ defines a *discrete Abel-Jacobi map* on the divisors of the universal covering $(\tilde{\Sigma}, \tilde{\Lambda}, p \circ z)$, and it does not depend on the choice of base point $\tilde{Q}$ for divisors of degree 0, that contain as many black as white vertices (counted with sign).

Remark. In the first remark of Section 3.4.3, we gave an example of a torus where the black and white periods of a canonical discrete holomorphic differential do not coincide. Looking at these periods $i\sqrt{3}/2$ and $i2/\sqrt{3}$, we see that L^W and L^B have a common sublattice L^{BW}, such that $\tilde{\mathcal{A}}_{\tilde{Q}}$ factors to a discrete Abel-Jacobi map $\mathcal{A}_Q : V(\Lambda) \cup V(\diamond) \to \mathbb{C}^g / L^{BW}$. However, such a common sublattice does not have to exist in general. To see this, consider a torus combinatorially composed of four quadrilaterals in the standard way, with a generic discrete complex structure.

Proposition 3.33. *The restriction of the discrete Abel-Jacobi map $\tilde{\mathcal{A}}_{\tilde{Q}}$ to $V(\tilde{\Lambda})$ is discrete holomorphic in each component.*

Proof. Let $Q \in F(\Lambda)$ and z_Q a chart of Q. Then, $\omega_k|_{\partial F_Q} = p_k dz_Q$ for some complex numbers p_k. If b_-, w_-, b_+, w_+ denote the vertices of Q in counterclockwise order, starting with a black vertex,

$$\left(\tilde{\mathcal{A}}_{\tilde{Q}}(b_+) - \tilde{\mathcal{A}}_{\tilde{Q}}(b_-)\right)_k = p_k(z_Q(b_+) - z_Q(b_-)),$$
$$\left(\tilde{\mathcal{A}}_{\tilde{Q}}(w_+) - \tilde{\mathcal{A}}_{\tilde{Q}}(w_-)\right)_k = p_k(z_Q(w_+) - z_Q(w_-)).$$

Thus, the quotient of the differences at white and black vertices equals to the ratio of diagonals, $(z_Q(w_+) - z_Q(w_-))/(z_Q(b_+) - z_Q(b_-))$, such that the discrete Cauchy-Riemann equation is fulfilled. ☐

Remark. In such a chart z_Q,

$$\left(\partial_\Lambda \tilde{\mathcal{A}}_{\tilde{Q}}\right)(Q) = p,$$

exactly as in the smooth case. In particular, the discrete Abel-Jacobi map is an injection unless there is $Q \in V(\diamond)$ such that all discrete holomorphic differentials vanish at Q. By the discrete Riemann-Roch Theorem 3.30, this would imply that there exists a discrete meromorphic function on $V(\Lambda)$ that has at most one pole (at Q), and is not essentially constant, i.e., a discrete meromorphic function with exactly one simple pole at Q. Though this can happen in the classical theory for $g = 0$ only, Proposition 3.32 shows that this does not have to be true in the discrete setup.

Chapter 4

Discrete complex analysis on planar parallelogram-graphs

This chapter deals with discrete complex analysis on planar parallelogram-graphs Λ that are assumed to be locally finite. The vertices Q of the dual $\diamond$ are placed at the centers of the parallelograms Q. Note that any planar parallelogram-graph is strongly regular and bipartite. As in Chapter 2, the induced graphs on black and white vertices are denoted by Γ and Γ^*, respectively.

In Propositions 2.1, 2.5, and 2.22, we have already seen that discrete complex analysis on parallelogram-graphs is closer to the classical theory than on general quad-graphs. For example, $f(v) = v^2$ is a discrete holomorphic function on $V(\Lambda)$, and $\partial_\Lambda f(Q) = 2Q$; $h(Q) = Q$ is a discrete holomorphic function on $V(\diamond)$, and $\partial_\diamond h \equiv 1$; and the discrete Laplacian $\triangle$ approximates the smooth one correctly up to order two.

In order to concentrate on the calculation of the asymptotics of a certain discrete Green's function and discrete Cauchy's kernels, we postpone the discussion of some necessary combinatorial and geometric results on planar parallelogram-graphs to Appendix A. Our setup is closely related to the quasicrystallic parallelogram-graphs discussed by Bobenko, Mercat, and Suris [BMS05]. In their case, the quad-graph can be lifted to $\mathbb{Z}^d$, and a discrete exponential defined on $\mathbb{Z}^d$ can be restricted to the quad-graph. We adapt their construction to our setting in Section 4.1. The discrete exponential is the basic building block for the construction of a discrete Green's function in Section 4.2, and discrete Cauchy's kernels in Section 4.3. These constructions base on the ideas of Kenyon in the case of rhombic quad-graphs [Ken02b]. The corresponding functions can be defined for general planar parallelogram-graphs, but we need more regularity of the graph to calculate their asymptotics. The two conditions we use are that all interior angles of the parallelograms are bounded (the same condition was used in the presentation of Chelkak and Smirnov in [CS11]), and that the ratio of side lengths of the parallelograms is bounded as well. For rhombic quad-graphs, the second condition is trivially fulfilled; for quasicrystallic graphs, there are only a finite number of interior angles. Note that instead of using boundedness of the ratio of side lengths of the parallelograms, we can assume that the side lengths themselves are bounded. This seems to be a stronger condition at first, but actually, both conditions are equivalent, see Proposition A.4 in Appendix A.

We conclude this chapter by a discussion of integer lattices in Section 4.4. On these graphs, discrete holomorphic functions can be discretely differentiated infinitely many times, and for all higher order discrete derivatives, discrete Cauchy's integral formulae with the right asymptotics hold true.

During this chapter, we use the following shorthand notation.

Definition. Let $v, v' \in V(\Lambda)$ and $Q, Q' \in V(\diamond)$.

(i) Choose any directed path of edges $e_1, \ldots, e_n$ on Λ going from v' to v. Define

$$J(v, v') := \sum_{j=1}^{n} e_j^{-1}.$$

(ii) Choose any directed path from v to Q that begins with a directed path on Λ with edges $e_1, \ldots, e_n$ to a vertex v_Q of the parallelogram Q, and ends with two half-edges $d_1/2, d_2/2$, where d_1, d_2 are emanating from v_Q. Define

$$-J(v, Q) = J(Q, v) := \sum_{j=1}^{n} e_j^{-1} + \frac{1}{2} d_1^{-1} + \frac{1}{2} d_2^{-1}.$$

Moreover, let $\tau(v, Q) = \tau(Q, v) := 1/(d_1 d_2)$ if v_Q, v are both in $V(\Gamma)$ or both in $V(\Gamma^*)$, and $\tau(v, Q) = \tau(Q, v) := -1/(d_1 d_2)$ otherwise.

(iii) Choose any directed path from Q' to Q consisting of half-edges $e_1/2, e_2/2$ connecting the center of the parallelogram Q' to one of its vertices $v_{Q'}$, a directed path on Λ with edges $e_3, \ldots, e_n$ going from $v_{Q'}$ to a vertex v_Q of the parallelogram Q, and two half-edges $d_1/2, d_2/2$ emanating from v_Q. Define

$$J(Q,Q') := \frac{1}{2}e_1^{-1} + \frac{1}{2}e_2^{-1} + \sum_{j=3}^{n} e_j^{-1} + \frac{1}{2}d_1^{-1} + \frac{1}{2}d_2^{-1}.$$

Furthermore, let $\tau(Q,Q') := 1/(e_1e_2d_1d_2)$ if $v_Q, v_{Q'}$ are both in $V(\Gamma)$ or both in $V(\Gamma^*)$, and $\tau(Q,Q') := -1/(e_1e_2d_1d_2)$ otherwise.

It is easy to see that these definitions do not depend on the choice of paths.

Remark. In the case that all parallelograms are rhombi of side length one, we have $J(v,v') = \overline{v-v'}$.

4.1 Discrete exponential function

Definition. Let v_0 be a vertex of the parallelogram-graph Λ. Then, the *discrete exponentials* $\mathrm{e}(\cdot,\cdot;v_0), \exp(\cdot,\cdot;v_0) : \mathbb{C} \times V(\Lambda) \to \mathbb{C}$ are defined by

$$\begin{aligned} \mathrm{e}(\lambda, v_0; v_0) &= 1 = \exp(\lambda, v_0; v_0), \\ \frac{\mathrm{e}(\lambda, v'; v_0)}{\mathrm{e}(\lambda, v; v_0)} &= \frac{\lambda + (v'-v)}{\lambda - (v'-v)}, \\ \frac{\exp(\lambda, v'; v_0)}{\exp(\lambda, v; v_0)} &= \frac{1 + \frac{\lambda}{2}(v'-v)}{1 - \frac{\lambda}{2}(v'-v)} \end{aligned}$$

for all vertices $v, v' \in V(\Lambda)$ adjacent to each other, and all $\lambda \in \mathbb{C}$.

For a quadrilateral $Q_0 \in V(\diamond)$ with incident vertices v_-, v'_-, v_+, v'_+ in counterclockwise order, and a vertex $v_0 \in V(\Lambda)$, we define the *discrete exponentials* $\mathrm{e}(\cdot,\cdot;Q_0) : \mathbb{C} \times V(\Lambda) \to \mathbb{C}$ and $\exp(\cdot,\cdot;v_0) : \mathbb{C} \times V(\diamond) \to \mathbb{C}$ as

$$\begin{aligned} \mathrm{e}(\lambda, v; Q_0) &:= \frac{\mathrm{e}(\lambda, v; v_\pm)}{(\lambda - (v_\pm - v'_+))\,(\lambda - (v_\pm - v'_-))}, \\ \exp(\lambda, Q_0; v_0) &:= \frac{\exp(\lambda, v_\pm; v_0)}{\left(1 - \frac{\lambda}{2}(v'_+ - v_\pm)\right)\left(1 - \frac{\lambda}{2}(v'_- - v_\pm)\right)}. \end{aligned}$$

Remark. Note that $\exp(\lambda,\cdot;v_0) = \mathrm{e}(2/\lambda,\cdot;v_0)$. Hence, e and exp are equivalent up to reparametrization. On square lattices, the discrete exponential was already considered by Ferrand [Fer44], and Duffin [Duf56]. The discrete exponential e on rhombic lattices was used in the work of Kenyon [Ken02b], Bobenko, Mercat, and Suris [BMS05], and Bücking [Bü08]. To be comparable to their work, we use their notion to perform our calculations of the asymptotic behavior. In contrast, Mercat [Mer07], and Chelkak and Smirnov [CS11], preferred the parametrization of exp that is closer to the smooth setting. Indeed, Mercat remarked that the discrete exponential exp in the rhombic setting is a generalization of the formula

$$\exp(\lambda x) = \left(\frac{1+\frac{\lambda x}{2n}}{1-\frac{\lambda x}{2n}}\right)^n + O\left(\frac{\lambda^3 x^3}{n^2}\right)$$

to the case when the path from the origin to x consists of $O(|x|/\delta)$ straight line segments of length δ of any directions [Mer07].

If v is fixed, $\exp(\cdot, v; v_0)$ is a rational function with poles at all edges of a shortest directed path connecting v_0 with v. It follows from Lemma A.2 that the arguments of all poles are contained in an interval of length less than π. If, in addition, the interior angles of parallelograms are bounded from below by α_0, the arguments of all poles lie even in an interval of length at most $\pi - \alpha_0$ by Corollary A.3.

Proposition 4.1. *For any* $\lambda \in \mathbb{C}$, $v_0 \in V(\Lambda)$, *and* $Q \in V(\diamond)$, $\exp(\lambda, \cdot; v_0)$ *is discrete holomorphic, and*

$$\left(\partial_\Lambda \exp\left(\lambda, \cdot; v_0\right)\right)(Q) = \lambda \exp(\lambda, Q; v_0).$$

Proof. Let v_-, v_-', v_+, v_+' be the vertices of Q, $a := v_+' - v_-$, and $b := v_-' - v_-$. Using Lemma 2.3, $v_+ - v_- = a + b$, and $v_+' - v_-' = a - b$, we get

$$\begin{aligned}
&\left(\bar{\partial}_\Lambda \exp(\lambda, \cdot; v_0)\right)(Q) \\
=&\frac{\exp(\lambda, v_-; v_0)}{8i\text{area}(F_Q)}\left((a+b)\left(\frac{1+\frac{\lambda}{2}b}{1-\frac{\lambda}{2}b} - \frac{1+\frac{\lambda}{2}a}{1-\frac{\lambda}{2}a}\right) + (a-b)\left(\frac{1+\frac{\lambda}{2}b}{1-\frac{\lambda}{2}b}\frac{1+\frac{\lambda}{2}a}{1-\frac{\lambda}{2}a} - 1\right)\right) \\
=&\frac{\exp(\lambda, v_-; v_0)}{8i\text{area}(F_Q)\left(1-\frac{\lambda}{2}a\right)\left(1-\frac{\lambda}{2}b\right)}\left((a+b)\,\lambda\,(b-a) + (a-b)\,\lambda\,(a+b)\right) = 0.
\end{aligned}$$

In a similar way as above, we calculate

$$\begin{aligned}
&\left(\partial_\Lambda \exp(\lambda, \cdot; v_0)\right)(Q) \\
=&\frac{\exp(\lambda, v_-; v_0)}{8i\text{area}(F_Q)\left(1-\frac{\lambda}{2}a\right)\left(1-\frac{\lambda}{2}b\right)}\left(-\overline{(a+b)}\lambda\,(b-a) + \overline{(b-a)}\lambda\,(a+b)\right) \\
=&\frac{\lambda\exp(\lambda, v_-; v_0)}{2i|a||b|\sin(\varphi_Q)\left(1-\frac{\lambda}{2}a\right)\left(1-\frac{\lambda}{2}b\right)}2i\,\mathrm{Im}(a\bar{b}) = \lambda\exp(\lambda, Q; v_0).
\end{aligned}$$

□

4.2 Asymptotics of the discrete Green's function

Following the presentation of Bobenko, Mercat, and Suris [BMS05], we first define a discrete logarithmic function on a certain branched covering $\tilde{\Lambda}_{v_0}$ of Λ.

Fix $v_0 \in V(\Lambda)$, and let $e_1, e_2, \ldots, e_n$ be the directed edges starting in v_0, ordered according to their slopes. To each of these edges e we assign the real angle $\theta_e := \arg(e) \in [0, 2\pi)$. We assume that $\theta_{e_1} < \theta_{e_n}$. Now, define $\theta_{a+bn} := \theta_a + 2\pi b$, where $a \in \{1, \ldots, n\}$, and $b \in \mathbb{Z}$.

Let $U_e \subset V(\Lambda)$ denote the set of all vertices to that v_0 can be connected by a directed path of edges whose arguments lie in $[\arg(e), \arg(e) + \pi)$, where e is one of the e_k. Lemma A.2 shows that the union of all these U_{e_k}, $k = 1, \ldots, n$, covers the whole quad-graph. It follows that

$$\tilde{U} := \bigcup_{m=-\infty}^{\infty} \tilde{U}_m$$

defines a parallelogram-graph $\tilde{\Lambda}_{v_0}$ that is a branched covering of Λ, branched over v_0. Here, $\tilde{U}_m$ is the set of all vertices of Λ to that v_0 can be connected by a directed path of edges whose arguments lie in $[\theta_m, \theta_m + \pi)$, equipped with the additional data of this interval. Then, all $\tilde{U}_{m+bn}$, $b \in \mathbb{Z}$ and $1 \leq m \leq n$, cover the same sector U_{e_m}, and $\tilde{U}_m \cap \tilde{U}_{m'} \neq \emptyset$ if and only if $|m - m'| < n$.

To each vertex $\tilde{v} \in V(\tilde{\Lambda}_{v_0})$ covering a vertex $v \neq v_0$ of Λ, we assign a real angle $\theta_{\tilde{v}} \equiv \arg(v - v_0) \mod 2\pi$ such that $\theta_{\tilde{v}} \in [\theta_m, \theta_m + \pi)$ if $\tilde{v} \in \tilde{U}_m$. Then, $\theta_{\tilde{v}}$ increases by 2π when $\tilde{v}$ winds once around v_0 in counterclockwise order; and if $\tilde{v}, \tilde{v}' \neq v_0$ are adjacent vertices of $\tilde{\Lambda}_{v_0}$, $|\theta_{\tilde{v}} - \theta_{\tilde{v}'}| < \pi$.

Note that the strip passing through an edge $\tilde{e}$ separates its two endpoints from each other in the quad-graph $\tilde{\Lambda}_{v_0}$. For the definition of a strip, see Appendix A. By construction, if we connect v_0 to some $\tilde{v} \neq v_0$ by a shortest directed path of edges of $\tilde{\Lambda}_{v_0}$, the angles assigned to the edges lie all in $(\theta_{\tilde{v}} - \pi, \theta_{\tilde{v}} + \pi)$.

Definition. Let $v_0 \in V(\Lambda)$, and let $\tilde{\Lambda}_{v_0}$ be the corresponding branched covering of Λ. The *discrete logarithmic function* on $V(\tilde{\Lambda}_{v_0})$ is given by

$$\log(\tilde{v}; v_0) := \frac{1}{2\pi i} \int\limits_{C_{\tilde{v}}} \frac{\log(\lambda)}{2\lambda} \mathrm{e}(\lambda, v; v_0) d\lambda,$$

where $C_{\tilde{v}}$ is a collection of counterclockwise oriented loops going once around each pole of $\mathrm{e}(\cdot, v; v_0)$, $v \in V(\Lambda)$ being the projection of $\tilde{v} \in V(\tilde{\Lambda}_{v_0})$. The arguments of all poles shall lie in $(\theta_{\tilde{v}} - \pi, \theta_{\tilde{v}} + \pi)$.

Remark. Let us suppose that v_0 is a black vertex. In the special case of a rhombic quasicrystallic quad-graph, Bobenko, Mercat, and Suris motivate the notion of the discrete logarithm as follows [BMS05]: The discrete logarithm is real-valued and does not branch on black vertices; and it is purely imaginary on white points and increases by $2\pi i$ if one goes once around v_0 in counterclockwise order. Therefore, the discrete logarithm models the behavior of the real and the imaginary part of the smooth logarithm if restricted to black and white vertices, respectively. As we will see later in the proof of Proposition 4.2, the values at vertices adjacent to v_0 coincide with the smooth logarithm.

If $\tilde{v}, \tilde{v}'$ cover the same point $v \in V(\Lambda)$, and $\theta_{\tilde{v}'} - \theta_{\tilde{v}} = 2\pi$, then

$$\log(\tilde{v}'; v_0) - \log(\tilde{v}; v_0) = \int\limits_{C_{\tilde{v}}} \frac{1}{2\lambda} \mathrm{e}(\lambda, v; v_0) d\lambda.$$

The function that is integrated is meromorphic on $\mathbb{C}$ with poles given by the one of $\mathrm{e}(\cdot, v; v_0)$, and zero. By residue formula, we can replace $C_{\tilde{v}}$ by an integration along a circle centered at 0 with large radius R (such that all poles lie inside the disk) in counterclockwise direction, and an integration along a circle centered at 0 with small radius r (such that all poles lie outside the disk) in clockwise direction. Now, $\mathrm{e}(\infty, v; v_0) = 1$. If v_0, v are both in $V(\Gamma)$ or both in $V(\Gamma^*)$, then $\mathrm{e}(0, v; v_0) = 1$, otherwise $\mathrm{e}(0, v; v_0) = -1$. Hence, $\log(\tilde{v}'; v_0) - \log(\tilde{v}; v_0) = 0$ in the first, and $\log(\tilde{v}'; v_0) - \log(\tilde{v}; v_0) = 2\pi i$ in the latter case.

In particular, the real part of the discrete logarithm $\log(\cdot; v_0)$ is a well defined function on $V(\Lambda)$. Divided by 2π, one actually obtains a discrete Green's function with respect to v_0. Bobenko, Mercat, and Suris showed that this function coincides with the one of Kenyon [Ken02b] in the rhombic case [BMS05].

Proposition 4.2. *Let $v_0 \in V(\Lambda)$. The function $G(\cdot; v_0) : V(\Lambda) \to \mathbb{R}$ defined by $G(v_0; v_0) = 0$ and*

$$G(v; v_0) = \frac{1}{2\pi} \operatorname{Re} \left(\frac{1}{2\pi i} \int\limits_{C_v} \frac{\log(\lambda)}{2\lambda} \mathrm{e}(\lambda, v; v_0) d\lambda \right),$$

is a (free) discrete Green's function with respect to v_0. Here, C_v is a collection of counterclockwise oriented loops going once around each pole of $\mathrm{e}(\cdot, v; v_0)$, where the arguments of all poles shall lie in $(\arg(v - v_0) - \pi, \arg(v - v_0) + \pi)$.

Proof. As we have seen above, the choice of the branch of the logarithm does not matter when evaluating the real part of the integral.

Let us consider the star of v_0 in Λ and its infinite branched covering. For any vertex $v'_s \in V(\Lambda)$ adjacent to v, $v'_s - v_0$ is the only pole of $\mathrm{e}(\cdot, v; v_0)$. The residue theorem shows that $\log(\tilde{v}'_s; v_0)$ and $\log(v'_s - v_0)$ coincide up to a multiple of $2\pi i$, if $\tilde{v}'_s$ covers v'_s. By a similar calculation for vertices v_s adjacent to v_0 in Γ or Γ^*, we finally get

$$\begin{aligned}
G(v'_s; v_0) &= \frac{1}{2\pi} \operatorname{Re}(\log(v'_s - v_0)),\\
G(v_s; v_0) &= \frac{1}{2\pi} \operatorname{Re}\left(\left(\log\left(v'_s - v_0\right) - \log\left(v'_{s-1} - v_0\right)\right) \frac{v_s - v_0}{v'_s - v'_{s-1}}\right),
\end{aligned}$$

where v_0, v'_{s-1}, v_s, v'_s are the vertices of a parallelogram Q_s.

As in Corollary 2.20, let

$$\rho_s := -i \frac{v'_s - v'_{s-1}}{v_s - v_0}.$$

In addition, we assign angles $\theta_{v'_s} \equiv \arg\left(v'_s - v_0\right) \mod 2\pi$ in such a way that $0 < \theta_{v'_s} - \theta_{v'_{s-1}} < \pi$. Due to $\operatorname{Re}\left(i/\rho_s\right) = \operatorname{Im}\left(\rho_s\right)/|\rho_s|^2$, $\operatorname{Im}\left(i/\rho_s\right) = \operatorname{Re}\left(\rho_s\right)/|\rho_s|^2$,

$$\begin{aligned}
&\frac{|\rho_s|^2 \left(G(v_s; v_0) - G(v_0; v_0)\right) + \operatorname{Im}\left(\rho_s\right)\left(G(v'_s; v_0) - G(v'_{s-1}; v_0)\right)}{\operatorname{Re}(\rho_s)}\\
=&\frac{|\rho_s|^2 \operatorname{Re}\left(-i\rho_s\right) + \operatorname{Im}\left(\rho_s\right)}{2\pi \operatorname{Re}\left(\rho_s\right)} \log|-i\rho_s| - \frac{|\rho_s|^2 \operatorname{Im}\left(-i\rho_s\right)}{2\pi \operatorname{Re}\left(\rho_s\right)} \left(\theta_{v'_s} - \theta_{v'_{s-1}}\right)\\
=&\frac{\theta_{v'_s} - \theta_{v'_{s-1}}}{2\pi}.
\end{aligned}$$

It follows that $\triangle G(v_0; v_0) = 1/(4\mathrm{area}(F_v))$.

Now, consider the star of some vertex $v \neq v_0$. We claim that the same contours of integration can be used for all vertices of the star, meaning that the arguments of common poles of $\mathrm{e}(\cdot, v; v_0)$ and $\mathrm{e}(\cdot, v'; v_0)$ are equal, where v' is a vertex of the star of v.

If v' is not adjacent to v, we can find a vertex v'' adjacent to both such that the common poles of $\mathrm{e}(\cdot, v; v_0)$ and $\mathrm{e}(\cdot, v'; v_0)$ are common poles of $\mathrm{e}(\cdot, v''; v_0)$ and $\mathrm{e}(\cdot, v'; v_0)$. Thus, it suffices to consider adjacent vertices v and v'.

Suppose that there is a common pole e of $\mathrm{e}(\cdot, v; v_0)$ and $\mathrm{e}(\cdot, v'; v_0)$ such that the argument of e is not contained in both $(\arg(v - v_0) - \pi, \arg(v - v_0) + \pi)$ and $(\arg(v' - v_0) - \pi, \arg(v' - v_0) + \pi)$. This can only happen if the edge vv' intersects the ray $v_0 - te$, $t \geq 0$. But since e is a pole of the discrete exponential, there is a strip with common parallel e that separates v_0 from both v and v' (see Appendix A for more details). Thus, the edge vv' cannot intersect this strip, and therefore not the ray $v_0 - te$, $t \geq 0$, contradiction.

Using the same contours of integration, the discrete Laplacian commutes with the integration. Thus, $G(\cdot; v_0)$ is discrete harmonic away from v_0 by Corollary 2.21, because $\mathrm{e}(\lambda, \cdot; v_0)$ is discrete holomorphic by Proposition 4.1. $\square$

Remark. With almost the same argument as in the proof of Proposition 4.2, we see that the discrete logarithm is a discrete holomorphic function on the vertices of $\tilde{\Lambda}_{v_0}$. In their paper [BMS05], Bobenko, Mercat, and Suris proved that the discrete logarithm on rhombic quasicrystallic quad-graphs is even more then discrete holomorphic, namely isomonodromic.

Theorem 4.3. *Assume that there are $\alpha_0, q_0 > 0$ such that $\alpha \geq \alpha_0$ and $e/e' \geq q_0$ for all interior angles α and the two side lengths e, e' of any parallelogram of Λ. Let $v_0 \in V(\Lambda)$ be fixed.*

Then, the discrete Green's function $G(\cdot; v_0)$ constructed in Proposition 4.2 has the following asymptotic behavior:

$$G(v; v_0) = \frac{1}{4\pi} \log \left| \frac{v - v_0}{J(v, v_0)} \right| + O\left(|v - v_0|^{-2}\right) \text{ if } v \text{ and } v_0 \text{ are of different color,}$$
$$G(v; v_0) = \frac{\gamma_{\text{Euler}} + \log(2)}{2\pi} + \frac{1}{4\pi} \log |(v - v_0) J(v, v_0)| + O\left(|v - v_0|^{-2}\right) \text{ otherwise.}$$

Here, γ_{Euler} denotes the Euler-Mascheroni constant.

The proof follows the ideas of Kenyon [Ken02b] and Bücking [Bü08]. Both considered just quasicrystallic rhombic quad-graphs. But the main difference to [Ken02b] is that we deform the path of integration into an equivalent one different from Kenyon's, since his approach does not generalize to parallelogram-graphs. As Chelkak and Smirnov did for rhombic quad-graphs with bounded interior angles in [CS11], Kenyon used that two points $v, v' \in V(\Lambda)$ can be connected by a directed paths of edges such that the angle between each directed edge and $v' - v$ is less than $\pi/2$, or the angle between the sum of two consecutive edges and $v' - v$ is less than $\pi/2$. This is true for rhombic quad-graphs, but not for parallelogram-graphs. Instead, we use essentially the same deformation of the path of integration as Bücking did.

Proof. The poles $e_1, \ldots, e_{k(v)}$ of $\mathrm{e}(\cdot, v; v_0)$ correspond to the directed edges of a shortest path from v_0 to v. By Corollary A.3 (i), there is a real θ_0 such that the angles associated to the directed edges above lie all in $[\theta_0, \theta_0 + \pi - \alpha_0]$. The claim is invariant under multiplication of the complex plane by $\exp(i\alpha)$. Indeed, neither the discrete exponential nor $d\lambda/\lambda$ change. To $\log(\lambda)$ we get an additive term $i\alpha$, such that

$$\frac{1}{2\pi} \operatorname{Re}\left(\frac{i\alpha}{2\pi i} \int\limits_{C_v} \frac{1}{2\lambda} \mathrm{e}(\lambda, v; v_0) d\lambda\right)$$

adds to $G(v; v_0)$. But we have already seen above that the integral term is either 0 or $2\pi i$; in any case, it is purely imaginary and the last expression is zero.

So without loss of generality, we assume $\theta_0 = -(\pi - \alpha_0)/2$. By definition,

$$G(v; v_0) = \operatorname{Re}\left(\frac{1}{8\pi^2 i} \int\limits_{C_v} \frac{\log \lambda}{\lambda} \mathrm{e}(\lambda, v; v_0) d\lambda\right),$$

where C_v is a collection of counterclockwise oriented loops going once around each $e_1, \ldots, e_{k(v)}$. The arguments of the poles correspond to the assigned angles. By residue theorem, we can deform C_v into a new path of integration C'_v that goes first along a circle centered at 0 with large radius $R(v)$ (such that all poles lie inside this disk) in counterclockwise direction starting and ending in $-R(v)$, then goes along the line segment $[-R(v), -r(v)]$ followed by the circle centered at 0 with small radius $r(v)$ (such that all poles lie outside this disk) in clockwise direction, and finally goes the line segment $[-R(v), -r(v)]$ backwards. Note that log differs by $2\pi i$ for the two integrations along $[-R(v), -r(v)]$.

By Proposition A.4, there are $E_1 > E_0 > 0$ such that $E_1 \geq e \geq E_0$ for all lengths e of edges of Λ. In particular, we can choose the radii of order

$$R(v) = \Omega(|v - v_0|^4) \text{ and } r(v) = \Omega(|v - v_0|^{-4}).$$

By construction, $k(v) \leq \operatorname{Re}(v - v_0)/(E_0 \cos(\theta_0))$, so $k(v) = O(|v - v_0|)$. Also,

$$|J(v, v_0)| = \left|\sum_{j=1}^{k(v)} e_j^{-1}\right| \leq \frac{k(v)}{E_0} \text{ and } \operatorname{Re}(J(v, v_0)) \geq \frac{\cos(\theta_0)|v - v_0|}{E_1^2}.$$

Hence, $|J(v, v_0)| = \Omega(|v - v_0|)$.

We first look at the contributions of the circles with radii $r(v)$ and $R(v)$ to $G(v;v_0)$. For $\lambda \to 0$,

$$-\frac{\lambda+e}{\lambda-e} = \frac{1+\frac{\lambda}{e}}{1-\frac{\lambda}{e}} = \left(1+\frac{\lambda}{e}\right)\left(1+\frac{\lambda}{e}+O(\lambda^2)\right) = 1+2\frac{\lambda}{e}+O(\lambda^2), \text{ so}$$

$$(-1)^{k(v)}\mathrm{e}(\lambda,v;v_0) = 1+O(\lambda|v-v_0|),$$

using $|J(v,v_0)| = \Omega(|v-v_0|)$. For $\lambda \to \infty$, $(\lambda+e)/(\lambda-e) = 1+2e/\lambda+O(\lambda^{-2})$, and

$$\mathrm{e}(\lambda,v;v_0) = 1+O(\lambda^{-1}|v-v_0|).$$

Thus, we get for the integration along the small circle with radius $r(v)$:

$$\begin{aligned}
&\mathrm{Re}\left(\frac{1}{8\pi^2 i}\int\limits_{\pi}^{-\pi}(-1)^{k(v)}\frac{\log(r(v))+i\theta}{r(v)\exp(i\theta)}\left(1+O(r(v)|v-v_0|)\right)d\left(r(v)\exp(i\theta)\right)\right)\\
&= -\mathrm{Re}\left(\frac{1}{8\pi^2}\int\limits_{-\pi}^{\pi}(-1)^{k(v)}\left(\log(r(v))+i\theta\right)\left(1+O(|v-v_0|^{-3})\right)d\theta\right)\\
&= -\frac{(-1)^{k(v)}\log(r(v))}{4\pi}\left(1+O(|v-v_0|^{-3})\right),
\end{aligned}$$

since $r(v) = \Omega\left(|v-v_0|^{-4}\right)$. Similarly, $\log(R(v))/(4\pi)\cdot(1+O(|v-v_0|^{-3}))$ is the contribution of the circle of radius $R(v)$. In total, the asymptotics for the real part of the integration along the two circles are

$$\frac{1}{4\pi}\left(\log(R(v))-(-1)^{k(v)}\log(r(v))\right)+O(|v-v_0|^{-2}).$$

The two integrations along $[-R(v),-r(v)]$ can be combined into the integral

$$\frac{1}{4\pi}\int\limits_{-R(v)}^{-r(v)}\frac{\mathrm{e}(\lambda,v;v_0)}{\lambda}d\lambda.$$

For $|v-v_0|\geq 1$ large enough, we split the integration into the three parts along

$$[-R(v),-E_1\sqrt{|v-v_0|}],[-E_1\sqrt{|v-v_0|},-\frac{E_1}{\sqrt{|v-v_0|}}], \text{ and } [-\frac{E_1}{\sqrt{|v-v_0|}},-r(v)].$$

We first consider the intermediate range $\lambda\in[-E_1\sqrt{|v-v_0|},-E_1/\sqrt{|v-v_0|}]$. Using that

$$\frac{|\lambda+e|^2}{|\lambda-e|^2} = 1+\frac{4\lambda\,\mathrm{Re}(e)}{\lambda^2-2\lambda\,\mathrm{Re}(e)+|e|^2} \leq 1+\frac{4\lambda\,\mathrm{Re}(e)}{(\lambda-|e|)^2} \leq \exp\left(\frac{4\lambda\,\mathrm{Re}(e)}{(\lambda-E_1)^2}\right),$$

$$|\mathrm{e}(\lambda,v;v_0)| \leq \exp\left(\frac{2\lambda\,\mathrm{Re}(v-v_0)}{(\lambda-E_1)^2}\right) \leq \exp\left(\frac{2\lambda\cos(\theta_0)|v-v_0|}{(\lambda-E_1)^2}\right).$$

Observing that $\lambda/(\lambda-E_1)^2$ attains its maximum on the boundary,

$$\frac{\lambda}{(\lambda-E_1)^2} \leq \frac{-\sqrt{|v-v_0|}}{E_1\left(1+\sqrt{|v-v_0|}\right)^2} \leq \frac{-1}{4E_1\sqrt{|v-v_0|}}.$$

As a consequence of the last two estimations,

$$\left|\frac{1}{4\pi}\int\limits_{-E_1\sqrt{|v-v_0|}}^{-E_1/\sqrt{|v-v_0|}} \frac{\mathrm{e}(\lambda,v;v_0)}{\lambda}d\lambda\right| \leq E_1\sqrt{|v-v_0|}\exp\left(-\frac{\cos(\theta_0)\sqrt{|v-v_0|}}{2E_1}\right).$$

The latter expression decays faster to zero than any power of $|v-v_0|$.

Now, let $\lambda\in[-E_1/\sqrt{|v-v_0|},-r(v)]$ be small. Having in mind that

$$-\frac{\lambda+e}{\lambda-e} = \left(1+\frac{\lambda}{e}\right)\left(1+\frac{\lambda}{e}+O(\lambda^2)\right) = 1+2\frac{\lambda}{e}+2\frac{\lambda^2}{e^2}+O(\lambda^3),$$
$$\log\left(-\frac{\lambda+e}{\lambda-e}\right) = 2\frac{\lambda}{e}+O(\lambda^3).$$

Together with $k(v)=O(|v-v_0|)$ this implies that

$$\begin{aligned}(-1)^{k(v)}\mathrm{e}(\lambda,v;v_0) &= \exp\left(2\lambda J(v,v_0)+O(k(v)\lambda^3)\right)\\ &= \exp\left(2\lambda J(v,v_0)\right)\left(1+O(|v-v_0|\lambda^3)\right).\end{aligned}$$

Thus, the integral near the origin is equal to

$$\frac{(-1)^{k(v)}}{4\pi}\int\limits_{-E_1/\sqrt{|v-v_0|}}^{-r(v)}\left(\frac{\exp\left(2\lambda J(v,v_0)\right)}{\lambda}+\exp\left(2\lambda J(v;v_0)\right)O(|v-v_0|\lambda^2)\right)d\lambda.$$

The integral of the second term gives $O(|v-v_0|^{-2})$. For the first term, we get

$$\frac{(-1)^{k(v)}}{4\pi}\left(\int\limits_{-2E_1J(v,v_0)/\sqrt{|v-v_0|}}^{-1}\frac{\exp(s)}{s}ds+\int\limits_{-1}^{-2r(v)J(v,v_0)}\frac{\exp(s)-1}{s}ds\right)$$
$$+\frac{(-1)^{k(v)}}{4\pi}\int\limits_{-1}^{-2r(v)J(v,v_0)}\frac{1}{s}ds$$

$$
\begin{aligned}
&=\frac{(-1)^{k(v)}}{4\pi}\left(\int_{-\infty}^{-1}\frac{\exp(s)}{s}ds+\int_{-1}^{0}\frac{\exp(s)-1}{s}ds\right)\\
&-\frac{(-1)^{k(v)}}{4\pi}\left(\int_{-\infty}^{-2E_1J(v,v_0)/\sqrt{|v-v_0|}}\frac{\exp(s)}{s}ds+\int_{-2r(v)J(v,v_0)}^{0}\frac{\exp(s)-1}{s}ds\right)\\
&+\frac{(-1)^{k(v)}}{4\pi}\log(2r(v)J(v,v_0))\\
&=\frac{(-1)^{k(v)}}{4\pi}\left(\gamma_{\text{Euler}}+\Omega(|v-v_0|^{-3})+\log(2r(v)J(v,v_0))\right).
\end{aligned}
$$

Here, we have used again that $J(v,v_0)=\Omega(|v-v_0|)$ and $r(v)=\Omega(|v-v_0|^{-4})$.

Finally, we consider $\lambda\in[-R(v),-E_1\sqrt{|v-v_0|}]$. Similar to the evaluation of the integral near zero,

$$
\begin{aligned}
\log\left(\frac{\lambda+e}{\lambda-e}\right)&=2\frac{e}{\lambda}+O(\lambda^{-3})\text{ implies}\\
\mathrm{e}(\lambda,v;v_0)&=\exp\left(\frac{2(v-v_0)}{\lambda}\right)\left(1+O(|v-v_0|\lambda^{-3})\right),
\end{aligned}
$$

so the integral near $-\infty$ gives 4π times

$$
\begin{aligned}
&\int_{-R(v)}^{-E_1\sqrt{|v-v_0|}}\frac{\exp(\lambda,v;v_0)}{\lambda}d\lambda=\int_{-R(v)}^{-E_1\sqrt{|v-v_0|}}\frac{\exp\left(\frac{2(v-v_0)}{\lambda}\right)}{\lambda}d\lambda+O(|v-v_0|^{-2})\\
&=\int_{-R(v)/(2(v-v_0))}^{-1}\frac{\exp\left(\frac{1}{s}\right)-1}{s}ds+\int_{-1}^{-E_1\sqrt{|v-v_0|}/(2(v-v_0))}\frac{\exp\left(\frac{1}{s}\right)}{s}ds\\
&+\int_{-R(v)/(2(v-v_0))}^{-1}\frac{1}{s}ds+O(|v-v_0|^{-2})\\
&=\gamma_{\text{Euler}}-\log\left(\frac{R(v)}{2(v-v_0)}\right)+O(|v-v_0|^{-2}).
\end{aligned}
$$

Summing up and taking the real part, we finally get

$$
\begin{aligned}
4\pi G(v;v_0)=&\log|R(v)|+(-1)^{k(v)}\left(-\log|r(v)|+\gamma_{\text{Euler}}+\log\left(|2r(v)J(v,v_0)|\right)\right)\\
&+\gamma_{\text{Euler}}-\log\left|\frac{R(v)}{2(v-v_0)}\right|+O(|v-v_0|^{-2})
\end{aligned}
$$

$$= \left(1 + (-1)^{k(v)}\right)(\gamma_{\text{Euler}} + \log(2))$$
$$+ \log|v - v_0| + (-1)^{k(v)} \log|J(v; v_0)| + O(|v - v_0|^{-2}).$$

□

Remark. Under the conditions of Theorem 4.3, it follows from Proposition A.4 and Theorem 2.31 that there is exactly one discrete Green's function having the same asymptotics as the one we constructed above up to terms of order $o(|v - v_0|^{-1/2})$.

Let us compare this result to the case of rhombi of side length one. Assume that $v_0 \in V(\Gamma)$. Then, the discrete logarithm is purely real and nonbranched on $V(\Gamma)$, and purely imaginary and branched on $V(\Gamma^*)$. Thus, $G(v; v_0) = 0$ if $v \in V(\Gamma^*)$, well fitting to the fact that $\triangle$ splits into two discrete Laplacians on Γ and Γ^*. Using $J(v, v_0) = \overline{v - v_0}$,

$$G(v; v_0) = \frac{1}{2\pi}(\gamma_{\text{Euler}} + \log(2) + \log|v - v_0|) + O(|v - v_0|^{-2}),$$

exactly as in the work of Bücking [Bü08], who slightly strengthened the error term in Kenyon's work [Ken02b]. In this paper, Kenyon showed that there is no further discrete Green's function with asymptotics $o(|v - v_0|)$.

4.3 Asymptotics of discrete Cauchy's kernels

Let $v_0 inV(\Lambda)$ and $Q_0 \in V(\Diamond)$. We first construct a discrete Cauchy's kernel K_{v_0} with respect to v_0 on $V(\Diamond)$ that has asymptotics $\Omega(|Q - v_0|^{-1})$. Then, we construct a discrete Cauchy's kernel K_{Q_0} with respect to Q_0 on $V(\Lambda)$ with asymptotics $\Omega(|v - Q_0|^{-1})$. In both cases, there are no further discrete Cauchy's kernels with asymptotics $o(|Q - v_0|^{-1/2})$ or $o(|v - Q_0|^{-1/2})$. In the end of this section, we determine the asymptotics of $\partial_\Lambda K_{Q_0}$.

Theorem 4.4. *Let $v_0 \in V(\Lambda)$, and $G(\cdot; v_0)$ be a discrete Green's function with respect to v_0.*

(i) $K_{v_0} := 8\pi\partial_\Lambda G(\cdot; v_0)$ is a discrete Cauchy's kernel with respect to v_0.

(ii) Assume additionally that there exist positive real numbers α_0, q_0 such that $\alpha \geq \alpha_0$ and $e/e' \geq q_0$ for all interior angles α and the two side lengths e, e' of any parallelogram of Λ. Suppose that $G(\cdot; v_0)$ is the discrete Green's

function constructed in Proposition 4.2 and K_{v_0} the discrete Cauchy's kernel given in (i). Then,

$$K_{v_0}(Q) = \frac{1}{Q - v_0} + \frac{\tau(Q, v_0)}{J(Q, v_0)} + O(|Q - v_0|^{-2}).$$

(iii) Under the conditions of (ii), there is exactly one discrete Cauchy's kernel with respect to v_0 with asymptotics $o(|Q - v_0|^{-1/2})$.

Proof. (i) By definition of a discrete Green's function and Corollary 2.20,

$$\bar{\partial}_\diamond K_{v_0}(v) = 8\pi \left(\bar{\partial}_\diamond \partial_\Lambda G(\cdot; v_0)\right)(v) = 2\pi \triangle G(v; v_0) = \delta_{vv_0} \frac{\pi}{2\text{area}(F_{v_0})}.$$

(ii) By Proposition A.4, all edge lengths of parallelograms are bounded.

For the ease of notation, we assume that $v_0 \in V(\Gamma)$, but note that the other case of a white vertex v_0 can be handled in the same manner. Using the notation of Figure 2.1, and the asymptotics of Theorem 4.3,

$$\begin{aligned}
4\pi G(b_+; v_0) - 4\pi G(b_-; v_0) &= \log\left(\frac{|J(b_+, v_0)||b_+ - v_0|}{|J(b_-, v_0)||b_- - v_0|}\right) + O(|Q - v_0|^{-2}),\\
4\pi G(w_+; v_0) - 4\pi G(w_-; v_0) &= \log\left(\frac{|J(w_-, v_0)||w_+ - v_0|}{|J(w_+, v_0)||w_- - v_0|}\right) + O(|Q - v_0|^{-2})
\end{aligned}$$

Let us introduce $a := w_+ - b_-$ and $b := w_- - b_-$. Using the standard formula $\log|1 + x| = (x + \bar{x})/2 + O(x^2)$, and $J(Q, v_0) = \Omega(|Q - v_0|)$ (compare with the proof of Theorem 4.3), we get

$$\begin{aligned}
\log|b_\pm - v_0| &= \log\left|(Q - v_0)\left(1 \pm \frac{a + b}{2(Q - v_0)}\right)\right|\\
&= \log|Q - v_0| \pm \text{Re}\left(\frac{a + b}{2(Q - v_0)}\right) + O(|Q - v_0|^{-2}),\\
\log|w_\pm - v_0| &= \log\left|(Q - v_0)\left(1 \pm \frac{a - b}{2(Q - v_0)}\right)\right|\\
&= \log|Q - v_0| \pm \text{Re}\left(\frac{a - b}{2(Q - v_0)}\right) + O(|Q - v_0|^{-2}),\\
\log|J(b_\pm, v_0)| &= \log\left|J(Q; v_0)\left(1 \pm \frac{a^{-1} + b^{-1}}{2J(Q, v_0)}\right)\right|\\
&= \log|J(Q, v_0)| \pm \text{Re}\left(\frac{a^{-1} + b^{-1}}{2J(Q, v_0)}\right) + O(|Q - v_0|^{-2}),
\end{aligned}$$

$$\begin{aligned}\log|J(w_\pm,v_0)| &= \log\left|J(Q;v_0)\left(1\pm\frac{a^{-1}-b^{-1}}{2J(Q,v_0)}\right)\right|\\ &= \log|J(Q,v_0)| \pm \operatorname{Re}\left(\frac{a^{-1}-b^{-1}}{2J(Q,v_0)}\right) + O(|Q-v_0|^{-2}).\end{aligned}$$

Therefore, we get for the discrete derivative of $8\pi G(\cdot;v_0)$ at the face Q:

$$\begin{aligned}K_{v_0}(Q) &= \lambda_Q \frac{\operatorname{Re}\left(\frac{a+b}{Q-v_0}\right)+\operatorname{Re}\left(\frac{a^{-1}+b^{-1}}{J(Q,v_0)}\right)}{a+b} + \bar{\lambda}_Q\frac{\operatorname{Re}\left(\frac{a-b}{Q-v_0}\right)-\operatorname{Re}\left(\frac{a^{-1}-b^{-1}}{J(Q,v_0)}\right)}{a-b}\\ &\quad + O(|Q-v_0|^{-2})\\ &= \frac{1}{Q-v_0}+\frac{1}{abJ(Q,v_0)} + O(|Q-v_0|^{-2})\\ &\quad + \frac{1}{2}\left(\lambda_Q\frac{\overline{a+b}}{a+b}+\bar{\lambda}_Q\frac{\overline{a-b}}{a-b}\right)\left(\frac{1}{\overline{Q-v_0}}+\frac{1}{\overline{abJ(Q,v_0)}}\right)\\ &= \frac{1}{Q-v_0}+\frac{\tau(Q,v_0)}{J(Q,v_0)}+O(|Q-v_0|^{-2}).\end{aligned}$$

Here, $\lambda_Q = \exp\left(-i(\varphi_Q-\pi/2)\right)$, and we have used the identity

$$-\frac{\lambda_Q}{\bar{\lambda}_Q} = \exp\left(-2i\varphi_Q\right) = \frac{a+b}{a-b}\cdot\frac{\overline{a-b}}{\overline{a+b}} \Leftrightarrow \lambda_Q\frac{\overline{a+b}}{a+b}+\bar{\lambda}_Q\frac{\overline{a-b}}{a-b}=0.$$

(iii) Let h be the difference of two discrete Cauchy's kernels with respect to v_0 with asymptotics $o(|Q-v_0|^{-1/2})$. Then, h is discrete holomorphic, and by Proposition 2.8, a discrete primitive f exists. By construction,

$$\frac{f(b_+)-f(b_-)}{b_+-b_-} = \frac{f(w_+)-f(w_-)}{w_+-w_-} = \partial_\Lambda f(Q) = h(Q) = o(|Q-v_0|^{-1/2}).$$

Since edge lengths are bounded by Proposition A.4, the conditions of Theorem 2.31 are fulfilled. Hence, f is essentially constant, so h vanishes identically. □

Since we do not have discrete Green's functions on $V(\diamond)$, we have to construct discrete Cauchy's kernels on $V(\Lambda)$ differently. To do so, we follow the original approach of Kenyon using the discrete exponential [Ken02b] that was reintroduced by Chelkak and Smirnov in [CS11].

Proposition 4.5. *Let $Q_0 \in V(\diamond)$. The function defined by*

$$K_{Q_0}(v) := \frac{1}{\pi i}\int\limits_{C_v}\log(\lambda)\mathrm{e}(\lambda,v;Q_0)d\lambda = 2\int\limits_{-(v-Q_0)\infty}^{0}\mathrm{e}(\lambda,v;Q_0)d\lambda$$

is a discrete Cauchy's kernel with respect to Q_0*. Here,* C_v *is a collection of counterclockwise oriented loops going once around each pole of* $\mathrm{e}(\cdot, v; Q_0)$*. The arguments of all poles shall lie in* $(\theta_v - \pi, \theta_v + \pi)$*, where* $\theta_v := \arg(v - Q_0)$*.*

Proof. Corollary A.3 (ii) assures that the arguments of all poles can be indeed chosen in such a way that they lie in $(\theta_v - \pi, \theta_v + \pi)$.

By residue theorem, we can replace C_v by an integration along a circle centered at 0 with large radius R (such that all poles lie inside the disk) in counterclockwise direction, an integration along a circle centered at 0 with small radius r (such that all poles lie outside the disk) in clockwise direction, and integrations along the two directions of the line segment $[-R, -r]$. Using

$$\lim_{\lambda \to \infty} \lambda \mathrm{e}(\lambda, v; Q_0) \log(\lambda) = 0,$$

and taking r arbitrarily small, there is a homotopy between the new integration path and the path from complex infinity to 0 and back along the ray spanned by $-(v - Q_0)$ that does not change the value of the integral. $\log(\lambda)$ differs by $2\pi i$ on the two sides of the ray, which shows

$$\frac{1}{\pi i} \int\limits_{C_v} \log(\lambda) \mathrm{e}(\lambda, v; Q_0) d\lambda = 2 \int\limits_{-(v-Q_0)\infty}^{0} \mathrm{e}(\lambda, v; Q_0) d\lambda.$$

In particular, the first integral is independent of the choice of θ_v modulo 2π.

Let us consider the parallelogram Q_0 with vertices b_-, w_-, b_+, w_+ as in Figure 2.1, and let denote $a := w_+ - b_-$, $b := w_- - b_-$. Without loss of generality, we rotate the complex plane in such a way that $\arg(b_+ - Q_0) = 0$. Then, $\arg(a)$ and $\arg(b)$ lie between 0 and $\pm\pi$, and using residue theorem,

$$\begin{aligned}
K_{Q_0}(b_-) &= 2\frac{(\log|a| + i\arg(a) + i\pi) - (\log|b| + i\arg(b) + i\pi)}{b - a},\\
K_{Q_0}(b_+) &= 2\frac{(\log|b| + i\arg(b)) - (\log|a| + i\arg(a))}{b - a},\\
K_{Q_0}(w_-) &= 2\frac{(\log|b| + i\arg(b) + 2i\pi) - (\log|a| + i\arg(a) + i\pi)}{a + b},\\
K_{Q_0}(w_+) &= 2\frac{(\log|a| + i\arg(a)) - (\log|b| + i\arg(b) + i\pi)}{a + b}.
\end{aligned}$$

Finally, we get by Lemma 2.3:

$$\bar{\partial}_\Lambda K_{Q_0}(Q_0) = \frac{(a - b)\left(K_{Q_0}(b_+) - K_{Q_0}(b_+)\right) + (a + b)\left(K_{Q_0}(w_-) - K_{Q_0}(w_+)\right)}{8i\mathrm{area}(F_{Q_0})}$$

$$= \frac{\pi}{2\mathrm{area}(F_{Q_0})}.$$

By a similar argument as in the proof of Proposition 4.2, we can choose the same contours of integration for all incident vertices v of a face $Q \neq Q_0$. Then, the discrete derivative $\bar{\partial}_\Lambda$ commutes with the integration, and $\bar{\partial}_\Lambda K_{Q_0}(Q) = 0$ because $\mathrm{e}(\lambda, \cdot; Q_0)$ is discrete holomorphic by Proposition 4.1. □

Theorem 4.6. *Assume that there are $\alpha_0, q_0 > 0$ such that $\alpha \geq \alpha_0$ and $e/e' \geq q_0$ for all interior angles α and the two side lengths e, e' of any parallelogram of Λ. Let $Q_0 \in V(\diamond)$ be fixed.*

(i) The discrete Cauchy's kernel K_{Q_0} constructed in Proposition 4.5 has the following asymptotics:

$$K_{Q_0}(v) = \frac{1}{v - Q_0} + \frac{\tau(v, Q_0)}{J(v, Q_0)} + O\left(|v - z_0|^{-3}\right).$$

(ii) There is no further discrete Cauchy's kernel with respect to Q_0 that has asymptotics $o(|v - Q_0|^{-1/2})$.

(iii) For K_{Q_0} constructed in Proposition 4.5, $\partial_\Lambda K_{Q_0}$ has asymptotics

$$\partial_\Lambda K_{Q_0}(Q) = -\frac{1}{(Q - Q_0)^2} - \frac{\tau(Q, Q_0)}{J(Q, Q_0)^2} + O\left(|Q - Q_0|^{-3}\right).$$

(iv) Up to two additive constants on Γ and Γ^*, there is no further discrete Cauchy's kernel with respect to Q_0 such that its discrete derivative has asymptotics $o(|Q - Q_0|^{-1/2})$.

The proof of the first part follows the ideas of Kenyon [Ken02b]. Similar to the proof of Theorem 4.3, we deform the path of integration into one different from $(-(v - Q_0)\infty, 0]$ that was used by Kenyon. For the same reasons as before, his approach does not generalize to parallelogram-graphs.

The second and the fourth part of the theorem are immediate consequences of Theorem 2.31; the third part is shown in a similar way as Theorem 4.4 (ii).

Proof. (i) The poles $d_1, d_2, e_1, \ldots, e_{k(v)}$ of $\mathrm{e}(\cdot, v; Q_0)$ correspond to the directed edges of a shortest path from Q_0 to v, starting with two half-edges $d_1/2, d_2/2$ from the center of the parallelogram Q_0 to an incident vertex. By Corollary A.3 (ii), there is a real θ_0 such that the arguments of all poles can be chosen in such a way that they lie in $[\theta_0, \theta_0 + \pi - \alpha_0]$. As in the proof of Theorem 4.3, the claim is invariant under multiplication of the complex plane by $\exp(i\alpha)$. Indeed, $\mathrm{e}(\lambda, v; Q_0)d\lambda$ scales with $\exp(-i\alpha)$, as $1/(v-Q_0)$ and $\tau(v, Q_0)/J(v, Q_0)$ do. Thus, we can restrict to $\theta_0 = -(\pi - \alpha_0)/2$. It follows that there are no poles in the left half-plane, such that

$$K_{Q_0}(v) = 2 \int\limits_{-(v-Q_0)\infty}^{0} \mathrm{e}(\lambda, v; Q_0)d\lambda = 2 \int\limits_{-\infty}^{0} \mathrm{e}(\lambda, v; Q_0)d\lambda.$$

By Proposition A.4, there are $E_1 > E_0 > 0$ such that $E_1 \geq e \geq E_0$ for all lengths e of edges of Λ. For $|v - Q_0| \geq 1$ large enough, we split the integration into the three parts along $(\infty, -E_1\sqrt{|v - Q_0|}]$, $[-E_1\sqrt{|v - Q_0|}, -E_1/\sqrt{|v - Q_0|}]$, and $[-E_1/\sqrt{|v - Q_0|}, 0]$.

By the same arguments as in the proof of Theorem 4.3, the contribution of the intermediate range $\lambda \in [-E_1\sqrt{|v - Q_0|}, -E_1/\sqrt{|v - Q_0|}]$ decays faster to zero than any power of $Q - v_0$. The proof of Theorem 4.3 also showed that

$$k(v) = O(v - Q_0) \text{ and } J(v, Q_0) = \Omega(v - Q_0).$$

For $\lambda \to 0$, we use the series expansions $\log(-(\lambda + e)/(\lambda - e)) = 2\lambda/e + O(\lambda^3)$ and $\log(-d/(\lambda - d)) = \lambda/d + O(\lambda^2)$ to get

$$\mathrm{e}(\lambda, v; Q_0) = \tau(v, Q_0) \exp\left(2\lambda J(v, Q_0)\right) \left(1 + O(\lambda^2) + O(|v - Q_0|\lambda^3)\right).$$

Thus, the integral near the origin behaves as

$$2 \int\limits_{-E_1/\sqrt{|v-Q_0|}}^{0} \mathrm{e}(\lambda, v; Q_0)d\lambda = \frac{\tau(v, Q_0)}{J(v, Q_0)} + O(|v - Q_0|^{-3}).$$

$\log((\lambda + e)/(\lambda - e)) = 2e/\lambda + O(\lambda^{-3})$ and $\log(\lambda/(\lambda - d)) = d/\lambda + O(\lambda^{-2})$ for $\lambda \to -\infty$ yield

$$\mathrm{e}(\lambda, v; Q_0) = \lambda^{-2} \exp\left(2(v - Q_0)/\lambda\right) \left(1 + O(\lambda^{-2}) + O(|v - Q_0|\lambda^{-3})\right).$$

Using the result of above,

$$2 \int\limits_{-\infty}^{-E_1\sqrt{|v-Q_0|}} \mathrm{e}(\lambda, v; Q_0)d\lambda$$

$$
\begin{aligned}
&=2\int_{-\infty}^{-E_1\sqrt{|v-Q_0|}} \lambda^{-2}\exp\left(2(v-Q_0)/\lambda\right)\left(1+O(\lambda^{-2})+O(|v-Q_0|\lambda^{-3})\right)d\lambda\\
&=2\int_{-1/(E_1\sqrt{|v-Q_0|})}^{0} \exp\left(2\lambda(v-Q_0)\right)\left(1+O(\lambda^{2})+O(|v-Q_0|\lambda^{3})\right)d\lambda\\
&=\frac{1}{v-Q_0}+O(|v-Q_0|^{-3}).
\end{aligned}
$$

Summing the contributions of the three ranges up gives the desired result.

(ii) The difference f of two discrete Cauchy's kernels with respect to Q_0 that have asymptotics $o(|v-Q_0|^{-1/2})$ is discrete harmonic and fulfills the conditions of Theorem 2.31. Hence, f is essentially constant, so $f\equiv 0$.

(iii) During the proof, we use that edge lengths are bounded by Proposition A.4.

In the notation of Figure 2.1, let us introduce $a:=w_+-b_-$ and $b:=w_--b_-$. Using $1/(1+x)=1-x+O(x^2)$, we get up to terms of order $O(|Q-Q_0|^{-3})$:

$$
\begin{aligned}
K_{Q_0}(b_\pm)&\sim\frac{1}{Q-Q_0}\mp\frac{a+b}{2(Q-Q_0)^2}+\frac{\tau(b_-,Q_0)}{J(Q,Q_0)}\mp\tau(b_-,Q_0)\frac{a^{-1}+b^{-1}}{2J(Q,Q_0)^2},\\
K_{Q_0}(w_\pm)&\sim\frac{1}{Q-Q_0}\mp\frac{a-b}{2(Q-Q_0)^2}+\frac{\tau(w_-,Q_0)}{J(Q,Q_0)}\mp\tau(w_-,Q_0)\frac{a^{-1}-b^{-1}}{2J(Q,Q_0)^2}.
\end{aligned}
$$

By definition, $\tau(b_-,Q_0)=-\tau(w_-,Q_0)$. Therefore,

$$
\begin{aligned}
\partial_\Lambda K_{Q_0}(Q)=&-\lambda_Q\frac{\frac{a+b}{(Q-Q_0)^2}-\tau(b_-,Q_0)\frac{a^{-1}+b^{-1}}{J(Q,Q_0)^2}}{a+b}+\bar{\lambda}_Q\frac{\frac{a-b}{(Q-Q_0)^2}-\tau(b_-,Q_0)\frac{a^{-1}-b^{-1}}{J(Q,Q_0)^2}}{a-b}\\
&+O(|Q-Q_0|^{-3})\\
=&-\frac{1}{(Q-Q_0)^2}-\frac{\tau(b_-,Q_0)}{abJ(Q,Q_0)^2}+O(|Q-Q_0|^{-3})\\
=&-\frac{1}{(Q-Q_0)^2}-\frac{\tau(Q,Q_0)}{J(Q,Q_0)^2}+O(|Q-Q_0|^{-3}).
\end{aligned}
$$

(iv) Let f be the difference of two discrete Cauchy's kernels with respect to Q_0 whose discrete derivatives have asymptotics $o(|Q-v_0|^{-1/2})$. Then, f is discrete holomorphic, and

$$\frac{f(b_+)-f(b_-)}{b_+-b_-}=\frac{f(w_+)-f(w_-)}{w_+-w_-}=\partial_\Lambda f(Q)=o(|Q-v_0|^{-1/2}).$$

Since edge lengths are bounded by Proposition A.4, the conditions of Theorem 2.31 are fulfilled. Hence, f is essentially constant. $\square$

Remark. In [CS11], Chelkak and Smirnov reintroduced Kenyon's construction of the discrete Cauchy's kernel on $V(\diamondsuit)$. On rhombic quad-graphs, their asymptotics coincide with ours in Theorem 4.6; disregarding the error term $O(|v-Q_0|^{-2})$ that was sufficient for their work. In the original paper of Kenyon, the computation was already performed up to terms of order $O(|v-Q_0|^{-3})$.

Note that Kenyon, and Chelkak and Smirnov proved that in the rhombic setting, there is a unique discrete Cauchy's kernel with asymptotics $o(1)$ [Ken02b, CS11].

4.4 Integer lattice

Let us consider a locally finite planar parallelogram-graph Λ such that each vertex has degree four. This happens if and only if $\diamondsuit$ is also a quad-graph, or if Λ has the combinatorics of the integer lattice $\mathbb{Z}^2$.

As usual, let Γ and Γ^* denote the subgraphs on black and white vertices, and let us place the vertices of $\diamondsuit$ at the centers of corresponding faces. Then, the vertices of $\diamondsuit$ lie at the midpoints of diagonals. Since any vertex of Γ or Γ^* is enclosed by a quadrilateral of Γ^* or Γ, respectively, the faces of $\diamondsuit$ are parallelograms by Varignon's theorem. Thus, $\diamondsuit$ becomes a planar parallelogram-graph as well.

Of particular interest is the case that the two notions of discrete holomorphicity on $\diamondsuit$, the one coming from $\diamondsuit$ being the dual of Λ, and the other coming from the quad-graph $\diamondsuit$ itself, coincide. We will first derive a condition when this is the case, and then investigate it.

Let $v \in V(\Lambda)$, and e_1, e_2, e_3, e_4 be the directed edges of Λ emanating in v, arranged in counterclockwise order. Let Q_1, Q_2, Q_3, Q_4 be the faces incident to v such that Q_j is incident to the edges e_j, e_{j+1} (indices of edges are considered modulo 4), see Figure 4.1. Finally, let $h : V(\diamondsuit) \to \mathbb{C}$.

Regarding h as a function on the vertices of the dual of Λ, h is discrete holomorphic at v if and only if

$$\sum_{j=1}^{4} (e_{j+1} - e_j) h(Q_j) = 0.$$

Regarding h as a function on the vertices of the quad-graph $\diamondsuit$, Lemma 2.3 shows

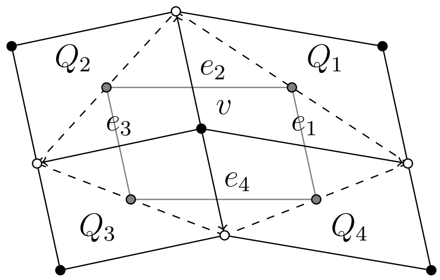


Figure 4.1: Dual quad-graph

that h is discrete holomorphic at v if and only if

$$\sum_{j=1}^{4}(e_{j+2}+e_{j+1}-e_j-e_{j-1})h(Q_j)=0.$$

Both notions are equivalent if and only if

$$(*) \quad e_1+e_3=e_2+e_4.$$

Given a vertex $v \in V(\Lambda)$, two directed edges e_1, e_2 emanating from v that belong to the same quadrilateral, and a complex number c such that the other two edges emanating from v are $c-e_1$ and $c-e_2$, the system of equations $(*)$ on $V(\Lambda)$ determines all edges of Λ, because Λ is combinatorially equivalent to $\mathbb{Z}^2$. Figure 4.2 shows the corresponding solution when all edges are oriented to the right or to the top in the combinatorial lattice $\mathbb{Z}^2$.

We get a parallelogram graph embedded in $\mathbb{C}$ if and only if for all integers n, the vectors e_1, e_2+nc, and $c-e_1$ appear in counterclockwise order. This can only be the case if c is a real multiple of e_1. But then, all e_1+nc, $n \in \mathbb{Z}$, have to be oriented the same way, which can only happen if $c=0$. The case of $c=0$ corresponds to the integer lattice of a skew coordinate system.

In the following, we restrict to Λ being the integer lattice of a skew coordinate system. If e_1, e_2 denote two spanning vectors, $\diamond$ is a parallel shift of Λ by $e_1/2+e_2/2$. Furthermore, the discrete differentials on $\diamond$ seen as the dual of Λ coincide with the discrete differentials on $\diamond$ seen as a parallelogram-graph.

Since corresponding notions coincide, and $\diamond$ and Λ are congruent, we can skip all subscripts Λ and $\diamond$ in the definitions of discrete derivatives. Moreover, the discrete Laplacian $\triangle$ is now defined for functions on $V(\Lambda)$ and functions on $V(\diamond)$

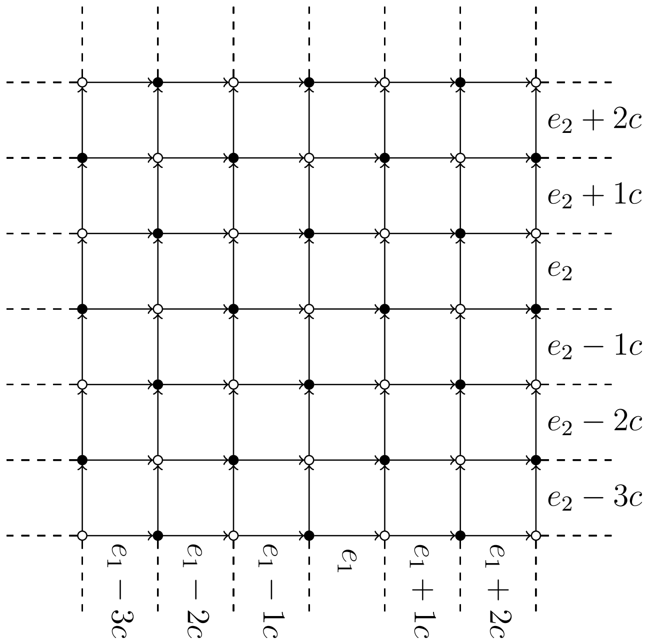


Figure 4.2: Equations ($*$) determine all edges

in the same way. Due to Corollary 2.20, $4\partial\bar{\partial} = \triangle = 4\bar{\partial}\partial$ is now true on both graphs. It follows that all discrete derivatives $\partial^n f$ of a discrete holomorphic function f are discrete holomorphic themselves. Conversely, a discrete primitive exists for any discrete holomorphic function on a simply-connected domain by Proposition 2.8.

Our main interest lies in giving discrete Cauchy's integral formulae for higher order derivatives of a discrete holomorphic function, and to determine the asymptotics of higher order discrete derivatives of the discrete Cauchy's kernel constructed in Section 4.3. Note that due to the uniqueness statements in Theorems 4.4 and 4.6, both constructions yield the same discrete Cauchy's kernel.

Without loss of generality, we restrict our attention to functions on $V(\Lambda)$. For the ease of notation, we introduce the *discrete distance* $D(\cdot,\cdot)$ on $V(\Lambda) \cup V(\Diamond)$ that is induced by the $|\cdot|_\infty$-distance on the integer lattice spanned by $e_1/2, e_2/2$.

Theorem 4.7. *Let $v_0 \in V(\Lambda)$, $Q_0 := v_0 + e_1/2 + e_2/2 \in V(\Diamond)$, and let f be a discrete holomorphic function on $V(\Lambda)$, and let K_{v_0} and K_{Q_0} be discrete Cauchy's kernels with respect to v_0 and Q_0, respectively. Let n be a nonnegative integer, and define $x_0 := v_0$ if n is even, and $x_0 := Q_0$ if n is odd. Similarly, let $x \in V(\Lambda)$*

if n is even, and $x \in V(\Diamond)$ if n is odd.

(i) For any counterclockwise oriented discrete contour C_{x_0} in the medial graph X enclosing all points $x' \in V(\Lambda) \cup V(\Diamond)$ with $D(x', v_0) \leq n/2$,

$$\partial^n f(x_0) = \frac{(-1)^n}{2\pi i} \oint\limits_{C_{x_0}} f \partial^n K_{x_0} dz.$$

(ii) If K_{Q_0} is the discrete Cauchy's kernel constructed in Proposition 4.5,

$$\frac{(-1)^n}{n!} \partial^n K_{Q_0}(x) = \frac{1}{(x-Q_0)^{n+1}} + \frac{\tau'(x,Q_0)}{(J(x,Q_0)e_1e_2)^{n+1}} + O(|x-Q_0|^{-n-3}),$$

where $\tau'(x,Q_0) = 1$ if x and Q_0 or $(x+e_1/2+e_2/2)$ and Q_0 can be connected by a path on $V(\Diamond)$ of even length, and $\tau'(x,Q_0) = -1$ otherwise.

Proof. (i) Let D be the discrete domain in X bounded by C_{x_0}. By the assumptions on C_{x_0}, the discrete one-form $\bar{\partial}\partial^{n-1}K_{x_0}d\bar{z}$ vanishes on C_{x_0}. Thus,

$$\oint\limits_{C_{x_0}} f \partial^n K_{x_0} dz = \oint\limits_{C_{x_0}} f d\left(\partial^{n-1} K_{x_0}\right) = \iint\limits_D d(fd(\partial^{n-1}K_{x_0})) = \iint\limits_D df \wedge d\left(\partial^{n-1}K_{x_0}\right)$$

by discrete Stokes' Theorem 2.9, Theorem 2.16, and Proposition 2.10. Now, f is discrete holomorphic, so $df \wedge d\left(\partial^{n-1}K_{x_0}\right) = \partial f \bar{\partial}\partial^{n-1}K_{x_0} dz \wedge d\bar{z}$. But since the discrete derivatives commute according to Corollary 2.11, $\bar{\partial}\partial^{n-1}K_{x_0} = \partial^{n-1}\bar{\partial}K_{x_0}$ vanishes outside C_{x_0}, so by subsequent use of Proposition 2.7,

$$\oint\limits_{C_{x_0}} f\partial^n K_{x_0} dz = -2i\langle \partial f, \bar{\partial}^{n-1}\partial \bar{K}_{x_0}\rangle = 2i(-1)^n \langle \partial^n f, \partial \bar{K}_{x_0}\rangle = 2\pi i (-1)^n \partial^n f(x_0).$$

(ii) Let us define the discrete exponential $\mathrm{e}(\cdot,\cdot;Q_0) : \mathbb{C} \times V(\Diamond) \to \mathbb{C}$ in the same way as $\mathrm{e}(\cdot,\cdot;v_0) : \mathbb{C} \times V(\Lambda) \to \mathbb{C}$. By inductive use of Proposition 4.1 and $\exp(\lambda,\cdot;Q_0) = \mathrm{e}(2/\lambda,\cdot;Q_0)$, we get for the discrete exponential $\mathrm{e}(\lambda,\cdot;Q_0)$ that is defined on $V(\Lambda)$:

$$(\partial^n \mathrm{e}(\lambda,\cdot;Q_0))(x) = \frac{(2\lambda)^n}{((\lambda-e_1)(\lambda-e_2)(\lambda+e_1)(\lambda+e_2))^{\lceil n/2 \rceil}} \mathrm{e}(\lambda,x;Q_0).$$

To compute $\partial^n K_{Q_0}(x)$, we need the values of K_{Q_0} only at vertices $v \in V(\Lambda)$ with $D(v,x) \leq n/2$. For these points,

$$K_{Q_0}(v) = 2\int_{-(v-Q_0)\infty}^{0} \mathrm{e}(\lambda,v;Q_0)d\lambda$$

by the construction in Proposition 4.5. If $D(x, Q_0)$ is sufficiently large, $\mathrm{e}(\cdot, v; Q_0)$ has at most three poles $\pm e_1, \pm e_2$, and by the same argument as in the proof of Theorem 4.6 (i), there are no poles in the convex hull of the rays $(-(v-Q_0)\infty, 0]$ if $D(x, Q_0)$ is large enough. By residue theorem, we can use the same path of integration for all relevant values of K_{Q_0}, and get

$$\partial^n K_{Q_0}(x) = 2 \int_{-(x-Q_0)\infty}^{0} (\partial^n \mathrm{e}(\lambda, \cdot; Q_0))(x) d\lambda.$$

Now, we follow the proof of Theorem 4.6 (i). Again, the claim is invariant under rotation of the complex plane, so we may assume $(x - Q_0) > 0$.

Let $E_1 := \max\{|e_1|, |e_2|\}$, and $E_0 := \min\{|e_1|, |e_2|\}$. For $|x - Q_0| \geq 1$ large enough, we split the integration into the three parts along $(\infty, -E_1\sqrt{|x - Q_0|}]$, $[-E_1\sqrt{|x - Q_0|}, -E_1/\sqrt{|x - Q_0|}]$, and $[-E_1/\sqrt{|x - Q_0|}, 0]$.

Essentially in the same way as in the proof of Theorem 4.3 it follows that the contribution of the intermediate range $\lambda \in [-E_1\sqrt{|x - Q_0|}, -E_1/\sqrt{|x - Q_0|}]$ decays faster to zero than any power of $Q - v_0$. In addition, we know

$$D(x, Q_0) = O(x - Q_0) \text{ and } J(x, Q_0) = \Omega(x - Q_0).$$

For $\lambda \to 0$, the series expansions $\log(-(\lambda + e)/(\lambda - e)) = 2\lambda/e + O(\lambda^3)$ and $\log(-e/(\lambda - e)) = \lambda/e + O(\lambda^2)$ show that $(\partial^n \mathrm{e}(\lambda, \cdot; Q_0))(x)$ equals

$$\frac{\tau'(x, Q_0)}{(e_1 e_2)^{n+1}} (2\lambda)^n \exp\left(2\lambda J(x, Q_0)\right) \left(1 + O(\lambda^2) + O(|x - Q_0|\lambda^3)\right).$$

Hence, the integral near the origin behaves as

$$\frac{(-1)^n n! \tau'(x, Q_0)}{(e_1 e_2 J(x, Q_0))^{n+1}} + O(|x - Q_0|^{-n-3}).$$

$\log((\lambda + e)/(\lambda - e)) = 2e/\lambda + O(\lambda^{-3})$ and $\log(\lambda/(\lambda - e)) = e/\lambda + O(\lambda^{-2})$ for $\lambda \to -\infty$ show that $(\partial^n \mathrm{e}(\lambda, \cdot; Q_0))(x)$ equals

$$(2\lambda)^n \lambda^{-2n-2} \exp\left(\frac{2(x - Q_0)}{\lambda}\right) \left(1 + O(\lambda^{-2}) + O(|x - Q_0|\lambda^{-3})\right).$$

Using the result of above,

$$2 \int_{-\infty}^{-E_1\sqrt{|x-Q_0|}} (\partial^n \mathrm{e}(\lambda, \cdot; Q_0))(x) d\lambda$$

$$
\begin{aligned}
=&2\int_{-\infty}^{-E_1\sqrt{|x-Q_0|}} \left(\frac{2}{\lambda}\right)^n \lambda^{-2}\exp\left(\frac{2(x-Q_0)}{\lambda}\right)\left(1+O(\lambda^{-2})+O(|x-Q_0|\lambda^{-3})\right)d\lambda\\
=&2\int_{-1/(E_1\sqrt{|x-Q_0|})}^{0} (2\lambda)^n \exp\left(2\lambda(x-Q_0)\right)\left(1+O(\lambda^{2})+O(|x-Q_0|\lambda^{3})\right)d\lambda\\
=&\frac{(-1)^n n!}{(x-Q_0)^{n+1}}+O(|x-Q_0|^{-n-3}).
\end{aligned}
$$

Summing up the contributions of the three ranges gives

$$\frac{(-1)^n}{n!}\partial^n K_{Q_0}(x)=\frac{1}{(x-Q_0)^{n+1}}+\frac{\tau'(x,Q_0)}{(J(x,Q_0)e_1e_2)^{n+1}}+O(|x-Q_0|^{-n-3}).$$

$\square$

Chapter 5

Variational principles of real (discrete) Laplace-type integrable equations

This chapter deals with the variational structure of certain discrete Laplace-type equations, namely these corresponding to discrete integrable quad-equations on a strongly regular and bipartite quadrilateral decomposition of an oriented surface. We start with the variational description of such equations in Section 5.1, and provide a nonintegrable generalization. Also, we explain which reality conditions on the field variables yield real generalized discrete action functionals.

The generalized discrete action functionals are studied in Section 5.2. We are particularly interested in conditions on the parameters of the quad-equations under which the corresponding generalized discrete action functional is strictly convex (or strictly concave). Strict convexity implies the uniqueness of solutions to Dirichlet boundary value problems, and helps to investigate their existence in Section 5.3. For the generalized discrete action functionals corresponding to any integrable quad-equation except (Q4), we give necessary and sufficient conditions for strict convexity; in the case of (Q4), we find at least sufficient conditions. These results are summarized in Section 5.2.1.

Having identified situations under which the generalized discrete action functional is strictly convex, we investigate in Section 5.4 when discrete integrable equations may lead to convex variational principles. We discuss the influence of combinatorial data, but our presentation is far from being complete.

Finally, we present a geometric interpretation of the solutions to the (generalized) discrete Laplace-type equations corresponding to (Q3) as Euclidean and hyperbolic circle patterns with conical singularities in Section 5.5. Note that the convexity of the functionals plays a crucial role in the variational description of circle patterns by Bobenko and Springborn in [BS04].

5.1 Discrete Laplace-type integrable equations

In Section 5.1.1, we first introduce basic definitions and notations we will use in the sequel, and comment on the discrete integrable quad-equations we are investigating. How these equations induce discrete Laplace-type equations is the topic of Section 5.1.2, as well as their variational description. In the end, we derive the reality conditions discussed later in Section 5.2, and explain why they are most likely the only reasonable conditions to that methods of real variational calculus can be applied.

5.1.1 Discrete integrable quad-equations

Similar to our study of discrete Riemann surfaces in Chapter 3, we consider a strongly regular and locally finite quadrilateral decomposition of an oriented surface described by a bipartite quad-graph Λ. But this time, we allow the surface to have a boundary. However, our main interest lies in planar quad-graphs, the same structure that was used for the discrete complex analysis discussed in Chapter 2.

To be able to apply variational principles later, we will need that the generalized discrete action functional is finite. For this reason, we restrict to finite quad-graphs Λ in this chapter. Again, let a 2-coloring of $V(\Lambda)$ be given, such that no two *black* and no two *white* vertices are incident to each other, and denote by Γ and Γ^* the corresponding graphs of black and white diagonals of faces of Λ. We assume that both Γ and Γ^* are connected, and that each graph contains at least two vertices.

On the set $E(\Lambda)$ of edges of Λ, we will investigate labelings with complex numbers such that opposite edges of a quadrilateral receive the same label. These labels will be the parameters of the quad-equations described below. For short, only such labelings are actually said to be *labelings* of $E(\Lambda)$. In the notation of

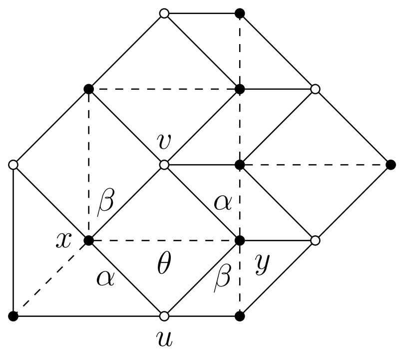


Figure 5.1: Bipartite quad-graph with black graph (dashed lines)

Figure 5.1, any such labeling induces a labeling of $E(\Gamma)$ by $\theta := \alpha - \beta$ that we say to be an *induced labeling.* Conversely, not any labeling of $E(\Gamma)$ is induced by some labeling of $E(\Lambda)$.

Definition. A labeling of $E(\Gamma)$ is said to be *integrable*, if it is induced by some labeling of $E(\Lambda)$.

Given a quad-graph Λ with a labeling of edges, we consider a collection of equations on each face of Λ of the type

$$Q(x, u, y, v; \alpha, \beta) = 0, \tag{5.1}$$

where α and β are associated to the edges of the quadrilateral as in Figure 5.1, and x, u, y, v are complex variables assigned to the four vertices. Note that x has the meaning of a vertex as well as of the field variable associated to it, but the meaning will be clear from the context.

Here, Q shall be affine linear in each variable, and Equation (5.1) is assumed to be invariant under the symmetry group of the square. Integrability of this quad-equation is identified by its three-dimensional consistency, see Figure 5.2. This means that the equation can be consistently imposed on the six faces of the cube, i.e., values x, x_1, x_2, x_3 uniquely define x_{123}. By assumption, x, x_1, x_2, x_3 uniquely determine x_{12}, x_{23}, x_{13}, but the three faces incident to x_{123} yield a priori three different values for x_{123}. Three-dimensional consistency assures that these three values agree.

In [ABS03], Adler, Bobenko, and Suris classified all these integrable equations under one additional assumption, called the tetrahedron property. This property

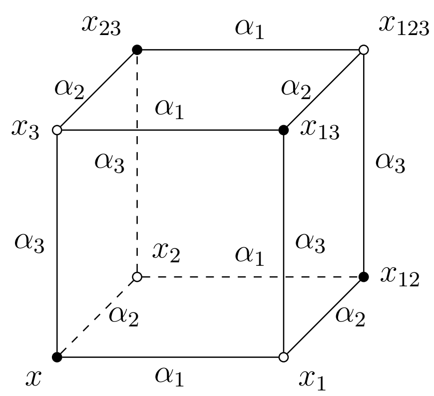


Figure 5.2: Three-dimensional consistency

demands that x_{123} depends on x_1, x_2, x_3, but not on x. They have showed that up to Möbius transformations of the field variables, any such equation is equivalent to one of nine canonical equations. These nine equations are listed in the so-called ABS list. The ABS list consists of a list Q containing four equations, a list H of three, and a list A of two equations. As a result of their subsequent paper [ABS09], the tetrahedron property can be replaced by certain nondegeneracy assumptions, just giving the list Q.

In our setting, we assume that Equation (5.1) is an equation of the ABS list. It turns out that for the investigation of discrete Laplace-type equations, one can restrict to the nondegenerate equations of the list Q.

Remark. Let us shortly comment how our definition of integrability is motivated, and how it is related to other uses. First, the way how a labeling of $E(\Lambda)$ induces a labeling of $E(\Gamma)$ is kind of a discrete derivative. In this sense, integrable labelings of $E(\Gamma)$ can be integrated to labelings of $E(\Lambda)$. But we are mainly interested in the connection to discrete integrable quad-equations. In the light of discrete Laplace-type equations and generalized discrete action functionals that we introduce in Section 5.1.2, any labeling of $E(\Gamma)$ defines a discrete Laplace-type equation, but only an integrable labeling gives a discrete Laplace-type equation that actually comes from a discrete integrable quad-equation.

Note that integrability of Q, i.e., three-dimensional consistency, is related to admitting a discrete zero curvature representation, as was discussed by Bobenko and Suris in [BS02], and by Nijhoff in [Nij02]. In the case of circle patterns in Section 5.5, our definition of integrability was denoted by *integrability of the* *corresponding cross-ratio system* in Bobenko's and Suris' book [BS08]. On the other hand, their definition of integrable circle patterns corresponds to isoradial realizations of them.

5.1.2 Discrete Laplace-type integrable equations and their variational structure

Adler, Bobenko, and Suris showed that each equation in the ABS list possesses a three-leg form [ABS03], which is additive for $(Q1)_{\delta=0}$, (H1), and $(A1)_{\delta=0}$, and multiplicative for the other equations. Here, an *additive three-leg form* of Equation (5.1) (centered at x) is an equivalent equation

$$i\psi(x,u;\alpha) - i\psi(x,v;\beta) = i\varphi(x,y;\alpha-\beta) = i\varphi(x,y;\theta)$$

for some functions ψ and φ called (additive) *three-leg functions*. If Equation (5.1) is equivalent to

$$\frac{\Psi(x,u;\alpha)}{\Psi(x,v;\beta)} = \Phi(x,y;\alpha,\beta) = \Phi(x,y;\theta)$$

for some functions Ψ and Φ, Equation (5.1) is said to possess a *multiplicative three-leg form*, and Ψ and Φ are called (multiplicative) *three-leg functions*. Clearly, $i\psi = \log\Psi$, $i\varphi = \log\Phi$ induce an additive three-leg form modulo 2π.

Definition. The functions φ and Φ appearing in the additive or multiplicative three-leg form, respectively, are called *long leg functions*; ψ and Ψ are called *short leg functions*.

Remark. Note that in most formulations of the three-leg forms, the factor of i does not enter. The reason why we add it here is that we want to get real functionals in Section 5.2. For the reality conditions discussed in Section 5.1.3, $i\varphi$ turns out to be (generally) a real function, leading to a real generalized discrete action functional later. Not taking the factor i into account would necessitate considering the imaginary part of the functionals.

For the discrete Laplace-type equations we define below, only the long leg functions are important. Since each equation from the lists A and H shares its long leg function with some equation from the list Q, we may restrict our attention to the latter. Note that $\psi = \varphi$ and $\Psi = \Phi$ for all equations from the list Q. In Appendix B, we provide a list of discrete integrable quad-equations of the list

Q together with their three-leg functions. There, we slightly adapt the formulation of the quad-equations to our setting, but mainly follow the presentation of Bobenko and Suris in the appendix of their paper [BS10b]. For the quad-equation $(\mathrm{Q3})_{\delta=0}$, we even give two three-leg functions, leading to equivalent three-leg forms. The reason for that is that under one reality condition, we see a remarkable analogy to Euclidean circle patterns if we not take the three-leg function as it comes up in the work [BS10b] of Bobenko and Suris.

In the original paper [ABS03] of Adler, Bobenko, and Suris, the notation of equation (Q4) used Weierstraß elliptic functions. In the end of Section 5.2.7, we will use this formulation to investigate rhombic period lattices. But the Jacobian formulation discovered by Hietarinta in [Hie05] seems to be more appropriate for rectangular period lattices, and will be used in this chapter. In particular, the limit of (Q4) to $(\mathrm{Q3})_{\delta=1}$ discussed in Section 5.2.8 becomes clearer.

The restriction of any solution of the system of discrete integrable equations on the quad-graph Λ to the black graph Γ satisfies for each inner black vertex x the *additive discrete Laplace equation*

$$\sum_{e=(x,y_k)\in E(\Gamma)} \varphi(x, y_k; \theta_e) = 0 \tag{5.2}$$

in the case of $(\mathrm{Q1})_{\delta=0}$, and the *multiplicative discrete Laplace equation*

$$\prod_{e=(x,y_k)\in E(\Gamma)} \Phi(x, y_k; \theta_e) = 1 \tag{5.3}$$

in the case of the other Q-equations. Here, θ_e denotes the label induced on the black edge e.

Equation (5.3) corresponds to the equivalent form

$$\sum_{e=(x,y_k)\in E(\Gamma)} \varphi(x, y_k; \theta_e) \equiv 0 \mod 2\pi, \tag{5.4}$$

where $i\varphi = i\varphi(x, y; \theta)$ is some branch of $\log \Phi$. Now, Equation (5.4) is satisfied if and only if there exists a real number $\Theta_x \equiv 0 \mod 2\pi$ such that

$$\sum_{e=(x,y_k)\in E(\Gamma)} \varphi(x, y_k; \theta_e) = \Theta_x \tag{5.5}$$

holds. The additive discrete Laplace Equation (5.2) is included in Equation (5.4) by taking $\Theta_x = 0$.

Even in the case that x is not an inner black vertex of Λ, we have a corresponding discrete Laplace equation (5.5), but the restriction $\Theta_x \equiv 0 \mod 2\pi$ does not have to be satisfied, and Θ_x could be nonreal. However, the reality conditions in Section 5.1.3 imply $\operatorname{Im}\varphi \equiv 0$, so $\Theta_x \in \mathbb{R}$.

Definition. For a general labeling of $E(\Gamma)$ and a general choice of Θ, Equation (5.5) is said to be a *discrete Laplace-type equation.* If the labeling is integrable, and $\Theta_x \equiv 0 \mod 2\pi$ (in the case of $(\mathrm{Q1})_{\delta=0}$, $\Theta_x = 0$) for all inner vertices $x \in V(\Gamma)$, Equation (5.5) is said to be a *discrete Laplace-type integrable equation.*

We name the discrete Laplace-type equations that are not induced by discrete integrable system still by their corresponding integrable cases (Q1), (Q2), (Q3), and (Q4). These generalized equations seem to be important. For example, the generalized discrete action functional of (Q3) describes general circle patterns with conical singularities, as we will see in Section 5.5. However, the integrable case is nevertheless important, and is investigated in Section 5.4.

According to Adler, Bobenko, and Suris [ABS03], there exists for any equation from the ABS list a change of variables $x = f(X)$, $u = f(U)$, $y = f(Y)$, $v = f(V)$ such that φ possesses a symmetric primitive $L = L(X, Y; \theta)$ in the new variables, i.e.,

$$L(X, Y; \theta) = L(Y, X; \theta), \tag{5.6}$$

$$\frac{\partial}{\partial X} L(X, Y; \theta) = \varphi(x, y; \theta). \tag{5.7}$$

Since any two branches of the logarithm differ by a multiple of $2\pi i$ only, Λ exists for any choice $i\varphi$ of the branch of $\log \Phi$. For convenience, we will use the notation $\varphi(X, Y; \theta)$ instead of $\varphi(x, y; \theta) = \varphi(f(X), f(Y); \theta)$, and consider φ as a function of X, Y, θ in the following.

Using the symmetry of L, the discrete Laplace-type Equations (5.5) for black vertices x are the discrete Euler-Lagrange equations for the *generalized discrete action functional*

$$S = \sum_{e=(x,y) \in E(\Gamma)} L(X, Y; \theta_e) - \sum_{x \in V(\Gamma)} \Theta_x X. \tag{5.8}$$

Thus, critical points of (5.8) correspond exactly to solutions of (5.5). Note that other choices of φ fulfilling $\exp(i\varphi) = \Phi$ lead to analogous results, but Θ may change. Hence, we can always restrict to some branch of $\log \Phi$ of our choice.

Remark. The presentations of Adler, Bobenko, and Suris in [ABS03] and of the last two authors in [BS10b], did not include the extra terms Θ. But these terms become important in our consideration, when we will restrict to real variables and extend φ smoothly. Since the manifold of solutions of Equation (5.1) is not connected in real space, φ can never be part of a global additive three-leg form unless we consider $(Q1)_{\delta=0}$. In contrast, the manifold of solutions of Equation (5.1) is connected in projective complex space $(\mathbb{CP}^1)^4$, and an additive three-leg form exists globally.

5.1.3 Reality conditions allowing variational principles

In this section, we will derive *reality conditions*, i.e., restrictions of the variables X, Y and the labels on edges of Λ to certain one-dimensional connected submanifolds of $\mathbb{C}$, under which the generalized discrete action functional takes values on an one-dimensional affine real subspace of $\mathbb{C}$. Only in such cases we can ask for convexity of S, and can apply the methods of real variational calculus. Furthermore, we will explain why we believe that there are no further reasonable reality conditions than the ones considered in Section 5.2.

If S takes its values on a line R in $\mathbb{C}$, $L(\cdot,\cdot;\theta)$ and thus its derivative $\varphi(\cdot,\cdot;\theta)$ take their values on lines parallel to R. Since the functions φ (or $\varphi = -i\log\Phi$) that are given in Appendix B are smooth functions with a discrete set of singularities, we can locally invert φ with respect to θ. So for fixed generic X and Y, θ lies on some one-dimensional submanifold of $\mathbb{C}$, at least locally. Using that $\theta = \alpha - \beta$ for an induced labeling, the set of all $\alpha - \beta$ is locally a one-dimensional submanifold. But then, the domains of α and β have to be lines parallel to each other. It is natural to demand that all labels have the same domain of definition, implying that the induced labels θ lie on a line through the origin.

In the cases of (Q1) and $(Q3)_{\delta=0}$, the three-leg functions depend just on $X - Y$ and θ. Applying the same reasoning as above, X and Y lie on lines parallel to each other.

To determine possible reality conditions for the discrete Laplace-type equation corresponding to $(Q1)_{\delta=0}$ is now quite simple. Since $X-Y$ lies on a line, $1/(X-Y)$ describes a circle unless $X-Y$ passes through the origin. But φ has to lie on a line, so all reasonable reality conditions are $X, Y \in \mathbb{R}\omega + c$, $\theta \in \mathbb{R}\omega'$ for some complex numbers $\omega, \omega' \neq 0$ and c. By rotating all three-leg functions appropriately, we may assume $\omega = \omega' = 1$. Furthermore, a variable transform $X \mapsto X - c$ for all X does not change φ. Thus, it suffices to consider $X, Y, \theta \in \mathbb{R}$ in Section 5.2.2.

For the other quad-equations of the list Q, we have to investigate the multiplicative three-leg functions Φ. Since $\Phi = \exp(i\varphi)$ and the values of φ lie on a line, the possible values of Φ lie either on the real line, a circle centered at the origin, or a spiral.

Let us first discuss the case that all variables X, Y lie on lines parallel to the one $i\theta$ lies. If we are in one of the two cases of (Q3), the condition that the values of φ lie on a line imply that either θ or $i\theta$ lies on the real line. For both $(\text{Q1})_{\delta=1}$ and $(\text{Q3})_{\delta=0}$, the same condition implies that the line of $X - Y$ passes through the origin. For (Q2) and $(\text{Q3})_{\delta=1}$, the same is true for the line of $X + Y$. Thus, we can assume that all $X, Y, i\theta$ lie on the same line.

In light of the fact that Equation (5.1) and generic three field variables on the vertices of a quadrilateral of Λ determine the fourth field variable uniquely, it is natural to demand that for a generic integrable labeling of $E(\Lambda)$, generic values of X, and generic values of all but one Y_k, $e_k = (x, y_k)$ an edge of Γ, the discrete Laplace Equation (5.5) determines the remaining Y_s uniquely. This is not fulfilled in the cases of (Q2) and $(\text{Q3})_{\delta=1}$ if $X, Y, i\theta$ lie on the same line, using that in this setting, $\Phi(X, \cdot; \theta)$ generally cannot attain a certain range of positive values. The same is true for $(\text{Q3})_{\delta=0}$ if $X, Y, i\theta \in \mathbb{R}$.

For the remaining two cases of (Q1) and $(\text{Q3})_{\delta=0}$ if $X, Y, i\theta \in \mathbb{R}i$, we observe that φ has a singularity if $X - Y = i\theta$. But to apply methods of real variational calculus, we need to restrict the variables to a domain where φ can be defined smoothly. Depending on the labels θ, $X - Y$ might take values only on a rather small interval, eventually contradicting the above condition that the discrete Laplace Equation (5.5) determines the remaining value Y_s uniquely in the generic case.

Therefore, $X, Y, i\theta$ do not all lie on parallel lines. Now, if both $X - Y$ and $X + Y$ avoid ε-balls of $i\theta + 2k\pi i$, $k \in \mathbb{Z}$, Φ is not only smooth for all X, Y, but also bounded in the cases of $(\text{Q1})_{\delta=1}$, (Q2), and (Q3). Thus, the values of Φ neither form a spiral, nor the real line in this case.

Let us fix X. If the values of Φ lie on a spiral or the real line, $X - Y$ or $X + Y$ approaches the set of $i\theta + 2k\pi i$, $k \in \mathbb{Z}$, for any θ. As above, we want to avoid singularities of φ, i.e., that $X - Y$ or $X + Y$ attains the set of $i\theta + 2k\pi i$, $k \in \mathbb{Z}$, for too many θs. In particular, there are θs such that no $i\theta + 2k\pi i$, $k \in \mathbb{Z}$, is attained. Noting that φ is locally invertible, $X - Y$ or $X + Y$ actually approaches some fixed $i\theta + 2k\pi i$ for $Y \to \infty$ or $Y \to -\infty$. Changing θ a tiny bit then gives a contradiction. As a consequence, the values of Φ indeed lie on a circle centered at the origin.

An explicit calculation for $(\mathrm{Q1})_{\delta=1}$ shows that this happens only if $X - Y$ lies on the same line as θ. Hence, all reality conditions for $(\mathrm{Q1})_{\delta=1}$ are given by the same as for $(\mathrm{Q1})_{\delta=0}$. By appropriate scaling and shifting, we can again restrict to $X, Y, \theta \in \mathbb{R}$.

For $(\mathrm{Q3})_{\delta=0}$, we get that θ is always real or purely imaginary, and $X \in \mathbb{R} + k\pi i$, $Y \in \mathbb{R} + k'\pi i$ in the first and $X, Y \in \mathbb{R}$ in the second case, for some integers k, k'. Again, there are just three essentially different reality conditions, namely $X, \theta \in \mathbb{R}$ and either Y or $Y + \pi i$ is real, or $X, Y, \theta \in \mathbb{R}i$.

Using that the multiplicative three-leg function of (Q2) or $(\mathrm{Q3})_{\delta=1}$ is (up to a factor of $\exp(i\theta)$ in the second case) just the one of $(\mathrm{Q1})_{\delta=1}$ or $(\mathrm{Q3})_{\delta=0}$, respectively, multiplied with the corresponding function if $X - Y$ is replaced by $X + Y$, the same reality conditions that we obtained for $(\mathrm{Q1})_{\delta=1}$ are applicable to (Q2); and for $(\mathrm{Q3})_{\delta=1}$, we can take the same as for $(\mathrm{Q3})_{\delta=0}$.

Let us now exclude further reality conditions for (Q2) and $(\mathrm{Q3})_{\delta=1}$. For this, we fix some $\theta \neq 0$, such that both $X - Y$ and $X + Y$ avoid $i\theta$. The existence of such a θ follows from our consideration above. Again by an explicit calculation using that the image of φ lies on a line parallel to the imaginary axis, θ is either real or purely imaginary in the case of $(\mathrm{Q3})_{\delta=1}$.

Now, it is not hard to see that the radius of the circle on which the values of Φ lie has to be one. Then, we can invert $\varphi(\cdot, \cdot; \theta)$ locally, and get a two-dimensional submanifold of $\mathbb{C}^2$ on which (X, Y) lies. But we already know that the image of φ of the planes (X, Y) coming from the reality conditions of $(\mathrm{Q1})_{\delta=1}$ and $(\mathrm{Q3})_{\delta=0}$ lie on the imaginary line. In particular, these planes are the corresponding preimages, and there are no further reality conditions.

In the case of (Q4), the situation is more complicated due to the theta functions appearing in the three-leg function. But we believe that for rectangular lattices with a half-period ratio of ti/π, $t \geq 0$, a similar reasoning as above can show that in addition to the reality conditions for (Q3), the only remaining reality condition is $X, \theta \in \mathbb{R}i$ and $Y + t$ is purely imaginary.

5.2 Generalized discrete action functionals

This section is devoted to the study of the generalized discrete action functionals defined in (5.8) of discrete Laplace-type equations as in (5.5). We stress again that in general, Equation (5.5) does not have to be induced by Equations (5.1) on the faces of Λ, although the three-leg functions come from integrable quad-equations. In particular, the labeling of $E(\Gamma)$ does not need to be integrable, and the extra terms $\Theta \in \mathbb{R}$ might be arbitrary. But if we consider a discrete Laplace-type integrable equation, the labeling of $E(\Gamma)$ is integrable, and $\Theta_x \in 2\pi\mathbb{Z}$ for inner black vertices x.

Having a quick look at the three-leg functions given in Appendix B, we see that $\theta = 0$ implies $\varphi \equiv 0$ in the case of $(\mathrm{Q1})_{\delta=0}$, and $\Phi \equiv 1$ for the other quad-equations; and $\theta \equiv 0 \mod \pi$ implies $\Phi \equiv 1$ for (Q3) and (Q4). In regard to the discrete Laplace-type equation, we may just delete edges e with $\theta_e = 0$ or $\theta_e \equiv 0 \mod \pi$, respectively. Then, Γ might split into several components, and each component can be treated separately. For simplicity, let us assume that already for all edges $e \in E(\Gamma)$, $\theta_e \neq 0$ or $\theta_e \not\equiv 0 \mod \pi$, depending on the discrete integrable quad-equation. By the same reason, we exclude $\theta \in \mathbb{Z}\pi\tau$ in the case of (Q4), where τ is the half-period ratio of the Jacobi theta functions.

To obtain a real functional S, we restrict the variables and labels to the reality conditions discussed in Section 5.1.3. Now, we can ask for convexity or concavity of S, factoring out the invariance under the global variable transformation $X \mapsto X + h$ in the cases of (Q1) and $(\mathrm{Q3})_{\delta=0}$. In Section 5.2.1, we summarize the main theorem of our subsequent consideration. We are able to give necessary and sufficient conditions on labels θ for strict convexity or concavity if the three-leg functions correspond to one of the integrable quad-equations (Q1), (Q2), and (Q3). These conditions will demand that θ or $i\theta$ is either always positive or always negative, or $\exp(i\theta)$ lies always in one of the semicircles $S^1_\pm := \{z \in \mathbb{C} | \, |z| = 1, \pm \operatorname{Im}(z) > 0\}$. In the case of a discrete Laplace-type equation corresponding to (Q4), we find at least sufficient conditions for strict convexity or concavity.

The actual proof of Theorem 5.1 is given in Sections 5.2.2 to 5.2.7. In Sections 5.2.5 and 5.2.6, we will see a remarkable analogy between the generalized discrete action functionals of (Q3) and the Euclidean and hyperbolic circle pattern functionals of Bobenko and Springborn in [BS04]. Finally, we show in Section 5.2.8 how the functionals of $(\mathrm{Q3})_{\delta=1}$ can be obtained as certain limits of the ones of (Q4).

5.2.1 Main theorem

We summarize the main result that will be proven in the following paragraphs. In the case of discrete Laplace-type equations corresponding to (Q4), we restrict to rectangular lattices, corresponding to a purely imaginary half-period ratio $\tau = ti/\pi$; the case of rhombic lattices ($|\tau| = 1$) will be discussed separately in Section 5.2.7.

Theorem 5.1. *Let Λ be a finite connected strongly regular bipartite quad-graph embedded in an oriented surface, and let Γ denote the black graph corresponding to a 2-coloring of $V(\Lambda)$. Moreover, let θ_e be a labeling of the edges $e \in E(\Gamma)$.*

For any discrete Laplace-type equation corresponding to a discrete integrable quad-equation of the list Q, we consider reality conditions under which the corresponding generalized discrete action functional is real. To be more precise, we restrict the variables X and labels θ to either the real or the imaginary line, assuming that θ is never zero (and $\theta \not\equiv 0 \mod \pi$ in the cases of (Q3) and (Q4)). Then, S is strictly concave on $\mathbb{R}^{|V(\mathcal{G}_W)|}$ or $i\mathbb{R}^{|V(\mathcal{G}_W)|}$, or on the subspace U or iU defined by

$$U = \left\{ \{X\}_{x \in V(\Gamma)} \subset \mathbb{R}^{|V(\Gamma)|} \mid \sum_{x \in V(\Gamma)} X = 0 \right\}, \tag{5.9}$$

if one of the following restrictions of the labels θ_e, $e \in E(\Gamma)$, is satisfied:

quad-equation	***functional***	***space***	***concavity condition***
$(\mathrm{Q1})_{\delta=0}$	*(5.10)*	U	$\theta_e \in \mathbb{R}^+$
$(\mathrm{Q1})_{\delta=1}$	*(5.11)*	U	$\theta_e \in \mathbb{R}^+$
(Q2)	*(5.13)*	$\mathbb{R}^{\|V(\Gamma)\|}$	$\theta_e \in \mathbb{R}^+$
$(\mathrm{Q3})_{\delta=0}$	*(5.16)*	U	$\exp(i\theta_e) \in S^1_+$
$(\mathrm{Q3})_{\delta=0}$	*(5.20)*	iU	$\theta_e \in i\mathbb{R}^+$
$(\mathrm{Q3})_{\delta=1}$	*(5.23)*	$\mathbb{R}^{\|V(\Gamma)\|}$	$\exp(i\theta_e) \in S^1_+$
$(\mathrm{Q3})_{\delta=1}$	*(5.26)*	$i\mathbb{R}^{\|V(\Gamma)\|}$	$\theta_e \in i\mathbb{R}^+$
(Q4)	*(5.29)*	$\mathbb{R}^{\|V(\Gamma)\|}$	$\exp(i\theta_e) \in S^1_+$
(Q4)	*(5.33)*	$i\mathbb{R}^{\|V(\Gamma)\|}$	$\theta_e \in i(0, r(t))$.

Here, $r(t)$ is a positive number depending on t. In the cases of (Q1), (Q2), *and* (Q3), *the conditions on θ for concavity are even necessary conditions.*

Replacing θ_e by $-\theta_e$ yields the same result with concavity replaced by convexity.

Note that the restriction to the subspace U is possible if and only if

$$\sum_{x\in V(\Gamma)} \Theta_x = 0$$

in the case of $(\mathrm{Q1})_{\delta=0}$ and the second reality condition for $(\mathrm{Q3})_{\delta=0}$,

$$\sum_{x\in V(\Gamma)} \Theta_x = 2\pi \sum_{e\in E(\Gamma)} \operatorname{sgn}(\theta_e)$$

in the case of $(\mathrm{Q1})_{\delta=1}$, and

$$\sum_{x\in V(\Gamma)} \Theta_x = \sum_{e\in E(\Gamma)} 2\theta_e^*$$

in the case of the first reality condition for $(\mathrm{Q3})_{\delta=0}$, where $\theta_e^* := \pi - \theta_e$, is satisfied.

Remark. For discrete Laplace-type equations corresponding to (Q3) and (Q4), there is another a priori different reality condition, that turns out to give almost the same generalized discrete action functional. We partition the vertex set of $V(\Gamma)$ into any two subsets V_1 and V_2. Then, $X \in \mathbb{R}$ if $x \in V_1$, $X \in \mathbb{R} - \pi i$ if $x \in V_2$, and $\theta \in \mathbb{R}$ is also a reality condition under which the generalized discrete action functional is real. But the functional in this case is essentially the same as the corresponding Functional (5.16), (5.23), or (5.29) given in Theorem 5.1: One just replaces $L(X, Y; \theta)$ by $L(X', Y'; \theta)$ if both x and y belong to the same vertex set, and $L(X, Y; \theta)$ by $L(X', Y'; \theta')$ if x and y belong to different vertex sets. Here, $\theta' := \theta + \pi$, $X' := X$ if $x \in V_1$, and $X' := X + \pi i$ if $x \in V_2$.

Then, the corresponding concavity conditions are given by $\exp(i\theta_e) \in S^1_+$ if the vertices x, y of e belong to the same vertex set, and $\exp(i\theta_e) \in S^1_-$ otherwise.

In the case of (Q4), a similar reasoning applies for the reality condition $X \in \mathbb{R}i$ if $x \in V_1$, $X \in \mathbb{R}i - t$ if $x \in V_2$, and $\theta \in \mathbb{R}i$. Then, $\theta' := \theta + it$, and the concavity condition transforms to $\theta_e \in (-t, -t + r(t))i$ for those edges e whose vertices lie in different vertex sets.

5.2.2 $(Q1)_{\delta=0}$

We have no change of variables, i.e., $X = x$, $Y = y$, etc. We will investigate the reality condition $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\Gamma)$, assuming that θ_e is never zero. In Appendix B, the three-leg function φ is given by

$$\varphi(X, Y; \theta) = \frac{\theta}{X - Y},$$

which is smooth as long $X \neq Y$. The generalized discrete action functional is given by

$$S = \sum_{e=(x,y)\in E(\Gamma)} \theta_e \log |X - Y| - \sum_{x\in V(\Gamma)} \Theta_x X. \tag{5.10}$$

Here, log is any fixed branch of the logarithm. S is smooth as long $X \neq Y$ for all edges $e = (x, y)$; otherwise, S is not defined. The Hessian of S can be easily calculated:

$$D^2 S = \sum_{e=(x,y)\in E(\Gamma)} \frac{-\theta_e}{(X - Y)^2}(dX - dY)^2.$$

Now, we can expect strict convexity or concavity only on the subspace U as defined in (5.9). But to be able to restrict to U, S has to be invariant under the global variable transformation $X \mapsto X + h$, $h \in \mathbb{R}$. This is the case if and only if

$$\sum_{x\in V(\Gamma)} \Theta_x = 0$$

is satisfied. Clearly, strict concavity on U is obtained if $\theta_e \in \mathbb{R}^+$ for all $e \in E(\Gamma)$, and strict convexity if $\theta_e \in \mathbb{R}^-$ for all $e \in E(\Gamma)$.

Conversely, let us assume that S is concave (or convex). For an edge e with vertices x and y, let us consider S as a function of X and Y. The other Z, $z \in V(\Gamma)\setminus \{x, y\}$, are just parameters. If $Z \to \pm\infty$ for all other vertices z,

$$D^2 S(X, Y) \to \frac{-\theta_e}{(X - Y)^2}(dX - dY)^2.$$

In particular, S is concave (or convex) only if $\theta_e > 0$ (or $\theta_e < 0$).

5.2.3 $(Q1)_{\delta=1}$

As before, we have no change of variables, and we investigate the reality condition $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\Gamma)$, $\theta_e \neq 0$. According to Appendix B, the three-leg function Φ is given by

$$\Phi(X,Y;\theta) = \frac{X-Y+i\theta}{X-Y-i\theta}.$$

Let us choose

$$\varphi(X,Y;\theta) = -i\log(X-Y+i\theta) + i\log(X-Y-i\theta),$$

where we consider log to be the principle branch of the logarithm. One can easily check that

$$\begin{aligned} L(X,Y;\theta) = &(iX-iY+\theta)\log(X-Y-i\theta) \\ &-(iX-iY-\theta)\log(X-Y+i\theta) + 2\pi Y\operatorname{sgn}(\theta) \end{aligned} \tag{5.11}$$

satisfies the Conditions (5.6) and (5.7). Then, the generalized discrete action functional S is just given by Formula (5.8), and S is smooth everywhere. An elementary calculation shows that its Hessian is

$$D^2 S = \sum_{e=(x,y)\in E(\Gamma)} \frac{-2\theta_e}{(X-Y)^2+\theta_e^2}(dX-dY)^2.$$

Again, we can only expect strict concavity or convexity if we restrict to the subspace U as defined in (5.9). With the same arguments as in Section 5.2.2, S is strictly concave if and only if $\theta_e \in \mathbb{R}^+$ for all $e \in E(\Gamma)$, and strictly convex if and only if $\theta_e \in \mathbb{R}^-$ for all $e \in E(\Gamma)$.

To be able to restrict to the subspace U, the functional S has to be invariant under the global variable transformation $X \mapsto X+h$, $h \in \mathbb{R}$, such that

$$\sum_{x\in V(\Gamma)} \Theta_x = 2\pi \sum_{e\in E(\Gamma)} \operatorname{sgn}(\theta_e). \tag{5.12}$$

Note that

$$\begin{aligned} 0 < \varphi(X,Y;\theta) &< 2\pi \text{ if } \theta > 0, \\ 0 > \varphi(X,Y;\theta) &> -2\pi \text{ if } \theta < 0. \end{aligned}$$

So according to the discrete Laplace-type Equation (5.5),

$$\begin{aligned} \forall x \in V(\Gamma) : 0 < \Theta_x &< 2\pi \deg_\Gamma(x) \text{ if } \theta_e > 0 \ \forall e \in E(\Gamma), \\ \forall x \in V(\Gamma) : 0 > \Theta_x &> -2\pi \deg_\Gamma(x) \text{ if } \theta_e < 0 \ \forall e \in E(\Gamma) \end{aligned}$$

is necessary to obtain solutions. Here, $\deg_\Gamma(x)$ denotes the *degree* of x in Γ, i.e., the number of adjacent black vertices.

5.2.4 (Q2)

Now, we change the variables by $x = X^2$, $y = Y^2$, etc. The reality condition is given by $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\Gamma)$, $\theta_e \neq 0$. In the case of (Q2), the three-leg function Φ is given by

$$\Phi(X, Y; \theta) = \frac{(X - Y + i\theta)(X + Y + i\theta)}{(X - Y - i\theta)(X + Y - i\theta)},$$

see Appendix B. We choose

$$i\varphi(X, Y; \theta) = \log(X - Y + i\theta) - \log(X - Y - i\theta) + \log(X + Y + i\theta) - \log(X + Y - i\theta),$$

where log is the principle branch of the logarithm. Then, a symmetric primitive of φ, i.e., a function L satisfying Conditions (5.6) and (5.7), is given by

$$\begin{aligned} L(X, Y; \theta) = &(iX - iY + \theta) \log(X - Y - i\theta) - (iX - iY - \theta) \log(X - Y + i\theta) \\ &+ (iX + iY + \theta) \log(X + Y - i\theta) - (iX + iY - \theta) \log(X + Y + i\theta) \\ &+ 2\pi Y \operatorname{sgn}(\theta). \end{aligned} \tag{5.13}$$

Now, the generalized discrete action functional is given by Formula (5.8). Using the result of Section 5.2.3,

$$D^2 S = \sum_{e=(x,y)\in E(\Gamma)} \left(\frac{-2\theta_e}{(X - Y)^2 + \theta_e^2} (dX - dY)^2 + \frac{-2\theta_e}{(X + Y)^2 + \theta_e^2} (dX + dY)^2 \right).$$

Clearly, S is strictly concave (or strictly convex) on $\mathbb{R}^{|V(\Gamma)|}$ if $\theta_e \in \mathbb{R}^+$ (or $\theta_e \in \mathbb{R}^-$) for all $e \in E(\Gamma)$. With a similar reasoning as in Section 5.2.3, these conditions are even necessary for concavity (or convexity), and

$$\begin{aligned} &\forall x \in V(\Gamma) : 0 < \Theta_x < 4\pi \deg_\Gamma(x) \text{ if } \theta_e > 0\ \forall e \in E(\Gamma), \\ &\forall x \in V(\Gamma) : 0 > \Theta_x > -4\pi \deg_\Gamma(x) \text{ if } \theta_e < 0\ \forall e \in E(\Gamma) \end{aligned}$$

is necessary to obtain solutions of the discrete Laplace-type equation.

5.2.5 $(Q3)_{\delta=0}$

We consider the following change of variables: $x = \exp(X)$, $y = \exp(Y)$, etc. Let us start with the reality condition $X, Y, \theta_e \in \mathbb{R}$ for all $e = (x, y) \in E(\Gamma)$. In addition, we assume that $\theta_e \not\equiv 0 \mod \pi$. Since the quad-equation given in

Appendix B does not change if we add multiples of 2π to α or β, θ is defined only up to a multiple of 2π. So let us assume that all labels $\theta \in (0, 2\pi)$. A change of the interval might only influence the terms Θ in the discrete Laplace-type Equation 5.5, but not the following arguments.

The three-leg function Φ is given by

$$\Phi(X, Y; \theta) = \exp(-i\theta)\frac{\sinh(\frac{X-Y+i\theta}{2})}{\sinh(\frac{X-Y-i\theta}{2})}, \tag{5.14}$$

so let us choose

$$\varphi(X, Y; \theta) = -i \log\left(-\frac{\sinh(\frac{X-Y+i\theta}{2})}{\sinh(\frac{X-Y-i\theta}{2})}\right) + (\pi - \theta), \tag{5.15}$$

where log is the principle branch of the logarithm. φ is smooth everywhere.

To find a symmetric primitive of φ, let us introduce the dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u} du.$$

The dilogarithm is an analytic function on the complex plane cut along $[1, \infty)$. Following the presentation of Bobenko and Springborn in [BS04], we have

$$\begin{aligned}\operatorname{Im}\mathrm{Li}_2(\exp(X-Y+i\theta)) &= \frac{1}{2i}\mathrm{Li}_2(\exp(X-Y+i\theta)) - \frac{1}{2i}\mathrm{Li}_2(\exp(X-Y-i\theta)) \\ &= \frac{1}{2i}\int_{-\infty}^{X-Y} \log\left(\frac{1-\exp(v-i\theta)}{1-\exp(v+i\theta)}dv\right),\end{aligned}$$

substituting $u = \exp(v + i\theta)$ into the integral representation of the dilogarithm.

Observing that

$$\frac{\sinh(\frac{X-Y+i\theta}{2})}{\sinh(\frac{X-Y-i\theta}{2})} = \exp(-i\theta)\frac{1-\exp(X-Y+i\theta)}{1-\exp(X-Y-i\theta)} = \exp(i\theta)\frac{1-\exp(Y-X-i\theta)}{1-\exp(Y-X+i\theta)},$$

we derive the generalized discrete action functional

$$\begin{aligned}S = &\sum_{e=(x,y)\in E(\Gamma)} - \left(\operatorname{Im}\mathrm{Li}_2(\exp(X-Y+i\theta_e)) + \operatorname{Im}\mathrm{Li}_2(\exp(Y-X+i\theta_e))\right) \\ &+ \sum_{e=(x,y)\in E(\Gamma)} \theta_e^*(X+Y) - \sum_{x\in V(\Gamma)} \Theta_x X,\end{aligned} \tag{5.16}$$

where $\theta^* := \pi - \theta$. $-S$ coincides with the Euclidean circle pattern functional of Bobenko and Springborn in [BS04], describing a Euclidean circle pattern combinatorially equivalent to Γ with logarithmic radii X, interior intersection angles $\theta_e^* \in (0, \pi)$, and cone (or boundary) angles $\Theta_x > 0$. We will investigate this interpretation again in Section 5.5.

An elementary calculation shows the same result for the Hessian of S as in [BS04],

$$D^2 S = \sum_{e=(x,y)\in E(\Gamma)} \frac{-\sin(\theta_e)}{\cosh(X - Y) - \cos(\theta_e)} (dX - dY)^2. \tag{5.17}$$

Clearly, we can only expect strict concavity or convexity if we restrict the variables X to the subspace U as defined in (5.9). Then, S is strictly concave if and only if $\theta_e \in (0, \pi)$ for all $e \in E(\Gamma)$, and strictly convex if and only if $\theta_e \in (\pi, 2\pi)$ for all $e \in E(\Gamma)$. Since θ was a priori only defined up to a multiple of 2π, we can state more generally that S is strictly concave if and only if $\exp(i\theta_e) \in S^1_+$ for all $e \in E(\Gamma)$, and strictly convex if and only if $\exp(i\theta_e) \in S^1_-$ for all $e \in E(\Gamma)$.

To be able to restrict to the subspace U, the functional S has to be invariant under the global variable transformation $X \mapsto X + h$, requiring

$$\sum_{x\in V(\Gamma)} \Theta_x = \sum_{e\in E(\Gamma)} 2\theta_e^*. \tag{5.18}$$

Let $s \in (0, \pi)$. Then,

$$\varphi(X, Y; \pi + s) = -\varphi(X, Y; \pi - s). \tag{5.19}$$

In particular, the discrete Laplace-type Equations (5.5) with $\theta_e = \pi - s_e$ and $\tilde{\theta}_e = \pi + s_e$ are equivalent if one chooses $\tilde{\Theta}_x = -\Theta_x$. Thus, we may restrict our attention to the concavity condition $\exp(i\theta_e) \in S^1_+$ for all $e \in E(\Gamma)$.

In this case, $0 < \varphi(X, Y; \theta) < 2\pi$, so $0 < \Theta_x < 2\pi \deg_\Gamma(x)$ for all vertices x is necessary to obtain solutions. Therefore, the terms Θ_x can be indeed seen as cone (or boundary) angles of a Euclidean circle pattern, and Functional (5.16) can be identified with the Euclidean circle pattern functional. Also, Condition 5.18 appeared in Bobenko's and Springborn's paper [BS04], and it will come again in Section 5.3. There, we will discuss when maxima of S exist, and adapt the results obtained by Bobenko and Springborn to our setting.

Remark. Let us shortly discuss the reality condition described by $X \in \mathbb{R}$ if $x \in V_1$, $X \in \mathbb{R} - \pi i$ if $x \in V_2$, and $\theta \in \mathbb{R}\backslash\mathbb{Z}\pi$, where $V_1 \dot{\cup} V_2$ is some partition of

$V(\Gamma)$ into two vertex sets. Now, define $X' := X$ if $x \in V_1$, $X' := X + \pi i$ if $x \in V_2$, $\theta'_e := \theta_e$ if the vertices of the edge e lie in the same vertex set, and $\theta'_e := \theta_e + \pi$ if the vertices of e are in different vertex sets. Then,

$$\Phi(X,Y;\theta) = \exp(-i\theta')\frac{\sinh(\frac{X'-Y'+i\theta'}{2})}{\sinh(\frac{X'-Y'-i\theta'}{2})}.$$

Therefore, we can argue exactly as above with variables X' instead of X, and parameters θ' instead of θ. In particular, the generalized discrete action functional is strictly concave if and only if $\exp(i\theta_e) \in S^1_+$ if the vertices x, y of e belong to the same vertex set, and $\exp(i\theta_e) \in S^1_-$ otherwise.

Note that due to Equation (5.19), another way to represent the generalized discrete action functional under this modified reality condition is to replace $L(X,Y;\theta_e)$ by $L(X',Y';\theta_e)$ if the vertices of the edge e belong to the same vertex set, and $L(X,Y;\theta_e)$ by $-L(X',Y';\theta_e^*)$ otherwise.

Let us now come to the reality condition $X, Y, \theta \in \mathbb{R}i$, $\theta \neq 0$. We introduce new variables X' and labels θ' by $X = iX'$, $\theta = i\theta'$, and consider the variational formulation with respect to these new variables. Then, we get for the three-leg function

$$\Phi'(X,Y;\theta) = \frac{\sin(\frac{X'-Y'+i\theta'}{2})}{\sin(\frac{X'-Y'-i\theta'}{2})},$$

and we can choose

$$\varphi(X,Y;\theta) = -i\log\left(\frac{\sin(\frac{X'-Y'+i\theta'}{2})}{\sin(\frac{X'-Y'-i\theta'}{2})}\right).$$

Note that we do not take the three-leg function Φ, but Φ' instead. The reason for taking Φ rather than Φ' before was to identify the generalized discrete action functional with the Euclidean circle pattern functional. Taking Φ instead of Φ' under the actual reality condition would give an additional imaginary summand of $-i\theta'$ to φ that we want to avoid. Furthermore, we cannot take one branch of the logarithm to define φ globally. But it is easy to see that there is a smooth extension of φ to all $X', Y' \in \mathbb{R}$. We just have to fix one value, and we take $\varphi(X',X';\theta) = \pi$.

In a similar way as before, we construct a symmetric primitive of φ, but we have to take into account that there is not just one branch of the logarithm in the definition of φ. Substituting $u = \exp(w + \theta')$ into the integral representation of the dilogarithm yields

$$\begin{aligned}
-\operatorname{Li}_2(\exp(iX' - iY' + \theta')) &= \int_{-\infty}^{i(X'-Y')} \log(1 - \exp(w + \theta'))dw \\
&= \int_{-\infty}^{0} \log(1 - \exp(w + \theta'))dw \\
&+ i \int_{0}^{X'-Y'} \log(1 - \exp(iw + \theta'))dw.
\end{aligned}$$

For $\theta' < 0$, $\operatorname{Li}_2(\exp(iX' - iY' + \theta'))$ is just the ordinary dilogarithm, but for $\theta > 0$, we extend $\operatorname{Li}_2(\exp(iX' - iY' + \theta'))$ to a smooth function of $X', Y' \in \mathbb{R}$ by the integral representation above, choosing appropriate branches of the logarithm during integration. We demand that for $X' = Y'$, the extended dilogarithm coincides with the ordinary. Again,

$$\begin{aligned}
\frac{\sin(\frac{X'-Y'+i\theta'}{2})}{\sin(\frac{X'-Y'-i\theta'}{2})} &= \exp(\theta')\frac{1 - \exp(i(X' - Y') - \theta')}{1 - \exp(i(X' - Y') + \theta')} \\
&= \exp(-\theta')\frac{1 - \exp(i(Y' - X') + \theta')}{1 - \exp(i(Y' - X') - \theta')},
\end{aligned}$$

and we derive the generalized discrete action functional

$$\begin{aligned}
S &= \frac{1}{2} \sum_{e=(x,y)\in E(\Gamma)} (\operatorname{Li}_2(\exp(iX' - iY' - \theta'_e)) - \operatorname{Li}_2(\exp(iX' - iY' + \theta'_e))) \\
&+ \frac{1}{2} \sum_{e=(x,y)\in E(\Gamma)} (\operatorname{Li}_2(\exp(iY' - iX' - \theta'_e)) - \operatorname{Li}_2(\exp(iY' - iX' + \theta'_e))) \\
&- \sum_{x\in V(\Gamma)} \Theta_x X' \\
&= \frac{1}{2} \sum_{e=(x,y)\in E(\Gamma)} (\operatorname{Li}_2(\exp(X - Y + i\theta_e)) - \operatorname{Li}_2(\exp(X - Y - i\theta_e))) \\
&+ \frac{1}{2} \sum_{e=(x,y)\in E(\Gamma)} (\operatorname{Li}_2(\exp(Y - X + i\theta_e)) - \operatorname{Li}_2(\exp(Y - X - i\theta_e))) \\
&+ \sum_{x\in V(\Gamma)} \Theta_x iX.
\end{aligned} \tag{5.20}$$

Using the analogy of this functional with (5.20) before, we obtain

$$D^2 S = \sum_{e=(x,y)\in E(\Gamma)} \frac{\sinh(\theta'_e)}{\cos(X'-Y') - \cosh(\theta'_e)} (dX' - dY')^2.$$

Again, we can only expect strict concavity or convexity if we restrict the variables X' to the subspace U as defined in (5.9). But now, S is strictly concave if and only if $\theta'_e > 0$ for all $e \in E(\Gamma)$, and S is strictly convex if and only if $\theta'_e < 0$ for all $e \in E(\Gamma)$. Thus, $X \in iU$, and the conditions for concavity and convexity correspond to $\theta_e \in \mathbb{R}^+ i$ and $\theta_e \in \mathbb{R}^- i$, respectively. In contrast to the reality conditions considered above, the restriction to the subspace iU is now possible if and only if

$$\sum_{x\in V(\Gamma)} \Theta_x = 0,$$

which corresponds to the same scaling condition as for $(\mathrm{Q1})_{\delta=0}$.

5.2.6 $(\mathrm{Q3})_{\delta=1}$

This time, we choose $x = \cosh(X)$, $y = \cosh(Y)$, etc., as change of variables. Again, we start with the reality condition $X, Y, \theta_e \in \mathbb{R}$, and assume that $\theta_e \notin \mathbb{Z}\pi$ for all edges $e = (x, y) \in E(\Gamma)$. Since the formulation of the quad-equation $(\mathrm{Q3})_{\delta=1}$ does not change if a multiple of 2π is added to a label, we may choose $\theta_e \in (0, 2\pi)$ for all edges e.

Appendix B gives us the three-leg function

$$\Phi(X, Y; \theta) = \frac{\sinh(\frac{X-Y+i\theta}{2})\sinh(\frac{X+Y+i\theta}{2})}{\sinh(\frac{X-Y-i\theta}{2})\sinh(\frac{X+Y-i\theta}{2})}, \tag{5.21}$$

and we choose

$$i\varphi(X, Y; \theta) = \log\left(-\frac{\sinh(\frac{X-Y+i\theta}{2})}{\sinh(\frac{X-Y-i\theta}{2})}\right) + \log\left(-\frac{\sinh(\frac{X+Y+i\theta}{2})}{\sinh(\frac{X+Y-i\theta}{2})}\right), \tag{5.22}$$

where log is the principle branch of the logarithm. Essentially the same calculations as in Section 5.2.5 yield

$$\begin{aligned} S = &- \sum_{e=(x,y)\in E(\Gamma)} \left(\operatorname{Im}\operatorname{Li}_2(\exp(X - Y + i\theta_e)) + \operatorname{Im}\operatorname{Li}_2(\exp(Y - X + i\theta_e))\right) \\ &- \sum_{e=(x,y)\in E(\Gamma)} \left(\operatorname{Im}\operatorname{Li}_2(\exp(X + Y + i\theta_e)) + \operatorname{Im}\operatorname{Li}_2(\exp(-X - Y + i\theta_e))\right) \end{aligned}$$

$$- \sum_{x \in V(\Gamma)} \Theta_x X \tag{5.23}$$

as the generalized discrete action functional with Hessian

$$\begin{aligned} D^2 S = & \sum_{e=(x,y)\in E(\Gamma)} \frac{-\sin(\theta_e)}{\cosh(X-Y)-\cos(\theta_e)}(dX-dY)^2 \\ & + \sum_{e=(x,y)\in E(\Gamma)} \frac{-\sin(\theta_e)}{\cosh(X+Y)-\cos(\theta_e)}(dX+dY)^2. \end{aligned} \tag{5.24}$$

It follows that S is strictly concave if and only if $\exp(i\theta_e) \in S^1_+$ for all edges e, and strictly convex if and only if $\exp(i\theta_e) \in S^1_-$ for all $e \in E(\Gamma)$. Since $|\varphi(X, Y; \theta)| < 2\pi$ if $\theta \not\equiv 0 \mod 2\pi$, $|\Theta_x| < 2\pi \deg_\Gamma(x)$ is necessary to obtain solutions of the discrete Laplace-type Equation (5.5).

Now, $-S$ coincides with the hyperbolic circle pattern functional of Bobenko and Springborn in [BS04], describing a hyperbolic circle pattern combinatorially equivalent to Γ with radii $2\,\mathrm{artanh}(\exp(X))$, interior intersection angles $\theta_e^* \in (0, \pi)$, and cone (or boundary) angles $\Theta_x > 0$. We refer in Section 5.5 below for some more details.

If $s \in (0, \pi)$,

$$\varphi(X, Y; \pi + s) = -\varphi(X, Y; \pi - s), \tag{5.25}$$

and the discrete Laplace-type Equations (5.5) with $\theta_e = \pi - s_e$ and $\tilde{\theta}_e = \pi + s_e$ can be identified, changing $\tilde{\Theta}_x = -\Theta_x$. Hence, we can restrict to the concavity condition $\exp(i\theta_e) \in S^1_+$ for all $e \in E(\Gamma)$. Then, $\varphi(X, Y; \theta) > 0$ if $X, Y < 0$, so $\Theta_x > 0$ for all vertices x is necessary to obtain solutions with $X < 0$ for all $x \in V(\Gamma)$. Hence, the terms Θ_x can be again seen as cone (or boundary) angles of a circle pattern, and Functional (5.16) can be indeed identified with the hyperbolic circle pattern functional.

Remark. For the reality condition $X \in \mathbb{R}$ if $x \in V_1$, $X \in \mathbb{R} - \pi i$ if $x \in V_2$, and $\theta \in \mathbb{R}\backslash\mathbb{Z}\pi$, where $V_1 \dot\cup V_2$ is some partition of $V(\Gamma)$ into two vertex sets, we can argue exactly as in Section 5.2.5. For this, observe that Equation (5.25) is the analog to Equation (5.19).

Considering the reality condition $X, Y, \theta \in \mathbb{R}i$, $\theta \neq 0$, we introduce new variables X' and labels θ' by $X = iX'$, $\theta = i\theta'$, and consider the variational formulation with respect to these new variables. With the same reasoning as in the end of Section 5.2.5, we get the generalized discrete action functional

$$\begin{aligned}
S = &\frac{1}{2} \sum_{e=(x,y)\in E(\Gamma)} \left(\mathrm{Li}_2(\exp(X-Y+i\theta_e)) - \mathrm{Li}_2(\exp(X-Y-i\theta_e))\right) \\
&+ \frac{1}{2} \sum_{e=(x,y)\in E(\Gamma)} \left(\mathrm{Li}_2(\exp(Y-X+i\theta_e)) - \mathrm{Li}_2(\exp(Y-X-i\theta_e))\right) \\
&+ \frac{1}{2} \sum_{e=(x,y)\in E(\Gamma)} \left(\mathrm{Li}_2(\exp(X+Y+i\theta_e)) - \mathrm{Li}_2(\exp(X+Y-i\theta_e))\right) \\
&+ \frac{1}{2} \sum_{e=(x,y)\in E(\Gamma)} \left(\mathrm{Li}_2(\exp(-X-Y+i\theta_e)) - \mathrm{Li}_2(\exp(-X-Y-i\theta_e))\right) \\
&+ \sum_{x\in V(\Gamma)} \Theta_x iX
\end{aligned} \tag{5.26}$$

with Hessian

$$\begin{aligned}
D^2 S = &\sum_{e=(x,y)\in E(\Gamma)} \frac{\sinh(\theta'_e)}{\cos(X'-Y') - \cosh(\theta'_e)} (dX' - dY')^2 \\
&+ \sum_{e=(x,y)\in E(\Gamma)} \frac{\sinh(\theta'_e)}{\cos(X'+Y') - \cosh(\theta'_e)} (dX' + dY')^2.
\end{aligned}$$

Again, S is strictly concave if and only if $\theta_e \in \mathbb{R}^+ i$ for all edges e of Γ, and strictly convex if and only if $\theta_e \in \mathbb{R}^- i$ for all edges e.

5.2.7 (Q4)

In this section, we will use several facts of complex analysis and the theory of elliptic functions without going into much detail; most of them can be found in the book of Hurwitz [Hur00]. But in contrast to the notation in his book, we define the Jacobi theta functions $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 = \vartheta_0$ with half-period ratio τ by the following, where $h^k := \exp(i\pi\tau k)$ for $k \in \mathbb{R}$:

$$\begin{aligned}
\vartheta_1(v) &= 2\left(h^{\frac{1}{4}}\sin(v) - h^{\frac{9}{4}}\sin(3v) + h^{\frac{25}{4}}\sin(5v) \mp \ldots\right), \\
\vartheta_2(v) &= 2\left(h^{\frac{1}{4}}\cos(v) + h^{\frac{9}{4}}\cos(3v) + h^{\frac{25}{4}}\cos(5v) + \ldots\right), \\
\vartheta_3(v) &= 1 + 2\left(h\cos(2v) + h^4\cos(4v) + h^9\cos(6v) + \ldots\right), \\
\vartheta_4(v) &= 1 - 2\left(h\cos(2v) - h^4\cos(4v) + h^9\cos(6v) \mp \ldots\right).
\end{aligned}$$

A scaling of the argument by the factor π relates these two standard formulations of Jacobi theta functions.

Coming back to the discrete integrable quad-equation (Q4), we consider the following change of variables: $x = \mathrm{sn}(\pi/2 - iX)$, $y = \mathrm{sn}(\pi/2 - iY)$, etc., where sn is the Jacobi elliptic function sn with modulus $\kappa = \vartheta_2^2(0)\vartheta_3^{-2}(0)$.

In the following, we mainly consider purely imaginary τ, i.e., rectangular period lattices. In the end, we will comment on the situation of rhombic lattices.

First, we investigate the reality condition $X, Y, \theta_e \in \mathbb{R}$, assuming that for all edges $e = (x, y) \in E(\Gamma)$, $\theta_e \not\equiv 0 \mod \pi$. According to Appendix B, the three-leg function Φ is given by

$$\Phi(X,Y;\theta) = \frac{\vartheta_1\left(\frac{X-Y+i\theta}{2i}\right)\vartheta_4\left(\frac{X-Y+i\theta}{2i}\right)\vartheta_1\left(\frac{X+Y+i\theta}{2i}\right)\vartheta_4\left(\frac{X+Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{-+Y-i\theta}{2i}\right)\vartheta_4\left(\frac{X-Y-i\theta}{2i}\right)\vartheta_1\left(\frac{X+Y-i\theta}{2i}\right)\vartheta_4\left(\frac{X+Y-i\theta}{2i}\right)}. \tag{5.27}$$

Since Φ is 2π-periodic in the θ-variable, we may choose the labels $\theta \in (0, 2\pi)$.

Now, fix for a moment some θ. Let $j \in \{1, 4\}$, and let U be an open strip along the real line such that for all $u \in U$, $\mathrm{Im}(u \pm i\theta) \not\equiv 0 \mod 2\pi$. Then, the function

$$u \mapsto (-1)^j \frac{\vartheta_j\left(\frac{u+i\theta}{2i}\right)}{\vartheta_j\left(\frac{u-i\theta}{2i}\right)}$$

is holomorphic and has no zeroes in U. Moreover, it takes the value 1 if $u = 0$. Thus, there is a holomorphic $s_j(u,\theta) : U \to \mathbb{C}$ such that $s_j(0;\theta) = 0$ and

$$\exp(s_j(u,\theta)) = (-1)^j \frac{\vartheta_j\left(\frac{u+i\theta}{2i}\right)}{\vartheta_j\left(\frac{u-i\theta}{2i}\right)}.$$

We now define

$$\log\left((-1)^j \frac{\vartheta_j\left(\frac{u+i\theta}{2i}\right)}{\vartheta_j\left(\frac{u-i\theta}{2i}\right)}\right) := s_j(u;\theta),$$

which is a smooth and purely imaginary function for real u. Indeed, that $s_j(u;\theta)$ is purely imaginary follows from $\vartheta_j(\bar{v}) = \overline{\vartheta_j(v)}$ due to the conjugate symmetry of the trigonometric functions. This allows us to choose

$$\begin{aligned} i\varphi(X,Y;\theta) = &\log\left(-\frac{\vartheta_1\left(\frac{X-Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{X-Y-i\theta}{2i}\right)}\right) + \log\left(\frac{\vartheta_4\left(\frac{X-Y+i\theta}{2i}\right)}{\vartheta_4\left(\frac{X-Y-i\theta}{2i}\right)}\right) \\ &+ \log\left(-\frac{\vartheta_1\left(\frac{X+Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{X+Y-i\theta}{2i}\right)}\right) + \log\left(\frac{\vartheta_4\left(\frac{X+Y+i\theta}{2i}\right)}{\vartheta_4\left(\frac{X+Y-i\theta}{2i}\right)}\right). \end{aligned} \tag{5.28}$$

For $j \in \{1,4\}$, we define the smooth function $I_j(\cdot;\theta) : \mathbb{R} \to \mathbb{R}$ by

$$I_j(u;\theta) = -i \int_0^u \log\left((-1)^j \frac{\vartheta_j(\frac{u+i\theta}{2i})}{\vartheta_j(\frac{u-i\theta}{2i})} \right) du,$$

where we integrate along the real line. Clearly, $I(0;\theta) = 0$, and I is even since $(-1)^j\vartheta_j$ is an even function. Therefore,

$$L(X,Y;\theta) = I_1(X-Y;\theta) + I_4(X-Y;\theta) + I_1(X+Y;\theta) + I_4(X+Y;\theta) \quad (5.29)$$

satisfies Conditions (5.6) and (5.7), and the generalized discrete action functional is given by Formula (5.8).

Let $\omega := \pi i$ and $\omega' := \tau\omega = -t < 0$. We introduce the meromorphic function

$$\tilde{\zeta}(u) := \frac{\pi}{2\omega}\frac{\vartheta_1'(v)}{\vartheta_1(v)} = \frac{1}{2i}\frac{d\log(\vartheta_1(v))}{dv}, \quad v = \frac{\pi u}{2\omega} = \frac{u}{2i}.$$

As ϑ_1, $\tilde{\zeta}$ is conjugate symmetric. Note that

$$\tilde{\zeta}(u) = \zeta(u) - \frac{\eta u}{\omega},$$

where $\eta := \zeta(\omega)$, and ζ is the Weierstraß zeta-function corresponding to the Weierstraß $\wp$-function with half-periods ω, ω'.

Using in addition that

$$\frac{\vartheta_4\left(\frac{X-Y+i\theta}{2i}\right)\vartheta_4\left(\frac{X+Y+i\theta}{2i}\right)}{\vartheta_4\left(\frac{X-Y-i\theta}{2i}\right)\vartheta_4\left(\frac{X+Y-i\theta}{2i}\right)} = \frac{\vartheta_1\left(\frac{X-Y+i\theta}{2i} - \frac{\pi\tau}{2}\right)\vartheta_1\left(\frac{X+Y+i\theta}{2i} + \frac{\pi\tau}{2}\right)}{\vartheta_1\left(\frac{X-Y-i\theta}{2i} - \frac{\pi\tau}{2}\right)\vartheta_1\left(\frac{X+Y-i\theta}{2i} + \frac{\pi\tau}{2}\right)},$$

we eventually obtain

$$\begin{aligned} D^2S = &-i \sum_{e=(x,y)\in E(\Gamma)} \left(\tilde{\zeta}(X-Y+i\theta_e) - \tilde{\zeta}(X-Y-i\theta_e)\right)(dX-dY)^2 \\ &-i \sum_{e=(x,y)\in E(\Gamma)} \left(\tilde{\zeta}(X-Y-t+i\theta_e) - \tilde{\zeta}(X-Y-t-i\theta_e)\right)(dX-dY)^2. \\ &-i \sum_{e=(x,y)\in E(\Gamma)} \left(\tilde{\zeta}(X+Y+i\theta_e) - \tilde{\zeta}(X+Y-i\theta_e)\right)(dX+dY)^2 \\ &-i \sum_{e=(x,y)\in E(\Gamma)} \left(\tilde{\zeta}(X+Y+t+i\theta_e) - \tilde{\zeta}(X+Y+t-i\theta_e)\right)(dX+dY)^2. \end{aligned} \quad (5.30)$$

As above, let $h^k := \exp(i\pi\tau k) = \exp(-tk)$ for any $k \in \mathbb{R}$. Moreover, define $z^{\pm 2} := \exp(\pm u)$. Then,

$$\begin{aligned}
\tilde{\zeta}(u) &= \frac{\pi}{2\omega}\cot\left(\frac{\pi u}{2\omega}\right) + \frac{i\pi}{\omega}\sum_{k=1}^{\infty}\left(\frac{h^{2k}z^{-2}}{1-h^{2k}z^{-2}} - \frac{h^{2k}z^{2}}{1-h^{2k}z^{2}}\right)\\
&= -\frac{i}{2}\cot\left(-\frac{iu}{2}\right) + \sum_{k=1}^{\infty}\frac{\exp(-2tk)\exp(-u)-\exp(-2tk)\exp(u)}{\exp(-4tk)-\exp(-2tk)(\exp(u)+\exp(-u))+1}\\
&= \frac{1}{2}\coth\left(\frac{u}{2}\right) + \sum_{k=1}^{\infty}\frac{\sinh(u)}{\cosh(u)-\cosh(2tk)}. \qquad (5.31)
\end{aligned}$$

Series (5.31) converges absolutely if the denominator never vanishes. In particular, the series converges for $u = Z \pm i\theta$ if $Z \in \mathbb{R}$ and $\theta \not\equiv 0 \mod \pi$. As a consequence, we obtain after a straightforward calculation:

$$\begin{aligned}
-\frac{\tilde{\zeta}(Z+i\theta)-\tilde{\zeta}(Z-i\theta)}{i\sin\theta} &= \frac{1}{\cosh(Z)-\cos(\theta)}+\\
2\sum_{k=1}^{\infty}&\frac{\cosh(Z)\cosh(2kt)-\cos(\theta)}{\sinh^2(Z)+\cosh^2(2kt)+\cos^2(\theta)-2\cosh(Z)\cosh(2kt)\cos(\theta)}. \qquad (5.32)
\end{aligned}$$

The right hand side is a positive real number if $\theta \not\equiv 0 \mod \pi$. For this, we observe that $\cos(\theta) < 1 \leq \cosh(Z) \leq \cosh(Z)\cosh(2kt)$, and use the estimation $(\cosh(Z)-\cosh(2kt))^2 \geq 0$ to obtain

$$\begin{aligned}
& \sinh^2(Z)+\cosh^2(2kt)+\cos^2(\theta)-2\cosh(Z)\cosh(2kt)\cos(\theta)\\
\geq\; & 2\cosh(Z)\cosh(2kt)(1-\cos(\theta))+\cos^2(\theta)-1\\
=\; & (1-\cos(\theta))(2\cosh(Z)\cosh(2kt)-\cos(\theta)-1)\\
>\; & 0.
\end{aligned}$$

It follows that S is strictly concave if $\exp(i\theta_e) \in S^1_+$ for all $e \in E(\Gamma)$, and strictly convex if $\exp(i\theta_e) \in S^1_-$ for all edges e.

In contrast to our consideration in the previous paragraphs, these conditions are not necessary for (strict) concavity or convexity, i.e., S might be strictly concave (or convex) also if there are labels θ_e on edges e satisfying $\exp(i\theta_e) \in S^1_+$ as well as labels $\theta_{e'}$ with $\exp(i\theta_{e'}) \in S^1_-$, even in the integrable case. For example, consider one black edge $e = (x, y)$, and assume that both x and y have exactly $4k+1$ other neighbors in Γ. In addition, let us suppose that x and y have no common neighbors. For simplicity, we now restrict to just this black graph. A corresponding quad-graph Λ can be easily constructed.

We label the edge e with $\theta_e = -\pi/2$, all other edges in the stars of the vertices x and y receive the label $\pi/2$. Since the labels around a vertex add up to $2k\pi$, the labeling is integrable. The actual reason why the conditions above are not necessary conditions for concavity or convexity is that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(u) := i\left(\tilde{\zeta}(u + \frac{\pi}{2}i) - \tilde{\zeta}(u - \frac{\pi}{2}i)\right),$$

is periodic and positive. Therefore, there exist positive real constants K_+, K_-, such that $K_+ \geq f(u) \geq K_-$ for all real u. Choosing the integer k such that $(4k+1)K_- > K_+$ yields a strictly concave action functional S.

Remark. Let us investigate the reality condition described by $X \in \mathbb{R}$ if $x \in V_1$, $X \in \mathbb{R} - \pi i$ if $x \in V_2$, and $\theta \in \mathbb{R}\backslash\mathbb{Z}\pi$, where $V_1 \dot{\cup} V_2$ is some partition of $V(\Gamma)$ into two vertex sets. Again, we define $X' := X$ if $x \in V_1$, $X' := X + \pi i$ if $x \in V_2$, $\theta'_e := \theta_e$ if the vertices of the edge e lie in the same vertex set, and $\theta'_e := \theta_e + \pi$ if the vertices of e are in different vertex sets. Then,

$$\Phi(X, Y; \theta) = \frac{\vartheta_1\left(\frac{X'-Y'+i\theta'}{2i}\right)\vartheta_4\left(\frac{X'-Y'+i\theta'}{2i}\right)\vartheta_1\left(\frac{X'+Y'+i\theta'}{2i}\right)\vartheta_4\left(\frac{X'+Y'+i\theta'}{2i}\right)}{\vartheta_1\left(\frac{X'-Y'-i\theta'}{2i}\right)\vartheta_4\left(\frac{X'-Y'-i\theta'}{2i}\right)\vartheta_1\left(\frac{X'+Y'-i\theta'}{2i}\right)\vartheta_4\left(\frac{X'+Y'-i\theta'}{2i}\right)}.$$

Therefore, we obtain the same generalized discrete action functional as above, if we replace X, Y, θ by X', Y', θ'.

We come now to the reality condition $X, Y, \theta \in \mathbb{R}i$, $\theta \notin \mathbb{Z}ti$, and introduce new variables X' and labels θ' by $X = iX'$, $\theta = i\theta'$. As before, we consider the variational formulation with respect to these new real variables.

Due to the $2ti$-periodicity of Φ in the θ-variable, we may restrict $\theta \in (-t, t)i$, still assuming that $\theta \neq 0$. Since the zeroes of ϑ_1 and ϑ_4 are given by $\mathbb{Z}\pi + \mathbb{Z}ti$ and $\mathbb{Z}\pi + \mathbb{Z}ti + ti/2$, respectively, there is an open strip U' containing the imaginary line, such that $(u \pm i\theta)/2i$ is not a zero of ϑ_1 or ϑ_4 for all $u \in U'$. Now, we can argue exactly as before, and get the generalized discrete action functional given by Formula (5.8) and

$$iL(X', Y'; \theta') = I_1(X-Y; \theta) + I_4(X-Y; \theta) + I_1(X+Y; \theta) + I_4(X+Y; \theta). \quad (5.33)$$

Here, we adapt the definitions of I_1 and I_4 appropriately. The factor of i is due to the different integration variable $X' = -iX$.

Now, the Hessian of S is essentially the same as the one in Formula (5.30), just ignore the factor $-i$ and replace $(dX \pm dY)^2$ by $(dX' \pm dY')^2$. By a similar argument as before, we cannot expect necessary and sufficient conditions for (strict) concavity or convexity, but at least sufficient ones. To find such conditions, we investigate the real-valued function

$$f(u;\theta) := \tilde{\zeta}(iu-\theta') - \tilde{\zeta}(iu+\theta') = 2\,\mathrm{Re}(\tilde{\zeta}(iu-\theta'))$$

for real u. If f is positive for all u and suitable θ', S is strictly convex, and if f is negative, S is strictly concave.

Since $\tilde{\zeta}$ is an odd function, it suffices to consider the case $\theta' > 0$. Using that $\tilde{\zeta}$ is 2ω-periodic by construction, $f(\cdot;\theta')$ is 2π-periodic. As a consequence, we can restrict to $u \in [0, 2\pi]$.

First, we want to show that for fixed u, there exists a positive real number $\rho(u)$, such that $f(u;\theta')$ has the same sign for all $\theta' \in (0, \rho(u))$. Indeed, if u is neither zero nor 2π, $f(u;\cdot)$ is holomorphic around zero, nonconstant, and has a zero at zero. Since zeroes of holomorphic functions lie discrete in $\mathbb{C}$, the existence for such a number ρ follows. If $u = 0$, $f(0;\cdot)$ has a pole at zero, but is holomorphic around zero, and the claim follows.

Noting that $f(\cdot;\cdot)$ is continuous in both variables (as long as we do not hit a pole), we can choose $\rho(u)$ depending continuously on u. Taking the minimum of $\rho(u)$ for all $u \in [0, 2\pi]$, we get a positive number $r = r(t)$ such that $f(u;\theta')$ does not change its sign for all real u and $\theta' \in (0, r(t))$. By investigating $f(0;\theta')$ one gets that $f(u;\theta') < 0$ in these cases. Therefore, S is strictly concave if $\theta' \in (0, r(t))$, i.e., if $\theta \in (0, r(t))i$.

Remark. We can deal with the reality condition $X \in \mathbb{R}i$ if $x \in V_1$, $X \in \mathbb{R}i - t$ if $x \in V_2$, and $\theta \in \mathbb{R}i \backslash \mathbb{Z}ti$, where $V_1 \dot{\cup} V_2$ is some partition of $V(\Gamma)$ into two vertex sets, in the same way as we did for $X \in \mathbb{R}$ if $x \in V_1$, $X \in \mathbb{R} - \pi i$ if $x \in V_2$, noting that Φ is $2ti$-periodic in θ.

To conclude this subsection, we want to comment on rhombic period lattices, i.e., a half-period ratio τ of length one. In this setting, the original formulation of the quad-equation (Q4) given by Adler, Bobenko, and Suris in [ABS03] seems to be easier to handle. Their three-leg function Φ is given by

$$\Phi(X, Y; \theta) = \frac{\sigma(X - Y + i\theta)\sigma(X + Y + i\theta)}{\sigma(X - Y - i\theta)\sigma(X + Y - i\theta)}.$$

Here, $i\theta := A - B$, where A, B are certain values fulfilling $\wp(A) = \alpha$, $\wp(B) = \beta$.

Let ω and $\omega' := \tau\omega$ be half-periods of $\wp$ such that $\Omega := \omega + \omega' \in \mathbb{R}^+$. Again, we start with the reality condition $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\Gamma)$, $\theta_e \notin \mathbb{Z}\Omega$. The derivation of the generalized discrete action functional is similar to our consideration above, so we just give its Hessian

$$\begin{aligned} D^2 S = &- i \sum_{e=(x,y)\in E(\Gamma)} \{\zeta(X - Y + \theta_e) - \zeta(X - Y - \theta_e)\} (dX - dY)^2 \\ &- i \sum_{e=(x,y)\in E(\Gamma)} \{\zeta(X + Y + \theta_e) - \zeta(X + Y - \theta_e)\} (dX + dY)^2. \end{aligned}$$

Now, there exists $r = r(\tau) > 0$, such that S is strictly concave if $\theta_e \in (0, r)$ for all $e \in E(\Gamma)$, and strictly convex if $\theta_e \in (-r, 0)$ for all edges e. The proof is almost identical to the one we gave in our investigation of strict concavity of the functional given by Formula (5.33) above. Just note that the periodicity of $\tilde{\zeta}$ is replaced by $\zeta(u + 2\Omega) = \zeta(u) + 2\zeta(\omega) + 2\zeta(\omega')$.

In the very same way as we did for the reality conditions in the case of rectangular period lattices, we can consider the reality condition $X, Y, \theta_e \in \mathbb{R}i$, and we may shift the variables for some of the vertices by $-\Omega$ or $\omega' - \omega$. The results are similar; however, we only get sufficient conditions for strict concavity for θ close to zero.

5.2.8 Limit cases

To conclude this section, we want to demonstrate how the situation of $(\mathrm{Q3})_{\delta=1}$ can be described as the limit of the setting in (Q4) when the (purely imaginary) half-period ratio τ goes to infinity.

As in Section 5.2.7, let $\tau = ti/\pi$. For simplicity, we just describe the reality condition $X, Y, \theta_e \in \mathbb{R}$ for all edges e. We already noted in Sections 5.2.6 and 5.2.7 that the shifting $X \to X - \pi i$ for some vertices x yields almost the same generalized discrete action functionals, and we observed, that the formulae for the reality condition $X, Y, \theta_e \in \mathbb{R}i$ were almost identical (however, the conditions for convexity or concavity were different).

Since sn $\to$ sin, the formulation of the quad-equation (Q4) given in Appendix B converges to $(\mathrm{Q3})_{\delta=1}$. The same is true for the three-leg functions Φ given in (5.27) and (5.21), observing that $\vartheta_1(v)/\vartheta_1(v') \to \sinh(iv)/\sinh(iv')$, and $\vartheta_4 \to 1$. Also, φ defined in (5.28) converges to φ defined in (5.22).

Equation (5.32) shows that

$$
\begin{aligned}
i\tilde{\zeta}(X \pm Y + i\theta) - i\tilde{\zeta}(X \pm Y - i\theta) &\to \frac{\sin(\theta)}{\cosh(X \pm Y) - \cos(\theta)}, \\
i\tilde{\zeta}(X \pm Y \pm t + i\theta) - i\tilde{\zeta}(X \pm Y \pm t - i\theta) &\to 0.
\end{aligned}
$$

Thus, the Hessian of the generalized discrete action functional $S_{(\mathrm{Q4})}$, given in (5.30), converges to the Hessian of $S_{(\mathrm{Q3})_{\delta=1}}$, given in (5.24).

Now, the limit of $L_{(\mathrm{Q4})}$ defined in (5.29) satisfies Conditions (5.6) and (5.7) for the additive three-leg function of $(\mathrm{Q3})_{\delta=1}$. But the same does $L_{(\mathrm{Q3})_{\delta=1}}$ defined as the corresponding summand in $S_{(\mathrm{Q3})_{\delta=1}}$ given in (5.23). In particular, their difference has to be constant in X and Y. Evaluating at $X = Y = 0$ we obtain

$$
\begin{aligned}
L_{(\mathrm{Q4})}(0, 0; i\theta) &= 0, \\
L_{(\mathrm{Q3})_{\delta=1}}(0, 0; i\theta) &= 4 \operatorname{Im} \operatorname{Li}_2(\exp(i\theta)).
\end{aligned}
$$

As a consequence,

$$
S_{(\mathrm{Q4})} \to S_{(\mathrm{Q3})_{\delta=1}} - \sum_{e=(x,y)\in E(\Gamma)} 4 \operatorname{Im} \operatorname{Li}_2(\exp(i\theta_e)).
$$

In the end, we comment shortly on the more delicate limit from $(\mathrm{Q3})_{\delta=1}$ to $(\mathrm{Q3})_{\delta=0}$. If we let all variables X go to $-\infty$, but preserve their distances, then

$$
\frac{\sinh\left(\frac{X+Y+i\theta}{2}\right)}{\sinh\left(\frac{X+Y-i\theta}{2}\right)} \to \exp(-i\theta),
$$

so the three-leg function of $(\mathrm{Q3})_{\delta=1}$ converges to the one of $(\mathrm{Q3})_{\delta=0}$. Also, the Hessian of $S_{(\mathrm{Q3})_{\delta=1}}$ converges to the Hessian of $S_{(\mathrm{Q3})_{\delta=0}}$, as becomes immediate from Formulae (5.24) and (5.17). Having the relation of the generalized discrete action functionals to the circle pattern functionals of Bobenko and Springborn in [BS04] in mind, $X \to -\infty$ means that the radii of the hyperbolic circle pattern go to zero. This corresponds to seeing the Euclidean plane as a limit of hyperbolic planes with curvature going to 0.

Note that for the reality condition $X, Y, \theta_e \in \mathbb{R}i$ for all $e \in E(\Gamma)$, we do not such have a limit since

$$
\frac{\sin\left(\frac{u+i\theta}{2}\right)}{\sin\left(\frac{u-i\theta}{2}\right)}
$$

does not converge if $u \to \pm\infty$.

5.3 Existence and uniqueness of solutions of (Q3)- and (Q4)-Dirichlet boundary value problems

This section is devoted to the study of Dirichlet boundary value problems for (Q3) and (Q4), considering the reality condition $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y)$, restricting to one of the cases $\exp(i\theta_e) \in S^1_+$ or $\exp(i\theta_e) \in S^1_-$. The reality conditions where some of the variables are shifted to $X - \pi i$ can be handled the same way.

The reason why we not consider the same problems for (Q1) and (Q2), or for the other reality conditions of (Q3) and (Q4), is that the conditions for convexity or concavity given in Theorem 5.1 cannot be achieved by an integrable labeling, as we will see in Proposition 5.5 of Section 5.4. In contrast, the conditions for (Q3) and (Q4) above might be realized by an integrable labeling. Still, we do not require the labeling to be integrable in the following.

Note that Theorem 5.2 below is due to Bobenko and Springborn [BS04], we just exploit the identification of the generalized discrete action functionals of (Q3) with the corresponding circle pattern functionals, as explained in Sections 5.2.5 and 5.2.6.

Theorem 5.2. *Let θ be a labeling of $E(\Gamma)$ such that $0 < \theta_e < \pi$ for all edges e. Moreover, let $\Theta_x \in \mathbb{R}^+$ be given. Then, the corresponding generalized discrete action functionals of* (Q3) *and* (Q4) *given by Formulae (5.16) and (5.23) have an extremum on the subspace U defined in (5.9) or $\mathbb{R}^{|V(\Gamma)|}$, respectively, if and only if the following two conditions are satisfied:*

(i) $\sum_{e \in E(\Gamma)} 2\theta^*_e - \sum_{x \in V(\Gamma)} \Theta_x$ *is equal to zero in the case of* $(\mathrm{Q3})_{\delta=0}$*, and greater than zero in the case of* $(\mathrm{Q3})_{\delta=1}$*;*

(ii) $\sum_{e \in E'} 2\theta^*_e - \sum_{x \in V'} \Theta_x > 0$ *for all nonempty $V' \subsetneq V(\Gamma)$, where E' denotes the set of all edges incident to some vertex in V'.*

The extremum is unique if it exists.

Proof. In Sections 5.2.5 and 5.2.6 we have seen that one can identify the generalized discrete action functionals in the case of (Q3) with the circle pattern functionals of [BS04] under certain conditions (e.g., $0 < \theta < \pi$ and $\Theta > 0$).

Thus, we can adapt Bobenko's and Springborn's result in [BS04] to our setting. They gave a complete answer to the question when extrema of these functionals exist. Their proof consisted of two steps. First, they showed that existence of an extremum is equivalent to the existence of so-called coherent angle systems; second, they used the feasible flow theorem of network theory to prove that coherent angle systems exist if and only if the conditions of the theorem hold. We refer the reader to their paper [BS04] for more details.

Note that the uniqueness of the extremum follows from strict concavity of the functionals. $\square$

Remark. Using Equations (5.19) and (5.25), we get an analogous theorem in the case of $\pi < \tilde{\theta}_e < 2\pi$, replacing $\tilde{\theta}_e = \pi + s_e$ by $\theta_e = \pi - s_e$, and choosing $\Theta_x = \tilde{\Theta}_x$.

Definition. We consider the generalized discrete action functional S corresponding to the reality condition $X, Y, \theta_e \in \mathbb{R}$ in the case of (Q3) or (Q4). A *Dirichlet boundary value problem* asks for the existence of a critical point of S under the constraint that the variables corresponding to some preassigned vertices of Γ are fixed.

Note that in the definition above, we do not require that the preassigned vertices lie on the boundary of Γ. In the nongeometric case, the situation simplifies compared to Theorem 5.2 if all Θ vanish. The proof follows the lines of Bobenko's and Springborn's proof of Theorem 5.2. But now, coherent angle systems are not necessary any more, since all Θ are zero.

Theorem 5.3. *Let θ be a labeling of $E(\Gamma)$ such that $\exp(i\theta_e) \in S^1_+$ for all edges e, or $\exp(i\theta_e) \in S^1_-$ for all edges e. Then, the Dirichlet boundary value problem corresponding to $(\mathrm{Q3})_{\delta=1}$ is uniquely solvable if $\Theta_x = 0$ for all vertices x corresponding to a nonfixed variable.*

Proof. The uniqueness follows from strict concavity or convexity of the generalized discrete action functional S due to Theorem 5.1. By Equation (5.25), we may restrict to $0 < \theta_e < \pi$ for all edges e, such that S is strictly concave. Then, for some constant C given by the contributions of $\Theta_x X$ for fixed variables X, $S + C$ is equal to

$$-\sum_{e=(x,y)\in E(\Gamma)} \{\operatorname{Im}\operatorname{Li}_2(\exp(X - Y + i\theta_e)) + \operatorname{Im}\operatorname{Li}_2(\exp(Y - X + i\theta_e))\}$$

$$- \sum_{e=(x,y)\in E(\Gamma)} \left\{ \operatorname{Im} \operatorname{Li}_2(\exp(X+Y+i\theta_e)) + \operatorname{Im} \operatorname{Li}_2(\exp(-X-Y+i\theta_e)) \right\}.$$

Now, for any $u \in \mathbb{R}$ and $0 < \theta < \pi$, we have

$$\operatorname{Im} \operatorname{Li}_2(\exp(u+i\theta)) + \operatorname{Im} \operatorname{Li}_2(\exp(-u+i\theta)) > (\pi - \theta)\, |u| \,.$$

As a consequence, we obtain

$$\begin{aligned} S &< - \sum_{e=(x,y)\in E(\Gamma)} (\pi-\theta_e)(|X-Y| + |X+Y|) - C \\ &\leq - \min_{e\in E(\Gamma)} (\pi - \theta_e) \sum_{e=(x,y)\in E(\Gamma)} 2\max(|X|\,, |Y|) - C. \end{aligned}$$

In particular, $S \to -\infty$ if the absolute value of some variable X goes to infinity. Thus, a maximum of the strictly concave functional X exists. □

Remark. In Section 5.2.5, we have seen that $\Theta_x > 0$ (or $\Theta_x < 0$) for all vertices $x \in V(\Gamma)$ is necessary to obtain solutions of the discrete Laplace-type Equation (5.5) corresponding to $(\mathrm{Q3})_{\delta=0}$ if $\exp(i\theta_e) \in S^1_+$ (or $\exp(i\theta_e) \in S^1_-$) for all edges e. On that account, Theorem 5.3 would not be reasonable in the case of $(\mathrm{Q3})_{\delta=0}$.

In the case of the most general quad-equation (Q4), it turns out that the corresponding Dirichlet boundary value problem is always solvable.

Theorem 5.4. *Let θ be a labeling of $E(\Gamma)$, such that $\exp(i\theta_e) \in S^1_+$ for all $e \in E(\Gamma)$, or $\exp(i\theta_e) \in S^1_-$ for all edges e. Then, the Dirichlet boundary value problem corresponding to* (Q4) *is uniquely solvable.*

Proof. By Theorem 5.1, the generalized discrete action functional S given by Formula (5.29) is strictly concave or strictly convex. Hence, a extremum is unique if it exists. In the following, we will restrict to the case that S is strictly concave, the other case can be dealt with in exactly the same way.

As in Section 5.2.7, let $\tau = ti/\pi$ denote the half-period ratio of the Jacobi theta functions. Then,

$$\begin{aligned} \frac{\partial \varphi}{\partial X}(X,Y;\theta) = &- i(\tilde{\zeta}(X+Y+i\theta) - \tilde{\zeta}(X+Y-i\theta)) \\ &- i(\tilde{\zeta}(X+Y+t+i\theta) - \tilde{\zeta}(X+Y+t-i\theta)) \end{aligned}$$

$$- i(\tilde{\zeta}(X - Y + i\theta) - \tilde{\zeta}(X - Y - i\theta))$$
$$- i(\tilde{\zeta}(X - Y - t + i\theta) - \tilde{\zeta}(X - Y - t - i\theta))$$

is a smooth and negative function that is $2t$-periodic in both variables X and Y. $\partial\varphi/\partial Y$ is also a smooth and double-periodic function. It follows that both derivatives of φ are bounded, so $\varphi(\cdot, \cdot; \theta)$ is Lipschitz-continuous. Since the set of edges is finite, we may choose the Lipschitz constant L uniformly for all θ_e.

A straightforward calculation shows that for all $\theta \in (0, \pi]$,

$$\Phi(X + 2t, Y; \theta) = \Phi(X, Y; \theta) \exp(8i\theta), \tag{5.34}$$
$$\Phi(X, Y + 2t; \theta) = \Phi(X, Y; \theta). \tag{5.35}$$

Let us choose $\mu = \mu(X, Y; \theta), \eta = \eta(X, Y; \theta)$ in such a way that

$$\varphi(X + 2t, Y; \theta) - \varphi(X, Y; \theta) = \mu(X, Y; \theta),$$
$$\varphi(X, Y + 2t; \theta) - \varphi(X, Y; \theta) = \eta(X, Y; \theta).$$

By continuity and Equations (5.34) and (5.35), η is constant, and $\mu = \mu(\theta)$ depends continuously only on $\theta \in (0, \pi]$. Now, $\partial\varphi/\partial X < 0$, so $\mu < 0$; and $\Phi(X, Y; \pi) \equiv 1$, so $\eta = 0$.

Let us suppose that the Dirichlet boundary value problem is not solvable. Then, there exists a sequence

$$(\{X_j(n)\}_{j=1}^{v})_{n=0}^{\infty}$$

of variables (some of them fixed), $v = |V(\Lambda)|$, such that at least one variable is unbounded, and such that the sequence

$$(S(n))_{n=0}^{\infty}, \quad S(n) := S(X_1(n), \ldots, X_v(n)),$$

is strictly increasing. In the following, we will subsequently pass to subsequences with certain properties, and for the ease of notation, we will use the same indexing for the subsequences as for the original sequence.

We enumerate the X-variables in such a way that the X_1-variable is unbounded. By passing to a subsequence, we can achieve that the sequence of $X_1(n)$ is strictly increasing or decreasing. Similarly, we can assume for any j that $X_j(n)$ either converges to a constant a_j, or strictly monotonously to $\pm\infty$. Note that fixed variables X_j are just constantly a_j.

For a variable X_j such that $X_j(n) \to \pm\infty$, consider the induced sequence on the compact circle $\mathbb{R}/2t\mathbb{Z}$. Then, we can choose a convergent subsequence, i.e., there exist $a_j \in \mathbb{R}$ and a sequence $(m_j(n))_{n=0}^{\infty}$ of integers, such that

$$X_j(n) - 2m_j(n)t \to a_j.$$

We may suppose that $m_j(n) \to \pm\infty$ strictly monotonously. Using the same notation, we can include the case that $X_j(n)$ converges to a constant by setting $m_j(n) = 0$ for all n.

Now, let $0 < \varepsilon < 2t$ be fixed. By passing to a subsequence, we can assume that

$$|X_j(n) - 2m_j(n)t - a_j| < \varepsilon$$

for all j and n. By our consideration above, we have for any edge $e = (x_j, x_k)$:

$$\begin{aligned}\varphi(X_j(n), X_k(n'); \theta_e) &= \varphi(a_j + 2m_j(n)t, a_k + 2m_k(n')t; \theta_e) + B \\ &= \varphi(a_j, a_k; \theta_e) + \mu(\theta_e)m_j(n) + B\end{aligned}$$

for some real number B depending on e, j, k, n, n' satisfying $|B| < L\sqrt{2\varepsilon^2}$. More general, we see that for any real $X = a + 2pt + q$, where $p \in \mathbb{Z}$ and $|q| < 2t$, that

$$\varphi(X, X_k(n'); \theta_e) = \varphi(a, a_k; \theta_e) + \mu(\theta_e)p + B,$$

where $|B| < L\sqrt{\varepsilon^2 + 4t^2}$.

For a positive real number M that we will fix later, we define

$$p_{\max} := \frac{1}{\min\limits_{e \in E(\Gamma)} |\mu(\theta_e)|} \left(L\sqrt{\varepsilon^2 + 4t^2} + M + \max_{X_j(n) \text{ diverges}} \max_{e=(x_j,x_k)} |\varphi(a_j, a_k; \theta_e)| \right).$$

By the results we obtained so far, we get for an edge $e = (x_j, x_k)$ such that $X_j(n)$ diverges, and $X = a + 2pt + q$ as above with $|p| > p_{\max}$:

$$\delta\varphi(X, X_k(n'); \theta_e) < -M,$$

where $\delta := \operatorname{sgn}(p)$. By passing to a subsequence, we suppose that $|m_j(n)| > p_{\max}$ for all j such that $X_j(n)$ diverges. Now, define

$$C := \sqrt{2}L\varepsilon + \max_{X_j(n) \text{ converges}} \max_{e=(x_j,x_k)} |\varphi(a_j, a_k; \theta_e)|.$$

Then, $|\varphi(X, X_k(n'); \theta_e)| < C$ for any edge $e = (x_j, x_k)$ such that $X_j(n)$ converges, provided that $|X - a_j| < \varepsilon$. Finally, define

$$E := \max_{x \in V(\Gamma)} |\Theta_x|, \tilde{C} := C + E, \tilde{M} := M - E.$$

We will choose M later in such a way that $\tilde{M} > 0$.

As a short summary of our consideration above, we have shown that

$$\begin{aligned}
&\delta \frac{\partial S}{\partial X_j}(X_1(n+1), \ldots, X_{j-1}(n+1), X_j, X_{j+1}(n), \ldots, X_v(n)) \\
=&\delta \left(\sum_{e=(x_j, x_k) \text{ edge}} \varphi(X_j, X_k(n \text{ or } n+1); \theta_e) - \Theta_{x_j} \right) \\
<&-\tilde{M}
\end{aligned}$$

if $X_j \in \operatorname{conv}\{X_j(n), X_j(n+1)\}$ and $X_j(n) \to \delta\infty$, $\delta = \pm 1$; and

$$\begin{aligned}
&\left| \frac{\partial S}{\partial X_j}(X_1(n+1), \ldots, X_{j-1}(n+1), X_j, X_{j+1}(n), \ldots, X_v(n)) \right| \\
=&\left| \sum_{e=(x_j, x_k) \text{ edge}} \varphi(X_j, X_k(n \text{ or } n+1); \theta_e) + \Theta_{x_j} \right| \\
<&\tilde{C}v
\end{aligned}$$

if $X \in \operatorname{conv}\{X_j(n), X_j(n+1)\}$ and $X_j(n) \to a_j$.

Without loss of generality, we enumerate the vertices in such a way that we have $X_j(n) \to \pm\infty$ if and only if $j \leq k$. If we integrate along the piecewise straight path

$$\begin{aligned}
(X_1(n), \ldots, X_v(n)) &- (X_1(n+1), X_2(n), \ldots, X_v(n)) \\
&- (X_1(n+1), X_2(n+1), \ldots, X_v(n)) \\
&- \ldots \\
&- (X_1(n+1), \ldots, X_v(n+1)),
\end{aligned}$$

we obtain

$$\begin{aligned}
S_{n+1} - S_n &< -\tilde{M} \sum_{j=1}^{k} |X_j(n+1) - X_j(n)| + \tilde{C}v \sum_{j=k+1}^{v} |X_j(n+1) - X_j(n)| \\
&< -\tilde{M} \, |X_1(n+1) - X_1(n)| + \tilde{C}v(v-k)2\varepsilon.
\end{aligned}$$

Again, we can achieve $|X_1(n+1) - X_1(n)| \geq 1$ for all n by passing to a subsequence. After all, we choose M large enough such that $M > E + 2\tilde{C}v(v-k)\varepsilon$. But then, $S_{n+1} - S_n < 0$, contradicting our assumption that $S(n)$ is increasing. Consequently, S has a (unique) maximum. $\square$

Remark. Although the concavity condition we gave for rhombic lattices in the end of Section 5.2.7 is not integrable, we would like to mention that Theorem 5.4 also holds for unitary half-periods τ. For this, we use that

$$\sigma(u + 2\Omega) = -\exp(2\eta'(u+\Omega))\sigma(u),$$

where ω and $\omega' := \tau\omega$ are half-periods of $\wp$ such that $\Omega := \omega + \omega' \in \mathbb{R}^+$, and $\eta' = \zeta(\omega) + \zeta(\omega')$. This gives results analogous to Equations (5.34) and (5.35), such that the same arguments work in the rhombic case as well.

5.4 Integrable cases

In this section, we want to discuss when a labeling of $E(\Lambda)$ fulfilling the conditions for concavity or convexity in Theorem 5.1 is integrable. First, we observe that the conditions $\pm\theta_e > 0$ and $\pm i\theta_e > 0$ are never integrable:

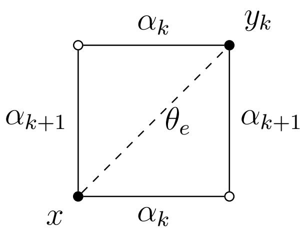


Figure 5.3: Induced labeling

Proposition 5.5. *If the quad-graph Λ contains at least one inner black vertex, there is no integrable labeling of $E(\Gamma)$ such that $\delta\theta_e > 0$ for some fixed nonzero $\delta \in \mathbb{C}$, and all edges e of Γ.*

Proof. If we consider all edges $e = (x, y_k)$ incident to an inner black vertex x, and $\theta_e = \alpha_k - \alpha_{k+1}$ is induced from a labeling of $E(\Lambda)$ as shown in Figure 5.3, then

$$\sum_{(x,y_k)\in E(\Gamma)} \theta_e = \sum_{(x,y_k)\in E(\Gamma)} (\alpha_k - \alpha_{k+1}) = 0. \tag{5.36}$$

Hence, not all $\delta\theta$ can be positive. $\square$

Therefore, only the conditions $\exp(i\theta_e) \in S^1_+$ for all edges e, or $\exp(i\theta_e) \in S^1_-$ for all $e \in E(\Gamma)$ can be integrable. Since the labeling $-\theta$ is integrable if and only if the labeling θ is integrable, we can restrict to the case $\exp(i\theta) \in S^1_+$, i.e., $\theta \equiv (0, \pi) \mod 2\pi$. To have a short notation, we say that such a labeling is *convex*.

In the following, we first determine all possible convex integrable labelings on some very special graphs, and then, we will discuss convex integrable labelings on rhombic-embeddable quad-graphs.

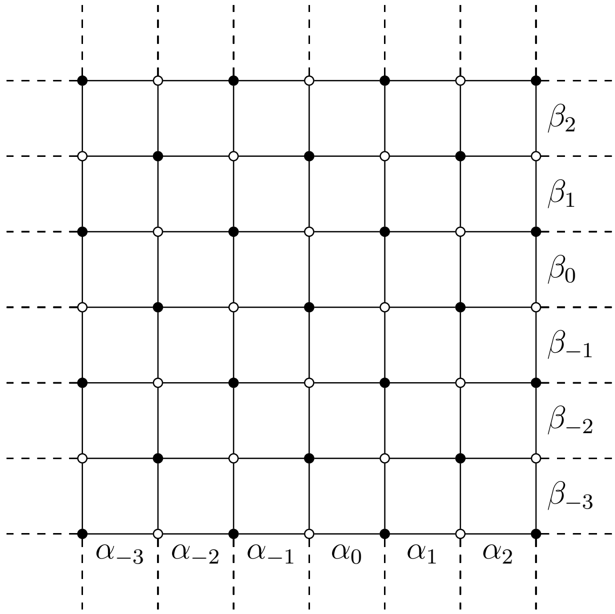


Figure 5.4: $\mathbb{Z}^2$

Given a finite sub-quad-graph of $\mathbb{Z}^2$, we consider only labelings of its edges that are actually induced from a labeling of all edges $\mathbb{Z}^2$, see Figure 5.4. It is easy to check that all such labelings that induce a convex labeling are given by the following (up to multiples of 2π and adding a fixed number to all labels):

$$\begin{aligned}
\beta_0 &= 0, \\
\alpha_{2j} &\in (0, \pi), \\
\alpha_{2j+1} &\in (\pi, 2\pi), \\
\beta_{2j+1} &\in (\max_{k\in M_j} \alpha_{2k}, \min_{k\in M_j} \alpha_{2k} + \pi) \cap (\max_{k\in N_j} \alpha_{2k+1} - \pi, \min_{k\in N_j} \alpha_{2k+1}), \\
\beta_{2j} &\in \{(\max_{k\in M'_j} \alpha_{2k} + \pi, 2\pi) \cup [0, \min_{k\in M'_j} \alpha_{2k})\}
\end{aligned}$$

$$\cap \{(\max_{k \in N_j'} \alpha_{2k+1}, 2\pi) \cup [0, \min_{k \in N_j'} \alpha_{2k+1} - \pi)\}$$

for all integers j. Here, M_j, M_j', N_j, N_j' denote the sets of all integers k such that there exists a quadrilateral in $\mathbb{Z}^2$ with labels $\alpha_{2k}, \beta_{2j+1}$; α_{2k}, β_{2j}; $\alpha_{2k+1}, \beta_{2j+1}$; $\alpha_{2k+1}, \beta_{2j}$, respectively.

Note that regardless which α_j we choose above, $\beta_{2j} = 0$ and $\beta_{2j+1} = \pi$ for all j will always satisfy the last two conditions. The result can be easily extended to infinite sub-quad-graphs of $\mathbb{Z}^2$, replacing minima and maxima by infima and suprema.

Definition. A *spider-graph* is the (infinite) quad-graph that is constructed in the following way: Take infinitely many concentric regular $2n$-gons, $n \geq 2$, that are equally spaced. Then, add the radial edges between two successive polygons, and divide the central polygon into quadrilaterals by adding $n-2$ parallel diagonals.

An example of a spider-graph is given in Figure 5.5 ($n = 4$).

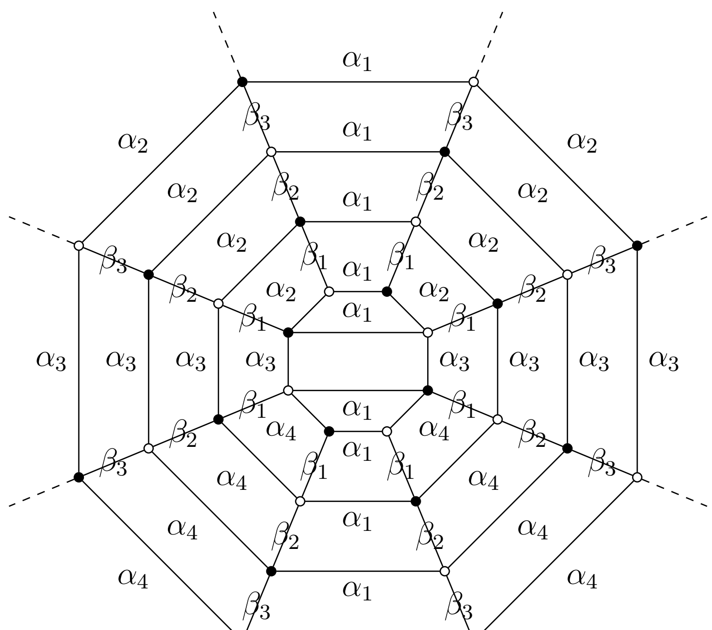


Figure 5.5: Spider-graph with central octagon

All labelings of the edges of a spider-graph that induce a convex labeling are given by the following (up to multiples of 2π and adding a fixed number to all labels):

$$\begin{aligned}
\alpha_1 &= 0,\\
\alpha_{2j+1} &\in (0,\pi),\\
\alpha_{2j} &\in (\pi, 2\pi),\\
\beta_{2j-1} &\in (\max\{\max_k \alpha_{2k} - \pi, \max_k \alpha_{2k+1}\}, \pi),\\
\beta_{2j} &\in (\max\{\max_k \alpha_{2k}, \max_k \alpha_{2k+1} + \pi\}, 2\pi)
\end{aligned}$$

for all nonnegative integers j.

Note that we encountered a subgraph of the spider-graph with a central square in Figure 3.4 in Chapter 3, where we gave examples of discrete Riemann surfaces of arbitrary genus such that there exists a discrete meromorphic function with exactly one pole.

Now, it follows from Proposition A.1 in Appendix A that spider-graphs possess no rhombic embedding into the complex plane. But rhombic-embeddable graphs are an interesting and important subclass of quad-graphs. Indeed, we have seen in Chapter 4 that the discrete theory of complex analysis on planar parallelogram-graphs is richer. Also, rhombic quad-graphs played an important role in the work [BMS05] of Bobenko, Mercat, and Suris on linear and nonlinear theories of discrete complex analysis.

Before we come to rhombic-embeddable quad-graphs, we will first prove a characterization of integrable labelings on *simply-connected* quad-graphs. These are quad-graphs where the part of the surface that $F(\Lambda)$ covers is simply-connected. For this, we first note that any labeling of $E(\Lambda)$ induces not only a labeling on the edges of the black graph Γ, but also on $E(\Gamma^*)$. Indeed, the induced label θ_{e^*} on an edge $e^* \in E(\Gamma^*)$ is just defined as $\theta_{e^*} = -\theta_e$, where θ_e is the label induced on the dual edge $e \in E(\Gamma)$. Since opposite edges of any quadrilateral carry the same label, this definition is consistent with our notion of an induced labeling.

The following three propositions are inspired by the paper [BMS05] of Bobenko, Mercat, and Suris.

Proposition 5.6. *Let Λ be simply-connected. Suppose that a labeling of $E(\Gamma)$ with real numbers is given, where the labels θ are considered the same if they differ by a multiple of 2π only.*

Then, this labeling is integrable if and only if the following two equations are *satisfied for all inner black vertices x and inner white vertices x^*:*

$$\prod_{e=(x,y_k)\in E(\Gamma)} \exp(i\theta_e) = 1, \tag{5.37}$$

$$\prod_{e^*=(x^*,y_k^*)\in E(\Gamma^*)} \exp(i\theta_{e^*}) = 1. \tag{5.38}$$

Proof. The forward implication is given by the multiplicative analog of (5.36). For the backward implication, we notice that (5.37) allows us to integrate the labeling on the edges of the star of any black vertex. Now, we just have to check that this consistently defines a labeling on $E(\Lambda)$. This is the case if and only if the sum of θ_e for all black edges e of a given cycle vanishes modulo 2π. Since Λ is simply-connected, this is true if it is true for any elementary cycle corresponding to the star of a white vertex. Using $\theta_{e^*} = -\theta_e$, we can replace the black edges by its dual white edges. Then, the sum of all θ_{e^*} around the white vertex vanishes if and only if Equation (5.38) is satisfied. □

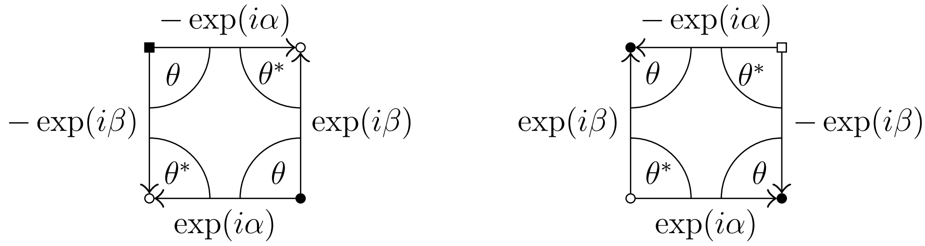
Proposition 5.7. *If the black graph Γ or the white graph Γ^* is bipartite, then any (possibly ramified) rhombic embedding of Λ into $\mathbb{C}$ yields a labeling of edges of Λ such that the induced labeling on $E(\Gamma)$ is convex. If Γ is bipartite, the induced labels are given by the angles θ of the rhombi at the black vertices; if Γ^* is bipartite, the induced labels are given by the angles $\theta^* = \pi - \theta$ of the rhombi at the white vertices.*

Proof. Since Γ (or Γ^*) is bipartite, we can choose a partition into two types of black (or white) vertices, say of type 1 and of type 2. We orient all edges of Λ in such a way that they always start in a black (or white) and end in a white (or black) vertex, see Figure 5.6.

To each oriented edge $\vec{e} = (x, y)$ we associate the complex number

$$\gamma(\vec{e}) := \frac{y-x}{\|y-x\|},$$

where we now consider the vertices as points in $\mathbb{C}$. If the black (or white) starting point of e is of type 1, we define $\exp(i\alpha) := \gamma(\vec{e})$, otherwise $\exp(i\alpha) := -\gamma(\vec{e})$. By construction, the numbers α (that are unique up to multiples of 2π) define labels on nonoriented edges of Λ such that opposite edges of a quadrilateral receive the same label.



(a) Γ bipartite

(b) Γ^* bipartite

Figure 5.6: Induced labeling of a rhombic embedding

Now, the induced labels are given by $\alpha - \beta$. Clearly, $\alpha - \beta \equiv \theta \mod 2\pi$ (or $\alpha - \beta \equiv \theta^* \mod 2\pi$), where θ (or θ^*) denotes the interior angle of the rhombus at the corresponding black (or white) vertex. $\square$

Remark. If Λ is simply-connected, Γ (or Γ^*) is bipartite if and only if all inner white (or black) vertices are of even degree.

For cellular decompositions of the disk, the converse of Proposition 5.7 is true:

Proposition 5.8. *Suppose that Λ corresponds to a cellular decomposition of the disk, and that Γ or Γ^* is bipartite. Then, any convex integrable labeling θ of $E(\Gamma)$ gives a (possibly ramified) rhombic embedding of Λ into the complex plane.*

Proof. Replacing θ by $-\theta$ if necessary, we can assume that $\exp(i\theta_e) \in S^1_+$ for all $e \in E(\Gamma)$. By assumption, we can choose labels α on the edges of Λ that induce the given labeling on $E(\Gamma)$. Then, we can define $\gamma(\vec{e})$ through the labels α exactly as in the proof of Proposition 5.7. Choosing one starting point, the $\gamma(\vec{e})$ give us consecutively the positions of the vertices of Λ in $\mathbb{C}$. By construction, any quadrilateral is mapped to a rhombus, and by Proposition 5.6 and the interpretation of θ in Proposition 5.7, the interior angles of the rhombi sum um to multiples of 2π. Thus, we obtain a possibly ramified rhombic realization of Λ. $\square$

The propositions above show that there is a large class of rhombic-embeddable quad-graphs for which convex integrable labelings exist. However, this is not true for all rhombic-embeddable quad-graphs.

Proposition 5.9. *There exist infinitely many rhombic-embeddable quad-graphs Λ such that every induced labeling on Γ is not convex.*

Proof. It is easy to see that the graph shown in Figure 5.7 possesses a rhombic embedding according to Proposition A.1, and that it can be easily extended to arbitrary large rhombic-embeddable quad-graphs. Let us show that every induced labeling on $E(\Gamma)$ is not convex.

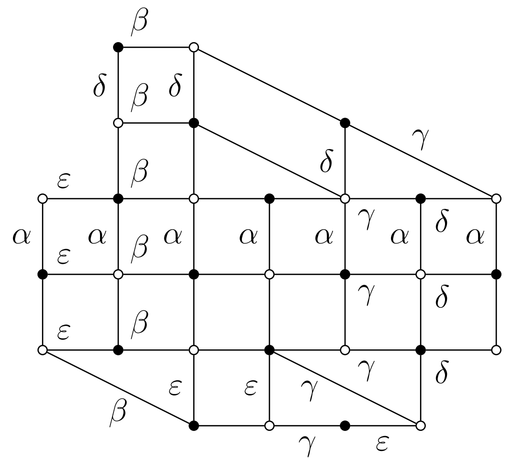


Figure 5.7: rhombic-embeddable graph from Proposition 5.9

Assume the contrary, and consider an appropriate labeling of $E(\Lambda)$. Again, we consider the labels modulo 2π, such that we can assume that they are all in $[0, 2\pi)$. Without loss of generality, we suppose that $\varepsilon = 0$.

Due to convexity of the labeling, $\exp(i(\varepsilon - \mu)) \in S^1_+$ for all $\mu \in \{\alpha, \beta, \gamma\}$. Thus, $\alpha, \beta, \gamma \in (\pi, 2\pi)$. Moreover, $\exp(i(\alpha - \beta)), \exp(i(\gamma - \alpha)) \in S^1_+$, which gives $\gamma > \alpha > \beta$. Again by convexity of the labeling, the following restrictions for δ hold: $\exp(i(\alpha - \delta)), \exp(i(\delta - \beta)), \exp(i(\delta - \gamma)) \in S^1_+$. Using all the results we have obtained so far,

$$\begin{aligned}
\delta &\in (\alpha - \pi, \alpha) \cap \{(\beta, 2\pi) \cup [0, \beta - \pi)\} \cap \{(\gamma, 2\pi) \cup [0, \gamma - \pi)\} \\
&= (\beta, \alpha) \cap \{(\gamma, 2\pi) \cup [0, \gamma - \pi)\} \\
&= (\beta, \alpha) \cap (\gamma, 2\pi) \\
&= \emptyset,
\end{aligned}$$

contradiction. $\square$

Remark. Note that the graph shown in Figure 5.7 is actually not that artificial. The proof why any labeling on its edges cannot induce a convex labeling mainly relies on the fact that there are a couple of triples of strips that are pairwise intersecting (for a definition of a strip, see Appendix A). The existence of such triples is reasonable to expect from a generic rhombic quad-graph. However, no three pairwise intersecting strips are all intersected by the same strip in the example above.

5.5 Integrable circle patterns and (Q3)

We conclude this chapter by continuing our discussion of the analogies between the discrete Laplace-type equations corresponding to (Q3) and circle patterns as described by Bobenko and Springborn in [BS04]. Also, we comment on our and other notions of integrability.

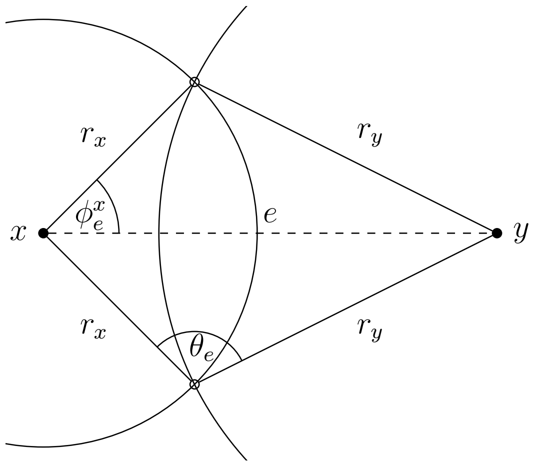


Figure 5.8: Intersection of circles

Figure 5.8 shows two circles, centered at x and y with radii r_x and r_y, that intersect at the angle θ_e (the circles being canonically oriented). Thus, the fact that the angles around an inner white vertex sum up to 2π (or a multiple of it, if we consider ramified patterns) corresponds to Equation (5.38) in Proposition 5.6.

Bobenko and Springborn observed in [BS04] that elementary geometric calculations yield

$$\exp(2i\phi_e^x) = \frac{r_x - r_y \exp(-i\theta_e)}{r_x - r_y \exp(i\theta_e)} \tag{5.39}$$

in the Euclidean case, and

$$\exp(2i\phi_e^x) = \log\left(\frac{\tanh\left(\frac{r_x}{2}\right) - \tanh\left(\frac{r_y}{2}\right)\exp(-i\theta_e)}{\tanh\left(\frac{r_x}{2}\right) - \tanh\left(\frac{r_y}{2}\right)\exp(i\theta_e)}\right)$$

$$-\log\left(\frac{1-\tanh\left(\frac{r_x}{2}\right)\tanh\left(\frac{r_y}{2}\right)\exp(-i\theta_e)}{1-\tanh\left(\frac{r_x}{2}\right)\tanh\left(\frac{r_y}{2}\right)\exp(i\theta_e)}\right) \tag{5.40}$$

in the hyperbolic case. Let us define

$$r_x = \exp(X), r_y = \exp(Y) \text{ in the Euclidean case,} \tag{5.41}$$

$$\tanh\left(\frac{r_x}{2}\right) = \exp(X), \tanh\left(\frac{r_y}{2}\right) = \exp(Y) \text{ in the hyperbolic case,} \tag{5.42}$$

$X, Y \in \mathbb{R}$. This is possible because all radii are positive. Straightforward calculations show in both cases that

$$2\phi_e^x = \varphi(X, Y; \theta_e),$$

where φ is defined in (5.15) for $(Q3)_{\delta=0}$, and in (5.22) for $(Q3)_{\delta=1}$, corresponding to the Euclidean and hyperbolic case, respectively.

In Sections 5.2.5 and 5.2.6, we have shown that in the case of $X \in \mathbb{R}$ and $\theta_e \in (0, \pi)$ for all vertices $x \in V(\Gamma)$ and edges $e \in E(\Gamma)$, the terms Θ_x are positive, such that they can be seen as the cone angles at vertices $x \in V(\Gamma)$. Thus, the discrete Laplace-type Equations (5.5) of type $(Q3)_{\delta=0}$ and $(Q3)_{\delta=1}$ are identical to the variational formulation of Euclidean and hyperbolic circle patterns, respectively, given by Bobenko and Springborn in [BS04].

Let us now come to integrability. By Proposition 5.6, the labeling θ of $E(\Gamma)$ is integrable if and only if

$$\prod_{e=(x,y_k)\in E(\Gamma)} \exp(i\theta_e) = 1 \tag{5.43}$$

is true for all inner black vertices x, having in mind that Equation (5.38) is satisfied since angles θ sum up to 2π around inner white vertices.

Assume now that the circle pattern can be partitioned into two sets of circles such that any two circles from one set have no points in common. Equivalently, Γ is bipartite. If we denote the decomposition by $V(\Gamma) = V_1 \dot\cup V_2$, there is another way to relate the discrete Laplace-type equations corresponding to (Q3) with such circle patterns. For this, we use the other reality condition that differs to the previous ones by subtracting πi to X if $x \in V_2$. If we replace X by $X - \pi i$ (or Y by $Y - \pi i$) in (5.41) or (5.42), we have to replace r_x by $-r_x$ (or r_y by $-r_y$). Now, $\exp(2i\phi_e^x)$ in Equations (5.39) and (5.40) then remains the same if we replace θ_e by $-\theta_e^*$.

In Sections 5.2.5 and 5.2.6, we have explained why the variational formulation of the discrete Laplace-type equation for the modified reality condition is essentially the same as for the original one. Consequently, we obtain another interpretation of the discrete Laplace-type equations corresponding to (Q3), now with labels $-\theta^*$ instead of θ, and circle patterns with intersection angles θ.

This new labeling of $E(\Gamma)$ is then integrable if and only if

$$\prod_{e=(x,y_k)\in E(\Gamma)} \exp(i\theta_e^*) = 1 \tag{5.44}$$

is true for all inner black vertices x.

Remark. Equation (5.43) and (5.44) are equivalent if and only if all vertices of Γ have even degree. In the case that Λ is simply-connected, we can equivalently state that the white graph Γ^* is bipartite.

If Λ corresponds to a cellular decomposition of a disk, circle patterns satisfying Equation (5.44) for all inner black vertices x are called *integrable* in the paper [BMS05] of Bobenko, Mercat, and Suris. If and only if this equation is satisfied, the circle patterns admit isoradial realizations. Assuming that Γ is bipartite (as it is necessary for the second reality condition), this statement is exactly the content of Propositions 5.7 and 5.8.

Note that our definition of integrability of the corresponding labeling of $E(\Gamma)$ is denoted by *integrability of the corresponding cross-ratio system* in [BMS05], meaning that the system of Equations (5.1), where Q is the cross-ratio equation $(\mathrm{Q1})_{\delta=0}$, is three-dimensional consistent.

In their paper [BMS05], Bobenko, Mercat, and Suris described how integrable circle patterns yield solutions of the corresponding cross-ratio system, and under which conditions solutions of the cross-ratio system describe integrable circle patterns.

Appendix A

Planar parallelogram-graphs

The aim of this appendix is to discuss some combinatorial and geometric properties of parallelogram-graphs that were used in Chapter 4. The following notion of a strip is standard, see for example the book of Bobenko and Suris [BS08].

Definition. A *strip* in a planar quad-graph Λ is a path on its dual $\diamond$ such that two successive faces share an edge, and the strip leaves a face in the opposite edge where it enters it. Moreover, strips are assumed to have maximal length, i.e., there are no strips containing it apart from itself.

Note that a strip is uniquely determined by the edges it passes through, meaning the edges two successive faces share.

Definition. For a strip S of a parallelogram-graph Λ, there exists a complex vector a_S such that any (nonoriented) edge through which S passes is $\pm a_S$. We call a_S a *common parallel.*

a_S is unique up to sign; the choice of the sign induces an orientation on all edges. The parallel edges of the strip can be rescaled to length $|a_S| = 1$, without changing the combinatorics. Hence, rhombic planar quad-graphs and planar parallelogram-graphs are combinatorially equivalent. Rhombic planar quad-graphs are characterized by the following proposition of Kenyon and Schlenker [KS05]:

Proposition A.1. *A planar quad-graph Λ admits a combinatorially equivalent embedding in $\mathbb{C}$ with all rhombic faces if and only if the following two conditions are satisfied:*

- *No strip crosses itself or is periodic.*
- *Two distinct strips cross each other at most once.*

That planar parallelogram-graphs fulfill these two conditions was already noted by Kenyon in [Ken02b]. The reason for that is, that any strip S is monotone with respect to the direction ia_S: The coordinates of the endpoints of the edges parallel to a_S, are strictly increasing or strictly decreasing if they are projected to ia_S. Whether the projections are decreasing or increasing depends on the direction in which the faces of S are passed through. Without loss of generality, we assume that the faces of S are passed through in such a way that the projections of the corresponding coordinates are strictly increasing. For $Q \in S$, let S^Q denote the semi-infinite part of S starting in the quadrilateral Q that passes through the faces of S in the same order.

As a consequence, no strip crosses itself or is periodic. Furthermore, S divides the complex plane into two unbounded regions, to one is a_S pointing, and to the other $-a_S$. When a distinct strip S' crosses S, it enters a different region determined by S, say it goes to the one to which a_S is pointing. Due to monotonicity, the angle between $ia_{S'}$ and a_S is less than $\pi/2$. It follows that S' cannot cross S another time, since it would then go to the region $-a_S$ is pointing to, contradicting that the angle between $ia_{S'}$ and $-a_S$ is greater and not less than $\pi/2$.

Note that if all interior angles of a rhombic quad-graph are uniformly bounded, Λ is locally finite, and for a strip S and a vertex $v \in V(\Lambda)$, the line $v + ta_S$, $t \in \mathbb{R}$, intersects S in exactly one line segment affinely equivalent to a_S.

In order to construct the discrete Green's function and the discrete Cauchy's kernels in Sections 4.2 and 4.3, we chose a particular directed path connecting two vertices (or a face and a vertex) by edges of the parallelogram-graph Λ; this path was monotone in one direction, and any angle between two consecutive was less than π. The existence of such a path follows from the following lemma, generalizing a result of Bobenko, Mercat and Suris [BMS05] to general parallelogram-graphs. The proof bases on the same ideas.

Lemma A.2. *Let Λ be a parallelogram-graph, and let $v_0 \in V(\Lambda)$ be fixed. For a directed edge e of Λ starting in v_0, consider the set $U_e \subset V(\Lambda)$ of all vertices to that v_0 can be connected by a directed path of edges whose arguments lie in $[\arg(e), \arg(e) + \pi)$.*

Then, the union of all U_e, e a directed edge starting in v_0, covers the whole quad-graph.

If there is $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all interior angles α of faces of Λ, the same statement holds true if $[\arg(e), \arg(e) + \pi)$ is replaced by $[\arg(e), \arg(e) + \pi - \alpha_0]$.

Proof. Let us rescale the edges such that all of them have length one. By this, we change neither the combinatorics of Λ nor the size of interior angles.

For a directed edge e starting in v_0, let U_e^- and U_e^+ denote the (directed) paths on Λ starting in v_0, obtained by choosing the directed edge with the least or largest argument in $[\arg(e), \arg(e) + \pi)$ (or $[\arg(e), \arg(e) + \pi - \alpha_0]$) at a vertex, respectively. We first show that all vertices in between U_e^- and U_e^+ are contained in U_e. Then, it follows that U_e is the conical sector with boundary U_e^- and U_e^+.

Suppose the contrary, i.e., suppose that there is a vertex v between U_e^- and U_e^+ to which v_0 cannot be connected by a directed path of edges whose arguments lie all in $[\arg(e), \arg(e) + \pi)$ (or $[\arg(e), \arg(e) + \pi - \alpha_0]$). Let the combinatorial distance between v_0 and v be minimal among all such vertices.

In the case that interior angles of rhombi are bounded by α_0 from below, they are bounded from above by $\pi - \alpha_0$. Hence, there is a vertex v' adjacent to v such that the argument of the directed edge $v'v$ lies in $[\arg(e), \arg(e) + \pi - \alpha_0]$. Even if interior angles are not uniformly bounded, v' can be chosen in such a way that the argument of $v'v$ lies in $[\arg(e), \arg(e) + \pi)$. By construction, v' is not in U_e, but still between U_e^- and U_e^+. Let us look at the strip S passing through $v'v$. Suppose that the common parallel a_S points from v' to v.

If S intersects U_e^- or U_e^+, an edge parallel to a_S is contained in U_e^- or U_e^+, respectively. By construction, v_0 and v' then lie on the same side of the strip S.

If S does neither intersect U_e^- nor U_e^+, it is completely contained in the left half space determined by the oriented line $v_0 + te$, $t \in \mathbb{R}$, as U_e^- and U_e^+ are. Suppose S intersects the ray $v_0 + ta_S$, $t \geq 0$. Again, it follows that v_0 and v' lie on the same side of S.

It remains the case that S neither intersects U_e^-, U_e^+, nor the ray $v_0 + ta_S$, $t \geq 0$. Consider the quadrilateral area P in between the parallels $v_0 + ta_S$, $v' + ta_S$ and $v_0 + te$, $v' + te$, $t \in \mathbb{R}$. By assumption, the semi-infinite part of S that starts with the edge $v'v$ and then goes into P does not intersect an edge of P incident to v_0, and by monotonicity, it does not intersect $v' + ta_S$ again. Now, Λ is locally finite, such that only finitely many quadrilaterals of S are inside P. Thus, S leaves P on the edge $v' + te$, $t \in \mathbb{R}$, and it follows that S separates v_0 and v.

So in any case, S separates v_0 from v. Any shortest path P connecting both points has to pass through S. Let w be the first point of P that lies on the same side of S as v does. Any strip passing through an edge on the shortest path connecting w and v on S has to intersect P as well. It follows that replacing all edges of P on the same side of S as v by the path from w to v does not change its length. But then, v' is combinatorially nearer to v_0 than v, contradiction.

Finally, we can cyclically order the directed edges starting in v_0 according to their slopes. Then, the sectors U_e are interlaced, i.e., U_e contains both $U_{e_-}^+$ and $U_{e_+}^-$, where e_-, e, e_+ are consecutive according to the cyclic order. As a consequence, the union of all these U_e covers the whole of $V(\Lambda)$. □

Corollary A.3. *Let Λ be a parallelogram-graph.*

(i) For any two vertices $v, v' \in V(\Lambda)$, there exists a directed edge e of Λ starting in v such that v and v' can be connected by a directed path such that the arguments of all edges lie in $[\arg(e), \arg(e) + \pi)$. If all interior angles of faces are bounded by α_0 from below, the arguments of all edges even lie in $[\arg(e), \arg(e) + \pi - \alpha_0]$.

(ii) For any center of a parallelogram $Q \in V(\diamond)$ and a vertex $v' \in V(\Lambda)$, there exists a directed edge e of Λ adjacent to Q such that Q and v can be connected by a directed path starting with two half-edges connecting Q with an incident vertex, and continuing with a directed path on Λ to v, in such a way that the arguments of the two half-edges and all the other edges lie in $[\arg(e), \arg(e) + \pi)$. If all interior angles of faces are bounded by α_0 from below, the arguments of all edges even lie in $[\arg(e), \arg(e) + \pi - \alpha_0]$.

Proof. The first part is an immediate consequence of Lemma A.2. For the second part, we apply Lemma A.2 to the subdivided parallelogram-graph Λ' obtained by splitting any parallelogram into four smaller parallelograms congruent to each other, such that the new vertex set is given by the vertices, the midpoints of edges, and the centers of faces of Λ. Clearly, all interior angles of faces are still bounded by α_0 from below. To get a directed path containing only two half-edges in the beginning, we modify the directed path on Λ' given by Lemma A.2.

If two consecutive edges of this path connect first a midpoint of an edge e of a parallelogram with its center, and then the center with the midpoint of an edge e' that is not the opposite (and parallel) edge, we can equivalently consider two edges on Λ' that pass over the vertex e and e' are sharing.

If three consecutive edges of this path connect first a midpoint of an edge e of a parallelogram with its center, and then the center with the midpoint of the edge that is the opposite (and parallel) edge, and finally this midpoint with a vertex, we can equivalently consider three edges on Λ' that connect the first midpoint with a vertex of $V(\Lambda)$, and then go along an edge e' of the parallelogram not parallel to e.

At each step, we do not change the number of edges of the path, but we reduce the number of $Q' \in V(\diamond)$ contained in the path by one. We cannot modify the path any more only if Q is the only remaining vertex of $V(\diamond)$. Then, the $(2k-1)$-th together with the $2k$-th edge, $k \geq 2$, form actually a directed edge on Λ. $\square$

To perform our computations in Sections 4.2 and 4.3, we needed not only that the interior angles were bounded, but also that the ratio of side lengths of all parallelograms were bounded. For general quad-graphs, we used instead boundedness of side lengths to show in Theorem 2.31 of Chapter 2 that essentially constant functions are the only discrete harmonic function whose difference functions on $V(\Gamma)$ and $V(\Gamma^*)$ have asymptotics $o(v^{-1/2})$. When the side lengths are bounded, the ratio of side lengths is trivially bounded as well. Under the assumption of bounded interior angles, the converse is true for parallelogram-graphs.

Proposition A.4. *Let Λ be a parallelogram-graph, and assume that there are $\alpha_0, q_0 > 0$ such that $\alpha \geq \alpha_0$ and $e/e' \geq q_0$ for all interior angles α and two side lengths e, e' of any parallelogram of Λ. Then, there exist positive numbers $E_1 > E_0$ such that $E_1 \geq e \geq E_0$ for all edge lengths e.*

Proof. Let $Q' \in V(\diamond)$ be fixed with edge lengths $e_1 \geq e_0$, and let $Q \in V(\diamond)$ be another parallelogram with center x. In the following, we construct a sequence of n strips such that any two consecutive strips are crossing each other, the first one contains Q', the last one contains Q, and $n \leq N := \lceil \lfloor 2\pi/\alpha_0 \rfloor /2 \rceil$. Then, it follows that the side lengths of X are bounded by $E_0 := e_0 q_0^N$ and $E_1 := e_1 q_0^{-N}$.

Let S_0 be a strip containing Q'. If $Q \in S_0$, we are done. Otherwise, we choose the common parallel a_{S_0} in such a way that x lies in the region $-a_{S_0}$ is pointing to. Since S_0 is monotone in the direction ia_{S_0} and interior angles are uniformly bounded, the ray $x + ta_{S_0}$, $t > 0$, intersects S in exactly one line segment. Let y_0 be the first intersection point, and Q_0 a quadrilateral of S containing y_0.

Because Λ is locally finite, the line segment connecting x and y_0 intersects only finitely many parallelograms. Through any such parallelogram at most two strips are passing. Thus, only a finite number of strips intersect this line segment. Therefore, we can choose a strip S_1 intersecting $S_0^{Q_0}$ in a parallelogram $Q_{0,1}$ such that S_1 does not contain Q, and does not intersect the line segment connecting x and y_0. Moreover, we require that $Q' \notin S_0^{Q_{0,1}}$. Now, choose the common parallel a_{S_1} of S_1 in such a way that $\pi + \arg(a_{S_0}) > \arg(a_{S_1}) > \arg(a_{S_0})$. By construction, x lies in the region $-a_{S_1}$ is pointing to. Note that $S_1^{Q_{0,1}}$ cannot cross S_0 a second time.

Suppose we have already constructed the strip S_k with common parallel a_{S_k}, $k > 0$, and x lies in the region $-a_{S_k}$ is pointing to. S_k shall not intersect the line segments connecting x and y_0, or connecting x and y_{k-1}. Moreover, assume that the semi-infinite part $S_k^{Q_{k-1,k}}$ starting in the intersection $Q_{k-1,k}$ of S_k with S_{k-1} does not cross S_0.

Let y_k be the first intersection of the ray $x + ta_{S_k}$, $t > 0$, with a quadrilateral $Q_k \in S_k$. By the same arguments as above, there exists a strip S_{k+1} intersecting $S_k^{Q_{k-1,k}} \cap S_k^{Q_k}$, that does not contain Q, and does not intersect the line segments connecting x and y_0, or connecting x and y_k. Choose its common parallel $a_{S_{k+1}}$ in such a way that $\pi + \arg(a_{S_k}) > \arg(a_{S_{k+1}}) > \arg(a_{S_k})$. By construction, x lies in the region $-a_{S_{k+1}}$ is pointing to. If the semi-infinite part $S_{k+1}^{Q_{k,k+1}}$ starting in the intersection $Q_{k,k+1}$ with S_k does not cross S_0, we continue this procedure.

After at most $l := \lceil 2\pi/\alpha_0 \rceil$ steps, we end up with a strip S_l such that $S_l^{Q_{l-1,l}}$ intersects S_0. Indeed, let us suppose the contrary, that is, let us suppose that all $S_2^{Q_{1,2}}, \ldots, S_l^{Q_{l-1,l}}$ do not cross S_0.

By assumption, $\arg(a_{S_k}) + \pi - \alpha_0 \geq \arg(a_{S_{k+1}}) \geq \arg(a_{S_k}) + \alpha_0$. It follows that the first j such that $\arg(a_{S_j})$ is greater or equal than $\arg(a_{S_0}) + 2\pi$ satisfies $j \leq \lceil 2\pi/\alpha_0 \rceil$. In addition, $\arg(a_{S_j}) < \arg(a_{S_0}) + 3\pi - \alpha_0$.

By construction, S_j does not intersect the line segment connecting x and y_{j-1}. Moreover, $S_j^{Q_{j-1,j}}$ cannot cross S_{j-1} a second time. It follows that $S_j^{Q_{j-1,j}}$ cannot

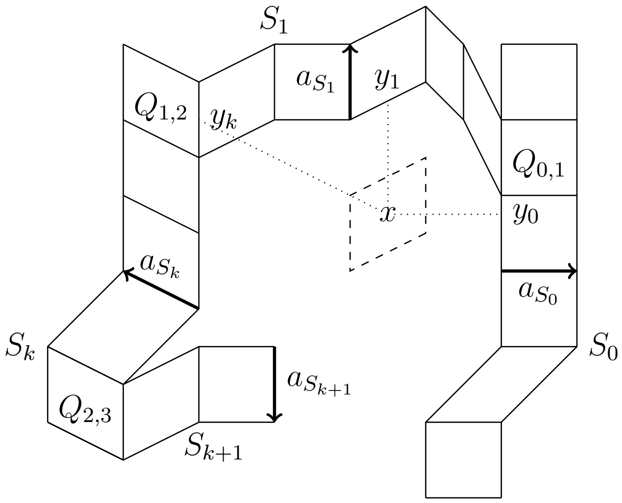


Figure A.1: Schematic picture of the proof of Proposition A.4

intersect the ray $x + ta_{S_{j-1}}$, $t > 0$.

Also, $S_j^{Q_{j-1,j}}$ does neither cross S_0 nor does it intersect the line segment connecting x and y_0, so it does not intersect the ray $x+ta_{S_0}$, $t > 0$. Thus, $S_j^{Q_{j-1,j}}$ is contained in the cone with tip x spanned by $a_{S_{j-1}}$ and a_{S_0} (with angle less than π). This contradicts the monotonicity of $S_j^{Q_{j-1,j}}$ into the direction ia_{S_j}, because the ray $x + ta_{S_j}$, $t > 0$, is not contained in the interior of the cone above.

In summary, we found a cycle of m strips $S_0, S_1, \dots, S_{m-1}$ surrounding x, where $m \le \lfloor 2\pi/\alpha_0 \rfloor + 1$. Actually, $m \le \lfloor 2\pi/\alpha_0 \rfloor$, because the a_{S_k} are cyclically ordered. Since only finitely many strips intersect the strip S_0 in between Q' and $Q_{0,1}$, we can assume that Q' is contained in $S_0^{Q_{m-1,0}}$.

These m strips determine a bounded region x is contained in. If $Q' \neq Q_{m-1,0}$, we look at the semi-infinite part of the strip $\tilde{S}_0$ different from S_0, that passes through Q' and goes into the interior of the bounded region above. It has to intersect one of the strips $S_1, \dots, S_{m-1}$, say S_k. Then, $S_0, \dots, S_k, \tilde{S}_0$ or $\tilde{S}_0, S_k, \dots, S_{m-1}, S_0$ determine a bounded region x is contained in (Q may be an element of $\tilde{S}_0$). Clearly, they are at most $\lfloor 2\pi/\alpha_0 \rfloor$ such strips, and Q' lies on an intersection.

If $Q \notin \tilde{S}_0$, a strip S_Q containing Q has to cross two different strips of the cycle due to local finiteness. In the same way as above, we can find a cycle of at most $m' \le \lfloor 2\pi/\alpha_0 \rfloor$ strips $S'_0, S'_1, \dots, S'_{m'-1}$ such that Q lies on one of the strips, say S'_k, and the intersection of S'_0 and $S'_{m'-1}$ is Q'. If $k \le \lceil m'/2 \rceil$, we choose the sequence of strips $S'_0, S'_1, \dots, S'_k$; otherwise, we take $S'_{m'-1}, S'_{m'-2}, \dots, S'_k$. Any two consecutive strips are crossing each other, Q' is on the first strip, Q on the last one, and there are at most $\lceil \lfloor 2\pi/\alpha_0 \rfloor /2 \rceil$ of them. $\square$

Remark. In general, the bound $\lceil \lfloor 2\pi/\alpha_0 \rfloor /2 \rceil$ in the proof is optimal. Indeed, consider n rays emanating from 0 such that the angle between any two neighboring rays is $2\pi/n$. In each of the n segments, choose the quad-graph combinatorially equivalent to the positive octant of the integer lattice that is spanned by two consecutive rays. For example, if $n = 4$, we obtain $\mathbb{Z}^2$. Then, any strip passes through exactly two adjacent segments, and $\lceil n/2 \rceil$ is the optimal bound.

In the case of a square lattice, the situation is much easier. For any two fixed orthogonal strips, any other strip is crossing one of them. If we could find a finite number of strips such that every other is crossing one of the chosen ones, this would directly imply Proposition A.4. However, this might be impossible, even for quasicrystallic rhombic quad-graphs. For example, Figure A.2 shows a cut-out of a counterexample, which is combinatorially equivalent to the two-dimensional subcomplex in the cubic lattice $\mathbb{Z}^3$ consisting of L-shapes translated by $(1, 1, 1)$.

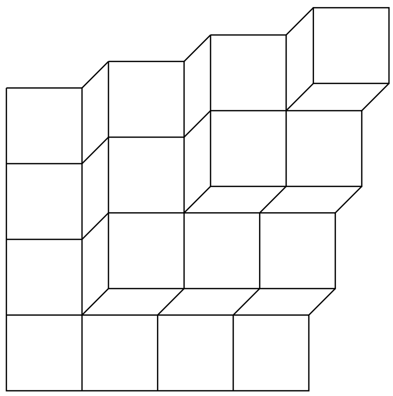


Figure A.2: Quasicrystallic rhombic quad-graph

Appendix B

ABS List

In the following, we will list the quad-equations of the ABS list that are of type Q, and present the additive and multiplicative long leg functions φ and Φ, respectively, that were elementary for our consideration in Chapter 5.

In contrast to the original formulation of the equation (Q4) by Adler, Bobenko, and Suris in [ABS03] that was based on the Weierstraß normalization of an elliptic curve, we use the Jacobian formulation that was first proposed by Hietarinta in [Hie05]. Bobenko and Suris used the Jacobi normalization in the appendix of their paper [BS10b]. Our formulation follows their presentation, but differs slightly. We will indicate which (simple) transformations of variables and parameters relate our notation to the one in [BS10b].

For parameters α, β, let $\theta := \alpha - \beta$. Note that the generalized action functionals of integrable quad-equations discussed in Section 5.2 correspond to a general choice of parameters θ.

List Q:

$(\mathrm{Q1})_{\delta=0}$: $\quad Q = \alpha(xu + yv) - \beta(xv + yu) - (\alpha - \beta)(xy + uv),$

$\quad i\varphi(x, y; \theta) = \frac{i\theta}{x-y}.$

Whereas the formulation of the quad-equation coincides with the one in [BS10b], their long leg function

$$i\varphi(x, y; \theta) = \frac{\theta}{x - y}$$

is related to ours by the transformation $\alpha \mapsto i\alpha$ for all parameters α.

$(\mathrm{Q1})_{\delta=1}$: $\quad Q = \alpha(xu+yv) - \beta(xv+yu) - (\alpha-\beta)(xy+uv) - \alpha\beta(\alpha-\beta),$

$\Phi(x,y;\theta) = \frac{x-y+i\theta}{x-y-i\theta}.$

In [BS10b], the term $+\alpha\beta(\alpha-\beta)$ instead of $-\alpha\beta(\alpha-\beta)$ appeared in the quad-equation, and their long leg function was given by

$$\Phi(x,y;\theta) = \frac{x-y+\theta}{x-y-\theta}.$$

The transformation $\alpha \mapsto i\alpha$ for all parameters α relates these two formulations.

$(\mathrm{Q2})$: $\quad Q = \alpha(xu+yv) - \beta(xv+yu) - (\alpha-\beta)(xy+uv)$
$\quad -\alpha\beta(\alpha-\beta)(x+u+y+v) - \alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2),$

$x = X^2,$

$\Phi(x,y;\theta) = \frac{(X-Y+i\theta)(X+Y+i\theta)}{(X-Y-i\theta)(X+Y-i\theta)}.$

As before, the transformation $\alpha \mapsto i\alpha$ for all parameters α relates the formulation in [BS10b] to ours. There, $+\alpha\beta(\alpha-\beta)(x+u+y+v)$ instead of $-\alpha\beta(\alpha-\beta(x+u+y+v))$ appeared in the quad-equation, and their long leg function was

$$\Phi(x,y;\theta) = \frac{(X-Y+\theta)(X+Y+\theta)}{(X-Y-\theta)(X+Y-\theta)}.$$

$(\mathrm{Q3})_{\delta=0}$: $\quad Q = \sin(\alpha)(xu+yv) - \sin(\beta)(xv+yu) - \sin(\alpha-\beta)(xy+uv),$

$x = \exp(X),$

$\Phi(x,y;\theta) = \exp(-i\theta)\frac{\sinh\left(\frac{X-Y+i\theta}{2}\right)}{\sinh\left(\frac{X-Y-i\theta}{2}\right)},$

$\Phi'(x,y;\theta) = \frac{\sinh\left(\frac{X-Y+i\theta}{2}\right)}{\sinh\left(\frac{X-Y-i\theta}{2}\right)}.$

Our quad-equation coincides with the one in [BS10b], but they used the variable transformation $x = \exp(iX)$ instead. Now, the transformation $X \mapsto -iX$ for all new variables X relates their long leg function given by

$$\Phi(x,y;\theta) = \frac{\sin\left(\frac{X-Y+\theta}{2}\right)}{\sin\left(\frac{X-Y-\theta}{2}\right)}$$

to ours if one does not take the factor $\exp(-i\theta)$ into account. However, multiplying the short leg functions by $\exp(-i\alpha)$ or $\exp(-i\beta)$, respectively, shows that these three-leg forms are equivalent.

$$
\begin{aligned}
(\mathrm{Q3})_{\delta=1}: \quad & Q = \sin(\alpha)(xu+yv) - \sin(\beta)(xv+yu) - \sin(\alpha-\beta)(xy+uv) \\
& \quad + \sin(\alpha-\beta)\sin(\alpha)\sin(\beta), \\
& x = \cosh(X), \\
& \Phi(x,y;\theta) = \frac{\sinh\left(\frac{X-Y+i\theta}{2}\right)\sinh\left(\frac{X+Y+i\theta}{2}\right)}{\sinh\left(\frac{X-Y-i\theta}{2}\right)\sinh\left(\frac{X+Y-i\theta}{2}\right)}.
\end{aligned}
$$

Whereas our quad-equation is formulated in the same way as in [BS10b], they used the variable transformation $x = \sin(X)$ instead. Their long leg function was given by

$$
\Phi(x,y;\theta) = \frac{\cos\left(\frac{X-Y+\theta}{2}\right)\sin\left(\frac{X+Y+\theta}{2}\right)}{\cos\left(\frac{X-Y-\theta}{2}\right)\sin\left(\frac{X+Y-\theta}{2}\right)};
$$

the transformation $X \mapsto \pi/2 - iX$ for all new variables X relates their long leg function to ours.

$$
\begin{aligned}
(\mathrm{Q4}): \quad & Q = \mathrm{sn}(\alpha)(xu+yv) - \mathrm{sn}(\beta)(xv+yu) - \mathrm{sn}(\alpha-\beta)(xy+uv) \\
& \quad + \mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\alpha)\,\mathrm{sn}(\beta)(1+\kappa^2 xuyv), \\
& x = \mathrm{sn}(\pi/2 - iX), \\
& \Phi(x,y;\theta) = \frac{\vartheta_1\left(\frac{X-Y+i\theta}{2i}\right)\vartheta_4\left(\frac{X-Y+i\theta}{2i}\right)\vartheta_1\left(\frac{X+Y+i\theta}{2i}\right)\vartheta_4\left(\frac{X+Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{X-Y-i\theta}{2i}\right)\vartheta_4\left(\frac{X-Y-i\theta}{2i}\right)\vartheta_1\left(\frac{X+Y-i\theta}{2i}\right)\vartheta_4\left(\frac{X+Y-i\theta}{2i}\right)}.
\end{aligned}
$$

The quad-equations are the same in both formulations, but $x = \mathrm{sn}(X)$ in [BS10b]. The transformation $X \mapsto \pi/2 - iX$ for all new variables X relates the long leg function in [BS10b],

$$
\Phi(x,y;\theta) = \frac{\vartheta_2\left(\frac{X-Y+\theta}{2}\right)\vartheta_3\left(\frac{X-Y+\theta}{2}\right)\vartheta_1\left(\frac{X+Y+\theta}{2}\right)\vartheta_4\left(\frac{X+Y+\theta}{2}\right)}{\vartheta_2\left(\frac{X-Y-\theta}{2}\right)\vartheta_3\left(\frac{X-Y-\theta}{2}\right)\vartheta_1\left(\frac{X+Y-\theta}{2}\right)\vartheta_4\left(\frac{X+Y-\theta}{2}\right)},
$$

to ours.

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