Q7i: We compute the commutator subgroup of D_6. Any element of the commutator subgroup is of the form

$$(\sigma^i\rho^j)(\sigma^k\rho^l)(\sigma^i\rho^j)^{-1}(\sigma^k\rho^l)^{-1}$$

We can use the relations on D_{2n} as outlined in chapter 1.2 of the textbook, to compute this product.

$$\begin{aligned}
(\sigma^i\rho^j)(\sigma^k\rho^l)(\sigma^i\rho^j)^{-1}(\sigma^k\rho^l)^{-1} &= (\sigma^i\rho^j\sigma^k\rho^l)(\rho^{-j}\sigma^{-i})(\rho^{-l}\sigma^{-k}) \\
&= (\sigma^i\rho^j\sigma^k\rho^l)(\rho^{-j}\sigma^{i})(\rho^{-l}\sigma^{k}) \\
&= \sigma^i\rho^{j-l}(\sigma^k\rho^{-j})(\sigma^i\rho^{-l})\sigma^k \\
&= \sigma^i\rho^{j-l}\rho^j\sigma^k\sigma^i\rho^{-l}\sigma^k \\
&= \sigma^i\rho^{2j-l}\sigma^{i+k}\rho^{-l}\sigma^k \\
&= \sigma^i\rho^{2l-2j}\sigma^{i+2k} \\
&= \sigma^{2i+2k}\rho^{2(j-l)} \\
&= \rho^{2(j-l)}
\end{aligned}$$

Hence the commutator subgroup of D_2n is simply just the set of all rotations. Therefore, D_6/D_6' is just $\{e, \sigma\}$. We can identify this subgroup with $\mathbb{Z}/2\,\mathbb{Z}$ with the isomorphism $\phi(\sigma) = 1$.

Q7ii: Similarly to 7i, we know that the commutator subgroup of D_8 is all the rotations, hence

$$D_8/D_8' \cong \{e, \sigma\} \cong \mathbb{Z}/2\,\mathbb{Z}$$

Q7iii: We compute the commutator subgroup of $\mathcal{Q}_8$. We compute all elements of the form $ghg^{-1}h^{-1}$. First assume that either g is equal to ± 1. We get that

$$ghg^{-1}h^{-1} = (\pm 1)h(\pm 1)^{-1}h^{-1} = (\pm 1)(\pm 1)^{-1}hh^{-1} = 1$$

If we had instead $h = \pm 1$ we would have the exact same conclusion. Now if $g = h$, we get that

$$\begin{aligned}
ghg^{-1}h^{-1} &= ggg^{-1}g^{-1} \\
&= gg^{-1} \\
&= 1
\end{aligned}$$

Now finally suppose that $g \neq h$ and both are not ± 1. Note that this implies that $g^{-1} = -g$ We compute:

$$\begin{aligned}
ghg^{-1}h^{-1} &= gh(-g)(-h) \\
&= g(hg)h \\
&= -g^2h^2 && \text{(from anticommutativity of } \mathcal{Q}_8\text{)} \\
&= -1 && \text{(since } g^2 = h^2 = -1\text{)}
\end{aligned}$$

Thus we have that $\mathcal{Q}_8' = \{\pm 1\}$. Hence

$$\mathcal{Q}_8/\mathcal{Q}_8' \cong \{1, i, j, k\}$$