5.5.63a: Let $\{x_n\}$ be a sequence of orthonormal vectors so that $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{H}$. By Bessel's Inequality, we have that

$$\sum_n |\langle x_n, y\rangle| \leq |y|^2$$

for all y. Thus the sequence $\{\langle x_n, y\rangle\}$ absolutely converges for any choice of y. We have that $\sum_{n=N} |\langle x_n, y\rangle| \to 0$ as $N \to \infty$, and so $\langle x, y\rangle = 0$ for all y. Therefore $x = 0$.

b: Let $x \in B$, let $\varepsilon > 0$. We wish to find a $y \in S$ so that $\langle x - y, x - y\rangle < \varepsilon$. We know that

$$\langle x - y, x - y\rangle = |x|^2 + 2\langle x, y\rangle + |y, y| \leq 2(1 + \langle x, y\rangle).$$

We choose $y \in S$ according to theorem 5.8 so that $f_y(x) = \langle x, y\rangle < \frac{\varepsilon - 1}{2}$. We now claim that any $x \in B$ is the weak limit of a sequence in S. Let $\varepsilon > 0$. Take $y \in S$ so that $|x - y| < \varepsilon$. Take a sequence $\{y_n\} \subset S$ so that $y_n \to y$ weakly. Then, we have that

$$\langle x - y_n, v\rangle = \langle x - y, v\rangle + \langle y - y_n, v\rangle \leq \langle x - y, v\rangle \leq |x - y| \cdot |v| < \varepsilon \cdot |v|.$$

As desired.