

Q5: We first claim that for all  $k \geq 0$ ,  $f_n \rightarrow 0$ . Let  $\varepsilon > 0$ , then for any  $x$ , take  $N > \frac{|x^k| - \varepsilon|x^2|}{\varepsilon}$ . Then, if we take  $n \geq N$ , we see that  $|f_n(x)| < \varepsilon$ . Now notice if  $f_n \rightrightarrows f$ , then  $f_n \rightarrow f$  as well. By uniqueness of limits, if sequence is uniformly convergent it must converge to 0. We claim that only for  $k = 0, 1$  the sequence  $f_n(x) = \frac{x^k}{x^2+1}$  converges uniformly. For  $k = 1$ , take  $N = \frac{1}{\varepsilon}$ . Then if  $n \geq N$ , then we have that  $|\frac{1}{x^2+n}| < \varepsilon$ . For  $k = 1$ , Take  $N > \max\{\frac{\varepsilon}{x} - x^2\}$ . Then if  $n > N$ ,  $n > \frac{x}{\varepsilon} - x^2$ , we have that  $\frac{x}{x^2+n} < \varepsilon$ . However, if  $k > 2$ , we would need  $\frac{x^k}{\varepsilon} - x^2 < n$  to hold for all  $x$ , for any given  $n$  sufficiently large. We see from properties of polynomials that this will not happen. We now claim that on any bounded subset of  $\mathbb{R}$ , for any  $k$ ,  $f_n$  converges. Without loss of generality, assume that  $0 < x < M$ . Then  $f_n(x) = \frac{x^k}{x^2+n} < \frac{M^k}{n}$ . If we take  $n$  sufficiently large we can make this as small as we desire. Hence  $f_n$  uniformly converges to 0.