Q6: $a \implies b$: Suppose that $f : M \to N$ is a diffeomorphism. Suppose that $\psi_1 : V \to \mathbb{R}^n$ is a coordinate chart of $\mathcal{B}$. Then we have that $\psi_1 \circ f$ is a homeomorphism from $f^{-1}(V)$ to $\mathbb{R}^n$ since f is a homeomorphism. Now given φ, coordinate chart of M, $(\psi_1 \circ f) \circ \varphi^{-1}$ is a diffeomorphism since f is. If ψ_2 is another chart of N, then

$$(\psi_2 \circ f) \circ (\psi_1 \circ f)^{-1} = \psi_2 \circ f \circ f^{-1} \circ \psi_1^{-1} = \psi_2 \circ \psi_1^{-1} \in \mathcal{C}^\infty .$$

Now if $\psi \circ f$ is a coordinate chart of $\mathcal{A}$, $\psi \circ f \circ f^{-1} = \psi$ is a homeomorphism on N into $\mathbb{R}^n$. We can see that for any other coordinate chart ψ_2 on N, $(\psi \circ f) \circ (\psi_2 \circ f)^{-1}$ is $\mathcal{C}^\infty$, since they are charts on $\mathcal{A}$, but

$$(\psi \circ f) \circ (\psi_2 \circ f)^{-1} = \psi \circ \psi_2^{-1}.$$

So ψ is $\mathcal{C}^\infty$ related to any other chart ψ.

$b \implies c$: Suppose $g \in \mathcal{C}^\infty$ on $(N, \mathcal{B})$. Then for any chart $\psi \in \mathcal{B}$, we have that $g \circ \psi^{-1} \in \mathcal{C}^\infty$. Then

$$g \circ f \circ \varphi^{-1} = (g \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}).$$

By assumption $\psi \circ f$ is a chart on $(M, \mathcal{A})$, so its composition with φ^{-1} is smooth. We have that $g \circ f \circ \varphi^{-1}$ is the composition of smooth functions hence smooth. Now suppose that $g \circ f$ is $\mathcal{C}^\infty$ on $(M, \mathcal{A})$. Therefore,

$$g \circ f \circ \varphi^{-1} = (g \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}).$$

We have that $(\psi \circ f \circ \varphi^{-1})$ is a diffeomorphism by assumption, so $g \circ \psi$ is $\mathcal{C}^\infty$.

$c \implies a$. Take $g = \psi \in \mathcal{B}$. We know that ψ is $\mathcal{C}^\infty$ on $(N, \mathcal{B})$, so we must have that $\psi \circ f$ is smooth on $(M, \mathcal{A})$ i.e. $\psi \circ f \circ \varphi^{-1}$ is smooth. Since f is a homeomorphism, we conclude that f is a diffeomorphism.