Q3: We will represent $U(2)$ as the zero set of a function $g : \mathbb{R}^8 \to \mathbb{R}^4$. Define g in the following way:

$$g(x) : x \mapsto \begin{pmatrix} x_1^2 - x_2^2 + x_3^2 - x_4^2 - 1 \\ x_1x_5 - x_1x_6 + x_2x_5 - x_2x_6 + x_3x_7 - x_3x_8 + x_4x_7 - x_4x_8 \\ x_1x_5 + x_1x_6 - x_2x_5 - x_2x_6 + x_3x_7 + x_3x_8 - x_4x_7 - x_4x_8 \\ x_5^2 - x_6^2 + x_7^2 - x_8^2 - 1 \end{pmatrix}$$

If we consider 2 by 2 complex matrices as

$$\begin{bmatrix} x_1 + x_2 & x_3 + x_4 \\ x_5 + x_6 & x_7 + x_8 \end{bmatrix}$$

Where the even indices correspond to complex coefficients and odd to real, then we have that g will be 0 on all of $U(2)$. We compute the differential of g as

$$Dg = \begin{bmatrix} 2x_1 & -2x_2 & 2x_3 & -2x_4 & 0 & 0 & 0 & 0 \\ x_5 - x_6 & x_5 - x_6 & x_7 - x_8 & x_7 - x_8 & x_1 + x_2 & -x_1 - x_2 & x_3 + x_4 & -x_3 - x_4 \\ x_5 + x_6 & -x_6 - x_5 & x_7 + x_8 & -x_7 - x_8 & x_1 - x_2 & x_1 - x_2 & x_3 - x_4 & x_3 - x_4 \\ 0 & 0 & 0 & 0 & 2x_5 & -2x_6 & 2x_7 & -2x_8 \end{bmatrix}$$

We see by inspection that Dg has 4 linearly independant columns on $U(2)$ and thus the rank of $Dg = 4$. We conclude that $U(2)$ is the 0 set of a smooth function, and hence it is a manifold of real dimension 4.