Q4: Letting

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2 + a^2},$$

and

$$g(z) = \frac{\pi}{a} \cdot \frac{\sinh(2\pi a)}{\cosh(2\pi a) - \cos(2\pi z)} = \frac{\pi}{a} \cdot \frac{-i\sin(2\pi i a)}{\cos(2\pi i a) - \cos(2\pi z)}.$$

Notice that $f(z)$ has simple poles of degree two at $z = -n \pm ia$. Note $g(z)$ does as well, following from the 2π periodicity of cos. We also note that $f(z+1) = f(z)$ and $g(z+1) = g(z)$. Finally the last property we will discuss is the boundary condition. Once can see that since

$$\cos(2\pi z) = \frac{1}{2}(e^{2\pi i z} + e^{-2\pi i z}),$$

if we write $z = x + iy$, we have that $g(z) \to 0$ uniformly as $y \to \infty$. Similarly, $f(z) \to 0$ as We now prove that $f = g$. In a neighbourhood of every pole $z = -n \pm ia$, we can write

$$f = \frac{1}{(z - n \pm ia)^2 + a^2} + \tilde{f}(z), g = \frac{1}{(z - n \pm ia)^2 + a^2} + \tilde{g}(z).$$

For some holomorphic functions $\tilde{f}, \tilde{g}$. Hence $f - g$ willl be holomorphic. On any strip given by $a_1 \leq z \leq a_2$ intersected with $|y| \leq b$, we have that $f - g$ is holomorphic and hence bounded. For $|y| > b$, the boundary condition tells us that $f - g \to 0$ and hence is bounded. Extending to $\mathbb{C}$ with periodicity yields that $f - g$ is bounded and hence constant by Liouvilles Theorem. Finally, the boundary condition tells us that the constant is 0 so in fact $f = g$.