Problem 12. *Ziqian*

(a) Let $\{y_i\}_{i=1,..,6}$ be the 6 neighbours of point x configured as below:

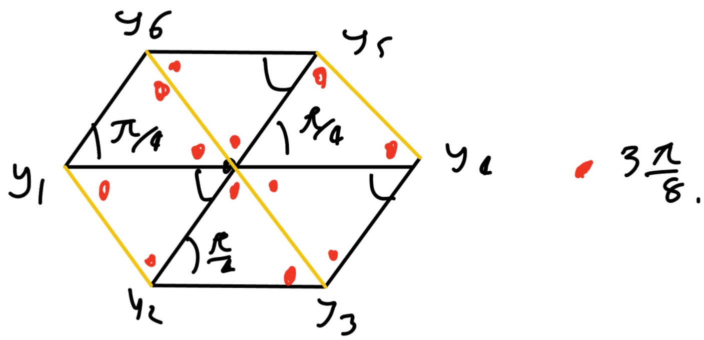

Figure 1

A discrete harmonic function must satisfy

$$\sum_{i=1}^{6} c(xy_i)\,(u(x) - u(y_i)) = 0.$$

Therefore

$$u(x) = \frac{1}{\sum_{i=1}^{6} c(xy_i)} \sum_{i=1}^{6} c(xy_i) u(y_i).$$

We compute that $c(xy_i) = \cot(\frac{3\pi}{8}) = \frac{1}{\sqrt{2}+1}$ when $i = 1, 2, 4, 5$. Else $c(xy_i) = 1$, so we write

$$u(x) = (4\sqrt{2} - 2) \sum_{i=1}^{6} c(xy_i) u(y_i).$$

(b) We characterize the behaviour of v, the harmonic conjugate to u. First since harmonic functions are defined up to a complex additive, we can set $v(l_{xy_4}) = 0$. Since discrete conjugate to u, we have

$$v(l_{xy_5}) - v(r_{xy_5}) = c(xy_5)(u(h_{xy_4}) - u(t_{xy_4})) = 0.$$

Thus $v(l_{xy_5}) = 0$. Now along xy_4, we have that

$$v(l_{xy_4}) - v(r_{xy_4}) = c(xy_4)(u(h_{xy_4}) - u(t_{xy_4})) = \frac{-1}{\sqrt{2}+1}.$$

Therefore $v(r_{xy_4}) = \frac{1}{\sqrt{2}+1}$. We repeat this, incrementing v by $c(e)(u(h_e) - u(e_t))$ whenever we pass over an edge. Therefore v will be constant on faces that are connected by rotating the oblique lines by $-\frac{5\pi}{8}$.