BASIC PROPERTIES OF DISCRETE ANALYTIC FUNCTIONS

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1. **Introduction.** Of concern in this paper are complex-valued functions defined at the points of the complex plane whose coordinates are integers. These points form a lattice which breaks up the plane into unit squares. A function f is said to be discrete analytic at one of these squares if the difference quotient across one diagonal is equal to the difference quotient across the other diagonal,

$$\text{(c)} \qquad [f(z + 1 + i) - f(z)]/(1 + i) = [f(z + i) - f(z + 1)]/(i - 1).$$

If $f = u + iv$ where u and v are real, then it is seen that discrete analyticity implies that u and v satisfy a pair of difference equations which are analogous to the Cauchy-Riemann equations. If f is analytic in a region it results that u and v are discrete harmonic in that region. That is, they satisfy the Laplacian difference equation.

The above definition of analyticity was introduced by Jacqueline Ferrand (Lelong) [5]. She developed several interesting analogies with ordinary analytic functions. This paper extends her work in several directions. These new developments include analogies of the function z^{-1}, the Cauchy integral formula, Liouville's theorem, Harnack's inequality, polynomial expansions, and Hilbert transforms. Some of the developments here have no direct analogy in the classical continuous theory. These include the notion of duality, bipolynomials, and an operational calculus.

The theory and application of discrete harmonic functions have received considerable attention in the literature. Much of this work may be brought to bear on the present problem. In particular the paper of H. A. Heilbronn [6] concerning discrete harmonic polynomials and the paper of A. C. Allen and B. H. Murdoch [1] concerning the analog of Poisson's integral formula have been of value in the preparation of this paper.

Rufus Isaacs [7, 8] developed a theory of discrete analytic functions based on the following definition of analyticity:

$$\text{(a)} \qquad f(z + 1) - f(z) = [f(z + i) - f(z)]/i.$$

He preferred this definition to

$$\text{(b)} \qquad f(z + 1) - f(z - 1) = [f(z + i) - f(z - i)]/i,$$

which he also considered. It appears that definition (b) is essentially equivalent to the Ferrand definition (c). It is not apparent that (a) and (c) have any

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direct relationship. Definition (a) is simpler than (c) and leads to simpler algebraic formulae. On the other hand the harmonic functions corresponding to (a) do not satisfy the standard Laplacian difference equation.

Isaacs defined a product operation for two analytic functions of type (a) provided one of them is a polynomial [8]. This product is also analytic. In the last section of this paper a similar theory is formulated for analytic functions of type (c). These considerations lead to certain interesting identities for the Green's function of the Laplacian difference equation.

The concept of a discrete Hilbert transform is far from new. Contributions to the theory of such transforms have been made by M. Riesz [14], O. A. Varsavsky [15], and A. P. Calderon and A. Zygmund [16]. In this paper it is shown that the discrete Hilbert transform relates the boundary values of the real and the imaginary parts of a function which is discrete analytic in the upper half-plane.

Synonyms for "discrete analytic" are "preholomorphic" and "monodiffric." "Preharmonic" is synonymous with "discrete harmonic." Various other discrete analogs of classical continuous concepts are to be introduced; when no confusion results, the qualifier "discrete" is dropped.

2. **Conjugates, duals, and derivatives.** The *lattice points* of the complex plane are the points $z = m + in$ where m and n take on the values $0, \pm 1, \pm 2, \cdots$. Of concern in this paper are complex-valued functions f defined at the lattice points. If z_0 is a lattice point, the points z_0, $z_0 + 1$, $z_0 + 1 + i$, $z_0 + i$ are the vertices of a *unit square* associated with the point z_0. A unit square is regarded as a closed two-dimensional point set. *Regions* are defined as the union of unit squares. A *simple region* R is a simply connected set which is the union of a finite number of unit squares. Thus the boundary of a simple region R is a simple closed curve B which is composed of edges of unit squares. For example, Figure 1 depicts the union of seven unit squares to form a simple

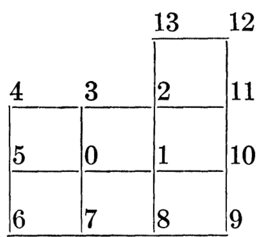

Fig. 1

region R'. The lattice points of R' are z_0, $z_1 = z_0 + 1$, $z_2 = z_0 + 1 + i$, $\cdots$, $z_{13} = z_0 + 1 + 2i$. If f is defined at these points then as a short notation let $f_k = f(z_k)$. The *even lattice* consists of those lattice points for which $n + m$ is even. The *odd lattice* consists of those lattice points for which $n + m$ is odd. Thus if z_0 is on the even lattice, so also are z_2, z_4, $\cdots$, z_{12} while z_1, z_3,

$\cdots$, z_{13} are on the odd lattice. The lattice points of a region which are not on the boundary are termed *interior points*. Thus in the example z_0 and z_1 are interior points, the other points are boundary points.

The scheme indicated in Figure 1 gives a method of listing the lattice points of the plane as a single sequence $z_0, z_1, z_2, \cdots$. This sequence may be termed the *spiral coordinate system*.

A function f is said to be *analytic* on the square associated with z_0 if

$$f_0 + if_1 + i^2f_2 + i^3f_3 = 0. \tag{1}$$

To handle such relations it is convenient to introduce translation operators X and Y defined by

$$X^nf(z) = f(z + n), \qquad Y^nf(z) = f(z + in) \tag{2}$$

for $n = 0, \pm 1, \pm 2, \cdots$. Let

$$L = I + iX - YX - iY. \tag{3}$$

Then

$$Lf(z_0) \equiv f_0 + if_1 + i^2f_2 + i^3f_3 \tag{4}$$

so that the analyticity condition may be expressed as $Lf = 0$. Let $f = u + iv$ where u and v are the real and imaginary parts of f. Then (1) is equivalent to

$$u_2 - u_0 = v_3 - v_1 \, ; \qquad u_3 - u_1 = v_0 - v_2 \, . \tag{5}$$

These are the analogs of the Cauchy-Riemann equations. A function is said to be analytic in a region if it is analytic in each unit square of the region.

The group of rigid motions which transform the lattice into itself may be generated by the translations X and Y, a 90° rotation, and a reflection in the x axis. Of course analyticity is preserved under translation. Given a function f then a 90° rotation defines a function $F(z) = f(-iz)$. If $Lf = w$ at a certain square then at the rotated square $LF = f_3 + if_0 - f_1 - if_2 = iw$. Thus 90° rotations preserve analyticity. To consider reflections let $F(z) = f^*(z^*)$ where the star denotes the complex conjugate. At the reflected square $LF = f_3^* + if_2^* - f_1^* - if_0^* = -iw^*$. Thus reflection plus conjugation preserves analyticity.

A function which is discrete analytic at every lattice point may be termed an entire function. *If a function f is prescribed arbitrarily at the lattice points of the x and y axis, then f has a unique continuation as an entire function.* To see this, attention is focused on the first quadrant. Then $Lf = 0$ evaluates $f(i + 1)$, $f(i + 2)$, $f(i + 3)$, $\cdots$ in succession. Then $f(2i + 1)$, $f(2i + 2)$, $\cdots$ are evaluated in succession. In this manner a continuation of f is determined in the first quadrant. A similar procedure applies to the other quadrants. This continuation is unique by construction.

The following device is useful to simplify the process of continuation. Given a function f, let an associated function $\mathbf{f}$ be defined as ,$(z) = i^{x-y}\mathbf{f}(z)$. Then

$$\mathbf{f}_0 + \mathbf{f}_1 - \mathbf{f}_2 + \mathbf{f}_3 = i^{x-y}Lf.$$

Thus $f(x, y)$ is analytic if and only if

$$\mathbf{f}(x+1, y+1) = \mathbf{f}(x, y) + \mathbf{f}(x+1, y) + \mathbf{f}(x, y+1). \tag{6}$$

In particular let ψ satisfy this relation and the conditions $\psi(0, y) = 1$ for all y and $\psi(x, 0) = 0$ for all x except $x = 0$. Then it is clear that $\psi(x, y) = 0$ for $x < 0$. The function ψ is a sort of two-dimensional Fibonacci series of integers.

Let $\psi(z)$ be the entire function associated with $\psi(z)$. Let $f(z)$ be an arbitrary entire function, then

$$\begin{aligned} f(z) = f(0) - f(0) \sum_{k=1}^{\infty} [\psi(z-k) &+ \psi(-z-k) + \psi(-iz-k) \\ &+ \psi(iz-k)] + \sum_{k=1}^{\infty} [f(k)\psi(z-k) + f(-k)\psi(-z-k) \\ &+ f(ik)\psi(-iz-k) + f(-ik)\psi(iz-k)]. \end{aligned} \tag{7}$$

To prove (7) it is noted that the series converges, because for any given z there are only a finite number of non-vanishing terms. It is clear that the expression on the right is an entire function. However, the left and right sides agree on the real and imaginary axes, and by the previous theorem this completes the proof of the expansion theorem (7).

Two functions f and F are said to be *dual* if $F = (-1)^{x+y}f^*$. Here the star denotes the complex conjugate. Then at $z_0 = x + iy$

$$(Lf)^* = f_0^* - if_1^* - f_2^* + if_3^* = (-1)^{x+y}LF. \tag{8}$$

It follows from (8) that a function and its dual have the same region of analyticity. The dual of a function f will be denoted by f^-.

Let

$$L' = I - iX^{-1} - Y^{-1}X^{-1} + iY^{-1} \tag{9}$$

then

$$L'f(z_0) = f_0 + if_7 + i^2f_6 + i^3f_5$$

and

$$(I - iX^{-1} - Y^{-1}X^{-1} + iY^{-1})(I + iX - YX - iY) = 4 - YX - Y^{-1}X - YX^{-1} - Y^{-1}X^{-1}.$$

This last relation may be written as

$$L'L = LL' = -D \tag{10}$$

where

$$Df(z_0) = f_2 + f_4 + f_6 + f_8 - 4f_0\,. \tag{11}$$

A function is said to be *harmonic* at a point z_0 if $Df(z_0) = 0$. It is seen from (10) that if f is analytic in a region R, then f is harmonic at the interior points of R. The Laplacian operator D is real, so the real and the imaginary parts of f are also harmonic. It is to be noted that the operator D does not interrelate functional values on the even lattice with functional values on the odd lattice.

The usual definition of the Laplacian difference operator is not (11) but

$$D_0 g(x, y) = g(x + 1, y) + g(x - 1, y) + g(x, y + 1) + g(x, y - 1) - 4g(x, y).$$

To relate D_0 to D let A be the transformation $x' = x + y$, $y' = x - y$. Then A is a rotation of 45° followed by a dilation of value $2^{\frac{1}{2}}$. It is clear that A gives a one-to-one mapping of the whole lattice onto the even lattice. Then $A \cdot (x \pm 1, y) = (x' \pm 1, y' \pm 1)$ and $A \cdot (x, y \pm 1) = (x' \pm 1, y' \mp 1)$. Let $f(x', y') = g(x, y)$ then clearly

$$Df(x', y') = D_0 g(x, y).$$

This relation shows that any function harmonic with respect to D_0 may be transformed into a function harmonic with respect to D simply by a change of coordinates. This transformation will be tactily employed in parts to follow.

Given a lattice function h, let f be defined by

$$f = L'h. \tag{12}$$

Then $Lf = LL'h = -Dh$. Thus if h is harmonic at a point z_0, f is analytic on the square associated with z_0.

Let u be a real function defined on a simple region R and harmonic at interior points of R. Then u is the real part of an analytic function f which is analytic in R. This may be proved by induction. The theorem is obviously true if R is a unit square. Suppose that it is true for all regions containing fewer than n squares, and consider a region R with n squares. By geometric intuition one sees that it is possible to delete one of the squares of R, say S, to leave a simple region R'. One of the edges of S must be on the boundary of R; otherwise R' would be doubly connected. Moreover, one of the edges of S is on the boundary, but the opposite edge is not on the boundary; otherwise R' would be disconnected. Referring to Figure 1 it may be supposed that S is the square whose lower left corner is at z_0. Suppose also that the edge (2, 3) is on the boundary of R but that (0, 1) is not on the boundary. This is no loss of generality, because any other case may be reduced to this by a rotation.

By assumption a conjugate function v may be assigned to the lattice points of R'. If the point z_3 is in R', then the square at z_5 is in R, for otherwise R would not be bounded by a simple closed curve. Also, the square at z_6 is in R, for otherwise R' would not be bounded by a simple closed curve. It follows that z_0 is an interior point of R, so $Du = 0$ at z_0. Thus

$$(u_2 - u_0) + (u_4 - u_0) + (u_6 - u_0) + (u_8 - u_0) = 0.$$

Making use of the Cauchy-Riemann relations for the squares at z_5, z_6, and z_7 gives

$$(u_2 - u_0) + (v_5 - v_3) + (v_7 - v_5) + (v_1 - v_7) = 0.$$

Thus $u_2 - u0 = v_3 - v_1$, and the first Cauchy-Riemann equation is satisfied. If z_3 is not in R', the value of v_3 may be assigned to satisfy this equation. A symmetrical argument applies to the point z_2 and the second Cauchy-Riemann equation. The function v so extended is conjugate to u in S and hence in all of R. Let $f = u + iv$, then f is analytic in R and the proof is completed. (The extension to infinite regions is not difficult.)

If f is harmonic in a finite region, then $|f|$ takes on its maximum on the boundary. This is an easy deduction from the relation $Df = 0$ at interior points. Likewise it is seen that the real and imaginary parts of f take on their maxima on the boundary. These facts may be termed the *maximum principle*.

Now let $z_0, z_1, z_2, \cdots, z_m$ denote any chain of lattice points, that is, $|z_0 - z_1| = 1$, $|z_1 - z_2| = 1$, etc. (This sequence is not necessarily the spiral coordinate system.) Let $a = z_0$ and $b = z_m$. Then the "line integral" $\int_a^b f\, \delta z$ is defined by

$$\int_a^b f\, \delta z = \sum_{n=1}^{m} (f_n + f_{n-1})(z_n - z_{n-1})/2. \tag{13}$$

If C is a closed chain, $a = b$, and in this case, (13) may be put in the form

$$\int_C f\, \delta z = \sum_{n=1}^{m} f_n(z_{n+1} - z_{n-1})/2. \tag{14}$$

In this formula z_{m+1} is interpreted to be z_1. Let the contour (chain) be taken in a counter-clockwise direction around a unit square. Then the right side of (14) gives

$$\begin{aligned} 2\int_C f\, \delta z &= (1 - i)f_0 + (1 + i)f_1 + (-1 + i)f_2 + (-1 - i)f_3 \\ &= (1 - i)(f_0 + if_1 - f_2 - if_3) \\ &= (1 - i)Lf. \end{aligned}$$

It is clear that the line integral around a simple region is a sum of the line integrals around the unit squares. Thus if B is the boundary of a simple region R, then

$$\int_B f\, \delta z = (1 - i)2^{-1} \sum_R Lf \tag{15}$$

where $\sum_R$ denotes a summation over the unit squares contained in R. If relation (15) is compared with the classical continuous case, then it is seen that $(1 + i)\, Lf/4\pi$ corresponds to the residue. There will be no confusion if Lf at a given unit square is termed the *residue* of f at that square.

Now suppose that f is analytic in a simple region R with boundary B, then

$$\int_B f\, \delta z = 0. \tag{16}$$

Let a and z be points of R and let

$$F(z) = \int_a^z f\, \delta z + c \tag{17}$$

where c is a constant. Then from (16) it is seen that $F(z)$ does not depend on the contour joining a and z provided that the contour is in R. If p and q are neighboring points, it follows from the definition of the integral that

$$\frac{F(p) - F(q)}{p - q} = \frac{f(p) + f(q)}{2}. \tag{18}$$

In particular this gives for a unit square

$$2(F_2 - F_0) = f_0 + f_1 + if_1 + if_2$$

and

$$2i(F_3 - F_1) = -f_1 - f_2 - if_2 - if_3\,.$$

Adding these relations gives

$$-2LF = Lf.$$

Hence $LF = 0$ for each square of R. It is thereby shown that if f is analytic in a simple region R, the line integral F is a single-valued analytic function in R.

Let $F(z)$ be a given analytic function in a simple region R. Let a and b be points of R and let k be an arbitrary constant. Then

$$f(z) = \left(4 \int_b^z F^- \,\delta z + k\right)^- \tag{19}$$

is analytic in R and

$$F(z) = \int_a^z f(z)\, \delta z + F(a). \tag{17a}$$

The integration paths are assumed to be in R.

To show this, take the dual of (19) getting

$$f^- = 4 \int_b^z F^- \,\delta z + k.$$

Then by (18) if p and q are neighboring points of R

$$f^-(p) - f^-(q) = 2(p - q)(F^-(p) + F^-(q)).$$

Take the conjugate of both sides. Note that $(p - q)^* = (p - q)^{-1}$ since $p - q = \pm 1$ or $\pm i$. Then

$$(f(p) + f(q))(p - q) = 2(F(p) - F(q)).$$

This is relation (18), and (17a) is a consequence, so the proof is completed.

The function f defined by (19) is termed the *derivative* of F. If k^- is termed a *biconstant,* the derivative is unique up to an arbitrary biconstant.

3. **Contour integrals and Cauchy's formula.** In this section more general types of line integrals are to be defined. These integrals also have the important property of vanishing around a closed contour if the functions concerned are analytic. To this end it is convenient to define other difference operators on the unit square. Let the operator Φ be defined as

$$\Phi = I + \epsilon X + \epsilon^2 XY + \epsilon^3 Y.$$

Specializing ϵ to be i, -1, $-i$ and 1 defines four specific operators L, T, M, and S respectively:

$$\begin{aligned} Lf &= f_0 + if_1 - f_2 - if_3 \\ Tf &= f_0 - f_1 + f_2 - f_3 \\ Mf &= f_0 - if_1 - f_2 + if_3 \\ Sf &= f_0 + f_1 + f_2 + f_3 . \end{aligned}$$

Let f and g be lattice functions and let $a = z_0, z_1, z_2, \cdots, z_m = b$ denote a chain of lattice points. Then three types of "line integrals" are defined by:

$$\int_a^b f:g \;\delta z = \sum_{n=1}^{m} (f_n + f_{n-1})(g_n + g_{n-1})(z_n - z_{n-1})/4 \tag{20}$$

$$\int_a^b f:g' \;\delta z = \sum_{n=1}^{m} (f_n + f_{n-1})(g_n - g_{n-1})/2 \tag{21}$$

$$\int_a^b f':g' \;\delta z = \sum_{n=1}^{m} (f_n - f_{n-1})(g_n - g_{n-1})/(z_n - z_{n-1}). \tag{22}$$

The notations on the left are merely symbolic for the precise definitions on the right. For a closed path B then $a = b$ and it is clear that these summations may be reformed as follows:

$$\int_B f:g \;\delta z = \sum_{n=1}^{m} f_n[g_{n-1}(z_n - z_{n-1}) + g_n(z_{n+1} - z_{n-1}) + g_{n+1}(z_{n+1} - z_n)]/4 \tag{23}$$

$$\int_B f:g' \;\delta z = \sum_{n=1}^{m} f_n(g_{n+1} - g_{n-1})/2 \tag{24}$$

$$\int_B f':g' \;\delta z = -\sum_{n=1}^{m} f_n\left[\frac{g_{n+1} - g_n}{z_{n+1} - z_n} - \frac{g_n - g_{n-1}}{z_n - z_{n-1}}\right]. \tag{25}$$

Here, as a matter of notation, $z_{m+1} = z_1$. Suppose B is the boundary of a simple region R and that the contour is described in the counter-clockwise sense. It is desired to prove that:

$$\int_B f:g \;\delta z = (1 - i)8^{-1} \sum_R (SgLf + SfLg) \tag{26}$$

$$\int_B f:g' \;\delta z = i4^{-1} \sum_R (MgLf - MfLg) \tag{27}$$

$$\int_B f':g' \;\delta z = (1 + i)2^{-1} \sum_R (TgLf + TfLg). \tag{28}$$

Here $\sum_R$ again denotes a summation over the unit squares of R. It is clear that it is sufficient to prove these relations for the case R is a single unit square. The general case then follows by juxtaposition. The line integral on edges common to two squares will cancel leaving only the line integral on B.

First consider (26). As a simplifying notation let

$$2\bar{f}_0 = f_0 + f_1\,, \quad 2\bar{f}_1 = f_1 + f_2\,, \quad 2\bar{f}_2 = f_2 + f_3\,, \quad \text{and} \quad 2\bar{f}_3 = f_3 + f_0\,.$$

Hence at the square under consideration

$$(1 - i)Lf = 2L\bar{f}, \qquad Sf = S\bar{f}, \quad \text{and} \quad T\bar{f} = 0. \tag{29}$$

If $\bar{g}$ is defined in a similar way then

$$\int f{:}g\ \delta z = L(\overline{fg}). \tag{30}$$

To proceed, an identity is needed for Lfg. Clearly

$$\begin{aligned} Lf^2 &= f_0^2 - f_2^2 + i(f_1^2 - f_3^2) \\ &= f_0^2 - f_2^2 + (f_1 + f_3)(f_2 - f_0) + (f_1 + f_3)Lf. \end{aligned}$$

$$Lf^2 = (f_0 - f_2)Tf + (f_1 + f_3)Lf. \tag{31}$$

Likewise

$$Lf^2 = i(f_1^2 - f_3^2) + i(f_3 - f_1)(f_0 + f_2) + (f_0 + f_2)Lf$$

$$Lf^2 = i(f_3 - f_1)Tf + (f_0 + f_2)Lf. \tag{32}$$

Adding (31) and (32) gives

$$2Lf^2 = MfTf + SfLf. \tag{33}$$

Note that $4fg = (f + g)^2 - (f - g)^2$. Thus (33) gives the desired identity:

$$4L(fg) = MfTg + MgTf + SfLg + SgLf. \tag{34}$$

Making use of (34) and (29) gives

$$8L\overline{fg} = (1 - i)(SfLg + SgLf). \tag{35}$$

Combining (35) and (30) gives

$$\int f{:}g\ \delta z = (1 - i)8^{-1}(SgLf + SfLg). \tag{36}$$

This proves (26) for a unit square region; the general case then follows by juxtaposition.

Now consider (28). Let F be the dual of f and let G be the dual of g. Then relation (26) holds for F and G. We take the complex conjugate of this relation. Note that $(SG)^* = \pm Tg$ and $(LF)^* = MF^* = \pm Lf$. Thus $(SGLF)^* = TgLf$. Again

$$(F_n + F_{n-1})^* = \pm(f_n - f_{n-1}) \quad \text{and} \quad (z_n - z_{n-1})^* = (z_n - z_{n-1})^{-1}.$$

Thus $(4\int_a^b F:G\ \delta z)^* = \int_a^b f':g'\ \delta z$. It is now clear that the complex conjugate of (26) for F and G yields (28) for f and g.

To prove (27) we first consider (24) applied to a unit square. Thus

$$2\int f:g'\ \delta z = f_0(g_1 - g_3) + f_1(g_2 - g_0) + f_2(g_3 - g_1) + f_3(g_0 - g_2)$$

$$= (f_0 - f_2)(g_1 - g_3) - (f_1 - f_3)(g_0 - g_2).$$

But $Lf + Mf = 2(f_0 - f_2)$ and $Lf - Mf = 2i(f_1 - f_3)$.

So

$$8i\int f:g'\ \delta z = (Lf + Mf)(Lg - Mg) - (Lg + Mg)(Lf - Mf)$$

$$= 2MfLg - 2MgLf.$$

This verifies (26) for a unit square. The general case then follows by juxtaposition.

Directly from definition (21) it is seen that

$$\int_a^b f:g'\ \delta z + \int_a^b g:f'\ \delta z = \sum_1^m (f_n g_n - f_{n-1}g_{n-1}).$$

Thus

$$\int_a^b (f:g' + g:f')\ \delta z = f(a)g(a) - f(b)g(b). \tag{37}$$

In particular for a closed circuit

$$\int_B f:g'\ \delta z = -\int_B g:f'\ \delta z. \tag{38}$$

Suppose that the path of integration is in a simple region where g is analytic. Then h the derivative of g exists. If z_{n-1} and z_n are two consecutive points on the path of integration

$$(h_n + h_{n-1})(z_n - z_{n-1}) = 2(g_n - g_{n-1}).$$

Comparing (20) and (21) gives

$$\int_a^b f:g'\ \delta z = \int_a^b f:h\ \delta z. \tag{39}$$

Comparing (21) and (22) gives

$$\int_a^b f':g'\ \delta z = \int_a^b f':h\ \delta z. \tag{40}$$

Let j be the second derivative of g. Considering a closed path of integration it follows from (38), (39) and (40) that

$$\int_B f':g'\ \delta z = -\int_B f:j\ \delta z. \tag{41}$$

Whether or not g is analytic, it is convenient to introduce the following notation

$$\text{(41a)} \qquad \int_B f':g' \,\delta z \equiv -\int_B f:g'' \,\delta z.$$

Analogs of the Cauchy integral formula are now to be obtained. To this end a lattice function q is supposed given such that

$$\text{(42)} \qquad Lq(z) = 0, \qquad z \neq 0 \quad \text{and} \quad Lq(0) = 1.$$

If h is a function which is analytic everywhere, then $q' = q + h$ also satisfies the condition (42). Let f be analytic in a simple region R which includes the unit square at $z = 0$. Then (26), (27), and (28) give

$$\text{(43)} \qquad \int_B f:q \quad \delta z = (1 - i)\, Sf/8 \qquad \text{(for } z = 0\text{)}$$

$$\text{(44)} \qquad \int_B f:q' \quad \delta z = -i\, Mf/4$$

$$\text{(45)} \qquad \int_B f:q'' \,\delta z = -(1 + i)\, Tf/2.$$

It is seen from the definitions of L, M, T, and S that

$$\text{(46)} \qquad \begin{aligned} 4f_0 &= (S + T + M + L)f \\ 4f_1 &= (S - T + iM - iL)f \\ 4f_2 &= (S + T - M - L)f \\ 4f_3 &= (S - T - iM + iL)f. \end{aligned}$$

It follows from relations (43) to (46) that

$$\text{(47)} \qquad \begin{aligned} f_0 &= \int_B f:[(1 + i)q + iq' - (1 - i)q''/4]\, \delta z \\ f_1 &= \int_B f:[(1 + i)q - q' + (1 - i)q''/4]\, \delta z \\ f_2 &= \int_B f:[(1 + i)q - iq' - (1 - i)q''/4]\, \delta z \\ f_3 &= \int_B f:[(1 + i)q + q' + (1 - i)q''/4]\, \delta z. \end{aligned}$$

Here $z_0 = 0$, $z_1 = 1$, $z_2 = 1 + i$, and $z_3 = i$. The "factoring" of f in relations (47) is a matter of symbolism.

Relations (47) are analogs of the Cauchy integral formula. It is of interest to obtain a more symmetric Cauchy type formula. To this end let

$$S' = I + X^{-1} + Y^{-1}X^{-1} + Y^{-1}$$

and

$$T' = I - X^{-1} + Y^{-1}X^{-1} - Y^{-1}.$$

Then

$$S'S = 4I + XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y + 2(X + X^{-1} + Y + Y^{-1}),$$

$$T'T = 4I + XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y - 2(X + X^{-1} + Y + Y^{-1}),$$

$$S'S + T'T = 16I + 2D. \tag{48}$$

Let

$$s(z) = (1 + i)S\, q(z)/4 \quad \text{and} \quad t(z) = (1 - i)T\, q(z)/16. \tag{49}$$

Let z_0 be an interior point of R then by (43)

$$4(1 + i) \int_B f(z):q(z - z_0)\ \delta z = Sf(z_0).$$

Operating with S' gives

$$16 \int_B f(z):s(z - z_0)\ \delta z = S'Sf(z_0).$$

Likewise (45) gives

$$16i \int_B f(z):t''(z - z_0)\ \delta z = T'Tf(z_0).$$

Relation (48) gives

$$\int_B f(z):[s(z - z_0) + it''(z - z_0)]\ \delta z = f(z_0). \tag{50}$$

Here use has been made of the fact that $Df = 0$ at an interior point z_0. This is another analog of the Cauchy formula.

Let $f(z)$ be a function which is analytic in every unit square of the upper half plane. Suppose that for each fixed z_0

$$f(z)q(z - z_0) = o\ |\ z\ |^{-1}, \qquad \operatorname{Im} z \geq 0. \tag{51}$$

Let R be a square region in the upper half-plane. One of the edges of R is the interval $(-\ m,\ m)$ of the x axis. If the unit square associated with z_0 is in R then formula (47a) applies with $q(z)$ being replaced by $q(z - z_0)$. If $m \to \infty$ a limiting relationship results. Because of (51) the contribution from the part of the contour not on the x axis tends to zero. Then (23), (24), (25), and (47a) give

$$
\begin{aligned}
4f(z_0) &= \sum_{-\infty}^{\infty} f_m(1 + i)[q(m + 1 - z_0) + 2q(m - z_0) + q(m - 1 - z_0)] \\
&\qquad + \sum_{-\infty}^{\infty} f_m 2i[q(m + 1 - z_0) - q(m - 1 - z_0)] \\
&\qquad + \sum_{-\infty}^{\infty} f_m(i - 1)[q(m + 1 - z_0) - 2q(m - z_0) + q(m - 1 - z_0)] \\
&= \sum_{-\infty}^{\infty} f_m 4[iq(m + 1 - z_0) + q(m - z_0)].
\end{aligned}
$$

Thus let

$$\theta(z) = q(-z) + iq(1 - z) \tag{52}$$

then

$$f(z_0) = \sum_{m=-\infty}^{\infty} f(m)\theta(z_0 - m), \qquad \operatorname{Im} z_0 \geq 0. \tag{53}$$

This is an analog of the Cauchy formula for the upper half-plane. It is seen that the right side of (53) vanishes if $\operatorname{Im} z_0 < 0$.

4. **An operational calculus.** Partial differential equations with constant coefficients have been treated by an operational calculus based on multiple Fourier integrals. Analogously partial difference equations with constant coefficients may be studied by an operational calculus based on multiple Fourier series. In a previous paper [3] such an operational calculus was applied to the discrete Poisson equation $Du = -w$ in three dimensions. Here it is desired to treat the equation

$$Lf = w. \tag{54}$$

More precisely, w is regarded as a given function at the lattice points $z = m + in$. Thus the problem presented by equation (54) is the study of discrete analytic functions with prescribed residues. In the operational calculus lattice functions are regarded as the Fourier coefficients of periodic functions. Thus let

$$W(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-i(mx+ny)} w(m, n). \tag{55}$$

For the present purpose it is sufficient to make the restrictive hypothesis

$$\sum \sum | w(m, n) | < \infty. \tag{56}$$

Then it is seen that $W(x, y)$ is a continuous function of period 2π in x and of period 2π in y.

Let the lattice function $f(m, n)$ be defined as

$$f(m, n) = (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{imx+iny} \rho^{-1}(x, y) W(x, y)\, dx\, dy, \tag{57}$$

$$\rho(x, y) = 1 + ie^{ix} - ie^{iy} - e^{ix+iy}.$$

It is seen that $|\rho|^2 = 4(1 - \cos x \cos y)$, so the only vanishing points of ρ in the region of integration are at $(0, 0)$, (π, π) and $(-\pi, -\pi)$. A Taylor series expansion of $\cos x \cos y$ at the vanishing points gives

$$|\rho(x, y)|^2 = 2(x_1^2 + y_1^2) + O(x_1^2 + y_1^2)^2. \tag{58}$$

Here $x_1 = x$ and $y_1 = y$ or $x_1 = x \pm \pi$ and $y_1 = y \pm \pi$. It follows that the integrand in (57) is absolutely integrable. Applying the operator L to both sides of (57) gives

$$Lf = (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{imx+iny} W(x, y)\, dx\, dy. \tag{59}$$

The expression on the right is simply the formula for the Fourier coefficients of W. Hence $Lf = w$ and so (57) gives a solution of (54). The Riemann-Lebesgue lemma applies to the integral in (57), so f vanishes as m and n approach infinity. Suppose that f' is another solution of (54) which vanishes at infinity. Then $L(f - f') = 0$, and it follows from the maximum principle that $f - f' = 0$. This proves the following statement:

If w is absolutely convergent in the sense of (56), *then the equation $Lf = w$ has a unique solution f which vanishes at infinity.*

For $W = 1$, let f, given by (57), be denoted by q. Then q so defined satisfies (42) and in addition vanishes at infinity. This specialization of q will be assumed in what follows.

It is of interest to obtain some properties of the function $\theta(z)$. Since $\theta(-z) = q(z) + iq(1 + z)$, it is seen that $L\theta(-z) = 0$ except at the points $z = 0$ and $z = -1$, where it has the values 1 and i respectively. Thus $\theta(-z)$ corresponds to $W(x, y) = 1 + ie^{ix}$ so

$$(2\pi)^2 \theta(m, n) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{-imx-iny}(1 + ie^{ix})}{1 + ie^{ix} - ie^{iy} - e^{ix+iy}}\, dx\, dy$$

$$= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{-imx-iny}}{1 - \Delta e^{iy}}\, dx\, dy,$$

where

$$\Delta = \frac{i + e^{ix}}{1 + ie^{ix}} = \frac{ie^{ix} - 1}{i(ie^{ix} + 1)} = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

Since the double integral for θ is absolutely convergent, it is permissible to evaluate θ as an iterated integral. The integration is first carried out with respect to y, but only the case $n \geq 0$ will be treated. Let

$$2\pi I = \int_{-\pi}^{\pi} e^{-iny}(1 - \Delta e^{iy})^{-1}\, dy.$$

If $|\Delta| < 1$, then $(1 - \Delta e^{iy})^{-1} = \sum_{i=0}^{\infty} \Delta^i e^{iiy}$ and it follows that $I = \Delta^n$. If $\Delta > 1$, then $(1 - \Delta e^{iy})^{-1} = -\sum_1^{\infty} \Delta^{-i} e^{-iiy}$, and it follows that $I = 0$. Thus

$$\begin{aligned} 2\pi\theta(m, n) &= \int_{-\pi}^{0} e^{-imx} \tan^n (x/2 + \pi/4)\, dx \\ &= 2\int_{-\pi/4}^{\pi/4} e^{im(\pi/2-2t)} \tan^n t\, dt. \end{aligned} \tag{60}$$

Let $h(m, n) = 2 \operatorname{Re} \theta(m, n)$ and $k(m, n) = 2 \operatorname{Im} \theta(m, n)$, then

$$h(m, n) = 2\pi^{-1} \int_{-\pi/4}^{\pi/4} \cos m(\pi/2 - 2t) \tan^n t\, dt \tag{61}$$

and

$$k(m, n) = 2\pi^{-1} \int_{-\pi/4}^{\pi/4} \sin m(\pi/2 - 2t) \tan^n t\, dt. \tag{62}$$

The values of θ for $n = 0$ and for $m = 0$ are readily determined. Thus

$$h(0, 0) = 1, \qquad h(m, 0) = 0 \qquad (m \neq 0). \tag{63}$$

$$k(m, 0) = 2/\pi m \quad (m \text{ odd}), \qquad = 0 \quad (m \text{ even}). \tag{64}$$

Obviously $k(0, n) = 0$ for $n \geq 0$. The relation $\int \tan^n t dt = (n-1)^{-1} \tan^{n-1} t - \int \tan^{n-2} t dt$ gives the recursion formula

$$h(0, n) = 4/\pi(n-1) - h(0, n-2) \qquad (n \text{ even}). \tag{65}$$

The Cauchy-Riemann equations now determine θ in the upper half-plane. The following partial table for the first quadrant has been prepared in this way.

n \ m	0	1	2	3
3	0; 0	−3; 5	9; −14	19; −89/3
2	−1; 2	2; −3	4; −6	−6; 29/3
1	0; 0	1; −1	−1; 2	−1; 5/3
0	1; 0	0; 1	0; 0	0; 1/3

Table of h and k

This table gives the value of h at the even points and the value of k at the odd points. It may be seen directly from formulas (61) and (62) that $h = 0$ at odd points and $k = 0$ at even points. The notation is explained by the following example

$$h(1, 3) = [-3; 5] = -3 + 5(2/\pi).$$

The formulas (61) and (62) reveal certain features of the asymptotic behavior of h and k at infinity. Thus integrating (61) by parts twice gives at a point (m, n) of the even lattice

$$m^2\pi h(m, n) = 2n - 2^{-1} \int_{-\pi/4}^{\pi/4} \cos (2mt - m\pi/2)(d^2 \tan^n t/dt^2)\, dt.$$

It follows that for fixed n

$$m^2\pi h(m, n) \to 2n \quad \text{as} \quad m \to \pm\infty. \tag{66}$$

A more powerful process for finding the asymptotic behavior will now be discussed.

An alternative procedure for analyzing functions with prescribed residues may be based on the Green's function for the operator D. The Green's function $g(m, n)$ is defined by the following conditions:

(a) $Dg = 0$ except at the origin, where $Dg = 1$.
(b) $g(m + 1, n + 1) - g(m, n)$ and $g(m + 1, n - 1) - g(m, n)$ vanish as m and n become infinite.
(c) $g(0, 0) = 0$.
(d) $g(m, n) \equiv 0$ on the odd lattice.

It is easy to check that the following function satisfies these conditions:

$$8\pi^2 g(m, n) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (1 - e^{imx+iny})(1 + (-1)^{m+n}) \mid \rho \mid^{-2} dx\, dy. \tag{67}$$

Suppose that there are two functions, say g and g_1 which satisfy the conditions. Then $f = g - g_1$ is everywhere harmonic. By (b) the "diagonal differences" of f vanish at infinity. By the maximum principle the diagonal differences vanish identically, so f is a constant. By (c) this constant must be zero. Thus the conditions (a), (b), (c), and (d) define a unique function g.

It is clear that $g(-m, n)$ satisfies the same conditions, so by the uniqueness property just shown it follows that $g(-m, n) = g(m, n)$. Likewise $g(m, -n) = g(m, n)$.

The function q may be defined in terms of g by the following formula:

$$q(m, n) = -g(m, n) + ig(m - 1, n) + g(m - 1, n - 1) - ig(m, n - 1). \tag{68}$$

To prove this it is first observed that q so defined vanishes at infinity because of property (b). Secondly $q = -L'g$, so $Lq = -LL'q = Dg$. Thus $Lq = 0$ except at the origin where $Lq = 1$, and the proof is complete. Since $\theta(z) =$

$q(-z) + iq(1 - z)$ it results that

$$h(m, n) = 2g(m + 1, n + 1) + 2g(m - 1, n + 1) - 4g(m, n), \tag{69}$$

$$k(m, n) = 2g(m + 1, n) - 2g(m - 1, n). \tag{70}$$

From relation (70) it is seen that $k(m, n)$ is an even function of n and an odd function of m. Likewise it is seen that $h(m, n)$ is an even function of m. It is noted that $h(m, n) + h(m, -n) = 2Dg(m, n)$ so h is an odd function of n except that $h(0, 0) = 1$.

The Green's function was introduced by Courant [2] in connection with random walk and diffusion problems on a lattice. Further studies of this function were given by McCrea and Whipple [9], Stöhr [10], and Duffin and Shaffer [4]. The asymptotic behavior is considered in the latter three papers. It is shown in [4] that if (x, y) is a point of the even lattice then

$$2\pi g(x, y) = \log 4r + \gamma + (\cos 4\theta)/6r^2 + O(r^{-4}). \tag{71}$$

Here $r^2 = x^2 + y^2$, $\theta = \arctan(y/x)$, and γ is Euler's constant. Substituting (71) in (69) and (70) gives asymptotic expressions for h and k. A straightforward but lengthy calculation gives

$$h(x, y) = 2y/\pi r^2 + O(r^{-3}) \tag{72}$$

$$k(x, y) = 2x/\pi r^2 + O(r^{-3}). \tag{73}$$

In the latter formula (x, y) is a point of the odd lattice. Further terms in the asymptotic expansion could be obtained by use of the procedures developed in [4].

5. **Polynomials and bipolynomials.** Of interest are polynomials which are discrete analytic everywhere. More generally we consider functions which take on the values of one polynomial on the even lattice and the values of another polynomial on the odd lattice. Such functions are termed bipolynomials. *Let f be a lattice function such that $f(m + 1, n) - f(m, n)$ and $f(m, n + 1) - f(m, n)$ are both polynomials, then f is a polynomial.* The proof this fundamental lemma in the difference calculus runs as follows: The first relation shows that for each value of n, f is a polynomial in m whose coefficients are functions only of n. Application of the second condition shows that these coefficients are polynomials in n. Considering a coordinate system rotated 45° gives the direct corollary: *Let f be a lattice function such that the diagonal differences $f(m + 1, n + 1) - f(m, n)$ and $f(m + 1, n - 1) - f(m, n)$ are both bipolynomials, then f is a bipolynomial.* The following result is a consequence of relation (18): *If f is a discrete analytic polynomial (bipolynomial), then the integral F is a discrete analytic polynomial (bipolynomial).*

Let a sequence of functions $\rho_0, \rho_1, \rho_2, \cdots$ be defined by the relations

$$\rho_{n+1}(z) = (n + 1) \int_0^z \rho_n(z)\, \delta z, \qquad \rho_0 = 1. \tag{74}$$

By what has just been proved this is a sequence of discrete analytic polynomials. It is to be shown that

$$\rho_n(z) = z^n + h_n \tag{75}$$

where h_n is a polynomial of degree $n - 2$ at most. It is clear that (75) is true for the case $n = 0$. Suppose it is true for the case $n = k$. Then by relation (18)

$$\rho_{k+1}(z + \epsilon) - \rho_{k+1}(z) = \epsilon(\rho_k(z + \epsilon) + \rho_k(z))(k + 1)/2$$

where $\epsilon = 1, -1, i$ or $-i$. Let δf denote $f(z + \epsilon) - f(z)$. Then $\delta h_{k+1} = \delta\rho_{k+1} - \delta z^{k+1} = \epsilon((z + \epsilon)^k + z^k)\ (k + 1)/2 - \delta z^{k+1} + \epsilon(h_k(z + \epsilon) + h_k(z))\ (k + 1)/2$. By hypothesis the last term is $O \mid z \mid^{k-2}$. The other terms may be evaluated by the binomial theorem to yield $\delta h_{k+1} = O \mid z \mid^{k-2}$, thus $h_{k+1} = O \mid z \mid^{k-1}$, so h_{k+1} is of degree $k - 1$ at most. This completes the proof of (75). It is not difficult to estimate the order of magnitude of h_n. Thus it may be shown that

$$\mid \rho_n(z) - z^n \mid \leq n!2^n \mid z \mid^{n-2}. \tag{76}$$

The proof is omitted. It is to be noted that (75) implies that the sequence ρ_n is linearly independent.

Let $p(x, y)$ *be a discrete analytic polynomial which vanishes at the points* $(0, 0)$, $(0, 1), \cdots, (0, b)$ *and* $(0, 0), (1, 0), \cdots, (a, 0)$ *where* a *and* b *are non-negative integers. If*

$$a + b \geq \text{degree } p \tag{77}$$

then p *vanishes identically.* To prove this lemma it is first noted that the relation $Lp = 0$ demands that $p = 0$ in the rectangle of lattice points (x, y) where $0 \leq x \leq a$ and $0 \leq y \leq b$. The statement is obviously true if $a + b = 0$. Proceeding by induction it is supposed that it is true for polynomials of lower degree. Suppose that $a \geq 1$ and $a \geq b$. Then $p_x = p(x + 1, y) - p(x, y)$ is a discrete analytic polynomial which vanishes at $(0, 0), \cdots, (0, b)$ and $(0, 0), \cdots, (a', 0)$ where $a' = a - 1$. Since degree $p_x <$ degree p it follows that $a' + b \geq$ degree p_x. Thus $p_x \equiv 0$ and p is independent of x. The only function independent of x which satisfies $Lp = 0$ is a constant. But $p(0, 0) = 0$, so $p \equiv 0$. The case $b \geq a$ is treated by a symmetrical argument, and the proof is complete.

Let f *be a discrete analytic function in a rectangular region* R *made up of unit squares. Let* a *and* b *be the lengths of the edges of* R. *Then there is a discrete analytic polynomial* p *of degree not exceeding* $a + b$ *such that* $f = p$ *at the lattice points of* R. *Such a* p *is unique.* To prove this theorem it is first observed that it is sufficient to consider the rectangle defined by $0 \leq x \leq a$ and $0 \leq y \leq b$. Let

$$p = c_0\rho_0 + c_1\rho_1 + \cdots + c_m\rho_m \tag{78}$$

where $c_0, c_1, \cdots c_m$ are constants and $m = a + b$. Then p is a discrete analytic polynomial of degree m at most. If $p = 0$ at the $m + 1$ points $(0, 0), (1, 0), \cdots (a, 0)$ and $(0, 1), \cdots (0, b)$ then $p \equiv 0$ by the previous lemma. Since the functions $\rho_0, \cdots, \rho_m$ are independent, it follows that $c_0 = c_1 = \cdots = c_m = 0$. The values of $\rho_0, \rho_1, \cdots, \rho_m$ at the above $m + 1$ edge points define the rows of a matrix. It has just been shown that this matrix is non-singular. It follows that the constants c_k may be determined so that $f - p = 0$ on these edges. But the condition $L(f - p) = 0$ then demands that $f - p = 0$ at all points of R. That such a p is unique follows from the previous lemma.

The vector space of discrete analytic polynomials of degree m or less has dimension $m + 1$. It is observed that the preceding theorem is valid for the degenerate rectangle with $a = m$ and $b = 0$. Then if p is a discrete analytic polynomial of degree not exceeding m the constants c_k may be determined so that (78) holds on the points $(0, 0), (1, 0), \cdots, (m, 0)$. By the lemma it follows that (78) is an identity at all lattice points. In other words, $\rho_0, \rho_1, \cdots, \rho_m$ form a basis for the space. Since they are independent, the dimension is $m + 1$.

Let p be a given discrete analytic polynomial of degree n. Then it follows from (74) and (78) that

$$p = \int_0^z p' \, \delta z + c \tag{79}$$

where c is a constant and p' is a discrete analytic polynomial of degree $n - 1$. It follows from (75) and (78) that

$$p = az^n + bz^{n-1} + h \tag{80}$$

where a and b are constants and h is a polynomial of degree $n - 2$.

Given a lattice function f, let

$$j = (f + f^-)/2 \quad \text{and} \quad k = (f - f^-)/2i. \tag{81}$$

Then clearly j and k are self-dual, $j = j^-$ and $k = k^-$. Moreover

$$f = j + ik. \tag{82}$$

Let the degree of a bipolynomial be the maximum of the degrees on the even and on the odd lattices. If f is a bipolynomial then so are j and k. The degrees of j and k do not exceed the degree of f. Suppose now that f is a discrete analytic bipolynomial. Then there are real polynomials u and v such that on the even lattice $j = u$ and on the odd lattice $j = iv$. The Cauchy-Riemann equations are

$$u(x + 1, y + 1) - u(x, y) - v(x, y + 1) + v(x + 1, y) = 0 \tag{83}$$

and

$$u(x, y) - u(x - 1, y + 1) - v(x, y + 1) + v(x - 1, y) = 0. \tag{84}$$

These relations are given to hold only for (x, y) an even point. However it is easy to show that if a polynomial vanishes on the even lattice then it vanishes identically. Hence (83) and (84) are identities for all lattice points. Let

$$J(x, y) = u(x, y) + iv(x, y). \tag{85}$$

Then J may be termed the *extension* of j. It is clear from (83) and (84) that J is a discrete analytic polynomial. It is clear that

$$j = (J + J^{-})/2. \tag{86}$$

Let K be the extension of k. Then from (82) we see that $2f = J + iK + (J - iK)^{-}$. *Thus a discrete analytic bipolynomial f may be written in the form*

$$f = s + t^{-} \tag{87}$$

where s and t are uniquely determined discrete analytic polynomials. The degrees of s and t do not exceed the degree of f. It remains to prove the statement that s and t are unique. It is sufficient to consider the case $f = 0$. Then $s + t^* = 0$ on the even lattice and $s - t^* = 0$ on the odd lattice. But since s and t are polynomials, these relations are identities for all x and y, so adding them gives $s = 0$ and hence $t = 0$. This completes the proof.

It is now seen that $\rho_0, \rho_1, \cdots, \rho_m$ and $\rho_0^{-}, \rho_1^{-}, \cdots, \rho_m^{-}$ form an independent set of $2m + 2$ functions. *Thus the vector space of discrete analytic bipolynomials of degree m or less has dimension $2m + 2$.*

If f is a discrete analytic bipolynomial of degree n, then it is seen from (80) and (87) that

$$f = az^n + A(z^n)^{-} + \text{terms of lower degree.} \tag{88}$$

It may be assumed that a and A are not both zero. It is then easy to check that *the degree of the real and imaginary parts of an analytic bipolynomial are equal.*

Let u be a real polynomial which is discrete harmonic at all lattice points. Then u has a conjugate v such that $u + iv$ is a discrete analytic polynomial. A function which is harmonic everywhere always has a conjugate. The Cauchy-Riemann equations (83) and (84) show that the conjugate is a polynomial on the even lattice. By the argument used before it is seen that the same polynomial will satisfy the equations (83) and (84) on the odd lattice.

A discrete analog of Holder's well-known inequality for harmonic functions was developed in a previous paper [3]. This theorem may be stated as follows:

Let $u(x, y)$ be non-negative in a region R and discrete harmonic at interior points of R. If the lattice point z is an interior point of R and z' denotes $z + 1 + i$, $z + 1 - i$, $z - 1 + i$, or $z - 1 - i$, then

$$| u(z') - u(z) | \leq Au(z)/d \tag{89}$$

where A is an absolute positive constant and d is the distance from z to the boundary of R. This inequality will now be employed to prove the following generalization of a theorem of Heilbronn [6]. (Heilbronn's result corresponds to using $| f |$ instead of Re f.)

Let $f(z)$ be discrete analytic at all lattice points. Then a necessary and sufficient condition that $f(z)$ be a bipolynomial of degree n or less is that

$$\operatorname{Re} f(z) \leq B \mid z \mid^n, \qquad z \neq 0 \tag{90}$$

for some positive constant B. The necessity of inequality (90) is obvious. To prove the sufficiency let k be a positive integer such that $k - n \geq 2$. Let j denote $0, 1, \cdots$, or n. Let R_j be the region composed of those unit squares contained in the circle with radius 3^{k-n+j} and with center at the origin. This radius is at least 9 and the radius for R_{j+1} is three times the radius for R_j. It is seen that the points of R_j are at a distance greater than 3^{k-n+j} from the boundary of R_{j+1}. Let $u(z) = \operatorname{Re} f(z)$ then $B(3^k)^n - u$ is non-negative in R_n. Inequality (89) gives for z in R_{n-1}

$$u(z') - u(z) \leq A(B3^{nk} - u(z))3^{-k+1}.$$

Let $u^{(1)}(z) = u(z') - u(z)\ (1 - A3^{-k+1})$ then

$$u^{(1)}(z) \leq AB3^{nk-k+1}, \qquad z \text{ in } R_{n-1}. \tag{91}$$

If z is in R_{n-2} then (89) gives

$$u^{(1)}(z') - u^{(1)}(z) \leq A(AB3^{nk-k+1} - u^{(1)}(z))3^{-k+2}.$$

Let $u^{(2)}(z) = u^{(1)}(z') - u^{(1)}(z)\ (1 - A3^{-k+2})$ then

$$u^{(2)}(z) \leq A^2B3^{nk-2k+3}, \quad \text{for} \quad z \text{ in } R_{n-2}. \tag{92}$$

This process may be repeated n times to obtain

$$u^{(n)}(z) \leq A^nB3^{n(n+1)/2}, \quad \text{for} \quad z \text{ in } R_0. \tag{93}$$

The right side of this last inequality is independent of k, so let $k \to \infty$. Then it is seen that in the limit $u^{(n)}(z)$ is an n-th difference of $u(z)$ on the even (odd) lattice if z is on the even (odd) lattice. At each application of inequality (89) the point z' may be any of four choices. Thus $u^{(n)}(z)$ is an arbitrary n-th difference. Again by (89)

$$\mid u^{(n)}(z') - u^{(n)}(z) \mid \leq A(A^nB3^{n(n+1)/2} - u^{(n)}(z))/d.$$

Now d can be made arbitrarily large so $u^{(n)}$ is constant on the even lattice and constant on the odd lattice. Thus $u^{(n)}$ is a biconstant. All n-th, diagonal differences of u are bipolynomials so $u(z)$ is a bipolynomial. Then by (90) its degree must not exceed n. The preceding lemmas on the conjugate of u complete the proof.

6. **Hilbert transforms and Poisson's formula.** Let $f(z)$ be discrete analytic in the upper half-plane and suppose that $f(z) \to 0$ uniformly as $\mid z \mid \to \infty$ in the upper half-plane. In formula (53) the function $\theta(z)$ is to be given the specific form defined by (60). Then it follows that $\theta(z) = O \mid z \mid^{-1}$ and so $f(z)\theta(z) = o$

$|z|^{-1}$. Thus the conditions justifying (53) are satisfied so if $\sum_m$ denotes $\sum_{m=-\infty}^{\infty}$

(53) $$f(z) = \sum_m f(m)\theta(z - m), \qquad \operatorname{Im} z \geq 0.$$

It is clear by the same argument that

(94) $$0 = \sum_m f(m)\theta(z - m), \qquad \operatorname{Im} z < 0.$$

Let Im $z > 0$ then by the symmetry properties of $2\theta(z) = h(z) + ik(z)$ we see that

(95) $$h(z - m) = \theta(z - m) - \theta(z^* - m).$$

Substracting (94) from (53) gives

(96) $$f(z) = \sum_m f(m)h(z - m).$$

The above proof applies for Im $z > 0$ but (96) is obviously true for Im $z = 0$ since $h(0) = 1$ and $h(m) = 0$ for $m \neq 0$. Multiplying (53) by 2 and subtracting (96) gives

(97) $$f(z) = i \sum_m f(m)k(z - m), \qquad \operatorname{Im} z \geq 0.$$

Let $f(z) = u(z) + iv(z)$ where u and v are real. Separating the real and imaginary parts of (97) gives

(98) $$v(z) = \sum_m u(m)k(z - m)$$

and

(99) $$u(z) = -\sum_m v(m)k(z - m).$$

In case z is real k has the remarkably simple form given by relation (64). Thus

(100) $$v(x) = \sum_m u(m)[1 - (-1)^{x-m}]/\pi(x - m)$$

and

(101) $$u(x) = -\sum_m v(m)[1 - (-1)^{x-m}]/\pi(x - m).$$

These are the analogs of the Hilbert transforms [13]. It follows from (96) that $u(z) = \sum u(m)h(z - m)$. Adding this to (99) multiplied by i gives

(102) $$f(z) = 2 \sum_m u(m)\theta(z - m), \qquad \operatorname{Im} z \geq 0.$$

Now consider these relationships from a different point of view. Instead of starting with a given analytic function suppose given a sequence of real numbers u_m, $m = 0, \pm 1, \cdots$ then seek to define an analytic function $f(z)$ whose real part $u(z)$ satisfies

$$u(m) = u_m . \tag{103}$$

First suppose that $u_m = 0$ for $| m | \geq M$. Let

$$f(z) = 2 \sum_m u_m \theta(z - m). \tag{104}$$

Clearly $f(z)$ so defined is discrete analytic for Im $z \geq 0$. By separating the real and imaginary parts in (104) it is obvious that (103) is satisfied. Moreover $f(z)$ is $0 \mid z \mid^{-1}$ at infinity, so all the relations derived above are valid.

Before considering a more general case it is desirable to obtain further properties of $f(z)$ defined by (104). It is seen from formulas (20), (21), and (22) that

$$\int_{-\infty+in}^{\infty+in} (f{:}f + f{:}f' + f'{:}f'/4)\ \delta z = \sum_m f^2(m + in).$$

The expression on the left is the limit of a contour integral around a square in the upper half-plane. Since f is analytic, this contour integral vanishes. Thus

$$\sum_m f^2(m + in) = 0. \tag{105}$$

Separating the real and imaginary parts of (105) gives

$$\sum_m u^2(m + in) = \sum_m v^2(m + in) \tag{106}$$

and

$$\sum_m u(m + in)v(m + in) = 0. \qquad n \geq 0. \tag{107}$$

The function $\theta(z)$ satisfies $\theta^*(x + iy) = \theta(- x + iy)$ so

$$f^*(x + iy) = \sum_m f^*(m)\theta(iy - x + m)$$

and

$$\sum_x | f(x + iy) |^2 = \sum_m f^*(m) \sum_x f(x + iy)\theta(iy - x + m)$$

so

$$\sum_x | f(x + iy) |^2 = \sum_m f^*(m)f(m + 2iy). \tag{108}$$

Making use of the Cauchy inequality gives

$$\sum_x | f(x + iy) |^2 \leq (\sum_m | f(m) |^2 \sum_m | f(m + 2iy) |^2)^{\frac{1}{2}}. \tag{109}$$

It is now to be shown that

$$\sum_x | f(x + iy) |^2 \leq \sum_x | f(x) |^2. \tag{110}$$

Let N_y denote the square root of the left side of (109). Then if (110) is false there is a maximum value of y, say k, such that $N_k > N_0$ because N_y vanishes as $y \to \infty$. According to (109) $N_k^2 \leq N_0 N_{2k} \leq N_0^2$; the contradiction completes the proof.

Consider now the Hilbert space of sequences of real numbers u_n such that

$$\sum_m (u_m)^2 < \infty. \tag{111}$$

An associated function is again defined by (104). The Cauchy inequality makes it clear that the series in (104) converges and defines a discrete analytic function. Now let $f_M(z)$ denote the expression (104) with $u_m = 0$ for $| m | > M$. Then at each point z, $f_M(z) \to f(z)$ as $M \to \infty$. It is seen from (106) that

$$\sum_x (v_M(x) - v_N(x))^2 = \sum_x (u_M(x) - u_N(x))^2. \tag{112}$$

Thus $v_M(x)$ converges in the Hilbert space norm to $v(x)$, moreover

$$\sum_x v^2(x) = \sum_x u^2(x). \tag{113}$$

It is clear therefore that (100) and (101) define an orthogonal transformation and its inverse in this Hilbert space. By similar considerations it follows that (110) remains true. It is then seen that relations (106), (107), (108), and (109) also hold. Presumably similar theorems could be developed for the case that the sequence u_n belongs to the space l_p , see [14].

Attention is now restricted to functions $f(z)$ which are discrete analytic in the upper half-plane and which have a non-negative real part there. Thus if $f = u + iv$

$$u(z) \geq 0 \quad \text{if} \quad \operatorname{Re} z \geq 0 \tag{114}$$

and $u(z)$ is discrete harmonic for $\operatorname{Re} z > 0$. Let

$$u_M(z) = \sum_{-M}^{M} u(m)h(z - m). \tag{115}$$

It follows from the maximum principle for harmonic functions that $0 \leq h(z - m) \leq 1$ in the upper half-plane. Thus $u_{M+1}(z) \geq u_M(z)$. Clearly $u(x) \geq u_M(x)$ and $u_M(z) \to 0$ as $z \to \infty$, so by the maximum principle $u(z) - u_M(z) \geq 0$ in the upper half-plane. It follows that $u_M(z)$ converges as $M \to \infty$. Let the limit be denoted by $u_\infty(z)$. A consequence is that

$$\sum_m u(m)(1 + m^2)^{-1} < \infty. \tag{116}$$

This follows from the asymptotic behavior of h. Let $w(z) = u(z) - u_\infty(z)$ then

$$w(z) \geq 0; \quad \operatorname{Re} z \geq 0 \quad \text{and} \quad w(z) = 0; \quad \operatorname{Re} z = 0. \tag{117}$$

Let $w_1 = w(x + i)$, $w_2 = w(x + 1 + 2i)$, $w_3 = w(x + 3i)$, $w_4 = w(x + 1 + 4i)$, etc. Making use of the Harnack inequality (89) gives

$$w_{k+1} \leq (1 + A/k)w_k \leq w_k \exp (A/k)$$

and so

$$w_j \leq w_1 \exp \left(\sum_1^{j-1} A/k \right) \leq w_1 j^A. \tag{118}$$

It is seen from (116) that $\sum_x w(x + i)(1 + x^2)^{-1} < \infty$. Thus there is a constant B such that

$$w(x + i) \leq (1 + x^2)B \text{ for all } x. \tag{119}$$

From (118) and (119) we see that

$$w(x + iy) \leq B_1(1 + x^2)y^A \text{ for some constant } B_1 . \tag{120}$$

Since w vanishes on the real axis it is clear that w can be continued as a harmonic function at all points of the plane by the relation $w(x, -y) = -w(x, y)$. Thus the extended function also satisfies (120). Making use of a previous theorem it follows that w is a bipolynomial of degree k and $k \leq A + 2$. It follows from (88) that on the even lattice w has the asymptotic behavior $w \sim \operatorname{Re}(az^k)$ for some constant a. This is incompatible with (117) unless $k \leq 1$. Thus on the even lattice

$$w(z) = by, \qquad b \geq 0. \tag{121}$$

The same relation must hold on the odd lattice with a different constant b. This proves

$$u(z) = \sum_m u(m)h(z - m) + Cy \tag{122}$$

where C is a non-negative biconstant. It is now to be shown that if $z - a$ is on the even lattice, then

$$v(z) = \sum_m u(m)[k(z - m) - k(a - m)] + c_1 x + c_2 . \tag{123}$$

The convergence of the series is assured by (116) and the asymptotic formula (73) for k. It is seen that the constants c_1 and c_2 can be selected so that u and v defined by (122) and (123) satisfy the Cauchy-Riemann equations, and this proves (123).

The above results concerning formula (122) can be expressed as a theorem: *If a lattice function $u(z)$ is discrete harmonic and non-negative in the upper half-plane, then relations* (116) *and* (122) *are satisfied. Conversely, if $u(m)$ is a sequence of non-negative constants which satisfies* (116), *relation* (122) *defines such a function* $u(z)$. This theorem is similar to a result of Allen and Murdoch [1]. They are concerned with an analog of the Poisson integral formula for discrete harmonic functions. In the coordinate system employed in this paper their result pertains to discrete harmonic functions which are non-negative in a half-plane bounded by a 45° line. In addition, they apply Phragmén-Lindelöf methods to obtain asymptotic behavior of such functions. Presumably their Phragmén-Lindelöf methods could be adapted to the present problem.

7. **The product of analytic functions.** This section is concerned with operators which correspond to multiplication of a continuous analytic function by z. The formulas prove to be somewhat more complicated than in a similar theory developed by Isaacs for type (a) analytic functions.

In the formula $4Lfg = MfTg + MgTf + SfLg + SgLf$ the function f is to be taken as z or z^*. Note that $Lz = 0$, $Tz = 0$, $Mz = -4\delta$, and $Sz = 4z + 4\delta$. Here δ denotes $(1 + i)/2$. Thus

$$Lzg = -\delta Tg + (z + \delta)Lg. \tag{124}$$

Also $Mz^* = 0$, $Tz^* = 0$, $Lz^* = -4\delta^*$; and $Sz^* = 4z^* + 4\delta^*$. Thus

$$Lz^*g = -\delta^* Sg + (z + \delta)^* Lg. \tag{125}$$

In (124) take $g = Sf$, in (125) take $g = iTf$, and subtract. Since $\delta^* i = \delta$ it results that

$$L(zS - iz^*T)f = [(z + \delta)S - i(z + \delta)^*T]Lf. \tag{126}$$

Let the operator Z be defined by

$$4Z = zS - iz^*T. \tag{127}$$

It follows from (126) that if f is analytic then Zf is analytic. It is seen that the operation Z has a correspondence with multiplication by z in the classical continuous theory.

It is of interest to express Z in other forms. Clearly $4Zf_0 = (z - iz^*)(f_0 + f_2) + (z + iz^*)(f_1 + f_3)$. Now $z + iz^* = (1 + i)(x + y) = 2\delta(x + y)$ and $z - iz^* = 2\delta^*(x - y)$. Thus

$$4\delta Zf_0 = (x - y)(f_0 + f_2) + i(x + y)(f_1 + f_3). \tag{128}$$

Suppose that $f = L'w$. Employing the spiral coordinate system gives:

$$f_0 = w_0 - w_6 + iw_7 - iw_5\,, \qquad f_1 = w_1 - w_7 + iw_8 - iw_0$$

$$f_2 = w_2 - w_0 + iw_1 - iw_3\,, \qquad f_3 = w_3 - w_5 + iw_0 - iw_4\,.$$

Substituting in (128)

$$\begin{aligned} 4\delta Zf_0 = (x - y)(w_2 - w_6) + (x + y)(w_4 - w_8) \\ + 2i[x(w_1 - w_5) + y(w_3 - w_7)]. \end{aligned} \tag{129}$$

Let w be a real harmonic function; then f is analytic. Thus it follows from (129) that

$$U = (x - y)(XY - X^{-1}Y^{-1})w + (x + y)(X^{-1}Y - XY^{-1})w \tag{130}$$

and

$$V = 2x(X - X^{-1})w + 2y(Y - Y^{-1})w \tag{131}$$

are a pair of harmonic conjugates.

Let g denote the Green's function and let $q = -L'g$. Thus $Lq = 0$ except at the origin where $Lq = 1$. Let $s = Zq$. It is clear from (126) that $Ls = 0$ everywhere except possibly at the points $z = 0, -1, -1 -i$, and $-i$. Testing these points by (126) gives $(Ls)_0 = \delta - i\delta^* = 0$, $(Ls)_{-1} = (-1 + \delta) + i(-1 + \delta)^* = 0$, $(Ls)_{-1-i} = (-1 - i + \delta) - i(-1 - i + \delta)^* = 0$, $(Ls)_{-i} = (-i + \delta) + i(-i + \delta)^* = 0$. Thus s is analytic everywhere. It was shown before that q vanishes at infinity. Thus $| zq | = o(| z |)$ so by a previous theorem, s is a bipolynomial. Since $| s | = o(| z |)$, clearly s must be a biconstant. On the even lattice $s = 0$ because $s(0) = 0$. To evaluate the constant value of s on the odd lattice, use may be made of (129) with $w = -g$ and $f = q$. Let $z_0 = 1$, then $4\delta s(1) = -2ig(2, 0)$, all other terms in (129) vanish. Now it is seen from (70) that $k(1, 0) = 2g(2, 0)$ and (64) gives $k(1, 0) = 2/\pi$ so $g(2, 0) = 1/\pi$. Thus $s(1) = -\delta/\pi$. *This shows that Zq vanishes on the even lattice and $Zq = -(1 + i)/2\pi$ on the odd lattice.*

From the above theorem it results that the Green's function satisfies some interesting identities. Thus (130) and (131) give

$$\begin{aligned} 0 = (x - y)[g(x + 1, y + 1) - g(x - 1, y - 1)] \\ + (x + y)[g(x - 1, y + 1) - g(x + 1, y - 1)] \end{aligned} \tag{132}$$

and

$$1/\pi = x[g(x + 1, y) - g(x - 1, y)] + y[g(x, y + 1) - g(x, y - 1)]. \tag{133}$$

In this last relation (x, y) denotes a point of the odd lattice. It is not difficult to show that identity (133) together with the difference equation defining g give a step by step procedure for explicitly evaluating g at any given point of the plane.

Let an operator $\mathbf{Z}$ be defined by $4\mathbf{Z} = ZS'$ where $S' = I + X^{-1} + Y^{-1} + X^{-1}Y^{-1}$. So

$$16\mathbf{Z} = z\mathbf{S} - iz^*\mathbf{T} \tag{134}$$

where $\mathbf{S} = SS'$ and $\mathbf{T} = TS'$. In terms of the spiral coordinate notation

$$\mathbf{S}f_0 = 4f_0 + 2f_1 + 2f_3 + 2f_5 + 2f_7 + f_2 + f_4 + f_6 + f_8 \tag{135}$$

$$\mathbf{T}f_0 = f_2 - f_4 + f_6 - f_8\,. \tag{136}$$

It follows from (126) that $16L\mathbf{Z}f = [(z + \delta)\mathbf{S} + (z + \delta)^*\,\mathbf{T}]\,Lf$. Thus $\mathbf{Z}f$ is analytic if f is analytic. The operators Z and $\mathbf{Z}$ are similar; however, $\mathbf{Z}$ has greater symmetry relative to the point of application.

To study the properties of $\mathbf{Z}$ it is useful to introduce an analog of the function

e^{zt}. Thus let $e(z, t)$ be defined by

$$e(z, t) = \left(\frac{2 + t}{2 - t}\right)^x \left(\frac{2 + it}{2 - it}\right)^y. \tag{137}$$

This function was introduced by Jacqueline Ferrand. A similar function was employed by Isaacs to study multiplication. It is easy to check that $Le(z, t) = 0$ and that

$$1 + t \int_0^z e(z, t) \, \delta z = e(z, t). \tag{138}$$

This relation brings out the analogy with e^{zt}.

If $| t | < 2$ a power series expansion in t is valid. Thus

$$e(z, t) = \sum_0^\infty \frac{z^{(n)} t^n}{n!}. \tag{139}$$

By applying the binomial theorem to (137) it would be possible to obtain a formula for the coefficients, $z^{(n)}$. For the present purpose it is sufficient to observe that they are polynomials in x and y. In fact it follows from (138) that $z^{(n+1)} = (n + 1) \int_0^z z^{(n)} \, \delta z$ and $z^{(0)} = 1$. Thus $z^{(n)}$ is identical with the polynomial ρ_n introduced before.

Straightforward calculation using (134) and (137) gives

$$\mathbf{Z}e(z, t) = e(z, t)(16z + 4t^2z^*)(16 - t^4)^{-1}.$$

It is readily verified that the same expression on the right is obtained by differentiating (137) with respect to t. Thus

$$\mathbf{Z}e(z, t) = \partial e(z, t)/\partial t. \tag{140}$$

Substituting the series (139) in (140) gives

$$\sum_0^\infty t^n \mathbf{Z} z^{(n)}/n! = \sum_0^\infty t^n z^{(n+1)}/n!$$

Equating the coefficients of t^n gives

$$\mathbf{Z} z^{(n)} = z^{(n+1)}. \tag{141}$$

No doubt there are other interesting relations of this sort corresponding to multiplication in the continuous case.

References

1. A. C. Allen and B. H. Murdoch, *A note on preharmonic functions*, Proceedings of the American Mathematical Society, vol. 4(1953), pp. 842–852.
2. R. Courant, *Über partielle Differenzengleichungen*, Atti Congresso Internazionale Dei Matematici-Bologna, vol. 3(1928), pp. 83–89.
3. R. J. Duffin, *Discrete potential theory*, this Journal, vol. 20(1953), pp. 233–251.
4. R. J. Duffin and D. H. Shaffer, *Asymptotic expansion of double Fourier transforms*, Technical Report 9, OOR Contract DA-36-061-ORD-277, Carnegie Institute of Technology (1953).

5. Jacqueline Ferrand, *Fonctions préharmoniques et fonctions préholomorphes*, Bull. Sci. Math., vol. 68(1944), second series, pp. 152–180.
6. H. A. Heilbronn, *On discrete harmonic functions*, Proceedings of the Cambridge Philosophical Society, vol. 45(1949), pp. 194–206.
7. Rufus Philip Isaacs, *A finite difference function theory*, Universidad Nacional Tucumán, Revista, vol. 2(1941), pp. 177–201.
8. R. Isaacs, *Monodiffric functions*, National Bureau of Standards Applied Mathematics Series no. 18(1952), pp. 257–266.
9. W. H. McCrea and F. J. W. Whipple, *Random paths in two and three dimensions*, Proceedings of the Royal Society of Edinburgh, vol. 60(1939–40), pp. 281–298.
10. Alfred Stöhr, *Über einige lineare partielle Differenzengleichungen mit konstanten Koeffizienten. III*, Mathematische Nachrichten, vol. 3(1950), pp. 330–357.
11. A. Terracini, *A first contribution to the geometry of monodiffric polynomials*, Actas de la Academia Nacional de Ciencias Exactas, Físicas y Naturales de Lima, vol. 8(1945), pp. 217–250.
12. A. Terracini, *On the geometry of monodiffric polynomials*, Revista de la Unión Matemática Argentina, vol. 12(1946), pp. 55–61.
13. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937, Chapter V.
14. M. Riesz, *Sur les fonctions conjuguées*, Mathematische Zeitschrift, vol. 27(1927), pp. 218–277.
15. O. A. Varsavsky, *Sobre la transformatión de Hilbert*, Revista de la Unión Matemática Argentina, vol. 14(1949), pp. 20–37.
16. A. P. Calderón and A. Zygmund, *Singular integrals and periodic functions*, Studia Mathematica, vol. 14(1954), pp. 249–271.

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