Q4a: We wish to compute the following integral using residue calculus,

$$\int_{[0,2\pi]} \cos^{2n}\theta d\theta.$$

Substituting $\cos(z) = \frac{1}{2}(z + \frac{1}{z})$ on S^1, we get the integral

$$\frac{-i}{2^{2n}} \int_{S^1} \frac{1}{z}\Big(\frac{z^2+1}{z}\Big)^{2n} dz.$$

Using the binomial expansion, we compute that

$$\frac{1}{z}\Big(\frac{z^2+1}{z}\Big)^{2n} = \frac{1}{z}\sum_{k=0}^{2n} \frac{(z^2)^{2n-k}}{z^{2n}}.$$

Notice that the only place in this expansion that we have a term of the form $\frac{1}{z}$ will be when $k = n$. Thus the laurent series for each polynomial will have a zero coefficient on a_{-1} except when $\binom{2n}{n}$ Hence the residue of the integral will be $\binom{2n}{n}$ Thus by the residue theorem, we have that

$$\int_{[0,2\pi]} \cos^{2n}(\theta)d\theta = \frac{2\pi i \cdot -i \cdot \binom{2n}{n}}{2^{2n}} = \frac{\pi\binom{2n}{n}}{2^{2n-1}}$$

Q4b: We wish to compute

$$\int_{[0,\infty]} \frac{x^2-a^2}{x^2+a^2}\frac{\sin(x)}{x}dx.$$

The integrand is even so it is equal to

$$\frac{1}{2}\int_{\mathbb{R}} \frac{x^2-a^2}{x^2+a^2}\frac{\sin(x)}{x}dx.$$

We will compute this integral as

$$Im\Big(\frac{1}{2}\int_{\mathbb{C}} \frac{z^2-a^2}{z^2+a^2}\frac{e^{iz}}{z}dz\Big).$$

This makes sense to integrate since

$$\Big|\frac{z^2-a^2}{z^2+a^2}\Big| < \infty, \forall z \in \mathbb{C}\,.$$

Note that the integrand has poles at $\pm ia, 0$. Thus by the residue formula we have

$$\frac{1}{2}\int_{\mathbb{C}} \frac{z^2-a^2}{z^2+a^2}\frac{e^{iz}}{z}dz = 2\pi i \sum Res(f) = \pi i\Big(\frac{(-a^2-a^2)e^{-a}}{-3a^2+a^2} + \frac{-a^2e^0}{2a^2} = \pi i e^{-a} - \frac{\pi}{2}i\Big)$$

Therefore

$$\int_{[0,\infty]} \frac{x^2-a^2}{x^2+a^2}\frac{\sin(x)}{x}dx = \pi e^{-a} - \frac{\pi}{2}.$$