Q4a: Suppose that $x(t) = t^\gamma$ solves $\sum_{k=0}^n a_k t^k x^{(k)}$. Then it must be that $\sum_{k=0}^n a_k t^k (t^\gamma)^{(k)} = 0$. Evaluating, we see that

$$\begin{aligned}
0 &= \sum_{k=0}^n a_k t^k (t^\gamma)^{(k)} \\
&= \sum_{k=0}^n a_k t^k t^{\gamma-k} \cdot (\gamma-1) \cdot \ldots (\gamma-k) \\
&= \sum_{k=0}^n a_k t^\gamma \frac{\gamma!}{(\gamma-k)!} \\
&= \sum_{k=0}^n a_k \frac{\gamma!}{(\gamma-k)!} \\
&= \sum_{k=0}^n b_k \gamma^k && \text{(for some } b_k\text{)}
\end{aligned}$$

We see that $x(t) = t^\gamma$ is a solution to our ODE if and only if it is a root to the polynomial $\sum_{k=0}^n b_k x^k$, where each $b_k = \sum_{j=0}^k (-1)^j \cdot j! \cdot a_{k+j}$

Q4b: Let $y(t) = t^\gamma Q(\log t)$ where Q is a polynomial of degree m. We compute

$$t \cdot y' - \alpha y = \gamma t^\gamma Q(\log t) + t^\gamma Q'(logt) - \alpha t^\gamma Q(\log t)$$

If $\alpha = \gamma$ we see that this expression evaluates to $t^\gamma Q'(\log t)$, with Q' being of degree $m-1$. If $\alpha \neq \gamma$, then this will be of the form $t^\gamma P(\log t)$ for some polynomial P.

Q4c: Suppose that γ is a root to $\sum_{k=0}^n b_k \gamma^k$. It suffices to check that $t^{\gamma_j} Q_j(\log t)$ satisfies our ODE, since it is linear and so any sum of functions of that form will work. We see that

$$\begin{aligned}
\sum_{k=0}^n a_k \cdot t^k (\frac{d}{dt})^k y(t) &= \sum_{k=0}^n b_k (t\frac{d}{dt})^k (y(t)) \\
&= \sum_{k=0}^n b_k ((t^\gamma Q_j'(\log t) + \alpha t^\gamma Q_j(\log t))^k && \text{(by 4b)} \\
&= \sum_{k=0}^n b_k (t^\gamma Q_j'(\log t))^k + \alpha \sum_{k=0}^n b_k (t^\gamma Q_j(\log t))^k \\
&= \alpha \sum_{k=0}^n (Q_j(\log t))^k \sum_{k=0} b^k \gamma^k \\
&= 0
\end{aligned}$$