Q4: Suppose $\|f_n - f\|_p \to 0$ as $n \to \infty$. Since $\|\cdot\|_p$ is a norm, we have that the reverse triangle inequality holds, *i.e.* we have

$$|\,\|f_n\|_p - \|f\|_p\,| \leq \|f_n - f\|_p\,.$$

Thus as $n \to \infty$,$|\,\|f_n\|_p - \|f\|_p\,| \to 0$ or equivalently $\|f_n\|_p \to \|f\|$. Conversely suppose that $\|f_n\|_p \to \|f\|_p$. We first claim that for any two functions f, g the following inequality holds:

$$2^{-p}|f+g|^p \leq |f|^p + g^p.$$

Indeed we have that

$$|f+g|^p \leq 2^p \max |f|^p, |g|^p \leq 2^p(|f|^p + |g|^p).$$

Dividing by 2^p gives the desired inequality. Therefore we can write that

$$2^{-p}|f_n - f|^p \leq |f|^p + |f_n|^p.$$

We now claim the Generalized Dominated Convergence Theorem. Given f_n, g_n with $f_n \to f, g_n \to g$ a.e. , $|f_n| \leq |g_n|$, and $\int g_n \to \int g$ then $\int f_n \to \int f$. Observe that by Fatou's lemma we have that

$$\int g - \int f \leq \liminf \int g_n - f_n = \lim_n \int g_n - \limsup \int f_n = \int g - \limsup \int f_n.$$

Similarly, we have that

$$\int g + \int f \leq \liminf \int g_n + f_n = \lim_{n\to\infty} \int g_n + \liminf \int f_n = \int g + \liminf \int f_n.$$

Thus we have that

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n.$$

And we conclude that

$$\int f = \lim_{n\to\infty} \int f_n.$$

Since $\|f_n\|_p \to \|f_n\|$, and $2^{-p}|f_n - f|^p \leq |f|^p + |f_n|^p$, the generalized DCT tells us that

$$\lim_{n\to\infty} \int 2^{-p}|f_n - f|^p d\mu = \int 2^{-p}\Big(\lim_{n\to\infty} |f_n - f|\Big)^p = 0,$$

and thus $|f_n - f| \to 0$ as $n \to \infty$.