Q2a: Suppose that f is uniformly continuous. Then we have that for each $\varepsilon > 0$ there exists some $\delta > 0$ such that $|x-t| < \delta \implies |f(x)-f(t)| < \varepsilon$. Now if we fix x in the domain of f, we have continuity of f at x. This is true for any x in the domain of f so f is continuous. We now claim that $f(x) : (0,1) \to \mathbb{R}$ defined by $x \mapsto \sin(\frac{1}{x})$ is continuous yet not uniformly continuous. It is easy to see that it is continuous, as it is the composition of two continuous maps. Choose $\varepsilon = 1$. Then for every $\delta > 0$, we can find $x, t \in (0, \delta)$ where $f(x) = 1$ and $f(t) = -1$ in the following way. Choose x so that $\frac{1}{x} > \frac{1}{\delta}$, and x is of the form $\frac{1}{x} = \frac{\pi}{2} + 2k\pi$ for sufficiently large k. Similarly, choose $\frac{1}{t} = \frac{\pi}{2} + (2k+1)\pi$ for sufficiently large k. We have that $|f(x)-f(t)| = 2 > \varepsilon$

Q2b: We claim $f(x) = 2x$ is uniformly continuous on $\mathbb{R}$. Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{2}$. For any $x, y \in \mathbb{R}$ we compute that

$$|x-y| < \frac{\varepsilon}{2} \implies |2x-2y| < \varepsilon \implies |f(x)-f(y)| < \varepsilon$$

Hence f is uniformly continuous.

Q2c: We claim $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Choosing $\varepsilon = 1$, and choose $x = y + \frac{\delta}{2}$. We have $|x-y| = |\frac{\delta}{2}| < \delta$. We see that for sufficiently large y, $|f(x)-f(y)| = |\frac{\delta^2}{4} + \delta y| > 1$. Hence f will not be uniformly continuous.