Q5: $\implies$

Suppose that an $n-1$ manifold M is orientable. Let μ_a be an orientation of $T_a\,\mathbb{R}^{n-1}$. Then we know that for all coordinate patches f, $f_*\mu_a = f_*\mu_b$. Now let $a \in W \subset \mathbb{R}^n$ corresponding to some $f : W \to M$. Consider the matrix given by

$$A = \begin{bmatrix} Df(a)\cdot e_1 \\ Df(a)\cdot e_2 \\ \vdots \\ Df(a)\cdot e_{n-1} \end{bmatrix}$$

This matrix is rank $n-1$ for all points a, and so by the fundamental theorem of linear maps it must have a kernel of dimension 1. Define $n(f(a))$ to be the nontrivial vector in the kernel such that

$$\det\begin{pmatrix} n(f(a)) \\ Df(a)\cdot e_1 \\ \vdots \\ Df(a)\cdot e_{n-1} \end{pmatrix} > 0$$

Note that we can always make the determinant positive, by appropriately scaling $n(f(a))$ and that $\langle n(a), Df(a)\cdot e_i\rangle = 0$ for all i. We now claim that the function $n(f(a))$ is a smooth vector field on M which vanishes nowhere. We first note that to obtain $n(f(a))$, we must compute the nullspace of a matrix of smooth entries. Computing null spaces amounts to arithemetic which is smooth, so our function is a composition of smooth functions and hence is smooth. Note that as we change between different coordinate patches, we can pre-compose with a smooth transition map to ensure smoothness. Note that by construction, the normal vector will not vanish. This function is our desired ν.

$\impliedby$

Suppose that there is a consistent non-zero normal field ν to M in $\mathbb{R}^n$. Let $p \in M$, define η_p on M by

$$\eta_p(v_1, \ldots v_{k-1}) = \omega(v_1 \ldots v_{k-1}, \nu(p))$$

Where ω is a volume form on $\mathbb{R}^n$. We can see that our choice of η_p does not vanish on M, since we have that $n(p)$ orthorgonal to all $v_i \in T_pM$. Hence we have a choice of a top form which does not vanish on M, so M is orientable.