Problem 1.

By Montels little theorem, $\{f_k\}$ is normal if and only if $\{f'_k\}$ is locally bounded and at some z_0, $f(z_0)$ is uniformly bounded. We see that

$$f'_k(z) = \cos(kz).$$

Therefore $\{f'_k\}$ is bounded since for $|z| < 1$,

$$|\cos(kz)| \leqslant \cos(k|z|) \leqslant 1.$$

Furthermore we have at $z_0 = 0$, $f_k(0) = \frac{\sin(kz)}{k} = 0$. Therefore $\{f_k\}$ is a normal family.

Problem 2.

(a) First note that by Harnacks Inequality, this is true for the real part of f, $i.e.$ we have that

$$\frac{1-|z|}{1+|z|} \leqslant \operatorname{Re}(f) \leqslant \frac{1+|z|}{1-|z|}.$$

A similar argument can be made for the imaginary part of f by adjusting constant so $im(f) > 0$. It follows that the inequality holds for any $f \in \mathcal{A}$.

(b) We claim that $\mathcal{A}$ is locally bounded. Take any $z \in D$. Then on any sufficiently small neighbourhood of z containing z we have that $|f(z)| \leqslant \frac{1+|z_0|}{1-|z_0|}$ at some z_0 in the disk, for all $f \in \mathcal{A}$. Thus $\mathcal{A}$ is locally bounded and hence normal.

(c) By Cauchys inequality, we have that for each f_k,

$$a_1^k \leqslant r^{-1} \sup_{|z|=r} |f_k(z)| \leqslant r^{-1} \sup_{|z|=r} \frac{1}{2\pi} \int_{|z|=r} |f(z)| dz \leqslant 1$$

Since this is true for all k, we have that $|f_k'(0)| \leqslant 1$.

Problem 3.

(a) By the Riemann Mapping Theorem, there exists a conformal $g : \Omega \to D$. For $a \in \Omega$, we define $h : D \to D$ by $h(z) = e^{i\theta}\frac{z-g(a)}{1-\overline{g(a)}g(z)}$ for some θ. We define $f = h \circ g$, so $f = e^{i\theta}\frac{g(z)-g(a)}{1-\overline{g(a)}g(z)}$. Notice that f is a conformal mapping of Ω to D, with $f(a) = 0$. It remains to show that $f'(a) > 0$. We compute that

$$f'(z) = e^{i\theta}\frac{g'(z)(1-\overline{g(a)}g(z)) + \overline{g(a)}(g(z)-g(a))}{(1-\overline{g(a)}g(z))^2}.$$

Evaluating at $z = a$ we get

$$f'(z) = e^{i\theta}\frac{g'(a)(1-|g(a)|^2)}{(1-|g(a)|^2)^2}.$$

This will be positive for some choice of θ, so that $e^{i\theta}g'(a) > 0$. We now claim that such f is unique. Suppose f_1, f_2 satisfy our desired properties. Then $f_2 \circ f_1^{-1} \in \mathrm{Aut}(D)$. Furthermore, $f_2 \circ f_1^{-1}(0) = f_2(a) = 0$. So by Schwartz' Lemma $f_1 = \lambda f_2$ for some $\lambda \in U(1)$. Since $f_1'(a), f_2'(a) > 0$ we have that $\lambda = 1$.

(b)
i) Let $\gamma \subset \Omega$ be a closed curve. There must be some minimal N so that $\gamma \subset \Omega_N$. Since Ω_N is simply connected γ can be deformed to a point in Ω_N and hence in Ω.

ii) Note that $\{f_n\}$ is a normal family, since $\{f_n'\}$ is locally bounded, and $f_n(0) = 0$ for all n. Therefore there is a uniformly convergent subseqence $\{f_{n_k}\}$ that converges to some $f : D \to \Omega$. Note that by uniform convergence, we have that $f(0) = 0$, and $f'(0) > 0$. By a previous result, we have that f is $1-1$ as well. It follows that f is a conformal mapping of $D \to \Omega$. Furthermore, it is unique by $3a$. This is true for every subsequence of $\{f_n\}$, since $\lim_{n\to\infty}\Omega_n = \Omega$. It follows that every subsequence converges uniformly to f so f_n converges uniformly to f.

Problem 4.

We identify $\mathbb{C} \setminus \{0\}$ with $S^2 \setminus \{S, N\}$, (Riemann Sphere without the poles). Therefore an automorphism of $\mathbb{C} \setminus \{0\}$ is an automorphism of S^2 which either fixes the poles or reverses them i.e. $f(0) = 0, f(\infty) = \infty$ or $f(0) = \infty, f(\infty) = 0$. We have that f must be a fractional linear transformation. If f fixes the poles, it must be of the form $f(z) = az$ for nonzero $a \in C$. If f swaps the poles it must be of the form $f(z) = \frac{c}{z}$ for nonzero $c \in \mathbb{C}$. This gives a complete description of $Aut(\mathbb{C} \setminus \{0\})$.

Problem 5.

(a) By the discussion from class, a rectangle with corners at $k, -k, -k+ik', k+ik'$ is the image of the Riemann mapping given by

$$F(w) = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

with $F(1) = k$. Therefore to have $F(\infty)$ be a corner, it is enough to find $g \in \mathrm{Aut}(\mathbb{H}^+)$ so $g(\infty) = 1$. Taking

$$g(z) = \frac{z+1}{z+2},$$

will suffice. Then $F \circ g$ will be a conformal mapping of $\mathbb{H}^+$ onto the rectangle, with $F \circ g(\infty) = k$.

(b) By tiling with the rectangles given by a), we obtain $\wp(z)$ generated by $\Gamma = \langle 4k, 2ik' \rangle$, corresponding to the elliptic curve $(\wp'(z), \wp(z)) \subset \mathbb{C}^2$.

Problem 6.

We claim the image of the map will be the attatched image. Note that each a_i gets sent to 0. We claim that on each arc between a_i, a_{i+1} the argument of f is constant. We have that

$$\log'(f(z)) = -\frac{1}{z} + \sum_{k=1}^{n} \frac{\lambda_k}{(z - a_k)}.$$

We claim that the imaginary part of this function is constant for $|z| = 1, z = e^{i\theta}, \theta \in (\arg(a_i), \arg(a_{i+1}))$. Then,

$$\log'(f(e^{i\theta})) = -e^{-i\theta} + \sum_{k=1}^{n} \frac{\lambda_k}{(e^{i\theta} - a_k)} = -e^{-i\theta} + \sum_{k=1}^{n} \frac{\lambda_k(e^{-i\theta} - \overline{a_k})}{2 - \mathrm{Re}(e^{-i\theta} a_k)}.$$

Since the imaginary part of this is constant, we have that this will map the arcs to 0.

Problem 7.

Let f be meromorphic, defined on S^2. Then the spherical derivative at z is given by:

$$f^{\#}(z) = \lim_{w \to z} \frac{d(f(z), f(w))}{d(z, w)},$$

where d is the chordal metric on S^2. First assume that z is not a pole of f. Then we have that

$$f^{\#}(z) = \lim_{w \to z} \frac{d(f(z), f(w))}{d(z, w)} = \lim_{w \to z} \frac{\rho(f(z), f(w)) + |f(z) - f(w)|^2}{d(z, w)} = \lim_{w \to z} \frac{\rho(f(z), f(w))}{|z - w|}.$$

If z is a pole, since $f^{\#} = \frac{1}{f^{\#}}$, we apply the previous computation and conclude that the desired equality holds.