Q2: Since $F(x)$ is right continuous, and defined on $[a,b]$ we have that $F(x) \in L^1$. Furthermore we have that each $F_j \in L^1$. By Folland 2.23 b, it is sufficient to show that

$$\int_{(a,x]} F'(t)dt = \int_{(a,x]} \sum_j F'(t)dt.$$

Note that by Lebesgue Radon Nikodyn Theorem, we can write

$$\mu_F = \nu_F + \rho_F = \nu_F + \int f dm,$$

and for each $F_j$ we have

$$\mu_{F_j} = \nu_{F_j} + \int f_j dm.$$

Summing over all $j$ we get that

$$\nu_F + \int f dm = \mu_F = \sum_j \mu_{F_j} = \sum_j \nu_{F_j} + \sum_j \int f_j dm.$$

We claim that this is a Lebesgue-Radon-Nikodyn decomposition of $\mu_F$. Note that if $\mu_F(E) = 0$, then for each $j$, $\mu_{F_j}(E) = 0$. Therefore for each $f_j, \int_E f_j dm = 0$. Hence their countable sum is 0 as well. We now claim that $\sum_j \nu_{F_j} \perp m$. For each $\nu_{F_j}$ let $E_j$ be the set which it is 0 on. We have that for $E_j^c$, $m|_{E_j^c} = 0$. Take the set $E = \bigcap E_j$. If this intersection is empty then the result holds trivially. If not, then we have that $\sum_j \nu_{F_j}|_E = 0$. Similarly, $m|_{E^c} = 0$. Hence this is a L-R-N decomposition. Mutual singularity of $\nu_{F_i}$ implies that on an $a.e.$ set

$$\int f dm = \sum_j \int f_j dm.$$

Furthermore, by DCT we have that

$$\int f dm = \int \sum_j f_j dm.$$

Hence we have that $f = \sum_j f_j$. Thus we compute that

$$\int_{(a,x]} F'(t)dt = \int_{(a,x]} d\mu_F = \int_{(a,x]} d\nu_F + f dm = \int_{(a,x]} d\sum_j d\nu_{F_j} + d\sum_j \mu_{F_j} = \int_{(a,x]} \sum_j F'(t)dt$$