

Q1: If v_1, v_2, v_3 is a basis for \mathbb{R}^3 , we let $\omega_1, \omega_2, \omega_3$ be a basis for $\Lambda^1 \mathbb{R}^3$ and let $\omega_1 \wedge \omega_2, \omega_2 \wedge \omega_3, \omega_1 \wedge \omega_3$ be a basis for $\Lambda^2 \mathbb{R}^3$. We can identify $\Lambda^1 \mathbb{R}^3$ with \mathbb{R}^3 by viewing v_i as ω_i . Similarly, we can identify $\Lambda^2 \mathbb{R}^3$ with \mathbb{R}^3 by $v_1 \rightarrow \omega_1 \wedge \omega_2, v_2 \rightarrow \omega_2 \wedge \omega_3, v_3 \rightarrow \omega_1 \wedge \omega_3$. We claim that the usual \wedge product on $\Lambda^1 \mathbb{R}^3 \times \Lambda^1 \mathbb{R}^3 \rightarrow \Lambda^2 \mathbb{R}^3$ can be viewed as a vector product operation on $\mathbb{R}^3 \times \mathbb{R}^3$. Let $\lambda = \alpha_1 \omega_1 + \alpha_2 \omega_2 + \alpha_3 \omega_3$ and let $\eta = \beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3$. We evaluate $\lambda \wedge \eta$ in the usual way;

$$\begin{aligned}
 \lambda \wedge \eta &= (\alpha_1 \omega_1 + \alpha_2 \omega_2 + \alpha_3 \omega_3) \wedge (\beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3) \\
 &= \alpha_1 \omega_1 \wedge (\beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3) + \alpha_2 \omega_2 \wedge \beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3 + \alpha_3 \omega_3 \wedge \beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3 \\
 &= \alpha_1 \omega_1 \wedge \beta_1 \omega_1 + \alpha_1 \omega_1 \wedge \beta_2 \omega_2 + \alpha_1 \omega_1 \wedge \beta_3 \omega_3 + \alpha_2 \omega_2 \wedge \beta_1 \omega_1 + \alpha_2 \omega_2 \wedge \beta_2 \omega_2 + \\
 &\quad \alpha_2 \omega_2 \wedge \beta_3 \omega_3 + \alpha_3 \omega_3 \wedge \beta_1 \omega_1 + \alpha_3 \omega_3 \wedge \beta_2 \omega_2 + \alpha_3 \omega_3 \wedge \beta_3 \omega_3 \\
 &= \alpha_1 \omega_1 \wedge \beta_2 \omega_2 + \alpha_1 \omega_1 \wedge \beta_3 \omega_3 - \beta_1 \omega_1 \wedge \alpha_2 \omega_2 + \alpha_2 \omega_2 \wedge \beta_3 \omega_3 - \beta_1 \omega_1 \wedge \alpha_3 \omega_3 - \beta_2 \omega_2 \wedge \alpha_3 \omega_3 \\
 &\quad \text{(by supercommutativity)} \\
 &= (\alpha_1 \beta_2 - \beta_1 \alpha_2) \omega_1 \wedge \omega_2 + (\alpha_2 \beta_3 - \beta_2 \alpha_3) \omega_2 \wedge \omega_3 + (\alpha_1 \beta_3 - \beta_1 \alpha_3) \omega_1 \wedge \omega_3 \quad \text{(by linearity of } \wedge \text{)}
 \end{aligned}$$

Therefore, we have a product operation on vectors in \mathbb{R}^3 defined in the following way:

$$(\alpha_1, \alpha_2, \alpha_3) \times (\beta_1, \beta_2, \beta_3) \mapsto (\alpha_1 \beta_2 - \alpha_2 \beta_1, \alpha_2 \beta_3 - \alpha_3 \beta_2, \alpha_1 \beta_3 - \alpha_3 \beta_1)$$

Note this product inherits properties of the wedge product. It is bilinear, alternating, supercommutative.