

Q2: Suppose not, that is suppose that in some neighbourhood of a point $p \in M$, we have that M is not measure 0 as a subset of \mathbb{R}^{22} . Let $U \ni p$ be a neighbourhood of p such that there exists an open $V \subset \mathbb{R}^{22}$, and a diffeomorphism $h : U \rightarrow V$ where $h(U \cap M) = V \cap (\mathbb{R}^{12} \times 0_{\mathbb{R}^{10}})$. However, we have that $V \cap (\mathbb{R}^{12} \times 0_{\mathbb{R}^{10}})$ is clearly a measure 0 set. We now claim that the image of a bounded measure 0 set under a diffeomorphism is measure 0.

pf: Let f be a diffeomorphism from A to B where A is measure 0. Note that on a bounded set, the mean value theorem implies that for all $x, y \in A$, we have that $|f(x) - f(y)| \leq M|x - y|$, for some $M \in \mathbb{R}$. Hence we have that f is Lipschitz. Thus, for all $\varepsilon > 0$, we can take an open covering of A by squares $\{S_k\}$ such that $\sum_k \text{vol}(S_k) < \varepsilon$. We have that the diameter (maximum distance between points) of $S_k \cap A$ will be less than or equal to the diameter of S_k . Therefore, by Lipschitz, we have that the diameter of $f(S_k \cap A)$ is less than $M \cdot \text{Diam}(S_k \cap A)$. Hence there exists some square S'_k containing $f(S_k \cap A)$ with diameter $M \cdot \text{Diam}(S_k)$. We can do this for every square covering A , and by continuity $\{S'_k\}$ forms a cover for B . We see that

$$\sum_k S'_k \leq M^n \sum_k \text{Diam}(S_k)^n \leq nM^n \sum_k \text{vol}(S_k) \leq nM^n \varepsilon$$

Since M, n are both fixed we have that $f(A)$ must be measure 0. ■

Therefore applying the inverse of h to our measure 0 set $V \cap (\mathbb{R}^{12} \times 0_{\mathbb{R}^{10}})$ should be measure 0, but by assumption $U \cap M$ is not measure 0. We obtain a contradiction and conclude that every 12 manifold in \mathbb{R}^{22} is measure 0.