Lie Groups and Lie Algebras

A.N.

June 16, 2023

Problem 1. *Exercise 1.7*

For $A \in Sp(n)$, A must belong to $GL(n, \mathbb{H}) \cap O(4n)$. We identify each $x = a + ib + jc + kd \in \mathbb{H}$ with a 4x4 real matrix, and a 2x2 complex matrix in the following way:

$$x \sim \begin{bmatrix} a & -d & c & b \\ d & a & -b & c \\ -c & b & a & d \\ -b & -c & -d & a \end{bmatrix} \sim \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}.$$

Where $\alpha = a + id, \beta = -c - ib$. Thus we can regard A as an element of $GL(2n, \mathbb{C}) \cap O(2n) = U(2n)$. Conjugation by J flips the order of the columns, and swaps signs This corresponds to conjugation of each x. So $\overline{A} = JAJ^{-1}$. Conversely orthogonality of A implies that it can be written as an invertible matrix over quaterions. So it must be in $Sp(n)$.

Problem 2. *Exercise 1.8*

Suppose $A \in SO(2m)$. A has determinant 1, preserves the norm, and hence preserves the inner product. From mat247 we know that there exists a change of basis matrix O so that OAO^{-1} is of the desired form. We claim that $O \in SO(2m)$. Since $(OAO^{-1})^T = (OAO^{-1})^{-1}$, and $A \in SO(2m)$ we must have that $O \in SO(2m)$. If $A \in SO(2m+1)$ then 1 must be a root of the characteristic polynomial, i.e. there is a 1-dim A stable subspace U. We can therefore write $\mathbb{R}^{2m} = U \oplus W$ for some even dimensional W. Apply the previous argument to $A|_W$. Therefore we have a continuous map which surjects onto $SO(n)$ given by

$$f(\theta_1, \dots, \theta_n) = \begin{bmatrix} R(\theta_1) & 0 & \cdots & 0 \\ 0 & R(\theta_2) & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & R(\theta_{\frac{n}{2}}) \end{bmatrix}$$

for even n, and for odd n we make the n'th row and column 0 except at (n, n) entry of the matrix. Since f is continous and $SO(n)$ is the image of f, we have that $SO(n)$ is connected.

Problem 3. *Exercise 1.9*

Since G is a connected manifold, it must be path connected as well. Let $\gamma : [0, 1] \to G$ be a path so $\gamma(0) = e$, $\gamma(1) = g$. Then the family $\{\gamma(t)U\}_{t\in[0,1]}$ gives an open covering of $\gamma([0, 1])$. By compactness, we have