Problem 1.

We have previously shown that

$$\frac{1}{z^2} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{(z-n)^2} = \left(\frac{\pi}{\sin(\pi z)}\right)^2.$$

Sufficiently near 0, the righthand side admits the following laurent series:

$$\left(\frac{\pi}{\sin(\pi z)}\right)^2 = \left(\frac{\pi}{\pi z - \frac{1}{6}\pi^3 z^3 + \dots}\right)^2 = \frac{1}{z^2} + \frac{\pi^2}{3} + z^2(\dots).$$

Therefore at $z = 0$ we have that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3},$$

and by symmetry

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Our intial expression is holomorphic so we apply the derivative twice to yield that

$$\frac{6}{z^4} + \sum_{n=-\infty}^{\infty} \frac{6}{(z-n)^4} = \frac{2\pi^4(1 + 2\cos^2(\pi z))}{\sin^4(\pi z)}.$$

Near $z = 0$ the righthand side has the following laurent expansion:

$$\frac{2\pi^4(1 + 2\cos^2(\pi z))}{\sin^4(\pi z)} = \frac{6}{z^4} + \frac{2\pi^4}{15} + z(\dots).$$

Therefore at $z = 0$ we have that

$$\sum_{n=-\infty}^{\infty} \frac{6}{(n)^4} = \frac{2\pi^4}{15}.$$

By symmetry we have

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Problem 2.

a) Let $\{a_k\}$ and $\{b_k\}$ be the zeros and poles of f respectively. They must all be of order 2 since f is an even elliptic function. Then the function

$$f(z) \cdot \prod_{k=1}^{n} \frac{\wp(z) - \wp(b_k)}{\wp(z) - \wp(a_k)}$$

is elliptic, has no zeros or poles since the zeros sum to 0 mod Γ. Hence it is constant by Liouvilles Theorem. Therefore we can write

$$f(z) = c \cdot \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}.$$

If 0 is a pole of order $2l$, then the function

$$f(z) \cdot \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)} \cdot \frac{1}{\wp(z)^l}$$

has no zeros nor poles and is hence constant. Similarly, if 0 is a zero of degree $2l$ we have that

$$f(z) \cdot \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)} \cdot \wp(z)^l$$

is constant. Therefore if f is an even elliptic function we can write

$$f(z) = R(\wp),$$

where R is a rational function.

b) If f is an odd elliptic function, then

$$\frac{f(z)}{\wp'(z)}$$

must be even and elliptic and so can be written as some rational function of $\wp$. So

$$f(z) = \wp' R(\wp).$$

c) We write

$$f(z) = \frac{1}{2}\Big(f(z) + f(-z)\Big) + \frac{1}{2}\Big(f(z) - f(-z)\Big).$$

The first summand on the right hand side is even and elliptic and so can be written as a rational function of $\wp$. The second summand is odd and so can be written as a rational function of $\wp, \wp'$. Therefore $f(z)$ can be written as a rational function of $\wp, \wp'$.

Problem 3.

By results from class it is enough to check that

1: $1 + z^{2^n}$ converges to 1 uniformly as $n \to \infty$

2: $\sum_{n=0}^{\infty} \log(1 + z^{2^n})$ is uniformly and absolutely convergent on compact subsets of $|z| < 1$.

For 1, we have that $|z^{2^n}| \to 0$ uniformly as $n \to \infty$ for $|z| < 1$. It remains to show 2 holds. From first year calculus we have that

$$|\log(1 + z^{2^n})| \leqslant |z|^{2^n}$$

for all z. We have that after finitely many n,

$$|z|^{2^n} < \frac{1}{n^2}.$$

Thus by the Weierstrass M test the series $\sum_{n=0}^{\infty} \log(1 + z^{2^n})$ will converge absolutely and uniformly, and thus so will $\prod_{n=0}^{\infty}(1 + z^{2^n})$.

Problem 4.

a) To show that

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

is an entire function we will show that

i) $\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \to 1$ as $n \to \infty$

ii) $\sum_{n=1}^{\infty} \log\left(\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right)$ converges uniformly and absolutely.

First i follows from using the taylor expansion of e, yielding that

$$\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} = 1 - \frac{z}{2n^2} + \frac{z^3}{3n^3} + \dots.$$

This clearly converges to 1 uniformly as $n \to \infty$. We now check ii. We have that for each z,

$$\left| \log\left(\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right) \right| \leqslant \frac{|z|^2}{n^2},$$

so by the Weierstrass M test ii converges uniformly and absolutely. Therefore $f(z)$ represents an entire function. The zeros of f occur exactly when $\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$ is zero, which happens only on the negative integers.

b) By part a),

$$\frac{1}{H(z)} = ze^z \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-\frac{z}{n}}$$

is holomorphic. By logarithmic differentiation, we compute that

$$-\frac{H'(z)}{H(z)} = \frac{d}{dz}\Big(\log\Big(\frac{1}{H(z)}\Big)\Big) = \frac{1}{z} + 1 + \sum_{n=1}^{\infty} \frac{-z}{(nz + n^2)}.$$

This function is holomorphic since it is the derivative of a holomorphic function. Taking the derivative once more we get that

$$\frac{d}{dz}\Big(-\frac{H'(z)}{H(z)}\Big) = \frac{d}{dz}\Big(\frac{1}{z} + 1 + \sum_{n=1}^{\infty} \frac{-z}{(nz + n^2)}\Big) = \sum_{n=0}^{\infty} \frac{-1}{(z + n)^2}.$$

As desired.

Problem 5.

(a) We define the following functions:

$$g(z) = z\prod_{i=1}^{\infty}\left(1-\frac{z}{n}\right)e^{-\frac{z}{n}}, \tilde{g}(z) = \prod_{i=1}^{\infty}\left(1-\frac{z+1}{n}\right)e^{-\frac{z+1}{n}}.$$

Both of these functions are holomorphic, so we can apply logarithmic differentiation to see that

$$\log(g(z))' = \frac{1}{z} + \sum \frac{1}{1-\frac{z}{n}} - \frac{1}{n} = \sum \frac{1}{1-\frac{z+1}{n}} - \frac{1}{n} = \log(\tilde{g}(z))'.$$

Therefore for some c we have that $g(z) = e^c\tilde{g}(z)$. Now define

$$f(z) = e^{cz}\prod_{i=1}^{\infty}\left(1-\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

It follows that

$$f(z+1) = e^c e^{cz}\prod_{i=1}^{n}\left(1-\frac{z+1}{n}\right)e^{-\frac{z+1}{n}} = e^c e^{cz}\tilde{g}(z) = e^{cz}g(z) = zf(z).$$

Our choice of f is holomorphic since constructed from holomorphic functions. It also satisfies the given properties.

(b) Let $p(z) = a_n z^n + \ldots a_1 z + a_0 = a_n(z-c_1)\ldots(z-c_n)$. We define

$$F(z) = a_n\prod_{i=1}^{n} f(z-c_i),$$

where f is defined as above. Then,

$$F(z+1) = a_n\prod_{i=1}^{n} f(z+1-c_i) = a_n\prod_{i=1}^{\infty} f((z-c_i)+1) = a_n\prod_{i=1}^{\infty}(z-c_i)f(z-c_i) = p(z)F(z)$$

Problem 6.

Let $n \in \mathbb{N}, 0 \neq f \in H(\mathbb{C})$ be given. We first suppose that there is some entire g so that $f = g^n$. If the zero set of f is empty the result is clear by A1Q4. Now suppose that $f(a) = 0$ with order of k. Then sufficiently close to a, we can write

$$f(z) = (z-a)^k \cdot \tilde{f}(z), g(z) = (z-a)^m \cdot \tilde{g}(z)$$

for nonzero $\tilde{f}, \tilde{g}$. By assumption we have

$$(z-a)^k \tilde{f}(z) = (z-a)^{nl} \tilde{g}^n(z).$$

Since $\tilde{f}, \tilde{g}^n$ nonzero, we have that $k | nl$ i.e. k is a multiple of n. Conversely suppose that every zero of f has order divisible by n. Let $\{a_i\}$ be the zero set with corresponding orders $\{nk_i\}$. We define

$$\tilde{g}(z) = z^{k_0} \prod_i^{\infty} \left[\left(1 - \frac{z}{a_i}\right) e^{p_i(z)} \right]$$

in such a way so that a_i is a root of $\tilde{g}(z)$ with order k_i for certain polynomials $p_i(z)$. It follows that the quotient $f/\tilde{g}^n$ is holomorphic and nonzero, so we can write

$$\frac{f(z)}{\tilde{g}^n(z)} = e^{h(z)}$$

for some entire $h(z)$. Thus we have that

$$f(z) = e^{h(z)} \tilde{g}^n(z) = \left(e^{\frac{h(z)}{n}} \tilde{g}(z)\right)^n = g^n(z).$$

Where we take $g(z) = e^{\frac{h(z)}{n}} \tilde{g}(z)$. This is exactly what we wanted to show.

Problem 7.

Let $\{a_i\}$ be the zero set of f_1. Let $\{b_i\}$ be the zero set of f_2. We can define a holomorphic function h so that $h(a_i) = 0, h(b_i) = 1$ and a_i is a root of $h(a_i) = 0$ with the same order as $f_1(a_i) = 0$, and b_i is a root of $h(b_i) - 1 = 0$ with the same order as $f_2(b_i) = 0$. Then we have that $\frac{h(z)-1}{f_2(z)} = g_2(z)$ is holomorphic. We also have that the quotient $\frac{h(z)}{f_1(z)}$ is holomorphic, since it has no poles. Letting $g_1(z) = \frac{h(z)}{f_1(z)}$ we see that

$$\frac{h(z)-1}{f_2(z)} = h_2(z) \implies f_1(z)g_1(z) = 1 - f_2(z)g_2(z) \implies f_1(z)g_1(z) + f_2(z)g_2(z) = 1.$$