Q2:
" $\implies$ " Suppose that $f$ is integrable, and let $P$ be a partition of $A$. Since $f$ integrable, by Spivak Theorem 3-8, it is continuous except on a set of measure 0. Let $E$ be such set. For any $S \in P$ we claim that $f$ is continous on it except perhaps on a set of measure 0. Clearly if $S$ is disjoint from $E$, then $f|_S$ will be continous on $S$ and hence integrable. If $S$ contains any points from $E$, then $S \cap E$ will be of measure 0 since any subset of a set of measure 0 is also measure 0. Hence $f|_S$ will be integrable on S. We now claim that $\int_A f = \sum_{S\in P} \int_S f|_S$. If we take $\chi_S$ to be the characteristic function of $S \in P$, we compute

$$\begin{aligned}
&\int_A f \\
&= \int_A \sum_{S\in P} \chi_S \cdot f && \text{rewriting f as sum of its restrictions} \\
&= \sum_{S\in P} \int_A \chi_S f && \text{by question 1} \\
&= \sum_{S\in P} \int_S f|_S
\end{aligned}$$

As desired.
" $\impliedby$ " Suppose that $P$ is a partition of $A$, and each $f|_S$ is integrable on $S$ for each $S \in P$. By spivak 3-8, we have that each $f|_S$ is continuous except on some measure 0 set $E_S$. Let $E = \bigcup_{S\in E} E_S$. By spivak theorem 3-4, $E$ will have measure 0. We can now express $f$ in terms of each $f|_S$. We define $\tilde{f}|_S : A \to \mathbb{R}$ as any function which is equal to $f|_S$ on $S$. We can write $f = \sum_{S\in P} \chi_S \tilde{f}|_S$, where $\chi_S$ is the characterstic function of $S$. Thus $f$ will be continous except on $E$, a set of measure 0 and perhaps along the finite union of the boundaries of each $S$. From discussion in class, we know that the boundary of a rectangle is of measure 0 so the union over all the boundaries of subrectangles of $P$ will be measure 0. Hence $f$ will be integrable. Using the linearity of the integral shown in question 1, we compute $\int_A f$ as

$$\int_A f = \int_A \sum_{S\in P} \chi_S \tilde{f}|_S = \sum_{S\in P} \int_A \chi_S \tilde{f}|_S = \sum_{S\in P} \int_S f|_S$$