Problem 1. *Griffiths 3.9*

Since the sphere has potential 0, we only need to add one point charge at the center that will make the sphere have a potential of V_0. A point charge placed at the center of the sphere will create a potential of $V(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r}$, so for our sphere to have a potential of V_0 we can take $q'' = 4\pi\varepsilon_0 V_0 R$. In a neutral sphere, where $q' + q'' = 0$, so the force acting on q will be

$$
\begin{aligned}
F(a) &= \frac{1}{4\pi\varepsilon_0} q \left[\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right] \\
&= \frac{qq'}{4\pi\varepsilon_0} \left[-\frac{1}{a^2} + \frac{1}{(a-b)^2}\right] \\
&= \frac{qq'}{4\pi\varepsilon_0} \left[\frac{b(b-2a)}{a^2(a-b)^2}\right] \\
&= \frac{q^2}{4\pi\varepsilon_0} \left(-\frac{R}{a}\right)^3 \left[\frac{R^2 - 2a^2}{(a^2 - R^2)^2}\right]
\end{aligned}
$$

Problem 2. *Griffiths 3.12*

Recall that the potential function of two wires with charge per length $\lambda, -\lambda$ at positions $a, -a$ is:

$$V_{total}(x, y, z) = \frac{\lambda}{4\pi\varepsilon_0} \log\left(\frac{z^2 + (y+a)^2}{z^2 + (y-a)^2}\right).$$

We wish to solve for the parameter a so that there will be an equipotential of V_0 and $-V_0$ at $(0, R+d, 0)$ and $(0, R-d, 0)$. We see that

$$e^{\frac{V_0 4\pi\varepsilon_0}{\lambda}} = \frac{(R+d+a)^2}{(R+d-a)^2}, e^{\frac{-V_0 4\pi\varepsilon_0}{\lambda}} = \frac{(R-d+a)^2}{(R-d-a)^2} \implies \frac{(R+d+a)^2}{(R+d-a)^2} = \frac{(R-d+a)^2}{(R-d-a)^2} \implies a = \sqrt{d^2 - R^2}.$$

We must therefore place the wires at $a = \pm\sqrt{d^2 - R^2}$. We now solve for λ. We have that

$$\lambda = 4V_0\pi\varepsilon_0 \log\left(\frac{(R+d+\sqrt{d^2-R^2})^2}{(R-d-\sqrt{d^2-R^2})^2}\right)$$

Thus we are done by uniqueness.

Problem 3. *Griffiths 3.13*

We wish to solve for the potential with the given boundary conditions:

$$\begin{cases} 1) V = 0 & y = 0 \\ 2) V = 0 & y = a \\ 3) V = V_0 & x = 0, y \in [0, \frac{a}{2}] \\ 4) V = -V_0 & x = 0, y \in (\frac{a}{2}, a] \\ 5) V \to 0 & x \to \infty \end{cases}$$

We write the potential $V(x, y, z) = X(x)Y(y)$, since this function must be independant of z. By a similar reasoning as in Griffiths, we must have that

$$X(x) = Ae^{kx} + Be^{-kx}, Y(y) = C\sin(kx) + D\cos(ky).$$

Condition 5 implies that $A = 0$, and condition 1 implies that $D = 0$. Thus we can write

$$V(x, y, z) = e^{-kx} C\sin(ky) = e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right).$$

Writing this as an infinite sum, we have

$$V(x, y, z) = \sum_{n=0}^{\infty} A_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right).$$

We now compute the coefficients on the parts with potential $V_0, -V_0$ respectively.

$$\begin{aligned} C_n &= \frac{2}{a}\int_0^{\frac{a}{2}} V_0 \sin\left(\frac{n\pi y}{a}\right) dy \\ &= -\frac{2}{a} \cdot \frac{aV_0}{n\pi} \left[\cos\left(\frac{n\pi y}{a}\right)\right]\Bigg|_0^{\frac{a}{2}} \\ &= \begin{cases} 0 & n \equiv 0 \mod (4) \\ \frac{2V_0}{n\pi} & n \equiv 1, 3 \mod (4) \\ \frac{4V_0}{n\pi} & n \equiv 2 \mod (4) \end{cases} \end{aligned}$$

Similarly,

$$\begin{aligned} C'_n &= \frac{2}{a}\int_{\frac{a}{2}}^{a} -V_0 \sin\left(\frac{n\pi y}{a}\right) dy \\ &= \begin{cases} 0 & n \equiv 0 \mod (4) \\ -\frac{2V_0}{n\pi} & n \equiv 1, 3 \mod (4) \\ \frac{4V_0}{n\pi} & n \equiv 2 \mod (4) \end{cases} \end{aligned}$$

Summing these together, we get that

$$A_n = \begin{cases} 0 & n \equiv 0, 1, 3 \mod (4) \\ \frac{8V_0}{\pi n} & n \equiv 2 \mod (4) \end{cases}.$$

Taking these coefficients gives us the desired potential function.

Problem 4. *Griffiths 3.20*

First note that on the boundary, the potential function must satisfy

$$V_0(R,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = \sum_{l}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta).$$

We know from Griffiths, that

$$A_l = \frac{2l+1}{2}\int_0^{\pi} V_0(\theta)\sin\theta P_l(\cos\theta)\,d\theta.$$

We now compute the electric field on the inside and outside, at $r = R$ and use the fact that $E_{out} - E_{in} = \frac{1}{\varepsilon_0}\sigma$. We see that

$$E_{out}(R) = -\nabla V|_{r=R} = \sum_{l=0}^{\infty} \frac{B_l(l+1)}{R^{l+2}} P_l(\cos\theta),$$

and

$$E_{in}(R) = -\sum_{l=0}^{\infty} l r^{l-1} A_l P_l(\cos\theta).$$

Computing their difference yields:

$$\begin{aligned}
\frac{1}{\varepsilon_0}\sigma(\theta) &= E_{out} - E_{in} \\
&= \sum_{l=0}^{\infty}\left(B_l \cdot \frac{l+1}{R^{l+2}} + lR^{l-1}A_l\right)P_l(\cos\theta) \\
&= \sum_{l=0}^{\infty} \frac{A_l}{R}(2l+1)P_l(\cos\theta) \\
&= \frac{1}{2R}\sum_{l=0}^{\infty}(2l+1)^2\left[\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta\right]P_l(\cos\theta).
\end{aligned}$$

Problem 5. *Griffiths 3.25*

The solution to the Laplace equation on cylidrical coordinates is

$$V(s, \phi) = A_0 + B_0 \log(s) + \sum_{k=1}^{\infty} \left(A_k s^k + B_k s^{-k}\right) \left(C_k \cos k\phi + D_k \sin k\phi\right),$$

with the conditions $V(R, \phi) = 0, V(s, \phi) \to -E_0 s \cos\phi$ as $s \to \infty$. We must have that $A_k = B_k = 0$ for all $k \neq 1$, since the potential must converge to the electric field. So we can write

$$V(s, \phi) = a_1 s \cos\phi + \frac{a_2}{s} \cos\phi.$$

The condition $V(R, \phi) = 0$ implies that

$$0 = E_0 a_1 R + \frac{a_2}{R} = 0 \implies a_2 = -a_1 R^2$$

Using the limit at infinity condition, we compute that

$$\lim_{s\to\infty} \frac{\partial V}{\partial s}(s, 0) = a_1 + \frac{a_1}{s^2} = -E_0 \implies a_1 = -E_0.$$

Therefore the potential function is $V(s, \phi) = -E_0 \left(s - \frac{R^2}{s}\right) \cos\phi$. We now compute the induced surface charge. We have that the potenial must vanish on the inside, since the boundary has 0 potential, and there is no enclosed charge. Therefore using the formula for surface charge, we compute that

$$\frac{1}{\varepsilon_0} \sigma(\theta) = \left(\frac{\partial V_{out}}{\partial s}\right)\Big|_{s=R} = -E_0 - E_0 \frac{R^2}{R^2} \cos\phi = -2E_0 \cos\phi.$$

Problem 6. *Griffiths 3.41*

Using the result of Griffiths 3.9, we can write the force as

$$F(a) = \frac{q}{4\pi\varepsilon_0}\left[\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right].$$

We impose the condition that $q'' + q' = q$, since we want the charge of the sphere to be q. Using this we can compute that

$$\begin{aligned} F(a) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{q-q'}{a^2} + \frac{q'}{(a-b^2}\right] \\ &= \frac{q}{4\pi\varepsilon_0}\left[\frac{q}{a^2} - \frac{q'}{a^2} + \frac{q'}{(a-b)^2}\right] \\ &= \frac{q^2}{4\pi\varepsilon_0}\cdot\frac{1}{a^2} + \frac{qq'}{4\pi\varepsilon_0}\cdot\frac{1}{a^2}\left[\frac{2a-b}{(a-b)^2}\right] \\ &= \frac{q^2}{4\pi\varepsilon_0 a^3}\left[a - \frac{R^3(2a^2-R^2)}{(a^2-R^2)^2}\right] && \text{(using } q' = q\text{)} \end{aligned}$$

The potential will be 0 exactly when $a - \left[\frac{R^3(2a^2-R^2)}{(a^2-R^2)^2}\right] = 0$, or when $a(a^2-R^2)^2 = R^3(2a^2-R^2)$. Using wolfram alpha, this has a solution exactly when $a = \varphi R$, where φ is the golden ratio. Approximately, we have that $a \approx 5.66$Å. The work to take a particle from ∞ to r is computed as the following integral:

$$W = \frac{q^2}{4\pi\varepsilon_0}\int_\infty^r \frac{1}{a^3}\left[a - \frac{R^3(2a^2-R^2)}{(a^2-R^2)^2}\right] = \frac{q^2}{8\pi\varepsilon_0 R}da,$$

According to wolfram alpha. We can explicitly compute the work as

$$W = \frac{q^2}{8\pi\varepsilon_0 R} = \frac{(1.6012\times10^{-19})^2}{8\pi 3.5\text{Å} * 8.85\times10^{-12}} = 2.03\times10^{-10}\text{J}$$