Q1a: Suppose that a solution to the ODE escapes to infinity in finite time, then we know from calculus that there must be a vertical asymptote of x. The slope of x will become arbitrarily large as you approach the asymptote hence $f(x)$ is unbounded, contradicting our assumption.

Q1b: Suppose that a solution to the ODE escapes to infinity in finite time. We have that

$$|x'(t)| = |x(0) + \int_0^t f(x(s))ds| \leq |x(0)| + |\int_0^t Cx(s)ds| + |Kt|$$

If x escapes to infinty in finite time then there must be a vertical asymptote at some point t_0. We can now apply Gronwall's inequality to $x(t)$, we see that on some neighborhood (a, b) sufficiently close to t_0, we have that $|x(t)| \leq \alpha e^{Ct_0}$, but since $x(t)$ is unbounded this can not happen.

Q1c: Simply by applying the norm to f, we have reduced it to the case on the real line, hence we know it is true by part a and b.

Q1d: We know by uniform continuity that there exists a $\delta > 0$ such that for all x, y where $|x - y| < \delta$, we have that $|f(x) - f(y)| < 1$. Let $z > 0$ be such that $0 < z < \delta$. It is true that for any $x \in \mathbb{R}^m$, there is a $C_x \in \mathbb{N}$ such that

$$\frac{|x|}{z} < C_x < \frac{|x|}{z} + 1$$

And so we have that $\frac{|x|}{C_x} < z < \delta$ Now by the reverse triangle inequality,

$$||f(x)| - |f(0)|| \leq |f(x) - f(0)| \leq \sum_{i=1}^{C_x} |f(\frac{ix}{C_x}) - f(\frac{(i-1)x}{C_x})| < C_x < \frac{|x|}{z} + 1$$

Hence we have that $|f(x)| \leq \frac{|x|}{z} + 1 + |f(0)|$ as desired. hence we can reason by 1c that the solution to this ODE will not escape to infinity.