Q3: We first show the forward implication. Suppose that $a_n x^n + \dots a_1 x + a_0$ is unital. Then for some polynomial $b_m x^m + \dots b_1 x + b_0$ we have that

$$1 = (a_n x^n + \dots a_1 x + a_0)(b_m x^m + \dots b_1 x + b_0) = (a_n b_m x^{m+n} + \dots a_0 b_0),$$

which implies that $a_0 b_0 = 1$ i.e. a_0 is unital. We now claim that $a_n x^n + \dots a_1 x$ is nilpotent. Observe that

$$1 + \mathcal{N}(\mathcal{R}) = (a_0 + P) + \mathcal{N}(\mathcal{R}) \cdot (b_0 + Q) + \mathcal{N}(\mathcal{R}) = a_0 b_0 + \mathcal{N}(\mathcal{R}) + a_0 Q \mathcal{N}(\mathcal{R}) + b_0 P + \mathcal{N}(\mathcal{R}) + PQ + \mathcal{N}(\mathcal{R}).$$

This implies that $P, Q \in \mathcal{N}(\mathcal{R})$ as desired. We now prove the converse direction. Suppose that $a_0 + P = a_0 + a_1 x + \cdots + a_n x^n$ with a_0 unital and P nilpotent. Then we have that for any nilpotent polynomial Q for some m,

$$1 = 1 - Q^m = (1 - Q)(1 + Q + \dots Q^{m-1}).$$

Setting $Q = -a_0^{-1} P$ we get that

$$1 = (1 + a_0^{-1} P)(1 + \dots (-a_0)^{m-1} P^{m-1}) \implies a_0 = (a_0 + P)(1 + \dots (-a_0)^{m-1} P^{m-1}),$$

which implies that

$$1 = (a_0 + P)[a_0(1 + \dots (-a_0)^{m-1} P^{m-1})].$$

Hence $a_0 + P$ has an inverse.