Problem 1.

(a) This is linear and homogenous. Consider the operator $\mathcal{L} = \partial_t + t^2\partial_x$. This is a linear operator, and we have that $\mathcal{L}u = 0$ is our desired PDE.

(b) This is a nonlinear PDE, since we have a term with uu_x. It is not fully nonlinear since it is linear in u_{xxx}.

(c) This is inhomogenous linear PDE, since it is of the form $Lu = g$, where $L = \partial_t^2 + \partial_x, g = t^2$.

(d) This is a totally nonlinear PDE. It is nonlinear in each of the partial derivatives.

Problem 2.

(a) Integrating u_x by x, we get that

$$\int u_x dx = \sin(xy) + \frac{A}{2}x^2y + F(y) \quad (1),$$

for some arbitrary function $F(y)$. Similarly, for u_y we compute that

$$\int u_y dy = sin(xy) + 3x^2y + arctan(y) + G(x). \quad (2)$$

Thus we must have that $A = 6$ for this PDE to have a solution.

(b) Given that $u(0,0) = B$, using our expressions from a) we get that

$$B = u(0,0) = F(0) = G(0).$$

By differentiating expression 2 with respect to x, we get that $G_x = 0$. So $G = B$. Therefore we have that $u(x,y) = sin(xy) + 3x^2y + \arctan(y) + B$ is the solution to the PDE.

(c) We claim that a necessary condition on f is $f_{1y} = f_{2x}$. Suppose that u solves $\nabla u = f$. Then on any closed curve γ with interiour D, we have that

$$0 = \int_\gamma \nabla u = \int_\gamma f = \int_D (f_{1y} - f_{2x}) dy dx$$

Since this holds for all γ, D, we have that $f_{1y} = f_{2x}$.

Problem 3.

Maxwells Equations tells us that $\frac{1}{c}E_t = \nabla \times B$. Applying ∂_t we get that

$$\begin{aligned}
\frac{1}{c}E_{tt} &= \partial_t(\nabla \times B) \\
&= \nabla \times B_t && \text{(since derivatives commute)} \\
&= -\frac{1}{c}\nabla \times (\nabla \times E) && \text{(Maxwells Equations)} \\
&= \frac{1}{c}(E_{xx}, E_{yy}, E_{zz}). && \text{(Vector Calc identity for } \nabla \cdot E = 0\text{)}
\end{aligned}$$

Similarly for B we compute that:

$$\begin{aligned}
\frac{1}{c}B_{tt} &= -\partial_t(\nabla \times E) \\
&= -\nabla \times E_t && \text{(since deriavtives commute)} \\
&= -\frac{1}{c}\nabla \times (\nabla \times B) && \text{(Maxwells Equations)} \\
&= \frac{1}{c}(B_{xx}, B_{yy}, B_{zz}) && \text{(Vector calc identity for } \nabla \cdot B = 0\text{)}
\end{aligned}$$

Therefore every component of E, B satisfy the wave equation.

Problem 4.

We wish to show that $\int_{\mathbb{R}^n} \nabla \cdot F(x) dx = 0$. Let $B(r)$ be the ball of radius r. We have by the divergence theorem, that

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n} \nabla \cdot F(x) dx \right| &= \lim_{r \to \infty} \left| \int_{B(r)} (F(x) \cdot n(x))\, dx \right| && \text{(Divergence theorem on } \mathbb{R}^n\text{)} \\
&\leqslant \lim_{r \to \infty} \int_{B(r)} |F(x)| \cdot |n(x)| dx && \text{(Integral ineqality + Cauchy-Schwartz)} \\
&\leqslant \lim_{r \to \infty} \int_{B(r)} |F(x)| dx && \text{(since } |n(x)| = 1\text{)} \\
&\leqslant \lim_{r \to \infty} \int_{B(r)} C|x|^{-n} dx && \text{(upper bound on } |F(x)|\text{)} \\
&\leqslant \lim_{r \to \infty} C|r|^{-n} \cdot \sigma(S^{n-1})|r|^{n-1} \\
& && \text{(since } \textstyle\int_{B(r)} 1 dx = \sigma(S^{n-1}) r^{n-1}, \text{ where } \sigma(S^{n-1}) = \int_{S^{n-1}} 1 dx\text{)} \\
&= \lim_{r \to \infty} \frac{C\sigma(S^{n-1})}{r} \\
&= 0.
\end{aligned}
$$

We conclude that $\int_{\mathbb{R}^n} \nabla \cdot F(x) dx = 0$.

Problem 5.

(a) We claim that $\int_D f = \int_{\partial D} g$ is necessary. Observe that if u is a solution to this PDE, then by the divergence theorem we have that

$$\int_D f = \int_D \Delta u = \int_D \nabla \cdot \nabla u = \int_{\partial D} \nabla u \cdot n(x) = \int_{\partial D} g.$$

Thus $\int_D f = \int_{\partial D} g$ is a necessary condition.

(b) Suppose that u solves this PDE. Then for any constant c we also have that

$$\Delta(u + c) = \Delta u = f, \frac{\partial(u + c)}{\partial n} = \frac{\partial u}{\partial n} + \frac{\partial c}{\partial n} = g$$