Problem 1.

We claim the existence of a vector field V satisfying $\langle V(0), \gamma'(0)\rangle = 0$ and that the if we parallel transport $V(0)$ along γ, then $P_{\gamma,t,1}(V(0)) = V(0)$ i.e. it gets parallel transported to itself along γ. Let W be the subspace orthogonal to $\gamma'(0)$. Let $\tilde{A}$ be the linear map given by parallel transporting $T_{\gamma(0)}M$ along γ to $\gamma(0)$ along γ. $\tilde{A}$ is an orientation preserving isometry, so it must have determinant of 1. Let A be the restriction of $\tilde{A}$ on W, it must also be an isometry and has determinant of 1, since $\tilde{A}\gamma'(0) = \gamma'(0)$. By Do Carmo Lemma 3.8 pg 203, we have that A must fix a subspace of W. Take a vector $v \in W$ and choose V so that $V(0) = v$. We compute the index as:

$$I(V, V) = \int \langle V', V'\rangle - \langle R(\gamma', V)\gamma', V\rangle dt = \int -\langle R(\gamma', V)\gamma', V\rangle dt < 0$$

Since curvature is positive. There is a subspace of positive dimension where I is negative definite so we conclude that the index of γ is at least 1.

Problem 2.

We first create a triangle, by letting γ_3 be the minimal geodesic from q_2 to q_1. Using criticality, we can find a minimal geodesic γ_4 so that the angle ϕ between γ_3, γ_4 is at most $\frac{\pi}{2}$. Since $\mathbb{H}^n$ is simply connected, complete and has curvature of $K = -1$, we can apply the Toponogov Comparison theorem. Consider the triangle with sides (a, b, c) where $|a| = |\gamma_4|$ and $|b| = |\gamma_3|$ and the angle between a and b is ϕ. Using hyperbolic trig. identities, we have that

$$\cosh|c| \leqslant \cosh|a|\cosh|b|.$$

Furthermore, the toponogov comparison theorem implies that $|\gamma_2| \leqslant |c|$. Since cosh is increasing on the positive reals we have

$$\cosh|\gamma_2| \leqslant \cosh|c| \leqslant \cosh|\gamma_3|\cosh|\gamma_4|.$$

Furthermore $|\gamma_4| = |\gamma_1|$ by minimality, so we get that $\cosh|\gamma_2| = \cosh|\gamma_3|\cosh|\gamma_1|$. We now consider the triangle in $\mathbb{H}^n$ formed by (d, c, b) where $|d| = |\gamma_1|$, $|b| = |\gamma_2|$ and the angle between d and b is θ. The hyperbolic law of cosines tells us that

$$\cosh|b| = \cosh|d|\cosh|c| - \sinh|d|\sinh|c|\cos\theta.$$

Similarly as before we apply toponogov's theorem to get that $|\gamma_3| \leqslant |b|$ and get that

$$\cosh|\gamma_3| \leqslant \cosh|\gamma_1|\cosh|\gamma_2| - \sinh|\gamma_1|\sinh|\gamma_2|\cos\theta.$$

Combining this with the inequality for the other triangle, we get:

$$\cosh|\gamma_2| \leqslant \cosh|\gamma_1|(\cosh|\gamma_1|\cosh|\gamma_2| - \sinh|\gamma_1|\sinh|\gamma_2|\cos\theta).$$

Dividing by $\cosh|\gamma_2|$ we get

$$1 \leqslant \cosh^2|\gamma_1| - \cosh|\gamma_1|\sinh|\gamma_1|\tanh|\gamma_2|\cos\theta.$$

Rearranging, we get

$$\cos\theta \leqslant \frac{\tanh|\gamma_1|}{\tanh|\gamma_2|}.$$

By assumption we had that $\alpha|\gamma_1| \leqslant |\gamma_2| \leqslant d$, so $\cos\theta \leqslant \frac{\tanh|\gamma_2|/\alpha}{\tanh|\gamma_2|}$ since tanh is increasing. Finally we get that $\cos\theta \leqslant \frac{\tanh d/\alpha}{\tanh d}$ since $\frac{\tanh x/\alpha}{\tanh x}$ is increasing.

Problem 3.

(a) It is not possible to provide $\mathbb{T}^n$ with a metric of negative curvature. The fundamental group is $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$ which admits a non cyclic subgroup. By Preissman's theorem (Do Carmo Thm 3.2) it can not have negative curvature. $\mathbb{T}^n$ cannot be given a constant positive curvature since if it could be, the universal covering space would be S^n, by Do Carmo thm 4.1.

(b) S^n is compact, simply connected for $n > 1$. If S^n admits non-positive sectional curvature it must be diffeomorphic to $\mathbb{R}^n$ by hadamards theorem. However this is absurd since S^n is compact and $\mathbb{R}^n$ is not.

(c) $S^1 \times \mathbb{R}P^2$ cannot have negative curvature since it is compact, and any compact manifold with negative curvature has non abelian fundamental group by Do Carmo thm 3.8. The fundamental group is $\mathbb{Z} \times \mathbb{Z}_2$ which is abelian. If $S^1 \times \mathbb{R}P^2$ has positive sectional curvature, it must be orientable by Synge. But this is clearly untrue since $\mathbb{R}P^2$ is not orientable.

(d) $S^2 \times S^2$ can not have non positive sectional curvature, since if it did it would be diffeomorphic to $\mathbb{R}^4$ by Hadamards theorem. We can endow $S^2 \times S^2$ with non negative curvature in the following way. Consider the product metric, where each S^2 is given the standard metric. At any point $(a, b) \in S^2 \times S^2$, we compute the scalar curvature as:

$$K(a, b) = \frac{1}{12} \sum_{i,j} \langle R(z_i, z_j) z_i, z_j \rangle.$$

Since we choose that the $\{z_i\}$ orthonormally, the sum above will be positive except for when $a = b$, then it will vanish.

Problem 4.

(a) First take a sequence $\{q_n\} \subset N$ such that $q_n \to q$ and $d(p_0, q_n) \to d(p_0, N)$. Since N is closed we must have that $q \in N$, and $d(p_0, q) = d(p_0, N)$. Choose geodesic $\gamma : [0, a] \to M$ so that $\gamma(0) = p_0, \gamma(a) = q_0$ by completeness of M. By the formula for the first variation of energy, given a variation f of γ, we have

$$\frac{1}{2}E'(0) = -\int_0^a \langle V(t), \frac{D}{dt}\gamma' \rangle dt - \sum_{i=1}^k \langle V(t_i), \gamma'(t_i^+) - \gamma'(t_i^-) \rangle - \langle V(0), \gamma'(0) \rangle + \langle V(a), \gamma'(a) \rangle.$$

Note however that since γ is C^1 the left and right derivatives are equal. Choose a vector field V so that $V(0) = 0$, and $V(a) \in T_{q_0}N$. Then by prop. 2.2, there exists a variation f of γ so that V is the variational field of f. First we must have that $E'(0) = 0$ since γ is a geodesic, and $\frac{D}{dt}\gamma' = 0$. We get that

$$0 = -\langle V(0), \gamma'(0) \rangle + \langle V(a), \gamma'(a) \rangle \implies \langle V(a), \gamma'(a) \rangle = 0.$$

Therefore γ is orthogonal to $T_{q_0}N$.

(b) We compute $\frac{1}{2}\Delta|\nabla u|^2$. Take a moving orthonormal frame $\{e_i\}$, so that $\nabla_{e_i} e_j(x) = 0$ for all x:

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla u|^2 &= \frac{1}{2}\sum_{i=1}^n e_i(e_i(\langle \nabla u, \nabla u \rangle)) && \text{(using the definitions of the operators)} \\
&= \sum_{i=1}^n e_i(\langle \nabla_{e_i}\nabla u, \nabla u \rangle) && \text{(applying } e_i \text{ to the inner product)} \\
&= \sum_{i=1}^n e_i H(u)(e_i, \nabla u) && \text{(by definition of hessian)} \\
&= \sum_{i=1}^n e_i H(u)(\nabla u, e_i) && \text{(by symmetry)} \\
&= \sum_{i=1}^n e_i \langle \nabla_{\nabla u}\nabla u, e_i \rangle \\
&= \sum_{i=1}^n \langle \nabla_{e_i}\nabla_{\nabla u}\nabla u, e_i \rangle && \text{(applying } \nabla_{e_i}) \\
&= \sum_{i=1}^n \langle \nabla_{\nabla u}\nabla_{e_i}\nabla u, e_i \rangle + \langle \nabla_{[e_i, \nabla u]}\nabla u, e_i \rangle + \langle R(e_i, \nabla u)\nabla u, e_i \rangle && \text{(by definition of curvature)} \\
&= \sum_{i=1}^n [\nabla u \langle \nabla_{e_i}\nabla u, e_i \rangle - \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u} e_i \rangle] + \sum_{i=1}^n H(u)([e_i, \nabla u], e_i) + \sum_{i=1}^n \langle R(e_i, \nabla u)\nabla u, e_i \rangle \\
& && \text{(expanding out first and second term in sum,)} \\
&= \nabla u(\Delta u) + \sum_{i=1}^n H(u)(e_i, \nabla_{e_i}\nabla u) + Ric(\nabla u, \nabla u) && \text{(simplyifying)} \\
&= \langle \nabla u, \nabla(\Delta u) \rangle + |H(u)|^2 + Ric(\nabla u, \nabla u).
\end{aligned}$$

Problem 5.

First take a set of points $\{p_i\} \subset B_{r-\varepsilon/2}(p)$ so that $d(p_i, p_j) \geqslant \frac{\varepsilon}{2}$ when $i \neq j$. We must then have that the $\frac{\varepsilon}{4}$ balls at each p_i are disjoint from one another. We have that

$$\begin{aligned}
N &\leqslant \frac{\mathrm{Vol}(B_r(p))}{\min_i \mathrm{Vol}(B_{\frac{\varepsilon}{4}}(p_i))} \\
&= \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_{\frac{\varepsilon}{4}}(p'))} && \text{(since we have a finite set of points, min attained at some } p'\text{)} \\
&\leqslant \frac{\mathrm{Vol}(B_{2r}(p'))}{\mathrm{Vol}(B_{\frac{\varepsilon}{4}}(p'))} && \text{(since } B_{2r}(p') \supset B_r(p)\text{)} \\
&\leqslant \frac{\mathrm{Vol}(B_{2r}(H))}{\mathrm{Vol}(B_{\frac{\varepsilon}{4}}(H))} && \text{(Do Carmo Rmk 2.7 pg 220)} \\
&= C_1(n, Hr^2, \frac{r}{\varepsilon})
\end{aligned}$$

We have that $d(p_i, p') \leqslant 2\varepsilon$, so by disjointness we have that the multiplicity satisfies:

$$\begin{aligned}
\text{mult.} &\leqslant \frac{\mathrm{Vol}(B_{3\varepsilon}(p'))}{\min_i \mathrm{Vol}(B_{\frac{\varepsilon}{4}}(p_i))} \\
&\leqslant \frac{\mathrm{Vol}(B_{3\varepsilon}(p'))}{\mathrm{Vol}(B_{\frac{\varepsilon}{4}}(p''))} && \text{(min attained at some } p''\text{)} \\
&\leqslant \frac{\mathrm{Vol}(B_{5\varepsilon}(p''))}{\mathrm{Vol}(B_{\frac{\varepsilon}{4}}(p''))} && \text{(since } B_{5\varepsilon}(p'') \supset B_{3\varepsilon}(p')\text{)} \\
&\leqslant \frac{\mathrm{Vol}(B_{5\varepsilon}(H))}{\mathrm{Vol}(B_{\frac{\varepsilon}{4}}(H))} && \text{(Do Carmo rmk 2.7 pg 220)} \\
&= C_2(n, H\varepsilon^2)
\end{aligned}$$