Q5:

We proceed by showing the negation of the definition of continuity is true. Let $\delta > 0$. It suffices to find some $x \in A$ such that $x \in B_\delta(0,0)$ but with $x \notin B_\epsilon(f(0,0))$ for some $\epsilon > 0$. Let $x = (x_1, x_2)$. Choose $x_1 = \frac{\delta}{2}$. For x_2, it must satisfy both $\frac{\delta}{4}^2 + x_2^2 < \delta^2$ and $0 < x_2 < \frac{\delta}{4}^2$. It is sufficent to choose $x_2 = \frac{1}{2}\min\{\frac{\delta^2}{4}, \frac{\sqrt{3}\delta}{2}\}$. This choice of x_2 will satisfy both inequalities, and so $x \in A$ and $x \in B_\delta(0,0)$. Taking $\epsilon = 1/2$, we notice that $f(0,0) = 0 \in B_\epsilon(f(0,0))$ but from our choice of x, $f(x) = 1 \notin B_\epsilon(0)$. The result follows $\blacksquare$.

To show that f is continuous on every straight line containing 0, we will consider 2 seperate cases. A straight line through 0 can either take the form of $y = \alpha x$ for some $\alpha \in \mathbb{R}$,which is the usual way to represent a line, or $x = 0$, the vertical line. We first consider the case when $x = 0$. Note that for every point of the form $(0, y)$ we have that $f(0, y) = 0$. Letting $\epsilon > 0$ and $C = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ consider the open ball $U = (-\epsilon, \epsilon)$. We claim its preimage under f can be written as follows. $f^{-1}|_C(U) = C \cap \mathbb{R}^2$.

pf:

Suppose that $x \in C \cap \mathbb{R}^2$. Then $x = (0, y)$ for some $y \in \mathbb{R}$. This point will not be in the support of f and so $f(x) \in U$. Now suppose that $x \in f^{-1}|_C(U)$. Since this is true for all choices of ϵ, it must be that $f(x) = 0$. Therefore $x \in C \subset C \cap \mathbb{R}^2$. Thus we have that for U open, $f^{-1}|_C(U)$ is open in C and so $f|_C$ is continuous.

Now we turn our attention to the set of all lines with slope, $B = \{(x, y) : y = \alpha x \text{ for some } \alpha \in \mathbb{R}\}$. Let $\epsilon > 0$ and let $U = (-\epsilon, \epsilon)$. Notice that this open ball is centered about $f(0,0)$. We will check continuity at $(0,0)$ for 2 possible cases, $\alpha \leq 0$ and $\alpha > 0$. We proceed first with the simpler case being when $\alpha \leq 0$. Consider $f^{-1}|_{B,\alpha\leq 0}(U)$. We claim that $f^{-1}|_{B,\alpha\leq 0}(U) = B_{\alpha\leq 0} \cap \mathbb{R}^2$. First suppose that some point $y \in B_{\alpha\leq 0} \cap \mathbb{R}^2$. It must be that y takes the form of $(x, \alpha x)$. From our choice of α, we will have that $\alpha x \leq 0 \leq x^2$ for $x \geq 0$ or some other inequality for $x < 0$. In either case, the point is not in A and so $f(y) \in U$ for any choice of ϵ. Now suppose that $y \in f^{-1}|_{B,\alpha\leq 0}(U)$. Since our choice of ϵ is arbitrary, it must be the case that $f(y) = 0$. Thus from the definition of the pre-image, $y \in B_{\alpha\leq 0} \subset B_{\alpha\leq 0} \cap \mathbb{R}^2$. Thus U is open in the set of all lines that have a slope which is not positive.

Finally we show that $f|_{B,\alpha>0}$ is continuous at $(0,0)$. Take the same set U as chosen above. We claim that for each $\alpha > 0$, $f^{-1}|_{B,\alpha>0}(U) = B_{\alpha>0} \cap (-\infty, \alpha) \times (-\infty, \alpha^2)$ We first suppose that $y \in f^{-1}|_{B,\alpha>0}(U)$. This means that $f(y) = 0$ from our arbitrary choice of ϵ. Equivalently, one of the following must hold. If $y = (x, \alpha x)$, then it must be that either $x < 0$ and $\alpha x < 0$ or $x > 0$ and $x^2 < \alpha x$. The second condition for inequality breaks when $x = \alpha$, hence we choose our open set as such. $R = (-\infty, \alpha) \times (-\infty, \alpha^2)$. Since we are choosing to restrict the domain of f to whenever $\alpha > 0$, $y \in R \cap B_{\alpha>0}$. Now consider when $y \in R \cap B_{\alpha>0}$. We know that if $y = (x, \alpha x)$, $x \in (-\infty, \alpha)$ and $\alpha x \in (-\infty, \alpha^2)$ It suffices to show that such a y will not be in A and hence be in the preimage of U for any ϵ. For when $x < 0$, the point y will not be in A. Now if $x \in (0, \alpha)$, and so $\alpha x \in (0, \alpha^2)$. $\alpha x < x^2 \iff (\alpha < x)$. Since we are taking $0 < x < \alpha$, point y will never fall into A and hence $f(y) = 0$. Thus $f^{-1}|_{B,\alpha>0}(U)$ is open in B_α. We have shown that f restricted through any straight line containing 0 is continuous as desired. $\blacksquare$