

5.5.65: Suppose $l^2(A)$ is unitarily isomorphic to $l^2(B)$. We have that $\{\phi_\alpha\}_{\alpha \in A}$ where $\phi_\alpha(x) = \delta_{x\alpha}$ forms an orthonormal basis for $l^2(A)$. If U is our unitary isomorphism, $\{U(\phi_\alpha)\}$ forms an orthonormal basis for $l^2(B)$ since U preserves the inner product, and is a bijection. There is a natural bijection between $\{\phi_\alpha\}$ and A by $\phi_\alpha \leftrightarrow \alpha$. The same reasoning with $\{\psi_\beta\}_{\beta \in B}$ shows that $\text{card}(A) = \text{card}(B)$. Now suppose that $\text{card}(A) = \text{card}(B)$. Construct $\{\phi_\alpha\}_{\alpha \in A}$ and $\{\psi_\beta\}_{\beta \in B}$ as above. If h is a bijection between A and B , define $U(\phi_\alpha) = \psi_{h(\alpha)}$ and extend linearly. We have that U sends an orthonormal basis to another orthonormal basis and is a bijection. Thus we are done.