**Problem 1.** *Q1 page 77 Do Carmo*

**(a)** We first show that $\varphi$ is an immersion. We compute

$$d\varphi = \begin{bmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{bmatrix}.$$

We compute the cross product of $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ as

$$\frac{\partial}{\partial u} \times \frac{\partial}{\partial v} = \left( f(v) \cdot \cos u \cdot f'(v), f(v) \cdot \sin u \cdot g'(v), -f(v) \cdot f'(v) \sin^2 u - \cos^2 \cdot u f(v) \cdot f'(v) \right).$$

We compute the norm as

$$\left\| \frac{\partial}{\partial u} \times \frac{\partial}{\partial v} \right\| = f^2(f'^2 + g'^2) \neq 0.$$

Therefore $d\varphi$ has rank 2 and hence an immersion. We compute the induced metric as:

$$g_{11} = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle = f^2, g_{12} = g_{21} = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle = 0, g_{22} = \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle = f'^2 + g'^2.$$

**(b)** We compute the Christoffel symbols, $\Gamma_{ij}^m$. We compute that:

$$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{1}{2}\left[\frac{\partial g_{21}}{\partial u} + \frac{\partial g_{11}}{\partial v} - \frac{\partial g_{12}}{\partial u}\right] g^{11} = \frac{ff'}{f^2}.$$

We also have that $\Gamma_{11}^1 = \Gamma_{22}^1 = 0$. Similarly, we compute that

$$\Gamma_{11}^2 = \frac{1}{2}\left[\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{21}}{\partial u} - \frac{\partial g_{11}}{\partial v}\right] g^{22} = \frac{-ff'}{f'^2 + g'^2}.$$

The last nonzero christoffel symbol is given by

$$\Gamma_{22}^2 = \frac{f'f'' + g'g''}{f'^2 + g'^2}.$$

Therefore the geodesic equations are given as:

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0,$$

and

$$\frac{d^2v}{dt^2} - \frac{ff'}{f'^2 + g'^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{f'^2 + g'^2}\left(\frac{dv}{dt}\right)^2 = 0,$$

by Do Carmo pg 62 eqn (1).

**(c)** We first show that the energy is constant. To make the calculations look nicer we will use the dot to represent a time derivative. Letting $\gamma = (u(t), v(t))$ in local coordiantes, We compute that:

$$\begin{aligned}
\frac{d}{dt}|\gamma'(t)|^2 &= \frac{d}{dt}\langle \gamma'(t), \gamma'(t) \rangle \\
&= \frac{d}{dt}\langle (\dot{u}, \dot{v}), (\dot{u}, \dot{v}) \rangle \\
&= \frac{d}{dt}\left(\dot{u}^2 f^2 + \dot{v}^2(f'^2 + g'^2)\right) \\
&= 2\dot{u}\ddot{u}f^2 + \dot{u}^2\dot{v}2ff' + 2\dot{v}\ddot{v}(f'^2 + g'^2) + 2\dot{v}^3(f'f'' + g'g'') \\
&= -4ff'\dot{u}^2\dot{v} + 2ff'\dot{u}^2\dot{v} + 2ff'\dot{v}\dot{u}^2 - 2\dot{v}^3(f'f'' + g'g'' + 2\dot{v}^3(f'f'' + g'g'') && \text{(by b))} \\
&= 0
\end{aligned}$$

We now wish to show Clairauts Relation holds. Let $P(s)$ by the parametrization of a parrellel, given by:

$$P(s) = (f(v)\cos u(s), f(v)\sin u(s), g(v))$$

so that $u$ has constant speed, say 1. Similarly, we can write a geodesic $\gamma$ as:

$$\gamma(t) = (f(v(t))\cos u(t), f(v(t))\sin u(t), g(v(t))).$$

At such a $t$, $s$ where $P(s) = \gamma(t)$, we know from basic linear algebra that

$$\langle \dot{P}, \dot{\gamma} \rangle = \|\dot{\gamma}\| \cdot \left\|\dot{P}\right\| \cos\beta(t).$$

We have that $\|\dot{\gamma}\|$ is constant by above. It is easy to see that $\left\|\dot{P}\right\| = f(v) = r$. Thus we want to show that the left hand side is constant, when $P = \gamma$. We compute:

$$\dot{P} = (-\sin u(s) \cdot f(v), \cos u(s) \cdot f(v), 0)$$

and

$$\dot{\gamma}(t) = (f'(v)\dot{v}\cos u(t) - f(v(t))\sin u(t)\dot{u}, f'(v)\dot{v}\sin u(t) + f(v)\cos u(t)\dot{u}, g'(v) \cdot \dot{v}).$$

Therfore at the points where $P = \gamma$ we have that

$$\langle \dot{P}, \dot{\gamma} \rangle = f^2\dot{u}.$$

This is constant since

$$\frac{d}{dt} f^2\dot{u} = 2ff'\dot{u}\dot{v} + f^2\ddot{u} = 0$$

by b).

**(d)** We first change to polar coordinates. The metric takes the form $ds^2 = (1 + 4v^2)dv^2 + v^2du^2$. Let $\gamma(t) = (v(t), u(t))$ be a unit speed parametrization of a geodesic which is not a meridian. We first claim that $u(t)$ is unbounded as $t \to \infty$. We first show that $\int_0^\infty u'(t)dt$ is infinity. We can write by Clairaut's relation that

$$v(t)\cos\beta(t) = C,$$

for some constant $C$. Since $\beta(t)$ is the angle between $\gamma'$, and $\frac{\partial}{\partial u}\gamma(t)$, we write that

$$\cos\beta(t) = \frac{\langle \gamma'(t), \frac{\partial}{\partial u} \rangle}{|\frac{\partial}{\partial u}|} = \frac{\langle v'(t)\frac{\partial}{\partial v} + u'(t)\frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle}{v(t)} = u'(t)v(t).$$

By Claurauts relation, we have

$$u'(t) = \frac{C}{v^2(t)}.$$

We now claim that $v^2(t) \leqslant v^2(0) + t$. We compute:

$$1 = |\gamma'(t)| = \sqrt{(1 + 4v^2)v'^2 + v^2u'^2} \geqslant \sqrt{1 + 4v^2}|v'| \geqslant 2|vv'| = |(v^2)'|.$$

Therefore $1 \leqslant |(v^2)'|$. Integrating from 0 to $t$, we get that $v^2(t) \leqslant v^2(0) + t$. Therefore we have that by Claurauts relations,

$$\int_0^\infty |u'(t)|dt \geqslant \int_0^\infty \frac{|C|}{v^2(0) + t} dt = \infty.$$

We now claim that $\nu(t) \to \infty$ as $t \to \infty$. First suppose that at some $t_0$, we have $\beta(t_0) > 0$. Then by Claurauts relation we have that $\cos\beta(t) \geqslant \cos\beta(t_0)$ and so $\nu' > C > 0$. Therefore $\nu(t) \to \infty$. Now suppose that $\beta(t) \leqslant 0$ for all $t$. By clairuts relation we must have that $\lim_{t\to\infty} \nu(t) = \nu_0 > 0$.. We claim that this cannot happen. By above, we can write

$$1 = (1+4\nu^2)(\nu')^2 + \nu^2(u')^2 = (1+4\nu^2)(\nu')^2 + \frac{C^2}{\nu^2}.$$

So we must have that $\lim_{t\to\infty} \nu'(t) = 0$, since if it were negative we would have that $\nu(t) \to -\infty$. We have by the geodesic equation that:

$$\nu'' = \frac{C^2}{(1+4\nu)\nu^4} - \frac{4\nu}{1+4\nu^2}(\nu')^2.$$

Therefore we have that

$$\nu'' > \frac{C^2}{2(1+4\nu_0^2)\nu_0^4}$$

as $t \to \infty$. Therefore $\nu'' > 0$ and so $\nu'(t) > 0$. Since $r(t)$ and $\beta(t)$ both go to $\infty$ as $t \to \infty$, we have that the curve must intercept itself an infinite number of times, since the angle with the parrellel is always increasing, and $\nu(t) \to \infty$.

**Problem 2.** *Do Carmo Q7,Q8 p.83*

Q7: Take an orthonormal basis $\{e_i\}$ of $T_pM$. By prop. 4.2, we can take a strongly convex neighbourhood $U$ of $p$. For any $q \in U$, let $\gamma$ be a geodesic from $p$ to $u$. We define $E_i(q) = P_{\gamma,t_0,t}(e_i)i$ i.e. the parrellel transport of the basis vectors. This is well defined, since there is only one choice of $\gamma$ by strong convexity. $E_i(q)$ will be smooth since parrellel transport is smooth. Furthermore parrellel transport is an isometry so we have that $\langle E_i(q), E_j(q)\rangle = \delta_{ij}$. We finally claim that $\nabla_{E_i}E_j(p) = 0$. By the previous problem set, we can write:

$$\nabla_{E_i}E_j(p) = \frac{d}{dt}P^{-1}_{\gamma,t_0,t}E_j(\gamma(t))\Big|_{t=0} = \frac{d}{dt}e_j|_{t=0} = 0.$$

Thus we are done.

Q8:

**(a)** We first write grad $f(p) = \sum_i g_iE_i(p)$ for some smooth functions $g_i$. Using orthonormality, we compute

$$\langle \text{grad } f(p), E_j(p)\rangle = g_j = df_p(E_j(p)) = E_j(f).$$

Thus we can write grad $f(p) = \sum_{i=1}^n E_i(f)E_i(p)$. Now let $X = \sum_i f_iE_i$. If $T$ is the linear mapping which assigns $Y(p) \to \nabla_Y X(p)$, then we have that

$$\begin{aligned}
\operatorname{div}X(p) &= \operatorname{Trace}(T) \\
&= \sum_{i=1}^n \langle TE_i(p), E_i(p)\rangle && \text{(definition of trace)} \\
&= \sum_{i=1}^n \langle \nabla_{E_i}X(p), E_i(p)\rangle \\
&= \sum_{i=1}^n \langle \nabla_{E_i}\sum_j f_jE_j(p), E_i(p)\rangle \\
&= \sum_{i=1}^n \langle \sum_{j=1}^n f_j\nabla_{E_i}E_j + E_i(f_j)E_j, E_i\rangle && \text{(property of affine connection)} \\
&= \sum_{i=1}^n \sum_{j=1}^n \langle E_i(f_j)E_j, E_i\rangle && \text{(by 7)} \\
&= \sum_{i=1}^n E_i(f_i) && \text{(by orthonormality)}
\end{aligned}$$

**(b)** Since in $\mathbb{R}^n$, geodesics are straight lines and parrellel transport along straight lines is just translation, we have that $E_i(p) = \frac{\partial}{\partial x_i} = e_i$. By the previous question, we can write

$$\text{grad } f(p) = \sum_{i=1}^n E_i(f)E_i(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}e_i.$$

Similarly we compute the divergence of $X = \sum_{i=1}^n f_iE_i(p)$ as:

$$\operatorname{div}X(p) = \sum_{i=1}^n E_i(f_i) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$$

**Problem 3.** *Do Carmo Q9,Q10, pg 83-84*

Q9:

**(a)** Using $8a$, we compute the laplacian of $f$ as:

$$\Delta f = \operatorname{div} \sum_{i=1}^{n} E_i(f)E_i(p) = \sum_{i=1}^{n} E_i(E_i(f)).$$

When $M = \mathbb{R}^n$, we have instead that $E_i(f) = \frac{\partial f}{\partial x_i}$, $E_i(E_i(f)) = \frac{\partial^2 f}{\partial x_i^2}$ so

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

**(b)** Using 8a, we compute:

$$\begin{aligned}
\Delta(f \cdot g) &= \operatorname{div} \text{ grad } (f \cdot g) \\
&= \operatorname{div} \sum_{i=1}^{n} E_i(f \cdot g)E_i(p) \\
&= \operatorname{div} \sum_{i=1}^{n} [fE_i(g) + gE_i(f)]\, E_i(p) && (E_i \text{ is a derivation}) \\
&= \sum_{i=1}^{n} E_i(fE_i(g) + gE_i(f)) && (\text{definition of div}) \\
&= \sum_{i=1}^{n} E_i(fE_i(g)) + \sum_{i=1}^{n} E_i(gE_i(f)) && (\text{linearity of } E_i) \\
&= \sum_{i=1}^{n} E_i(f)E_i(g) + fE_i(E_i(g)) + \sum_{i=1}^{n} + \sum_{i=1}^{n} E_i(g)E_i(f) + gE_i(E_i(f)) && (E_i \text{ is a derivation}) \\
&= f\Delta g + g\Delta f + 2\langle \text{grad } f, \text{grad } g\rangle
\end{aligned}$$

As desired.

Q10: We wish to show that

$$\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle = 0.$$

We compute that

$$\begin{aligned}
\frac{d}{ds}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle &= \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle + \langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{\partial f}{\partial t}\rangle && (\text{by symmetry of connection}) \\
&= \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s}\rangle && (\text{since f is geodesic in s}) \\
&= \frac{1}{2}\frac{d}{dt}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\rangle \\
&= 0 && (\text{since f parametrized by arclength})
\end{aligned}$$

Therefore $\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle$ is independant of $s$ and $t$. Thus it must always be 0.

**Problem 4.** *Do Carmo Q4 pg 104 + Find expontential map on* $S^1$*.*

**(a)** Consider a parametrized surface $f : U \subset \mathbb{R}^2 \to M$, with

$$U = \{(s,t) \in \mathbb{R}^2 : -\varepsilon < t < 1+\varepsilon, -\varepsilon < s < 1+\varepsilon, \varepsilon > 0\},$$

so that $f(s,0) = f(0,0)$. Take $V_0 \in T_{f(0,0)}M$, and define the vector field $V$ along $f$ by $V(s,0) = V_0$ and for $t \neq 0$, $V(s,t)$ is the parrellel transport of $V_0$ along $t \mapsto f(s,t)$. By lemma 4.1 Do Carmo, we have that

$$\frac{D}{\partial s}\frac{D}{\partial t}V - \frac{D}{\partial t}\frac{D}{\partial s}V = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V.$$

Since $V$ is a parrellel transport we have that $\frac{D}{\partial t}V = 0$. By assumption we have that parallel transport is independant of choice of curve. Therefore $V(s,1)$ is a parallel transport of $V(0,1)$ along $s \mapsto f(s,1)$. Thus $\frac{D}{\partial s}V(s,1) = 0$, and so we have that

$$R_{f(0,1)}(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1))V(0,1) = 0.$$

Since $V_0$ arbitrary, and $f$ was arbitrary, we have that $R(X,Y)Z = 0$ for all $X, Y, Z$.

**(b)** By prop 2.7 there is a unique geodesic $\gamma(t, p, \nu_p)$ defined for $t \in (-2,2)$ so that at $t = 0$ $\gamma$ passes through $p = (\cos\phi, \sin\phi)$ with velocity $\nu_p = (-\theta\sin\phi, \theta\cos\phi)$ in some coordinate chart. Consider the curve given by

$$\gamma(t) = e^{i(\phi+\theta t)}.$$

We have that $\gamma(0) \cong p$, $\gamma'(0) \cong \nu_p$, and $\exp_p(\nu_p) = \gamma(1) = (\sin\phi + \theta, \cos\phi + \theta)$. It remains to show that $\gamma$ is geodesic. We can take a coordinate system $x : (-\pi, \pi) \to S^1$ defined as $x(t) = (\cos\phi + t, \sin\phi + t)$. We have that $c(t) = x(\theta t)$, so $c = \theta t$ in local coordinates. We compute that:

$$g_{11} = \langle x_t, x_t \rangle = \|(-\sin\theta + t, \cos\theta + t)\|^2 = 1.$$

Therefore the christoffel symbol $\Gamma^1_{11} = 0$, and so the geodesic equation becomes:

$$\frac{d^2}{dt^2}c = \frac{d^2}{dt^2}\theta t = 0.$$

Thus $\gamma$ is a geodesic with corresponding exp given as $\gamma(1)$.

**Problem 5.** *Do Carmo Q9 pg 107.*

First we define an orthonormal basis $\{e_i\}$ so that if $x = \sum_{i=1}^n x_i e_i$, then $\mathrm{Ric}_p(x) = \sum_{i=1}^n \lambda_i x_i^2$. Since $|x| = 1$, we have that $x$ defines an outward pointing normal on $S^{n-1}$. We define the vector field

$$V = (\lambda_1 x_1, \dots, \lambda_n x_n).$$

We compute using Stoke's Theorem,

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} (\sum \lambda x_i^2) dS^{n-1} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \langle V, x \rangle dS^{n-1} = \frac{1}{\omega_{n-1}} \int_{B^n} \nabla \cdot V dB^n.$$

Since we know that $\nabla \cdot V = \sum_{i=1}^n \lambda_i$, We have that

$$\frac{\sum_{i=1}^n \lambda_i}{\omega_{n-1}} \int_{B^n} dB^n = \frac{\sum_{i=1}^n \lambda_i}{n} = \frac{\sum_{i=1}^n \mathrm{Ric}_p(e_i)}{n} = K(p),$$

where we use the fact that $\frac{vol(B^n)}{\omega_n} = \frac{1}{n}$. Thus we have $K(p) = \frac{1}{\omega} \int_{S^{n-1}} \mathrm{Ric}_p(x) dS^{n-1}$.

**Problem 6.** *Is there a closed Riemannian manifold diffeomorphic to* $S^2$*, such that a shortest geodesic loop in* $M$ *is not a periodic geodesic?*