Q4a: We first will deal with the case that $m(E) < \infty$.We suppose for the sake of contradiction that there is some $\alpha \in (0,1)$ such that for all intervals I such that

$$m(E \cap I) < \alpha m(I)$$

From the measurability of E, for any $\varepsilon > 0$, we can find some open cover of E by intervals $\{I_n\}$ with

$$m(E) \leq \sum_n m(I_n) \leq m(E) + \varepsilon$$

We mupltiply this inequality with α to get that

$$\alpha m(E) \leq \sum_n \alpha m(I_n) \leq \alpha m(E) + \alpha\varepsilon$$

Using our assumption, we get that

$$\sum_n m(E \cap I_n) < \sum_n \alpha m(I_n)$$

Using subadditivity we get

$$m(\bigcup_n E \cap I_n) \leq \sum_n m(E \cap I_n)$$

And since $E \subset \bigcup_n E \cap I_n$, we get

$$m(E) < m(\bigcup_n E \cap I_n)$$

Combining these inequalities together we get that

$$m(E) < \alpha m(E) + \alpha\varepsilon$$

We assume that $\alpha \in (0,1)$ and ε is arbitrary, hence we obtain a contradiction. Now if $m(E) = \infty$, we can cover it with countable many E_n which all have some I^n corresponding to them, along with some α_n. Take $I = \bigcup_n I_n$ and we are done.

Q4b: Suppose for the sake of contradiction that $E - E$ contains no interval centered about 0. Then for all $\varepsilon > 0$, there exists some $p \in (-\varepsilon, \varepsilon)$ so that $E + p \cap E = \emptyset$. Take $\alpha \in (\frac{3}{4}, 1)$. From 4a, there must exist some interval I with

$$m(E \cap I) \geq \alpha m(I)$$

Translation invariance of the lebesgue measure also guarantees that

$$m(E \cap I + p) = m(E \cap I)$$

and our choice of p guarantees that

$$(E \cap I + p) \cap (E \cap I) = \emptyset$$

Hence using the property of the measure, we get that

$$m((E \cap I + p) \cup (E \cap I)) \geq 2\alpha m(I)$$

From properties of sets it is clear that we have

$$(E \cap I + p) \cup (E \cap I) \subset (I + p) \cup I$$

and so

$$m((E \cap I + p) \cup (E \cap I)) \leq m((I + p) \cup I)$$

Since I is an interval, it follows that $m(I + p \cup I) = \mu(I) + |p|$. Since $|p| < \varepsilon$, along with the inequalities from above, and taking $\varepsilon < m(I)$ we get

$$m(I) + \varepsilon \geq 2\alpha m(I)$$

If we take $\varepsilon = \frac{1}{2}m(I)$, We get that

$$\frac{3}{2}m(I) \geq 2\alpha m(I) > \frac{3}{2}m(I)$$

A contradiction. We conclude that there exists an interval cenetered around 0 contained in $E - E$.