Q1a: Note by $A1Q1$ we have that for $|\overline{a}z| \neq 1$,

$$1 - |\frac{a-z}{1-\overline{a}z}| = \frac{(1-|z^2|)(1-|a|^2)}{|1-\overline{a}z|^2}.$$

When $|z| = 1$ this will hold, since $|a| < 1$, and we have that it will be 0, or equivalently, we have that

$$|\frac{a-z}{1-\overline{a}z}| = 1.$$

Hence this fractional linear transformation will map S^1 to S^1. We can check that it maps the interiour to the interiour since $g_a(a) = 0$. Continuity implies that this holds for all $z \in D$. Thus this is a homeomorphism of D.

Q1b: Let $a = f^{-1}(0)$. Consider the mapping $f \circ g_a$. We claim that it is scaling by some $\lambda \in S^1$. First we have that

$$f(g_a(0)) = f(a) = 0.$$

We now claim that for some $z \neq 0$ we have

$$|f(g_a(z))| = |z|.$$

Applying Schwartz' lemma to $f \circ g_a$ and $(f \circ g_a)^{-1}$ we get that for all z,

$$|f(g_a(z))| = |z|.$$

Thus we have that

$$f(g_z(z)) = \lambda z \implies \frac{1}{\lambda} f(g_a(z)) = z \implies \frac{1}{\lambda} f = g_a^{-1}(z) \implies f(z) = \lambda g_a^{-1}(z)$$

We have that

$$g_a^{-1}(z) = \frac{z-a}{\overline{a}z - 1} = g_a(z).$$

Thus we are done.