

Q3: Note that we can write that

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \frac{z^n}{(1-z)(1+z+\dots+z^{n-1})}$$

We will compute the ratio test.

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{z^{n+1}}{(1-z)(1+z+\dots+z^n)}}{\frac{z^n}{(1-z)(1+z+\dots+z^{n-1})}} \right| = \lim_{n \rightarrow \infty} \frac{|z + z^2 + \dots + z^n|}{|1 + z + \dots + z^n|}$$

We let $r = z + z^2 + \dots + z^n$ and rewrite the equation as

$$\lim_{n \rightarrow \infty} \frac{|r|}{|1+r|}$$

If we have that $|z| < 1$, then this ratio will converge to a quantity less than one. If we have that $|z| \geq 1$, then it will approach 1 and hence diverge.