Q3a: We first claim that $\mu^*(A) \leq \inf_{E \in \mathcal{M}_{\mu^*} A \subset E} \mu(E)$. This follows immediately, since μ^* and μ agree on all elements of $\mathcal{M}_{\mu^*}$, hence for any $E \supset A$, we have that

$$\mu^*(A) \leq \mu^*(E) = \mu(E)$$

We now claim that $\mu^*(A) \geq \inf_{E \in \mathcal{M}_{\mu^*} A \subset E} \mu(E)$. It is a fact that

$$\mu^* = \inf\{\sum_i \mu_0(A_i) : \bigcup_i A_i \supset E, A_i \in \mathcal{A}\}$$

Now since

$$\sum_i \mu_0(A_i) \geq \bigcup_i A_i$$

We now apply Folland Prop 1.13, which was proven during the lectures, to tell us that $\mu^*|\mathcal{A} = \mu_0 = \mu|\mathcal{A}$, and that every set in $\mathcal{A}$ is μ^* measurable. Hence we obtain the inequality

$$\inf\{\sum_i \mu_0(A_i) : \bigcup_i A_i \supset E, A_i \in \mathcal{A}\} \geq \inf\{\mu_0(\bigcup_i A_i) : A_i \in \mathcal{A}, \bigcup_i A_i \supset A\}$$

Since the union of A_i are in $\mathcal{A}$, they must also be measurable by the proposition, and hence belong to $\mathcal{M}_{\mu^*}$. Therefore

$$\inf\{\mu_0(\bigcup_i A_i) : A_i \in \mathcal{A}, \bigcup_i A_i \supset A\} \geq \inf\{\mu(E) : E \supset A, E \in \mathcal{M}_\mu^*\}$$

As desired

Q3b: First note that $\mu_0(X) = \mu^*(X)$ by the definition of μ^*. We now suppose that A is measureable. We have that

$$\mu_*(A) = \mu_0(A) - \mu^*(A^c) = \mu^*(X) - \mu^*(A^c) = \mu^*(X \setminus A^c) = \mu^*(A)$$

As desired. Now suppose that $\mu_*(A) = \mu^*(A)$. It is sufficient to prove that for any test set E,

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

By $3a$, for $\varepsilon > 0$, there must exist some measurable set F with $\mu^*(F \setminus E) < 2\varepsilon$. By assumption, we have that

$$\mu^*(X) = \mu^*(A) + \mu^*(A^c) = \mu^*(F \cap A) + \mu^*(F^c \cap A) + \mu^*(F \cap A^c) + \mu^*(F^c \cap A^c)$$

Using 3a again, we create sets E_1, E_2, E_3, E_4 respectively which are all measurable and contain the four sets on the right hand sum of the inequality and the sum of their measures differs by ε. We therefore have that

$$\sum_i \mu^*(E_i) \leq \mu^*(X) + \varepsilon$$

Expanding out, we get

$$\mu^*(E_1) + \mu^*(E_2) + \mu^*(E_3) + \mu^*(E_4) \leq \mu^*(F) + \mu^*(F^c) + \varepsilon$$

And so we see that

$$\mu^*(E_1) + \mu^*(E_3) \leq \mu^*(F) + \varepsilon$$

Taking the infimums on the left hand side we get that

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(F \cap A) + \mu^*(F \cap A^c) \leq \mu^*(F)$$

Taking infimums on the righthand side we get that

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E)$$

We reach our desired conclusion.