Q5: The maximum order of an element of $C_m \times C_n$ is $lcm(n, m)$, since if g generates C_m and h generates C_n, we have that (g, h) generates $C_m \times C_n$. The order of this element must be both a multiple of n and m, so the largest it can be is $lcm(n, m)$. We now claim that

$$C_m \times C_n \cong C_{lcm(m,n)} \times C_{gcd(m,n)}.$$

First note that from basic number theory we have that $m \cdot n = lcm(m, n) \cdot gcd(m, n)$. Hence these groups must have the same order. We will construct an injective homeomorphism between them and conclude that they are isomorphic. Note that for every element $([x]_m, [y]_n)$ we can find some integer a so that $[a]_m = [x]_m$ and $[a]_n = [y]_n$. Define $\varphi : C_m \times C_n \to C_{lcm(m,n)} \times C_{gcd(m,n)}$ by

$$\varphi([a]_m, [a]_n) = ([a]_{lcm(m,n)}, [a]_{gcd(m,n)}).$$

Note that from properties of integers mod n, we have that φ is a homeomorphism. We claim that this is an injective mapping. First suppose that for some $(x, y) \in C_m \times C_n$, we have that $\varphi(x, y) = (0, 0)$. Find a $b \in \mathbb{Z}$ so that $[b]_m = [x]_m$ and $[b]_n = [y]_n$. We have that $\varphi(b, b) = (0, 0)$. We have that $b|nk, lm$ for some $k, l \in \mathbb{Z}$ and so $[b]_m = [b]_n = e$. Hence $(x, y) = (e, e)$ and φ is an isomorphism. It is clear from elementary number theory that $gcd(n, m)|lcm(n, m)$. We now claim that if for any r, s with $s|r$, and $C_m \times C_n \cong C_r \times C_s$ we must have that $r = lcm(m, n)$ and $r = gcd(m, n)$. Note that we must have that $r \cdot s = m \cdot n$, furthermore $gcd(m, n) = gcd(r, s)$ and $lcm(r, s) = kcm(m, n)$. If $s|r$. This uniquely determines s, r as $r = lcm(m, n)$ from number theory.