Q4a: By the Weierstrass Approximation Theorem, for each $\varepsilon > 0$, there exists some polynomial $p(x)$ where $|p(x) - f(x)| < \varepsilon$. We first claim that for such an ε and $p(x)$,

$$\lim_{n\to\infty} \int_0^1 x^n p(x)dx = \lim_{n\to\infty} \int_0^1 x^n f(x)dx$$

Let $\varepsilon > 0$ be given, let $p(x)$ be such that $|p(x) - f(x)| < \varepsilon$, then we evaluate that

$$\lim_{x\to\infty} |\int_0^1 x^n f(x) - x^n f(x)dx| \leq \lim_{n\to\infty} \int_0^1 x^n |p(x) - f(x)|dx < \lim_{n\to\infty} \int_0^1 x^n \varepsilon dx = \varepsilon \lim_{n\to\infty} \Big[\frac{x^{n+1}}{n+1}\Big]\Big|_{x=0}^{x=1} = 0$$

Thus these limits are equal. To find $\lim_{x\to\infty} \int_0^1 x^n f(x)$ it suffices to compute $\lim_{x\to\infty} \int_0^1 x^n p(x)$. Let $p(x) = \sum_{k=0}^m a_k x^k$ We compute:

$$\begin{aligned}
\lim_{x\to\infty} \int_0^1 x^n p(x)dx &= \lim_{x\to\infty} \int_0^1 x^n \sum_{k=0}^m a_k x^k dx \\
&= \lim_{x\to\infty} \sum_{k=0}^m a_k \int_0^1 x^{n+k} dx \\
&= \lim_{x\to\infty} \sum_{k=0}^n a_k \Big[\frac{x^{n+k+1}}{n+k+1}\Big]\Big|_{x=0}^{x=1} \\
&= \lim_{x\to\infty} \sum_{k=0}^m a_k \frac{1}{n+k+1} \\
&= 0
\end{aligned}$$

Therefore $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$

Q4b: By Q5, for each ε there exists a polynomial $p(x)$ such that $|p(x) - f(x)| < \varepsilon$ and $p(1) = f(1)$. We claim that

$$\lim_{n\to\infty} n \int_0^1 x^n p(x)dx = \lim_{n\to\infty} n \int_0^1 x^n f(x)dx$$

By a similar computation to 4a, we see that

$$\lim_{n\to\infty} n \int_0^1 x^n |p(x) - f(x)|dx < \lim_{n\to\infty} n \int_0^1 x^n \varepsilon dx = \varepsilon \Big[\frac{x^{n+1}}{n+1}\Big]\Big|_{x=0}^{x=1} = \varepsilon$$

The limits can be made within epsilon of eachother, hence they are equal. Let $p(x) = \sum_{k=0}^m a_k x^m$ .We now will evaluate $\lim_{n\to\infty} n \int_0^1 x^n p(x)dx$ as :

$$\begin{aligned}
\lim_{n\to\infty} n \int_0^1 x^n p(x)dx &= \lim_{n\to\infty} n \int_0^1 x^n \sum_{k=0}^m a_k x^k dx \\
&= \lim_{n\to\infty} n \sum_{k=0}^m a_k \int_0^1 x^{n+k} \\
&= \lim_{n\to\infty} n \sum_{k=0}^m a_k \Big[\frac{x^{n+k+1}}{n+k+1}\Big]\Big|_{x=0}^{x=1} \\
&= \lim_{n\to\infty} \sum_{k=0}^m a_k \frac{n}{n+k+1} \\
&= \sum_{k=0}^m a_k \\
&= p(1)
\end{aligned}$$

Therefore this limit converges to $p(1) = f(1)$.