Q5a: We wish to compute the integral

$$\int_{[0,\infty]} \frac{\log(x^2+1)}{x^2+1} dz$$

Substituting for $z = \arctan(x)$, we compute that

$$\begin{aligned}
\int_{[0,\infty]} \frac{\log(x^2+1)}{x^2+1} dx &= \int_{[0,\frac{\pi}{2}]} \log(\cos^{-2}(z)) dz && \text{(by change of variables)} \\
&= -2 \int_{[0,\frac{\pi}{2}]} \log(\cos(z)) dz \\
&= -\int_{[-\frac{\pi}{2},\frac{\pi}{2}]} \log\Big(\frac{e^{iz}+e^{-iz}}{2}\Big) dz \\
&= \int_{[-\frac{\pi}{2},\frac{\pi}{2}]} \log(2) dz + \int_{[-\frac{\pi}{2},\frac{\pi}{2}]} \log(e^{iz}+e^{-iz}) dz \\
&= \pi \log(2) + \int_{[-\frac{\pi}{2},\frac{\pi}{2}]} \log(e^{iz}+e^{-iz}) dz.
\end{aligned}$$

We now claim that the integral $\int_{[-\frac{\pi}{2},\frac{\pi}{2}]} \log(e^{iz}+e^{-iz}) dz = 0$. Note that $|e^{-iz}+e^{iz}| > 1$ on the domain of integration, hence the function will integrate to be a positive number. However, if we take a rectangle given by $[-\frac{\pi}{2},\frac{\pi}{2}] \times [0,R]$ for sufficiently large R, we have that the integral will be 0 along this rectangle by the residue formula. Furthermore, the value of the function is positive along the boundary. Hence the integral on the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ will be 0.

Q5b: We wish to compute the integral

$$\int_{[0,\infty]} \frac{\sin^2(kx)}{x^2} dx.$$

Since this is an even function, we can write this as

$$\frac{1}{2}\int_{\mathbb{R}} \frac{\sin^2(x)}{kx} dx.$$

Substituting for $y = kx$, get

$$\frac{k}{2}\int_{\mathbb{R}} \frac{\sin^2(y)}{y^2} dy.$$

Taking half circles of radius r as γ_1 such that it encloses 0 and γ_2 does not, we integrate this over $\mathbb{C}$ in the following way:

$$\begin{aligned}
\frac{k}{2}\int_{\mathbb{R}} \frac{\sin^2(z)}{z^2} dz &= \frac{k}{2}\int_{\mathbb{R}} \frac{1-\cos(2z)}{2z^2} dz \\
&= \frac{k}{2}\int_{\mathbb{R}} \frac{(1-e^{2iz})+(1-e^{-2iz})}{4z^2} \\
&= \frac{k}{2}\Big[\int_{\gamma_1} \frac{1-e^{2iz}}{4z^2} dz + \int_{\gamma_2} \frac{1-e^{-2ix}}{}\Big] \\
&= \frac{k}{2}\int_{\gamma_1} \frac{1-e^{2iz}}{4z^2} dz
\end{aligned}$$

We now compute the laurent expansion of $\frac{1-e^{2iz}}{4z^2}$ in an annulus containing 0. We have that

$$\frac{1-e^{2iz}}{4z^2} = \frac{1}{4z^2}(-(2iz) - \dots) = -\frac{i}{2z}.$$

Thus we have by the residue formula that

$$\int_{[0,\infty]} \frac{\sin^2(kx)}{x^2} dx = \frac{k}{2} \cdot 2\pi i \cdot \frac{-i}{2} = \frac{k\pi}{2}.$$