Problem 1.

Suppose both f satisfies $f'(l) = cf(l), f'(0) = -cf(0)$, and g does as well. We compute:

$$\begin{aligned} [f'(x)g(x) - f(x)g'(x)]\Big|_0^l &= f'(l)g(l) - f(l)g'(l) - f'(0)g(0) + f(0)g'(0) \\ &= cf(l)g(l) - cf(l)g(l) + cf(0)g(0) - cf(0)g(0) \\ &= 0 \end{aligned}$$

Therefore for f, g with identical boundary conditions, we have that they are symmetric. This implies eigenfunctions with the same boundary conditions but different eigenvalues of $\frac{d^2}{dx^2}$ are orthogonal, and hence form an orthonormal basis of the Hilbert space.

Problem 2.

Recall from Assignment 6 that for any $f \in C^2$ the following holds:

$$c_n(f'') = \frac{-n^2}{2\pi} c_n(f).$$

Consider now the Fourier Series of f, given as $\sum_{n\geqslant 1} e^{inx} c_n(f)$. We can bound each term of the sum in the following way:

$$|e^{inx} c_n(f)| = \frac{2\pi}{n^2} |c_n(f'')| = \frac{1}{n^2} \left| \int_0^{2\pi} e^{inx} f(x)\, dx \right| \leqslant \frac{1}{n^2} \int_0^{2\pi} |f''(x)| dx \leqslant \frac{2\pi \sup |f''|}{n^2}.$$

We have that the sum $\sum_{n\geqslant 1} \frac{2\pi \sup |f''|}{n^2}$ converges, so by the M-Test the Fourier series of f converges absolutely and uniformly.

Problem 3.

We wish to solve the following boundary value problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on } (0, l) \\ u(0,t) = u_t(l,t) = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = x \end{cases}$$

Writing $u(x,t) = X(x)T(t)$, we know that X, T take the following forms:

$$T(t) = A_n \sin(c\lambda_n t) + B_n \cos(c\lambda_n t), X(x) = A'_n \sin(\lambda_n x) + B'_n \cos(\lambda_n x).$$

We first look at $X(x)$. Note that since $X(0) = 0$, we have that $B'_n = 0$. Since $X'(l) = \lambda_n A'_n \cos(\lambda_n l) = 0$, we have that $\lambda_n = \frac{n\pi}{l} + \frac{\pi}{2l}$. So

$$X(x) = A'_n \sin\left(\left(\frac{n\pi}{l} + \frac{\pi}{2l}\right) x\right).$$

Thus we write

$$u(x,t) = \sum_{n \geqslant 1} \left(A_n \sin\left(ct\left(\frac{n\pi}{l} + \frac{\pi}{2l}\right)\right) + B_n \cos\left(ct\left(\frac{n\pi}{l} + \frac{\pi}{2l}\right)\right)\right) \sin\left(x\left(\frac{n\pi}{l} + \frac{\pi}{2l}\right)\right)$$

Now using the initial conditions we have that

$$\begin{cases} u(x,0) = x^2 = \sum B_n \sin\left(x\left(\frac{n\pi}{l} + \frac{\pi}{2l}\right)\right) \\ u_t(x,0) = x = \sum A_n \cdot c \cdot \left(\frac{n\pi}{l} + \frac{\pi}{2l}\right) \sin\left(x\left(\frac{n\pi}{l} + \frac{\pi}{2l}\right)\right). \end{cases}$$

Orthongonality of the basis implies that

$$B_n = \frac{2}{l}\int_0^l x^2 \sin\left(x\left(\frac{n\pi}{l} + \frac{\pi}{2l}\right)\right) dx = \frac{(-1)^n \cdot 4l^2 \cdot (8\pi n + 4\pi n)}{\pi^3(2n+1)^3} = \frac{(-1)^n 16 l^2}{\pi^2(2n+1)^2}.$$

Similarly for the A'_ns, we have that

$$A_n = \frac{1}{c} \cdot \frac{2}{l} \cdot \frac{1}{\frac{n\pi}{l} + \frac{\pi}{2l}} \cdot \frac{(-1)^n 4l^3}{\pi^2(2n+1)^2} = \frac{(-1)^n 16 l^3}{c\pi^3(2n+1)^3}$$

Problem 4.

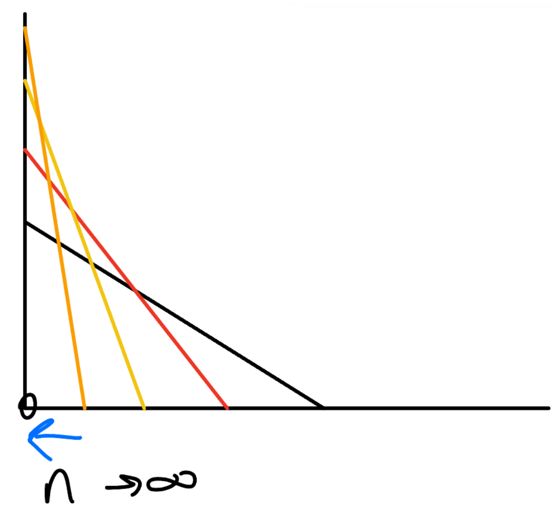
First for $f_n = x^n$ on $[0, 1]$, pointwise this does not converge to a continuous function. It converges to $\delta_1(x)$. Furthermore it does not converge uniformly, since it does not converge pointwise. We claim that it converges to 0 in L^2 however. Observe:

$$\int_0^1 |x^n|^2 dx = \int_0^1 x^{2n} dx = \frac{1}{2n+1},$$

which goes to 0 as $n \to \infty$. Now for $g_n(x) = nx^n$. This does not converge pointwise or uniformly since $g_n(1) \to \infty$ as $n \to \infty$. We now verify if $g_n(x)$ converges in L^2:

$$\int_0^1 |nx^n|^2 dx = \int_0^1 n^2 x^{2n} dx = \frac{n^2}{2n+1},$$

which diverges in L^2. The following picture demonstrates a family of functions which converge to 0 pointwise, but does not converge to 0 in L^2.



where the functions are normalized so that their integral is 1. The converse is not true. Suppose we have a family of functions $\{f_n\}$ which converge to 0 in L^2. We have that $\int_0^1 |f_n|^2 dx \to 0$. This means that $|f_n|^2 \to 0$ almost everywhere, and so $f_n \to 0$ almost everywhere.

Problem 5.

(a) Recall that it has been computed in Lectures that the Fourier sin series of $\phi(x) = x$ on $(0, l)$ is

$$\phi(x) = \sum_n (-1)^{n+1} \frac{2l}{n\pi} \sin \frac{n\pi x}{l}.$$

Since $x^2 = 2\int_0^x y\,dy$, we can easily obtain the Fourier cosine series of x^2 by integrating $\phi(x)$ term by term. So,

$$\begin{aligned} x^2 &= A_0 + 2\int_0^x \sum_n (-1)^{n+1} \frac{2l}{n\pi} \sin \frac{n\pi y}{l}\,dy \\ &= A_0 + \sum_n (-1)^{n+1} \frac{4l}{n\pi} \int_0^x \sin \frac{n\pi y}{l}\,dy \\ &= A_0 + \sum_n (-1)^n \frac{4l^2}{n^2\pi^2} \cos \frac{n\pi x}{l} \end{aligned}$$

Where $A_0 = \frac{1}{l}\int_0^l x^2 dx = \frac{1}{3}l^2$. At $x = 0$, the above gives us:

$$\frac{1}{3}l^2 = \sum_n (-1)^{n+1} \frac{4l^2}{n^2\pi^2} \implies \sum \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

(b) We now determine the Fourier series of x^3, x^4 in the same way. We first find the sin series of x^3. We have that 4

$$x^3 = \sum_{n \geqslant 1} A_n \sin \frac{n\pi x}{l}.$$

Orthogonality tells us that

$$A_n = \frac{2}{l}\int_0^l x^3 \sin \frac{n\pi x}{l} = \frac{2l^3(-1)^{n+1}}{\pi n} - \frac{12l^3(-1)^{n+1}}{\pi^3 n^3}.$$

Therefore

$$x^3 = \sum_{n \geqslant 1} \left(\frac{2l^3(-1)^{n+1}}{\pi n} - \frac{12l^3(-1)^{n+1}}{\pi^3 n^3} \right) \sin \frac{x\pi n}{l}.$$

We perform the same trick as above to obtain the Fourier series for x^4, integrating. We have that

$$x^4 = A_0 + 4\int_0^x y^3 dy$$

with

$$A_0 = \frac{1}{2l}\int_0^l x^4 = \frac{l^4}{10}.$$

Therefore

$$
\begin{aligned}
x^4 &= \frac{l^4}{10} + 4\int_0^x y^3 dy \\
&= \frac{l^4}{10} + 4\int_0^x \sum_{n\geqslant 1} A_n \sin\frac{n\pi y}{l} dy \\
&= \frac{l^4}{10} + 4\sum_{n\geqslant 1} A_n \int_0^x \sin\frac{n\pi y}{l} dy \\
&= \frac{l^4}{10} + 4\sum_{n\geqslant 1} A_n \frac{l}{n\pi} \cdot -\cos\frac{n\pi x}{l}
\end{aligned}
$$

(c) At $x = 0$, we have that

$$
0 = \frac{l^4}{10} + 4\sum_{n\geqslant 1} A_n \frac{l}{n\pi} = \frac{l^4}{10} + 4\sum_{n\geqslant 1} \frac{2l^4(-1)^{n+1}}{\pi^2 n^2} - \frac{12l^4(-1)^{n+1}}{\pi^4 n^4} \implies \frac{\pi^4}{45} = \sum_{n\geqslant 1} \frac{(-1)^n}{n^4}
$$