Q1a: We define $F(x) : M \to M \times \mathbb{R}$ by $F(x) = (x, f(x))$. This is the composition of continuous functions, and hence is continuous. The image of $F$ is the graph of $f$, and is connected, since the image of a connected set under a continous mapping is connected.

Q1b: Consider the set $\{(x, \sin\frac{1}{x}) : x \in (0,1)\} \cup \{(0,0)\}$. This is a connected graph of the function $f(x) = \begin{cases} \sin(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$ We have that this graph is connected, yet $f$ is not continuous at 0.

Q1c: Let $a, b \in M$. Let $\gamma : [0,1] \to M$ be a path between $a$ and $b$. We define $F$ in the same way as in Q1a. Then for any $(a, f(a))$ and $(b, f(b))$ in the graph of $f$, we have that $F \circ \gamma$ will be a path. This will be continous, as it is the composition of continuous mappings. Thus the graph of $f$ is path connected.

Q1d: Suppose that $\Gamma(f)$ the graph of $f$ is path connected. Suppose that $M$ has 2 points, $a, b$ and there does not exist some continuous path between them. Then there must exist some continuous $\gamma : [0,1] \to \Gamma(f)$ such that $\gamma(0) = (a, f(a))$ and $\gamma(1) = (b, f(b))$. However, if $\pi_M$ is defined to be the projection map onto $M$, then we can create a path from $a$ to $b$ in $M$ by composing $\pi_M \circ \gamma$. This contradicts that $M$ is not path connected. Now suppose that $\Gamma(f)$ path connected, but $f$ is not continuous. If $\pi_{\mathbb{R}}$ is the projection onto the real space, we take note that $\pi_{\mathbb{R}} \circ \gamma$ is a continuous map with $\pi_{\mathbb{R}} \circ \gamma(0) = f(b)$ and $\pi_{\mathbb{R}} \circ \gamma(1) = f(b)$. Since $a, b$ were chosen arbitrarily, $f$ is continuous on all of $M$. A contradiction.