MA 200 – Last Homework

Due: Friday, December 9th (10pm)

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Prove each of the following statements.

<u>**Section 4.3**</u>

1) (12pt) Prove $f(x) = x^2$ is continuous at $x = 2$.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = min\{1, \frac{\epsilon}{5}\}$ since $\delta \geq 1$, $1 < x < 3$ and $3 < x + 2 < 5$. Now, if $0 < |x - 2| < \delta$, we have $|f(x) - f(2)| = |x^2 - 4| = |x - 2||x + 2| < 5|x - 2| < 5\delta < 5\frac{\epsilon}{5} = \epsilon$ □

2) (12pt) Prove there are no integer solutions x and y for $x^2 - y^2 = 26$.

Proof. We will prove this is true by factoring the left hand side of the equation and testing each of the possible cases. The left hand side $x^2 - y^2$ factors into $(x+y)(x-y)$. The right hand side can factor two ways, into (1, 26) or (2, 13). The factorization of 26 into those factors are the possible cases for $(x+y)$ and $(x - y)$. By testing each case as follows it is proven that there is not a case for which $x^2 - y^2 = 26$:

$x + y = 26, x - y = 1$

$x + y = 1, x - y = 26$

$x + y = 13, x - y = 2$

$x + y = 2, x - y = 13$

None of these systems of equations yields integer solutions. This completes the proof. $\square$

Prove for $n \geq 1$ that $x^{2n} - y^{2n}$ is divisible by $x + y$.

Proof. We will proceed by induction. First, we must prove that this statement is true for the base case of n=1. This means that $x^{2(1)} - y^{2(1)}$ will be divisible by $x + y$.

$x^{2(1)} - y^{2(1)} = x^2 - y^2 = (x + y)(x - y)$. This is divisible by $(x + y)$ because $(x + y)$ is a factor. Now we will assume that this holds for some $n = k$ and attempt to prove for $n = k + 1$.

Assume $(x + y)$ divides $x^{2k} - y^{2k}$.

Now look at $x^{2(k+1)} - y^{2(k+1)} =$

$x^{2k+2} - y^{2k+2} =$

$x^{2k+2} - x^{2k}y^2 - y^{2k+2} + x^{2k}y^2 =$

$x^{2k}(x^2 - y^2) - y^2(x^{2k} - y^{2k}) =$

Thus, both sides are divisible by $x + y$ by the induction hypothesis and the base case. This completes the proof. $\square$

4) (12pt) Prove the center Z of a group G is abelian. (You may assume Z is a subgroup of G since it was proven on Test 2.)

Proof. To prove that Z is abelian, we must prove that it is a group where $ab = ba$. We know from Test 2 that Z is a subgroup of G. Now we must prove that the center is abelian. By definition, the center of a group is the set of elements that commute with every element in the group such that $Z(G) = z \in G | \forall g \in G, zg = gz$. This means that by definition, the center of a group is also abelian because for some $a \in Z, ag = ga$. $\square$

5) (12pt) Prove the following is an equivalence relation.

For $a, b \in \mathbb{Z}$, define $a \sim b$ iff $a + b$ is even.

Proof. To prove something is an equivalence relation you must prove that it is symmetric, transitive, and reflexive. For reflexive $a \sim a$ because if $a \sim b$ is even then a must either be even or odd and an odd plus an odd is an even and an even plus an even is also an even therefore $a + a$ must be even. For symmetric $a \sim b$ implies $b \sim a$ because $a + b = b + a$ therefore if $a + b$ is even, so is $b + a$. For transitivity if $a \sim b$ and $b \sim c$ then $a \sim c$ because if $a \sim b$ then that implies a, b are either both even or both odd. If also $b \sim c$ then c must be the same (even or odd) as b and therefore a. This means that they are an equivalence relation.

$\square$

6) (12pt) Prove that for any $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} (-1)^i = \frac{(-1)^n - 1}{2}$$

Proof. We will proceed by induction.

Base case $n = 1$

$$\sum_{i=1}^{1}(-1)^1 = \frac{(-1)^1 - 1}{2}$$

$$-1 = -1$$

Induction Case assume for some $n = k$ that

$$\sum_{i=1}^{k}(-1)^i = \frac{(-1)^k - 1}{2}$$

and prove that it holds for $n = k + 1$ such that

$$\sum_{i=1}^{k+1}(-1)^i = \frac{(-1)^{k+1} - 1}{2}$$

$$\sum_{i=1}^{k}(-1)^i + (-1)^{k+1} = \frac{(-1)^{k+1} - 1}{2}$$

$$\sum_{i=1}^{k}(-1)^i = \frac{(-1)^{k+1} - 1}{2} - (-1)^{k+1}$$

$$\sum_{i=1}^{k}(-1)^i = \frac{-(-1)^k - 1}{2} + (-1)^k$$

$$\sum_{i=1}^{k}(-1)^i = \frac{-(-1)^k - 1 + 2(-1)^k}{2}$$

$$\sum_{i=1}^{k}(-1)^i = \frac{(-1)^k - 1}{2}$$

Thus by the induction hypothesis, the claim holds. This concludes the proof.

$\square$