（装订线内不要答题）

复旦大学大数据学院
2021 ~ 2022 学年第二学期期末考试试卷

☒ **A 卷**　　☐ **B 卷**

课程名称：线性规划
课程代码：DATA130058.01
开课院系：大数据学院 考试形式：闭卷

姓　名： __________ **学　号：** __________ **专　业：** ______________

提示：请同学们秉持诚实守信宗旨，谨守考试纪律，摒弃考试作弊。学生如有违反学校考试纪律的行为，学校将按《复旦大学学生纪律处分条例》规定予以严肃处理。

题　目	一	二	三	四	五	六	七	八	总　分
得　分									

PLEASE READ THE FOLLOWING REMARKS FIRST!

- There are **100** points in total. Solve as many problems as you can in the two-hour period. Please note that the problems are *not* necessarily arranged in terms of difficulty. Take a look at all of them first.
- Please provide **necessary steps** in your answers unless otherwise stated.

You may need the following definitions or theorems. In this test, $\|\cdot\|$ denotes the Euclidean norm, i.e., $\|x\| := \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ for any $x \in \Re^n$.

Definition 1 (Polyhedron) *A polyhedron is a set that can be described in the form* $\{x \in \Re^n \mid Ax \geq b\}$, *where* $A \in \Re^{m\times n}$ *and* $b \in \Re^m$ *are given.*

Definition 2 (Local and global minimizer) *Let* $f : \Re^n \to \Re$ *be a given function and* $S \subset \Re^n$ *is a subset of* $\Re^n$. *We say* $x^* \in S$ *is a local minimizer to the optimization problem* $\min_{x\in S} f(x)$ *if there exists an* $\epsilon > 0$ *such that* $f(x^*) \leq f(x)$ *for all* $x \in S \cap \{x \in \Re^n \mid \|x - x^*\| < \epsilon\}$.
A point $x^* \in S$ *is said to be a global minimizer to the optimization problem* $\min_{x\in S} f(x)$ *if* $f(x^*) \leq f(x)$ *for all* $x \in S$.

Definition 3 (Convex set) *A set* $C \subset \Re^n$ *is said to be convex if*

$$\lambda x + (1-\lambda)y \in C \quad \forall \lambda \in [0,1],\ x, y \in C.$$

Theorem 1 (Optimal value of an LP) *Consider the linear programming problem of minimizing* $\langle c, x\rangle$ *over a nonempty polyhedron. Then, either the optimal cost is equal to* $-\infty$ *or there exists an optimal solution.*

Theorem 2 (Farkas' Lemma) *Let* $A \in \Re^{m\times n}$ *and* $b \in \Re^m$ *be given data. Then, exactly one of the following two alternatives holds:*

1. *There exists some* $x \geq 0$ *such that* $Ax = b$.
2. *There exists some vector* p *such that* $A^T p \geq 0$ *and* $p^T b < 0$.

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一、(10 points)

1. A constraint is said to be redundant for the system of linear inequalities $Ax \leq b$ if its addition or removal does not alter the set of feasible solutions. Set up a linear programming problem to determine if the constraint $p^T x \leq q$ is redundant for the system $Ax \leq b$, and explain.

2. Consider the following problem involving absolute values:

$$\begin{aligned} \min \quad & 3|x_1| + 2|x_2 - 3| \\ \text{s.t.} \quad & x_1 + 2x_2 \geq -1, \\ & x_2 \geq 0. \end{aligned}$$

   Transform the above problem to a equivalent linear programming problem in standard form (do not solve the problem).

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二、(15 points) For each claim below, state first whether it is true or false. Then prove it (briefly in 1–3 sentences, enough to convince us that you are not guessing the answer) or provide a counterexample (for the false case). An unjustified True or False answer gets no credit.

1. There dose not exist nonempty and bounded polyhedron of the form $\{x \in \Re^n \mid Ax \leq b\}$, in which every basic solution is also a vertex.
2. Let $A = -A^T$, then the linear programming problem:
$$\max\left\{-b^T x \mid Ax \leq b,\, x \geq 0\right\},$$
has nonempty optimal solution set if it is feasible.
3. Consider a linear programming (minimization) problem in standard form, with the matrix A having full row rank. If the optimal cost is $-\infty$, then the right hand side vector b can be adjusted in some way to make the optimal cost finite.

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三、(10 points) Given $A \in \Re^{m\times n}$ and $b \in \Re^m$, consider the following optimization problem

$$\min_{x\in\Re^n} \|Ax - b\|_\infty, \tag{1}$$

where $\|z\|_\infty = \max_{i=1,\ldots,n} |z_i|$ for any $z \in \Re^n$.

1. Show that problem (1) has a nonempty optimal solution set.
2. Let v^* be the optimal value to problem (1). Show that

$$v^* \geq \max_{p\in\Omega} p^T b \text{ with } \Omega = \left\{ p \in \Re^m \mid A^T p = 0, \sum_{i=1}^m |p_i| \leq 1 \right\}.$$

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四、(15 points) Consider the liner programming problem

$$\text{(P)} \quad \min\left\{c^T x \mid Ax = b,\ l \le x \le u\right\}.$$

Here, $A \in \Re^{m\times n}$, $b \in \Re^m$ and $c, l, u \in \Re^n$ are given data.

1. Write down the dual of (P).

2. Let
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad c = [c_1, 1, -1, c_4]^T$$
and $l = 0$, $u = +\infty$. Are there values of b_1, b_2, c_1 and c_4 such that $\hat{x} = [3, 2, 1, 0]^T$ is optimal to (P)? If so, determine all such values.

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五、(10 points) Consider the following liner programming problem

$$
\begin{aligned}
\min \quad & 2x_1 + 3x_2 + 2x_3 + 2x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 + x_3 + 2x_4 = 3, \\
& x_1 + x_2 + 2x_3 + 4x_4 = 5, \\
& x_i \geq 0, \quad i = 1, 2, 3, 4.
\end{aligned}
\tag{2}
$$

1. Using the simplex method solve (2) (You can choose to use either the primal or the dual simplex method).
2. Write down an optimal solution to the dual of (2).

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六、(15 points) Consider the standard form polyhedron $P = \{x \in \Re^n \mid Ax = b,\ x \geq 0\}$ with $A \in \Re^{m\times n}$ and $\operatorname{rank}(A) = m$. Assume that every basic feasible solution is nondegenerate. Let $\bar{x} \in P$ and $\mathcal{N} = \{i \in \{1, \ldots, n\} \mid \bar{x}_i > 0\}$. Show that the cardinality of $\mathcal{N}$ (i.e., the number of elements in $\mathcal{N}$), denoted by $|\mathcal{N}|$, satisfies $|\mathcal{N}| \geq m$.

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七、(10 points) Given $A \in \Re^{m\times n}$, $x, s \in \Re^n$, assume that $x_i > 0$ and $s_i > 0$ for all $i = 1, \ldots, n$. Let $X = \mathrm{diag}(x) \in \Re^{n\times n}$ and $S = \mathrm{diag}(s) \in \Re^{n\times n}$. Let

$$M := \begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \in \Re^{(2n+m)\times(2n+m)}$$

and consider the following linear system

$$M \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1. Show that any solution $(\Delta x; \Delta y; \Delta s)$ to the above linear system satisfies $\Delta x = 0$ and $\Delta s = 0$.

2. Show that M is nonsingular if and only if $\mathrm{rank}(A) = m$.

八、(15 points) Given $A \in \Re^{m\times n}$, let $P = \{x \in \Re^n \mid Ax = 0,\ x \geq 0\}$ be a nonempty set. Consider the following problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} y_i \\ \text{s.t.} \quad & A(z+y) = 0, \\ & y_i \leq 1, \quad i = 1, \ldots, n, \\ & y \geq 0,\ z \geq 0. \end{aligned} \tag{3}$$

1. Show that problem (3) has a nonempty optimal solution set.
2. Show that any optimal solution y^* to problem (3) satisfies $y_i^* \in \{0, 1\}$ for $i = 1, \ldots, n$.
3. Explain how problem (3) can be used to find $x \in P$ such that the number of positive components of x is maximized.

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