Chapter 3: The Simplex Method

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Scribe: Eric Chen

Throughout this chapter, we consider the standard form problem

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and let P be the corresponding feasible set. We assume that the dimensions of matrix A are $m \times n$ and that its rows are linearly independent. Also, denote A_i as the ith column of the matrix A and a_i^T its ith row.

1 Optimality Conditions

Definition 1.1. Let x be an element of a polyhedron P. A vector $d \in \mathbb{R}^n$ is said to be a **feasible direction** at x, if there exists a positive scalar θ for which $x + \theta d \in P$.

Now let x be a basic feasible solution to the standard form problem, let $B(1), \dots, B(m)$ be the indices of the basic variables, and let $B = [A_{B(1)}, \cdots, A_{B(m)}]$ be the corresponding basis matrix, then we have

$$x_B = B^{-1}b.$$

Now consider moving away from x to a new vector $x + \theta d$ by selecting a variable x_j and increasing it to a positive value θ, while **keeping the remaining nonbasic variables at zero**. Algebraically, $d_j = 1$ and $d_i = 0$ for every nonbasic index i other than j. At the same time, we change x_B to $x_B + \theta d_B$ where $d_B = (d_{B(1)}, d_{B(2)}, \dots, d_{B(m)})$.
Let the new solution be basic feasible solution, we have $A(x + \theta d) = b$ and since x is feasible, we also have $Ax = b$, thus we have $Ad = 0$. Notice that $d_j = 1$ and $d_i = 0$ for all other nonbasic indices i. Then

$$0 = Ad = \sum_{i=1}^{n} A_i d_i = \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j = B d_B + A_j.$$

Thus we can obtain

$$d_B = -B^{-1} A_j.$$

The direction vector d that we have just constructed will be referred to as the jth **basic iteration**. As for the nonnegativity constraints, we only worry about the basic variables, and we distinguish two cases:

(a) Suppose x is a nondegenerate BFS, and when θ is sufficiently small, we can guarantee $x_B + \theta d_B \geq 0$ for any d.

(b) Suppose x is degenerate. Then d is not always a feasible direction. Indeed, it is possible that a basic variable $x_{B(i)}$ is zero while the corresponding component $d_{B(i)}$ of $d_B = -B^{-1}A_j$ is negative. In this case, the nonnegativity constraint will be violated if we follow the jth basic solution.

Definition 1.2. Let x be a basic solution, let B be an associated basis matrix, and let c_B be the vector of costs of the basic variables. For each j, we define the **reduced cost** $\bar{c}_j$ of the variable x_j according to the formula

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j.$$

Theorem 1.3. *Consider a basic feasible solution x associated with a basis matrix B, and let $\bar{c}$ be the corresponding vector of reduced costs.*

(a) If $\bar{c} \geq 0$, then x is optimal.

(b) If x is optimal and nondegenerate, then $\bar{c} \geq 0$.

Proof. Assume that $\bar{c} \geq 0$, let y be an arbitrary feasible solution, and define $d = y - x$. As $Ax = Ay = b$, so $Ad = 0$. The latter can be written in the form

$$Bd_B + \sum_{i \in N} A_i d_i = 0,$$

where N is the set of indices corresponding to the nonbasic variables under the given basis. Since B is invertible, we obtain

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i,$$

and

$$c^T d = c_B^T d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c_B^T B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i.$$

For any nonbasic index $i \in N$, we must have $x_i = 0$ and since y is feasible, we have $y_i \geq 0$. Thus $d_i \geq 0$ and $\bar{c}_i d_i \geq 0$ for all $i \in N$. Then we can conclude that $c^T(y - x) = c^T d \geq 0$, so x is optimal.

Suppose x is a nondegenerate basic feasible solution and that $\bar{c}_j < 0$ for some j. Since the reduced cost of a basic variable is always zero, x_j must be a nonbasic variable. Since x is nondegenerate, the jth basic direction is a feasible direction of cost decrease. By moving in that direction we obtain feasible solutions whose cost is less than that of x, so x is not optimal. $\square$

Definition 1.4. A basis matrix B is said to be **optimal** if:

(a) $B^{-1}b \geq 0$,

(b) $\bar{c}^T = c^T - c_B^T B^{-1} A \geq 0$.

2 Development of the Simplex Method

Throughout this section, we assume that all basic feasible solutions are nondegenerate. We start moving away from x along the direction d, that is on the line $x + \theta d$, where $\theta > 0$, then we must find θ^* satisfying:

$$\theta^* = \max\{\theta > 0 \mid x + \theta d \in P\}.$$

At the point where nonnegativity constrain is about to be violated, we have

(a) If $d \geq 0$, then $x + \theta d \geq 0$ for all $\theta \geq 0$, we have $\theta^* = +\infty$.

(b) If $d_i \leq 0$, the constraint $x_i + \theta d_i \geq 0$ becomes $\theta \leq -x_i/d_i$. Then we have

$$\theta^* = \min_{\{i|d_i<0\}} \left(-\frac{x_i}{d_i} \right).$$

Theorem 2.1.

(a) The columns of $A_{B(i)}, i \neq l$ and A_j are linearly independent and therefore $\bar{B}$ is a basis matrix.

(b) The vector $y = x + \theta^* d$ is a basic feasible solution associated with the basis matrix $\bar{B}$.

Proof. For (a), if the vectors $A_{\bar{B}(i)}, i = 1, \ldots, m$ are linearly dependent, then there exist coefficients $\lambda_1, \ldots, \lambda_m$ not all of them are zero, such that

$$\sum_{i=1}^{m} \lambda_i A_{\bar{B}(i)} = 0,$$

which implies that

$$\sum_{i=1}^{m} \lambda_i B^{-1} A_{\bar{B}(i)} = 0$$

and $B^{-1}A_{\bar{B}(i)}$ are also linearly dependent. To show the contradiction, we need to prove that $B^{-1}A_{B(i)}, i \neq l$ and $B^{-1}A_j$ are linearly independent. Since $A_{B(i)}$ is the ith column of B, it follows that $B^{-1}A_{B(i)}$ are all the unit vectors except the lth column. On the other hand, $B^{-1}A_j = -d_B$. Its lth entry, $-d_{B(l)}$ is nonzero by the definition of l. Thus, $-B^{-1}A_j$ is linearly independent from the unit vectors $B^{-1}A_{B(i)}, i \neq l$.
For (b), we have $y_i = 0$ for $i \neq \bar{B}(1), \ldots, \bar{B}(m)$. Also, $A_{\bar{B}(1)}, \ldots, A_{\bar{B}(m)}$ are linearly independent. It follows that y is a basic feasible solution associated with the basis matrix $\bar{B}$. $\square$

An Iteration of the Simplex Method

1. Start with a basic columns $A_{B(1)}, \ldots, A_{B(m)}$ and an associated basic feasible solution x.

2. Compute the reduced costs $\bar{c}_j = c_j - c_B^T B^{-1} A_j$ for all non-basic indices j. If all are positive, then terminates; else, choose some j for which $\bar{c}_j < 0$.

3. Compute $u = B^{-1}A_j$, if no component of u is positive, then optimal cost is $-\infty$, algorithm terminates.

4. Let
$$\theta^* = \min_{\{i=1,\ldots,m \mid u_i>0\}} \frac{x_{B(i)}}{u_i}.$$

5. Form a new basis by replacing $A_{B(l)}$ with A_j. Denote new BFS y as $y_{B(i)} = x_{B(i)} - \theta^* u_i, i \neq l$.

Theorem 2.2. *Assume that the feasible set is nonempty and that every basic feasible solution is nondegenerate. Then the simplex method terminates after a finite number of iterations. At termination, there are the following two possibilities:*

(a) We have an optimal basis B and an associated basic feasible solution which is optimal.

(b) We have found a vector d satisfying $Ad = 0, d \geq 0$, and $c^T d < 0$, and the optimal cost is $-\infty$.

Naive Implementation

1. Given starting basic indices $B(1), \ldots, B(m)$, compute $p^T = c_B^T B^{-1}$;
2. Compute the reduced cost $\bar{c}_j = c_j - c_B^T B^{-1} A_j = c_j - p^T A_j$;
3. Select an out basis column A_j and get $u = B^{-1}A_j$;
4. Run one iteration of Simplex Method described above.

Remark. *We need $O(m^3)$ arithmetic operations to solve the systems $p^T B = c_B^T$ and $Bu = A_j$; computing the reduced costs need $O(mn)$ flops, so the total computational effort per iteration is $O(m^3 + mn)$.*

Revised Simplex Method

Let
$$B = \begin{bmatrix} A_{B(1)} & \cdots & A_{B(m)} \end{bmatrix}$$
and
$$\bar{B} = \begin{bmatrix} A_{B(1)} & \cdots & A_{B(l-1)} & A_j & A_{B(l+1)} & \cdots & A_{B(m)} \end{bmatrix}$$
be the basis matrix at the beginning of the next iteration. Notice that $B^{-1}B = I$, so $B^{-1}A_{B(i)}$ is the ith unit vector. Under this observation, we have
$$B^{-1}\bar{B} = \begin{bmatrix} 1 & & u_1 & & \\ & \ddots & \vdots & & \\ & & u_l & & \\ & & \vdots & \ddots & \\ & & u_m & & 1 \end{bmatrix},$$

then we apply the following elementary row operations:

1. For each $i \neq l$, we add lth row times $-u_i/u_l$ to the ith row.

2. Divide lth row by u_l, this replaces u_l by one.

By following the procedures above, we have

$$QB^{-1}\bar{B} = I,$$

this yields

$$QB^{-1} = \bar{B}^{-1},$$

thus we can compute $\bar{B}^{-1}$ easily with B^{-1}.

An Iteration of the Revised Simplex Method

1. We start with basis $A_{B(1)}, \ldots, A_{B(m)}$, an associated basic feasible solution x and the inverse B^{-1} of the basis matrix.

2. Compute $p^T = c_B B^{-1}$ and then compute reduced costs $\bar{c}_j = c_j - p^T A_j$. If all nonnegative, then terminates; otherwise choose some j for which $\bar{c}_j < 0$.

3. Compute $u = B^{-1}A_j$, if no component of u is positive, the optimal cost is $-\infty$, and algorithm terminates.

4. Let
$$\theta^* = \min_{\{i=1,\ldots,m \mid u_i>0\}} \frac{x_{B(i)}}{u_i}.$$

5. Form a new basis by replacing $A_{B(l)}$ with A_j, and the values of new basic variables are $y_j = \theta^*$ and $y_{B(i)} = x_{B(i)} - \theta^* u_i, i \neq l$.

6. Form the $m \times (m+1)$ matrix $[B^{-1} \mid u]$. Make the last column equal to the unit vector e_l. The first m columns of the result is the matrix $\bar{B}^{-1}$.

The Full Tableau Implementation

An Iteration of the Full Tableau Implementation

1. Starts with the tableau associated with a basis matrix B and the corresponding basic feasible solution x.

2. Examine the reduced costs in the zeroth row of the tableau. If all are nonnegative, the current extreme point is optimal and terminates, otherwise choose some j for which $\bar{c}_j < 0$.

3. Consider jth colunm of the tableau, if no component is positive, then it's unbounded.

4. For each u_i that is positive, compute $x_{B(i)}/u_i$. Let l be the index of a row that corresponds to the smallest ratio. The column $A_{B(l)}$ exits the basis and A_j enters the basis.

5. Update the tableau.

3 Anticycling

3.1 Lexicography

Definition 3.1. A vector $u \in \mathbb{R}^n$ is said to be **lexicographically larger** than another vector $v \in \mathbb{R}^n$ if $u \neq v$ and the first nonzero component of $u - v$ is positive. Symbolically, we write

$$u \overset{L}{>} v.$$

Lexicographic Pivoting Rule

1. Choose an entering column A_j arbitrarily, as long as its reduced cost $\bar{c}_j$ is negative. Let $u = B^{-1}A_j$ be the jth column of the tableau.

2. For each i with $u_i > 0$, divide the ith row of the tableau by u_i and choose the lexicographically smallest row. If row l is lexicographically smallest, then the lth basic variable $x_{B(l)}$ exits the basis.

Also, we have the following theorem without proof.

Theorem 3.2. *Suppose that the simplex algorithm starts with all the rows in the simplex tableau, other than the zeroth row, lexicographically positive. Suppose that the lexicographic pivoting rule is followed. Then:*

(a) Every row of the simplex tableau, other than the zeroth row, remains lexicographically positive throughout the algorithm.

(b) The zeroth row strictly increases lexicographically at each iteration.

(c) The simplex method terminates after a finite number of iterations.

3.2 Bland's Rule

1. Find the smallest j for which the reduced cost $\bar{c}_j$ is negative and have the column A_j enter the basis.

2. Out of all variables x_j that are tied in the test for choosing an exiting variable, select the one with the smallest value of i.

4 Finding an Initial Basic Feasible Solution

Use the simplex method to solve the auxiliary problem

$$\begin{aligned} \min \quad & y_1 + y_2 + \cdots + y_m \\ \text{s.t.} \quad & Ax + y = b \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

By solving this auxiliary problem, we can develop the two phase simplex method.

Phase 1

1. By multiplying some of the constraints by -1 so that $b \geq 0$.
2. Solve the auxiliary problem, if the optimal value is positive, then the original problem is infeasible and the algorithm terminates.
3. If the optimal cost in the auxiliary problem is zero, a feasible solution to the original problem has been found. If no artificial basis is in the final basis, then return the BFS; otherwise drive the artificial variable out of the basis.

Phase 2

1. Let the final basis and tableau obtained from Phase 1 be the initial basis and tableau for Phase 2.
2. Compute the reduced costs of all variables for this initial basis, using the cost coefficients of the original problem.
3. Apply the simplex method to the original problem.