

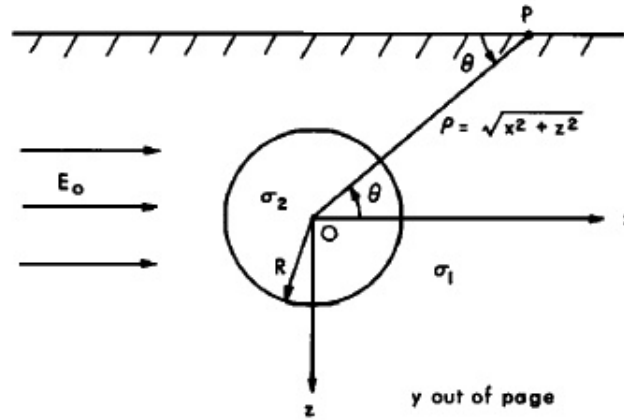
ANALYTICAL RESULTS AND PHYSICAL UNDERSTANDING

TEAM C

1. UNIFORM ELECTRIC FIELD ILLUMINATING A SPHERE IN A UNIFORM EARTH (ANALYTIC SOLUTION, REFERENCE)

Let consider a resistive uniform half-space, of conductivity σ_1 enclosing a conductive sphere σ_2 . Let assume a uniform, unidirectional static electric field E_0 going through this half-space.

FIGURE 1. Uniform electric field illuminating a sphere in a uniform earth



2. MAXWELL EQUATIONS

In this case, we need:

$$(2.1) \quad \begin{aligned} \nabla \times E &= 0 \\ J &= \sigma E. \end{aligned}$$

The first equation gives $E = -\nabla V$.

The primary field E_0 can then be expressed by:

$$(2.2) \quad E^p_0 = -\frac{dV^p}{dx}.$$

Assuming a primary potential null at the origin:

$$(2.3) \quad V^p = E_0 x = E_0 r \cos \theta.$$

As the primary potential respects $\nabla^2 V = 0$, as only a dependence in x direction, the anomalous or secondary field can be expressed as (using spherical coordinates):

$$(2.4) \quad V^s = (Ar + Br^{-2}) \cos \theta.$$

If we assume finite values of the potential everywhere, we can divide the anomalous potential in two domain:

$$(2.5) \quad \begin{aligned} V_e^s &= Br^{-2} \cos \theta & \text{if } r > R, \\ V_i^s &= Ar \cos \theta & \text{if } r < R. \end{aligned}$$

The total external potential is then:

$$(2.6) \quad V_e = V_e^s + V^p = (-E_0 r + Br^{-2}) \cos \theta.$$

On the surface of the sphere, both the normal current density and potential have to be continuous across the interface.

Using the continuity of current density, we got:

$$(2.7) \quad \begin{aligned} \sigma_1 E_e &= \sigma_2 E_i \\ \sigma_1 \frac{dV_e}{dr} &= \sigma_2 \frac{dV_e}{dr}. \end{aligned}$$

$$(2.8) \quad 2\sigma_1 BR^{-3} + \sigma_1 E_0 = -\sigma_2 A$$

Using the continuity of potential, we got:

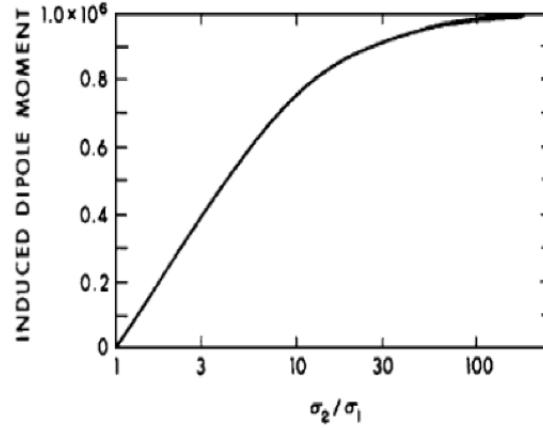
$$(2.9) \quad \begin{aligned} V_e &= V_i^s. \\ -E_0 R + BR^{-2} &= AR. \end{aligned}$$

From equations 2.8 and 2.9, we get:

$$(2.10) \quad \begin{aligned} A &= -\frac{3\sigma_1}{\sigma_2 + 2\sigma_1} E_0 \\ B &= E_0 R^3 \frac{\sigma_2 - \sigma_1}{\sigma_2 + 2\sigma_1}. \end{aligned}$$

From equations (10) and (12), we get:

FIGURE 2. Induced Dipole moment P in a sphere in a uniform earth



$$A = -\frac{3\sigma_1}{\sigma_2+2\sigma_1}E_0 \quad (13)$$

$$B = E_0 R^3 \frac{\sigma_2 - \sigma_1}{\sigma_2 + 2\sigma_1} \quad (14)$$

And the anomalous electric field is:

$$(2.11) \quad \mathbf{E}_s = -\nabla V_e^s = \mathbf{E}_0 R^3 \frac{\sigma_2 - \sigma_1}{\sigma_2 - 2\sigma_1} \frac{(2x^2 - y^2 - z^2)\mathbf{u}_x + 3xy\mathbf{u}_y + 3xz\mathbf{u}_z}{r^5}.$$

3. CONTINUITY OF CURRENT AND CHARGE ACCUMULATION

We assume to be here in a steady state with direct current. The current entering a cylinder through an interface as in figure 4 consists both in tangential and normal components.

As the cylinder height is collapsed to zero, we can write the normal component as:

$$(3.1) \quad I = \mathbf{J}_1 \cdot \mathbf{n} \Delta a, \text{ or as } I = \mathbf{J}_2 \cdot \mathbf{n} \Delta a.$$

Note: Otherwise in steady state we would have an infinite built up of charges at the interface. Then

$$(3.2) \quad \mathbf{J}_2 \cdot \mathbf{n} = \mathbf{J}_1 \cdot \mathbf{n},$$

so we have

$$(3.3) \quad \mathbf{J}_1^n = \mathbf{J}_2^n.$$

FIGURE 3. Anomalous field of a sphere in a uniform earth illuminated by a uniform electric field

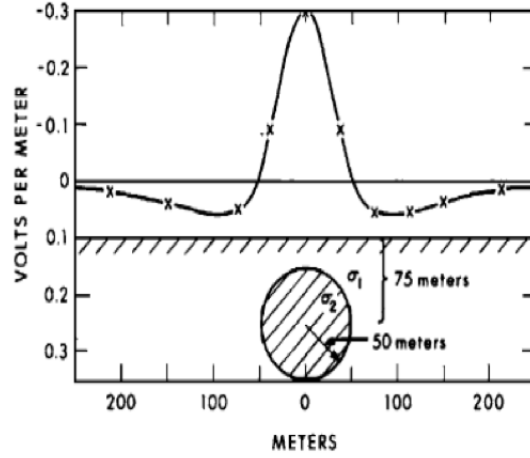
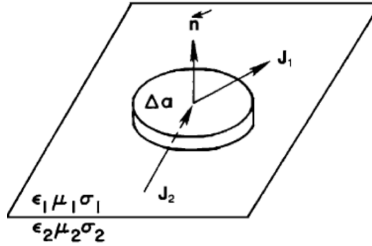


FIGURE 4. Uniform electric field illuminating a sphere in a uniform earth



Note: This is only true in a direct current case in a steady state. It appears to be satisfactory up to 10^5 Hz, as long as displacement currents can be considered negligible.

4. CHARGES, COULOMB'S LAW AND POTENTIALS.

Electric charge produces an electric potential; the Coulomb's electrostatic potential is

$$(4.1) \quad V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}.$$

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5. ANOMALOUS CURRENTS AND ELECTRIC FIELDS

Anomalous current density is defined as

$$(5.1) \quad \mathbf{J}_a = \sigma_a \mathbf{E}.$$

Here \mathbf{E} is total electric field and σ_a is the difference between wholespace conductivity σ_1 and conductivity of the target σ_2 : $\sigma_a = \sigma_2 - \sigma_1$.

6. DC APP FOR LOOKING AT CURRENTS, CHARGES ETC WITH A CURRENT SOURCE AT THE SURFACE

7. ANALYTIC SOLUTION FOR A BURIED SPHERE IN A UNIFORM SPACE

Extensive studies have been carried out concerning the solution for a buried spherical body in a homogeneous earth due to a point source on the earth's surface. We summarize a general procedure to this problem based on image method. A bispherical coordinate system is usually used considering the boundary conditions on both the spherical and planar surface. Figure 5 shows the bispherical system for a conducting sphere with conductivity ρ_1 buried in a uniform earth with conductivity ρ_0 , A is the current point located on the x axis with bispherical coordinate (r_A, θ_A, ϕ_A) , here $r_A = D, \theta_A = \arccos(h_0/D), \phi_A = \pi$. D is the distance between the current point A and center of buried sphere, R is the distance between A and an arbitrary potential site M . For convenience, the earth is divided into exterior region 1 and interior region 2 to the sphere. In a bispherical coordinate system, the total potential V can be expressed as the sum of a primary potential V_p , a secondary potential V_s caused by the sphere, and a virtual potential V_i due to the image of the spherical body. The potentials for sites in region 1 and 2 are

$$(7.1) \quad V_1 = V_p + V_{s1} + V_i$$

$$(7.2) \quad V_2 = V_p + V_{s2} + V_i$$

In regions free of charge, the potential is governed by the Laplace's equation

$$(7.3) \quad \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

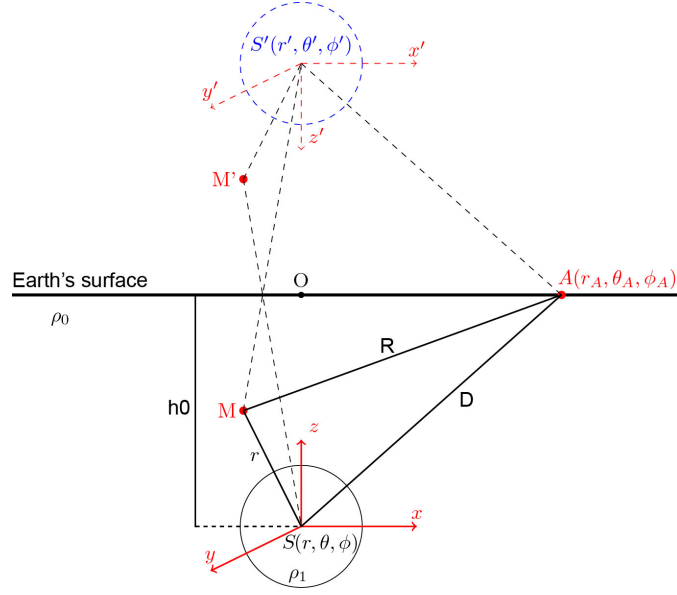


FIGURE 5. Sketch of a spherical body in a uniform earth.

The general solution for potential V can be obtained by applying the method of separation of variables

(7.4)

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^n (A_{mn} r^n + B_{mn} r^{-n-1}) \times [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos \theta$$

Considering the boundary condition at surface of sphere

$$(7.5a) \quad \frac{\partial V_1}{\partial r} \Big|_{r=h_0} = 0$$

$$(7.5b) \quad V_1 = V_2 \quad \text{for } r = r_0$$

$$(7.5c) \quad \frac{1}{\rho_1} \frac{\partial V_1}{\partial r} = \frac{1}{\rho_2} \frac{\partial V_2}{\partial r} \Big|_{r=r_0}$$

where r_0 is the radius of the sphere. Applying boundary conditions (7.5a) to (7.4), we obtains

$$(7.6) \quad V_{s1} = \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^{n+1} \sum_{m=0}^n [A_{mn} \cos(m\phi) + B_{mn} \sin(m\phi)] P_n^m \cos \phi$$

$$(7.7) \quad V_{s2} = \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n \sum_{m=0}^n [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos \phi$$

where A_{mn}, B_{mn}, C_{mn} and D_{mn} are unknown coefficients. P_n^m is the Legendre function of the first kind. The primary potential may be also expanded in bispherical coordinates as

(7.8)

$$V_p = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} \{ P_n \cos \theta_A P_n \cos \theta + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} [\cos(m\phi_A) \cos(m\phi) + \sin(m\phi_A) \sin(m\phi)] \times P_n^m \cos \theta_A P_n^m \cos \theta \}$$

when $r > D$, the position of r and D in above equation should be exchanged.

The virtual potential due to image of the sphere S' is

$$(7.9) \quad V_i(r, \theta, \phi) = \sum_{n=0}^{\infty} \left(\frac{r_0}{r_i} \right)^{n+1} \sum_{m=0}^n [A_{mn} \cos(m\phi_i) + B_{mn} \sin(m\phi_i)] P_n^m \cos \phi_i$$

where (r_i, θ_i, ϕ_i) is for spherical coordinate S' . Considering the equivalence of potential site $M(r, \theta, \phi)$ in spherical coordinate S with virtual potential site $M'(r_i, \theta_i, \phi_i)$ in coordinate S' , we have

$$(7.10a) \quad P_n^m(\cos \theta_i) r_i^{-n-1} = \sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r_i'^k}{h_0^{n+k+1}} P_k^m \cos \theta_i'$$

$$(7.10b) \quad \phi = \phi_i'$$

Substitute equation (7.10) into equation (7.9),

(7.11)

$$V_i = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_0^{n+1} [A_{mn} \cos(m\phi) + B_{mn} \sin(m\phi)] \times \sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r^k}{h_0^{n+k+1}} P_k^m \cos \phi$$

Rearranging equations (7.6)(7.7)(7.8)(7.11), we have

$$(7.12) \quad V_1 = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} \left(\frac{r_0}{r} \right)^{n+1} Y_m(\theta, \phi)$$

$$(7.13) \quad V_2 = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} \left(\frac{r_0}{r} \right)^n Y_m'(\theta, \phi)$$

where

$$(7.14) \quad Y_m(\theta, \phi) = \sum_{m=0}^{\infty} [A_{mn} \cos(m\phi) + B_{mn} \sin(m\phi)] P_n^m \cos(\phi)$$

$$(7.15) \quad Y_m'(\theta, \phi) = \sum_{m=0}^{\infty} [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos(\phi)$$

and

(7.16)

$$L_m(\theta_A, \phi_A, \theta, \phi) = P_n \cos \theta_A P_n \cos \theta + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} [\cos(m\phi_A) \cos(m\phi) + \sin(m\phi_A) \sin(m\phi)] \times P_n^m \cos \theta_A P_n^m \cos \theta$$