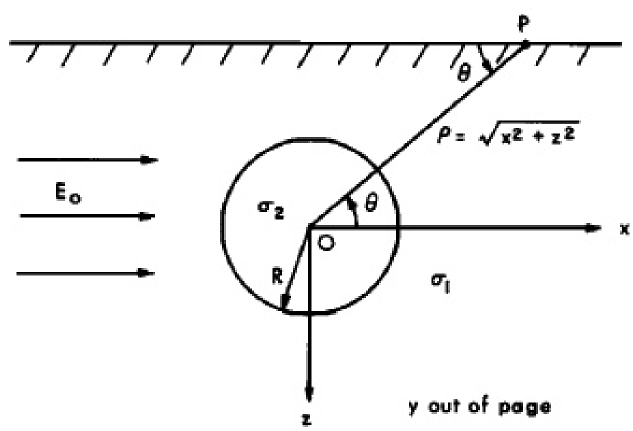
ANALYTICAL RESULTS AND PHYSICAL UNDERSTANDING

TEAM C

1. Uniform electric field illuminating a sphere in a uniform earth (analytic solution, reference)

Let consider a resistive uniform half-space, of conductivity $\sigma_1$enclosing a conductive sphere σ_2. Let assume a uniform, unidirectional static electric field E_0 going through this half-space.

Figure 1. Uniform electric field illuminating a sphere in a uniform earth



2. Maxwell equations

In this case, we need:

$$
\begin{aligned}
\nabla \times E &= 0 \\
J &= \sigma E.
\end{aligned}
\tag{2.1}
$$

The first equation gives $E = -\nabla V$.

The primary field E_0 can then be expressed by:

$$E^p{}_0 = -\frac{dV^p}{dx}. \tag{2.2}$$

Assuming a primary potential null at the origin:

$$V^p = E_0 x = E_0 r \cos\theta. \tag{2.3}$$

As the primary potential respects $\nabla^2 V = 0$, as only a dependence in x direction, the anomalous or secondary field can be expressed as (using spherical coordinates):

$$V^s = (Ar + Br^{-2})\cos\theta. \tag{2.4}$$

If we assume finite values of the potential everywhere, we can divide the anomalous potential in two domain:

$$\begin{aligned} V_e^s &= Br^{-2}\cos\theta \quad \text{if } r > R, \\ V_i^s &= Ar\cos\theta \qquad \text{if } r < R. \end{aligned} \tag{2.5}$$

The total external potential is then:

$$V_e = V_e^s + V^p = (-E_0 r + Br^{-2})\cos\theta. \tag{2.6}$$

On the surface of the sphere, both the normal current density and potential have to be continuous across the interface.

Using the continuity of current density, we got:

$$\begin{aligned} \sigma_1 E_e &= \sigma_2 E_i \\ \sigma_1 \frac{dV_e}{dr} &= \sigma_2 \frac{dV_e}{dr}. \end{aligned} \tag{2.7}$$

$$2\sigma_1 BR^{-3} + \sigma_1 E_0 \tag{2.8}$$

Using the continuity of potential, we got:

$$\begin{aligned} V_e &= V^s{}_i. \\ -E_0 R + BR^{-2} &= AR. \end{aligned} \tag{2.9}$$

From equations 2.8 and 2.9, we get:

$$\begin{aligned} A &= -\frac{3\sigma_1}{\sigma_2 + 2\sigma_1} E_0 \\ B &= E_0 R^3 \frac{\sigma_2 - \sigma 1}{\sigma_2 + 2\sigma_1}. \end{aligned} \tag{2.10}$$

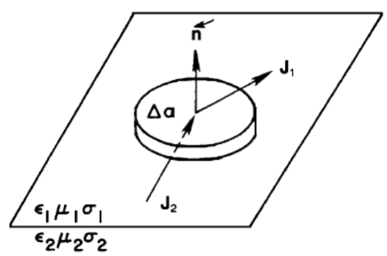
And the anomalous electric field is:

(2.11)
$$\mathbf{E_s} = -\nabla \mathbf{V^s}_\mathbf{e} = \mathbf{E_0 R^3} \frac{\sigma_\mathbf{2} - \sigma_\mathbf{1}}{\sigma_\mathbf{2} - \mathbf{2}\sigma_\mathbf{1}} \frac{(\mathbf{2x^2 - y^2 - z^2})\mathbf{u_x} + \mathbf{3xyu_y} + \mathbf{3xzu_z}}{\mathbf{r^5}}.$$

3. Continuity of current and charge accumulation

We assume to be here in a steady state with direct current. The current entering a cylinder through an interface as in figure 2 consists both in tangential and normal components.

Figure 2. Uniform electric field illuminating a sphere in a uniform earth



As the cylinder height is collapsed to zero, we can write the normal component as:

(3.1)
$$I = J_1 \cdot \mathbf{n}\Delta a, \text{or as} I = J_2 \cdot \mathbf{n}\Delta a.$$

Note: Otherwise in steady state we would have an infinite built up of charges at the interface. Then

(3.2)
$$J_2 \cdot \mathbf{n} = J_1 \cdot \mathbf{n},$$

so we have

(3.3)
$$\mathbf{J_1}^\mathbf{n} = \mathbf{J_2}^\mathbf{n}.$$

Note: This is only true in a direct current case in a steady state. It appears to be satisfactory up to 10^5 Hz, as long as displacement currents can be considered negligeable.

4. Charges, Coulomb's law and potentials.

Electric charge produces an electric potential; the Coulomb's electrostatic potential is

(4.1)
$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}.$$

Electric charge produces an electric potential: the Coulomb's electrostatic potential is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}. \tag{4.2}$$

5. Anomalous currents and electric fields

Anomalous current density is defined as

$$\mathbf{J}_a = \sigma_a \mathbf{E}. \tag{5.1}$$

Here $\mathbf{E}$ is total electric field and σ_a is the difference between wholespace conductivity σ_1 and conductivity of the target σ_2: $\sigma_a = \sigma_2 - \sigma_1$.

6. DC app for looking at currents, charges etc with a current source at the surface

7. Analytic solution for a buried sphere in a uniform space

Extensive studies have been carried out concerning the solution for a buried spherical body in a homogeneous earth due to a point source on the earth's surface. We summarize a general procedure to this problem based on image method. A bispherical coordinate system is usually used considering the boundary conditions on both the spherical and planar surface. Figure 3 shows the bispherical system for a conducting sphere with conductivity ρ_1 buried in a uniform earth with conductivity ρ_0, A is the current point located on the x axis with bispherical coordinate (r_A, θ_A, ϕ_A), here $r_A = D, \theta_A = arcos(h_0/D), \phi_A = \pi$. D is the distance between the current point A and center of buried sphere, R is the distance between A and an arbitrary potential site M. For convenience, the earth is divided into exterior region 1 and interior region 2 to the sphere. In a bispherical coordiante system, the total potential V can be expressed as the sum of a primary potential V_p, a secondary potential V_s caused by the sphere, and a virtual potential V_i due to the image of the spherical body. The potentials for sites in region 1 and 2 are

$$V_1 = V_p + V_{s1} + V_i \tag{7.1}$$

$$V_2 = V_p + V_{s2} + V_i \tag{7.2}$$

In regions free of charge, the potential is governed by the Laplace's equation

$$\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{7.3}$$

The general solution for potential V can be obtained by applying the method of separation of variables

(7.4)

$$V = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(A_{mn}r^n + B_{mn}r^{-n-1}) \times [C_{mn}\cos(m\phi) + D_{mn}\sin(m\phi)]P_n^m\cos\theta$$

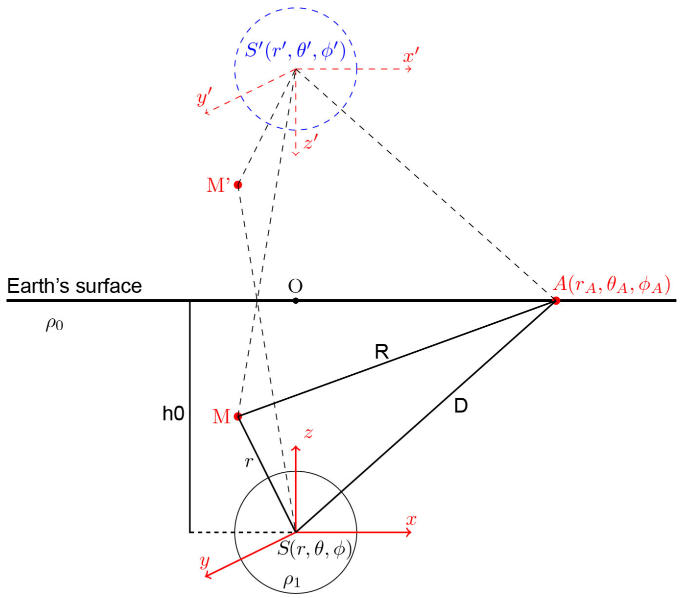


FIGURE 3. Sketch of a spherical body in a uniform earth.

Considering the boundary condition at surface of sphere

$$\frac{\partial V_1}{\partial r}|_{r=h_0} = 0 \tag{7.5a}$$

$$V_1 = V_2 \quad \text{for } r = r_0 \tag{7.5b}$$

$$\frac{1}{\rho_1}\frac{\partial V_1}{\partial r} = \frac{1}{\rho_2}\frac{\partial V_2}{\partial r}|_{r=r_0} \tag{7.5c}$$

where r_0 is the radius of the sphere. Applying boundary conditions (7.5a) to (7.4), we obtains

$$V_{s1} = \sum_{n=0}^{\infty}(\frac{r_0}{r})^{n+1}\sum_{m=0}^{n}[A_{mn}\cos(m\phi) + B_{mn}\sin(m\phi)]P_n^m \cos\phi \tag{7.6}$$

$$V_{s2} = \sum_{n=0}^{\infty}(\frac{r_0}{r})^{n}\sum_{m=0}^{n}[C_{mn}\cos(m\phi) + D_{mn}\sin(m\phi)]P_n^m \cos\phi \tag{7.7}$$

where A_{mn}, B_{mn}, C_{mn} and D_{mn} are unknown coefficients. P_n^m is the Legendre function of the first kind. The primary potential may be also expanded in bispherical coordinates as

(7.8)

$$V_p = \frac{\rho_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{r^n}{D^{n+1}}\{P_n\cos\theta_A P_n\cos\theta + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}[\cos(m\phi_A)\cos(m\phi)$$
$$+ \sin(m\phi_A)\sin(m\phi)] \times P_n^m\cos\theta_A P_n^m\cos\theta\}$$

when $r > D$, the position of r and D in above equation should be exchanged.

The virtual potential due to image of the sphere S' is

$$V_i(r,\theta,\phi) = \sum_{n=0}^{\infty}(\frac{r_0}{r_i})^{n+1}\sum_{m=0}^{n}[A_{mn}\cos(m\phi_i) + B_{mn}\sin(m\phi_i)]P_n^m\cos\phi_i \tag{7.9}$$

where(r_i,θ_i,ϕ_i) is for spherical coordinate S'. Considering the equivalence of potential site $M(r,\theta,\phi)$ in spherical coordinate S with virtual potential site $M'(r_i,\theta_i,\phi_i)$ in coordinate S', we have

$$P_n^m(\cos\theta_i)r_i^{-n-1} = \sum_{k=m}^{\infty}\frac{(n+k)!}{(n-m)!(m+k)!}\frac{r_i'^k}{h_0^{n+k+1}}P_k^m\cos\theta_i' \tag{7.10a}$$

$$\phi = \phi_i' \tag{7.10b}$$

Substitute equation (7.10) into equation (7.9),

(7.11)

$$V_i = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}r_0^{n+1}[A_{mn}\cos(m\phi)+B_{mn}\sin(m\phi)]\times\sum_{k=m}^{\infty}\frac{(n+k)!}{(n-m)!(m+k)!}\frac{r^k}{h_0^{n+k+1}}P_k^m\cos\phi$$

Rearranging equations (7.6)(7.7)(7.8)(7.11), we have

$$V_1 = \frac{\rho_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{r^n}{D^{n+1}}L_m(\theta_A,\phi_A,\theta,\phi) + V_i + \sum_{n=0}^{\infty}(\frac{r_0}{r})^{n+1}Y_m(\theta,\phi) \tag{7.12}$$

$$V_2 = \frac{\rho_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{r^n}{D^{n+1}}L_m(\theta_A,\phi_A,\theta,\phi) + V_i + \sum_{n=0}^{\infty}(\frac{r_0}{r})^{n}Y_m'(\theta,\phi) \tag{7.13}$$

where

$$Y_m(\theta,\phi) = \sum_{m=0}^{\infty}[A_{mn}\cos(m\phi) + B_{mn}\sin(m\phi)]P_n^m\cos(\phi) \tag{7.14}$$

$$Y_m'(\theta,\phi) = \sum_{m=0}^{\infty}[C_{mn}\cos(m\phi) + D_{mn}\sin(m\phi)]P_n^m\cos(\phi) \tag{7.15}$$

and

(7.16)

$$L_m(\theta_A,\phi_A,\theta,\phi) = P_n\cos\theta_A P_n\cos\theta + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}[\cos(m\phi_A)\cos(m\phi)+ \\ \sin(m\phi_A)\sin(m\phi)]\times P_n^m\cos\theta_A P_n^m\cos\theta$$