ANALYTICAL RESULTS AND PHYSICAL UNDERSTANDING

TEAM C

[redacted]< HEAD

Uniform electric field illuminating a sphere in a uniform earth (analytic solution, reference)

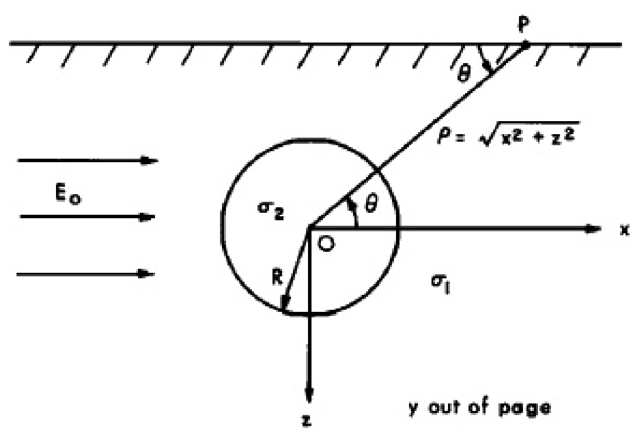
[redacted]< HEAD ======= [redacted]> 32636add57830d02d6c2cf0eb89e5ae979a27b1a
=======

1. Uniform electric field illuminating a sphere in a uniform earth (analytic solution, reference)

[redacted]> 9a1f053371e64e01759b9956df188d9d5181280e

Let consider a resistive uniform half-space, of conductivity $\sigma_1$enclosing a conductive sphere σ_2. Let assume a uniform, unidirectional static electric field E_0 going through this half-space.

Figure 1. Uniform electric field illuminating a sphere in a uniform earth



2. Maxwell equations

< HEAD In this case, we need:
< HEAD $\nabla \times E = 0$,so $E = -\nabla V$
$J = \sigma E$
The primary field E_0 can then be expressed by:
$E^p{}_0 = -\frac{dV^p}{dx}$
Assuming a primary potential null at the origin:
$V^p = E_0 x = E_0 r cos\theta$
As [...], the anomalous or secondary field is expressed as:
$V^s = (Ar + Br^{-2}) cos\theta$

======= ======= > 9a1f053371e64e01759b9956df188d9d5181280e

In this case, we need:

$$\begin{aligned} \nabla \times E &= 0 \\ J &= \sigma E. \end{aligned} \tag{2.1}$$

The first equation gives $E = -\nabla V$.

The primary field E_0 can then be expressed by:

$$E^p{}_0 = -\frac{dV^p}{dx}. \tag{2.2}$$

Assuming a primary potential null at the origin:

$$V^p = E_0 x = E_0 r \cos\theta. \tag{2.3}$$

As the primary potential respects $\nabla^2 V = 0$, as only a dependence in x direction, the anomalous or secondary field can be expressed as (using spherical coordinates):

$$V^s = (Ar + Br^{-2}) \cos\theta. \tag{2.4}$$

If we assume finite values of the potential everywhere, we can divide the anomalous potential in two domain:

$$\begin{aligned} V_e^s &= Br^{-2} \cos\theta \quad && \text{if } r > R, \\ V_i^s &= Ar \cos\theta && \text{if } r < R. \end{aligned} \tag{2.5}$$

The total external potential is then:

$$V_e = V_e^s + V^p = (-E_0 r + Br^{-2}) \cos\theta. \tag{2.6}$$

On the surface of the sphere, both the normal current density and potential have to be continuous across the interface.

Using the continuity of current density, we got:

$$\begin{aligned} \sigma_1 E_e &= \sigma_2 E_i \\ \sigma_1 \frac{dV_e}{dr} &= \sigma_2 \frac{dV_e}{dr}. \end{aligned} \tag{2.7}$$

[redacted]> 32636add57830d02d6c2cf0eb89e5ae979a27b1a

$$2\sigma_1 B R^{-3} + \sigma_1 E_0 = -\sigma_2 A \tag{2.8}$$

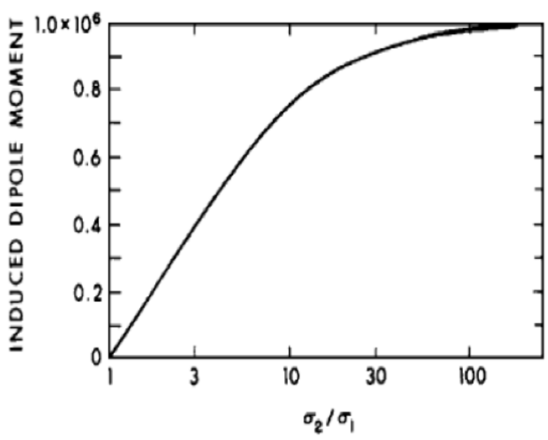
Using the continuity of potential, we got:

$$\begin{aligned} V_e &= V^s{}_i. \\ -E_0 R + B R^{-2} &= AR. \end{aligned} \tag{2.9}$$

From equations 2.8 and 2.9, we get:

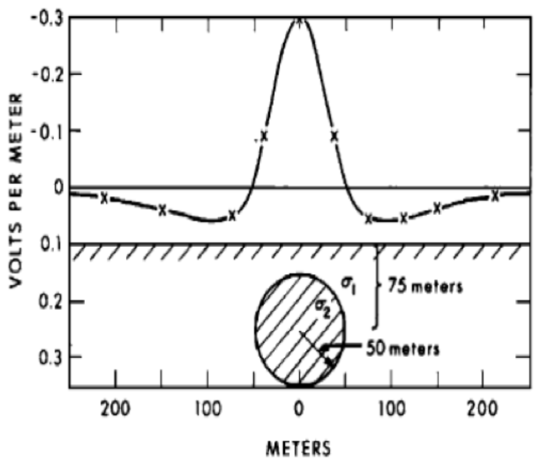
$$\begin{aligned} A &= -\frac{3\sigma_1}{\sigma_2 + 2\sigma_1} E_0 \\ B &= E_0 R^3 \frac{\sigma_2 - \sigma 1}{\sigma_2 + 2\sigma_1}. \end{aligned} \tag{2.10}$$

FIGURE 2. Induced Dipole moment P in a sphere in a uniform earth



And the anomalous electric field is:

FIGURE 3. Anomalous field of a sphere in a uniform earth illuminated by an uniform electric field



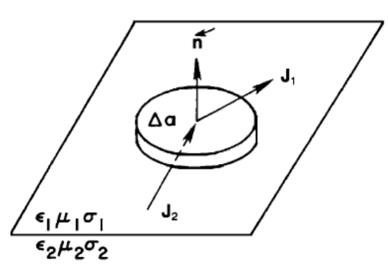
(2.11)

$$\mathbf{E_s} = -\nabla \mathbf{V^s}_\mathbf{e} = \mathbf{E_0 R^3} \frac{\sigma_\mathbf{2} - \sigma_\mathbf{1}}{\sigma_\mathbf{2} - \mathbf{2}\sigma_\mathbf{1}} \frac{\mathbf{(2x^2 - y^2 - z^2)u_x + 3xyu_y + 3xzu_z}}{\mathbf{r^5}}.$$

3. CONTINUITY OF CURRENT AND CHARGE ACCUMULATION

We assume to be here in a steady state with direct current. The current entering a cylinder through an interface as in figure 4 consists both in tangential and normal components.

FIGURE 4. Uniform electric field illuminating a sphere in a uniform earth



As the cylinder height is collapsed to zero, we can write the normal component as:

$$I = J_1 \cdot \mathbf{n}\Delta a, \text{or as} I = J_2 \cdot \mathbf{n}\Delta a. \tag{3.1}$$

Note: Otherwise in steady state we would have an infinite built up of charges at the interface. Then

$$J_2 \cdot \mathbf{n} = J_1 \cdot \mathbf{n}, \tag{3.2}$$

so we have

$$\mathbf{J_1}^{\mathbf{n}} = \mathbf{J_2}^{\mathbf{n}}. \tag{3.3}$$

Note: This is only true in a direct current case in a steady state. It appears to be satisfactory up to 10^5 Hz, as long as displacement currents can be considered negligeable.

4. Charges, Coulomb's law and potentials.

Electric charge produces an electric potential; the Coulomb's electrostatic potential is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}. \tag{4.1}$$

Electric charge produces an electric potential: the Coulomb's electrostatic potential is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}. \tag{4.2}$$

5. Anomalous currents and electric fields

Anomalous current density is defined as

$$\mathbf{J}_a = \sigma_a \mathbf{E}. \tag{5.1}$$

Here $\mathbf{E}$ is total electric field and σ_a is the difference between wholespace conductivity σ_1 and conductivity of the target σ_2: $\sigma_a = \sigma_2 - \sigma_1$.
[redacted]< HEAD

[redacted]< HEAD

Charges, Coulomb?s law and potentials.
[redacted]< Updated upstream

Anomalous currents and electric fields ======= Anomalous current density is defined as

$$\mathbf{J}_a = \sigma_a \mathbf{E}.$$

Here $\mathbf{E}$ is total electric field and σ_a is the difference between wholespace conductivity σ_1 and conductivity of the target σ_2: $\sigma_a = \sigma_2 - \sigma_1$.
[redacted]> Stashed changes

DC app for looking at currents, charges etc with a current source at the surface.

6. Analytic solution for a buried sphere in a uniform space

Extensive studies have been carried out concerning the solution for a buried spherical body in a homogeneous earth due to a point source on the earth's surface ((Large, 1971, Merkel and Alexander, 1971, Singh and Espindola, 1976, Snyder and Merkel, 1973, Tang and Yuan, 2012, Van Nostrand and Cook, 1966)). We summarize a general procedure to this problem based on image method. A bispherical coordinate system is usually used considering the boundary conditions on both the spherical and planar surface. Figure 6 shows the bispherical system for a conducting sphere with conductivity ρ_1 buried in a uniform earth with conductivity ρ_0, A is the current point located on the x axis with bispherical coordinate (r_A, θ_A, ϕ_A), here $r_A = D, \theta_A = arcos(h_0/D), \phi_A = \pi$. D is the distance between the current point A and center of buried sphere, R is the distance between A and an arbitrary potential site M. For convenience, the earth is divided

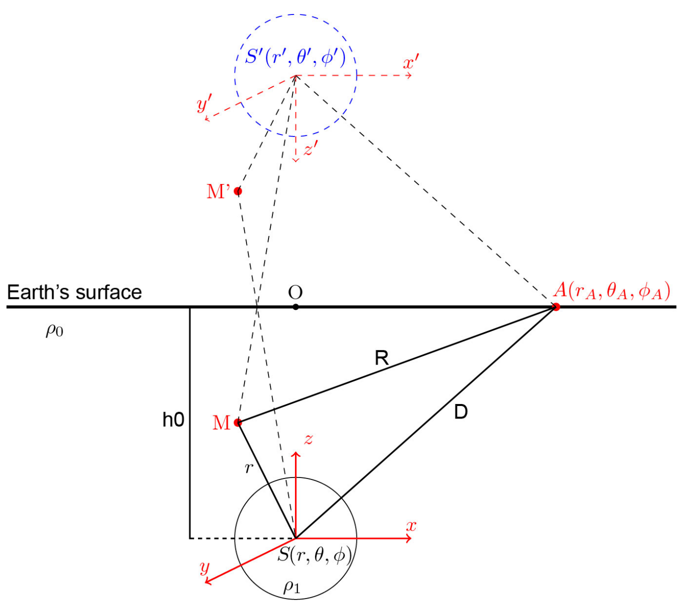


Figure 5. Sketch of a spherical body in a uniform earth.

into exterior region 1 and interior region 2 to the sphere. In a bispherical coordiante system, the total potential V can be expressed as the sum of a primary potential V_p, a secondary potential V_s caused by the sphere, and a virtual potential V_i due to the image of the spherical body. The potentials for sites in region 1 and 2 are

$$V_1 = V_p + V_{s1} + V_i \tag{6.1}$$

$$V_2 = V_p + V_{s2} + V_i \tag{6.2}$$

In regions free of charge, the potential is governed by the Laplace's equation

$$\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{6.3}$$

The general solution for potential V can be obtained by applying the method of separation of variables

(6.4)

$$V = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(A_{mn}r^n + B_{mn}r^{-n-1}) \times [C_{mn}\cos(m\phi) + D_{mn}\sin(m\phi)]P_n^m\cos\theta$$

Considering the boundary condition at surface of sphere

$$\frac{\partial V_1}{\partial r}|_{r=h_0} = 0 \tag{6.5a}$$

$$V_1 = V_2 \quad \text{for } r = r_0 \tag{6.5b}$$

$$\frac{1}{\rho_1}\frac{\partial V_1}{\partial r} = \frac{1}{\rho_2}\frac{\partial V_2}{\partial r}|_{r=r_0} \tag{6.5c}$$

where r_0 is the radius of the sphere. Applying boundary conditions (8.5a) to (8.4), we obtains

$$V_{s1} = \sum_{n=0}^{\infty}(\frac{r_0}{r})^{n+1}\sum_{m=0}^{n}[A_{mn}\cos(m\phi) + B_{mn}\sin(m\phi)]P_n^m\cos\phi \tag{6.6}$$

$$V_{s2} = \sum_{n=0}^{\infty}(\frac{r_0}{r})^{n}\sum_{m=0}^{n}[C_{mn}\cos(m\phi) + D_{mn}\sin(m\phi)]P_n^m\cos\phi \tag{6.7}$$

where A_{mn}, B_{mn}, C_{mn} and D_{mn} are unknown coefficients. P_n^m is the Legendre function of the first kind. The primary potential may be also expanded in bispherical coordinates as

(6.8)

$$V_p = \frac{\rho_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{r^n}{D^{n+1}}\{P_n\cos\theta_A P_n\cos\theta + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}[\cos(m\phi_A)\cos(m\phi)$$
$$+ \sin(m\phi_A)\sin(m\phi)] \times P_n^m\cos\theta_A P_n^m\cos\theta\}$$

when $r > D$, the position of r and D in above equation should be exchanged.

The virtual potential due to image of the sphere S' is

$$V_i(r,\theta,\phi) = \sum_{n=0}^{\infty}(\frac{r_0}{r_i})^{n+1}\sum_{m=0}^{n}[A_{mn}\cos(m\phi_i) + B_{mn}\sin(m\phi_i)]P_n^m\cos\phi_i \tag{6.9}$$

where(r_i, θ_i, ϕ_i) is for spherical coordinate S'. Considering the equivalence of potential site $M(r, \theta, \phi)$ in spherical coordinate S with virtual potential site $M'(r_i, \theta_i, \phi_i)$ in coordinate S', we have

$$P_n^m(\cos\theta_i)r_i^{-n-1} = \sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r_i'^k}{h_0^{n+k+1}} P_k^m \cos\theta_i' \tag{6.10a}$$

$$\phi = \phi_i' \tag{6.10b}$$

Substitute equation (8.10) into equation (8.9),

$$V_i = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_0^{n+1}[A_{mn}\cos(m\phi)+B_{mn}\sin(m\phi)]\times\sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r^k}{h_0^{n+k+1}} P_k^m \cos\phi \tag{6.11}$$

Rearranging equations (8.6)(8.7)(8.8)(8.11), we have

$$V_1 = \frac{\rho_0 I}{4\pi}\sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A,\phi_A,\theta,\phi) + V_i + \sum_{n=0}^{\infty}(\frac{r_0}{r})^{n+1} Y_m(\theta,\phi) \tag{6.12}$$

$$V_2 = \frac{\rho_0 I}{4\pi}\sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A,\phi_A,\theta,\phi) + V_i + \sum_{n=0}^{\infty}(\frac{r_0}{r})^{n} Y_m'(\theta,\phi) \tag{6.13}$$

where

$$Y_m(\theta,\phi) = \sum_{m=0}^{\infty}[A_{mn}\cos(m\phi)+B_{mn}\sin(m\phi)]P_n^m\cos(\phi) \tag{6.14}$$

$$Y_m'(\theta,\phi) = \sum_{m=0}^{\infty}[C_{mn}\cos(m\phi)+D_{mn}\sin(m\phi)]P_n^m\cos(\phi) \tag{6.15}$$

and

$$L_m(\theta_A,\phi_A,\theta,\phi) = P_n\cos\theta_A P_n\cos\theta + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}[\cos(m\phi_A)\cos(m\phi)+ \sin(m\phi_A)\sin(m\phi)]\times P_n^m\cos\theta_A P_n^m\cos\theta \tag{6.16}$$

=======
DC app for looking at currents, charges etc with a current source at the surface.

Analytic solution for a buried sphere in a uniform space [redacted]> 32636add57830d02d6c2cf0eb89e5ae979a27b1a

=======

7. DC app for looking at currents, charges etc with a current source at the surface

8. Analytic solution for a buried sphere in a uniform space

Extensive studies have been carried out concerning the solution for a buried spherical body in a homogeneous earth due to a point source on the earth's surface. We summarize a general procedure to this problem based on image method. A bispherical coordinate system is usually used considering the boundary conditions on both the spherical and planar surface. Figure 6 shows the bispherical system for a conducting sphere with conductivity ρ_1 buried in a uniform earth with conductivity ρ_0, A is the current point located on the x axis with bispherical coordinate (r_A, θ_A, ϕ_A), here $r_A = D, \theta_A = arcos(h_0/D), \phi_A = \pi$. D is the distance between the current point A and center of buried sphere, R is the distance between A and an arbitrary potential site M. For convenience, the earth is divided into

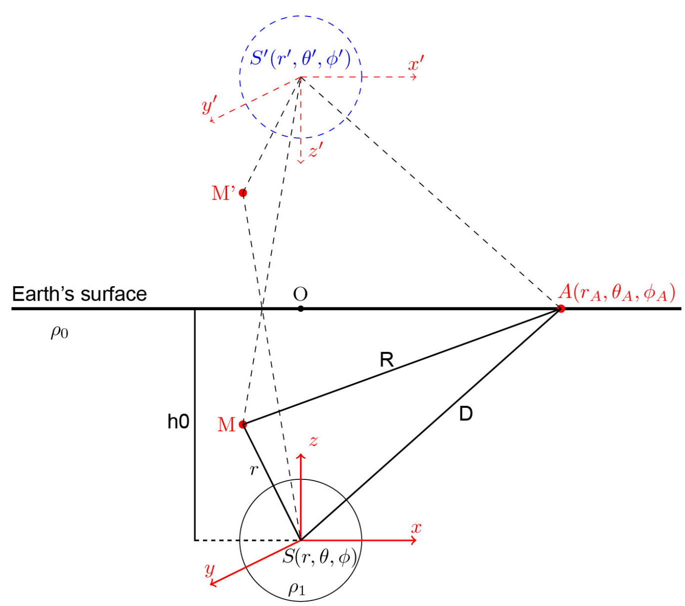


Figure 6. Sketch of a spherical body in a uniform earth.

exterior region 1 and interior region 2 to the sphere. In a bispherical coordiante system, the total potential V can be expressed as the sum of a primary potential V_p, a secondary potential V_s caused by the sphere, and a virtual potential V_i due to the image of the spherical body. The potentials for sites in region 1 and 2 are

$$V_1 = V_p + V_{s1} + V_i \tag{8.1}$$

$$V_2 = V_p + V_{s2} + V_i \tag{8.2}$$

In regions free of charge, the potential is governed by the Laplace's equation

$$\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{8.3}$$

The general solution for potential V can be obtained by applying the method of separation of variables
(8.4)

$$V = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(A_{mn}r^n+B_{mn}r^{-n-1})\times[C_{mn}\cos(m\phi)+D_{mn}\sin(m\phi)]P_n^m\cos\theta$$

Considering the boundary condition at surface of sphere

$$\frac{\partial V_1}{\partial r}|_{r=h_0} = 0 \tag{8.5a}$$

$$V_1 = V_2 \quad \text{for } r = r_0 \tag{8.5b}$$

$$\frac{1}{\rho_1}\frac{\partial V_1}{\partial r} = \frac{1}{\rho_2}\frac{\partial V_2}{\partial r}|_{r=r_0} \tag{8.5c}$$

where r_0 is the radius of the sphere. Applying boundary conditions (8.5a) to (8.4), we obtains

(8.6)

$$V_{s1} = \sum_{n=0}^{\infty} (\frac{r_0}{r})^{n+1} \sum_{m=0}^{n} [A_{mn} \cos(m\phi) + B_{mn} \sin(m\phi)] P_n^m \cos\phi$$

(8.7)

$$V_{s2} = \sum_{n=0}^{\infty} (\frac{r_0}{r})^{n} \sum_{m=0}^{n} [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos\phi$$

where A_{mn}, B_{mn}, C_{mn} and D_{mn} are unknown coefficients. P_n^m is the Legendre function of the first kind. The primary potential may be also expanded in bispherical coordinates as

(8.8)

$$V_p = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} \{P_n \cos\theta_A P_n \cos\theta + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} [\cos(m\phi_A)\cos(m\phi)$$
$$+ \sin(m\phi_A)\sin(m\phi)] \times P_n^m \cos\theta_A P_n^m \cos\theta\}$$

when $r > D$, the position of r and D in above equation should be exchanged.

The virtual potential due to image of the sphere S' is

(8.9)

$$V_i(r,\theta,\phi) = \sum_{n=0}^{\infty} (\frac{r_0}{r_i})^{n+1} \sum_{m=0}^{n} [A_{mn} \cos(m\phi_i) + B_{mn} \sin(m\phi_i)] P_n^m \cos\phi_i$$

where(r_i, θ_i, ϕ_i) is for spherical coordinate S'. Considering the equivalence of potential site $M(r,\theta,\phi)$ in spherical coordinate S with virtual potential site $M'(r_i, \theta_i, \phi_i)$ in coordinate S', we have

(8.10a)

$$P_n^m(\cos\theta_i)r_i^{-n-1} = \sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r_i'^k}{h_0^{n+k+1}} P_k^m \cos\theta_i'$$

(8.10b) $$\phi = \phi_i'$$

Substitute equation (8.10) into equation (8.9),

(8.11)

$$V_i = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_0^{n+1}[A_{mn}\cos(m\phi)+B_{mn}\sin(m\phi)]\times\sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r^k}{h_0^{n+k+1}} P_k^m$$

Rearranging equations (8.6)(8.7)(8.8)(8.11), we have

(8.12)

$$V_1 = \frac{\rho_0 I}{4\pi}\sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} (\frac{r_0}{r})^{n+1} Y_m(\theta, \phi)$$

(8.13)

$$V_2 = \frac{\rho_0 I}{4\pi}\sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} (\frac{r_0}{r})^{n} Y_m'(\theta, \phi)$$

where

(8.14)

$$Y_m(\theta, \phi) = \sum_{m=0}^{\infty}[A_{mn}\cos(m\phi) + B_{mn}\sin(m\phi)]P_n^m \cos(\phi)$$

(8.15)

$$Y_m'(\theta, \phi) = \sum_{m=0}^{\infty}[C_{mn}\cos(m\phi) + D_{mn}\sin(m\phi)]P_n^m \cos(\phi)$$

and
(8.16)

$$L_m(\theta_A, \phi_A, \theta, \phi) = P_n \cos\theta_A P_n \cos\theta + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!}[\cos(m\phi_A)\cos(m\phi)+$$
$$\sin(m\phi_A)\sin(m\phi)] \times P_n^m \cos\theta_A P_n^m \cos\theta$$

[redacted]> 9a1f053371e64e01759b9956df188d9d5181280e

References

David B Large. Electric potential near a spherical body in a conducting half-space. *Geophysics*, 36(4):763–767, 1971. ISSN 0016-8033.

R H Merkel and S S Alexander. Resistivity analysis for models of a sphere in a half-space with buried current sources. *Geophysical Prospecting*, 19(4):640–651, 1971. ISSN 1365-2478.

S K Singh and J M Espindola. Apparent resistivity of a perfectly conducting sphere buried in a half-space. *Geophysics*, 41(4):742–751, 1976. ISSN 0016-8033.

D D Snyder and R M Merkel. Analytic models for the interpretation of electrical surveys using buried current electrodes. *Geophysics*, 38(3):513–529, 1973. ISSN 0016-8033.

J J Tang and Y Yuan. Comparison of approximate solution and exact solution of buried sphere model at point source with different arrays. *Journal of Central South University (Science and Technology)*, 43(3):1057–1064, 2012.

G R. Van Nostrand and L K. Cook. Intepretation of resistivity data. *Professional Paper 499, U.S. Geological Survey*, 1966.