

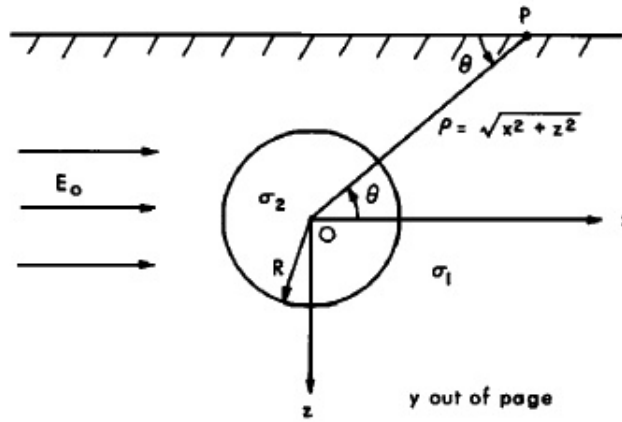
# ANALYTICAL RESULTS AND PHYSICAL UNDERSTANDING

TEAM C

## 1. UNIFORM ELECTRIC FIELD ILLUMINATING A SPHERE IN A UNIFORM EARTH (ANALYTIC SOLUTION, REFERENCE)

Let consider a resistive uniform half-space, of conductivity  $\sigma_1$  enclosing a conductive sphere  $\sigma_2$ . Let assume a uniform, unidirectional static electric field  $E_0$  going through this half-space.

FIGURE 1. Uniform electric field illuminating a sphere in a uniform earth



### Maxwell equations

In this case, we need:

$$\nabla \times E = 0 \quad , \text{so} \quad E = -\nabla V \quad (1)$$

$$J = \sigma E \quad (2)$$

The primary field  $E_0$  can then be expressed by:

$$E^p_0 = -\frac{dV^p}{dx} \quad (3) \text{ Assuming a primary potential null at the origin:}$$

$$V^p = E_0x = E_0r\cos\theta \quad (4)$$

As the primary potential respects  $\nabla^2V = 0$ , as only a dependence in  $x$  direction, the anomalous or secondary field can be expressed as (using spherical coordinates):

$$(1.1) \quad V^s = (Ar + Br^{-2})\cos\theta$$

If we assume finite values of the potential everywhere, we can divide the anomalous potential in two domain:

$$(1.2) \quad \begin{aligned} V_e^s &= Br^{-2}\cos\theta & \text{if } r > R, \\ V_i^s &= Arcos\theta & \text{if } r < R. \end{aligned}$$

The total external potential is then:

$$V_e = V_e^s + V^p = (-E_0r + Br^{-2})\cos\theta \quad (8)$$

On the surface of the sphere, both the normal current density and potential have to be continuous across the interface.

$$\text{Using the continuity of current density, we got: } \sigma_1 E_e = \sigma_2 E_i \quad \sigma_1 \frac{dV_e}{dr} = \sigma_2 \frac{dV_i}{dr} \quad (9)$$

$$2\sigma_1 BR^{-3} + \sigma_1 E_0 \quad (10)$$

Using the continuity of potential, we got:

$$V_e = V_i \quad (11)$$

$$-E_0R + BR^{-2} = AR \quad (12)$$

From equations (10) and (12), we get:

$$A = -\frac{3\sigma_1}{\sigma_2 + 2\sigma_1} E_0 \quad (13)$$

$$B = E_0 R^3 \frac{\sigma_2 - \sigma_1}{\sigma_2 + 2\sigma_1} \quad (14)$$

And the anomalous electric field is:

$$(1.3) \quad \mathbf{E}_s = -\nabla V^s_e = \mathbf{E}_0 \mathbf{R}^3 \frac{\sigma_2 - \sigma_1}{\sigma_2 - 2\sigma_1} \frac{(2\mathbf{x}^2 - \mathbf{y}^2 - \mathbf{z}^2)\mathbf{u}_x + 3\mathbf{xy}\mathbf{u}_y + 3\mathbf{xz}\mathbf{u}_z}{\mathbf{r}^5}$$

FIGURE 2. Induced Dipole moment  $P$  in a sphere in a uniform earth

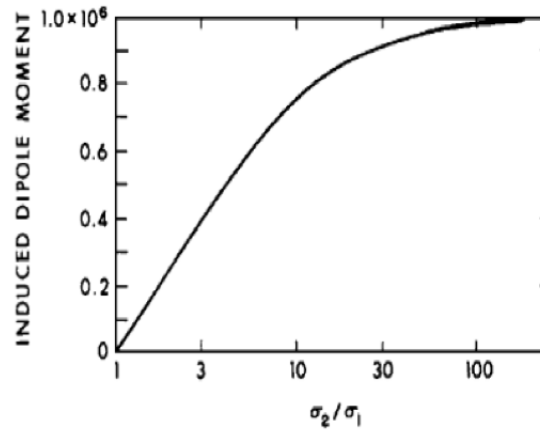
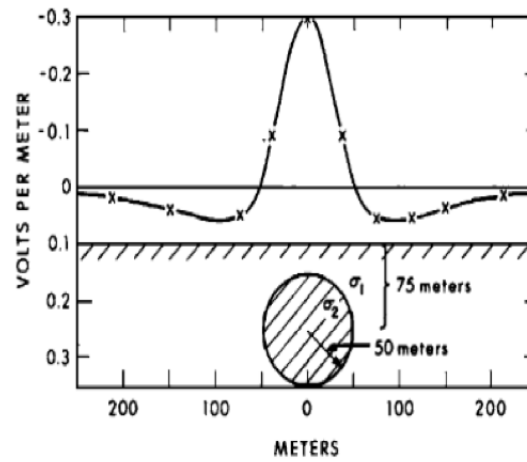


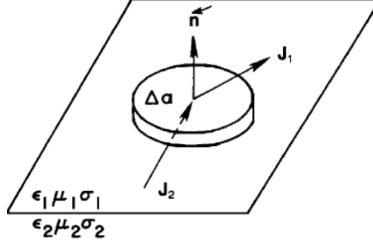
FIGURE 3. Anomalous field of a sphere in a uniform earth illuminated by an uniform electric field



Continuity of current and charge accumulation

We assume to be here in a steady state with direct current. The current entering a cylinder through an interface as in figure 4 consists both in tangential and normal components.

FIGURE 4. Uniform electric field illuminating a sphere in a uniform earth



As the cylinder height is collapsed to zero, we can write the normal component as:

$$I = \mathbf{J}_1 \cdot \mathbf{n} \Delta a \quad (16) \text{ or as } I = \mathbf{J}_2 \cdot \mathbf{n} \Delta a \quad (17)$$

Note: Otherwise in steady state we would have an infinite built up of charges at the interface. Then  $\mathbf{J}_2 \cdot \mathbf{n} = \mathbf{J}_1 \cdot \mathbf{n}$  (18)

$$\text{so } \mathbf{J}_1 \cdot \mathbf{n} = \mathbf{J}_2 \cdot \mathbf{n} \quad (19)$$

Note: This is only true in a direct current case in a steady state. It appears to be satisfactory up to  $10^5$  Hz, as long as displacement currents can be considered negligible.

#### Charges, Coulomb's law and potentials.

■< Updated upstream Electric charge produces an electric potential; the Coulomb's electrostatic potential is

$V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$ . (20) ===== Electric charge produces an electric potential: the Coulomb's electrostatic potential is

$$(1.4) \quad V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}.$$

■> Stashed changes

#### Anomalous currents and electric fields

Anomalous current density is defined as  $\mathbf{J}_a = \sigma_a \mathbf{E}$ . (21)

Here  $\mathbf{E}$  is total electric field and  $\sigma_a$  is the difference between wholespace conductivity  $\sigma_1$  and conductivity of the target  $\sigma_2$ :  $\sigma_a = \sigma_2 - \sigma_1$ .

DC app for looking at currents, charges etc with a current source at the surface.

## 2. ANALYTIC SOLUTION FOR A BURIED SPHERE IN A UNIFORM SPACE

Extensive studies have been carried out concerning the solution for a buried spherical body in a homogeneous earth due to a point source on the earth's surface. We summarize a general procedure to this problem based on image method. A bispherical coordinate system is usually used considering the boundary conditions on both the spherical and planar surface. Figure 5 shows the bispherical system for a conducting sphere with conductivity  $\rho_1$  buried in a uniform earth with conductivity  $\rho_0$ ,  $A$  is the current point located on the x axis with bispherical coordinate  $(r_A, \theta_A, \phi_A)$ , here  $r_A = D$ ,  $\theta_A = \arccos(h_0/D)$ ,  $\phi_A = \pi$ .  $D$  is the distance between the current point  $A$  and center of buried sphere,  $R$  is the distance between  $A$  and an arbitrary potential site  $M$ . For convenience, the earth is divided into exterior region 1

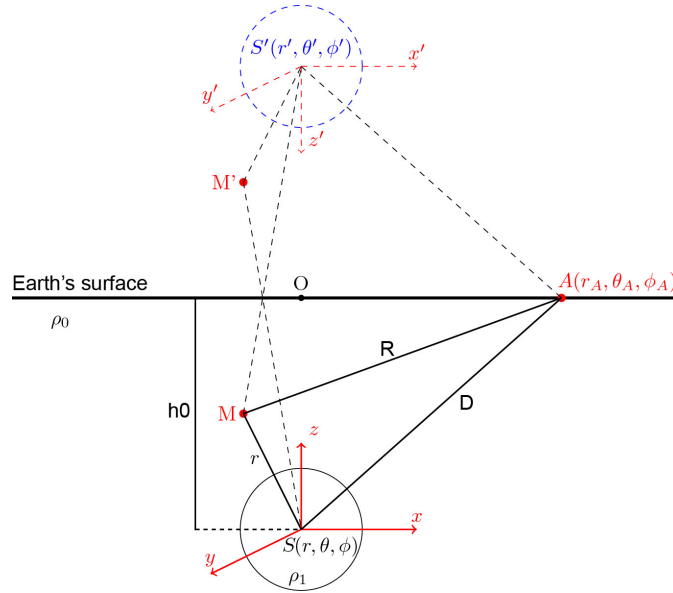


FIGURE 5. Sketch of a spherical body in a uniform earth.

and interior region 2 to the sphere. In a bispherical coordinate system, the total potential  $V$  can be expressed as the sum of a primary potential  $V_p$ , a secondary potential  $V_s$  caused by the sphere, and a virtual potential  $V_i$  due to the image of the spherical body. The potentials for sites in region 1 and

2 are

$$(2.1) \quad V_1 = V_p + V_{s1} + V_i$$

$$(2.2) \quad V_2 = V_p + V_{s2} + V_i$$

In regions free of charge, the potential is governed by the Laplace's equation

$$(2.3) \quad \frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}(\sin \theta \frac{\partial V}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

The general solution for potential  $V$  can be obtained by applying the method of separation of variables

$$(2.4) \quad V = \sum_{n=0}^{\infty} \sum_{m=0}^n (A_{mn} r^n + B_{mn} r^{-n-1}) \times [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos \theta$$

Considering the boundary condition at surface of sphere

$$(2.5a) \quad \frac{\partial V_1}{\partial r} \Big|_{r=h_0} = 0$$

$$(2.5b) \quad V_1 = V_2 \quad \text{for } r = r_0$$

$$(2.5c) \quad \frac{1}{\rho_1} \frac{\partial V_1}{\partial r} = \frac{1}{\rho_2} \frac{\partial V_2}{\partial r} \Big|_{r=r_0}$$

where  $r_0$  is the radius of the sphere. Applying boundary conditions (2.5a) to (2.4), we obtains

$$(2.6) \quad V_{s1} = \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^{n+1} \sum_{m=0}^n [A_{mn} \cos(m\phi) + B_{mn} \sin(m\phi)] P_n^m \cos \phi$$

$$(2.7) \quad V_{s2} = \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n \sum_{m=0}^n [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos \phi$$

where  $A_{mn}, B_{mn}, C_{mn}$  and  $D_{mn}$  are unknown coefficients.  $P_n^m$  is the Legendre function of the first kind. The primary potential may be also expanded in bispherical coordinates as

$$(2.8) \quad V_p = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} \{ P_n \cos \theta_A P_n \cos \theta + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} [\cos(m\phi_A) \cos(m\phi) + \sin(m\phi_A) \sin(m\phi)] \times P_n^m \cos \theta_A P_n^m \cos \theta \}$$

when  $r > D$ , the position of  $r$  and  $D$  in above equation should be exchanged.

The virtual potential due to image of the sphere  $S'$  is

$$(2.9) \quad V_i(r, \theta, \phi) = \sum_{n=0}^{\infty} \left(\frac{r_0}{r_i}\right)^{n+1} \sum_{m=0}^n [A_{mn} \cos(m\phi_i) + B_{mn} \sin(m\phi_i)] P_n^m \cos \phi_i$$

where  $(r_i, \theta_i, \phi_i)$  is for spherical coordinate  $S'$ . Considering the equivalence of potential site  $M(r, \theta, \phi)$  in spherical coordinate  $S$  with virtual potential site  $M'(r_i, \theta_i, \phi_i)$  in coordinate  $S'$ , we have

$$(2.10a) \quad P_n^m(\cos \theta_i) r_i^{-n-1} = \sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r_i'^k}{h_0^{n+k+1}} P_k^m \cos \theta_i'$$

$$(2.10b) \quad \phi = \phi_i'$$

Substitute equation (2.10) into equation (2.9),

$$(2.11) \quad V_i = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_0^{n+1} [A_{mn} \cos(m\phi) + B_{mn} \sin(m\phi)] \times \sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r^k}{h_0^{n+k+1}} P_k^m \cos \phi$$

Rearranging equations (2.6)(2.7)(2.8)(2.11), we have

$$(2.12) \quad V_1 = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_n(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^{n+1} Y_n(\theta, \phi)$$

$$(2.13) \quad V_2 = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_n(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n Y_n'(\theta, \phi)$$

where

$$(2.14) \quad Y_n(\theta, \phi) = \sum_{m=0}^{\infty} [A_{mn} \cos(m\phi) + B_{mn} \sin(m\phi)] P_n^m \cos(\phi)$$

$$(2.15) \quad Y_n'(\theta, \phi) = \sum_{m=0}^{\infty} [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos(\phi)$$

and

$$(2.16) \quad L_m(\theta_A, \phi_A, \theta, \phi) = P_n \cos \theta_A P_n \cos \theta + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} [\cos(m\phi_A) \cos(m\phi) + \sin(m\phi_A) \sin(m\phi)] \times P_n^m \cos \theta_A P_n^m \cos \theta$$