# ANALYTICAL RESULTS AND PHYSICAL UNDERSTANDING

TEAM C

1. Uniform electric field illuminating a sphere in a uniform earth (analytic solution, reference)

Let consider a resistive uniform half-space, of conductivity σ_1 enclosing a conductive sphere σ_2. Let assume a uniform, unidirectional static electric field E_0 going through this half-space.

FIGURE 1. Uniform electric field illuminating a sphere in a uniform earth
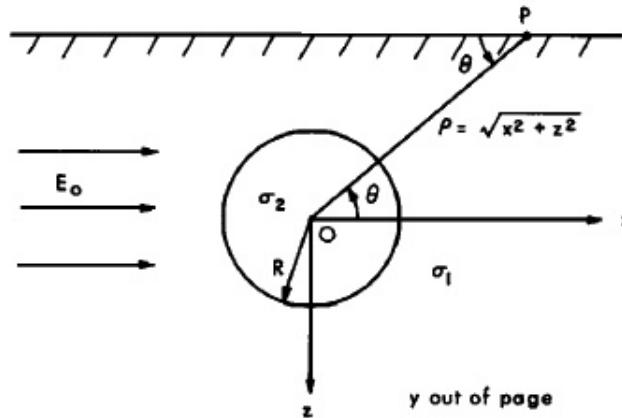
2. Maxwell equations

In this case, we need:

$\nabla \times E = 0$,so $E = -\nabla V$

$J = \sigma E$

The primary field E_0 can then be expressed by:

$E^p{}_0 = -\frac{dV^p}{dx}$

Assuming a primary potential null at the origin:

$V^p = E_0 x = E_0 r cos\theta$

As [...], the anomalous or secondary field is expressed as:

$V^s = (Ar + Br^{-2})cos\theta$

In this case, we need:

$$\nabla \times E = 0$$
$$J = \sigma E.$$
(2.1)

The first equation gives $E = -\nabla V$.

The primary field E_0 can then be expressed by:

$$E^p{}_0 = -\frac{dV^p}{dx}.$$
(2.2)

Assuming a primary potential null at the origin:

$$V^p = E_0 x = E_0 r \cos\theta.$$
(2.3)

As the primary potential respects $\nabla^2 V = 0$, as only a dependence in x direction, the anomalous or secondary field can be expressed as (using spherical coordinates):

$$V^s = (Ar + Br^{-2})\cos\theta.$$
(2.4)

If we assume finite values of the potential everywhere, we can divide the anomalous potential in two domain:

$$V_e^s = Br^{-2}\cos\theta \quad \text{if } r > R,$$
$$V_i^s = Ar\cos\theta \quad \text{if } r < R.$$
(2.5)

The total external potential is then:

$$V_e = V_e^s + V^p = (-E_0 r + Br^{-2})\cos\theta.$$
(2.6)

On the surface of the sphere, both the normal current density and potential have to be continuous across the interface.

Using the continuity of current density, we got:

$$\sigma_1 E_e = \sigma_2 E_i$$
$$\sigma_1 \frac{dV_e}{dr} = \sigma_2 \frac{dV_e}{dr}.$$
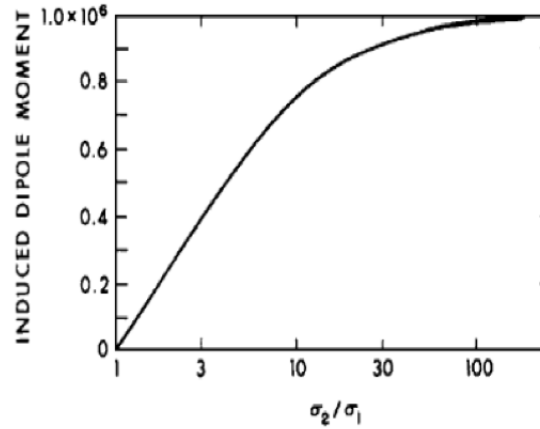(2.7)

$$2\sigma_1 BR^{-3} + \sigma_1 E_0 = -\sigma_2 A$$
(2.8)

Using the continuity of potential, we got:

$$V_e = V_i^s.$$

(2.9)
$$-E_0 R + BR^{-2} = AR.$$

From equations 2.8 and 2.9, we get:

(2.10)
$$A = -\frac{3\sigma_1}{\sigma_2 + 2\sigma_1} E_0$$

$$B = E_0 R^3 \frac{\sigma_2 - \sigma_1}{\sigma_2 + 2\sigma_1}.$$

FIGURE 2. Induced Dipole moment P in a sphere in a uniform earth

And the anomalous electric field is:

(2.11)
$$\mathbf{E_s} = -\nabla \mathbf{V^s}_\mathbf{e} = \mathbf{E_0 R^3} \frac{\sigma_2 - \sigma_1}{\sigma_2 - 2\sigma_1} \frac{(2\mathbf{x}^2 - \mathbf{y}^2 - \mathbf{z}^2)\mathbf{u_x} + 3\mathbf{xyu_y} + 3\mathbf{xzu_z}}{\mathbf{r}^5}.$$

3. CONTINUITY OF CURRENT AND CHARGE ACCUMULATION

We assume to be here in a steady state with direct current. The current entering a cylinder through an interface as in figure 4 consists both in tangential and normal components.

FIGURE 3. Anomalous field of a sphere in a uniform earth illuminated by an uniform electric field
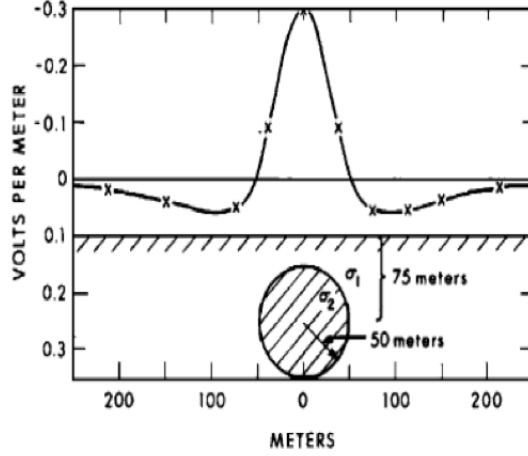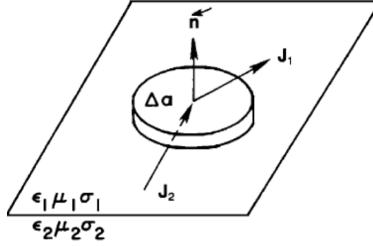

FIGURE 4. Uniform electric field illuminating a sphere in a uniform earth

As the cylinder height is collapsed to zero, we can write the normal component as:

(3.1) $$I = J_1 \cdot \mathbf{n}\Delta a, \text{or as} I = J_2 \cdot \mathbf{n}\Delta a.$$

Note: Otherwise in steady state we would have an infinite built up of charges at the interface. Then

(3.2) $$J_2 \cdot \mathbf{n} = J_1 \cdot \mathbf{n},$$

so we have

(3.3) $$\mathbf{J_1^n} = \mathbf{J_2^n}.$$

Note: This is only true in a direct current case in a steady state. It appears to be satisfactory up to 10^5 Hz, as long as displacement currents can be considered negligible.

4. Charges, Coulomb's law and potentials.

Electric charge produces an electric potential: the Coulomb's electrostatic potential is

(4.1) $$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}.$$

5. Anomalous currents and electric fields

Anomalous current density is defined as

(5.1) $$\mathbf{J}_a = \sigma_a\mathbf{E}.$$

Here $\mathbf{E}$ is total electric field and σ_a is the difference between wholespace conductivity σ_1 and conductivity of the target σ_2: $\sigma_a = \sigma_2 - \sigma_1$.

6. Analytic solution for a buried sphere in a uniform space

Extensive studies have been carried out concerning the solution for a buried spherical body in a homogeneous earth due to a point source on the earth's surface ((**??????**)). We summarize a general procedure to this problem based on image method. A bispherical coordinate system is usually used considering the boundary conditions on both the spherical and planar surface. Figure 6 shows the bispherical system for a conducting sphere with conductivity ρ_1 buried in a uniform earth with conductivity ρ_0, A is the current point located on the x axis with bispherical coordinate (r_A, θ_A, ϕ_A), here $r_A = D, \theta_A = arcos(h_0/D), \phi_A = \pi$. D is the distance between the current point A and center of buried sphere, R is the distance between A and an arbitrary potential site M. For convenience, the earth is divided into exterior region 1 and interior region 2 to the sphere. In a bispherical coordiante system, the total potential V can be expressed as the sum of a primary potential V_p, a secondary potential V_s caused by the sphere, and a virtual potential V_i due to the image of the spherical body. The potentials for sites in region 1 and 2 are

(6.1) $$V_1 = V_p + V_{s1} + V_i$$

(6.2) $$V_2 = V_p + V_{s2} + V_i$$

In regions free of charge, the potential is governed by the Laplace's equation

(6.3) $$\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$
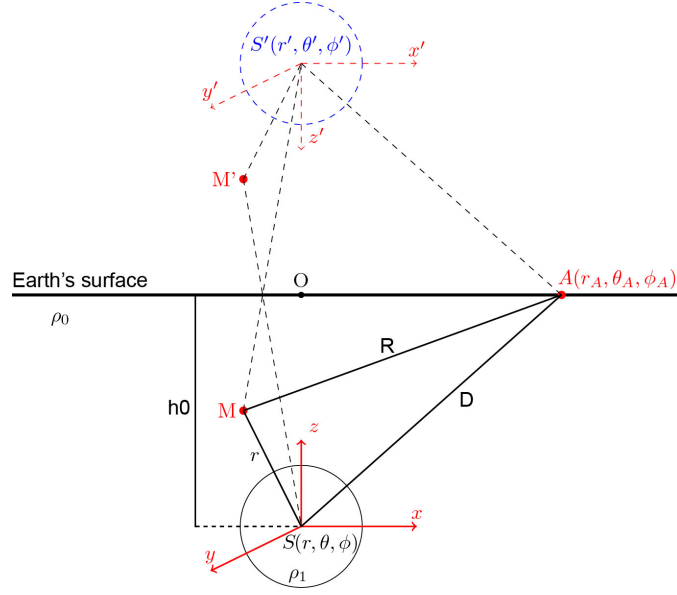
FIGURE 5. Sketch of a spherical body in a uniform earth.

The general solution for potential V can be obtained by applying the method of separation of variables

(6.4)
$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (A_{mn} r^n + B_{mn} r^{-n-1}) \times [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos\theta$$

Considering the boundary condition at surface of sphere

(6.5a)
$$\frac{\partial V_1}{\partial r}\Big|_{r=h_0} = 0$$

(6.5b)
$$V_1 = V_2 \quad \text{for } r = r_0$$

(6.5c)
$$\frac{1}{\rho_1} \frac{\partial V_1}{\partial r} = \frac{1}{\rho_2} \frac{\partial V_2}{\partial r}\Big|_{r=r_0}$$

where r_0 is the radius of the sphere. Applying boundary conditions (7.5a) to (7.4), we obtains

(6.6)
$$V_{s1} = \sum_{n=0}^{\infty} (\frac{r_0}{r})^{n+1} \sum_{m=0}^{n} [A_{mn} \cos(m\phi) + B_{mn} \sin(m\phi)] P_n^m \cos\phi$$

(6.7)
$$V_{s2} = \sum_{n=0}^{\infty} (\frac{r_0}{r})^n \sum_{m=0}^{n} [C_{mn} \cos(m\phi) + D_{mn} \sin(m\phi)] P_n^m \cos\phi$$

where A_{mn}, B_{mn}, C_{mn} and D_{mn} are unknown coefficients. P_n^m is the Legendre function of the first kind. The primary potential may be also expanded in bispherical coordinates as

(6.8)

$$V_p = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} \{P_n \cos\theta_A P_n \cos\theta + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!}[\cos(m\phi_A)\cos(m\phi)$$
$$+ \sin(m\phi_A)\sin(m\phi)] \times P_n^m \cos\theta_A P_n^m \cos\theta\}$$

when $r > D$, the position of r and D in above equation should be exchanged.

The virtual potential due to image of the sphere S' is

(6.9) $\quad V_i(r, \theta, \phi) = \sum_{n=0}^{\infty} (\frac{r_0}{r_i})^{n+1} \sum_{m=0}^{n} [A_{mn}\cos(m\phi_i) + B_{mn}\sin(m\phi_i)]P_n^m \cos\phi_i$

where(r_i, θ_i, ϕ_i) is for spherical coordinate S'. Considering the equivalence of potential site $M(r, \theta, \phi)$ in spherical coordinate S with virtual potential site $M'(r_i, \theta_i, \phi_i)$ in coordinate S', we have

(6.10a) $\qquad P_n^m(\cos\theta_i)r_i^{-n-1} = \sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r_i'^k}{h_0^{n+k+1}} P_k^m \cos\theta_i'$

(6.10b) $\qquad\qquad\qquad\qquad \phi = \phi_i'$

Substitute equation (7.10) into equation (7.9),

(6.11)

$$V_i = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_0^{n+1}[A_{mn}\cos(m\phi) + B_{mn}\sin(m\phi)] \times \sum_{k=m}^{\infty} \frac{(n+k)!}{(n-m)!(m+k)!} \frac{r^k}{h_0^{n+k+1}} P_k^m \cos\phi$$

Rearranging equations (7.6)(7.7)(7.8)(7.11), we have

(6.12) $\quad V_1 = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} (\frac{r_0}{r})^{n+1} Y_m(\theta, \phi)$

(6.13) $\quad V_2 = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} (\frac{r_0}{r})^{n} Y'_m(\theta, \phi)$

where

(6.14) $\qquad Y_m(\theta, \phi) = \sum_{m=0}^{\infty} [A_{mn}\cos(m\phi) + B_{mn}\sin(m\phi)]P_n^m \cos(\phi)$

(6.15) $\qquad Y'_m(\theta, \phi) = \sum_{m=0}^{\infty} [C_{mn}\cos(m\phi) + D_{mn}\sin(m\phi)]P_n^m \cos(\phi)$

and

(6.16)

$$L_m(\theta_A, \phi_A, \theta, \phi) = P_n \cos\theta_A P_n \cos\theta + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!}[\cos(m\phi_A)\cos(m\phi)+$$
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7. ANALYTIC SOLUTION FOR A BURIED SPHERE IN A UNIFORM SPACE

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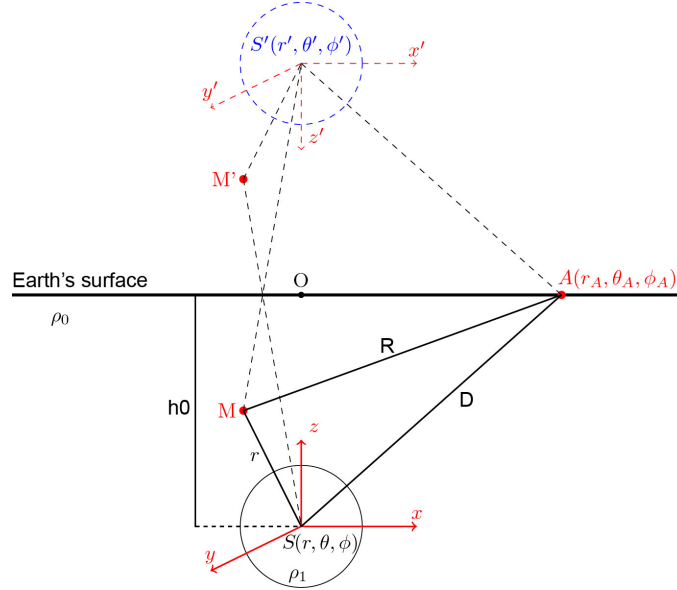

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(7.13) $\qquad V_2 = \frac{\rho_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{r^n}{D^{n+1}} L_m(\theta_A, \phi_A, \theta, \phi) + V_i + \sum_{n=0}^{\infty} (\frac{r_0}{r})^n Y_m'(\theta, \phi)$

where

(7.14) $\qquad Y_m(\theta, \phi) = \sum_{m=0}^{\infty} [A_{mn}\cos(m\phi) + B_{mn}\sin(m\phi)] P_n^m \cos(\phi)$

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$$\sin(m\phi_A)\sin(m\phi)] \times P_n^m \cos\theta_A P_n^m \cos\theta$$

8. DC app for looking at currents, charges etc with a current source at the surface