CSE 105: Computation

by Jy

Contents

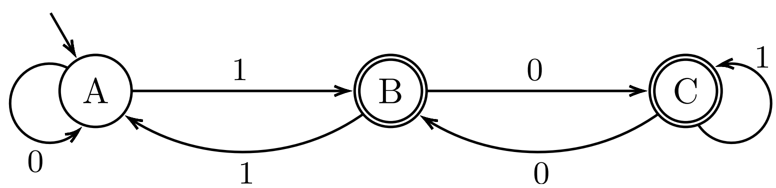
1 Deterministic Finite Automaton (DFA)

A machine consists of different states drawn in circles with names. Often a state drawn as a double circle is an "acceptive state," and a plain circle indicates a "rejective state." A machine receives a string consisted of '1's and '0's as input and the states change as the machine reads through input digits. An arrow is used to indicate which state is to start with. See Example 1.1 for detailed information.

1.1 Expressions of DFA's

Example 1.1: A DFA

Let's first look at the DFA below which starts at state A.



If the string "010110" is input to the machine, will it result in true or false? Will the state be acceptive or rejective?

$$\xrightarrow{010110} \boxed{\text{M}} \xrightarrow{1/0 \text{ (True / False, Accept / Reject)}}$$

There are two arrows leaving state A: one with a label reading '1' which points to state B and one reading '0' which goes back to state A itself. That means, if an input digit reads '1,' the state changes to B, and if '0' the state stays in A.

Now step through the procedure:

1. The machine starts off at state A with input '0,' which, as explained above, changes the state to A itself.
2. Next, the second digit '1' is read so the state is changed to B.
3. The next digit '0' makes state B to switch to state C.
4. Then state C reads '1' so no state change occurs.
5. The next digit is '1' again so the state remains still on C.
6. Last, the digit '0' switches the state from C to B.

Thus the input string "010110" changes the machine to state B, which is an acceptive state.

Definition 1.1 DFA. *A DFA is a 5-tuple*

$$M = (\, Q, \Sigma, \delta, s, F\,)$$

where

Q *is a finite set, for states*

Σ *is a finite set, for input alphabet*

s $\in Q$, for start states

F $\subseteq Q$, for accepting states

δ $Q \times \Sigma \mapsto Q$, a function that specifies the transition between states

Example 1.2: Denoting machine in Example 1.1

According to definition 1.1, the machine in Example 1.1 can be denoted by

$$M = (Q, \Sigma, \delta, s, F)$$

where

- $Q = \{A, B, C\}$
- $\Sigma = \{0, 1\}$
- $s = \{A\}$
- $F = \{B, C\}$

And function δ can be described by the table below.

δ	0	1
A	A	B
B	C	A
C	B	C

Definition 1.2 f_M. *For any DFA $M = (Q, \Sigma, \delta, s, F)$, let*

$$f_M : \Sigma^* \mapsto \{\text{True}, \text{False}\}$$

where Σ^* is a set of strings over Σ.

$$f_M(w) = \begin{cases} \text{True}, & \delta^*(s, w) \in F \\ \text{False}, & \textit{else} \end{cases}$$

Definition 1.3 δ^*.

$$\delta^* : Q \times \Sigma^* \mapsto Q$$

which is an inductive function defined as

$$\begin{cases} \delta^*(q, \varepsilon) = q \\ \delta^*(q, aw) = \delta^*(\delta(q, a), w) \end{cases}$$

where $varepsilon$ is an empty string and $q \in Q, a \in \Sigma, w \in \Sigma^*$).

1.2 Configurations of DFA's

Definition 1.4 Configurations.

$$\text{Conf} = Q \times \Sigma^*$$

Definition 1.5 Initial Configurations. *The initial configuration of a machine* $I_M(w) \in \text{Conf}$

$$I_M(w) = (s, w)$$

Definition 1.6 Final Configurations. *The final configuration of a machine* $H_M(w) \subseteq \text{Conf}$

$$H_M(w) = \{ (q, u) \mid q \in Q, u = \varepsilon \}$$

Definition 1.7 Output. *The output of a machine is a function that returns either "*True*" or "*False.*"*

$$O_M \colon H_M \mapsto \{ \text{True}, \text{False} \}$$

defined as

$$O_M(q, \varepsilon) = \begin{cases} \text{True}, & q \in F \\ \text{False}, & \textit{else} \end{cases}$$

Definition 1.8 . $R_M \subseteq \text{Conf}$

$$R_M = \rightarrow_M = \{(q, aw) \rightarrow (\delta(q, a), w) \mid q \in Q, a \in \Sigma, w \in \Sigma^*\}$$

Example 1.3: Configurations of machine in Example 1.1

With input "10010" write in mathematical language the configurations of machine in Example 1.1:

$$I_M(10010) = (A, 10010) \rightarrow (B, 0010) \rightarrow (C, 010) \rightarrow (B, 10) \rightarrow (A, 0) \rightarrow (A, \varepsilon) \in H_M$$

And thus the output

$$O_F(A, \varepsilon) = \text{False}$$

.

The machine in fact will only accept integers that are *not* multiples of 3.

Definition 1.9 .

$$f'_n(w) = O_F(C_n)$$

A subset of Σ^* of a DFA that contains all inputs to which the output of the machine is True is called the *language* of the machine.

Definition 1.10 Regularity of Language. $L \subseteq \Sigma^*$ *is regular if*

$$\exists \text{DFA} M \mid L(M) = L$$

Example 1.4

Given that $\varepsilon^* = \{\varepsilon\}$ and $\Sigma^* = \{\varepsilon, 1, 0, 10, 101, \cdots\} = \{0,1\}^*$, which of the following languages are regular?

- $L_1 = \left\{ w \in \{0,1\}^* \mid w \text{ is not a multiple of } 3 \right\}$,
- $L_2 = \left\{ w \in \{0,1\}^* \mid w \text{ is a power of } 2 \right\}$, and
- $L_2 = \left\{ w \in \{0,1\}^* \mid w \text{ is a power of } 3 \right\}$.

Definition 1.11 Operations on Languages.

Complement $L^C = \{ w \in \Sigma^* \mid w \notin L \}$

Union $L_1 \cup L_2 = \{ w \in \Sigma^* \mid w \in L_1 \vee w \in L_2 \}$

Intersection $L_1 \cap L_2 = \{ w \mid w \in L_1 \wedge \in L_2 \}$

Dot Product $L_1 \cdot L_2 = \{ w_1 \cdot w_2 \mid w \in L_1, w_2 \in L_2 \}$

Example 1.5: If L is regular, is L^C also regular?

Yes.

Proof of statement $\mathbb{R}$ is closed under complement. Let $L \in \mathbb{R}$, prove $L^C \in \mathbb{R}$:

By definition,

$$\exists M = (Q, \Sigma, \delta, s, F) \text{ s.t. } L(M) = L.$$

Let $M' = \left(Q, \Sigma, \delta, s, F^C\right)$,
then $L(M') = L(M)^C = L^C$.
$L^C \in \mathbb{R}$ because $L^C = L(M')$. $\square$

Example 1.6: $\forall L_1, L_2$
$$L_1 \in \mathbb{R} \vee L_2 \in \mathbb{R} \implies L_1 \cup L_2 \in \mathbb{R}$$

Yes, $\mathbb{R}$ is closed under union.