Brief revision guide

- Revise all the theory (see document on Learn in Analysis/Revision)
- Revise all assignment problems.
- Revise all problems discussed at workshops.
- Solve as many of the remaining problems as you have time for.

FAQ's.

Q. What is the format of the exam?

A. The exam has two parts. Part A has 6 short questions worth 40 marks in total. Part B has 4 long questions worth 20 marks each. The best three count.

Q. In part A, what does *brief proof* or *brief explanation* mean?

A. All questions in Part A have very short answers/proofs. If you find yourselves writing too much, stop and think whether you are on the right track.

Here are two Examples:

Problem 1: Let $A = (0, \sqrt{2})$. Prove that $\sup A = \sqrt{2}$.

Problem 2: Is the following statement True or False? Give a proof or a counterexample.

The supremum of a non-empty bounded subset of $\mathbb{R}$ is always a rational number.

Answer to Q2: False. Counterexample: Let $A = (0, \sqrt{2})$. Then $\sup A = \sqrt{2}$, which is not rational.

In Q2 there is no need to write a detailed proof that $\sup A = \sqrt{2}$. In Q1 you MUST write a proof that $\sup A = \sqrt{2}$.

Q. Do we have to show `Rough Work`?

A. You MUST NOT show `Rough Work`. Use the blank pages at the end of the booklet for `Rough Work`.

Q. How frequently are these questions asked?

A. Almost never! FAQ stands for Frequently Answered Questions.

Questions from the Revision Lecture (plus a few we didn't have time to work on).

1. Prove that the equation $\cos x = 2x$ has exactly one real solution. (Assignment 8)

   Solution

   The proof has two steps:

   Step 1: The equation has at least one solution.

   The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - \cos x$ is continuous and $f(0) = -1 < 0$, $f(\pi/2) = \pi > 0$. By the IVT, there exists at least one c (between 0 and $\pi/2$) such that $f(c) = 0$.

   Step 2: The equation can not have more than one solution.

   If there existed two solutions c_1, c_2 with $c_1 \neq c_2$ then, by Rolle's thm, f' would vanish somewhere (between c_1 and c_2). However,
   $$f'(x) = 2 + \sin x \geq 2 - 1 = 1$$
   doesn't vanish anywhere.

   It follows that there is exactly one solution.

2. Is the following statement True or False? Give a proof or a counterexample.

   If $a_n \xrightarrow[n\to+\infty]{} 0$ *then* $\sum_n a_n$ *converges.*

   Solution False. Counterexample: $\frac{1}{n} \to 0$ but $\sum_n \frac{1}{n}$ doesn't converge.

3. (FPM Exam, August 2015, Problem 10c). Is the following statement True or False? Give a proof or a counterexample.

   If $a_n \xrightarrow[n\to+\infty]{} 0$ *then then there is a subsequence* $(a_{n_k})_{k\in\mathbb{N}}$ *such that* $\sum_{k=1}^{\infty} a_{n_k}$ *converges.*

   Solution

   True.

   Proof: It is enough to find a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\left|a_{n_k}\right| < \dfrac{1}{k^2}$ for all k.

   For each $k \in \mathbb{N}$ we shall define an index n_k in such a way that
   $$\forall k \geq 2 \quad \left|a_{n_k}\right| < 1/k^2 \quad \text{(so that } \sum_k a_{n_k} \text{ converges)} \tag{1}$$
   $$\forall k \geq 2 \;\; n_k > n_{k-1} \qquad \text{(so that } (a_{n_k})_{k\in\mathbb{N}} \text{ is a subsequence of } (a_n)_{n\in\mathbb{N}}) \tag{2}$$

   We'll do this by induction on k.

   $\underline{k=1}$: Since $a_n \xrightarrow[n\to+\infty]{} 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n| < 1$. Let n_1 be any index $\geq N$. Then
   $$|a_{n_1}| < \frac{1}{1^2}.$$

   $\underline{k \text{ to } k+1}$: Assume that n_k has already been chosen so that (1) and (2) are satisfied. Let $\varepsilon = \frac{1}{(k+1)^2}$. Since $a_n \xrightarrow[n\to+\infty]{} 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have
   $$|a_n| < \varepsilon = \frac{1}{(k+1)^2}.$$

Choose n_{k+1} to be any positive integer larger than both N and n_k, for example $n_{k+1} = 1 + N + n_k$. Then

$$\left|a_{n_{k+1}}\right| < \frac{1}{(k+1)^2},$$

and

$$n_{k+1} > n_k,$$

i.e. (1) and (2) are satisfied with k replaced by $k+1$.

4. Is the following statement True or False? Give a proof or a counterexample.

   If all elements of a non-empty bounded subset of $\mathbb{R}$ *are* ≤ 10 *then* $\sup A \leq 10$.

   Solution True. Proof: 10 is an upper bound of A. $\sup A$ is the smallest upper bound of A. Therefore $\sup A \leq 10$.

5. Is the following statement True or False? Give a proof or a counterexample.

   If A, B *are non-empty bounded subsets of* $\mathbb{R}$ *then* $\sup(A \cap B) \leq \sup A$.

   Solution True.

   Proof:

   $A \cap B \subseteq A$ therefore (monotonicity property of the supremum) $\sup(A \cap B) \leq \sup A$.

6. Is the following statement True or False? Give a proof or a counterexample.

   If A, B *are non-empty bounded subsets of* $\mathbb{R}$ *and* $A \cdot B$ *is defined by*

$$A \cdot B = \{ab : a \in A, b \in B\},$$

   then $\sup(A \cdot B) = (\sup A)(\sup B)$.

   Solution False.

   Counterexample: Let $A = \{-1\}$, $B = \{1, 2, 3\}$. Then $A \cdot B = \{-1, -2, -3\}$, $\sup A = -1$, $\sup B = 3$, $\sup(A \cdot B) = -1$, $(\sup A)(\sup B) = (-1) \cdot 3 = -3$..

   Exercise: Add a hypothesis to make it True.

7. Is the following statement True or False? Give a proof or a counterexample.

   If $a_n b_n \to 0$ *then* $a_n \to 0$ *or* $b_n \to 0$.

   Solution False.

   Counterexample:

$$a_n = \begin{cases} 0, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}, \quad b_n = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

   Then $a_n b_n = 0$ for all n, therefore $a_n b_n \to 0$, but neither (a_n) nor (b_n) converges to zero (they don't converge at all).

8. Is the following statement True or False? Give a proof or a counterexample.

   The sequence $\dfrac{\sin(n)}{2+\sin(n)}$ *has a convergent subsequence.*

   Solution True.

   Proof:

   The sequence is bounded because

   $$\left|\frac{\sin(n)}{2+\sin(n)}\right| = \frac{|\sin(n)|}{2+\sin(n)} \leq \frac{1}{2-1} = 1.$$

   By Bolzano-Weierstrass it has at least one convergent subsequence.

   The sequence $\dfrac{\sin(n)}{2+\sin(n)}$ *has a convergent subsequence.*

9. Is the following statement True or False? Give a proof or a counterexample.

   If $a_n \xrightarrow[n\to+\infty]{} +\infty$ *then* $\dfrac{1}{2^{a_n}} \xrightarrow[n\to+\infty]{} 0$.

   Solution True.

   Proof: Let $\varepsilon > 0$. Since $a_n \to +\infty$ there exists $N \in \mathbb{N}$ such that

   $$a_n > \frac{\ln\frac{1}{\varepsilon}}{\ln 2}.$$

   For all $n \geq N$ we then have

   $$a_n \ln 2 > \ln\frac{1}{\varepsilon} \;\rightsquigarrow\; \ln\left(2^{a_n}\right) > \ln\frac{1}{\varepsilon} \;\rightsquigarrow\; 2^{a_n} > \frac{1}{\varepsilon} \;\rightsquigarrow\; \frac{1}{2^{a_n}} < \varepsilon.$$

10. Is the following statement True or False? Give a proof or a counterexample.

    The equation $x^7+x^6+x^5+x^4+x^3+x^2+x+1=0$ *has at least one real root.*

    Solution True.

    Proof: Every polynomial with real coefficients and odd degree has at least one real root. (This is a consequence of the IVT)

    Exercise: Find all seven roots (some of them are real and some are complex).

11. Are there any points where the following function is continuous and what are they?

    $$f(x) = \begin{cases} -3x, & x \text{ is rational} \\ 2x+5, & x \text{ is irrational} \end{cases}$$

    Solution

    We Claim that f is continuous[1] at -1 and discontinuous everywhere else.

    Proof that f is continuous at -1:

[1] The lines $y=-3x$ and $y=2x+5$ intersect at a point with $x=-1$.

We wish to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x-(-1)| < \delta$ we have $|f(x) - f(-1)| < \varepsilon$.

Let $\varepsilon > 0$ be given and choose a positive number δ such that $\delta < \frac{\varepsilon}{3}$, for example $\delta = \frac{\varepsilon}{4}$. For all $x \in \mathbb{R}$ with $|x+1| < \delta$ we have:

if x is rational then

$$|f(x) - f(-1)| = |-3x-3| = 3|x+1| < 3\delta < \varepsilon,$$

and if x is irrational then

$$|f(x) - f(-1)| = |2x+5-3| = |2x+2| = 2|x+1| < 2\delta < 3\delta < \varepsilon.$$

<u>Proof that f is discontinuous at all points $\neq -1$:</u>

Fix $a \neq -1$.

If a is rational pick a sequence $(a_n)_{n\in\mathbb{N}}$ of irrationals with $a_n \to a$. Then $f(a_n) \not\to f(a)$ because

$$f(a_n) = 2a_n + 5 \to 2a + 5 \neq -3a = f(a).$$

By the sequential characterization of continuity, f is not continuous at the point a.

If a is irrational pick a sequence $(a_n)_{n\in\mathbb{N}}$ of rationals with $a_n \to a$. Then $f(a_n) \not\to f(a)$ because

$$f(a_n) = -3a_n \to -3a \neq 2a + 5 = f(a).$$

By the sequential characterization of continuity, f is not continuous at the point a.

12. Is the following statement True or False? Give a proof or a counterexample.

    If $a, b \in \mathbb{R}$, $a < b$ and $f : (a,b) \to \mathbb{R}$ is continuous then f is bounded.

    <u>Solution</u> False.

    Counterexample: $f : (-\pi/2, \pi/2) \to \mathbb{R}$, $f(x) = \tan x$.

13. Is the following statement True or False? Give a proof or a counterexample.

    If $a, b \in \mathbb{R}$, $a < b$ and $f : (a,b) \to \mathbb{R}$ is continuous and the limits $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to b-} f(x)$ exist, then f is bounded.

    <u>Solution</u> True.

    Proof:

    The function $g : [a,b] \to \mathbb{R}$ given by

    $$g(x) = \begin{cases} \lim\limits_{x\to a+} f(x), & x = a \\ f(x), & a < x < b \\ \lim\limits_{x\to b-} f(x), & x = b \end{cases}$$

    is continuous, therefore g is bounded, therefore f is bounded.

14. Is the following statement True or False? Give a proof or a counterexample.

If $f : \mathbb{R} \to \mathbb{R}$ is such that f^2 is differentiable then f is differentiable.

Solution False. Counterexample: Let $f(x) = |x|$. Then $f(x)^2 = |x|^2 = x^2$ is differentiable but f itself isn't.

15. Does the sequence (a_n) given by

$$a_n = (2^n + 3^n + 4^n)^{1/n}$$

converge, and if it does what is the limit?

`Rough Work:  The dominant term in the sum` $2^n + 3^n + 4^n$ `is` 4^n`, therefore`

$$(2^n + 3^n + 4^n)^{1/n} \simeq (4^n)^{1/n} = 4.$$

`We expect` $a_n \to 4$.

Solution We claim that the sequence converges and the limit is 4.

We use the squeeze theorem: for all n we have

$$(2^n + 3^n + 4^n)^{1/n} \geq (0 + 4^n)^{1/n} = 4$$

and

$$(2^n + 3^n + 4^n)^{1/n} \leq (4^n + 4^n + 4^n)^{1/n} = (3 \cdot 4^n)^{1/n} = 3^{1/n} \cdot 4$$

We have shown that for all n,

$$4 \leq a_n \leq 3^{1/n} \cdot 4.$$

Since $3^{1/n} \cdot 4 \to 3^0 \cdot 4 = 4$, the squeeze theorem implies that $a_n \to 4$ as well.