Fundamentals of Pure Mathematics 2015-16
Analysis Problems for weeks 9 and 10

I welcome your feedback on these problems and solutions. If you have any comments, if there is anything that needs more explaining, if you have any questions on any of the material, please come and see me or email me at n.bournaveas@ed.ac.uk or ask a question on Piazza.

Suggested problems for the Analysis workshop in week 10: 83, 86, 87, 88. If time permits or if you have already solved these Problems at home then work on any of the following: 84, 85, 89, 90, 91, 92, 93,

Problem 83 a, c,d and e is on Assignment 10.

Infinite Series (Wade, Chapter 6)

82. (a) Show that
$$\frac{1}{pq} = \frac{1}{q-p}\left(\frac{1}{p} - \frac{1}{q}\right),$$
for all $p, q \in \mathbb{R}$ such that $p, q, p-q \neq 0$.

(b) Compute the partial sum $S_n = \sum_{k=2}^{n} \frac{1}{k^2-1}$. Compute the infinite sum $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$.

Solution:

(a) An easy calculation.

(b) From the first bit, the n-th partial sum is

$$\begin{aligned}
S_n &= \sum_{k=2}^{n} \frac{1}{k^2-1} = \sum_{k=2}^{n} \frac{1}{(k-1)(k+1)} = \sum_{k=2}^{n} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\
&= \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots\right. \\
&\quad \left. \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right] \\
&= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right].
\end{aligned}$$

Since $\lim_{n\to\infty} s_n = \frac{3}{4}$, we have $\sum_{k=2}^{\infty} \frac{1}{k^2-1} = 3/4$.

83. Establish the convergence or divergence of each of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3n-2}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+\sin(n)}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+3}$,

(d) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{1}{n(n+1)(n+2)}}$, (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$, (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{5^n}$, (g) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$.

Solution:

(a) The sequence $\frac{n}{3n-2}$ doesn't converge to 0 (it converges to $\frac{1}{3}$) therefore the series $\sum_n \frac{n}{3n-2}$ diverges.

(b) The sequence $\frac{1}{4+\sin(n)}$ doesn't converge to 0 because all its terms are $\geq \frac{1}{5}$. Indeed, $\frac{1}{4+\sin(n)} \geq \frac{1}{4+1} = \frac{1}{5}$. Therefore the series $\sum_n \frac{1}{4+\sin(n)}$ diverges.

(c) `Rough Work:` `For large n we expect` $\frac{n}{n^3+3}$ `to behave like` $\frac{n}{n^3} = \frac{1}{n^2}$. `Since` $\sum_n \frac{1}{n^2}$ `converges, we expect the series` $\sum_n \frac{n}{n^3+3}$ `to converge.`

`END OF ROUGH WORK`

<u>Solution 1:</u> We use the limit comparison test. We have

$$\frac{\frac{n}{n^3+3}}{\frac{1}{n^2}} = \frac{n^3}{n^3+3} \xrightarrow[n\to\infty]{} 1.$$

Since $\sum_n \frac{1}{n^2}$ converges, $\sum_n \frac{n}{n^3+3}$ converges as well.

<u>Solution 2:</u> We use the comparison test. We have

$$\frac{n}{n^3+3} \leq \frac{n}{n^3+0} = \frac{1}{n^2}.$$

Since $\sum_n \frac{1}{n^2}$ converges, $\sum_n \frac{n}{n^3+3}$ converges as well.

(d) `Rough Work:` `For large n we expect` $\sqrt{\frac{1}{n(n+1)(n+2)}}$ `to behave like` $\sqrt{\frac{1}{n^3}} = \frac{1}{n^{3/2}}$. `Since` $3/2 > 1$ `the series` $\displaystyle\sum_n \frac{1}{n^{3/2}}$ `converges, therefore we expect the series` $\displaystyle\sum_n \sqrt{\frac{1}{n(n+1)(n+2)}}$ `to converge.`

`END OF ROUGH WORK`

<u>Solution 1:</u> We use the limit comparison test. We have

$$\frac{\sqrt{\frac{1}{n(n+1)(n+2)}}}{\frac{1}{n^{3/2}}} = \sqrt{\frac{\frac{1}{n(n+1)(n+2)}}{\frac{1}{n^3}}} = \sqrt{\frac{n^3}{n(n+1)(n+2)}} \xrightarrow[n\to\infty]{} \sqrt{1} = 1.$$

Since $3/2>1$ the series $\sum_n \frac{1}{n^{3/2}}$ converges, therefore the series $\sum_n \sqrt{\frac{1}{n(n+1)(n+2)}}$ converges as well.

Solution 2: We use the comparison test. We have

$$\sqrt{\frac{1}{n(n+1)(n+2)}} \leq \sqrt{\frac{1}{n(n+0)(n+0)}} = \frac{1}{n^{3/2}}.$$

Since $3/2>1$ the series $\sum_n \frac{1}{n^{3/2}}$ converges, therefore the series $\sum_n \sqrt{\frac{1}{n(n+1)(n+2)}}$ converges as well.

(e) `Rough Work: The numerator` $\mathtt{n}^{10}$ `grows fast thanks to the large exponent, but` $2^{\mathtt{n}}$ `grows even faster. This is because` $\mathtt{n}^{10}$ `is only a polynomial of n while` $2^{\mathtt{n}}$ `is an exponential (geometric progression with common ratio` $2>1$`). We could work as in Assignment 2 to show` $2^{\mathtt{n}} \geq \mathtt{n}^{12}$ `eventually for all n. This would give` $\frac{\mathtt{n}^{10}}{2^{\mathtt{n}}} \leq \frac{1}{\mathtt{n}^2}$`. By the comparison test,` $\sum_{\mathtt{n}} \frac{\mathtt{n}^{10}}{2^{\mathtt{n}}}$ `converges. Another thing to try is the ratio test or the root test.`

`END OF ROUGH WORK`

Solution 1: We use the ratio test. We have

$$\frac{\frac{(n+1)^{10}}{2^{n+1}}}{\frac{n^{10}}{2^n}} = \frac{1}{2}\left(\frac{n+1}{n}\right)^{10} \xrightarrow[n\to\infty]{} \frac{1}{2}\cdot 1^{10} = \frac{1}{2} < 1,$$

therefore the series $\sum_n \frac{n^{10}}{2^n}$ converges.

Solution 2: We use the root test. We have

$$\sqrt[n]{\frac{n^{10}}{2^n}} = \frac{\left(\sqrt[n]{n}\right)^{10}}{2} \xrightarrow[n\to\infty]{} \frac{1^{10}}{2} = \frac{1}{2} < 1,$$

therefore the series $\sum_n \frac{n^{10}}{2^n}$ converges. We have used: $\sqrt[n]{n} \xrightarrow[n\to\infty]{} 1$ (See Problem **??**).

Solution 3 We use the comparison test. First, we show by induction that for all[1] $n \geq 84$ we have

$$2^n \geq n^{12}. \tag{1}$$

[1] Wolfram Alpha says $n \geq 75$ is ok as well. We use 84 because it is a multiple of 12 and makes (2) easy.

Indeed, for $n = 84$ we have

$$2^n = 2^{7\cdot 12} = \left(2^7\right)^{12} = 128^{12} > 84^{12} = n^{12}. \tag{2}$$

If $2^n \geq n^{12}$ is true for some $n \geq 84$ then

$$2^{n+1} = 2 \cdot 2^n \geq 2n^{12}.$$

We are done if we can show $2n^{12} \geq (n+1)^{12}$. Indeed,

$$\frac{(n+1)^{12}}{n^{12}} = \left(\frac{n+1}{n}\right)^{12} = \left(1+\frac{1}{n}\right)^{12} \leq \left(1+\frac{1}{84}\right)^{12} = 1.1525\cdots < 2.$$

This completes the proof by induction of (1).

Back to our series, we have

$$\frac{n^{10}}{2^n} \leq \frac{n^{10}}{n^{12}} = \frac{1}{n^2}, \quad n \geq 84.$$

By the comparison test, $\sum_n \frac{n^{10}}{2^n}$ converges.

(f) `Rough Work`

We can think of $\frac{n2^n}{5^n}$ as $n\left(\frac{2}{5}\right)^n$. The series $\sum_n \left(\frac{2}{5}\right)^n$ converges (it's a geometric series with ratio $\frac{2}{5} < 1$). Multiplying by n makes the general term larger, however, we expect the geometric decay of $\left(\frac{2}{5}\right)^n$ to win over the linear growth of n. There are several elementary inequalities we can use to express this. For example, we can try to show by induction that

$$\left(\frac{2}{5}\right)^n \leq \frac{1}{n^3}, \quad \text{eventually for all } n,$$

which gives

$$n\left(\frac{2}{5}\right)^n \leq \frac{1}{n^2}, \quad \text{eventually for all } n.$$

Since $\sum_n \frac{1}{n^2}$ converges, $\sum_n n\left(\frac{2}{5}\right)^n$ converges as well.

An easier option is to use $n \leq 2^n$, which gives

$$n\left(\frac{2}{5}\right)^n \leq 2^n\left(\frac{2}{5}\right)^n = \left(\frac{4}{5}\right)^n.$$

Since $\sum_n \left(\frac{4}{5}\right)^n$ converges, $\sum_n n\left(\frac{2}{5}\right)^n$ converges as well.

Another thing to try is the root and ratio tests.

END OF ROUGH WORK

Solution 1: We use the ratio test:

$$\frac{\frac{(n+1)2^{n+1}}{5^{n+1}}}{\frac{n2^n}{5^n}} = \frac{2}{5}\frac{n+1}{n} \xrightarrow[n\to\infty]{} \frac{2}{5} < 1,$$

therefore the series $\sum_n \frac{n2^n}{5^n}$ converges.

Solution 2: We use the root test:

$$\sqrt[n]{\frac{n2^n}{5^n}} = \frac{2}{5}\sqrt[n]{n} \xrightarrow[n\to\infty]{} \frac{2}{5} < 1,$$

therefore the series $\sum_n \frac{n2^n}{5^n}$ converges. We have used $\sqrt[n]{n} \xrightarrow[n\to\infty]{} 1$ (Problem **??**).

Solution 3: We use the comparison test. We need: $n \leq 2^n$ for all n, which is easily proved by induction. Using this we have

$$n\left(\frac{2}{5}\right)^n \leq 2^n\left(\frac{2}{5}\right)^n = \left(\frac{4}{5}\right)^n.$$

Since $\sum_n \left(\frac{4}{5}\right)^n$ converges, $\sum_n n\left(\frac{4}{5}\right)^n$ converges as well.

(g) Rough Work:

We know that 10^n grows fast. But n! grows even faster. Indeed, $10^n = 10\cdot 10\cdots 10$ is a product of n tens, while n! is the product $1\cdot 2\cdot\cdots\cdot n$. The first few factors $1,2,3,..,$ are smaller than 10, but the factors eventually overtake 10. For example, for $n > 20$ we have

$$\begin{aligned} n! &= 1\cdot 2\cdot 3\cdots 19\cdot 20\cdot 21\cdots n \geq 1\cdot 2\cdot 3\cdots 19\cdot(2\cdot 10)\cdot(2\cdot 10)\cdots(2\cdot 10) \\ &= (19!)(2\cdot 10)^{n-19} = \frac{19!}{(2\cdot 10)^{19}}2^n 10^n = C2^n 10^n, \quad (3) \end{aligned}$$

where we have set $C = \frac{19!}{(2\cdot 10)^{19}}$ (junk constant). This shows that n! grows[2,3] faster than 10^n at least by a factor of 2^n. Therefore, we expect the series $\sum_n \frac{10^n}{n!}$ to converge.

Another thing to try is the root and ratio tests.

END OF ROUGH WORK

Solution 1: We use the ratio test.

$$\frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \frac{10}{n+1} \xrightarrow[n\to\infty]{} 0 < 1,$$

therefore the series $\sum_n \frac{10^n}{n!}$ converges.

Solution 2: We use the comparison test. We know from (3) that $n! \geq C2^n 10^n$, eventually for all n, where C is a numerical constant (i.e. it has no n in it). Therefore,

$$\frac{10^n}{n!} \leq \frac{1}{C}\frac{1}{2^n}.$$

Since $\sum_n \frac{1}{2^n}$ converges, $\sum_n \frac{10^n}{n!}$ converges as well.

Solution 3: We use the root test.

$$\sqrt[n]{\frac{10^n}{n!}} = \frac{10}{\sqrt[n]{n!}} \xrightarrow[n\to\infty]{} 0 < 1,$$

therefore the series $\sum_n \frac{10^n}{n!}$ converges.

We have used

$$\sqrt[n]{n!} \xrightarrow[n\to\infty]{} +\infty, \tag{4}$$

which is easy to see. We simply observe that there are at least $\frac{n}{2}$ factors in the product $1 \cdot 2 \cdot 3 \cdots n$ which are $\geq \frac{n}{2}$, therefore

$$n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}. \tag{5}$$

Indeed, if n is even, write $n = 2k$. Then

$$n! = \underbrace{1 \cdot 2 \cdots (k-1) \cdot k}_{\geq 1} \cdot \underbrace{(k+1) \cdots (2k)}_{\geq k \cdots k = k^k} \geq k^k = \left(\frac{n}{2}\right)^{\frac{n}{2}}. \tag{6}$$

[2] See also (5) below.

[3] Stirling's formula shows that n! behaves like $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. James Stirling was an 18th c. Scottish mathematician.

If n is odd, write $n = 2k+1$. Then

$$n! = \underbrace{1\cdot 2\cdots(k-1)\cdot k}_{\geq 1}\cdot \underbrace{(k+1)\cdots(2k+1)}_{\geq (k+1)\cdots(k+1)=(k+1)^{k+1}} \geq (k+1)^{k+1} =$$

$$\left(\frac{n}{2}+\frac{1}{2}\right)^{\frac{n}{2}+\frac{1}{2}} \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}. \quad (7)$$

This completes the proof of (5). Taking the n-th root of both sides gives

$$\sqrt[n]{n!} \geq \left(\frac{n}{2}\right)^{\frac{1}{2}}. \quad (8)$$

Since $\left(\frac{n}{2}\right)^{\frac{1}{2}} = \frac{\sqrt{n}}{\sqrt{2}} \xrightarrow[n\to\infty]{} +\infty$, (4) follows.

84. Show that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges and the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges.

Solution:

The function $f:[2,\infty)\to\mathbb{R}$ given by $f(x)=\frac{1}{x\ln x}$ is continuous, decreasing and non-negative. By the integral test, the series $\sum_{n=2}^{\infty}\frac{1}{n\ln(n)} = \sum_{n=2}^{\infty} f(n)$ converges iff the generalized integral $\int_2^{\infty} f(x)dx$ converges. We have (change variables $\ln x = y$)

$$\int_2^b f(x)dx = \int_2^b \frac{1}{x\ln x}dx = \int_{\ln 2}^{\ln b}\frac{1}{y}dy = \ln y\Big|_{\ln 2}^{\ln b} = \ln(\ln b) - \ln(\ln 2) \xrightarrow[b\to+\infty]{} +\infty.$$

It follows that the generalized integral $\int_2^{\infty} f(x)dx$ diverges, therefore the series $\sum_n \frac{1}{n\ln(n)}$ diverges.

The function $g:[2,\infty)\to\mathbb{R}$ given by $g(x)=\frac{1}{x(\ln x)^2}$ is continuous, decreasing and non-negative. By the integral test, the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2} = \sum_{n=2}^{\infty} g(n)$ converges iff the generalized integral $\int_2^{\infty} g(x)dx$ converges. We have (change variables $\ln x = y$)

$$\int_2^b g(x)dx = \int_2^b \frac{1}{x(\ln x)^2}dx = \int_{\ln 2}^{\ln b}\frac{1}{y^2}dy = \frac{-1}{y}\Big|_{\ln 2}^{\ln b} = \frac{1}{\ln 2} - \frac{1}{\ln b} \xrightarrow[b\to+\infty]{} \frac{1}{\ln 2}.$$

It follows that the generalized integral $\int_2^{\infty} g(x)dx$ converges, therefore the series $\sum_n \frac{1}{n(\ln(n))^2}$ converges.

85. ([Wade], Exercise 6.1.4) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers converging to a real number L. Prove that the series

$$\sum_{n=2}^{\infty}(a_{n+1}-2a_n+a_{n-1})$$

converges and find its sum.

Solution:

Write

$$a_{n+1}-2a_n+a_{n-1}=(a_{n+1}-a_n)-(a_n-a_{n-1})\,.$$

The series $\sum_n (a_{n+1}-a_n)$ and $\sum_n (a_n-a_{n-1})$ converge because they are telescopic and (a_n) is convergent. Therefore their difference converges. Finally,

$$\begin{aligned}\sum_{n=2}^{\infty}(a_{n+1}-2a_n+a_{n-1})=&\sum_{n=2}^{\infty}(a_{n+1}-a_n)-\sum_{n=2}^{\infty}(a_n-a_{n-1})\\ =&(L-a_2)-(L-a_1)\\ =&a_1-a_2.\end{aligned}$$

86. ([Wade], Exercise 6.1.6) If a series $\sum_n a_n$ converges, prove that the sequence (S_n) of partial sums is bounded. Give a counterexample to show that the converse is not true.

Solution:

If $\sum_n a_n$ converges then the sequence (S_n) of partial sums converges, therefore it is bounded.

The converse is not true. Take for example the series $\sum_{n=1}^{\infty}(-1)^n$. Its partial sums are bounded (each S_n is either -1 or 0) but the series doesn't converge.

87. True or False? Give a proof or a counterexample.

(a) If $\sum_n a_n$ converges absolutely then $\sum_n a_n^2$ converges.

(b) If $\sum_n a_n$ diverges then $\sum_n a_n^2$ diverges.

(c) If $\sum_n a_n^2$ converges and all a_n are non-negative then $\sum_n \frac{a_n}{n}$ converges.

Solution:

(a) `Rough Work:`

`If` $\sum_n |a_n|$ `converges then` $|a_n|\to 0$`, therefore the` a_n`'s are small for large` n`. Squaring makes them even smaller. We expect` $\sum_n a_n^2$ `to converge.` `END OF ROUGH WORK`

True.

Proof 1:

Assume that the series $\sum_n |a_n|$ is convergent. It follows that the sequence $(|a_n|)_{n\in\mathbb{N}}$ converges to zero, therefore $|a_n| \le 1$ eventually for all n. It follows that

$$a_n^2 = |a_n|^2 = |a_n|\,|a_n| \le |a_n|,$$

eventually for all n. By the comparison test, $\sum_n a_n^2$ converges.

Proof 2:

Assume that the series $\sum_n |a_n|$ is convergent. It follows that the sequence $(|a_n|)_{n\in\mathbb{N}}$ converges (to zero), therefore it is bounded. Let M be such that $|a_n| \le M$ for all n. Then

$$a_n^2 = |a_n|^2 = |a_n|\,|a_n| \le M|a_n|,$$

for all n. By the comparison test, $\sum_n a_n^2$ converges.

(b) False.

Counterexample: Take $a_n = \frac{1}{n}$. Then $\sum_n a_n = \sum_n \frac{1}{n}$ diverges but $\sum_n a_n^2 = \sum_n \frac{1}{n^2}$ converges.

(c) True.

Proof: Using $2ab \le a^2 + b^2$ we have

$$0 \le \frac{a_n}{n} \le \frac{1}{2}\left(a_n^2 + \frac{1}{n^2}\right),$$

for all n. Both $\sum_n a_n^2$ and $\sum_n \frac{1}{n^2}$ converge, therefore their sum $\sum_n \left(a_n^2 + \frac{1}{n^2}\right)$ converges. By the comparison test, $\sum_n \frac{a_n}{n}$ converges as well.

Things to think about:
Can we remove from part (c) the hypothesis that all a_n are ≥ 0?

88. True or False? Give a proof or a counterexample. In all cases assume $a_n > 0$ for all n.

(a) If $\frac{a_{n+1}}{a_n} \ge 1$ for all n, then the series $\sum_n a_n$ diverges.

(b) If $\frac{a_{n+1}}{a_n} < 1$ for all n, then the series $\sum_n a_n$ converges.

(c) If there exists an $r < 1$ such that $\frac{a_{n+1}}{a_n} \le r$ for all n, then the series $\sum_n a_n$ converges.

(NB: We are not assuming that $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists.)

Solution:

(a) True.

Proof: The hypothesis $\frac{a_{n+1}}{a_n} \geq 1$ for all n means that the sequence (a_n) is increasing, therefore $a_n \geq a_1$ for all n. Since $a_1 > 0$ it follows that (a_n) doesn't converge to zero, therefore $\sum_n a_n$ diverges.

(b) False.

Counterexample: Take $a_n = \frac{1}{n}$. Then $a_{n+1} < a_n$, therefore $\frac{a_{n+1}}{a_n} < 1$ for all n, but the series $\sum_n a_n$ diverges.

(c) True.

Proof: For all n we have

$$\begin{aligned} a_2 &\leq r a_1, \\ a_3 &\leq r a_2, \\ &\vdots \\ a_{n-1} &\leq r a_{n-2}, \\ a_n &\leq r a_{n-1}. \end{aligned}$$

Multiplying them together we find $a_n \leq r^{n-1} a_1$ for all n. Since $0 < r < 1$, the geometric series $\sum_n r^{n-1}$ converges. By the comparison test, $\sum_n a_n$ converges.

89. Which of the following series converge absolutely? conditionally? not at all?

$$a) \sum \frac{(-1)^n}{n^{1.001}} \ , \quad b) \sum (-1)^n \frac{1}{n^{0.001}} \ .$$

Solution:

(a) The series $\sum_n \frac{(-1)^n}{n^{1.001}}$ converges absolutely because the series $\sum_n \left|\frac{(-1)^n}{n^{1.001}}\right| = \sum_n \frac{1}{n^{1.001}}$ converges (by the p-test).

(b) The series $\sum_n \frac{(-1)^n}{n^{0.001}}$ doesn't converge absolutely because the series $\sum_n \left|\frac{(-1)^n}{n^{0.001}}\right| = \sum_n \frac{1}{n^{0.001}}$ diverges (by the p-test).

The series $\sum_n \frac{(-1)^n}{n^{0.001}}$ converges because the sequence $\frac{1}{n^{0.001}}$ is decreasing and converges to 0 (by the alternating series test).

It follows that the series $\sum_n \frac{(-1)^n}{n^{0.001}}$ converges conditionally.

90. (FPM Exam, May 2015, Problem 6)

For what values of $x \in \mathbb{R}$ is the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n} x^n$$

absolutely convergent? For what values of $x \in \mathbb{R}$ is the series convergent?

Solution:

We compute the radius of convergence of the series. We have

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{(n+1)\log(n+1)}\right|}{\left|\frac{(-1)^n}{n\log n}\right|} = \lim_{n\to\infty} \frac{n \log n}{(n+1)\log(n+1)} = 1.$$

It follows that for $|x| < R = 1$ the series converges absolutely, for $|x| > R = 1$ diverges.

It remains to consider points $x = -1$ and $x = 1$. At $x = 1$ the series $\sum_n \frac{(-1)^n}{n \log n}$ converges by the alternating series test. It does not converge absolutely because the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n},$$

diverges (Problem 84).

At $x = -1$ the series $\sum_n \frac{(-1)^n}{n\log n}(-1)^n = \sum_n \frac{1}{n \log n}$ diverges. (no need to examine it 'absolutely' as it has positive terms)

91. (FPM Exam, August 2015, Problem 10)

   (a) Suppose that $(a_n)_{n\in\mathbb{N}}$ is sequence of real numbers such that $a_n \neq 0$ for all n, and

   $$\frac{|a_{n+1}|}{|a_n|} \to a \tag{9}$$

   where $a > 0$. Prove that for all x with $|x| < 1/a$ the series

   $$\sum_{k=1}^{\infty} a_k (\sin x)^k \tag{10}$$

   is absolutely convergent.

(b) For which positive p is the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \tag{11}$$

convergent? Absolutely convergent?

(c) State, with a brief justification, whether the following statements are true or false.

i. If $|a_k| \leq b_k$ for all k, and the series $\sum_k b_k$ is absolutely convergent then $\sum_k a_k$ is convergent.

ii. If $a_n \to 0$ then there is a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} a_{n_k}$$

converges.

iii. If $\sum_k c_k = \infty$ then $\lim\limits_{k \to \infty} c_k$ can't be zero.

Solution:

(a) Recall that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$. Therefore, for all x with $|x| < 1/a$ we have

$$\left| a_k (\sin x)^k \right| \leq |a_k| \, |x|^k .$$

Apply the ratio test to the series $\sum_k |a_k| \, |x|^k$ to find

$$\frac{\left|a_{k+1}\right| |x|^{k+1}}{|a_k| \, |x|^k} = \frac{\left|a_{k+1}\right|}{|a_k|} |x| \to a \, |x| < 1.$$

Therefore, the series $\sum_k |a_k||x|^k$ converges. By the comparison test, the series $\sum_k \left| a_k (\sin x)^k \right|$ converges as well, which means that the series $\sum_k a_k (\sin x)^k$ converges absolutely.

(b) The series $\sum_k \frac{(-1)^k}{k^p}$ converges for any $p > 0$ because the sequence $\frac{1}{k^p}$ is decreasing and converges to 0 (alternating series test).

It remains to examine the series $\sum_k \left| \frac{(-1)^k}{k^p} \right|$, i.e. the series $\sum_k \frac{1}{k^p}$. By the p-test, it converges iff $p > 1$.

(c) i. True. Observe first that the hypothesis $|a_k| \leq b_k$ implies $b_k \geq 0$, therefore to say $\sum_k b_k$ *is absolutely convergent* is the same as saying $\sum_k b_k$ *is convergent*.

If $\sum_k b_k$ converges and $|a_k| \leq b_k$ for all k, then (comparison test) the series $\sum_k |a_k|$ converges, therefore $\sum_k a_k$ converges as well.

ii. True. If $a_n \to 0$ then we can find an index n_1 such that $|a_{n_1}| < \frac{1}{2}$, we can find an index $n_2 > n_1$ such that $|a_{n_2}| < \frac{1}{2^2}$, and so on. Working in this way we can construct a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $|a_{n_k}| < \frac{1}{2^k}$ for all k. By the comparison test the series $\sum_{k=1}^{\infty} |a_{n_k}|$ converges, therefore the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.

iii. False. Take $c_k = \frac{1}{k}$.

92. Is it possible for a power series of the form $\sum\limits_{n=0}^{\infty} a_n x^n$ to converge at $x = 2$ and diverge at $x = -1$?

Solution: No it isn't. Let R be the radius of convergence. Then the power series converges for all $x \in (-R,R)$ and diverges for all $x \notin [-R,R]$. If the series converges at $x = 2$ then $R \geq 2$, therefore $-1 \in (-R,R)$ therefore the series converges at $x = -1$ as well.

93. ([Wade], Exercise 6.4.2) For each of the following, find all values of $x \in \mathbb{R}$ for which the given series converges.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$, (b) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{3k}}{2k}$, (c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{\sqrt{k^2+1}}$, (d) $\displaystyle\sum_{k=1}^{\infty} \frac{(x+2)^k}{k\sqrt{k+1}}$.

Solution: All these series are power series. Recall that the radius of convergence of $\displaystyle\sum_k a_k (x-x_0)^k$ is given by $R = \frac{1}{L}$, where $L = \displaystyle\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$, provided that the limit exists.

(a) First we find the radius of convergence. We have

$$\lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1,$$

therefore $R = 1$. It follows that the series $\sum_k \frac{x^k}{k}$ converges for all $x \in (-1,1)$ and diverges for all $x \in (-\infty,-1) \cup (1,+\infty)$.

It remains to examine the endpoints $x = \pm 1$.

When $x = 1$ the series is $\sum\limits_{k=1}^{\infty} \frac{1}{k}$. This is the harmonic series and we know that it diverges.

When $x = -1$ the series is $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$. This series converges by the alternating series test.

(b) Set $x^3 = y$ and think of the series as $\sum\limits_{k=1}^{\infty} \frac{y^k}{2k}$. We first find the radius of convergence. We have

$$\lim_{k\to\infty} \frac{\frac{1}{2(k+1)}}{\frac{1}{2k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1,$$

therefore $R = 1$. It follows that the series $\sum\limits_{k=1}^{\infty} \frac{y^k}{2k}$ converges for all $y \in (-1,1)$ and diverges for all $y \in (-\infty,-1) \cup (1,+\infty)$. Recall that $y = x^3$ and observe that $x^3 \in (-1,1) \iff x \in (-1,1)$. Therefore, the series $\sum\limits_{k=1}^{\infty} \frac{x^{3k}}{2k}$ converges for all $x \in (-1,1)$ and diverges for all $x \in (-\infty,-1) \cup (1,+\infty)$.

It remains to examine the endpoints $x = \pm 1$.

At $x = 1$, the series $\sum\limits_{k=1}^{\infty} \frac{1}{2k}$ diverges (because it's just $\frac{1}{2}\sum_k \frac{1}{k}$).

At $x = -1$, the series $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{2k}$ converges by the alternating series test.

(c) We have

$$\frac{\left|\frac{(-1)^{k+1}}{\sqrt{(k+1)^2+1}}\right|}{\left|\frac{(-1)^k}{\sqrt{k^2+1}}\right|} = \sqrt{\frac{k^2+1}{(k+1)^2+1}} = \sqrt{\frac{k^2+1}{k^2+2k+2}} = \sqrt{\frac{\frac{k^2+1}{k^2}}{\frac{k^2+2k+2}{k^2}}}$$

$$= \sqrt{\frac{1+\frac{1}{k^2}}{1+\frac{2}{k}+\frac{2}{k^2}}} \xrightarrow[k\to\infty]{} \sqrt{\frac{1+0}{1+0+0}} = 1,$$

therefore the radius of convergence is $R = 1$.

Therefore, the series converges for all $x \in (-1,1)$ and diverges for all $x \in (-\infty,-1) \cup (1,+\infty)$.

It remains to examine the endpoints $x = \pm 1$.

At $x=1$, the series $\sum\limits_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k^2+1}}$ converges by the alternating series test.

At $x=-1$, the series $\sum\limits_{k=1}^{\infty}\frac{1}{\sqrt{k^2+1}}$ diverges. We can prove this by comparing it to the divergent series $\sum_k \frac{1}{k}$. Indeed,

$$\frac{\frac{1}{\sqrt{k^2+1}}}{\frac{1}{k}}=\sqrt{\frac{k^2}{k^2+1}}\xrightarrow[k\to\infty]{}1,$$

therefore, by the limit comparison test, the series $\sum_k \frac{1}{\sqrt{k^2+1}}$ diverges.

(d) The center is $x_0=-2$. For the radius of convergence we calculate

$$\lim_{k\to+\infty}\frac{\frac{1}{(k+1)\sqrt{k+2}}}{\frac{1}{k\sqrt{k+1}}}=\lim_{k\to+\infty}\frac{k\sqrt{k+1}}{(k+1)\sqrt{k+2}}=\left(\lim_{k\to+\infty}\frac{k}{k+1}\right)\sqrt{\lim_{k\to+\infty}\frac{k+1}{k+2}}=1$$

therefore $R=1$. It follows that the series $\sum\limits_{k=1}^{\infty}\frac{(x+2)^k}{k\sqrt{k+1}}$ converges for all $x\in(x_0-R,x_0+R)=(-3,-1)$ and diverges for all $x\in(-\infty,-3)\cup(-1,+\infty)$. It remains to examine the endpoints $x=-3$ and $x=-1$.

When $x=-1$ the series is $\sum\limits_{k=1}^{\infty}\frac{1}{k\sqrt{k+1}}$. It converges because

$$\frac{1}{k\sqrt{k+1}}\leq\frac{1}{k\sqrt{k}}=\frac{1}{k^{3/2}}$$

and $\sum\limits_{k=1}^{\infty}\frac{1}{k^{3/2}}$ converges (comparison test).

When $x=-3$ the series is $\sum\limits_{k=1}^{\infty}\frac{(-1)^k}{k\sqrt{k+1}}$. This series converges by the alternating series test. Alternatively, we may observe that the series $\sum\limits_{k=1}^{\infty}\frac{(-1)^k}{k\sqrt{k+1}}$ converges absolutely, as $\sum\limits_{k=1}^{\infty}\left|\frac{(-1)^k}{k\sqrt{k+1}}\right|$ is the series $\sum\limits_{k=1}^{\infty}\frac{1}{k\sqrt{k+1}}$ we studied just above.

94. ([Wade], Exercise 6.1.8) Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of non-negative real numbers such that the series $\sum_n a_n$ converges. Prove that $na_n \xrightarrow[n\to+\infty]{} 0$. Is the result still true if we remove the hypothesis that $(a_n)_{n\in\mathbb{N}}$ is decreasing?

Solution:

We show first that
$$2n\,a_{2n} \xrightarrow[n\to\infty]{} 0. \tag{12}$$
Let $\varepsilon > 0$ be given. By Cauchy's criterion there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ with $m \geq n$ we have
$$a_{n+1} + a_{n+2} + \cdots + a_m < \frac{\varepsilon}{2}, \tag{13}$$
which gives (take $m = 2n$ and use the fact that the smallest term in the sum is a_m)
$$n\,a_{2n} < \frac{\varepsilon}{2}, \tag{14}$$
therefore
$$2n\,a_{2n} < \varepsilon. \tag{15}$$
This completes the proof of (12).

Next we show that
$$(2n+1)\,a_{2n+1} \xrightarrow[n\to\infty]{} 0. \tag{16}$$
Indeed, using $a_{2n+1} \leq a_{2n}$ we find,
$$(2n+1)a_{2n+1} = 2na_{2n+1} + a_{2n+1} \leq 2na_{2n} + a_{2n+1}, \tag{17}$$
for all n. Now $2na_{2n} \to 0$ by (12). On the other hand, $a_n \to 0$ because the series $\sum_n a_n$ converges, therefore $a_{2n+1} \to 0$ as well ((a_{2n+1}) is a subsequence of (a_n), therefore it has the same limit). It follows that $(2n+1)a_{2n+1} \to 0$.

From (12), (16) and Problem **??** it follows that $na_n \to 0$.

If the hypothesis that (a_n) is decreasing is removed then it doesn't follow that $na_n \to 0$. Take for example the series $\sum_n a_n$ where $a_n = \frac{1}{k^2}$ when $n = 2^k$ for some $k \in \mathbb{N}$, and $a_n = 0$ when n is not a power of 2. That's the series
$$0 + \underbrace{\frac{1}{1^2}}_{\text{2nd term}} + 0 + \underbrace{\frac{1}{2^2}}_{\text{4th term}} + 0 + 0 + 0 + \underbrace{\frac{1}{3^2}}_{\text{8th term}} + 0 + \cdots + 0 + \underbrace{\frac{1}{4^2}}_{\text{16th term}} + 0 + \cdots \tag{18}$$
This series converges (proof left to the reader) but (na_n) doesn't converge to 0 because when n is a power of 2 we have $2^k a_{2^k} = 2^k \frac{1}{k^2} \xrightarrow[k\to\infty]{} +\infty$.

95. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $(b_n)_{n\in\mathbb{N}}$ be the sequence defined by
$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n},$$
(that's the sequence a_1, $\frac{a_1+a_2}{2}$, $\frac{a_1+a_2+a_3}{3}$, ...).

(a) If $a_n \to L$, where $L \in \mathbb{R}$, prove that $b_n \to L$ as well. (Hint: Do the case $L = 0$ first. The general case can be reduced to that special case by considering the sequence $a_n - L$.)

(b) Give an example where $(b_n)_{n\in\mathbb{N}}$ converges but $(a_n)_{n\in\mathbb{N}}$ doesn't.

(c) If $a_n \to \pm\infty$, prove that $b_n \to \pm\infty$ as well.

(d) If (a_n) is monotone and $b_n \to L$ then $a_n \to L$.

Solution:

(a) Assume first that $L = 0$. The sequence $(a_n)_{n\in\mathbb{N}}$ converges, therefore it is bounded. Let $M > 0$ be such that for all n, $|a_n| \leq M$.

Let $\varepsilon > 0$. Since $a_n \to L = 0$, there is an n_1 such that for all $n \geq n_1$ we have $|a_n| < \frac{\varepsilon}{2}$. Pick a positive integer n_2 such that

$$n_2 > \frac{(n_1 - 1)M}{\frac{\varepsilon}{2}}. \tag{19}$$

Set $n_0 = \max\{n_1, n_2\}$. For all $n \geq n_0$ we have:

$$\begin{aligned}
&\left|\frac{a_1 + a_2 + \cdots + a_n}{n}\right| \leq \left|\frac{a_1 + a_2 + \cdots + a_{n_1-1}}{n}\right| + \left|\frac{a_{n_1} + \cdots + a_n}{n}\right| \\
&\leq \frac{|a_1| + |a_2| + \cdots + |a_{n_1-1}|}{n} + \frac{|a_{n_1}| + \cdots + |a_n|}{n} \\
&\leq \frac{(n_1 - 1)M}{n} + \frac{(n - n_1 + 1)\frac{\varepsilon}{2}}{n}
\end{aligned} \tag{20}$$

Using $n \geq n_0 \geq n_2$ and (19) we have

$$\frac{(n_1 - 1)M}{n} \leq \frac{(n_1 - 1)M}{n_2} < \frac{\varepsilon}{2}.$$

On the other hand,

$$\frac{(n - n_1 + 1)\frac{\varepsilon}{2}}{n} = \left(1 - \frac{n_1 - 1}{n}\right)\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}.$$

Therefore, we can continue from (20) as follows:

$$\frac{(n_1 - 1)M}{n} + \frac{(n - n_1 + 1)\frac{\varepsilon}{2}}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof in the special case $L = 0$.

Suppose now that $a_n \to L$, where L is any real number. Then $a_n - L \to 0$. By the result of the special case

$$\frac{(a_1-L)+(a_2-L)+\cdots+(a_n-L)}{n} \to 0,$$

therefore

$$\frac{a_1+a_2+\cdots+a_n-nL}{n} \to 0,$$

therefore

$$\frac{a_1+a_2+\cdots+a_n}{n} - L \to 0$$

therefore

$$\frac{a_1+a_2+\cdots+a_n}{n} \to L.$$

(b)

(c) Let $a_n = (-1)^n$. Then

$$b_n = \begin{cases} 0, & n \ even \\ \frac{1}{n}, & n \ \ odd \end{cases}$$

therefore $b_n \to 0$ but (a_n) doesn't converge.

(d) Suppose (a_n) is monotone and $b_n \to L$. Let $A \in \overline{\mathbb{R}}$ be the limit of (a_n) in the extended real number system. Then $b_n \to A$ therefore $A = L$.

96. ([Wade], Exercise 6.1.10) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $L \in \mathbb{R}$. The infinite series $\sum\limits_{n=1}^{\infty} a_n$ is said to be *Cesàro summable*, with Cesàro sum equal to L, iff the sequence $(S_n)_{n\in\mathbb{N}}$ of partial sums has the property

$$\frac{S_1+S_2+\cdots+S_n}{n} \xrightarrow[n\to\infty]{} L.$$

(a) If the series $\sum\limits_{n=1}^{\infty} a_n$ converges and its infinite sum is L, show that it is Cesàro summable with Cesàro sum equal to L.

(b) (Tauber's Theorem) If $a_n \geq 0$ for all n, and $\sum\limits_{n=1}^{\infty} a_n$ is Cesàro summable with Cesàro sum equal to L, then $\sum\limits_{n=1}^{\infty} a_n$ converges and its infinite sum is L.

(c) (Grandi's Series) Prove that the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1}$ doesn't converge but it is Cesàro summable with Cesàro sum equal to $\frac{1}{2}$. (This shows that the converse of 96a is false.)

Solution:

(a) If the series $\sum_n a_n$ converges and its sum is $S \in \mathbb{R}$ then $S_n \to S$, therefore $\frac{S_1+\cdots+S_n}{n} \to S$ as well, therefore $\sum_n a_n$ is Cèsaro summable with Cèsaro sum S.

(b) If $a_n \geq 0$ for all n and the series is summable to L, then (S_n) is increasing therefore it has a limit S in the extended real number system. Then $\frac{S_1+\cdots S_n}{n} \to S$ as well (by parts (a) and (b)). Therefore $S = L$.

(c) Let $a_n = (-1)^{n+1}$. Then $\sum_n a_n$ doesn't converge as the sequence of partial sums is $1,0,1,0,\ldots$. However, if n is even then $\frac{S_1+\cdots+S_n}{n} = \frac{1+0+1+0+\cdots+1+0}{n} = \frac{\frac{n}{2}}{n} = \frac{1}{2}$, and if n is odd then $\frac{S_1+\cdots+S_n}{n} = \frac{1+0+1+0+\cdots+1+0+1}{n} = \frac{\frac{n-1}{2}+1}{n} = \frac{n+1}{2n}$, therefore $\frac{S_1+\cdots+S_n}{n} \to \frac{1}{2}$.

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