1 The Real Numbers

1.1 Algebraic structure of real numbers

- **Closure properties:** $a+b$ and $a \cdot b$ belong to $\mathbb{R}$.
- **Associative properties:** $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Commutative properties:** $a+b = b+a$ and $a \cdot b = b \cdot a$.
- **Existence of the Additive Identity:** There is a unique element $0 \in \mathbb{R}$ such that $0+a = a$ for all $a \in \mathbb{R}$.
- **Existence of the Multiplicative Identity:** There is a unique element $1 \in \mathbb{R}$ such $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbb{R}$.
- **Existence of Additive Inverses:** For every $a \in \mathbb{R}$ there is a unique element $-a \in \mathbb{R}$ such that $a+(-a) = 0$.
- **Existence of Multiplicative Inverses:** For every $a \in \mathbb{R}$ and $a \neq 0$ there is a unique element $a^{-1} \in \mathbb{R}$ such that $a \cdot (a^{-1}) = 1$.
- **Distributive Law:** (This is the only law connecting addition with multiplication) $a \cdot (b+c) = a \cdot b + a \cdot c$.

Order Axioms: There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ that has the following properties:

- **Trichotomy Property:** Given $a, b \in \mathbb{R}$ one and only one of the following statements hold:
$$a < b, \qquad b < a, \qquad a = b.$$
- **Transitive Property:** $a < b$ and $b < c$ imply $a < c$.
- **Additive Property:** $a < b$ and $c \in \mathbb{R}$ imply $a+c < b+c$.
- **Multiplicative Properties**
$$a < b \quad \text{and} \quad c > 0 \quad \text{imply} \quad ac < bc$$
and
$$a < b \quad \text{and} \quad c < 0 \quad \text{imply} \quad bc < ac.$$

Definition 1.1. *We shall call a real number $a \in \mathbb{R}$ positive if $a > 0$. We shall call $a \in \mathbb{R}$ nonnegative if $a \geq 0$.*

The set $\mathbb{R}$ contains certain special subsets: the set of natural numbers $\mathbb{N}$
$$\mathbb{N} = \{1, 2, 3, \dots\},$$
the set of integers $\mathbb{Z}$
$$\mathbb{Z} = \{\cdots -2, -1, 0, 1, 2, 3, \dots\},$$
and the set of rationals
$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}.$$

1.2 Mathematical proof

There are three main methods of proof: mathematical induction (which we will introduce in the next section), direct deduction and proof by contradiction. The proof by direct deduction goes as follows. We assume the hypotheses of the statement are true and proceed step by step to the conclusion. Each step is justified by hypothesis, one of the axioms or a mathematical result that has already been proved.

The proof by contradiction has the following construction. We assume the the hypotheses of the statement we want to establish are true and that the conclusion we want to establish is false. Then we work step by step (like when doing direct deduction) until we obtain a statement that is obviously false. At this point we are done and using mathematical logic we can deduce that the conclusion we wanted to establish must be true (since assuming the opposite lead us to a contradiction).

1.3 The absolute value and intervals

Definition 1.2. *Let $a \in \mathbb{R}$. The absolute value of a is the number $|a|$ defined by*

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

Theorem 1.1. *The following statements are true:*

- *The absolute value is multplicative, i.e., $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.*
- *Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M$ if and only if $-M \leq a \leq M$.*

Theorem 1.2. *The absolute value satisfies the following three properties:*

(i) (Positive definiteness) $|a| \geq 0$. $|a| = 0$ if and only if $a = 0$.

(ii) (Symmetry) $|a - b| = |b - a|$.

(iii) (Triangle inequality) $|a + b| \leq |a| + |b|$.

Theorem 1.3. *Let $x, y \in \mathbb{R}$.*

(i) $x < y + \varepsilon$ for all $\varepsilon > 0$ if an only if $x \leq y$.

(ii) $x > y - \varepsilon$ for all $\varepsilon > 0$ if an only if $x \geq y$.

(iii) $|x| < \varepsilon$ for all $\varepsilon > 0$ if an only if $x = 0$.

1.4 Well ordering principle

Definition 1.3. *A number x is the least element of a set $E \subset \mathbb{R}$ if and only if $x \in E$ and $x \leq a$ for all $a \in E$.*

Theorem 1.4. *(On mathematical Induction). Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition (a verbal statement or a formulae) that satisfies the following two properties:*

(i) $A(1)$ is true

(ii) For every $k \in \mathbb{N}$ for which $A(k)$ is true, $A(k+1)$ is also true.

The for every integer $n \in \mathbb{N}$ the $A(n)$ is true.

1.5 Completeness Axiom

Definition 1.4. *Let $E \subset \mathbb{R}$ be nonempty.*

- *The set E is said to be bounded above if there is $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$.*
- *A number M is called an upper bound of the set E if $a \leq M$ for all $a \in E$.*
- *A number s is called the* **supremum** *of the set E if*
  - *s is an upper bound of E*
  - *$s \leq M$ for all upper bounds M of the set E.*

  If number s exists, we shall say that E has a supremum and write $s = \sup E$.

Theorem 1.5. *(Approximation Property for Suprema). Let the set $E \subset \mathbb{R}$ has a supremum. Then for any positive number $\varepsilon > 0$ there exists $a \in E$ such that*

$$\sup E - \varepsilon < a \leq \sup E.$$

Completeness Axiom. If $E \subset \mathbb{R}$ is nonempty and is bounded above, then E has a finite supremum.

Theorem 1.6. *(Archimedean Principle). Given positive real numbers $a, b \in \mathbb{R}$ there is an integer $n \in \mathbb{N}$ such that $b < na$.*

Theorem 1.7. *(Density of Rational numbers). Let $a < b$ be real numbers. Then there is $q \in \mathbb{Q}$ such that $q \in (a, b)$.*

Definition 1.5. *Let $E \subset \mathbb{R}$ be nonempty.*

- *The set E is said to be bounded below if there is $m \in \mathbb{R}$ such that $m \leq a$ for all $a \in E$.*
- *A number m is called an lower bound of the set E if $m \leq a$ for all $a \in E$.*
- *A number t is called the* **infimum** *of the set E if*
  - *t is a lower bound of E*
  - *$m \leq t$ for all lower bounds m of the set E.*

  If number t exists, we shall say that E has an infimun and write $t = \inf E$.

We observe that supremum and infimum is related via the following (reflection) principle. Here the set $-E$ is defined as

$$-E = \{x \in \mathbb{R} : x = -e \text{ for some } e \in E\}.$$

Theorem 1.8. *Let $E \subset \mathbb{R}$ be nonempty.*

- *Set E has a supremum if and only if the set $-E$ has an infimum. Also*
$$\inf(-E) = -\sup E.$$
- *Set E has an infimum if and only if the set $-E$ has a supremum. Also*
$$\sup(-E) = -\inf E.$$

1.6 Countability.

Definition 1.6. *Let f be a function from a set X into a set Y.*

(i) f is said to be **one-to-one** *(1-1) on X if and only if each element $y \in Y$ is assigned to at most one $x \in X$. That is*

$$\text{If } x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

(ii) f is said to take X onto Y if for each $y \in Y$ there is an $x \in X$ such that $y = f(x)$.

Definition 1.7. *Let E be a set.*

(i) E is said to be finite if either $E = \emptyset$ or there is an integer $n \in \mathbb{N}$ and a bijection $f : \{1, 2, 3, \ldots, n\} \to E$. (we say that the set E has n elements).

(ii) E is said to be countable if there is a bijection function $f : \mathbb{N} \to E$.

(iii) E is said to be at most countable if E is finite or countable.

(iv) E is said to be uncountable if E is neither finite not countable.

Theorem 1.9. *(Cantor) The open interval $(0, 1)$ is uncountable.*

Theorem 1.10. *A nonempty set E is at most countable if there is an onto (surjective) function $f : \mathbb{N} \to E$.*

Theorem 1.11. *Let A, B are sets.*

(i) If $A \subset B$ and B is at most countable, then A is at most countable.

(ii) If $A \subset B$ and A is uncountable, then B is uncountable.

(iii) The set of real numbers $\mathbb{R}$ is uncountable.

Theorem 1.12. *Let $A_1, A_2, A_3, \ldots$ be at most countable sets.*

(i) $A_1 \times A_2$ is at most countable.

(ii) $E = \bigcup_{j=1}^{\infty} A_j = \{x : x \in A_j \text{ for some } j \in \mathbb{N}\}$ is at most countable.

Definition 1.8. *Let X, Y be two sets and $f : X \to Y$. The image of a set $E \subset X$ under f is the set*

$$f(E) = \{(y = f(x) : \text{for some } x \in E\}.$$

The inverse image if a set $E \subset Y$ under f is the set

$$f^{-1}(E) = \{x \in X : \text{ there is } y \in E \text{ such that } f(x) = E\}.$$

2 Real Sequences

2.1 Introduction

Definition 2.1. *A sequence of real numbers (x_n) is said to converge to a real number a if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ (N usually depends on ε) such that*

$$\text{for all } n \geq N \quad \text{we have that} \quad |x_n - a| < \varepsilon.$$

Definition 2.2. *By a subsequence of a sequence $(x_n)_{n \in N}$ we shall mean a sequence of the form $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ (written shortly as $(x_{n_k})_{k \in \mathbb{N}}$) where $n_1 < n_2 < n_3 < \ldots$ is an increasing sequence of positive integers.*

Definition 2.3. *Let (x_n) be a sequence of real numbers.*

- *$(x_n)_{n \in \mathbb{N}}$ is said to be bounded above if $x_n \leq M$ for some $M \in \mathbb{R}$ and all $n \in \mathbb{N}$.*
- *$(x_n)_{n \in \mathbb{N}}$ is said to be bounded below if $x_n \geq m$ for some $m \in \mathbb{R}$ and all $n \in \mathbb{N}$.*
- *$(x_n)_{n \in \mathbb{N}}$ is said to be bounded it is bounded both above and below.*

Theorem 2.1. *Every convergent sequence is bounded.*

2.2 Limit Theorems

Theorem 2.2 (Squeeze Theorem)**.** *Suppose that (x_n), (y_n) and (w_n) are real sequences.*

- *If both $x_n \to a$ and $y_n \to a$ (the SAME a) as $n \to \infty$ and if*

  $$x_n \leq w_n \leq y_n, \qquad \text{for all } n \geq N_0,$$

  then $w_n \to a$ as $n \to \infty$.
- *If $x_n \to 0$ and (y_n) is bounded then the product $x_n y_n \to 0$ as $n \to \infty$.*

Theorem 2.3. *Let $E \subset \mathbb{R}$. If E has a finite supremum then there is a sequence $x_n \in E$ such that $x_n \to \sup E$ as $n \to \infty$. Analogous statement holds if E has a finite infimum as well.*

Proof. From the definition of supremum for each $n \in \mathbb{N}$ there exists $x_n \in \mathbb{N}$ such that

Theorem 2.4. *Suppose that (x_n), (y_n) are real sequences and $\alpha \in \mathbb{R}$. If both (x_n), (y_n) are convergent then*

(i)
$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

(ii)
$$\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n.$$

(iii)
$$\lim_{n\to\infty}(x_n \cdot y_n) = (\lim_{n\to\infty} x_n) \cdot (\lim_{n\to\infty} y_n).$$

(iv) If in addition $\lim_{n\to\infty} y_n \neq 0$ and $y_n \neq 0$ then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.$$

Definition 2.4. *Let (x_n) be a sequence of real numbers.*

(i) (x_n) *is said to diverge to* $+\infty$ *(notation* $x_n \to +\infty$ *as* $n \to \infty$ *or* $\lim_{n\to\infty} x_n = +\infty$*) if for each* $M \in \mathbb{R}$ *there is* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *we have* $x_n > M$*.*

(ii) (x_n) *is said to diverge to* $-\infty$ *(notation* $x_n \to -\infty$ *as* $n \to \infty$ *or* $\lim_{n\to\infty} x_n = -\infty$*) if for each* $M \in \mathbb{R}$ *there is* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *we have* $x_n < M$*.*

Theorem 2.5. *Suppose that* (x_n), (y_n) *are real sequences. If both* $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ *exist (and belong to the set of extended real numbers* $\mathbb{R}^*$*) then*

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$

Theorem 2.6 (Comparisson theorem for sequences)**.** *Suppose that* (x_n), (y_n) *are real sequences. If both* $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ *exist (and belong to the set of extended real numbers* $\mathbb{R}^*$*) and if*

$$x_n \leq y_n \qquad \text{for all } n \geq N \qquad \text{for some } N \in \mathbb{N}$$

then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

2.3 Bolzano-Weierstrass Theorem

Definition 2.5. *Let (x_n) be a sequence of real numbers.*

(i) (x_n) *is said to be increasing (respectively strictly increasing) if* $x_1 \leq x_2 \leq x_3 \leq \dots$ $(x_1 < x_2 < x_3 < \dots$ *for strictly increasing).*

(ii) (x_n) *is said to be decreasing (respectively strictly decreasing) if* $x_1 \geq x_2 \geq x_3 \geq \dots$ $(x_1 > x_2 > x_3 > \dots$ *for strictly increasing).*

(iii) (x_n) *is said to be monotone if it is either increasing or decreasing.*

Theorem 2.7 (On monotone convergence)**.** *If* (x_n) *is increasing and bounded above or if it is decreasing and bounded below, then* (x_n) *is convergent (and has a finite limit).*

Definition 2.6. *A sequence* $(I_n)_{n\in\mathbb{N}}$ *of sets is said to ne nested if*

$$I_1 \supset I_2 \supset I_3 \supset \dots.$$

Theorem 2.8 (Nested Interval Property)**.** *If* (I_n) *is a nested sequence of nonempty bounded intervals then*

$$E = \bigcap_{n\in\mathbb{N}} I_n = \{x \in \mathbb{R} : x \in I_n \text{ for all } n \in \mathbb{N}\}$$

is nonempty (i.e. it contains at least one number). Moreover if $|I_n| \to 0$ *(*$|I_n|$ *denotes the length of interval* I_n*) then* E *contains exactly one number.*

Theorem 2.9 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

2.4 Cauchy Sequences

Definition 2.7. *A sequence of points* $x_n \in \mathbb{R}$ *is said to be Cauchy if for every* $\varepsilon > 0$ *there is* $N \in \mathbb{N}$ *such that*

$$|x_n - x_m| < \varepsilon \qquad \text{for all } n, m \geq N.$$

Theorem 2.10 (Cauchy)**.** *Let* (x_n) *be a sequence of real numbers. Then* (x_n) *is a Cauchy sequence if and only if* (x_n) *is a convergent sequence.*

3 Continuity

3.1 Introduction

Definition 3.1. *Let $a \in \mathbb{R}$, let I be an open interval that contains a and let f be a real function defined everywhere on I except possibly at a. Then f is said to converge to L as x approaches a if for every $\varepsilon > 0$ there is a $\delta > 0$ (δ in general depends on ε, function f, interval I and a) such that*

$$\text{if} \quad 0 < |x-a| < \delta \qquad \text{then} \qquad |f(x)-L| < \varepsilon.$$

In this case we write

$$L = \lim_{x\to a} f(x).$$

Theorem 3.1 (Sequential characterization of limits). *Let $a \in \mathbb{R}$, let I be an open interval containing a and let f be a real function defined everywhere on I except possibly at a. Then*

$$L = \lim_{x\to a} f(x)$$

exists if and only if

$$\lim_{n\to\infty} f(x_n) = L$$

for every sequence (x_n) such that $x_n \in I \setminus \{a\}$ and $x_n \to a$ as $n \to \infty$.

Theorem 3.2 (Comparison Theorem for functions). *Let $a \in \mathbb{R}$, let I be an open interval containing a and let f, g be real functions defined everywhere on I except possibly at a. If both f and g have limits as x approaches a and*

$$f(x) \le g(x), \qquad \text{for all } x \in I \setminus \{a\},$$

then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

3.2 One sided limits and limits at infinity

Definition 3.2. *Let $a \in \mathbb{R}$, let I be an open interval with left end-point a.*
We say that f converges to L as x approaches a from the right, and write

$$\lim_{x\to a+} f(x) = L$$

or $f(x) \to L$ as $x \to a+$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{if } a < x < a+\delta \text{ and } x \in I, \quad \text{then} \quad |f(x)-L| < \varepsilon.$$

The value L of the limit we shall denoted as $f(a+)$.

Same goes for a-

Theorem 3.3. *Let f be a real function. The limit*

$$\lim_{x\to a} f(x)$$

exists and equals to L if and only if

$$L = \lim_{x\to a+} f(x) = \lim_{x\to a-} f(x).$$

3.3 Continuity

Definition 3.3. *Let $\emptyset \neq E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$.*

(i) f is continuous at a point $a \in E$ if for any $\varepsilon > 0$ there is $\delta > 0$ (depending on ε, f, a) such that

$$\text{if } |x-a| < \delta \text{ and } x \in E \quad \text{then} \quad |f(x) - f(a)| < \varepsilon.$$

(ii) f is continuous E (we write $f : E \to \mathbb{R}$ is continuous) if f is continuous at every point $a \in E$.

Theorem 3.4. *Let I be an open interval that contains a point a and $I \subset E$ for a function $f : E \to \mathbb{R}$. Then f is continuous at a if and only if*

$$f(a) = \lim_{x \to a} f(x).$$

Theorem 3.5. *Let $E \subset \mathbb{R}$ be nonempty and $f : E \to \mathbb{R}$. Then f is continuous at $a \in E$ if and only if*

$$\text{for all sequences } (x_n) \text{ such that } x_n \to a \text{ and } x_n \in E\text{: } \lim_{n \to \infty} f(x_n) = f(a).$$

Theorem 3.6. *Let $E \subset \mathbb{R}$ be nonempty and $f, g : E \to \mathbb{R}$. If f and g are continuous at a point $a \in E$ then so are $f + g$, fg and αf for any $\alpha \in \mathbb{R}$. Moreover, if $g(a) \neq 0$ the function f/g is continuous at $a \in E$ as well. Similar statements holds for continuity on the set E.*

Definition 3.4. *Let $A, B \subset \mathbb{R}$ be nonempty, let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subset B$. The composition of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by*

$$(g \circ f)(x) = g(f(x)), \qquad \text{for all } x \in A.$$

Theorem 3.7. *Let $A, B \subset \mathbb{R}$ be nonempty, let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subset B$.*

(i) If f is continuous at $a \in A$ and g is continuous at the point $f(a) \in B$ then $g \circ f$ is continuous at the point $a \in A$.

(ii) Let $I \setminus \{a\} \subset A$ where I is either a non-degenerate interval containing a or has a as one of its endpoints. If

$$\lim_{x \to a, x \in I} f(x) = L$$

exists, $L \in B$ and g is continuous at the point $L \in B$ then

$$\lim_{x \to a, x \in I} (g \circ f)(x) = g\left(\lim_{x \to a, x \in I} f(x) \right).$$

Definition 3.5. *Let $E \subset \mathbb{R}$ be nonempty. A function $f : E \to \mathbb{R}$ is said to be bounded on E if*

$$|f(x)| \leq M, \qquad \text{for all } x \in E,$$

where M is some (large) real number.

Theorem 3.8. *Let $I \subset \mathbb{R}$ be a closed bounded interval. Let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on the interval I.*

Denote by

$$m = \inf_{x \in I} f(x), \qquad M = \sup_{x \in I} f(x).$$

Then there exist points $x_m, x_M \in I$ such that

$$f(x_m) = m \qquad \text{and} \qquad f(x_M) = M.$$

Lemma 3.9. *Let $f : I \to \mathbb{R}$ where I is an open nonempty interval. If f is continuous at a point $a \in I$ and $f(a) > 0$ then for some $\delta, \varepsilon > 0$ we have that*

$$f(x) > \varepsilon, \qquad \text{for all } x \in (a - \delta, a + \delta).$$

Theorem 3.10 (Intermediate Values Theorem)**.** *Let I be a non-degenerate interval and $f : I \to \mathbb{R}$ continuous. If $a, b \in I$, $a < b$ then f attains on the interval (a, b) all values between $f(a)$ and $f(b)$. That is given y_0 between $f(a)$ and $f(b)$ there exists $x_0 \in (a, b)$ such that*

$$f(x_0) = y_0.$$

Example. The Riemann function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational}, \\ 1/q, & \text{if } x = p/q \text{ where } q \in \mathbb{N},\ p \in \mathbb{Z} \text{ and } (p,q) = 1), \end{cases}$$

is continuous on irrational numbers and discontinuous at all rational numbers.

4 Differentiability on $\mathbb{R}$

4.1 Introduction

Definition 4.1. *A real function f is said to be differentiable at a point $a \in \mathbb{R}$ if f is defined at some open interval containing a and*

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists. The number $f'(a)$ is called the derivative of f at the point a.

We give two different characterizations of derivative. One is using "chord" function (see below) and another is is in terms of linear approximation. These two notions are equivalent.

The chord function F is defined by

$$F(x) = \frac{f(x)-f(a)}{x-a}, \qquad \text{for } x \neq a.$$

Theorem 4.1. *A real function f is differentiable at a point $a \in \mathbb{R}$ if and only if there exists and open interval I and a function $F : I \to \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous at the point a and*

$$f(x) = F(x)(x-a) + f(a) \qquad \textit{holds for all} x \in I.$$

The number $F(a)$ is equal to $f'(a)$.

Theorem 4.2. *Let $f : I \to \mathbb{R}$ where I is an open interval containing point $a \in I$. Then f is differentiable at a if and only if there is a linear function T of the form $T(x) = mx$ such that*

$$\lim_{h\to 0} \frac{|f(a+h)-f(a)-T(h)|}{|h|} = 0.$$

If this limit exists then $f'(a) = m$.

Theorem 4.3. *If f is differentiable at a, then f is continuous at a.*

Definition 4.2. *Let I be a non-degenerate interval.*

(i) A function $f : I \to \mathbb{R}$ is said to be differentiable on I if

$$f_I'(a) := \lim_{x\to a, x\in I} \frac{f(x)-f(a)}{x-a}$$

exists and is finite at every $a \in I$.

(ii) f is said to be continuously differentiable on I if f_I' exists and is continuous on I.

4.2 Differentiability Theorems

Theorem 4.4. *Let f, g be real functions and $\alpha \in \mathbb{R}$. If f and g are differentiable at a, then $f+g$, $f\cdot g$, αf and (if $g(a) \neq 0$) also f/g are all differentiable at a. We have*

$$(f+g)'(a) = f'(a) + g'(a), \qquad (f\cdot g)'(a) = f'(a)g(a) + f(a)g'(a),$$

$$(\alpha f)'(a) = \alpha f'(a), \qquad \left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Theorem 4.5 (Chain Rule). *Let f, g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Theorem 4.6 (Power Rule). *Show that*

(i) If $n \in \mathbb{N}$ then $(x^n)' = nx^{n-1}$ for all $x \in \mathbb{R}$.

(ii) If $q \in \mathbb{Q}$ then $(x^q)' = qx^{q-1}$ for all $x > 0$.

4.3 Mean Value Theorem

Theorem 4.7 (Rolle's Theorem). *Suppose $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$ then $f'(c) = 0$ for some $c \in (a, b)$.*

Theorem 4.8 (Mean Value Theorem). *Suppose $a, b \in \mathbb{R}$ with $a < b$.*

(i) If f is continuous on $[a, b]$ and differentiable on (a, b) then there is $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

(ii) If f, g are continuous on $[a, b]$ and differentiable on (a, b) then there is $c \in (a, b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

Theorem 4.9 (L'Hopital's Rule). *Let a be an extended real number and I and interval that either contains a or has I as an endpoint.*

Let f, g be differentiable on $I \setminus \{a\}$ and $g(x) \neq 0 \neq g'(x)$ for all $x \in I \setminus \{a\}$. Suppose further that

$$A = \lim_{x \to a, x \in I} f(x) = \lim_{x \to a, x \in I} g(x)$$

is either 0 or ∞.

If $B = \lim_{x \to a, x \in I} \dfrac{f'(x)}{g'(x)}$ exists as an extended real number, then

$$\lim_{x \to a, x \in I} \frac{f(x)}{g(x)} = \lim_{x \to a, x \in I} \frac{f'(x)}{g'(x)}.$$

4.4 Monotone functions and the Inverse function theorem

Definition 4.3. *Let $E \subset \mathbb{R}$, $E \neq \emptyset$ and $f : E \to \mathbb{R}$.*

(i) f is called increasing (strictly increasing) on E if for all $x_1, x_2 \in E$, $x_1 < x_2$ we have $f(x_1) \leq f(x_2)$ ($f(x_1) < f(x_2)$ in the strictly increasing case).

(ii) f is called decreasing (strictly decreasing) on E if for all $x_1, x_2 \in E$, $x_1 < x_2$ we have $f(x_1) \geq f(x_2)$ ($f(x_1) > f(x_2)$ in the strictly decreasing case).

(iii) f is called monotone (strictly monotone) on E if f is either increasing or decreasing (respectively, either strictly increasing or strictly decreasing) on E.

Theorem 4.10. *Let $a < b$ are real and f be continuous on $[a,b]$ and differentiable on (a,b).*

(i) If $f'(x) > 0$ for all $x \in (a,b)$ then f is strictly increasing on $[a,b]$.

(ii) If $f'(x) < 0$ for all $x \in (a,b)$ then f is strictly decreasing on $[a,b]$.

(iii) $f'(x) = 0$ for all $x \in (a,b)$ then f is constant on $[a,b]$.

Theorem 4.11. *Let f be 1-1 and continuous on an interval I. Then f is strictly monotone on I and the inverse function f^{-1} is continuous and strictly monotone on $f(I)$.*

Theorem 4.12 (Inverse function theorem)**.** *Let f be 1-1 and continuous on an interval I. If $a \in f(I)$ and f' at the point $f^{-1}(a)$ exists and is nonzero, then f^{-1} is differentiable at a and*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

4.5 Taylor's theorem

Definition 4.4. *Let $n \in \mathbb{N}$ and $a < b$ be extended real numbers. If $f : (a,b) \to \mathbb{R}$ is a function differentiable n-times at a point $x_0 \in (a,b)$ we call the polynomial*

$$P_n^{f,x_0}(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^n$$

Taylor's polynomial of degree n at x_0.

Theorem 4.13 (Taylor's formula)**.** *Let $n \in \mathbb{N}$ and $a < b$ be extended real numbers. If $f : (a,b) \to \mathbb{R}$ and if $f^{(n+1)}$ exists on (a,b), then for each $x, x_0 \in (a,b)$ there exists a number c between x and x_0 such that*

$$\begin{aligned} f(x) &= f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \\ &= P_n^{f,x_0}(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

Example. The Taylor's polynomial of function e^x of degree n at 0 is

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

The "error term" equals to

$$e^x - (1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}) = \frac{e^c}{(n+1)!}x^{n+1},$$

5 Infinite series of real numbers

5.1 Introduction

Definition 5.1. *Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series with terms a_k. For each n the partial sum of S of order n is the defined by*

$$s_n = \sum_{k=1}^{\infty} a_k.$$

S is said to converge if an only if its sequence of partial sums (s_n) converge to some $s \in \mathbb{R}$ as $n \to \infty$. That is for any $\varepsilon > 0$ the is $N \in \mathbb{N}$ such that if $n \geq N$ we have

$$|s_n - s| = \left| \sum_{k=1}^{n} a_k - s \right| < \varepsilon.$$

In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call the number s the **sum** *or* **value** *of the series $\sum_{k=1}^{\infty} a_k$.*

S is said to diverge if its sequence of partial sums (s_n) does not converge as $n \to \infty$. When (s_n) diverge to ∞ (or $-\infty$) we shall write

$$\sum_{k=1}^{\infty} a_k = \pm\infty.$$

(Example: $\sum_{n=1}^{\infty} n! = \infty$.)

Theorem 5.1 (Divergence test)**.** *Let (a_k) be a sequence of real numbers. If a_k does not converge to zero then the series*

$$\sum_{k=1}^{\infty} a_k \qquad \text{diverges.}$$

Theorem 5.2 (Telescopic series)**.** *Let (a_k) be a convergent sequence of real numbers. Then*

$$\sum_{k=1}^{\infty} (a_k - a_{k+1}) = a_1 - \lim_{k \to \infty} a_k.$$

Theorem 5.3 (Geometric series)**.** *Let $x \in \mathbb{R}$ and $N \in \{0, 1, 2, \dots\}$. Then the series*

$$\sum_{k=N}^{\infty} x^k \qquad \text{converges if and only if } |x| < 1.$$

In this case

$$\sum_{k=N}^{\infty} x^k = \frac{x^N}{1-x}.$$

In particular,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad |x| < 1.$$

Theorem 5.4 (Cauchy criterion). *Let (a_k) be a real sequence. The infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m \geq n \geq N$ we have*

$$\left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$

Theorem 5.5. *Let (a_k) and (b_k) be a real sequences. If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent then*

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

and

$$\sum_{k=1}^{\infty} (\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k.$$

5.2 Series with nonnegative terms

Theorem 5.6. *Suppose that $a_k \geq 0$ for large k. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums (s_n) is bounded. That is there exists $M > 0$ such that*

$$\left| \sum_{k=1}^{n} a_k \right| \leq M, \qquad \text{for all } n \in \mathbb{N}.$$

Theorem 5.7 (Integral test). *Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Let $a_k = f(k)$, $k = 1, 2, 3, \ldots$. Then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral*

$$\int_1^{\infty} f(x)\, dx < \infty.$$

Example [p-series test] The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Theorem 5.8 (Comparison test). *Suppose that $0 \leq a_k \leq b_k$ for large k*

- *If $\displaystyle\sum_{k=1}^{\infty} b_k < \infty$ then $\displaystyle\sum_{k=1}^{\infty} a_k < \infty$.*
- *If $\displaystyle\sum_{k=1}^{\infty} a_k = \infty$ then $\displaystyle\sum_{k=1}^{\infty} b_k = \infty$.*

Theorem 5.9 (Limit Comparison test). *Suppose that $0 \leq a_k$, $0 < b_k$ for large k and that*

$$L = \lim_{n \to \infty} \frac{a_n}{b_n} \qquad \text{exists as an extended real number.}$$

- *If* $L \in (0,\infty)$ *then* $\sum_{k=1} a_k$ *converges if and only if* $\sum_{k=1} b_k$ *converges.*

- *If* $L = 0$ *and* $\sum_{k=1}^{\infty} b_k$ *converges then* $\sum_{k=1}^{\infty} a_k$ *converges.*

- *If* $L = \infty$ *and* $\sum_{k=1}^{\infty} b_k$ *diverges then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

5.3 Absolute convergence

Definition 5.2. *Let* $S = \sum_{k=1}^{\infty} a_k$ *be an infinite series.*

We say that the series S *converges absolutely if* $\sum_{k=1}^{\infty} |a_k| < \infty$.

We say that the series S *converges conditionally if* S *converges but* $\sum_{k=1}^{\infty} |a_k|$ *diverges.*

Theorem 5.10. *If* $\sum_{k=1}^{\infty} a_k$ *converges absolutely, then* $\sum_{k=1}^{\infty} a_k$ *converges, but not conversely. There exists a conditionally convergent series.*

Theorem 5.11 (Root test). *Let* $a_k \in \mathbb{R}$ *and assume that* $r = \lim_{k\to\infty} |a_k|^{1/k}$ *exists. If*

- $r < 1$ *then the series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

- $r > 1$ *then the series* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Usually easier to check is the following test

Theorem 5.12 (Ration test). *Let* $a_k \in \mathbb{R}$ *and assume that* $r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ *exists as an extended real number. If*

- $r < 1$ *then the series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

- $r > 1$ *then the series* $\sum_{k=1}^{\infty} a_k$ *diverges.*

5.4 Series with alternating signs

Theorem 5.13 (Alternating sign series). *Let* (a_k) *be a decreasing sequence on nonnegative numbers such that* $a_k \to 0$ *as* $k \to \infty$. *Then the series*

$$\sum_{k=1}^{\infty} (-1)^k a_k \qquad \text{is convergent.}$$