Fundamentals of Pure Mathematics 2014-15
Analysis Notes

These are last year's notes by Martin Dindos. If you find any misteaks please email the current lecturer at n.bournaveas@ed.ac.uk

Contents

The goal of this course is to present Analysis with proofs. This is **not** yet another Calculus course; our focus will be less on computational aspects and more on understanding of the theory including many subtler points such as completeness of the real numbers, notions of continuity and convergence and uniform convergence. We will develop the theory from basis principles and will teach you how to develop your own proofs.

1 The Real Numbers

Every rigorous study on mathematics begins with certain undefined concepts; primitive notions on which the theory is based and certain postulates that are assumed to be true and do not need proof. In our case these primitive notions will be notions of sets and real numbers with certain postulated properties.

As a side note we mention, that the sets of numbers (integers, rational and real numbers) are constructed in another theory; a Set Theory. Hence what we take here as a primitive notion and postulate its properties without proof is in another part of mathematics proven from yet simpler notions and concepts.

1.1 Algebraic structure of real numbers

We shall denote the set of real numbers by $\mathbb{R}$. The set $\mathbb{R}$ is equipped with two algebraic operations $+$ and $\cdot$ (these are functions with domain $\mathbb{R}\times\mathbb{R}$ and range in $\mathbb{R}$). We shall assume that the triple $(\mathbb{R},+,\cdot)$ is a **field**; that is it satisfies the following properties for every $a,b,c\in\mathbb{R}$:

- **Closure properties:** $a+b$ and $a\cdot b$ belong to $\mathbb{R}$.

- **Associative properties:** $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Commutative properties:** $a+b = b+a$ and $a \cdot b = b \cdot a$.
- **Existence of the Additive Identity:** There is a unique element $0 \in \mathbb{R}$ such that $0+a = a$ for all $a \in \mathbb{R}$.
- **Existence of the Multiplicative Identity:** There is a unique element $1 \in \mathbb{R}$ such $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbb{R}$.
- **Existence of Additive Inverses:** For every $a \in \mathbb{R}$ there is a unique element $-a \in \mathbb{R}$ such that $a+(-a) = 0$.
- **Existence of Multiplicative Inverses:** For every $a \in \mathbb{R}$ and $a \neq 0$ there is a unique element $a^{-1} \in \mathbb{R}$ such that $a \cdot (a^{-1}) = 1$.
- **Distributive Law:** (This is the only law connecting addition with multiplication) $a \cdot (b+c) = a \cdot b + a \cdot c$.

Remark: Properties 1-6 mean that $(\mathbb{R},+)$ is a commutative group. They also mean that $(\mathbb{R} \setminus \{0\}, \cdot)$ is also a commutative group. The last property ties these two algebraic operators $+$ and $\cdot$ together.

Remark 2: To simplify our notation in what follows we shall usually write

- $a-b$ instead of $a+(-b)$
- ab instead of $a \cdot b$
- $1/a$ or $\frac{1}{a}$ instead of a^{-1}
- $\frac{a}{b}$ or a/b instead of $a \cdot b^{-1}$

From the postulated properties one might derive all algebraic laws of $\mathbb{R}$.

Exercise: Show that

- $-a = (-1) \cdot a$ and $-(-a) = a$
- $-(a-b) = b-a$
- If $a, b \in \mathbb{R}$ and $ab = 0$ then either $a = 0$ or $b = 0$.

The fact that $\mathbb{R}$ is a **field** does not completely describe the real number system. The set of real numbers is also **ordered**, i.e. it has a concept of “less than”.

Order Axioms: There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ that has the following properties:

- **Trichotomy Property:** Given $a, b \in \mathbb{R}$ one and only one of the following statements hold:

$$a < b, \qquad b < a, \qquad a = b.$$

- **Transitive Property:** $a < b$ and $b < c$ imply $a < c$.
- **Additive Property:** $a < b$ and $c \in \mathbb{R}$ imply $a + c < b + c$.
- **Multiplicative Properties**

$$a < b \qquad \text{and} \qquad c > 0 \qquad \text{imply} \qquad ac < bc$$

and

$$a < b \qquad \text{and} \qquad c < 0 \qquad \text{imply} \qquad bc < ac.$$

Remark: By $b > a$ we shall mean $a < b$. By $a \leq b$ and $b \geq a$ we shall mean $a < b$ or $a = b$. If $a < b$ and $b < c$ we shall simply write $a < b < c$.

Definition 1.1. *We shall call a real number $a \in \mathbb{R}$ positive if $a > 0$. We shall call $a \in \mathbb{R}$ nonnegative if $a \geq 0$.*

The set $\mathbb{R}$ contains certain special subsets: the set of natural numbers $\mathbb{N}$

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$

the set of integers $\mathbb{Z}$

$$\mathbb{Z} = \{\cdots - 2, -1, 0, 1, 2, 3, \ldots\},$$

and the set of rationals

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}.$$

We also call the numbers belonging to the set $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ irrational numbers.

Remark. Note that we have not really defined the sets $\mathbb{N}$ and $\mathbb{Z}$ since it is not clear what number 2 or 3 really mean without a proper definition. We define 2 as $1 + 1$ and 3 as $2 + 1$ or $1 + 1 + 1$. The sets $\mathbb{N}$ and $\mathbb{Z}$ are characterized by the following properties:

- Given $n \in \mathbb{Z}$ then one and only one of the following holds:

$$n \in \mathbb{N}, \qquad -n \in \mathbb{N} \qquad \text{or} \qquad n = 0.$$

- If $n \in \mathbb{N}$ then $n \geq 1$ and $n + 1 \in \mathbb{N}$.
- If $n \in \mathbb{N}$ and $n \neq 1$ then $n - 1 \in \mathbb{N}$.
- If $n \in \mathbb{Z}$ and $n > 0$ then $n \in \mathbb{N}$.

Observe that $\mathbb{N}$ and $\mathbb{Z}$ are not fields, the set $\mathbb{N}$ fails to contain additive identity 0 and also has no additive inverses. Also $(\mathbb{Z}, +)$ is a group but $\mathbb{Z}$ fails to contain multiplicative inverses (for example $1/2$ which is a multiplicative inverse of $2 \in \mathbb{Z}$ fails to be in $\mathbb{Z}$). Finally, $\mathbb{Q}$ is as **field**. The same ordering we have on $\mathbb{R}$ also works for $\mathbb{Q}$ which means that so far all axioms we stated for the set of real numbers are also satisfied by the set of rationals. We shall introduce two further sets of axioms that will make distinction between these two sets.

1.2 Mathematical proof

What is a mathematical proof? Every mathematical statement has a list of hypotheses and a conclusion(s). Mathematical proof is a sequence of logical steps that shows from the hypotheses of the statement and axioms (in our case axioms of real numbers) that the conclusion must be true.

There are three main methods of proof: mathematical induction (which we will introduce in the next section), direct deduction and proof by contradiction. The proof by direct deduction goes as follows. We assume the hypotheses of the statement are true and proceed step by step to the conclusion. Each step is justified by hypothesis, one of the axioms or a mathematical result that has already been proved.

The proof by contradiction has the following construction. We assume the the hypotheses of the statement we want to establish are true and that the conclusion we want to establish is false. Then we work step by step (like when doing direct deduction) until we obtain a statement that is obviously false. At this point we are done and using mathematical logic we can deduce that the conclusion we wanted to establish must be true (since assuming the opposite lead us to a contradiction).

Example. Prove that $a \neq 0$ implies that $a^2 > 0$.

Proof. By the Trichotomy Property either $a > 0$ or $a < 0$. *Case 1:* We first consider the case when $a > 0$. Then by the first Multiplicative property if we multiply both sides of inequality $0 < a$ by a we obtain $0 = 0 \cdot a < a \cdot a = a^2$.

Case 2: If $a < 0$ then $-a > 0$. Since

$$a^2 = a \cdot a = (-a) \cdot (-a) = (-a)^2$$

we again obtain that $a^2 > 0$ since by the *Case 1* we have already established does apply to $-a > 0$.

Exercise: Show that $\sqrt{2}$ is an irrational number. *Hint:* Use proof by contradiction. Assume that $\sqrt{2}$ is a rational number.

1.3 The absolute value and intervals

Definition 1.2. *Let $a \in \mathbb{R}$. The absolute value of a is the number $|a|$ defined by*

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

Theorem 1.1. *The following statements are true:*

- *The absolute value is multplicative, i.e., $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.*
- *Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M$ if and only if $-M \leq a \leq M$.*

Theorem 1.2. *The absolute value satisfies the following three properties:*

(i) (Positive definiteness) $|a| \geq 0$. $|a| = 0$ if and only if $a = 0$.

(ii) (Symmetry) $|a-b| = |b-a|$.

(iii) (Triangle inequality) $|a+b| \leq |a| + |b|$.

The absolute value is closely related with the notion of distance. If a, b are two real numbers the define the distance between these two points by $|b-a|$. We might also denote the distance between a and b by the symbol $d(a,b)$. Hence $d(a,b) = |b-a|$.

Notice that the property (i) of the last Theorem assures that the distance between two points is *never* negative. The property (ii) says that the distance between a and b is the same as the distance between b and a. Finally, the (iii) can be interpreted as follows:

$$d(a,b) \leq d(a,c) + d(b,c).$$

Hint: Use (iii) to prove $|b-a| = |(b-c)+(c-a)| \leq |b-c| + |c-a|$.

Intervals: Let a, b are real numbers. A *closed* interval is a set of the form:

$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}, \qquad [a,\infty) = \{x \in \mathbb{R} : a \leq x\},$$

$$(\infty, b] = \{x \in \mathbb{R} : x \leq b\}, \quad \text{or} \quad (-\infty, \infty) = \mathbb{R}.$$

Note that if $a > b$ then $[a,b]$ is an empty set. An *open* interval is a set of the form

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}, \qquad (a,\infty) = \{x \in \mathbb{R} : a < x\},$$

$$(\infty, b) = \{x \in \mathbb{R} : x < b\}, \quad \text{or} \quad (-\infty, \infty) = \mathbb{R}.$$

Finally a *half-open* interval is a set of the form

$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}, \quad \text{or} \quad [a,b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

By saying *interval* we mean a closed interval, an open interval or half-open interval as defined above.

If $a < b$ the intervals (a,b), $[a,b]$, $(a,b]$ and $[a,b)$ correspond to line segment on a real line (with then end-points a and b either belonging or not belonging to the set). As we already observed of $b > a$ these "intervals" are all the empty set.

Theorem 1.3. *Let $x, y \in \mathbb{R}$.*

(i) $x < y + \varepsilon$ for all $\varepsilon > 0$ if an only if $x \leq y$.

(ii) $x > y - \varepsilon$ for all $\varepsilon > 0$ if an only if $x \geq y$.

(iii) $|x| < \varepsilon$ for all $\varepsilon > 0$ if an only if $x = 0$.

Proof. (i) We prove the statement by contradiction. Assume that $x < y + \varepsilon$ for all $\varepsilon > 0$ but $x > y$. Set $\varepsilon_0 = x - y > 0$. Then $x = y + \varepsilon_0$, hence the hypothesis (i) is not satisfied for $\varepsilon = \varepsilon_0$. This is a contradiction.

(ii) follows from (i) (by multiplication of both sides by -1).

Finally, If $|a| < \varepsilon$ for all $\varepsilon > 0$ it follows that $-\varepsilon < x < \varepsilon$. Using parts (i) and (ii) (for $y = 0$) we then conclude that $0 \leq a \leq 0$. Then by the Trichotomy property $a = 0$.

1.4 Well ordering principle

Definition 1.3. *A number x is the least element of a set $E \subset \mathbb{R}$ if and only if $x \in E$ and $x \leq a$ for all $a \in E$.*

Well ordering Principle: Every nonempty subset of $\mathbb{N}$ has a least element.

Notice that neither $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ satisfy the well ordering principle.

Theorem 1.4. *(On mathematical Induction). Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition (a verbal statement or a formulae) that satisfies the following two properties:*

(i) $A(1)$ is true

(ii) For every $k \in \mathbb{N}$ for which $A(k)$ is true, $A(k+1)$ is also true.

The for every integer $n \in \mathbb{N}$ the $A(n)$ is true.

Proof. Suppose that this Theorem is false. Denote by E the set $\{n \in \mathbb{N} : A(n) \text{ is false}\}$. It follows that E is nonempty. Thus by the Well Ordering Principle E has a least element, say x. Also by (i) clearly $x \neq 1$. It follows from the properties of the set $\mathbb{N}$ that $x-1 \in \mathbb{N}$. Also $x-1 < x$and hence $x-1 \notin E$ so $A(x-1)$ is true. Applying hypothesis (ii) to $k = x-1$ we see that $A(x) = A(k+1)$ must also be true. It follows that $x \notin E$ which is a contradiction.

1.5 Completeness Axiom

Definition 1.4. *Let $E \subset \mathbb{R}$ be nonempty.*

- *The set E is said to be bounded above if there is $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$.*
- *A number M is called an upper bound of the set E if $a \leq M$ for all $a \in E$.*
- *A number s is called the* **supremum** *of the set E if*
  - *s is an upper bound of E*
  - *$s \leq M$ for all upper bounds M of the set E.*

  If number s exists, we shall say that E has a supremum and write $s = \sup E$.

We observe that if the supremum s exists then s is the least upper bound of the set E.

Example. If $E = [0,1]$ show that $\sup E = 1$. Clearly 1 is an upper bound of E as by the definition of an interval $x \leq 1$ for all $\in E$. Let M be any other upper bound of E. Then $x \leq M$ for all $x \in E$. Since $1 \in E$ it follows that $1 \leq M$. Thus 1 is the smallest upper bound of the set E.

Remark. If set has a supremum, it only has one supremum.

Theorem 1.5. *(Approximation Property for Suprema). Let the set $E \subset \mathbb{R}$ has a supremum. Then for any positive number $\varepsilon > 0$ there exists $a \in E$ such that*

$$\sup E - \varepsilon < a \leq \sup E.$$

Proof. Suppose this theorem is false for some $\varepsilon > 0$. It follows that no element of E lies between $\sup E - \varepsilon$ and $\sup E$. But then $a \leq \sup E - \varepsilon$ for all $a \in E$ and hence $\sup E - \varepsilon$ must be an upper bound of E. Since $\sup E$ is the smallest upper bound it must be true that

$$\sup E \leq \sup E - \varepsilon, \qquad \text{or} \qquad 0 \leq -\varepsilon.$$

This is a contradiction with the fact that $\varepsilon > 0$. □

Remark. If $E \subset \mathbb{N}$ has a supremum then $\sup E \in E$.

Proof. Let $s = \sup E$. Apply the Approximation Property (choosing $\varepsilon = 1$). If follows that there is $x_0 \in E$ such that

$$s - 1 < x_0 \leq s = \sup E.$$

If $x_0 = s$ then $s \in E$ as promised and we are done.

Otherwise $s - 1 < x_0 < s$ and we can apply the Approximation property again to choose $x_1 \in E$ such that $x_0 < x_1 < s$. By subtracting x_0 we obtain $0 < x_1 - x_0 < s - x_0$. Since $x_1 - x_0$ is a positive integer we have that $x_1 - x_0 \geq 1$. But since $s - 1 < x_0$ then

$$x_1 - x_0 < s - (s - 1) = 1$$

which is a contradiction. □

Completeness Axiom. If $E \subset \mathbb{R}$ is nonempty and is bounded above, then E has a finite supremum.

Theorem 1.6. *(Archimedean Principle). Given positive real numbers $a, b \in \mathbb{R}$ there is an integer $n \in \mathbb{N}$ such that $b < na$.*

Proof. If $b < a$ we set $n = 1$ and we are done.

Otherwise, denote by E the set $\{k \in \mathbb{N} : ka \leq b\}$. Clearly, E is nonempty as $1 \in E$. The set E is also bounded as $ka \leq b$ is equivalent to $k \leq b/a$ so b/a is an upper bound of E. Then by the completeness axiom $s = \sup E$ exists. Also by the Remark we made above, since $E \subset \mathbb{N}$ we have that $s \in E$. We set $n = s + 1$. Then $n \in \mathbb{N}$ and $na > b$ hence the claim holds. □

Example 1: Show that for any real number $r > 0$ there is an integer $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < r.$$

Proof. Set $b = 1/r$ and $a = 1$ in the Archimedean Principle. It follows that there is $n \in \mathbb{N}$ such that $1/r < na = n$. But $1/r < n$ is equivalent to $1/n < r$. □

Example 2: Show that if $B = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ then $\sup B = 1$.

Proof. Clearly, 1 is an upper bound of B. Assume that there exist a smaller upper bound of B, say M such that $M < 1$. Then $1 - M > 0$ and hence by Example 1 we can find an integer $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < 1 - M.$$

It follows that

$$M < 1 - \frac{1}{n} = \frac{n-1}{n}.$$

Hence the number $(n-1)/n$ is bigger than M, but it also belong to the set B as B contains all numbers of the form $(n-1)/n$ where n is an integer $n \geq 2$. This is a contradiction as M is supposed to be an upper bound of B. Hence $M < 1$ cannot be an upper bound of B. From this $1 = \sup B$. □

Theorem 1.7. *(Density of Rational numbers). Let $a < b$ be real numbers. Then there is $q \in \mathbb{Q}$ such that $q \in (a,b)$.*

Proof. Since $b - a > 0$ by Example 1 one can find an integer such that $\frac{1}{n} < b - a$. Consider two cases.

(Case 1). If $b > 0$ the set $E = \{k \in \mathbb{N} : b \leq k/n\}$ is nonempty (By the Archimedean principle). By the Well Ordering Principle E has the least element, say k_0. Set $m = k_0 - 1$ and $q = m/n$. Clearly, $m \notin E$. Either $m \leq 0$ or $b > m/n = q$. Hence in either case $q < b$. On the other hand since $b \leq k_0/n$ if follows that

$$a = b - (b-a) < \frac{k_0}{n} - \frac{1}{n} = \frac{k_0 - 1}{n} = q.$$

The second case is if $b \leq 0$. By the Archimedean principle one can find integer k such that $k + b > 0$. Then using Case 1 there is $q \in \mathbb{Q}$ such that $a + k < q < b + k$. Therefore $a < q - k < b$, and thus $q - k$ is a rational number between a and b. □

Definition 1.5. *Let $E \subset \mathbb{R}$ be nonempty.*

- *The set E is said to be bounded below if there is $m \in \mathbb{R}$ such that $m \leq a$ for all $a \in E$.*
- *A number m is called an lower bound of the set E if $m \leq a$ for all $a \in E$.*
- *A number t is called the* **infimum** *of the set E if*
  - *t is a lower bound of E*
  - *$m \leq t$ for all lower bounds m of the set E.*

  If number t exists, we shall say that E has an infimun and write $t = \inf E$.

We observe that supremum and infimum is related via the following (reflection) principle. Here the set $-E$ is defined as

$$-E = \{x \in \mathbb{R} : x = -e \text{ for some } e \in E\}.$$

Theorem 1.8. *Let $E \subset \mathbb{R}$ be nonempty.*

- *Set E has a supremum if and only if the set $-E$ has an infimum. Also*
$$\inf(-E) = -\sup E.$$
- *Set E has an infimum if and only if the set $-E$ has a supremum. Also*
$$\sup(-E) = -\inf E.$$

Exercise. (Monotone Property). If $A \subset B$ are two nonempty subsets of $\mathbb{R}$. If B is bounded above then $\sup A \leq \sup B$. If B is bounded below then $\inf A \geq \inf B$.

1.6 Countability.

Let us first recall what is a function $f : X \to Y$ between two sets X and Y. Each element $x \in X$ is assigned a unique (meaning one and only one) $y = f(x) \in Y$. Notice that same y can be assigned to two (or more) different x's (An example can be a function $f(x) = x^2$ where both $x = -1, 1$ are assigned same value 1.). Also not every $y \in Y$ must be assigned (Again $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ does not assign values $y < 0$ to any x).

For this reason we introduce the following terminology.

Definition 1.6. *Let f be a function from a set X into a set Y.*

(i) f is said to be **one-to-one** *(1-1) on X if and only if each element $y \in Y$ is assigned to at most one $x \in X$. That is*

$$\text{If } x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

(ii) f is said to take X onto Y if for each $y \in Y$ there is an $x \in X$ such that $y = f(x)$.

Functions that are 1-1 are also called *injective*, functions that are *onto* are also called *surjective*. Functions that are both 1-1 and onto are called *bijective*.

For example $f(x) = x^2$ bijective as a function $f : [0,\infty) \to [0,\infty)$ but is neither 1-1 or onto as a function $f : \mathbb{R} \to \mathbb{R}$.

Function $f : X \to Y$ that is 1-1 and onto (bijection) has an *inverse* function. Inverse function to f is a unique function $g : Y \to X$ such that

$$g(f(x)) = x \text{ for all } x \in X \qquad \text{and} \qquad f(g(y)) = y \text{ for all } y \in Y.$$

The function g is usually denoted as f^{-1}. This is a bit unfortunate notation since f^{-1} also denotes the function $1/f$. Since we are using the same notation for two very different things you must deduce from the context whether we mean an inverse function or a reciprocal $1/f$.

Functions that have inverse can be used to "count" number of elements in finite and infinite sets. Intuitively, here is how it works. When we count "E" we assign a number $n \in \mathbb{N}$ to each element of E. This means that we construct a function from a subset of $\mathbb{N}$ to E. For example. if E contains 3 objects the counting function takes $\{1,2,3\}$ to E. Since we don't want to count any element E more than once f should be 1-1. We also do not want to miss any element of E and hence the counting function f must be onto E. This intuition leads us to the following definition.

Definition 1.7. *Let E be a set.*

(i) E is said to be finite if either $E = \emptyset$ or there is an integer $n \in \mathbb{N}$ and a bijection $f : \{1,2,3,\dots,n\} \to E$. (we say that the set E has n elements).

(ii) E is said to be countable if there is a bijection function $f : \mathbb{N} \to E$.

(iii) E is said to be at most countable if E is finite or countable.

(iv) *E is said to be uncountable if E is neither finite not countable.*

Observe a peculiar property. Even though the set of even positve integers $\{2,4,6,\dots\}$ is proper subset of $\mathbb{N}$, by our definition it has the same size as $\mathbb{N}$ since there is a bijection $f(n) = 2n$ between the set $\mathbb{N}$ and the set of even positive integers.

Not every set is countable, for example open nonempty intervals are uncountable.

Theorem 1.9. *(Cantor) The open interval* $(0,1)$ *is uncountable.*

Proof. Suppose to the contrary that $(0,1)$ is countable. Then there is a bijection $f : \mathbb{N} \to (0,1)$. Let us write each number $f(j)$, $j \in \mathbb{N}$ in decimal notation (using finite expansion if possible). That is

$$f(1) = 0.\alpha_{11}\alpha_{12}\alpha_{13}\dots,$$
$$f(2) = 0.\alpha_{21}\alpha_{22}\alpha_{23}\dots,$$

and so on for each $f(j)$. The number α_{ij} represents the j-th digit in the decimal expansion of $f(i)$.

We now show that the function f is not onto the interval $(0,1)$. We construct a real number x from this interval not equal to $f(j)$ for any $j \in \mathbb{N}$. We consider

$$x = 0.\beta_1\beta_2\beta_3\dots,$$

such that we take $\beta_k = \alpha_{kk} + 1$ if $\alpha_{kk} \le 5$ and $\beta_k = \alpha_{kk} - 1$ otherwise. Observe that if $x = f(j)$ it would have to be true that $\beta_k = \alpha_{jk}$ for every $k = 1,2,3,\dots$. That however is false as we defined β_j such that it is different from α_{jj}. Hence $x \ne f(j)$ for any j and therefore f is not onto $(0,1)$. This is a contradiction and therefore the set $(0,1)$ is uncountable. □

Theorem 1.10. *A nonempty set E is at most countable if there is an onto (surjective) function* $f : \mathbb{N} \to E$.

Theorem 1.11. *Let A, B are sets.*

(i) *If* $A \subset B$ *and B is at most countable, then A is at most countable.*

(ii) *If* $A \subset B$ *and A is uncountable, then B is uncountable.*

(iii) *The set of real numbers* $\mathbb{R}$ *is uncountable.*

Theorem 1.12. *Let* $A_1, A_2, A_3, \dots$ *be at most countable sets.*

(i) $A_1 \times A_2$ *is at most countable.*

(ii) $E = \bigcup_{j=1}^{\infty} A_j = \{x : x \in A_j \textit{ for some } j \in \mathbb{N}\}$ *is at most countable.*

Remark. The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable, the set of all irrational numbers is uncountable.

Definition 1.8. *Let* X, Y *be two sets and* $f : X \to Y$. *The image of a set* $E \subset X$ *under* f *is the set*

$$f(E) = \{(y = f(x) : \textit{for some } x \in E\}.$$

The inverse image if a set $E \subset Y$ *under* f *is the set*

$$f^{-1}(E) = \{x \in X : \ \textit{there is } y \in E \textit{ such that } f(x) = E\}.$$

This definition make sense whether or not f is 1-1or onto.

2 Real Sequences

2.1 Introduction

An infinite sequence is a function whose domain is $\mathbb{N}$. A sequence whose terms are $x_n = f(n)$ we will denote by $x_1, x_2, x_3, \ldots$, or $(x_n)_{n\in\mathbb{N}}$ or $(x_n)_{n=1}^{\infty}$ or just (x_n).

For example sequence $1, 2, 3, 4, \ldots$ shall be written as $(n)_{n\in\mathbb{N}}$.

Note that a sequence $(x_n)_{n\in N}$ should not be mistaken with the set $\{x_n : n \in \mathbb{N}\}$. These are entirely different concepts.

Definition 2.1. *A sequence of real numbers (x_n) is said to converge to a real number a if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ (N usually depends on ε) such that*

$$\text{for all } n \geq N \quad \text{we have that} \quad |x_n - a| < \varepsilon.$$

We shall use the following phrases and notations interchangeably;

- (x_n) *converges to* a
- x_n *converges to* a
- $x_n \to a$ *as* $n \to \infty$
- $\lim\limits_{n\to\infty} x_n = a$
- *the* **limit** *of* (x_n) *exists and equals to* a.

Example 1. Show that $\frac{1}{n} \to 0$ as $n \to \infty$. **Proof.** Pick any $\varepsilon > 0$. From Archimedean principle it follows that there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$ or $1/N < \varepsilon$. It follows that if $n \geq N$ then

$$|1/n - 0| = 1/n \leq 1/N < \varepsilon.$$

Hence the limit of this sequence is 0. □

Example 2. Sequence $((-1)^n)_{n\in\mathbb{N}}$ has no limit. **Proof.** Suppose that $(-1)^n \to a$ as $n \to \infty$. Then for $\varepsilon = 1$ there is $N \in \mathbb{N}$ such that for $n \geq N$ we have:

$$|(-1)^n - a| < 1.$$

These are two different inequalities depending whether n is even or odd. If n is even we have $|1-a| < 1$ and for n odd $|-1-a| = |1+a| < 1$. Hence,

$$2 = |(1-a) + (1+a)| < |1-a| + |1+a| < 1 + 1 = 2.$$

This is a contradiction. □

Remark. A sequence has at most one limit.

We introduce a notion of subsequence. Informally, a subsequence is obtained from $x_1, x_2, x_3, \ldots$ by "deleting" some x_n's. For example if the original sequence is $(-1)^n$ by deleting every odd term the resulting subsequence is $1, 1, 1, \ldots$. Formally,

Definition 2.2. *By a subsequence of a sequence $(x_n)_{n\in N}$ we shall mean a sequence of the form $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ (written shortly as $(x_{n_k})_{k\in\mathbb{N}}$) where $n_1 < n_2 < n_3 < \ldots$ is an increasing sequence of positive integers.*

Remark. Any subsequence of a convergent sequence is also convergent and has the same limit.

We also introduce the notion of a sequence *bounded above*, *bounded below* and *bounded*. We remark that these notions are identical with notions of boundedness of the set $\{x_n : n \in \mathbb{N}\}$ (the notion of boundedness for sets was introduced in Chapter 1).

Definition 2.3. *Let (x_n) be a sequence of real numbers.*

- *$(x_n)_{n\in\mathbb{N}}$ is said to be bounded above if $x_n \le M$ for some $M \in \mathbb{R}$ and all $n \in \mathbb{N}$.*
- *$(x_n)_{n\in\mathbb{N}}$ is said to be bounded below if $x_n \ge m$ for some $m \in \mathbb{R}$ and all $n \in \mathbb{N}$.*
- *$(x_n)_{n\in\mathbb{N}}$ is said to be bounded it is bounded both above and below.*

Theorem 2.1. *Every convergent sequence is bounded.*

Proof. Given $\varepsilon = 1$ the is $N \in \mathbb{N}$ such that $|x_n - a| < 1$ for all $n \ge N$. Here a denotes the limit of the given sequence. By triangle inequality this implies that $|x_n| < |a| + 1$ for $n \ge N$. So all terms of the sequence (x_n) with “large” are bounded. On the other hand

$$|x_n| \le \max\{|x_1|, |x_2|, \ldots, |x_N|\} = M, \qquad \text{if } 1 \le n \le N.$$

Therefore whole sequence (x_n) is dominated by $\max\{M, 1 + |a|\}$. □

2.2 Limit Theorems

Usually it is a challenge to decide whether a given sequence converges (or does not). To make this task easier we often compare a sequence whose convergence is in doubt with another sequence whose convergence property is already known.

Theorem 2.2 (Squeeze Theorem)**.** *Suppose that (x_n), (y_n) and (w_n) are real sequences.*

- *If both $x_n \to a$ and $y_n \to a$ (the SAME a) as $n \to \infty$ and if*

  $$x_n \le w_n \le y_n, \qquad \textit{for all } n \ge N_0,$$

  then $w_n \to a$ as $n \to \infty$.
- *If $x_n \to 0$ and (y_n) is bounded then the product $x_n y_n \to 0$ as $n \to \infty$.*

Example. Use the squeeze theorem to show that the sequence $\left(\frac{1}{n^2+n}\right)_{n\in\mathbb{N}}$ is convergent.

Proof. Recall that $\frac{1}{n} \to 0$ as $n \to \infty$. Since $(\frac{1}{n^2})$ is a subsequence of $(\frac{1}{n})$ it must converge to 0 as well. Now,

$$\frac{1}{2n^2} = \frac{1}{n^2+n^2} \le \frac{1}{n^2+n} \le \frac{1}{n^2}, \qquad \text{for all } n \ge 1.$$

Since both $(\frac{1}{n^2})$ and $(\frac{1}{2n^2})$ converge to 0 the result follows. We have that

$$\lim_{n\to\infty} \frac{1}{n^2+n} = 0.$$

$\square$

Theorem 2.3. *Let $E \subset \mathbb{R}$. If E has a finite supremum then there is a sequence $x_n \in E$ such that $x_n \to \sup E$ as $n \to \infty$. Analogous statement holds if E has a finite infimum as well.*

Proof. From the definition of supremum for each $n \in \mathbb{N}$ there exists $x_n \in \mathbb{N}$ such that

$$\sup E - 1/n < x_n \leq \sup E.$$

Hence by the squeeze theorem such $x_n \to \sup E$. $\square$

Theorem 2.4. *Suppose that (x_n), (y_n) are real sequences and $\alpha \in \mathbb{R}$. If both (x_n), (y_n) are convergent then*

(i)

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

(ii)

$$\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n.$$

(iii)

$$\lim_{n\to\infty}(x_n \cdot y_n) = (\lim_{n\to\infty} x_n) \cdot (\lim_{n\to\infty} y_n).$$

(iv) If in addition $\lim_{n\to\infty} y_n \neq 0$ *and* $y_n \neq 0$ *then*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.$$

Example. Calculate $\lim_{n\to\infty} \frac{n^3+n^2-1}{1-3n^3}$.
Solution:

$$\frac{n^3+n^2-1}{1-3n^3} = \frac{1+(1/n)+(1/n^3)}{(1/n^3)-3}.$$

Since $1/n^k \to 0$ as $n \to \infty$ for any $k \in \mathbb{N}$ (as this is a subsequence of $(1/n)_{n\in\mathbb{N}}$) we have by the previous Theorem:

$$\lim_{n\to\infty} \frac{n^3+n^2-1}{1-3n^3} = \frac{1+0+0}{0-3} = -\frac{1}{3}.$$

It is convenient to introduce the notation $x_n \to +\infty$ and $x_n \to -\infty$. Note that these sequences DO NOT converge, but diverge!

Definition 2.4. *Let (x_n) be a sequence of real numbers.*

(i) (x_n) *is said to diverge to* $+\infty$ *(notation* $x_n \to +\infty$ *as* $n \to \infty$ *or* $\lim_{n\to\infty} x_n = +\infty$*) if for each* $M \in \mathbb{R}$ *there is* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *we have* $x_n > M$*.*

(ii) (x_n) *is said to diverge to* $-\infty$ *(notation* $x_n \to -\infty$ *as* $n \to \infty$ *or* $\lim_{n\to\infty} x_n = -\infty$*) if for each* $M \in \mathbb{R}$ *there is* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *we have* $x_n < M$*.*

We note that previous Theorems can be extended to infinite limits.

We introduce the set of so-called extended real numbers $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ and some permitted algebraic operators. Let $a \in \mathbb{R}$,

$$\infty + \infty = \infty + a = \infty, \quad -\infty - \infty = -\infty + a = -\infty,$$
$$\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty \cdot a = \infty \qquad a > 0$$
$$\infty \cdot (-\infty) = \infty \cdot a = -\infty \qquad a < 0 \quad \text{and} \quad a/\infty = 0.$$

Not all algebraic operation are allowed, the forbidden operations are

$$\infty - \infty, \qquad 0 \cdot \infty, \qquad \infty/\infty.$$

We also introduce ordering on the set $\mathbb{R}^*$ (keeping the usual ordering on $\mathbb{R}$):

$$-\infty < a, \qquad a < \infty \qquad -\infty < \infty \qquad a \in \mathbb{R}.$$

With the notation we can generalize Theorem **??** to incorporate divergent limits (to $\pm\infty$). For example the part (i) has the following generalization:

Theorem 2.5. *Suppose that* (x_n), (y_n) *are real sequences. If both* $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ *exist (and belong to the set of extended real numbers* $\mathbb{R}^*$*) then*

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$

provided the forbidden algebraic operation $\infty - \infty$ *(or its commutative analogue* $-\infty + \infty$*) does not occur.*

There are similar statements for parts (ii)-(iv) of Theorem **??**.

Theorem 2.6 (Comparisson theorem for sequences)**.** *Suppose that* (x_n), (y_n) *are real sequences. If both* $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ *exist (and belong to the set of extended real numbers* $\mathbb{R}^*$*) and if*

$$x_n \leq y_n \qquad \text{for all } n \geq N \qquad \text{for some } N \in \mathbb{N}$$

then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Proof. We limit ourselves here to the case $\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n \in \mathbb{R}$. Assume by contradiction that

$$\varepsilon = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n > 0.$$

If follows from the definition of limit of a sequence that there are natural numbers N_1, N_2 such that

$$\left|(\lim_{n\to\infty} x_n) - x_k\right| < \varepsilon/2, \qquad \text{for all } k \geq N_1$$

and

$$\left|(\lim_{n\to\infty} y_n) - y_k\right| < \varepsilon/2, \qquad \text{for all } k \geq N_2.$$

Hence

$$\varepsilon = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = \left[(\lim_{n\to\infty} x_n) - x_k\right] + (x_k - y_k) + \left[y_k - (\lim_{n\to\infty} y_n)\right].$$

If we take $k \geq \max\{N, N_1, N_2\}$ we have that $x_k - y_k \leq 0$ by our assumption and hence by triangle inequality:

$$\varepsilon \leq \left|(\lim_{n\to\infty} x_n) - x_k\right| + \left|y_k - (\lim_{n\to\infty} y_n)\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which is a contradiction. Hence our claim holds. $\square$

Remark. A version of theorem above with strict inequality "$<$" fails to hold. It is false to claim that $x_n < y_n$ implies $\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$. Consider as a counterexample sequences $x_n = 0$ and $y_n = 1/n$.

2.3 Bolzano-Weierstrass Theorem

We have already seen that some subsequences behave better than the original sequence. For example we have seen that even though the sequence $((-1)^n)$ does not converge, one might find a convergent subsequence. This observation is a special case of a general theorem (called Bolzano-Weierstrass) which states that any bounded sequence has a convergent subsequence.

We begin with a special case about monotone sequences.

Definition 2.5. *Let (x_n) be a sequence of real numbers.*

(i) (x_n) is said to be increasing (respectively strictly increasing) if $x_1 \leq x_2 \leq x_3 \leq \ldots$ ($x_1 < x_2 < x_3 < \ldots$ for strictly increasing).

(ii) (x_n) is said to be decreasing (respectively strictly decreasing) if $x_1 \geq x_2 \geq x_3 \geq \ldots$ ($x_1 > x_2 > x_3 > \ldots$ for strictly increasing).

(iii) (x_n) is said to be monotone if it is either increasing or decreasing.

The first observation we make is that monotone bounded sequences are convergent.

Theorem 2.7 (On monotone convergence)**.** *If (x_n) is increasing and bounded above or if it is decreasing and bounded below, then (x_n) is convergent (and has a finite limit).*

Proof. Suppose that (x_n) is increasing and bounded above. By the Completness Axiom the supremum $a = \sup\{x_n : n \in \mathbb{N}\}$ exists and is finite. Let $\varepsilon > 0$. By the definition of the supremum there exists $N \in \mathbb{N}$ such that

$$a - \varepsilon < x_N \leq a.$$

Since (x_n) is monotone $x_N \leq x_n$ for all $n \geq N$. It follows that $a - \varepsilon < x_n \leq a$ for all $n \geq N$. Hence $|x_n - a| < \varepsilon$ from which convergence follows. The proof in the case (x_n) is decreasing and bounded below is analogous (try it as an exercise). □

Example. Show that $a^{1/n} \to 1$ as $n \to \infty$, provided $a > 0$.
Hint of a solution. We shall only consider the case $a > 1$. Observe that $(a^{1/n})$ is decreasing. Indeed, $a^{n+1} > a^n$. Taking $n(n+1)$ root then yields $a^{1/n} > a^{1/(n+1)}$. Since $a > 1$ we also have $a^{1/n} > 1$ so the sequence is also bounded below. Thus by the previous theorem this sequence has a limit (and the limit is ≥ 1). Denote this limit by L. We see that

$$L^2 = \left(\lim_{n\to\infty} a^{1/(2n)}\right)^2 = \lim_{n\to\infty} a^{1/n} = L.$$

Here we have use the fact that the limit of $(a^{1/(2n)})$ is L since it is a subsequence of $(a^{1/n})$. Now $L^2 = L$ implies that either $L = 0$ or $L = 1$. Since $L \geq 1$ it must be that $L = 1$. □

We also have “monotone” property also for intervals.

Definition 2.6. *A sequence $(I_n)_{n\in\mathbb{N}}$ of sets is said to ne nested if*

$$I_1 \supset I_2 \supset I_3 \supset \dots.$$

Theorem 2.8 (Nested Interval Property)**.** *If (I_n) is a nested sequence of nonempty bounded intervals then*

$$E = \bigcap_{n\in\mathbb{N}} I_n = \{x \in \mathbb{R} : x \in I_n \textit{ for all } n \in \mathbb{N}\}$$

is nonempty (i.e. it contains at least one number). Moreover if $|I_n| \to 0$ ($|I_n|$ denotes the length of interval I_n) then E contains exactly one number.

Remark 1. The Nested Interval Property might not hold if “closed” is omitted from the statement of the theorem. Consider for example $I_n = (0, 1/n)$.

Remark 2. The Nested Interval Property might not hold if “bounded” is omitted from the statement of the theorem. Consider for example $I_n = [n, \infty)$.

Theorem 2.9 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Assume that a is the lower and b the upper bound of the given sequence. Let $I_0 = [a, b]$. We divide I_0 into two halves, I' and I''. ($I' = [a, (a+b)/2]$ and $I'' = [(a+b)/2, b]$). Since $I_0 = I' \cup I''$ at least one of these intervals contain x_n for infinitely many values of n. We denote interval with this property I_1. Let $n_1 > 1$ be such that $x_{n_1} \in I_1$.

We proceed by induction. Having constructed an interval I_m $(m \geq 0)$ we divide it into two halves (like we did with I_0). At least one of the halves (which we will denote I_{m+1}) will contain infinite many x_n. We choose $n_{m+1} > n_m$ such that $x_{n_{m+1}} \in I_{m+1}$.

Observe that (I_n) is a nested sequence of bounded and closed intervals. Hence there is $x \in \mathbb{R}$ that belongs to every interval I_k. Hence

$$|x - x_{n_k}| \leq |I_k| = \frac{b-a}{2^k}.$$

By the squeeze theorem $x_{n_k} \to x$ as $k \to \infty$. □

2.4 Cauchy Sequences

It follows from the definition a convergent sequence that if $x_n \to a$ as $n \to \infty$, then x_n's are close to number a for n large. Hence, these x_n's are also close to each other.

Definition 2.7. *A sequence of points $x_n \in \mathbb{R}$ is said to be Cauchy if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that*

$$|x_n - x_m| < \varepsilon \qquad \text{for all } n, m \geq N.$$

The simple observation we have made few line above implies that if a sequence is convergent it must be Cauchy.

Remark. If (x_n) is convergent, then (x_n) is Cauchy.

Remarkably, the converse is also true for sequences real numbers.

Theorem 2.10 (Cauchy). *Let (x_n) be a sequence of real numbers. Then (x_n) is a Cauchy sequence if and only if (x_n) is a convergent sequence.*

Proof. Suppose that (x_n) is Cauchy. Pick $\varepsilon = 1$. Then there is $N \in \mathbb{N}$ such that $|x_n - x_m| < 1$ for all $n, m \geq N$. In particular,

$$|x_m| < 1 + |x_N|, \qquad \text{for all } m \geq N.$$

From this we can see that the sequence (x_n) must be bounded. Hence by the Bolzano-Weierstrass Theorem (x_n) has a convergent subsequence, say $x_{n_k} \to a$ as $k \to \infty$.

Choose any $\varepsilon > 0$. Since (x_n) is Cauchy, we can choose $N_1 \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon/2, \qquad \text{for all } n, m \geq N_1.$$

Also, since $x_{n_k} \to a$ as $k \to \infty$ we can find $N_2 \in \mathbb{N}$ such that

$$|x_{n_k} - a| < \varepsilon/2, \qquad \text{for all } k \geq N_2.$$

Pick an index k such that $k \geq N_2$ and also $n_k \geq N_1$. Then for all $n \geq N_1$

$$|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $x_n \to a$ as $n \to \infty$. □

Example 1. A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessary Cauchy. (Consider $x_n = \log n$).

Example 2. A sequence that satisfies $|x_{n+1} - x_n| \leq 2^{-n}$ for all n is Cauchy (and hence convergent).

Proof. Consider any $m > n$. Clearly, by triangle inequality

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n_2} - x_{n+1}| + |x_{n+1} - x_n|.$$

The terms on the right hand side are by our assumption bounded from above by $2^{1-m} + 2^{2-m} + \cdots + 2^{-n}$. It is a simple exercise to see that is sum is at most 2^{1-n}. Hence:

$$|x_m - x_n| \leq 1/2^{n-1}.$$

It follows that given $\varepsilon > 0$ one can find N sufficiently large such that if $n \geq N$, then $1/2^{n-1} < \varepsilon$. Clearly, for $n, m \geq N$ we then have $|x_m - x_n| < \varepsilon$. And therefore (x_n) must be a Cauchy sequence as desired. $\square$

3 Continuity

3.1 Introduction

In elementary calculus the notion of limit was intuitively defined. A function $f(x)$ converges to a limit L, as x approaches a if all values $f(x)$ are near L when x is near a. We now make this precise:

Definition 3.1. *Let $a \in \mathbb{R}$, let I be an open interval that contains a and let f be a real function defined everywhere on I except possibly at a. Then f is said to converge to L as x approaches a if for every $\varepsilon > 0$ there is a $\delta > 0$ (δ in general depends on ε, function f, interval I and a) such that*

$$\textit{if} \quad 0 < |x-a| < \delta \qquad \textit{then} \qquad |f(x) - L| < \varepsilon.$$

In this case we write

$$L = \lim_{x \to a} f(x).$$

We might also use the notation $f(x) \to L$ as $x \to a$. We call the number L the **limit** *of function f at the point a.*

Example. If $f(x) = x^2 + 1$ show that $f(x) \to 5$ as $x \to 2$.

Let $\varepsilon > 0$ and $L = 5$. Observe that

$$f(x) - L = x^2 + 1 - 5 = (x-2)(x+2).$$

If $0 < \delta \leq 1$ then $|x-2| < \delta$ implies that $1 < x < 3$ and therefore for such x: $|x+2| < 5$. If follows that

$$|f(x) - L| = |x-2||x+2| < \delta \cdot 5 = 5\delta.$$

Hence if $5\delta \leq \varepsilon$ the desired estimate $|f(x) - L| < \varepsilon$ will hold. It follows that $\delta = \min\{\varepsilon/5, 1\}$ has the required properties and that $\lim_{x \to 2}(x^2+1) = 5$.

It is important to realize that f does not have to be defined at the point a. For example a function $f(x) = \frac{\sin x}{x}$ is defined on the set $\mathbb{R} \setminus \{0\}$. Yet, the limit $\lim_{x \to 0} f(x)$ exists and can be calculated.

Moreover, the value of a limit at a point a never depends on the value $f(a)$ even if f is defined at the point a. Consider a function $g(x) = 1$ for $x \in (-3,-1) \cup (-1,0)$ and $g(-1) = 5$. The domain of this function is the interval $(-3,0)$. Clearly

$$\lim_{x \to -1} g(x) = 1 \neq g(-1) = 5.$$

This observation can be generalized as follows. Let $a \in \mathbb{R}$ and I be an open interval containing a. If $\lim_{x \to a} f(x) = L$ and f and g are two functions defined everywhere on I except possibly at a such that

$$f(x) = g(x) \quad \text{for all } x \in I \setminus \{a\} \quad \text{then} \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L.$$

Example. Let $g(x) = \dfrac{x^3 - x^2 + x - 1}{x - 1}$. Find the limit of g at the point 1.

Solution: Since

$$g(x) = \frac{x^3 - x^2 + x - 1}{x - 1} = \frac{(x-1)(x^2+1)}{x-1} = x^2 + 1 \quad \text{for all } x \neq 1$$

if we define $f(x) = x^2 + 1$ on $\mathbb{R}$ we see that the functions f and g have the same limit at the point $a = 1$. It follows that

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x) = \lim_{x \to a} (x^2 + 1) = 2.$$

We now make connection between limits of functions and limits of sequences.

Theorem 3.1 (Sequential characterization of limits)**.** *Let $a \in \mathbb{R}$, let I be an open interval containing a and let f be a real function defined everywhere on I except possibly at a. Then*

$$L = \lim_{x \to a} f(x)$$

exists if and only if

$$\lim_{n \to \infty} f(x_n) = L$$

for every sequence (x_n) such that $x_n \in I \setminus \{a\}$ and $x_n \to a$ as $n \to \infty$.

Example. Show that the function $f(x) = \sin(1/x)$ defined on $\mathbb{R} \setminus \{0\}$ has no limit at $a = 0$.

Solution: The simplest is to examine graph of this function near zero. We can see that this function seems to oscillate infinitely many times between values -1 and $+1$. In fact if we consider

$$a_n = \frac{2}{(4n+1)\pi}, \quad b_n = \frac{2}{(4n+3)\pi}, \quad \text{for } n \in \mathbb{N},$$

then clearly $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$ but $f(a_n) = 1$ and $f(b_n) = -1$ for all $n \in \mathbb{N}$. It follows that $f(a_n) \to 1$ and $f(b_n) \to -1$ as $n \to \infty$. By the previous Theorem if $\lim_{x \to 0} f(x)$ existed then the limits of $(f(a_n))_{n \in \mathbb{N}}$ and $(f(b_n))_{n \in \mathbb{N}}$ would have to be equal,but they are not.

The previous Theorem also implies that the results we had about sums, products, and divisions of limits of sequences do also apply to limits of functions. Hence for example we have that if both

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist, then so does $\lim\limits_{x \to a} (f+g)(x)$ and

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

There is also Squeeze theorem for functions than can be formulated in a same way as for sequences. In fact we have also the following

Theorem 3.2 (Comparison Theorem for functions)**.** *Let $a \in \mathbb{R}$, let I be an open interval containing a and let f,g be real functions defined everywhere on I except possibly at a. If both f and g have limits as x approaches a and*

$$f(x) \leq g(x), \qquad \text{for all } x \in I \setminus \{a\},$$

then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Remark. If

$$f(x) < g(x), \qquad \text{for all } x \in I \setminus \{a\},$$

the conclusion of previous theorem still is only true with $\leq$ not $<$. That is

$$\lim_{x\to a} f(x) < \lim_{x\to a} g(x),$$

might be false. See for example functions $f(x) = 0$ and $g(x) = x^2$ and $a = 0$.

3.2 One sided limits and limits at infinity

In the preceding section we considered limits of functions f defined everywhere around a point a. What about functions like $\sqrt{x+1}$ that are only defined for $x \geq -1$? Can we consider limit as $x \to -1$? Clearly, we can but only "from the right" if we think about points x in the domain of the function approaching -1 on the real line. This motivates to consider "one sided" limits.

Definition 3.2. *Let $a \in \mathbb{R}$, let I be an open interval with left end-point a.*

We say that f converges to L as x approaches a from the right, and write

$$\lim_{x\to a+} f(x) = L$$

or $f(x) \to L$ as $x \to a+$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{if } a < x < a+\delta \text{ and } x \in I, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

The value L of the limit we shall denoted as $f(a+)$.

Let $a \in \mathbb{R}$, let I be an open interval with left right-point a.

We say that f converges to L as x approaches a from the left, and write

$$\lim_{x\to a-} f(x) = L$$

or $f(x) \to L$ as $x \to a-$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{if } a-\delta < x < a \text{ and } x \in I, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

The value L of the limit we shall denoted as $f(a-)$.

The connection between limit and one-sided limits is straightforward.

Theorem 3.3. *Let f be a real function. The limit*

$$\lim_{x \to a} f(x)$$

exists and equals to L if and only if

$$L = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x).$$

The definition of limits of real functions can be expanded to include extended real numbers. We say that $f(x) \to L$ as $x \to \infty$ (respectively, $x \to -\infty$) if $(c, \infty) \subset \mathrm{Dom}(f)$ for some $c \in \mathbb{R}$ (respectively, $(-\infty, c) \subset \mathrm{Dom}(f)$) and

for every $\varepsilon > 0$ there is $M \in \mathbb{R}$ such that $x > M$(respectively $x < M$) $\implies |f(x) - L| < \varepsilon$.

Exercise: Think about correctly formulating definitions of $\lim_{x \to a} f(x) = +\infty$, $\lim_{x \to a+} f(x) = -\infty$ and $\lim_{x \to -\infty} f(x) = +\infty$. (There are further combinations possible).

3.3 Continuity

Again, you have already seen in elemental calculus an intuitive definition of a limit. A function was called continuous at a if $a \in \mathrm{Dom}(f)$ and $f(x) \to f(a)$ as $x \to a$. (This tacitly assumes that $\mathrm{Dom}(f)$ is on both sides of a, hence a modification is needed for functions defined on closed intervals, for example).

Definition 3.3. *Let $\emptyset \neq E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$.*

(i) f is continuous at a point $a \in E$ if for any $\varepsilon > 0$ there is $\delta > 0$ (depending on ε, f, a) such that

$$\text{if } |x - a| < \delta \text{ and } x \in E \quad \text{then} \quad |f(x) - f(a)| < \varepsilon.$$

(ii) f is continuous E (we write $f : E \to \mathbb{R}$ is continuous) if f is continuous at every point $a \in E$.

This definition allows to consider continuity of functions with domains like $E = \mathbb{Z}$ or $E = \mathbb{Q}$. However, if $a \in I \subset E$ for some open interval I the connection with limit of function we mentioned at the beginning will hold.

Theorem 3.4. *Let I be an open interval that contains a point a and $I \subset E$ for a function $f : E \to \mathbb{R}$. Then f is continuous at a if and only if*

$$f(a) = \lim_{x \to a} f(x).$$

Theorem 3.5. *Let $E \subset \mathbb{R}$ be nonempty and $f : E \to \mathbb{R}$. Then f is continuous at $a \in E$ if and only if*

$$\text{for all sequences } (x_n) \text{ such that } x_n \to a \text{ and } x_n \in E\text{: } \lim_{n \to \infty} f(x_n) = f(a).$$

Theorem 3.6. *Let $E \subset \mathbb{R}$ be nonempty and $f, g : E \to \mathbb{R}$. If f and g are continuous at a point $a \in E$ then so are $f+g$, fg and αf for any $\alpha \in \mathbb{R}$. Moreover, if $g(a) \neq 0$ the function f/g is continuous at $a \in E$ as well. Similar statements holds for continuity on the set E.*

Definition 3.4. *Let $A, B \subset \mathbb{R}$ be nonempty, let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subset B$. The composition of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by*

$$(g \circ f)(x) = g(f(x)), \qquad \text{for all } x \in A.$$

Theorem 3.7. *Let $A, B \subset \mathbb{R}$ be nonempty, let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subset B$.*

(i) If f is continuous at $a \in A$ and g is continuous at the point $f(a) \in B$ then $g \circ f$ is continuous at the point $a \in A$.

(ii) Let $I \setminus \{a\} \subset A$ where I is either a non-degenerate interval containing a or has a as one of its endpoints. If

$$\lim_{x \to a, x \in I} f(x) = L$$

exists, $L \in B$ and g is continuous at the point $L \in B$ then

$$\lim_{x \to a, x \in I} (g \circ f)(x) = g\left(\lim_{x \to a, x \in I} f(x) \right).$$

Definition 3.5. *Let $E \subset \mathbb{R}$ be nonempty. A function $f : E \to \mathbb{R}$ is said to be bounded on E if*

$$|f(x)| \leq M, \qquad \text{for all } x \in E,$$

where M is some (large) real number.

Continuous functions on closed bounded interval are always bounded. Actually, more is true. The function attains both its maximum and minimum on such interval.

Theorem 3.8. *Let $I \subset \mathbb{R}$ be a closed bounded interval. Let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on the interval I.*

Denote by

$$m = \inf_{x \in I} f(x), \qquad M = \sup_{x \in I} f(x).$$

Then there exist points $x_m, x_M \in I$ such that

$$f(x_m) = m \qquad \text{and} \qquad f(x_M) = M.$$

Proof. We first establish that f is bounded. If it were not, then one can find a sequence $(x_n) \subset I$ such that $|f(x_n)| > n$. By the Bolzano-Weierstrass theorem (x_n) has a convergent subsequence, say (x_{n_k}). Let a be the limit of such subsequence. Clearly, $a \in I$ and

$$|f(a)| = \lim_{k \to \infty} |f(x_{n_k})| = \infty.$$

This is a contradiction since $f(a)$ must be a real number as $a \in I$. Thus f is bounded.

We now show the existence of a minimum. By the properties of inf there exists a sequence $(y_n) \subset I$ such that $f(y_n) \to m$. Again by Bolzano-Weierstrass find a convergent subsequence, say (y_{n_k}). Let x_m be the limit of (y_{n_k}). It follows that

$$f(x_m) = \lim_{k\to\infty} f(y_{n_k}) = m.$$

□

This theorem does not hold if either "closed" or "bounded" is omitted from the hypotheses about the interval I.

Lemma 3.9. *Let $f : I \to \mathbb{R}$ where I is an open nonempty interval. If f is continuous at a point $a \in I$ and $f(a) > 0$ then for some $\delta, \varepsilon > 0$ we have that*

$$f(x) > \varepsilon, \qquad \textit{for all } x \in (a-\delta, a+\delta).$$

Theorem 3.10 (Intermediate Values Theorem)**.** *Let I be a non-degenerate interval and $f : I \to \mathbb{R}$ continuous. If $a, b \in I$, $a < b$ then f attains on the interval (a,b) all values between $f(a)$ and $f(b)$. That is given y_0 between $f(a)$ and $f(b)$ there exists $x_0 \in (a,b)$ such that*

$$f(x_0) = y_0.$$

Proof. To simplify the notation assume that $f(a) < f(b)$ and that $f(a) < y_0 < f(b)$. Let

$$x_0 = \sup E, \qquad \text{where: } E = \{x \in [a,b] : f(x) < y_0\}.$$

Clearly, $a \in E$, and $E \subset [a,b)$ as $b \notin E$. A this point using continuity show that $f(x_0) = y_0$. □

Example. The Riemann function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ 1/q, & \text{if } x = p/q \text{ where } q \in \mathbb{N},\ p \in \mathbb{Z} \text{ and } (p,q) = 1), \end{cases}$$

is continuous on irrational numbers and discontinuous at all rational numbers.

Solution: First consider a rational number $x \in \mathbb{Q}$. If f were continuous at x then for any sequence $x_n \to x$ we would have $f(x_n) \to f(x)$ as $n \to \infty$. If we choose $x_n \in \mathbb{Q}^c$ then $f(x_n) = 0 \to 0$ but $f(x) > 0$ as x is rational. Hence, f is not continuous at x.

Consider now a irrational number x. To prove continuity at x we have to show that for any sequence (y_n), $y_n \to x$ we have $f(y_n) \to 0 = f(x)$. If suffices to consider $y_n \in \mathbb{Q}$ since for irrational numbers $f(y_n) = 0$ hence there is nothing left to prove in such case.

Hence, let $y_n = p_n/q_n$ where the quotient is in the reduced form ($q_n \in \mathbb{N}$, $p_n \in \mathbb{Z}$ and $(p_n, q_n) = 1$)). We have to prove that

$$f(y_n) = \frac{1}{q_n} \to 0, \qquad \text{as } n \to \infty.$$

Assume by contradiction that this is false. Then for some subsequence $n_1 < n_2 < n_3, \dots$ we have that

$$\frac{1}{q_{n_k}} \geq \varepsilon > 0 \qquad \text{or} \qquad q_{n_k} \leq \frac{1}{\varepsilon} = M, \qquad k = 1, 2, 3, \dots.$$

Consider a set E of rational numbers such that

$$E = \{x = p/q; p \in \mathbb{Z} \text{ and } q \in \mathbb{N}, q \leq M\}.$$

We claim that Cauchy sequences of elements from E have the following property. If $(y_n) \subset E$ is a Cauchy sequence, then (y_n) is eventually constant (that is $y_n = y_{n+1}$ for all $n \geq N$). Indeed, claim like this was presented at the tutorial for $E = \mathbb{Z}$. The argument here is similar, we claim that if $x, y \in E$ and $x \neq y$ then

$$|x - y| \geq \frac{1}{M(M-1)}.$$

Having this, since $p_{n_k}/q_{n_k} \to x$ as $n \to \infty$ and $p_{n_k}/q_{n_k} \in E$ it follow that (p_{n_k}/q_{n_k}) is eventually constant and hence its limit must belong to E. So $x \in E \subset \mathbb{Q}$. This is a contradiction as we have assumed that $x \in \mathbb{Q}^c$. Hence we have shown that $f(y_n) \to 0$ as desired and so f is continuous at x. $\square$

4 Differentiability on $\mathbb{R}$

4.1 Introduction

Definition 4.1. *A real function f is said to be differentiable at a point $a \in \mathbb{R}$ if f is defined at some open interval containing a and*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The number $f'(a)$ is called the derivative of f at the point a.

If we plot the function f as a graph $(x, f(x))$ then $f'(a)$ is the slope of the tangent line to the graph at the point $(a, f(a))$. To see this we notice that $\frac{f(a+h)-f(a)}{h}$ is the slope of the chord passing though points $(a, f(a))$ and $(a+h, f(a+h))$ on the graph of f. If we let $h \to 0$ the slope of the chord will approximate the slope of the tangent line.

If f is differentiable at every point of a set E then f' is a function with domain E. This function can be denoted several ways (you will encounter all of these):

$$f' = f^{(1)} = \frac{df}{dx} = D_x f.$$

When $y = f(x)$ the notation dy/dx or y' is also used.

The differentiation can be performed more that once. The higher derivatives are defined by induction, if $n \in \mathbb{N}$ then

$$f^{(n+1)}(a) = (f^{(n)}(a))',$$

provided, the derivative exists. Again various notation is used for higher derivatives such as

$$f^{(n)} = \frac{d^n f}{dx^n} = D^n_x f \qquad \text{or if } y = f(x) \text{ also } y^{(n)} \text{ or } \frac{d^n y}{dx^n}.$$

The second derivative, for example can be written as $f^{(2)}$ or f''. We say that f is twice differentiable at a point a if $f''(a)$ exists.

We give two different characterizations of derivative. One is using "chord" function (see below) and another is is in terms of linear approximation. These two notions are equivalent.

The chord function F is defined by

$$F(x) = \frac{f(x) - f(a)}{x - a}, \qquad \text{for } x \neq a.$$

Theorem 4.1. *A real function f is differentiable at a point $a \in \mathbb{R}$ if and only if there exists and open interval I and a function $F : I \to \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous at the point a and*

$$f(x) = F(x)(x - a) + f(a) \qquad \textit{holds for all} x \in I.$$

The number $F(a)$ is equal to $f'(a)$.

Notice that this theorem simply states that the chord function we defined above by $F(x) = \frac{f(x)-f(a)}{x-a}$ for $x \neq a$ can be extended to the point a by taking $F(a) = f'(a)$ and that with this extension the function F is continuous at a.

Here is the second characterization of derivative using linear approximation.

Theorem 4.2. *Let $f : I \to \mathbb{R}$ where I is an open interval containing point $a \in I$. Then f is differentiable at a if and only if there is a linear function T of the form $T(x) = mx$ such that*

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0.$$

If this limit exists then $f'(a) = m$.

Differentiability and continuity are related. Every differentiable function is continuous but the reverse is not true. There are continuous functions $f : \mathbb{R} \to \mathbb{R}$ that are NOT differentiable at any point. It quite complicated to show such function, but it's easy to find function that is continuous but not differentiable at a single point. One such function is $f(x) = |x|$ that is NOT differentiable at $x = 0$.

Indeed, since $x \to 0$ implies that $|x| \to 0$ f is continuous at 0. Considering these two limits

$$\lim_{h\to 0+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0+} \frac{h}{h} = 1, \qquad \lim_{h\to 0-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0-} \frac{-h}{h} = -1,$$

we see that they are not equal and hence $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ does not exists. Hence f is NOT differentiable at 0.

Theorem 4.3. *If f is differentiable at a, then f is continuous at a.*

Proof. Suppose that f is differentiable at a. Then by the Theorem **??** there is and open interval I and a function F continuous a a such that $f(x) - f(a) = F(x)(x-a)$ for all $x \in I$. Taking the limit $x \to a$ we see that

$$\lim_{x\to a} [f(x) - f(a)] = \lim_{x\to a} F(a) \cdot 0 = 0.$$

It follows that $f(x) \to f(a)$ as $x \to a$. □

As with continuity it is convenient to define "one sided" derivatives to deal with functions whose domains are closed (or half-open) intervals. Here is a brief discussion what it means for a function to be differentiable **on** an interval I.

Definition 4.2. *Let I be a non-degenerate interval.*

(i) A function $f : I \to \mathbb{R}$ is said to be differentiable on I if

$$f_I'(a) := \lim_{x\to a, x\in I} \frac{f(x) - f(a)}{x - a}$$

exists and is finite at every $a \in I$.

(ii) f is said to be continuously differentiable on I if f_I' exists and is continuous on I.

Notice that if $a \in I$ is not an endpoint of I then $f_I'(a)$ is just $f'(a)$. The difference is at the endpoints, if $I = [a,b]$ then the limits taken at the endpoints are

$$\lim_{h\to 0+} \frac{f(a+h)-f(a)}{h}, \quad \text{and} \quad \lim_{h\to 0-} \frac{f(b+h)-f(b)}{h}.$$

In what will follow we usually drop the subscript I from f_I' and just write f' (this is slightly "sloppy" at the endpoints).

Example. Show that $f(x) = x^{3/2}$ is differentiable on $[0,\infty)$.

For $x > 0$ the usual Power Rule (see exercises) we have that $f'(x) = 3/2x^{3/2-1} = 3/2\sqrt{x}$. At $a = 0$ by the definition:

$$f'(0) = \lim_{h\to 0+} \frac{h^{3/2}-0}{h} = \lim_{h\to 0+} \sqrt{h} = 0.$$

In conjunction with the definition we just gave the following notation is widely used. Let I be a non-degenerate interval. For each $n \in \mathbb{N} \cup \{0\}$, $C^n(I)$ denotes the collection of real functions whose n-th derivatives exist are continuous on I. Hence $C^1(I)$ is the collection of all real functions that are continuously differentiable on I. $C^0(I)$ also denoted just by $C(I)$ is the collection of all real functions that are continuous on I (0-derivative just means that no operation is performed on f).

$C^\infty(I)$ means the intersection $\bigcap_{n\in\mathbb{N}} C^n(I)$. We have that $C^m(I) \subset C^n(I)$ if $m > n$.

These classes are bit restrictive, for example NOT every function that is differentiable on $\mathbb{R}$ belongs to $C^1(\mathbb{R})$.

Example. The function $f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is differentiable on I but is not continously differentiable on any interval containing 0.

Solution: Using the definition,

$$f'(0) = \lim_{h\to 0} h\sin(1/h) = 0, \quad \text{and} \quad f'(x) = 2x\sin(1/x) - \cos(1/x), x \neq 0.$$

It means that f is differentiable on $\mathbb{R}$, but the function f' is not continuous at 0 since the limit $\lim_{x\to 0} f'(x)$ does not exists (this is due to oscillation between -1 and 1 of $\cos(1/x)$ as $x \to 0$).

Also, differentiability on two sets does not necessary mean differentiability on their union (consider $f(x) = |x|$ on $[-1,0]$ and $[0,1]$ and then on $[-1,1] = [-1,0] \cup [0,1]$).

4.2 Differentiability Theorems

We shall establish several familiar theorems (from calculus) about derivatives.

Theorem 4.4. *Let f,g be real functions and $\alpha \in \mathbb{R}$. If f and g are differentiable at a, then $f+g$, $f \cdot g$, αf and (if $g(a) \neq 0$) also f/g are all differentiable at a. We have*

$$(f+g)'(a) = f'(a) + g'(a), \qquad (f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a),$$

$$(\alpha f)'(a) = \alpha f'(a), \qquad \left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof. We do the product rule here. By Theorem **??** there exists an open interval $I \ni a$ and and functions $F : I \to \mathbb{R}$, $G : I \to \mathbb{R}$ continuous at a such that

$$f(x) = F(x)(x-a) + f(a), \qquad g(x) = G(x)(x-a) + g(a), \quad x \in I.$$

If follows that

$$f(x)g(x) = f(a)g(a) + [F(x)g(a) + G(x)f(a) + F(x)G(x)(x-a)](x-a).$$

Let $H(x) = F(x)g(a) + G(x)f(a) + F(x)G(x)(x-a)$ for $x \in I$. It follows that H is continuous at a as sums and products of continuous functions at a point are also continuous there. Hence, since

$$f(x)g(x) = f(a)g(a) + H(a)(x-a),$$

by Theorem **??** we have that

$$(fg)'(a) = H(a) = F(a)g(a) + G(a)f(a) + F(a)G(a)(a-a) = f'(a)g(a) + g'(a)f(a).$$

□

Theorem 4.5 (Chain Rule)**.** *Let f,g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proof. By Theorem **??** there exist open intervals $I \ni a$ and $J \ni f(a)$ and functions $F : I \to \mathbb{R}$ continuous at a, $G : J \to \mathbb{R}$ continuous at $f(a)$ such that

$$f(x) = F(x)(x-a) + f(a), \quad x \in I, \qquad g(y) = G(y)(y - f(a)) + g(f(a)), \quad y \in J.$$

By making I smaller (if necessary) we may assume that $f(x) \in J$ for all $x \in I$ (due to continuity of f at a). Fix $x \in I$. We can write:

$$(g \circ f)(x) = g(f(x)) = G(f(x))(f(x) - f(a)) + g(f(a)) = G(f(x))F(x)(x-a) + (g \circ f)(a).$$

We see that if we set $H(x) = G(f(x))F(x)$ for $x \in I$, then

$$(g \circ f)(x) = H(x)(x-a) + (g \circ f)(a).$$

Moreover, since F is continuous at a and G is continuous at $f(a)$ the function H is continuous at a. If follows that

$$(g \circ f)'(a) = H(a) = G(f(a))F(a) = g'(f(a))f'(a).$$

□

Theorem 4.6 (Power Rule). *Show that*

(i) If $n \in \mathbb{N}$ then $(x^n)' = nx^{n-1}$ for all $x \in \mathbb{R}$.

(ii) If $q \in \mathbb{Q}$ then $(x^q)' = qx^{q-1}$ for all $x > 0$.

Proof. (i) Recall the algebraic formula

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}), \qquad \text{for } n \in \mathbb{N}.$$

Let $f(x) = x^n$. Using the definition of derivative and formula above we have

$$f'(a) = \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} [x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}] = na^{n-1},$$

since the brackets $[\dots]$ contain exactly n terms and each of them has limit a^{n-1} as $x \to a$.
(ii) First consider $q = 1/n$ where $n \in \mathbb{N}$. Then again from the definition of derivative and formula above we have for $f(x) = x^{1/n}$, $x > 0$:

$$f'(a) = \lim_{x \to a} \frac{x^{1/n} - a^{1/n}}{x - a}$$

$$= \lim_{x \to a} \frac{x^{1/n} - a^{1/n}}{(x^{1/n} - a^{1/n})[x^{(n-1)/n} + x^{(n-2)/n}a^{1/n} + \cdots + x^{1/n}a^{(n-2)/n} + a^{(n-1)/n}]} = \frac{1}{na^{(n-1)/n}}.$$

This hold, since term $x^{1/n} - a^{1/n}$ cancels out and what remains in the denominator are n terms and each of them has limit $a^{(n-1)/n}$ as $x \to a$.

Similarly, we can verify the formula for $q = -1/n$, $n \in \mathbb{N}$, since $f(x) = x^{-1/n}$ can be written as a composition $g \circ h$ where $g(z) = 1/z$ and $h(x) = x^{1/n}$. We can differentiate both of the functions (g using the quotient rule and h we have already done above).

Finally consider any $q = m/n$ where $m \in \mathbb{N}$ and $n \in \mathbb{Z} \setminus \{0\}$. We claim that $f(x) = x^{m/n}$ is differentiable for every $x > 0$. This follows from the fact that we can write $f = g \circ h$ where $g(z) = z^m$ and $h(x) = x^{1/n}$. From (i) we have that g is differentiable and from (ii) (the parts just above) we have that h is also differentiable for $x > 0$. Hence by the Chain rule f must be differentiable. Applying the Chain Rule we have:

$$f'(a) = g'(f(a))f'(a) = m(a^{1/n})^{m-1}\frac{1}{na^{(n-1)/n}} = (\tfrac{m}{n})a^{\frac{m-1}{n} - \frac{n-1}{n}} = (\tfrac{m}{n})a^{m/n-1} = qa^{q-1}.$$

□

4.3 Mean Value Theorem

Mean value theorem makes precise the observation we have made about the relationship between the derivative of a function and the slope of one of its chords. We begin with a special case of this theorem.

Theorem 4.7 (Rolle's Theorem). *Suppose $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$ then $f'(c) = 0$ for some $c \in (a,b)$.*

Proof. By the extreme value theorem f has a fine minimum m and maximum M it attains somewhere on $[a,b]$. If $m = M$ then f is constant, hence $f'(c) = 0$ everywhere on (a,b) and the claim follows.

If $m < M$ then either $f(a) = f(b) \neq m$ or $f(a) = f(b) \neq M$ or both, hence at least one of these values is attained at a point $c \in (a,b)$. Let us assume that $f(c) = M$, the other case is analogous. Since M is the largest value we have for all h such that $c+h \in [a,b]$:

$$f(c+h) - f(c) = f(c+h) - M \leq 0.$$

It follows that

$$f'(c) = \lim_{h\to 0+} \frac{f(c+h)-f(c)}{h} \leq 0, \quad \text{and} \quad f'(c) = \lim_{h\to 0-} \frac{f(c+h)-f(c)}{h} \geq 0,$$

hence $f'(c) = 0$ from these two inequalities. $\square$

Remark. Both continuity and differentiability hypothesis in this theorem cannot be relaxed.

If $f(a) \neq f(b)$ the Rolle's theorem does not hold but instead we have the following Mean value theorem:

Theorem 4.8 (Mean Value Theorem)**.** *Suppose $a, b \in \mathbb{R}$ with $a < b$.*

(i) If f is continuous on $[a,b]$ and differentiable on (a,b) then there is $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

(ii) If f, g are continuous on $[a,b]$ and differentiable on (a,b) then there is $c \in (a,b)$ such that

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a)).$$

Proof. (i) is a special case of (ii) for $g(x) = x$. Hence it suffices to prove (ii). Consider a function

$$h(x) = f(x)(g(b)-g(a)) - g(x)(f(b)-f(a)).$$

Due to our assumptions on continuity and differentiability of f, g we see that h must satisfy the same assumptions. Also $h(a) = h(b)$and hence Rolle's Theorem applies. It follows that there is a point $c \in (a,b)$ where $h'(c) = 0$, i.e.,

$$h'(c) = f'(c)(g(b)-g(a)) - g'(c)(f(b)-f(a)) = 0.$$

$\square$

Mean value theorem has many uses and applications in analysis. It can be used to prove certain inequalities.

Example 1. Show that $e^x > 1 + x$ for all $x > 0$.

Solution. Consider $f(x) = e^x - 1 - x$ for $x > 0$. By the mean value theorem

$$f(x) = f(x) - f(0) = f'(c)(x-0) = xf'(c).$$

Since $f'(c) = e^c - 1 > 0$ for $c > 0$ and $x > 0$ we see that $f(x) > 0$ which gives us the desired inequality.

Example 2.[Bernoulli's inequality] Let $\alpha > 0$, $\delta \geq -1$. Then

$$(1+\delta)^\alpha \leq 1 + \delta\alpha, \qquad \text{if } \alpha \in (0,1]$$

$$(1+\delta)^\alpha \geq 1 + \delta\alpha, \qquad \text{if } \alpha \in [1,\infty).$$

Solution. Consider only the case $\alpha \in (0,1]$. Let $f(x) = x^\alpha$. By the mean value theorem

$$f(1+\delta) = f(1) + \alpha\delta c^{\alpha-1}$$

for some c between values 1 and $1+\delta$. There are two case to consider. If $\delta > 0$ then $c > 1$ and hence $c^{\alpha-1} < 1$ as $\alpha - 1 < 0$. This gives

$$\alpha\delta c^{\alpha-1} \leq \alpha\delta.$$

On the other hand if $-1 \leq \delta \leq 0$ then $c < 1$ and $c^{\alpha-1} > 1$. Multiplying by negative or zero delta yields again

$$\alpha\delta c^{\alpha-1} \leq \alpha\delta.$$

Theorem 4.9 (L'Hopital's Rule)**.** *Let a be an extended real number and I and interval that either contains a or has I as an endpoint.*

Let f,g be differentiable on $I \setminus \{a\}$ and $g(x) \neq 0 \neq g'(x)$ for all $x \in I \setminus \{a\}$. Suppose further that

$$A = \lim_{x\to a, x\in I} f(x) = \lim_{x\to a, x\in I} g(x)$$

is either 0 or ∞.

If $B = \lim_{x\to a, x\in I} \frac{f'(x)}{g'(x)}$ exists as an extended real number, then

$$\lim_{x\to a, x\in I} \frac{f(x)}{g(x)} = \lim_{x\to a, x\in I} \frac{f'(x)}{g'(x)}.$$

Proof. Since there are lot of cases to consider we only outline only the basic one when $B \in \mathbb{R}$ and $A = 0$, $a \in \mathbb{R}$. Consider arbitrary sequence $x_k \to a$ as $k \to \infty$ such that $x_k \in I \setminus \{a\}$. By the sequential characterization of limits it suffices to show that

$$\frac{f(x_k)}{g(x_k)} \to B, \qquad \text{as } k \to \infty.$$

In general, functions f,g might not be defined at the point $x = a$ but since

$$0 = \lim_{x\to a, x\in I} f(x) = \lim_{x\to a, x\in I} g(x)$$

if we set $f(a) = g(a) = 0$ then f, g are defined on $I \cup \{a\}$ and both are continuous at the point a. It follows the the part (ii) of the mean value theorem that there is c between a and x_k such that

$$\frac{f(x_k)}{g(x_k)} = \frac{f(x_k) - f(a)}{g(x_k) - g(a)} = \frac{f'(c)}{g'(c)}.$$

(Notice that c depends on k here). If we let $k \to \infty$ on both sides, because c lies between x_k and a then $c \to a$ as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} \frac{f(x_k)}{g(x_k)} = \lim_{c\to a,\, c\in I} \frac{f'(c)}{g'(c)} = B.$$

□

Example 1. Find $\lim_{x\to 0+} x \log x$.

Solution. To use L'Hopital's Rule we write x as $1/x$ in the denominator. It becomes a limit of type ∞/∞.

$$\lim_{x\to 0+} x \log x = \lim_{x\to 0+} \frac{\log x}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = 0.$$

4.4 Monotone functions and the Inverse function theorem

Definition 4.3. *Let $E \subset \mathbb{R}$, $E \neq \emptyset$ and $f : E \to \mathbb{R}$.*

(i) f is called increasing (strictly increasing) on E if for all $x_1, x_2 \in E$, $x_1 < x_2$ we have $f(x_1) \leq f(x_2)$ ($f(x_1) < f(x_2)$ in the strictly increasing case).

(ii) f is called decreasing (strictly decreasing) on E if for all $x_1, x_2 \in E$, $x_1 < x_2$ we have $f(x_1) \geq f(x_2)$ ($f(x_1) > f(x_2)$ in the strictly decreasing case).

(iii) f is called monotone (strictly monotone) on E if f is either increasing or decreasing (respectively, either strictly increasing or strictly decreasing) on E.

For example $f(x) = x^2$ is strictly monotone on $[-1, 0]$ or $[0, 1]$ but it is non monotone on $[-1, 1]$.

Theorem 4.10. *Let $a < b$ are real and f be continuous on $[a, b]$ and differentiable on (a, b).*

(i) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$.

(ii) If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing on $[a, b]$.

(iii) $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.

Proof. In each case use the Mean value theorem.

Theorem 4.11. *Let f be 1-1 and continuous on an interval I. Then f is strictly monotone on I and the inverse function f^{-1} is continuous and strictly monotone on $f(I)$.*

Proof. We may assume that I contains at least two points, otherwise the claim is trivial. Let $a,b \in I$, $a<b$. Since f is 1-1 this implies that $f(a)<f(b)$ or $f(a)>f(b)$. Thus if f is not strictly monotone on I there exists three points $a,b,c \in I$, $a<c<b$ such that $f(c)$ does not lie between points $f(a)$ and $f(b)$. Hence either $f(a)$ lies between $f(b)$ and $f(c)$ or $f(b)$ lies between $f(a)$ and $f(c)$. Hence by the Intermediate value theorem (for continuous functions) there is $x_1 \in (a,b)$ such that $f(x_1)=a$ or $f(x_1)=b$. Since f is 1-1 either $x_1=a$ or $x_1=b$, but neither can happen as $x_1 \in (a,b)$. This is a contradiction. Hence f is strictly monotone. We leave the claim about f^{-1} to the reader. □

Theorem 4.12 (Inverse function theorem)**.** *Let f be 1-1 and continuous on an interval I. If $a \in f(I)$ and f' at the point $f^{-1}(a)$ exists and is nonzero, then f^{-1} is differentiable at a and*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Proof. By the previous theorem we already know that f^{-1} exists, is continuous and strictly monotone. Let $x_0 = f^{-1}(a) \in I$. Since we assume that I is open it follows that there are $c,d \in \mathbb{R}$ such that $x_0 \in (c,d) \subset I$. Hence a is a point between $f(c)$, $f(d)$ and therefore we can choose $h \neq 0$ sufficiently small such that $a+h$ is still between $f(c)$, $f(d)$ and hence $f^{-1}(a+h)$ is well defined and belongs to interval (c,d).

Let $x = f^{-1}(a+h)$. Then $f(x)-f(a) = a+h-a = h$. Since f^{-1} is continuous $x \to x_0$ if and only if $h \to 0$. Hence

$$\lim_{h\to 0}\frac{f^{-1}(a+h)-f^{-1}(a)}{h} = \lim_{x_0\to a}\frac{x-x_0}{f(x)-f(a)} = \frac{1}{f'(x_0)}.$$

4.5 Taylor's theorem

Definition 4.4. *Let $n \in \mathbb{N}$ and $a<b$ be extended real numbers. If $f:(a,b)\to\mathbb{R}$ is a function differentiable n-times at a point $x_0 \in (a,b)$ we call the polynomial*

$$P_n^{f,x_0}(x) = f(x_0) + \sum_{k=1}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^n$$

Taylor's polynomial of degree n at x_0.

The idea is that near the point x_0 this polynomial approximates well the function f. How good is the approximation is is implied by the following theorem:

Theorem 4.13 (Taylor's formula)**.** *Let $n \in \mathbb{N}$ and $a<b$ be extended real numbers. If $f:(a,b)\to\mathbb{R}$ and if $f^{(n+1)}$ exists on (a,b), then for each $x,x_0 \in (a,b)$ there exists a number c between x and x_0 such that*

$$\begin{aligned} f(x) &= f(x_0) + \sum_{k=1}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \\ &= P_n^{f,x_0}(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

Proof. Assume that $x < x_0$ (the other case is similar). Consider two functions

$$F(t) = \frac{(x-t)^{n+1}}{(n+1)!}, \qquad G(t) = f(x) - f(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^n.$$

We claim that F, G are differentiable on (a,b) and that

$$F'(t) = -\frac{(x-t)^n}{n!}, \qquad G'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

By the generalized mean value theorem there exists $c \in (x, x_0)$ such that

$$(F(x) - F(x_0))G'(c) = (G(x) - G(x_0))G'(c).$$

Since $F(x) = G(x) = 0$ this implies

$$-F(x_0)G'(c) = -G(x_0)G'(c) \quad \text{or} \quad G(x_0) = F(x_0) \cdot \frac{G'(c)}{F'(c)}.$$

Hence

$$f(x) - P_n^{f,x_0}(x) = f(x) - f(x_0) - \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^n = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

□

Example. The Taylor's polynomial of function e^x of degree n at 0 is

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

The "error term" equals to

$$e^x - (1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}) = \frac{e^c}{(n+1)!}x^{n+1},$$

for some c between 0 and x.

Calculate the error we make if we estimate the value of the number e using Taylor's polynomial of degree 10.

Solution: The error is less than

$$\left| e - \sum_{k=0}^{10} \frac{1}{k!} \right| = \left| \frac{e^c}{(10+1)!} 1^{10+1} \right| \le \frac{3}{11!} \approx 7.8 \times 10^{-8},$$

since $c \in (0,1)$ and we estimate $e^c \le e^1 = e < 3$.

5 Infinite series of real numbers

5.1 Introduction

This is one of the most used tools of analysis. Series are used to approximate numbers and define functions (ln, sin, cos, exp, ...). What is a series? It is a formal expression of the form

$$\sum_{k=1}^{\infty} a_k,$$

where (a_k) is a sequence of numbers. No convergence is assumed at this points. For example $\sum_{n=1}^{\infty} n!$ is one example of such formal sum.

Definition 5.1. *Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series with terms a_k. For each n the partial sum of S of order n is the defined by*

$$s_n = \sum_{k=1}^{\infty} a_k.$$

S is said to converge if an only if its sequence of partial sums (s_n) converge to some $s \in \mathbb{R}$ as $n \to \infty$. That is for any $\varepsilon > 0$ the is $N \in \mathbb{N}$ such that if $n \geq N$ we have

$$|s_n - s| = \left| \sum_{k=1}^{n} a_k - s \right| < \varepsilon.$$

In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call the number s the **sum** *or* **value** *of the series $\sum_{k=1}^{\infty} a_k$.*

S is said to diverge if its sequence of partial sums (s_n) does not converge as $n \to \infty$. When (s_n) diverge to ∞ (or $-\infty$) we shall write

$$\sum_{k=1}^{\infty} a_k = \pm\infty.$$

(Example: $\sum_{n=1}^{\infty} n! = \infty$.)

We have already encountered one type of infinite convergent series in this course, namely decimal expansions. We write $x \in [0,1)$ in the form

$$x = 0.a_1a_2a_3\ldots, \qquad \text{where } a_i \in \{0,1,2,\ldots,9\}$$

and we mean by that

$$x = \sum_{k=1}^{\infty} \frac{a_k}{10^k}.$$

For example $1/3 = 0.33333\dots$. The partial sums are $s_1 = 0.3$, $s_2 = 0.33$, $s_3 = 0.333, \dots$. These are approximations of $1/3$ and they get closer and closer to $1/3$ as more terms of the decimal expansion are taken.

The main question we shall consider in the chapter is how to determine whether a given series converges or diverges.

One way to determine if a given series converges is to find a formula for its partial sums. I most cases this is not possible but there are few simple examples where it works.

Example 1. Show that $\sum_{k=1}^{\infty} 2^{-k} = 1$.

Solution: This series is simple enough such that we can show that the partial sums $s_n = \sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}$ for $n \in \mathbb{N}$. Thus $s_n \to 1$ as $n \to \infty$.

Example 2. Show that $\sum_{k=1}^{\infty} (-1)^k$ diverges.

Solution: We have for the partial sums s_n:

$$s_n = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Thus (s_n) does not converge as $n \to \infty$.

Sometimes we are not able to able to calculate the partial sums exactly, but we are able to estimate them and the estimate is enough to determine convergence or divergence.

Example 3 (Harmonic series). Show that the series $\sum_{k=1}^{\infty} 1/k$ diverges. (Note that the sequence $(1/k)$ converges! This is the most common mistake, to confuse convergence of sequence with convergence of a series. They are different things!)

Solution: We compare $1/k$ with an integral $\displaystyle\int_k^{k+1} 1/x\,dx$. It follows that

$$s_n = \sum_{k=1}^{n} 1/k \geq \sum_{k=1}^{n} \int_k^{k+1} 1/x\,dx = \int_1^{n+1} 1/x\,dx = \log(n+1) \to \infty,$$

as $n \to \infty$.

The example above shows that terms of divergent series **might** converge to zero. However, if the terms if a series DO NOT converge to zero the series itself is ALWAYS divergent.

Theorem 5.1 (Divergence test)**.** *Let (a_k) be a sequence of real numbers. If a_k does not converge to zero then the series*

$$\sum_{k=1}^{\infty} a_k \qquad \textit{diverges.}$$

This is an important result, but as the Example 3 have already shown it can never be use to test convergence, only divergence. If (a_k) converges to zero, this theorem gives no information and the series

$$\sum_{k=1}^{\infty} a_k$$

might converge or diverge. Some other method has to be used instead. Here are two cases when partial sum can determined explicitly.

Theorem 5.2 (Telescopic series)**.** *Let (a_k) be a convergent sequence of real numbers. Then*

$$\sum_{k=1}^{\infty} (a_k - a_{k+1}) = a_1 - \lim_{k\to\infty} a_k.$$

Example 4. Show that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

Solution: We can write

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1}\right] = 1 - \lim_{k\to\infty} \frac{1}{k} = 1.$$

Theorem 5.3 (Geometric series)**.** *Let $x \in \mathbb{R}$ and $N \in \{0,1,2,\dots\}$. Then the series*

$$\sum_{k=N}^{\infty} x^k \qquad \text{converges if and only if } |x| < 1.$$

In this case

$$\sum_{k=N}^{\infty} x^k = \frac{x^N}{1-x}.$$

In particular,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad |x| < 1.$$

Proof. Using mathematical induction we can establish that for $x \neq 1$ the partial sum $s_n = \sum_{k=N}^{n+N} x^k$ is equal to

$$s_n = x^N \left[\frac{1}{1-x} - \frac{x^{n+1}}{1-x}\right].$$

Taking the limit $n \to \infty$ we see that this converges if and only if $x^{n+1} \to 0$. □

Theorem 5.4 (Cauchy criterion)**.** *Let (a_k) be a real sequence. The infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m \geq n \geq N$ we have*

$$\left|\sum_{k=n}^{m} a_k\right| < \varepsilon.$$

Theorem 5.5. *Let (a_k) and (b_k) be a real sequences. If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent then*

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

and

$$\sum_{k=1}^{\infty} (\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k.$$

5.2 Series with nonnegative terms

We focus here on series with nonnegative terms. This simplifies the situation and allows us to establish several criteria on convergence and divergence.

Theorem 5.6. *Suppose that $a_k \geq 0$ for large k. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums (s_n) is bounded. That is there exists $M > 0$ such that*

$$\left| \sum_{k=1}^{n} a_k \right| \leq M, \qquad \textit{for all } n \in \mathbb{N}.$$

Proof. If $\sum_{k=1}^{\infty} a_k$ converges then the sequence if partial sums is convergent, hence bounded. Conversely, assume that (s_n) is bounded. Due to our assumption, this sequence is monotone for $n \geq N$, that is $s_n \leq s_{n+1}$ for all such n. By the monotone convergence theorem the sequence (s_n) must be convergent as it is monotone and bounded. $\square$

If $a_k \geq 0$ for large k we shall introduce the following notation. We write

$$\sum_{k=1}^{\infty} a_k < \infty \qquad \text{if the series is convergent and}$$

$$\sum_{k=1}^{\infty} a_k = \infty \qquad \text{if the series is divergent.}$$

Theorem 5.7 (Integral test)**.** *Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Let $a_k = f(k)$, $k = 1, 2, 3, \ldots$. Then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral*

$$\int_1^{\infty} f(x)\, dx < \infty.$$

Proof. Due to monotonicity of function f we have that

$$a_{k+1} = f(k+1) \leq \int_k^{k+1} f(x)\, dx \leq f(k) = a_k,$$

for $k \in \mathbb{N}$. Let us sum this inequality over all $k = 1, 2, \ldots, n$. We get for the partial sums s_n:

$$s_n - a_1 \leq \sum_{k=2}^{n+1} a_k \leq \int_1^{n} f(x)\, dx \leq \sum_{k=1}^{n} a_k = s_n.$$

This inequality implies that (s_n) is bounded if and only if $\int_1^\infty f(x)\,dx < \infty$. $\square$

Example [p-series test] The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Solution: Set $f(x) = x^{-p}$ on $[1,\infty)$. Then $f'(x) = -px^{-p-1} < 0$ for all x provided $p > -1$. Hence f is nonnegative and decreasing on $[1,\infty)$. Since

$$\int_1^\infty x^{-p} dx = \lim_{n\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^n = \lim_{n\to\infty} \frac{n^{1-p}-1}{1-p},$$

the integral has finite limit if and only if $n^{1-p} \to 0$ or $p > 1$. Note that for $p = 1$ we get the harmonic series, when $p < 0$ the series diverges trivially as $1/n^p$ does not converge to 0.

Integral test is a powerful tool but what limit its usefulness is the condition that f is monotone. Instead in many applications it is very useful to compare two series

Theorem 5.8 (Comparison test)**.** *Suppose that* $0 \le a_k \le b_k$ *for large* k

- *If* $\sum_{k=1}^{\infty} b_k < \infty$ *then* $\sum_{k=1}^{\infty} a_k < \infty$.
- *If* $\sum_{k=1}^{\infty} a_k = \infty$ *then* $\sum_{k=1}^{\infty} b_k = \infty$.

Example. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{3k}{k^2+k}\sqrt{\frac{\log k}{k}}$$

is convergent of divergent.

Solution: It is known that for large k, $\log k$ grows slower than any positive power k^ε, $\varepsilon > 0$. Because of this we can estimate

$$\frac{3k}{k^2+k}\sqrt{\frac{\log k}{k}} = \frac{1}{k}\frac{3k}{k+1}\sqrt{\frac{\log k}{k}} \le 3\frac{1}{k}\sqrt{\frac{k^\varepsilon}{k}} = \frac{3}{k^{3/2-\varepsilon/2}},$$

for k large. By choosing $\varepsilon > 0$ small we see that $3/2 - \varepsilon/2 > 1$ and hence by the p-series test $\sum \frac{3}{k^{3/2-\varepsilon/2}}$ is convergent. If follows from the comparison test that our original series is also convergent.

Sometimes comparison test requires a bit delicate inequalities and it might be easier to consider the ratio a_n/b_n of two sequences as $n \to \infty$. This leads to

Theorem 5.9 (Limit Comparison test)**.** *Suppose that* $0 \le a_k$*,* $0 < b_k$ *for large* k *and that*

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} \qquad \textit{exists as an extended real number.}$$

- *If $L \in (0,\infty)$ then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.*
- *If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges.*
- *If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges.*

In general, whenever limit comparison test works, the comparison test also does, but the limit comparison test might be a bit easier to verify.

5.3 Absolute convergence

Definition 5.2. *Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series.*

We say that the series S converges absolutely if $\sum_{k=1}^{\infty} |a_k| < \infty$.

We say that the series S converges conditionally if S converges but $\sum_{k=1}^{\infty} |a_k|$ diverges.

A series converges absolutely if and only if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m \geq n \geq N$

$$\sum_{k=n}^{m} |a_k| < \varepsilon.$$

(This is a consequence that the sequence of partial sums must be Cauchy).

Theorem 5.10. *If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then $\sum_{k=1}^{\infty} a_k$ converges, but not conversely. There exists a conditionally convergent series.*

Proof. If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m \geq n \geq N$

$$\sum_{k=n}^{m} |a_k| < \varepsilon.$$

It follows that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < \varepsilon,$$

and hence by Cauchy criterion the series $\sum_{k=1}^{\infty} a_k$ converges.

An example of a series that converges conditionally is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$. □

Note that to test convergence of $\sum_{k=1}^{\infty} |a_k|$ we may use every test of the previous section (integral, comparison, limit comparison).

We finish by introducing two additional tests for convergence.

Theorem 5.11 (Root test). *Let $a_k \in \mathbb{R}$ and assume that $r = \lim_{k\to\infty} |a_k|^{1/k}$ exists. If*

- *$r < 1$ then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.*
- *$r > 1$ then the series $\sum_{k=1}^{\infty} a_k$ diverges.*

Usually easier to check is the following test

Theorem 5.12 (Ration test). *Let $a_k \in \mathbb{R}$ and assume that $r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ exists as an extended real number. If*

- *$r < 1$ then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.*
- *$r > 1$ then the series $\sum_{k=1}^{\infty} a_k$ diverges.*

Proofs. Both of these results follow from comparison test. In both cases if $r < 1$ one can prove that for all k large

$$|a_k| \leq \left(\frac{1+r}{2}\right)^k.$$

It follows that $\sum_{k=1}^{\infty} |a_k|$ converges as the geometric series $\sum_{k=1}^{\infty} ((1+r)/2)^k$ is convergent.

For $r > 1$ one can show that for all k large

$$|a_k| \geq \left(\frac{1+r}{2}\right)^k.$$

Thus again one can use comparison test to show divergence. □

Remark. Both root and ratio test are inconclusive if $r = 1$. If $r = 1$ the series might diverge, absolutely converge or conditionally converge. Examples are $\sum_{k=1}^{\infty} 1/k$, $\sum_{k=1}^{\infty} 1/k^2$ and $\sum_{k=1}^{\infty} (-1)^k/k$.

5.4 Series with alternating signs

It has been stated that the series $\sum_{k=1}^{\infty}(-1)^k 1/k$ is conditionally convergent. This series is an example of a series with alternating signs. We now establish a general theorem showing convergence for this and similar series.

Theorem 5.13 (Alternating sign series)**.** *Let (a_k) be a decreasing sequence on nonnegative numbers such that $a_k \to 0$ as $k \to \infty$. Then the series*

$$\sum_{k=1}^{\infty}(-1)^k a_k \qquad \textit{is convergent.}$$

Proof. As before, let (s_n) be the sequence of partial sums, i.e.,

$$s_n = \sum_{k=1}^{n}(-1)^k a_k.$$

We claim the following: The subsequence of odd terms $s_1 \le s_3 \le s_5 \le \dots$ is increasing and bounded above (by s_2). The subsequence of even terms $s_2 \ge s_4 \ge s_6 \ge \dots$ is decreasing and bounded below (by s_1).

Indeed, Let n be odd. Then

$$s_{n+2} = s_n + a_{n+1} - a_{n+2} \ge s_n,$$

since $a_{n+1} \ge a_{n+2}$. Similarly, if n is even then

$$s_{n+2} = s_n - a_{n+1} + a_{n+2} \le s_n.$$

Also since for n odd $s_n < s_n + a_{n+1} = s_{n+1} \le s_{n-1} \le \dots \le s_2$ the subsequence of odd terms is bounded above. Similarly for n even we have $s_n > s_n - a_{n+1} = s_{n+1} \ge s_{n-1} \ge \dots \ge s_1$ the subsequence of even terms is bounded below.

By the monotone convergence theorem it follows that these two limits exist:

$$e = \lim_{k\to\infty} s_{2k}, \qquad \text{and} \qquad o = \lim_{k\to\infty} s_{2k+1}.$$

Moreover, we claim that $e = o$ as $a_k \to 0$, hence the limit of the full sequence $\lim_{k\to\infty} s_k = e = o$ exists. To see this consider

$$o - e = \lim_{k\to\infty} s_{2k+1} - \lim_{k\to\infty} s_{2k} = \lim_{k\to\infty}[s_{2k+1} - s_{2k}] = \lim_{k\to\infty} a_{2k+1} = 0.$$

$\square$

Corollary. The series

$$\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}, \qquad \sum_{k=2}^{\infty}(-1)^k \frac{1}{\log k}, \qquad \sum_{k=2}^{\infty}(-1)^k \frac{1}{k \log k}$$

are all convergent.