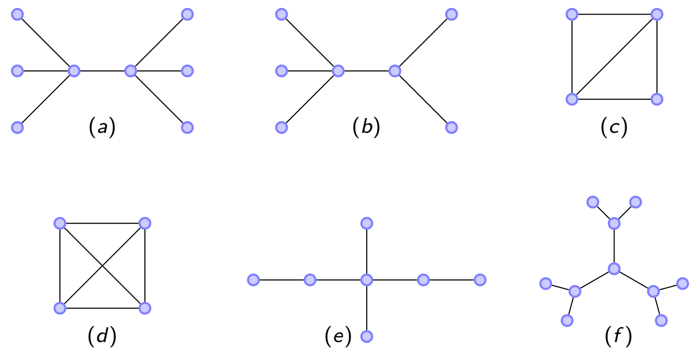
**Fundamentals of Pure Mathematics**
**Workshop Week 1. From Graphs to Product Groups.**

The point of this workshop to practice determining symmetry groups of graphs. This will give us many examples of groups, and amongst other things will also naturally motivate the study of product groups.

> Let $V$ denote the set of vertices of a graph. Recall that a *symmetry* of the graph is a bijection $f\colon V \to V$ such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if $v_1$ and $v_2$ are joined by an edge.

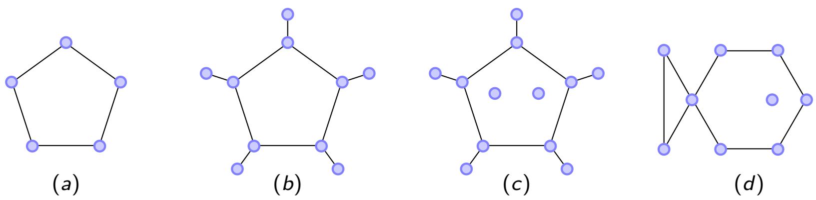
The symmetries of a graph form a group, under composition.

**Question 1.** Determine the number of symmetries of the following graphs.




In your answer, you must argue that you have *all* symmetries.

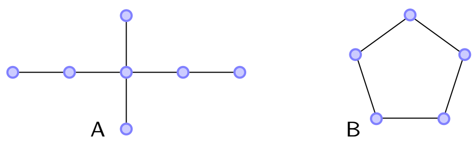
**Question 2.** Determine the number of symmetries of the following graphs.



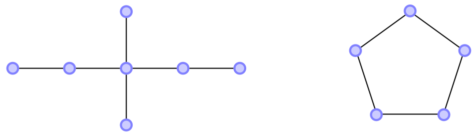

In Question 1 and Question 2(a) and 2(b), the graphs are *connected*, whereas in Question 2(c) and 2(d) the graphs are not connected.

**Question 3.** Write down a mathematical definition of what it means for a graph to be *connected*.

**Question 4.** (motivates product groups) Let $A, B$ be two different connected graphs with symmetry groups $G, H$. To make this concrete, these could be



We could then take the union of the graphs $A$ and $B$, which just means that we write the two graphs down next to each other, without any edges going between them. So, in the case above the union is just



(considered as one graph which is not connected).

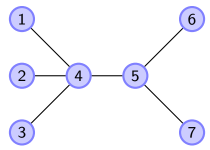
(a) What is the relationship between the number of symmetries of the union, $|G|$ and $|H|$?

(b) Can you make sense of the following statement: 'the group of symmetries of the union is just $G$ and $H$'? Can you write this down mathematically, and provide a proof?

(c) Why do we insist that the two graphs $A$ and $B$ are different?

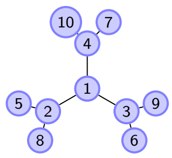
**Solutions.**

1. (b) If we number the vertices




then 4 is the only vertex with valency four, and 5 is the only vertex with valency three, hence any symmetry must fix both 4 and 5. Thus a symmetry is determined by specifying a permutation of the vertices 1, 2 and 3, and a permutation of vertices 6 and 7. There are precisely $6 \times 2$ possibilities, and so precisely 12 symmetries.

(f) If we number the vertices




then 1,2,3 and 4 are the only vertices with valency three so any symmetry must permute them. Since $f(1)$ must be connected to $f(2)$, $f(3)$ and $f(4)$, necessarily $f(1)$ must be fixed. Thus by looking at valencies, a symmetry is specified by some permutation of vertices 2, 3 and 4 (there are 6 possibilities), together with a permutation of 5,8 (there are 2 possibilities), a permutation of 6,9 (there are 2 possibilities), and a permutation of 10,7 (there are 2 possibilities). Hence there are $6 \times 2 \times 2 \times 2 = 48$ symmetries.

Other answers, in brief:

(a) 72 (you can separately permute the three danglers at each end giving $6 \times 6$ and you can also swap the two ends over, giving another $\times 2$).

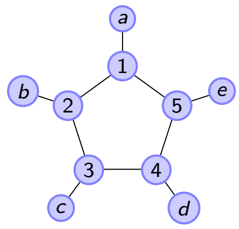
(c) 4 (you can swap the two valence three vertices and separately the two valence 2 ones, so $2 \times 2$)

(d) 24 (in this graph every edge is connected to every other, so every permutation of the vertices is a symmetry).

(e) 4 (reflect top to bottom and/or left to right, so $2 \times 2$)

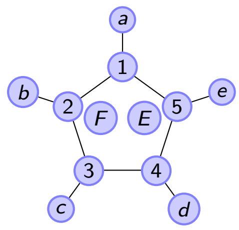
2. (a) Label the vertices $1, \dots, 5$. Any symmetry must preserve vertices of valency two, must take the vertex 1 to a member of the set $\{1, 2, \dots, 5\}$. Any such choice is possible, and after making this choice (there are five possible), everything else is determined by whether the neighbours of 1 are fixed or swapped (of which there are two choices). Hence there are 10 symmetries in total.

(b) For convenience, label




Any symmetry must preserve vertices of valency three, so must take the vertex 1 to a member of the set $\{1, 2, \dots, 5\}$. Any such choice is possible, and after making this choice (there are five possible), everything else is determined by whether the neighbours of 1 are fixed or swapped (of which there are two choices). This shows that there are $2 \times 5 = 10$ symmetries in total.

(c) For convenience, label

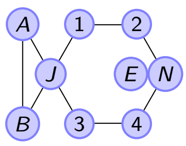



Any symmetry

- must either fix the eyes $E$ and $F$ or swap them (since they are the only vertices connected to no others). This gives two options.
- Any symmetry must preserve vertices of valency three, so must take the vertex 1 to a member of the set $\{1, 2, \dots, 5\}$. Any such choice is possible, and after making this choice (there are five possible), everything else is determined by whether the neighbours of 1 are fixed or swapped (of which there are two choices).

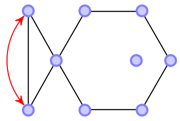
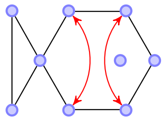
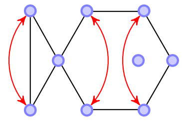
This shows that there are $2 \times 5 \times 2 = 20$ symmetries in total.

(d) For convenience, label




Any symmetry must fix the eye $E$ (since it is the only vertex connected to no others), the join $J$ (since it is the only vertex of valency four), and the nose $N$ (since it is the only vertex a distance three from the fixed $J$). Now any symmetry must either fix both 2 and 4, or interchange 2 and 4 (since they are connected to fixed $N$). In either case, this determines the action on 1 and 3 — if 2 and 4 are fixed, so must be 1 and 3, whereas if 2 and 4 swap, so must 1 and 3. Hence there are two options for the "body" of the fish.

Similarly, there are two options for the "tail" — either $A$ and $B$ are fixed, or they are interchanged. Hence, overall there are $2 \times 2 = 4$ symmetries of the graph. Explicitly, they are the identity, together with

 and  and 

$g_1$ $g_2$ $g_3$

3. One way is the following: First, a *path* in a graph is a sequence of vertices such that for each vertex in the sequence there is an edge to the next vertex in the sequence. We then say that a graph is *connected* if, for every two vertices in the graph, there is a path between them.

4. (b) The group of symmetries of the union is the product group $G \times H := \{(g, h) \mid g \in G, h \in H\}$, with group structure $(g, h) * (g', h') := (g *_G g', h *_H h')$ (where we write $*_G$ for the group operation in $G$, etc).
To see this, denote $C$ to be the union of $A$ and $B$, denote $V$ to be the set of vertices of $A$, and denote $W$ to be the set of vertices of $B$. Hence the vertices of $C$ are $V \cup W$. A symmetry of $C$ is specified by a bijection $f : V \cup W \to V \cup W$ that preserves edges. Since $A$ and $B$ are different, we claim $f$ must map $V$ to $V$, and map $W$ to $W$. To verify the claim, note that there are no edges between members of $V$ and members of $W$, so if $f$ maps one element of $V$ to an element of $W$, it must map all members of $V$ to $W$ (since $A$ is connected). This would imply that $A$ is a subgraph of $B$. But then, since $f$ is a bijection, at least one element of $W$ must get mapped to an element of $V$ and hence (arguing as above) every member of $W$ must get sent to a member of $V$. Since $f$ is a bijection, this would imply that $|V| = |W|$ and so since $A$ is a subgraph of $B$, necessarily $A = B$, which is a contradiction.
This shows that a symmetry of $C$ is specified by a symmetry of $A$ together with a symmetry of $B$, so the symmetry group is $G \times H$.
(a) follows from (b) since $|G \times H| = |G||H|$.
(c) If $A = B$ then the above fails, since there is an extra symmetry of $C$ given by swapping $A$ and $B$.