Fundamentals of Pure Mathematics 2015-16
Assignment 6
Due-date: Thursday of week 7
(Submit your work to your tutor at the workshop)

1. (5 marks) (This is Problem 46 from the week 6 workshop)

   Prove directly from the ε-δ definition of continuity[1] (without using any Limit Theorems) that the function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=3x^2$ is continuous at $x_0=1$.

   Rough Work:

   Given $\varepsilon>0$ we are looking for $\delta>0$ such that for all x with

   $$|x-1|<\delta \tag{1}$$

   we have

   $$\big|f(x)-f(1)\big|<\varepsilon. \tag{2}$$

   We have

   $$\big|f(x)-f(1)\big|=\left|3x^2-3\right|=3|x+1|\,|x-1|. \tag{3}$$

   Infront of $|x-1|$ we have the variable quantity $3|x+1|$ instead of a constant. Since we are only interested in x near 1 we expect $x+1$ to be near 2, and therefore we expect $3|x+1|$ to be bounded. We can replace it in (3) by one of its bounds, which is a constant, and then work as in (**??**), i.e.

   $$\begin{aligned}\big|f(x)-f(1)\big|&=\left|3x^2-3\right|=3|x+1|\,|x-1|\\&\le(\text{a bound for }3|x+1|)\,|x-1|=(\text{constant})\,|x-1|.\end{aligned} \tag{4}$$

   Indeed, for x with $|x-1|<\delta$ we have

   $$3|x+1|=3|x-1+2|\le3\big(|x-1|+2\big)<3(\delta+2) \tag{5}$$

   therefore we can continue (3) as follows:

   $$\big|f(x)-f(1)\big|=\left|3x^2-3\right|=3|x+1|\,|x-1|<3(\delta+2)\delta. \tag{6}$$

   It is enough now to ensure that

   $$3(\delta+2)\delta<\varepsilon. \tag{7}$$

[1] Wade, Definition 3.19. Dindos, Definition 3.3.

This is not as bad as it looks. We are not trying to find all δ that satisfy (7), we are not trying to solve for delta. All we need is to find a single δ that satisfies (7). To make our task easier let us try to find a δ with

$$\delta < 1 \tag{8}$$

that satisfies (7). We have $3(\delta+2)\delta < 3(1+2)\delta = 9\delta$. It is enough now to ensure that $9\delta < \varepsilon$ and this is easily done by choosing a δ with

$$\delta < \frac{\varepsilon}{9}. \tag{9}$$

An example of a positive number δ that satisfies both (8) and (9) is $\delta = \frac{1}{2}\min\{1, \frac{\varepsilon}{9}\}$.

Remark: There is nothing special about the bound 1 in (8). We could have said: let's try to find a δ with $\delta < 2016$ that satisfies (7). We have $3(\delta+2)\delta < 3(2016+2)\delta = 6054\delta$. It is enough now to ensure that $6054\delta < \varepsilon$ and this is easily done by choosing a δ with $\delta < \frac{\varepsilon}{6054}$, etc etc .

(Reminder: Do not hand in Rough Work.) END OF ROUGH WORK

Solution Let $\varepsilon > 0$ be given. Let δ be a positive number such that[2] $\delta < 1$ and $\delta < \frac{\varepsilon}{9}$. For all $x \in \mathbb{R}$ with $|x-1| < \delta$ we have

$$|x+1| = |x-1+2| \leq |x-1| + 2 < \delta + 2 < 1 + 2 = 3,$$

therefore,

$$\left|f(x) - f(1)\right| = \left|3x^2 - 3\right| = 3\,|x+1|\,|x-1| < 9\delta < \varepsilon.$$

2. (5 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

(a) Prove that f is discontinuous at all points $x_0 \neq 0$.

(b) Prove that f is continuous at $x_0 = 0$.

[2] An example of such a δ is $\delta = \frac{1}{2}\min\{1, \frac{\varepsilon}{9}\}$.

(Comments: You are not asked to prove (a) and (b) directly from ε-δ definition of continuity. You may use any results from Chapters 1-3 provided that you state clearly which ones you are using.

If you decide to use the sequential characterization of continuity[3] you may need the result of Problem 37: for every real number x there exist a sequence of rationals and a sequence of irrationals converging to x.

If you decide to use ε's and δ's you may need the following result from Liebeck: between any two real numbers there exist at least one rational and at least one irrational.)

Support is available:

- From the lecturer (n.bournaveas@ed.ac.uk, room 4614)
- On Piazza
- At the Hub (Tuesdays, 1-3pm)

Solution 1 (Using sequences)

(a) Fix $x_0 \neq 0$. We wish to show that f is not continuous at x_0.

According to the sequential characterization of continuity, f is continuous at x_0 iff for every sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x_0, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$.

Therefore, to show that f is not continuous at x_0 it is enough to find a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x_0, such that the sequence $(f(x_n))_{n\in\mathbb{N}}$ doesn't converge to $f(x_0)$.

If x_0 is rational then $f(x_0) = x_0$. Pick a sequence $(x_n)_{n\in\mathbb{N}}$ of irrationals converging to x_0 (see Problem 37). Then $f(x_n) = 0$ for all n, therefore $f(x_n) \to 0 \neq f(x_0)$ ($f(x_0)$ is not zero because it is equal to x_0 and x_0 isn't zero).

If x_0 is irrational then $f(x_0) = 0$. Pick a sequence $(x_n)_{n\in\mathbb{N}}$ of rationals converging to x_0. Then $f(x_n) = x_n$ for all n, therefore $f(x_n) \to x_0 \neq f(x_0)$ (because $f(x_0)$ is zero and x_0 isn't).

[3] Wade, Thm 3.21. Dindos, Thm 3.5.

(b) We prove that f is continuous at $x_0 = 0$. Using the sequential characterization of continuity, we need to show that for every sequence $(x_n)_{n\in\mathbb{N}}$ that converges to 0, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(0)$.

Let $(x_n)_{n\in\mathbb{N}}$ be any sequence of real numbers converging to zero. Observe that $\big|f(x_n)\big| \le |x_n|$ for all n (because the left-hand side is $|x_n|$ or $|0|$). By the Squeeze Theorem, $\big|f(x_n)\big| \to 0$, therefore $f(x_n) \to 0 = f(0)$.

<u>Solution 2</u> (Using ε's and δ's.)

(a) Fix $x_0 \neq 0$. We show that f is discontinuous at x_0. Since f is odd, it is enough to consider the case $x_0 > 0$. ()

Recall from the solution to Problem 5 that the negation of a statement of the form

$$\forall x \;\; p(x)\,,$$

is

$$\exists x \;\; \textit{not} \;\; p(x)\,,$$

and that the negation of a statement of the form

$$\exists x \;\; p(x)\,,$$

is

$$\forall x \;\; \textit{not} \;\; p(x)\,.$$

Now 'f is continuous at x_0' is the same as:

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in (x_0 - \delta, x_0 + \delta) \;\; \big|f(x) - f(x_0)\big| < \varepsilon. \tag{10a}$$

Therefore, 'f is discontinous at x_0' is the same as

$$\exists \varepsilon > 0 \;\; \textit{not} \left[\;\exists \delta > 0 \;\; \forall x \in (x_0 - \delta, x_0 + \delta) \;\; \big|f(x) - f(x_0)\big| < \varepsilon \right], \tag{10b}$$

which is the same as

$$\exists \varepsilon > 0 \;\; \forall \delta > 0 \;\; \textit{not} \left[\forall x \in (x_0 - \delta, x_0 + \delta) \;\; \big|f(x) - f(x_0)\big| < \varepsilon \right], \tag{10c}$$

which is the same as

$$\exists \varepsilon > 0 \;\; \forall \delta > 0 \;\; \exists x \in (x_0 - \delta, x_0 + \delta) \;\; \textit{not} \left[\;\; \big|f(x) - f(x_0)\big| < \varepsilon \right], \tag{10d}$$

which is the same as

$$\exists\varepsilon>0 \quad \forall\delta>0 \quad \exists x\in(x_0-\delta,x_0+\delta) \quad \bigl|f(x)-f(x_0)\bigr|\geq\varepsilon. \tag{10e}$$

Back to our f, we fix $x_0>0$ and prove (10e). Set $\varepsilon=\frac{1}{2}x_0$. Let $\delta>0$ be given.

If x_0 is rational, pick an irrational x in $(x_0-\delta,x_0+\delta)$. Then $\bigl|f(x)-f(x_0)\bigr|=|0-x_0|=x_0\geq\frac{1}{2}x_0=\varepsilon$.

If x_0 is irrational, pick a rational x in $(x_0,x_0+\delta)$. Then $\bigl|f(x)-f(x_0)\bigr|=|x-0|=x\geq x_0\geq\frac{1}{2}x_0\geq\varepsilon$.

(b) We show that f is continuous at $x_0=0$. Let $\varepsilon>0$ be given and set $\delta=\varepsilon$. For all x with $|x-0|<\delta$ we have $\bigl|f(x)-f(0)\bigr|=\bigl|f(x)\bigr|\leq|x|<\delta=\varepsilon$.

Another way to say the same thing: We observe that $\bigl|f(x)\bigr|\leq|x|$ for all x. Clearly, $|x|\xrightarrow[x\to 0]{}0$. By the Squeeze Theorem, $\bigl|f(x)\bigr|\xrightarrow[x\to 0]{}0$ as well, therefore $f(x)\xrightarrow[x\to 0]{}0=f(0)$. It follows that f is continuous at $x_0=0$.

Things to think about:
Write down a rigorous proof of the following fact that was used in the solution to part (a):

Let $f:\mathbb{R}\to\mathbb{R}$ be an odd function and $x_0\neq 0$. Then f is discontinuous at x_0 iff f is discontinuous at $-x_0$.

Is the same true if we replace 'discontinuous' by 'continuous'?
Is the same true for even functions?