Student Examination Number ....

The University of Edinburgh
College of Science and Engineering

FOR ANONYMOUSLY MARKED EXAMINATIONS
FOLD AND SEAL WITH BARCODE

Surname............................

Pre-Names............................

Signature............................

Enrolment Number............................
(as shown on your Matriculation card)

Pre – Honours 2

MATH08064 Fundamentals of Pure Mathematics

Tuesday, 20th August 2013

2:30pm – 5:30pm

Chairman of Examiners – Professor B Leimkuhler

External Examiner – Dr N D Gilbert

All six questions from **Section A** count (40% total mark). The **best three of four** answers from **Section B** will count (60% of total mark).

In the examination it is permitted to have;
A copy of the course text book
An Introduction to Analysis by W.R.Wade (for analysis)
Groups by C.R.Jordan and D.A.Jordan (for group theory)
Any other notes (printed or hand-written)

Section A	**Mark**
1	
2	
3	
4	
5	
6	
Total	
Section B	**Mark**
7	
8	
9	
10	
Total	

Calculators and other electronic aids
Only calculators from the list maintained by the College of Science and Engineering may be used in this examination.

Make and Model
Casio fx85 (any version, e.g. fx85WA, fx85MS)
Casio fx83 (any version, e.g. fx83ES)
Casio fx82 (any version)

This examination will be marked anonymously.

Section A (40 points). Solve all 6 problems in this section

(1) Show that no two of the groups

$$A_4, \quad D_4, \quad D_6, \quad \mathbb{Z}_{12}, \quad \mathbb{Z}_2 \times \mathbb{Z}_6$$

are isomorphic. **[6 marks]**

(2) In each of the following, give an example of a group G acting on a set X for which

(a) The action is both transitive and faithful.

(b) The action is transitive but not faithful.

(c) The action is faithful but not transitive.

(d) The action is not transitive and not faithful.

(e) The action has precisely two orbits.

(f) The action has precisely three orbits.

[7 marks]

(3) Consider the set

$$G := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

(a) Show that G is a subgroup of $\mathrm{GL}(2, \mathbb{R})$. **[4 marks]**

(b) Show that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. You may use any theorems from lectures, provided that they are clearly stated. **[3 marks]**

(4) Are the following statements TRUE or FALSE? If TRUE give a proof, if FALSE give a counterexample or justification.

(a) Let $\alpha > 0$, A be a nonempty bounded subset of $\mathbb{R}$ and $B = \{\alpha x : x \in A\}$. Then $\sup B = \alpha(\sup A)$.

(b) Let A and B be two nonempty bounded subsets of $\mathbb{R}$. If $A - B = \{a - b : a \in A \text{ and } b \in B\}$ then $\sup(A - B) = \sup A - \sup B$.

(c) Suppose that E is a set. If there exists a function from E onto $\mathbb{N}$, then E is at most countable.

[7 marks]

[Please turn over]

(5) Let (x_n) be a sequence. Suppose that for all $n \in \mathbb{N}$ we have that

$$|x_{n+1} - x_n| \leq \frac{1}{n^2}.$$

Show that (x_n) is a Cauchy sequence. Explain briefly why (x_n) is convergent.

[7 marks]

(6) Let (y_n) be a convergent sequence with limit 2. Using the definition of a convergent sequence show that the sequence $(y_n^2 + 1)$ is also convergent and has limit 5.

[6 marks]

Section B (60 points). Solve any three problems in this section

(7) (a) Recall that $\mathbb{R}^*_+ := \{r \in \mathbb{R} \mid r > 0\}$, which is a group under multiplication. Show that the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}^*_+$ is a group isomorphism. **[4 marks]**

(b) (i) Justify why the function $\mathbb{Z}_4 \to \mathbb{Z}_4$ sending $z \mapsto 1$ (for all $z \in \mathbb{Z}_4$) is not a group homomorphism. **[3 marks]**

(ii) Show that the function $\mathbb{Z}_4 \to \mathbb{Z}_4$ sending $z \mapsto 2z$ (for all $z \in \mathbb{Z}_4$) is a group homomorphism. **[3 marks]**

(c) Give an example of a group G and a group homomorphism $\theta\colon G \to G$ for which $\operatorname{Ker}\theta = \operatorname{Im}\theta$. **[4 marks]**

(d) If G is a group, then is the function $\phi\colon G \to G$ defined by $\phi(g) = g^2$ always a group homomorphism? Give a proof or counterexample. **[6 marks]**

(8) (a) Consider the symmetric group S_4.

(i) List all the elements of S_4, in cycle notation. **[3 marks]**

(ii) Find the conjugacy classes of S_4, and write down all the elements in each. You may use any theorem from lectures, provided that it is clearly stated. **[3 marks]**

(iii) Consider the set $H := \{e, (24)(13), (14)(23), (34)(12)\} \subseteq S_4$. Prove that H a normal subgroup of S_4. Is $H \cong \mathbb{Z}_4$? **[4 marks]**

(b) Consider now the alternating group A_4.

(i) List all the elements of A_4, in cycle notation. **[3 marks]**

(ii) Justify why A_4 is a normal subgroup of S_4. **[2 marks]**

(iii) Does A_4 contain a normal subgroup other than $\{e\}$ and A_4? **[2 marks]**

(iv) Is A_4 isomorphic to $D_3 \times \mathbb{Z}_2$? Justify your answer. **[3 marks]**

(9) (a) Let (a_k) be a bounded sequence with $a_k \geq 0$ for all k. Prove that

$$\sum_{k=1}^{\infty} \frac{a_k}{(k+2)^p}$$

converges for all $p > 1$. **[7 marks]**

(b) For what values of $p \in \mathbb{R}$ is the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$$

convergent? **[7 marks]**

(c) State, with a brief justification, whether the following statements are TRUE or FALSE:

(i) If the series $\sum_{k=1}^{\infty} a_k$ is convergent then the sequence (a_k) is convergent.

(ii) If the sequence (a_k) is convergent then the series $\sum_{k=1}^{\infty} a_k$ is convergent.

[6 marks]

[Please turn over]

(10) (a) Use the mean value theorem to prove that for all $x, y \in \mathbb{R}$

$$|\sin x - \sin y| \leq |x - y|.$$

[6 marks]

(b) Find the Taylor polynomial of degree 2 at $x = 0$ for the function $f(x) = \sqrt{1+x}$. Show that, if the Taylor polynomial of degree 2 is used to estimate $\sqrt{1.1}$, the error is at most $3/48000$. **[7 marks]**

(c) Let I be an open interval, $f : I \to \mathbb{R}$ continuous and $a \in I$ a point. Use the definition of continuity to show that if $f(a) < M$ there is $\delta > 0$ such that

$$\text{if } x \in (a - \delta, a + \delta), \quad \text{then} \quad f(x) < M.$$

[7 marks]

[End of Paper]

[End of paper]