Fundamentals of Pure Mathematics
Workshop Week 7: Group Actions

The point of this workshop is to practice working with group actions, the orbit–stabilizer theorem, and Cayley's Theorem.

Let G be a group, and let X be a nonempty set. Then a (left) action of G on X is a map $G \times X \to X$, written $(g, x) \mapsto g \cdot x$, such that

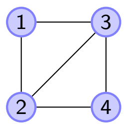
$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad \text{and} \quad e \cdot x = x$$

for all $g_1, g_2 \in G$ and all $x \in X$.

Warmup Question 1. Consider the group D_4. Give as many examples as you can of D_4 acting on a set. Compare with the next table to see how many you got.

If $x \in X$, the *orbit* of x is $\mathrm{Orb}_G(x) = \{g \cdot x | g \in G\}$, and the *stabilizer* of x is $\mathrm{Stab}_G(x) = \{g \in G | g \cdot x = x\}$.

In the next three questions, let G be the symmetry group of the graph



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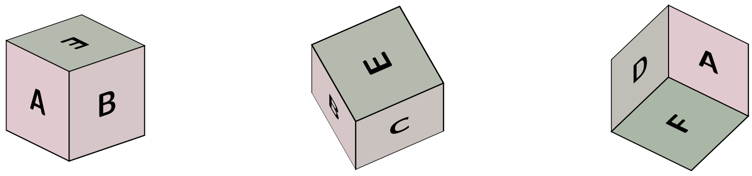
Let G on the set $X = \{1, 2, 3, 4\}$ of vertices in the obvious way.

Warmup Question 2.
(a) Write down the elements of G.
(b) Calculate $\mathrm{Orb}_G(2)$ and $\mathrm{Stab}_G(2)$. Verify that $|\mathrm{Orb}_G(2)| \times |\mathrm{Stab}_G(2)| = |G|$.

Warmup Question 3.
(a) What is the kernel of the action of G on X?
(b) Verify that G also acts on $\mathrm{Orb}_G(2)$. What is the kernel of this action?

Continued over...

Question 4. Consider the rotational symmetry group G of the cube



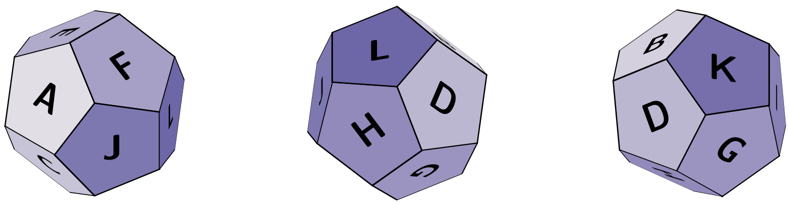
(a) G acts on the set of vertices V of the cube. Pick $v \in V$. Describe the stabilizer $\mathrm{Stab}_G(v)$ and the orbit $\mathrm{Orb}_G(v)$. By using the orbit–stabilizer theorem, deduce $|G|$.

(b) G acts on the set of faces F of the cube. Pick $f \in F$. Describe the stabilizer $\mathrm{Stab}_G(f)$ and the orbit $\mathrm{Orb}_G(f)$. Deduce $|G|$.

(c) G acts on the set of edges E of the cube. Pick $e \in E$. Describe the stabilizer $\mathrm{Stab}_G(e)$ and the orbit $\mathrm{Orb}_G(e)$. Deduce $|G|$.

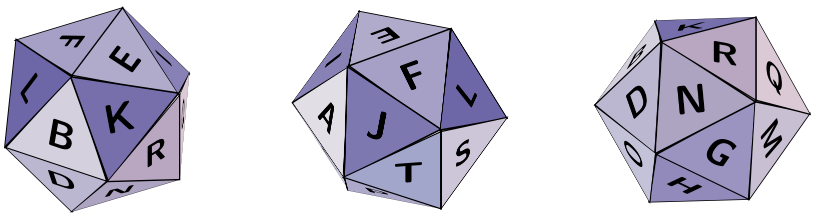
Question 5.

(a) Consider the rotational symmetry group G of the dodecahedron:



Then G acts on the set of faces F. By a similar strategy as in Q4, calculate $|G|$. From this, deduce the number of vertices, and deduce the number of edges on the dodecahedron.

(b) Consider the rotational symmetry group H of the icosahedron:



Then H acts on the set of faces F. By a similar strategy as in Q4, calculate $|H|$. From this, deduce the number of vertices, and deduce the number of edges on the icosahedron.

Question 6. This question is about Cayley's Theorem. Recall that a group G always acts on itself by left multiplication, and that this gives rise to an injective group homomorphism $G \to S_{|G|}$. Identify the subgroup of S_4 that arises in this process applied to $G := C_2 \times C_2$. Do the same for when $G := C_4$.

Solutions.

1. As in lectures, D_4 acts on itself (i.e. take $X := G$) in three ways, namely left action, right action and conjugacy action. Further D_4 acts on the set of vertices of the square, and it acts on the set of edges of the square. It also acts on the two faces ('top' and 'bottom') of the square.

   Since D_4 acts on any nonempty set X by $g \cdot x = x$ for all $g \in D_4, x \in X$, there are infinitely many examples!

2. (a) $G = \{e, (14), (23), (14)(23)\}$.

   (b) $\mathrm{Orb}_G(2) = \{2, 3\}$ and $\mathrm{Stab}_G(2) = \{e, (14)\}$. Yes, $2 \times 2 = 4$.

3. (a) This action is faithful, so the kernel is $\{e\}$.

   (b) Let N be the kernel. Then $g \in N \iff g(2) = 2$ and $g(3) = 3$. Thus $N = \{e, (14)\} = \mathrm{Stab}_G(2)$.

4. (a) The stabilizer of a vertex v is the 3-element group containing the rotational symmetries about the line through the vertex and its opposite vertex. Since the action is transitive on vertices (any vertex can be taken to any other vertex by some rotational symmetry), $\mathrm{Orb}_G(v) = V$, which has 8 elements. Hence

   $$|G| = |\mathrm{Stab}_G(v)| \times |\mathrm{Orb}_G(v)| = 3 \times 8 = 24.$$

   (b) The stabilizer is the 4-element group containing the rotational symmetries about the line through the centre of the face and its opposite face. Since the action is transitive on faces (any face can be taken to any other face by some rotational symmetry), $\mathrm{Orb}_G(f) = F$, which has 6 elements. Hence

   $$|G| = |\mathrm{Stab}_G(f)| \times |\mathrm{Orb}_G(f)| = 4 \times 6 = 24.$$

   (c) The stabilizer is the 2-element group containing the half-turn about the line through the centre of the edge and its opposite. Since the action is transitive on edges (any edge can be taken to any other edge by some rotational symmetry), $\mathrm{Orb}_G(e) = E$, which has 12 elements. Hence

   $$|G| = |\mathrm{Stab}_G(e)| \times |\mathrm{Orb}_G(e)| = 2 \times 12 = 24.$$

5. (a) Pick a face $f \in F$. Then since the action is transitive on faces (as in Q4), $\mathrm{Orb}_G(f) = F$, which has 12 elements. The orbit containing f is the 5-element group consisting of all rotations about the line through the centre of the face f and its opposite face. Hence

   $$|G| = |\mathrm{Stab}_G(f)| \times |\mathrm{Orb}_G(f)| = 12 \times 5 = 60.$$

   Now G acts transitively on the edges, and $|\mathrm{Stab}_G(e)| = 2$ for any edge e. Hence

   $$\text{The number of edges } = |\mathrm{Orb}_G(e)| = \frac{|G|}{|\mathrm{Stab}_G(e)|} = \frac{60}{2} = 30.$$

   Similarly G acts transitively on the vertices, and $|\mathrm{Stab}_G(v)| = 3$ for any vertex v. Hence

   $$\text{The number of verticies } = |\mathrm{Orb}_G(v)| = \frac{|G|}{|\mathrm{Stab}_G(v)|} = \frac{60}{3} = 20.$$

(b) Pick a face $f \in F$. Then since the action is transitive on faces (as in Q4), $\mathrm{Orb}_G(f) = F$, which has 20 elements. The orbit containing f is the 3-element group consisting of all rotations about the line through the centre of the face f and its opposite face. Hence

$$|H| = |\mathrm{Stab}_H(f)| \times |\mathrm{Orb}_H(f)| = 20 \times 3 = 60.$$

Now G acts transitively on the edges, and $|\mathrm{Stab}_G(e)| = 2$ for any edge e. Hence

$$\text{The number of edges } = |\mathrm{Orb}_G(e)| = \frac{|G|}{|\mathrm{Stab}_G(e)|} = \frac{60}{2} = 30.$$

Similarly G acts transitively on the vertices, and $|\mathrm{Stab}_G(v)| = 5$ for any vertex v. Hence

$$\text{The number of verticies } = |\mathrm{Orb}_G(v)| = \frac{|G|}{|\mathrm{Stab}_G(v)|} = \frac{60}{5} = 12.$$

6. (a) Consider $G = C_2 \times C_2 = \{e := (1,1), (g,1), (1,h), (g,h)\}$. Note $G = \langle (g,1), (1,h) \rangle$. Now relabel $X = G$ by $e \leftrightarrow 1$, $(g,1) \leftrightarrow 2$, $(1,h) \leftrightarrow 3$ and $(g,h) \leftrightarrow 4$. Then multiplication by $(g,1)$ sends $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 4$ and $4 \mapsto 3$, hence it corresponds to

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Similarly multiplication by $(1,h)$ sends 1 to 3, 3 to 1, 2 to 4 and 4 to 2, hence it corresponds to

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

Thus

$$G = \langle (g,1), (h,1) \rangle \cong \langle \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \rangle \leq S_4.$$

This group is $\{e, (12)(34), (13)(24), (14)(23)\}$.

(b) In a similar way

$$C_4 = \langle g \rangle \cong \langle \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \rangle \leq S_4.$$

This group is $\{e, (1234), (13)(24), (1432)\}$.