Fundamentals of Pure Mathematics 2015-16
Analysis Problems for weeks 3-4

I welcome your feedback on these problems and solutions. If you have any comments, if there is anything that needs more explaining, if you have any questions on any of the material, please come and see me or email me at n.bournaveas@ed.ac.uk or ask a question on Piazza.

Suggested problems for the Analysis workshop in week 4: 23, 24, 25. If time permits or if you have already solved these Problems at home then work on any of the following: 28, 30, 31 , 33a, 35.

Sequences in $\mathbb{R}$ (Wade, Chapter 2)

22. ([Wade], Exercise 2.1.4a) Prove that a sequence $(a_n)_{n\in\mathbb{N}}$ is bounded[1] iff and only it is absolutely bounded, i.e. there exists $M \geq 0$ such that for all $n \in \mathbb{N}$ we have $|a_n| \leq M$.

    Solution:

    Assume that the sequence is bounded. Then there exist $m, M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have $m \leq a_n \leq M$. Set $M' = \max\{|m|, |M|\}$. Then $a_n \leq M \leq |M| \leq M'$ and $a_n \geq m \geq -|m| \geq -M'$, for all n. Therefore $|a_n| \leq M'$ for all n. We have shown that the sequence is absolutely bounded.

    Conversely, if the sequence is absolutely bounded then there is an $M \geq 0$ such that for all $n \in \mathbb{N}$ we have $|a_n| \leq M'$. Set $m = -M'$ and $M = M'$. Then $m \leq a_n \leq M$ for all n, therefore the sequence is bounded.

23. Prove that the inequality
$$\frac{3n+1}{n^2+\sqrt{17n}+\pi} < \frac{1}{100} \tag{1}$$
    is true eventually for all $n \in \mathbb{N}$, i.e. that there exists a positive integer N such that the inequality holds true for all $n \geq N$. (Any N that works will do. We are <u>not</u> trying to find the 'best' N or the smallest N. We are <u>not</u> trying to find all n for which the inequality is true.)

[1] Recall that *bounded* means that there exist $m, M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have $m \leq a_n \leq M$.

Rough Work: We want $\frac{3n+1}{n^2+\sqrt{17n}+\pi} < \frac{1}{100}$. Cross multiply to make it

$$300n + 100 < n^2 + \sqrt{17n} + \pi.$$

This looks horrible!

Let's try and get a simpler fraction before cross multiplying. We can afford to be a little wasteful in our estimates as we are not trying to find the 'best' N.

We have

$$\frac{3n+1}{n^2+\sqrt{17n}+\pi} \leq \frac{3n+n}{n^2+0+0} = \frac{4}{n}.$$

We are done if we can ensure that $\frac{4}{n} < \frac{1}{100}$. Cross multiply to make it $400 < n$. Taking $N = 500$ should work.

(Reminder: Do <u>not</u> hand in rough work) END OF ROUGH WORK

Solution: Let $N = 500$. Then for all $n \geq N$ we have

$$\frac{3n+1}{n^2+\sqrt{17n}+\pi} \leq \frac{4n}{n^2} = \frac{4}{n} \leq \frac{4}{N} = \frac{4}{500} < \frac{1}{100}.$$

□

Remark 1. (a) $N = 401$ works as well, but who cares!

(b) Observe that the formal proof proceeds almost in reverse order to Rough Work. Rough Work starts with some inequalities and ends with discovering a value for N. The formal proof starts by defining N and then does the inequalities.

24. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of real numbers given by $a_n = \frac{n+2}{5n^2+4}$. Prove directly from the ε-N definition of the limit that $a_n \to 0$.

Rough Work:

Given $\varepsilon > 0$ we want to prove that the inequality

$$|a_n - 0| < \varepsilon \tag{2}$$

is true eventually for all n, i.e. we are looking for an index N such that all terms a_n with indices $n \geq N$ satisfy $|a_n - 0| < \varepsilon$.

Inequality (2) is the same as $\frac{n+2}{5n^2+4} < \varepsilon$. Cross multiply to make it $n+2 < 5n^2\varepsilon + 4\varepsilon$ which looks horrible! Let's try to get a simpler fraction before cross multiplying. We have

$$|a_n - 0| = \frac{n+2}{5n^2+4} \le \frac{n+2n}{1 \cdot n^2 + 0} = \frac{3n}{n^2} = \frac{3}{n}. \tag{3}$$

It is enough now to ensure that $\frac{3}{n} < \varepsilon$. Rearrange to make it $n > \frac{3}{\varepsilon}$.

Choose for N any[2] positive integer $> \frac{3}{\varepsilon}$. For all $n \ge N$ we then have $\frac{3}{n} \le \frac{3}{N} < \varepsilon$, as required.

(Reminder: Do not hand in rough work.) END OF ROUGH WORK

Solution We wish to show that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \ge N$ we have $|a_n| < \varepsilon$.

Let $\varepsilon > 0$ be given. By the Archimedean property[3] of the Real Number System there exists a positive integer N such that $N > \frac{3}{\varepsilon}$.

For all $n \ge N$ we have

$$|a_n| = a_n = \frac{n+2}{5n^2+4} \le \frac{n+2n}{n^2} = \frac{3n}{n^2} = \frac{3}{n} \le \frac{3}{N} < \varepsilon,$$

as required.

25. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of real numbers given by $a_n = \frac{7n+5}{3n+1}$. Prove directly from the ε-N definition of the limit that $a_n \to \frac{7}{3}$.

Rough Work 1:

Given $\varepsilon > 0$ we want to prove that the inequality

$$\left| a_n - \frac{7}{3} \right| < \varepsilon \tag{4}$$

is true eventually for all n, i.e. we are looking for an index N such that all terms a_n with indices $n \ge N$ satisfy $\left| a_n - \frac{7}{3} \right| < \varepsilon$.

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[2] We can't take $N = \frac{3}{\varepsilon} + 1$ because that's not necessarily an integer. We can take $N = \lfloor \frac{3}{\varepsilon} \rfloor + 1$, where $\lfloor x \rfloor$ denotes the largest integer $\le$ x. Since this is not very enlightening and all we are ever going to use is the inequality $N > \frac{3}{\varepsilon}$ we won't bother with an explicit formula for N.

[3] [Wade], Theorem 1.16.

We calculate

$$\mathtt{a_n} - \frac{7}{3} = \frac{7\mathtt{n}+5}{3\mathtt{n}+1} - \frac{7}{3} = \frac{8}{9\mathtt{n}+3}$$

Inequality (4) is the same as

$$\frac{8}{9\mathtt{n}+3} < \varepsilon. \tag{5}$$

Cross multiply to make it

$$8 < 9\varepsilon\mathtt{n} + 3\varepsilon. \tag{6}$$

This doesn't look too complicated, so rearrange to make it

$$\mathtt{n} > \frac{8-3\varepsilon}{9\varepsilon}. \tag{7}$$

Choose for N any positive integer $> \frac{8-3\varepsilon}{9\varepsilon}$. For all $\mathtt{n} \geq \mathtt{N}$ inequality (7) will be true, therefore (4) will be true, as required.

(Reminder: Do <u>not</u> hand in rough work.) END OF ROUGH WORK

Solution 1:

We wish to show that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\left|a_n - \frac{7}{3}\right| < \varepsilon$.

Let $\varepsilon > 0$ be given. By the Archimedean property of the Real Number System there exists a positive integer N such that $N > \frac{8-3\varepsilon}{9\varepsilon}$. For all $n \geq N$ we then have $n > \frac{8-3\varepsilon}{9\varepsilon}$, therefore

$$\left|a_n - \frac{7}{3}\right| = \left|\frac{7n+5}{3n+1} - \frac{7}{3}\right| = \frac{8}{9n+3} < \varepsilon.$$

□

Remark 2. The following solution is essentially the same as the one above but handles the elementary inequalities in a slightly different way.

When we cross multiplied in (5) to get (6) we jumped too soon. Simple as it is, the fraction $\frac{8}{9n+3}$ can be simplified further.

Rough Work 2:

Given $\varepsilon > 0$ we want to prove that the inequality

$$\left|\mathtt{a_n} - \frac{7}{3}\right| < \varepsilon \tag{8}$$

is true eventually for all n, i.e. we are looking for an index N such that all terms a_n with indices $n \geq N$ satisfy $\left|a_n - \frac{7}{3}\right| < \varepsilon$.

We have

$$a_n - \frac{7}{3} = \frac{7n+5}{3n+1} - \frac{7}{3} = \frac{8}{9n+3}$$

therefore

$$\left|a_n - \frac{7}{3}\right| = \frac{8}{9n+3} \leq \frac{8}{1 \cdot n + 0} = \frac{8}{n}.$$

It is enough now to ensure that $\frac{8}{n} < \varepsilon$. Rearrange to make it

$$n > \frac{8}{\varepsilon}. \tag{9}$$

Choose for N any positive integer $> \frac{8}{\varepsilon}$. For all $n \geq N$ inequality (9) will be true, therefore (8) will be true, as required.

(Reminder: Do <u>not</u> hand in rough work.) END OF ROUGH WORK

Solution 2: We wish to show that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\left|a_n - \frac{7}{3}\right| < \varepsilon$.

Let $\varepsilon > 0$ be given. By the Archimedean property of $\mathbb{R}$ there exists a positive integer N such that $N > \frac{8}{\varepsilon}$. For all $n \geq N$ we have

$$\left|a_n - \frac{7}{3}\right| = \left|\frac{7n+5}{3n+1} - \frac{7}{3}\right| = \frac{8}{9n+3} \leq \frac{8}{n} \leq \frac{8}{N} < \varepsilon,$$

as required.

26. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers that converges to 1. Prove directly from the ε-N definition of the limit that $2x_n + x_n^2 \to 3$.

Solution:

Since (x_n) converges, it is bounded, therefore there is an $M \in \mathbb{R}$ such that for all n we have $|x_n| \leq M$. (see Problem 22).

Let ε be a given positive real number. Set $\varepsilon' = \frac{\varepsilon}{M+3}$. Since $x_n \to 1$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - 1| < \varepsilon'$.

Then for all $n \geq N$ we have

$$\left|2x_n + x_n^2 - 3\right| = \left|2(x_n - 1) + x_n^2 - 1\right| \leq 2\,|x_n - 1| + \left|x_n^2 - 1\right| \leq 2\,|x_n - 1| + |x_n + 1|\,|x_n - 1|$$

$$\leq 2|x_n - 1| + \big(|x_n| + 1\big)|x_n - 1| \leq 2|x_n - 1| + (M+1)|x_n - 1| < 2\varepsilon' + (M+1)\varepsilon'$$
$$= (M+3)\varepsilon' = (M+3)\frac{\varepsilon}{M+3} = \varepsilon.$$

27. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Prove directly from the ε-N definition of the limit that
$$\frac{x_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0.$$

`Rough Work:`

`Since` $(x_n)_{n\in\mathbb{N}}$ `is bounded there exists` $M > 0$ `such that for all` $n \in \mathbb{N}$ `we have` $|x_n| \leq M$ `(see Problem 22).`

`Given` $\varepsilon > 0$ `we want to show that the inequality`
$$\left|\frac{\mathtt{x_n}}{\sqrt{\mathtt{n}}}\right| < \varepsilon \tag{10}$$
`is true eventually for all` n`, i.e. that there exists an index N such that for all indices` $\mathtt{n} \geq \mathtt{N}$ `we have` $\left|\frac{\mathtt{x_n}}{\sqrt{\mathtt{n}}}\right| < \varepsilon$`. We have`
$$\left|\frac{\mathtt{x_n}}{\sqrt{\mathtt{n}}}\right| = \frac{|\mathtt{x_n}|}{\sqrt{\mathtt{n}}} \leq \frac{\mathtt{M}}{\sqrt{\mathtt{n}}}.$$
`It is enough to ensure that` $\frac{M}{\sqrt{n}} < \varepsilon$`, which is the same as`
$$n > \left(\frac{M}{\varepsilon}\right)^2. \tag{11}$$

`Pick a positive integer` $N > \left(\frac{M}{\varepsilon}\right)^2$`. For all` $n \geq N$ `inequality (11) will be true, therefore (10) will be true as well.`

Solution: Since $(x_n)_{n\in\mathbb{N}}$ is bounded there exists $M > 0$ such that for all $n \in \mathbb{N}$ we have $|x_n| \leq M$ (see Problem 22).

We wish to show that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have
$$\left|\frac{x_n}{\sqrt{n}}\right| < \varepsilon.$$

Let $\varepsilon > 0$ be given. By the Archimedean property of the reals there exists a positive integer N such that $N > \left(\frac{M}{\varepsilon}\right)^2$. For all $n \geq N$ we have

$$\left|\frac{x_n}{\sqrt{n}}\right| = \frac{|x_n|}{\sqrt{n}} \leq \frac{M}{\sqrt{N}} < \varepsilon.$$

28. Prove that each of the following sequences converges[4] and find its limit. State clearly which limit theorems you are using.

(a) $\dfrac{3n^2+n+6}{5n^2-7n+1}$, (b) $\sqrt{n+1}-\sqrt{n}$, (c) $\left(3^n+5^n+7^n\right)^{1/n}$, (d) $\dfrac{2^n+3^n}{4^n+5^n}$, (e) $\left(\frac{n+3}{3n}\right)^n$.

Solution:

(a) Write

$$\frac{3n^2+n+6}{5n^2-7n+1} = \frac{\frac{3n^2+n+6}{n^2}}{\frac{5n^2-7n+1}{n^2}} = \frac{3+\frac{1}{n}+\frac{6}{n^2}}{5-\frac{7}{n}+\frac{1}{n^2}}.$$

Both the numerator and the denominator of the last fraction are sums of convergent sequences, therefore they are themselves convergent.

The limit of a sum is equal to the sum of the limits, therefore the limit of the numerator is

$$\lim_{n\to\infty}\left(3+\frac{1}{n}+\frac{6}{n^2}\right) = 3+0+0 = 3,$$

and the limit of the denominator is

$$\lim_{n\to\infty}\left(5-\frac{7}{n}+\frac{1}{n^2}\right) = 5-0+0 = 5 \neq 0.$$

A quotient of two convergent sequences is a convergent sequence, provided that the quotient of the limits is meaningful. Therefore the sequence $\frac{3+\frac{1}{n}+\frac{6}{n^2}}{5-\frac{7}{n}+\frac{1}{n^2}}$ is convergent and

$$\lim_{n\to\infty}\frac{3+\frac{1}{n}+\frac{6}{n^2}}{5-\frac{7}{n}+\frac{1}{n^2}} = \frac{\lim\limits_{n\to\infty}\left(3+\frac{1}{n}+\frac{6}{n^2}\right)}{\lim\limits_{n\to\infty}\left(5-\frac{7}{n}+\frac{1}{n^2}\right)} = \frac{3}{5}.$$

[4] Recall that the terms *converges* and *convergent* mean that the limit is a real number.

(b)

$$\sqrt{n+1}-\sqrt{n} = \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0.$$

(c) Rough Work:

When n is large we expect 7^n to dominate both 3^n and 5^n, therefore $3^n+5^n+7^n \simeq 7^n$, therefore $(3^n+5^n+7^n)^{\frac{1}{n}} \simeq 7$. We guess

$$\lim_{n\to\infty}\left(3^n+5^n+7^n\right)^{\frac{1}{n}} = 7.$$

END OF ROUGH WORK

Solution

We use the sandwich theorem. For all n we have $7^n \le 3^n+5^n+7^n \le 3\cdot 7^n$, therefore

$$7 \le \left(3^n+5^n+7^n\right)^{1/n} \le 3^{1/n}\cdot 7.$$

Since[5] $3^{1/n} \to 3^0 = 1$, we have $3^{1/n}\cdot 7 \to 7$. By the sandwich theorem, our sequence converges and its limit is 7.

(d) Rough Work:

In the numerator we expect $2^n+3^n \simeq 3^n$ for large n, and in the denominator $4^n+5^n \simeq 5^n$. Therefore $\frac{2^n+3^n}{4^n+5^n} \simeq \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n \to 0$.

END OF ROUGH WORK

Solution 1:

We use the sandwich theorem. For all n we have

$$0 \le \frac{2^n+3^n}{4^n+5^n} \le \frac{3^n+3^n}{0+5^n} = 2\left(\frac{3}{5}\right)^n.$$

Since $\frac{3}{5} < 1$ we have $\left(\frac{3}{5}\right)^n \xrightarrow[n\to\infty]{} 0$. By the sandwich theorem our sequence converges to 0 as well.

[5] [Wade], Example 3.21

Solution 2: Divide top and bottom by the dominant term 5^n.

$$\frac{2^n+3^n}{4^n+5^n} = \frac{\frac{2^n+3^n}{5^n}}{\frac{4^n+5^n}{5^n}} = \frac{\left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n}{\left(\frac{4}{5}\right)^n + 1} \xrightarrow[n\to\infty]{} \frac{0+0}{0+1} = 0.$$

(e) `Rough Work:  For large` n `we have`

$$\frac{n+3}{3n} = \frac{1}{3} + \frac{1}{n} \simeq \frac{1}{3} \tag{12}$$

`therefore`

$$\left(\frac{n+3}{3n}\right)^n \simeq \left(\frac{1}{3}\right)^n. \tag{13}$$

`Since` $\frac{1}{3} < 1$ `we have` $\left(\frac{1}{3}\right)^n \to 0$`, therefore we expect` $\left(\frac{n+3}{3n}\right)^n$ `to converge to zero as well.`

`END OF ROUGH WORK`

Solution For $n \geq 2$ we have

$$\frac{n+3}{3n} = \frac{1}{3} + \frac{1}{n} \leq \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \tag{14}$$

therefore

$$0 \leq \left(\frac{n+3}{3n}\right)^n \leq \left(\frac{5}{6}\right)^n. \tag{15}$$

Since $\frac{5}{6} < 1$ we have $\left(\frac{5}{6}\right)^n \xrightarrow[n\to\infty]{} 0$, therefore (Sandwich Theorem) $\left(\frac{n+3}{3n}\right)^n \xrightarrow[n\to\infty]{}$ 0 as well.

Remark 3. The only thing that matters in (14) is that we end up with a bound < 1. We could have said:

For all $n \geq 20$ we have

$$\frac{n+3}{3n} = \frac{1}{3} + \frac{1}{n} \leq \frac{1}{3} + \frac{1}{20} = \frac{23}{60} \tag{16}$$

therefore

$$0 \leq \left(\frac{n+3}{3n}\right)^n \leq \left(\frac{23}{60}\right)^n. \tag{17}$$

Since $\frac{23}{60} < 1$ we have $\left(\frac{23}{60}\right)^n \xrightarrow[n\to\infty]{} 0$, therefore (Sandwich Theorem) $\left(\frac{n+3}{3n}\right)^n \xrightarrow[n\to\infty]{}$ 0 as well.

Things to think about:
Is there anything wrong with the following proof?

$$\left(\frac{n+3}{3n}\right)^n \xrightarrow[n\to\infty]{} \left(\frac{1}{3}\right)^n \xrightarrow[n\to\infty]{} 0\,.$$

How about this one?

$$\lim_{n\to\infty}\left(\frac{n+3}{3n}\right)^n = \lim_{n\to\infty}\left(\frac{1}{3}\right)^n = 0\,.$$

29. ([Wade], Exercise 2.2.5) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers that converges to a real number l. Show that $\sqrt{x_n} \to \sqrt{l}$.

Solution 1

Since $x_n \geq 0$ for all n, we have $l \geq 0$ ([Wade], Thm 2.17).

Case 1: $l > 0$.

Let $\varepsilon > 0$ be given and define $\varepsilon' = \varepsilon\sqrt{l}$. Since $x_n \to l$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - l| < \varepsilon'$. Then for all $n \geq N$ we have

$$\left|\sqrt{x_n} - \sqrt{l}\right| = \left|\frac{\left(\sqrt{x_n} - \sqrt{l}\right)\left(\sqrt{x_n} + \sqrt{l}\right)}{\sqrt{x_n} + \sqrt{l}}\right| = \frac{|x_n - l|}{\sqrt{x_n} + \sqrt{l}} \leq \frac{|x_n - l|}{\sqrt{l}} < \frac{\varepsilon'}{\sqrt{l}} = \varepsilon. \quad (18)$$

(All denominators in (18) are $\neq 0$ thanks to the hypothesis $l > 0$)

Case 2: $l = 0$

We have $x_n \to 0$ and we wish to show $\sqrt{x_n} \to \sqrt{0} = 0$.

Let $\varepsilon > 0$ be given and define $\varepsilon' = \varepsilon^2$. Since $x_n \to 0$ there exists an $N \in \mathbb{N}$ such that $x_n < \varepsilon'$, which gives $\sqrt{x_n} < \sqrt{\varepsilon'} = \varepsilon$.

Remark 4. Here is another solution that doesn't have any denominators and doesn't need to be split into two cases. We need to prove an elementary inequality first.

For any two real numbers x, y with $x \geq y \geq 0$ we have

$$\sqrt{x} - \sqrt{y} \leq \sqrt{x - y}. \quad (19)$$

To see this, move $\sqrt{y}$ to the right hand side and square both sides to obtain $x \leq (x-y) + y + 2\sqrt{x-y}\sqrt{y}$, which is clearly true.

It follows that for any two real numbers x, y with $x, y \geq 0$ (we are not assuming $x \geq y$ now) we have

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x-y|}. \tag{20}$$

Indeed, if $x \geq y$ this is the same as (19). If $y \geq x$ then (20) can be written as

$$\sqrt{y} - \sqrt{x} \leq \sqrt{y-x}, \tag{21}$$

which is the same as (19) with the roles of x and y reversed.

Solution 2 Let $\varepsilon > 0$ be given and set $\varepsilon' = \varepsilon^2$. Since $x_n \to l$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - l| \leq \varepsilon'$. Using (20) we see that for all $n \geq N$,

$$\left|\sqrt{x_n} - \sqrt{l}\right| \leq \sqrt{|x_n - l|} < \sqrt{\varepsilon'} = \varepsilon.$$

Remark 5. Let $\alpha > 0$. A function f is said to be Hölder continuous with exponent α iff there exists a positive constant C such that for all x, y in the domain of f we have

$$\left|f(x) - f(y)\right| \leq C\left|x-y\right|^{\alpha}. \tag{22}$$

Inequality (20) says that the square root function is Hölder continuous with exponent $\alpha = \frac{1}{2}$ (and $C = 1$).

30. In each of the following cases give a proof if the statement is true or a counterexample if it is false. In all cases $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences of real numbers and $x, y \in \mathbb{R}$.

    (a) If $x_n \to x$, $y_n \to y$ and $x_n > y_n$ for all n, then $x > y$.

    (b) If $x_n \to x$, $y_n \to y$ and $x_n > y_n + \frac{1}{n}$ for all n, then $x > y$.

    (c) If $x_n \to x$, $y_n \to y$ and $x_n > 2y_n$ for all n, then $x > y$.

    (d) If $x_n \to x$, $y_n \to y$ and $x_n > y_n + 1$ for all n, then $x > y$.

    Solution:

    The first three are false. Counterexamples:

    (a) $x_n = \frac{1}{n}$, $y_n = 0$, $x = y = 0$.

    (b) $x_n = \frac{1}{n}$, $y_n = -\frac{1}{n}$, $x = y = 0$.

    (c) $x_n = \frac{1}{n}$, $y_n = -\frac{1}{n}$, $x = y = 0$.

The last one is true.

Proof:

By the comparison theorem ([Wade], Theorem 2.17), $\lim_{n\to\infty} x_n \geq \lim_{n\to\infty}(y_n+1)$, therefore $x \geq y+1$, hence $x > y$.

31. (FPM Exam, August 2015, Problem 6)

    Consider the sequence defined recursively by

    $$a_1 = 0\ ,\ \ a_n = \sqrt{6+a_{n-1}}\ ,\ \text{for } n = 2,3,\ldots \tag{23}$$

    Prove that the sequence is convergent and find its limit. [Hint: Write down a few terms.]

    Solution:

    An easy induction argument shows that $a_n \geq 0$ for all n.

    We claim that the sequence is increasing, i.e. that $a_n \geq a_{n-1}$ for all $n \geq 2$. Indeed, $a_2 = \sqrt{6} \geq 0 = a_1$, and if $a_n \geq a_{n-1}$ for some $n \geq 2$, then $6+a_n \geq 6+a_{n-1}$, therefore $\sqrt{6+a_n} \geq \sqrt{6+a_{n-1}}$, therefore $a_{n+1} \geq a_n$.

    Next, we show that $a_n \leq 3$ for all n. This can be done by yet another induction argument. For $n=1$ we trivially have $a_1 = 0 \leq 3$, and if $a_n \leq 3$ for some $n \geq 1$, then $a_{n+1} = \sqrt{6+a_n} \leq \sqrt{6+3} = 3$.

    We know now that our sequence is increasing and bounded, therefore it is convergent. Call its limit l. Since all $a_n \geq 0$ we have $l \geq 0$. Letting $n \to \infty$ in $a_n = \sqrt{6+a_{n-1}}$ we find $l = \sqrt{6+l}$ which has one positive solution, namely $l = 3$.
    Remark 6. We have shown that the sequence

    $$\sqrt{6},\ \sqrt{6+\sqrt{6}},\ \sqrt{6+\sqrt{6+\sqrt{6}}},\ \ldots$$

    converges to 3. The third number above is equal to $2.9844\cdots$.

32. For all $a, x > 0$ show that

    $$\frac{1}{2}\left(a+\frac{x}{a}\right) \geq \sqrt{x}. \tag{24}$$

    [Hint: Multiply by $2a$ and rearrange. We'll need this inequality in the next problem.]

    Solution: Multiply by $2a$ and rearrange to find

    $$a^2 + x - 2a\sqrt{x} \geq 0,$$

which is the same as

$$\left(a-\sqrt{x}\right)^2 \geq 0.$$

33. Let $x>0$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

   (i) a_1 is any positive real number, and

   (ii) for all $n \geq 1$,

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right). \tag{25}$$

   Prove that:

   (a) (FPM Exam, May 2014, Problem 6)

   The sequence $(a_n)_{n\in\mathbb{N}}$ is eventually monotonic, bounded and converges to $\sqrt{x}$.

   (b) The 'error' $e_n = a_n - \sqrt{x}$ has the property

$$e_{n+1} = \frac{e_n^2}{2a_n}. \tag{26}$$

   Moreover, we have the error estimate

$$0 \leq e_{n+1} \leq \frac{e_n^2}{2\sqrt{x}}, \quad n \geq 2. \tag{27}$$

   Solution: An easy induction argument shows that $a_n > 0$ for all n. Moreover, inequality (24) implies that $a_n = \frac{1}{2}\left(a_{n-1} + \frac{x}{a_{n-1}}\right) \geq \sqrt{x}$ for all $n \geq 2$. ([6])

   For all $n \geq 2$ we have

$$a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) - a_n = \frac{1}{2}\frac{x}{a_n} - \frac{1}{2}a_n = \frac{1}{2}\frac{x-a_n^2}{a_n}. \tag{28}$$

   In the last fraction above we have $x - a_n^2 \leq 0$ and $a_n > 0$, therefore $\frac{x-a_n^2}{a_n} \leq 0$. Therefore, $a_{n+1} \leq a_n$ for $n \geq 2$. Therefore the sequence (a_n) is eventually decreasing.

[6] Remark: The term a_1 is any positive real number. It can be less than $\sqrt{x}$, or more than $\sqrt{x}$, or even equal to $\sqrt{x}$. From a_2 onwards, all terms are $\geq \sqrt{x}$.

Clearly, the sequence is bounded below as all its terms are positive. It follows that the sequence converges to some real number L. Moreover, $L \geq \sqrt{x}$ because $a_n \geq \sqrt{x}$ for all $n \geq 2$, and since $x > 0$ it follows that $L > 0$.

Taking the limits as $n \to \infty$ of both sides of

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right)$$

we find

$$L = \frac{1}{2}\left(L + \frac{x}{L}\right) \tag{29}$$

which gives

$$L^2 = x, \tag{30}$$

which implies

$$L = \sqrt{x}, \tag{31}$$

as required.

For the sequence (e_n) we have

$$\begin{aligned} e_{n+1} = a_{n+1} - \sqrt{x} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) - \sqrt{x} = \frac{a_n^2 + x}{2a_n} - \sqrt{x} &= \frac{a_n^2 + x - 2a_n\sqrt{x}}{2a_n} \\ &= \frac{(a_n - \sqrt{x})^2}{2a_n} = \frac{e_n^2}{2a_n}. \end{aligned}$$

For $n \geq 2$ we know that $a_n \geq \sqrt{x}$, therefore

$$e_{n+1} = \frac{e_n^2}{2a_n} \leq \frac{e_n^2}{2\sqrt{x}}.$$

Remark 7. Let $x = 2$ and take $a_1 = 1$. An easy induction argument shows that all a_n are rational. We have here an example of a sequence of rationals that converges to $\sqrt{2}$.

Remark 8. The idea behind the recursion (25) is very simple and goes back to the Babylonians. The Babylonian clay tablet YBC 7289 (approx. 1800 to 1600 BC), now held at Yale, gives an approximation to $\sqrt{2}$ accurate to seven significant digits.

Suppose we have a positive real number x and we wish to find an approximation to $\sqrt{x}$. Observe that, if the positive number a is an overestimate for $\sqrt{x}$, then $\frac{x}{a}$ is an underestimate, meaning simply that if $a > \sqrt{x}$ then $\frac{x}{a} < \sqrt{x}$. Similarly, if a is an underestimate, then $\frac{x}{a}$ is an overestimate, i.e. if $a < \sqrt{x}$ then $\frac{x}{a} > \sqrt{x}$. One hopes therefore that their arithmetic mean $\frac{1}{2}\left(a + \frac{x}{a}\right)$ is a better approximation to $\sqrt{x}$ than a itself (by (24) this arithmetic mean is always an overestimate for $\sqrt{x}$).

The sequence (a_n) starts with a positive real number a_1. Thinking of a_1 as a first approximation to $\sqrt{x}$, we define a_2 to be the arithmetic mean of a_1 and $\frac{x}{a_1}$, we define a_3 to be the arithmetic mean of a_2 and $\frac{x}{a_2}$, and so on. The error estimate (27) shows that (a_n) provides a quadratically convergent approximation to $\sqrt{x}$, meaning that at every step of the iteration the number of accurate digits roughly doubles. Try it out with $x = 100$, $a_1 = 1$.

You can learn more at these places:

Square roots with pencil and paper at The Math Less Traveled blog

Ancient square roots at Mathpages

34. In each of the following cases give a proof if the statement is true or a counterexample if it is false. In all cases (x_n) and (y_n) are sequences of real numbers.

    (a) If $x_n \to +\infty$ and $y_n \to -\infty$ then $x_n + y_n \to 0$.

    (b) If the sequence (x_n) converges and each term x_n is an integer then the sequence is eventually constant (i.e. there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $x_n = x_m$).

    (c) If (x_n) is a Cauchy sequence and (y_n) is bounded then $(x_n y_n)$ is a Cauchy sequence.

    Solution:

    (a) False.

    Counterexample: Let $x_n = 2n$, $y_n = -n$. Then $x_n \to +\infty$, $y_n \to -\infty$ but $x_n + y_n = 2n - n = n \not\to 0$.

    (b) True.

    Proof: Since (x_n) is a convergent sequence it is also a Cauchy sequence, therefore, for $\varepsilon = \frac{1}{10}$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|x_n - x_m| < \frac{1}{10}$, and since x_n, x_m are integers, it follows that $x_n = x_m$.

Thinks to think about:
Under the hypotheses in (b) does it follow that the limit of the sequence is an integer?

(c) False.

Counterexample: Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be the sequences given by $x_n = 1$ and $y_n = (-1)^n$. Then (x_n) is Cauchy and (y_n) is bounded. On the other hand, $x_n y_n = (-1)^n$ is not convergent, hence not Cauchy.

35. (FPM Exam, May 2014, Problem 4)

True of False? Give a proof or a counterexample.

(a) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers then $(|x_n|)_{n\in\mathbb{N}}$ is a Cauchy sequence.

(b) If $(x_n)_{n\in\mathbb{N}}$ and $(x_n y_n)_{n\in\mathbb{N}}$ are convergent then $(y_n)_{n\in\mathbb{N}}$ is convergent.

(c) If $|x_n| \leq y_n$ for all $n \in \mathbb{N}$ and (y_n) is convergent then $(x_n)_{n\in\mathbb{N}}$ is bounded.

Solution:

(a) True.

Proof 1: If (x_n) is Cauchy then it is convergent, say $x_n \to l$. Then $|x_n| \to |l|$, therefore $(|x_n|)$ is convergent, therefore $(|x_n|)$ is Cauchy.

Proof 2. Assume (x_n) is Cauchy. We prove that $(|x_n|)$ is Cauchy straight from the Definition.

Let $\varepsilon > 0$. Since (x_n) is Cauchy there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|x_n - x_m| < \varepsilon$, which implies

$$\big||x_n| - |x_m|\big| \leq |x_n - x_m| < \varepsilon.$$

Therefore $(|x_n|)$ is Cauchy.

(b) False.

Counterexample. Take $y_n = (-1)^n$ and $x_n = 0$ for all n.

(c) True.

Proof: Assume $|x_n| \leq y_n$ for all n and (y_n) is convergent. Then (y_n) is bounded. Therefore there exists an M such that for all n, $y_n \leq M$. It follows that $|x_n| \leq M$ for all n, therefore $-M \leq x_n \leq M$ for all n. Therefore (x_n) is bounded.

36. Prove that $\sqrt[n]{n} \to 1$.

Solution:

Clearly, $\sqrt[n]{n} \geq 1$ for all n. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > 1 + \frac{2}{\varepsilon^2}$. Then $\frac{N-1}{2}\varepsilon^2 > 1$. For all $n \geq N$ we have

$$(1+\varepsilon)^n \geq \frac{n(n-1)}{2}\varepsilon^2 \geq n\frac{N-1}{2}\varepsilon^2 > n$$

therefore

$$\sqrt[n]{n} < 1 + \varepsilon,$$

therefore

$$\left|\sqrt[n]{n} - 1\right| = \sqrt[n]{n} - 1 < \varepsilon.$$

References

[Hardy] G. H. Hardy, A course of Pure Mathematics.

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