**Fundamentals of Pure Mathematics 2015-16**
**Assignment 10**
**Due-date: Thursday of week 11**
**(Submit your work to your tutor at the workshop)**
**You will be able to collect your marked assignment from MTO. The course secretary will email the class when the assignments are ready for collection.**

1. (5 marks) (This is Problem 83 a, c,d,e from the week 10 workshop)

   Establish the convergence or divergence of each of the following series.

   (a) $\sum_{n=1}^{\infty} \frac{n}{3n-2}$ , (c) $\sum_{n=1}^{\infty} \frac{n}{n^3+3}$ , (d) $\sum_{n=1}^{\infty} \sqrt{\frac{1}{n(n+1)(n+2)}}$ , (e) $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$ .

2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with non-negative terms. Prove the following:

   (a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=2}^{\infty} \sqrt{a_n a_{n-1}}$ converges.

   (b) If $\sum_{n=2}^{\infty} \sqrt{a_n a_{n-1}}$ converges and the sequence $(a_n)_{n\in\mathbb{N}}$ is decreasing, then $\sum_{n=1}^{\infty} a_n$ converges.

Support is available:

- From the lecturer (n.bournaveas@ed.ac.uk, room 4614)
- On Piazza
- At the Hub (Tuesdays, 1-3pm)

**Solutions**

1. (a) The sequence $\frac{n}{3n-2}$ doesn't converge to 0 (it converges to $\frac{1}{3}$) therefore the series $\sum_n \frac{n}{3n-2}$ diverges.

   (c) `Rough Work: For large n we expect` $\frac{n}{n^3+3}$ `to behave like` $\frac{n}{n^3} = \frac{1}{n^2}$. `Since` $\sum_n \frac{1}{n^2}$ `converges, we expect the series` $\sum_n \frac{n}{n^3+3}$ `to converge.`

   `END OF ROUGH WORK`

   <u>Solution 1:</u> We use the limit comparison test. We have

$$\frac{\frac{n}{n^3+3}}{\frac{1}{n^2}} = \frac{n^3}{n^3+3} \xrightarrow[n\to\infty]{} 1.$$

Since $\sum_n \frac{1}{n^2}$ converges, $\sum_n \frac{n}{n^3+3}$ converges as well.

Solution 2: We use the comparison test. We have

$$\frac{n}{n^3+3} \leq \frac{n}{n^3+0} = \frac{1}{n^2}.$$

Since $\sum_n \frac{1}{n^2}$ converges, $\sum_n \frac{n}{n^3+3}$ converges as well.

(d) `Rough Work: For large n we expect` $\sqrt{\frac{1}{n(n+1)(n+2)}}$ `to behave like` $\sqrt{\frac{1}{n^3}} = \frac{1}{n^{3/2}}$. `Since` $3/2 > 1$ `the series` $\sum_n \frac{1}{n^{3/2}}$ `converges, therefore we expect the series` $\sum_n \sqrt{\frac{1}{n(n+1)(n+2)}}$ `to converge.`

`END OF ROUGH WORK`

Solution 1: We use the limit comparison test. We have

$$\frac{\sqrt{\frac{1}{n(n+1)(n+2)}}}{\frac{1}{n^{3/2}}} = \sqrt{\frac{\frac{1}{n(n+1)(n+2)}}{\frac{1}{n^3}}} = \sqrt{\frac{n^3}{n(n+1)(n+2)}} \xrightarrow[n\to\infty]{} \sqrt{1} = 1.$$

Since $3/2 > 1$ the series $\sum_n \frac{1}{n^{3/2}}$ converges, therefore the series $\sum_n \sqrt{\frac{1}{n(n+1)(n+2)}}$ converges as well.

Solution 2: We use the comparison test. We have

$$\sqrt{\frac{1}{n(n+1)(n+2)}} \leq \sqrt{\frac{1}{n(n+0)(n+0)}} = \frac{1}{n^{3/2}}.$$

Since $3/2 > 1$ the series $\sum_n \frac{1}{n^{3/2}}$ converges, therefore the series $\sum_n \sqrt{\frac{1}{n(n+1)(n+2)}}$ converges as well.

(e) `Rough Work: The numerator` $n^{10}$ `grows fast thanks to the large exponent, but` $2^n$ `grows even faster. This is because` $n^{10}$ `is only a polynomial of n while` $2^n$ `is an exponential (geometric progression with common ratio` $2>1$`). We could work as in Assignment 2 to show` $2^n \geq n^{12}$ `eventually for all n. This would give` $\frac{n^{10}}{2^n} \leq \frac{1}{n^2}$`. By the comparison test,` $\sum_n \frac{n^{10}}{2^n}$ `converges. Another thing to try is the ratio test or the root test.`

END OF ROUGH WORK

Solution 1: We use the ratio test. We have

$$\frac{\frac{(n+1)^{10}}{2^{n+1}}}{\frac{n^{10}}{2^n}} = \frac{1}{2}\left(\frac{n+1}{n}\right)^{10} \xrightarrow[n\to\infty]{} \frac{1}{2}\cdot 1^{10} = \frac{1}{2} < 1,$$

therefore the series $\sum_n \frac{n^{10}}{2^n}$ converges.

Solution 2: We use the root test. We have

$$\sqrt[n]{\frac{n^{10}}{2^n}} = \frac{\left(\sqrt[n]{n}\right)^{10}}{2} \xrightarrow[n\to\infty]{} \frac{1^{10}}{2} = \frac{1}{2} < 1,$$

therefore the series $\sum_n \frac{n^{10}}{2^n}$ converges. We have used: $\sqrt[n]{n} \xrightarrow[n\to\infty]{} 1$ (Problem 36).

Solution 3 We use the comparison test. First, we show by induction that for all[1] $n \geq 84$ we have

$$2^n \geq n^{12}. \tag{1}$$

Indeed, for $n = 84$ we have

$$2^n = 2^{7\cdot 12} = \left(2^7\right)^{12} = 128^{12} > 84^{12} = n^{12}. \tag{2}$$

If $2^n \geq n^{12}$ is true for some $n \geq 84$ then

$$2^{n+1} = 2\cdot 2^n \geq 2n^{12}.$$

We are done if we can show $2n^{12} \geq (n+1)^{12}$. Indeed,

$$\frac{(n+1)^{12}}{n^{12}} = \left(\frac{n+1}{n}\right)^{12} = \left(1+\frac{1}{n}\right)^{12} \leq \left(1+\frac{1}{84}\right)^{12} = 1.1525\cdots < 2.$$

This completes the proof by induction of (1).

Back to our series, we have

$$\frac{n^{10}}{2^n} \leq \frac{n^{10}}{n^{12}} = \frac{1}{n^2}, \quad n \geq 84.$$

By the comparison test, $\sum_n \frac{n^{10}}{2^n}$ converges.

[1] Wolfram Alpha says $n \geq 75$ is ok as well. We use 84 because it is a multiple of 12 and makes (2) easy.

2. (a) By the AGM inequality[2] we have $\sqrt{a_n a_{n-1}} \leq \frac{1}{2}(a_n + a_{n-1})$ for all $n \geq 2$. The series $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} a_{n-1}$ converge, therefore their sum $\sum_{n=2}^{\infty} (a_n + a_{n-1})$ converges. By the comparison test, $\sum_{n=2}^{\infty} \sqrt{a_n a_{n-1}}$ converges as well.

(b) Suppose that $(a_n)$ is decreasing, i.e. $a_n \leq a_{n-1}$ for all $n \geq 2$, and that $\sum_{n=2}^{\infty} \sqrt{a_n a_{n-1}}$ converges. Then $a_n = \sqrt{a_n^2} = \sqrt{a_n a_n} \leq \sqrt{a_n a_{n-1}}$ for all $n \geq 2$. By the comparison test, $\sum_{n=2}^{\infty} a_n$ converges, therefore $\sum_{n=1}^{\infty} a_n$ converges.

[2] Arithmetic-Geometric Mean inequality: $\sqrt{xy} \leq \frac{x+y}{2}$, $x, y \geq 0$. (Warm-up Problem 8)