Fundamentals of Pure Mathematics 2015-16
Analysis Problems for weeks 5 and 6

I welcome your feedback on these problems and solutions. If you have any comments, if there is anything that needs more explaining, if you have any questions on any of the material, please come and see me or email me at n.bournaveas@ed.ac.uk or ask a question on Piazza.

Suggested Problems for the workshop in week 6: 45, 46, 48, 49. If time permits or if you have already solved these problems at home then work on any of the following: 50-55.

Problem 46 is part of Assignment 6.

Sequences (cont.)

37. Prove that for every real number there exist a sequence of rationals and a sequence of irrationals converging to it.

    Solution: For each $n \in \mathbb{N}$ choose a rational between $x - \frac{1}{n}$ and $x + \frac{1}{n}$ and call it r_n. Then $x - \frac{1}{n} < r_n < x + \frac{1}{n}$ for all n. Now $x + \frac{1}{n} \xrightarrow[n\to\infty]{} x$ and $x - \frac{1}{n} \xrightarrow[n\to\infty]{} x$. By the Squeeze Theorem, $r_n \xrightarrow[n\to\infty]{} x$.

    Choosing an irrational between $x - \frac{1}{n}$ and $x + \frac{1}{n}$, instead of a rational, leads to a sequence of irrationals converging to x.

    Things to think about:
    Can we find a sequence $(r_n)_{n\in\mathbb{N}}$ of rationals that converges to x from the right? (i.e. $r_n \to x$ and $r_n > x$ for all n). How about from the left?

38. How many convergent subsequences does the sequence $(-1)^n$ have? [Hint: the correct answer is not *two*.]

    Solution: The sequence $(a_n)_{n\in\mathbb{N}}$ given by $(-1)^n$ has infinitely many convergent subsequences. Here are infinitely many examples, all converging to 1:

$$
\begin{aligned}
(a_1, a_2, a_4, a_6, a_8, a_{10}, \ldots) &= (-1, 1, 1, 1, 1, 1, \ldots)\\
(a_1, a_3, a_4, a_6, a_8, a_{10}, \ldots) &= (-1, -1, 1, 1, 1, 1, \ldots)\\
(a_1, a_3, a_5, a_6, a_8, a_{10}, \ldots) &= (-1, -1, -1, 1, 1, 1, \ldots)\\
&\vdots\\
(a_1, a_3, \ldots, a_{2k-1}, a_{2k}, a_{2k+2}, a_{2k+4}, \ldots) &= (-1, -1, \ldots, -1, 1, 1, 1, 1, \ldots)\\
&\vdots
\end{aligned}
$$

Things to think about:
1. Can you find infinitely many subsequences of $(-1)^n$ that converge to -1?
2. The following result seems obvious[a]. Write down a rigorous proof.
If $(a_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(-1)^n$ then $\lim\limits_{k\to\infty} a_{n_k}$ is equal to 1 or -1.

[a]According to [Gowers], a statement is obvious if a proof instantly springs to mind.

39. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $L \in \mathbb{R}$. If $a_{2n} \xrightarrow[n\to\infty]{} L$ and $a_{2n+1} \xrightarrow[n\to\infty]{} L$, show that $a_n \xrightarrow[n\to\infty]{} L$.

Solution: It is convenient in this problem to denote the two subsequences by $(a_{2k})_{k\in\mathbb{N}}$ and $(a_{2k+1})_{k\in\mathbb{N}}$, and the sequence itself by $(a_n)_{n\in\mathbb{N}}$.

Let $\varepsilon > 0$. Since $a_{2k} \to L$ as $k \to \infty$, there exists $K_1 \in \mathbb{N}$ such that for all $k \geq K_1$ we have $|a_{2k} - L| < \varepsilon$.

Since $a_{2k+1} \to L$ as $k \to \infty$, there exists $K_2 \in \mathbb{N}$ such that for all $k \geq K_2$ we have $\left|a_{2k+1} - L\right| < \varepsilon$.

Let $N = \max\{2K_1, 2K_2 + 1\}$. We claim that for all $n \geq N$ we have $|a_n - L| < \varepsilon$.

Fix $n \geq N$. If n is even, write $n = 2k$ for some k. Then $2k = n \geq N \geq 2K_1$, therefore $k \geq K_1$, therefore $|a_{2k} - L| < \varepsilon$, i.e. $|a_n - L| < \varepsilon$.

If n is odd, write $n = 2k+1$ for some k. Then $2k+1 = n \geq N \geq 2K_2 + 1$, therefore $k \geq K_2$, therefore $\left|a_{2k+1} - L\right| < \varepsilon$, i.e. $|a_n - L| < \varepsilon$.

40. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$|x_n - x_{n-1}| \leq \left(\frac{1}{2}\right)^n,$$

for all $n \geq 2$. Prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. [Hint: For $n \geq m$ the difference $x_n - x_m$ can be written as $(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{m+1} - x_m)$.]

Solution:

Let $\varepsilon > 0$. The sequence $\left(\frac{1}{2}\right)^n$ converges to zero, therefore there exists a positive integer N such that for all $n \geq N$,

$$\left(\frac{1}{2}\right)^n < \varepsilon. \tag{1}$$

We claim that for all $n, m \geq N$ we have

$$|x_n - x_m| < \varepsilon.$$

Fix $n, m \geq N$. We may assume $n > m$. Then

$$\begin{aligned}
|x_n - x_m| &= \left|(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{m+1} - x_m)\right| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\leq \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \cdots + \left(\frac{1}{2}\right)^{m+1} \\
&= \left(\frac{1}{2}\right)^{m+1} \left[\left(\frac{1}{2}\right)^{n-m-1} + \left(\frac{1}{2}\right)^{n-m-2} + \cdots + 1\right] \\
&= \left(\frac{1}{2}\right)^{m+1} \frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}},
\end{aligned}$$

where we have used the identity

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}, \quad (x \neq 1).$$

Therefore (use (1) with m in the place of n),

$$|x_n - x_m| \leq \left(\frac{1}{2}\right)^{m+1} \frac{1}{1 - \frac{1}{2}} = \left(\frac{1}{2}\right)^m < \varepsilon.$$

Functions, Limits and Continuity (Wade, Chapter 3)

41. Formulate the definitions of the following limits. In (a) and (b) x_0 is a real number. Your definition should start by specifying a suitable domain for f (as in [Wade], Definition 3.1).

(a) $\lim\limits_{x \to x_0} f(x) = +\infty$,

(b) $\lim\limits_{x \to x_0+} f(x) = -\infty$,

(c) $\lim\limits_{x \to -\infty} f(x) = +\infty$.

Solution:

(a) Let $x_0 \in \mathbb{R}$, let I be an open interval containing x_0 and let $f : I \setminus \{x_0\} \to \mathbb{R}$ be a given function. We say that $\lim\limits_{x \to x_0} f(x) = +\infty$ if and only if for every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for all $x \in I \setminus \{x_0\}$ with $|x - x_0| < \delta$ we have $f(x) > M$.

(b) Let $x_0 \in \mathbb{R}$, let I be an open interval of the form (x_0, b), and let $f : I \to \mathbb{R}$ be a given function. We say that $\lim\limits_{x \to x_0+} f(x) = -\infty$ if and only if for every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for all $x \in I$ with $x - x_0 < \delta$ we have $f(x) < M$.

(c) Let J be an open interval of the form $(-\infty, b)$ and let $f : J \to \mathbb{R}$ be a given function. We say that $\lim_{x\to-\infty} f(x) = +\infty$ if and only if for every $M \in \mathbb{R}$ there exists an $M' \in \mathbb{R}$ such that for all $x \in J$ with $x < M'$ we have $f(x) > M$.

42. ([Wade], Exercise 3.3.1d) Show that the following function[1] is continuous.

$$f(x) = \begin{cases} \sqrt{x}\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

State clearly which limit theorems or properties of continuous functions you are using.

Solution: Both $\sqrt{x}$ and $\sin\frac{1}{x}$ are continuous in $\mathbb{R} \setminus \{0\}$, therefore so is their product. Therefore f is continuous in $\mathbb{R} \setminus \{0\}$.

It remains to show that f is continuous at $x_0 = 0$, i.e. $\lim_{x\to 0} f(x) = f(0)$, i.e. $\lim_{x\to 0}\left(\sqrt{x}\sin\frac{1}{x}\right) = 0$. Since the sine function takes values in $[-1, 1]$ we have

$$-\sqrt{x} \leq \sqrt{x}\sin\frac{1}{x} \leq \sqrt{x},$$

for all $x \neq 0$. Since $\pm\sqrt{x} \xrightarrow[x\to 0]{} 0$, the Squeeze Theorem gives $\sqrt{x}\sin\frac{1}{x} \xrightarrow[x\to 0]{} 0$.

Things to think about:
True or False? If $f(x) \xrightarrow[x\to x_0]{} 0$ and $g(x)$ is bounded, then $f(x)g(x) \xrightarrow[x\to x_0]{} 0$.

43. Prove that the function $\cos(1/x)$ has no limit as $x \to 0$.

Solution: We argue by contradiction. Suppose $\cos(1/x)$ converged to a limit L as $x \to 0$. For every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to 0$ we would then have $\cos(1/x_n) \to L$ as $n \to \infty$.

Take $x_n = \frac{1}{2n\pi}$. Then $x_n \to 0$, therefore $\cos(1/x_n) \to L$. But $\cos(1/x_n) = \cos(2n\pi) = 1 \to 1$. Therefore $L = 1$.

Now take $x_n = \dfrac{1}{2n\pi + \frac{\pi}{2}}$. Then $x_n \to 0$, therefore $\cos(1/x_n) \to L$. But $\cos(1/x_n) = \cos(2n\pi + \pi/2) = 0 \to 0$. Therefore $L = 0$. Contradiction.

[1] Strictly speaking we haven't rigorously introduced trigonometric functions yet. We'll do that later in Chapter 6. In the meantime they will appear here and there and we'll pretend we all know what they are and what basic properties they have.

44. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function with $\lim_{x \to 0} f(x) = 7$. Prove that there exists $\delta > 0$ such that for all $x \in (-\delta, \delta) \setminus \{0\}$ we have $f(x) > 6.9$.

Solution: Let $\varepsilon = \frac{1}{10}$. Since $f(x) \to 7$ as $x \to 0$, there exists a $\delta > 0$ such that for all $x \in (-\delta, \delta) \setminus \{0\}$ we have $|f(x) - 7| < \varepsilon = \frac{1}{10}$, which gives $-\frac{1}{10} < f(x) - 7 < \frac{1}{10}$, hence (use the left inequality) $f(x) > 7 - \frac{1}{10} = 6.9$.

45. Prove directly from the ε-δ definition[2] of continuity that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5x + 7$ is continuous at $x_0 = 1$.

Solution: Let $\varepsilon > 0$ be given. Set $\boxed{\delta = \frac{\varepsilon}{5}}$. Then for all $x \in \mathbb{R}$ such that $|x - 1| < \delta$ we have

$$\left|f(x) - f(1)\right| = |5x + 7 - 12| = 5\,|x - 1| < \boxed{5\delta = \varepsilon}. \tag{2}$$

46. Prove directly from the ε-δ definition of continuity that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x^2$ is continuous at $x_0 = 1$.

Rough Work:

Given $\varepsilon > 0$ we are looking for $\delta > 0$ such that for all x with

$$|x - 1| < \delta \tag{3}$$

we have

$$\left|f(x) - f(1)\right| < \varepsilon. \tag{4}$$

We have

$$\left|f(x) - f(1)\right| = \left|3x^2 - 3\right| = 3\,|x + 1|\,|x - 1|. \tag{5}$$

Comparing (5) to (2) we see that infront of $|x - 1|$ we have the variable quantity $3|x + 1|$ instead of a constant. Since we are only interested in x near 1 we expect $x + 1$ to be near 2, and therefore we expect $3|x + 1|$ to be bounded. We can replace it in (5) by one of its bounds, which is a constant, and then work as in (2), i.e.

$$\begin{aligned}\left|f(x) - f(1)\right| &= \left|3x^2 - 3\right| = 3\,|x + 1|\,|x - 1| \\ &\leq (\text{a bound for } 3|x + 1|)\,|x - 1| = (\text{constant})\,|x - 1|. \end{aligned} \tag{6}$$

Indeed, for x with $|x - 1| < \delta$ we have

$$3\,|x + 1| = 3\,|x - 1 + 2| \leq 3\left(|x - 1| + 2\right) < 3(\delta + 2) \tag{7}$$

[2] [Wade, Definition 3.19] , [Dindos, Definition 3.3]

therefore we can continue (5) as follows:

$$\left|\mathtt{f}(\mathtt{x})-\mathtt{f}(1)\right| = \left|3\mathtt{x}^2-3\right| = 3\left|\mathtt{x}+1\right|\left|\mathtt{x}-1\right| < 3(\delta+2)\delta. \tag{8}$$

It is enough now to ensure that

$$3(\delta+2)\delta < \varepsilon. \tag{9}$$

This is not as bad as it looks. We are not trying to find all δ that satisfy (9), we are not trying to solve for delta. All we need is to find a single δ that satisfies (9). To make our task easier let us try to find a δ with

$$\delta < 1 \tag{10}$$

that satisfies (9). We have $3(\delta+2)\delta < 3(1+2)\delta = 9\delta$. It is enough now to ensure that $9\delta < \varepsilon$ and this is easily done by choosing a δ with

$$\delta < \frac{\varepsilon}{9}. \tag{11}$$

An example of a positive number δ that satisfies both (10) and (11) is $\delta = \frac{1}{2}\min\{1, \frac{\varepsilon}{9}\}$.

<u>Remark</u>: There is nothing special about the bound 1 in (10). We could have said: let's try to find a δ with $\delta < 2016$ that satisfies (9). We have $3(\delta+2)\delta < 3(2016+2)\delta = 6054\delta$. It is enough now to ensure that $6054\delta < \varepsilon$ and this is easily done by choosing a δ with $\delta < \frac{\varepsilon}{6054}$, etc etc .

(Reminder: Do <u>not</u> hand in Rough Work.) END OF ROUGH WORK

<u>Solution:</u> Let $\varepsilon > 0$ be given. Let δ be a positive number such that[3] $\boxed{\delta < 1}$ and $\boxed{\delta < \frac{\varepsilon}{9}}$. For all $x \in \mathbb{R}$ with $|x-1| < \delta$ we have

$$3\,|x+1| = 3\,|x-1+2| \le 3\left(|x-1|+2\right) < \boxed{3\left(\delta+2\right) < 3(1+2) = 9},$$

therefore,

$$\left|f(x)-f(1)\right| = \left|3x^2-3\right| = 3\,|x+1|\,|x-1| \le 9\,|x-1| < \boxed{9\delta < \varepsilon.}$$

[3]An example of such a δ is $\delta = \frac{1}{2}\min\{1, \frac{\varepsilon}{9}\}$. Since this is not very enlightening and all we are ever going to use are the inequalities $\delta < 1$ and $\delta < \frac{\varepsilon}{9}$, we won't bother with an explicit formula for δ.

Things to think about:

What is wrong with the following proof?

Let $\varepsilon > 0$ be given. Take $\delta = \frac{\varepsilon}{3|x+1|}$. For all $x \in \mathbb{R}$ with $|x-1| < \delta$ we have

$$\left|f(x) - f(1)\right| = \left|3x^2 - 3\right| = 3|x+1||x-1| < 3|x+1|\delta = \varepsilon.$$

47. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function given by the formula $f(x) = \frac{1}{x}$. Prove directly from the ε-δ definition of continuity that f is continuous at $x_0 = 1$.

    Solution:

    Let $\varepsilon > 0$ be given. Choose a positive number δ such that[4] $\delta < \frac{1}{2}$ and $\delta < \frac{\varepsilon}{2}$. For all $x \in \mathbb{R} \setminus \{0\}$ with $|x-1| < \delta$ we have $|x-1| < \frac{1}{2}$, therefore $x > \frac{1}{2}$, therefore

    $$\left|f(x) - f(1)\right| = \left|\frac{1}{x} - 1\right| = \frac{|x-1|}{x} \leq 2|x-1| < 2\delta < \varepsilon.$$

48. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all rational x. Prove that $f(x) = 0$ for all real x.

    Solution: Fix $x \in \mathbb{R}$. By Problem 37 there is a sequence $(r_n)_{n\in\mathbb{N}}$ of rationals with $\lim_{n\to\infty} r_n = x$. Since f is continuous,

    $$f(x) = f(\lim_{n\to\infty} r_n) = \lim_{n\to\infty} f(r_n).$$

    But $f(r_n) = 0$ for all n, therefore $\lim_{n\to\infty} f(r_n) = 0$, therefore $f(x) = 0$.

49. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = x^2$ for all rational x. Find $f(\sqrt{2})$.

    Solution: Apply the result of Problem 48 to the function $f(x) - x^2$ to find that $f(x) - x^2 = 0$ for all real x, therefore $f(x) = x^2$ for all real x. In particular, $f(\sqrt{2}) = 2$.

50. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{10} - x^4 - 10$ has at least two real roots.

    Solution:

    The function f is a polynomial function, therefore it is continuous. We have $f(0) = -10 < 0$ and[5] $f(2) = 2^{10} - 2^4 - 10 = 2^4(2^6 - 1) - 10 \geq 16 \cdot 1 - 10 > 0$.

[4] An example of such a δ is $\delta = \frac{1}{2}\min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$.

[5] My calculator says $f(2) = 998$.

By the IVT there exists a real number c between 0 and 2 such that $f(c) = 0$. Since f is even we have $f(-c) = 0$ as well. The roots c and $-c$ are distinct because the first one is positive and other negative. We have shown that f has has at least two real roots.

51. Let $f, g : [a,b] \to \mathbb{R}$ be continuous with $f(a) > g(a)$ and $f(b) < g(b)$. Prove that there exists at least one real number c between a and b such that $f(c) = g(c)$.

    Solution: Apply the Intermediate Value Theorem to the function $f - g$.

52. ([Stewart], Section 1.5, Problem 51) A monk leaves the monastery at 7.00 a.m. and takes his usual path to the top of the mountain, arriving at 7.00 p.m. The following morning, he starts at 7.00 a.m. at the top and takes the same path back, arriving at the monastery at 7.00 p.m. Show that there is a point on the path that the monk will cross at exactly the same time of day on both days. [Hint: Find a way to use Problem 51]

    Solution: Let $g(t)$, $0 \le t \le 12$, be the monk's distance from the monastery t hours after leaving the monastery at 7.00 am on the first day. Let $f(t)$, $0 \le t \le 12$, be his distance from the monastery t hours after leaving the top of the mountain at 7.00 am on the second day. Let D be the distance between the monastery and the top of the mountain. Then $g(0) = 0$, $g(12) = D$, $f(0) = D$ and $f(12) = 0$.

    We are assuming that f and g are continuous. By Problem 51 there exists a t_0 between 0 and 12 such that $f(t_0) = g(t_0)$. This means that the monk's distance from the monastery at time 7.00am + t_0 hours is the same on both days.

53. (The Meteorologist's Theorem) Prove that at any given moment there exist two diametrically opposite points on the Earth's equator with exactly the same temperature. (We are assuming that temperature is a continuous function of position)

    Solution: Let $T(\theta)$, $0 \le \theta \le 2\pi$, be the temperature at longitudinal angle θ. We are assuming that T is a continuous function. Notice that $T(0) = T(2\pi)$ because $\theta = 0$ and $\theta = 2\pi$ correspond to the same point on the equator.

    Diametrically opposite points correspond to angles θ and $\theta + \pi$. We wish to show that there exists a $\theta_0 \in [0, \pi]$ such that $T(\theta_0) = T(\theta_0 + \pi)$.

    Consider the function $f : [0, \pi] \to \mathbb{R}$ given by $f(\theta) = T(\theta) - T(\theta + \pi)$. Then $f(0) = T(0) - T(\pi)$ and $f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) = -(T(0) - T(\pi))$. If $T(0) - T(\pi) = 0$ we are done (with $\theta_0 = 0$). If $T(0) - T(\pi) \neq 0$, then one of the numbers $f(0)$ and $f(\pi)$ is positive and the other negative. By the IVT, there exists a θ_0 between 0 and π such that $f(\theta_0) = 0$, i.e. $T(\theta_0) = T(\theta_0 + \pi)$.

Things to think about:
Is the result still true if we replace the equator by any other great circle? Any other closed curve?
Learn more here: The Borsuk-Ulam Theorem explained by a Youtube nerd, and here: The Borsuk-Ulam Theorem

54. ([Wade], Exercise 3.3.4) If $f : [a,b] \to [a,b]$ is continuous, then f has a *fixed point*; that is, there is a $c \in [a,b]$ such that $f(c) = c$. [Hint: Draw a picture. The point $(c, f(c))$ is on the graph of f and on ...]

    Solution:

    Since f takes values in $[a,b]$ we have $f(a) \geq a$ and $f(b) \leq b$. If $f(a) = a$, take $c = a$. If $f(b) = b$, take $c = b$.

    Assume now that $f(a) \neq a$ and $f(b) \neq b$. Then $f(a) > a$ and $f(b) < b$. Let $g : [a,b] \to \mathbb{R}$ be the function given by $g(x) = f(x) - x$. Then g is continuous with $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$, therefore there exists at least one c between a and b such that $g(c) = 0$, which is the same as $f(c) = c$.

55. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f(0) = 1$ and such that $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. Prove that f has a global maximum. [Hint: Drawing pictures helps.]

    Solution:

    Since $\lim_{x\to+\infty} f(x) = 0$ there exists an $M > 0$ such that for all $x > M$ we have $f(x) < \frac{1}{2}$.

    Since $\lim_{x\to-\infty} f(x) = 0$ there exists an $M' > 0$ such that for all $x < -M'$ we have $f(x) < \frac{1}{2}$.

    On $[-M', M]$ the function f has a maximum attained at some point $x_0 \in [-M', M]$ (because it is a continuous function and the interval $[-M', M]$ is closed and bounded).

    We claim that $f(x_0)$ is the global maximum of f. We already know that $f(x_0) \geq f(x)$ for all $x \in [-M', M]$. The same is true for all $x \notin [-M', M]$. Indeed, if $x \notin [-M', M]$ then $f(x) \leq \frac{1}{2} < 1 = f(0) \leq f(x_0)$.

56. Let $f : [a,b] \to \mathbb{R}$ be continuous and $x_1, x_2, \ldots, x_n \in [a,b]$. Prove that there exists $c \in [a,b]$ such that

    $$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = f(c).$$

    (This says: the average of a finite number of values of f is another value of f.)

Solution:

The function f is continuous on the closed and bounded interval $[a,b]$, therefore it has a minimum m and a maximum M. We have $m \leq f(x_i) \leq M$ for all $i \in \{1,2,...,n\}$. Adding up these n inequalities we find $nm \leq f(x_1)+\cdots f(x_n) \leq nM$, therefore $m \leq \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \leq M$. By the IVT, there exists $c \in [a,b]$ such that $f(c) = \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$.

Things to think about:

1. Integration (Chapter 5) is not on the FPM syllabus, but we all know what the definite integral $\int_a^b f(x)dx$ is and we are familiar with its basic properties. Let $f : [a,b] \to \mathbb{R}$ be continuous. Its average over $[a,b]$ is defined to be $\frac{1}{b-a}\int_a^b f(x)dx$. Prove that there exists $c \in [a,b]$ such that $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$. This says: the average of f is a value of f.

2. The domain of f in Problem 56 was a closed and bounded interval $[a,b]$. This was used in the proof to guarantee that f had a minimum and a maximum. Would the result still be true if we replaced $[a,b]$ by $\mathbb{R}$? In other words, is the following statement true?
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $x_1, x_2, ..., x_n \in \mathbb{R}$. Then there exists $c \in \mathbb{R}$ such that

$$\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} = f(c).$$

57. ([Wade], Exercise 3.3.0 a) Give a proof if the following statement is true, otherwise give a counterexample.

    If $f : [a,b] \to \mathbb{R}$ is continuous then its range is a closed and bounded interval.

    Solution:

    The given statement is *True*.

    Proof:

    Since the function f is continuous and the interval $[a,b]$ is closed and bounded, it has a minimum m and a maximum M. By the IVT all values between m and M are attained. Therefore the range of f is the closed and bounded interval $[m,M]$.

58. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz continuous* iff there exists a positive constant L such that for all $x, y \in \mathbb{R}$ we have

    $$\big|f(x) - f(y)\big| \leq L|x-y|. \tag{12}$$

    Prove that a Lipschitz continuous function is continuous.

Solution:

Fix $x_0 \in \mathbb{R}$. We show that f is continuous at x_0.

Let ε be a given positive number and define $\delta = \frac{\varepsilon}{L}$. For all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ we have

$$\left|f(x) - f(x_0)\right| \leq L|x - x_0| < L\delta = \varepsilon.$$

59. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a *contraction* iff it is Lipschitz continuous with constant $L < 1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction and fix $a \in \mathbb{R}$. Define a sequence $(x_n)_{n\in\mathbb{N}}$ by $x_1 = a$ and $x_{n+1} = f(x_n)$ for $n \geq 1$. Prove that:

(a) $|x_n - x_{n-1}| \leq L|x_{n-1} - x_{n-2}|$, for all $n \geq 3$.

(b) $|x_n - x_{n-1}| \leq L^{n-2}|x_2 - x_1|$, for all $n \geq 2$.

(c) $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

(d) $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence.

(e) The limit l of $(x_n)_{n\in\mathbb{N}}$ is a *fixed point* of f, i.e. $f(l) = l$.

(f) f has no other fixed points.

(g) The limit l of $(x_n)_{n\in\mathbb{N}}$ is always the same, irrespective of the starting point $x_1 = a$.

Solution:

(a) Fix $n \geq 3$. Using the recursion formula that defines the sequence (x_n) and (12) we have

$$|x_n - x_{n-1}| = \left|f(x_{n-1}) - f(x_{n-2})\right| \leq L|x_{n-1} - x_{n-2}|.$$

(b) We use induction on n and part (a). For $n = 2$ the result is trivial. Assuming that the result is true for some $n \geq 2$ then it is true for $n+1$ as well because

$$|x_{n+1} - x_n| \leq L|x_n - x_{n-1}| \leq LL^{n-2}|x_2 - x_1| = L^{n-1}|x_2 - x_1|.$$

(c) (This step is essentially the same as Problem 40)

Let $\varepsilon > 0$. The sequence $\left(L^{n-1}\right)_{n\in\mathbb{N}}$ converges to zero because $0 < L < 1$, therefore there exists a positive integer N such that for all $n \geq N$,

$$L^{n-1} < \frac{\varepsilon(1-L)}{1 + |x_2 - x_1|}. \tag{13}$$

We claim that for all $n,m \geq N$ we have

$$|x_n - x_m| < \varepsilon.$$

Fix $n,m \geq N$. We may assume $n > m$. Then

$$\begin{aligned}
|x_n - x_m| &= \left|(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{m+1} - x_m)\right| \\
&\leq \left(L^{n-2} + L^{n-3} + \cdots + L^{m-1}\right)|x_2 - x_1| \\
&= L^{m-1}\left(L^{n-m-1} + L^{n-m-2} + \cdots + 1\right)|x_2 - x_1| \\
&= L^{m-1}\frac{1-L^{n-m}}{1-L}|x_2 - x_1|,
\end{aligned}$$

where we have used the identity

$$1 + x + x^2 + \cdots + x^k = \frac{1-x^{k+1}}{1-x}, \quad (x \neq 1).$$

Therefore,

$$\begin{aligned}
|x_n - x_m| \leq L^{m-1}\frac{1-L^{n-m}}{1-L}|x_2 - x_1| &\leq L^{m-1}\frac{1}{1-L}|x_2 - x_1| \\
&< \frac{\varepsilon(1-L)}{1+|x_2 - x_1|}\frac{1}{1-L}|x_2 - x_1| < \varepsilon, \quad (14)
\end{aligned}$$

where we have used (13) in the last step (with n replaced by m) .

(d) We know from above that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Every Cauchy sequence of real numbers is convergent.

(e) Let l be the limit of (x_n). Since f is continuous, $f(x_n) \xrightarrow[n\to\infty]{} f(l)$. On the other hand, (x_{n+1}) converges to l. Letting $n \to \infty$ on both sides of $x_{n+1} = f(x_n)$ we find $l = f(l)$.

(f) We argue by contradiction. Suppose f has another fixed point $l' \neq l$. Then

$$\left|l - l'\right| = \left|f(l) - f(l')\right| \leq L\left|l - l'\right|$$

therefore $1 \leq L$; contradiction.

(g) We know from above that (x_n) converges to a fixed point of f and that f has a unique fixed point. Therefore, no matter what the starting point $x_1 = a$ is, the sequence (x_n) converges to the unique fixed point of f.

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