Fundamentals of Pure Mathematics

MATH08064 **Solutions and comments** **June13**

Section A (40 points). Solve all 6 problems in this section

1. State, with a brief justification, whether the following statements are TRUE or FALSE:

(a) D_3 is the smallest possible non-cyclic group.

(b) Every group of order p^3 (where p is a prime) is abelian.

(c) $\mathbb{Z}_9 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

[6 marks]

Solution:

(a) FALSE. $\mathbb{Z}_2 \times \mathbb{Z}_2$ is non-cyclic (it does not have an element of order 4), and has only four elements. D_3 has six elements.

(b) FALSE. D_4 has order $8 = 2^3$ and is non-abelian.

(c) FALSE. $\mathbb{Z}_9$ has an element of order 9, whereas $\mathbb{Z}_3 \times \mathbb{Z}_3$ does not.

2.

(a) Write down four groups, each with twelve elements, that are mutually non-isomorphic. You must justify why they are mutually non-isomorphic. **[4 marks]**

(b) Let G be a group with twelve elements. State, with a brief justification, whether the following statements are TRUE or FALSE:

(i) G has an element of order 6.

(ii) G has an element of order 4.

(iii) G has an element of order 2.

[3 marks]

Solution:

(a) Consider $\mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, D_6 and A_4. Only $\mathbb{Z}_{12}$ has an element of order 12, so none of the others can be isomorphic to it. Next, $\mathbb{Z}_2 \times \mathbb{Z}_6$ is abelian, so it cannot be isomorphic to either D_6 or A_4 (since they are non-abelian). Lastly, D_6 has an element of order 6, whilst A_4 does not, hence they cannot be isomorphic.

(b) (i) FALSE. A_4 does not have an element of order 6.

(ii) FALSE. D_6 does not have an element of order 4.

(iii) TRUE, by Cauchy's theorem.

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3. Let H and K be subgroups of a group G. State, with a brief justification, whether the following statements are TRUE or FALSE:

(a) $gH = Hg$ for all $g \in G$.

(b) If $g_1, g_2 \in G$ with $g_1H = g_2H$, then $g_1 = g_2$.

(c) The union $H \cup K$ is a subgroup of G.

[7 marks]

Solution:

(a) FALSE. This only holds for normal subgroups. An example of a non-normal subgroup is $\langle h \rangle$ in D_4.

(b) FALSE. Consider $H = \{0, 2\}$ inside $G := \mathbb{Z}_4$. Then $0H = 2H = \{0, 2\}$ but $0 \neq 2$.

(c) FALSE. Consider $H := \langle 2 \rangle = \{0, 2, 4\}$ and $K := \langle 3 \rangle = \{0, 3\}$ inside $G := \mathbb{Z}_6$. Then $H \cup K = \{0, 2, 3, 4\}$ which cannot be a subgroup since its order does not divide $|\mathbb{Z}_6| = 6$.

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4. State, with a brief justification, whether the following statements are TRUE or FALSE.

(a) Let $\alpha \in \mathbb{R}$, A be a nonempty bounded subset of $\mathbb{R}$ and $B = \{\alpha x : x \in A\}$. Then $\sup B = \alpha(\sup A)$.

(b) If $A + B = \{a + b : a \in A \text{ and } b \in B\}$ then $\sup(A + B) = \sup A + \sup B$.

(c) A dyadic rational is a point $x \in \mathbb{R}$ such that $x = n/2^m$ for some $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. The set of dyadic rationals is uncountable.

[7 marks]

Solution: (Similar true-false questions done at the tutorial)

(a) FALSE. Take $\alpha < 0$ and $E = \{1, 2, 3\}$.

(b) TRUE. The proof has 2 steps. First of all, $\sup A + \sup B$ is an upper bound of the set $A + B$ since $a + b \leq \sup A + \sup B$ for all $a \in A$ and $b \in B$. Next, for any $\varepsilon > 0$ there exit $a_0 \in A$ and $b_0 \in B$ such that

$$a_0 > \sup A - \varepsilon/2, \qquad b_0 > \sup B - \varepsilon/2.$$

Hence $A + B \ni a_0 + b_0 > \sup A + \sup B - \varepsilon$. It follows that $\sup A + \sup B - \varepsilon$ is not an upper bound of the set $A + B$, hence $\sup A + \sup B$ is the least upper bound.

(c) FALSE. The set is countable, as there is a bijection between the set of dyadic rationals and the set $\mathbb{Z} \times \mathbb{N}$. However a cartesian product of countable sets is countable.

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5. Suppose that $x_1 > 5$ and $x_{n+1} = (2 + x_n)/3$. Use the Monotone Convergence Theorem to prove that the sequence (x_n) is convergent and find its limit. **[7 marks]**

Solution: (Similar to assigned exercises) We shall establish by induction that $x_n \geq 1$. The statement if obviously true for x_1. Assume that $x_n \geq 1$. Then for x_{n+1} we have that

$$x_{n+1} = (2 + x_n)/3 \geq (2 + 1)/3 = 1,$$

as desired. Next we show that the sequence (x_n) is decreasing. Indeed,

$$x_{n+1} = (2 + x_n)/3 \leq (2x_n + x_n)/3 = x_n,$$

hence $x_{n+1} \leq x_n$. Here we used the inequality $x_n \geq 1$ established above. By the Monotone Convergence Theorem, since the sequence (x_n) is bounded below and decreasing, it must have a limit. Let $a = \lim_{n \to \infty} x_n$. Then

$$a = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} (2 + x_n)/3 = (2 + a)/3.$$

Solving for a yields $a = 1$.

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6. Let (x_n) be an arbitrary sequence of real numbers. Show that the following sequence is bounded. Does it have a convergent subsequence?

$$y_n = \frac{3x_n + 2}{1 + x_n^2}, \qquad n = 1, 2, 3, \ldots$$

[6 marks]

Solution: The sequence (y_n) is bounded, since

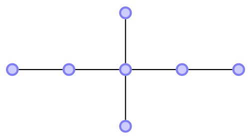
$$\left| \frac{3x_n + 2}{1 + x_n^2} \right| \le 2 + \frac{3|x_n|}{1 + x_n^2} \le 2 + 3 = 5.$$

Thus by the Bolzano-Weierstrass theorem a convergent subsequence of (y_n) must exit.

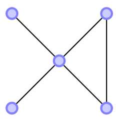
Section B (60 points). Solve any three problems in this section

7.

(a) Let G denote the symmetry group of the graph

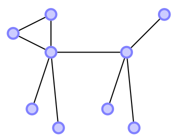


and let H denote the symmetry group of the graph



Show that $G \cong H$. **[6 marks]**

(b) Show that the symmetry group of the graph



is isomorphic to $S_2 \times S_2 \times S_3$. **[5 marks]**

(c) A stick-dog manufacturer wants to make stick-dogs, based on the design in (b). The manufacturer has two colours, and wants to colour each node of the above graph. How many different stick-dogs can be made? (Two stick-dogs are regarded as identical if they differ by an element of the symmetry group of the graph.) **[9 marks]**

Solution:

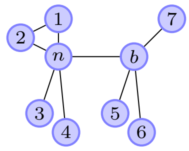
(a) (The students have seen graph 1 in Workshops, and many times in the Exercise Sheets). We show that both groups are isomorphic to $S_2 \times S_2$.

Any element of G must fix the middle vertex (being the only one of valency four). Given this, it is completely determined by whether it fixes or swaps the left/right arms, and whether it fixes or swaps the up/down arms. These are independent, so a symmetry is specified by a pair (a, b) with $a, b \in S_2$. This shows that $G \cong S_2 \times S_2$.

Now any element of H must fix the middle vertex (being the only one of valency four). Given this, it is completely determined by whether it fixes or swaps two left arms (they are the only vertices with valency one), and whether it fixes or swaps the right arms (they are the only vertices with valency two). These are independent, so a symmetry is specified by a pair (a, b) with $a, b \in S_2$. This shows that $H \cong S_2 \times S_2$.

Hence $G \cong S_2 \times S_2 \cong H$.

(b) (This is new, but very similar in style to Workshop 5) For convenience, label the vertices



Any symmetry must fix n (being the only vertex with valency five) and must fix b (being the only vertex with valency four). A symmetry must then either fix 1 and 2 or swap them (since they are the only vertices of valency two attached to n), it must fix 3 and 4 or swap them (since they are the only vertices of valency one attached to n), and it must permute the vertices 5, 6, 7 (since they are the only vertices of valency one attached to b). These are all independent of each other, so a symmetry is given by a triple (x, y, z) where x is a permutation of $\{1, 2\}$, y is a permutation of $\{3, 4\}$ and z is a permutation of $\{5, 6, 7\}$. This shows that the symmetry group is $S_2 \times S_2 \times S_3$.

(c) (This is again new, but very similar in style to Workshop 5). By Pólya counting, if a finite group G acts on a finite set X, then

$$\text{the number of } G\text{-orbits in } X = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|.$$

To solve the problem in the question, let G be the symmetry group of the graph. This acts on the set X of all possible (i.e. 2^9) colourings. The question asks for the number of G-orbits, hence we must analyse the fixed points.

- The identity $(0, 0, e)$ fixes every coloured stick dog, so has 2^9 fixed points.
- For a stick dog to be fixed under $(1, 0, e)$, 1 and 2 must have the same colour, whereas all other vertices can be arbitrary. Thus there are 2^8 fixed points.
- A similar analysis shows that there are 2^8 fixed points for the group elements $(0, 1, e)$ and $(0, 0, \sigma)$ where σ is any 2-cycle.
- A similar analysis shows that there are 2^7 fixed points for the group elements $(0, 0, \sigma)$ where σ is any 3-cycle.

Continuing in this way, the fixed point analysis is

Type of element	Number	Fixed points
Identity	1	2^9
Swapping of only two vertices	5	2^8
Swapping of two pairs of two vertices	7	2^7
Swapping of three pairs of two vertices	3	2^6
Swapping of three vertices only	2	2^7
Swapping of two vertices and three vertices	4	2^6
Swapping of two vertices, two vertices, and three vertices	2	2^5

Hence the number of colourings, i.e. the number of orbits,

$$\begin{aligned}
&= \tfrac{1}{24}\left(2^9 + (5\times 2^8) + (9\times 2^7) + (7\times 2^6) + (2\times 2^5)\right)\\
&= \tfrac{1}{2^3\times 3}\left(2^9 + (5\times 2^8) + (9\times 2^7) + (8\times 2^6)\right)\\
&= \tfrac{1}{2^3\times 3}\left(4\times 2^7 + 10\times 2^7 + 9\times 2^7 + 4\times 2^7\right)\\
&= \tfrac{1}{2^3\times 3}\left(27\times 2^7\right)\\
&= \tfrac{2^7\times 3^3}{2^3\times 3}\\
&= 2^4\times 3^2\\
&= 144.
\end{aligned}$$

8.

(a) In each of the following, give a specific example of a group G and a subset $H \subseteq G$ such that

(i) H is a subgroup of G.

(ii) H is not a subgroup of G.

(iii) H is a normal subgroup of G.

(iv) H is a subgroup of G that is not normal.

[8 marks]

(b) Consider now $G := \mathbb{Z}_2 \times \mathbb{Z}_2$ acting on itself via conjugation. Calculate the conjugacy classes (i.e. the orbits of this action). **[4 marks]**

(c) Now suppose that G acts on a set X, and let $x \in X$. Consider

$$\phi : G/\operatorname{Stab}_G(x) \to \operatorname{Orb}_G(x) \quad \text{which sends} \quad g\operatorname{Stab}_G(x) \mapsto g\cdot x.$$

Show that

(i) ϕ is a function. **[4 marks]**

(ii) ϕ is a bijection. **[4 marks]**

Solution:

(a) (This type of question has not been asked directly before, but tests basic knowledge)

(i) Consider the set $H := \{e, g, g^2\} = \langle g\rangle$ inside $G := D_3$.

(ii) Consider the set $H := \{e, g\}$ inside $G := D_3$. Since $gg \notin H$, H is not closed.

(iii) Consider the set $H := \{e, g, g^2\} = \langle g\rangle$ inside $G := D_3$. Since $\frac{|G|}{|H|} = 2$, H is normal in G.

(iv) Consider the set $H := \{e, h\} = \langle h\rangle$ inside $G := D_3$. This is a subgroup (it is a cyclic subgroup). If H were normal it would be a union of conjugacy classes, so $\{h\}$ would be a conjugacy class. This would force h to be central, but $gh \neq hg$. Hence H is not normal.

(b) (They have calculated conjugacy classes in Exercise Sheet 5. This example is new, but easy since the group is abelian). The group G is abelian, and the question asks to calculate the conjugacy classes. But since G is abelian, the conjugacy classes are clearly just one-element sets. Thus the conjugacy classes (=the orbits of this action) are precisely $\{(0,0)\}$, $\{(1,0)\}$, $\{(0,1)\}$ and $\{(1,1)\}$.

(c) (Although this is orbit-stabilizer, the book proves it using a map in the other direction, so they have not seen this question. However, they have seen how to prove something is a function in Problem 1.1 in the exercise sheets; showing something is bijective is also a basic skill)

(i) Denote $H := \mathrm{Stab}_G(x)$. If $g_1H = g_2H$ then $g_2 = g_1h$ for some $h \in H$. This implies that
$$g_2 \cdot x = (g_1h) \cdot x = g_1 \cdot (h \cdot x) = g_1 \cdot x,$$
hence $\phi(g_1H) = \phi(g_2H)$ and so ϕ is a function.

(ii) Clearly the image of ϕ is the whole of the orbit of x, and so ϕ is surjective. To see that ϕ is injective, suppose that $\phi(g_1H) = \phi(g_2H)$. Then $g_1 \cdot x = g_2 \cdot x$, so acting with g_1^{-1} on both sides we get
$$(g_1^{-1}g_2) \cdot x = (g_1^{-1}g_1) \cdot x = e \cdot x = x.$$
Hence $g_1^{-1}g_2 \in H$ and so $g_1H = g_2H$. Thus ϕ is injective.

9.

(a) Suppose that (a_k) is a sequence such that $\sqrt[k]{|a_k|} \to a > 0$ for some $a \in \mathbb{R}$. Prove that for all $|x| < 1/a$ the series
$$\sum_{k=1}^{\infty} a_k x^k \qquad \text{is absolutely convergent.}$$
[7 marks]

(b) Is the series
$$\sum_{k=2}^{\infty} \frac{(-1)^k}{\log k}$$
convergent? Is it absolutely convergent? **[7 marks]**

(c) State, with a brief justification, whether the following statements are TRUE or FALSE:

(i) If $a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ is absolutely convergent then $\sum_{k=1}^{\infty} a_k$ is convergent.

(ii) If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent then $\sum_{k=1}^{\infty} a_k^2$ is absolutely convergent. **[6 marks]**

Solution:

(a) By the Root test for convergence
$$\lim_{k\to\infty} \sqrt[k]{|a_k x^k|} = \left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right) |x| = a|x| < 1, \qquad \text{for all } |x| < 1/a.$$
Hence the series is absolutely convergent.

(b) Since $1/\log k > 1/k$ for large k and $\sum_{k=2}^{\infty} 1/k = \infty$ the series $\sum_{k=2}^{\infty} \frac{1}{\log k}$ diverges and hence our series is not absolutely convergent. It however converges conditionally as $1/\log k$ is monotone and converges to 0 as $k \to \infty$.

(c) (i) FALSE. Take $a_k = -k$ and $b_k = 0$.

(ii) TRUE. The convergence of $\sum_{k=1}^{\infty} a_k$ implies that $a_k \to 0$ as $k \to \infty$, hence for some $k \geq N$ we have that $|a_k| \leq 1$. It follows that for $k \geq N$: $a_k^2 = |a_k|^2 \leq |a_k|$ and hence by the comparison theorem for sequences $\sum_{k=1}^{\infty} a_k^2$ is convergent. As $a_k^2 \geq 0$ we also have absolute convergence.

10.

(a) Use the mean value theorem to prove that for all $x, y \geq 0$

$$|e^{-x} - e^{-y}| \leq |e^x - e^y|.$$

[6 marks]

(b) Use the Taylor's theorem to prove that for all $x > 0$ and $n \in \mathbb{N}$ odd

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots - \frac{x^n}{n!} < e^{-x}.$$

[6 marks]

(c) Show that the function

$$f(x) = \begin{cases} x^p \cos(1/x), & \text{for } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

is continuous for $p = 1$ at the point 0 and discontinuous for $p = 0$ at 0. **[8 marks]**

Solution:

(a) The function e^x is differentiable on $\mathbb{R}$ with derivative e^x. Hence, by the mean value theorem

$$|e^{-x} - e^{-y}| = |e^{-c}||x - y|, \qquad |e^x - e^y| = |e^d||x - y|,$$

for some c, d between x and y, hence nonnegative. Therefore

$$|e^{-c}| \leq 1 \leq |e^d|$$

and the claim follows by multiplying both sides by $|x - y|$.

(b) By the Taylor's theorem

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

Since $x > 0$, we see that $c > 0$. $f^{(n+1)}(x) = (-1)^{n+1} e^{-x}$, hence for $n+1$ even $f^{(n+1)}(c) > 0$ and hence the right hand side is positive. The left hand side is equal to $e^x - (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots - \frac{x^n}{n!})$.

(c) Let $\varepsilon > 0$ and take $\delta = \varepsilon$. When $p = 1$ then for $|x| = |x - 0| < \delta = \varepsilon$ we have that

$$|f(x) - f(0)| = |f(x)| \leq |x| < \varepsilon.$$

Hence the continuity follows.

If $p = 0$ consider a sequence $x_n = (2\pi n)^{-1}$. As $x_n \to 0$ we see that $f(x_n) = 1 \to 1 \neq 0 = f(0)$. Hence f is not continuous at 0.
