Fundamentals of Pure Mathematics 2015-16
Assignment 4
Due-date: Thursday of week 5
(Submit your work to your tutor at the workshop)

1. (5 marks) (This is Problem 25 from the week 4 workshop)

   Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of real numbers given by $a_n = \frac{7n+5}{3n+1}$. Prove directly from the ε-N definition of convergence[1] that $a_n \xrightarrow[n\to\infty]{} \frac{7}{3}$.

   Solution

   We wish to show that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\left|a_n - \frac{7}{3}\right| < \varepsilon$.

   Let $\varepsilon > 0$ be given. By the Archimedean property of $\mathbb{R}$ there exists a positive integer N such that[2] $N > \frac{8}{\varepsilon}$. For all $n \geq N$ we have

   $$\left|a_n - \frac{7}{3}\right| = \left|\frac{7n+5}{3n+1} - \frac{7}{3}\right| = \frac{8}{9n+3} \leq \frac{8}{n} \leq \frac{8}{N} < \varepsilon.$$

2. (5 marks) Prove that

   $$\frac{n}{3^n} \xrightarrow[n\to\infty]{} 0. \tag{1}$$

   (A few Comments and a Hint: You are <u>not</u> asked to prove (1) directly from the definition. You may use any Limit Theorems provided that you state clearly which ones you are using. Use material from Chapters 1 and 2 only. Both the numerator and the denominator of $\frac{n}{3^n}$ go to infinity but you are <u>not</u> allowed to convert the sequence $\frac{n}{3^n}$ to the function $\frac{x}{3^x}$ and use L'Hospital's Rule. And now the Hint: you are looking for something that says that the geometric progression in the denominator grows faster than the numerator.)

   Solution

   We claim that

   $$3^n \geq n^2 \tag{2}$$

[1] Wade, Definition 2.1

[2] An example of such an N is $\lfloor \frac{8}{\varepsilon} \rfloor + 1$.

for all $n \in \mathbb{N}$. We'll prove (2) by induction on n. The cases $n = 1, 2$ are easy to check, and if (2) is true for some $n \geq 2$ then it is true for $n+1$ as well because

$$3^{n+1} = 3 \cdot 3^n \geq 3n^2 \geq (n+1)^2.$$

To prove the last inequality above expand the square and rearrange to make it $2n^2 - 2n - 1 \geq 0$, which is easily seen to be true:

$$2n^2 - 2n - 1 = 2n(n-1) - 1 \geq 2 \cdot 2 \cdot (2-1) - 1 = 3 > 0.$$

This completes the proof of (2).

Using (2) we have

$$0 \leq \frac{n}{3^n} \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. Since $\frac{1}{n} \xrightarrow[n\to\infty]{} 0$, the Squeeze Thm implies that $\frac{n}{3^n} \xrightarrow[n\to\infty]{} 0$.

Remark 1. There are several variants of (2) that we can use to prove (1). For example, using the Binomial Theorem and mimicking the second proof of Bernoulli's inequality in Problem 8 we have

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k \geq \binom{n}{2} 2^2 = 2n(n-1), \ (n \geq 2) \tag{3}$$

therefore,

$$0 \leq \frac{n}{3^n} \leq \frac{n}{2n(n-1)} = \frac{1}{2(n-1)}.$$

We can then use the Squeeze Theorem as above.

Remark 2. Yet another option is to estimate the numerator instead of the denominator. For all $n \in \mathbb{N}$ we have

$$n < 2^n, \tag{4}$$

(Exercise: prove this by induction) therefore,

$$\frac{n}{3^n} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

Since $\frac{2}{3} < 1$ we have $\left(\frac{2}{3}\right)^n \xrightarrow[n\to\infty]{} 0$, therefore $\frac{n}{3^n} \xrightarrow[n\to\infty]{} 0$.