Fundamentals of Pure Mathematics 2015-16
Analysis problems for weeks 7 and 8

I welcome your feedback on these problems and solutions. If you have any comments, if there is anything that needs more explaining, if you have any questions on any of the material, please come and see me or email me at n.bournaveas@ed.ac.uk or ask a question on Piazza.

Suggested Problems for the workshop in week 8: 60, 61, 62. If time permits or if you have already solved these problems at home then work on any of the following: 64, 66, 68, 69, 76, 77, 78.

Problem 61 is part of Assignment 8.

Differentiability on $\mathbb{R}$ (Wade, Chapter 4)

60. ([Wade], Exercise 4.1.1) For each of the following real functions use directly the definition of the derivative to prove that $f'(a)$ exists.

    (a) $f(x) = x^2 + x$, $a \in \mathbb{R}$,

    (b) $f(x) = \frac{1}{x}$, $a \neq 0$.

    Solution:

    (a) The difference quotient is

    $$\frac{f(x)-f(a)}{x-a} = \frac{x^2+x-a^2-a}{x-a} = \frac{(x-a)(x+a)+(x-a)}{x-a} = x+a+1$$

    therefore

    $$\frac{f(x)-f(a)}{x-a} \xrightarrow[x\to a]{} 2a+1.$$

    It follows that f is differentiable at a and $f'(a) = 2a+1$.

    (b) The difference quotient is

    $$\frac{f(x)-f(a)}{x-a} = \frac{\frac{1}{x}-\frac{1}{a}}{x-a} = \frac{\frac{a-x}{ax}}{x-a} = -\frac{1}{ax}$$

    therefore

    $$\frac{f(x)-f(a)}{x-a} \xrightarrow[x\to a]{} -\frac{1}{a^2}.$$

    It follows that f is differentiable at a and $f'(a) = -\frac{1}{a^2}$.

61. Prove directly from the definition of the derivative that $f(x) = \sqrt[3]{x}$ is differentiable at $x_0 = 1$ and find $f'(x_0)$. (You may need: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$)

This Problem is on Assignment 8. The solution will be posted here later.

62. Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|^3$ differentiable? Prove your claim.

Solution: Yes, it is differentiable everywhere (even at $x_0 = 0$).

Proof: Fix $x_0 \in \mathbb{R}$. We prove that $f(x) = |x|^3$ is differentiable at x_0.

If $x_0 \neq 0$ then the function $g(x) = |x|$ is differentiable at x_0 and our f is g^3 (f is the product of three functions, each differentiable at x_0), therefore f is differentiable at x_0.

If $x_0 = 0$ we use[1] the definition of differentiability at a point. The difference quotient is

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x) - f(0)}{x - 0} = \frac{|x|^3}{x} = \frac{|x|^2\,|x|}{x} = \frac{x^2\,|x|}{x} = x\,|x| \xrightarrow[x \to 0]{} 0|0| = 0,$$

therefore f is differentiable at $x_0 = 0$ (and $f'(0) = 0$).

Things to think about:
In the second part of the solution we said that the function $|x|$ wasn't differentiable at $x_0 = 0$. What property of that function was used in $x|x| \xrightarrow[x \to 0]{} 0|0| = 0$?

63. Find all points where the function $f(x) = |x-1| + |x-2|$ is differentiable. (NB: Prove all your claims. In particular, if you claim that f is not differentiable at a point, you have to give a proof.)

Solution: Fix $x_0 \in \mathbb{R}$. We examine whether f is differentiable at x_0.

Suppose first that $x_0 \neq 1, 2$. Then the functions $g(x) = |x-1|$ and $h(x) = |x-2|$ are both differentiable at x_0, and since $f = g + h$, it follows that f is differentiable at x_0.

Consider now the case $x_0 = 1$. Then g is not differentiable at x_0 and h is differentiable at x_0. We claim that f is not differentiable at x_0. Indeed, if f was differentiable at x_0 we would write g as $g = f - h$. It would follow from this that g is differentiable at x_0 as the difference of two differentiable at x_0 functions.

Similarly, f is not differentiable at $x_0 = 2$.

[1] We can't use the same argument as for $x_0 = 0$ because $g(x) = |x|$ is not differentiable at $x_0 = 0$.

Things to think about:
Give an example of a function f : $\mathbb{R} \to \mathbb{R}$ that fails to be differentiable at exactly n points.

64. ([Wade], Exercise 4.1.8) (Fermat's Theorem) Let I be an open interval, $f : I \to \mathbb{R}$, and $c \in I$. The function f is said to have a local maximum at c if and only if there is a $\delta > 0$ such that $f(c) \geq f(x)$ holds for all $x \in I$ with $|x - c| < \delta$.

a) If f has a local maximum at c, prove that

$$\frac{f(c+u) - f(c)}{u} \leq 0 \quad \text{and} \quad \frac{f(c+t) - f(c)}{t} \geq 0$$

for $u > 0$ and $t < 0$ sufficiently small.

b) If f is differentiable at c and has a local maximum at c, prove that $f'(c) = 0$.

c) Make and prove analogous statements for local minima.

d) Show by example that the converses of the statements in parts b) and c) are false. Namely, find an f such that $f'(c) = 0$ but f has neither a local maximum nor a local minimum at c.

Solution:

(a) For all positive u which are sufficiently small so that $c + u \in I$ (so that $f(c+u)$ makes sense) and $c + u \in (c - \delta, c + \delta)$ (so that we can compare $f(c+u)$ with $f(c)$) we have $f(c+u) \leq f(c)$, therefore

$$\frac{f(c+u) - f(c)}{u} \leq 0 \tag{1}$$

(the numerator is ≤ 0 and the denominator is > 0).

For all negative t with sufficiently small size so that $c + t \in I$ (so that $f(c+t)$ makes sense) and $c + t \in (c - \delta, c + \delta)$ (so that we can compare $f(c+t)$ with $f(c)$) we have $f(c+t) \leq f(c)$, therefore

$$\frac{f(c+t) - f(c)}{t} \geq 0 \tag{2}$$

(the numerator is ≤ 0 and the denominator is < 0).

(b) Assume now that f is differentiable at c. Letting $u \to 0+$ in (1) we obtain $f'(c) \leq 0$. Letting $t \to 0-$ in (2) we obtain $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

(c) The corresponding results for minima are:

- If f has a local minimum at c, prove that

$$\frac{f(c+u)-f(c)}{u} \geq 0 \quad \text{and} \quad \frac{f(c+t)-f(c)}{t} \leq 0$$

for $u > 0$ and $t < 0$ sufficiently small.

- If f is differentiable at c and has a local minimum at c, then $f'(c) = 0$.

To prove these we can argue as in parts (a) and (b), or apply the results of (a) and (b) to $-f$.

(d) Take $f(x) = x^3$ and $c = 0$. Then $f'(c) = 0$ but f has neither a local maximum nor a local minimum at c.

65. The function $f(x) = 1 - |x|$ has $f(-1) = f(1) = 0$, yet there is no $x \in (-1, 1)$ such that $f'(x) = 0$. Does this show Rolle's Theorem is wrong?

Solution: No. The function does not satisfy all the hypotheses of Rolle's Theorem as it is not differentiable everywhere in $(-1, 1)$ (it fails to be differentiable at 0).

66. Show that $f(x) = x^3 + ax + 1$ has a single real root when $a > 0$.

Solution:

We know that f has at least one real root as it is a cubic polynomial.

If there is a second root then, by Rolle's Theorem, there is a $c \in (a, b)$ such that $f'(c) = 0$. But f' never vanishes because $f'(x) = 3x^2 + a \geq a > 0$ for all x. Therefore there can't be a second root.

67. Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Prove that the equation $x^n + ax + b = 0$ has at most two distinct real roots when n is even, and at most three distinct real roots when n is odd.

Solution:

Let $f(x) = x^n + ax + b$. Clearly f is differentiable (it is a polynomial function).

Case 1: n is even.

We argue by contradiction. Suppose that the equation $f(x) = 0$ has more than two distinct solutions. Take three of them and call them x_1, x_2 and x_3 in increasing order, i.e. $x_1 < x_2 < x_3$. Since $f(x_1) = f(x_2) = 0$, there exists a $c \in (x_1, x_2)$ such that $f'(c) = 0$ (Rolle's Thm). Similarly, there exists a $d \in (x_2, x_3)$ such that $f'(d) = 0$. This shows that f' has at least two distinct real roots. But that's impossible because $f'(x) = nx^{n-1} + a$ and $n - 1$ is odd, therefore the equation $f'(x) = 0$ has exactly one root, namely $\sqrt[n-1]{-\frac{a}{n}}$.

Case 2: n is odd.

If $n = 1$ the result is obvious. Assume from now on that $n \geq 3$.

We argue by contradiction. Suppose that the equation $f(x) = 0$ has more than three distinct solutions. Take four of them and call them x_1, x_2, x_3 and x_4 in increasing order, i.e. $x_1 < x_2 < x_3 < x_4$. Since $f(x_1) = f(x_2) = 0$, there exists a $c_1 \in (x_1, x_2)$ such that $f'(c_1) = 0$ (Rolle's Thm). Similarly, there exist $c_2 \in (x_2, x_3)$, $c_3 \in (x_3, x_4)$ such that $f'(c_2) = f'(c_3) = 0$. This shows that f' has at least three distinct real roots. But that's impossible because $f'(x) = nx^{n-1} + a$ and $n-1$ is even. If $a > 0$ then f' has no real roots at all, if $a = 0$ then f' has only one real root, namely $x = 0$, and if $a < 0$ it has exactly two real roots, namely $\pm \sqrt[n-1]{-\frac{a}{n}}$.

68. What is the largest number in the sequence $1, \sqrt[2]{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots, \sqrt[n]{n}, \ldots$?

    Solution: Let $f : (0, \infty) \to \mathbb{R}$ be the function given by $f(x) = x^{\frac{1}{x}} = e^{\frac{\ln x}{x}}$, so this sequence is $\big(f(n)\big)_{n \in \mathbb{N}}$.

    f is differentiable with $f'(x) = \frac{1 - \ln x}{x^2} e^{\frac{\ln x}{x}}$, which vanishes at $x = e$. Moreover, $f'(x) > 0$ for $x < e$, $f'(x) < 0$ for $x > e$. Therefore $f(x)$ increases on $(0, e)$, decreases on (e, ∞) and $x = e$ is a global maximum point.

    Now $1, 2 \in (0, e)$ and $3, 4, \ldots \in (e, \infty)$, therefore $f(1) \leq f(2)$ and $f(3) \geq f(4) \geq \cdots$. Therefore, the largest number in the sequence $\big(f(n)\big)_{n \in \mathbb{N}}$ is $f(2)$ or $f(3)$. It is easy to check that $f(3) > f(2)$, i.e. $\sqrt[3]{3} > \sqrt[2]{2}$ (just raise both sides to the sixth power to find $3^2 > 2^3$).

    (My calculator says: $\sqrt[3]{3} = 1.4422\cdots$, $\sqrt[2]{2} = 1.4142\cdots$)

69. Prove that $\log x \leq x - 1$ for all $x > 0$.

    Solution: Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = x - 1 - \log x$. Then f is differentiable with $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. Since f' is negative in $(0, 1)$, vanishes at $x = 1$, and is positive in $(1, \infty)$, the function f is strictly decreasing in $(0, 1)$ and strictly increasing in $(1, \infty)$. It follows that f has a global minimum at $x = 1$ equal to $f(1) = 0$. Therefore, for all $x \in (0, \infty)$ we have $f(x) \geq f(0) = 0$, i.e. $x - 1 - \log x \geq 0$, which gives $x - 1 \geq \log x$. See figures 1 and 2 (graphs created at fooplot.com).

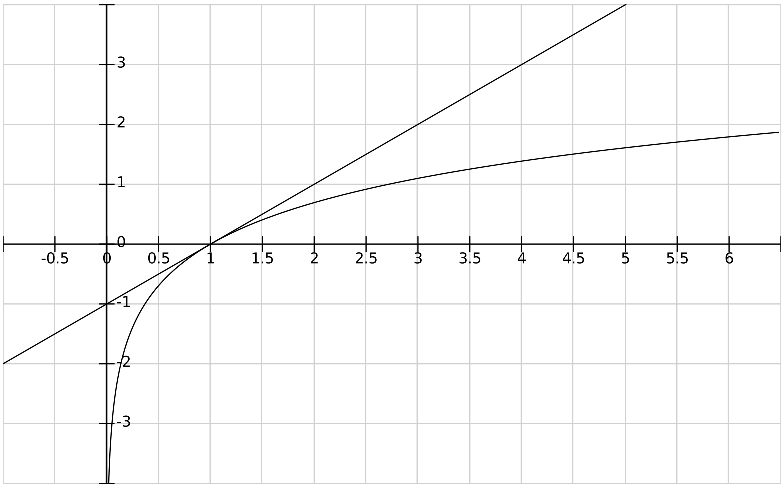

Figure 1: The graphs of $x-1$ and $\log x$.

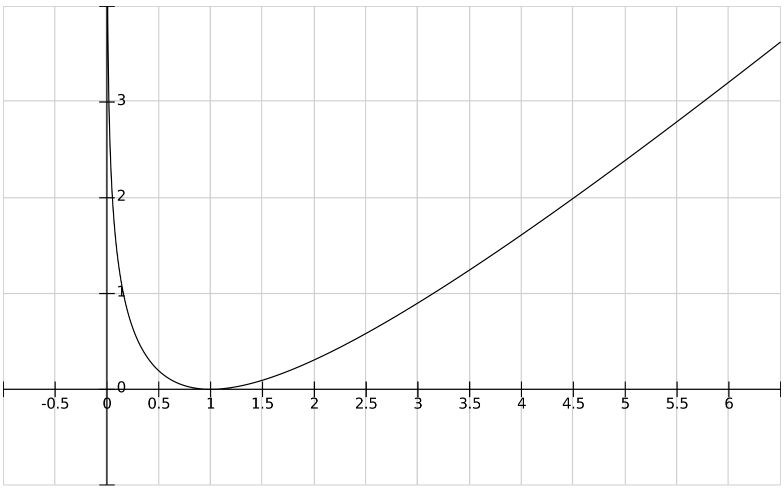

Figure 2: The graph of $x-1-\log x$.

Things to think about:
What is the equation of the tangent to the graph of $\log x$ at the point with $x=1$? Is the graph of $\log x$ below all its tangents? If so, why?

70. Show that $1+\frac{1}{4}x > \sqrt[4]{1+x}$ for $x>0$.

Solution:

Let $f:(-1,\infty)\to\mathbb{R}$ be given by $f(x)=1+\frac{1}{4}x-\sqrt[4]{1+x}$. Then f is differentiable.

In addition,

$$f'(x) = \frac{1}{4} - \frac{1}{4}(1+x)^{-\frac{3}{4}} = \frac{1}{4}\left(1 - \frac{1}{(1+x)^{3/4}}\right) > 0,$$

for all $x > 0$. Therefore f is strictly increasing on $(0,\infty)$, so $0 = f(0) < f(x)$ for all $x > 0$. This proves that

$$1 + \frac{1}{4}x > \sqrt[4]{1+x}$$

for all $x > 0$.

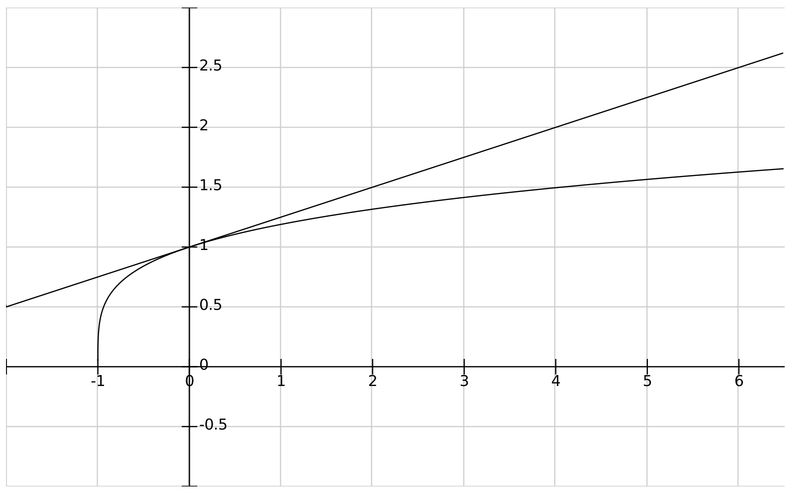

Figure 3: The graphs of $1 + \frac{1}{4}x$ and $\sqrt[4]{1+x}$. (graph done at fooplot.com)

71. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with f' bounded above, i.e. there exists a real number M such that for all $x \in \mathbb{R}$ we have $f'(x) \leq M$. Prove that for any $a, b \in \mathbb{R}$ with $a < b$ we have $f(b) - f(a) \leq M(b-a)$. (One way to prove this inequality is by integrating both sides of $f'(x) \leq M$ from a to b and using the Fundamental Theorem of Calculus. Since we haven't done any integration in FPM try to prove it using only the material in Wade's Chapters 1-4).

    Solution 1:

    By the MVT there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, therefore $f(b) - f(a) = f'(c)(b-a) \leq M(b-a)$.

    Solution 2: (Rough Work: Think of a as fixed and let b vary. Better replace b by x and try to show: for all $x \geq a$ we have $f(x) - f(a) \leq M(x-a)$, which is the same as $M(x-a) - f(x) + f(a) \geq 0$. The function

`M(x−a)−f(x)+f(a)` vanishes at $x = a$ and is increasing because its derivative is `M−f′(x) ≥ 0`. Done. END OF ROUGH WORK)

Fix a and b with $a < b$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = M(x-a) - f(x) + f(a)$. Then $g'(x) = M - f'(x) \geq 0$ for all $x \in \mathbb{R}$. Therefore g is increasing, therefore for all $x \geq a$ we have $g(x) \geq g(a) = 0$, which gives $M(x-a) \geq f(x) - f(a)$. Setting $x = b$ we obtain the required estimate $M(b-a) \geq f(b) - f(a)$.

72. If $f : (1, +\infty) \to \mathbb{R}$ is differentiable and $\left|f'(x)\right| \leq \frac{1}{x}$ for all x, show that

$$\lim_{x \to +\infty} \big(f(x+1) - f(x)\big) = 0.$$

Solution: By the Mean Value Theorem, for each x there is a c between x and $x+1$ such that $f(x+1) - f(x) = f'(c)\,\big((x+1) - x\big) = f'(c)$. Therefore,

$$\left|f(x+1) - f(x)\right| = \left|f'(c)\right| \leq \frac{1}{c} \leq \frac{1}{x} \to 0$$

as $x \to +\infty$.

73. ([Wade], Exercise 4.3.5) Suppose that f is differentiable on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

Solution:

The desired estimate is clearly true for $x = 0$. Fix $x \neq 0$. By the MVT there exists c between 0 and x such that $f'(c) = \frac{f(x)-f(0)}{x-0}$, therefore $f(x) = f(0) + f'(c)x$. It follows that $|f(x)| \leq |f(0)| + |f'(c)||x| \leq 1 + |x|$.

74. ([Wade], Exercise 4.3.6) Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b), continuous on $[a,b]$, and that $f(a) = f(b) = 0$. Prove that if $f(c) \neq 0$ for some $c \in (a,b)$, then there exist $x_1, x_2 \in (a,b)$ such that $f'(x_1)$ is positive and $f'(x_2)$ is negative.

Solution: We may assume that $f(c) > 0$ (if not, change f to $-f$). By the MVT there is an x_1 between a and c such that $f'(x_1) = \frac{f(c)-f(a)}{c-a} = \frac{f(c)}{c-a} > 0$, and an x_2 between c and b such that $f'(x_2) = \frac{f(b)-f(c)}{b-c} = \frac{-f(c)}{b-c} < 0$.

75. Is the 'converse' of the mean value theorem true? More precisely, let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Is it true that for every $c \in \mathbb{R}$ there exist $a, b \in \mathbb{R}$ with $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$?

Solution: No, it isn't. Take for example $f(x) = x^3$ and $c = 0$. We have $f'(0) = 0$ but for any $a, b \in \mathbb{R}$ with $0 \in (a,b)$ we have $\frac{f(b)-f(a)}{b-a} = \frac{b^3-a^3}{b-a} = b^2 + ba + a^2 > 0$.

Things to think about:
Why is $b^2 + ba + a^2$ positive when $0 \in (a, b)$? Is $b^2 + ba + a^2$ ever negative? Is it ever equal to zero?

76. (FPM Exam, May 2015, Problem 9)

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows

$$f(x) = \begin{cases} x^p \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Show that f is continuous at $x = 0$ iff $p > 0$.

(b) Show that f is differentiable at $x = 0$ if $p > 1$, and find $f'(0)$.

(c) Calculate $f'(x)$ for $x \neq 0$.

(d) Consider $p = 2$. Is f' continuous at $x = 0$?

Solution:

(a) Assume first that $p > 0$. Then $\left|f(x)\right| = |x|^p |\sin(1/x^2)| \leq |x|^p$. Since $|x|^p \xrightarrow[x\to 0]{} 0$, it follows that $f(x) \xrightarrow[x\to 0]{} 0 = f(0)$, therefore f is continuous at $x = 0$.

If $p = 0$ then $f(x) = \sin(1/x^2)$ for $x \neq 0$. Let

$$x_k = \left(\frac{1}{\frac{\pi}{2} + 2k\pi}\right)^{1/2}, \quad (k = 1, 2, 3, \ldots).$$

Then $x_k \to 0$ as $k \to \infty$, but $f(x_k) \not\to f(0)$, because

$$f(x_k) = \sin\left(\frac{\pi}{2} + 2k\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1, \ (k = 1, 2, 3 \ldots).$$

Therefore f is not continuous at $x = 0$.

Finally, assume $p < 0$ and let (x_k) be as above. Then $x_k \to 0$ as $k \to \infty$, but $f(x_k) \not\to f(0)$, because

$$f(x_k) = x_k^p \sin(x_k) = x_k^p = \left(\frac{\pi}{2} + 2k\pi\right)^{-p/2} \xrightarrow[k\to\infty]{} +\infty.$$

Therefore f is not continuous at $x = 0$.

(b) Assume $p > 1$. Then

$$\frac{f(x)-f(0)}{x-0} = x^{p-1}\sin(1/x^2) \xrightarrow[x\to 0]{} 0$$

because $x^{p-1} \to 0$ and $\sin(1/x^2)$ is bounded. Therefore f is differentiable at 0 and the derivative is

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0.$$

(c) For $x \neq 0$ we have

$$f'(x) = px^{p-1}\sin(1/x^2) + x^p\cos(1/x^2)(-2x^{-3}) = px^{p-1}\sin(1/x^2) - 2x^{p-3}\cos(1/x^2).$$

(d) Let $p = 2$. From (b) and (c) we know that

$$f'(x) = \begin{cases} 2x\sin(1/x^2) - \frac{2}{x}\cos(1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Let

$$x_k = \left(\frac{1}{2k\pi}\right)^{1/2}, \ k = 1,2,3,..$$

Then $x_k \to 0$ but $f'(x_k) \not\to f'(0) = 0$. Indeed, $\sin(1/x_k^2) = 0$ and $\cos(1/x_k^2) = 1$ for all k, therefore

$$f'(x_k) = -\frac{2}{x_k} \to -\infty \ as \ k \to \infty.$$

Therefore, f' is not continuous at $x = 0$.

77. Let $\alpha > 0$. A real valued function f defined on some interval I is said to be *Hölder continuous with exponent* α iff there exists a constant $C > 0$ such that for all $x, y \in I$ we have

$$\big|f(x) - f(y)\big| \leq C|x-y|^{\alpha}.$$

Prove the following statements:

(a) If f is Hölder continuous then f is continuous.

(b) The function $f : [0,\infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is Hölder continuous with exponent $1/2$.

(c) If f is Hölder continuous with exponent α and $\alpha > 1$ then f is constant.

Solution:

(a) Fix $x_0 \in I$. For all $x \in I$ we have

$$0 \leq \left|f(x) - f(x_0)\right| \leq C\left|x - x_0\right|^\alpha.$$

By the Sandwich Theorem, $\left|f(x) - f(x_0)\right| \to 0$ as $x \to x_0$, therefore $f(x) \to f(x_0)$ as $x \to x_0$.

(b) This says: for all $x, y \geq 0$, $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x-y|}$. The proof is easy and is left to the reader.

(c) Fix $x \in I$. For all $h \neq 0$ with $x + h \in I$ we have

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq \frac{C\left|(x+h) - x\right|^\alpha}{|h|} = C\left|h\right|^{\alpha-1}.$$

Since $\alpha > 1$ we have $|h|^{\alpha-1} \to 0$ as $h \to 0$. Therefore $\frac{f(x+h)-f(x)}{h} \to 0$ as $h \to 0$. It follows that f is differentiable at x with $f'(x) = 0$. Since x was arbitrary, this argument shows that f is differentiable everywhere and that its derivative is zero. Therefore f is constant.

78. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz continuous* iff it is Hölder continuous with exponent $\alpha = 1$, i.e. iff there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}$ we have

$$\left|f(x) - f(y)\right| \leq C|x-y|. \tag{3}$$

Prove that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous iff f' is bounded, i.e. there exists a constant C' such that for all $x \in \mathbb{R}$ we have

$$\left|f'(x)\right| \leq C'. \tag{4}$$

Solution: Suppose that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Then there exists a positive constant C such that for all x, y we have $\left|f(x) - f(y)\right| \leq C|x-y|$. Fix $x \in \mathbb{R}$. For all $h \neq 0$ we then have

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq \frac{C\left|(x+h) - x\right|}{|h|} = C.$$

Letting $h \to 0$ we find $\left|f'(x)\right| \leq C$. This shows that f' is a bounded function ((4) is satisfied with $C' = C$).

Conversely, suppose that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ has bounded f', i.e. that there is a positive constant C' such that for all x, $\left|f'(x)\right| \leq C'$. By the MVT, for all $x, y \in \mathbb{R}$ there is a c between them such that $f(x) - f(y) = f'(c)(x-y)$, therefore $\left|f(x) - f(y)\right| = \left|f'(c)\right| |x-y| \leq C' |x-y|$. This shows that f is Lipschitz continuous ((3) is satisfied with $C = C'$).

Remark 1. Observe that in both parts of the proof we ended up with $C = C'$.

79. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose graph doesn't pass through the origin. Suppose that the point $P = (a, f(a))$ on the graph of f is closest to the origin. Show that the straight line from the origin to P is perpendicular to the curve $y = f(x)$ (meaning that the angle between that straight line and the tangent to the graph at P is $90°$).

    Solution:

    The distance from the point $(x, f(x))$ on the graph of f to the origin $(0,0)$ is $d(x) = \sqrt{x^2 + f(x)^2}$. At the point $P = (a, f(a))$ the distance is minimized therefore the square of the distance is minimized as well[2]. Therefore the derivative of $d(x)^2$ at $x = a$ is zero. We have

    $$\left(d(x)^2\right)' = \left(x^2 + f(x)^2\right)' = 2x + 2f(x)f'(x).$$

    Therefore,

    $$2a + 2f(a)f'(a) = 0. \tag{5}$$

    Assume for the moment that $a \neq 0$. The straight line through the origin and the point $P = (a, f(a))$ is given by $y = \frac{f(a)}{a}x$ and has slope $\frac{f(a)}{a}$. The tangent to the graph of f at P has slope $f'(a)$. Using (5), the product of these two slopes is $f'(a)\frac{f(a)}{a} = -1$, therefore the two lines meet at $90°$.

    If $a = 0$ then the point $P = (0, f(0))$ is on the y-axis. Equation (5) becomes $f(0)f'(0) = 0$. Since $f(0) \neq 0$ (because the graph of f doesn't pass through the origin), it follows that $f'(0) = 0$, therefore the tangent to the graph of f at P is horizontal, therefore it meets the y-axis at $90°$.

80. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable with $f(0) = f'(0) = 0$ and satisfying the ordinary differential equation $f'' + f = 0$. Show that $f = 0$. (Hint 1: Calculate the derivatives of the functions $g(x) = f(x)\cos x - f'(x)\sin x$ and $h(x) = f(x)\sin x + f'(x)\cos x$. Hint 2: Consider the function $f^2 + (f')^2$.)

    Solution 1: Let $g(x) = f(x)\cos x - f'(x)\sin x$. Then

    $$g'(x) = f'(x)\cos x + f(x)(-\sin x) - f''(x)\sin x - f'(x)\cos x = -(f(x) + f''(x))\sin x = 0$$

    for all x, therefore g is constant. Since $g(0) = 0$, we have $g(x) = 0$ for all x, i.e.

    $$f(x)\cos x - f'(x)\sin x = 0, \tag{6}$$

[2] We use the square of the distance in order to avoid differentiating the square root.

for all x. Now let $h(x) = f(x)\sin x + f'(x)\cos x$. Then

$$h'(x) = f'(x)\sin x + f(x)\cos x + f''(x)\cos x + f'(x)(-\sin x) = (f(x) + f''(x))\cos x = 0$$

for all x. Therefore h is constant. Since $h(0) = 0$ it follows that

$$f(x)\sin x + f'(x)\cos x = 0 \tag{7}$$

for all x. Multiply (6) by $\cos x$ and (7) by $\sin x$ and add up to find

$$f(x)(\cos^2 x + \sin^2 x) = 0,$$

i.e. $f(x) = 0$, for all x.

Solution 2: Multiply the equation $f + f'' = 0$ by f' to get $ff' + f'f'' = 0$, i.e. $\left(\frac{1}{2}f^2 + \frac{1}{2}(f')^2\right)' = 0$. It follows that $f^2 + (f')^2$ is constant, and since $f(0)^2 + f'(0)^2 = 0$, we have $f^2 + (f')^2 = 0$, hence $f = 0$.

81. (A Uniqueness Theorem for the ODE $f'' + f = 0$.) Let $f, g : \mathbb{R} \to \mathbb{R}$ be twice differentiable functions that satisfy $f'' + f = 0$ and $g'' + g = 0$, and have the same 'initial conditions', i.e. $f(0) = g(0)$ and $f'(0) = g'(0)$. Show that $f = g$.

    Solution: Apply the result of Problem 80 to $f - g$.

Remark 2. The ODE $x'' + x = 0$ describes a simple harmonic oscillator. In Physics it is usually written in the form $mx'' = -kx$, where $x = x(t)$ denotes displacement as a function of time, the positive constant m represents mass and the positive constant k is known as the spring constant (if we are talking about a mass attached to one end of a spring). Pure mathematicians shamelessly set all constants equal to 1. It is easy to check that one solution to

$$mx'' = -kx,\ x(0) = x_0,\ x'(0) = x_1, \tag{8}$$

where $x_0, x_1 \in \mathbb{R}$ are two given nunbers, is

$$x(t) = A\cos(\omega t - \phi)$$

where $\omega = \sqrt{\frac{k}{m}}$, $A = \sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}$ and ϕ is calculated from $\cos\phi = \frac{x_0}{A}$, $\sin\phi = \frac{x_1}{A\omega}$. Problem 81 says that there are no other solutions to (8).

Things to think about:
What property of the ODE $f'' + f = 0$ allows us to reduce the proof of the uniqueness result of Problem 81 to that of Problem 80?

References

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