**True or False? Give a proof or a counterexample.**

1. If $A \subseteq \mathbb{R}$ is non-empty and bounded and $\inf A = \sup A$ then $A$ has exactly one element.

   True.

   Proof: We argue by contradiction. Suppose that $\inf A = \sup A$ but $A$ has more than one elements. Take two of them, call them $a$ and $b$ and assume (without loss of generality) that $a < b$. Then $\inf A \leq a$ (because $\inf A$ is a lower bound of $A$) and $b \leq \sup A$ (because $\sup A$ is an upper bound of $A$). Therefore $\inf A < \sup A$; contradiction

2. If $A \subseteq \mathbb{R}$ has ten elements then $\sup A \in A$.

   True.

   Proof: If $A$ has ten elements then $A$ has a maximum element, call it $a_0$. Then $\sup A$ is equal to $a_0$ therefore $\sup A \in A$.

3. If $A \subseteq \mathbb{R}$ is non-empty and bounded and every element of $A$ is an integer then $\sup A \in A$.

   True.

   Proof: If $A \subseteq \mathbb{R}$ is non-empty and bounded and every element of $A$ is an integer then $A$ is a finite set. Working as in the last question we can show that $\sup A \in A$.

4. If $a_n^2 \to a^2$ then $a_n \to a$.

   False.

   Counterexample: $a_n = (-1)^n$, $a = 1$.

5. If $a_n^3 \to a^3$ then $a_n \to a$.

   True.

   Proof: (Now that we have developed all the theory we have more tools available than when this question was first asked several months ago)

   Assume $a_n^3 \to a^3$. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt[3]{x}$ is continuous, therefore $f(a_n^3) \to f(a^3)$, i.e. $a_n \to a$.

6. If $a_n \to a$ then $|a_n| \to |a|$.

   True. This is a well known property of limits.

7. If $|a_n| \to |a|$ then $a_n \to a$.

   False. Counterexample: $a_n = (-1)^n$, $a = 1$.

8. If $a_n \to a$ then for every $\varepsilon > 0$ the interval $(a-\varepsilon, a+\varepsilon)$ contains infinitely many terms of $(a_n)$.

   True.

   Proof: If $a_n \to a$ and $\varepsilon > 0$ then there is an index $N$ such that for all $n \geq N$ we have $a_n \in (a-\varepsilon, a+\varepsilon)$, therefore the interval $(a-\varepsilon, a+\varepsilon)$ contains infinitely many terms of the sequence.

9. If for every $\varepsilon > 0$ the interval $(a-\varepsilon, a+\varepsilon)$ contains infinitely many terms of $(a_n)$, then $a_n \to a$.

   False.

   Counterexample: $a_n = (-1)^n$, $a = 1$. For every $\varepsilon > 0$ the interval $(1-\varepsilon, 1+\varepsilon)$ contains infinitely many terms of the sequence, namely $a_2, a_4, a_6, \ldots$, but $a_n \not\to 1$.

10. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}. \tag{1}$$

   Is the following proof that $a_n \to 0$ correct?

   Proof: Since the limit of a sum is equal to the sum of the limits we have

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right) && (2)\\
&= \lim_{n\to\infty}\frac{1}{n+1} + \lim_{n\to\infty}\frac{1}{n+2} + \lim_{n\to\infty}\frac{1}{n+3} + \cdots + \lim_{n\to\infty}\frac{1}{2n} && (3)\\
&= 0+0+0+\cdots+0\\
&= 0
\end{aligned}$$

   Answer: The proof is wrong. The Theorem that says that the limit of a sum is equal to the sum of the limits talks about sums with a fixed number of terms. Here the number of terms varies with $n$.

11. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is discontinuous then $f+g$ is discontinuous.

   True.

Proof: If $f+g$ were continuous then $g=(f+g)-f$ would be continuous as the difference of two continuous function.

12. If $f:\mathbb{R}\to\mathbb{R}$ is continuous and $g:\mathbb{R}\to\mathbb{R}$ is discontinuous then $fg$ is discontinuous.

    False.

    Counterexample: Take $f=0$ and $g$ any discontinuous functions. Then $fg=0$ is continuous.