Introductory Applied Machine Learning

Naïve Bayes

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Overview

- Naïve Bayes classifier
  - components and their function
  - independence assumption
  - dealing with missing data
- Continuous example
- Discrete example
- Pros and cons

Bayesian classification

- Goal: learning function *f(x)* → *y*
  - y ... one of k classes (e.g. spam/ham, digit 0-9)
  - $x = x_1 \ldots x_n$ – values of attributes (numeric or categorical)
- Probabilistic classification:
  - most probable class given observation: $\hat{y} = \arg\max_y P(y|x)$
- Bayesian probability of a class:

$$P(y|x) = \frac{\overbrace{P(x|y)}^{\text{class model}}\overbrace{P(y)}^{\text{prior}}}{\underbrace{\sum_{y'} P(x|y')P(y')}_{\text{normalizer } P(x)}}$$

Bayesian classification: components

$$P(y|x) = \frac{P(x|y)P(y)}{\sum_{y'} P(x|y')P(y')}$$

Example:
y … patient has Avian flu
x … observed symptoms

- P(*y*): prior probability of each class
  - encodes how which classes are common, which are rare
  - apriori much more likely to have common cold than Avian flu
- P(*x*|*y*): class-conditional model
  - describes how likely to see observation *x* for class y
  - assuming it's Avian flu, do the symptoms look plausible?
- P(*x*): normalize probabilities across observations
  - does not affect which class is most likely (**arg max**)

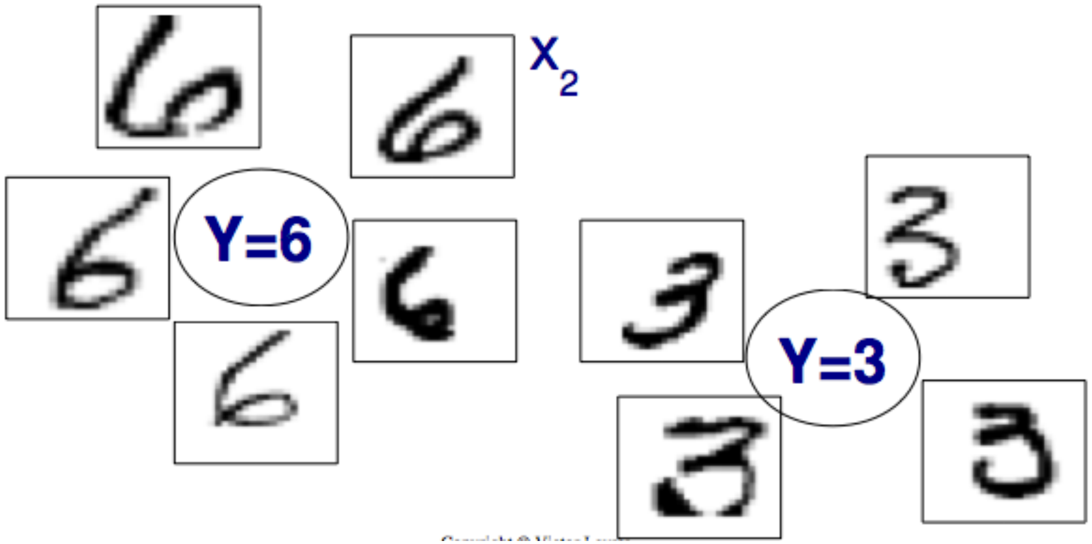
Bayesian classification: normalization

Normalizer: $P(x) = \sum_{y'} P(x|y')P(y')$

- an "outlier" has a low probability under every class

$P(X=x_1 \mid Y=3) < P(X=x_2 \mid Y=3)$

x_1



normalizer makes *$P(Y=3|X=x_1)$* comparable to non-outliers

Naïve Bayes: a generative model

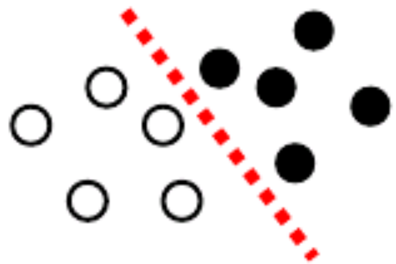
- A complete probability distribution for each class
  - defines likelihood for any point *x*
  - P(class) via P(observation)

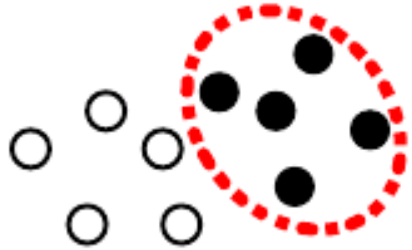
$$P(y|x) \propto P(x|y)\,P(y)$$

  - can "generate" synthetic observations
    - will share many properties of the original data
- Not all probabilistic classifiers do this
  - possible to estimate P(*y*|*x*) directly
  - e.g. logistic regression:

$$P(y|x) = \frac{1}{z_y} \exp\left(\sum_i \lambda_i g_i(y,x)\right)$$

discriminative

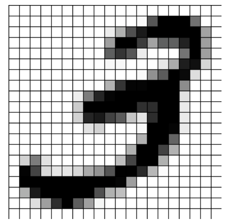




generative

Independence assumption

- Compute $P\,(x_1 \ldots x_n | y)$ for every observation $x_1 \ldots x_n$
  - class-conditional "counts", based on training data
  - problem: may not have seen every $x_1 \ldots x_n$ for every y
    - digits: 2^{400} possible black/white patterns (20x20)
    - spam: every possible combination of words: $2^{10,000}$
  - often have observations for individual x_i for every class



- idea: assume $x_1 \ldots x_n$ conditionally independent given y

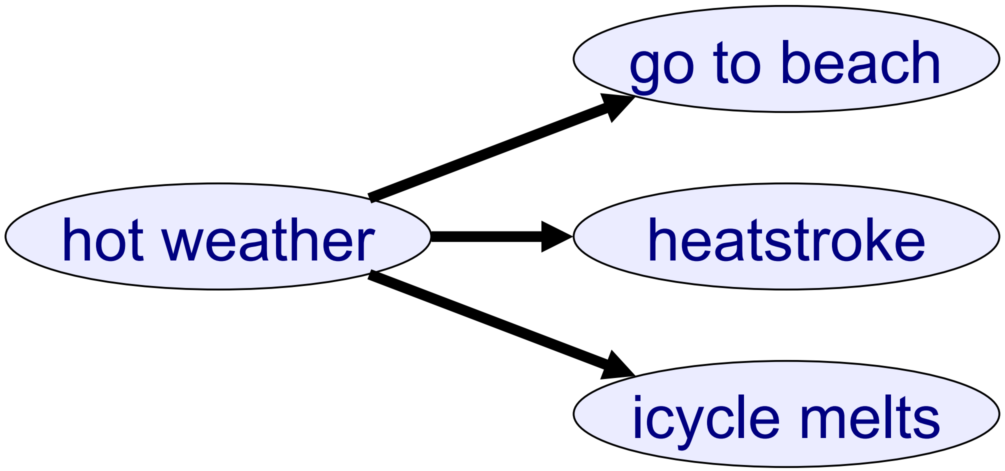
$$P(x_1 \ldots x_d | y) = \underbrace{\prod_{i=1}^{d} P(x_i | x_1 \ldots x_{i-1}, y)}_{\text{chain rule (exact)}} = \underbrace{\prod_{i=1}^{d} P(x_i | y)}_{\text{independence}}$$

Conditional independence

- Probabilities of going to the beach and getting a heat stroke are not independent: $P(B,S) > P(B)\,P(S)$
- May be independent if we know the weather is hot

$$P(B,S|H) = P(B|H)\,P(S|H)$$

- Hot weather "explains" all the dependence between beach and heatstroke
- In classification:
  - class value explains all the dependence between attributes



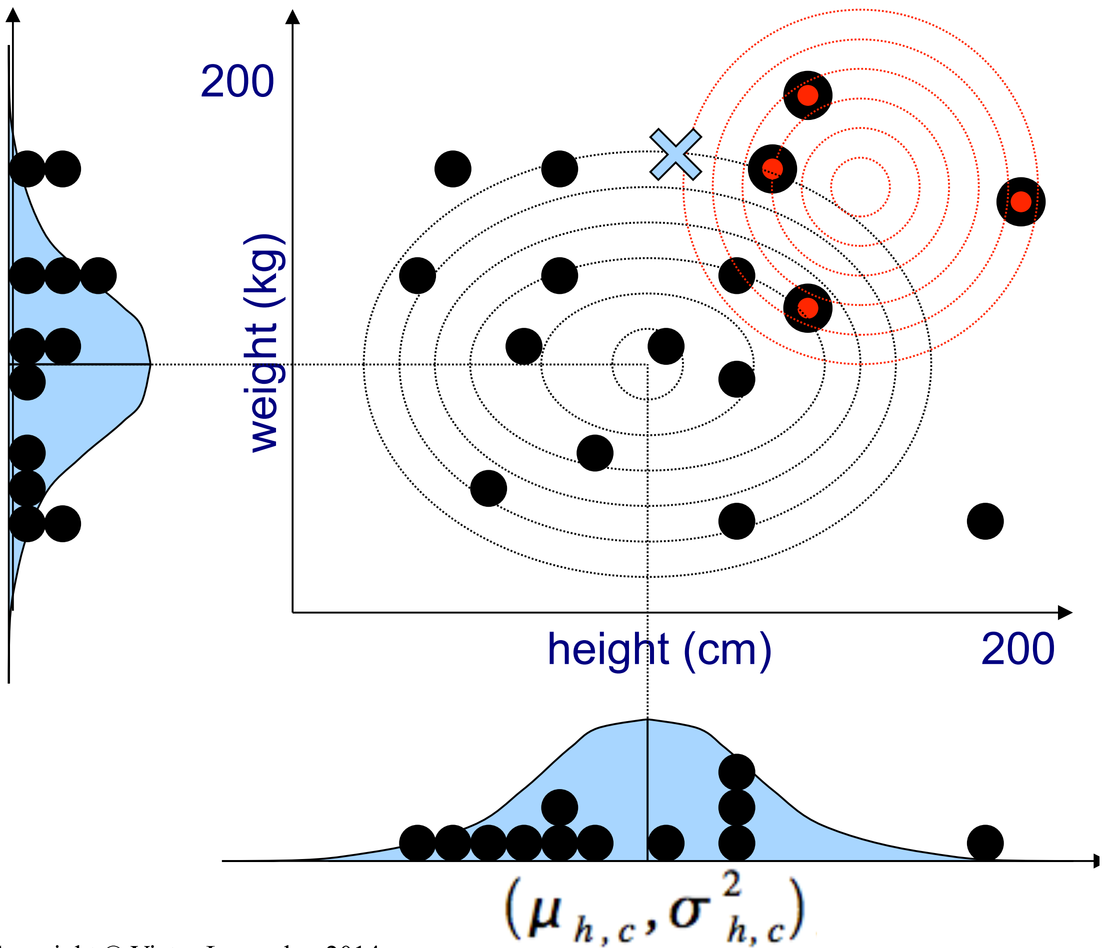
Example © Amos Storkey, 2007

Continuous example

- Distinguish children from adults based on size
  - classes: {*a,c*}, attributes: height [cm], weight [kg]
  - training examples: {h_i, w_i, y_i}, 4 adults, 12 children
- Class probabilities: $P(a)=\frac{4}{4+12}=0.25\ ; P(c)=0.75$
- Model for adults:
  - height ~ Gaussian with mean, variance $\begin{cases} \mu_{h,a}=\frac{1}{4}\sum_{i:y_i=a} h_i \\ \sigma^2_{h,a}=\frac{1}{4}\sum_{i:y_i=a} (h_i-\mu_{h,a})^2 \end{cases}$
  - weight ~ Gaussian $\left(\mu_{w,a},\sigma^2_{w,a}\right)$
  - assume height and weight independent
- Model for children: same, using $(\mu_{h,c},\sigma^2_{h,c}),(\mu_{w,c},\sigma^2_{w,c})$

Continuous example

$$P(a) = \frac{4}{4+12} = 0.25 \,;\, P(c) = 0.75$$



$$p(h_x|c) = \frac{1}{\sqrt{2\pi\sigma^2_{h,c}}} \exp - \frac{1}{2}\left(\frac{(h_x - \mu_{h,c})^2}{\sigma^2_{h,c}}\right)$$

$$p(w_x|c) = \frac{1}{\sqrt{2\pi\sigma^2_{w,c}}} \exp - \frac{1}{2}\left(\frac{(w_x - \mu_{w,c})^2}{\sigma^2_{w,c}}\right)$$

$$p(h_x|a) = \frac{1}{\sqrt{2\pi\sigma^2_{h,a}}} \exp - \frac{1}{2}\left(\frac{(h_x - \mu_{h,a})^2}{\sigma^2_{h,a}}\right)$$

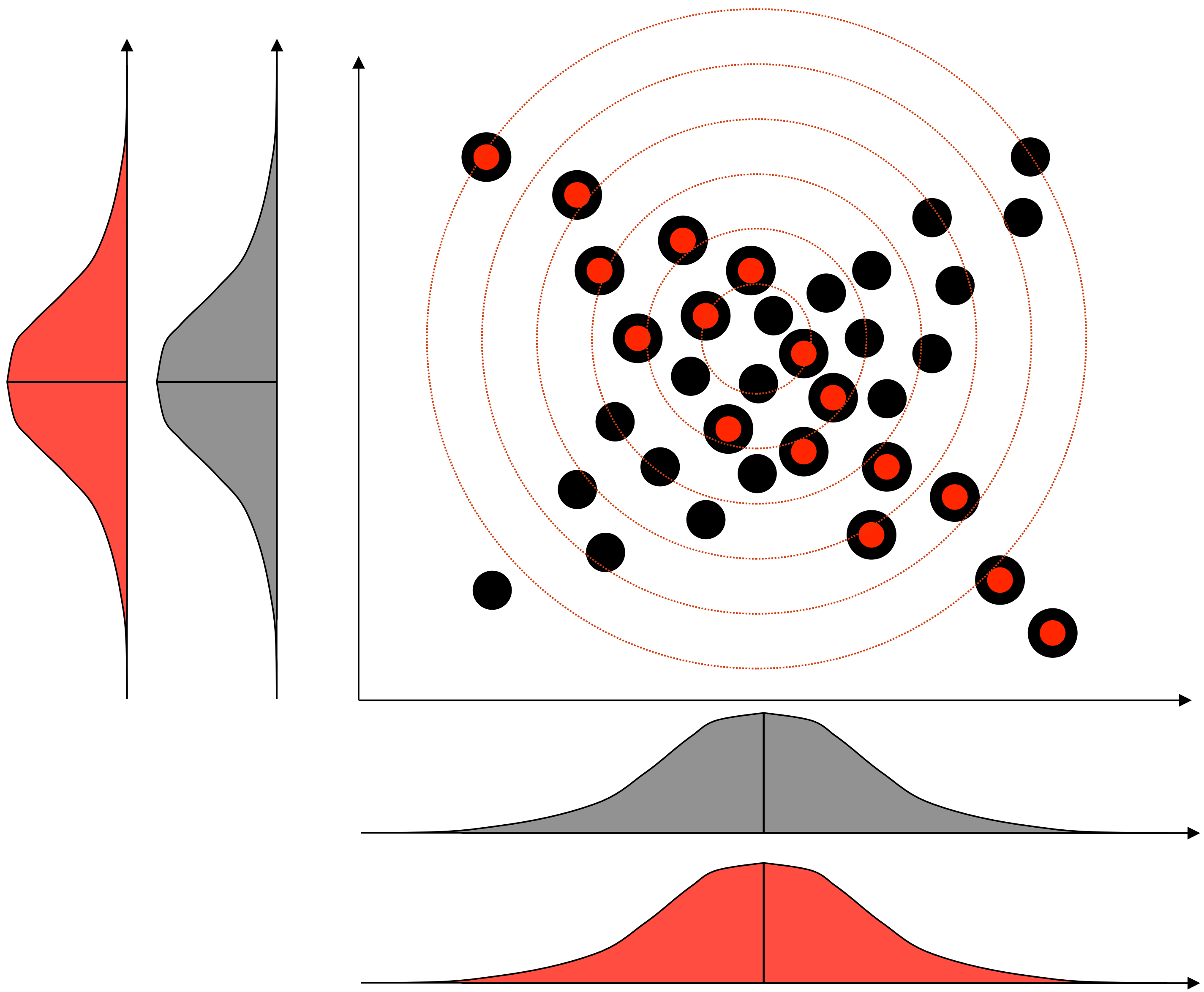
$$p(w_x|a) = \frac{1}{\sqrt{2\pi\sigma^2_{w,a}}} \exp - \frac{1}{2}\left(\frac{(w_x - \mu_{w,a})^2}{\sigma^2_{w,a}}\right)$$

$$P(x|a) = p(h_x|a)\, p(w_x|a)$$

$$P(x|c) = p(h_x|c)\, p(w_x|c)$$

$$P(a|x) = \frac{P(x|a)P(a)}{P(x|a)P(a) + P(x|c)P(c)}$$

Problems with Naïve Bayes



Discrete example: spam

- Separate spam from valid email, attributes = words

D1: "send us your password"	spam
D2: "send us your review"	ham
D3: "review your password"	ham
D4: "review us"	spam
D5: "send your password"	spam
D6: "send us your account"	spam

P (spam) = 4/6   P (ham) = 2/6

spam	ham	
2/4	1/2	password
1/4	2/2	review
3/4	1/2	send
3/4	1/2	us
3/4	1/2	your
1/4	0/2	account

new email: "review us now"

$$P(\text{review us} \mid \text{spam}) = P(0,1,0,1,0,0 \mid \text{spam}) = (1-\frac{2}{4})(\frac{1}{4})(1-\frac{3}{4})(\frac{3}{4})(1-\frac{3}{4})(1-\frac{1}{4})$$

$$P(\text{review us} \mid \text{ham}) = P(0,1,0,1,0,0 \mid \text{ham}) = (1-\frac{1}{2})(\frac{2}{2})(1-\frac{1}{2})(\frac{1}{2})(1-\frac{1}{2})(1-\frac{0}{2})$$

$$P(\text{ham} \mid \text{review us}) = \frac{0.0625 \times 2/6}{0.0625 \times 2/6 + 0.0044 \times 4/6} = 0.87$$ (note identical example)

Problems with Naïve Bayes

- Zero-frequency problem
  - any mail containing “account” is spam: P(account|ham) = 0/2
  - solution: never allow zero probabilities
    - Laplace smoothing: add a small positive number to all counts: $$P(w|c)=\frac{num(w,c)+\epsilon}{num(c)+2\epsilon}$$
    - may use global statistics in place of ε: *num(w) / num*
  - very common problem (Zipf's law: 50% words occur once)
- Assumes word independence
  - every word contributes independently to P(spam|email)
  - fooling NB: add lots of “hammy” words into spam email

Missing data

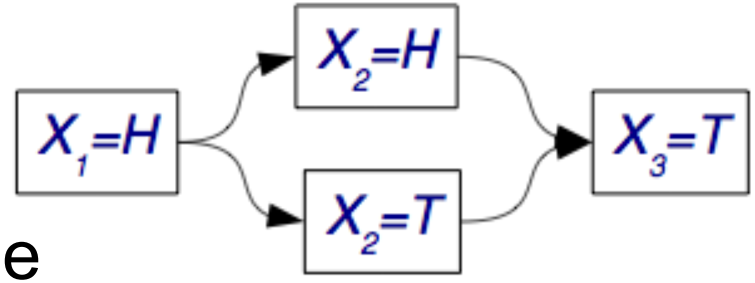
- Suppose don't have value for some attribute X_i
  - applicant's credit history unknown
  - some medical test not performed on patient
  - how to compute $P(X_1=x_1 \ldots X_j=? \ldots X_d=x_d \mid y)$
- Easy with Naïve Bayes
  - ignore attribute in instance where its value is missing

$$P(x_1 \ldots \boxed{X_j} \ldots x_d | y) = \prod_{\boxed{i \neq j}}^{d} P(x_i | y)$$

  - compute likelihood based on observed attribtues
  - no need to "fill in" or explicitly model missing values
  - based on conditional independence between attributes

Missing data (2)

- Ex: three coin tosses: Event = $\{X_1$=H, X_2=?, X_3=T$\}$
  - event = head, unknown (either head or tail), tail
  - event = {H,H,T} + {H,T,T}
  - P(event) = P(H,H,T) + P(H,T,T)



- General case: X_j has missing value

$$P(x_1 \ldots \boxed{x_j} \ldots x_d | y) = P(x_1|y) \cdots \boxed{P(x_j|y)} \cdots P(x_d|y)$$

$$\sum_{\boxed{x_j}} P(x_1 \ldots \boxed{x_j} \ldots x_d | y) = \sum_{x_j} P(x_1|y) \cdots \boxed{P(x_j|y)} \cdots P(x_d|y)$$

$$= P(x_1|y) \cdots \boxed{\left[\sum_{x_j} P(x_j|y)\right]} \cdots P(x_d|y)$$

$$= P(x_1|y) \cdots \boxed{[1]} \cdots P(x_d|y)$$