Fundamentals of Pure Mathematics 2015-16
Analysis Problems for weeks 1-2.

Suggested problems for the Analysis workshop in week 2: 12, 13, 14. If time permits, any of the following: 15, 16, 18, 19.

Mathematical Logic

1. Decide whether the following statements are true or false. Prove the true ones and give counterexamples for the false ones. Here x denotes a real number.

   (a) $x > 1 \Rightarrow x^2 > 1$;

   (b) $x^2 > 1 \Rightarrow x > 1$;

   (c) $x^2 \leq 1 \Rightarrow x \leq 1$;

   (d) $x \leq 1 \Rightarrow x^2 \leq 1$.

   Solution:

   (a) True. If $x > 1$ then $x^2 - 1 = (x+1)(x-1)$ is the product of two positive reals, therefore it is positive.

   (b) False. Counterexample: $x = -2$.

   (c) True. If $x^2 \leq 1$ then $\sqrt{x^2} \leq 1$, i.e. $|x| \leq 1$, therefore $x \leq 1$.

   (d) False. Counterexample: $x = -2$.

2. Let P and Q be mathematical statements (e.g. $x > 1$). The converse of the statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$. The contrapositive of $P \Rightarrow Q$ is $(not\ Q) \Rightarrow (not\ P)$. If an implication is true, is its converse necessarily true? What about its contrapositive?

   Solution: If an implication is true, its converse is not necessarily true. See for example (a) and (b) in Problem 1 above.

   The contrapositive of an implication is equivalent to the implication itself; they are both true or both false (Liebeck, Chapter 1).

3. Decide whether each of the following statements is True or False.

$$4 \text{ is even } \Rightarrow \text{ 7 is prime;}$$

$$4 \text{ is even } \Rightarrow \text{ 6 is prime;}$$

$$4 \text{ is odd } \Rightarrow 7 \text{ is prime;}$$
$$4 \text{ is odd } \Rightarrow 6 \text{ is prime.}$$

Solution:

The truth table for $\Rightarrow$ is:

P	Q	P $\Rightarrow$ Q
T	T	T
T	F	F
F	T	T
F	F	T

Therefore, the statement

$$4 \text{ is even} \Rightarrow 6 \text{ is prime}$$

is False, the rest of them are True.

4. Decide whether each of the following statements is True or False.

   (i) For all real numbers x, there exists a real number y such that $x+y>0$.

   (ii) There exists a real number x such that for all real numbers y, $x+y>0$.

   Solution:

   (i) True. Given any real number x, set $y=-x+1$ to find $x+y=1>0$.

   (ii) False. We argue by contradiction. Suppose the statement is true, i.e. suppose that there is an $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$ we have $x+y>0$. Set $y=-x-1$. Then $x+y=-1<0$; contradiction.

5. Write down the negations of the statements in Problem 4.

   Solution: The negation of a statement of the form

   $$\forall x \; p(x)$$

   is

   $$\exists x \;\; \overline{p(x)},$$

   where $\overline{p(x)}$ is denotes *not* $p(x)$, the negation of $p(x)$.

   For example, the negation of the statement $\forall x \, x^2>1$ is $\exists x \, x^2 \leq 1$.

The negation of a statement of the form

$$\exists x\ p(x)$$

is

$$\forall x\ \ \overline{p(x)}.$$

For example, the negation of the statement $\exists x\ x > 0$ is $\forall x\ x \leq 0$.

Combining these two cases we see that the negation of a statement of the form

$$\forall x\ \exists y\ \ p(x,y)$$

is

$$\exists x\ \ \overline{\exists y\ \ p(x,y)}$$

which is the same as

$$\exists x\ \ \forall y\ \ \overline{p(x,y)}.$$

Similarly, the negation of a statement of the form

$$\exists x\ \forall y\ \ p(x,y)$$

is

$$\forall x\ \ \overline{\forall y\ \ p(x,y)}$$

which is the same as

$$\forall x\ \ \exists y\ \ \overline{p(x,y)}.$$

The negation of statement (i) is: there exists a real number x such that for all real numbers y we have $x+y \leq 0$.

The negation of statement (ii) is: for all real numbers x there exists a real number y such that $x+y \leq 0$.

Real Numbers (Wade, Chapter 1)

6. (Triangle Inequalities) Prove that for all real numbers a, b we have

$$\big||a|-|b|\big| \leq |a-b| \leq |a|+|b| \tag{1}$$

and

$$\big||a|-|b|\big| \leq |a+b| \leq |a|+|b|\,. \tag{2}$$

Solution: Recall first that for all real numbers x we have $x \leq |x|$. We first prove

$$|a-b| \leq |a|+|b|\,. \tag{3}$$

If $a \geq b$ then $|a-b| = a-b = a+(-b) \leq |a|+|-b| = |a|+|b|$. The proof in the case $a \leq b$ is similar (or simply observe that (3) is symmetric with respect to a and b).

Next we prove

$$\big||a|-|b|\big| \leq |a-b|\,. \tag{4}$$

This follows easily from (3). Indeed, if $|a| \geq |b|$ then (4) is the same as

$$|a|-|b| \leq |a-b|\,,$$

which is the same as

$$|a| \leq |b|+|a-b|\,.$$

To prove the last inequality simply write a as $(a-b)+b$ and apply (3):

$$|a| = \big|(a-b)+b\big| \leq |a-b|+|b|\,.$$

If $|b| \leq |a|$ the proof is similar (or just change the roles of a and b). This completes the proof of (1). Changing b to $-b$ in (1) gives (2).

7. ([Wade], Exercise 1.2.3) For $x \in \mathbb{R}$ we define x^+ and x^- as follows:

$$x^+ = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}, \quad x^- = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

x^+ and x^- are known as the positive and negative parts of x respectively. Prove that:

(a) $x^+ \geq 0$ and $x^- \geq 0$ (NB: both the positive part and the negative part are ≥ 0),

(b) $x = x^+ - x^-$ (as a consequence, every real number can be written as the difference of two non-negative real numbers),

(c) $|x| = x^+ + x^-$.

(d) $x^+ = \frac{|x|+x}{2}$ and $x^- = \frac{|x|-x}{2}$.

Solution:

(a) If $x \geq 0$ then $x^+ = x \geq 0$ and $x^- = 0 \geq 0$. If $x < 0$ then $x^+ = 0 \geq 0$ and $x^- = -x > 0$. Therefore in all cases both x^+ and x^- are ≥ 0.

(b) If $x \geq 0$ then $x^+ - x^- = x - 0 = x$. If $x < 0$ then $x^+ - x^- = 0 - (-x) = x$.

(c) If $x \geq 0$ then $x^+ + x^- = x + 0 = x = |x|$. If $x < 0$ then $x^+ + x^- = 0 + (-x) = -x = |x|$.

(d) We have shown above that $|x| = x^+ + x^-$ and $x = x^+ - x^-$. Adding them together gives $|x| + x = 2x^+$, therefore $x^+ = \frac{|x|+x}{2}$. Subtracting them gives $|x| - x = 2x^-$, therefore $x^- = \frac{|x|-x}{2}$.

8. (Bernoulli's inequality) Let $a \geq 0$. Prove that for all $n \in \mathbb{N}$ we have

$$(1+a)^n \geq 1 + na. \tag{5}$$

(Hint 1: Induction. Hint 2: Binomial Theorem.)

Solution 1. We use induction on n. For $n = 1$ inequality (5) is trivially true. If (5) is true for some $n \in \mathbb{N}$ then it's true for $n+1$ as well because

$$(1+a)^{n+1} = (1+a)^n(1+a) \geq (1+na)(1+a) = 1 + a + na + na^2 \geq 1 + (n+1)a.$$

Solution 2. We use the binomial theorem.

$$(1+a)^n = 1 + na + \frac{n(n-1)}{2}a^2 + \cdots + na^{n-1} + a^n \geq 1 + na.$$

Things to think about:
True or False? $3^n > n^3$ eventually for all $n \in \mathbb{N}$. (Hint: $3 = 1 + 2$).

9. Find all real numbers x such that $|x-1| < |x+1|$.

Hint: Inequalities like the one above can be solved in various ways (e.g squaring both sides) and we have seen quite a few of them in PPS. There is a very easy solution using distances. The distance on the real line between two real numbers a and b is $|a-b|$. The inequality $|x-1| < |x+1|$ says that the distance of x from 1 is smaller than the distance of x from -1 (draw a picture). In other words, x is closer to 1 than to -1. Therefore, ...

Solution: The inequality $|x-1| < |x+1|$ says that the distance of x from 1 is smaller than the distance of x from -1. In other words, x is closer to 1 than to -1. This is true if and only if $x > 0$.

Things to thing about:
1. Which complex numbers z satisfy $|z-1| < |z+1|$?
2. Let a, b be two real numbers with $a \neq b$. Which real numbers x satisfy $|x-a| < |x-b|$?

10. Let $a, b \in \mathbb{R}$, $a < b$. Let $c = \frac{a+b}{2}$ and $R = \frac{b-a}{2}$. We call c the *center* of the interval (a, b) and R the *radius* of (a, b).

    (a) Prove that
    $$(a,b) = (c-R, c+R) \ .$$

    (b) Prove that a real number x belongs to (a, b) if and only if its distance from the center c is strictly smaller than the radius R, i.e.
    $$x \in (a,b) \iff |x-c| < R.$$

    Solution: We have $c - R = \frac{a+b}{2} - \frac{b-a}{2} = a$ and $c + R = \frac{a+b}{2} + \frac{b-a}{2} = b$, therefore $(a, b) = (c-R, c+R)$. For all real numbers x we have
    $$x \in (a,b) \iff x \in (c-R, c+R) \iff c-R < x < c+R$$
    $$\iff -R < x-c < R \iff |x-c| < R.$$

    Things to think about:
    True or False? $x \notin (a, b)$ iff the distance of x from the center of the interval is $\geq$ the radius.

11. Let $a, b \in \mathbb{R}$ with $a < b$. Is there a smallest open interval that contains the closed interval $[a, b]$? Prove your claim.

    Solution: No there isn't. If (c, d) is an open interval that contains the closed interval $[a, b]$, then $a, b \in (c, d)$ (draw a picture), therefore $c < a$ and $b < d$. If we set $c' = \frac{c+a}{2}$ and $d' = \frac{b+d}{2}$ then $(c, d) \supsetneq (c', d') \supseteq [a, b]$, i.e. (c', d') is a strictly smaller open interval than the one we started with and it still contains the closed interval $[a, b]$.

12. Fill in the following table.

    (You are not asked for proofs in this problem. Drawing pictures helps.)

A	$\max A$	$\sup A$	$\min A$	$\inf A$
$(-1,1)$	*doesn't exist*	1	*doesn't exist*	-1
$[-1,1]$				
$[1,\sqrt{2})$				
$\{x \in (1,\sqrt{2}] : x \text{ is irrational}\}$				
$\left(-\sqrt{7},\sqrt{7}\right) \cap \mathbb{Q}$				
$(0,1) \cup (2,3]$				

Solution:

A	$\max A$	$\sup A$	$\min A$	$\inf A$
$(-1,1)$	*does not exist*	1	*does not exist*	-1
$[-1,1]$	1	1	-1	-1
$[1,\sqrt{2})$	*does not exist*	$\sqrt{2}$	1	1
$\{x \in (1,\sqrt{2}] : x \text{ is irrational}\}$	$\sqrt{2}$	$\sqrt{2}$	*does not exist*	1
$\left(-\sqrt{7},\sqrt{7}\right) \cap \mathbb{Q}$	*does not exist*	$\sqrt{7}$	*does not exist*	$-\sqrt{7}$
$(0,1) \cup (2,3]$	3	3	*does not exist*	0

13. In each of the following cases give a proof if the statement is true or a counterexample if the statement is false.

    (a) If A is a non-empty bounded subset of $\mathbb{R}$ and x is a real number between $\inf A$ and $\sup A$ then $x \in A$.

    (b) If A and B are bounded non-empty subsets of $\mathbb{R}$ such that $\inf A = \inf B$ and $\sup A = \sup B$ then $A = B$.

    Solution:

    (a) False. Counterexample: Let $A = \{1,3\}$, $x = 2$. Then $\inf A = 1$, $\sup A = 3$, x is between $\inf A$ and $\sup A$ but $x \notin A$.

    (b) False. Counterexample: Let $A = \{1,2,3\}$, $B = \{1,3\}$. Then $\inf A = \inf B = 1$, $\sup A = \sup B = 3$ but $A \neq B$.

Things to think about:
The examples in this problem show that a set does not always consist of everything between its infimum and supremum. Can you think of any examples of sets A that do contain all reals between $\inf A$ and $\sup A$?

14. Let $A = \left\{ a \in \mathbb{R} : a^2 > 5 \text{ and } a \text{ is a positive irrational} \right\}$. Prove that A is non-empty, bounded below, and that $\inf A = \sqrt{5}$.

    (This problem is on Assignment 2. The solution will be posted here later.)

15. Let
$$A = \left\{ \frac{n^2}{n^2+1} : n \in \mathbb{N} \right\} = \left\{ \frac{1}{2}, \frac{4}{5}, \frac{9}{10}, \ldots \right\}.$$
    Prove that $\sup A = 1$.

    Solution 1 Every element of A is smaller than 1, therefore 1 is an upper bound of A.

    It remains to show that 1 is the least (i.e. smallest) upper bound of A. It is enough to show that any real number $M < 1$ is not an upper bound of A.

    Fix $M < 1$. To prove that M is not an upper bound of A it is enough to find an element of A larger than M. If $M \leq 0$ then every element of A is larger than M. Assume $M > 0$. ([1])

    By the Archimedean property of the reals there exists a natural number n such that $n > \sqrt{\frac{M}{1-M}}$. Squaring both sides and rearranging we find $\frac{n^2}{n^2+1} > M$. The number $\frac{n^2}{n^2+1}$ is an element of A larger than M, as required.

    Solution 2 Every element of A is smaller than 1, therefore 1 is an upper bound of A.

    Since $\sup A$ is the smallest upper bound we have $\sup A \leq 1$. We wish to show that $\sup A = 1$.

    We argue by contradiction. Suppose that $\sup A < 1$. Since $\frac{1}{2} \in A$ we have $\frac{1}{2} \leq \sup A$, therefore $\sup A > 0$.

[1] Rough work: The elements of A are of the form $\frac{n^2}{n^2+1}$, so we need an n such that $M < \frac{n^2}{n^2+1}$. Cross-multiplying and rearranging we find $M < n^2(1-M)$, and dividing by the positive number $1-M$ we find $n^2 > \frac{M}{1-M}$, which is the same as $n > \sqrt{\frac{M}{1-M}}$. Now back to the formal proof.
(reminder: do not hand in rough work).

By the Archimedean property of the reals there exists a natural number n such that $n > \sqrt{\frac{\sup A}{1-\sup A}}$ (the quantity under the root is positive thanks to $0 < \sup A < 1$). Squaring both sides and rearranging we find $\frac{n^2}{n^2+1} > \sup A$. We have discovered an element of A larger than $\sup A$; contradiction.

16. ([Wade], Exercise 1.3.4) Let A be a non-empty bounded below subset of $\mathbb{R}$. Prove that the infimum of A is unique.

    Solution: We argue by contradiction. Suppose that A has more than one infima. Take two of them and call them m and m'. Since m is a lower bound and m' is a greatest lower bound we have $m \leq m'$. Since m' is a lower bound and m is a greatest lower bound we have $m' \leq m$. Therefore $m = m'$.

17. ([Wade], Exercise 1.3.8) Let A, B be two non-empty bounded above subsets of $\mathbb{R}$. Show that $A \cup B$ is non-empty and bounded above and that

$$\sup(A \cup B) = \max\{\sup A, \sup B\} \ . \tag{6}$$

Solution:

Since A is non-empty there exists an element $a_0 \in A$. Then $a_0 \in A \cup B$, therefore $A \cup B$ is a non-empty set.

Next we show that $A \cup B$ is bounded above. Since the sets A and B are bounded above there exist real numbers M and M' such that for all $a \in A$ we have $a \leq M$, and for all $b \in B$ we have $b \leq M'$. If x is any element of $A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $x \leq M$. If $x \in B$ then $x \leq M'$. In either case we can say that $x \leq \max\{M, M'\}$. We have shown that $A \cup B$ is bounded above and that $\max\{M, M'\}$ is an upper bound.

It remains to prove (6). First, we apply the result in the last paragraph with two specific upper bounds M and M', namely $M = \sup A$ and $M' = \sup B$. We have shown above that $\max\{M, M'\}$ is an upper bound of $A \cup B$, and since the smallest upper bound of $A \cup B$ is $\sup(A \cup B)$, we have

$$\sup(A \cup B) \leq \max\{M, M'\} = \max\{\sup A, \sup B\}.$$

Next, $A \subseteq A \cup B$, therefore (monotone property of the supremum, [Wade], Theorem 1.21)

$$\sup A \leq \sup(A \cup B).$$

Similarly,

$$\sup B \leq \sup(A \cup B).$$

Now $\max\{\sup A, \sup B\}$ is one of $\sup A$ and $\sup B$, therefore

$$\max\{\sup A, \sup B\} \leq \sup(A \cup B).$$

This completes the proof of (6).

18. (Approximation property for infima, [Wade], Exercise 1.3.6a) Let A be a non-empty bounded below subset of $\mathbb{R}$. Prove that for every $\varepsilon > 0$ there exists an $a \in A$ such that $\inf A \leq a < \inf A + \varepsilon$.

    Remark: $\inf A + \varepsilon$ is just an Analyst's way of denoting an arbitrary point on the real line to the right of $\inf A$. The use of the Greek letter ε indicates that only small values of it are of any real interest, but observe that we are not actually assuming that ε is small in any way. The approximation property says: between $\inf A$ and any point to its right we can always find an element of A.

    Solution 1 Among all lower bounds of A, $\inf A$ is the largest. The number $\inf A + \varepsilon$ is larger than $\inf A$ therefore it is not a lower bound of A, therefore there exists $a \in A$ such that $a < \inf A + \varepsilon$. On the other hand, $\inf A \leq a$ simply because $\inf A$ is a lower bound of A and a is an element of A.

    Solution 2 We wish to show there is an element of A in the interval $[\inf A, \inf A + \varepsilon)$. We argue by contradiction. Suppose that there are no elements of A in $[\inf A, \inf A + \varepsilon)$. There are no elements of A in the $(-\infty, \inf A)$ either, because $\inf A$ is a lower bound of A. It follows that all elements of A are in $[\inf A + \varepsilon, +\infty)$, which implies that $\inf A + \varepsilon$ is a lower bound of A. We have discovered a lower bound of A larger than its infimum; contradiction.

19. Let A and B be non-empty subsets of $\mathbb{R}$ such that for every $a \in A$ and $b \in B$ we have $a \leq b$.

    (a) Show that $\sup A \leq \inf B$.

    (b) Give an example with $\sup A < \inf B$ and an example with $\sup A = \inf B$.

    (c) If, moreover, for every $\varepsilon > 0$ there exist $a \in A$ and $b \in B$ such that $b - a < \varepsilon$, then $\sup A = \inf B$.

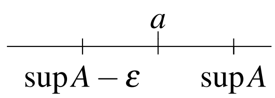
    (This Problem is related to *Dedekind cuts*. You can learn more about this topic in [Hardy]).

    Solution:

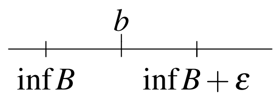
(a) Before we start with the proof, a comment on Notation:

We are going to need the approximation property for suprema and infima. Wade uses the following notation:

- Let A be non-empty bounded above. For every $\varepsilon > 0$ there is an $a \in A$ such that $\sup A - \varepsilon < a \leq \sup A$.



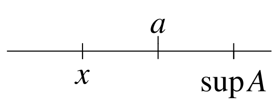
- Let A be non-empty bounded below. For every $\varepsilon > 0$ there is an $a \in A$ such that $\inf A \leq a < \inf A + \varepsilon$.



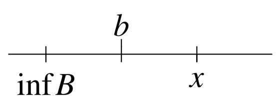
We'll use Wade's notation in the first proof below.

We observed in class that these can be rephrased as follows:

- Let A be non-empty bounded above. For every $x < \sup A$ there is an $a \in A$ such that $x < a \leq \sup A$.



- Let $A \subseteq \mathbb{R}$ be non-empty bounded below. For every $x > \inf A$ there is an $a \in A$ such that $\inf A \leq a < x$.



We'll use this notation in the second proof below. It results in a much shorter argument.

<u>Proof 1:</u> We argue by contradiction. Suppose $\inf B < \sup A$. Pick an ε such that

$$0 < \varepsilon < \frac{1}{2}\left(\sup A - \inf B\right), \tag{7}$$

for example $\varepsilon = \frac{1}{4}(\sup A - \inf B)$. Then

$$\inf B + \varepsilon < \sup A - \varepsilon. \tag{8}$$

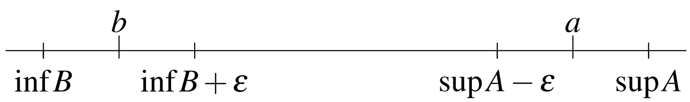
By the approximation property of suprema there exists an $a \in A$ such that

$$\sup A - \varepsilon < a \leq \sup A. \tag{9}$$

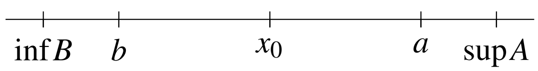
By the approximation property of infima there exists $b \in B$ such that

$$\inf B \leq b < \inf B + \varepsilon. \tag{10}$$

It follows that $b < a$; contradiction.



Proof 2: We argue by contradiction. Suppose $\inf B < \sup A$. Fix any number x_0 with $\inf B < x_0 < \sup A$. By the approximation property of infima the interval $(\inf B, x_0)$ contains at least one element b of B. By the approximation property of suprema the interval $(x_0, \sup A)$ contains at least one element a of A. But now $b < x_0$ and $x_0 < a$, therefore $b < a$ contradicting one of our hypothesis.



Proof 3: (This proof uses material from Chapter 2)

There exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A such that $a_n \to \sup A$. There exists a sequence $(b_n)_{n\in\mathbb{N}}$ of elements of B such that $b_n \to \inf B$. By hypothesis, $a_n \leq b_n$ for all n. Therefore, $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$, i.e. $\sup A \leq \inf B$.

(b) Let $A = \{1,2,3\}$ and $B = \{4,5,6\}$. Then every element of A is smaller than every element of B and $\sup A = 3 < 4 = \inf B$.

Let $A = (0,1)$ and $B = (1,2)$. Then every element of A is smaller than every element of B and $\sup A = 1 = \inf B$.

(c) We argue by contradiction. Suppose that $\sup A < \inf B$. Let $\varepsilon = \inf B - \sup A$. Then $\varepsilon > 0$ but there are no $a \in A$, $b \in B$ with $b - a < \varepsilon$. This is because all $a \in A$ are $\leq \sup A$ and all $b \in B$ are $\geq \inf B$, therefore $b - a \geq \inf B - \sup A = \varepsilon$.

20. ([Wade], Exercise 1.6.6a) Suppose that $n \in \mathbb{N}$ and $\phi : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$. Prove that if ϕ is 1-1 then ϕ is onto.

    Solution: We prove the claim by induction on n. If $n = 1$ then $\phi(1) = 1$ and we are done as such ϕ is onto.

    Assume that we have established the result for $n = k$. We want to show that it holds for $n = k+1$. Assume therefore that $\phi : \{1,2,\ldots,k,k+1\} \to \{1,2,\ldots,k,k+1\}$ is 1-1. We shall consider 2 cases. If $\phi(k+1) = k+1$ it follows that $\phi : \{1,2,\ldots,k\} \to \{1,2,\ldots,k\}$ and is 1-1 as the original map was. Now we can apply the induction hypothesis and get that $\phi : \{1,2,\ldots,k\} \to \{1,2,\ldots,k\}$ is onto and hence $\phi : \{1,2,\ldots,k,k+1\} \to \{1,2,\ldots,k,k+1\}$ is also onto as $\phi(k+1) = k+1$.

    The second possibility is that $\phi(k+1) = j < k+1$. Consider then a new map $\psi : \{1,2,\ldots,k\} \to \{1,2,\ldots,k\}$ defined as follows:

    $$\psi(i) = \phi(i), \qquad \text{provided } \phi(i) < j, \qquad \psi(i) = \phi(i) - 1, \qquad \text{provided } \phi(i) > j.$$

    Clearly, ψ must be 1-1 since ϕ was. Also

    $$\psi : \{1,2,\ldots,k\} \to \{1,2,\ldots,k\}$$

    and hence by the induction assumption ψ is onto. It follows that

    $$\phi : \{1,2,\ldots,k\} \to \{1,2,\ldots,j-1,j+1,\ldots,k,k+1\}$$

    is also onto. As $\phi(k+1) = j$ it must follows that the original map $\phi : \{1,2,\ldots,k,k+1\} \to \{1,2,\ldots,k,k+1\}$ is also onto.

21. ([Wade], Exercise 1.6.7) A number $x_0 \in \mathbb{R}$ is called *algebraic* if it is a root of a polynomial

    $$P(x) = a_n x^n + \cdots + a_1 x + a_0, \qquad \text{where } a_i \in \mathbb{Z} \text{ and } a_n \neq 0.$$

    A number $x_0 \in \mathbb{R}$ is called *transcendental* if x_0 is not algebraic.

    (a) Prove that if $n \in \mathbb{N}$ and $q \in Q$ then n^q is algebraic.

    (b) Prove that for each $n \in \mathbb{N}$ the collection of algebraic numbers of degree n is countable.

    (c) Prove that the collection of transcendental numbers is uncountable.

    Solution:

(a) Let $q = a/b$ and $a, b \in \mathbb{Z}$, $b > 0$. If $a > 0$ then the equation

$$x^b - n^a = 0$$

has one of the roots n^q and hence this number is algebraic. If $a < 0$ the the equation

$$n^{-a} x^b - 1 = 0$$

works.

(b) Each polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ has at most n real roots. The set of all $(n+1)$-tuples $(a_n, a_{n-1}, \ldots, a_1, a_0)$, where all a_i are integers, is countable as it is a cartesian product of countable sets. Hence there are countably many different polynomials $P(x)$ of degree n with integer coefficients and each has at most n real roots. From this it follows that the set of their roots is also countable.

(c) If the set of transcendental numbers were countable then $\mathbb{R}$ would also be countable as a union of two countable sets is also countable. It follows that the set of transcendental numbers must be uncountable.

References

[Hardy] G. H. Hardy, A course of Pure Mathematics.

[Wade] William R. Wade, Introduction to Analysis, 4th ed., Pearson New International Edition.