**Fundamentals of Pure Mathematics 2015-16**
**Assignment 8**
**Due-date: Thursday of week 9**
**(Submit your work to your tutor at the workshop)**

1. (5 marks) (This is Problem 61 from the week 8 workshop)

   Prove directly from the definition of the derivative[1] that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt[3]{x}$ is differentiable at $x_0 = 1$ and find $f'(x_0)$. (You may need: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$)

2. (5 marks) Prove that the equation $\cos x = 2x$ has exactly one real solution.

## Solutions

1. We examine the difference quotient of $f$ at $x_0 = 1$.

$$\frac{f(x) - f(1)}{x-1} = \frac{\sqrt[3]{x} - 1}{x-1} = \frac{\left(\sqrt[3]{x} - 1\right)\left(x^{2/3} + x^{1/3} + 1\right)}{(x-1)(x^{2/3} + x^{1/3} + 1)} = \frac{\left(\sqrt[3]{x}\right)^3 - 1^3}{(x-1)(x^{2/3} + x^{1/3} + 1)}$$
$$= \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x^{2/3} + x^{1/3} + 1)} = \frac{1}{x^{2/3} + x^{1/3} + 1} \xrightarrow[x \to 1]{} \frac{1}{3}.$$

   It follows that $f$ is differentiable at $x_0 = 1$ and $f'(1) = \frac{1}{3}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 2x - \cos x$. Then $f$ is continuous with $f(0) = -1 < 0$ and $f(\pi/2) = \pi > 0$, therefore, by the IVT, $f$ has at least one real root $c$.

   To prove that $f$ has exactly one root we argue by contradiction. Suppose $f$ has a second root $c' \neq c$. By Rolle's Theorem, $f'$ vanishes somewhere between $c$ and $c'$. But
$$f'(x) = 2 + \sin x \geq 2 - 1 = 1 > 0, \tag{1}$$
   for all $x$. Contradiction.

[1] Wade, Definition 4.1. Dindos, Definition 4.1

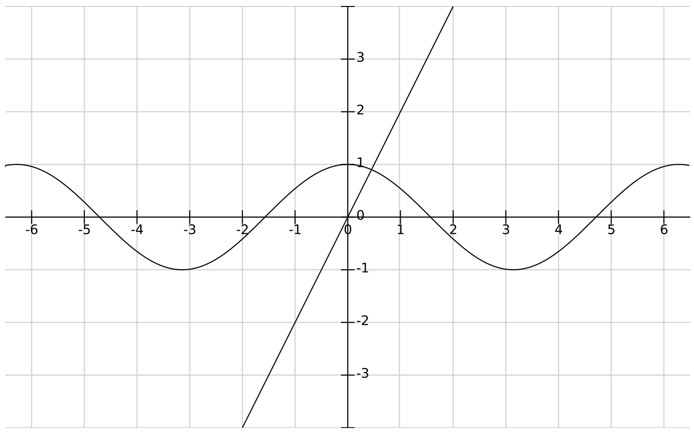


Figure 1: The graphs of $\cos x$ and $2x$.

Things to think about:
1. Is it true that the equation $\cos x = x$ has exactly one real solution? If we tried to run the same argument as above using $f(x) = x - \cos x$, then the second step wouldn't work because the derivative would be

$$f'(x) = 1 + \sin x, \tag{2}$$

which does vanish at several points ($x = -\frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$).
2. How many real solutions does $\cos x = \frac{1}{4}x$ have?