Student Examination Number ....

The University of Edinburgh

College of Science and Engineering

FOR ANONYMOUSLY MARKED EXAMINATIONS
FOLD AND SEAL WITH BARCODE

Surname..

Pre-Names..

Signature..

Enrolment Number..
(as shown on your Matriculation card)

Pre – Honours 2

MATH08064 Fundamentals of Pure Mathematics

Friday, 2nd May 2014

9:30am – 12:30pm

Chairman of Examiners – Professor B Leimkuhler

External Examiner – Dr S D Theriault

Section A	Mark
1	
2	
3	
4	
5	
6	
Total	
Section B	**Mark**
7	
8	
9	
10	
Total	

All six questions from **Section A** count (40% total mark).
The **best three of four** answers from **Section B** will count (60% of total mark).

In the examination it is permitted to have;
A copy of the course text book
An Introduction to Analysis by W.R.Wade (for analysis)
Groups by C.R.Jordan and D.A.Jordan (for group theory)
Any other notes (printed or hand-written)

Calculators and other electronic aids
Only calculators from the list maintained by the College of Science and Engineering may be used in this examination.

Make and Model
Casio fx85 (any version, e.g. fx85WA, fx85MS)
Casio fx83 (any version, e.g. fx83ES)
Casio fx82 (any version)

This examination will be marked anonymously.

Section A (40 points). Solve all 6 problems in this section.

(1) In each of the following, give a specific example of:

(a) A group with eleven elements.

(b) An infinite group that is cyclic.

(c) An infinite group that is not cyclic.

(d) A group G for which $C(G) = \{e\}$.

[6 marks]

[Please turn over]

(2) State, with a brief justification, whether the following statements are TRUE or FALSE:

(a) $\mathbb{Z}_2 \times \mathbb{Z}_3$ is a cyclic group.

(b) $D_3 \times \mathbb{Z}_2$ is a cyclic group.

(c) The group $\mathbb{Q}_+ := \{q \in \mathbb{Q} \mid q > 0\}$ (under multiplication) is a cyclic group.

[7 marks]

(3) Suppose that G is a group with subgroup H.

(a) Prove that $HH = H$. **[3 marks]**

(b) If $g \in G$, prove that gHg^{-1} is also a subgroup of G. **[4 marks]**

(4) Are the following statements TRUE or FALSE? If TRUE give a proof, if FALSE give a counterexample or justification.

(a) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers, then $(|x_n|)_{n\in\mathbb{N}}$ is also a Cauchy sequence.

(b) If $(x_n)_{n\in\mathbb{N}}$ and $(x_n y_n)_{n\in\mathbb{N}}$ are convergent sequences of real numbers, then $(y_n)_{n\in\mathbb{N}}$ must also be convergent.

(c) If $|x_n| \leq y_n$ for all n and $(y_n)_{n\in\mathbb{N}}$ is a convergent sequence, then $(x_n)_{n\in\mathbb{N}}$ is bounded.

[6 marks]

(5) Let $A, B \subset \mathbb{R}$ be nonempty and bounded, and let

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Prove that

$$\sup(A + B) = \sup A + \sup B.$$

[7 marks]

(6) Consider the sequence $x_1 = 2$ and

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}, \qquad n = 1, 2, 3, \ldots.$$

Use the monotone convergence theorem to prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent, and find its limit. **[7 marks]**

Section B (60 points). Solve any three problems in this section.

(7) (a) List the eight elements of D_4, and for each element calculate its order. **[6 marks]**

(b) There is a binary operation, $*$, on $\mathbb{R}$ defined by

$$a * b := a + b - ab$$

for all $a, b \in \mathbb{R}$. Consider $A := \{r \in \mathbb{R} \mid r \neq 1\}$.

(i) Prove that A is a group under $*$ with identity $e = 0$. **[7 marks]**

(ii) Consider now $\mathbb{R}^* := \{r \in \mathbb{R} \mid r \neq 0\}$ considered as a group under multiplication. Prove that

$$\theta\colon A \to \mathbb{R}^*$$

defined by $\theta(a) := \frac{1}{1-a}$ is a group homomorphism. **[3 marks]**

(iii) By using your answer to (ii), or otherwise, show that $A \cong \mathbb{R}^*$. **[4 marks]**

(8) (a) Consider the symmetric group S_5.

(i) Consider the cycle $\sigma := (52314)$. Write down σ^3 and σ^{-1} in cycle notation. **[2 marks]**

(ii) List all the elements of S_5 of cycle type 2,2. **[3 marks]**

(iii) List the conjugacy classes of S_5, writing down an example element from each, and giving the number of elements in each conjugacy class. You may use any theorem from the lectures, provided that it is clearly stated. **[6 marks]**

(b) Consider now the alternating group A_5.

(i) List all the possible cycle types of elements of A_5. **[2 marks]**

(ii) State, with a brief justification, whether the following statements are TRUE or FALSE:

(A) A_5 contains a cyclic subgroup of order 5.

(B) A_5 contains a cyclic subgroup of order 7.

(C) A_5 contains a subgroup of order 6.

[7 marks]

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$f(x) = \begin{cases} x^p \sin(1/x^2), & \text{for } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Show that f is continuous at 0 if and only if $p > 0$. **[5 marks]**

(b) Show that f is differentiable at 0 for $p > 1$ and calculate $f'(0)$. **[5 marks]**

(c) Calculate $f'(x)$ for $x \neq 0$. **[5 marks]**

(d) Consider $p = 2$. Is the derivative f' continuous at 0? **[5 marks]**

(10) Let $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{for } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$

(a) Find the first three nonzero terms of Taylor's polynomial of f at zero.
[6 marks]

(b) Estimate the largest possible error you make if you use the first three nonzero terms of Taylor's polynomial of f to evaluate the function $f(x)$ for $x \in [-2, 2]$.
[7 marks]

(c) Prove that $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. (*Hint:* Use the mean value theorem).
[7 marks]

[End of Paper]