Student Examination Number ....

FOR ANONYMOUSLY MARKED EXAMINATIONS FOLD AND SEAL WITH BARCODE

Surname..

Pre-Names..

Signature..

Enrolment Number (as shown on your Matriculation card)..

The University of Edinburgh

College of Science and Engineering

Pre – Honours 2

MATH08064 Fundamentals of Pure Mathematics

Thursday, 14th August 2014

9:30am – 12:30pm

Chairman of Examiners – Professor B Leimkuhler

External Examiner – Dr S D Theriault

Section A	Mark
1	
2	
3	
4	
5	
6	
Total	
Section B	**Mark**
7	
8	
9	
10	
Total	

All six questions from **Section A** count (40% total mark). The **best three of four** answers from **Section B** will count (60% of total mark).

In the examination it is permitted to have;
A copy of the course text book
An Introduction to Analysis by W.R.Wade (for analysis)
Groups by C.R.Jordan and D.A.Jordan (for group theory)
Any other notes (printed or hand-written)

Calculators and other electronic aids
Only calculators from the list maintained by the College of Science and Engineering may be used in this examination.

Make and Model
Casio fx85 (any version, e.g. fx85WA, fx85MS)
Casio fx83 (any version, e.g. fx83ES)
Casio fx82 (any version)

This examination will be marked anonymously.

Section A (40 points). Solve all 6 problems in this section.

(1) State, with a brief justification, whether the following statements are TRUE or FALSE:

(a) $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

(b) $\mathbb{Z}_{12} \cong \mathbb{Z}_2 \times \mathbb{Z}_6$.

(c) $D_6 \cong A_4$.

[7 marks]

(2) Let G be a group and p be a prime. State, with a brief justification, whether the following statements are TRUE or FALSE:

(a) If p divides $|G|$, then G contains an element of order p.

(b) If p^2 divides $|G|$, then G contains an element of order p^2.

(c) If p^3 divides $|G|$, then G contains an element of order p^3.

[6 marks]

(3) Suppose that in a finite group G, there exist subgroups H and K such that $|H| = 25$ and $|K| = 36$.

(a) Using Lagrange's theorem, or otherwise, prove that $H \cap K = \{e\}$. **[4 marks]**

(b) Prove that 900 divides $|G|$. **[3 marks]**

(4) Are the following statements TRUE or FALSE? If TRUE give a proof, if FALSE give a counterexample or justification.

(a) Let f be a real function continuous at a point $a \in \mathbb{R}$. Then $f'(a)$ exists.

(b) Let f be a real function differentiable at a point $a \in \mathbb{R}$. Then $\lim_{x \to a} f(x)$ exists.

(c) Let $a < b$ be real numbers. If $f : [a, b] \to \mathbb{R}$ is such that $f([a, b])$ is a closed bounded interval, then f is continuous on $[a, b]$.

[6 marks]

(5) Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. Consider new sequences

$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}, \quad \text{and} \quad z_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

(a) Prove that both sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are convergent. **[3 marks]**

(b) Prove that if $(x_n)_{n\in\mathbb{N}}$ is convergent then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

[4 marks]

(6) For what values of $x \in \mathbb{R}$ is the series

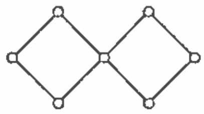
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n} x^n$$

absolutely convergent? For what values of $x \in \mathbb{R}$ is the series convergent?

[7 marks]

Section B (60 points). Solve any three problems in this section.

(7) (a) Determine the precise number of symmetries of the graph



[4 marks]

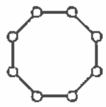
(b) Give an example of

(i) Two different graphs that have isomorphic symmetry groups.

(ii) Two different graphs that have the same number of symmetries, but their symmetry groups are not isomorphic.

[6 marks]

(c) A jewellery manufacturer wants to make necklaces, where each necklace contains eight coloured beads, in the shape of a regular 8-gon



The jeweller has n colours. How many different coloured necklaces can be made? Two coloured necklaces are regarded as identical if they differ by an element of the symmetry group of the graph. **[10 marks]**

(8) (a) Let G and H be groups. Prove that $G \times H \cong H \times G$. **[6 marks]**

(b) Give a specific example of a group in which there are two conjugacy classes whose union is not a subgroup. **[4 marks]**

(c) State, with a brief justification, whether the following statements are TRUE or FALSE:

(i) The subgroup $\{0, 3\}$ of $\mathbb{Z}_6$ is a normal subgroup.

(ii) The subgroup $\{e, h\}$ (where h is a reflection) of D_3 is a normal subgroup.

(iii) The intersection of two normal subgroups is also a normal subgroup.

(iv) The union of two normal subgroups is also a normal subgroup.

[10 marks]

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Show that if $f : \mathbb{R} \to \mathbb{Z}$ then f must be constant. **[5 marks]**

(b) Show that if f is a polynomial of odd degree then there exists $c \in \mathbb{R}$ such that $f(c) = 0$. **[5 marks]**

(c) Show that if f is differentiable and f' is a bounded function (that is $|f'(x)| \leq M$ for all $x \in \mathbb{R}$) then

$$|f(x) - f(y)| \leq M|x - y|, \qquad \text{for all } x, y \in \mathbb{R}.$$

[5 marks]

(d) Find an example of a function f that is differentiable and bounded on $\mathbb{R}$ but f' is not bounded. (*Hint:* You do not have to give an explicit formula of such a function, a sketch/graph of it will suffice) **[5 marks]**

(10) Consider the following sequence of real numbers: $x_1 = \sqrt{2}$ and

$$x_{n+1} = \sqrt{2 + x_n}, \qquad n = 1, 2, 3, \ldots.$$

(a) Prove that the sequence is bounded. **[3 marks]**

(b) Prove that the sequence is convergent, and find its limit. **[4 marks]**

(c) Consider the function $f(x) = \sqrt{2 + x}$. Show that

$$|f(x) - 2| \leq \frac{1}{2\sqrt{2}}|x - 2|, \qquad \text{for all } x \geq 0.$$

[4 marks]

(d) Use part (c) to show that

$$|x_{n+1} - 2| \leq \left(\frac{1}{2\sqrt{2}}\right)^n |x_1 - 2|.$$

[4 marks]

(e) Is the series $\sum_{n=1}^{\infty}(x_n - 2)$ absolutely convergent? **[5 marks]**

[End of Paper]