Fundamentals of Pure Mathematics
Workshop 5: Pólya Counting

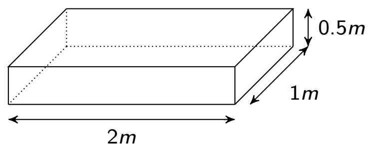
The point of this workshop to illustrate the power of Pólya counting.

Recall that if a finite group G acts on a finite set X, then

$$\text{the number of } G\text{-orbits in } X = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|.$$

Question 1. (Mattress turning)

(a) Determine the group of rotational symmetries of the solid

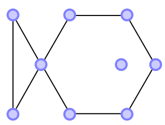


(b) Customers complain to mattress manufacturers that they want their mattresses to wear evenly. To achieve this, customers would like a simple 'turning' rule which, if performed on the mattress every month, would cycle through all possible configurations of the mattress. Explain, using your answer to (a), why this is not possible.

(c) A mattress manufacturer wants to produce mattresses, based on the design in (a). They can colour the faces of the mattress with three possible colours. How many different mattresses can be produced? (Two mattresses are regarded as identical if they differ by an element of the rotational symmetry group.)

Question 2.

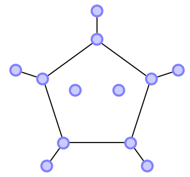
(a) (revision of Workshop 1) Determine all symmetries of the graph



(b) A plastic fish manufacturer wants to produce model fish, based on the design in (a). The manufacturer has two colours, and wants to colour each node of the above graph. How many different fish can be made? (Two coloured fish are regarded as identical if they differ by an element of the symmetry group of the graph.)

Question 3.

(a) (revision of Workshop 1) Show that the symmetry group of the graph



is $D_5 \times \mathbb{Z}_2$.

(b) A toy manufacturer wants to make aliens, based on the design in (c). The manufacturer has two colours, and wants to colour each node of the above graph. How many different aliens can be made? (Two aliens are regarded as identical if they differ by an element of the symmetry group of the graph.)

Solutions.

1. (a) Consider the group of rotational symmetries acting on the set of faces. Pick the top face T, then $\mathrm{Orb}_G(T) = \{T, B\}$ where B is the bottom face. Also $\mathrm{Stab}_G(T)$ consists of precisely those rotational symmetries that fix T, which since the mattress is strictly rectangular, consists of the identity and the rotation through 180 degrees. Thus, by orbit–stabilizer,

$$|G| = |\mathrm{Orb}_G(T)| \times |\mathrm{Stab}_G(T)| = 2 \times 2 = 4.$$

Explicitly, these are the identity e, flip T and B keeping the left and right hand faces fixed (denoted g), rotate by 180 degrees fixing T and B (denoted h), and $g \circ h$. These are four distinct symmetries, so since $|G| = 4$ these are them all. Now since $|G| = 4$ and no element has order four (they all have order 2), we must have $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) The problem translates into asking whether the group G is cyclic. By (a) it is not — there is no element of order four.

(c) By Pólya counting, if a finite group G acts on a finite set X, then

$$\text{the number of } G\text{-orbits in } X = \frac{1}{|G|}\sum_{g\in G} |\operatorname{Fix}(g)|.$$

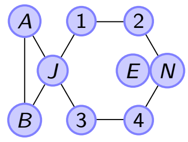
To solve the problem in the question, let G be the symmetry group of the graph. This acts on the set X of all possible (i.e. 3^6) colourings. The question asks for the number of G-orbits, hence we must analyse the fixed points.

- The identity fixes every mattress, so $|\mathrm{Fix}(e)| = 3^6$.
- Consider the element g from (a). The only way a mattress is fixed under the action of g is if the colour of the top and bottom faces are the same, and the colours of the front and back faces are the same. The remaining faces (left and right) can have arbitrary colour. There are 3^4 such diagrams.
- Consider the element h from (a). The only way a coloured mattress is fixed under the action of h is if the colour of the front and back faces are the same, and the colour of the left and right faces are the same. The top and bottom faces can be an arbitrary colour. There are 3^4 such diagrams.
- Consider the element $g \circ h$ from (a). The only way a mattress is fixed under the action of g is if the colour of the top and bottom faces are the same, and the colours of the left and right faces are the same. The remaining faces (front and back) can have arbitrary colour. There are 3^4 such diagrams.

Hence the number of orbits (and so the number of different mattresses) is equal to

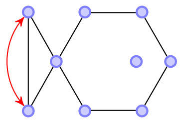
$$\tfrac{1}{4}(3^6 + 3^4 + 3^4 + 3^4) = \frac{1}{4}(12 \times 3^4) = 3^5 = 243.$$

2. (a) For convenience, label



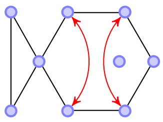
Any symmetry must fix the eye E (since it is the only vertex connected to no others), the join J (since it is the only vertex of valency four), and the nose N (since it is the only vertex a distance three from the fixed J). Now any symmetry must either fix both 2 and 4, or interchange 2 and 4 (since they are connected to fixed N). In either case, this determines the action on 1 and 3 — if 2 and 4 are fixed, so must be 1 and 3, whereas if 2 and 4 swap, so must 1 and 3. Hence there are two options for the "body" of the fish.

Similarly, there are two options for the "tail" — either A and B are fixed, or they are interchanged. Hence, overall there are $2 \times 2 = 4$ symmetries of the graph. Explicitly, they are the identity, together with



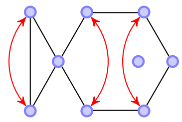
g_1

and



g_2

and



g_3

(b) By Pólya counting, if a finite group G acts on a finite set X, then

$$\text{the number of } G\text{-orbits in } X = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|.$$

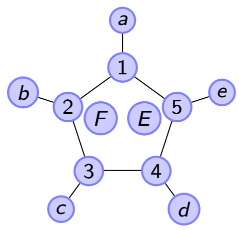
To solve the problem in the question, let G be the symmetry group of the graph. This acts on the set X of all possible (i.e. $2^9 = 512$) colourings. The question asks for the number of G-orbits, hence we must analyse the fixed points.

- The identity fixes every coloured fish, so $|\operatorname{Fix}(e)| = 2^9$.
- Consider the element g_1 from (a). The only way a coloured fish is fixed under the action of g_1 is if the colour of the vertices A and B are the same. All other vertices can have an arbitrary colour. There are 2^8 such diagrams.
- Consider the element g_2 from (a). The only way a coloured fish is fixed under the action of g_2 is if the colour of the vertices 1 and 3 are the same, and the colour of the vertices 2 and 4 are the same. All other vertices can have an arbitrary colour. There are 2^7 such diagrams.
- Consider the element g_3 from (a). The only way a coloured fish is fixed under the action of g_3 is if the colour of the vertices A and B are the same, the colour of the vertices 1 and 3 are the same, and the colour of the vertices 2 and 4 are the same. All other vertices can have an arbitrary colour. There are 2^6 such diagrams.

Hence the number of orbits (and so the number of different fish) is equal to

$$\tfrac{1}{4}(2^9 + 2^8 + 2^7 + 2^6) = \tfrac{1}{4}(15 \times 2^6) = 15 \times 16 = 240.$$

3. (a) For convenience, label

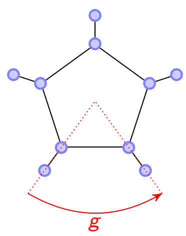


Any symmetry

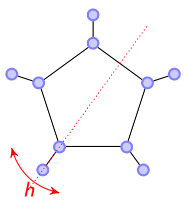
- must either fix the eyes E and F or swap them (since they are the only vertices connected to no others). This gives two options.
- Any symmetry must preserve vertices of valency three, must take the vertex 1 to a member of the set $\{1, 2, \ldots, 5\}$. Any such choice is possible, and after making this choice (there are five possible), everything else is determined by whether the neighbours of 1 are fixed or swapped (of which there are two choices).

This shows that there are $2 \times 5 \times 2 = 20$ symmetries in total.

Now consider $H := \{e, g, g^2, \ldots, g^4, h, gh, \ldots, g^4h\}$ and $K := \{e, z\}$, where



and



both fixing the eyes, and z is the element that swaps the eyes and fixes everything else.

Then clearly $H \cap K = \{e\}$ and the elements of H and K commute (since swapping the eyes is independent of the other options). Hence, by result in lectures, HK is a subgroup of G with $HK \cong H \times K$. Since both G and $H \times K$ have order 20, it follows that

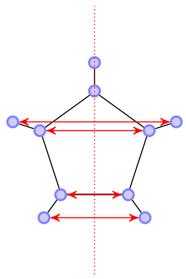
$$G = HK \cong H \times K \cong D_5 \times \mathbb{Z}_2$$

(b) By Pólya counting, if a finite group G acts on a finite set X, then

$$\text{the number of } G\text{-orbits in } X = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|.$$

To solve the problem in the question, let G be the rotational symmetry group of the mattress. This acts on the set X of all possible (i.e. $2^{12} = 4096$) colourings. The question asks for the number of G-orbits, hence we must analyse the fixed points.

- The identity fixes every coloured alien, so $|\mathrm{Fix}(e)| = 4096$.
- The element (e, z) fixes every coloured alien whose eyes are both the same colour. There are $2^{11} = 2048$ such aliens.
- Consider a non-trivial rotation (g^t, e). Since 5 is prime, for a colouring to be fixed the colours of the vertices $\{1, 2, \ldots, 5\}$ all have to be the same, as do the colours of the vertices $\{a, b, \ldots, e\}$. The two eyes can be an arbitrary colour. There are $2^4 = 16$ such diagrams.
- Consider a non-trivial rotation (g^t, z). Since 5 is prime, for a colouring to be fixed the colours of the vertices $\{1, 2, \ldots, 5\}$ all have to be the same, as do the colours of the vertices $\{a, b, \ldots, e\}$. The two eyes now have to be the same colour. There are $2^3 = 8$ such diagrams.
- Consider a reflection $(g^t h, e)$. Since 5 is odd, the colour of the two vertices through which the line of reflection passes can be arbitrary, but the rest must match up as in

  The two eyes can be an arbitrary colour. There are $2^8 = 256$ such diagrams.
- Consider a reflection $(g^t h, z)$. The analysis is the same as above, but now the two eyes must have the same colour. There are $2^7 = 128$ such diagrams.

Hence the number of orbits (and so the number of different aliens) is equal to

$$\frac{1}{20}(4096 + 2048 + 4 \times 16 + 4 \times 8 + 5 \times 256 + 5 \times 128) = 408.$$