Fundamentals of Pure Mathematics

MATH08064 **Solutions and comments** **May 2015**

1. Suppose that G is a finite group, and let p be a prime. State, with a brief justification, whether the following statements are TRUE or FALSE.

(a) If G contains an element of order p, then p divides $|G|$.

(b) If p divides $|G|$, then G contains an element of order p.

(c) If p divides $|G|$, then G contains a subgroup of order p.

[6 marks]

Solution: This tests basic understanding of some of the main theorems of the course.

(a) TRUE. By Lagrange's Theorem, the order of any element must divide $|G|$.

(b) TRUE. This is Cauchy's Theorem.

(c) TRUE. Take the element g in (b) of order p, and consider the subgroup $\langle g \rangle$ it generates. This has p elements.

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2. Suppose that $\theta \colon G \to H$ is a group homomorphism, where G and H are finite groups. State whether the following statements are TRUE or FALSE. If they are TRUE give a brief proof, whereas if they are FALSE give a counterexample.

(a) $|\operatorname{Im}\theta| \leq |H|$.

(b) $|\operatorname{Im}\theta| \leq |G|$.

(c) $|\operatorname{Ker}\theta| \leq |H|$.

[7 marks]

Solution: This is testing basic understanding of functions, and of group homomorphisms.

(a) TRUE. $\operatorname{Im}\theta$ is a subgroup of H, so in particular the number of its elements is less than or equal to the number of elements in H.

(b) TRUE. By construction the function $G \to \operatorname{Im}\theta$ is surjective, and so the number of elements in $\operatorname{Im}\theta$ is less than or equal to the number of elements in G.

(c) FALSE. Consider $\mathbb{Z}_4 \to \{e\}$ sending everything to e. Then the kernel has four elements, which is larger than 1, the number of elements in $\{e\}$.

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3. Suppose that G is a group with subgroup H.

(a) Let $g_1, g_2 \in G$. Prove that $Hg_1 = Hg_2$ if and only if $g_1 = hg_2$ for some $h \in H$.

(b) Give an explicit example of a group G, with subgroup H and element $g \in G$, such that Hg is not a subgroup of G.

[7 marks]

Solution: Part (a) is very similar to Rules for Cosets (in the lectures we did left cosets, and in that setting showed the equivalence with $g_2 = g_1h$). Part (b) is new, but is similar to questions about cosets and Lagrange.

(a) Let $g_1, g_2 \in G$.
($\Rightarrow$) If $Hg_1 = Hg_2$ then $g_1 = eg_1 \in Hg_1 = Hg_2$ and so there exists $h \in H$ such that $g_1 = hg_2$.
($\Leftarrow$) If $g_1 = hg_2$, then since $Hh = H$, we have

$$Hg_1 = H(hg_2) = (Hh)g_2 = Hg_2,$$

as required.

(b) The point is you can take any G, any H provided that $g \notin H$. For example consider $G = \mathbb{Z}_4$ with $H = \{0, 2\}$ and $g = 1$. The $Hg = \{1, 3\}$ which is not a subgroup since it does not contain the identity 0.

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4. Are the following statements TRUE or FALSE? If TRUE give a proof, if FALSE give a counterexample or justification.

(a) If $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that $|x_{n+1} - x_n| \to 0$ as $n \to \infty$ then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent.

(b) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence then $(x_n)_{n\in\mathbb{N}}$ is bounded.

(c) A subsequence of a bounded sequence is convergent.

[6 marks]

Solution:

(a) FALSE. Take $x_n = \log n$. Then $x_{n+1} - x_n \to 0$ but clearly the sequence diverges to $+\infty$.

(b) TRUE. A Cauchy sequence is convergent. By the theorem from lectures any convergent sequence is bounded, hence the claim holds.

(c) FALSE. This is not true for ANY subsequence, provided the original sequence does not converge. The Bolzano-Weierstrass theorem only claims the "THERE EXISTS" a convergent subsequence of a bounded sequence.

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5. Let $\sum_{n=1}^{\infty} a_n$ be a nonnegative series ($a_n \geq 0$). Prove that the series is convergent if and only if

$$s = \sup\left\{\sum_{n=1}^{k} a_n : k \in \mathbb{N}\right\} < \infty.$$

Furthermore prove that s is equal to the sum of the series, that is $s = \displaystyle\sum_{n=1}^{\infty} a_n$. **[7 marks]**

Solution: Let (s_k) be the sequence of partial sums $s_k = \sum_{n=1}^{k} a_n$. By definition of convergence of a series, the series converges if and only if the sequence (s_k) is convergent. However, since $a_n \geq 0$ the series is monotone (increasing). It follows that this sequence converges if and only if it is bounded above which is exactly when $s < \infty$. It follows (by theorem from lecture) that $s = \lim_{n\to\infty} s_n$ and hence

$$s = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n.$$

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6. For what values of $x \in \mathbb{R}$ is the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n} x^n$$

absolutely convergent? For what values of $x \in \mathbb{R}$ is the series convergent? **[7 marks]**

Solution: We compute the radius of convergence of the series. We have

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{(n+1)\log(n+1)}\right|}{\left|\frac{(-1)^n}{n \log n}\right|} = \lim_{n\to\infty} \frac{n \log n}{(n+1)\log(n+1)} = 1.$$

It follows that for $|x| < R = 1$ the series converges absolutely, for $|x| > R = 1$ diverges. It remains to consider points $x = -1$ and $x = 1$. At $x = 1$ the series is conditionally convergent by the alternating series test but not absolutely convergent as

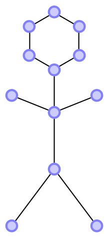
$$\sum_{n=2}^{\infty} \frac{1}{n \log n},$$

diverges (for example by the integral test). For the same reason the series diverges at $x = -1$. To summarize, the series is absolutely convergent for $x \in (-1, 1)$ and convergent for $x \in (-1, 1]$.

Section B (60 points). Solve any three problems in this section.

7.

(a) Determine the precise number of symmetries of the graph



You must argue that you have all symmetries. **[4 marks]**

(b) To each vertex in the above graph, we have to place the number 1, 2 or 3. With this rule, how many different ways are there of numbering the above graph? Two choices of numberings are considered identical if they differ by an element of the symmetry group of the graph. **[9 marks]**

(c) Suppose that G is a finite group.

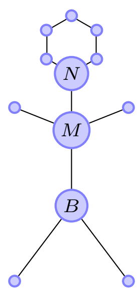
(i) Prove that if $g^2 = e$ for all $g \in G$, then $|G| = 2^n$ for some $n \in \mathbb{N}$. You may use any result from lectures, provided that it is clearly stated.

(ii) Give an explicit example of a group of order 8 in which $g^2 = e$ for all $g \in G$.

[7 marks]

Solution: Parts (a) and (b) are standard Pólya counting questions. There are twelve easy marks in (a) and (b) (and (c)(ii)). Part (c) is new.

(a) For convenience, label the vertices



Symmetries preserve valencies, and M is the only vertex of valency four, so it must remain fixed. Vertex B is the only vertex with valency three attached to two vertices of valency one, so it too must remain fixed. The only other vertex with valency three is N, so it too must remain fixed under any symmetry.

Hence N, M and B are fixed under any symmetry, and so a symmetry is specified by giving a permutation of the feet (2 options), a permutation of the arms (2 options), and whether the two vertices above N get fixed or swapped (forcing the head to be fixed, or reflected in the y axis) (2 options). Hence overall there are $2 \times 2 \times 2 = 8$ symmetries.

(b) By Pólya counting, if a finite group G acts on a finite set X, then

$$\text{the number of } G\text{-orbits in } X = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|.$$

To solve the problem in the question, let G be the symmetry group of the graph. This acts on the set X of all possible (i.e. 3^{12}) numberings. The question asks for the number of G-orbits, hence we must analyse the fixed points.

We let g denote the element that swaps the head but fixes everything else, h the element that swaps the arms but fixes everything else, and j the element that swaps the legs but fixes everything else.

- The identity fixes every numbered human, so has 3^{12} fixed points.
- Consider the j element that fixes the head and arms, but swaps the legs. For a numbered human to be fixed under this element, the legs have to have the same number, whereas all other vertices can be arbitrary. Thus there are 3^{11} fixed points. The analysis for the element h is identical.
- Consider g. To be fixed under g, the vertices from N downwards can be arbitrary, but the numbers on the two left hand vertices of the head have to match the two right hand vertices. Thus there are 3^{10} fixed points.

Continuing in this way, with identical arguments the fixed point analysis is

Element	Fixed points
Identity	3^{12}
j	3^{11}
h	3^{11}
g	3^{10}
hj	3^{10}
gj	3^{9}
gh	3^{9}
ghj	3^{8}

Hence the number of colourings, i.e. the number of orbits,

$$\begin{aligned}
&= \tfrac{1}{8} \times 3^8 \left(3^4 + 2 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 1\right) \\
&= \tfrac{1}{8} \times 3^8 \times 160 \\
&= 3^8 \times 20 \\
&= 2^2 \times 3^8 \times 5 \\
&= 131220.
\end{aligned}$$

(the students are not required to reduce the formula; the top line is sufficient)

(c) (i) If $G = \{e\}$ then the result is clearly true by taking $n = 0$. Hence assume $|G| \geq 2$. Then, if $|G|$ is not a power of 2, there must be some other prime p dividing $|G|$. By Cauchy's Theorem, G must then contain an element g (say) of order p. But the property $g^2 = e$ forces the order of g to be less than or equal to 2, which since $p \neq 2$ is a contradiction. Hence $|G|$ must be a power of 2.

(ii) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has this property. This is just the group in part (a).

8.

(a) Consider the alternating group A_4.

(i) List all the elements of A_4, in cycle notation.

(ii) Justify why A_4 is a subgroup of S_4.

(iii) Justify why A_4 is a normal subgroup of S_4.

[6 marks]

(b) State, with justification, whether the following statements are TRUE or FALSE. If they are TRUE provide a brief proof, whereas if they are FALSE provide a counterexample.

(i) Every group of order 9 is abelian.

(ii) Every group of order 10 is abelian.

(iii) If G is a group where $g^2 = e$ for all $g \in G$, then G is abelian.

(iv) If G is a group and $g \in G$, then $\langle g^2 \rangle$ is always strictly smaller than $\langle g \rangle$.

(v) If G is a finite group with $|G| \geq 2$, then G contains a non-trivial cyclic subgroup.

(vi) If G is a finite group with $|G| \geq 2$, then G contains a non-trivial non-cyclic subgroup.

[14 marks]

Solution: Part (a) was mostly on the 2013 resit exam (for which the solutions are not available). Question (b)(i)(ii) similar but different to previous years questions, and (b)(iii) was a Suggested Problem on the exercise sheets. The rest are new, but (b)(iv)(v) is similar(ish) to a question on Workshop 2.

(a) (i) The group A_4 consists of the identity, eight 3-cycles and three elements of cycle type $2, 2$. Explicitly, these are

$$\begin{gathered} e, \\ (12)(34), (13)(24), (14)(23), \\ (123), (132), (124), (142), (134), (143), (234), (243). \end{gathered}$$

(ii) The fact that A_4 is a subgroup of S_4 can be seen directly from the list in (i) above, using test for a subgroup. Alternatively, it can be seen theoretically: let $n \in \mathbb{N}$ and set

$$P = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Let $X = \{P, -P\}$. Then S_n acts on X by

$$\sigma \cdot P = \prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)})$$

If $\sigma \in S_n$ has the property that $\sigma \cdot P = P$, we say that σ is *even*. We define A_n to be the set of all even permutations in S_n. Note that $A_n = \mathrm{Stab}_{S_n}(P)$ and so it is a subgroup of S_n, since stabilizers of group actions are always subgroups.

(iii) By (ii), we next claim that A_n is normal. By the orbit-stabilizer theorem $|A_n| = \frac{|S_n|}{2}$. Now if H is any subgroup of a finite group G with $|H| = \frac{|G|}{2}$, necessarily H is normal. Hence it follows that A_n is normal in S_n.

(b) (i) TRUE. $9 = 3^2$, and every group of order p^2 is abelian, where p is a prime.

(ii) FALSE. D_5 has ten elements, and is not abelian.

(iii) TRUE. Let $g, h \in G$ and consider gh. Since $gh \in G$, by the assumption $(gh)^2 = e$, so $ghgh = e$, so $gh = h^{-1}g^{-1}$. But $g^{-1} = g$ (since $g^2 = e$) and similarly $h^{-1} = h$. Thus $gh = hg$ for all $g, h \in G$, so G is abelian.

(iv) FALSE. Consider $G = \mathbb{Z}_3 = \langle g \rangle$. Since both $\langle g \rangle$ and $\langle g^2 \rangle$ are non-trivial subgroups of G, and $|G|$ is prime, necessarily $\langle g \rangle = G = \langle g^2 \rangle$.

(v) TRUE. Since $|G| \geq 2$, there exists some prime p which divides $|G|$. By Cauchy's Theorem, there exists an element of that order. This element generates a cyclic subgroup with p elements, so in particular it is non-trivial.

(vi) FALSE. Consider $\mathbb{Z}_3$. By Lagrange, the only non-trivial subgroup is $\mathbb{Z}_3$. It is cyclic.

9.

(a) Use the mean value theorem to prove that for all $x, y \in \mathbb{R}$

$$|\cos x - \cos y| \leq |x - y|.$$

[6 marks]

(b) Find the Taylor polynomial of degree 2 at $x = 0$ for the function $f(x) = \sqrt{1-x}$. Estimate the error you will make if you use the Taylor polynomial of degree 2 to calculate $\sqrt{0.9}$.

[7 marks]

(c) Let I be an open interval, $f : I \to \mathbb{R}$ a continuous function and $a \in I$ a point. Use the definition of continuity to show that if $|f(a)| > \eta$ then there is $\delta > 0$ such that

$$\text{if } x \in (a - \delta, a + \delta), \quad \text{then} \quad |f(x)| > \eta.$$

[7 marks]

Solution:

(a) The function $\cos x$ is differentiable on $\mathbb{R}$ with derivative $-\sin x$. Hence, by the mean value theorem

$$|\cos x - \cos y| = |\sin c||x - y|,$$

for some c between x and y. Since $|\sin c| \leq 1$ the claim follows.

(b) We have

$$f(x) = \sqrt{1-x}, \quad f'(x) = -\tfrac{1}{2}(1-x)^{-1/2}, \quad f''(x) = -\tfrac{1}{4}(1-x)^{-3/2} \quad f'''(x) = -\tfrac{3}{8}(1-x)^{-5/2}.$$

Hence the Taylor polynomial of degree two at zero is

$$1 - \frac{1}{2}x - \frac{1}{8}x^2.$$

The error made is less than

$$\left| \frac{f'''(c)}{3!} x^3 \right| \leq \frac{3}{48}(1-c)^{-5/2}|x|^3.$$

Here c is between 0 and 0.1 hence $(1-c)^{-5/2} \leq 0.9^{-5/2}$ and the error is at most 8.2×10^{-5}.

(c) Pick $\varepsilon = |f(a)| - \eta > 0$. Then there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset I$ and for $x \in (a - \delta, a + \delta)$ we have

$$|f(x) - f(a)| < \varepsilon. \quad \text{Hence: } |f(x)| > |f(a)| - \varepsilon = |f(a)| - |f(a)| + \eta = \eta.$$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Formulate precise definitions of the following two limits.

$$\text{i)} \lim_{x\to-\infty} f(x) = -\infty, \qquad \text{and} \qquad \text{ii)} \lim_{x\to\infty} f(x) = -\infty.$$

[4 marks]

(b) Assume that f satisfies $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = -\infty$. Prove that there exists a point $x_M \in \mathbb{R}$ such that

$$f(x_M) = \sup\{f(x) : x \in \mathbb{R}\}.$$

[6 marks]

(c) Assume now that f satisfies $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. Is there a point $z \in \mathbb{R}$ such that

$$f(z) = \pi\,?$$

Justify your answer. **[5 marks]**

(d) Assume that f is differentiable on $\mathbb{R}$ and that $f(0) = e$ and $f(100) = \sqrt{2}$. Does there exist a point $w \in \mathbb{R}$ such that

$$f'(w) = \tfrac{1}{100}(\sqrt{2} - e)\,?$$

Justify your answer. **[5 marks]**

Solution:

(a) i) For every $N \in \mathbb{R}$ there exists $M \in \mathbb{R}$ such that if $x < M$ then $f(x) < N$. ii) For every $N \in \mathbb{R}$ there exists $M \in \mathbb{R}$ such that if $x > M$ then $f(x) < N$.

(b) Recall that a continuous function on a closed bounded interval attains is maximum. We might not use this theorem directly, but it is still useful after we use the extra assumptions on our function. It follows from part (a) that there exists numbers $a, b \in \mathbb{R}$ such that

$$\text{if } x < a \text{ or } x > b \quad \text{then} \quad f(x) < f(0).$$

It follows that if we denote by x_M the point in the interval $[a, b]$ such that

$$f(x_M) = \sup\{f(x) : x \in [a, b]\},$$

then $f(x_M) \geq f(0) > f(x)$ for all $x \notin [a, b]$ as well. (Here x_M exists as $[a, b]$ is a closed bounded interval). Hence x_M is the global maximum of the function f.

(c) Using $\lim_{x\to-\infty} f(x) = -\infty$ we see that there exists $a \in \mathbb{R}$ such that $f(x) < \pi - 1$ for all $x < a$. Similarly, using $\lim_{x\to\infty} f(x) = \infty$ we see that there exists $b \in \mathbb{R}$ such that $f(x) > \pi + 1$ for all $x > b$. Therefore $f(a) \leq \pi - 1 < \pi < \pi + 1 \leq f(b)$. As π is an *intermediate value* between $f(a)$ and $f(b)$ there exists $z \in (a, b)$ such that $f(z) = \pi$ by the Intermediate value theorem.

(d) Yes such w exists. This is a consequence of the mean value theorem which implies the existence of $w \in (0, 100)$ such that

$$\sqrt{2} - e = f(100) - f(0) = f'(w)[100 - 0] = 100f'(w).$$

Hence the result follows.