Student Examination Number ....

The University of Edinburgh

College of Science and Engineering

FOR ANONYMOUSLY MARKED EXAMINATIONS
FOLD AND SEAL WITH BARCODE

Surname..........................

Pre-Names..........................

Signature..........................

Enrolment Number (as shown on your Matriculation card)..........................

Pre – Honours 2

MATH08064 Fundamentals of Pure Mathematics

Tuesday, 18th August 2015

9:30am – 12:30pm

Chairman of Examiners – Professor B Leimkuhler

External Examiner – Dr S D Theriault

All six questions from **Section A** count (40% total mark).
The **best three of four** answers from **Section B** will count (60% of total mark).

In the examination it is permitted to have;
A copy of the course text book
An Introduction to Analysis by W.R.Wade (for analysis)
Groups by C.R.Jordan and D.A.Jordan (for group theory)
Any other notes (printed or hand-written)

Calculators and other electronic aids
A scientific calculator is permitted in this examination.
It must not be a graphical calculator.
It must not be able to communicate with any other device.

Section A	Mark
1	
2	
3	
4	
5	
6	
Total	
Section B	**Mark**
7	
8	
9	
10	
Total	

This examination will be marked anonymously.

Section A (40 points). Solve all 6 problems in this section.

(1) State, with a brief justification, whether the following statements are TRUE or FALSE:

(a) $\mathbb{R}$ is a group under multiplication.

(b) $M_2(\mathbb{R}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \, a, b, c, d \in \mathbb{R} \right\}$ is a group under matrix multiplication.

(c) $\mathbb{R}^*_+ = \{r \in \mathbb{R} \mid r > 0\}$ is a group under multiplication.

[7 marks]

(2) State, with a brief justification, whether the following statements are TRUE or FALSE:

(a) $D_3 \cong \mathbb{Z}_6$.

(b) $D_3 \cong S_3$.

(c) $D_{12} \cong S_4$.

[7 marks]

(3) Suppose that G is a group with subgroup H, and let $g \in G$. Prove that

$$gH \text{ is a subgroup of } G \iff g \in H.$$

[6 marks]

(4) Are the following statements TRUE or FALSE? If TRUE give a proof, whereas if FALSE give a counterexample or justification.

(a) Let $f : \mathbb{Z} \to \mathbb{R}$. Then f is a continuous function.

(b) Let $f : \mathbb{R} \to \mathbb{Q}$ be a continuous function. Then f is a constant function.

(c) Let $f : [a, b] \to \mathbb{R}$ be a continuous function with maximum at a point $c \in (a, b)$. Then $f'(c) = 0$.

[6 marks]

(5) Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence. Then the number

$$\lim_{n\to\infty} (\inf\{a_k : k \geq n\})$$

is called *limit inferior* and is denoted by $\liminf_{n\to\infty} a_n$.

(a) Show that the limit inferior is well defined, that is the limit $\lim_{n\to\infty}(\inf\{a_k : k \geq n\})$ exists and is finite for any bounded sequence (a_n).

(b) Show that the sequence $(a_n)_{n=1}^{\infty}$ has a subsequence that converges to $\liminf_{n\to\infty} a_n$.
Hint: Argue that for any $n \in \mathbb{N}$ one can find $i \geq n$ such that

$$\inf\{a_k : k \geq n\} \leq a_i < \inf\{a_k : k \geq n\} + \frac{1}{n}.$$

Use this to construct the subsequence we are looking for. **[7 marks]**

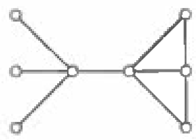
(6) Consider the sequence defined recursively by

$$a_1 = 0, \qquad a_{n+1} = \sqrt{6 + a_n}, \quad \text{for } n = 2, 3, \ldots.$$

Prove that this sequence is convergent and find its limit. **[7 marks]**

Section B (60 points). Solve any three problems in this section.

(7) (a) Prove that the symmetry group of the graph



is isomorphic to $S_3 \times S_2$. **[6 marks]**

(b) For the above graph, consider its set of vertices V, and its group of symmetries G. Then G acts on V.

(i) The set V is partitioned into orbits. Determine this partition.

(ii) For each vertex $v \in V$, describe the stabilizer $\mathrm{Stab}_G(v)$.

[10 marks]

(c) State, with a brief justification, whether the following statements are TRUE or FALSE.

(i) The group action in (b) is transitive.

(ii) The group action in (b) is faithful.

[4 marks]

(8) (a) Recall that $\mathbb{C}^* := \{z \in \mathbb{C} \mid z \neq 0\}$, which is a group under multiplication. Show that the exponential function $\exp\colon \mathbb{C} \to \mathbb{C}^*$ sending $z \mapsto e^z$ is a group homomorphism, and determine its image and kernel. **[6 marks]**

(b) State whether the following claims are TRUE or FALSE. If they are TRUE give a proof, whereas if they are FALSE give a counterexample.

(i) The function $D_3 \to \mathbb{Z}_6$ sending all elements to 1 is a group homomorphism.

(ii) The function $D_3 \to \mathbb{Z}_6$ sending all elements to 0 is a group homomorphism.

(iii) The function $\mathbb{Z}_{10} \to \mathbb{Z}_{10}$ sending $z \mapsto 2z$ (for all $z \in \mathbb{Z}_{10}$) is a group homomorphism.

[9 marks]

(c) Give an explicit example of a group homomorphism $\theta\colon G \to H$ for which $\operatorname{Im}\theta$ is not a normal subgroup of H. **[5 marks]**

(9) Let $f(x) = \exp(-x^2)$. Consider the sequence defined recursively by

$$x_{n+1} = f(x_n), \quad n = 1, 2, 3, \ldots, \qquad x_1 = \frac{1}{3}.$$

(a) Show that $f : [0, 1] \to [0, 1]$. **[3 marks]**

(b) Use the intermediate value theorem to show that there exists $u \in (0, 1)$ such that $\exp(-u^2) = u$. **[7 marks]**

(c) Show that $|f'(x)| \leq \sqrt{2\exp(-1)} < 1$ for all $x \in [0, 1]$. **[3 marks]**

(d) Let $c = \sqrt{2\exp(-1)}$. Show using the mean value theorem that

$$|x_{n+1} - u| \leq c|x_n - u|, \qquad \text{for all } n \in \mathbb{N}.$$

[5 marks]

(e) Prove that the limit $\lim_{n\to\infty} x_n$ exists and equals u. **[2 marks]**

[Please turn over]

(10) (a) Suppose that $(a_k)_{k=1}^{\infty}$ is a sequence such that $\frac{|a_{k+1}|}{|a_k|} \to a > 0$ for some $a \in \mathbb{R}$. Prove that for all $|x| < 1/a$ the series

$$\sum_{k=1}^{\infty} a_k (\sin x)^k$$

is absolutely convergent. **[7 marks]**

(b) For which $p > 0$ is the series

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k^p}$$

convergent? Absolutely convergent? **[7 marks]**

(c) State, with a brief justification, whether the following statements are TRUE or FALSE:

(i) If $|a_k| \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ is absolutely convergent then $\sum_{k=1}^{\infty} a_k$ is convergent.

(ii) If the sequence $(a_n)_{n\in\mathbb{N}} \to 0$ as $n \to \infty$ then there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} a_{n_k}$$

is convergent.

(iii) If $\sum_{k=1}^{\infty} a_k = \infty$ then the limit $\lim_{k\to\infty} a_k$ cannot equal to zero. **[6 marks]**

[End of Paper]