Fundamentals of Pure Mathematics 2015-16
Assignment 2
Due-date: Thursday of week 3
(submit your work to your tutor at the workshop)

1. (5 marks) (This is Problem 14 from the week 2 workshop)

   Let $A = \left\{x \in \mathbb{R} : x^2 > 5 \text{ and } x \text{ is a positive irrational}\right\}$. Prove that A is non-empty, bounded below, and $\inf A = \sqrt{5}$.

2. (5 marks) If A is a non-empty bounded above subset of $\mathbb{R}$ and $x > 0$ prove that the set[1] xA is non-empty, bounded above, and $\sup(xA) = x\sup(A)$.

Solutions

1. Clearly[2], $\sqrt{7} \in A$ therefore A is non-empty.

   All elements of A are positive, therefore A is bounded below.

   It remains to show that the greatest lower bound of A is $\sqrt{5}$.

   For all $x \in A$ we have $x > \sqrt{5}$, therefore $\sqrt{5}$ is a lower bound of A.

   We claim that $\sqrt{5}$ is the greatest lower bound of A. It is enough to show that any real number $m > \sqrt{5}$ is not a lower bound of A. Pick an irrational x such that[3] $m > x > \sqrt{5}$. Then x is an element of A (because $x^2 > 5$ and x is a positive irrational) smaller than m, which shows that m is not a lower bound of A.

2. Since A is non-empty there is a real number a_0 with $a_0 \in A$. Then $xa_0 \in xA$, therefore xA is non-empty.

   Since the set A is bounded above there exists $M \in \mathbb{R}$ such that for all $a \in A$ we have $a \le M$. Multiplying by x we find $xa \le xM$. This shows that xM is an upper bound of the set xA. Therefore the set xA is bounded above.

[1] For $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, we denote by xA the set $\{xa : a \in A\}$. Example 1: If $A = \{1,2,3\}$ and $x = 4$ then $xA = \{4\cdot 1, 4\cdot 2, 4\cdot 3\} = \{4,8,12\}$. Example 2: If $A = [-1,2]$ and $x = 3$ then $xA = [-3,6]$.

[2] We assume we all know that $\sqrt{7}$ is irrational and that we all know how to prove it.

[3] Between any two real numbers there exists at least one irrational (Liebeck, Chapter 2). Actually, between any two real numbers there are infinitely many irrationals, but we don't need that.

It remains to show that the least upper bound of the set xA is $x\sup A$. We already know that $x\sup A$ is an upper bound of xA (just set $M = \sup A$ in the argument of the last paragraph).

On the other hand, if M' is any upper bound of xA then for all $a \in A$ we have $xa \leq M'$, therefore $a \leq \frac{M'}{x}$. This shows that $\frac{M'}{x}$ is an upper bound of A, and since $\sup A$ is the smallest upper bound of A, we have $\sup A \leq \frac{M'}{x}$. Therefore, $x\sup A \leq M'$.

We have shown that among all upper bounds of xA, $x\sup A$ is the smallest, in other words $x\sup A = \sup(xA)$.

Things to think about:
Where was the hypothesis $x > 0$ used in the proof? Is the result true for $x < 0$? How about $x = 0$?
What is the corresponding property for infima?