8. SOLVING OBLIQUE TRIANGLES: THE LAW OF COSINES

When two sides and the included angle (SAS) or three sides (SSS) of a triangle are given, we cannot apply the law of sines to solve the triangle. In such cases, the law of cosines may be applied.
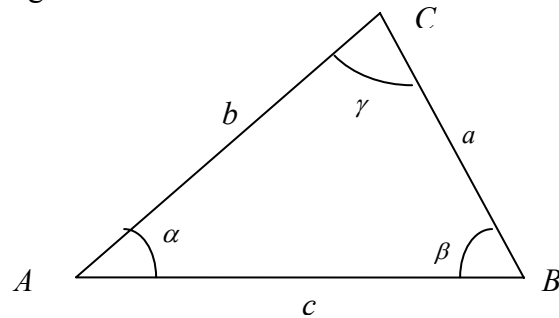
Theorem 8.1: The Law of Cosines

In the general triangle $\triangle ABC$, the square of the length of any side is equal to the sum of the squares of the lengths of the other two sides minus twice the product of those side lengths times the cosine of the angle between them.

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
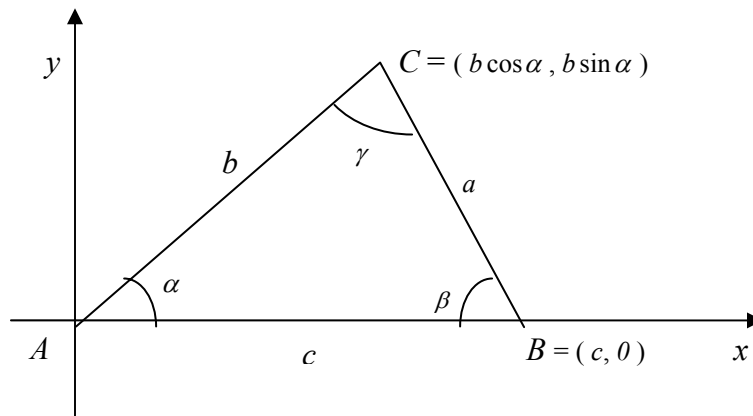$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

To prove the theorem, we place triangle $\triangle ABC$ in a coordinate plane with vertices labeled counterclockwise and so that one side lies on the positive x axis and one vertex is at O.

Suppose that A is at $(0,0)$. Then $B = (c,0)$ and $C = (b\cos\alpha, b\sin\alpha)$.

Thus, $\left| \overline{BC} \right|^2 = (b\cos\alpha - c)^2 + (b\sin\alpha)^2 = a^2$.

$$b^2\cos^2\alpha - 2bc\cos\alpha + c^2 + b^2\sin^2\alpha = a^2$$

So $a^2 = b^2 + c^2 - 2bc\cos\alpha$.

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Now rotate the triangle so that B is at the origin and C is on the positive x axis.

An analogous argument now gives
$$b^2 = a^2 + c^2 - 2ac\cos\beta .$$

When C is at the origin, we find
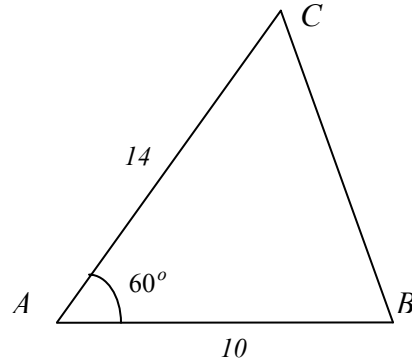$$a^2 = b^2 + c^2 - 2bc\cos\alpha .$$

Example 8.1

SAS case: Solve the triangle $\triangle ABC$ if $\alpha = 60^\circ$, $b = 14$, $c = 10$.

Since $a^2 = b^2 + c^2 - 2bc\cos\alpha$
$\quad = (14)^2 + (10)^2 - 2(14)(10)\cos(60^o)$
$\quad = 196 + 100 - 140$
$\quad a^2 = 156$
$\Rightarrow a = 12.49.$

It is geometrically evident that
β is acute and by the law of sines

$\dfrac{\sin 60^\circ}{12.49} = \dfrac{\sin\beta}{14} \Rightarrow \sin\beta = \left(\dfrac{14}{12.49}\right)\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{7\sqrt{3}}{12.49}$

$\Rightarrow \beta = \arcsin\left(\dfrac{7\sqrt{3}}{12.49}\right) = 76.10^o$.

Then $\gamma = 180^\circ - 60^\circ - 76.10^\circ = 43.90^\circ$.

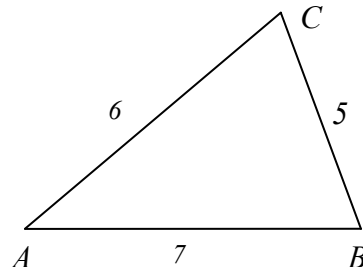*Example 8.2*

SSS case: Solve the triangle $\triangle ABC$ if $a = 5$, $b = 6$, $c = 7$.

$a^2 = b^2 + c^2 - 2bc\cos\alpha$
$25 = 36 + 49 - 2(6)(7)\cos\alpha$

$25 = 85 - 84\cos\alpha \Rightarrow \cos\alpha = \dfrac{60}{84} = \dfrac{5}{7}$

$\Rightarrow \alpha = \arccos\left(\dfrac{5}{7}\right) = 44.42^o$

Note: there is no other angle θ for which:

$0^o \le \theta < 180^o$ and $\cos\theta = \dfrac{5}{7}$.

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Then by the law of sines

$$\frac{\sin 44.42^{\text{o}}}{5} = \frac{\sin \beta}{6} \Rightarrow \sin\beta = \left(\frac{6}{5}\right)\sin 44.42^{\text{o}}$$
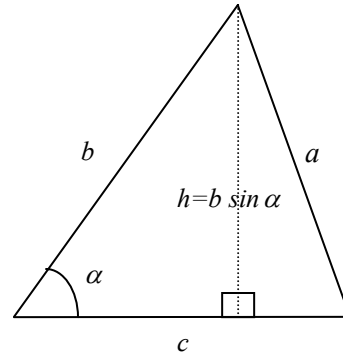
$$\Rightarrow \beta = \arcsin\left(\left(\frac{6}{5}\right)\sin 44.42^{\text{o}}\right) = 57.13^{o}$$. Clearly, β must be acute.

Then $\gamma = 180^{\text{o}} - 44.42^{o} - 57.13^{o} = 78.45^{\text{o}}$.

Theorem 8.2: Heron's Area Formula

The area of a triangle with sides a, b, and c and semiperimeter $s = \dfrac{a+b+c}{2}$ has area A given by $A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$.

The proof follows from the law of cosines expressed in the form:

$$2bc\cos\alpha = b^2 + c^2 - a^2$$

Note that $A = \dfrac{1}{2}ch = \dfrac{1}{2}bc\sin\alpha \Rightarrow A^2 = \dfrac{1}{4}b^2c^2\sin^2\alpha$.

Now we may obtain the desired formula by algebraic manipulation.

$$A^2 = \frac{1}{4}b^2c^2\sin^2\alpha = \frac{1}{4}b^2c^2(1-\cos^2\alpha)$$

$$= \frac{1}{16}(2bc)(1+\cos\alpha)(2bc)(1-\cos\alpha)$$

$$= \frac{1}{16}(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)$$

$$A^2 = \frac{1}{16} \times \left[(b+c)^2 - a^2 \right] \left[(a^2 - (b-c)^2 \right]$$

$$= \frac{(b+c+a)}{2} \frac{(b+c-a)}{2} \frac{(a-b+c)}{2} \frac{(a+b-c)}{2}$$

$$= \left[\frac{a+b+c}{2} \right]\left[\frac{a+b+c}{2} - a \right]\left[\frac{a+b+c}{2} - b \right]\left[\frac{a+b+c}{2} - c \right]$$

$$A^2 = s(s-a)(s-b)(s-c).$$

Section 8 Problems

In problems 1 to 5 use the law of cosines to find the specified part of the triangle $\triangle ABC$. Round off angles to the nearest hundredth of a degree and side lengths to four significant digits.

1. Find c if $a = 3$, $b = 10$, $\gamma = 60°$.

2. Find a if $b = 3.2$, $c = 2.4$, $\alpha = 117°$.

3. Find β if $a = 200$, $b = 50$, $c = 177$.

4. Find a if $b = 68$, $c = 14$ and $\alpha = 24.5°$.

5. Find γ if $a = 2$, $b = 3$ and $c = 4$.

In problems 6 through 8 use Heron's Formula to find the area of the triangle.

6. Find the area of a right triangle with sides 3, 4, and 5.

7. Find the area of the triangle with sides 31, 42, and 53.

8. Find the area of the triangle with sides 5.9, 6.7, and 10.3.

9. Use the answer you obtained in problem 7 to find the length h of the shortest altitude of the triangle with sides 31, 42, and 53.

10. A ship takes a sighting on two buoys. At a certain instant, the bearing of buoy A is N 44.23° W, and that of buoy B is N 62.17° E. The distance between the buoys is 3.60 km, and the bearing of B from A is N 87.87° E. Find the distance of the ship from each buoy.
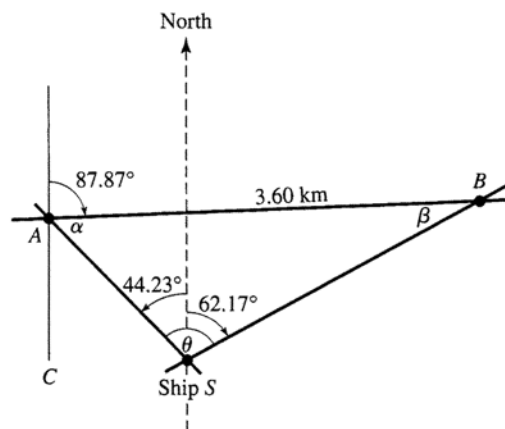
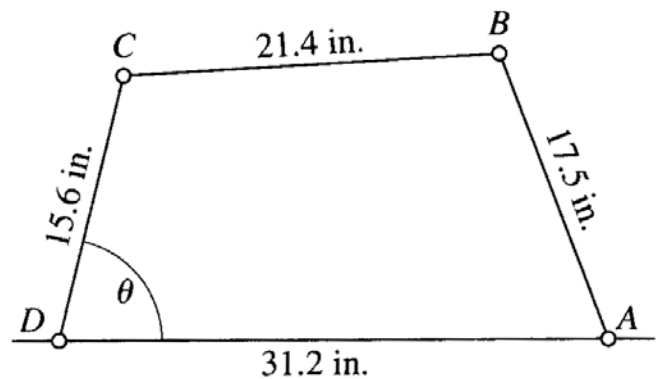**11.** A 71.6-m-high antenna mast is to be placed on sloping ground, with the cables making an angle of 42.5° with the top of the mast. Find the length of each cable.
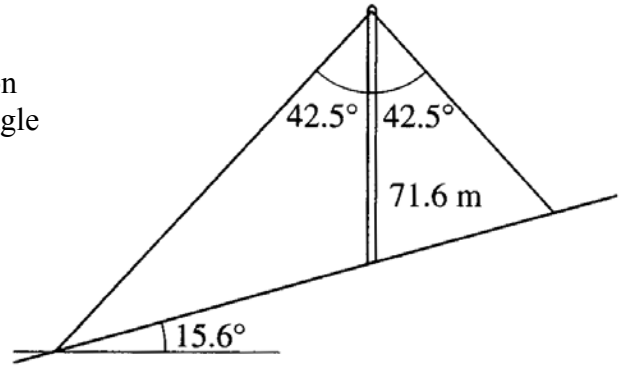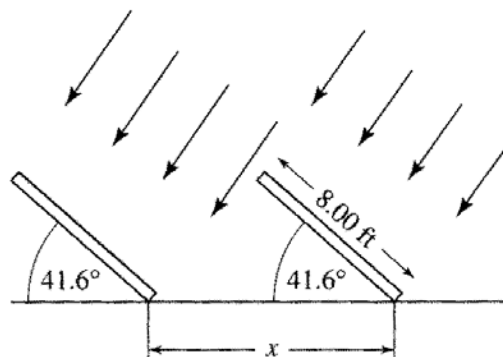
12. In the roof truss shown in the figure, find the lengths of the members *AB*, *BD*, *AC*, and *AD*.
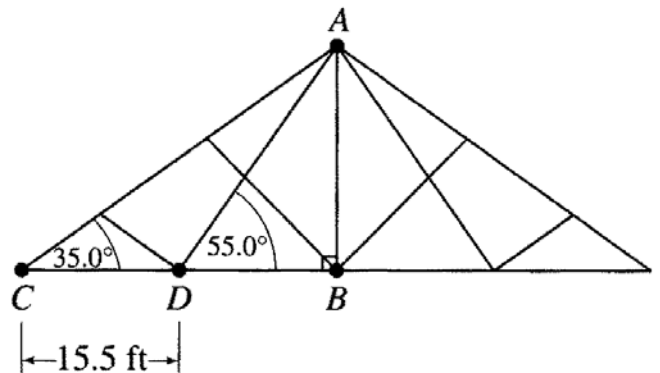
13. In the four-bar linkage shown in the figure, find the angle θ when $\angle BAD$ is 41.5°.

14. Two solar panels are to be placed as shown in the figure. Find the minimum distance x so that the first panel will not cast a shadow on the second when the angle of elevation of the sun is 18.5°.

15. Find the length of side *AB* in the quadrilateral shown here.