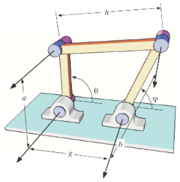
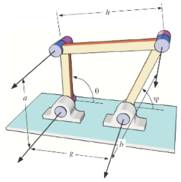


Four-Bar Linkage Analysis:
Slider Crank Linkage

J. Michael McCarthy

July 14, 2009

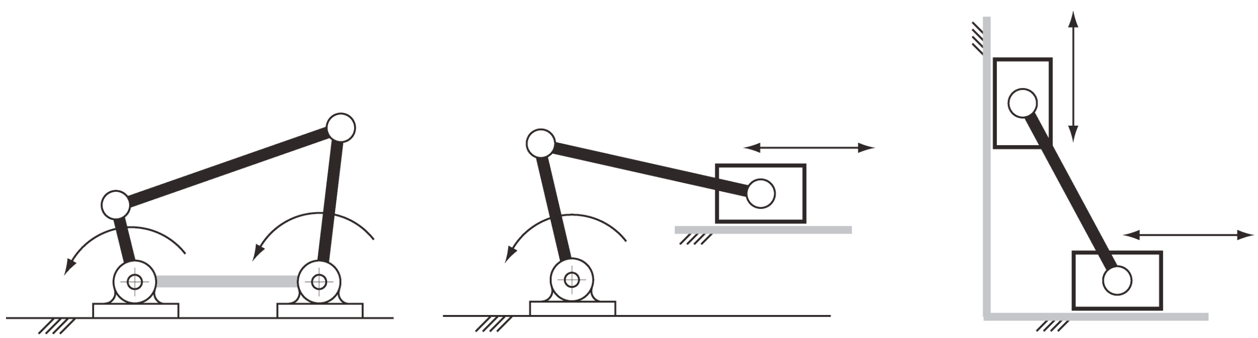


The Slider-Crank Linkage

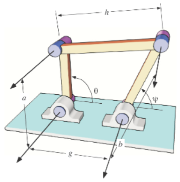
Let R denote a hinged joint and P a sliding joint (prismatic). These are the initials in "revolute joint" and "prismatic joint," which are the technical names of these joints.

A four-bar linkage with four revolute joints and forms a quadrilateral. A four-bar linkage that has a prismatic joint forms a triangle.

These figures show an RRRR, RRRP, and PRRP linkages:

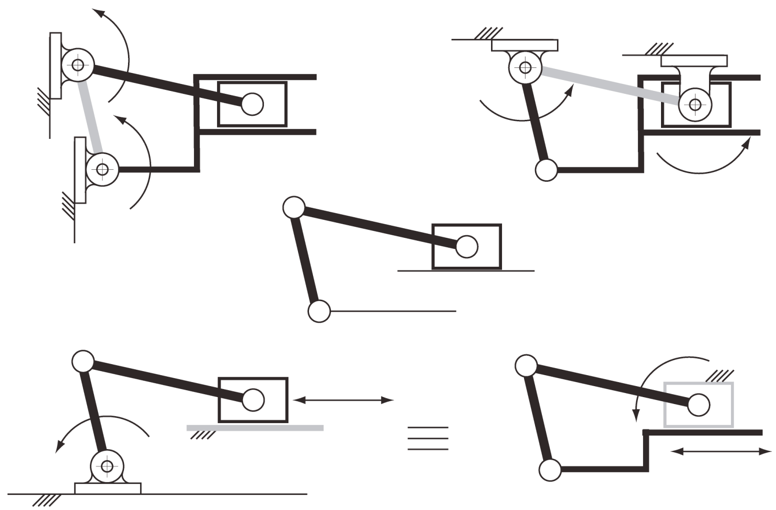


The RRRP is the *slider-crank* linkage, and the PRRP is called a *double slider* linkage.

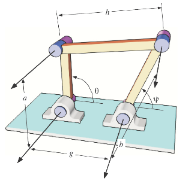


Inversions of the Slider-Crank

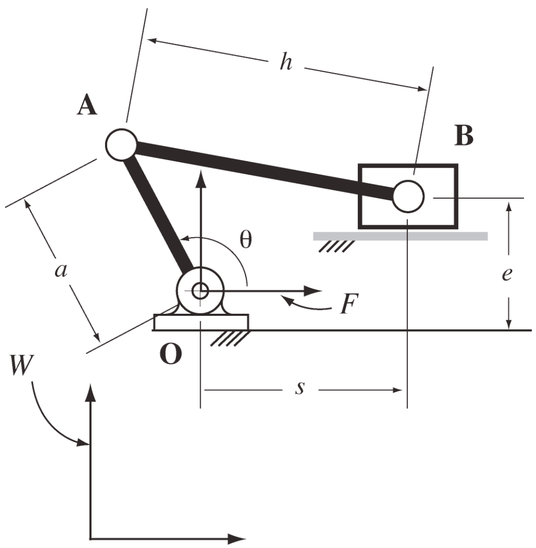
Cyclic permutation of the joints RRRP to obtain RRPR, RPRR, and PRRR is the same as changing which link of the four-bar loop is to be the ground link. Each of these is called an *inversion* of the slider-crank linkage.

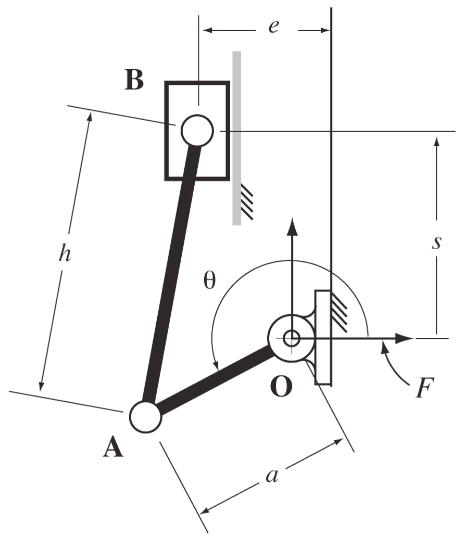


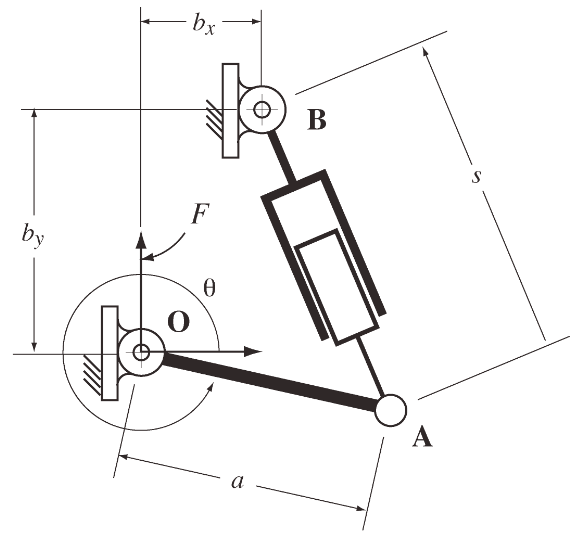
The RRPR called an “inverted slider-crank,” and the RPRR is called a “turning block.” The PRRR is actually the same as the RRRP.



Linkage Frames for Slider-Cranks



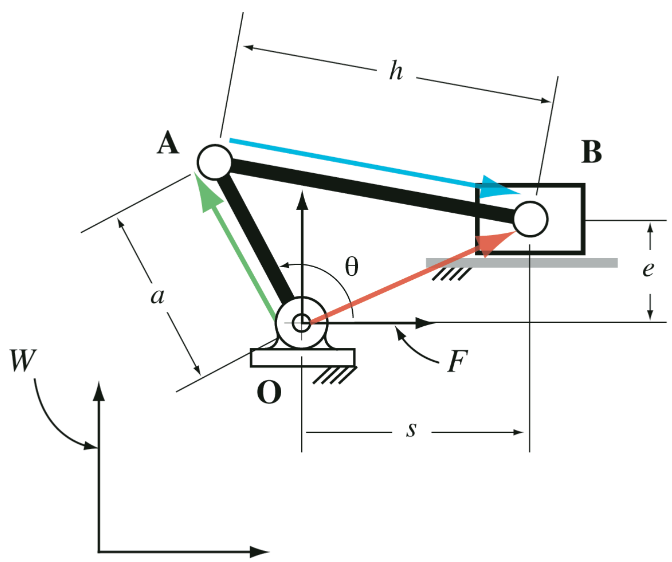




For a slider-crank and its inversions, it is generally convenient to consider the slider movement s as either the input or output parameter, even if it is not moving relative to the ground link. Then, let define as θ the rotation of the crank.

Constraint Equation for a Slider-Crank

For **slider-crank linkages** the constraint equation can take several forms. If the slider is an input or output link, then the constraint equation is given by the distance between the two moving pivots of the connecting rod **AB**.



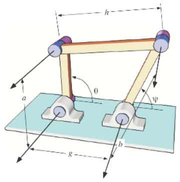
$$\mathcal{C}: \quad (\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A}) - h^2 = 0,$$

where

$$\mathbf{A} = \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{Bmatrix} s \\ e \end{Bmatrix}.$$

$$\mathcal{C}: \; s^2 + (-2a\cos\theta)s + (a^2 + e^2 - h^2 - 2ae\sin\theta) = 0.$$

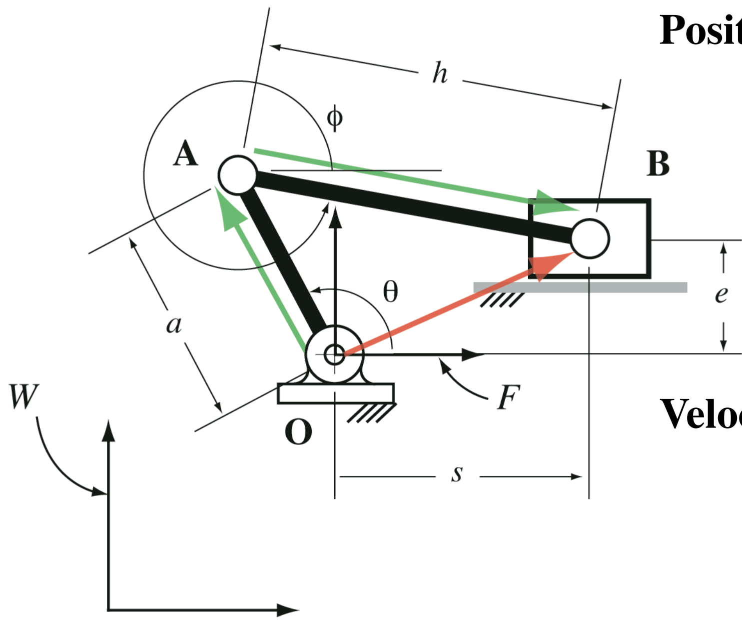
The quadratic formula yields two values of slider position s for each value of the input crank angle θ. The range of crank angles that yield real values for s is defined by the discriminant of quadratic formula.



Loop Equations for a Slider-Crank

Once the output slide s has been determined using the slider-crank constraint equation, the position loop equations are used to compute the coupler angle ϕ.

The position and velocity loop equations for a slider-crank are obtained in the same way as for a 4R linkage.



Position Loop Equations:

$$\mathbf{A} + (\mathbf{B} - \mathbf{A}) = \mathbf{B},$$

$$\begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} + \begin{Bmatrix} h\cos\phi \\ h\sin\phi \end{Bmatrix} = \begin{Bmatrix} s \\ e \end{Bmatrix}.$$

yields the angle $\phi = \arctan\left(\dfrac{e - a\sin\theta}{s - a\cos\theta}\right).$

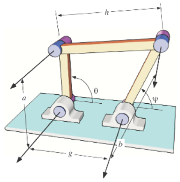
Velocity Loop Equations

$$\dot{\mathbf{A}} + \frac{d}{dt}(\mathbf{B} - \mathbf{A}) = \dot{\mathbf{B}},$$

$$\begin{Bmatrix} -a\sin\theta \\ a\cos\theta \end{Bmatrix} \dot\theta + \begin{Bmatrix} -h\sin\phi \\ h\cos\phi \end{Bmatrix} \dot\phi = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \dot s.$$

The velocity loop equations can be written as the matrix equation and solved using Cramer's rule.

$$\begin{bmatrix} 1 & h\sin\phi \\ 0 & -h\cos\phi \end{bmatrix} \begin{Bmatrix} \dot s \\ \dot\phi \end{Bmatrix} = \begin{Bmatrix} -a\sin\theta \\ a\cos\theta \end{Bmatrix} \dot\theta.$$



Acceleration Loop Equations

The values of θ, $d\theta/dt = \omega_\theta$ and $d^2\theta/dt^2 = \alpha_\theta$ are known input parameters to the motion of the slider-crank linkage, so the acceleration loop equations can be rearranged into the form,

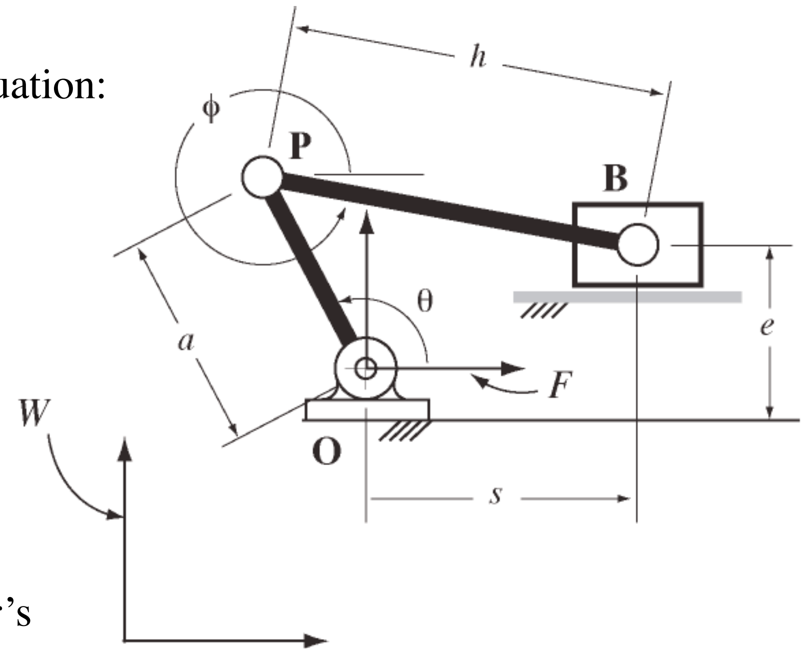
Acceleration Loop Equations:

$$\ddot{\mathbf{A}} + \frac{d^2}{dt^2}(\mathbf{B} - \mathbf{A}) = \ddot{\mathbf{B}},$$

$$\begin{Bmatrix} -a\sin\theta \\ a\cos\theta \end{Bmatrix}\ddot{\theta} - \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix}\dot{\theta}^2 + \begin{Bmatrix} -h\sin\phi \\ h\cos\phi \end{Bmatrix}\ddot{\phi} - \begin{Bmatrix} h\cos\phi \\ h\sin\phi \end{Bmatrix}\dot{\phi}^2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}\ddot{s}.$$

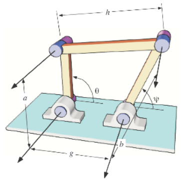
These equations can be assembled into the matrix equation:

$$\begin{bmatrix} 1 & h\sin\phi \\ 0 & -h\cos\phi \end{bmatrix}\begin{Bmatrix} \ddot{s} \\ \ddot{\phi} \end{Bmatrix} = \begin{Bmatrix} K_x \\ K_y \end{Bmatrix}.$$



- The values for ϕ, s, and $d\phi/dt = \omega_\phi$, ds/dt are determined by solving the *position* and *velocity* loop equations,

- Thus, the parameters K_x and K_y are known and the *acceleration* loop equations are solved using Cramer's rule.



Inverted Slider Crank

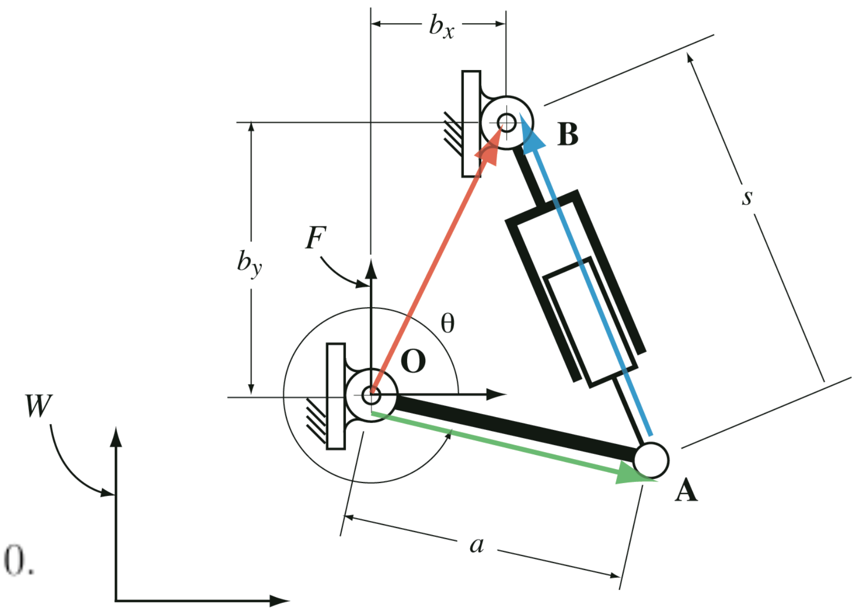
The constraint equation for the **inverted slider-crank linkage** is the distance between the pivots **AB** on either side of the slider. In this case, the distance is not constant. Instead, it equals the amount of slider movement.

$$\mathcal{C}: \quad (\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A}) - s^2 = 0,$$

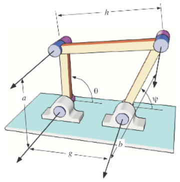
where

$$\mathbf{A} = \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} \text{ and } \mathbf{B} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}.$$

$$\mathcal{C}: \; s^2 + 2ab_x\cos\theta + 2ab_y\sin\theta - a^2 - b_x^2 - b_y^2 = 0.$$



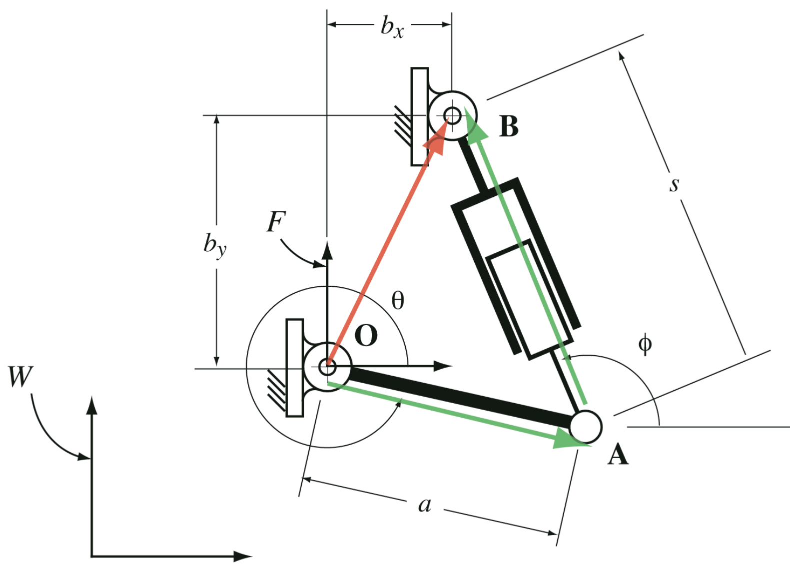
The slider position s for each value of the input crank angle θ is computed using the quadratic formula. The discriminant of quadratic formula defines the range values for the crank angle θ.



Loop Equations: Inverted Slider-Crank

Once the output slide s has been determined for the inverted slider crank, we use its loop equations to compute the coupler angle ϕ.

The position and velocity loop equations for the inverted slider-crank are obtained in the same way as for a slider-crank linkage.



Position Loop Equations

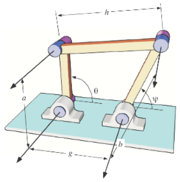
$$\mathbf{A} + (\mathbf{B} - \mathbf{A}) = \mathbf{B},$$

$$\begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} + \begin{Bmatrix} s\cos\phi \\ s\sin\phi \end{Bmatrix} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}.$$

Velocity Loop Equations

$$\dot{\mathbf{A}} + \frac{d}{dt}(\mathbf{B} - \mathbf{A}) = 0,$$

$$\begin{Bmatrix} -a\sin\theta \\ a\cos\theta \end{Bmatrix}\dot{\theta} + \begin{Bmatrix} \cos\phi \\ \sin\phi \end{Bmatrix}\dot{s} + \begin{Bmatrix} -s\sin\phi \\ s\cos\phi \end{Bmatrix}\dot{\phi} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$



Mechanical Advantage

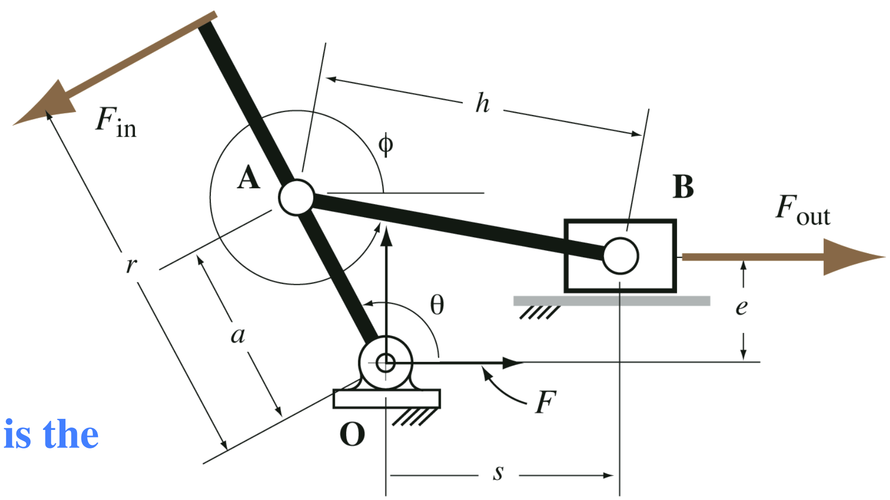
The derivative of the constraint equations yields a relationship between the input angular velocity and output velocity:

$$\dot{C} = 2(\mathbf{B} - \mathbf{A}) \cdot (\dot{\mathbf{B}} - \dot{\mathbf{A}})\delta t = (-ae\cos\theta + as\sin\theta)\dot{\theta} + (s - a\cos\theta)\dot{s} = 0.$$

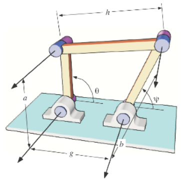
This yields the speed ratio: $$R = \frac{\theta}{\dot{s}} = \frac{s - a\cos\theta}{-ae\cos\theta + as\sin\theta}.$$

Let the input torque be $T_{in} = F_{in}\, r$, then "power in equals power out" yields the relationship" $$F_{in} r\dot{\theta} = F_{out}\dot{s},$$

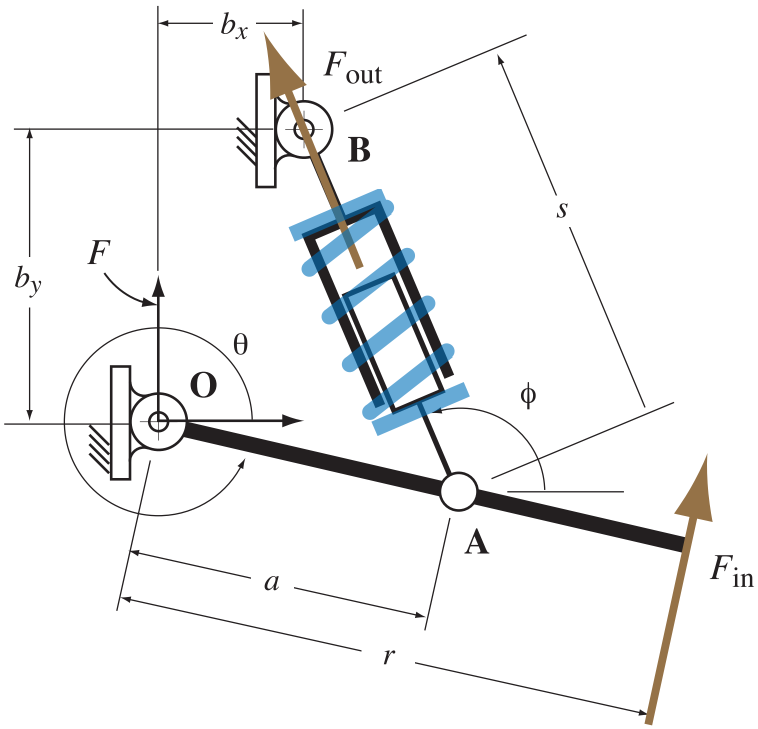
$$\frac{F_{out}}{F_{in}} = \frac{r\dot{\theta}}{\dot{s}} = rR.$$



In this case, the mechanical advantage is the distance r times the speed ratio.



Spring Suspension



The suspension formed by using a spring to support a crank form an inverted slider-crank linkage.

Recall that the constraint equation of this linkage is:

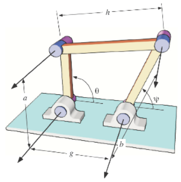
$$\mathcal{C}: \quad s^2 + 2ab_x\cos\theta + 2ab_y\sin\theta - a^2 - b_x^2 - b_y^2 = 0.$$

From this constraint equation, we can compute the mechanical advantage of this system to be:

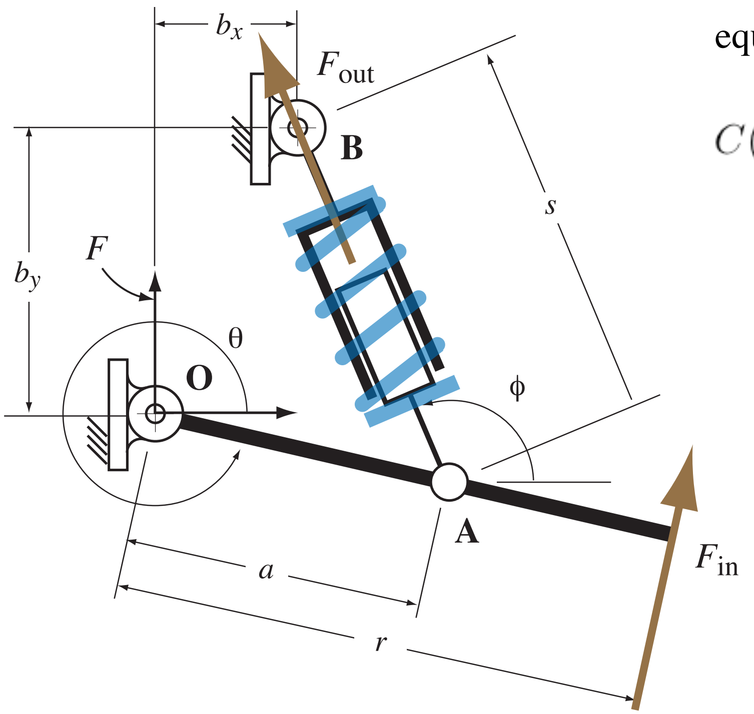
$$\frac{F_{out}}{F_{in}} = \frac{r\dot{\theta}}{\dot{s}} = \frac{rs}{a(-b_y\cos\theta + b_x\sin\theta)}.$$

Special case

Assemble the suspension so $b_x = a$, and $b_y = s_0$, then for $\theta=0$, $\frac{F_{out}}{F_{in}} = \frac{r}{a}$



Equivalent Spring Rate



In the vicinity of a reference configuration, θ_0, s_0 , the constraint equation of the inverted slider crank can be written as the Taylor series:

$$C(\theta_0+\Delta\theta, s_0+\Delta s) = C(\theta_0, s_0) + \frac{\partial C}{\partial \theta}\Delta\theta + \frac{\partial C}{\partial s}\Delta s + \ldots = 0,$$
$$= (-ab_y\cos\theta_0 + ab_x\sin\theta_0)\Delta\theta + (-s_0)\Delta s = 0.$$

Now force on the spring is $F_{in} = k\,\Delta s$, so from the formula for mechanical advantage and the equation above, we have that in the vicinity of the reference configuration, θ_0, s_0:

$$F_{\text{out}} = k\left(\frac{rs_0}{a(-b_y\cos\theta_0 + b_x\sin\theta_0)}\right)^2 r\Delta\theta = k_{\text{eq}}\Delta y.$$

Here Δy is the vertical displacement of the wheel.

Therefore the equivalent spring rate is given by

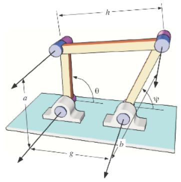
$$k_{\text{eq}} = k\left(\frac{rs_0}{a(-b_y\cos\theta_0 + b_x\sin\theta_0)}\right)^2$$

Special case

For $\theta=0$, and $b_x = a$, and $b_y = s_0$,

then $k_{eq} = k(r/a)^2$.

UCIrvine The Henry Samueli School of Engineering



Summary

- An *ideal linkage* is a collection of rigid links connected by *ideal joints*. These assumptions allow the use of geometry to analyze the movement of the linkage and evaluate its mechanical advantage. Real linkage systems will flex and lose energy through friction and wear.

- The *constraint equation* of a linkage is obtained from the distance specified between the input and output moving pivots. This equation is solved to determine the relationship between the input and output variables. Differentiation of this constraint yields the velocities needed to compute mechanical advantage.

- The *position, velocity, and acceleration loop equations* are used to compute the other configuration variables in the linkage, such as the coupler angle and its angular velocity.