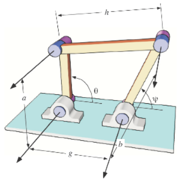

Four-Bar Linkage Analysis:
The 4R Quadrilateral

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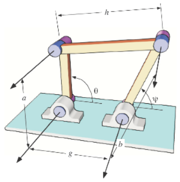
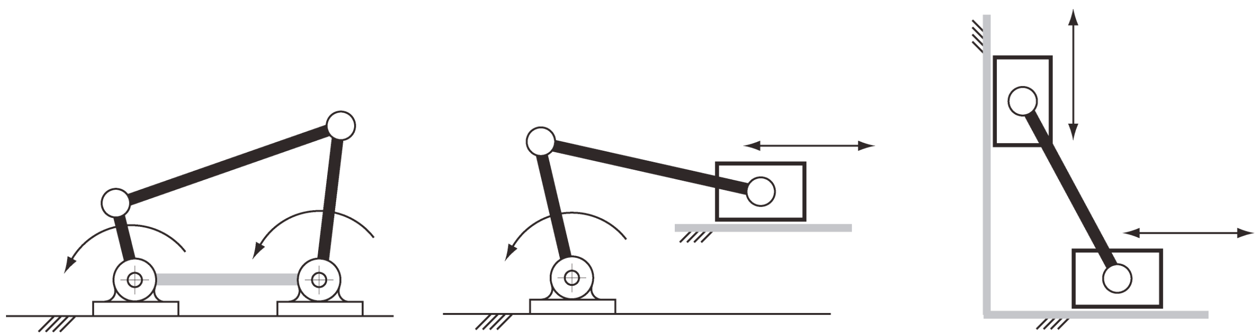
July 13, 2009



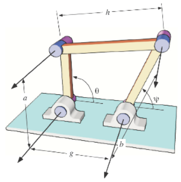
The Four-bar Linkage

Let R denote a hinged joint and P a sliding joint (prismatic). These are the initials in “revolute joint” and “prismatic joint,” which are the technical names of these joints.

Now let the sequence $j_1 j_2 j_3 j_4$ be the sequence of four joints in a four-bar linkage beginning and ending with the joints j_1 and j_4 connected to the ground frame, one of which is connected to the input and the other to the output.

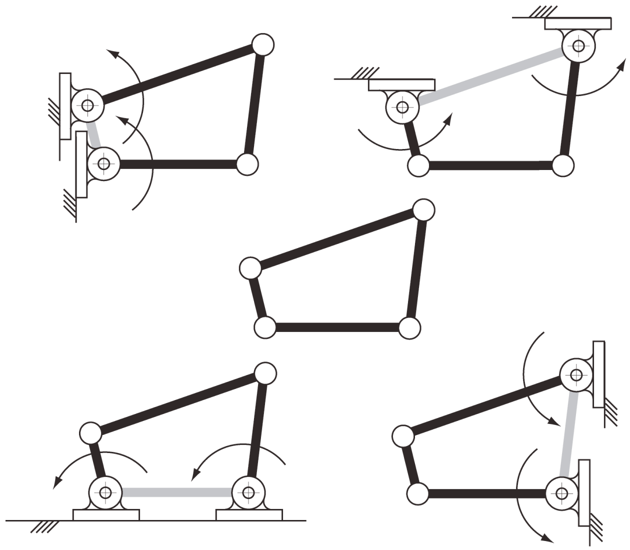


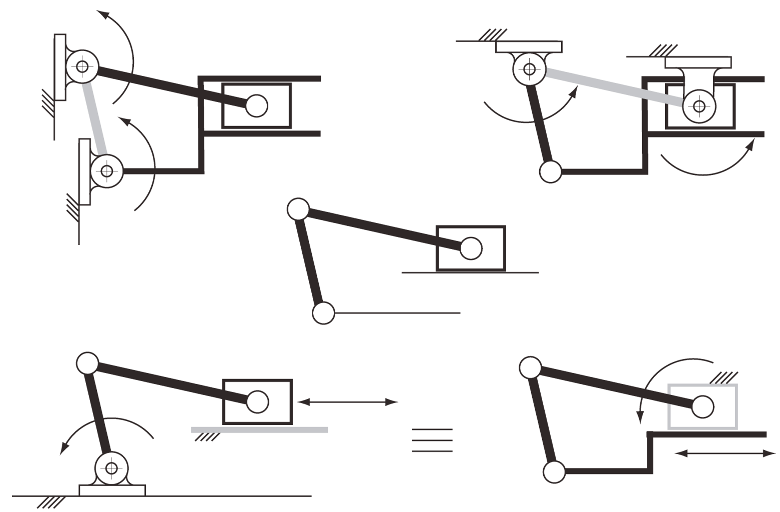
For example, the RRRR is the *four-bar* quadrilateral, the RRRP is the *slider-crank* linkage, and the PRRP is called a *double slider* linkage.



Inversions of the Four-Bar Linkage

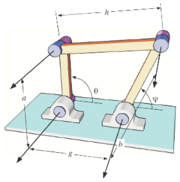
Cyclic permutation of the joints $j_1j_2j_3j_4$ to obtain $j_2j_3j_4j_1$, $j_3j_4j_1j_2$, and $j_4j_1j_2j_3$ is the same as changing which link of the loop is to be the ground link. Each of these is called an *inversion* of the linkage.





These are the four inversions of the 4R linkage. They are also called crank-rocker, rocker-crank, double rocker and drag-link mechanisms, to describe how they move.

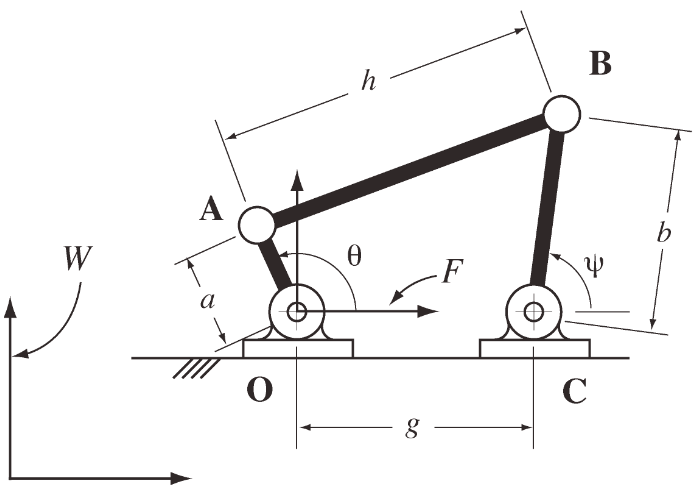
The inversions of the 3RP slider-crank consist of the RRPR called an "inverted slider-crank," and the RPRR is called a "turning block." The PRRR is the same as the RRRP.

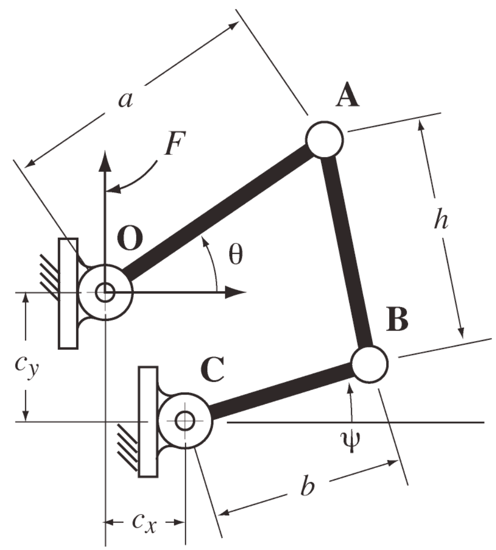


Linkage Analysis

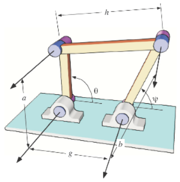
Linkage analysis determines the each of the joint angles of the linkage for a given values of the input angle.

The first step is to position a local linkage frame at the fixed pivot for the input crank and then identify the configuration parameters and dimensions of the linkage in terms of the local frame.



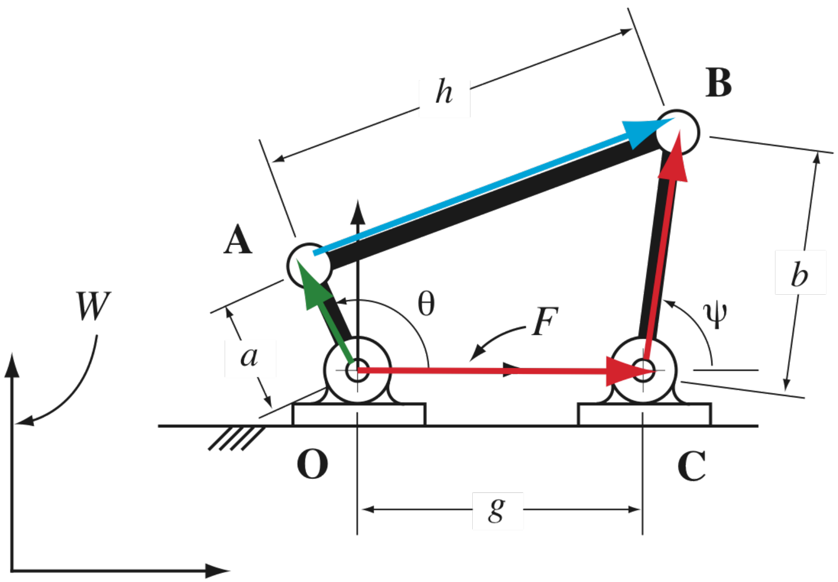


For a four-bar quadrilateral, the two links joined to the ground frame are generally the input and output cranks. For convenience, let θ be the input crank rotation and ψ be the output crank rotation.



Constraint Equation: 4R Linkage

The constraint equation for a **4R linkage** is obtained from the requirement that the distance *h* between the moving pivots **A** and **B** be constant throughout the movement of the linkage.



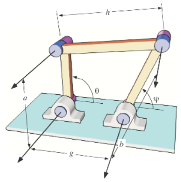
$$\mathcal{C}: \quad (\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A}) - h^2 = 0,$$

where

$$\mathbf{A} = \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{Bmatrix} g \\ 0 \end{Bmatrix} + \begin{Bmatrix} b\cos\psi \\ b\sin\psi \end{Bmatrix}.$$

$$\mathcal{C}: \ (2bg - 2ab\cos\theta)\cos\psi - (2ab\sin\theta)\sin\psi + (a^2 + b^2 + g^2 - h^2 - 2ag\cos\theta) = 0.$$

This equation has the form $A(\theta)\cos\psi + B(\theta)\sin\psi + C(\theta) = 0$.



Solving the Constraint Equation

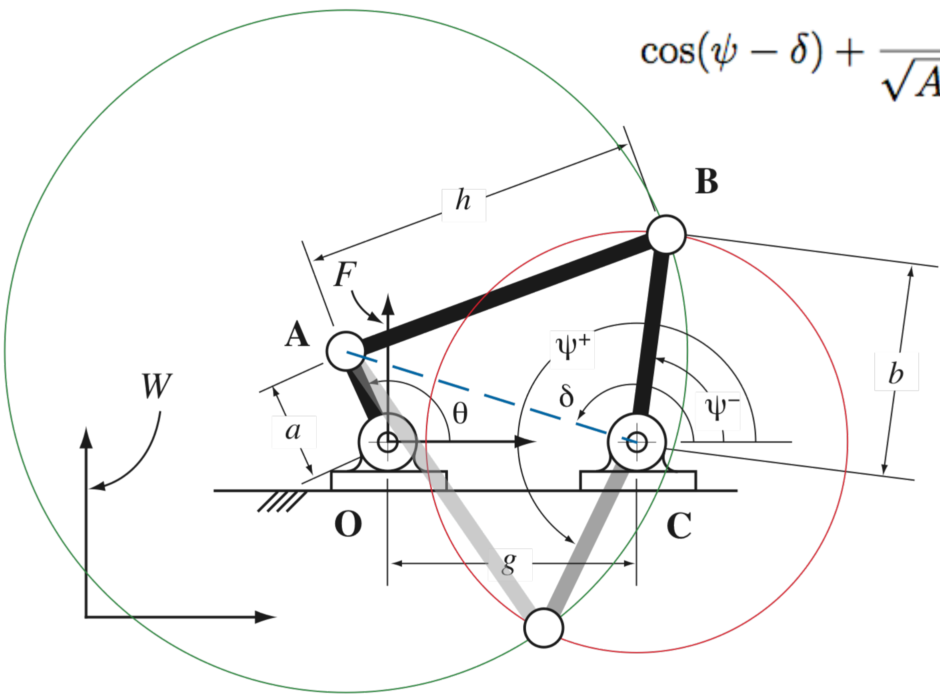
The 4R linkage constraint equation $A(\theta)\cos\psi + B(\theta)\sin\psi + C(\theta) = 0$ is solved to determine the follower angle ψ, for each value of the input crank θ as follows:

- introduce the angle δ, defined such that

$$\cos\delta = A/(A^2+B^2)^{1/2} \text{ and } \sin\delta = B/(A^2+B^2)^{1/2}, \text{ that is } \delta = \arctan(B/A);$$

- the constraint equation becomes $\cos\delta\cos\psi + \sin\delta\sin\psi + C/(A^2+B^2)^{1/2} = 0;$
- the formula for the cosine of the difference of two angles yields:

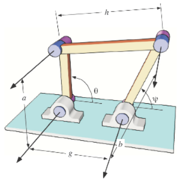
$$\cos(\psi-\delta) + \frac{C}{\sqrt{A^2+B^2}} = 0, \quad \text{or} \quad \psi = \delta \pm \arccos\left(\frac{-C}{\sqrt{A^2+B^2}}\right).$$



Notice the arccos() function yields

i. two values (like a square root), which are plus and minus the same number; and

ii. real values when $-1 \le -C/(A^2+B^2)^{1/2} \le 1$, which defines the range of values for the input angle θ.

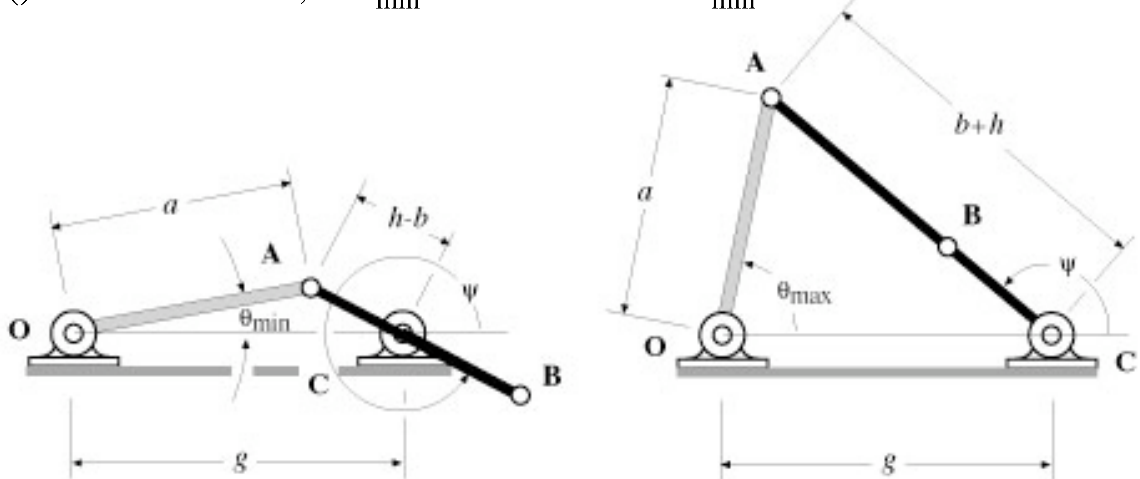


Range of Input Angles

• Limits to the input angle θ occur in the configurations below. Using the cosine law we obtain:

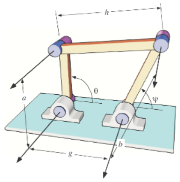
$\cos\theta_{min} = ((h-b)^2 - a^2 - g^2)/2ag$, and $\cos\theta_{max} = ((h+b)^2 - a^2 - g^2)/2ag$.

• It is possible for one, both or neither limit to exist depending on the dimensions of the linkage. Also because arccos() has two values, if θ_{min} is a limit then $-\theta_{min}$ is as well.



There are four cases:

1. Neither limit exists and the crank can make a full rotation;
2. only θ_{min} exists and the crank oscillates through the range $\theta_{min} \leq \theta \leq \pi$, and $-\pi \leq \theta \leq -\theta_{min}$;
3. only θ_{max} exists and the crank oscillates though the range $-\theta_{max} \leq \theta \leq \theta_{max}$; or
4. both limits exist and the crank oscillates in either range $\theta_{min} \leq \theta \leq \theta_{max}$ or $-\theta_{max} \leq \theta \leq \theta_{min}$.

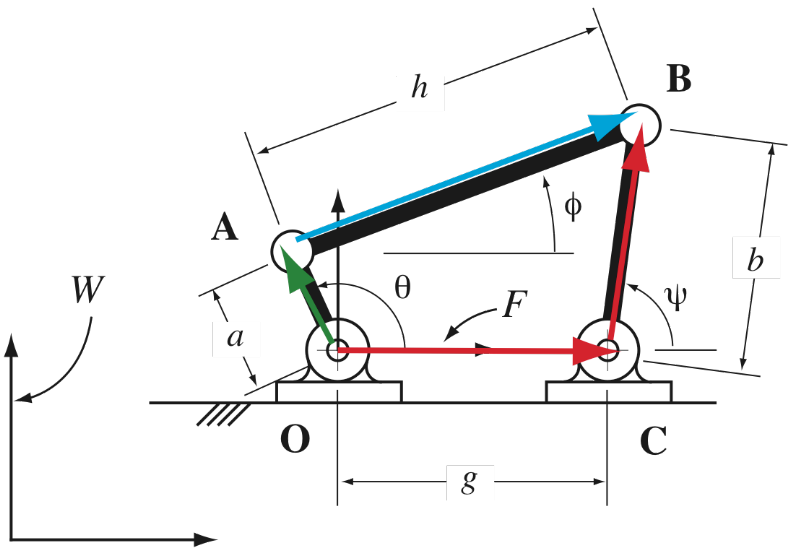


Position Loop Equations: 4R Chain

The *position loop equations* of a **4R linkage** equate two ways to define the moving pivot **B**, that is

$$\mathbf{A} + (\mathbf{B} - \mathbf{A}) = \mathbf{C} + (\mathbf{B} - \mathbf{C}),$$

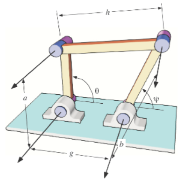
$$\begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} + \begin{Bmatrix} h\cos\phi \\ h\sin\phi \end{Bmatrix} = \begin{Bmatrix} g \\ 0 \end{Bmatrix} + \begin{Bmatrix} b\cos\psi \\ b\sin\psi \end{Bmatrix}.$$



For a given input angle θ and a known value of the output angle ψ, the position loop equations are solved to determine the coupler angle ϕ:

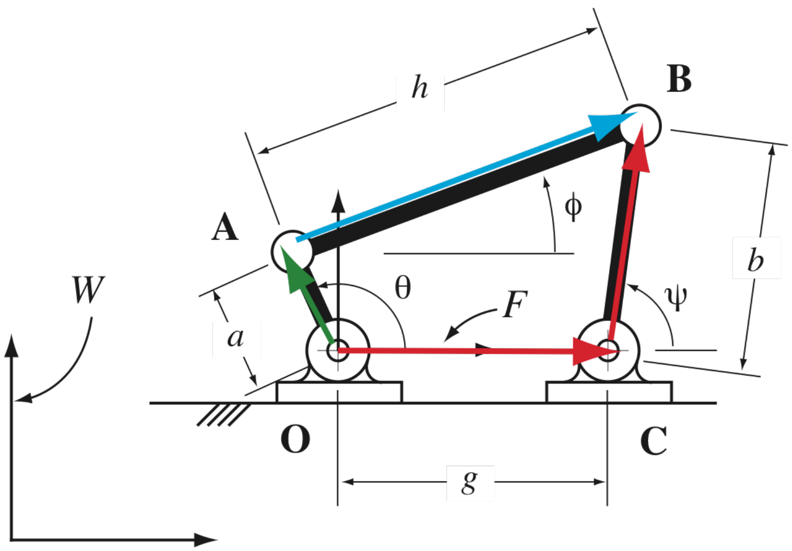
$$\phi = \arctan\left(\frac{b\sin\psi - a\sin\theta}{g + b\cos\psi - a\cos\theta}\right).$$

This formula yields two values of the coupler angle ϕ^{+} and ϕ^{-} obtained from the two values of the output angle ψ^{+} and ψ^{-}.



Velocity Loop Equations

The angular velocity of the coupler and output links in a **4R linkage** are determined from the velocity loop equations obtained by computing the derivative of the position loop equations:



Velocity loop equations:

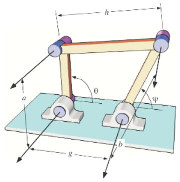
$$\dot{\mathbf{A}} + \frac{d}{dt}(\mathbf{B} - \mathbf{A}) = \frac{d}{dt}(\mathbf{B} - \mathbf{C}),$$

$$\begin{Bmatrix} -a\sin\theta \\ a\cos\theta \end{Bmatrix}\dot{\theta} + \begin{Bmatrix} -h\sin\phi \\ h\cos\phi \end{Bmatrix}\dot{\phi} = \begin{Bmatrix} -b\sin\psi \\ b\cos\psi \end{Bmatrix}\dot{\psi}.$$

Given values for θ and $d\theta/dt$ and the computed values for ψ, and ϕ the velocity loop equations can be written as the matrix equation:

$$\begin{bmatrix} -b\sin\psi & h\sin\phi \\ b\cos\psi & -h\cos\phi \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\phi} \end{Bmatrix} = \begin{Bmatrix} -a\sin\theta \\ a\cos\theta \end{Bmatrix}\dot{\theta}.$$

This matrix equation is easily solved using Cramer's rule.



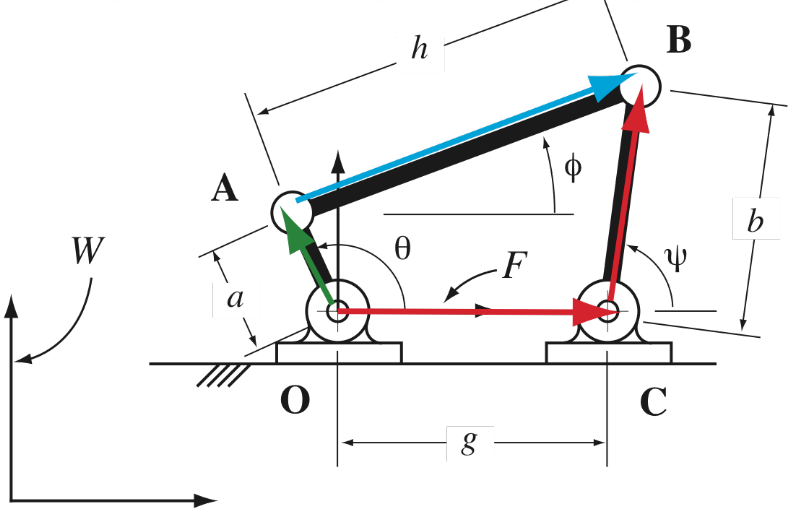
Acceleration Loop Equations

The angular acceleration of the coupler and output links in a **4R linkage** are determined from the acceleration loop equations obtained by computing the derivative of the velocity loop equations:

Acceleration loop equations:

$$\ddot{\mathbf{A}} + \frac{d^2}{dt^2}(\mathbf{B}-\mathbf{A}) = \frac{d^2}{dt^2}(\mathbf{B}-\mathbf{C}),$$

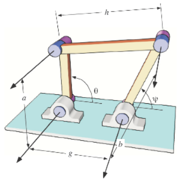
$$\begin{Bmatrix}-a\sin\theta\\ a\cos\theta\end{Bmatrix}\ddot{\theta} - \begin{Bmatrix}a\cos\theta\\ a\sin\theta\end{Bmatrix}\dot{\theta}^2 + \begin{Bmatrix}-h\sin\phi\\ h\cos\phi\end{Bmatrix}\ddot{\phi} - \begin{Bmatrix}h\cos\phi\\ h\sin\phi\end{Bmatrix}\dot{\phi}^2 = \begin{Bmatrix}-b\sin\psi\\ b\cos\psi\end{Bmatrix}\ddot{\psi} - \begin{Bmatrix}b\cos\psi\\ b\sin\psi\end{Bmatrix}\dot{\psi}^2.$$



Given values for θ, $d\theta/dt$ and $d^2\theta/dt^2$ the computed values for ψ, $d\psi/dt$ and ϕ, $d\phi/dt$ the acceleration loop equations become:

$$\begin{bmatrix}-b\sin\psi & h\sin\phi\\ b\cos\psi & -h\cos\phi\end{bmatrix}\begin{Bmatrix}\ddot{\psi}\\ \ddot{\phi}\end{Bmatrix} = \begin{Bmatrix}K_1\\ K_2\end{Bmatrix}.$$

The parameters K_1 and K_2 are known, so the *acceleration* loop equations are solved using Cramer's rule.



Mechanical Advantage

The time derivative of the constraint equation, *C=0,* of a 4R linkage yields a relationship between the input and output angular velocities

$$\dot{C} = 2(\mathbf{B} - \mathbf{A}) \cdot (\dot{\mathbf{B}} - \dot{\mathbf{A}}) = 0,$$
$$= (ag \sin\theta + ab \cos\psi \sin\theta - ab \cos\theta \sin\psi)\dot{\theta} + (-ab\cos\psi\sin\theta - bg\sin\psi + ab\cos\theta\sin\psi)\dot{\psi} = 0.$$

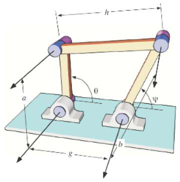
which defines the speed ratio *R* of the linkage,

$$R = \frac{\dot{\theta}}{\dot{\psi}} = \frac{(-ab\cos\psi\sin\theta - bg\sin\psi + ab\cos\theta\sin\psi)}{(ag\sin\theta + ab\cos\psi\sin\theta - ab\cos\theta\sin\psi)}.$$

The principle of virtual work, or "power in equals power out," $T_{\text{in}}\dot{\theta} = T_{\text{out}}\dot{\psi},$

yields the relationship: $\frac{T_{\text{out}}}{T_{\text{in}}} = \frac{\dot{\theta}}{\dot{\psi}} = R.$

Thus, the mechanical advantage of a 4R linkage equals its speed ratio.



Summary

- A *4R linkage* is a movable quadrilateral formed by links connected by hinged or *revolute joints*. Analysis of the linkage yields its configuration, as well as the velocity and angular velocities of the links. This analysis uses the idealization that the links do not flex during the movement, and can be considered to be rigid.

- The *constraint equation* of a linkage is obtained from the requirement that coupler link maintains a constant distance between the moving pivots of the input and output cranks. Differentiation of this constraint yields the speed ratio of the linkage which defines its mechanical advantage in a particular configuration.

- The *position loop equations* are used to compute the coupler angle. The first and second derivatives of the loop equations define the velocity and acceleration loop equations, which are used to compute the angular velocities and angular accelerations of the output crank and coupler link.

- The velocity and acceleration loop equations are linear equations, and are solved using Cramer's rule.