8.4 多元复合函数的求导法则

8.4.1 多元复合函数的求导法则

复合函数的中间变量均为一元函数的情形

定理 4.1 (链式求导法则). 若函数 $u=\varphi(t), v=\psi(t)$ 都在点 t 可导, 函数 $z=f(u,v)$ 在对应点 $(u,v)=(\varphi(t),\psi(t))$ 可微, 则复合函数 $z=f(\varphi(t),\psi(t))$ 在点 t 可导, 且有

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}, \tag{8.4.1}$$

公式(8.4.1)中的导数称为全导数.

例 4.1. 设 $z=uv^2+\arctan w, u=\sin t, v=\ln t, w=e^t$, 求全导数 $\frac{dz}{dt}$.

解:

$$\begin{aligned}\frac{dz}{dt}&=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}+\frac{\partial z}{\partial w}\cdot\frac{dw}{dt}\\&=v^2\cdot\cos t+2uv\cdot\frac{1}{t}+\frac{1}{1+w^2}\cdot e^t\\&=\cos t\ln^2 t+\frac{2}{t}\sin t\ln t+\frac{e^t}{1+e^{2t}}.\end{aligned}$$

例 4.2. 设 $z=(\sin t)^{\cos t}$, 求 $\frac{dz}{dt}$.

解: 令 $u=\sin t, v=\cos t$, 则 $z=u^v$. 由求导公式得

$$\begin{aligned}\frac{dz}{dt}&=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}=vu^{v-1}\cdot\cos t+u^v\ln u\cdot(-\sin t)\\&=\cos^2 t(\sin t)^{\cos t-1}-(\sin t)^{\cos t+1}\ln\sin t.\end{aligned}$$

中间变量多于两个的情形

若函数 $z=f(x_1,x_2,\cdots,x_n)$ 在点 $(x_1,x_2,\cdots,x_n)$ 处可微, 而 $x_k=\varphi_k(t)$ 在点 t 处可导 $(k=1,2,\cdots,n)$, 则复合函数 $z=f[\varphi_1(t),\varphi_2(t),\cdots,\varphi_n(t)]$ 在点 t 处可导, 且

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x_1}\cdot\frac{\mathrm{d}x_1}{\mathrm{d}t}+\frac{\partial z}{\partial x_2}\cdot\frac{\mathrm{d}x_2}{\mathrm{d}t}+\cdots+\frac{\partial z}{\partial x_n}\cdot\frac{\mathrm{d}x_n}{\mathrm{d}t}.$$

中间变量均为二元函数的情形

定理 4.2 (链式求导法则). 若函数 $u=\varphi(x,y), v=\psi(x,y)$ 在点 (x,y) 处偏导数存在, 函数 $z=f(u,v)$ 在对应点 $(u,v)=(\varphi(x,y),\psi(x,y))$ 处可微, 则复合函数

$$z=f(\varphi(x,y),\psi(x,y))$$

在点 (x,y) 处偏导数也存在, 且有

$$\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\\\frac{\partial z}{\partial y}&=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.\end{aligned}$$

中间变量既有一元函数又有多元函数的情形

定理 4.3 (链式求导法则). 设三元函数 $z=f(u,x,y)$ 在点 (u,x,y) 处可微, $u=\varphi(x,y)$ 在点 (s,t) 处偏导数存在, 则 $z=f(\varphi(x,y),x,y)$ 有

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x},$$
$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}.$$

注
这里 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial f}{\partial x}$ 是不同的:

- $\frac{\partial z}{\partial x}$ 是把复合函数 $z=f(\varphi(x,y),x,y)$ 中的 y 看作不变而对 x 求偏导数;
- $\frac{\partial f}{\partial x}$ 是把函数 $z=f(u,x,y)$ 中的 u 及 y 看作不变而对 x 求偏导数.

例 4.3. 设 $z=e^u\sin v$, $u=xy$, $v=\frac{x}{y}$, 求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

解:

$$\frac{\partial z}{\partial u}=e^u\sin v,\quad \frac{\partial z}{\partial v}=e^u\cos v,$$
$$\frac{\partial u}{\partial x}=y,\ \frac{\partial u}{\partial y}=x,\ \frac{\partial v}{\partial x}=\frac{1}{y},\ \frac{\partial v}{\partial y}=\frac{-x}{y^2}.$$

根据链式法则可得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=ye^u\sin v+\frac{1}{y}e^u\cos v=ye^{xy}\sin\frac{x}{y}+\frac{1}{y}e^{xy}\cos\frac{x}{y},$$
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=xe^u\sin v-\frac{x}{y^2}e^u\cos v=xe^{xy}\sin\frac{x}{y}-\frac{x}{y^2}e^{xy}\cos\frac{x}{y}.$$

例 4.4. 设 $z=f(x,y)$ 有连续偏导数, 且 $x=r\cos\theta$, $y=r\sin\theta$, 证明:

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

证明: 因为

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta,$$
$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta),$$

所以

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

例 4.5. 设 $z=(3x+2y)^{xy^2}$, 求 z_x, z_y.

解: 令 $u = 3x + 2y, v = xy^2$, 则 $z = u^v$. 于是

$$
\begin{aligned}
z_x =& z_u u_x + z_v v_x = vu^{v-1} \cdot 3 + u^v \ln u \cdot y^2 \\
=& 3xy^2 (3x + 2y)^{xy^2-1} + y^2 (3x + 2y)^{xy^2} \ln(3x + 2y),
\end{aligned}
$$

$$
\begin{aligned}
z_y =& z_u u_y + z_v v_y = vu^{v-1} \cdot 2 + u^v \ln u \cdot 2xy \\
=& 2xy^2 (3x + 2y)^{xy^2-1} + 2xy (3x + 2y)^{xy^2} \ln(3x + 2y).
\end{aligned}
$$

例 4.6. 设 $z = f(x^2 - y^2, y^2 - x^2)$, f 有对各个变量的连续偏导数, 证明

$$
y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.
$$

解: 令 $u = x^2 - y^2, v = y^2 - x^2$, 则 $z = f(u, v)$. 于是

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = 2x\frac{\partial z}{\partial u} - 2x\frac{\partial z}{\partial v}, \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = -2y\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}.
\end{aligned}
$$

所以

$$
y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = y\left(2x\frac{\partial z}{\partial u} - 2x\frac{\partial z}{\partial v}\right) + x\left(-2y\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right) = 0.
$$

例 4.7. 设 $z = f(x + y, xy)$, 其中 f 有二阶连续偏导数, 求 $\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$.

解: 令 $u = x + y, v = xy$, 则 $z = f(u, v)$. 于是

$$
\frac{\partial z}{\partial x} = f_u u_x + f_v v_x = f_u + y f_v,
$$

$$
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(f_u + y f_v) = \frac{\partial f_u}{\partial y} + f_v + y\frac{\partial f_v}{\partial y}.
$$

又 $f_u = f_u(x + y, xy)$, $f_v = f_v(x + y, xy)$, 故

$$
\begin{aligned}
\frac{\partial f_u}{\partial y} &= f_{uu} u_y + f_{uv} v_y = f_{uu} + x f_{uv}, \\
\frac{\partial f_v}{\partial y} &= f_{vu} u_y + f_{vv} v_y = f_{vu} + x f_{vv}.
\end{aligned}
$$

因此

$$
\frac{\partial^2 z}{\partial x \partial y} = f_{uu} + x f_{uv} + f_v + y(f_{vu} + x f_{vv}).
$$

由于 f 有二阶连续偏导数, 有 $f_{uv} = f_{vu}$, 所以

$$
\frac{\partial^2 z}{\partial x \partial y} = f_{uu} + (x + y) f_{uv} + xy f_{vv} + f_v.
$$

在本例中复合函数 $f(x+y,xy)$ 的中间变量没有明显写出, 为了简便起见, 通常可 用 f'_i 表示 f 对第 i 个中间变量的偏导数, 用 f''_{ij} 表示 f 先对第 i 个中间变量后对第 j 个中间变量的二阶偏导数. 这样, 上述 $f_u, f_v, f_{uv}, f_{vu}, f_{uu}, f_{vv}$ 就可以写成 $f'_1, f'_2, f''_{12}, f''_{21}, f''_{11}, f''_{22}$. 于是本例的结果可改写成

$$\frac{\partial z}{\partial x} = f'_1 + yf'_2,$$

$$\frac{\partial^2 z}{\partial x \partial y} = f''_{11} + (x+y)f''_{12} + xyf''_{22} + f'_2.$$

例 4.8. 设 $z = e^{x^2+y^2+u^2}$, $u = x\sin y$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解: 令 $z = f(u,x,y) = e^{x^2+y^2+u^2}$, 则有

$$\begin{aligned}\frac{\partial z}{\partial x} =& \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial x} = e^{x^2+y^2+u^2} 2u \cdot \sin y + e^{x^2+y^2+u^2} \cdot 2x \\ =& 2x(\sin^2 y + 1)e^{x^2+y^2+x^2\sin^2 y},\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} =& \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial y} = e^{x^2+y^2+u^2} 2u \cdot x\cos y + e^{x^2+y^2+u^2} \cdot 2y \\ =& 2(x^2 \sin y \cos y + y)e^{x^2+y^2+x^2\sin^2 y}.\end{aligned}$$

例 4.9. 设 $z = uv + \sin t$, $u = e^t$, $v = \cos t$, 求全导数 $\frac{\mathrm{d}z}{\mathrm{d}t}$.

解:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} =& \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t} \\ =& ve^t - u\sin t + \cos t = e^t(\cos t - \sin t) + \cos t.\end{aligned}$$

多元函数链式法则可以推广到两个以上中间变量的情形.

例 4.10. 设 $z = uv\ln w$, $u = x+y$, $v = y-x$, $w = 1+xy$, 求 $\frac{\partial z}{\partial x}$.

解:

$$\begin{aligned}\frac{\partial z}{\partial x} =& \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x} \\ =& v\ln w \cdot 1 + u\ln w\cdot(-1) + \frac{uv}{w}\cdot y \\ =& -2x\ln(1+xy) + \frac{y(y^2-x^2)}{1+xy}.\end{aligned}$$

例 4.11. 设 $F = f(x, xy, xyz)$, 求 $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$.

解: 令 $u = x, v = xy, w = xyz$, 有 $F = f(u, v, w)$. 于是

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = f_1' + yf_2' + yzf_3',$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = xf_2' + xzf_3',$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = xyf_3'.$$

8.4.2 一阶全微分形式不变性

设 $z = f(u, v)$ 具有连续偏导数, 则全微分

$$\mathrm{d}z = \frac{\partial z}{\partial u}\,\mathrm{d}u + \frac{\partial z}{\partial v}\,\mathrm{d}v.$$

如果 $z = f(u, v)$ 具有连续偏导数, 而 $u = \varphi(x, y), v = \psi(x, y)$ 也具有连续偏导数, 则全微分

$$\begin{aligned}
\mathrm{d}z &= \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y \\
&= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y \\
&= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y\right) = \frac{\partial z}{\partial u}\,\mathrm{d}u + \frac{\partial z}{\partial v}\,\mathrm{d}v.
\end{aligned}$$

无论 u, v 是自变量还是中间变量, z 的全微分形式是一样的. 这个性质叫做一阶全微分形式不变性.

例 4.12. 设 $z = e^{xy}\sin(x + y)$, 求 $\mathrm{d}z$, 并由此导出 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

解: 令 $u = xy, v = x + y$, 则 $z = f(u, v) = e^u \sin v$. 于是

$$\begin{aligned}
\mathrm{d}z &= \mathrm{d}(e^u \sin v) = e^u \sin v\,\mathrm{d}u + e^u \cos v\,\mathrm{d}v \\
&= e^u \sin v\,\mathrm{d}(xy) + e^u \cos v\,\mathrm{d}(x + y) \\
&= e^u \sin v(y\,\mathrm{d}x + x\,\mathrm{d}y) + e^u \cos v(\,\mathrm{d}x + \,\mathrm{d}y) \\
&= e^u(y \sin v + \cos v)\,\mathrm{d}x + e^u(x \sin v + \cos v)\,\mathrm{d}y \\
&= e^{xy}(y \sin(x + y) + \cos(x + y))\,\mathrm{d}x + e^{xy}(x \sin(x + y) + \cos(x + y))\,\mathrm{d}y.
\end{aligned}$$

所以

$$\frac{\partial z}{\partial x} = e^{xy}(y \sin(x + y) + \cos(x + y)), \quad \frac{\partial z}{\partial y} = e^{xy}(x \sin(x + y) + \cos(x + y)).$$

高阶微分没有微分形式不变性这一性质.

设 $z=u+v, u=x^2y, v=x+y$, 则

$$
\begin{aligned}
\mathrm{d}^2z &= \mathrm{d}\left(\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y\right)=\mathrm{d}\left(\frac{\partial z}{\partial x}\right)\mathrm{d}x+\mathrm{d}\left(\frac{\partial z}{\partial y}\right)\mathrm{d}y\\
&=\frac{\partial^2 z}{\partial x^2}\,\mathrm{d}x^2+2\frac{\partial^2 z}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y+\frac{\partial^2 z}{\partial y^2}\,\mathrm{d}y^2=2y\,\mathrm{d}x^2+4x\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

以上为 u,v 是中间变量的结果. 若 u,v 为自变量, 则

$$
\mathrm{d}^2z=\mathrm{d}^2(u+v)=\mathrm{d}^2u+\mathrm{d}^2v=0.
$$

8.4.3 思考与练习

练习 214. 设 $z=f\left(\arctan\frac{y}{x}\right)$, 其中 $f(x)$ 为可微函数, 且 $f(x)$ 是 x^2 的一个原函数, 则 $\frac{\partial z}{\partial x}\big|_{(1,1)}=$ ____.

$-\frac{\pi^2}{32}$

练习 215. 已知 $f(x,y)|_{y=x^2}=1$, $f_1'(x,y)|_{y=x^2}=2x$, 求 $f_2'(x,y)|_{y=x^2}$.

解: 由 $f(x,x^2)=1$ 两边对 x 求导, 得

$$
f_1'(x,x^2)+2xf_2'(x,x^2)=0.
$$

故 $f_2'(x,x^2)=-1$.

练习 216. 设函数 $z=f(x,y)$ 在点 $(1,1)$ 处可微, 且

$$
f(1,1)=1,\quad \frac{\partial f}{\partial x}|_{(1,1)}=2,\quad \frac{\partial f}{\partial y}|_{(1,1)}=3,
$$

$\varphi(x)=f(x,f(x,x))$, 求 $\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)|_{x=1}$.

解: 由题设 $\varphi(1)=f(1,f(1,1))=f(1,1)=1$. 故

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)|_{x=1}=3\varphi^2(x)\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\bigg|_{x=1}\\
=&3\left[f_1'(x,f(x,x))+f_2'(x,f(x,x))(f_1'(x,x)+f_2'(x,x))\right]|_{x=1}\\
=&3[2+3(2+3)]=51.
\end{aligned}
$$

练习 217. 设 $z=f(x,y)$ 有连续偏导数, 且 $x=r\cos\theta, y=r\sin\theta$, 证明:

$$
\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}=\frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)+\frac{\partial^2 z}{\partial \theta^2}\right].
$$

练习 218. 设 $w=f(x+y+z,xyz)$, f 具有二阶连续偏导数, 求 $\frac{\partial w}{\partial x}$, $\frac{\partial^2 w}{\partial x\partial z}$.

$$\frac{\partial w}{\partial x} = f_1'(x+y+z, xyz) + yzf_2'(x+y+z, xyz),$$

$$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' + y(x+z)f_{12}'' + xy^2zf_{22}'' + yf_2'.$$

练习 219. 设 $f(u,v)$ 具有二阶连续偏导数,且满足 $\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 1$,设 $g(x,y) = f(xy, (x^2 - y^2)/2)$,求 $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$.

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = (x^2+y^2)(f_{11}'' + f_{22}'') = x^2 + y^2.$$

练习 220. 设 $g(y)$ 在实轴上有二阶连续导数,$\phi(\xi,\eta)$ 有二阶连续偏导数,$f(x,y) = \phi(x, x+y) + g(xy)$,求 $\frac{\partial^2 f(x,y)}{\partial y \partial x}$.

$$\frac{\partial f(x,y)}{\partial x} = \phi_1' + \phi_2' + g'(xy)y, \quad \frac{\partial^2 f(x,y)}{\partial y \partial x} = \phi_{12}'' + \phi_{22}'' + g'(xy) + g''(xy)xy.$$

练习 221. 设 $u = f(x, xy, xyz)$,其中 f 具有连续的二阶偏导数,求 $\frac{\partial^2 f}{\partial y \partial z}$.

$xf_3' + x^2yf_{32}'' + x^2yzf_{33}''$

练习 222. 设 $z = y^2 f\left(\frac{y}{x}, \frac{x}{y^2}\right)$,其中 $f(u,v)$ 具有连续的二阶偏导数,求 $\frac{\partial^2 z}{\partial x \partial y}$.

$-\frac{3y^2}{x^2}f_1' - \frac{y^3}{x^3}f_{11}'' + \frac{3}{x}f_{12}'' - \frac{2x}{y^3}f_{22}''$

练习 223. 设函数 $z = f(xy, yg(x))$,其中函数 f 具有二阶连续偏导数,函数 $g(x)$ 可导且在 $x=1$ 处取得极值 $g(1) = 1$,求 $\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{\substack{x=1\\y=1}}$.

由题意知 $g'(1) = 0$. $\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{\substack{x=1\\y=1}} = f_{11}''(1,1) + f_{12}''(1,1) + f_1'(1,1)$