☐PHYS115 ☐PHYS121 ☒PHYS123
☐PHYS116 ☐PHYS122 ☐PHYS124
Lab Cover Letter

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Date Performed October 25, 2023 Date Submitted October 31, 2023

Lab (*such as #1: UNC*) #5: RKE

TA: Olivia Green

GRADE (*to be filled in by your TA*) See your TA for detailed feedback.
An '*x*' next to a subcategory means you need to improve this aspect of your work.

***Paper Subtotals* (points)**

( ) ***General* (6)**
_____ Sig. figs.
_____ Units
_____ Clarity of Presentation
_____ Format

( ) ***Abstract* (4)**
_____ Quantity or principle
_____ How measurement was made
_____ Numerical Results
_____ Conclusion

( ) ***Intro & Theory* (9)**
_____ Basic principle
_____ Main equations to be used
_____ Apparatus
_____ What will be plotted
_____ Fitting parameters related

( ) ***Exp. Procedures* (15)**
_____ Description
_____ Stating and justifying uncertainties
_____ Data Record
_____ Quality of Lab Work

( ) ***Analysis & Error Analysis* (20)**
_____ Discussion
_____ Equations & Calculations
_____ Presentation inc. Graphs, Tables
_____ Results Reported & Reasonable
_____ Underlined items addressed

( ) ***Discussion & Conclusions* (6)**
_____ Numerical comparison of results
_____ Logical conclusions
_____ Discussion of pos. errors
_____ Suggestions to reduce errors

( ) **Paper Total (60 points)**
(30 points for CME or EPF)

( ) **Notebook (10 points)**
_____ Format (*proper style, following directions*)
_____ Apparatus (*brief description of equipment, including sketches*)
_____ Data (*including computer file names and manually recorded data*)
_____ Experimental Technique (*describing your procedures; stating & justifying uncerts.*)
_____ Analysis (*results and errors*)

( ) **Worksheet(s)/Fill-in-the-Blank-Report (30 points)** *if applicable*

( ) **Adjustments** – late submissions, improper procedures, etc. – or bonus points for exceptional work.

( ) Total Grade

Graded by __________ *(TA's initial)*

Rotational Kinetic Energy Lab

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October 31, 2023

Abstract

Our experiment studied the law of conservation of energy by using it to experimentally determine the moment of inertia of a Roto-Dyne wheel. We attempted to guess a reasonable value based on similar shapes, such as a disc and ring, and then performed a Monte Carlo simulation to apply some degree of uncertainty to our estimate. The Monte Carlo simulation estimate for I, which we call I_M, was 0.07182 ± 0.00104 kg m^2. We then experimentally measured the moment of inertia for the system, and determined that I, which we called I_1 for the given system, was 0.06663 ± 0.00017kg m^2. Using this information we finally attempted to find the specific moment of inertia for load masses discluding the Roto-Dyne wheel. To do this we also experimentally measured the moment of inertia of the Roto-Dyne wheel without the masses to be 0.03482 ± 0.00026 kg m^2, and then subtracted I_2 from I_1 to get 0.03181 ± 0.00070, which we denoted as I_E. We found that predicting moments of inertia using different simple, known approximations, such as discs, hoops, and point masses, prove to be wildly varying in accuracy. Additionally, our analysis produces various reusable useful equations to quickly compute moments of inertia for arbitrary nonuniform systems.

Contents

1 Introduction

Background information, including that about moment of inertia and conservation of energy, has been largely adapted from source [2], but has been rephrased and modified notably.

1.1 Moment of Inertia

Moment of inertia, often denoted by I, is a function of the specific geometry and mass distribution of an object. Moment of inertia is implicitly relative to the axis of rotation. Where R is the distance from the axis of rotation and M is the mass of the object, for a point mass the moment of inertia, I, is given by

$$I = MR^2 \tag{1}$$

The entire moment of inertia of an object can be computed by thinking of it as a collection of tiny masses. As the masses' volumes shrink down to some small volume with some proportionately small mass, the can then be eventually said to be the differential dM. Integrating all the small point masses across the entire object implies that the entire moment of inertia of an object, I_{tot}, is given by

$$I = \sum_{k=1}^{\infty} \frac{m}{k} R^2 \tag{2}$$

$$= \int R^2 dM \tag{3}$$

This is conceptually helpful in understanding moment of inertia for arbitrary shapes, but is not practically useful for shapes that are not simple (i.e. circles, squares, collections of discreet point masses), such as the mass-loaded, spoked wheel that we used in our experiment, or those that have nonuniform mass distribution. As a result of such, it is often helpful to actually measure moment of inertia instead of attempting to compute it.

1.2 Conservation of Energy

The translational kinetic energy of an object in motion with mass M moving at speed v is given to be

$$K_T = \frac{1}{2}Mv^2 \tag{4}$$

Since we know that

$$\frac{\theta}{2\pi} = \frac{s}{2\pi R}$$
$$s = \theta R \tag{5}$$

and

$$\frac{d}{dt}s = \frac{d}{dt}\theta R$$
$$v = \omega r \tag{6}$$

We can then derive from equation 4 that the rotational kinetic energy, K_R, is

$$\begin{aligned} K_R &= \frac{1}{2}M(\omega R)^2 \\ &= \frac{1}{2}(MR^2)\omega^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned} \tag{7}$$

where I is defined to be the moment of inertia about the axis of rotation.

As the mass in figure 2 descends downwards due to gravity, it begins to lose its gravitational potential energy, U_W. The total energy of the system is internally conserved, however a small amount of energy is lost due to friction. So, where ΔU_W is the change in the gravitational potential energy of the counterweight, K_T is the translational kinetic energy of the counterweight, and K_R is the rotational kinetic energy of the counterweight, we state that

$$\Delta U_W + K_T + K_R = W_f \tag{8}$$

Which, using equations 7 and 4, implies that

$$\Delta U_W + (\frac{1}{2}Mv^2) + (\frac{1}{2}I\omega^2) = W_f \tag{9}$$

ΔU_W should be negative, and K_T & K_R positive because the mass is falling, and, thus, losing gravitational kinetic energy, whilst simultaneously proportionately gaining kinetic energy.

Using the fact that gravitational potential energy for an object at height h of mass m, in an environment where gravity can be approximated to some constant g, is given to be

$$U_G = (M \cdot g \cdot h) \tag{10}$$

Plugging this in, and renaming h be y, we get the final equation

$$W_f = -(M \cdot g \cdot y) + (\frac{1}{2}Mv^2) + (\frac{1}{2}I\omega^2) \tag{11}$$

Figure 1: Visual representation of k and r

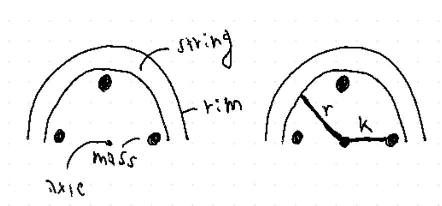
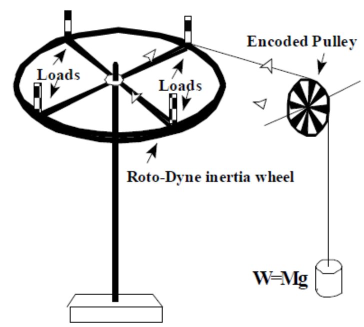


Figure 2: Roto-Dyne wheel Inertia Wheel Apparatus [2]



1.3 Working Equation

Since for our specific experiment we used paperclips attached to the counterweight to cancel out friction, we can instead rewrite equation 11 to be

$$-(M \cdot g \cdot y) + (\frac{1}{2}Mv^2) + (\frac{1}{2}I\omega^2) = 0 \tag{12}$$

Here we must note that we have discluded the energy of the moving paperclips, as it is negligible in comparison to the other energies of the system. To further simplify things, we will define y to be vertically positive, so as to make the equation into

$$-(M \cdot g \cdot y) + (\frac{1}{2}Mv^2) + (\frac{1}{2}I\omega^2) = 0 \tag{13}$$

Then, using relationship 6, we plug in $\frac{v}{r}$ for ω, resulting in the equation

$$\begin{aligned} -(M \cdot g \cdot y) + (\frac{1}{2}Mv^2) + (\frac{1}{2}I(\frac{v}{r})^2) = 0 \\ = \frac{1}{2}v^2 \cdot (M + \frac{I}{r^2}) \end{aligned} \tag{14}$$

Or, as will be used later for our computations, the equivalent equation in the form

$$gy = \frac{1}{2}(1 + \frac{I}{Mr^2}) \cdot v^2 \tag{15}$$

Where v is a value that is determined by the *Logger Pro*™ software that we used. It is computed with an advanced proprietary algorithm, but is similar to

$$v \approx \frac{\Delta s}{\Delta T} \tag{16}$$

2 Procedure

2.1 Taking Measurements

Preliminary Measurements Before conducting our experiment, we took measurements of various parts of our setup. First we obtained r from the lab manual, the distance from the axle of the wheel to the string, and then we measured k, the distance from the axle of the wheel to the masses. Our measurement for k was originally 0.073, but we shifted it to 0.173 because the original value did not make sense given the radius. After careful deliberations we are very certain it was the result of misreading the ruler, but have used a fairly large uncertainty interval for k, 0.01m, as a result of this. For r we used the provided uncertainty of ± 0.002m.

$$r = 0.200 \pm 0.002 \text{ m} \tag{17}$$
$$k = 0.173 \pm 0.01 \text{ m} \tag{18}$$

We used a counterweight with a given mass of 0.06kg to provide a torque to spin our Roto-Dyne wheel, as can be seen in figure 2. To account for friction, we incrementally added paperclips to the bottom of the counterweight. We continued to add paperclips up until the mass would fall at a constant speed to counteract the force of friction, using *Logger Pro*™ software and an encoded pulley to monitor acceleration and velocity during this process. Let M_c be the mass of the counterweight and M_p be the mass of the paperclips opposing friction, not used in our computations but still important to the experimental design and recorded below.

$$M_p = 0.0015 \pm 0.0001 \text{ kg} \tag{19}$$
$$M_c = 0.06 \text{ kg} \tag{20}$$

Also, we were provided with the mass of the Roto-Dyne wheel, M_R and the mass loads, M_L, of which there were 4.

$$M_R = 1.5 \text{ kg} \tag{21}$$
$$M_L = 0.225 \pm 0.002 \text{ kg} \tag{22}$$

For the encoded pulley that we used to measure velocity and length of unrolled string it was given that the gaps between intervals of measurement, Δs, was

$$\Delta s = 0.015 \text{ m} \tag{23}$$

2.2 Rough Estimation of I_1

Before measuring the moment of inertia, we decided to make a rough approximation. To do this, we used two different formulas from common models for moments of inertia, the moment of inertia of a disc, $\frac{1}{2}Mr^2$, and the moment of inertia of a ring, Mr^2. These computations will not be accurate since the actual Rote-Dyne disc is neither a perfect disc nor a hoop, and it does not have uniform density even if it were. For our rough estimates, we took the radius to be r, and the mass to be

$$\begin{aligned} M_{tot} &= M_R + 4M_L \\ &= 2.4 \text{ kg} \end{aligned} \tag{24}$$

For the disc estimate, we got

$$\begin{aligned} I_{\text{disc}} &= \frac{1}{2}Mr^2 \\ &= 0.048 \text{ kg m}^2 \end{aligned} \tag{25}$$

For the hoop estimate, we got

$$\begin{aligned} I_{\text{hoop}} &= Mr^2 \\ &= 0.096 \text{ kg m}^2 \end{aligned} \tag{26}$$

To determine an overall estimate, I_{est}, we averaged I_{disc} and I_{hoop}, and then set the uncertainty to be that average. Assuming that the wheel is a hoop is an under-estimate, and assuming that the wheel is a disc is an over-estimate, the actual moment of inertia should be somewhere between the two.

$$\begin{aligned} I_{\text{est}} &= \frac{I_{\text{disc}} + I_{\text{hoop}}}{2} \\ &= 0.072 \pm 0.072 \text{ kg m}^2 \end{aligned} \tag{27}$$

2.3 Monte Carlo Simulation

Before actually measuring the moment of inertia, we performed a Monte Carlo Simulation using our estimated I so that we could compare a graph of data of the estimated value to a graph of data for the actual value later on. This simulation involves pretending that our estimated value for I_1, $I_{1,\text{est}}$, is the value for I_1, and then artificially inducing uncertainty with simulated randomness.

To perform this simulation, we began by creating a new *Origin Pro*™ document, and arranged a table including a column for Δs, the overall displacement (vertical distance it has fallen) of the string, and ΔT_0, the time elapsed since dropping the counterweight to arrive at that overall displacement. The values for Δs were computed using the equation

$$y_i = i\Delta s \tag{28}$$

Which is simply stating that the total displacement of the rope is equal to the amount of a single displacement, a notch in the encoded pulley, times the number of increments, which is the total number of notches that passed by the laser at a given time point. To compute the ΔT_0 from the Δs we derived the following equation from equations 16 and 15.

First, we manipulated equation 15 by solving for v.

$$\begin{aligned} gy &= \frac{1}{2}(1 + \frac{I}{Mr^2}) \cdot v^2 \\ v^2 &= \frac{\frac{1}{2}(1 + \frac{I}{Mr^2})}{gy} \\ v &= \sqrt{\frac{\frac{1}{2}(1 + \frac{I}{Mr^2})}{gy}} \end{aligned} \tag{29}$$

Next we solved equation 16 for t. For our use case here, we will allow ΔT to be ΔT_0. ΔT_0 is simply the amount of time elapsed since dropping the counterweight (or, in this case, since the start of the point at which a counterweight would have been dropped, since this is a simulation).

$$v \approx \frac{\Delta s}{\Delta T} \tag{16}$$

$$\Delta T \approx \frac{\Delta s}{v}$$

$$\Delta T_0 = \frac{\Delta s}{v} \tag{30}$$

Then, we solved for ΔT_0 by plugging in v from equation 30 and Δs into equation 30.

$$\Delta T_0 = \frac{\Delta s}{v}$$

$$\Delta T_0 = \frac{\Delta s}{\sqrt{\frac{\frac{1}{2}(1+\frac{I}{Mr^2})}{gy}}}$$

$$\Delta {T_0}^2 = \frac{\Delta s^2}{\frac{\frac{1}{2}(1+\frac{I}{Mr^2})}{gy}}$$

$$\Delta {T_0}^2 = \frac{\Delta s^2(1+\frac{I}{Mr^2})}{2gy}$$

$$\Delta T_0 = \Delta S\sqrt{\frac{(1+\frac{I}{Mr^2})}{2gy}} \tag{31}$$

With a simple script we had *Origin Pro*™ apply this equation to each row, utilizing that respective row's Δs value. Now, since this data is purely based on an estimated moment of inertia value, we applied Monte Carlo randomization. To do this, we shifted each $\Delta T0_i$ by some $\Delta T0_G i$ obtained from a Gaussian distribution G with the given mean 0 and σ equal to the estimated uncertainty for Δt, $\delta_{\Delta t}$, which was given to be 0.0002s, resulting with a column with values for δt_R. This process can be expressed formulaically as

$$\delta T_R = \Delta T0 + \Delta t \cdot \mathrm{grnd}() \tag{32}$$

After running this computation, for the first three values of ΔT_r we got

Table 1: Samples of random Monte Carlo data generation with seed `1016`.

trial #	s
1	0.15409
2	0.10925
3	0.08891
1	0.15381
2	0.10883
3	0.08919
1	0.01540
2	0.10891
3	0.08891
1	0.15385
2	0.10918
3	0.08894

Figure 3: Monte Carlo Simulation of Rotational Kinetic Energy Experiment plot

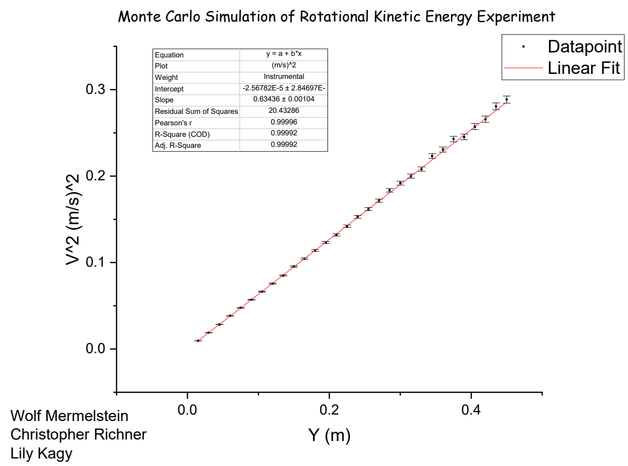
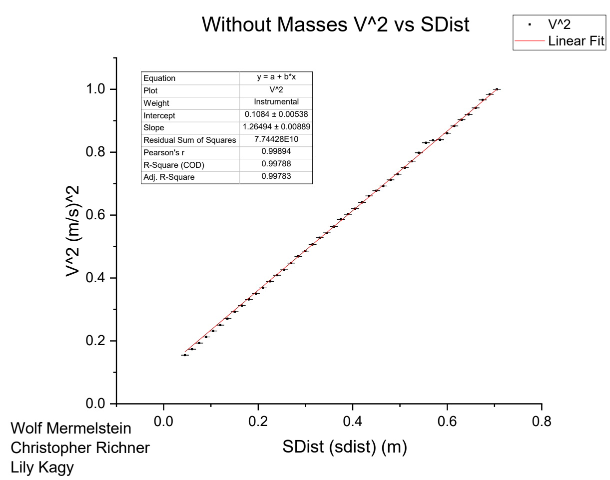


Figure 4: Masses v^2 vs sDist plot



3 Results

3.1 Plotting Data

For the actual results, we measured the velocity, acceleration, time (duration), and position of the falling counterweight mass using the encoded pulley and *Logger Pro*™. We then imported the data into *Origin Pro*™ for analysis and to help us compile plots. To keep our data consistent, we decided to trim the first three rows and all rows after row 44. Data beyond that in either direction was problematic because of us abruptly setting up and stopping the counterweight from hitting the floor.

To visualize our data, we plotted v^2 against displacement for both the system with and without load masses, and did the same for our simulation. For our

v^2 column in *Origin Pro*™ we used

$$\begin{aligned} v &= \frac{\Delta s}{\Delta T} \\ v^2 &= (\frac{\Delta s}{\Delta T})^2 \\ v^2 &= (\frac{0.015}{\Delta T_i})^2 \end{aligned} \tag{33}$$

To include error bars in our *Origin Pro*™ plot, we applied the derivative method row-wise to equation 33, using equation 35. We obtained the value

$$\delta_{\Delta T} = 0.0002\text{s} \tag{34}$$

from our lab manual, which is a consequence of the intrinsic inaccuracy of *Logger Pro*™ and our recording hardware. Solving for δ_{v^2}, we got

$$\begin{aligned} \delta_{v^2} &= |\frac{\partial}{\partial \Delta T_i}((\frac{0.015}{\Delta T_i})^2) \cdot \delta_{\Delta t}| \\ &= |0.015^2 \cdot \frac{\partial}{\partial \Delta T_i}(\Delta T^2) \cdot \delta_{\Delta t}| \\ &= |0.000225 \cdot -2\Delta T^{-3} \cdot \delta_{\Delta t}| \\ &= \frac{.00000009}{\Delta T^3} \end{aligned} \tag{35}$$

Then we used *Origin Pro*™ to compute a line of best fit, of which the regression and slope has been superposed onto our plot figures, figure 3, **??**, and 4.

These aforementioned plots can be found in the appendix, section 6. Additionally, tables 3 and 2 contain the actual datatables with raw *Logger Pro*™ data used for generating the plots.

4 Analysis

Objective For the analysis of our data, our objective was to compute I_E, the measured moment of inertia for just the mass loads on the wheel. In order to do this, we took many intermediate steps to prevent mistakes and measurements. This included the previously discussed Monte Carlo simulation process to verify our computation for I_1, the moment of the Roto-Dyne wheel including the mass loads, and an careful estimate of I_E using the definition of moment of inertia for point masses.

4.1 Computing I_M (Monte Carlo)

Before examining our experimental values we decided to compute the moment of inertia resultant from our Monte Carlo system. The idea behind this is that for the Monte Carlo system we used our estimated value for I, ran a simulation, and applied a randomized skew to the data, so that when computing I_1, the moment of inertia for the system with masses, we would have a point of value to compare to.

In order to find I_M we first needed to find an equation for it. To begin, using the elementary equation for some slope y' of a linear function plotted on an x-y graph, it is well known that

$$\begin{aligned} y' &= \frac{x_2 - x_1}{y_2 - y_1} \\ &= \frac{\Delta y}{\Delta x} \end{aligned} \tag{36}$$

We determined that for our vertical axis v^2 and horizontal axis y,

$$B = \frac{v^2}{y} \tag{37}$$

Also, we note here that the unit for B is $\frac{\text{m}}{\text{s}^2}$ because, when $v = 1$ & $y = 1$

$$\begin{aligned} B &= \frac{v^2}{y} \\ &= \frac{\frac{\text{m}}{\text{s}}^2}{m} \\ &= \frac{\text{m}}{\text{s}^2} \end{aligned} \tag{38}$$

First, we solved equation 15 for v^2 as a function of y

$$gy = \frac{1}{2}(1 + \frac{I}{Mr^2}) \cdot v^2 \tag{15}$$

$$v^2 = \frac{2gy}{1 + \frac{I}{Mr^2}} \tag{39}$$

General equation for I Then we plugged this value for v^2 into equation 37, finally solving for I. This equation will be useful shortly, as once we have a general equation for I in terms of M, r, g, and y, and a complementary uncertainty equation, we can easily compute moment of inertias for all of our systems.

$$B = \frac{\frac{2gy}{1+\frac{I}{Mr^2}}}{y} = \frac{\frac{2gy}{1+\frac{I}{Mr^2}}}{y} \cdot \frac{1+\frac{I}{Mr^2}}{1+\frac{I}{Mr^2}} = \frac{2g}{(1+\frac{I}{Mr^2})} \tag{40}$$

$$By\frac{I}{Mr^2} = 2g - B$$
$$I = Mr^2 \cdot \frac{2g - B}{B}$$
$$I = Mr^2 \cdot (\frac{2g}{B} - 1) \tag{41}$$

General equation for δ_I To compute the error, we applied the derivative method to equation 41, only factoring the error on I_1 due to B as strictly requested by our lab manual. It is, however, worth noting that we believe that the uncertainties due to M and r could be significant, especially given that the r term is squared. These uncertainties will, however, be taken into account for computation of I_E later, but not for I_1 and I_2.

$$\delta_I = |\frac{\partial}{\partial B}(Mr^2 \cdot (\frac{2g}{B} - 1)) \cdot \delta_B| = |\frac{\partial}{\partial B}(Mr^2 \cdot (2gB^{-1} - 1)) \cdot \delta_B| = 2Mgr^2\delta_B \cdot |\frac{\partial}{\partial B}(B^{-1} - 1) \cdot \delta_B| = 2Mgr^2\delta_B \cdot |\frac{-1}{B^2}| = 2\delta_B\frac{Mgr^2}{B^2} \tag{42}$$

Now, armed with these equations we could then plug in values to find I_M, the moment of inertia for the Monte Carlo system, and the corresponding δ_{I_M}, the uncertainty in I_M. This value will be useful in the next section when evaluating the reasonable-ness of our measured value.

For the Monte Carlo system, according to our *Origin Pro*™ linear regression model,

$$B_M = 0.63436 \pm 0.00889\ \frac{\text{m}}{\text{s}^2} \tag{43}$$

We will also henceforth use the generally accepted near-earth gravitational acceleration constant value

$$g = 9.81 \pm 0.01\ \text{ms}^2 \tag{44}$$
$$\tag{45}$$

Plugging in this and other aforestated values into equations 42 and 41, we get

$$I_M = Mr^2 \cdot (\frac{2g}{B} - 1) \pm 2\delta_B\frac{Mgr^2}{B^2} = 0.07182 \pm 0.00104\ \frac{\text{m}}{\text{s}^2} \tag{46}$$

In the next section we will use this to determine whether our value I_1, the moment of inertia for the system with masses, is reasonable

4.2 Computing I_1 (w/masses)

The first step towards accomplishing this goal was to determine whether our estimate for the moment of inertia of the Roto-Dyne wheel was reasonably close to our measured moment value. In order to make this determination we first worked to derive a formula for I, the measured value of the moment of inertia. To start, we note the equation for the slope of the Roto-Dyne wheel with the load masses, B_1 and B_2, the slope without load masses, obtained through *Origin Pro*™'s provided linear fit line.

Solving for I_1 Using the same equations that we derived for the Monte Carlo simluation, equations 41 and 42 we then solved for I_1 for the system including the load masses. Most of the values have already been discussed earlier in our procedure. The two novel values are B_1 and its uncertainty, δ_B.

$$B_1 = 0.68204 \pm 0.00168\ \frac{\text{m}}{\text{s}^2} \tag{47}$$

So, applying equations 41, we get

$$I_1 = Mr^2 \cdot (\frac{2g}{B} - 1) \pm 2\delta_B\frac{Mgr^2}{B^2} = 0.06663 \pm 0.00017\ \text{kg m}^2 \tag{48}$$

Looking back at our Monte Carlo value for I, I_M and placing it inline with our measured value, we have

$$I_M = 0.07182 \pm 0.00104\ \text{kg m}^2$$
$$I_1 = 0.06663 \pm 0.00017\ \text{kg m}^2 \tag{48}$$

To determine if our measured value falls within two uncertainty intervals of our Monte Carlo value, we first determine the distance of a double uncertainty interval to be

$$\begin{aligned}\delta_{I_1} \cdot 2 \\ &= 0.00017 \cdot 2 \\ &= 0.00034\end{aligned}$$

So, mathematically, we are checking

$$\begin{aligned}0.06663 \notin \\ &\notin [0.07182 - 0.00034, 0.07182 + 0.00034] \\ &\notin [0.07148, 0.07216] \quad (49)\end{aligned}$$

Which is false, so our measured indeed value **does not** fall within one to two uncertainty intervals from our estimate. It is, however, close enough that we do not believe that we completely screwed up. The value makes sense; our approximations are just unideal.

4.3 Computing I_2 (w/o masses)

Solving for I_2 Next we compute the value of I_2, the moment of inertia of the Roto-Dyne wheel without mass loads, using the same equations for I_1, equation 41 and for δ_{I_2}, equation 42.

For the system without masses, *Origin Pro*™ produced the linear regression value

$$B_2 = 1.26494 \pm 0.00889 \ \frac{\mathrm{m}}{\mathrm{s}^2} \quad (50)$$

So, plugging our values into equations 41 and 42, we get

$$\begin{aligned}I_2 &= Mr^2 \cdot (\frac{2g}{B} - 1) \pm 2\delta_B \frac{Mgr^2}{B^2} \\ &= 0.03482 \pm 0.00026 \text{ kg m}^2 \quad (51)\end{aligned}$$

for the massless system. It makes intuitive sense that the moment of inertia is lesser. Inertia is a measure of all the infinite infinitesimally small point masses of an object. There are less masses in the system without mass loads, so the moment of inertia indeed should be notably lesser.

4.4 Computing I_E (mass loads)

We are now very close to being able to find I_E, the moment of just the mass loads. Knowing the experimental moment of inertia for the system with the mass loads and the system without the mass loads, we can compute I_E by subtracting the moment of inertia of the system with the mass loads by the system without the mass loads.

$$\begin{aligned}I_E &= I_1 - I_2 \quad (52) \\ &= 0.06663 - 0.03482 \\ &= 0.03181 \text{ kg m}^2 \quad (53)\end{aligned}$$

We will base the uncertainty of this measurement on the literal, complete equation, subbing I into equation 52 for equation 41.

$$\begin{aligned}I_E &= I_1 - I_2 \quad (52) \\ &= Mr^2 \cdot (\frac{2g}{B_1} - 1) - Mr^2 \cdot (\frac{2g}{B_2} - 1) \\ &= Mr^2((\frac{2g}{B_1} - 1) - (\frac{2g}{B_2} - 1)) \\ &= Mr^2 \frac{2g(B_2 - B_1)}{B_1 B_2} \quad (54)\end{aligned}$$

Writing the equation like this does not help us solve for I_E, in fact if anything it makes it more convoluted. Rather, the purpose of this literal expression is to allow us to compute an uncertainty based on the values that it is based on. We believe that $\delta_{I_E,M}$ and $\delta_{I_E,g}$ are negligible in comparison to $\delta_{I_E,r}$, δ_{I_E,B_1}, and δ_{I_E,B_2}, since, besides the fact that the actual value for δ_M is 0.0001kg, r is squared which makes its uncertainty even more significant. To start, we recall

$$\begin{aligned}B_1 &= 0.68204 \pm 0.00168 \ \frac{\mathrm{m}}{\mathrm{s}^2} \quad (47) \\ B_2 &= 1.26494 \pm 0.00889 \ \frac{\mathrm{m}}{\mathrm{s}^2} \quad (50) \\ r &= 0.200 \pm 0.002 \text{ m} \quad (17) \\ M &= 0.0600 \pm 0.0001 \text{ kg} \quad (20) \\ g &= 9.81 \pm 0.01 \frac{m}{\mathrm{s}^2} \quad (44)\end{aligned}$$

To compute the overall uncertainty δ_{I_E}, we will first compute $\delta_{I_E,r}$ using the derivative method, and δ_{I_E,B_1}, and δ_{I_E,B_2} using the computational method, and then will add all the uncertainties in quadrature.

$$\begin{aligned}\delta_{I_E,r} &= |\frac{\partial}{\partial r}(Mr^2 \frac{2g(B_2 - B_1)}{B_1 B_2}) \cdot \delta_r| \\ &= |2rM \frac{2g(B_2 - B_1)}{B_1 B_2} \cdot \delta_r| \\ &\approx 0.00063 \text{ kg m}^2 \quad (55)\end{aligned}$$

$$\begin{aligned}\delta_{I_E,B_2} = &|Mr^2\frac{2g(B_2-B_1)}{B_1B_2} - \\ &Mr^2\frac{2g((B_2+\delta_{B_2})-B_1)}{B_1(B_2+\delta_{B_2})}| \\ &\approx 0.00026 \text{ kg m}^2\end{aligned} \tag{56}$$

$$\begin{aligned}\delta_{I_E,B_1} = &|Mr^2\frac{2g(B_2-B_1)}{B_1B_2} - \\ &Mr^2\frac{2g((B_1+\delta_{B_1})-B_1)}{(B_1+\delta_{B_1})B_2}| \\ &\approx 0.00017 \text{ kg m}^2\end{aligned} \tag{57}$$

So, to find the overall uncertainty we add in quadrature, getting

$$\delta_{I_E} = \tag{58}$$

$$\begin{aligned}&= \sqrt{\delta^2_{I_E,r} + \delta^2_{I_E,B_1} + \delta^2_{I_E,B_2}} \\ &= \sqrt{0.00063^2 + 0.00026^2 + 0.00017^2} \\ &\approx 0.00070 \text{ kg m}^2\end{aligned} \tag{59}$$

We can then succinctly state that

$$I_E = 0.03181 \pm 0.00070 \text{ kg m}^2 \tag{60}$$

4.4.1 Estimating I_E

Now, like how we came up with an approximate value for the Roto-Dyne wheel with masses and loads by approximating it to be a disc and a ring and averaging the two, we wanted to come up with a estimate in similar fashion for I_E. To do this, we treated the four loads M_L, of 0.225 ± 0.002kg each, as point masses for computing an estimate for I_E. To do this, we applied equation 1 to all 4 point masses. Note that we use k here instead of r since the masses are at distance k from the center of the Roto-Dyne wheel.

$$\begin{aligned}I_{\text{est},E} &= \sum Mk^2 \\ &= 4(M_Lk^2) \\ &= 4 \cdot 0.225 \cdot 0.173^2 \\ &= 0.02693 \text{ kg m}^2\end{aligned} \tag{61}$$

Computing $\delta_{I_{\text{est},E}}$ is fairly trivial, requiring us to propagate δ_k and δ_{M_L}'s effect on $\delta_{I_{\text{est},E}}$ and then sum in quadrature. For this we used the derivative method.

$$\begin{aligned}&\delta_{I_{\text{est},E},k} \\ &= |\frac{\partial}{\partial k}(4(M_Lk^2)) \cdot \delta_k| \\ &= |8M_Lk\delta_k| \\ &= |8 \cdot 0.225 \cdot 0.173 \cdot 0.01| \\ &\approx 0.00311 \text{ kg m}^2\end{aligned} \tag{62}$$

$$\begin{aligned}&\delta_{I_{\text{est},E},M_L} \\ &= |\frac{\partial}{\partial M_L}(4M_Lk^2\delta_{M_L})| \\ &= |4k^2\delta_{M_L}| \\ &= |4 \cdot 0.173^2 \cdot 0.002| \\ &\approx 0.00023 \text{ kg m}^2\end{aligned} \tag{63}$$

Adding the uncertainties $\delta_{I_{\text{est},E},M_L}$ and $\delta_{I_{\text{est},E},r}$ in quadrature we get

$$\begin{aligned}\delta_{I_{\text{est},E}} &= \sqrt{\delta^2_{I_{\text{est},E},r} + \delta^2_{I_{\text{est},E},M_L}} \\ &= \sqrt{0.00311^2 + 0.00023^2} \\ &\approx 0.00311 \text{ kg m}^2\end{aligned} \tag{64}$$

Succinctly, we can thus state

$$I_{\text{est},E} = 0.02693 \pm 0.00311 \text{ kg m}^2 \tag{65}$$

Now we can compare this to our value for I_E to see if it is reasonable and agrees two within two uncertainty intervals. We recall that our value for I_E is

$$I_E = 0.03181 \pm 0.00070 \text{ kg m}^2 \tag{60}$$

Two uncertainty intervals from I_E is

$$\begin{aligned}&\delta_{I_E} \cdot 2 \\ &= 0.00311 \cdot 2 \\ &= 0.00622\end{aligned} \tag{66}$$

So, upon inspection of the mathematical statement

$$\begin{aligned}0.02693 &\in [0.03181 - 0.00622, 0.03181 + 0.00622] \\ 0.02693 &\in [0.02559, 0.03803]\end{aligned} \tag{67}$$

we determine that indeed our estimated uncertainty lies within two uncertainty intervals. This provides us assurance that our value for I_E is indeed reasonable.

What we have shown is that estimates for point masses seem to be more reliable than estimates based off of disc or ring approximations for Roto-Dyne wheels.

5 Conclusion

Ultimately, using the law of conservation of angular momentum we were able to show that moments of inertia are able to be experimentally measured. These specific equations that we obtained based on the law of conservation of rotational inertia and the equation that we found for the respective uncertainty are very useful. In certain applications, such as those of objects with nonuniform density or shape, the technique presented proves particularly useful in determining the moment of inertia, since in such cases estimates involving integrals are very difficult and often rely on approximations.

We also showed that although Monte Carlo simulations are useful in inducing uncertainty in this specific use case, it still greatly hinges on the estimated value for the moment that you use for determining whether the simulated I_M is close to the actual value of I. This, we noticed, greatly depends on the specific approximation used. In our case, we found that using disc and hoop approximations for the Roto-Dyne wheel was much less ideal than the point mass approximations that we used for the load masses in computing moments, which is an interesting and potentially unintuitive finding.

Acknowledgments

I would like to thank Christopher Richner and Lily Kagy, CWRU Department of Physics, for their help in obtaining the experimental data, collaborating on preparation of the figures, and checking calculations. Additionally, I would like to thank Olivia Green, CWRU Department of Physics, for helping facilitate our lab. Also, I would like to thank the authors of our lab manual, which was of great aid in writing this report.

References

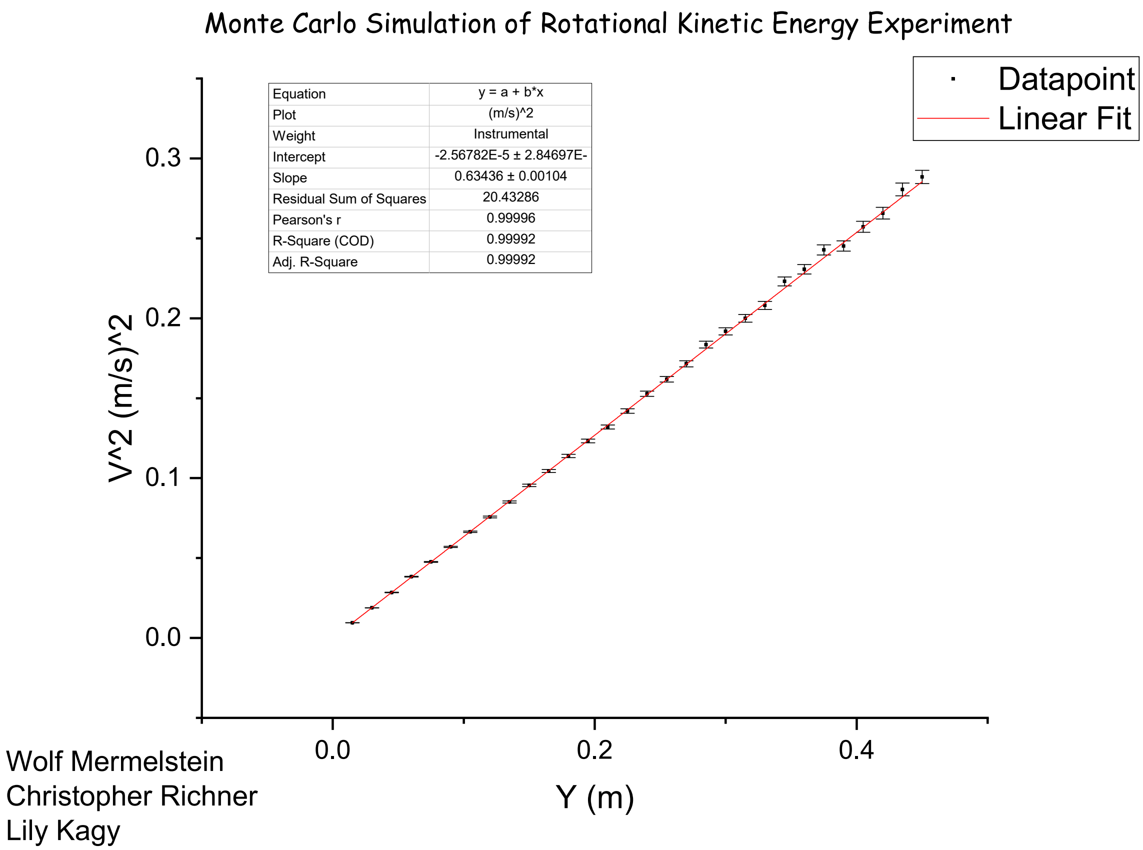
[1] Resnick Halliday and Walker. *Fundamentals of Physics*. Addison-Wesley Professional, 6 edition, 2023.

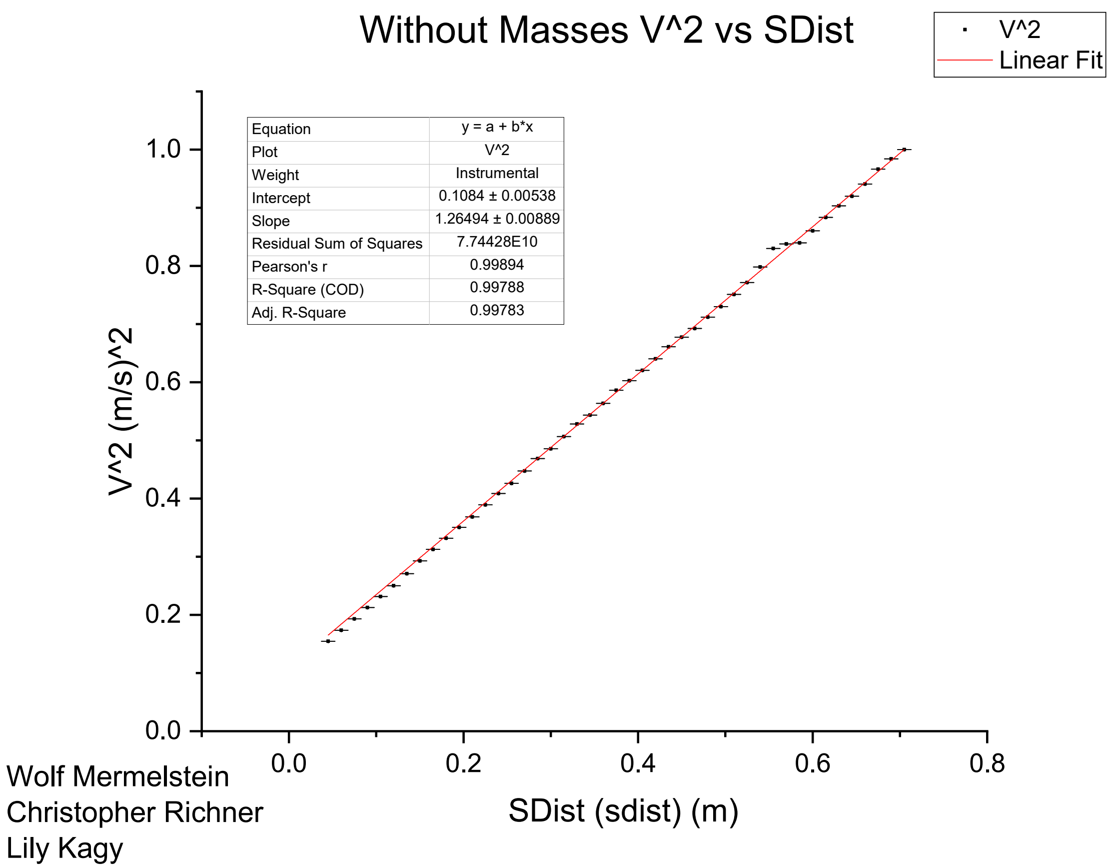
[2] D. Schultz. *General Physics I: Mechanics Lab Manual*. CWRU Bookstore, Spring 2004.

6 Appendix

As this lab was compiled with LaTeX, the resources and files used for the lab are all stored within the same folder as the submission, which is located in a Github repository at `https://github.com/404wolf/phys123-lab5-rke`. This appendix contains enlarged plots for our lab, along with the raw data tables obtained from *Logger Pro*™ and used by *Origin Pro*™.

6.1 Enlarged figures and Datatables




Without Masses V^2 vs SDist
V^2
Linear Fit
Equation
y = a + b*x
Plot
V^2
Weight
Instrumental
Intercept
0.1084 ± 0.00538
Slope
1.26494 ± 0.00889
Residual Sum of Squares
7.74428E10
Pearson's r
0.99894
R-Square (COD)
0.99788
Adj. R-Square
0.99783
V^2 (m/s)^2
SDist (sdist) (m)
Wolf Mermelstein
Christopher Richner
Lily Kagy

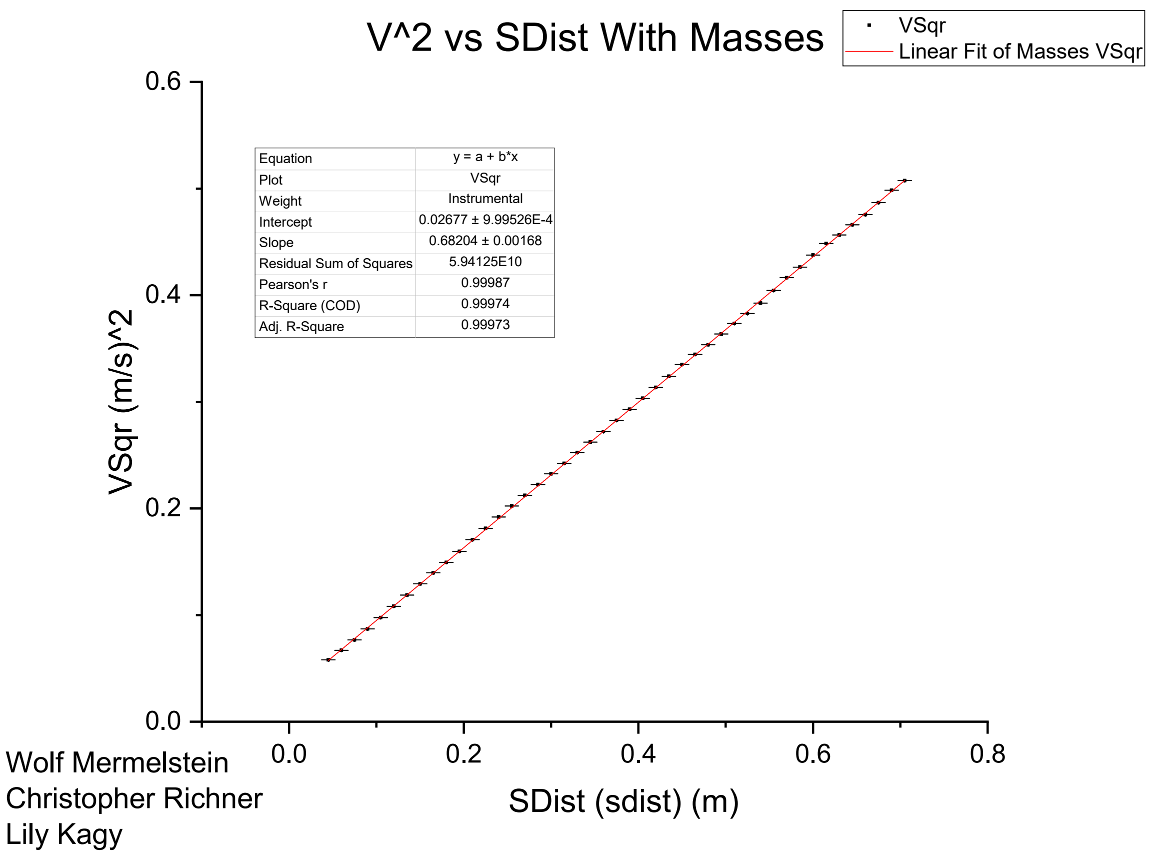

V^2 vs SDist With Masses
VSqr
Linear Fit of Masses VSqr
Equation y = a + b*x
Plot VSqr
Weight Instrumental
Intercept 0.02677 ± 9.99526E-4
Slope 0.68204 ± 0.00168
Residual Sum of Squares 5.94125E10
Pearson's r 0.99987
R-Square (COD) 0.99974
Adj. R-Square 0.99973
VSqr (m/s)^2
SDist (sdist) (m)
0.0
0.2
0.4
0.6
0.8
Wolf Mermelstein
Christopher Richner
Lily Kagy

Table 2: Data for falling counterweight **without** mass loads in place

Time	STime	SDist	SVel	SAccel
t	stime	sdist	svel	saccel
s	s	m	m/s	m/s2̂
0.2003832	0.3409468	0.045	0.240637469	0.2974361002
0.2421024	0.4010488	0.06	0.2587705469	0.3059740303
0.2727832	0.4571542	0.075	0.2768823709	0.3396616489
0.3099096	0.5096392	0.09	0.2950082819	0.3510465914
0.3375832	0.5590336	0.105	0.3125067287	0.3574728706
0.371984	0.6057826	0.12	0.3289729322	0.3469787101
0.3977972	0.6503324	0.135	0.3445762028	0.3535079613
0.4301136	0.6929332	0.15	0.3594867431	0.3465042602
0.4541972	0.7338524	0.165	0.3734799691	0.3374400019
0.4841948	0.7733118	0.18	0.3866063477	0.3278706007
0.506816	0.8114928	0.195	0.3997294453	0.3595444818
0.5350836	0.8484054	0.21	0.4130797945	0.3638046842
0.5563028	0.8841552	0.225	0.4258449397	0.3503333039
0.5829836	0.9188832	0.24	0.4381633461	0.3590888604
0.6031832	0.9526498	0.255	0.4497965385	0.3299465981
0.6285816	0.9856	0.27	0.4607194593	0.3330491259
0.6478832	1.0177834	0.285	0.4715773455	0.3417015924
0.6720832	1.0492332	0.3	0.4820423673	0.3238048217
0.6905832	1.0800322	0.315	0.492183323	0.3347201767
0.7137832	1.1101992	0.33	0.5023979233	0.3424834134
0.7315832	1.1397582	0.345	0.5121068853	0.3144374509
0.7539216	1.1687902	0.36	0.5216114938	0.3403302933
0.7710904	1.1972824	0.375	0.5315253346	0.355568289
0.792702	1.2252416	0.39	0.5413072074	0.3441565057
0.8092096	1.2527124	0.405	0.5507273167	0.3416705778
0.8302836	1.2797228	0.42	0.559880448	0.3360780896
0.8462136	1.3063022	0.435	0.5692255573	0.3671062769
0.8665272	1.3324336	0.45	0.5787253594	0.359973184
0.8820832	1.358147	0.465	0.5869743702	0.2816386499
0.9017832	1.383547	0.48	0.5946036744	0.3190939627
0.9168836	1.4086054	0.495	0.602973345	0.3489192067
0.9359832	1.4333056	0.51	0.6110845183	0.3078506391
0.9506832	1.4577022	0.525	0.6186489591	0.3122719003
0.9693164	1.481802	0.54	0.6266088147	0.3483025149
0.983612	1.5055832	0.555	0.6359267169	0.4353334838
1.0018836	1.5289834	0.57	0.6452122961	0.3582989806
1.0158992	1.5520834	0.585	0.6529222225	0.3092271113
1.0336832	1.5749334	0.6	0.661597301	0.4500795393
1.0474092	1.5974336	0.615	0.669666613	0.2671862678
1.0647832	1.6197336	0.63	0.6756628186	0.27059002
1.0782148	1.6418362	0.645	0.6826017458	0.3572933224
1.0952812	1.663686	0.66	0.6896131877	0.2844921273
1.1083832	1.6853406	0.675	0.69766979	0.4596086524
1.1251172	1.7066906	0.69	0.7060389196	0.32438475
1.1379832	1.7278332	0.705	0.7123894853	0.2763517435

Table 3: Data for falling counterweight **without** mass loads in place

Time t	STime stime	SDist sdist	SVel svel	SAccel saccel
s	s	m	m/s	m/s2̂
0.0942928	0.1784214	0.045	0.3931293508	0.6331416361
0.118666	0.2154714	0.06	0.4167682347	0.6429114748
0.13679	0.2505154	0.075	0.4395279097	0.6560084549
0.1593652	0.2838154	0.09	0.4611842509	0.6446726969
0.1763652	0.3156328	0.105	0.4813227422	0.6212064342
0.1974776	0.3461942	0.12	0.5002408825	0.6168350372
0.2134652	0.375645	0.135	0.5204425281	0.7550560866
0.2334652	0.403888	0.15	0.5411476689	0.7111579009
0.2485668	0.4311196	0.165	0.5590171297	0.6012483335
0.2675656	0.457576	0.18	0.5759222213	0.6767102446
0.2819648	0.4832344	0.195	0.5919181069	0.5701239701
0.3000652	0.5082738	0.21	0.6071150987	0.6437223597
0.3138984	0.5326654	0.225	0.6238645636	0.7296573993
0.3312004	0.5563802	0.24	0.6392812271	0.5705149372
0.3445048	0.5796034	0.255	0.6528166664	0.5951675979
0.361188	0.602345	0.27	0.668869986	0.8166345302
0.3740644	0.624472	0.285	0.6847467198	0.618420718
0.390102	0.6461654	0.3	0.6968292431	0.4955147032
0.4024884	0.6675292	0.315	0.7117831029	0.9044103837
0.417674	0.6883282	0.33	0.7267542228	0.53518959
0.4292784	0.7088134	0.345	0.737173249	0.4820351523
0.4445652	0.7290278	0.36	0.7508551143	0.8716399856
0.4560652	0.7487786	0.375	0.7655336953	0.6147383898
0.4705868	0.768221	0.39	0.7764021555	0.5032779268
0.4818648	0.7874214	0.405	0.7876149561	0.664697796
0.495882	0.8063156	0.42	0.8001386413	0.6609667181
0.5067812	0.8249194	0.435	0.8130346045	0.72541275
0.5206656	0.8432194	0.45	0.8231695853	0.3822354223
0.5311832	0.8613652	0.465	0.8321622217	0.6089180588
0.5446652	0.8792732	0.48	0.8437710657	0.6875800459
0.554982	0.8969236	0.495	0.8544283269	0.520014252
0.5680952	0.9143864	0.51	0.8665462466	0.8678410374
0.5782656	0.931549	0.525	0.8783503608	0.5077214323
0.5911116	0.948543	0.54	0.8933737653	1.260361948
0.6010652	0.965139	0.555	0.9110416678	0.8688140546
0.6135784	0.9814764	0.57	0.915255149	-0.3530060006
0.6232648	0.9979174	0.585	0.9163732773	0.4890230607
0.6353656	1.0142154	0.6	0.9275233967	0.8792576405
0.6447648	1.0302654	0.615	0.9399816885	0.673177482
0.6569652	1.046133	0.63	0.9503827681	0.6378058508
0.6663652	1.0618334	0.645	0.9590269548	0.4633363769
0.6780932	1.0774156	0.66	0.9699853551	0.9431916229
0.6870656	1.0927652	0.675	0.9830297701	0.7564507123
0.6985632	1.1079356	0.69	0.9919658955	0.4216494522
0.7075648	1.1230088	0.705	0.999952488	0.6380581699