VE203 Discrete Math

Spring 2022 — HW7 Solutions

April 19, 2022

Exercise 7.1

i) For

$$f : B \to U, \quad B = \{1,2,3\}, U = \{1,2,3,4,5\}$$

we have

Domain	Codomin	Any	Injectve	surjective
distinguishable	distinguishable	5^3	$\binom{5}{3} \cdot 3!$	0
indistinguishable	distinguishable	$\binom{7}{3}$	$\binom{5}{3}$	0
distinguishable	indistinguishable	5	1	0
indistinguishable	indistinguishable	3	1	0

ii) For

$$f : B \to U, \quad B = \{1,2,3,4,5\}, U = \{1,2,3\}$$

we have

Domain	Codomain	Any	Injective	Surjective
distinguishable	distinguishable	3^5	0	$25 \times 3!$
indistinguishable	distinguishable	$\binom{7}{5}$	0	$\binom{4}{2}$
distinguishable	indistinguishable	41	0	25
indistinguishable	indistinguishable	5	0	2

Exercise 7.2

By the fundamental theorem of arithmetic there is a unique decomposition for n in product of primes numbers : $n = p_1^{a_1} p_2^{a_2} \dots p_k^{ak}$. Thus we have :

$$\varphi(n) = \varphi\left(p_1^{a_1}\right)\varphi\left(p_2^{a_2}\right)\dots\varphi\left(p_k^{a_k}\right)$$

$$\varphi(n) = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right)$$

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right)$$

We obtain the formula stated before.

$$\begin{aligned}\prod_{i=1}^{n}(1-x_i) &= 1-\sum_{i=1}^{n}x_i+\sum_{i,j=1}^{n}x_ix_j-\sum_{i,j,k=1}^{n}x_ix_jx_k+\cdots+(-1)^n x_1x_2\cdots x_n\\ &= \sum_{I\subset 1,2,\ldots,n}(-1)^{|I|}\prod_{i\in I}x_i\end{aligned}$$

when you consider the numbers which are multiple of p_1 or p_2, if you want to count them you have to compute $\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p_1p_2}$, you subtract the number of integers which are in the same time a multiple of p_1 and p_2. Following this reasoning we have :

$$\begin{aligned}\varphi(n) &= n-\sum_{p_i \text{ prime } p_i|n}\frac{n}{p_i}+\sum_{p_i,p_j \text{ prime } p_i,p_j|n}\frac{n}{p_ip_j}-\sum_{p_i,p_jp_k \text{ prime } p_i,p_jp_k|n}\frac{n}{p_ip_jp_k}+\cdots\\ &+(-1)^{|\,\mathrm{Pr}\,|}\frac{n}{p_1p_2\cdots p}\\ &= n\left(1-\sum\frac{1}{p_i}+\sum\frac{1}{p_ip_j}-\sum\frac{1}{p_ip_jp_k}+\cdots+(-1)^{|Pr|}\frac{1}{p_1p_2\cdots p}\right)\\ &= n\prod_{p\in P_r}\left(1-\frac{1}{p}\right)\end{aligned}$$

Exercise 7.3

i) If $x_i > 0$ and the equality holds. Namely, we have

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7=100$$

with $x_i>0$, or equivalently, $x_i\geq 1$. Then we can divide 100 into

$$\underbrace{1+1+\cdots+1}_{100\text{terms}}=100$$

Since we have 7 variables, then the number of integer solutions is given by

$$\begin{pmatrix}99\\6\end{pmatrix}$$

which stands for choosing 6 of + sign out of 99+ sign to form a 7− partition of 100 .

ii) If $x_i \geq 0$ and the equality holds. Namely, we have

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7=100$$

with $x_i\geq 0$, or equivalently, we have

$$x_1'+x_2'+x_3'+x_4'+x_5'+x_6'+x_7'=107$$

with $x_i'>0$, or equivalently, $x_i'\geq 1$ since $x_i' := x_i+1$. Then by the same argument from (i), we know that the number of integer solutions is given by

$$\begin{pmatrix}106\\6\end{pmatrix}$$

iii) If $x_i > 0$ and the equality does not hold. We introduce another variable $x_8 > 0$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 100$$

which is equivalent to the original one. Then by the same argument from (i), we know that the number of integer solutions is given by

$$\begin{pmatrix} 99 \\ 7 \end{pmatrix}$$

iv) If $x_i \geq 0$ and the equality does not hold. We introduce another variable $x_8 > 0$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 100,$$

which is equivalent to the original one. By the same transformation of variables as in (ii), we have

$$x_1' + x_2' + x_3' + x_4' + x_5' + x_6' + x_7' + x_8 = 107,$$

then now all the variables are strictly greater than 0 . By the same argument as in (i), we know that the number of integer solution is given by

$$\begin{pmatrix} 106 \\ 7 \end{pmatrix}$$

v) If $x_i \geq 0$. Then the number of integer solution is given by the results in (ii) and (iv) combined. Since we know that the solution sets in the case of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 100$$

and the case of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 < 100$$

are mutually disjoint, hence the number of solution can be combined, namely

$$\begin{pmatrix} 106 \\ 6 \end{pmatrix} + \begin{pmatrix} 106 \\ 7 \end{pmatrix} = \begin{pmatrix} 107 \\ 7 \end{pmatrix}$$

by the recursive identity for binomial coefficient.

Exercise 7.4

i) From the Master Theorem, we see that the recurrence relation is in the form of

$$T(n) = aT(n/b) + O\left(n^d\right)$$

for constants $a \geq 1, b > 1, d \geq 0$. Specifically, we see that

$$a = 4, \quad b = 4, \quad d = 1$$

since $5n \in O(n)$. Then, we find out

$$\log_b a = \log_4 4 = 1 = d,$$

hence from Master Theorem we conclude that

$$T(n) = O\left(n^d \log n\right) = O(n \log n).$$

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$$a = 5, \quad b = 4, \quad d = 1$$

since $4n \in O(n)$. Then, we find out

$$\log_b a = \log_4 5 > 1 = d,$$

hence from Master Theorem we conclude that

$$T(n) = O\left(n^{\log_b a}\right) = O\left(n^{\log_5 4}\right).$$

iv)

$$\begin{aligned}
&\text{Let } n = 2^m \text{ and } S(m) = T\left(2^m\right) \\
&\Rightarrow S(m) = 4S\left(\frac{m}{2}\right) + m^5 \\
&a = 4, b = 2, d = 5, \Rightarrow \log_b a = 2 < d \\
&\Rightarrow S(m) = O\left(m^5\right) \\
&\Rightarrow T(n) = S(\log m) = O\left((\log n)^5\right)
\end{aligned}$$

v)

$$\begin{aligned}
&\text{Let } n = 2^m \text{ and } S(m) = T\left(2^m\right) \\
&\Rightarrow S(m) = 4S\left(\frac{m}{2}\right) + m^2 \\
&a = 4, b = 2, d = 2, \Rightarrow \log_b a = 2 = d \\
&\Rightarrow S(m) = O\left(m^2 \log m\right) \\
&\Rightarrow T(n) = S(\log m) = O\left((\log n)^2 \log \log n\right)
\end{aligned}$$

Exercise 7.5

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