Ve203 Discrete Mathematics (Fall 2020)

Assignment 12: Partial Orderings

This assignment will not be graded. Marks assigned to the exercises are for reference only.

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Exercise 12.1
Use induction in $(\mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}, \preccurlyeq)$, the set of pairs of natural numbers with lexicographic ordering induced by the ordering $\leq$ of $\mathbb{N} \setminus \{0\}$, to show that if the numbers $a_{m,n}$, $m, n \in \mathbb{Z}_+$, are defined recursively by

$$a_{m,n} = \begin{cases} 5 & m = n = 1, \\ a_{m-1,n} + 2 & n = 1 \wedge m > 1, \\ a_{m,n-1} + 2 & n > 1, \end{cases}$$

then $a_{m,n} = 2(m+n) + 1$ for all $m, n \in \mathbb{Z}_+$.
(3 Marks)

Exercise 12.2
The original Ackermann function $\varphi(m, n, p)$ is defined for $m, n, p \in \mathbb{N}$ by the following rules:

$$\begin{aligned}
&\varphi(m, n, 0) = m + n, \\
&\varphi(m, 0, 1) = 0, \\
&\varphi(m, 0, 2) = 1, \\
&\varphi(m, 0, p) = m \qquad \text{for } p > 2, \\
&\varphi(m, n, p) = \varphi(m, \varphi(m, n-1, p), p-1) \qquad \text{for } n > 0 \text{ and } p > 0.
\end{aligned}$$

Use some variant of induction to show that

$$\varphi(m, n, 1) = m \cdot n, \qquad \text{and} \qquad \varphi(m, n, 2) = m^n.$$

What is $\varphi(m, n, 3)$? (No proof needed.)
(4 + 2 Marks)

Exercise 12.3
Consider the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

i) Draw the Hasse diagram for this poset.

ii) Find all maximal and minimal elements.

iii) Find the least and greatest elements of the poset, if they exist.

iv) Find all upper bounds of $\{2, 9\}$ and $\sup\{2, 9\}$, if it exists.

v) Find all lower bounds of $\{60, 72\}$ and $\inf\{60, 72\}$, if it exists.

(5 × 1 Marks)