Ve203 Discrete Mathematics (Fall 2020)

Assigment 1: Logic and Set Theory

Date Due: 12:10 PM, Thursday, the 18$^{\text{th}}$ of September 2020

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This assignment has a total of **(30 Marks)**.

Exercise 1.1 De Morgan's Rules

i) Let a, b be statements. Write out the truth tables to prove *de Morgan's rules*:

$$\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b, \qquad \neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b.$$

(2 Marks)

ii) Let M be a set and $A, B \subset M$. Prove the following equalities by writing out the sets in terms of predicates and applying de Morgan's rules.

$$(A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}}, \qquad (A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}}.$$

(2 Marks)

Exercise 1.2 Disjunctive Normal Form

Suppose that a truth table in n propositional variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations, with one conjunction for each combination of values for which the compound proposition is true. The resulting compound proposition is said to be in *disjunctive normal form*.
(2 Marks)

Exercise 1.3 Functional Completeness

A collection of logical operators is called *functionally complete* if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

i) Show that $\{\wedge, \vee, \neg\}$ is a functionally complete collection of logical operators. (Hint: use the disjunctive normal form.)

ii) Show that $\{\wedge, \neg\}$ is a functionally complete collection of logical operators. (Hint: use a de Morgan law.)

iii) Show that $\{\vee, \neg\}$ is a functionally complete collection of logical operators.

(3 Marks)

Exercise 1.4 Exclusive Disjunction

In every-day language, the phrase "A or B" is generally taken to mean "A or B, but not both A and B." The corresponding binary operation is called the *exclusive or*, written as $\oplus$ in logic or XOR in logic gate design. It is defined by the truth table

A	B	$A \oplus B$
T	T	F
T	F	T
F	T	T
F	F	F

i) Express $\oplus$ by logical conjunction, disjunction and negation, i.e., through the operations $\{\wedge, \vee, \neg\}$.
(2 Marks)

ii) As shown in question 3, any binary operation can be represented through $\{\wedge, \vee, \neg\}$. For technical reasons, it is preferable in computer design to represent logical operations using $\{\wedge, \oplus, \neg\}$ instead. Write $\vee$ using $\{\wedge, \oplus, \neg\}$.
(2 Marks)

iii) Explain why ii) proves that $\{\wedge, \oplus, \neg\}$ is functionally complete.
(1 Mark)

Exercise 1.5 Functional Completeness with a Single Operator

In computer design, the logical operations NAND and NOR play an important role.[1] In logic, NAND is represented by the *Scheffer stroke* $|$ while NOR is represented by the *Peirce arrow* $\downarrow$. They are defined as

$$A \,|\, B :\equiv \neg(A \wedge B), \qquad\qquad A \downarrow B :\equiv \neg(A \vee B).$$

i) Give the truth tables for $A \,|\, B$ and $A \downarrow B$.
(1 Mark)

ii) Prove that $A \downarrow A \equiv \neg A$ and $(A \downarrow B) \downarrow (A \downarrow B) \equiv A \vee B$.
(2 Marks)

iii) Deduce that $\{\downarrow\}$ is a functionally complete collection of logical operators.
(1 Mark)

iv) Represent the exclusive or $\oplus$ solely through $\downarrow$.
(1 Mark)

v) Prove that $\{|\}$ is a functionally complete collection of logical operators.
(1 Mark)

Exercise 1.6 Some Algebraic Properties

In algebra, basic properties of a binary operation $\boxplus\colon M \times M \to M$ on a set M are the following:

- Associativity: $(a \boxplus b) \boxplus c = a \boxplus (b \boxplus c)$ for any $a, b, c \in M$;
- Existence of a neutral element: there exists an $e \in M$ such that $a \boxplus e = e \boxplus a = a$ for any $a \in M$;
- Existence of an inverse element: for any $a \in M$ there exists an element $a' \in M$ such that $a' \boxplus a = a \boxplus a' = e$;
- Commutativity: $a \boxplus b = b \boxplus a$ for any $a, b \in M$.

If a second operation $\boxtimes\colon M \times M \to M$ is given, another important property is

- Distributivity: $(a \boxplus b) \boxtimes c = (a \boxtimes b) \boxplus (a \boxtimes c)$ for any $a, b, c \in M$.

To illustrate, we consider the following examples:

i) Let S be a set and let $\cap$ and $\cup$ be the usual operations on the power set $\mathcal{P}(S)$. Which of the above five properties are satisfied? **(2 Marks)**

ii) Show that the Scheffer stroke $|$ acting on logical statements is not associative, i.e., $(A \,|\, B) \,|\, C \not\equiv A \,|\, (B \,|\, C)$.
(2 Marks)

Exercise 1.7 $2 + 2 = 4$

Let $(\mathbb{N}, \mathrm{succ})$ be a realization of the natural numbers with successor function succ. We define addition of the numbers 0 and $1 := \mathrm{succ}(0)$ by setting

$$n + 0 := n, \qquad\qquad n + 1 := \mathrm{succ}(n), \qquad\qquad n \in \mathbb{N}.$$

i) Formulate an inductive definition for $n + m$, where $m, n \in \mathbb{N}$.
(2 Marks)

ii) Set $2 := \mathrm{succ}(1)$, $3 := \mathrm{succ}(2)$, $4 := \mathrm{succ}(3)$. Verify that[2]

$$2 + 2 = 4.$$

(2 Marks)

iii) Prove by induction that $n + m = m + n$ for all $m, n \in \mathbb{N}$.
(2 Marks)

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[1] According to https://en.wikipedia.org/wiki/Logical_NOR, "The computer used in the spacecraft that first carried humans to the moon, the Apollo Guidance Computer, was constructed entirely using NOR gates with three inputs." A reference for this claim is is given in that article. See also https://en.wikipedia.org/wiki/Flash_memory for a discussion of NAND and NOR flash memory.

[2] The proof of 2+2=4 is due to the German philosopher and mathematician Gottfried Wilhelm Leibniz (1646–1716).