$X_1 + \cdots X_6 < 538$

$X_1 + \cdots X_7 = 538$

Ve203 Discrete Mathematics (Fall 2022)

Assignment 8

$X_i > 0$ $\quad C_{537}^{6}$

Date Due: None

Exercise 8.1 Consider the functions $f : B \to U$, count the number of functions and fill in the blanks below.

Elements of Domain	Elements of Codomain	Any f	Injective f	Surjective f
distinguishable	distinguishable			
indistinguishable	distinguishable			

where

(i) $B = [3]$ and $U = [5]$. (ii) $B = [5]$ and $U = [3]$.

Exercise 8.2 Derive the following formula for the Euler's totient function φ

$$\varphi(n) = n \prod_{\substack{p \in \mathbb{P} \\ p \mid n}} \left(1 - \frac{1}{p}\right)$$

by applying the inclusion-exclusion principle to the set $[n]$.

Exercise 8.3 Consider $\quad x_1 - 5 - 1 - - - x_7 - 5 = 65$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 100$$

What are the number of integer solutions if

(i) $x_i > 0$ and $=$ holds; C_{99}^{6} (ii) $x_i \geq 0$ and $=$ holds; C_{106}^{6} (iii) $x_i > 0$ and $<$ holds; C_{99}^{7}

(iv) $x_i \geq -1$ and $<$ holds; C_{114}^{7} (v) $1 \leq x_i \leq 5$ and $<$ holds; $C_{99}^{7} - C_{65}^{7}$ (vi) $1 \leq x_i \leq 5$ and $=$ holds; $C_{99}^{6} - C_{65}^{6}$

Exercise 8.4 Given a formal power series $A(x) = \sum_{n \geq 0} a_n x^n$, show that

(i) $A(x) = a_0$ if $DA = 0$. (ii) $A(x) = c \exp(x)$ if $DA = A$, where c is a constant and $\exp(x) := \sum_{n \geq 0} x^n / n!$.

Exercise 8.5 Given a formal power series $A(x) = \sum_{n \geq 0} a_n x^n$, show that

(i) If $k \in \mathbb{N} \setminus \{0\}$, then

$$\sum_{n \geq 0} a_{n+k} x^n = \frac{1}{x^k} \left[A(x) - \sum_{n=0}^{k-1} a_n x^n \right]$$

(ii) If p is a polynomial, then

$$(p(xD)A)(x) = \sum_{n \geq 0} p(n) a_n x^n$$

Exercise 8.6 Find closed formulas (in the sense that there is no infinite sums) for the generating function $A(x)$ of the following sequences $(a_n)_{n \geq 0}$. You may want to consult tables for z-transform from signal and systems. α and ω are fixed scalars.

(i) $a_n = n$ (ii) $a_n = n^2$ (iii) $a_n = \alpha^n$

(iv) $a_n = n\alpha^n$ (v) $a_n = n^2 \alpha^n$ (vi) $a_n = \cos \omega n$

(vii) $a_n = \alpha^n \sin \omega n$ (viii) $a_n = (n)_2 = n(n-1)$ (ix) $a_n = (n)_3 = n(n-1)(n-2)$

Exercise 8.7 Find the general solution a_n to the the following recurrence equations using formal power series.

(i) $a_n = a_{n-1} + 3a_{n-2} + 2^{n+1} - n^2$, $n \geq 2$. (ii) $(T-2)^2(T-1)a_n = (1 + 3n^2)2^n$, $n \geq 3$.

Exercise 8.8 Find the Θ bound of $T(n)$ for the following recurrence relation.

Excercise 8.3

(i) $X_1 + \cdots X_7 = 100$

$n = 100$, $\left(\!\binom{100}{7}\!\right) = \binom{107}{7}$

Exrercise 8.7.1

$a_n = a_{n-1} + 3a_{n-2} + 2^{n+1} - n^2, \quad n \geq 2$

$t^2 = t + 3$

$t^2 - t - 3 = 0$

$t = \frac{1 \pm \sqrt{10}}{2}$

$a_n = c_1\left(\frac{1+\sqrt{10}}{2}\right)^n + C_2\left(\frac{1-\sqrt{10}}{2}\right)^n$

Inhomo

① $-n^2 \Rightarrow C_3 n^2 + C_4 n + C_5$.

② $2 \cdot 2^n \Rightarrow C_6 \cdot 2^n$

So $a_n = C_1\left(\frac{1+\sqrt{10}}{2}\right)^n + C_2\left(\frac{1-\sqrt{10}}{2}\right)^n + C_3 n^2 + C_4 n + C_5 + C_6 \cdot 2^n$.

Excercise 8.12, prove pascal's identity by using two generating equations

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

① $$\sum_{k\geq 0} \binom{n+1}{k+1} x^k = \sum_{k\geq 0} \left[\binom{n}{k} + \binom{n}{k+1}\right] x^k$$

$$\frac{(n+1)!}{(k+1)!(n-k)!} - \frac{n!}{k!(n-k)!} - \frac{n!}{(k+1)!(n-k-1)!}$$

$$= \frac{(n+1)!}{(k+1)!(n-k)!} - \frac{n!(k+1)}{(k+1)!(n-k)!} - \frac{n!(n-k)}{(k+1)!(n-k)!}$$

$$= \frac{n!(n+1-k-1-n+k)}{(k+1)!(n-k)!} = 0$$, first equation proved.

② $$\sum_{n\geq 0} \binom{n+1}{k+1} x^n = \sum_{n\geq 0} \left[\binom{n}{k} + \binom{n}{k+1}\right] x^n$$

This part is the last part because the coefficient is the same, then the pascal's identity is proved.

(i) $T(n) = 4T(n/4) + 5n$ (ii) $T(n) = 4T(n/5) + 5n$ (iii) $T(n) = 5T(n/4) + 4n$

(iv) $T(n) = 4T(\sqrt{n}) + \log^5 n$ (v) $T(n) = 4T(\sqrt{n}) + \log^2 n$

Exercise 8.9 Let $a \geq 1$ and $b > 1$ be constants, and $T(n)$ satisfies the recurrence

$$T(n) = aT(n/b) + f(n)$$

Show that if $f(n) = \Theta(n^{\log_b a} \lg^k n)$, $k \geq 0$, then the recurrence has solution $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$. Assume n is integer power of b for simplicity.

Exercise 8.10 (Series Multisection) Show that for $s, t \in \mathbb{N}$ with $0 \leq t < s$,

$$\sum_{m\geq 0} \binom{n}{t+sm} = \frac{1}{s} \sum_{j=0}^{s-1} 2^n \cos^n\left(\frac{\pi j}{s}\right) \cos\frac{\pi(n-2t)j}{s} .$$

Exercise 8.11 Verify the following identities (if you like)

(i) $\displaystyle\sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$ (ii) $\displaystyle\sum_{n\geq 0} \binom{3n}{n} x^n = \frac{2\cos\left(\frac{1}{3}\arcsin\left(\frac{3}{2}\sqrt{3x}\right)\right)}{\sqrt{4-27x}}$

Exercise 8.12 For integers $n, k \geq 0$, prove Pascal's identity

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

by verifying the following equalities of generating functions.

(i) $\sum_{k\geq 0} \binom{n+1}{k+1} x^k = \sum_{k\geq 0} \left[\binom{n}{k} + \binom{n}{k+1}\right] x^k$ (ii) $\sum_{n\geq 0} \binom{n+1}{k+1} x^n = \sum_{n\geq 0} \left[\binom{n}{k} + \binom{n}{k+1}\right] x^n$

Exercise 8.13 [Gal11, p. 440] Given two matrices $A = (a_{ij}), B = (b_{ij}) \in M_{m\times m}(\mathbb{F}_2)$, i.e., A, B are $m \times m$ matrices with entries either 0 or 1, define

$$(A+B)_{ij} := a_{ij} \vee b_{ij}$$
$$(AB)_{ij} := \bigvee_{k=1}^{m} (a_{ik} \wedge b_{kj}) = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{im} \wedge b_{mj})$$

that is, interpret 0 as **FALSE**, 1 as **TRUE**, $+$ as **OR**, and $\cdot$ as **AND**. Let $B^k := A + A^2 + \cdots + A^k$. Show that there is some $k_0 \in \mathbb{N}$ such that

$$B^{n+k_0} = B^{k_0}$$

for all $n \geq 1$. Describe the graph associated with the adjacency matrix B^{k_0}.

Exercise 8.14 [BBN05] Let

$$E_n := |\{\sigma \in S_n \mid \sigma(i) \neq i \ \forall i \in [n], \operatorname{sgn}(\sigma) = +1\}|$$
$$O_n := |\{\sigma \in S_n \mid \sigma(i) \neq i \ \forall i \in [n], \operatorname{sgn}(\sigma) = -1\}|$$

Show that $E_n - O_n = (-1)^{n-1}(n-1)$.

hint: The Leibniz formula for determinants is given by

$$\det A = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where $A = (a_{ij}) \in \operatorname{Mat}_n(\mathbb{C})$.

References

[BBN05] Arthur T. Benjamin, Curtis T. Bennett, and Florence Newberger. "Recounting the Odds of an Even Derangement". In: *Mathematics Magazine* 78.5 (2005), pp. 387–390 (Cited on page 2).

[Gal11] J. Gallier. *Discrete Mathematics*. Universitext. Springer, 2011 (Cited on page 2).

Excercise 8.9

(i) $T(n) = 4T(n/4) + 5n$