Ve203 Discrete Mathematics (Spring 2022)

Assigment 2

Date Due: 21:00 PM, Tuesday, Mar. 08, 2022
This assignment has a total of **(44 points)**.

Exercise 2.1 (2 pts)
Show by induction that every nonempty finite set of real numbers has a smallest element.

Exercise 2.2 (2 pts)
What is wrong with the following **proof** of the "theorem"?

Theorem. *Given any positive number a, then for all positive integer n, we have $a^{n-1} = 1$.*

Proof. If $n = 1$, $a^{n-1} = a^{1-1} = a^0 = 1$. By induction, assume that the theorem is true for $n = 1, 2, \ldots, k$, then for $n = k + 1$,

$$a^{(k+1)-1} = a^k = \frac{a^{k-1} \times a^{k-1}}{a^{(k-1)-1}} = \frac{1 \times 1}{1} = 1$$

therefore the theorem is true for all positive integers n. $\square$

Exercise 2.3 (2 pts)
Define a *nonempty sorted list* as either

- $\langle x, \langle\rangle\rangle$; or
- $\langle x, \langle y, L\rangle\rangle$ where $x \leq y$ and $\langle y, L\rangle$ is a nonempty sorted list.

Prove by structural induction that in a nonempty sorted list $\langle x, L\rangle$, every element z in L satisfies $z \geq x$.

Exercise 2.4 (4 pts)
Show that for any logical proposition φ using the connectives $\{\neg, \wedge, \vee, \rightarrow\}$, i.e., wffs, there exists a proposition that is logically equivalent to φ using only

(i) (2 pts) $\{\downarrow\}$, where $\downarrow$ is the Peirce arrow (NOR), with $p \downarrow q \Leftrightarrow \neg(p \vee q)$.
(ii) (2 pts) $\{|\}$, where $|$ is the Sheffer stroke (NAND), with $p \mid q \Leftrightarrow \neg(p \wedge q)$.

Exercise 2.5 (4 pts)
Show by induction that the following two algorithms `mergeSort` and `merge` are correct.

Input: $A[1 \ldots n]$, unsorted array
Output: all the $A[i], 1 \leq i \leq n$ in increasing order
Function `mergeSort`$(A[1 \ldots n])$:
  if $n = 1$ **then**
    return A
  else
    $L \leftarrow$ `mergeSort`$(1 \ldots \lfloor \frac{n}{2} \rfloor)$
    $R \leftarrow$ `mergeSort`$(\lfloor \frac{n}{2} \rfloor + 1 \ldots n)$
    return `merge`(L, R)
  end
end

Input: $X[1 \ldots n]$, $Y[1 \ldots m]$, 2 sorted arrays
Output: $X \cup Y$ sorted with elements in increasing order
Function `merge`$(X[1 \ldots n], Y[1 \ldots m])$:
  if $n = 0$ **then**
    return Y
  else if $m = 0$ **then**
    return X
  else if $X[1] < Y[1]$ **then**
    return $X[1]$ *followed by* `merge`$(X[2 \ldots n], Y)$
  else
    return $Y[1]$ *followed by* `merge`$(X, Y[2 \ldots m])$
  end
end

Exercise 2.6 (10 pts)
Let

$$m \sim n \qquad :\Leftrightarrow \qquad 2 \mid (n-m), \qquad\qquad m, n \in \mathbb{Z}.$$

(i) `(1 pt)` Show that $\sim$ is an equivalence relation.

(ii) `(1 pt)` What partition $\mathbb{Z}_2 := \mathbb{Z}/\sim$ is induced by $\sim$?

(iii) `(2 pts)` Define addition and multiplication on $\mathbb{Z}_2$ by the addition and multiplication of class representatives, i.e.,

$$[m] + [n] := [m+n], \qquad\qquad [m] \cdot [n] := [m \cdot n].$$

Show that these operations are well-defined, i.e., independent of the representatives m and n of each class.

(iv) `(6 pts)` Verify that $(\mathbb{Z}_2, +, \cdot)$ is a field, i.e.,

(a) Closure under addition, i.e., $\forall m, n \in \mathbb{Z}_2$, $\exists m+n \in \mathbb{Z}_2$;
(b) Closure under multiplication, i.e., $\forall m, n \in \mathbb{Z}_2$, $\exists m \cdot n \in \mathbb{Z}_2$;
(c) Commutativity of the addition "+", i.e., $m+n = n+m$ for all $m, n \in \mathbb{Z}_2$;
(d) Commutativity of the multiplication "·", i.e., $m \cdot n = n \cdot m$ for all $m, n \in \mathbb{Z}_2$;
(e) Associativity of the addition "+", i.e., $(m+n)+k = n+(m+k)$ for all $m, n, k \in \mathbb{Z}_2$;
(f) Associativity of the multiplication "·", i.e., $(m \cdot n) \cdot k = n \cdot (m \cdot k)$ for all $m, n, k \in \mathbb{Z}_2$;
(g) Distributivity: $k \cdot (m+n) = k \cdot m + k \cdot n$ for all $k, m, n \in \mathbb{Z}_2$;
(h) Existence of an additive identity, i.e., $\exists 0 \in \mathbb{Z}_2$, $\forall m \in \mathbb{Z}_2$: $0+m = m+0 = m$;
(i) Existence of a multiplicative identity, i.e., $\exists 1 \in \mathbb{Z}_2$, $\forall m \in \mathbb{Z}_2$: $1 \cdot m = m \cdot 1 = m$;
(j) Existence of an additive inverse, i.e., $\forall m \in \mathbb{Z}_2$, $\exists n \in \mathbb{Z}_2$ such that $m+n = n+m = 0$;
(k) Existence of a multiplicative inverse, i.e., $\forall m \in \mathbb{Z}_2$, $m \neq 0$, $\exists n \in \mathbb{Z}_2$ such that $m \cdot n = n \cdot m = 1$;
(l) The additive and multiplicative identity elements are different, i.e., $0 \neq 1$.

Exercise 2.7 (8 pts)
Determine whether the relation R on the set of all integers is reflexive, symmetric and/or transitive, where $(x, y) \in R$ iff

(i) $x + y = 0$ (ii) $2 \mid (x-y)$ (iii) $xy = 0$ (iv) $x = 1$ or $y = 1$
(v) $x = \pm y$ (vi) $x = 2y$ (vii) $xy \geq 0$ (viii) $x = 1$

Exercise 2.8 (12 pts)
Let $f : X \to Y$ be any function. Show that for all $A, B \subset X$, we have

(i) `(2 pts)` $f(A \cup B) = f(A) \cup f(B)$.
(ii) `(2 pts)` $f(A \cap B) \subset f(A) \cap f(B)$, where equality holds if f is injective.
(iii) `(2 pts)` $f(A) - f(B) \subset f(A - B)$, where equality holds if f is injective.

Show that for all $C, D \subset Y$, we have

(iv) `(2 pts)` $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
(v) `(2 pts)` $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
(vi) `(2 pts)` $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.

Note that the function $f^{-1} : 2^Y \to 2^X$ has better behavior than $f : 2^X \to 2^Y$ with respect to union, intersection, and complementation. (Note that $f : 2^X \to 2^Y$ is induced by $f : X \to Y$, which is a not uncommmon overloading/abusing of notation.)