Ve203 Discrete Mathematics (Fall 2022)

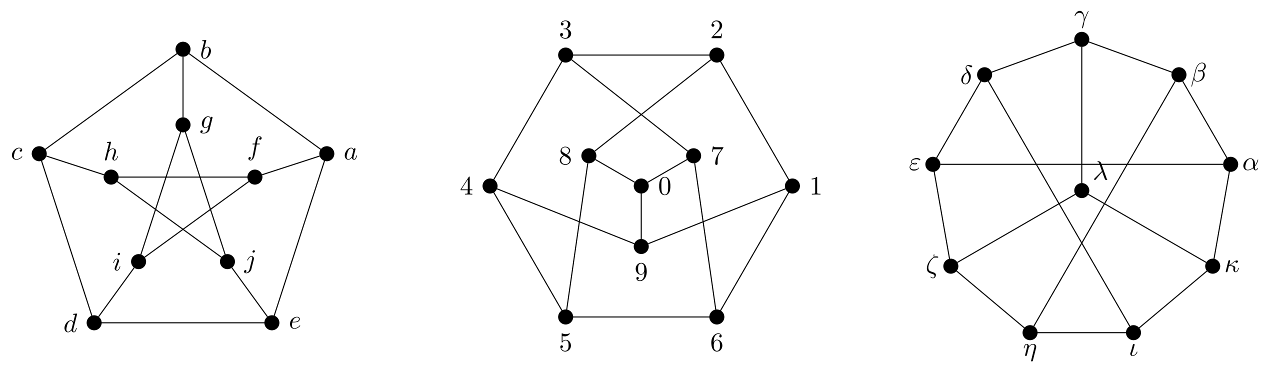
Assignment 7

Date Due: See canvas

This assignment has a total of **(44 points)**.

Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**. **Explain** (briefly) if you claim something is trivial or straightforward. Provide a counterexample if you are trying to disprove something. It is **NOT OK** to write something like "how do we know that blahblahblah is even true..." In addition, be careful that some problems might be ill-defined.

Exercise 7.1 (10 pts) Show that the following 3 drawings (of the Petersen graph) are isomorphic.



Exercise 7.2 (4 pts)

(i) (2 pts) Sketch all non-isomorphic simple graphs with 3 vertices.

(ii) (2 pts) Sketch all 11 non-isomorphic simple graphs with 4 vertices and identify all pairs of complement graphs (including self-complementary ones, i.e., one that is isomorphic to its own complement).

Exercise 7.3 (2 pts) Show that two simple graphs are isomorphic iff their complement graphs are isomorphic.

Exercise 7.4 (2 pts) How many cycles of length n are there in the complete graph K_n, $n \geq 1$?

Exercise 7.5 (2 pts) Consider the complete graph K_n, $n > 0$. Show that if $\sum_i n_i = n$, $n_i \in \mathbb{N}$, then $\binom{n}{2} \geq \sum_i \binom{n_i}{2}$.

Exercise 7.6 (2 pts) For $m, n \in \mathbb{N}$, let C_{2m+1} and C_{2n+1} be two cycles, show that there exists a graph homomorphism $f : C_{2m+1} \to C_{2n+1}$ iff $m \geq n$.

Exercise 7.7 (2 pts) Given a finite graph G with the degrees of every vertex at least 2, show that G contains a cycle (as a subgraph).

Exercise 7.8 (4 pts) A graph G is called **k-regular** if all vertices of G have the same degree k.

(i) (2 pts) Show that a k-regular bipartite graph has no cut-edge for $k \geq 2$.

(ii) (2 pts) Show that a k-regular bipartite graph has a perfect matching for $k \geq 1$.

Exercise 7.9 (2 pts) Given graph G, show that G is a tree iff G is connected and e is a cut-edge for all $e \in E(G)$.

Exercise 7.10 (2 pts) (Birkhoff–von Neumann theorem) A *permutation matrix* is a square binary matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere.

A *doubly stochastic matrix* (also called *bistochastic matrix*), is a square matrix $A = (a_{ij})$ of nonnegative real numbers, each of whose rows and columns sums to 1, i.e., $\sum_i a_{ij} = \sum_j a_{ij} = 1$.

Show that every doubly stochastic matrix is a convex combination of permutation matrices. In other words, if A is a doubly stochastic matrix, then there exists $\alpha_1, \ldots, \alpha_k \geq 0$ with $\sum_{i=1}^{k} \alpha_i = 1$, and permutation matrices $P_1, \ldots, P_k$ such that $A = \sum_{i=1}^{k} \alpha_i P_i$.

Exercise 7.11 (2 pts) Sketch all 8 spanning trees of the following graph.[1]

[1] For counting the number of spanning trees, check *Kirchhoff's matrix tree theorem.*

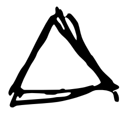
Excercise 7.1

To prove isomorphism, only need to find a bijection

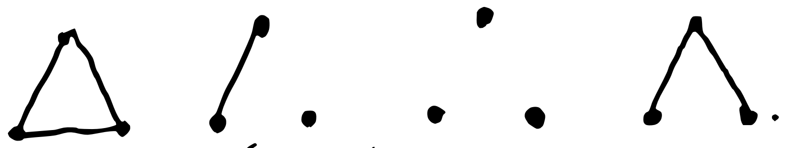
$f: V(G) \to V(H)$ such that $uv \in E(G)$, $f(u)f(v) \in E(H)$

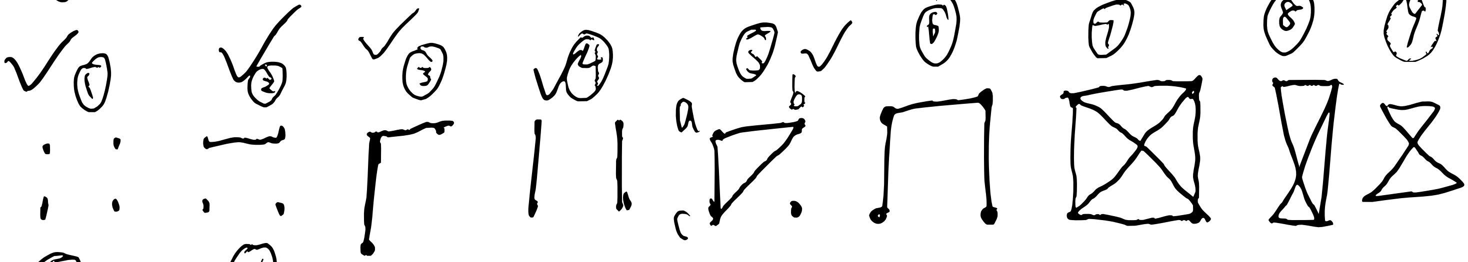
The bijection is follows

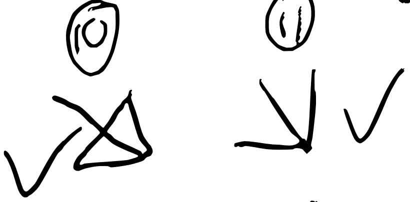
$0 \to j$	$5 \to i$	$\alpha \to a$	$\lambda \to h$
$1 \to a$	$6 \to f$	$\beta \to b$	$\kappa \to f$
$2 \to b$	$7 \to h$	$\gamma \to c$	$\iota \to i$
$3 \to c$	$8 \to g$	$\delta \to d$	$\eta \to g$
$4 \to d$	$9 \to e$	$\varepsilon \to e$	$\xi \to j$



Excercise 7.2

(i) 

(ii) 



complement (1, 7), (2, 8), (3, 10), (4, 9), (5, 11),

(6)

Excercise 7.3

A, B are isomorphic $\Leftrightarrow uv \in E(A)$ iff $f(v)f(v) \in E(H)$

$\Leftrightarrow uv \notin E(A)$ iff $f(u)f(v) \in E(H)$

$\Leftrightarrow$ comp(A), comp(B) are isomorphic

Excercise 7.4

$\frac{1}{2k} \cdot \frac{n!}{(n-k)!}$ total number of cycle of length k.

take $k=n$ $\quad \frac{n!}{2n} = (n-1)!/2$

Excercise 7.5

$\binom{n}{2}$ is $|E|$ of K_n. it is bigger than $\sum_i \binom{n_i}{2}$

because $n = n_1 + \cdots n_n$, for each n_k $(1 \le k \le n)$, $\binom{n_k}{2}$ represents the connection of verticies within these n_k points. However, it lacks the connection between n_k and $n - n_k$ points, which is contaired in K_n graph.

Excercise 7.6

This kind of homomorphisim can be found by

$a_1 \to b_1 \qquad a_{2n+1} \to b_1$

$a_2 \to b_2 \qquad a_{2n+2} \to b_2$

$\vdots \qquad\qquad \vdots$

$a_n \to b_{2n} \qquad a_{2m+1} \to b_1$

Excercise 7.7

We start from any arbitrary vertex and move forward, we will always be able to continue. If we ever return to a vertex we've been, we get a cycle. otherwise, the graph. is infinite, and that leads to a contradiction.

Excercise 7.8

i) Removing a bridge from such a graph would create connected components with all degrees except one equal to k, and the remaining degree equal to $k-1$. In such a component the number of edges incident to one independent set is a multiple of k, but the number of edges incident to the other independent set is not, a contradiction.

ii) Since the graph is regular and edges go from X to Y. Without loss of generality, consider $A \subseteq X$ to be an arbitrary subset and denote by $N(A)$ the set of neighbors of elements of A.

Every edge with an endpoint in A has an endpoint in $N(A)$, let E_A and $E_{N(A)}$ denote the respective edge sets.

Then since G is regular, $|E_A| = d|A|$, $|E_{NA}| = d|N(A)|$, hence $|A| \leq |N(A)|$. By Hall's theorem, there is a complete matching.

But $|x|=|y|$, and therefore it is a perfect matching.

Excercise 7.9.

$\Rightarrow$ G is a tree, no cycle, every edge is a cut edge

$\Leftarrow$ every edge cut edge , completely no cycle, tree

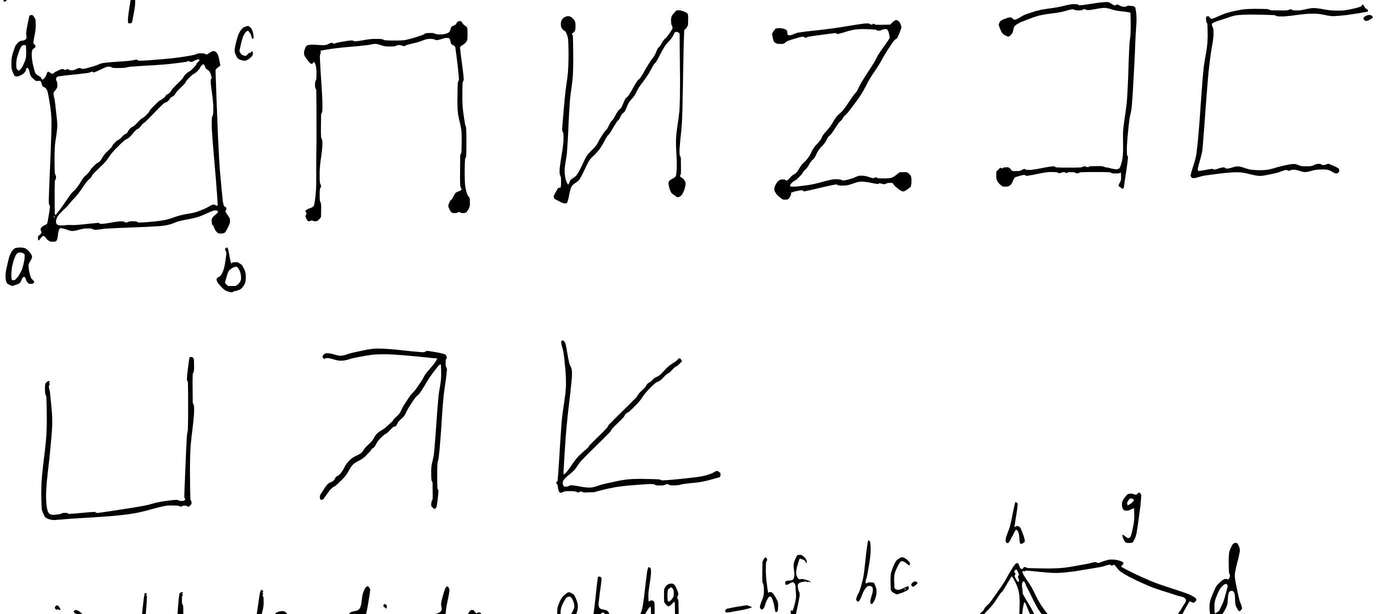
$\hookrightarrow$ e connects A and B

$\forall a \in A, b \in B$ all paths from A to B pass through e.

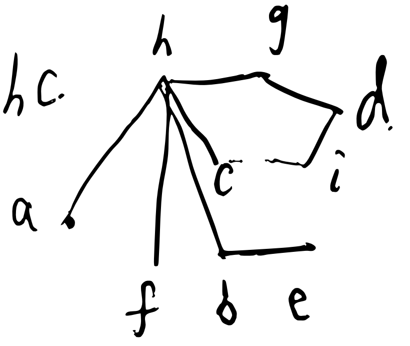
no cycle in G, so G is a tree.

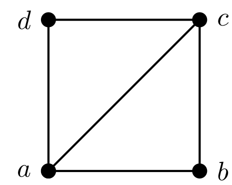
10. skip

11.

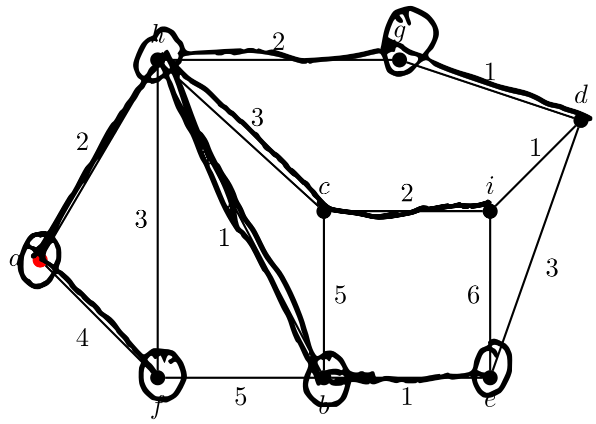


12. (i) hb be di dg -ah hg -hf hc.





Exercise 7.12 (10 pts) Given the following simple connected graph, with weighted edges specified as follows,

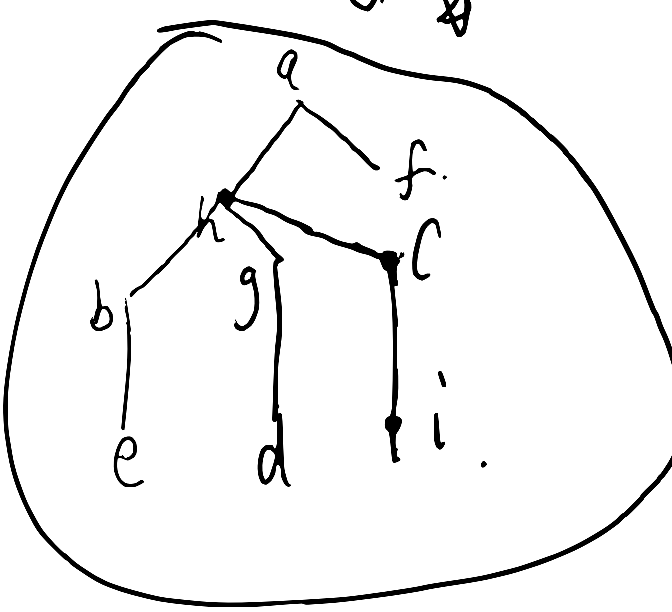


(i) `(5 pts)` Find a minimum-weight spanning tree via Kruskal's algorithm. List the edges chosen in order and sketch the tree.

(ii) `(5 pts)` Given the root vertex a, find a shortest-path spanning tree via Dijkstra's algorithm. List the edges chosen in order, list the shortest path distance (from root vertex) to each vertex. Sketch the tree.

a h b e f g

a	0	0	0	0
b	∞	3	3	3
c	∞	5	5	5
d	∞	∞	∞	7
e	∞	∞	4	4
f	4	4	4	4
g	∞	4	4	4
h	2	2	2	
i	∞	∞	∞	



Order is a, h, b, e, f, g, d, c, i