上 海 交 通 大 学 试 卷

2021 – 2022 Academic Year (Fall Term)

Ve203 Discrete Mathematics
Final Exam

Exercise 1 (10 points)
Let $f(n,k)$ denote the number of ways to partition a set of n objects into k subsets, with each subset containing **at least** 2 elements. Write down a 3-term recurrence relation for $f(n,k)$. (Similar to $\left\{ {n+1 \atop k} \right\} = k\left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}$ or $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. No details are required.)

Solution:

$$f(n+1,k) = kf(n,k) + nf(n-1,k-1) \tag{1}$$

Exercise 2 (10 points)
Find the **general solution** to the following inhomogeneous linear recurrence equation

$$y_{n+2} - 5y_{n+1} + 6y_n = n^2 \cdot 3^n$$

(Show your work. If you happen to be able to guess one, then verification of its correctness is required.)

Solution: First note that the homogeneous solution is given by $y_n^{(h)} = c_1 \cdot 2^n + c_2 \cdot 3^n$. Next assume that a particular solution is given by $y_n^{(p)} = (an + bn^2 + cn^3) \cdot 3^n$, then

$$\begin{aligned} y_{n+2} - 5y_{n+1} + 6y_n = {} & [a(n+2) + b(n+2)^2 + c(n+2)^3] \cdot 3^{n+2} \\ & - 5[a(n+1) + b(n+1)^2 + c(n+1)^3] \cdot 3^{n+1} \\ & + 6[an + bn^2 + cn^3] \cdot 3^n & (2) \\ = {} & [a + 7b + 19c + (2b + 21c)n + 3cn^2]3^{n+1} & (3) \\ = {} & [3(a + 7b + 19c) + 3(2b + 21c)n + 9cn^2]3^n & (4) \end{aligned}$$

therefore we have

$$\begin{cases} a + 7b + 19c = 0 \\ 2b + 21c = 0 \\ 9c = 1 \end{cases} \tag{5}$$

which yields $a = \frac{109}{18}$, $b = -\frac{7}{6}$, $c = \frac{1}{9}$. Therefore we have a particular solution given by

$$y_n^{(p)} = \left(\frac{109}{18}n - \frac{7}{6}n^2 + \frac{1}{8}n^3\right)3^n \tag{6}$$

hence the general solution is given by

$$y_n = y_n^{(h)} + y_n^{(p)} = c_1 2^n + \left(c_2 + \frac{109}{18}n - \frac{7}{6}n^2 + \frac{1}{8}n^3\right)3^n \tag{7}$$

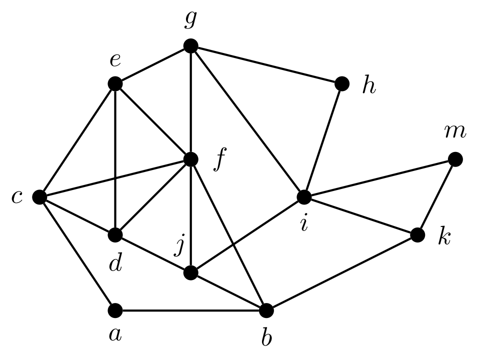
where c_1, c_2 are arbitrary constants.

Exercise 3 (10 points)
Apply the **master theorem** to get asymptotic estimates (as tight as possible) of $T(n)$ for the following recurrence equations.

$$T(n) = 4T(n/2) + n^3$$

Solution: By the master theorem, we have $a = 4$ and $b = 2$, thus $n^{\log_b a} = n^2$. Note that $f(n) = n^3 = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$, and $4(n/2)^3 \leq cn^3$ for $c = \frac{1}{2}$ (regularity condition), we have $T(n) = \Theta(n^3)$.

Exercise 4 (20 points)
Given a graph G as follows:



List the corresponding vertices or edges of G for the following questions. (**DO NOT** leave the question unanswered if the required task happens to be impossible.)

(i) (4 points) Find a clique of size 4 if possible.

Solution: $\{c, d, e, f\}$

(ii) (4 points) Find a maximal clique that is not maximum.

Solution: e.g., $\{i, k, m\}$

(iii) (4 points) Find an induced cycle of size 5 if possible.

Solution: e.g., $\{e, d, j, i, g\}$

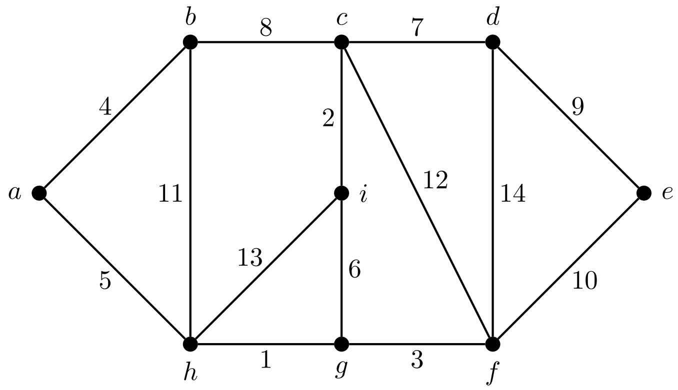
(iv) (4 points) Find the induced cycle of largest size.

Solution: $\{a, b, k, i, g, e, c\}$

(v) (4 points) What is the girth (length of a shortest cycle contained in the graph) of the graph? Explain.

Solution: The girth is 3. For example $\{i, k, m\}$ is a shortest cycle, which is of size 3.

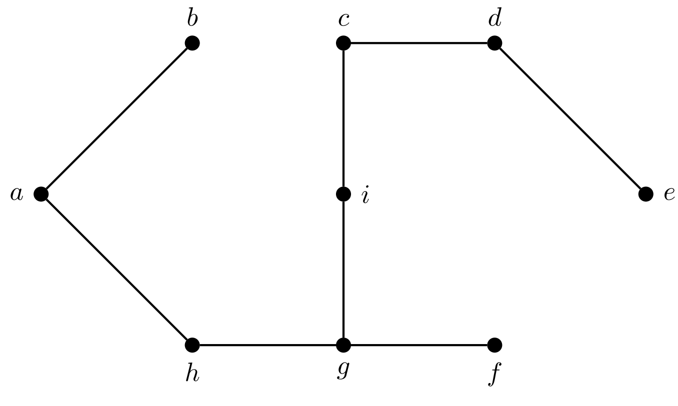
Exercise 5 (10 points)
Given the following weighted graph G:



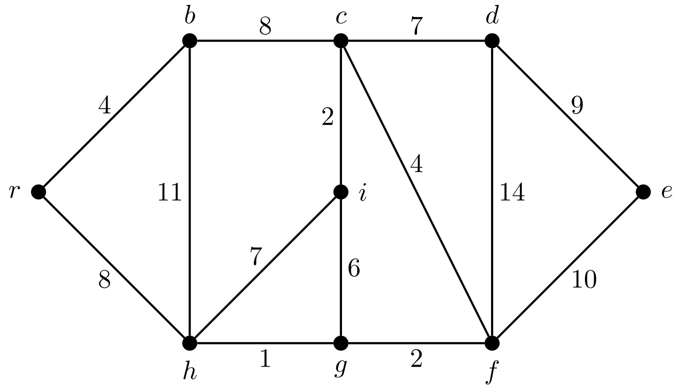
Find a minimum-weight spanning tree using Kruskal's algorithm (avoiding cycles). List the edges with their weights chosen in order and sketch the tree by connecting the vertices in the following figure.

Solution:

- hg, 1
- ci, 2
- gf, 3
- ab, 4
- ah, 5
- ig, 6
- cd, 7
- de, 9



Exercise 6 (10 points)
Given the following weighted graph G:

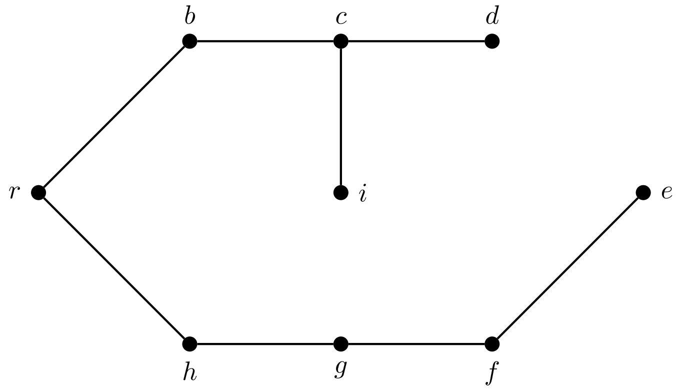


and the root vertex r, find a shortest-path spanning tree using Dijkstra's algorithm. List the edges chosen in order, list the shortest path distance from the root vertex r to each vertex in G (including r). Sketch the tree by connecting the vertices in the following figure.

Solution: Selected edge in order: rb, rh, hg, gf, bc, ci, cd, fe.

Distance from root vertex:

- r: 0
- b: 4
- c: 12
- d: 19
- e: 21
- f: 11
- g: 9
- h: 8
- i: 14



Exercise 7 (10 points)
Consider The following variant of Hall's matching theorem. Given a sequence of (not necessarily distinct) sets $S_1, S_2, \ldots, S_m$, there exists a sequence of distinct elements $x_1, x_2, \ldots, x_m$ such that $x_i \in S_i$ for all $i = 1, 2, \ldots, m$ if and only if **Hall's condition** holds. State Hall's condition in this context.

Solution: For every $k = 1, 2, \ldots, m$, the union of any k sets has at least k elements, that is,

$$\left|\bigcup_{i\in I} S_i\right| \geq |I| \text{ for all } I \subset \{1, \ldots, m\} \tag{8}$$