Mid 1 Practice

Tips: ① Manage your time carefully: Browse the exam, solve the easy problem first.

② No multiple choice in this exam

③ Do the quiz!

④ Though it's open book exam, you still need to review: * properties of relation; $A \not\subseteq P(A)$.

⑤ You can use any off-line materials, so you can prepare some notes that you want to use.

And remeber to download the slides.

⑥ Make sure you have an offline PDF reader software. Using web browser is not allowed!

⑦ Make sure you know the symbols well for example: "|" (for divide) "[n]" "≤" "2^N" "→"

Part I: Set and Logic:

Tips: To prove two sets are equal, you can use $A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B$ Or truthtable.

1. $2^{A\cap B} = 2^A \cap 2^B$ (2^A meanings the power set of A or P(A)).

1° $2^{A\cap B} \subseteq 2^A \cap 2^B$, Let $X \in 2^{A\cap B} \Rightarrow X \subseteq A\cap B \Rightarrow X \subseteq A \wedge X \subseteq B \Rightarrow X \in 2^A \cap 2^B$

2° $2^A \cap 2^B \subseteq 2^{A\cap B}$, Let $X \in 2^A \cap 2^B \Rightarrow X \subseteq A \wedge X \subseteq B \Rightarrow X \subseteq A \cap B \Rightarrow X \in 2^{A\cap B}$.

By $1^\circ, 2^\circ$ $2^{A\cap B} = 2^A \cap 2^B$

X Y

2. $(A-B) \Delta (A-C) = \phi \Leftrightarrow A\cap B = A \cap C$ ($X \Delta Y$ means $(X-Y)\cup(Y-X)$)

$a: x\in A$, $b: x \in B$, $a \wedge \neg b: x \in A-B$, $a \wedge \neg c: x \in A-C$.

Let $(a\wedge \neg b) \oplus (a \wedge \neg c): x \in (A-B) \Delta (A-C)$

XOR

Tips: Please refer to the slides table to review logic operation properties (De Morgan's Law e.g)

Concepts important: CNF & DNF, "$\rightarrow$" means $\neg p \vee q$

Two methods: use logic operation property table directly or use truth table.

When it's hard to prove, try truthtable

3. Prove: $(a \rightarrow c) \wedge (b \rightarrow c) \rightarrow (a \vee b \rightarrow c)$

$$(\neg a \vee c) \wedge (\neg b \vee c) \Leftrightarrow (\neg a \wedge \neg b) \vee c$$
$$\Leftrightarrow \neg(a \vee b) \vee c$$
$$\Leftrightarrow a \vee b \rightarrow c$$

4. Write CNF and DNF for $(a \leftrightarrow b) \rightarrow (b \wedge c)$ Truth Table

$(a \leftrightarrow b) \rightarrow (b \wedge c)$

$((a \rightarrow b) \wedge (b \rightarrow a)) \rightarrow (b \wedge c)$

$\neg((\neg a \vee b) \wedge (\neg b \vee a)) \vee (b \wedge c)$

$\neg(\neg a \vee b) \vee \neg(\neg b \vee a) \vee (b \wedge c)$

$(a \wedge \neg b) \vee (b \wedge \neg a) \vee (b \wedge c)$ CNF

a	b	c	$a \leftrightarrow b$	$b \wedge c$	$(a \leftrightarrow b) \rightarrow (b \wedge c)$	CNF	DNF
0	0	0	1	0	0	✓	
0	0	1	1	0	0	✓	
0	1	0	0	0	1		✓
0	1	1	0	1	1		✓
1	0	0	0	0	1		✓
1	0	1	0	0	1		✓
1	1	0	1	0	0	✓	
1	1	1	1	1	1		✓

Part II Induction :

$P(n)$ $P(n-1)$

Strong Induction: (I) $P(n_0)$ is true (II) $P(n+1)$ is true whenever $P(n_0), P(n_0+1), \cdots P(n)$ is true

Example: Prime Factorization (slides)

Recursively Defined Structures: Structures defined recursively

Structural Induction: (I) Statement is true for basis element

(II) If Statement is true for all the elements constructing new elements then Statement is true for new element.

※ structural Induction is used in recursively defined objects

Concepts important: strings

5. Given a string w over the alphabet Σ, let $w_R = \begin{cases} \lambda & \text{if } w=\lambda \\ ax^R & \text{if } w=xa \text{ with } x \in \Sigma^*, a \in \Sigma \end{cases}$

A symmetric string is a string w such that $w = w^R$ (e.g. A, ABA, ABBA). Empty string is symmetric.

(i) Provide a recursive definition of a symmetric string (x^R means x reverse)

(ii) Show that $w = w^R$ iff w is a symmetric string based on your recursive definition.

(You may use the fact that $(x \cdot y)^R = y^R \cdot x^R$ and $(x^R)^R = x$ for all $x, y \in \Sigma^*$)

(From previous mid 1)

(i) Base 1: $l(w)=0$, $w=w^R$

Base 2: $l(w)=1$, $w=a$, $w_R = a \cdot \lambda = a$

Inductive case when x is a symmetric, $w=xa$, $w_R = axa$.

(ii) $w=w^R \Rightarrow w=\lambda$ or $w=a$ or $w=axa$ x is symmetric.

($\Rightarrow$) Base 1: $l(w)=0$, $w=\lambda$, it's symmetric.

Base 2: $l(w)=1$, $w=a$ it's symmetric.

Inductive: $w=w^R$, $w=xa$? $w=axa$.

6. ($\Rightarrow$) A binary relation $\rightarrow A \subseteq \text{dom}A \times \text{range}A$ is a set of ordered pairs.

($\Rightarrow$)

1° $A = \phi$, $\text{dom}A = \phi$, $\text{range}A = \phi$ $A \subseteq \text{dom}A \times \text{range}A = \phi$

2° $A \neq \phi$, for set $1 \subseteq$ set $2 \Leftrightarrow \forall x \in$ set $1 \rightarrow x \in$ set 2.

$\forall (x,y) \in A$ there exists y such that $(x,y) \in A$, $x \in \text{dom}A$

there exists x such that $(x,y) \in A$, $y \in \text{range}A$

($\Leftarrow$) if $A \subseteq \text{dom}A \times \text{range}A \rightarrow A$ is a binary relation

Relation Ordered pairs

Part III. Function and Relation

Concept important: definition of function and relation, (Relation are just set of ordered pairs)

Domain and Range of function

6. Given a set A, show that A is a binary relation iff $A \subseteq \text{dom}A \times \text{range}A$. (Previous mid 1)

$aRa \qquad aRb \to bRa \qquad aRb \wedge bRc \to aRc$

Concept important: Equivalence Relation: reflexive, symmetric, transitive

Equivalence Class: $[x]_R := \{t \in A \mid xRt\}$ R is a equivalence relation.

if xRy then $[x]_R = [y]_R$

$\Pi \qquad A/R$

Partition / Quotient: set of all equivalence relation of A

Tips: to prove an equivalence relation, prove reflexive, symmetric, transitive seperately.

Unfamiliar relation: refer to slides.

7. Let R, S be relations on A. $S = \{(a,b) \mid (\exists c)(aRc \wedge cRb)\}$

Prove: if R is an equivalence relation, then S is an equivalence relation.

If R is an equivalence relation.

$aRa \to T$

$aRb \wedge bRc \to aRc$

$aRb \to bRa$

for any $c \in A$ $aRc \wedge cRa \to aRa \to T$ so $aSa \to T$ (reflexitivity)

$aSb \Rightarrow (\exists c)\ aRc \wedge cRb \to T$

$\Rightarrow aRb \to T$

$\Rightarrow bRa \to T$

$\Rightarrow bRc \wedge cRa \to T$

$\Rightarrow bSa \to T$

$aRc = \underline{aRd \wedge dRc}$

$\exists d\ aRd \wedge dRb \to T$

$aSb \wedge bSc \to \exists e\ bRe \wedge eRc \quad \cancel{\exists} f = e$

Part IV Equinnumerous & Cardinality.

Important statements you might use in exam:

① $A \not\approx P(A)$. ✡

② The set of all sets does not exist. ✡

③ $\{x \mid \text{card}\, x = K\}$ is not a set. ✡

If you are interested, you can search for how to prove ②, ③.

Tips: Three ways to prove equinumerous 1° find a bijection.

2° Prove two injections $(A \preceq B \wedge B \preceq A \Leftrightarrow A \approx B)$ (or cantor-schröder-theorem)

3° To prove $A \approx B$ You can find C so that $A \approx C \wedge C \approx B$

Review the injection given in slides.

8. Given an infinite set A, show that $|A| < |P(A)|$ (previous mid 1)

1° $A \preceq P(A)$, $f: A \to P(A)$ $x \mapsto \{x\}$

2° $A \not\approx P(A)$ Use contradiction

assume there's injection from $A \to P(A)$. f.

Then let set B be all the elements

$B = \{a \mid a \mapsto C \wedge a \notin C\}$

Let's suppose f is bijective. f is onto so $f(b) = B$

If $b \in B$, then $b \notin f(b) = B$ contradiction

If $b \notin B$, $b \in f(b) = B$ contradiction

f is not a bijection $A \to P(A)$ $A \not\approx P(A)$

Part V Partial Order

Concept important: maximum element, maximum chain, maximal element, maximal chain.

minimum element, minimum chain, minimal element, minimal chain.

Comparable: $x<y$ or $y<x$ in P then x,y is comparable ✲ an element is not comparable to itself!

Height: maximum size of a chain.

Width: maximum size of an antichain.

✲ 4 Theorems:

Observation: if P can be partitioned into t ~~chains~~ antichains, then height of P is at most t.

pigenhole principle if P can be partitioned into s chains, then width of P is at most s.

Mirskey Theorem: a poset of height h can be partitioned into h antichains.

Dilworth Theorem: a poset of width w can be partitioned into w chains.