Ve203 Discrete Mathematics

Runze Cai

University of Michigan - Shanghai Jiao Tong University
Joint Institute

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Part III

Selected Topics in Graph Theory

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Graphs

Definition

A ***graph*** G consists of a set of ***vertices***, denoted by $V(G)$, a set of edges, denoted by $E(G)$, and a relation called ***incidence*** so that each edge is incident with either one or two vertices, called ***ends*** (or ***endpoints***). For convenience, we sometimes write $G = (V, E)$ to indicate that G is a graph with ***vertex set*** V and ***edge set*** E.

Definition

Two distinct vertices u, v in a graph G are ***adjacent*** if there is an edge with ends u, v. We also call u, v neighbors in G.

Remark

- Vertices are also called nodes, points, locations, stations, etc.
- Edges are also called arcs, lines, links, pipes, connectors, etc.

Loops, Parallel Edges, and Simple Graphs

Definition

An edge with just one end is called a ***loop***. Two distinct edges with the same ends are are ***parallel*** (called "parallel edges" or "multiple edges"). A graph without loops or parallel edges is called ***simple***.

Remark

We specify a simple graph (V, E) by its ***vertext set*** V, and ***edge set*** E, where $E \subset \binom{V}{2}$. We write $e = uv$ or $e = vu$ for an edge $e \in E$ with ends $u, v \in V$. (That is, $e = \{u, v\}$.)

Isomorphism

Definition

An ***isomorphism*** from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$. We say "G is ***isomorphic*** to H", donoted $G \cong H$, if there is an isomorphism from G to H.

Remark

The ***relation isomorphism***, consisting of the set of ordered pairs (G, H) such that G is isomorphic to H is an equivalence relation on the class of simple graphs.

Representing Graph

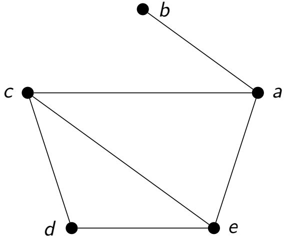
Example

- By specifying the vertex and edge sets of the graph.
- By using database structure.
- By showing a "drawing" of the graph.

Ajacency Tables

An ajacency table lists all the vertices of the graph and the vertices adjacent to them. Consider $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{c, d\}, \{c, e\}, \{d, e\}\}$.

Vertex	Adjacent Vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d



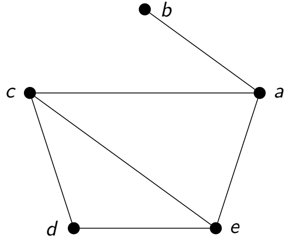
Using Graphs as Models

Example

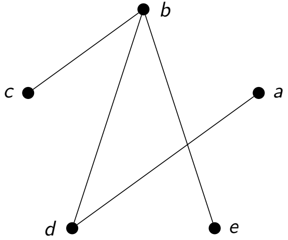
- Acquaintance relations.

Definition

The ***complement*** $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ iff $uv \notin E(G)$. Note that given graph $G = (V, E)$, we have $\overline{G} = \left(V, \binom{V}{2} - E\right)$.



G



$\overline{G}$

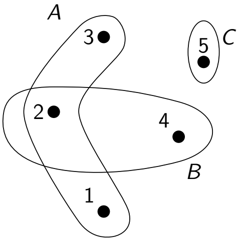
Using Graphs as Models

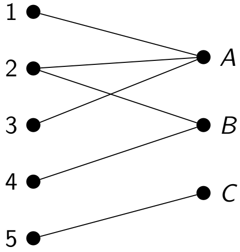
Example

- Job assignments.

Definition

A graph (not necessarily simple) is ***bipartite*** if $V(G)$ is the union of two disjoint (possibly empty) independent sets (i.e., a set of pairwise nonadjacent vertices), called ***partite sets*** of G.





	A	B	C
1	1	0	0
2	1	1	0
3	1	0	0
4	0	1	0
5	0	0	1

Using Graphs as Models

Example

- Maps and coloring.
- Routes in road networks.
- ...

Standard Graphs

Definition

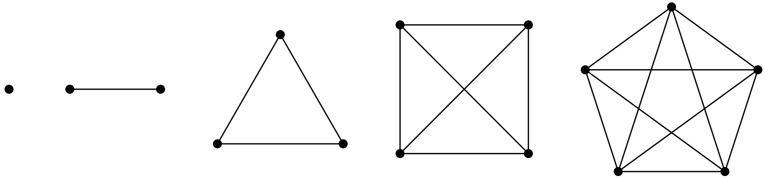
The ***null graph*** is the graph whose vertex set and edge set are empty.

Definition

A graph G is ***complete*** if it is simple and all pairs of distnct vertices are adjacent. A complete graph on n vertices is denoted by K_n.

Definition

A ***clique*** in a graph is a set of pairwise adjacent vertices.

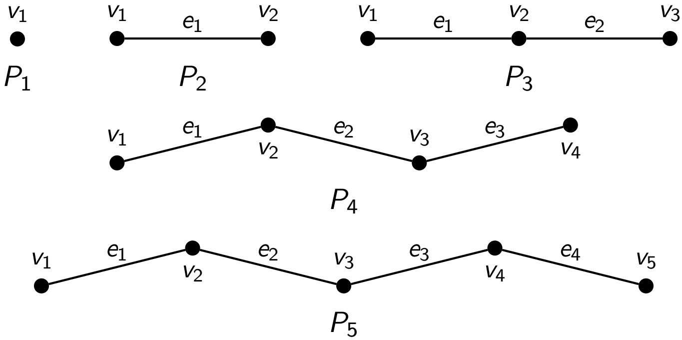


K_1 K_2 K_3 K_4 K_5

Standard Graphs

Definition

A graph G is called a ***path*** if the vertices can be ordered as $v_1, \ldots, v_n$, and edges can be ordered as $e_1, \ldots, e_{n-1}$ such that $e_i = v_i v_{i+1}$, $i = 1, \ldots, n$. A path on n vertices is denoted by P_n.



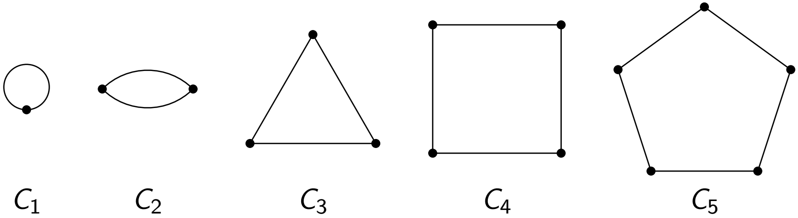
Standard Graphs

Definition

A graph G is a ***cycle*** if $V(G)$ can be ordered as $v_1, \ldots, v_n$, and $E(V)$ can be ordered as $e_1, \ldots, e_n$, where

$$e_i = \begin{cases} v_i v_{i+1}, & 1 \leq i \leq n-1 \\ v_n v_1, & i = n \end{cases}$$

A cycle on n vertices is denoted by C_n.



Subgraphs

Definition

If G, H have $V(H) \subset V(G)$, and $E(H) \subset E(G)$ with incidence in H the same as G, then H is a ***subgraph*** of G, denoted by $H \subset G$.

Obviously, given $H_1, H_2 \subset G$, then

- $H_1 \cap H_2 \subset G$, with

$$V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$$
$$E(H_1 \cap H_2) = E(H_1) \cap E(H_2)$$

- $H_1 \cup H_2 \subset G$, with

$$V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$$
$$E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$$

Subgraphs

Remark

When we name a graph without naming its vertices, we often mean its isomorphsim class. Technically, "H is a subgraph of G" means that some subgraph of G is isomorphic to H (we say "G contains a **copy** of G").

Example

- C_3 is a subgragh of K_5.
- P_1 is a subgragh of K_5.
- K_5 is a subgragh of K_6.

Degree of Vertices

Definition

The ***degree*** of a vertex v in a graph G, denoted $\deg(v)$ is the number of incident edges (loops counted twice). We write $\deg_G(v)$ if G is not clear (i.e., we are not sure if $v \in V(G)$).

Theorem

For all $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof.

By double counting. $\square$

Corollary (Handshaking lemma/degree sum formula)

Every graph has an even number of odd degree vertices.

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Walks

Definition

A **walk** W in a graph G is a sequence $v_0, e_1, v_1, \ldots, e_n, v_n$ such that every e_i has ends v_{i-1} and v_i. If $v_0 = v_n$, we say that W is closed.

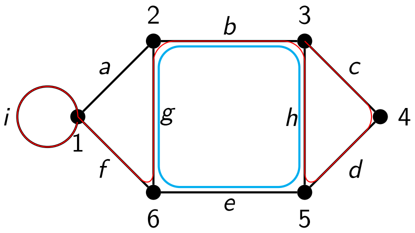
Definition

The **length** of a walk, path, or cycle is its number of edges. A walk is **closed** if its ends are the same.

Remark

- A walk is **NOT** a graph in general.
- A path is a graph.
- If $v_0, \ldots, v_n$ in a walk are distinct, we also call this walk a path.
- A walk with only 1 vertex has length 0.

Walks



Example

- $W = 3, c, 4, d, 5, h, 3, b, 2, g, 6, f, 1, i, 1$. W is NOT a closed walk (b/c 3 is not the same vertex as 1). The length of W is 7.
- $W' = 3, b, 2, g, 6, e, 5, h, 3$. W' is a closed walk, with with length 4. Note that $W' \cong C_4$.

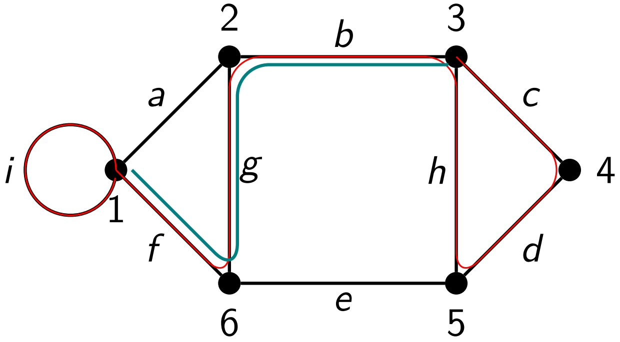
Connected Graph

Definition

A graph G is **connected** if for all $u, v \in V(G)$, there is a walk from u to v (also called a u, v-walk). Otherwise, G is **disconnected**.

Theorem

If there is a walk from u to v, then there is a path from u to v.



Connected Graph

Proof.

Claim: The path from u to v is the shortest walk from u to v (i.e., the walk of minimum length.)

Indeed. Let W be a walk of minimum length from u to v, say $v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n$, with $u = v_0$ and $v = v_n$.

Suppose this is is **NOT** a path, then there exsits $v_i = v_j$ such that $0 \leq i < j \leq n$. Therefore $v_0 e_1 v_1 \cdots v_i e_{j+1} v_{j+1} \cdots e_n v_n$ is a shorter walk from u to v, which is a contradiction. □

Connected Graph

Theorem

G is disconnected iff there is a partition $\{X, Y\}$ of $V(G)$ such that no edge has an end in X and an end in Y.

Proof.

($\Leftarrow$) True by definition of connnectivity.
($\Rightarrow$) Choose $x, y \in V(G)$ such that no walk from x to y exists, define

$$X := \{z \mid \exists \text{ a walk from } x \text{ to } z\}$$
$$Y := V(G) \setminus X$$

Claim: no edge has an end in X and an end in Y, which is obvious. □

Connected Graph

Theorem

Given $H_1, H_2 \subset G$, H_1, H_2 connected graphs, and $V(H_1) \cap V(H_2) \neq \varnothing$, then $H_1 \cup H_2$ is connected.

Proof.

Let $u, v \in V(H_1 \cup H_2)$. Choose $w \in V(H_1 \cap H_2)$ $(\neq \varnothing)$, note that u, v is either in H_1 or H_2, w.l.o.g., let $u \in V(H_1)$, $v \in V(H_2)$. For $i = 1, 2$, H_i is connected, so there is a u, w-walk W_i. Now concatenate W_1 and W_2, we have a u, v-walk. Since u, v are arbitrary, therefore $H_1 \cup H_2$ is connected. □

A Result From Analysis (iii)⇒(ii)⇒(i)⇒(iii)

Let U be an open subset of a normed space over $\mathbb{R}$, TFAE,

(i) U is connected.

(ii) Any two points of U can be joined by a path in U (path connected).

(iii) Any two points of U can be joined by a polygonal path in U.

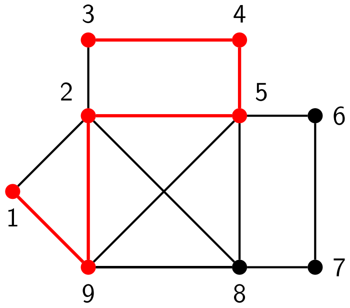
Connected Graph

Definition

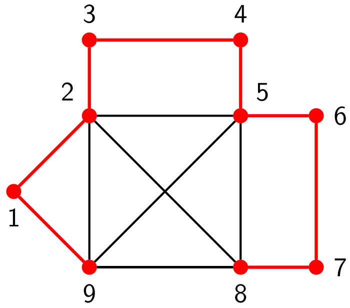
A maximal connected subgraph of G is a subgraph that is connected and is ***not*** contained in any other connected subgraph of G.

Remark

A path/subgraph in G is maximal if it cannot be enlarged.



a path that is maximal

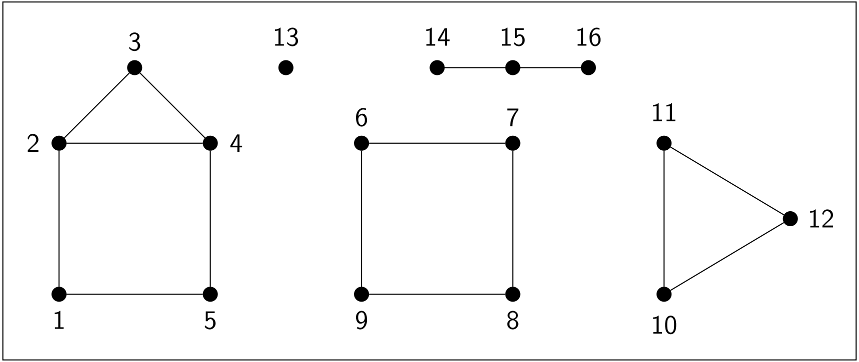


a path that is both maximal and maximum

Connected Graph

Definition

A **component** of a graph G is a maximal non-empty connected subgraph of G. The number of components of G is denoted $\text{comp}(G)$.



$$\text{comp}(G) = 5$$

Connected Graph

Theorem

Every vertex is in a ***unique*** *component.*

Proof.

Let $v \in V(G)$. Note that v is in a connected subgraph $(\{v\}, \varnothing)$, which consists of only v and no other vertices or edges. If H_1 and H_2 are connected subgraphs containing v, then $H_1 \cap H_2 \neq \varnothing$, thus $H_1 \cup H_2$ is connected. Therefore v is in a unique component. □

Remark

- Components are pairwise disjoint;
- No two components share a vertex;
- Adding an edge with endpoints in distinct components combine the two components into one.
- Adding/Deleting an edge decreases/increases the number of components by at most 1.

Connected Graph

Deleting Edges

Given graph G, $S \subset E(G)$, then $G - S$ is the graph obtained from G by deleting S.

Deleting Vertices

Given graph G, $X \subset V(G)$, then $G - X$ is the graph obtained from G by deleting every vertex in X and every edge incident to a vertex in X.

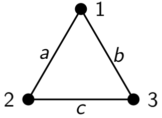
Notation

If $e \in E(G)$ or $v \in V(G)$, we define

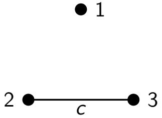
- $G - e := G - \{e\}$;
- $G - v := G - \{v\}$.

For example, $G - v - w = G - \{v, w\}$.

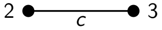
$C_3 =$



$C_3 - \{a, b\} =$



$C_3 - \{1\} =$



Connected Graph

Definition

An edge $e \in E(G)$ is called a ***cut-edge*** or ***bridge*** if no cycle contains e.

Theorem

Given graph G and $e \in E(G)$, then

- *either e is a cut-edge and* $\text{comp}(G - e) = \text{comp}(G) + 1$*;*
- *or e is NOT a cut-edge and* $\text{comp}(G - e) = \text{comp}(G)$*.*

Proof.

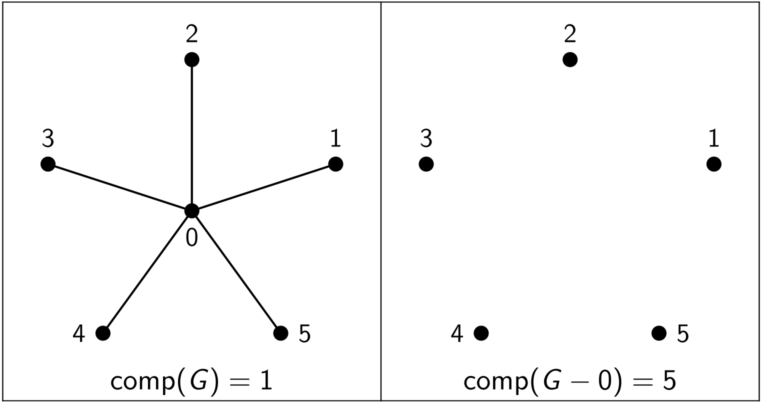
Let u, v be the ends of e ($u = v$ if e is a loop). Note that G has a cycle containing e, iff $G - e$ contains a path from u to v, iff u, v are in the same component of $G - e$. Now

- If u, v are in the same component H of $G - e$, then $H + e$ is a component of G, so $\text{comp}(G - e) = \text{comp}(G)$.
- If u, v are in distinct components, say H_1, H_2 of $G - e$, then $H_1 \cup H_2 + e$ is a component of G, so $\text{comp}(G - e) = \text{comp}(G) + 1$. □

Connected Graph

Definition

A vertex $v \in V(G)$ is called a ***cut-vertex*** whose deletion increases the number of components.



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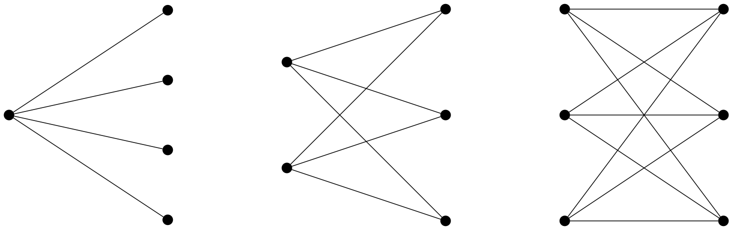
Bipartation

Definition

A ***bipartation*** of a graph G is a pair (A, B) where $A, B \subset V(G)$ with $A \cap B = \varnothing$, $A \cup B = V(G)$ such that every edge has an end in A and an end in B. G is ***bipartite*** if it admits a bipartation.

Definition

A ***complete bipartite graph*** or ***biclique***, denoted $K_{m,n}$, is a simple bipartite graph with bipartation (A, B) with $|A| = m$ and $|B| = n$ such that every vertex in A is adjacent to every vertex in B.



$K_{1,4}$ $K_{2,3}$ $K_{3,3}$

Bipartation

Theorem

For every graph G, TFAE

(i) *G is bipartite.*

(ii) *G has no cycle of odd length.*

(iii) *G has no closed walk of odd length.*

Proof.

(i)$\Rightarrow$**(ii):** Assume that $G = (A \cup B, E)$ is bipartite and let $C \subset G$ be a cycle. Then every other vertex of C is in A and every other vertex is in B, hence C must have even length. (It takes even number of steps in a bipartite graph to return to the starting point.)

Bipartation

Proof (Cont.)

(ii)$\Rightarrow$(iii): We show the contrapositive, i.e., $\neg$(iii) $\Rightarrow$ $\neg$(ii). Let G have a closed walk of odd length, and choose such a walk $v_0, e_1, v_1, \ldots, v_n$ of minimum length. If there exist $1 \leq i < j \leq n$ with $v_i = v_j$, then

- either $j - i$ is odd and $v_i, e_i, \ldots, v_j$ is a shorter closed walk of odd length,
- or $j - i$ is even and $v_0, e_1 \ldots v_i, e_{j+1}, v_{j+1}, \ldots, v_n$ is a shorter closed walk of odd length.

It follows that $v_1, \ldots, v_n$ must be distinct ($v_0 = v_n$), hence $(\{v_1, \ldots, v_n\}, \{e_1, \ldots, e_n\})$ is an odd cycle.

Bipartation

Proof (Cont.)

(iii)⇒(i): Let G be a graph with no closed walk of odd length, w.l.o.g., we may assume that G is connected. Choose a "base point" $u \in V(G)$, observe that for every vertex $v \in V(G)$,

- either all u, v-walks have even length,
- or all all u, v-walks have odd length.

(Note that otherwise can concatenate and odd and an even walk to form a closed walk of odd length.) Now define

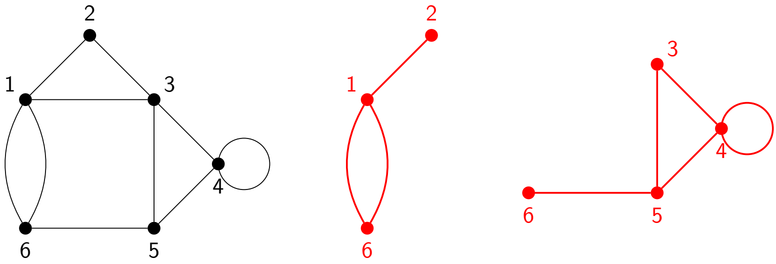
$$A := \{v \in V(G) \mid \exists u, v\text{-walk of even length}\}$$
$$B := \{v \in V(G) \mid \exists u, v\text{-walk of odd length}\}$$

It follows that $A \cap B = \varnothing$. Since G is connected, we have $A \cup B = V(G)$. It follows that (A, B) is a bipartition of G, hence (i) is satisfied. $\square$

Induced Subgraph

Definition

A subgraph $H \subset G$ is **induced** if every edge of G with both ends in $V(H)$ is in $E(H)$. Equivalently, H is induced if $H = G - (V(G) \setminus V(H))$.



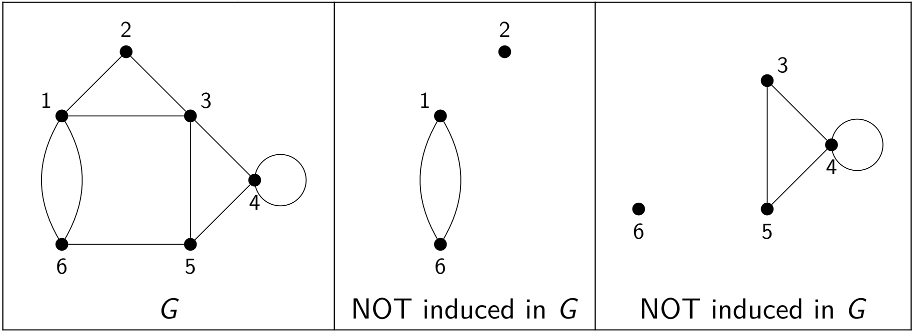
G — induced in G — induced in G

Remark

- An ***induced path*** is sometimes called a ***snake***.
- An ***induced cycle*** is sometimes called a ***chordless cycle*** or a ***hole***.

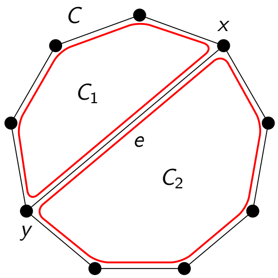
Induced Subgraph

NOT induced subgraph



G | NOT induced in G | NOT induced in G

Bipartation



Theorem

For every graph G, TFAE

(i) *G is bipartite.*

(ii) *G has no cycle of odd length.*

(iii) *G has no closed walk of odd length.*

(iiii) *G has no induced cycle of odd length.*

Proof.

(ii)⇒(iiii). Immediate.

(iiii)⇒(ii). We show the contrapositive, i.e., $\neg$(ii) $\Rightarrow$ $\neg$(iiii). Suppose G has a cycle of odd length, choose a shortest cycle $C \subset G$. Note that C is induced, otherwise $\exists e \in E(G) \setminus E(C)$, with ends x, y. But now either C_1 or C_2 is is an odd cycle of shorter length, contradiction. □

Remark

The ***girth*** of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.

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Matching

Definition

A **matching** in a graph $G = (V, E)$ is a subset of edges M such that M does not contain a loop and no two edges in M are incident with a common vertex. (i.e., the graph (V, M) has all vertices of degree < 2)

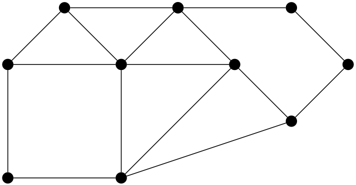
Definition

- A matching M is maximal if there is no matching M' such that $M \subsetneq M'$.
- A matching M is maximum if there is no matching M' such that $|M| < |M'|$.
- A ***perfect matching*** is a matching M such that every vertex of G is incident with an edge in M.

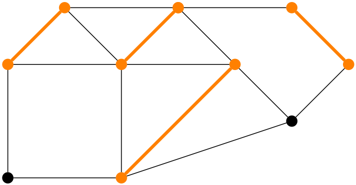
Example

- $K_{n,n}$ has $n!$ perfect matchings.
- K_{2n+1} has 0 perfect matchings.
- K_{2n} has $(2n-1)(2n-3)\cdots(3)(1) = (2n-1)!!$ perfect matchings.

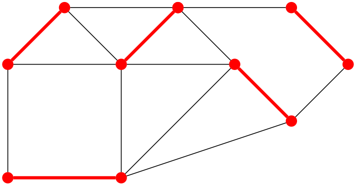
Matching



G



The matching M is maximal in G

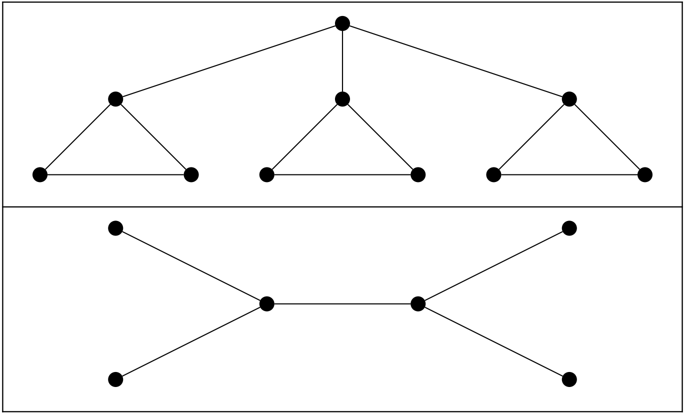


The matching M' is maximal, maximum, and perfect in G

Matching

Remark

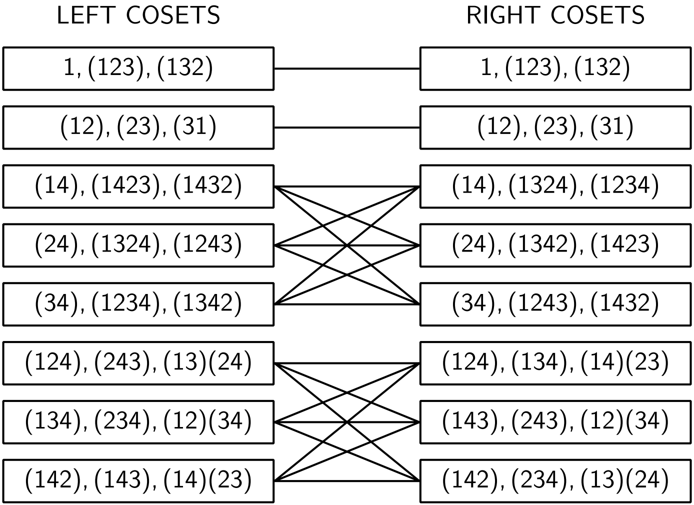
A necessary but not sufficent condition for a graph G to have a perfect matching is that $|V(G)|$ is even.



Group Transversals

Example

Consider $G = S_4$, and subgroup $H = \langle (123) \rangle \cong C_3$.



Group Transversals

Definition

Let $H, K \leq G$. We define the coset intersection graph $\Gamma^G_{H,K}$ to be a graph with vertex set consisting of all left cosets of H, i.e., $\{\ell_i H\}_{i \in I}$, together with all right cosets of K, i.e., $\{Kr_j\}_{j \in J}$, where I, J are index sets. If a left coset of H and right coset of K correspond, they are included twice. Edges (undirected) are included whenever any two of these cosets have non-empty intersection, and an edge aH-Kb corresponds to the nonempty set $aH \cap Kb$.

Theorem

The coset intersection graph $\Gamma^G_{H,K}$ is always a disjoint union of complete bipartite graphs.

Remark

The common transversals are given by a perfect matching in $\Gamma^G_{H,K}$.

Group Transversals

Proof (Button, Choido, and Laris).

We first show that for $a, b, c, d \in G$, if aH-Kb-cH-Kd is a path in $\Gamma^G_{H,K}$, then there is an edge aH-Kd. Note that there exist $h_1, h_2, h_3 \in H$ and $k_1, k_2, k_3 \in K$ such that $ah_1 = k_1 b$, $k_2 b = ch_2$, $ch_3 = k_3 d$. Rearranging yields $c = k_3 d h_3^{-1}$, so $b = k_2^{-1} k_3 d h_3^{-1} h_2$, so $a = k_1 k_2^{-1} k_3 d h_3^{-1} h_2 h_1^{-1}$, and thus $ah_1 h_2^{-1} h_3 = k_1 k_2^{-1} k_3 d$. Hence aH-Kd as required.

Take any $\ell_i H$ and some Kr_j in the connected component of $\ell_i H$ in $\Gamma^G_{H,K}$. There must be at least one such Kr_j, we show that $\ell_i H$ and Kr_j are connected by an edge. For if not, then there must be at least one finite path of length > 1 connecting them; take a minimal such path γ from $\ell_i H$ to Kr_j. Then γ begins with $\ell_i H$-Ka-bH-Kc-$\cdots$, where $Ka \neq Kr_j$. But we have showed previously that $\ell_i H$ and Kr_j must be joined by an edge. contradicting the minimality of γ. So $\ell_i H$ and Kr_j are joined by an edge for every Kr_j in the connected component of $\ell_i H$. $\square$

Group Transversals

Now suppose that G is finite with $|H| = |K| = m$. Since the connected component is isomorphic to $K_{s,t}$ for some $s, t \in \mathbb{N}$, and note that the cosets of H (or K) are disjoint and have the same size $|H|$ (or $|K|$), we have both $s|H| \leq t|K|$ and $t|H| \leq s|K|$, hence $s = t$. Therefore a perfect matching exists for each component of $\Gamma^{G}_{H,K}$. Each perfect matching corresponds to a set of commmon transversals. The results also follow if $H = K$.

Hall's Theorem

Definition

If $X \subset V(G)$, the ***neighbors*** of X is

$$N(X) := \{v \in V(G) \setminus X \mid v \text{ is adjacent to a vertex in } X\}$$

For simplicity, we write $N(x) := N(\{x\})$.

Definition

The edges $S \subset E(G)$ ***covers*** $X \subset V(G)$ if every $x \in X$ is incident to some $e \in S$.

Definition

The vertices $X \subset V(G)$ ***covers*** $S \subset E(G)$ if every $e \in S$ is incident to some $v \in X$.

Theorem (Hall)

Let G be a finite bipartite graph with bipartation (A, B). There exists a matching covering A iff there does not exist $X \subset A$ with $|N(X)| < |X|$.

Hall's Theorem

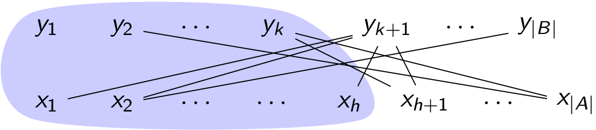
Proof via Dilworth's Theorem.

Necessity. Immediate by pigeonhole principle.

Sufficiency. Let $G = (A \cup B, E)$ be a bipartite graph satisfying Hall's condition that $|N(X)| \geq |X|$ for all $X \subset A$. Define a poset $(P, \leq)$ by letting $P = A \cup B$, and $x < y$ if $x \in A$, $y \in B$, and $xy \in E$. Suppose that the largest antichain is $S = \{x_1, \ldots, x_h, y_1, \ldots, y_k\}$, then

$$N(\{x_1, \ldots, x_h\}) \subset B \setminus \{y_1, \ldots, y_k\}$$

(for otherwise S would not be an antichain if $y \in \{y_1, \ldots, y_k\}$ were the neighbor of some $x \in \{x_1, \ldots, x_h\}$.) Thus Hall's condition implies $|B| - k \geq h$, i.e., $|B| \geq k + h$.



Hall's Theorem

Proof (Cont.)

By Dilworth's theorem, P can be partitioned into $k + h$ chains, denote the matching by M, then

$$|M| + (|A| - |M|) + (|B| - |M|) = k + h \leq |B|$$

that is,

$$|A| + |B| - |M| \leq |B|$$

thus

$$|M| \geq |A|$$

i.e., there is a matching M covering A. □

Hall's Theorem

Theorem (Hall, balanced version)

Let G be a finite bipartite graph with bipartation (A, B), then G has a perfect matching iff $|A| = |B|$ and $|N(X)| \geq |X|$ for all $X \subset A$.

Proof.

We only prove the sufficiency of Hall's condition.
First we show that Hall's condition is symmetric, i.e., $|N(Y)| \geq |Y|$ for all $Y \subset B$. Indeed, take $Y \subset B$. Note that $N(A - N(Y)) \subset B - Y$, thus $|N(A - N(Y))| \leq |B - Y|$. By Hall's condition,
$|A - N(Y)| \leq |N(A - N(Y))| \leq |B - Y|$, thus $|A| - |N(Y)| \leq |B| - |Y|$, and it follows that $N(Y) \geq |Y|$.
We proceed by induction on $|A|$.
Base case: $|A| = |B| = 1$. Trivial.
Inductive case: Assume the IH that a perfect matching exists for $|A| = |B| < n$. Let $|A| = |B| = n$, take $a \in A$ and $b \in B$ that are connected by an edge. If $G - \{a, b\}$ satisfies Hall's condition, we are done.

Hall's Theorem

Proof (Cont.)

Otherwise, we can find a subset $X \subset A - a$ such that $|N(X) - b| < |X|$, and thus $|N(X)| = |X|$. Let H and H' be the subgraphs induced by $X \cup N(X)$ and $(A - X) \cup (B - N(X))$, respectively. Note that both H and H' are balanced bipartite graphs (of smaller size). Now H satisfies Hall's condition by restriction, and H' satisfies Hall's condition by argument prior to the induction. This completes the proof. $\square$

Kőnig-Egerváry Theorem

Definition

A **vertex cover** of a graph G is a set $X \subset V(G)$ if every $e \in E(G)$ is incident with a vertex in X. The vertices in X ***cover*** $E(G)$.

Remark

The size of the smallest vertex cover is denoted $\beta(G)$, and the size of the largest matching is denoted $\alpha'(G)$.

Theorem (Kőnig-Egerváry)

Given a finite bipartite graph G, $\alpha'(G) = \beta(G)$.

Proof.

First of all, it is clear that $\alpha'(G) \leq \beta(G)$. For the other direction. Let (A, B) be a bipartitation of G, and X a vertex cover of minimum size, and $Y = V(G) - X$. Let H_1 and H_2 be the subgraphs induced by $(A \cap X) \cup (B \cap Y)$ and $(A \cap Y) \cup (B \cap X)$ respectively. Note that H_1 and H_2 have bipartations $(A \cap X, B \cap Y)$ and $(A \cap Y, B \cap X)$ respectively, and there is no edge between $A \cap Y$ and $B \cap Y$.

Kőnig-Egerváry Theorem

Proof (Cont.)

Now we claim that H_1 has a matching covering $A \cap X$, and H_2 has a matching covering $B \cap X$.
Indeed, consider H_1, suppose there is no such matching, i.e., $|N(Z)| < |Z|$ for some $Z \subset A \cap X$, then we can switch Z and $N(Z)$ for a smaller vertex cover $X' = ((A \cap X) - Z) \cup N(Z)$, a contradiction. It remains to show that X' is indeed a vertex cover. The part for H_2 is similar. $\square$

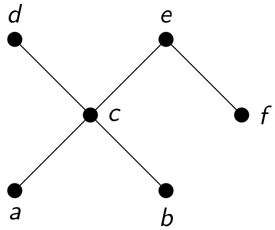
Kőnig-Egerváry implies Dilworth

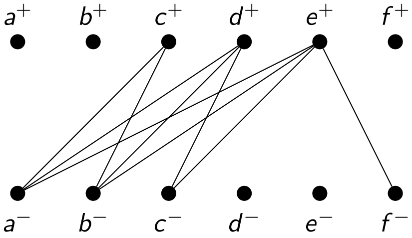
Theorem (Fulkerson, 1956)

Kőnig-Egerváry theorem implies Dilworth theorem (and vice versa).

Proof.

Given a finite poset $P = (X, \leq)$, define a bipartite graph $B_P = (X^- \cup X^+, E)$ with bipartitation (X^-, X^+), where X^- and X^+ are copies of X, and $x^- y^+ \in E$ iff $x < y$ in P.





Kőnig-Egerváry implies Dilworth

Proof (Cont.)

For every matching M of B_P (not necessarily maximum or maximal), we can associate with it a chain partition $\mathcal{C}_M$ of P, then $|\mathcal{C}_M| = |X| - |M|$. Take a minimum vertex cover R of B_P, let $A_R = \{x \mid x^-, x^+ \notin R\}$, then

- A_R is an antichain.
- $\{x^-, x^+\} \not\subset R$ for all $x \in V(B_P)$.

As a consequence, $|A_V| = |X| - |R|$. Take M and R of the same size, we get $\mathcal{C}_M$ and A_R of the same size. This is Dilworth theorem. □

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Trees

Definition

A **forest** is a graph with no cycles. A **tree** is a connected forest.

Theorem

If G is is a forest, then $\mathrm{comp}(G) = |V(G)| - |E(G)|$*. In particular, if T is a tree, then* $|V(T)| = |E(T)| + 1$*.*

Proof.

Induction on $|E(G)|$.

Base case. If $|E(G)| = 0$, G has no edge, thus $\mathrm{comp}(G) = |V(G)|$.

Inductive case. $|E(G)| > 0$. Choose $e \in E(G)$, since G has no cycle, then e is a cut-edge, thus

$$\begin{aligned}\mathrm{comp}(G) &= \mathrm{comp}(G - e) - 1 \\ &= |V(G - e)| - |E(G - e)| - 1 \qquad \text{(by IH)} \\ &= |V(G)| - |E(G)|\end{aligned}$$

□

Trees

Definition

A **leaf** is a vertex of degree 1.

Theorem

Let T be a tree with $|V(T)| \geq 2$, then T has at least 2 leaves, and if there are only 2 leaves, then T is a path.

Proof.

Note that $2 = 2|V(T)| - 2|E(T)| = \sum_{v \in V(T)} (2 - \deg(v))$. Since T is connected, and $|V(T)| \geq 2$, all vertices have degree > 0. This means there are at least 2 leaves.

Further if there are exactly 2 leaves, then all other vertices have degree 2, therefore T is a path. (Take any maximal path in T, note that any extra edge would increase the degree of interior vertices to 3, or increases the degree of leaves to 2) $\square$

Trees

Lemma
If T is a tree and v is a leaf, then $T - v$ is a tree.

Proof.
Observe that $T - v$ has no cycle and is connected. $\square$

Theorem
If T is a tree, and $u, v \in V(T)$, then there is a unique u, v-path.

Proof.
Induction on $|V(T)|$.
Base case. $|V(T)| = 1$. We have $u = v$.
Inductive case.

- If there is a leaf $w \neq u, v$, apply induction to $T - w$;
- Otherwise T is a path with ends u, v, which is unique by IH. $\square$

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Spanning Trees

Definition

If T is a subgraph of a graph G, and T is a tree with $V(T) = V(G)$, then we call T a ***spanning tree*** of G.

Notation

If G is a graph, $H \subset G$ and $e \in E(G)$, then we define $H + e$ to be the subgraph of G obtained from H by adding e and its ends.

Spanning Trees

Theorem

Let G be a connected graph with $|V(G)| \geq 2$. If H is a subgraph satisfying

(i) *either H is minimal such that $V(H) = V(G)$ and H is connected,*

(ii) *or H is maximal such that H has no cycles,*

then H is a spanning tree of G.

Proof.

(i) It suffices to show that H is a tree. Suppose that H has a cycle C, choose $e \in E(C)$. Now $H - e$ is connected (b/c e is not a cut-edge), but this contradicts that H is minimal.

(ii) Note that $V(H) = V(G)$ by maximality of H. It remains to show that H is connected. Suppose not, choose a partition $\{X, Y\}$ of $V(H) = V(G)$ such that no edge of H has one end in X and the other in Y. Choose $e \in E(G)$ such that e has one end in X and the other in Y (b/c G connected), but now $H + e$ contradicts that H is maximal. $\square$

Trees

Theorem

If $|V(G)| = |E(G)| + 1$, and

(i) *either G has no cycles,*

(ii) *or G is connected,*

then G is a tree.

Proof.

(i) Since G is a forest, then $1 = |V(G)| - |E(G)| = \mathrm{comp}(G)$.

(ii) Choose a spanning tree T of G (possible b/c G connected), then

$$|E(G)| = |V(G)| - 1 = |V(T)| - 1 = |E(T)|$$

thus $G = T$, so G is a tree. $\square$

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Weighted Graph

Definition

A ***weighted graph*** is a graph G with a weight function $w : E(G) \to \mathbb{R}$. A ***minimum-cost tree*** (or ***minimum-weight spanning tree***) of G is a spanning tree T for which

$$\sum_{e \in E(T)} w(e)$$

is minimum.

Kruskal's Algorithm

Kruskal's Algorithm

- Input: A connected weighted graph $G = (V, E)$.
- Output: A minimum-cost tree T.
- Procedure: Choose a sequence of edges $e_1, e_2, \ldots, e_m$ according to the rule that e_i is an edge of minimum weight in $E(G) \setminus \{e_1, \ldots, e_{i-1}\}$ so that $\{e_1, \ldots, e_{i-1}\}$ does not contain the edge set of a cycle. When no such edge exists, stop and return the subgraph $T = (V, \{e_1, \ldots, e_m\})$.

Spanning Trees v. Vector Space Bases

Finite dimensional vector spaces

A basis for a finite dimensional vector space is any of the following

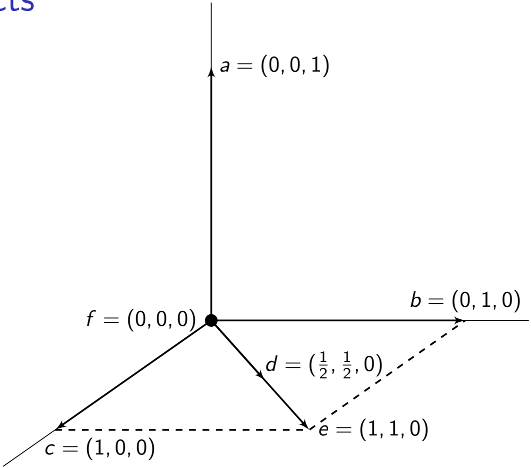
- A minimal spanning/generating set.
- A maximal linearly independent set.
- Every element of the vector space is uniquely represented by as a linear combination of the basis vectors.

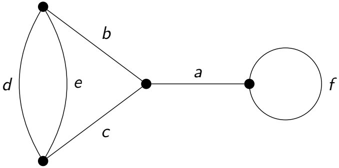
Finite connected graphs

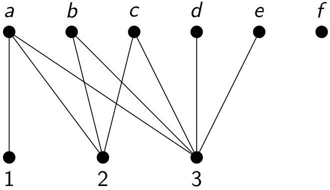
A spanning tree is any of the following

- A minimal subgraph maintaining the same vertex set and connectedness.
- A maximal subgraph without cycles.
- For any two vertices, there is a unique path between them in the tree.

A Few Objects







Matroid

Definition

A matroid is a pair $(E, \mathcal{I})$ where

- E is a finite set (i.e., "ground set")
- $\mathcal{I}$ is a collection of subsets of E (i.e., "independent sets") such that
  - **(I1)** $\varnothing \in \mathcal{I}$.
  - **(I2)** If $J \in \mathcal{I}$ and $I \subset J$, then $I \in \mathcal{I}$.
  - **(I3)** If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $j \in J - I$ such that $I \cup j \in \mathcal{I}$. (exchange axiom)

Steinitz exchange lemma

Theorem

Let U and W be finite subsets of a vector space V. If U is a set of linearly independent vectors, and W spans V, then:

- $|U| \leq |W|$;
- *There is a set $W' \subseteq W$ with $|W'| = |W| - |U|$ such that $U \cup W'$ spans V.*

The result is often called the Steinitz–Mac Lane exchange lemma, also recognizing the generalization by Saunders Mac Lane of Steinitz's lemma to matroids.

> *A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.*
>
> — *Stefan Banach*

Examples of Matroids

Theorem

Let E be a finite set of vectors in a vector space V. Let $\mathcal{I}$ be a collection of linearly independent subsets of E, then $(E, \mathcal{I})$ is a matroid.

Theorem

Let $G = (V, E)$ be a finite graph. Let $\mathcal{I}$ be the collection of independent set of edges (i.e., they do not form any cycles), then $(E, \mathcal{I})$ is a matroid.

Proof.

We need to verify axioms **(I1)–(I3)**. **(I1)** and **(I2)** are clear. To see **(I3)**, recall that if $I \subset E$ is independent, then (V, I) has $|V| - |I|$ connected components. Let $I, J \in \mathcal{I}$ with $|I| < |J|$, suppose (for contradiction) that every edge $j \in J - I$ forms a cycle in (V, I), then $(V, I \cup J)$ also has $|V| - |I|$ components. Now removing I from $I \cup J$ recovers (V, J), and as a result (V, J) has at least $|V| - |I|$ components. But (V, J) has $|V| - |J|$ components to start with, this implies that $|V| - |J| \geq |V| - |I|$, hence $|I| \geq |J|$, a contradiction! $\square$

Bases

Definition

Given a finite matroid M, a basis/base of M is a maximal independent set.

The following is immediate.

Lemma

All the basis of a finite matroid have the same size/cardinality.

Proof.

Suppose B_1 and B_2 are bases with $|B_1| < |B_2|$, then by exchange axiom we can find $b_2 \in B_2$ such that $B_1 \cup b_2 \in \mathcal{I}$. This implies that $B_1 \subsetneq B_1 \cup b_2$, hence B_1 is not a basis, a contradiction! $\square$

Spanning Trees

Theorem

Given a finite graph $G = (V, E)$, then a basis for the graphical matroid $M(G)$ is a spanning tree of G.

Theorem

The spanning trees of a connected graph have the same number of edges.

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Distance Function

Definition

Given graph G, and $u, v \in V(G)$, the distance from u to v, denoted $\text{dist}(u, v)$, is the shortest length of a walk from u to v in G.

Remark

For $u, v, w \in V(G)$, the triangle inequality holds,

$$\text{dist}(u, v) + \text{dist}(v, w) \geq \text{dist}(u, w).$$

Weighted Distance

Definition

Given a simple connected graph with weight function $w : E(G) \to \mathbb{R}_{\geq 0}$, the length of a walk $v_1 e_1 v_2 e_2 \cdots e_k v_{k+1}$ is given by

$$w(e_1) + w(e_2) + \cdots + w(e_k).$$

Then the distance from u to v is the length of the shortest walk from u to v.

Definition

Given graph G, and $r \in V(G)$, a tree $T \subset G$ with $r \in V(T)$ is a **shortest path tree** for r if

$$\mathrm{dist}_G(r, v) = \mathrm{dist}_T(r, v)$$

for every $v \in V(T)$.

Dijkstra's Algorithm

Dijkstra's Algorithm

- Input: A simple connected graph $G = (V, E)$ with root vertex r and nonnegative weight function $w : E(G) \to \mathbb{R}_{\geq 0}$.
- Output: A shortest path spanning tree for r.
- Procedure:
  1. $i = 1$. Set T_1 to be the tree consisting of only the root vertex r.
  2. $i \geq 2$. Choose an edge uv such that $u \in V(T_{i-1})$, $v \in V(G) \setminus V(T_{i-1})$, and $\text{dist}_T(r, u) + w(uv)$ is minimum. Let $T_i := T_{i-1} + uv$. If no such choice is possible, return the present tree.