上 海 交 通 大 学 试 卷

2021 – 2022 Academic Year (Spring Term)

Ve203 Discrete Mathematics
Final Exam

Name (Hanzi) ______________ Name (Pinyin) ______________

Student No. ______________ Class No. ______________

You have **100 minutes** to complete this evaluation. Please write your answers in this booklet. Remember to write neatly and clearly, so your answers can be fully understood. Make sure that you **explain your reasoning** in as detailed a manner as possible.

- You **may** bring a calculator of type "`Casio fx-991CN X`" or "`Casio fx-82`".
- You **may** use pencil, pen, eraser, ruler, compass and other non-electronic writing materials.
- You **may** use an English monolingual dictionary in book form — no electronic translators are allowed.
- The exam is **closed-book**. You may use the internet only for
  - Maintaining connection to Feishu;
  - Downloading exam paper from Feishu;
  - Uploading your answer files to canvas (or email to the instructor in case canvas malfunctions).

Pledge of Honor

The University of Michigan – Shanghai Jiao Tong University Joint Institute trusts its students to participate in examinations in an honorable and respectful manner, following a spirit of fairness and equality. Cheating, seeking unfair advantage and disturbing the safe and harmonious environment of examinations are contrary to the ethical principles of students of the Joint Institute. The letter and spirit of the Honor Code shall guide the behavior of students, faculty and all members of the Joint Institute. Therefore, I hereby declare that

(i) I will neither give nor receive unauthorized aid during the present examination, nor will I conceal any violations of the Honor Code by others or myself.

(ii) I confirm that I have read and understood the rules and procedures for examination set out by SJTU. I will follow them to the best of my ability.

(iii) I understand that violating the rules and procedures for examinations or the Honor Code will lead to administrative and/or academic sanctions.

Please sign the Pledge of Honor on the other file.

Exercise	Points	Score	Signature
1	10		
2	10		
3	10		
4	10		
5	20		
6	20		
7	20		
Total	100		

Twelvefold Way Distribute k balls into n urns. ($f : B \to U$, $|B| = k$, $|U| = n$)

Balls (domain)	Urns (codomain)	unrestricted (any function)	≤ 1 (injective)	≥ 1 (surjective)
labeled	labeled	n^k	$n^{\underline{k}}$	$n!\left\{ {k \atop n} \right\}$
unlabeled	labeled	$\left(\!\binom{n}{k}\!\right)$	$\binom{n}{k}$	$\left(\!\binom{n}{k-n}\!\right)$
labeled	unlabeled	$\sum_{i=1}^{n} \left\{ {k \atop i} \right\}$	$[k \leq n]$	$\left\{ {k \atop n} \right\}$
unlabeled	unlabeled	$\sum_{i=1}^{n} p_i(k)$	$[k \leq n]$	$p_n(k)$

Master Theorem If $T(n) = aT(n/b) + f(n)$ (for constants $a \geq 1$, $b > 1$, $d \geq 0$), then

(i) $T(n) = \Theta(n^{\log_b a})$ if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.

(ii) $T(n) = \Theta(n^{\log_b a} \lg^{d+1} n)$ if $f(n) = \Theta(n^{\log_b a} \lg^d n)$.

(iii) $T(n) = \Theta(f(n))$, if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.

Exercise 1 (10 points)
Apply the **master theorem** to get asymptotic estimates (as tight as possible) of $T(n)$ for the following recurrence equation.

$$T(n) = T(\sqrt{n}) + \lg \lg n$$

Exercise 2 (10 points)
Find the number of integer solutions to the equation

$$\sum_{i=1}^{10} x_i = 2022$$

such that $0 \leq x_1 \leq 1011$ and $x_i \geq 0$ for $i = 2, \ldots, 10$. Express the answer in terms of binomial coefficients. Show your work.

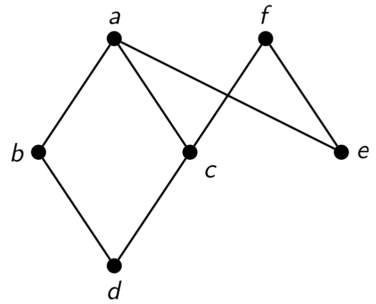
Exercise 3 (10 points)
Given a finte graph $G = (V, E)$ with $|V| = n$ and $|E| \geq n$, $n \in \mathbb{N} \setminus \{0\}$. Show that G contains a cycle.

Exercise 4 (10 points)
Given finite graphs G and H. Define the graph K as follows,

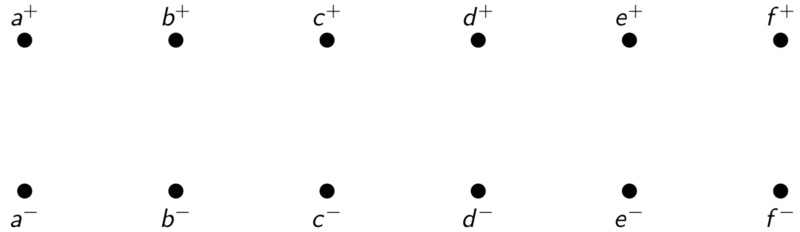
- $V(K) \coloneqq V(G) \times V(H)$;
- $\{(u, u'), (v, v')\} \in E(K)$ iff
  - either $u = v$ and $u'v' \in H$,
  - or $u' = v'$ and $uv \in G$.

Draw the graph K below if $G = C_4$ and $H = P_2$. (Note that $|V(P_2)| = 2$.)

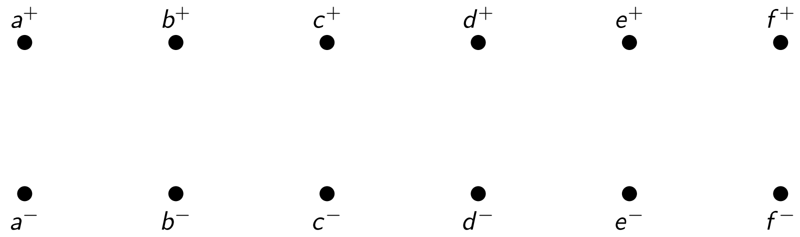
Exercise 5 (20 points)
Given a poset P with the following Hasse diagram,



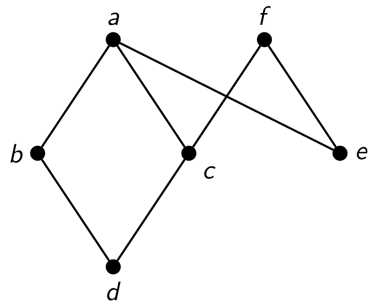
(i) (5 points) Construct the bipartite graph B_P induced by P according to the rule that $x^- y^+ \in E(B_P)$ iff $x < y$ in P. Complete the edges in the following diagram.



(ii) (5 points) Find a maximum matching in B_P. Draw the matching in the following diagram.



(iii) (10 points) Write down explicitly the chain partition according to the maximum matching found in (ii), as well as indicating it in the following diagram.



Exercise 6 (20 points)
Let $k, n \in \mathbb{N}$ with $k \leq (n-1)/2$.

(i) (10 points) Show that there exists an injection $f : \binom{[n]}{k} \to \binom{n}{k+1}$ such that $A \subset f(A)$ for all $A \in \binom{[n]}{k}$.

(ii) (10 points) Use (i) to show that the width of the poset $P = (2^{[n]}, \subset)$ is at most $\binom{n}{\lfloor n/2 \rfloor}$. (You may use the fact that P is self-dual, i.e., $P = P^{\mathrm{d}}$.)

Exercise 7 (20 points)

Let G be a finite graph.

(i) (10 points) Let T be a spanning tree of G, $e \in E(T)$, and $f \in E(G) - E(T)$. Let $P \subset T$ be the unique path connecting the ends of f, and $e \in P$. Show that $T - e + f$ is a spanning tree.

(ii) (10 points) Given two **distinct** cycles $C, D \subset G$, and an edge $e \in C \cap D$. Show that $C \cup D - e$ contains a cycle.