Problem 1

From the definition, we only need to prove that there doesn't exist an injection from N to $\{0,1\}$

We let A be the set of infinite binary sequences

On every digit, there can be two choices 1, 0

so $\text{card} A = 2^{\aleph_0} = \text{card} \mathbb{R}$.

Since there doesn't exist an injection from $N \to \mathbb{R}$ (Cantor's theorem)

We prove that the set is uncountable.

Problem 2

(i) To be a poset, the order needs to be reflexive, antisymmetric and transitive.

Reflexivity

$A \preceq A \Leftrightarrow (\forall a \in A)(\exists a \in A)(a \leq a)$

which is T because $(P, \leq)$ is a poset that has reflexivity which satisfies $a \leq a \rightarrow T$.

Antisymmetric

$A \preceq B \wedge B \preceq A \Leftrightarrow (\forall a \in A) \exists b \in B (a \leq b)$ ①

$\wedge (\forall b \in B) \exists a \in A (b \leq a)$ ②

This will ① $\wedge$ ② will not be true unless $A = B$, since if $A \neq B$

if we consider the maximum element of b_t

if ① is satisfied, we can n't find any element that satisfies

$a \geq b_t$ since for $\forall a \in A$. $a \leq b$, therefore

$A \preceq B \wedge B \preceq A \Rightarrow A = B$

Transitivity

$A \preceq B \Leftrightarrow (\forall a \in A)(\exists b \in B)(a \leq b)$

$B \preceq C \Leftrightarrow (\forall b \in B)(\exists c \in C)(b \leq c)$

We take the same b, denoted as b_0 here.

$\forall a \in A$, $\exists b_0 (\in B)$, $a \leq b_0$

$\forall b \in B$ (which includes b_0) $\exists c \in C$ $c \geq b_0$

then, there always exits a c (denoted as c_0) that satisfies

$c_0 \geq b_0 \geq a$ (any element in A)

therefore $A \preceq B \wedge B \preceq C \Rightarrow A \preceq C$

To conclude $(\mathcal{A}(P), \preceq)$ is a poset

(ii) $\Leftarrow$ when $A \preceq B$

$(\forall a \in A)(\exists b \in B)(a \leq b)$ pick any a.

Then is $x \in D(A)$, we can always find an element in set B

that satisfies $b \geq a$ and $x \in D(B)$ too since $\underline{x \leq a \leq b}$.

Then the leftward is proved.

$\Rightarrow$ if for any $x \in D(A)$ also belong to $D(B)$

This indicates that:

for $\forall x \in P$, $x \leq a$ for any arbitrary $a \in A$.

it alway satisfies $x \leq b$ for some $b \in B$.

And this is just the definition of $A \leq B$

And the right ward direction is proved.

Therefore $D(A) \subseteq D(B) \Leftrightarrow A \leq B$

Problem 3

$|M| < \infty$, $c * a = c * b \Rightarrow a = b$

We need to show that for all $x, y, z \in M$

$x * z = y * z$ implies $x = y$

when $x = e$ $z = y * z$

when $y = e$ $z = x * z$.

$$
\begin{aligned}
x * (z * x) &= x * (y * z * x) \\
&= (x * y) * (z * x) \\
&= x * (z * y).
\end{aligned}
$$

Since $c * a = c * b \Rightarrow a = b$

therefore $z * x = z * y$.

$\Rightarrow x = y$.

And the proof is done.

Problem 4

R, relation on A. R is transitive and irreflexive, then R is asymmetric.

Transitive:

$\forall x, y, z \in A,\ xRy \wedge yRz \rightarrow xRz$

Irreflexive:

$\forall x \in A,\ xRx \rightarrow \bot$

We want to show that

$\forall x, y \in A\ \ xRy \wedge yRx \rightarrow \bot.$

Irreflexivity

By transitivity $xRy \wedge yRx \rightarrow xRx \rightarrow \bot$

And the proof is done.

Problem 5

$(A, \leq A)$ is total order, for all $x, y \in A$ either $x \leq y$ or $y \leq x$

We want to show that f^{-1} is no decreasing, we only need to show that.

$y \leq_B x$ implies $f^{-1}(y) \leq_A f^{-1}(x)$

Since we know that f is non decreasing

we have that.

if $x \leq_A y$ then $f(x) \leq_B f(y)$

Since function f is a bijection, we have.

if $f^{-1}(f(x)) \leq_A f^{-1}(f(y))$ then $f(x) \leq_B f(y)$

therefore $f(x) \leq_B f(y)$ implies $f^{-1}(f(x)) \leq_A f^{-1}(f(y))$

And that concludes our proof.

Problem 7

Base case when $w = \varepsilon$.

$l(\delta(w)) = l(\varepsilon) = 0 = 2l(\varepsilon) = 2l(w)$ By IH $l(\delta(w)) = 2l(w)$
$= 2n.$

Inductive case, suppose $w^* = aw$, $l(w) = n$, (we need to know whether $l(w^*) = n+1$ satisfies)

$l(\delta(w^*)) = l(a(a(\delta(w))))$

$= l(aa\delta(w))$

$= 2n+2.$

$= 2(n+1)$

$= 2l(w^*)$

And that concludes our proof.

Problem 6

$h : (C^B)^A \to C^{A \times B}$, $f \mapsto h(f)$

$h(f)(x,y) := f(x)(y)$, $x \in A$, $y \in B$

(i) to prove that h is injective.

we only have to prove that when $x \neq y$ $h(x) \neq h(y)$