Assignment 1

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Problem 1.1.

i) The truth table is as follows:

A	B	C	$B \to C$	$A \to C$	$A \to (B \to C)$	$B \to (A \to C)$	φ
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	1	1	1
1	0	1	1	1	1	1	1
1	1	0	0	0	0	0	1
1	1	1	1	1	1	1	1

ii) This is the disjunctive normal form of φ :

$$\begin{aligned}
\varphi &= (A \to (B \to C)) \to (B \to (A \to C)) \\
&= \neg(A \to (B \to C)) \vee (B \to (A \to C)) \\
&= \neg(\neg A \vee (B \to C)) \vee (\neg B : (A \to C)) \\
&= (A \wedge (\neg(B \to C)) \vee (\neg B) \vee (\neg A) \vee C \\
&= (A \wedge \neg(\neg B \vee C)) \vee (\neg B) \cup (\neg A) \vee C \\
&= (A \wedge B \wedge \neg C) \vee (\neg B) \vee (\neg A) \vee C
\end{aligned}$$

iii) This is the conjunctive normal form of φ :

$$\begin{aligned}
\varphi &= A \wedge B \wedge \neg C) \vee \neg(B \wedge A) \vee C \\
&= \neg\neg(A \wedge B \wedge \neg C) \wedge (B \wedge A)) \vee C \\
&= \neg(\neg(A \wedge B) \vee C \wedge (B \wedge A)) \vee C \\
&= (A \wedge B) \wedge \neg(C \wedge B \wedge A)) \vee C \\
&= (A \wedge B) \wedge (\neg C \vee \neg(B \wedge A)) \vee C \\
&= (A \wedge B) \wedge (\neg C \vee \neg B \vee \neg A) \vee C \\
&= (\vee((A \wedge B) \wedge (\neg C \vee \neg B \vee \neg A)) \\
&= (C \vee (A \wedge B)) \wedge (C \vee \neg C \vee \neg B \vee \neg A). \\
&= (C \vee A) \wedge (C \vee B) \wedge (C \vee \neg C \vee \neg B \vee \neg A)
\end{aligned}$$

Problem 1.2.

i) Since:

$$
\begin{aligned}
&\varphi_0 = p \wedge \neg p\\
&\varphi_1 = p \wedge q\\
&\varphi_2 = \neg(\neg p \vee q) = p \wedge (\neg q)\\
&\varphi_3 = p\\
&\varphi_4 = \neg(\neg q \vee p) = q \wedge (\neg p)\\
&\varphi_5 = q\\
&\varphi_6 = (\neg p \wedge q) \vee (p \vee \neg q)\\
&\varphi_7 = p \vee q\\
&\varphi_8 = \neg(p \vee q)\\
&\varphi_q = (p \wedge q) \vee (\neg q \wedge \neg p)\\
&\varphi_{10} = \neg q\\
&\varphi_{11} = \neg q \vee p\\
&\varphi_{12} = \neg p\\
&\varphi_{13} = \neg p \vee q\\
&\varphi_{14} = \neg(p \wedge q)\\
&\varphi_{15} = p \vee \neg p
\end{aligned}
$$

Therefore,$\{\vee, \wedge, \neg\}$ is a functionally complete set.

ii) We know that when $p, q = 1$,$p \wedge q$, $p \vee q$ equals to 1, if there does not exist $\neg$, $\varphi_i \neq 0$ when $p, q = 1$.
Therefore, at least φ_0,φ_2,φ_4....φ_{14} can not be represented by S.
Therefore, S is not functionally complete.

iii)

$$\psi_0(p,q) = \begin{cases} 1 & \text{if } a_0 p + b_0 q + c_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} a_0 = -1 \\ b_0 = -1 \\ c_0 = -1 \end{cases}$$

$$\psi_1(p,q) = \begin{cases} 1 & \text{if } a_1 p + b_1 q + c_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} a_1 = 1 \\ b_1 = 1 \\ c_1 = -1 \end{cases}$$

$$\psi_2(p,q) = \begin{cases} 1 & \text{if } a_2 p + b_2 q + c_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} a_2 = 2 \\ b_2 = -2 \\ c_2 = -1 \end{cases}$$

$$\psi_3(p,q) = \begin{cases} 1 & \text{if } a_3 p + b_3 q + c_3 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} a_3 = 3 \\ b_3 = -2 \\ c_3 = 0 \end{cases}$$

$$\begin{cases} a_4 = -2 \\ b_4 = 3 \\ c_4 = -1 \end{cases} \qquad \begin{cases} a_5 = -2 \\ b_5 = 3 \\ c_5 = 0 \end{cases}$$

$$\varphi_6(p,q) = \begin{cases} 1 & \text{if } a_6 p + b_6 q + c_6 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \begin{cases} c \leq 0 \\ a_6 + c > 0 \\ b_6 + c > 0 \\ a_6 + b_6 + c \leq 0 \end{cases}$$

$$0 > a_6 + b_6 + c > -c \geq 0$$

Therefore, a_6, b_6, c_6 doesn't exist.

$$\begin{cases} a_7 = 2 \\ b_7 = 2 \\ c_7 = -1 \end{cases} \qquad \begin{cases} a_8 = -2 \\ b_8 = -2 \\ c_8 = 1 \end{cases}$$

$$\varphi_9(p,q) = \begin{cases} 1 & \text{if } a_9 p + b_9 q + c_9 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{cases} c_9 > 0 \\ a_9 + c_9 \leq 0 \\ b_9 + c_9 \leq 0 \\ a_9 + b_9 + c_9 > 0 \end{cases} \Rightarrow 0 > -c \geqslant a_9 + b_9 + c_9 > 0$$

therefore, a_9, b_9, c_9 dosn't exist.

$$\begin{cases} a_{10} = 2 \\ b_{10} = -100 \\ c_{10} = 2 \end{cases}, \begin{cases} a_{11} = 4 \\ b_{11} = -2 \\ c_{11} = 1 \end{cases}, \begin{cases} a_{12} = -100 \\ b_{12} = 1 \\ c_{12} = 1 \end{cases}, \begin{cases} a_{13} = -2 \\ b_{13} = 3 \\ c_{13} = 1 \end{cases}, \begin{cases} a_{14} = -1 \\ b_{14} = -1 \\ c_{14} = 3 \end{cases}, \begin{cases} a_{15} = 1 \\ b_{15} = 1 \\ c_{15} = 1 \end{cases}$$

Problem 1.3.

i) For any element $a \in X\Delta Y$, a either belongs to $X - Y$ or $Y - X$, which means $a \in X, a \notin Y$ or $a \in Y, a \notin X$.

For any element $b \in (X \cup Y) - (X \cap Y)$, b satisfies that $b \in X \cup Y$ and $b \notin X \cap Y$.

For any $a \in X\triangle Y$, a satisfies $a \in X \cup Y$ and $a \notin X \cap Y$ for any $b \in (X \cup Y - X \cap Y)$, b satisfies $b \in X, b \notin Y$ or $b \in Y, b \notin X$.

Therefore $X\Delta Y = (X \cup Y) - (X \cap Y)$.

ii) $(M - X)\Delta(M - Y) = ((M - X) - (M - Y)) \cup ((M - Y) - (M - X))$

For any element that belongs to $(M - X) - (M - Y), a \in M$ and $a \in Y$ and $a \notin X$.

For any element that belongs to $(M - Y) - (M - X), a \in M$ and $a \in X$ and $a \notin Y$.

Then for any element $t \in (M - x)\Delta(M - x), t \in Y, t \notin X$ or $t \in X, t \notin Y$. which is the definition of $X\Delta Y$, and that ends the proof.

iii)

$$\begin{aligned} (X\Delta Y)\Delta Z &= ((X - Y) \cup (Y - X))\Delta Z = ((X - Y) \cup (Y - X) - Z) \cup (Z - (X - Y) \cup (Y - X)) \\ X\Delta(Y\Delta Z) &= X\Delta((Y - Z) \cup (Z - Y)) \\ &= (X - (Y - Z) \cup (Z - Y))((Y - Z) \cup (Z - Y) - X) \end{aligned}$$

For any $a \in (X \triangle Y) \triangle Z, a \in X, a \notin Y, a \notin Z$ or $a \notin X a \in Y \notin Z$ or $a \in Z, a \in X, a \in Y$ or $a \in Z, a \notin X, a \notin Y$.

For any $b \in X\Delta(Y\Delta Z)$, b satisfies that $b \in X, b \in Y, b \in Z$ or $b \in X, b \notin Y, b \notin Z$, or $b \in Y, b \notin Z, b \notin X$ or $b \in Z, b \notin Y, b \notin X$.

Therefore, for any $a \in (X\Delta Y)\Delta Z$, if also satisfies $a \in X\Delta(Y\Delta Z)$. For any $b \in X\Delta(Y\Delta Z)$, if also satisfies $b \in (X\Delta Y)\Delta Z$.

So, $(X\Delta Y)\Delta Z = X\Delta(Y\Delta Z)$.

iv) For any $a \in X \cap (Y\Delta Z), a \in X$ and $a \in Y, a \notin Z$.

For any $b \in (X \cap Y)\Delta(X \cap Z), b \in (X \cap Y)$ but $b \notin (X \cap Z)$.

Since $b \in X$, therefore $b \notin Z. \Rightarrow b \in X, b \in Y, b \notin Z$.

Therefore $a \in (X \cap Y)\Delta(X \cap Z), b \in X \cap (Y\Delta Z)$, proving that $X \cap (Y\Delta Z) = (X \cap Y)\Delta(X \cap Z)$.

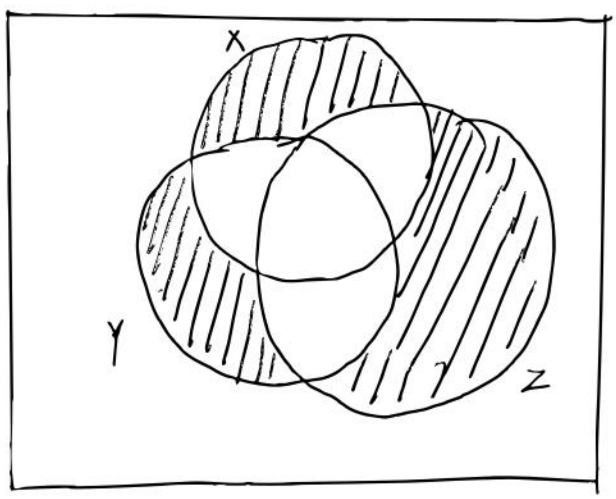
v)

$$
\begin{aligned}
& X\Delta Y = Z\Delta W \\
\Leftrightarrow & Z\Delta(X\Delta Y) = Z\Delta(Z\Delta W) = (Z\Delta Z)\Delta W = \phi\Delta W = W \\
\Leftrightarrow & (Z\Delta X\Delta Y)\Delta Y = W\Delta Y. \\
\Leftrightarrow & Z\Delta X\Delta(Y\Delta Y) = W\Delta Y \\
\Leftrightarrow & Z\Delta X = W\Delta Y. \\
\Leftrightarrow & X\Delta Z = Y\Delta W
\end{aligned}
$$

Thus the theorem is proved.

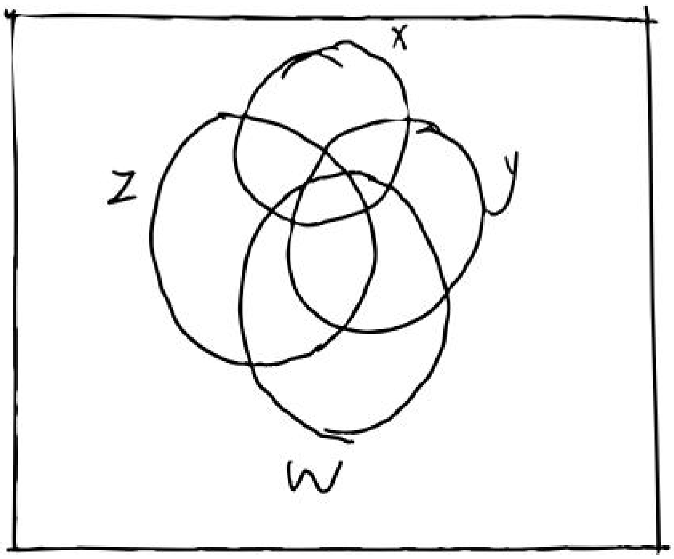
vi) The diagram is as follows:

Figure 1: Diagram for $X\Delta Y\Delta Z$



vii) The diagram is as follows:

Figure 2: A Venn diagram for 4 (distinct) sets



Problem 1.4.

We get $\exists y \forall x (xy = 0) \Rightarrow \forall x \exists y (xy = 0)$ for free because when we found such a y_0 in (ii), this y_0 can also be replaced in (i).

$$\begin{aligned}
&\exists y \forall x P(x, y) \to \forall x \exists y P(x, y) \\
&\Leftrightarrow \exists y \forall x P(x, y) \to \forall z \exists w P(z, w) \\
&\Leftrightarrow \forall y \forall z \exists x \exists w (P(x, y) \to P(z, w))
\end{aligned}$$

which is indeed true by taking $x = z$ and $w = y$.

And $\forall x \exists y (xy = 0) \Rightarrow \exists y \forall x (xy = 0)$ is also a simple proof, because we can find a fixed $y = 0$ in $\forall xzy(xy = 0)$ that is applicable in $\exists y \forall x (xy = 0)$. And that ends the proof.