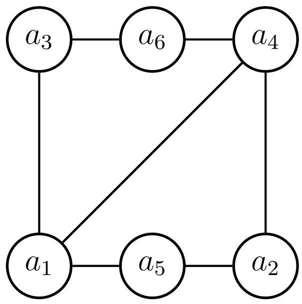
# Ve203 Discrete Mathematics

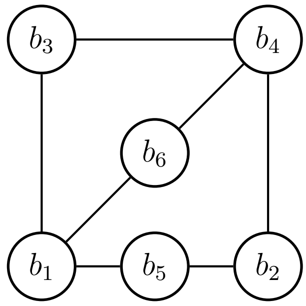
## Sample Exercises for the Final Exam - Solutions

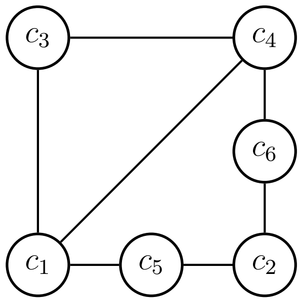
**Exercise 0.1.** Let $G$ be a simple connected graph with $n \geq 3$ vertices such that every vertex is adjacent to exactly two other vertices. Show that $G = C_n$, the cycle with $n$ vertices.
**(3 Marks)**

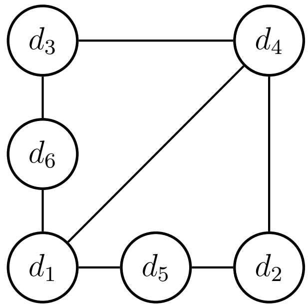
*Solution.* Let $G = (V, e)$. Pick an arbitrary vertex, call it $v_1$. It is connected to two other vertices; pick one, call it $v_2$. This vertex is connected to one other vertex, which we denote by $v_3$. Again, $v_3$ is connected to one other vertex, which can not be $v_1$ (since otherwise the vertices $v_1, v_2, v_3$ would constitute a subgraph that is disconnected from the rets of $G$). **(1 Mark)** So, $v_3$ is connected to a vertex which we denote by $v_4$. This vertex can not be connected to $v_2$ (since $v_2$ is already connected to two other vertices) or $v_1$ (since $G$ is connected). **(1 Mark)** We continue in this way. From the previous arguments it is clear that this procedure will enumerate all vertices of the graph, denoted $v_1, \ldots, v_n$. Once we add $v_n$, there is only one other possible vertex to which $v_n$ can be connected, and that is $v_1$. Thus, for $k = 1, \ldots, n$, $v_k$ is connected to $v_{(k+1) \bmod n}$, which is the definition of a cycle. **(1 Mark)**

**Exercise 0.2.** Determine which of the following graphs A,B,C,D are isomorphic to each other. If they are isomorphic, give a graph isomorphism and prove that it actually is an isomorphism. If they are not isomorphic, prove this.

A)



B)



C)



D)



**(6 Marks)**

*Solution.*

i) $C$ has a simple circuit of length 3, which none of the other graphs have. Hence, $C$ is not isomorphic to $A$, $B$ or $D$. **(2 Marks)**

ii) $B$ has a simple circuit of length 5, which $A$ and $D$ do not have. Hence, $B$ is not isomorphic to $A$ or $D$. $B$ is not isomorphic to $C$ either by (i).) **(2 Marks)**

iii) $A$ and $D$ are isomorphic, since the map

$$f(a_k) = \begin{cases} d_k & k = 1, 2, 4, 5, \\ d_6 & k = 3, \\ d_3 & k = 6. \end{cases}$$

is a graph isomorphism. This can be seen by writing out the adjacency matrix for $D$ given the ordering $(d_1, d_2, d_6, d_4, d_5, d_3)$, which is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

and equals the matrix for $A$. **(2 Marks)**

**Exercise 0.3.** Ackerman's function is defined for $m, n \in \mathbb{N}$ by

$$A(m, n) = \begin{cases} 2n & \text{if } m = 0, \\ 0 & \text{if } m \geq 1 \text{ and } n = 0, \\ 2 & \text{if } m \geq 1 \text{ and } n = 1, \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2. \end{cases}$$

Use induction based on the lexicographic ordering in $\mathbb{N}^2$ to show that $A(m, n) \geq n$ for all $m, n \in \mathbb{N}$.
**(3 Marks)**

*Solution.* We show the stronger statement that $A(m, n) \geq 2n$ for all $m, n \in \mathbb{N}$. **(1 Mark)** Fix $m, n \in \mathbb{N}$ and assume that the statement is true for all $(m', n')$ preceding $(m, n)$. **(1/2 Mark)** Suppose that $m = 0$. Then the statement is true because $A(0, n) = 2n$. Suppose that $n = 0$. Then $A(m, 0) = 0 \geq 2 \cdot 0$, so the statement is true. Suppose that $n = 1$ and $m \geq 1$. Then $A(m, n) = 2 \geq 2 \cdot 1$, so the statement is true. **(1/2 Mark)** Lastly, suppose that $n \geq 2$ and $m \geq 1$. Since $(m, n-1)$ precedes $(m, n)$, we have $A(m, n-1) \geq 2(n-1)$. **(1/2 Mark)** Since $A(m-1, k)$ precedes $(m, k)$ for all $k \in \mathbb{N}$, we have

$$A(m, n) = A(m-1, A(m, n-1)) \geq 2A(m, n-1) \geq 4(n-1) = 2n + 2(n-2) \geq 2n.$$

**(1/2 Mark)**

**Exercise 0.4.** Use Huffman coding to encode these symbols with given frequencies:

$a$: 0.20, $b$: 0.10, $c$: 0.15, $d$: 0.25, $e$: 0.30.

What is the average number of bits required to encode a character?
**(4 Marks)**

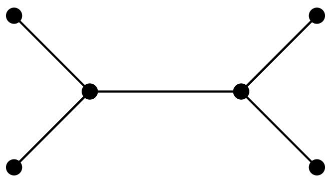
*Solution.*

$a$: 11, $b$: 101, $c$: 100, $d$: 01, $e$: 00

The average number of bits is 2.25 bits. (Note: This coding depends on how ties are broken, but the average number of bits is always the same.)

**Exercise 0.5.** Could a graph with six vertices, of which four have degree 1 and two have degree 3, be a tree? If not, prove this. If so, give an example.
**(2 Marks)**

*Solution.* Yes:



**Exercise 0.6.** Consider the arithmetic expression

$$a \cdot d \cdot (b/(c+e))$$

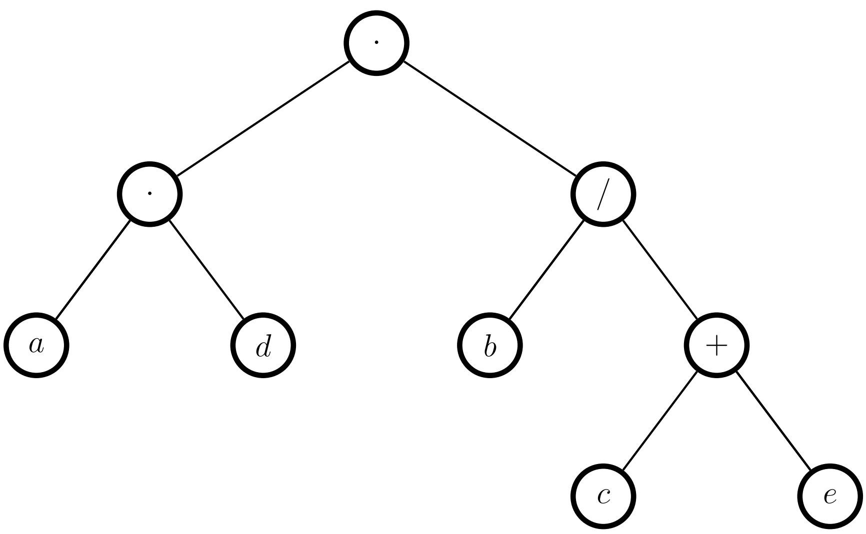
i) Represent the expression as a binary tree.

ii) Give the postfix and prefix notations for this expression.

**(2+2 Marks)**

*Solution.*

i)



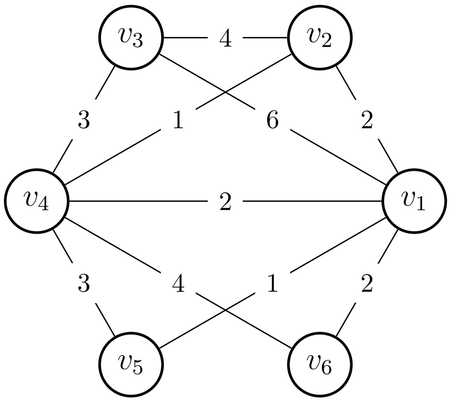
ii) postfix notation: $a, d, \cdot, b, c, e, +, /, \cdot$.

prefix notation: $\cdot, \cdot, a, d, /, b, +, e, c$.

**Exercise 0.7.** Find the least number of comparisons needed to sort four elements and devise an algorithm that sorts these elements using this number of comparisons.
**(4 Marks)**

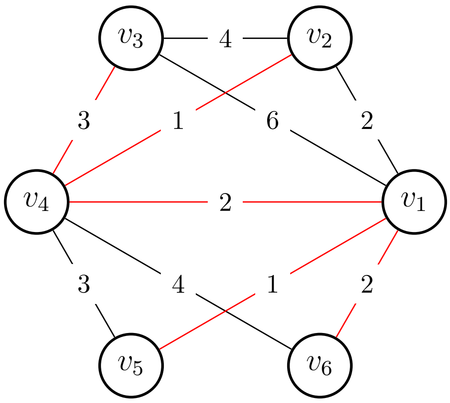
*Solution.* The least number is five. Call the elements $a$, $b$, $c$, and $d$. First compare $a$ and $b$; then compare $c$ and $d$. Without loss of generality, assume that $a < b$ and $c < d$. Next compare a and $c$. Whichever is smaller is the smallest element of the set. Again without loss of generality, suppose $a < c$. Finally, compare $b$ with both $c$ and $d$ to completely determine the ordering.

**Exercise 0.8.** Find and draw a minimum spanning tree for the following graph:

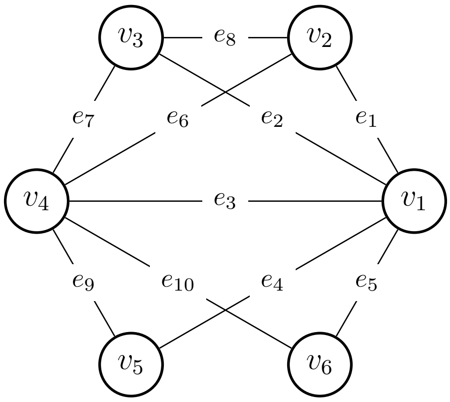



**(2 Marks)**

*Solution.*



**Exercise 0.9.** Starting at the vertex $v_1$, find a minimum spanning tree of the following graph through a depth-first search. Assume that an edge $e_i$ is added before an edge $e_j$ if $i < j$.




**(2 Marks)**

*Solution.*