Ve203 Discrete Mathematics (Fall 2020)

Assigment 2: Induction, Relations, Algebraic Structures

Date Due: 12:10 PM, Thursday, the 24th of September 2020

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This assignment has a total of **(34 Marks)**.

Exercise 2.1 Straightforward Induction
Let (a_n) be the sequence defined by

$$a_1 = 1, \qquad a_2 = 8 \qquad \text{and} \qquad a_n = a_{n-1} + 2a_{n-2}, \qquad n \geq 3.$$

Prove that for all $n > 0$, $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$.
(2 Marks)

Exercise 2.2 The Fifth Peano Axiom
Prove that the induction axiom implies the well-ordering principle.
(3 Marks)

Exercise 2.3 Is a direct induction approach always successful?
Try to prove by induction that for any real number $x > -1$ and any $n \in \mathbb{N}$, $(1 + x)^n \geq nx$. If you encounter difficulties, modify your approach.
(2 Marks)

Exercise 2.4 Strong Induction
Use strong induction to show that every $n \in \mathbb{N} \setminus \{0\}$ can be written as a sum of distinct powers of 2, i.e., as a sum of a subset of integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$ etc.

(*Hint*: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2 \in \mathbb{N}$.)
(3 Marks)

Exercise 2.5 Structural Induction
Let $S \subset \mathbb{N}^2$ be defined by

- $(0, 0) \in S$,
- $(a, b) \in S \Rightarrow \big(((a + 2, b + 3) \in S) \wedge ((a + 3, b + 2) \in S)\big)$.

Use structural induction to show that $(a, b) \in S$ implies $5 \mid (a + b)$.
(3 Marks)

Exercise 2.6 Some easy practice of relation properties
Determine whether the relation R on the set of all integers is reflexive, symmetric and/or transitive, where $(x, y) \in R$ if and only if

i) $x + y = 0$
ii) $2 \mid (x - y)$
iii) $xy = 0$
iv) $x = 1$ or $y = 1$
v) $x = \pm y$
vi) $x = 2y$
vii) $xy \geq 0$
viii) $x = 1$

(8 Marks)

Exercise 2.7 Roots of Unity
For this question, you may use everything you know about complex numbers from calculus.

i) Show that the set $S = \{z \in \mathbb{C} : |z| = 1\}$ is a group $(S, \cdot)$ with the group operation being the usual multiplication of complex numbers.

   (2 Marks)

ii) Show that for any $n \in \mathbb{N} \setminus \{0\}$ the set $S(n) = \{z \in \mathbb{C} : z^n = 1\}$ is a group with the usual multiplication of complex numbers.

   (2 Marks)

Exercise 2.8 Matrix Groups

For this question, you may use everything you know about matrices and real numbers from linear algebra or calculus. The set of $n \times n$ matrices with real coefficients is denoted by $\mathrm{Mat}(n \times n; \mathbb{R})$.

i) The matrix representing a rotation of $\mathbb{R}^2$ by the angle φ is given by

$$A(\varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}.$$

Show that the set $S = \{A(\varphi) \colon \varphi \in \mathbb{R}\}$ is a group, with the group operation being the usual matrix multiplication.

(2 Marks)

ii) Show that the following sets of matrices are groups (with group operation being matrix multiplication):

(a) The *special linear group* $\mathrm{SL}(n, \mathbb{R}) := \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \colon \det A = 1\}$.

(b) The *orthogonal group* $\mathrm{O}(n, \mathbb{R}) := \{A \in \mathrm{GL}(n, \mathbb{R}) \colon A^T = A^{-1}\}$.

(c) The *special orthogonal group* $\mathrm{SO}(n, \mathbb{R}) := \{A \in \mathrm{O}(n, \mathbb{R}) \colon \det A = 1\}$.

(3 Marks)

Exercise 2.9 A Finite Field

Let

$$m \sim n \qquad :\Leftrightarrow \qquad 2 \mid (n - m), \qquad\qquad m, n \in \mathbb{Z}.$$

i) Show that $\sim$ is an equivalence relation.

(1 Mark)

ii) What partition $\mathbb{Z}_2 := \mathbb{Z}/\sim$ is induced by $\sim$?

(1 Mark)

iii) Define addition and multiplication on $\mathbb{Z}_2$ by the addition and multiplication of class representatives, i.e.,

$$[m] + [n] := [m + n], \qquad\qquad [m] \cdot [n] := [m \cdot n].$$

Show that these operations are well-defined, i.e., independent of the representatives m and n of each class.

(2 Marks)

iv) Show that $(\mathbb{Z}_2, +, \cdot)$ is a field.

(2 Marks)

Remark: Everything that you may have learned about vector fields over the real numbers or complex numbers remains valid for vector spaces over general fields, such as the one introduced here.