

# Ve203 Discrete Mathematics (Spring 2022)

## Assignment 7

### Exercise 7.1

Consider the functions  $f : B \rightarrow U$ , count the number of functions and fill in the blanks below. Express the results in binomial coefficients, factorials, or powers (**AVOID** double bracket notation).

Elements of Domain	Elements of Codomain	Any $f$	Injective $f$	Surjective $f$
distinguishable	distinguishable			
indistinguishable	distinguishable			

where

1.  $B = \{1, 2, 3\}$  and  $U = \{1, 2, 3, 4, 5\}$ .
2.  $B = \{1, 2, 3, 4, 5\}$  and  $U = \{1, 2, 3\}$ .

### Exercise 7.2

Derive the following formula for the Euler's totient function  $\varphi$

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

by applying the inclusion-exclusion principle to the set  $\{1, 2, \dots, n\}$ .

### Exercise 7.3

Consider

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 100$$

What are the number of integer solutions if

- (i)  $x_i > 0$  and  $=$  holds;
- (ii)  $x_i \geq 0$  and  $=$  holds;
- (iii)  $x_i > 0$  and  $<$  holds;
- (iv)  $x_i \geq 0$  and  $<$  holds;
- (v)  $x_i \geq 0$ .

**AVOID** double bracket notation in the final solution.

### Exercise 7.4

Find the  $\Theta$  bound of  $T(n)$  for the following recurrence relation.

- (i)  $T(n) = 4T(n/4) + 5n$ .
- (ii)  $T(n) = 4T(n/5) + 5n$ .
- (iii)  $T(n) = 5T(n/4) + 4n$ .
- (iv)  $T(n) = 4T(\sqrt{n}) + \log^5 n$
- (v)  $T(n) = 4T(\sqrt{n}) + \log^2 n$

### Exercise 7.5

The purpose of this problem is to prove that the number of spanning trees of the complete graph  $K_n$ ,  $n \geq 2$ , is  $n^{n-2}$ , a formula due to Cayley (1889).<sup>1</sup>

- (i) Let  $T(n; d_1, \dots, d_n)$  be the number of trees with  $n \geq 2$  vertices  $v_1, \dots, v_n$ , and degrees  $d(v_1) = d_1$ ,  $d(v_2) = d_2$ ,  $\dots$ ,  $d(v_n) = d_n$ , with  $d_i \geq 1$ . Show that

$$T(n; d_1, \dots, d_n) = \binom{n-2}{d_1-1, d_2-1, \dots, d_n-1}$$

- (ii) Show that  $d_1, \dots, d_n$ , with  $d_i \geq 1$ , are degrees of a tree with  $n$  vertices iff

$$\sum_{i=1}^n d_i = 2(n-1)$$

- (iii) Use (i) and (ii) prove that the number of spanning trees of  $K_n$  is  $n^{n-2}$ .

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<sup>1</sup>For hints, see Gallier, p. 254