Ve203 Discrete Mathematics (Fall 2020)

Assigment 6: Master Theorem, Generating Functions, Counting

Date Due: 12:10 PM, Thursday, the 29th of October 2020

This assignment has a total of **(28 Marks)**.

Exercise 6.1

In this question, assume that f is an increasing function satisfying the recurrence relation $f(n) = af(n/b) + cn^d$ with $a \geq 1$, $b \in \mathbb{N} \setminus \{0, 1\}$, $c, d \in \mathbb{R}_+$. Our goal is to prove the Master Theorem 13.19 of the lecture.

i) Show that if $a = b^d$ and n is a power of b, then $f(n) = f(1)n^d + cn^d \log_b n$.
(2 Marks)

ii) Show that if $a = b^d$, then $f(n) = O(n^d \log n)$.
(1 Mark)

iii) Show that if $a \neq b^d$ and n is a power of b, then

$$f(n) = C_1 n^d + C_2 n^{\log_b a}, \qquad C_1 = \frac{b^d c}{b^d - a}, \qquad C_2 = f(1) + \frac{b^d c}{a - b^d}.$$

(2 Marks)

iv) Show that if $a < b^d$, then $f(n) = O(n^d)$.
(1 Mark)

v) Show that if $a > b^d$, then $f(n) = O(n^{\log_b a})$.
(1 Mark)

Exercise 6.2

Let a_n be the number of words of length n that comprise the letters $0, 1, 2$ but which never include two successive zeroes. For example, $a_1 = 3$, $a_2 = 8$, $a_3 = 22$ etc.

i) Derive a recurrence relation for a_n.

ii) Find the generating function for the sequence (a_n).

iii) Use the generating function to obtain an explicit formula for a_n.

(8 Marks)

Exercise 6.3

Show that the countable union of countable sets is countable, i.e., if $\{A_i\}_{i=0}^{\infty}$ is an infinite family of sets, each of which is countable, then $\bigcup_{i \in \mathbb{N}} A_i$ is countable. *Hint:* Let a_{ij} denote the ith element of A_j...
(2 Marks)

Exercise 6.4

Let M, N be finite sets with $\operatorname{card} M = \operatorname{card} N$ and $M \subset N$. Prove that $M = N$.
(2 Marks)

Exercise 6.5

Consider the scheme for counting fractions shown below and let

$$\varphi\colon \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*, \qquad\qquad \varphi(p,q) = \frac{(p+q-1)(p+q-2)}{2} + p,$$

where $N^* = \mathbb{N} \setminus \{0\}$. The goal of this question is to prove that, in the scheme shown, traversing successive diagonals from top to bottom, the fraction p/q is indeed the $\varphi(p,q)$th fraction encountered and that φ gives a bijection $N^* \times \mathbb{N}^* \to \mathbb{N}^*$.

i) Write the traversal of the fractions as a recursively defined sequence of pairs (p_n, q_n).

(1 Mark)

ii) Prove that for any $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*$ there exists an $n \in \mathbb{N}$ such that $(p,q) = (p_n, q_n)$.

(2 Marks)

iii) Prove (by induction in n) that $\varphi(p_n, q_n) = n$.

(2 Marks)

iv) Deduce that φ is bijective.

(0.5 Marks)

v) Find the inverse of φ.

(2 Marks)

vi) How can φ be modified to give a bijective map $\mathbb{N} \times \mathbb{N}^* \to \mathbb{N}$?

(1 Mark)

vii) How can φ be modified to give a bijective map $\mathbb{Z} \times \mathbb{N}^* \to \mathbb{N}$?

(0.5 Marks)