Question 6

$T(n) = 9\,T(n/3) + n$

Therefore, $k=9$, $b=3$, $d=1$

$k/b^d = 3 > 1$

So $T(n) = O(n^{\log_3 9}) = O(n^2)$.

$-3+4\cdot 4.$

Question 2.

$a_n = 5a_{n-1} - 6a_{n-2} + n^2 2^n$, $n \geq 2$, $a_0 = a_1 = 3$.

We first solve the homogeneous part.

$t^2 - 5t + 6 = 0$

$(t-2)(t-3) = 0 \Rightarrow a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$.

Then for the inhomogeneous part,

$F(n) = n^2 \cdot 2^n$

We assume the solution to be. $(C_3 + C_4 n + C_5 n^2) \cdot n \cdot 2^n$

plug in and then we have.

$a_n - 5a_{n-1} + 6a_{n-2} = (C_3 + C_4 n + C_5 n^2) \cdot n \cdot 2^n$

Solution = hom + par.

$= C_1 \cdot 2^n + C_2 \cdot 3^n + (C_3 + C_4 n + C_5 n^2) \cdot n \cdot 2^n$.

$$\begin{cases} C_1 + C_2 = 3 \\ 2C_1 + 3C_2 + (C_3 + C_4 + C_5) \cdot 2 = 3 \\ 8C_1 + 27C_2 + 24C_3 + 72C_4 + 64C_5 = 13 \\ \vdots \end{cases}$$

solve for coefficient and we gain the final result.

Question 1.

A generating series of the LHS of $\sum_{i=0}^{n} F_i = F_{n+2}-1$ is

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} F_i\right)x^n = \frac{1}{1-x}\cdot\frac{1}{1-x-x^2}$$

A generating series of the RHS of (1) is

$$\sum_{n=0}^{\infty}(F_{n+2}-1)x^n = \sum_{n=2}^{\infty} F_n x^{n-2} - \frac{1}{1-x}$$

$$= \frac{1}{x^2}\left(\frac{1}{1-x-x^2}-1-x\right)-\frac{1}{1-x}$$

$$= \frac{1}{(1-x)(1-x-x^2)}$$

And we are done.

Question 3,

Suppose G has parallel edges, then we deduce that it has a cycle of length 2 and we are done.

If G has no parallel edges, it is simple.

Assume K to be a path of G with maximum length of walk. $v_0 e_1 \cdots v_n$. Since $\deg(v_0 \geq 3)$, so there must exist $n \geq i, j \geq 2$, with $v_0 v_i$, $v_0 v_j \in E(G)$

No matter whether i or j is odd $P + v_0 v_i / v_j$ contains a unique even cycle. If both even, then $P - v_0 v_1 + v_0 v_i + v_0 v_j$ is even.

Question 7

We first suppose that G is not connected with bipartition (A, B)
Then we define T as a component of G.

Set $X = V(T)$, $A^c = A \cap X$, $B^c = B \cap X$, and we find that
(A^c, B^c) is a bipartion of T.
Meanwhile $A^{cc} = A \setminus X$ and $B^{cc} = B \setminus X$ is a bipartition of $G - X$.
After that, we'll have no edges between X and rest of the graph.
so that
$(A, B) = (A^c \cup A^{cc}, B^c \cup B^{cc})$ and $(A^c \cup B^{cc}, B^c \cup A^{cc})$ are not
the same. And then we are done.

Question 5:

$(X_1+1)+(X_2-1)+(X_3+1)+(X_4+1)\leq n+2$.

$|X\setminus(A\cup B)| = |X|-|A|-|B|+|A\cap B|$

1° $X_1\geq 0,\ X_2\geq 2,\ X_3\geq 0,\ X_4\geq 0$

The total number is $\binom{n+2}{3}$.

2° $X_1\geq 0,\ X_2\geq 2,\ 4\nmid X_3,\ X_4\geq 0$

Let $(n+2)\equiv p \pmod 4$

$\binom{n+1}{2}+\binom{n-3}{2}+\cdots\binom{n+2-4p}{2}=\sum_{k=1}^{p}\binom{n+2-k}{2}$.

The total number is $\binom{n+2}{3}-\sum_{k=1}^{p}\binom{n+2-k}{2}$.

3° $X_1\geq 0,\ X_2\geq 2,\ X_3\geq 0,\ X_4>3$

$(X_1+1)+(X_2-1)+(X_3+1)+(X_4-3)\leq n-2$

$\binom{n-2}{3}$.

4° $X_1\geq 0,\ X_2\geq 2,\ X_4>3,\ 4\nmid X_3.$ $(n-2)\equiv p'\pmod 4$.

$\binom{n-2}{3}-\sum_{k=1}^{p'}\binom{n-2-k}{2}$

So the final result is $\binom{n+2}{3}+\sum_{k=1}^{p}\binom{n+2-k}{2}-\sum_{k=1}^{p'}\binom{n-2-k}{2}$

Question 4.

(i) R is reflexive, this is true because we simply match every single point of the graph to itself. Then there is a homomorphism $f: G \to G$.
So that the relation is reflexive.

(ii) R is also transitive, take any two points $u, v \in G$,
if $uv \in E(G)$, then $f_1(u)f_1(v) \in E(H)$
$f(u)f(v) \in E(H)$, then $f_2(f_1(u))f_2(f_1(v)) \in E(T)$

So we can simply take the projection $f_2(f_1(\cdot))$ and the relation is transitive.

(iii) R is not antisymetric, we simply take the counter example.

and we are done.