Ve203 Discrete Mathematics (Fall 2022)

Assignment 1

Date Due: See canvas
This assignment has a total of **(21 points)**.
Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**. **Explain** (briefly) if you claim something is trivial or straightforward. Provide a counterexample if you are trying to disprove something. It is **NOT OK** to write something like "how do we know that blahblahblah is even true…"

Exercise 1.1 (6 pts)
Given $\varphi = (A \to (B \to C)) \to (B \to (A \to C))$,

(i) `(2 pts)` Write the truth table for φ.

(ii) `(2 pts)` Write φ in disjunctive normal form.

(iii) `(2 pts)` Write φ in conjunctive normal form.

Exercise 1.2 (6 pts)
The following shows the truth table for all $2^{2^2} = 16$ different binary logical operators φ_i, $i = 0, \ldots, 15$.

p	q	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Using infix notation, for example, φ_{13} can be represented as $\varphi_{13} = \to(p, q) = p \to q$.

A set S of logical operators is called *functionally complete* if every compound proposition is logically equivalent to a compound proposition involving only these logical operators in S. In this exercise, in order to show S is a functionally complete set, it suffices to verify that for all $i = 0, \ldots, 15$, φ_i over logical variables p and q can be represented using only operators in S.

(i) `(1 pt)` Show that $\{\wedge, \vee, \neg\}$ is functionally complete.

(ii) `(1 pt)` Show that $\{\vee, \wedge\}$ is *not* functionally complete.

(iii) `(4 pts)` Suppose that the logical variables take on numerical values 0 and 1 as in the table above, and consider $\varphi_i : \{0,1\}^2 \to \{0,1\}$ given by

$$\varphi_i(p, q) = \begin{cases} 1, & \text{if } a_i p + b_i q + c_i > 0 \\ 0, & \text{otherwise} \end{cases}, \qquad a_i, b_i, c_i \in \mathbb{R}$$

Find valid a_i, b_i, c_i for each $i = 0, \ldots, 15$.

Exercise 1.3 (7 pts)
Let M be a set and let $X, Y, Z, W \subset M$. We define the *symmetric difference*:

$$X \triangle Y := (X - Y) \cup (Y - X)$$

(i) `(1 pt)` Prove that $X \triangle Y = (X \cup Y) - (X \cap Y)$.

(ii) `(1 pt)` Prove that $(M - X) \triangle (M - Y) = X \triangle Y$.

(iii) `(1 pt)` Show that the symmetric difference is associative, i.e., $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$.

(iv) `(1 pt)` Prove that $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$.

(v) `(1 pt)` Show that $X \triangle Y = Z \triangle W$ iff $X \triangle Z = Y \triangle W$.

(vi) `(1 pt)` Indicate the region of $X \triangle Y \triangle Z$ in a Venn diagram.

(vii) `(1 pt)` Sketch a Venn diagram for 4 (distinct) sets using circles (maybe with different radii).

Exercise 1.4 (2 pts)
Let $x, y \in \mathbb{R}$, show that $\forall x \exists y (xy = 0) \Leftrightarrow \exists y \forall x (xy = 0)$

利用真值表，求命题的主合取范式和主析取范式.（主合取范式）

No 1. 写命题公式真值表

No 2 对于使A取0的指派，写出对应的最大项，

No 3 A等值于所有最大项的合取.

$\neg\varphi_{CNF} = (\neg a \wedge \neg b \wedge \neg c) \vee \cdots \vee \cdots$

$\varphi_{CNF} = \neg(\cdots) \cdots \vee \neg(\cdots) \cdots \vee \neg(\cdots)$

De Morgan's Law

$C-(A \cup B) = (C-A) \cap (C-B)$

$C-(A \cap B) = (C-A) \cup (C-B)$

Excercise 1.1

$\varphi = (A \to (B \to C)) \to (B \to (A \to C))$

i) The truth table is as follows:

A	B	C	$B\to C$	$A \to C$	$A\to(B\to C)$	$B\to(A\to C)$	φ
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	1	1	1
1	0	1	1	1	1	1	1
1	1	0	0	0	0	0	1
1	1	1	1	1	1	1	1

ii) $\varphi = (A \to (B \to C)) \to (B \to (A \to C))$

$= \neg(A \to (B\to C)) \vee (B \to (A \to C))$

$= \neg(\neg A \vee (B \to C)) \vee (\neg B \vee (A \to C))$

$= (A \wedge (\neg(B \to C))) \vee (\neg B) \vee (\neg A) \vee C$

$= (A \wedge \neg(\neg B \vee C)) \vee (\neg B) \vee (\neg A) \vee C$

$= (A \wedge B \wedge \neg C) \vee (\neg B) \vee (\neg A) \vee C$

This is the disjunctive normal form of φ

iii) $\varphi = (A \wedge B \wedge \neg C) \vee \neg(B \wedge A) \vee C$

$= \neg(\neg(A\wedge B \wedge \neg C) \wedge (B \wedge A)) \vee C$

$= \neg(\underline{\overset{p}{\neg(A \wedge B)}} \vee \underline{\overset{q}{C \wedge (B \wedge A)}}) \vee C$

$= (A \wedge B) \wedge \neg(C \wedge B \wedge A)) \vee C$

$= (A \wedge B) \wedge (\neg C \vee \neg(B \wedge A)) \vee C$

$= (A \wedge B) \wedge (\neg C \vee \neg B \vee \neg A) \vee C$

$= C \vee ((A \wedge B) \wedge (\neg C \vee \neg B \vee \neg A))$

$= (C \vee (A \wedge B)) \wedge (C \vee \neg C \vee \neg B \vee \neg A)$

$= (C \vee A) \wedge (C \vee B) \wedge (C \vee \neg C \vee \neg B \vee \neg A)$

This is the conjunctive normal form of φ.

iii)

$$
\begin{aligned}
\varphi &= A \wedge B \wedge \neg C) \vee \neg (B \wedge A) \vee C. \\
&= \neg(\neg(A \wedge B \wedge \neg C) \wedge (B \wedge A)) \vee C \\
&= \neg(\neg(A \wedge B) \vee C \wedge (B \wedge A)) \vee C \\
&= (A \wedge B) \wedge \neg(C \wedge (B \wedge A))) \vee C \\
&= (A \wedge B) \wedge (\neg C \vee \neg (B \wedge A)) \vee C \\
&= (A \wedge B) \wedge (\neg C \vee \neg B \vee \neg A) \vee C \\
&= C \vee ((A \wedge B) \wedge (\neg C \vee \neg B \vee \neg A)). \\
&= (C \vee (A \wedge B)) \wedge (C \vee \neg C \vee \neg B \vee \neg A). \\
&= ((C \vee A) \wedge (C \vee B)) \wedge ((C \vee \neg C \vee \neg B \vee \neg A
\end{aligned}
$$

$\wedge$

Excercise 2.1

i) $\varphi_0 = p \wedge \neg p$

$\varphi_1 = p \wedge q$

$\varphi_2 = \neg(\neg p \vee q) = p \wedge (\neg q)$

$\varphi_3 = p$

$\varphi_4 = \neg(\neg q \vee p) = q \wedge (\neg p)$

$\varphi_5 = q$

$\varphi_6 = (\neg p \wedge q) \vee (p \vee \neg q)$

$\varphi_7 = p \vee q$

$\varphi_8 = \neg(p \vee q)$

$\varphi_9 = (p \wedge q) \vee (\neg q \wedge \neg p)$.

$\varphi_{10} = \neg q$

$\varphi_{11} = \neg q \vee p$

$\varphi_{12} = \neg p$

$\varphi_{13} = \neg p \vee q$

$\varphi_{14} = \neg(p \wedge q)$

$\varphi_{15} = p \vee \neg p$

Therefore, $\{\wedge, \vee, \neg\}$ is a functionally complete set.

ii) We know that when $p, q = 1$. $p \vee q$. $p \wedge q = 1$, if there doesn't exist $\neg$, $\varphi_i \neq 0$ when $p, q = 1$.

Therefore, at least $\varphi_0, \varphi_2, \varphi_4 \cdots \varphi_{14}$ cannot be represented by S.

Therefore, S is not functionally complete.

$$\varphi_0(p,q)=\begin{cases}1 & \text{if } a_0p+b_0q+c_0>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_0=-1\\ b_0=-1\\ c_0=-1\end{cases}$$

iii)

$$\varphi_0(p,q)=\begin{cases}1 & \text{if } a_0p+b_0q+c_0>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_0=-1\\ b_0=-1\\ c_0=-1\end{cases}$$

$$\varphi_1(p,q)=\begin{cases}1 & \text{if } a_1p+b_1q+c_0>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_1=1\\ b_1=1\\ c_1=-1\end{cases}$$

$$\varphi_2(p,q)=\begin{cases}1 & \text{if } a_2p+b_2q+c>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_2=2\\ b_2=-2\\ c_2=-1\end{cases}$$

$$\varphi_3(p,q)=\begin{cases}1 & \text{if } a_3p+b_3q+c>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_3=3\\ b_3=-2\\ c_3=0\end{cases}$$

$$\varphi_4(p,q)=\begin{cases}1 & \text{if } a_4p+b_4q+c>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_4=-2\\ b_4=3\\ c_4=-1\end{cases}$$

$$\varphi_5(p,q)=\begin{cases}1 & \text{if } a_5p+b_5q+c>0\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{cases}a_5=-2\\ b_5=3\\ c_5=0\end{cases}$$

$$\varphi_6(p,q)=\begin{cases}1 & \text{if } a_6p+b_6q+c>0\\ 0 & \text{otherwise}\end{cases}$$

$$\Rightarrow\begin{cases}c\le 0\\ a_6+c>0\\ b_6+c>0\\ a_6+b_6+c\le 0\end{cases}$$

$$0>a_6+b_6+c>-c\ge 0$$

therefore, a_6, b_6, c_6 doesn't exsist.

$$\varphi_7(p,q)=\begin{cases}1 & \text{if } a_7p+b_7q+c_7>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_7=2\\ b_7=2\\ c_7=-1\end{cases}$$

$$\varphi_8(p,q)=\begin{cases}1 & \text{if } a_8p+b_8q+c_8>0\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{cases}a_8=-2\\ b_8=-2\\ c_8=1\end{cases}$$

$$\varphi_9(p,q)=\begin{cases}1 & \text{if } a_9p+b_9q+c_9>0\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{cases}c_9>0\\ a_9+c_9\le 0\\ b_9+c_9\le 0\\ a_9+b_9+c_9>0\end{cases}\Rightarrow 0>-c\ge a_9+b_9+c>0$$

therefore, a_9, b_9, c_9 doesn't exist.

$$\varphi_{10}(p,q)=\begin{cases}1 & \text{if } a_{10}p+b_{10}q+c_{10}>0\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{cases}a_{10}=2\\ b_{10}=-100\\ c_{10}=2\end{cases}$$

$$\varphi_{11}(p,q)=\begin{cases}1 & \text{if } a_{11}p+b_{11}q+c_{11}>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_{11}=4\\ b_{11}=-2\\ c_{11}=1\end{cases}$$

$$\varphi_{12}(p,q)=\begin{cases}1 & \text{if } a_{12}p+b_{12}q+c_{12}>0\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{cases}a_{12}=-100\\ b_{12}=1\\ c_{12}=1\end{cases}$$

$$\varphi_{13}(p,q)=\begin{cases}1 & \text{if } a_{13}p+b_{13}q+c_{13}>0\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{cases}a_{13}=-2\\ b_{13}=3\\ c_{13}=1\end{cases}$$

$$\varphi_{14}(p,q)=\begin{cases}1 & \text{if } a_{14}p+b_{14}q+c_{14}>0\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{cases}a_{14}=-1\\ b_{14}=-1\\ c_{14}=3\end{cases}$$

$$\varphi_{15}(p,q)=\begin{cases}1 & \text{if } a_{15}p+b_{15}q+c_{15}>0\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{cases}a_{15}=1\\ b_{15}=1\\ c_{15}=1\end{cases}$$

Excercise 1.3

Let M be a set and let $X, Y, Z, W \subset M$. $X \triangle Y := (X-Y) \cup (Y-X)$

(i) for any element $a \in X \triangle Y$

a either belongs to $X-Y$ or $Y-X$

$\Rightarrow$ which means $a \in X, a \notin Y$ or $a \in Y, a \notin X$

for any element $b \in (X \cup Y) - (X \cap Y)$

b satisfies that $b \in X \cup Y$ and $b \notin X \cap Y$.

for any $a \in X \triangle Y$, a satisfies $a \in X \cup Y$ and $a \notin X \cap Y$

for any $b \in (X \cup Y - X \cap Y)$, b satisfies $b \in X, b \notin Y$ or $b \in Y, b \notin X$

Therefore $X \triangle Y = (X \cup Y) - (X \cap Y)$

(ii) $(M-X) \triangle (M-Y)$

$= ((M-X) - (M-Y)) \cup ((M-Y) - (M-X))$

for any element that belongs to $(M-X) - (M-Y)$, $a \in M$ and $a \in Y$ and $a \notin X$

for any element that belongs to $(M-Y) - (M-X)$, $a \in M$ and $a \in X$ and $a \notin Y$

then for any element $t \in (M-X) \triangle (M-Y)$, $t \in Y \notin X$ or $t \in X \notin Y$,

which is the definition of (since $X, Y \subset M$)

$X \triangle Y$, and that ends the proof.

iii) $(X \triangle Y) \triangle Z = ((X-Y) \cup (Y-X)) \triangle Z = (((X-Y) \cup (Y-X)) - Z) \cup (Z - ((X-Y) \cup (Y-X)))$

$X \triangle (Y \triangle Z) = X \triangle ((Y-Z) \cup (Z-Y))$

$= (X - ((Y-Z) \cup (Z-Y))) ((Y-Z) \cup (Z-Y)) - X)$

for any $a \in (X \triangle Y) \triangle Z$, $a \in X \notin Y \notin Z$ or $a \notin X\, a \in Y \notin Z$ or

$a \in Z, a \in X, a \in Y$ or $a \in Z\, a \notin X \notin Y$.

for any $b \in X \triangle (Y \triangle Z)$

b satisfies that $b \in X, b \in Y, b \in Z$ or $b \in X, b \notin Y, b \notin Z$.

or $b \in Y\ b \notin Z\ b \notin X$ or $b \in Z\ b \notin Y\ b \notin X$.

therefore, for any $a \in (X \triangle Y) \triangle Z$, it also satisfies $a \in X \triangle (Y \triangle Z)$.

for any $b \in X \triangle (Y \triangle Z)$, it also satisfies $b \in (X \triangle Y) \triangle Z$.

so $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$

iv) $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$

for any $a \in X \cap (Y \triangle Z)$, $a \in X$ and $a \in Y \notin Z$

for any $b \in (X \cap Y) \triangle (X \cap Z)$. $b \in (X \cap Y)$ but $b \notin (X \cap Z)$

since $b \in X$, therefore $b \notin Z$.

$\Rightarrow b \in X, b \in Y, b \notin Z$.

therefore $a \in (X \cap Y) \triangle (X \cap Z)$, $b \in X \cap (Y \triangle Z)$

proving that $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$

v) $X \triangle Y = Z \triangle W \Leftrightarrow X \triangle Z = Y \triangle W$

$X \triangle Y = Z \triangle W$

$\Leftrightarrow Z \triangle (X \triangle Y) = Z \triangle (Z \triangle W) = (Z \triangle Z) \triangle W = \phi \triangle W = W$

$\Leftrightarrow (Z \triangle X \triangle Y) \triangle Y = W \triangle Y$.

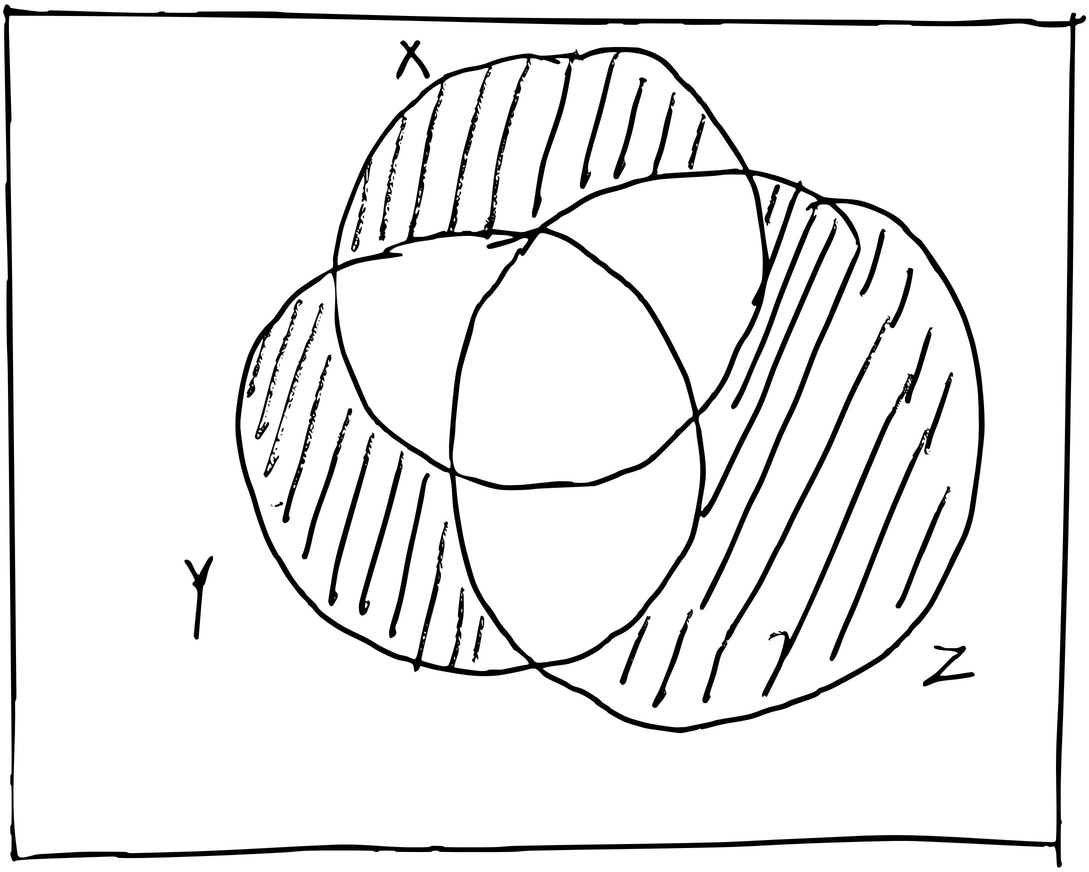
$\Leftrightarrow Z \triangle X \triangle (Y \triangle Y) = W \triangle Y$

$\Leftrightarrow Z \triangle X = W \triangle Y$.

$\Leftrightarrow X \triangle Z = Y \triangle W$.

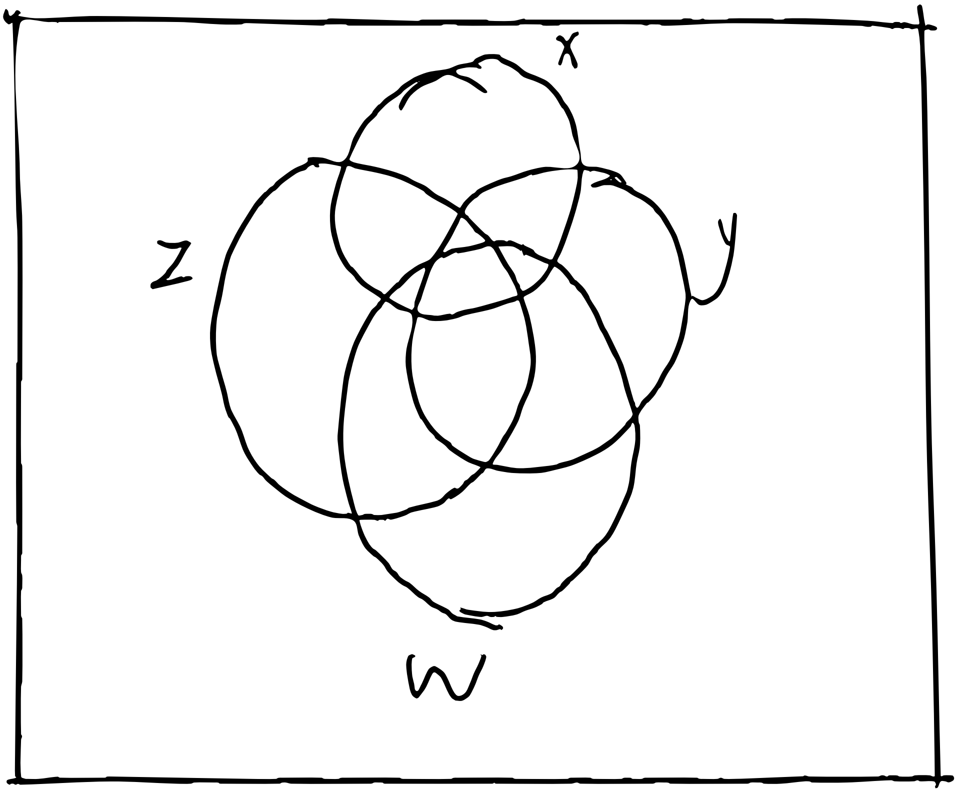
Thus the theorem is proved.

vi)



This is the Venn diagram of $X \triangle Y \triangle Z$.

vii)



Excercise 4

$\forall x \exists y (xy=0) \Leftrightarrow \exists y \forall x (xy=0)$

We get $\underset{(ii)}{\exists y \forall x (xy=0)} \Rightarrow \underset{(i)}{\forall x \exists y (xy=0)}$ for free because.

when we found such a y_0 in (ii), this y_0 can also be replaced in (i)

which can be viewed as a full blown version of $(y_0=0)$

$\exists y (P_{x_1}(y) \wedge P_{x_2}(y)) \Rightarrow \exists y P_{x_1}(y) \wedge \exists y P_{x_2}(y)$.

$\exists y \forall x P(x,y) \rightarrow \forall x \exists y P(x,y)$

$\Leftrightarrow \exists y \forall x P(x,y) \rightarrow \forall z \exists w P(z,w)$

$\Leftrightarrow \forall y \forall z \exists x \exists w (P(x,y) \rightarrow P(z,w))$

which is indeed true by taking $x=z$ and $w=y$.

And $\forall x \exists y (xy=0) \Rightarrow \exists y \forall x (xy=0)$ is also a simple proof.

because the we can find a fixed $y=0$ in $\forall x \exists y (xy=0)$ that is applicable in $\exists y \forall x (xy=0)$.

And that ends the proof.