Ve203 Discrete Mathematics (Fall 2022)

Assignment 3

This assignment has a total of **(28 points)**.

Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**. **Explain** (briefly) if you claim something is trivial or straightforward. Provide a counterexample if you are trying to disprove something. It is **NOT OK** to write something like "how do we know that blahblahblah is even true..." In addition, be careful that some problems might be ill-defined.

Exercise 3.1 (2 pts) Assume that Π is a partition of a set A. Define the relation R_Π as follows:

$$xR_\Pi y \Leftrightarrow (\exists B \in \Pi)(x \in B \wedge y \in B).$$

Show that R_Π is an equivalence relation on A.

Exercise 3.2 (2 pts)

(i) `(1 pt)` Let $\pi : \mathbb{N}^2 \to \mathbb{N}$ be Cantor's pairing function. Find $m, n \in \mathbb{N}$ such that $\pi(m,n) = 99$. You may do this question however you wish.

(ii) `(1 pt)` Give an explicit formula that defines a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. You do not have to prove that this formula works!

Exercise 3.3 (6 pts) Show that the following are posets.

(i) `(2 pts)` Let J be the set of closed intervals of the real line, with the partial order defined on J by

$$[a,b] \leq_{\text{int}} [c,d] \Leftrightarrow b \leq c \text{ or } [a,b] = [c,d].$$

(ii) `(2 pts)` The set $\mathbb{N}^n$, $n \in \mathbb{N}$, with the lexicographic order defined on $\mathbb{N}^n$ by

$$\begin{aligned}(x_1,\dots,x_n) \preceq (y_1,\dots,y_n) \Leftrightarrow{}& (x_1,\dots,x_n) = (y_1,\dots,y_n)\\ &\text{or } \exists k \in \{1,\dots,n\} \text{ with } x_i = y_i \text{ for } i < k \text{ and } x_k < y_k\end{aligned}$$

(iii) `(2 pts)` Given a poset $(P, \leq_P)$, the dual of of P, denoted by P^{d}, with the dual order defined on P by

$$\leq_{P^{\mathrm{d}}} := \{(a,b) \mid b \leq_P a\}.$$

Exercise 3.4 (2 pts) Let $0 < a_1 < a_2 < \cdots < a_{sr+1}$ be $sr+1$ integers, $s, r \in \mathbb{N}$. Show that we can select either $s+1$ of them, no one of which divides any other, or $r+1$ of them, each dividing the following one.

Exercise 3.5 (2 pts) Given Dilworth's Theorem as follows, find out what goes wrong with the proof.

Theorem (Dilworth's Theorem)**.** *Let P be a finite poset of width k. Then P can be partitioned into k chains.*

"Poof". Induction on $n := |P|$, $n = 1$ is obvious. For the induction step n to step $n+1$, assume that Dilworth's theorem holds for posets with n elements and let P be a poset with $(n+1)$ elements. Let $m \in P$ be a maximal element. Then $|P - \{m\}| = n$. By induction hypothesis, there are chains $C_1, \dots, C_{w(P-\{m\})}$ forming a partition of $P - \{m\}$. If $w(P - \{m\}) = k - 1$, set $C_k := \{m\}$ and we are done. Otherwise, m has strict lower bounds and thus for some $i_0 \in \{1, \dots, w(P - \{m\}\}$, we have that m is an upper bound of C_{i_0}. Then $C_{i_0} \cup \{m\}$ is a chain, and $C_1, \dots, C_{i_0-1}, C_{i_0} \cup \{m\}, C_{i_0+1}, \dots, C_{w(P-\{m\})}$ form a chain partition of P. □

Remark:

- The proof mentioned above is indeed problematic. No tricks here.
- As we learned that an induction proof in general correspond to an algorithm and vice versa, it is also the case here. Try to provide an example on which the above induction proof fails. It is **NOT** acceptable to just answer this question with something like "how do we know blahblahblah is even true?", since it is not a definite statement. Provide a concrete example/counterexample for the steps that does not go, and of course, explain why.

Exercise 3.6 (8 pts)

(i) `(2 pts)` Prove that the function $f : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ defined by

$$f(a_0a_1\cdots a_n\cdots, b_0b_1\cdots b_n\cdots) = a_0b_0a_1b_1\cdots a_nb_n\cdots$$

is a bijection, where $a_i, b_i \in \{0,1\}$, and $\{0,1\}^{\mathbb{N}}$ is the set of countably infinite sequences of 0 and 1.

Excercise 3.1

To show R_Π is an equivalence relation on A, we need to show that it is reflexive, symmetric and transitive.

Reflexisive $x R_\Pi x \Leftrightarrow (\exists B \in \Pi)\underbrace{(x \in B \wedge x \in B)}_{\text{This is always true.}}$

So the relationship is reflexive.

Symmetric

$x \in B \wedge y \in B \Leftrightarrow y \in B \wedge x \in B$

So the relationship is symmetric.

transitive

$x R_\Pi y \Rightarrow \exists B \in \Pi,\ x \in B \wedge y \in B$

$y R_\Pi z \Rightarrow \exists C \in \Pi,\ y \in C \wedge z \in C$

Since $B \cap C = \emptyset$, if $x R_\Pi y$ and $y R_\Pi z$, then $\underline{B = C}$

Therefore, $z \in B$

then $x R_\Pi y \wedge y R_\Pi z \Rightarrow x R_\Pi z$.

Therefore, R_Π is an equivalence relation on A.

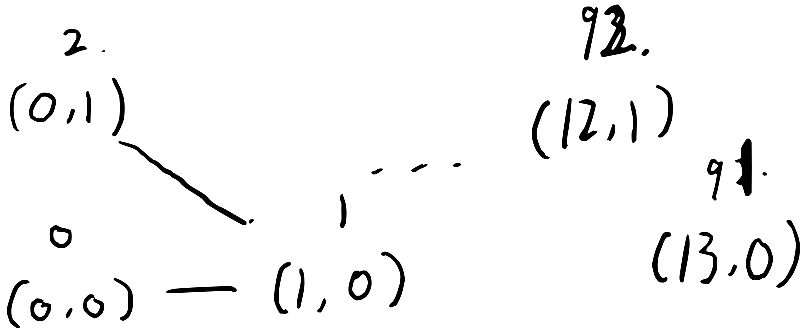
Excercise 3.2

i)

$$J: N^2 \to N \quad J(x,y) = \frac{(x+y+1)(x+y)}{2!} + y$$

$$\frac{(m+n+1)(m+n)}{2} + n = 99$$

$(5, 8)$



$\Rightarrow m=5,\ n=8$

(ii) $$f(x,y,z) = \frac{\left(\frac{(x+y+1)(x+y)}{2} + y + z + 1\right)\left(\frac{(x+y+1)(x+y)}{2} + y + z\right)}{2} + z.$$

We simply use Cantor's pairing function twice to derive this function.

Excercise 3.3.

(i) We need to show that $\leq$ is reflexive, antisymmetric and transitive.

Reflexive:

$[a,b] \leq_{int} [a,b]$, because $[a,b]=[a,b]$

Antisymmetric

$[a,b] \leq_{int} [c,d] \Leftrightarrow b \leq c$ or $[a,b]=[c,d]$

$[c,d] \leq_{int} [a,b] \Leftrightarrow d \leq a$ or $[a,b]=[c,d]$

if $[a,b] \neq [c,d]$, then $b \leq c$ and $d \leq a$.

since $b \geq a$, $d \geq c$ which is contradictory.

we have $a \leq b \leq c \leq d$

Therefore, we have $[a,b]=[c,d]$

Transitive.

$[a,b] \leq_{int} [c,d] \Leftrightarrow b \leq c$ or $[a,b]=[c,d]$ if $[a,b]=[c,d]=[e,f]$

$[c,d] \leq_{int} [e,f] \Leftrightarrow d \leq e$ or $[c,d]=[e,f]$ $\Rightarrow [a,b]=[e,f]$

if $b \leq c$, $d \leq e$, because $d \geq c \Rightarrow e \geq b$ Therefore, it is transitive.

if $b \leq c$ $[c,d]=[e,f] \Rightarrow e \geq b$

if $[a,b]=[c,d]$, $d \leq e \Rightarrow e \geq b$ Then, it's a poset.

(ii) The set N^n, $n \in N$, with the lexicographic order by

$(x_1, \cdots x_n) \preceq (y_1 \cdots y_n) \Leftrightarrow (x_1 \cdots x_n) = (y_1 \cdots y_n)$
or $\exists k \in \{1 \cdots n\}$ with $x_i = y_i$ for $i < k$
and $x_k < y_k$

Reflexive

$(x_1 \cdots x_n) \preceq (x_1 \cdots x_n) \Rightarrow T$ because $(x_1 \cdots x_n) = (x_1 \cdots x_n)$

Antisymmetric

$(x_1 \cdots x_n) \preceq (y_1 \cdots y_n) \Leftrightarrow (x_1 \cdots x_n) = (y_1 \cdots y_n)$ or
$\exists k \in \{1 \cdots n\}$ with $x_i = y_i$ for $i < k$
and $x_k < y_k$. (1)

$(y_1 \cdots y_n) \preceq (x_1 \cdots x_n) \Leftrightarrow (y_1 \cdots y_n) = (x_1 \cdots x_n)$ or
$\exists k \in \{1 \cdots n\}$ with $x_i = y_i$ for $i < k$
and $y_k < x_k$ (2)

if $(x_1 \cdots x_n) \neq (y_1 \cdots y_n)$, then $\exists k_1, k_2$ that satisfies statement (1), (2). , let $k_1 \geq k_2$.

for $i < k_1$ $y_i > x_i$

however for $i' < k_2 < k_1$ $y_i < x_i$

and that's a contradiction

$k_2 > k_1$ is similar.

Therefor $(x_1 \cdots x_n) = (y_1 \cdots y_n)$.

$\Rightarrow$ antisymmetric.

transitive.

$(x_1 \cdots x_n) \preceq (y_1 \cdots y_n) \Leftrightarrow \overset{(1)}{(x_1 \cdots x_n) = (y_1 \cdots y_n)}$ or

$\exists k \in \{1, \cdots n\}$ with $x_i = y_i$ for $\overset{(2)}{i<k}$ and $x_k < y_k$.

$(y_1 \cdots y_n) \preceq (z_1 \cdots z_n) \Leftrightarrow \overset{(3)}{(y_1 \cdots y_n) = (z_1 \cdots z_n)}$ or (4)

$\exists k \in \{1 \cdots n\}$ with $y_i = z_i$ for $i<k$ and

$y_k < z_k$

if (1) and (3), clearly transitive.

if (1) and (4), we take the same k that satisfies 4 and we are done prooving transitivity.

if (2) and (3), take the k that satisfies (2).

if (2) and (4) $\exists k_1$ with $x_{k_1} < y_{k_1}$, $x_i = y_i$ when $i < k_1$

$\exists k_2$ with $y_{k_2} < z_{k_2}$, $y_i = z_i$ when $i < k_2$

take $k_3 = \min\{k_1, k_2\}$, then k_3 satisfies $x_{k_3} < z_{k_3}$, with $x_i = z_i$ when $i < k_3$

Therefore, the relation is transitive.

(iii) $\leq_{P^d} := \{(a,b) \mid b \leq_P a\}$

$a \leq_{P^d} b$ if and only if $b \leq_P a$

Reflexive $a \leq_{P^d} a$, because $a \leq_P a$ $(P, \leq_P)$ $\leq_P$ is an order with the three properties.

Antisymmetric:

$$\left.\begin{array}{l} a \leq_{P^d} b \Rightarrow b \leq_P a \\ b \leq_{P^d} a \Rightarrow a \leq_P b \end{array}\right\} \Rightarrow a = b.$$

Transitive.

$$\left.\begin{array}{l} a \leq_{P^d} b \Rightarrow b \leq_P a \\ b \leq_{P^d} c \Rightarrow c \leq_P b \end{array}\right\} \Rightarrow c \leq_P a \Rightarrow a \leq_P c.$$

Therefore, the set P^d is a poset.

Excercise 3.4

For each i, $1 \le i \le sr+1$, be $sr+1$ integers. Let n_i be the length of the longest sequence starting with a_i and each dividing the following one. $(a_i, a_{i+1} \cdots a_{sr+1})$, if $n_i > r$ then the problem is solved. otherwise, by the pigenhole principle, there are at least $s+1$ values of n_i that are equal. (Then the integers a_i corresponding to these n_i cannot divide each other.)

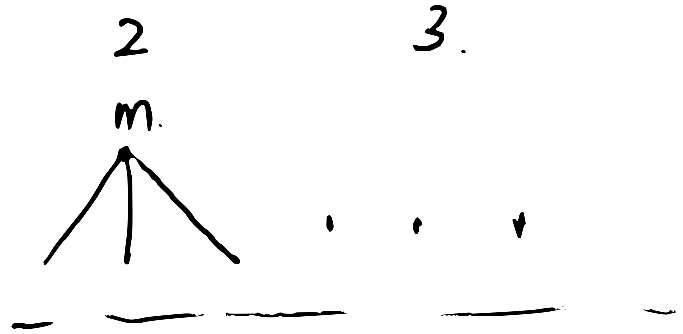
↓ This is because the sequence is monotically increasing.

Excercise 3.5

$w(P-\{m\})$ can be larger than k.

Before m is removed, The width of P is 2.

After m is removed, the width of $P-\{m\}$ is 3.



Exrercise 3.6

(i)

Surjectivity.

For any $\underbrace{k_1 \cdots k_n}$ $\in \{0,1\}^N$

We only need to take $f(a_0 \cdots a_n, b_0 \cdots b_n)$, that satisfies

$$\begin{cases} a_0 = k_1 \\ a_1 = k_3 \\ \vdots \\ a_n = k_{2n+1} \end{cases} \wedge \begin{cases} b_0 = k_2 \\ b_1 = k_4 \\ \vdots \\ b_n = k_{2n+2} \end{cases}$$ and we are done.

Injectivity If $f(a_0 \cdots a_n, b_0 \cdots b_n \cdots) = f(a_0' \cdots a_n', \cdots b_0' \cdots b_n')$

Then $a_0 b_0 a_1 b_1 \cdots a_n b_n = a_0' b_0' a_1' b_1' \cdots a_n' b_n' \cdots$

$\cdots$ So $a_i' = a_i$, $b_i' = b_i$, we get the proof.

(ii) Not surjective: $0.1919 \cdots 19$ cannot be represented as $s_0 \cdots s_n = 0.99 \cdots$

Injective: If $f(a_0 \cdots a_n \cdots, b_0 \cdots b_n \cdots) = f(a_0' \cdots a_n' \cdots b_0' \cdots b_n')$

So $\forall i \in N$, $a_i = a_i'$, $b_i = b_i'$, we get the proof.

(iii) Take $0.lololo\cdots$ and we are done.

(iv) Take $0.r_0\cdots r_n$, $0.s_0\cdots s_n$.

Since we have the Cantor pairing function that maps $N \times N \to N$.

we take $r_0\cdots r_n$, $s_0\cdots s_n$ to be two natural numbers and map it to $k_1\cdots k_w$.

the we obtain the bijection.

$$f(0.r_0\cdots r_n, 0.s_0\cdots s_n) = 0.k_1\cdots k_w$$

Excercise 3.7

(i) To show that the relation is a Total order.

We need to prove that it is reflexive, antisymmetric, transitive and total

Reflexivity $m \preceq m \to T$ because $m=m$.

Antisymmetric. $m \preceq n \wedge n \preceq m$

m even n odd ①

m, n both even or both odd and $m<n$. ②

m odd n even ③

m, n both even, odd $n<m$ ④

① and ③ ×

① and ④ × therefore.

② and ③ × $m=n$ is the only option.

② and ④ ×

Transitive.

$m \leq n \Leftrightarrow \underset{①}{(m=n)} \vee \underset{②}{(m \text{ even and } n \text{ odd})} \vee \underset{③}{(m,n \text{ both even or both odd, and } m<n)}$

$n \leq t \Leftrightarrow \underset{④}{(n=t)} \vee \underset{⑤}{(n \text{ even and } t \text{ odd})} \vee \underset{⑥}{(n,t \text{ both even, odd and } n<t)}$

① + ④ $\Rightarrow m \leq t$.

① + ⑤ $\Rightarrow m \leq t$

① + ⑥ $\Rightarrow m \leq t$.

② + ④ $\Rightarrow m \leq t$

② + ⑤ doesn't exist.

② + ⑥ doesn't exist.

③ + ④ $\Rightarrow m \leq t$.

③ + ⑤ doesn't exist

③ + ⑥ $\Rightarrow m \leq t$.

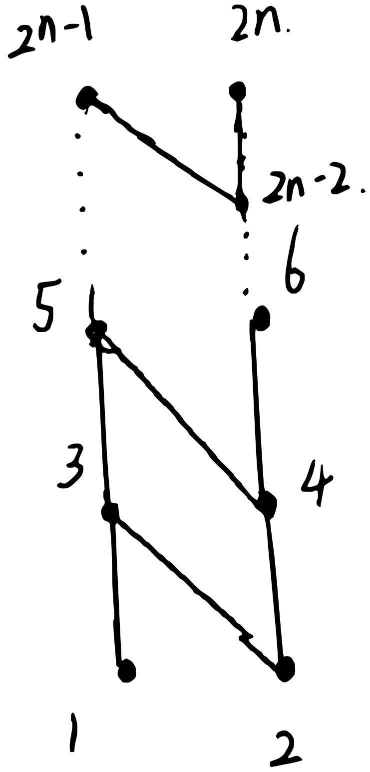
Therefore, the relation is transitive.

Total:

m even	n even	$m \leq n$ (if $m \leq n$) $n \leq m$ (if $m > n$)
m even	n odd	$n \leq m$.
m odd	n even	$m \leq n$.
m odd	n odd	$m \leq n$ (if $m \leq n$) $n \leq m$ (if $m > n$)

Therefore, it is a total order.

(ii)



(iii) Let $S=\{x \mid \operatorname{card} x = 1\}$, $P=\{\{x\} \mid x\in R\}$

$\forall x\in R$, $\operatorname{card}\{x\}=1$, so $\{x\}\in S$, $P\subseteq S$

$f: \mathbb{R}\to P$, $f(x)=\{x\}$, f is a bijection.

$\operatorname{card} P = \operatorname{card} R$, as $P\subseteq S$, so S is not a set.

(ii) (2 pts) Represent the reals in $(0,1)$ by their decimal expansions **WITHOUT** the infinite suffix $99999\cdots$. Define the function $h : (0,1) \times (0,1) \to (0,1)$ by

$$h(0.r_0r_1\cdots r_n\cdots, 0.s_0s_1\cdots s_n\cdots) = 0.r_0s_0r_1s_1\cdots r_ns_n\cdots$$

with $r_i, s_i \in \{0, 1, 2, \ldots, 9\}$. Prove that h is injective but not surjective.

(iii) (2 pts) If we pick in (ii) the decimal representations ending **WITH** the infinite suffix $99999\cdots$ rather that an infinite string of 0's, prove that h is also injective but still not surjective.

(iv) (2 pts) Show that there exists a bijection between $(0,1) \times (0,1)$ and $(0,1)$.

Exercise 3.7 (4 pts) Define a relation $\preceq$ on $\mathbb{N}$ by

$$m \preceq n \Leftrightarrow (m = n) \vee (m \text{ even and } n \text{ odd}) \vee (m, n \text{ both even or both odd, and } m < n)$$

(i) (2 pts) Show that $(\mathbb{N}, \preceq)$ is a total order.

(ii) (2 pts) Sketch a Hasse diagram for $(\mathbb{N}, \preceq)$, and provide the explicit coordinates for each element in $\mathbb{N}$.

Exercise 3.8 (2 pts) Briefly explain why the collection $\{x \mid \operatorname{card} x = 1\}$ is not a set.