Excercise 2.1

(i) $F_n = \frac{1}{\sqrt{5}}(\phi^n - \bar{\phi}^n)$ $\qquad\qquad$ $\bar{\phi} + \bar{\phi}^2 =$

$F_{m+n} = \frac{1}{\sqrt{5}}(\phi^{m+n} - \bar{\phi}^{m+n})$

$\phi^m F_n + \bar{\phi}^n F_m = \phi^m \cdot \frac{1}{\sqrt{5}}(\phi^n - \bar{\phi}^n) + \bar{\phi}^n(\phi^m - \bar{\phi}^m) \cdot \frac{1}{\sqrt{5}}$

$= \frac{1}{\sqrt{5}}(\phi^{m+n} - \bar{\phi}^{m+n}) = F_{m+n}$

(ii) Let $P(x)$ be the statement that $F_{x+y} = \phi^x F_y + \bar{\phi}^y F_x$, whenever $y \in \mathbb{N}$.

Base Case: $P(0)$ is true, since $F_y = F_y + \bar{\phi}^y F_0$

Inductive Case: Assume the IH (Inductive Hypothesis) that $P(x)$ is true, we'll show that $P(x+1)$ is also true.

$$F_{x+1+y} = F_{x+y} + F_{x+y-1} = \phi^x F_y + \bar{\phi}^y F_x + \phi^x F_{y-1} + \bar{\phi}^{y-1} F_x$$

$$= (\bar{\phi}^y + \bar{\phi}^{y-1}) F_x + \phi^x (F_y + F_{y-1})$$

$$= \phi^x F_y + \phi^{y+1} F_x$$

Therefore, we get the induction proof.

Excercise 2.2

When $k=1$, $k-1=0$ which doesn't belong to $\{1 \cdots k\}$, so that the base case should be proved when $n=0$, however $a^{-1}=1$ is not true all the time. Therefore the proof is wrong.

Excercise 2.3

We need to proove the associativity.

$(w \cdot x) \cdot y = w \cdot (x \cdot y)$ for all strings $w, x, y$

Assume that $(z \cdot x) \cdot y = z \cdot (x \cdot y)$ for every string $z$ such that $|z| < |w|$, we use the induction on $w$. ( $|\,|$ is the length of string ( ).)

Suppose $w = \varepsilon$ $\quad (w \cdot x) \cdot y = (\varepsilon \cdot x) \cdot y = x \cdot y$

$= \varepsilon (x \cdot y)$

$= w (x \cdot y)$

Suppose $w = az$ $(a \in \Sigma)$ $\quad (w \cdot x) \cdot y = (az \cdot x) \cdot y$

$= (a(z \cdot x)) \cdot y$

$= a((z \cdot x) \cdot y)$

$= a(z \cdot (x \cdot y))$ by the inductive hypothethis

$= az \cdot (x \cdot y)$

$= w \cdot (x \cdot y)$.

The we conclude that $(w \cdot x) \cdot y = w \cdot (x \cdot y)$

Since the notation of $w, x, y$ can be arbitrary

We deduce that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Excercise 2.4

(i) We also prove this by using induction:

We rewrite the statement to be $w \cdot y = w$ then $y = \varepsilon$ $(l(w) < l(x))$

Suppose $1^\circ$ $w = \varepsilon$ $\quad \varepsilon \cdot y = \varepsilon \Rightarrow y = \varepsilon$.

$2^\circ$ $x = aw$ $\qquad x \cdot y = x$

$\Rightarrow (aw) \cdot y = aw$

$\Rightarrow a(w \cdot y) = aw$

$\Rightarrow (a, w \cdot y) = (a, w)$

$\Rightarrow w \cdot y = w \Rightarrow y = \varepsilon$

Therefore, by induction, the property is proved.

Similarly, we write

(ii) $w \cdot y = w \cdot z$ $\quad (l(w) < l(x))$

Therefore, by induction, the property is proved.

$1^\circ$ $w = \varepsilon$ $\quad \varepsilon \cdot y = \varepsilon \cdot z \Rightarrow y = z$.

$2^\circ$ $x = aw$ $\quad aw \cdot y = aw \cdot z \Rightarrow w \cdot y = w \cdot z \Rightarrow y = z$.

Excercise 2.5

(i) For a logical proposition $\varphi$, let $A(\varphi)$ denote the property that there exists a $\{\downarrow\}$, only proposition logically equivalent to $\varphi$.

Base case: $\varphi$ is a variable, $A(\varphi)$ is vacously true.

Inductive Case I: $\varphi$ is a negation, say $\varphi = \neg p$, Assume $A(p)$, we'll show $A(\neg p)$ is true, By IH, there exists a $\downarrow$ only proposition $q$ that $q \Leftrightarrow p$, $\neg p \Leftrightarrow \neg q \Leftrightarrow q \downarrow q$.

$A(\neg p)$ is true.

Case II: $\varphi$ is a conjunction, say $\varphi = p_1 \wedge p_2$.

As $p_1 \wedge p_2 \Leftrightarrow \neg(\neg p_1 \vee \neg p_2) \Leftrightarrow \neg(q_1 \vee q_2) \Leftrightarrow q_1 \downarrow q_2$.

$A(p_1 \wedge p_2)$ is true.

Case III. $\varphi$ is a disjunction, $\varphi = p_1 \vee p_2$.

$p_1 \vee p_2 \Leftrightarrow \neg(\neg(p_1 \vee p_2)) \Leftrightarrow \neg(p_1 \downarrow p_2) \Leftrightarrow q$

$A(p_1 \vee p_2)$ is true.

Case IV $\varphi$ is an implication, say $\varphi = p_1 \to p_2$

By IHL $q_1 \Leftrightarrow \neg p_1$, $q_2 \Leftrightarrow p_2$.

As $p_1 \to p_2 \Leftrightarrow \neg p_1 \wedge p_2 \Leftrightarrow q_1 \wedge q_2$ from Case II, $A(p_1 \to p_2)$ is true.

(ii) Base case $\varphi$ is a variable $\varphi = x$, vacously true.

Inductive Case I $\varphi = \neg p$, Assume a proposition $q$ only contains $q$ $p \Leftrightarrow q$, $\neg p \Leftrightarrow \neg q \Leftrightarrow q | q$

Case II $\varphi = p_1 \vee p_2$

As $p_1 \vee p_2 \Leftrightarrow \neg(\neg p_1 \wedge \neg p_2) \Leftrightarrow \neg(q_1 \wedge q_2) \Leftrightarrow q_1 | q_2$.

Case III $\varphi = p_1 \wedge p_2$.

As $p_1 \wedge p_2 \Leftrightarrow \neg(\neg(p_1 \wedge p_2)) \Leftrightarrow \neg(p_1 | p_2) \Leftrightarrow q$

Case IV $\varphi = p_1 \to p_2$.

$p_1 \to p_2 \Leftrightarrow \neg p_1 \wedge p_2 \Leftrightarrow q_1 \wedge q_2$, from Case III. $A(p_1 \to p_2)$ is true.

Excercise 2.6

i) merge

Let $P(n)$ be the statement that $X[1 \cdots n]$ and $Y[1 \cdots m]$ can be sorted by merge whenever $m \in N$.

Base Case: $n=1$, $X[1]$ has only one element and is put right after the largest element in $Y$ that is smaller than $X[1]$, so $P(1)$ true.

Inductive Case: Assume $P(n)$, we'll show $P(n+1)$ is true.

Consider $X[1]$, $X[1]$ is put right after the largest element in $Y$ but is smaller than $X[1]$. Then set $X[1]$ aside, because all elements in front of $X[1]$ is in $Y$, and is sorted. Then sort $X[2, \cdots n]$ and rest of $Y$. From IH, all elements are sorted.

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ii) msort

Base Case: $n=1$ $A[1]$ has only one element, so it is true.

Inductive case assume that $A[1 \cdots n]$ is true, we'll show that $A[1 \cdots n+1]$ is true.

As $\lfloor \frac{n}{2} \rfloor < n$ for $n \in N, n \geq 2$, so $L$ is sorted.

As $n - (\lfloor \frac{n}{2} + 1) + 1 = n - \lfloor \frac{n}{2} \rfloor < n$ for $n \in N, n \geq 2$

so $R$ is sorted.

Because $L, R$ are sorted, their combination using merge is sorted.

7.(i) Reflective $m \sim m \Leftrightarrow 2 | 0$ is true

Symmetric As $2 | m-n \Leftrightarrow 2 | n-m$, So $m \sim n \Leftrightarrow n \sim m$.

Transitive: If $m \sim n, n \sim p$ then $2 | m-n$, $2 | n-p$

So $2 | m-p$, which shows $m \sim p$.

(ii) As $m \sim n \Leftrightarrow 2 | m-n$, all odd numbers form the set $[1]_R$ and all even numbers form the set $[0]_R$.

$Z_2 = Z/\sim = \{[0], [1]\}$

iii) We arbitrary take 2 representatives $m_1, m_2$ from the first equivalent class $[m]$, and $n_1, n_2$ from $[n]$.

$$[m_1]+[n_1]=[m_1+n_1]=\{t\in Z \mid (m_1+n_1)\sim t\}$$

For $t\in[m_1+n_1]$, $2\mid(t-m_1-n_1)$. Let $t-m_1-n_1=2k\ (k\in Z)$, then $t=2k+m_1+n_1$. Since $m_1, m_2\in[m]$, $m_1\sim m_2$. Let $m_2-m_1=2k_m\ (k_m\in Z)$. Similarly, let $n_2-n_1=2k_n\ (k_n\in Z)$.

Thus, $t-m_2-n_2=\cdots=2(k-k_m-k_n)\in Z$, $2\mid(t-m_2-n_2)$

which means that $m_2+n_2\sim t$, it shows that $t\in[m_2+n_2]$, which implies that $[m_1+n_1]\subset[m_2+n_2]$

The proof of $[m_2+n_2]\subset[m_1+n_1]$ is similar.

Therefore $[m_1+n_1]=[m_2+n_2]$. This implies that the definition of addition on $Z$ is independent of $m, n$.

For any $t \in [m_1]\cdot[n_1] = [m_1 \cdot n_1]$, $2 \mid (t - m_1 n_1)$. Let

$t - m_1 n_1 = 2k$ $(k \in \mathbb{Z})$, then $t = 2k + m_1 n_1$

$$t - m_2 n_2 = 2(k - k_m n_1 - k_n m_1 - 2k_m k_n)$$

Since $k - k_m n_1 - k_n m_1 - 2k_m k_n \in \mathbb{Z}$, $2 \mid (t - m_2 n_2)$. This shows that $t \in [m_2 \cdot n_2]$, which implies that $[m_1 \cdot n_1] \subset [m_2 \cdot n_2]$.

The proof of $[m_2 \cdot n_2] \subset [m_1 \cdot n_1]$ is done with the same process shown above.

Therefore, $[m_1 \cdot n_1] = [m_2 \cdot n_2]$. This implies that the definition of multiplication on $\mathbb{Z}$ is independent of $m$ and $n$.

Excercise 2.8

i) $R = \{(a,a), (a,b), (b,a), (b,b)\}$ on the set

$A = \{a, b, c\}$, not reflexive because $(c,c)$ not in $R$.

ii) if $R$ is transitive, then if $xRy$ and $yRz$, then $xRz$. In particular, if $xRy$ and $yRx$ then $xRx$. If $R$ is irreflective. if $xRy$ then not-$(yRx)$ ie $R$ is asymmetric.

$$
\begin{tikzcd}
xRy \wedge yRx \arrow[r] \arrow[r, bend left, "\text{transitive}"] \arrow[rr, bend right, "\text{asymmetric.}"'] & x \wedge x \arrow[r] \arrow[r, bend left, "\text{irreflective.}"] & \perp
\end{tikzcd}
$$