Ve203 Discrete Mathematics (Fall 2020)

Assigment 5: Algorithms and Recurrence Relations

Date Due: 12:10 PM, Thursday, the 22nd of October 2020

This assignment has a total of **(49 Marks)**.

Exercise 5.1 Landau Symbols

Interpret and prove the following relations for integer sequences (a_n) and (b_n). In all cases, the Landau symbol is as $n \to \infty$.

$$O(a_n) + O(b_n) = O(|a_n| + |b_n|), \quad O(a_n)O(b_n) = O(a_n b_n), \quad O(O(a_n)) = O(a_n),$$
$$O(n) = O(n^2), \quad O(n^2) \neq O(n), \quad O(\log_x(n)) = O(\log_y(n)), \quad \text{for all } x, y > 0.$$

(6 Marks)

Exercise 5.2 Binary Insertion Sort

The *binary insertion sort* is a variation of the insertion sort that uses a binary search technique rather than a linear search technique to insert the i element in the correct place among the previously sorted elements.

i) Express the binary insertion sort in pseudocode.

   (2 Marks)

ii) Compare the number of comparisons of elements used by the insertion sort and the binary insertion sort when sorting the list $7, 4, 3, 8, 1, 5, 4, 2$.

   (2 Marks)

iii) Show that the insertion sort uses $O(n^2)$ comparisons of elements.

   (2 Marks)

iv) Find the complexity of the binary insertion sort. Is it significantly faster?

   (2 Marks)

Exercise 5.3

Order the letters M,I,C,H,I,G,A,N alphabetically using

i) merge sort,

   (2 Marks)

ii) insertion sort,

   (2 Marks)

iii) bubble sort

   (2 Marks)

algorithms. (Note that it does not matter that the letter "I" is repeated.) For each algorithm, show what the arrangement is after each pass/merge. How many comparisons of letters are made using each algorithm?

Exercise 5.4

The sums of the digits of numbers can be used to obtain a variety of results about the numbers:

i) Show that a positive integer is divisible by 3 if and only if the sum of its decimal digits is divisible by 3.

   (2 Marks)

ii) Show that a positive integer is divisible by 11 if and only if the difference of the sum of its decimal digits in even-numbered positions and and the sum of its decimal digits in odd-numbered positions is divisible by 11.

   (2 Marks)

iii) Show that a positive integer is divisible by 3 if and only if the difference of the sum of its binary digits in even-numbered positions and and the sum of its binary digits in odd-numbered positions is divisible by 3.

(2 Marks)

Exercise 5.5 Modular Exponentiation

Find $4102^{1042} \bmod 2014$ using the algorithm for modular exponentiation given in the lecture. Show all the steps in the algorithm.

(2 Marks)

Exercise 5.6 Discretization of Differential Equations

Let (a_n) be a sequence of real numbers. We define the sequences of *backward differences* $(\nabla^k a_n)$ as follows:

- The *first (backward) difference* is given by $\nabla a_n = a_n - a_{n-1}$.
- The *kth (backward) difference* is given by $\nabla^k a_n = \nabla^{k-1} a_n - \nabla^{k-1} a_{n-1}$.

i) Find (∇a_n) and $(\nabla^2 a_n)$ for the following sequences:

a) $a_n = 4$, b) $a_n = 2n$, c) $a_n = n^2$, d) $a_n = 2^n$.

(1 Mark)

ii) Show that $a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$. Use this to express the recurrence relation $a_n = a_{n-1} + a_{n-2}$ in terms of a_n, ∇a_n and $\nabla^2 a_n$.

(2 Marks)

iii) Prove that a_{n-k} can be expressed in terms of $a_n, \nabla a_n, \ldots, \nabla^k a_n$. Deduce that any recurrence relation for the sequence a_n can be written in terms of backward differences. The resulting equation is called a *difference equation*. Such equations occur when "discretizing" differential equations, for example, in numerical solution algorithms.

(3 Marks)

Exercise 5.7

Solve the following recurrence relations:

$$
\begin{aligned}
a_n &= a_{n-1} + 6a_{n-2}, && n \geq 2, && a_0 = 3,\ a_1 = 6, \\
a_{n+2} &= -4a_{n+1} + 5a_n, && n \geq 0, && a_0 = 2,\ a_1 = 8.
\end{aligned}
$$

(4 Marks)

Exercise 5.8

The *Lucas numbers* are defined by

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2, \qquad L_1 = 1.$$

i) Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, 4, \ldots$, where f_n is the nth Fibonacci number.

(1 Mark)

ii) Find an explicit formula for the Lucas numbers.

(2 Marks)

Exercise 5.9

Prove Theorem **??** of the lecture, which states that all solution to a linear homogeneous recurrence relation of degree two are of the form

$$a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot n r_0^n, \qquad \alpha_1, \alpha_2 \in \mathbb{R},\ n \in \mathbb{N}.$$

if there is only a single characteristic root r_0.

(2 Marks)

Exercise 5.10

Find all solutions of the following recurrence relations:

$$
\begin{aligned}
a_n &= 5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n, \\
a_n &= -5a_{n-1} - 6a_{n-2} + 2^n + 3n, \\
a_n &= 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n.
\end{aligned}
$$

(6 Marks)