Ve203 Discrete Mathematics (Fall 2022)

Assignment 2

Date Due: See canvas
This assignment has a total of **(24 points)**.
Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**. **Explain** (briefly) if you claim something is trivial or straightforward. Provide a counterexample if you are trying to disprove something. It is **NOT OK** to write something like "how do we know that blahblahblah is even true..." In addition, be careful that some problems might be ill-defined.

Exercise 2.1 (2 pts) Let (F_n) be the Fibonacci sequence with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \bar{\phi} = \frac{1-\sqrt{5}}{2}$$

Show that $F_{m+n} = \phi^m F_n + \bar{\phi}^n F_m$ for all $m, n \in \mathbb{N}$, using

(i) `(0 pts)` the explicit formula (i.e., using powers of ϕ and $\bar{\phi}$) for F_n.

(ii) `(2 pts)` induction.

Exercise 2.2 (2 pts) What is wrong with the following **proof** of the "theorem"?

"Theorem". Given any positive number a, then for all positive integer n, we have $a^{n-1} = 1$.

Proof. If $n = 1$, $a^{n-1} = a^{1-1} = a^0 = 1$. By induction, assume that the theorem is true for $n = 1, 2, \ldots, k$, then for $n = k+1$,

$$a^{(k+1)-1} = a^k = \frac{a^{k-1} \times a^{k-1}}{a^{(k-1)-1}} = \frac{1 \times 1}{1} = 1$$

therefore the theorem is true for all positive integers n. $\square$

Exercise 2.3 (2 pts) Show that concatenation of string is associative, i.e., $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \Sigma^*$.

Exercise 2.4 (4 pts) Based on the recursive definition of string and string concatenation $\cdot$, use structural induction to show that for all strings $x, y, z \in \Sigma^*$,

(i) `(2 pts)` If $x \cdot y = x$, then $y = \varepsilon$.

(ii) `(2 pts)` If $x \cdot y = x \cdot z$, then $y = z$.

Exercise 2.5 (4 pts) Show that for any logical proposition φ using the connectives $\{\neg, \wedge, \vee, \rightarrow\}$, i.e., wffs, there exists a proposition that is logically equivalent to φ using only

(i) `(2 pts)` $\{\downarrow\}$, where $\downarrow$ is the Peirce arrow (NOR), with $p \downarrow q \Leftrightarrow \neg(p \vee q)$.

(ii) `(2 pts)` $\{|\}$, where $|$ is the Sheffer stroke (NAND), with $p \mid q \Leftrightarrow \neg(p \wedge q)$.

Exercise 2.6 (4 pts) Show by induction that the following two algorithms `msort` and `merge` are correct.

Input: $A[1 \ldots n]$, unsorted array
Output: all the $A[i], 1 \leq i \leq n$ in increasing order
Function `msort`$(A[1 \ldots n])$:
  if $n = 1$ **then**
    return A
  else
    $L \leftarrow$ `msort`$(1 \ldots \lfloor \frac{n}{2} \rfloor)$
    $R \leftarrow$ `msort`$(\lfloor \frac{n}{2} \rfloor + 1 \ldots n)$
    return `merge`(L, R)
  end
end

```
msort :: Ord a => [a] -> [a]
msort []  = []
msort [x] = [x]
msort xs  = merge (msort ys) (msort zs)
    where (ys, zs) = halve xs

halve :: [a] -> ([a], [a])
halve xs = (take n xs, drop n xs)
    where n = length xs `div` 2
    -- splitAt n xs
```

Input: $X[1 \dots n]$, $Y[1 \dots m]$, 2 sorted arrays
Output: $X \cup Y$ sorted with elements in increasing order

Function `merge`($X[1 \dots n]$, $Y[1 \dots m]$):
- **if** $n = 0$ **then**
  - **return** Y
- **else if** $m = 0$ **then**
  - **return** X
- **else if** $X[1] < Y[1]$ **then**
  - **return** $X[1]$ *followed by* `merge`$(X[2 \dots n], Y)$
- **else**
  - **return** $Y[1]$ *followed by* `merge`$(X, Y[2 \dots m])$
- **end**

end

```
merge :: Ord a => [a] -> [a] -> [a]
merge [] ys = ys
merge xs [] = xs
merge (x:xs) (y:ys)
    | x <= y    = x:merge xs (y:ys)
    | otherwise = y:merge (x:xs) ys
```

Exercise 2.7 (4 pts) Let

$$m \sim n \qquad \Leftrightarrow \qquad 2 \mid (n - m), \qquad\qquad m, n \in \mathbb{Z}.$$

(i) `(1 pt)` Show that $\sim$ is an equivalence relation.

(ii) `(1 pt)` What partition $\mathbb{Z}_2 := \mathbb{Z}/\sim$ is induced by $\sim$?

(iii) `(2 pts)` Define addition and multiplication on $\mathbb{Z}_2$ by the addition and multiplication of class representatives, i.e.,

$$[m] + [n] := [m + n], \qquad\qquad [m] \cdot [n] := [m \cdot n].$$

Show that these operations are well-defined, i.e., independent of the representatives m and n of each class.

Exercise 2.8 (2 pts) Given a relation R on a nonempty set A, show that

(i) `(0 pts)` If R is transitive and symmetric, then R is reflexive.

(ii) `(2 pts)` If R is transitive and asymmetric, then R is irreflexive.