VE203 Discrete Math
Spring 2022 — HW1 Solutions

March 6, 2022

Exercise 1.1.1

a	b	$a \wedge b$	$\neg(a \wedge b)$	$\neg a$	$\neg b$	$\neg a \vee \neg b$	$\neg(a \wedge b) \leftrightarrow \neg a \vee \neg b$
0	0	0	1	1	1	1	1
0	1	0	1	1	0	1	1
1	0	0	1	0	1	1	1
1	1	1	0	0	0	0	1

$$\Rightarrow \neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b$$

a	b	$a \vee b$	$\neg(a \vee b)$	$\neg a$	$\neg b$	$\neg a \wedge \neg b$	$\neg(aVb) \leftrightarrow \neg a \wedge \neg b$
0	0	0	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	1	0	0	1	0	1
1	1	1	0	0	0	0	1

$$\Rightarrow \neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b$$

Exercise 1.1.2
For sets $A, B \subset M$, we write out them in terms of predicates $P_1(x)$ and $P_2(x)$ as $A = \{x \in M \mid P_1(x)\}$ and $B = \{x \in M \mid P_2(x)\}$ Therefore, $A \cap B = \{x \in M \,|P_1|\, x \mid \wedge P_2(x)\}$, $A \cup B = \{x \in M \mid P_1(x) \vee P_2(x)\}$

$$M - A = \{x \in M \mid \neg P_1(x)\}, \quad M - B = \{x \in M \mid \neg P_2(x)\}$$

$M - (A \cap B) = \{x \in M \mid \neg\left(P_1(x) \wedge P_2(x)\right)\}, M - (A \cup B) = \{x \in M \mid \neg\left(P_1(x) V P_2 \mid x)\right)\}$
Applying de Morgan's rules,

$$\begin{aligned} M - (A \cap B) &= \{x \in M \mid \neg P_1(x) \vee \neg P_2(x)\} = (M - A) \cup (M - B) \\ M - (A \cup B) &= \{x \in M \mid \neg P_1(x) \wedge \neg P_2(x)\} = (M - A) \cap (M - B) \end{aligned}$$

Exercise 1.2.1

A	B	C	$A \to (B \to C)$	$B \to (A \to C)$	$(A \to (B \to C)) \to (B \to (A \to C))$
1	1	1	1	1	1
1	1	0	0	0	1
1	0	1	1	1	1
1	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	1	1
0	0	0	1	1	1

Exercise 1.2.2

Since φ is tautology, we simply write the disjunctive normal form of φ as

$$\varphi = A \vee \neg A$$

where A is a predicate.

Exercise 1.2.3

Since φ is tautology, we simply write the conjunctive normal form of φ as

$$\varphi = (A \vee \neg A)$$

where A is a predicate.

Exercise 1.3.1

$$\varphi_0 : p \wedge \neg p$$
$$\varphi_1 : p \wedge q$$
$$\varphi_2 : p \wedge \neg q$$
$$\varphi_3 : p$$
$$\varphi_4 : \neg p \wedge q$$
$$\varphi_5 : q$$
$$\varphi_6 : (p \wedge \neg q) \vee (\neg p \wedge q)$$
$$\varphi_7 : p \vee q$$
$$\varphi_8 : \neg(p \vee q)$$
$$\varphi_9 : (p \wedge q) \vee \neg(p \vee q)$$
$$\varphi_{10} : \neg q$$
$$\varphi_{11} : p \vee \neg q$$
$$\varphi_{12} : \neg p$$
$$\varphi_{13} : \neg p \vee q$$
$$\varphi_{14} : \neg(p \wedge q)$$
$$\varphi_{15} : p \vee \neg p$$

Exercise 1.3.2

$$\varphi_0 : p \wedge \neg p$$
$$\varphi_1 : p \wedge q$$
$$\varphi_2 : p \wedge \neg q$$
$$\varphi_3 : p$$
$$\varphi_4 : \neg p \wedge q$$
$$\varphi_5 : q$$
$$\varphi_6 : \neg(\neg(p \wedge \neg q)) \wedge \neg(\neg p \wedge q)$$
$$\varphi_7 : \neg(\neg p \wedge \neg q)$$
$$\varphi_8 : \neg p \wedge \neg q$$
$$\varphi_9 : \neg(\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q))$$
$$\varphi_{10} : \neg q$$
$$\varphi_{11} : \neg(\neg p \wedge q)$$
$$\varphi_{12} : \neg p$$
$$\varphi_{13} : \neg(p \wedge \neg q)$$
$$\varphi_{14} : \neg(p \wedge q)$$
$$\varphi_{15} : \neg(\neg p \wedge p)$$

Exercise 1.3.3

$$\varphi_0 : \neg(\neg p \vee p)$$
$$\varphi_1 : \neg(\neg p \vee \neg q)$$
$$\varphi_2 : \neg(\neg p \vee q)$$
$$\varphi_3 : p$$
$$\varphi_4 : \neg(p \vee \neg q)$$
$$\varphi_5 : q$$
$$\varphi_6 : \neg(\neg p \vee q) \vee \neg(p \vee \neg q)$$
$$\varphi_7 : p \vee q$$
$$\varphi_8 : \neg(p \vee q)$$
$$\varphi_9 : \neg(\neg p \vee \neg q) \vee \neg(p \vee q)$$
$$\varphi_{10} : \neg q$$
$$\varphi_{11} : p \vee \neg q$$
$$\varphi_{12} : \neg p$$
$$\varphi_{13} : \neg p \vee q$$
$$\varphi_{14} : \neg p \vee \neg q$$
$$\varphi_{15} : p \vee \neg p$$

Exercise 1.3.4 Assume that $\{\vee, \wedge\}$ is functionally complete Then their combination can express φ_{12}, which is $\neg p$ The following will prove $\{\wedge, \vee\}$ is self-closed, which means they cannot generate other functions, including $\neg$ Consider all possible combinations with only one input p. Combinations of $\{0,1\}$ with $\{\wedge,\vee\}$ obviously cannot generate $\neg$.

$$\begin{aligned}
p \vee p &\Leftrightarrow p \\
p \vee 1 \Leftrightarrow 1 \vee p &\Leftrightarrow 1 \\
p \vee 0 \Leftrightarrow 0 \vee p &\Leftrightarrow p \\
p \wedge p &\Leftrightarrow p \\
p \wedge 1 \Leftrightarrow 1 \vee p &\Leftrightarrow p \\
p \wedge 0 \Leftrightarrow 0 \vee p &\Leftrightarrow 0
\end{aligned}$$

They also cannot generate $\neg$, so all combinations of $\{\wedge,\vee\}$ cannot generate $\neg$, which contradicts. So $\{\vee,\wedge\}$ is not functionally complete.

Exercise 1.4.1

A	B	$A \wedge B$	$A \vee B$	$A \mid B$	$A \downarrow B$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	1	0
1	1	1	1	0	0

Exercise 1.4.2

$$\begin{aligned}
A \downarrow A &\Leftrightarrow \neg(A \vee A) \\
&\Leftrightarrow \neg A \\
(A \downarrow B) \downarrow (A \downarrow B) &\Leftrightarrow \neg(A \downarrow B) \\
&\Leftrightarrow A \vee B
\end{aligned}$$

Exercise 1.4.3 By $A \downarrow A \Leftrightarrow \neg A$, $\{\neg\}$ can be expressed by $\{\downarrow\}$ By $(A \downarrow B) \downarrow (A \downarrow B) \Leftrightarrow A \vee B$, $\{\vee\}$ can be expressed by $\{\downarrow\}$ Since we previously proved that $\{\neg,\vee\}$ is functionally complete, $\downarrow$ is functionally complete

Exercise 1.4.4

$$\oplus = (A \vee B) \wedge (\neg A \vee \neg B) = (A \downarrow B) \downarrow ((A \downarrow A) \downarrow (B \downarrow B))$$

Exercise 1.4.5 We here prove $\mid$ can denote $\downarrow$ that is functionally complete.

$$A \mid B = \neg(\neg A \downarrow \neg B) = ((A \downarrow A) \downarrow (B \downarrow B)) \downarrow ((A \downarrow A) \downarrow (B \downarrow B))$$

Exercise 1.4.6 Here we can raise a counter-example:

A	B	C	$A \mid B$	$B \mid C$	$(A \mid B) \mid C$	$A \mid (B \mid C)$
1	1	0	0	1	1	0

Therefore, associate law does not apply to $\downarrow$.

Exercise 1.5.1 For any X that satisfies $X \in 2^A \cap 2^B$,

$$\begin{aligned} X \in 2^A \cap 2^B &\Rightarrow X \in 2^A \wedge X \in 2^B \\ &\Rightarrow X \subset A \wedge X \subset B \\ &\Rightarrow X \subset A \cap B \\ &\Rightarrow X \in 2^{A \cap B} \\ 2^A \cap 2^B &\subset 2^{A \cap B} \end{aligned}$$

For any Y that satisfies $Y \in 2^{A \cap B}$,

$$\begin{aligned} Y \in 2^{A \cap B} &\Rightarrow Y \subset A \cap B \\ &\Rightarrow Y \subset A \wedge Y \subset B \\ &\Rightarrow Y \in 2^A \wedge Y \in 2^B \\ &\Rightarrow Y \in 2^{A \cap B} \\ 2^{A \cap B} &\subset 2^A \cap 2^B \end{aligned}$$

Since $2^A \cap 2^B \subset 2^{A \cap B}$ and $2^A \cap 2^B \supset 2^{A \cap B}, 2^A \cap 2^B = 2^{A \cap B}$ is proved.

Exercise 1.5.2 For any Z that satisfies $Z \in 2^A \cup 2^B$,

$$\begin{aligned} Z \in 2^A \cup 2^B &\Rightarrow Z \in 2^A \vee Z \in 2^B \\ &\Rightarrow Z \subset A \vee Z \subset B \\ &\Rightarrow Z \subset A \cup B \\ &\Rightarrow Z \in 2^{A \cup B} \\ 2^A \cup 2^B &\subset 2^{A \cup B} \end{aligned}$$

Exercise 1.6.1

$$\begin{aligned} X \Delta Y &= (X - Y) \cup (Y - X) \\ &= (X \cup Y - Y) \cup (X \cup Y - X), \text{ apply de Morgan's rules:} \\ &= (X \cup Y) - (X \cap Y) \end{aligned}$$

Exercise 1.6.2

$$\begin{aligned} &(M - X)\Delta(M - Y) \\ =&(M - X) - (M - Y)) \cup ((M - Y) - (M - X)) \\ =&(Y - X) \cup (X - Y) = (X - Y) \cup (Y - X) = X \Delta Y \end{aligned}$$

Exercise 1.6.3

X	Y	Z	$X\Delta Y$	$(X\Delta Y)\Delta Z$	$Y\Delta Z$	$X\Delta(Y\Delta Z)$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	0	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	0	1	0
1	1	1	0	1	0	1

Exercise 1.6.4

X	Y	Z	$X\Delta Y$	$(X\Delta Y)\Delta Z$	$Y\Delta Z$	$X\Delta(Y\Delta Z)$	$X\cap(Y\Delta Z)$	$X\cap Y$	$X\cap Z$	$(X\cap Y)\Delta(X\cap Z$
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	0	0	0	0
0	1	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	0	0	0	0	0
1	0	0	1	1	0	1	0	0	0	0
1	0	1	1	0	1	0	1	0	1	1
1	1	0	0	0	1	0	1	1	0	1
1	1	1	0	1	0	1	0	1	1	0

Exercise 1.6.5 $X\Delta Y = Z\Delta W \Leftrightarrow X\Delta Z = Y\Delta W$ if $x \in X\Delta Z. x \in X - Z$ or $Z - x$. suppose $x \in X - Z$. prove $x \in Y - w$ or $W - Y$. if $x \in X \cap Y$. then $x \notin X\Delta Y$. so $x \notin Z\Delta W$. as $x \notin z$, x can't $\in W$. so $x \in Y, x \notin w$. $x \in Y\Delta w$ So for $x \in X\Delta Z$ we know $x \in Y\Delta W$. Similarly if $x \in Y\Delta\omega$ we can get $x \in X\Delta Z$. So $X\Delta Y = Z\Delta\omega \Rightarrow X\Delta Z = Y\Delta\omega$ Substitude Yand Z, we get the inverse and therefore $X\Delta Y = Z\Delta W \Leftrightarrow X\Delta Z = Y\Delta W$.

Exercise 1.6.6

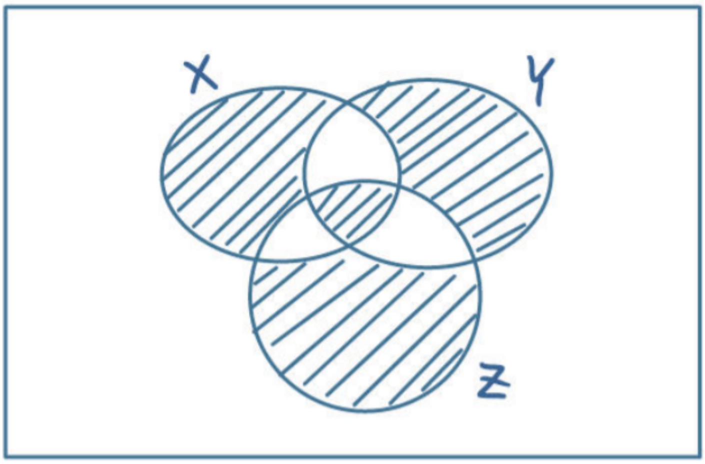

Figure 1: problem 1.6.6

Exercise 1.7.1

$$\begin{aligned}\varrho(A,B)=0 &\Leftrightarrow A\Delta B=\phi\\ &\Leftrightarrow (A-B)\cup(B-A)=\phi\\ &\Leftrightarrow (A-B)=\phi,(B-A)=\phi \quad \Leftrightarrow A=B\end{aligned}$$

Proved.

Exercise 1.7.2

$$\varrho(A,B)=|A\Delta B|=|B\Delta A|=\varrho(B,A)$$

Proved.

Exercise 1.7.3 let $a=A-B-C, b=B-A-C, c=C-A-B, ab=A\cap B, bc=B\cap C$, $ac=A\cap C, abc=A\cap B\cap C$. $\varrho(A,C)=|A\Delta C|=|a\cup c\cup ab\cup bc|$ Also $\varrho(A,B)+\varrho(B,C)=|A\Delta B|+|B\Delta C|=|a\cup b\cup ac\cup bc|+|b\cup c\cup ab\cup ac|$ Since $|a\cup b\cup ac\cup bc|+|b\cup c\cup ab\cup ac|\geq |a\cup c\cup ab\cup bc|$ Then $\varrho(A,C)\leq\varrho(A,B)+\varrho(B,C)$ Proved.