Ve203 Discrete Mathematics (Spring 2022)

Assignment 1

Date Due: 21:00 PM, Tuesday, Mar. 01, 2022
This assignment has a total of **(31 points)**.

Exercise 1.1

(i) `(1 point)` Let a, b be statements. Write out the truth tables to prove *de Morgan's rules*:

$$\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b, \qquad \neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b.$$

(ii) `(1 point)` Let M be a set and $A, B \subset M$. Prove the following equalities by writing out the sets in terms of predicates and applying de Morgan's rules.

$$M - (A \cap B) = (M - A) \cup (M - B), \qquad M - (A \cup B) = (M - A) \cap (M - B).$$

(2 points)

Exercise 1.2 Given $\varphi = (A \to (B \to C)) \to (B \to (A \to C))$,

(i) `(2 points)` Write the truth table for φ.

(ii) `(2 points)` Write φ in disjunctive normal form.

(iii) `(2 points)` Write φ in conjunctive normal form.

(6 points)

Exercise 1.3 The following shows the truth table for all $2^{2^2} = 16$ different binary logical operators φ_i, $i = 0, \ldots, 15$.

p	q	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Using infix notation, for example, φ_{13} can be represented as $\varphi_{13} = \to(p, q) = p \to q$.

A set S of logical operators is called *functionally complete* if every compound proposition is logically equivalent to a compound proposition involving only these logical operators in S. In this exercise, in order to show S is a functionally complete set, it suffices to verify that for all $i = 0, \ldots, 15$, φ_i over logical variables p and q can be represented using only operators in S.

(i) `(1 point)` Show that $\{\wedge, \vee, \neg\}$ is functionally complete.

(ii) `(1 point)` Show that $\{\wedge, \neg\}$ is functionally complete.

(iii) `(1 point)` Show that $\{\vee, \neg\}$ is functionally complete.

(iv) `(1 point)` Show that $\{\vee, \wedge\}$ is *not* functionally complete.

(4 points)

Exercise 1.4 In computer design, the logical operations NAND and NOR play an important role.[1] In logic, NAND is represented by the *Scheffer stroke* $|$ while NOR is represented by the *Peirce arrow* $\downarrow$. They are defined as

$$A \mid B :\!= \neg(A \wedge B), \qquad A \downarrow B :\!= \neg(A \vee B).$$

(i) `(1 point)` Give the truth tables for $A \mid B$ and $A \downarrow B$.

(ii) `(2 points)` Prove that $A \downarrow A \Leftrightarrow \neg A$ and $(A \downarrow B) \downarrow (A \downarrow B) \Leftrightarrow A \vee B$.

(iii) `(1 point)` Deduce that $\{\downarrow\}$ is functionally complete.

(iv) `(1 point)` Represent the exclusive or $\oplus$ solely through $\downarrow$.

[1] According to `https://en.wikipedia.org/wiki/Logical_NOR`, "The computer used in the spacecraft that first carried humans to the moon, the Apollo Guidance Computer, was constructed entirely using NOR gates with three inputs." A reference for this claim is is given in that article. See also `https://en.wikipedia.org/wiki/Flash_memory` for a discussion of NAND and NOR flash memory.

(v) `(1 point)` Prove that $\{|\}$ is functionally complete.

(vi) `(1 point)` Is the Scheffer stroke $|$ acting on logical statements is associative? That is, is it correct that $(A|B)|C \Leftrightarrow A \,|\, (B \,|\, C)$?

(7 points)

Exercise 1.5 For any sets A and B, show that

(i) `(1 point)` $2^A \cap 2^B = 2^{A\cap B}$.

(ii) `(1 point)` $(2^A \cup 2^B) \subset 2^{A\cup B}$.

(2 points)

Exercise 1.6 Let M be a set and let $X, Y, Z, W \subset M$. We define the *symmetric difference*:

$$X \triangle Y := (X - Y) \cup (Y - X)$$

(i) `(1 point)` Prove that $X \triangle Y = (X \cup Y) - (X \cap Y)$.

(ii) `(1 point)` Prove that $(M - X) \triangle (M - Y) = X \triangle Y$.

(iii) `(1 point)` Show that the symmetric difference is associative, i.e., $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$.

(iv) `(1 point)` Prove that $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$.

(v) `(1 point)` Show that $X \triangle Y = Z \triangle W$ iff $X \triangle Z = Y \triangle W$.

(vi) `(1 point)` Indicate the region of $X \triangle Y \triangle Z$ in a Venn diagram.

(6 points)

Exercise 1.7 Let X be a finite set, define the distance/metric $\varrho(A, B)$ of two sets $A, B \in 2^X$ by

$$\varrho(A, B) := |A \triangle B|.$$

Show that $(2^X, \varrho)$ is a *metric space* by verifying that for all $A, B, C \in 2^X$,

(i) `(1 point)` $\varrho(A, B) = 0$ iff $A = B$;

(ii) `(1 point)` $\varrho(A, B) = \varrho(B, A)$;

(iii) `(2 points)` $\varrho(A, C) \leq \varrho(A, B) + \varrho(B, C)$.

(4 points)