Ve203 Discrete Mathematics (Fall 2020)

Assignment 10: Graphs

Date Due: 12:10 PM, Thursday, the 26th of November 2020

This assignment has a total of **(36 Marks)**.

Exercise 10.1
For which values of n do the following graphs have an Euler circuit?

i) K_n ii) W_n iii) C_n iv) Q_n

(4 Marks)

Exercise 10.2
Show that a graph with at least two vertices is bipartite if and only if all simple circuits have even length.

Hint for the "if" part: For two vertices u and v, let the *distance* $d(u, v)$ be the length of the shortest path joining u to v. Show that choosing an arbitrary vertex u and setting $S = \{v : 2 \mid d(u, v)\}$ and $T = \{v : 2 \nmid d(u, v)\}$ defines a bipartition of the graph.
(4 Marks)

Exercise 10.3
Is the hypercube Q_n bipartite for all $n \in \mathbb{N}$? If so, give a bipartition. Otherwise, give a proof that it is not.
(3 Marks)

Exercise 10.4
The *complement* of a simple graph $G = (V, E)$ is given by $G^c = (V, E^c)$, where $E^c = V \times V \setminus E$, i.e., the complement has the same vertex set and an edge is in E^c if and only if it is not in E. A graph G is said to be *Self-complementary* if G is isomorphic to G^c.

i) Show that a self-complementary graph must have either $4m$ or $4m + 1$ vertices, $m \in \mathbb{N}$.

   (3 Marks)

ii) Find all self-complementary graphs with 8 or fewer vertices.

   (2 Marks)

Exercise 10.5
The parts of this question outline a proof of Ore's Theorem. Suppose that G is a simple graph with n vertices, $n > 3$, and $\deg(x) + \deg(y) > n$ whenever x and y are nonadjacent vertices in G. Ore's Theorem states that under these conditions, G has a Hamilton circuit.

i) Show that if G does not have a Hamilton circuit, then there exists another graph H with the same vertices as G, which can be constructed by adding edges to G such that the addition of a single edge would produce a Hamilton circuit in H.

   Hint: Add as many edges as possible at each successive vertex of G without producing a Hamilton circuit.

   (2 Marks)

ii) Show that there is a Hamilton path in H.

   (1 Mark)

iii) Let $v_1, v_2, \ldots, v_n$ be a Hamilton path in H. Show that $\deg(v_1) + \deg(v_n) > n$ and that there are at most $\deg(v_1)$ vertices not adjacent to v_n (including v_n itself).

   (2 Marks)

iv) Let S be the set of vertices preceding each vertex adjacent to v_1 in the Hamilton path. Show that S contains $\deg(v_1)$ vertices and $v_n \notin S$.

   (2 Marks)

v) Show that S contains a vertex v_k, which is adjacent to v_n, implying that there are edges connecting v_1 and v_{k+1} and v_k and v_n.

(1 Mark)

vi) Show that part (iii) implies that $v_1, v_2, \ldots, v_{k-1}, v_k, v_n, v_{n-1}, \ldots, v_{k+1}, v_1$ is a Hamilton circuit in G. Conclude from this contradiction that Ore's Theorem holds.

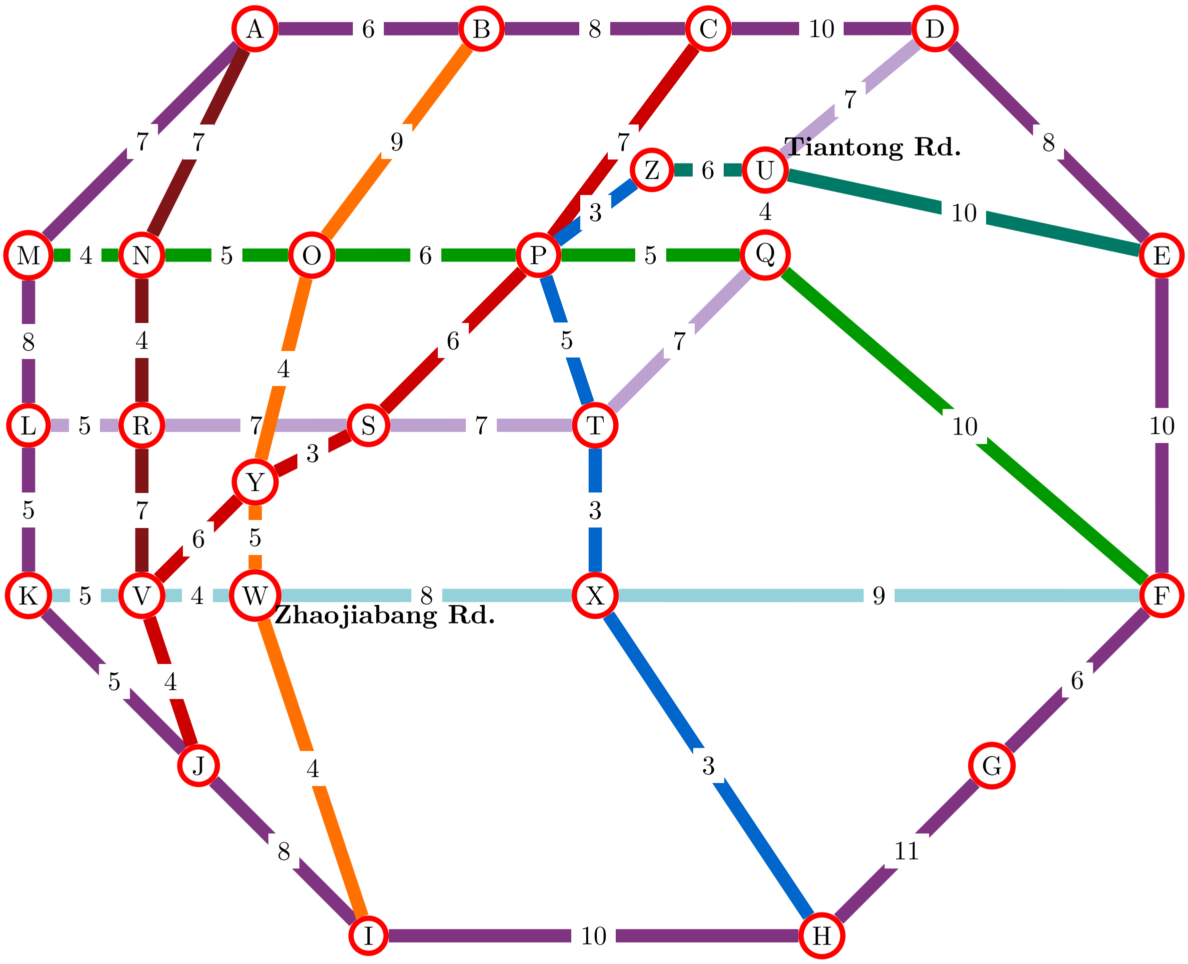
(1 Mark)

Exercise 10.6

The following graph shows a simplified map (ca. 2015) of the Shanghai metro within the inner circle, leaving out all stations that are not intersections of lines. Each edge is weighted with the travel time between these intersections.

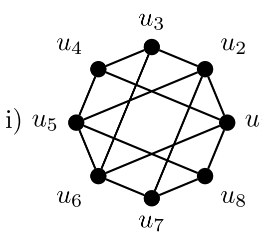
Use Dijkstras algorithm to find the shortest path between Tiantong Rd. station (vertex U) and Zhaojiabang Rd. station (vertex W). Give the distinguished set of vertices S_k and the labels of all vertices at every step of the algorithm.

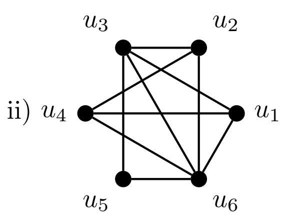
(5 Marks)

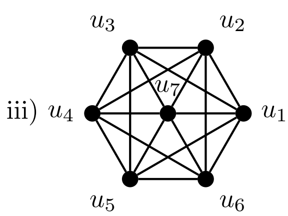


Exercise 10.7

Determine whether each given graph is planar. Either draw an isomorphic graph without crossing edges, or prove that the graph is non-planar.







(6 Marks)