VE203 Discrete Math

Spring 2022 — HW3 Solutions

March 30, 2022

Exercise 3.1

reflexive: Since $\bigcup \Pi = A$, for any $x \in A$, there exists $B \in \Pi$ such that $x \in B$.
Therefore, $xR_\Pi x \Rightarrow T$.

symmetric: $xR_\Pi y \Leftrightarrow (\exists B \in \Pi)(x \in B \wedge y \in B) \Leftrightarrow (\exists B \in \Pi)(y \in B \wedge x \in B) \Leftrightarrow yR_\Pi x$,
so $(xR_\Pi y) \wedge (yR_\Pi x) \Rightarrow \top$

transitive: $xR_\Pi y \Leftrightarrow (\exists B_1 \in \Pi)\,(x \in B_1 \wedge y \in B_1)$
$yR_\Pi z \Leftrightarrow (\exists B_2 \in \Pi)\,(y \in B_2 \wedge z \in B_2)$
Since $B_1 \cap B_2 \neq \phi$, $B_1 = B_2 = B$ according to the property of partition.
So $(\exists B \in \Pi)(x \in B \wedge z \in B)$, then $xR_\Pi y \wedge yR_\Pi z \Rightarrow xR_\Pi z$.

Exercise 3.2

(i) We derive Eq.(1) definition according to the definition.

$$\frac{(x+y+1)(x+y)}{2} + y = 333$$

where $x, y \in \mathbb{N}$.

Reduce Eq.(1)

$$x^2 + (2y+1)x + \left(y^2 + 3y - 666\right) = 0$$

whose $\Delta x = -8y + 2665$ is a perfect square. Then we use Python to traverse y from 1 to 333 to find all possible pair(x,y).

We finally derive only one pair that satisfies all limitation factor.

$$x = 17, y = 8$$

(ii)

$$f(x,y,z) = \left(\binom{x+y+1}{2} + y + z + 1\right) + z$$

Exercise 3.3

i) (a) self-reflexive
Since $[a,b] = [a,b]$,
$[a,b] \leq_{\text{int}} [a,b]$

(b) anti-symmetric
Since both $[a,b]$ and $[c,d]$ are intervals,

$$a \leq b \text{ and } c \leq d$$

Then for

$$[a,b] \leq_{\text{int}} [c,d] \wedge \quad [c,d] \leq_{\text{int}} [a,b]$$

if $b \leq c \wedge d \leq a$ exist, we derive

$$a \leq b \leq c \leq d \text{ while } \quad d \leq a$$

contradiction $\not\ast$ Therefore, we can only deduce that

$$[a, b] = [c, d]$$

(c) transitive
Let $[a, b] \leq_{\text{int}} [c, d]$ and $[c, d] \leq_{\text{int}} [e, f]$. - If any of the $\leq$ int is valid due to $=$, the transitivity is trivial. - When $b \leq c$ and $d \leq e$,

$$b \leq c \leq d \leq e$$

where $c \leq d$ since $[c, d]$ is an interval and hence

$$[a, b] \leq_{\text{int}} [c, d] \wedge [c, d] \leq_{\text{int}} [e, f] \Rightarrow [a, b] \leq_{\text{int}} [e, f]$$

ii) (a) self-reflexive
It is trivial that $(x_1, \ldots, x_n) \leq (x_1, \ldots, x_n)$.

(b) anti-symmetric
If $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$, and

$$\begin{aligned} (x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \quad &\Rightarrow \quad \ldots x_{k1} < y_{k1} \\ (y_1, \ldots, y_n) \leq (x_1, \ldots, x_n) \quad &\Rightarrow \quad \ldots y_{k2} < y_{k3} \end{aligned}$$

Whatever the relative value of k_1 and k_2, there is contradiction. Therefore

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \wedge (y_1, \ldots, y_n) \leq (x_1, \ldots, x_n) \quad \Rightarrow \quad (x_1, \ldots, x_n) = (y_1, \ldots, y_n)$$

(c) transitive
If there is a "=", then it is trivial.
Let the k-value for x and y are k_1 and k_2 correspondingly. Then let $k_3 = \min\{k_1, k_2\}$.
Then the disjunction (V) of the two following formula must be true.

$$\begin{aligned} x_{k3} < y_{k3} = z_{k3} \\ x_{k3} = y_{k3} < z_{k3} \end{aligned}$$

Namely

$$(x_1, \ldots, x_n) \leq (z_1, \ldots, z_n)$$

iii) We know

$$\begin{cases} \forall a \in P, a \leq_{\text{P}} \\ a \leq_{\text{P}} b \wedge b \leq_{\text{P}} a \Rightarrow a = b \\ a \leq_{\text{P}} b \wedge b \leq_{\text{P}} c \Rightarrow a \leq_{\text{P}} c \end{cases}$$

Then, for $\left(P, \leq_{\text{p}^d}\right)$

$$a \leq_{\text{p}^d} a$$

since $a \leq_{\text{P}} a$.

$$a \leq_{\text{p}^d} b \wedge b \leq_{\text{p}^d} a \Leftrightarrow b \leq_{\text{p}} a \wedge a \leq_{\text{P}} b \Rightarrow a = b$$

Additionally

$$(a, b) \wedge (b, c), \text{ i.e., } b \leq_{\text{p}^d} a \wedge a \leq_{\text{p}^d} b \Rightarrow c \leq_{\text{p}^d} a := (a, c)$$

Exercise 3.4

We define a function $\{a_1, a_2, \ldots, a_{sr+1}\} \to \{0, 1, \ldots, n\}$ with $a_i \mapsto (x_i, y_i)$, where x_i is the number of elements after a_i that are not divisible by a_i, and y_i is the length of the longest sequence of elements, each dividing the following one. Then for any $i, j \in \{1, 2, \ldots, n\}$, we have $y_i > y_j$ when $a_j \mid a_i$, or $x_i > x_j$ when $a_i \nmid a_j$. The rest follows from the Pigeonhole Principle.

Exercise 3.5

For the case when $w(P-\{m\}) \neq k-1$, there may not exist any $i_0 \in \{1, \ldots, w(P-\{m\}\}$, such that m is an upper bound of C_{i_0}. We'll give an counter example using the following figure.

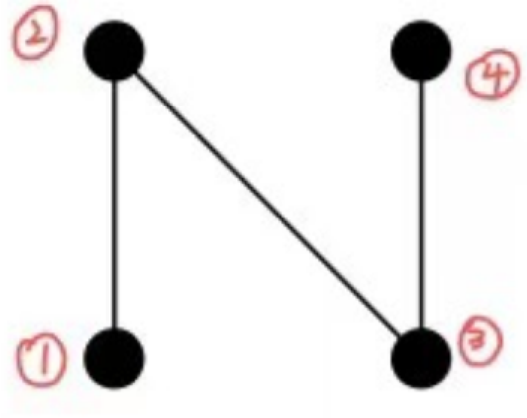
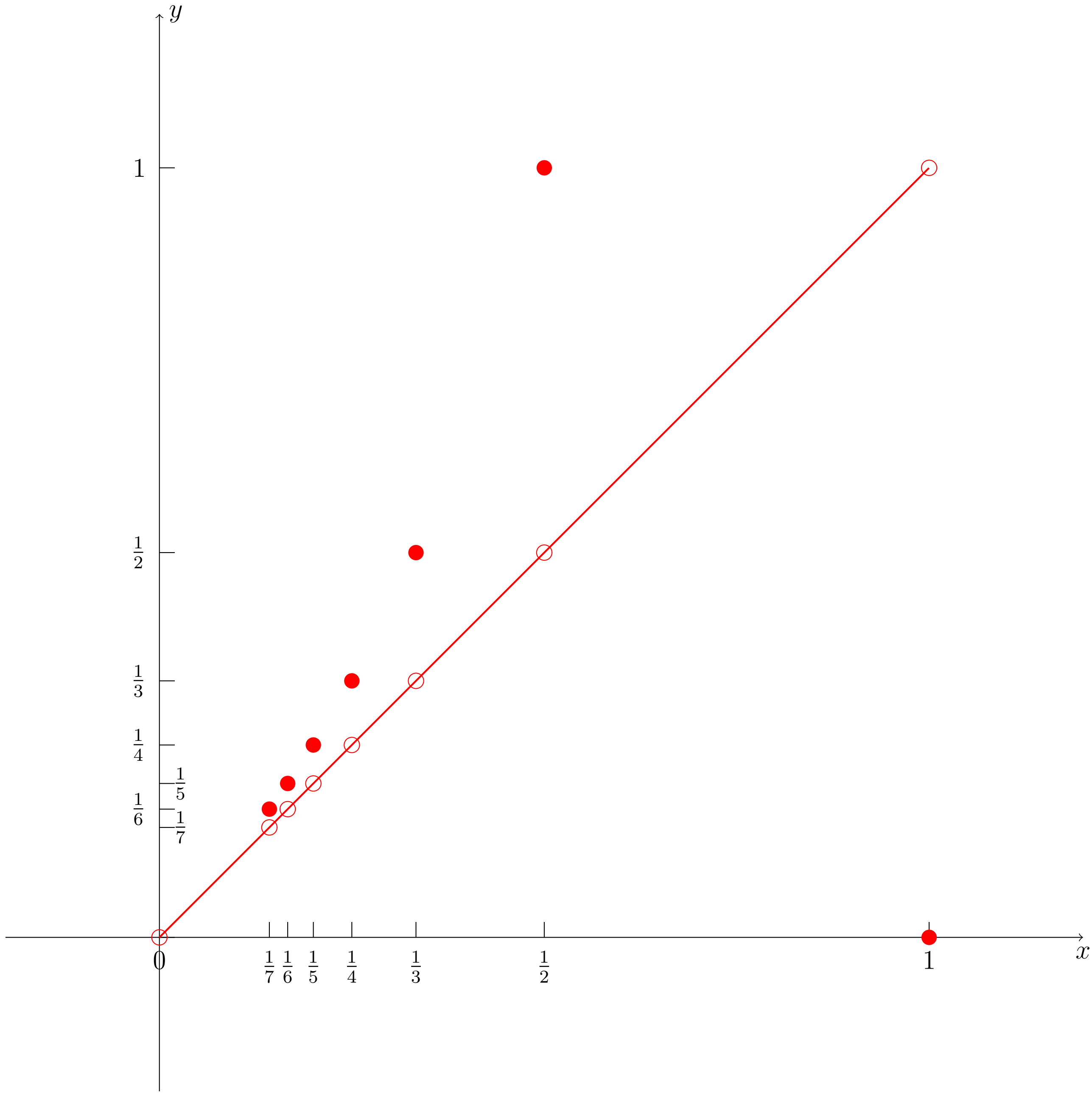


Figure 1: problem 3.5

Exercise 3.6

$$f(x) = \begin{cases} 0 & x = 1 \\ \frac{1}{\frac{1}{x}-1} & x = \frac{1}{n}(n \in N^+) \\ x & \text{otherwise} \end{cases}$$



Exercise 3.7

(i) (1) injective

Suppose that there are two functions $f(A_1, B_1) = C, f(A_2, B_2) = D$, in which $C = c_1c_2c_3 \ldots c_n \ldots c_{2n}$. and $D = d_1d_2d_3 \ldots d_n \ldots d_{2n}$. If we let $C = D$, which means $c_1c_2c_3 \ldots c_n \ldots c_{2n}.. = d_1d_2d_3 \ldots d_n \ldots d_{2n}..$, then we have $c_1 = d_1, c_2 = d_2 \ldots c_{2n} = d_{2n} \ldots$ Since $A_1 = c_1c_3 \ldots c_{2n+1} \ldots, A_2 = d_1d_3 \ldots d_{2n+1} \ldots$, we can deduct that $A_1 = A_2$, similarly $B_1 = B_2$. Thus the function is injective.

(2) surjective

For every result of the function C, it can be written as $C = c_1c_2c_3 \ldots c_n \ldots c_{2n}.\{0,1\}^{\mathbb{N}}$. Then we can find $A = c_1c_3 \ldots c_{2n+1} \ldots, B = c_2c_4 \ldots c_{2n} \ldots$, that $f(A, B) = C$. Thus the function is surjective. In this way, the function is a bijection.

(ii) For every element A $\in \{0,1\}^N$, since $\{0,1\}^N$ is a set of countably infinite sequence of 0 and 1 , it can be written as $A = a_1a_2a_3 \ldots a_n$, where $a_i \in \{0,1\}$. As is proved in (i), it is a bijection.

(iii) (1) injective

Similar to (i), let $h(R_1, S_1) = A, h(R_2, S_2) = B$, where $A = 0.r_0s_0r_1s_1 \ldots r_ns_n \ldots, B = 0.p_0q_0p_1q_1 \ldots p_nq_n \ldots$ If $A = B$ then $0.r_0s_0r_1s_1 \ldots r_ns_n \ldots = 0.p_0q_0p_1q_1 \ldots p_nq_n \ldots$, then $r_0 = p_0, s_0 = q_0, r_n = p_n, s_n = q_n$. Since $R_1 = 0.r_0r_1 \ldots r_n \ldots, R_2 = 0.p_0p_1 \ldots p_n \ldots$, we can deduct that $R_1 = R_2$. Similarly $S_1 = S_2$. Thus the function is injective.

(2) not surjective

Let $h(A, B) = C$, where $C = 0.9191919191 \ldots (r_i = 9, s_i = 1)$. Since $A = 0.999999 \ldots$ does not exist, it is not surjective.

(iv) (1) injective

almost the same as proved in (iii)

(2) not surjective

Let $h(A, B) = C$, where $C = 0.9190909090 \ldots (r_i = 9, s_0 = 1, s_i = 0 \text{ for } i > 0)$. $B = 0.10000\ldots$ does not exist since it would be written as 0.0999999.., so it is not surjective.

(v) Let $f : (0, 1) \times (0, 1) \to (0, 1) \Leftrightarrow I \times I \to I$, and $g : (0, 1) \to (0, 1) \times (0, 1) \Leftrightarrow I \to I \times I$.

Consider $x = (x_0x_1x_2x_3 \cdots)$ and $y = (y_0y_1y_2y_3 \cdots)$.

By the definition in (i), $f((x, y)) = z = x_0y_0x_1y_1x_2y_2x_3y_3 \cdots x_ny_n \ldots$. If $(x, y) \neq (x', y')$, then their decimal expansions are also $z \neq z'$.

Therefore, f is injective. Also, there exists natural injection from $I \to I \times I$. Therefore, g is injective. By injecion of f and g, there exists bijection between $(0, 1) \times (0, 1) \to (0, 1)$.

Exercise 3.8

(i) linear poset: (1)

(ii) (iii)

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
width	3	2	3	3	2	2	3	3	2	2	4	1	2	2	2	2
height	2	3	2	2	2	2	2	2	3	3	1	4	3	2	3	3