Ve203 Discrete Mathematics (Spring 2021)

Assignment 4

Date Due: 21:00 PM, Tuesday, Mar. 29, 2021
This assignment has a total of **(50 points)**.

Exercise 4.1 (2 pts)
Given $a, b, c \in \mathbb{N} \setminus \{0\}$, show that $a \mid bc$ iff $\left.\dfrac{a}{\gcd(a,b)}\ \right|\ c$.

Exercise 4.2 (4 pts)
Show that

(i) `(2 pts)` There exist infinitely many primes of the form $3n+2$, $n \in \mathbb{N}$.

(ii) `(2 pts)` There exist infinitely many primes of the form $6n+5$, $n \in \mathbb{N}$.

Exercise 4.3 (4 pts)
The numbers $F_n = 2^{2^n} + 1$ are called the *Fermat numbers*.

(i) `(2 pts)` Show that $\gcd(F_n, F_{n+1}) = 1$, $n \in \mathbb{N}$.

(ii) `(2 pts)` Use (i) to show that there are infinitely many primes.

Exercise 4.4 (2 pts)
Show that

(i) `(1 pt)` If a is even and b is odd, then $\gcd(a,b) = \gcd(a/2, b)$.

(ii) `(1 pt)` If both a and b are even, then $\gcd(a,b) = 2\gcd(a/2, b/2)$.

Exercise 4.5 (4 pts)
Find all $x, y \in \mathbb{Z}$ such that

(i) `(2 pts)` $56x + 72y = 39$,

(ii) `(2 pts)` $84x - 439y = 156$.

Exercise 4.6 (2 pts)
Given a group $G = (S, \cdot)$, where S is the underlying set, and $\cdot$ is the groups law. Define a new function

$$\begin{aligned} \boxtimes : S \times S &\to S \\ (a,b) &\mapsto a \boxtimes b := b \cdot a \end{aligned}$$

Show that $(S, \boxtimes)$ is a group.

Exercise 4.7 (4 pts)
Given a group G, show that

(i) `(2 pts)` If the order of every nonidentity element of G is 2, then G is Abelian.

(ii) `(2 pts)` If $a, b \in G$, then $|ab| = |ba|$, i.e., ab and ba have the same order.

Exercise 4.8 (6 pts)
Given $f : (\mathbb{R}, +) \to (\mathbb{C} \setminus \{0\}, \times)$, $x \mapsto e^{ix}$.

(i) `(2 pts)` Show that f is a homomorphism.

(ii) `(2 pts)` Find $\ker f$.

(iii) `(2 pts)` Find $\operatorname{im} f$.

Exercise 4.9 (4 pts)
Given groups G, G', and $f : G \to G'$ a surjective homomorphism. Show that

(i) `(2 pts)` G' is cyclic if G is cyclic.

(ii) `(2 pts)` G' is abelian if G is abelian.

Exercise 4.10 (2 pts)
Given group G and a function $f : G \to G$, $x \mapsto x^{-1}$. Show that the following are equivalent,

(a) G is abelian.

(b) f is a homomorphism.

Exercise 4.11 (2 pts)
Show that $\{1, (12)(34), (13)(24), (14)(23)\}$ is a subgroup of A_4.

Exercise 4.12 (2 pts)
Given group G with $|G|$ even, show that G contains an element of order 2.

Exercise 4.13 (6 pts)

(i) `(2 pts)` Show that the normal subgroup property is not transitive.

(ii) `(2 pts)` Show that a subgroup of index 2 is normal.

(iii) `(2 pts)` Show that a subgroup of index 3 is not necessarily normal.

Exercise 4.14 (4 pts)
Let G be a group of order p^2, with p prime. Show that

(i) `(2 pts)` G has at least one subgroup of order p.

(ii) `(2 pts)` If G contains only one subgroup of order p, then G is cyclic.

Exercise 4.15 (2 pts)
State a converse of Lagrange's theorem. If the statement is true, find a reference, otherwise provide a counterexample.