Modeling of a Trebuchet with a Sling and Counterweight

MEAM 211 Project 3, Phase III

April 4, 2007

Background

In Phase III, you are required to analyze the dynamics of a trebuchet with a sling and develop a dynamic model of the system that allows you to predict the range of the trebuchet. This handout provides information that will help you complete this exercise.

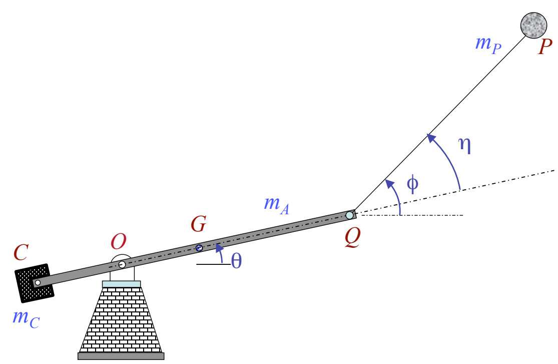

Figure 1: A trebuchet with the projectile in a sling (not shown).

Freebody Diagrams and Equations of Motion

You will derive the equations of motion for a trebuchet with a sling and fixed counterweight, pictured in Figure 1. You will make two simplifying assumptions. First, assume the projectile is small and can be modeled as a point mass. In other words, assume that its mass moment of inertia about the center of mass is neglible. Second, while the sling (more correctly the string) is in tension, you can model the sling as a massless rigid link that is attached to a point-mass projectile of mass m_P. Note that the arm with the pivot point O together with the sling constitute a double pendulum, with the arm having mass and the sling assumed to be massless except for the mass of the projectile.

You will use force balance and moment balance equations to obtain the equations of motion. Separate the trebuchet into two components for this analysis. The free body diagram of the arm (component 1) is shown in Figure 2. The free body diagram of the string with the projectile at the end (in the sling), component 2 of our system, is shown in Figure 3.

Report 1. Draw the inertia response diagrams for each component.

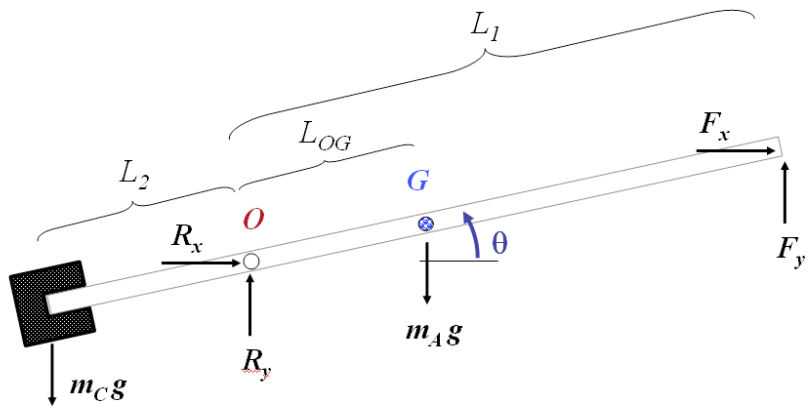

Figure 2: Free body diagram (FBD) of the arm (Component 1)

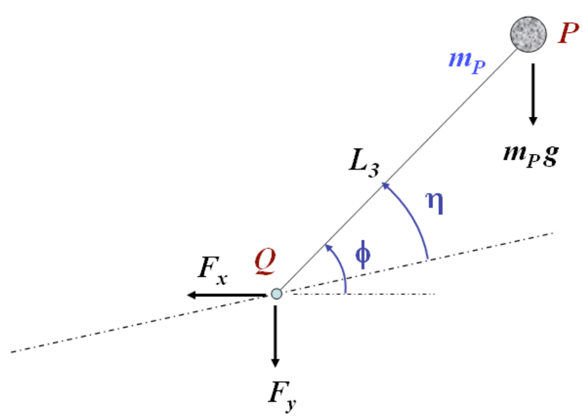

Figure 3: Free Body Diagram of Component 2 (the massless string with the projectile at the end). The center of mass of Component 2 is at P.

Report 2. Derive the force and moment balance equations (1-4) below.

Write the force and moment balance equations from both sets of FBDs and IRDs to derive the following equations.

$$\begin{aligned} F_y L_1 \cos(\theta) - F_x L_1 \sin(\theta) + m_C g L_2 \cos(\theta) & \\ -m_A g L_{OG} \cos(\theta) &= (I_G + m_A L_{OG}^2 + m_C L_2^2)\ddot{\theta} \quad (1) \\ -F_x &= m_P \ddot{x}_P \quad (2) \\ -F_y - m_P g &= m_P \ddot{y}_P \quad (3) \\ -F_x L_3 \sin(\phi) + F_y L_3 \cos(\phi) &= 0 \quad (4) \end{aligned}$$

Report 3. Acceleration Analysis

From the geometry of Figure 1, using point O as the origin of a Cartesian coordinate system, derive the following equations relating the accelerations of point P $(\ddot{x}_P, \ddot{y}_P)$ to the position (θ, ϕ), velocity ($\dot{\theta}$, $\dot{\phi}$) and accelerations ($\ddot{\theta}$, $\ddot{\phi}$) of the trebuchet:

$$\ddot{x}_P = -L_1(\dot{\theta}^2 \cos(\theta) + \ddot{\theta}\sin(\theta)) - L_3(\dot{\phi}^2 \cos(\phi) + \ddot{\phi}\sin(\phi)) \quad (5)$$

$$\ddot{y}_P = L_1(\ddot{\theta}\cos(\theta) - \dot{\theta}^2 \sin(\theta)) + L_3(\ddot{\phi}\cos(\phi) - \dot{\phi}^2 \sin(\phi)) \quad (6)$$

Equations of motion for the trebuchet

You now will use equations (5 - 6) with equations (1 - 4) to derive two equations of motion.
First, substitute $\ddot{x}_P$ and $\ddot{y}_P$ into equations (2) and (3) and solve for F_x and F_y:

$$F_x = \ldots \quad (7)$$

$$F_y = \ldots \quad (8)$$

Report 4. Derive the two equations of motion below.

Substitute the resulting expressions for F_x and F_y into equations (1) and (4) to get:

$$\begin{aligned} (I_G + m_P L_{OG}^2 + m_C L_2^2 + m_P L_1^2)\ddot{\theta} &+ m_P L_1 L_3(\ddot{\phi}\cos(\theta - \phi) - \dot{\phi}^2 \sin(\phi - \theta)) \\ &- g\cos(\theta)(m_C L_2 - m_P L_1 - m_A L_{OG}) = 0 \quad (9) \\ m_P L_3^2 \ddot{\phi} + m_P L_1 L_3(\ddot{\theta}\cos(\theta - \phi) &+ \dot{\theta}^2 \sin(\phi - \theta)) + g m_P L_3 \cos(\phi) = 0 \quad (10) \end{aligned}$$

Non-dimensionalized equations of motion

Let us define the effective inertia I_e, the relative angle η, and the nondimensional inertia γ as follows:

$$
\begin{aligned}
I_e &= I_G + m_P L_{OG}^2 + m_C L_2^2 + m_P L_1^2 && (11)\\
\eta &= \phi - \theta && (12)\\
\gamma &= \frac{(m_C L_2 - m_P L_1 - m_A L_{OG}) L_1}{I_e} && (13)
\end{aligned}
$$

Use the shorthand notation,

$$c_\phi = \cos(\phi),\ s_\phi = \sin(\phi),\quad s_\eta = \sin(\eta),\ c_\eta = \cos(\eta),\quad s_\theta = \sin(\theta),\ c_\theta = \cos(\theta).$$

Report 5. Rewrite equations (9) and (10) in the form:

$$
\begin{aligned}
I_e \ddot{\theta} + m_P L_1 L_3 c_\eta \ddot{\phi} - m_P L_1 L_3 s_\eta\, \dot{\phi}^2 &= \frac{\gamma g I_e}{L_1} c_\theta && (14)\\
m_P L_1 L_3 c_\eta \ddot{\theta} + m_P L_3^2 \ddot{\phi} + m_P L_1 L_3 s_\eta \dot{\theta}^2 &= -m_P g L_3 c_\phi && (15)
\end{aligned}
$$

You will now define three additional nondimensional quantities for the next part:

$$
\begin{aligned}
\mu &= \frac{m_P L_1 L_3}{I_e} && (16)\\
\rho &= \frac{L_3}{L_1} && (17)\\
\sigma &= \rho - \mu c_\eta^2 && (18)
\end{aligned}
$$

Report 6. Obtain the Solve for $\ddot{\theta}$ and $\ddot{\phi}$ to obtain the equations of motion:

$$
\begin{aligned}
\ddot{\theta} &= \mu s_\eta \left(\frac{c_\eta}{\sigma} \dot{\theta}^2 + (1 + \frac{\mu c_\eta^2}{\sigma}) \dot{\phi}^2 \right) + \frac{g}{L_1} \left(\gamma c_\theta + \frac{\mu c_\eta}{\sigma} (c_\phi + \gamma c_\eta c_\theta) \right) && (19)\\
\ddot{\phi} &= \frac{-1}{\sigma} \left(s_\eta (\dot{\theta}^2 + \mu c_\eta \dot{\phi}^2) + \frac{g}{L_1} (c_\phi + \gamma c_\eta c_\theta) \right) && (20)
\end{aligned}
$$

These are the two equations of motion that allow us to calculate the accelerations ($\ddot{\theta}$, $\ddot{\phi}$) for any given position (θ, ϕ) and velocity ($\dot{\theta}$, $\dot{\phi}$) of the trebuchet. The tension in the trebuchet can be calculated by observing that the force on the string at Q must act along the string.

Report 7. Find an expression for the tension as a function of the position (θ, ϕ), velocity ($\dot{\theta}$, $\dot{\phi}$) and acceleration ($\ddot{\theta}$, $\ddot{\phi}$).
The above equations of motion (19-20) are valid while the tension is positive. We assume that projectile leaves the sling, *i.e.*, release or separation occurs, when this tension goes to zero. After this point, the range can be calculated the way it was done in Phase 1.

Report 8. Use the simulator based on the equations above (provided to you — see website for matlab files) to develop several candidate designs and explore the effect of varying the nondimensional parameters, γ, μ and ρ on the range of the trebuchet.