Lecture 2: Kinematics of Particles

- Straight-Line motion [2.1]
- Analytical solutions for position/velocity [2.1]
- Solving equations of motion
  - Analytical solutions (1 – D review) [2.1]
  - Numerical solutions [2.1]
- Numerical integration [Appendix A, 2.1]
- Position vectors [2.2]
- Cartesian coordinate system [2.2]
- Velocity/Acceleration vectors [2.2]
- State vector, state space: Extensions to 2 and 3 dimensions

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Rectilinear motion

1-dimensional motion

- Position, $x(t)$
- Velocity, $v(t)$
- Acceleration, $a(t)$
- Jerk, $j(t)$
- Snap, $s(t)$

Two types of problems

- Given forces, find motion
- Given motion, find forces

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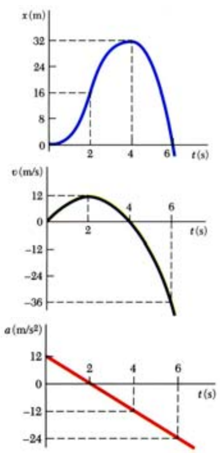
External motion is known, find force

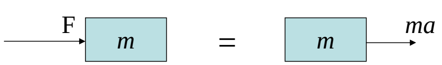


- Consider particle with motion given by

$$x = 6t^2 - t^3$$

$$v = \frac{dx}{dt} = 12t - 3t^2$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 12 - 6t$$



Free body diagram *Inertia response diagram*

$$F = ma = 12m - 6mt$$

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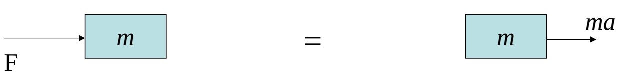
External forces are known, find motion



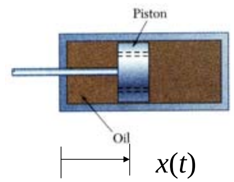
Free body diagram Inertia response diagram

Special cases

1. Acceleration is a given function of time, $a(t) = f(t)$
   - $a(t) = f(t) = \text{constant}$
2. Acceleration is a given function of position, $a(x) = f(x)$
3. Acceleration is a given function of velocity, $a(v) = f(v)$

Special case 3
$a(v) = f(v)$

Example: Viscous Damping
[BJ, 11.3]



Brake mechanism used to reduce gun recoil consists of piston attached to barrel moving in fixed cylinder filled with oil. As barrel recoils with initial velocity v_0, piston moves and oil is forced through orifices in piston, causing piston and cylinder to decelerate at rate proportional to their velocity.

Determine $v(t)$, $x(t)$, and $v(x)$.

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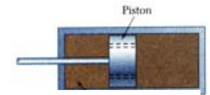
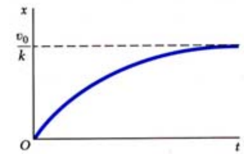
Solution



- Integrate $a = dv/dt = -kv$ to find $v(t)$.

$$a = \frac{dv}{dt} = -kv \qquad \int_{v_0}^{v(t)} \frac{dv}{v} = -k\int_0^t dt \qquad \ln\frac{v(t)}{v_0} = -kt$$

$$\boxed{v(t) = v_0 e^{-kt}}$$



- Integrate $v(t) = dx/dt$ to find $x(t)$.

$$v(t) = \frac{dx}{dt} = v_0 e^{-kt}$$

$$\int_0^{x(t)} dx = v_0 \int_0^t e^{-kt}\,dt \qquad x(t) = v_0\left[-\frac{1}{k}e^{-kt}\right]_0^t$$

$$\boxed{x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)}$$

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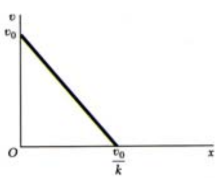
Solution (continued)



- Integrate $a = v\,dv/dx = -kv$ to find $v(x)$.

$$a = v\frac{dv}{dx} = -kv \qquad dv = -k\,dx \qquad \int_{v_0}^{v} dv = -k\int_0^x dx$$

$$v - v_0 = -kx$$

$$\boxed{v = v_0 - kx}$$

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Special case 2: force $=F(x)$, $a(x) = f(x)$

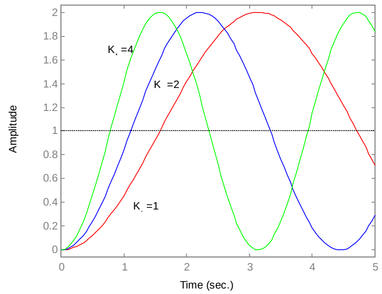
Many "passive" systems

- Simple pendulum
- Spring-mass system

Control systems for positioning

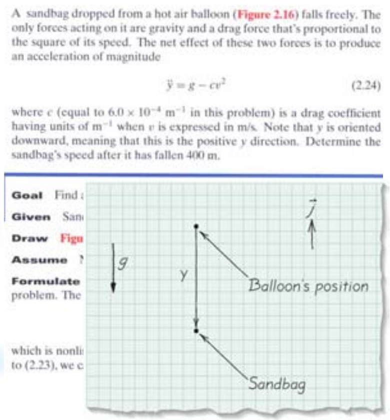
- Guidance systems for missiles
- Car
  - Imagine a car being accelerated (or decelerated) toward an intersection

$F(x) = kx$, Increasing k



What $a(t)$ is an arbitrary function of t?

Example 2.6 [TS]


A sandbag dropped from a hot air balloon (Figure 2.16) falls freely. The only forces acting on it are gravity and a drag force that's proportional to the square of its speed. The net effect of these two forces is to produce an acceleration of magnitude

$$\ddot{y} = g - c\dot{y}^2 \qquad (2.24)$$

where c (equal to 6.0×10^{-4} m^{-1} in this problem) is a drag coefficient having units of m^{-1} when v is expressed in m/s. Note that y is oriented downward, meaning that this is the positive y direction. Determine the sandbag's speed after it has fallen 400 m.

Numerical Integration

See Appendix A

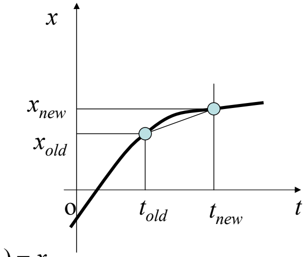
Example: Suppose velocity is known function of time

Given

- $v(t)$, initial condition $x(t_0)$

Find

- $x(t)$



Need to solve

$$\frac{dx}{dt} = v(t), \quad x(t_0) = x_0$$

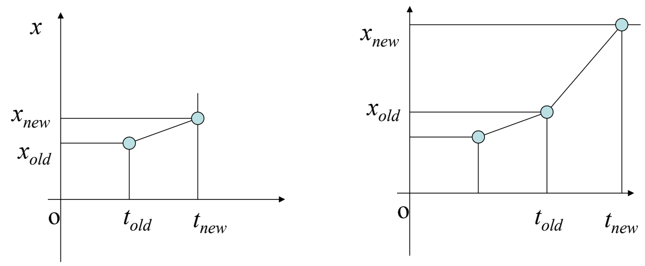
Basic idea:

Approximate $$\frac{dx}{dt} \approx \frac{x_{new} - x_{old}}{t_{new} - t_{old}}$$

Algorithm

In steps of δt seconds,

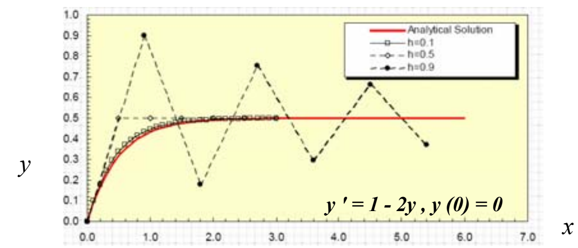
$$x_{new} = x_{old} + v(t_{old})\delta t$$



Numerical Integration of ODEs

$$\frac{dy}{dx} = f(x\ ,\ y)$$

- Initial value problem: Given the initial state at $y_0 = y(x_0)$, to compute the whole trajectory $y(x)$



Explicit Euler or Backward Euler (Appendix A)

$$\boxed{y_{i+1} = y_i + h \cdot f(x_i, y_i)}$$

Euler's method

- **Explicit**: evaluate derivative using values at the beginning of the time step
  - Not very accurate, requires small time steps for stability

$$y_{i+1} = y_i + h \cdot f(x_i, y_i) + O(h^2)$$

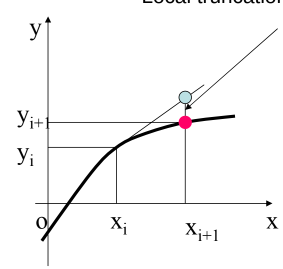
  - Global accuracy $O(h)$
- **Implicit:** Evaluate derivative using values at the end of the time step

$$y_{i+1} = y_i + h \cdot f(x_{i+1}, y_{i+1}) + O(h^2)$$

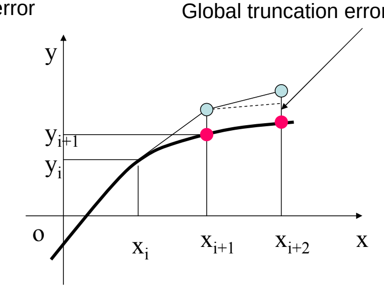
  - May require iteration since the answer depends upon what is calculated at the end.
  - Still not very accurate (global accuracy $O(h)$).
  - Unconditionally stable for all time step sizes.

Truncation errors

Local truncation error



Global truncation error



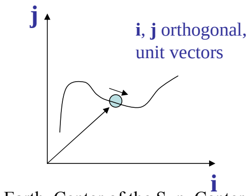
Stability

- A numerical method is ***stable*** if errors occurring at one stage of the process do not tend to be magnified at later stages.
- A numerical method is ***unstable*** if errors occurring at one stage of the process tend to be magnified at later stages.
- In general, the stability of a numerical scheme depends on the step size. Usually, large step sizes lead to unstable solutions.
- Implicit methods are in general more stable than explicit methods.

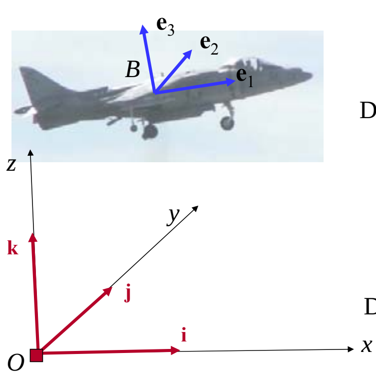
2-Dimensional motion

Need more powerful representation

- Positions require two coordinates (Cartesian coordinates)
  - $x(t)$
  - $y(t)$
- Need position vectors
  - $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$
- Need a reference frame
  - Fixed reference frames
    - Frame fixed to
      - Classroom, Earth, Center of the Earth, Center of the Sun, Center of the Universe, …
  - Moving reference frames
    - Frames fixed to moving bodies
- Need to be able to differentiate vectors
  - $\mathbf{v}(t)$, $\mathbf{a}(t)$, $\mathbf{j}(t)$, $\mathbf{s}(t)$, …



Differentiation of Vectors



$$\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{i} + u_z\,\mathbf{k}$$

Differentiate with respect to time:

$$\mathbf{u} = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2 + u_3\,\mathbf{e}_3$$

Differentiate with respect to time:

Transformations between unit vectors

- Understanding the relationship between sets of unit vectors is very important
  - Visualize
  - Write down the dot products

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
i	$\mathbf{i.e}_1$	$\mathbf{i.e}_2$	$\mathbf{i.e}_3$
j	$\mathbf{j.e}_1$	$\mathbf{j.e}_2$	$\mathbf{j.e}_3$
k	$\mathbf{k.e}_1$	$\mathbf{k.e}_2$	$\mathbf{k.e}_3$

Dot product of unit vectors = Cosine of angle between vectors