Analysis of forces in linkages
using conservation of power

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Introduction

Earlier, we discussed the kinematic analysis of planar mechanisms which involved the position analysis, followed by the velocity analysis and then the acceleration analysis. Assuming you have already done the kinematic analysis of a linkage, what can you learn about the forces acting on the linkage? In general, you cannot infer anything about the cause of the motion from observing the motion itself. However, under special cases you can leverage your understanding of the kinematics to make some statements about forces and moments acting on a linkage.

The basis of the discussion is the *principle of virtual work*, which is a very fundamental statement about static equilibrium of systems. It allows us to formulate the equations of static equilibrium of a mechanical system without having to draw free body diagrams of each member of the system and having to consider internal forces and moments. Its origins date back to Galileo (1594) and Bernoulli (1717), and its extensions to systems with dynamics are due to D'Alembert and Lagrange (1788). But rather than derive the principle from fundamentals, we will try to approach the principle from the intuition that comes with the principle of conservation of energy. To facilitate this informal derivation, we will first make a few assumptions.

Assumptions

Consider a mechanical system in which the following two assumptions can be made:

1. There is no energy loss due to friction or due to any other cause. Thus mechanical energy is conserved.

2. Inertial effects can be neglected which means either the masses, velocities and accelerations are such that all terms involving the product of mass and acceleration or the mass and the square of

the velocity are small compared to other forces (applied forces, gravitational forces) acting on the system.

At this point you might want to re-read Assumption 2. Even though the linkage or mechanism is in motion, we will assume that we can use static analysis to analyze any snapshot of the linkage.

Kinematic analysis

Recall that in most problems involving kinematic analysis, there are typically the following two steps:

1. First, we determine the values for all the joint positions (angles for revolute joints and linear displacements for prismatic joints). Typically, the values of some of the joint positions are known and the **position analysis** step involves writing down vector equations or closure equations to solve for the unknown joint positions.

2. The next step involves **velocity analysis**, which involves differentiating the closure equations to obtain equations that allow you to solve for the unknown velocities.

Assume that both these steps have been performed and we now know all the velocities for the system. Thus, for a single degree of freedom system like the four bar linkage, we assume we know all the velocities for a given crank rotational speed. Alternatively, for a slider crank linkage, we know the velocity of each point on the linkage as a function of the piston translational speed.

The key idea

Suppose there are many forces (later we will also include moments) acting on the system. These forces do work. With each force we can also define the power which is the rate at which work is being done. Because of the principle of conservation of energy, the total work done equals the change in kinetic energy. But if inertial effects are assumed to be negligible (Assumption 2), the change in kinetic energy is zero. In other words, the total work done must be zero. If we instead consider power, the total power associated with all the forces acting on the system is zero.

Formally, consider n forces $\mathbf{F}_1$, $\mathbf{F}_2$, ..., $\mathbf{F}_n$ acting on a mechanical system at points P_1, P_2, ..., P_n respectively. If the velocities of these points are known through kinematic analysis, then the conservation principle tells us:

$$\mathbf{F}_1.\mathbf{v}_{P_1} + \mathbf{F}_2.\mathbf{v}_{P_2} + \ldots \mathbf{F}_n.\mathbf{v}_{P_n} = 0 \tag{1}$$

Thus, in Figure 1,

$$\mathbf{F}_1.\mathbf{v}_C + \mathbf{F}_2.\mathbf{v}_P = 0 \tag{2}$$

If $\mathbf{F}_1$ is perpendicular to the crank and acting at its midpoint, this gives us:

$$-F_1\frac{r_2}{2}\dot{\theta}_2 - F_2\dot{r}_1 = 0 \tag{3}$$

But we know from our previous work that:

$$\dot{r}_1 = r_2\frac{sin(\theta_3 - \theta_2)}{cos(\theta_3)}\dot{\theta}_2 \tag{4}$$

Substituting (4) into (3) we get:

$$-F_1\frac{r_2}{2}\dot{\theta}_2 - F_2\, r_2\frac{sin(\theta_3 - \theta_2)}{cos(\theta_3)}\dot{\theta}_2 = 0$$

or,

$$F_1 = -2F_2\frac{sin(\theta_3 - \theta_2)}{cos(\theta_3)}.$$

For the example in Figure 2,

$$\mathbf{F}_1.\mathbf{v}_{P_1} + \mathbf{F}_2.\mathbf{v}_{P_2} + \mathbf{F}_3.\mathbf{v}_{P_3} + \mathbf{F}_4.\mathbf{v}_{P_4} = 0 \tag{5}$$
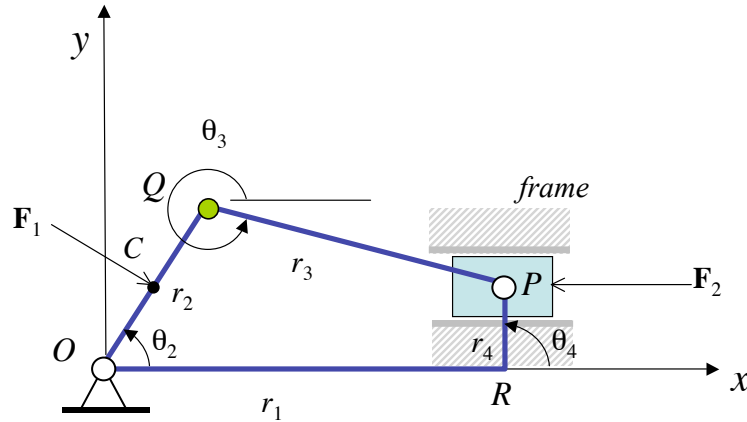

Figure 1: A slider crank linkage subject to two external forces

When moments are applied (instead of just a system of forces) in addition to forces, we must consider the work done by moments on rigid bodies. For example, in Figure 3, we need to consider the power generated by the resistance on the crank which is a moment, in addition to the work done by forces acting on the handle bars and the pedals. But this requires only a minor fix to Equation 1.
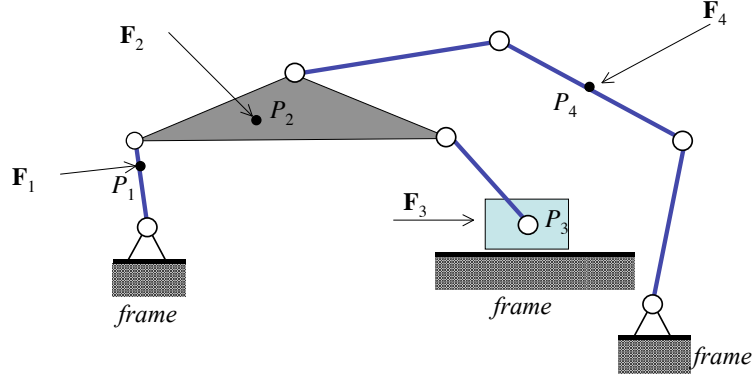
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Figure 2: A linkage subject to three forces. Can we determine if the system is in static equilibrium?

Consider n forces $\mathbf{F}_1$, $\mathbf{F}_2$, ..., $\mathbf{F}_n$ acting on a mechanical system at points P_1, P_2, ..., P_n respectively, and m moments $\mathbf{M}_1$, $\mathbf{M}_2$, ..., $\mathbf{M}_m$ acting on a mechanical system on rigid bodies B_1, B_2, ..., B_m respectively. Let the angular velocity of rigid body B_i be denoted by ω_i. If the velocities of these points and rigid bodies are known through kinematic analysis, then the conservation principle tells us:

$$\mathbf{F}_1.\mathbf{v}_{P_1} \; + \; \mathbf{F}_2.\mathbf{v}_{P_2} \; + \; \ldots \; \mathbf{F}_n.\mathbf{v}_{P_n} \; + \mathbf{M}_1.\omega_1 \; + \; \mathbf{M}_2.\omega_2 \; + \; \ldots \; \mathbf{M}_m.\omega_m \; = 0 \tag{6}$$

Thus in Figure 3, if the angular velocity of the (hidden) crank is ω, and the forces $\mathbf{F}_i$ act on points P_i, we know that

$$\mathbf{F}_1.\mathbf{v}_{P_1} \; + \; \mathbf{F}_2.\mathbf{v}_{P_2} \; + \; \mathbf{F}_3.\mathbf{v}_{P_3} \; + \mathbf{F}_4.\mathbf{v}_{P_4} \; + \mathbf{M}.\omega = 0 \tag{7}$$

Figure 3: A PRECOR Elliptical Fitness Machine. For a preset resistance, $\mathbf{M}$, what can we say about the forces that must be applied at the handle bars and at the feet?