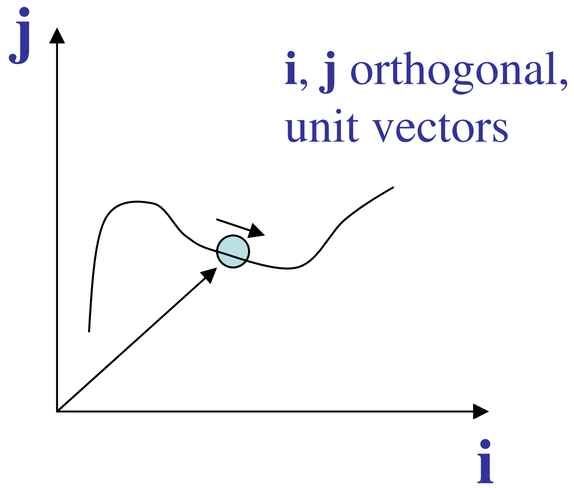
Lecture 3: Kinematics of Particles

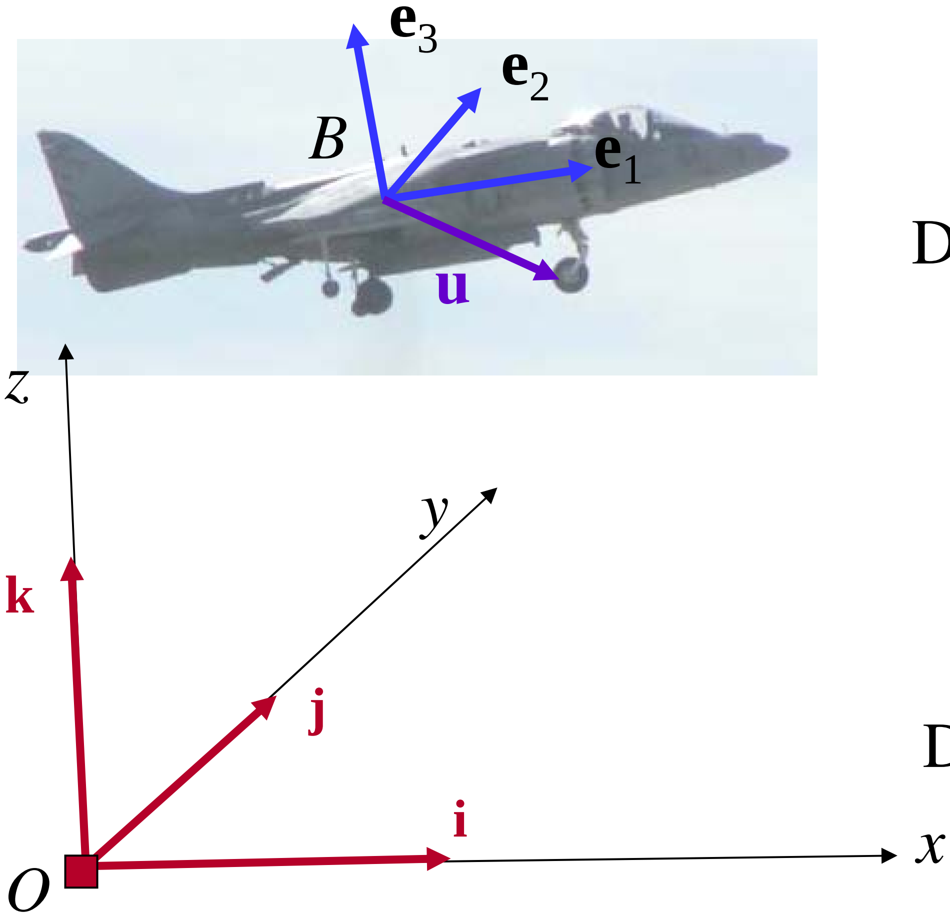
- Recap: Numerical integration [Appendix A, 2.1]
- 2 and 3-Dimensional Motion
- Resolving vectors in 3-Dimensions
- Differentiation of vectors leading into a review of
  - Polar coordinates [2.3]
  - Path coordinates [2.4]
- State vector, state space: Extensions to 2 and 3 dimensions
  - Reducing higher order differential equations to first order differential equations

2-Dimensional motion

- Positions require two coordinates (Cartesian coordinates)
  - $x(t)$
  - $y(t)$
- Need position vectors
  - $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$
- Need a reference frame
  - Fixed reference frames
    - Frame fixed to
      - Classroom, Earth, Center of the Earth, Center of the Sun, Center of the Universe, …
  - Moving reference frames
    - Frames fixed to moving bodies
- Need to be able to differentiate vectors
  - $\mathbf{v}(t)$, $\mathbf{a}(t)$, $\mathbf{j}(t)$, $\mathbf{s}(t)$, …



Differentiation of Vectors



$$\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{i} + u_z\,\mathbf{k}$$

Differentiate with respect to time:

$$\mathbf{u} = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2 + u_3\,\mathbf{e}_3$$

Differentiate with respect to time:

Transformations between unit vectors

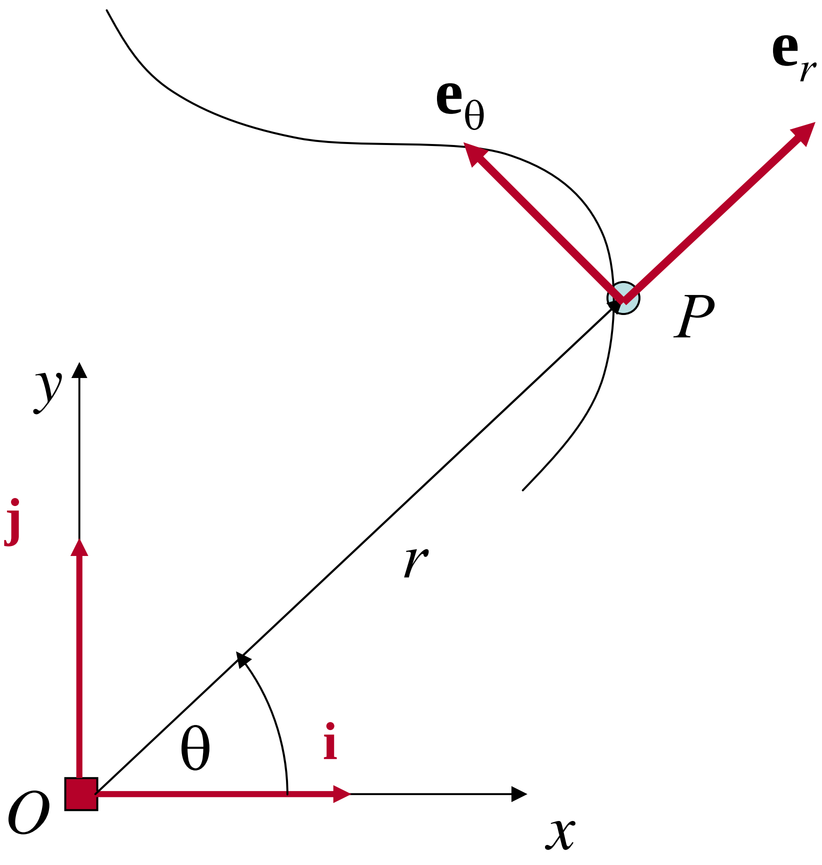
- Understanding the relationship between sets of unit vectors is very important
  - Visualize
  - Write down the dot products

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
i	$\mathbf{i}.\mathbf{e}_1$	$\mathbf{i}.\mathbf{e}_2$	$\mathbf{i}.\mathbf{e}_3$
j	$\mathbf{j}.\mathbf{e}_1$	$\mathbf{j}.\mathbf{e}_2$	$\mathbf{j}.\mathbf{e}_3$
k	$\mathbf{k}.\mathbf{e}_1$	$\mathbf{k}.\mathbf{e}_2$	$\mathbf{k}.\mathbf{e}_3$

Dot product of unit vectors =
Cosine of angle between vectors

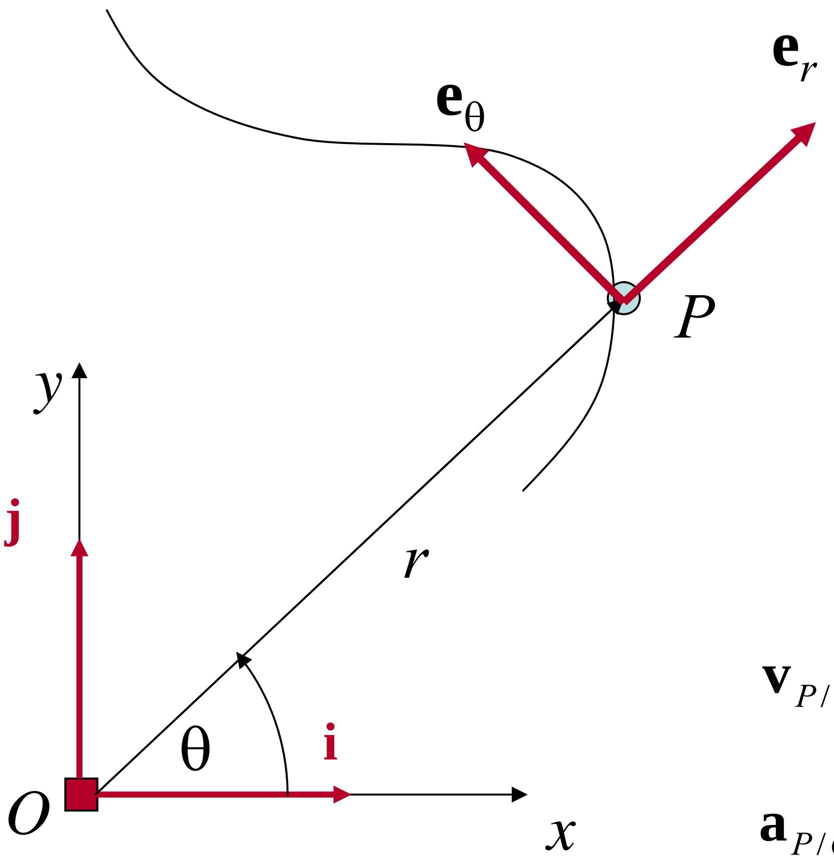
Polar coordinates

	$\mathbf{e}_r$	$\mathbf{e}_\theta$
$\mathbf{i}$	$\cos\theta$	$-\sin\theta$
$\mathbf{j}$	$\sin\theta$	$\cos\theta$



Polar coordinates

	$\mathbf{e}_r$	$\mathbf{e}_\theta$
$\mathbf{i}$	$\cos\theta$	$-\sin\theta$
$\mathbf{j}$	$\sin\theta$	$\cos\theta$

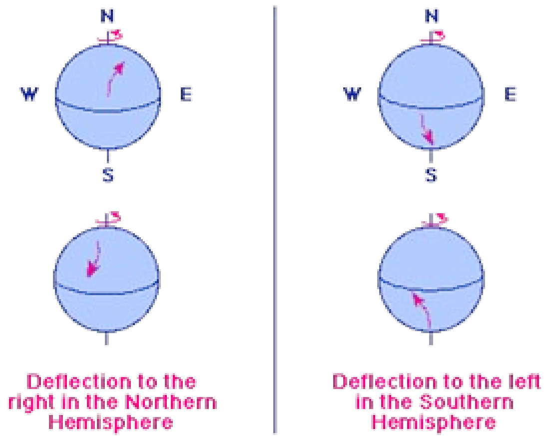


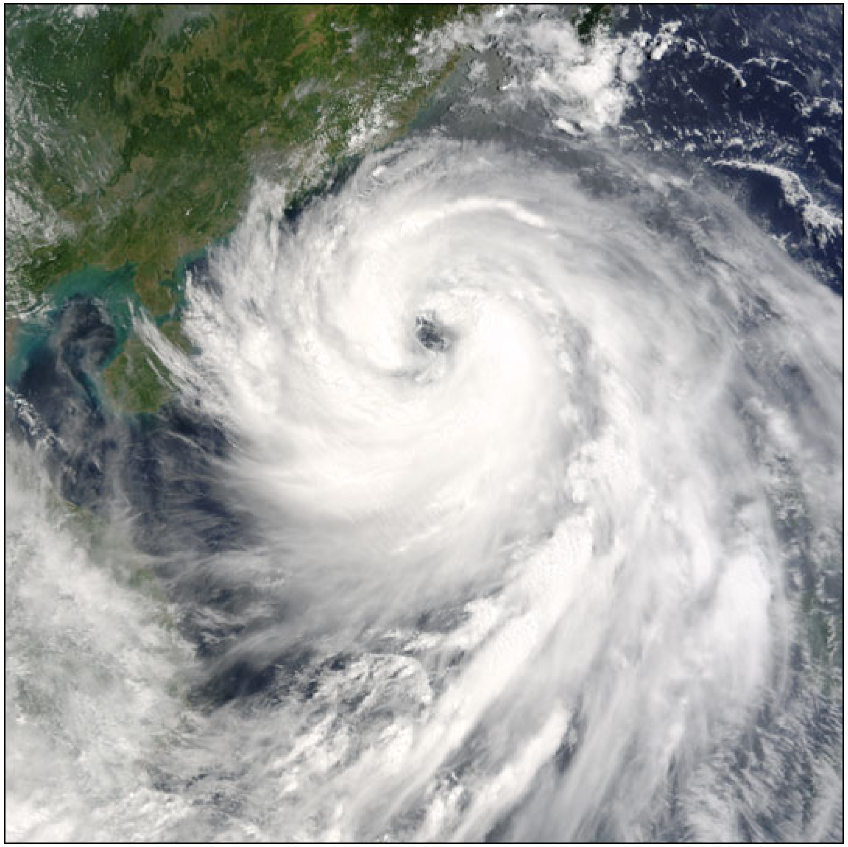
$$\frac{d\mathbf{e}_r}{dt} = -\sin\theta\dot{\theta}\ \ \mathbf{i} + \cos\theta\dot{\theta}\ \ \mathbf{j} = \mathbf{e}_\theta\dot{\theta}$$

$$\frac{d\mathbf{e}_\theta}{dt} = -\cos\theta\dot{\theta}\ \ \mathbf{i} + -\sin\theta\dot{\theta}\ \ \mathbf{j} = -\mathbf{e}_r\dot{\theta}$$

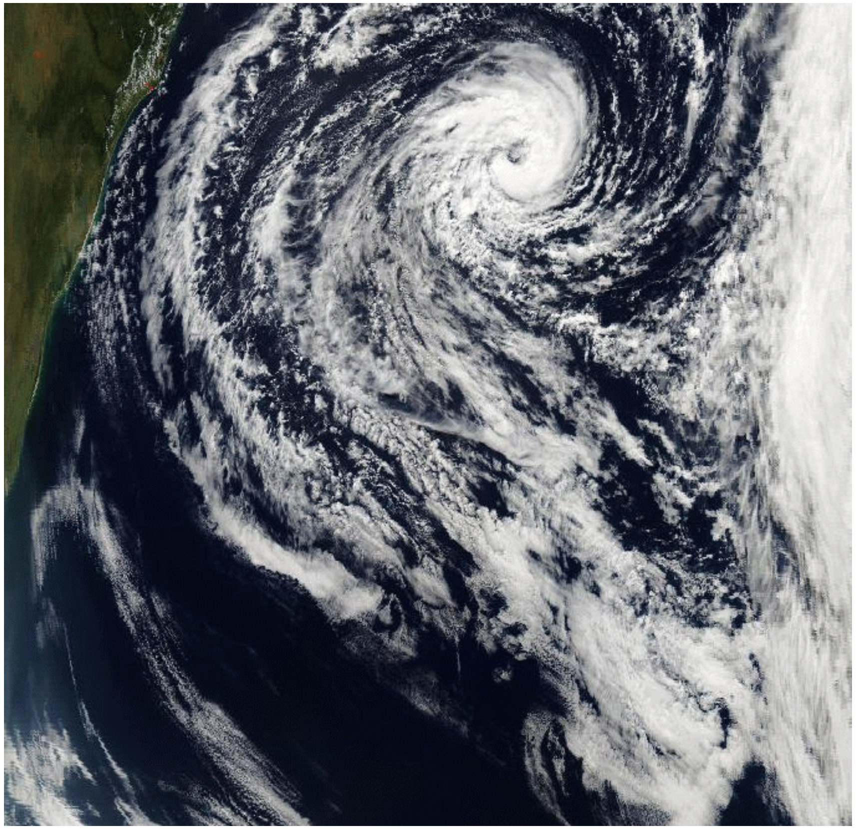
$$\mathbf{v}_{P/O} = \frac{d}{dt}(r\mathbf{e}_r) = \dot{r}\ \ \mathbf{e}_r + r\dot{\theta}\ \ \mathbf{e}_\theta$$

$$\mathbf{a}_{P/O} = (\ddot{r} - r\dot{\theta}^2)\ \ \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\ \ \mathbf{e}_\theta$$





Typhoon Imbudo, July 2003

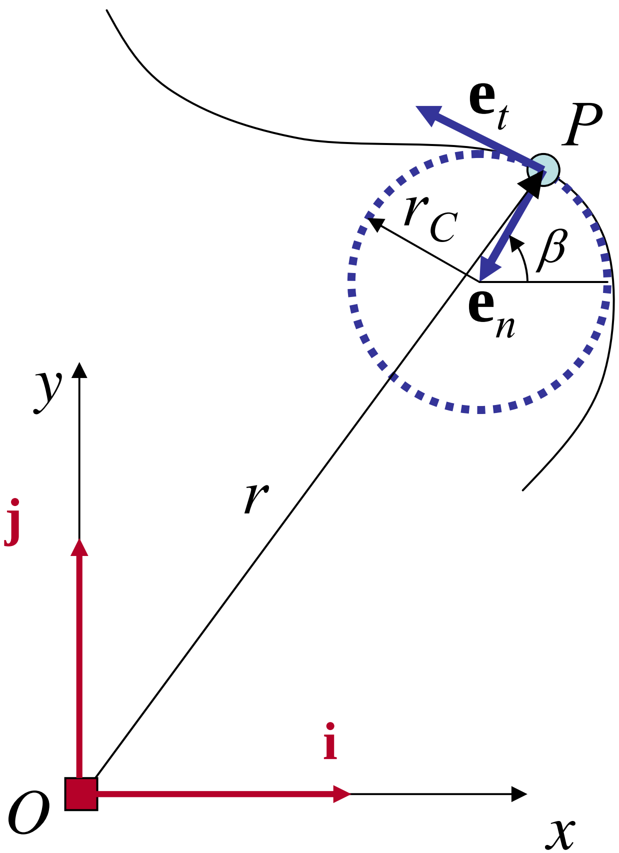


Gafilo, March 2004

- Hurricanes, anti-cyclones (later)

Path coordinates

	$\mathbf{e}_n$	$\mathbf{e}_t$
$\mathbf{i}$	$-\cos\beta$	$-\sin\beta$
$\mathbf{j}$	$-\sin\beta$	$\cos\beta$



$$\mathbf{v}_P = v\ \mathbf{e}_t = r_c\dot{\beta}\ \mathbf{e}_t$$

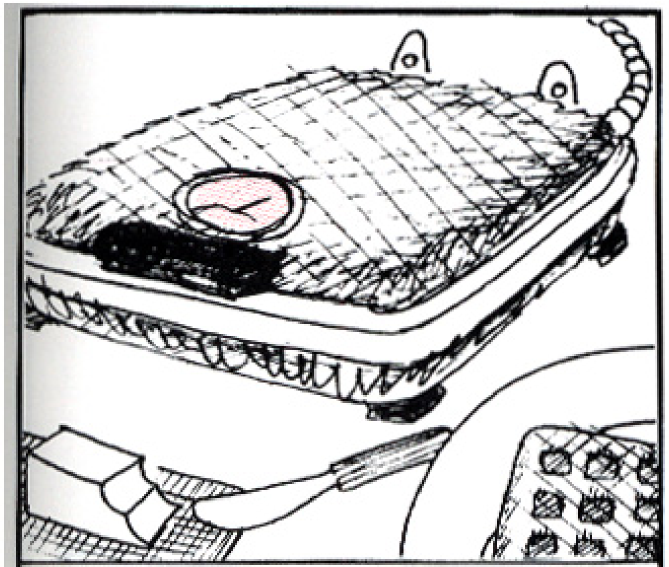
$$\mathbf{a}_p = \underbrace{r_c\dot{\beta}^2}_{\frac{v^2}{r_C}}\ \mathbf{e}_n + \underbrace{r_c\ddot{\beta}}_{\dot{v}}\ \mathbf{e}_t$$

Normal acceleration

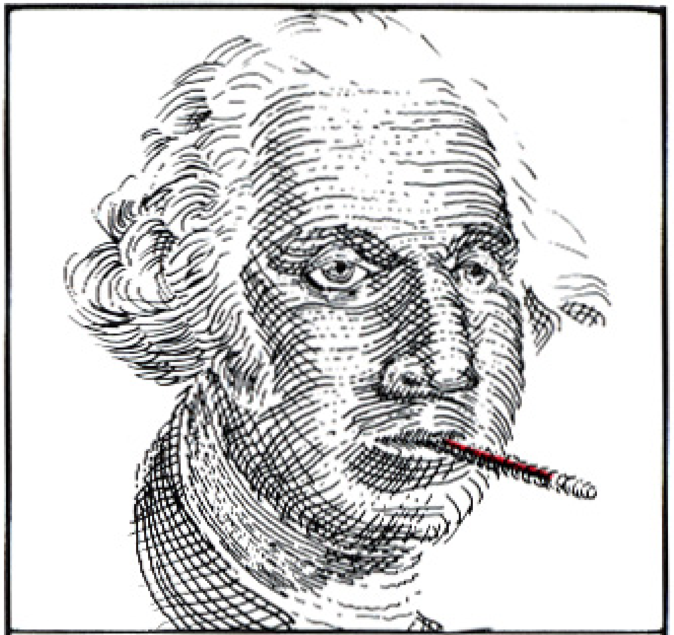
Tangential acceleration

State Space

State



1.1.1. The actual state of this waffle iron cannot be described completely by a single observable parameter, such as the temperature. But usually we find it convenient to pretend that it can. This pretense is an agreement, the *conventional interpretation*, within the modeling process. It is justified by its usefulness in describing the behavior of the device.



1.1.2. The correlation between the internal state of a complex system, such as a mammal, and a single observed parameter may be very good or very bad, depending on the context. In the case of George Washington, the oral temperature correlates better with his health than his honesty.

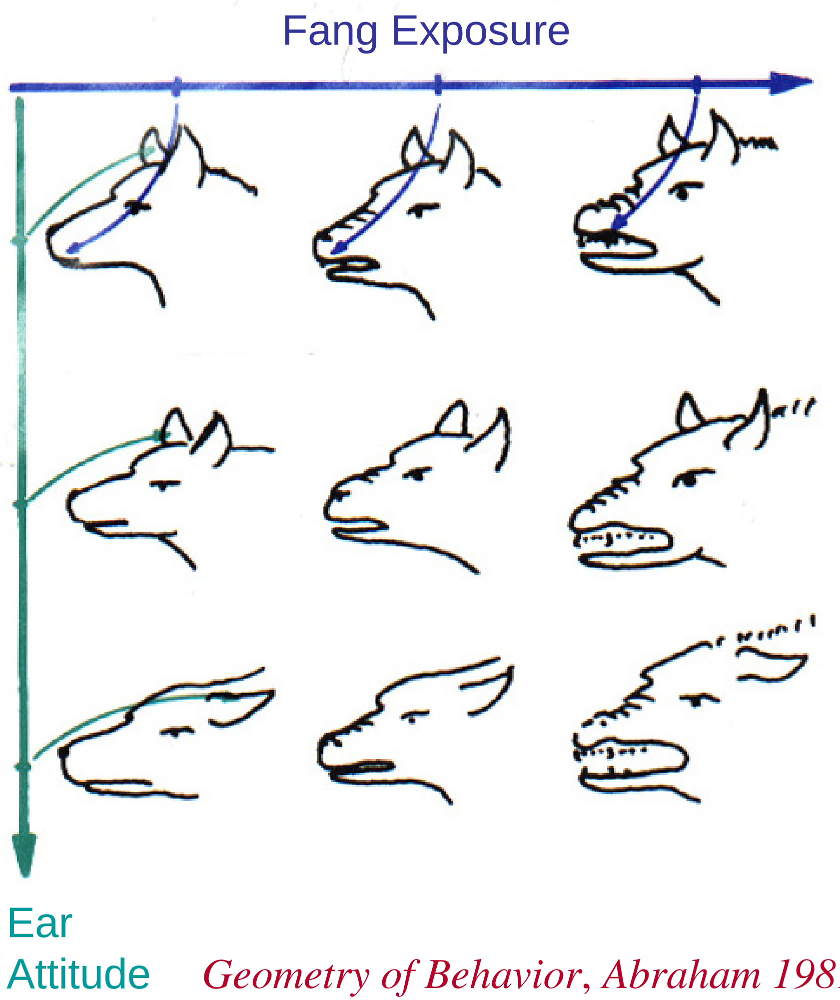
Geometry of Behavior, Abraham 1981

- Actual (internal) state of the system
- Mathematical model requires an idealized state
- The idealized state must be observable (in order for results to be practical)

Modeling: State

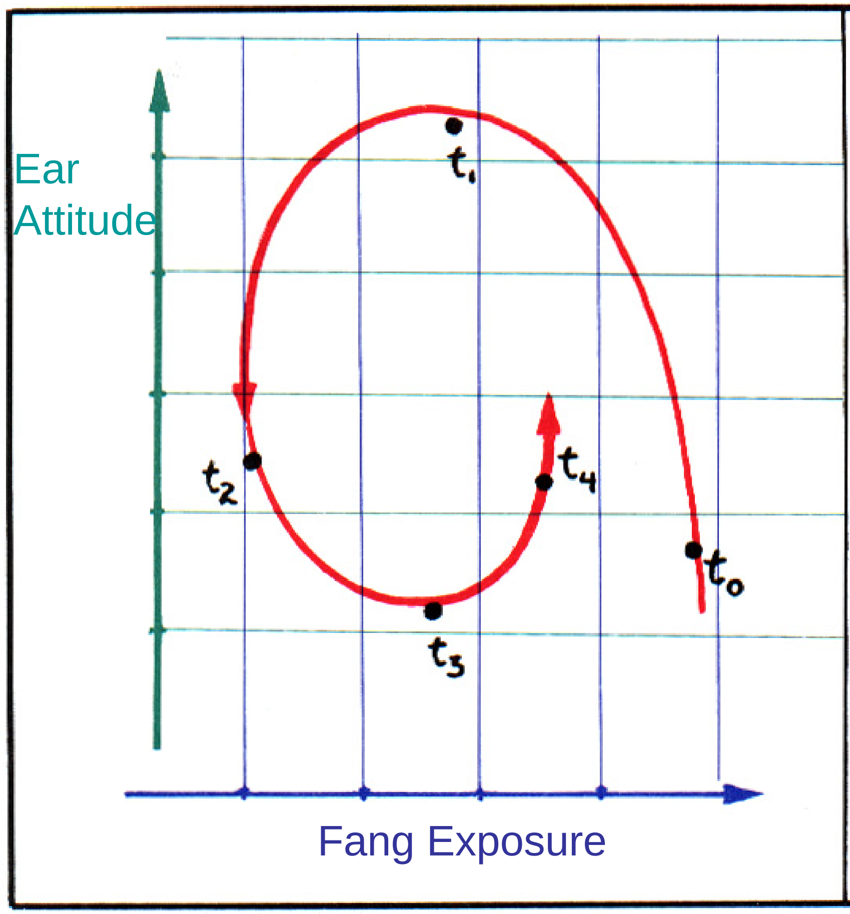
$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

1.1.6. In this modeling scheme of Konrad Lorenz and Christopher Zeeman,[1] two parameters are used for the emotional state of a dog. The two observed parameters are *ear attitude*, which correlates with the emotional state of fear, and *fang exposure*, corresponding to the degree of rage.



Geometry of Behavior, Abraham 1981

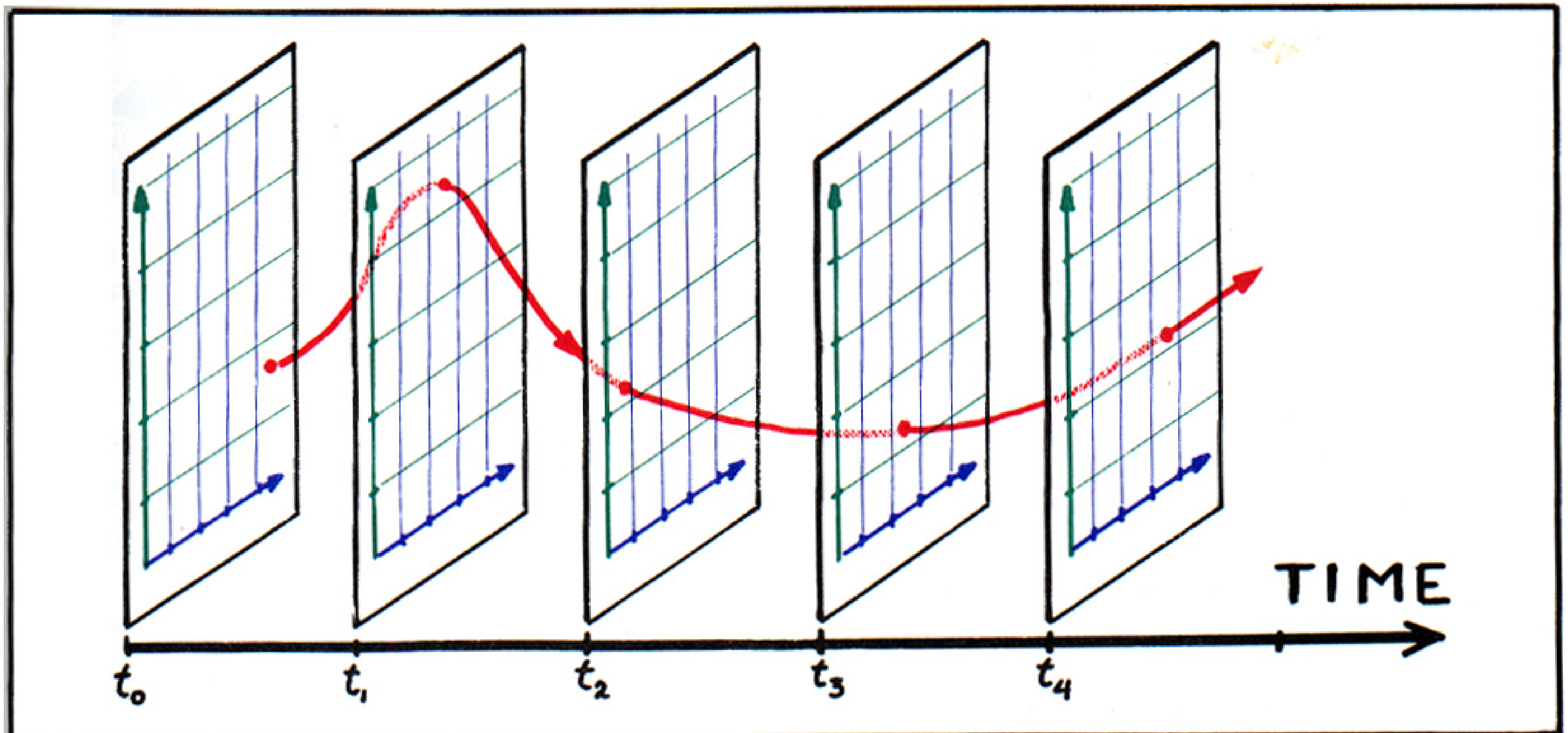
Modeling: State



Geometry of Behavior, Abraham 1981

1.1.9. For example, if the two parameters representing the emotional state of a dog, or the internal state of an electronic black box, are observed at successive times and recorded in the plane with labels, a trajectory of the model is obtained.

State Space and Time



1.1.10. Here the vertical plane represents the state space, and the horizontal axis represents the time of observation. The parameters observed at a given time are plotted in the vertical plane passing through the appropriate point on the time axis.

Geometry of Behavior, Abraham 1981

Integration in State Space

- ODE in one scalar variable
  - We know how to integrate

$$\dot{y} = f(x, y), \;\; y(x_0) = y_0$$

or

$$\dot{x} = f(x, t), \;\; x(t_0) = x_0$$

State Space Notation

- Motivation
  - We know how to integrate

$$\dot{y} = f(x, y), \;\; y(x_0) = y_0$$

or

$$\dot{x} = f(x, t), \;\; x(t_0) = x_0$$

- Can we do 2nd or higher order differential equations?

$$m\ddot{x} = f(x, t) - c\dot{x}|\dot{x}|, \;\; x(t_0) = x_0, \; \dot{x}(t_0) = v_0$$

State Space Notation

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- Yes - n^{th} order ODEs can be converted into n first order ODEs

Converting 2nd order ODEs to 2 1st order ODEs

- Example

$$m\ddot{x} = f(x,t) - c\dot{x}|\dot{x}|, \;\; x(t_0) = x_0, \; \dot{x}(t_0) = v_0$$

Converting 2nd order ODEs to 2 1st order ODEs

- Example

$$m\ddot{x} = f(x,t) - c\dot{x}|\dot{x}|, \;\; x(t_0) = x_0, \; \dot{x}(t_0) = v_0$$

- Define the **state vector**

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Converting 2nd order ODEs to 2 1st order ODEs

- Example

$$m\ddot{x} = f(x,t) - c\dot{x}|\dot{x}|, \quad x(t_0) = x_0, \ \dot{x}(t_0) = v_0$$

- Define the **state vector**

$$X = \left[\begin{array}{c} X_1 \\ X_2 \end{array}\right] = \left[\begin{array}{c} x \\ \dot{x} \end{array}\right]$$

- Write **state equations**

$$\dot{X} = \left[\begin{array}{c} \dot{x} \\ \ddot{x} \end{array}\right] = \left[\begin{array}{c} X_2 \\ \frac{f(X_1,t) - cX_2|X_2|}{m} \end{array}\right]$$

Converting 2nd order ODEs to 2 1st order ODEs

- Example

$$m\ddot{x} = f(x,t) - c\dot{x}|\dot{x}|, \quad x(t_0) = x_0, \; \dot{x}(t_0) = v_0$$

- Define the **state vector**

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

- Write **state equations**

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} X_2 \\ \frac{f(X_1,t) - cX_2|X_2|}{m} \end{bmatrix}$$

- With initial conditions

$$X(0) = \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

Solving Equations of Motions for Particles

Example

- Particle in 3-D subject to thrust, gravity and drag

$$\mathbf{F}_T(\mathbf{x},\mathbf{v}) + \mathbf{F}_D(\mathbf{v}) + \mathbf{F}_G = m\mathbf{a}$$

- State vector is 6x1

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

- 6 State Equations

$$\dot{X} = \begin{bmatrix} X_4 \\ X_5 \\ X_6 \\ \frac{F_{T,x}+F_{D,x}+F_{G,x}}{m} \\ \frac{F_{T,y}+F_{D,y}+F_{G,y}}{m} \\ \frac{F_{T,z}+F_{D,z}+F_{G,z}}{m} \end{bmatrix}$$

MATLAB Example

A ball is thrown upward against gravitational attraction and air resistance with an initial velocity of 30 meters/second. The air resistance opposes the velocity and is proportional to the square of the velocity. The acceleration is:

$$a = -g - cv^2 \operatorname{sign}(v)$$

where g = 9.81 meter/sec^2 and c = 0.001 1/meter. Solve for the position and velocity of the particle as a function of time through a six second time interval.

main.m

```
timeInterval=[0, 6];  % interval for integration
x0=[0; 30];           % initial position = 0, velocity = 30


intFn ('vertical', timeInterval, x0);
```

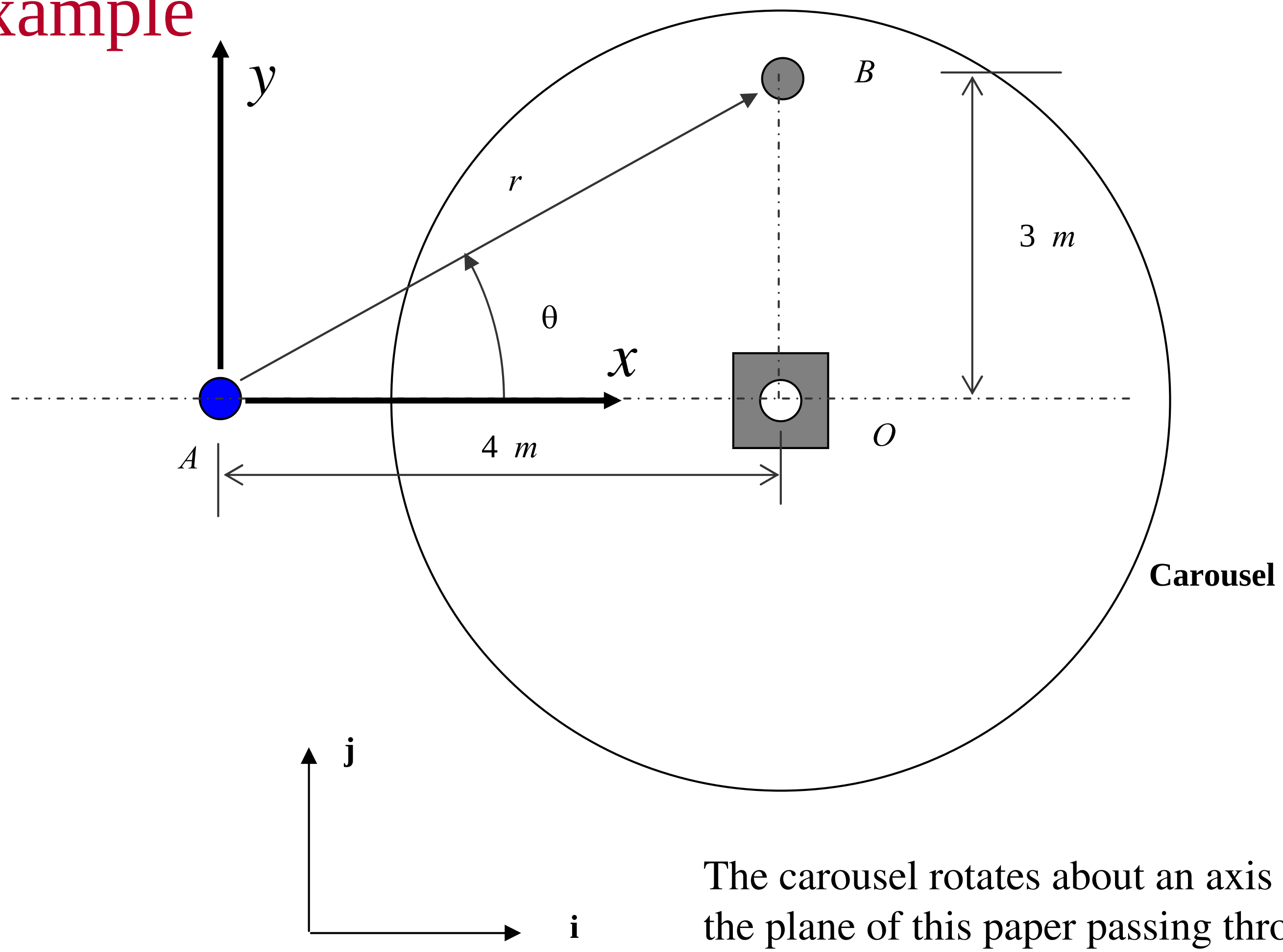
vertical.m

```
function xdot=vertical(t, x);

c=0.001;          % c=0.001 1/m
g=9.81;           % g=9.81 m/sec/sec

xdot1 = x(2);     % derivative of position = velocity
xdot2 = -g - c*x(2)*abs(x(2)); % acceleration

xdot=[xdot1; xdot2];
```

Example on particle kinematics

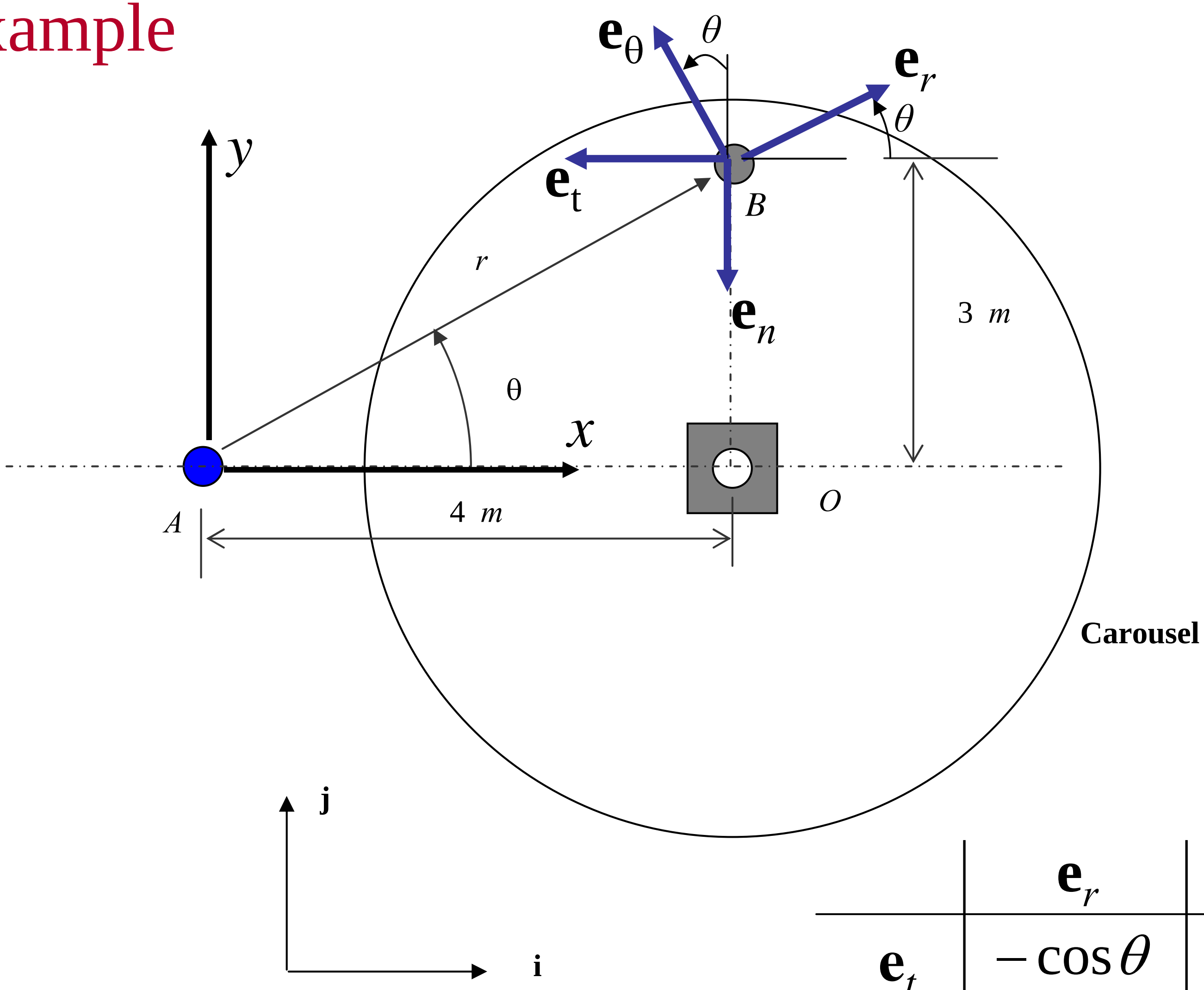
Path and Polar Coordinates

Example



The carousel rotates about an axis perpendicular to the plane of this paper passing through *O* at a constant rate of 3 rotations per minute. Your friend is standing (feet fixed to the carousel and not moving relative to the carousel) on the carousel at point *B*. You are standing at *A* with your feet on firm ground and at rest with respect to the Earth.

Example



	$\mathbf{e}_r$	$\mathbf{e}_\theta$
$\mathbf{e}_t$	$-\cos\theta$	$\sin\theta$
$\mathbf{e}_n$	$-\sin\theta$	$-\cos\theta$