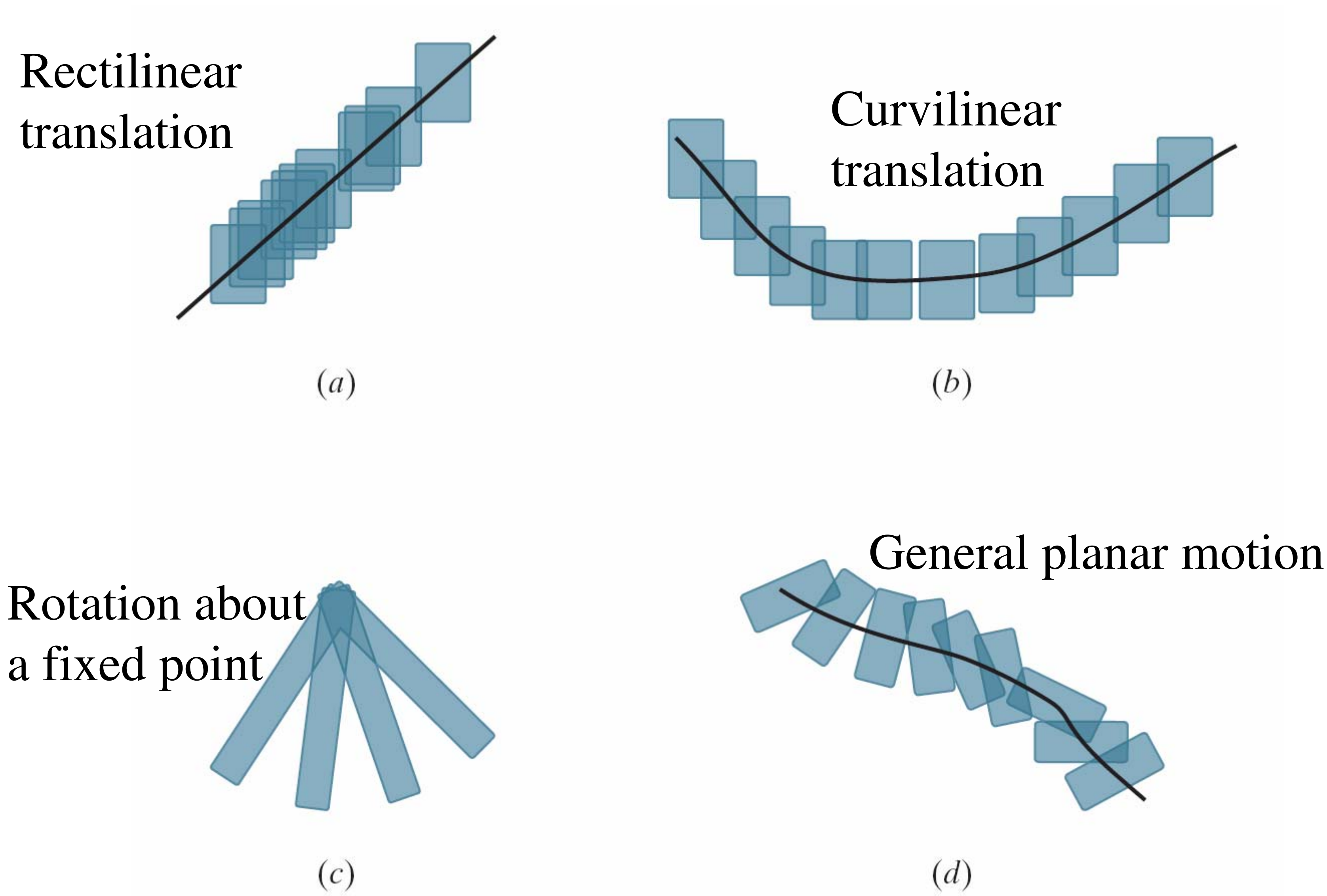
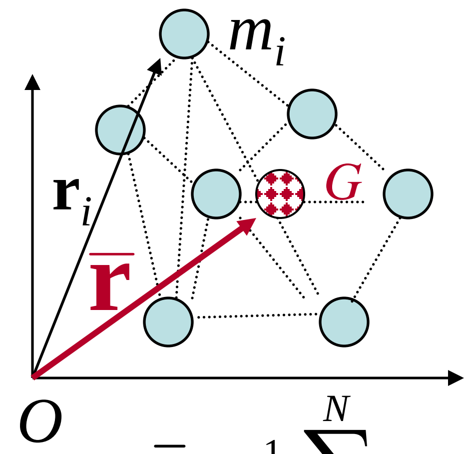
# Kinetics of Rigid (Planar) Bodies

# Types of motion



# Kinetics of a System of Particles

The center of mass for a system of particles accelerates in an inertial frame as if it were a single particle with mass *m* (equal to the total mass of the system) acted upon by a force equal to the net external force.




$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\ddot{\bar{\mathbf{r}}} = m\,\frac{d\,\mathbf{v}_G}{dt}$$

$$\bar{\mathbf{r}} = \frac{1}{m}\sum_{i=1}^{N} m_i \mathbf{r}_i$$

The rate of change of angular momentum of the *system* about a fixed point *O* is equal to the resultant moment of all external forces acting on the *system* about *O*

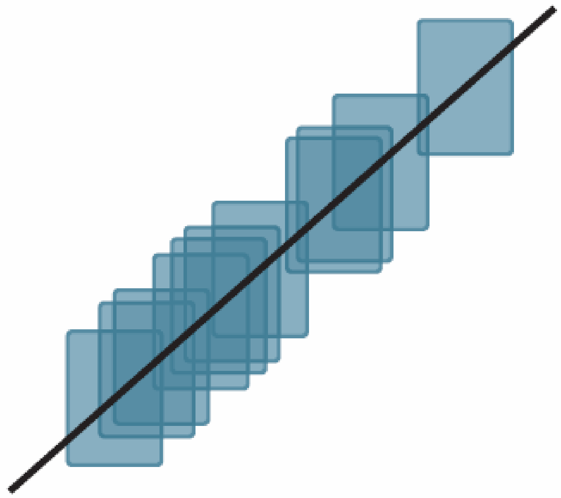
$$\frac{d\,\mathbf{H}_O}{dt} = \mathbf{M}_O$$

The rate of change of angular momentum of the *system* about the center of mass *G* is equal to the resultant moment of all external forces acting on the *system* about *G*

$$\frac{d\,\mathbf{H}_G}{dt} = \mathbf{M}_G$$

**Apply these equations to a single rigid body (really a system of infinite particles)**

# Rectilinear Translation

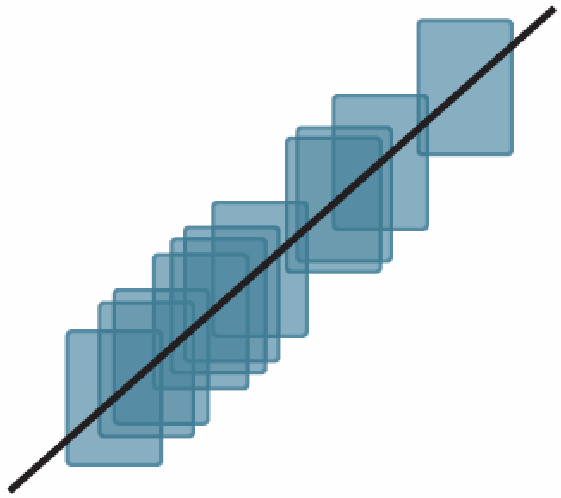


(a)

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\dot{\mathbf{r}} = m\,\frac{d\,\mathbf{v}_G}{dt}$$

$$\frac{d\,\mathbf{H}_G}{dt} = \mathbf{M}_G$$

# Rectilinear Translation



(*a*)

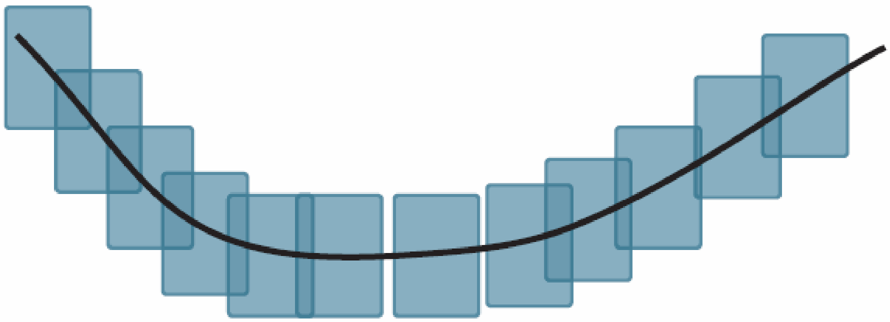
$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\ddot{\mathbf{r}} = m\,\frac{d\,\mathbf{v}_G}{dt}$$

The rigid body can be treated as a single particle of mass $m$ at the CM

$$\frac{d\,\mathbf{H}_G}{dt} = \mathbf{M}_G$$

No rotation ⇒ No change in angular momentum ⇒ No net moment about CM

# Curvilinear Translation

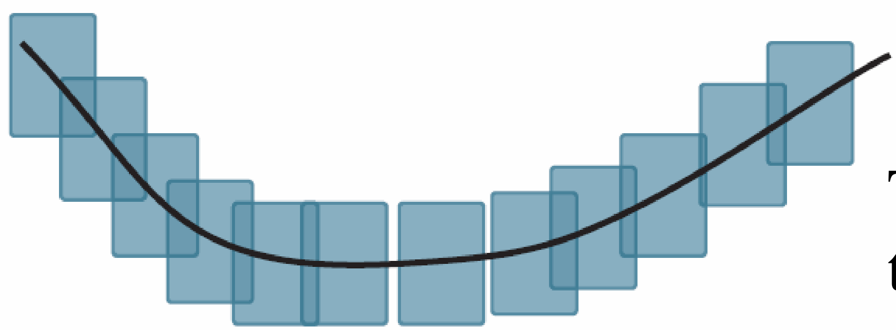


(b)

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\ddot{\mathbf{r}} = m\,\frac{d\,\mathbf{v}_G}{dt}$$

$$\frac{d\,\mathbf{H}_G}{dt} = \mathbf{M}_G$$

# Curvilinear Translation



(b)

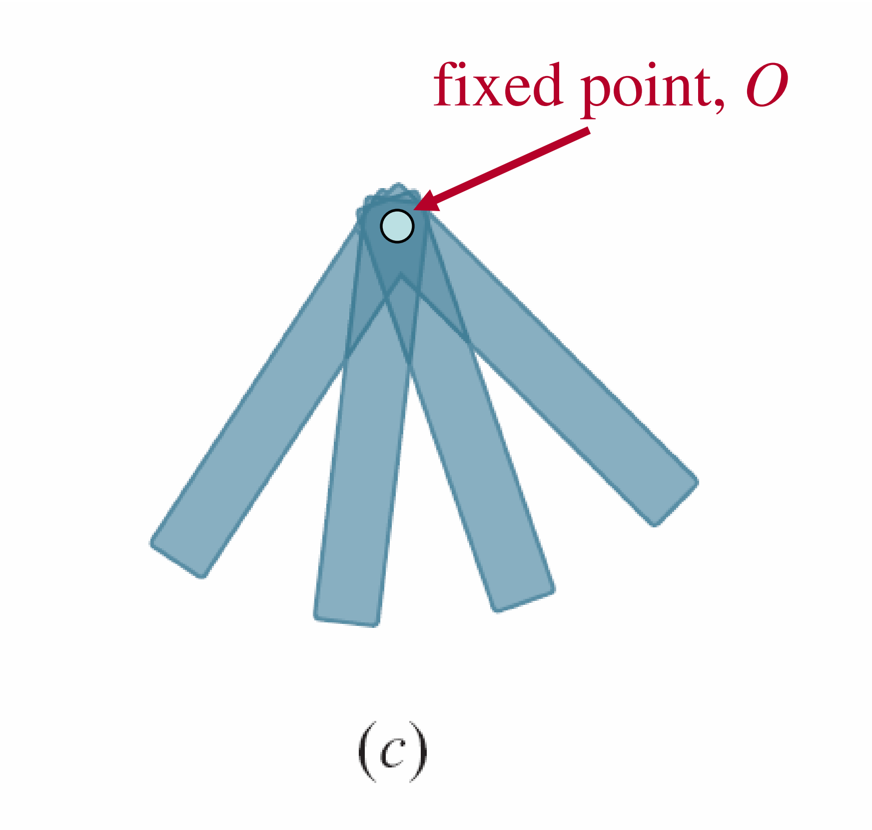
$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\dot{\mathbf{r}} = m\,\frac{d\,\mathbf{v}_G}{dt}$$

The rigid body can be treated as a single particle of mass $m$ at the CM

$$\frac{d\,\mathbf{H}_G}{dt} = \mathbf{M}_G$$

No rotation ⇒ No change in angular momentum ⇒ No net moment about CM

# Rotation about a fixed point

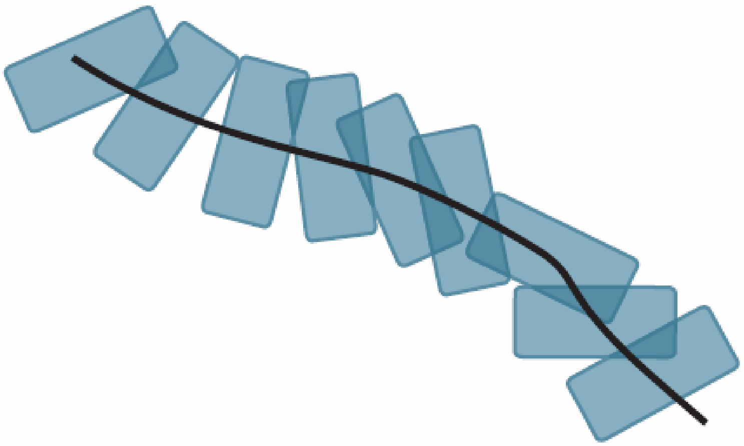



$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\dot{\mathbf{r}} = m\,\frac{d\,\mathbf{v}_G}{dt}$$

$$\cancel{\frac{d\,\mathbf{H}_G}{dt} = \mathbf{M}_G}$$

$$\frac{d\,\mathbf{H}_O}{dt} = \mathbf{M}_O$$

# General Planar Motion



(d)

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\dot{\mathbf{r}} = m\,\frac{d\,\mathbf{v}_G}{dt}$$

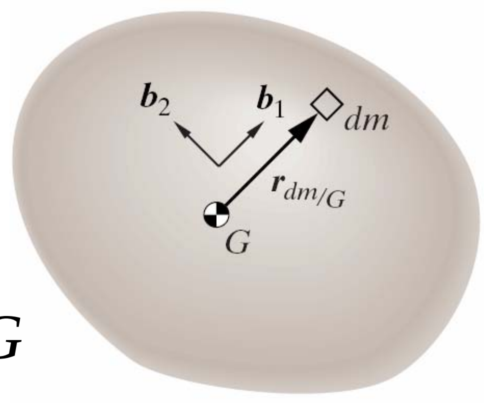
$$\frac{d\,\mathbf{H}_G}{dt} = \mathbf{M}_G$$

# Angular Momentum of a Rigid Body about its Center of Mass

Recall

- Expression for angular momentum about *O*
  - $\mathbf{H}_O = I_O \boldsymbol{\omega}$

$$I_O = \int_{body} \left(r_{dm/O}\right)^2 dm$$




A similar derivation …

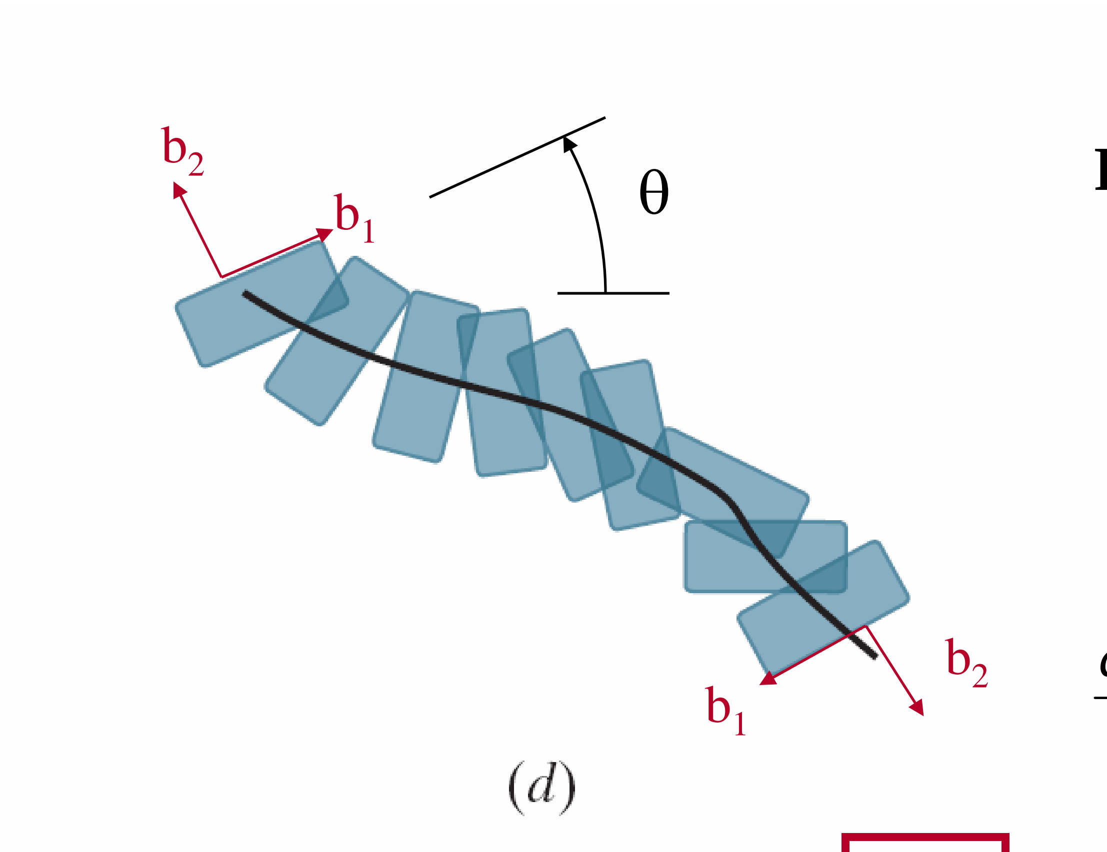
- Expression for angular momentum about *G*
  - $\mathbf{H}_G = I_G \boldsymbol{\omega}$

$$I_G = \int_{body} \left(r_{dm/G}\right)^2 dm$$

$I_G$ usually available in tables

– it is a property of the rigid body (unlike $I_O$ which also depends on the choice of *O*)

# General Planar Motion




$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\ddot{\mathbf{r}} = m\,\frac{d\,\mathbf{v}_G}{dt}$$

$$\frac{d\,\mathbf{H}_G}{dt} = \mathbf{M}_G$$

$$\mathbf{H}_G = I_G\omega = I_G\dot{\theta}\mathbf{b}_3$$

$$\boxed{\mathbf{M}_G} = I_G\dot{\omega} = \boxed{I_G}\,\boxed{\ddot{\theta}\mathbf{b}_3}$$

net moment of all external forces about *G*

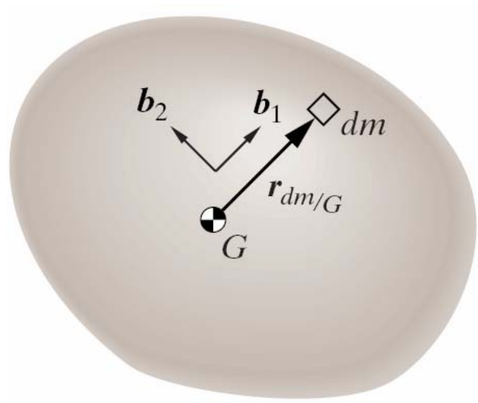
Mass moment of inertia about G

Angular acceleration

# General Planar Motion for a Rigid Body

- Force balance

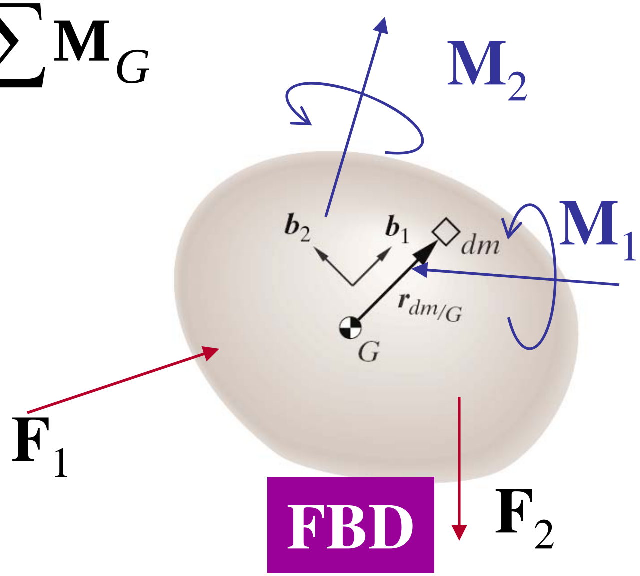
$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m\,\mathbf{a}_G$$



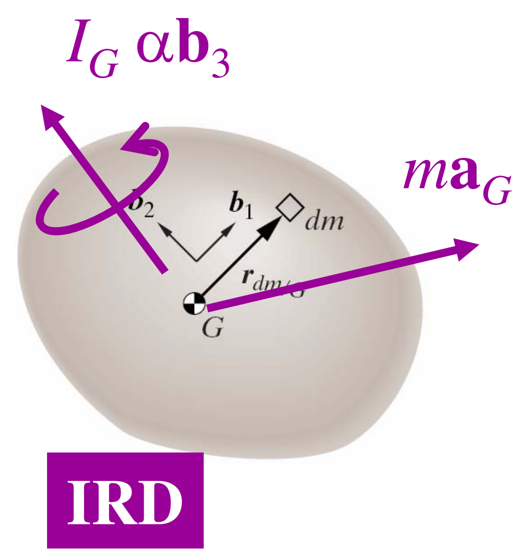

$$I_G = \int_{body} \left(r_{dm/G}\right)^2 dm$$

- Moment balance

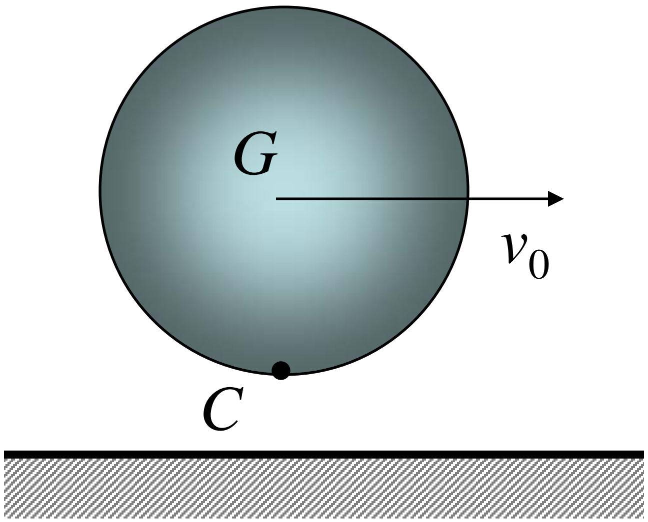
$$\frac{d\,\mathbf{H}_G}{dt} = \sum \mathbf{M}_G$$




$$\equiv$$



# Example 1



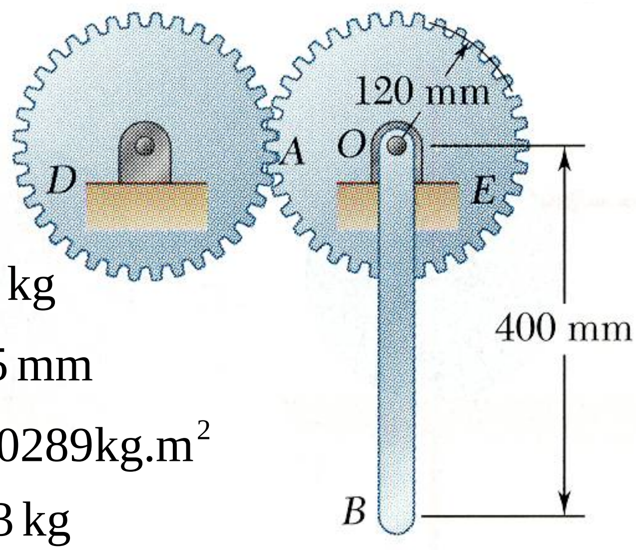

A uniform sphere of mass $m$ and radius $r$ is projected along a rough horizontal surface with a linear velocity $v_0$. The coefficient of kinetic friction between the sphere and the surface is $\mu_k$.

Determine: (*a*) the time $t_1$ at which the sphere will start rolling without sliding, and (*b*) the linear and angular velocities of the sphere at time $t_1$.

SOLUTION:

1. Draw the FBD with external (including reaction) forces on the sphere, and the IRD, and write the three scalar equations.
2. Solve the three corresponding scalar equilibrium equations for the normal reaction from the surface and the linear and angular accelerations of the sphere.
3. Apply the kinematic relations for uniformly accelerated motion to determine the time at which the tangential velocity of the sphere at the surface is zero, i.e., when the sphere stops sliding.

# Example 2




$m_E = 4\,\text{kg}$

$\bar{k}_E = 85\,\text{mm}$

$I_E = 0.0289\text{kg.m}^2$

$m_{OB} = 3\,\text{kg}$

The portion *AOB* of the mechanism is actuated by gear *D* and at the instant shown has a clockwise angular velocity of 8 rad/s and a counterclockwise angular acceleration of 40 rad/s$^2$.

Determine: a) tangential force exerted by gear *D*, and b) components of the reaction at shaft *O*.

SOLUTION:

1. Draw the FBD with external (including reaction) forces on *AOB*, and the IRD, and write the three scalar equations.

2. Evaluate the external forces due to the weights of gear *E* and arm *OB* and the effective forces associated with the angular velocity and acceleration.

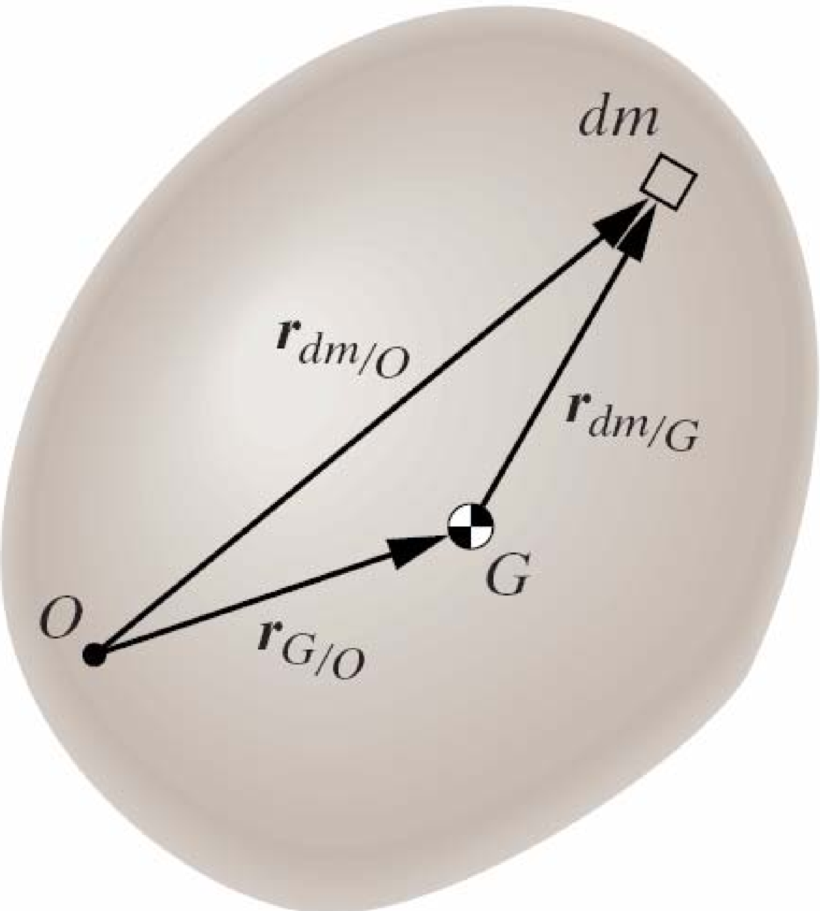
3. Solve the three scalar equations derived from the free-body-equation for the tangential force at *A* and the horizontal and vertical components of reaction at shaft *O*.

# How to calculate mass moments of inertia?

## Fundamental Result

## (Parallel Axis Theorem)

$$I_O = I_G + m\left(r_{dm/G}\right)^2$$




| | | | | |
|---|---|---|---|---|
| Mass moment of inertia about **any** point *O* | = | Mass moment of inertia about center of mass *G* | + | Mass moment of inertia about *O* of a single particle of mass *m* concentrated at the center of mass *G* |

# Free vectors and bound vectors

## Bound vectors

- Characterized by an axis
- Wrong to associate with a single point

## Free vectors

- Not characterized by an axis
- But… referenced to a point

### Examples

- Angular velocity of a rigid body (characterized by an axis)
- Force applied to a rigid body (tied to a line of action of the force)

### Examples

Linear velocity of a point *P*

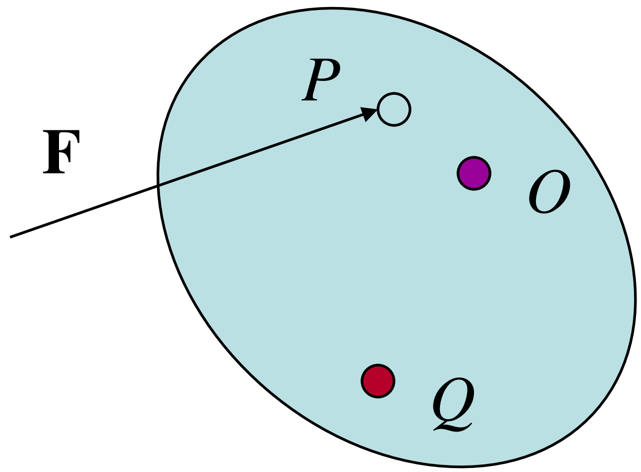
Moment about a point *P*

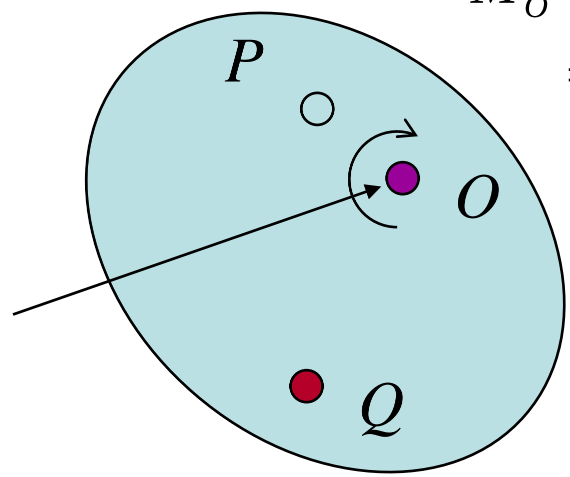
In general,

$$\mathbf{v}_P \neq \mathbf{v}_Q$$

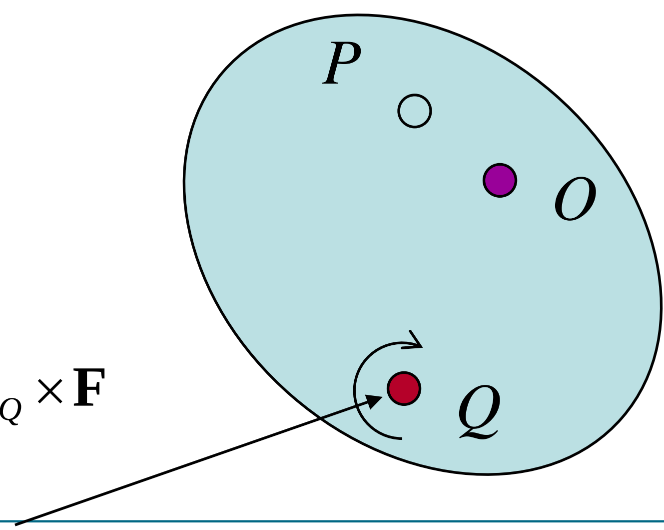
$$\mathbf{M}_P \neq \mathbf{M}_Q$$

# Relating moments about two different points




$$M_O = M_P + \mathbf{r}_{P/O} \times \mathbf{F}$$
$$= \mathbf{r}_{P/O} \times \mathbf{F}$$




$$M_Q = M_O + \mathbf{r}_{O/Q} \times \mathbf{F}$$

or

$$M_Q = M_P + \mathbf{r}_{P/Q} \times \mathbf{F} = \mathbf{r}_{P/Q} \times \mathbf{F}$$