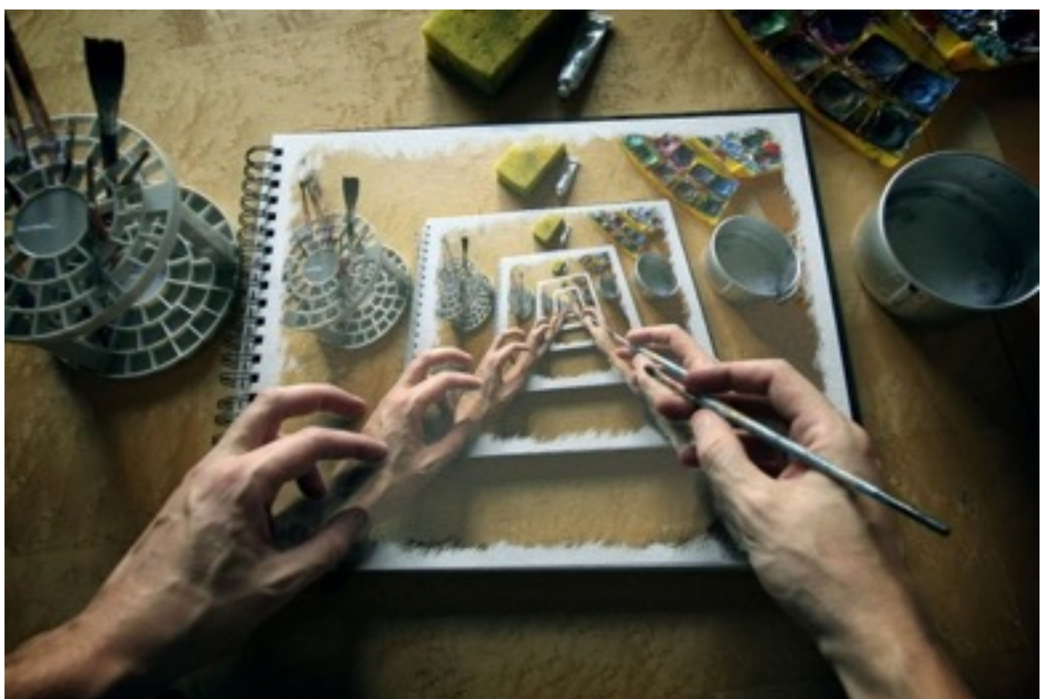
15-112
Fundamentals of Programming

Week 9 - Lecture 1a:
Recursion



March 15, 2016

What is recursion?

Recursion:

To understand recursion, you have to first understand recursion.

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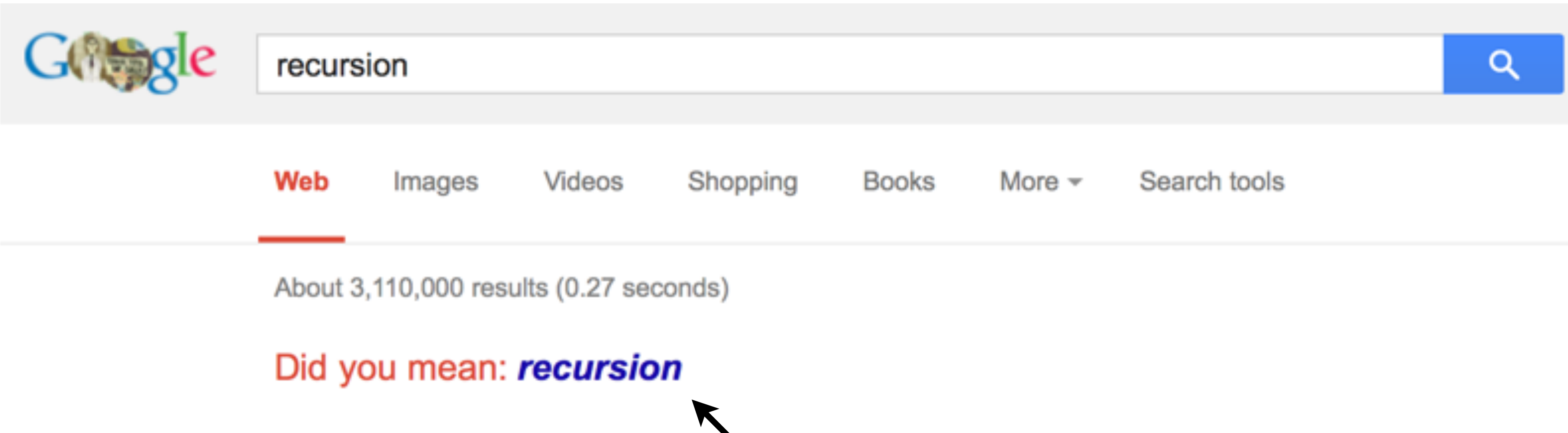
To understand recursion, you have to first understand recursion.

Not making progress. Let's ask Google.

What is recursion?



?!!?

Let’s see what my dictionary says.

What is recursion?

recursion (n):

See *recursion*

What is recursion in programming?

We say that a function is recursive if at some point, it calls itself.

```
def test():
    test()
```

Can we do something more meaningful?

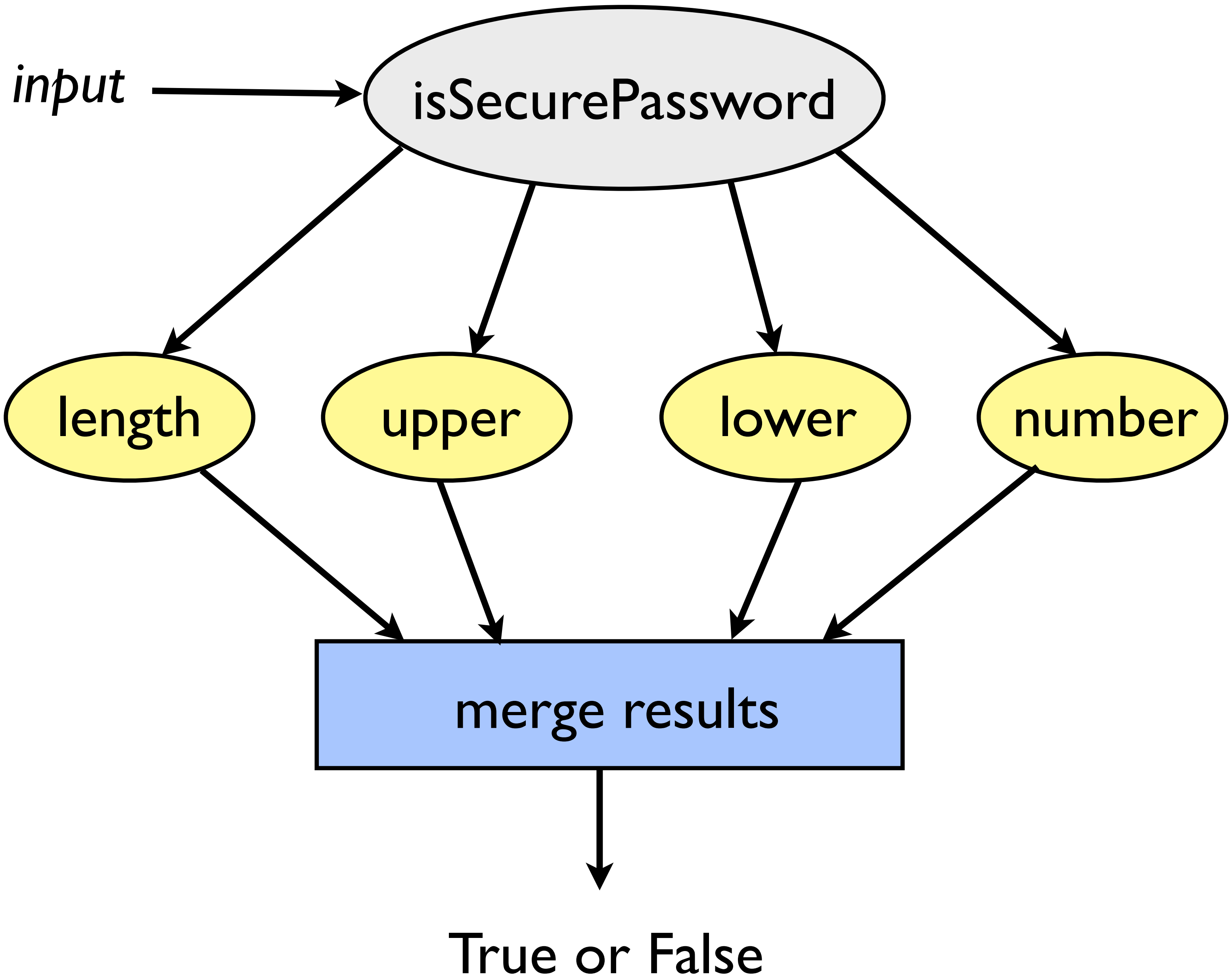
Warning:
Recursion can be weird and counter-intuitive at first!

Motivation: break a problem into smaller parts

Example: Figuring out if a given password is secure.

- Is string length at least 10?
- Does the string contain an upper-case letter?
- Does the string contain a lower-case letter?
- Does the string contain a number?

Motivation: break a problem into smaller parts



Motivation: break a problem into smaller parts

isSecurePassword:

The problem is split into smaller but **different** problems.

Recursion:

The smaller problems are **not** different.
They are smaller versions of the original problem.

Recursion Example: Sorting

Sorting the midterms by name.

Sort:

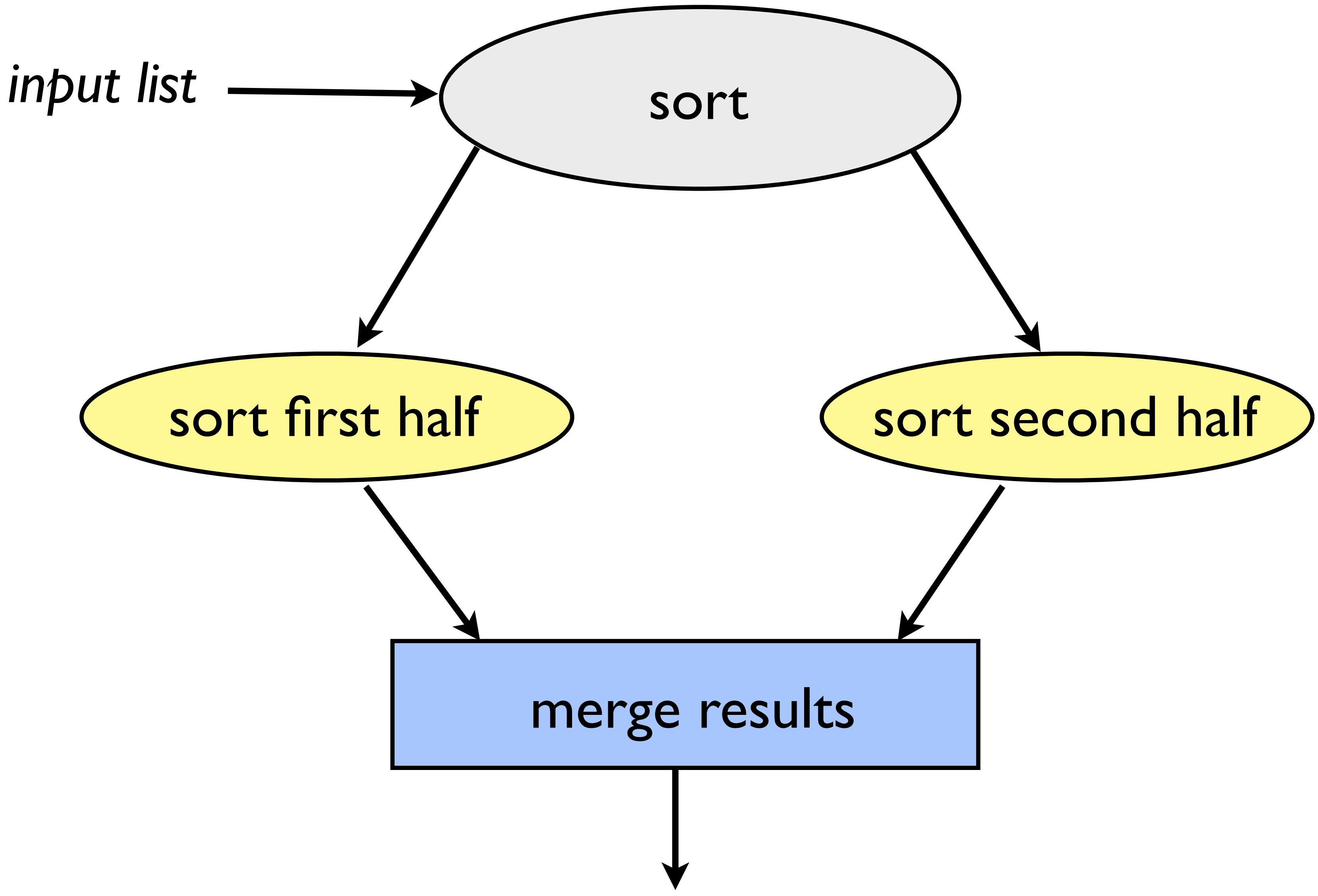
Divide the pile in half.

Sort the first half.

Sort the second half.

Merge the sorted piles.

Recursion Example: Sorting



Recursion Example: Sorting

Sort:

Divide the pile in half.

Sort the first half.

Sort the second half.

Merge the sorted piles.

What if my pile consists of just a single exam?

Recursion Example: Sorting

Sort:

If the pile consists of one element, do nothing.

Else:

Divide the pile in half.

Sort the first half.

Sort the second half.

Merge the sorted piles.

Recursion Example: Sorting

```
def merge(a, b):
    # We have already seen this.


def sort(a):
    if (len(a) <= 1):
        return a
    leftHalf = a[0 : len(a)/2]
    rightHalf = a[len(a)/2 : len(a)]
    return merge(sort(leftHalf),  sort(rightHalf))
```

This works!

To understand how recursion works,
let’s look at simpler examples.

Simple Example: Factorial

n factorial is the product of integers from 1 to n.

$1! = 1$

$2! = 2 \times 1$

$3! = 3 \times 2 \times 1$

$4! = 4 \times 3 \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

...

$n! = n \times (n-1) \times (n-2) \times \ldots \times 1$

Simple Example: Factorial

Finding the recursive structure in factorial:

Can we express n! using a smaller factorial ?

$n! = n \times (n - 1) \times (n - 2) \times \ldots \times 1$

Simple Example: Factorial

Finding the recursive structure in factorial:

Can we express n! using a smaller factorial ?

$$n! = n \times \underbrace{(n-1) \times (n-2) \times \ldots \times 1}_{(n-1)!}$$

$$n! = n \times (n-1)!$$

Simple Example: Factorial

```
def factorial(n):
    return n * factorial(n - 1)
```

"Unwinding" the code when n = 4:

factorial(4)

4 * factorial(3)

3 * factorial(2)

2 * factorial(1)

1 * factorial(0)

0 * factorial(-1)

...

No stopping condition

Simple Example: Factorial

```
def factorial(n):
    if (n == 1): return 1
    else: return n * factorial(n - 1)
```

factorial(4)

    4 * factorial(3)

        3 * factorial(2)

            2 * factorial(1)

                1

Simple Example: Factorial

```
def factorial(n):
        if (n == 1): return 1
        else: return n * factorial(n - 1)
```

factorial(4)

4 * factorial(3)

3 * factorial(2)

2 * 1

Simple Example: Factorial

```
def factorial(n):
        if (n == 1): return 1
        else: return n * factorial(n - 1)
```

factorial(4)

4 * factorial(3)

3 * 2

Simple Example: Factorial

```
def factorial(n):
    if (n == 1): return 1
    else: return n * factorial(n - 1)
```

factorial(4) ⟶ evaluates to 24

4 * 6

Simple Example: Factorial

```
def factorial(n):
    if (n == 1): return 1
    else: return n * factorial(n - 1)
```

factorial(4) ⟶ evaluates to 24

4 * 6

Recursive calls make their way *down* to the base case.

The solution is then built *up* from base case.

Simple Example: Factorial

```
def factorial(n):
    if (n == 1): return 1
    else: return n * factorial(n - 1)
```

<u>Another way of convincing ourselves it works:</u>

Does factorial(1) work (base case) ? ✓

Does factorial(2) work ? ✓

returns 2*factorial(1)

Does factorial(3) work ? ✓

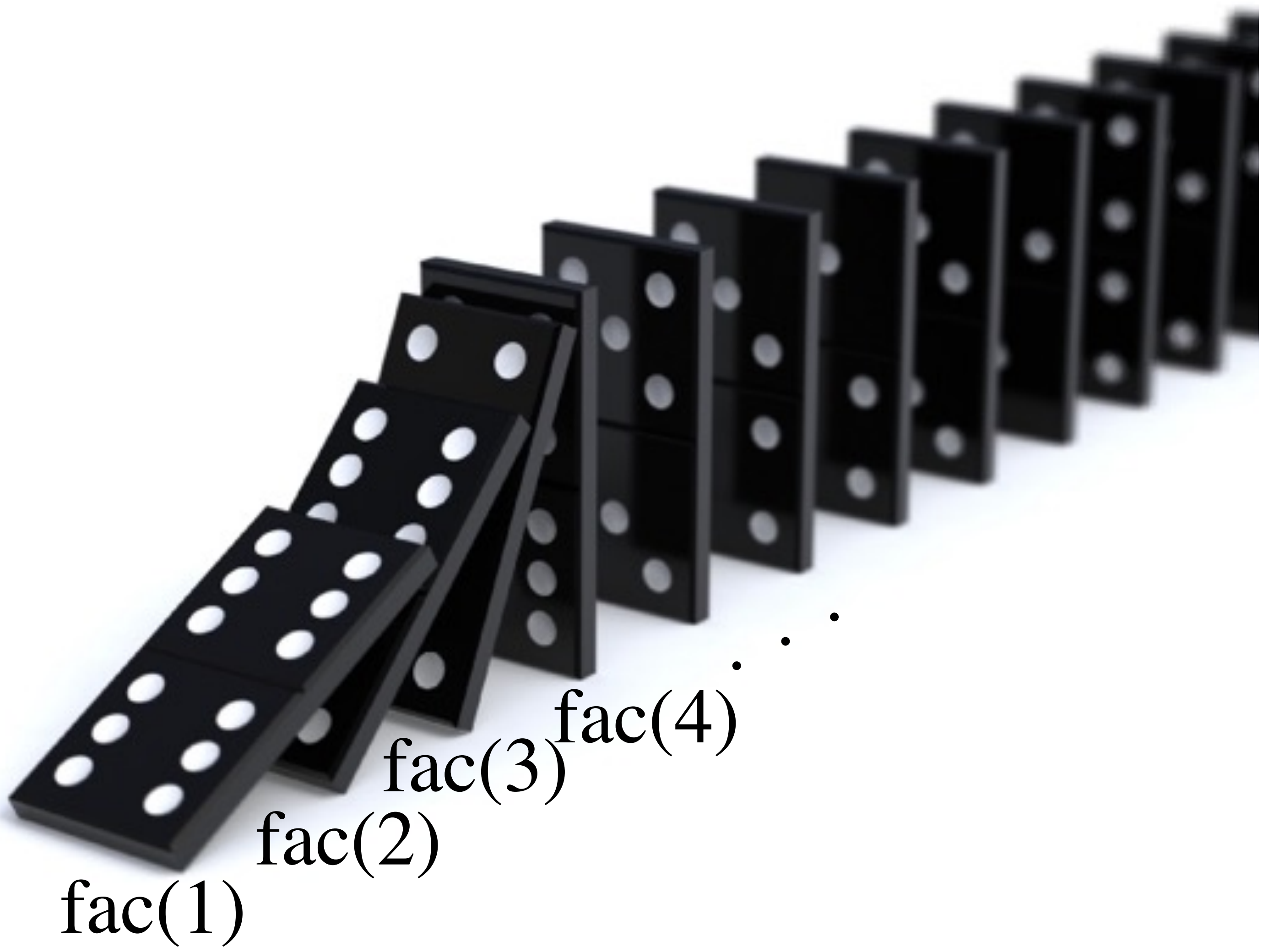
returns 3*factorial(2)

Does factorial(4) work ? ✓

returns 4*factorial(3)

How recursion works

fac(1) —> fac(2) —> fac(3) —> fac(4) —> ...



2 important properties of recursive functions

1. “Base case”

There should be a **base case**
(a case which does not make a recursive call)

2. “Progress”

The recursive call(s) should make **progress** towards the base case.

Another example: Fibonacci

Fibonacci Sequence: 1 1 2 3 5 8 13 21 ...

```
def fib(n):
    if (n == 0): return 1
    else: return fib(n-1) + fib(n-2)
```

What happens when we call fib(1) ?

Another example: Fibonacci

Fibonacci Sequence: 1 1 2 3 5 8 13 21 ...

```
def fib(n):
    if (n == 0 or n == 1): return 1
    else: return fib(n-1) + fib(n-2)
```

Another example: Fibonacci

Fibonacci Sequence: 1 1 2 3 5 8 13 21 ...

```
def fib(n):
    if (n == 0 or n == 1): return 1
    else: return fib(n-1) + fib(n-2)
```

Base case

Another example: Fibonacci

Fibonacci Sequence: 1 1 2 3 5 8 13 21 ...

```
def fib(n):
    if (n == 0 or n == 1): return 1
    else: return fib(n-1) + fib(n-2)
```

Each recursive call makes progress towards the base case
(and doesn't skip it!!!)

Unwinding the code

fib(4)

Unwinding the code

fib(4)

$\downarrow$

fib(3) + fib(2)

Unwinding the code

fib(4)

↓

fib(3) + fib(2)

↓

fib(2) + fib(1)

Unwinding the code

fib(4)

fib(3) + fib(2)

fib(2) + fib(1)

fib(1) + fib(0)

Unwinding the code

fib(4)

fib(3) + fib(2)

fib(2) + fib(1)

1 + 1

Unwinding the code

fib(4)

↓

fib(3) + fib(2)

↓

2 + fib(1)

Unwinding the code

fib(4)

↓

fib(3)   +   fib(2)

↓

2   +   1

Unwinding the code

fib(4)

↓

3 + fib(2)

Unwinding the code

fib(4)

↓

3 + fib(2)

↓

fib(1) + fib(0)

Unwinding the code

fib(4)

3 + fib(2)

1 + fib(0)

Unwinding the code

fib(4)

↓

3 + fib(2)

↓

1 + 1

Unwinding the code

fib(4)

↓

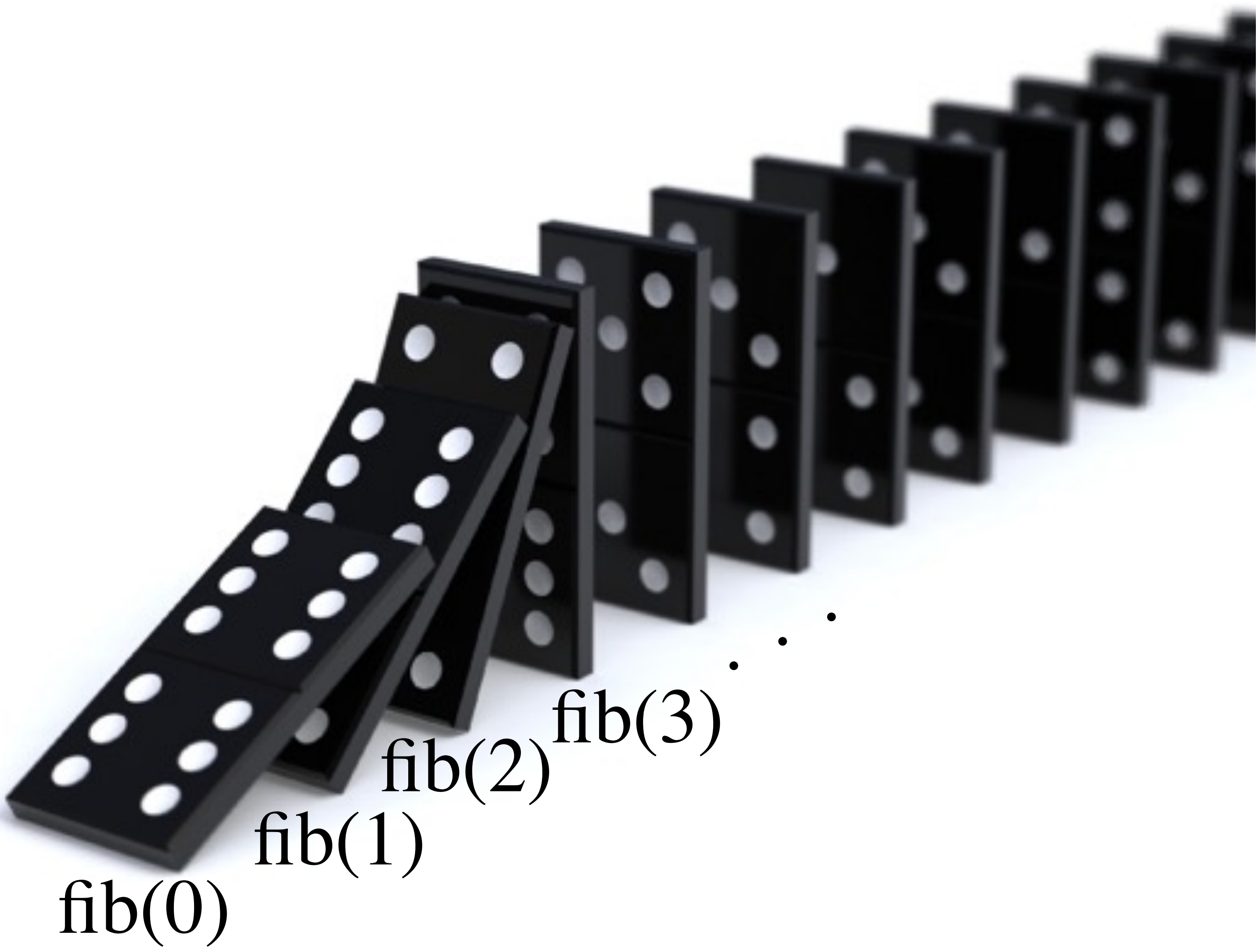
3 + 2

Unwinding the code

5

Recursion

fib(0), fib(1) —> fib(2) —> fib(3) —> fib(4) —> ...



The sweet thing about recursion

Do these 2 steps:

1. Base case:
Solve the "smallest" version of the problem (with no recursion).

2. Recursive call(s):
Correctly write the solution to the problem in terms of "smaller" version(s) of the same problem.

Your recursive function will always work!

Unwinding vs Trusting

Unwinding recursive functions:

- OK at first (for simple examples)
- Not OK once you understand the logic

Over time, you will start trusting recursion.

This trust is very important!

Recursion **<u>will</u>** earn your trust.

Unwinding vs Trusting

```
def fib(n):
    if (n == 1 or n == 2): return 1
    else: return fib(n-1) + fib(n-2)
```

You have to trust these will return the correct answer.

This is why recursion is so powerful.

You can assume every subproblem is solved for free!

Getting comfortable with recursion

1. See **lot's** of examples

2. Practice yourself

Getting comfortable with recursion

1. See **lot's** of examples

Recursive function design

Ask yourself:
If I had the solutions to the smaller instances for free, how could I solve the original problem?

Write the recursive relation:
e.g. $fib(n) = fib(n-1) + fib(n-2)$

Handle the base case:
A small version of the problem that does not require recursive calls.

Double check:
All your recursive calls make progress towards the base case(s) and they don't miss it.

Examples

Example: sum

Write a function that takes an integer n as input, and returns the sum of all numbers from 1 to n.

$$sum(n) = n + (n-1) + (n-2) + (n-3) + ... + 3 + 2 + 1$$

Example: sum

Write a function that takes an integer n as input, and returns the sum of all numbers from 1 to n.

$$sum(n) = n + (n-1) + (n-2) + (n-3) + \ldots + 3 + 2 + 1$$

$$sum(n) = n + sum(n-1)$$

Example: sum

Write a function that takes an integer n as input, and returns the sum of all numbers from 1 to n.

```
def sum(n):
    if (n == 0): return 0
    else: return n + sum(n-1)
```

Example: sum in range

Write a function that takes integers n and m as input
(n <= m),
and returns the sum of all numbers from n to m.

$$\text{sum}(n, m) = n + (n+1) + (n+2) + \ldots + (m-1) + m$$

Example: sum in range

Write a function that takes integers n and m as input
(n <= m),
and returns the sum of all numbers from n to m.

$$\mathrm{sum}(n, m) = n + (n+1) + (n+2) + \ldots + (m-1) + m$$

$$\mathrm{sum}(n, m) = \mathrm{sum}(n, m-1) + m$$

Example: sum in range

Write a function that takes integers n and m as input (n <= m),
and returns the sum of all numbers from n to m.

$$\text{sum}(n, m) = n + \underbrace{(n+1) + (n+2) + \ldots + (m-1) + m}_{\text{sum}(n+1,\, m)}$$

Example: sum in range

Write a function that takes integers n and m as input
(n <= m),
and returns the sum of all numbers from n to m.

$$sum(n, m) = n + (n+1) + (n+2) + \ldots + (m-1) + m$$

$$sum(n, m) = n + sum(n+1, m)$$

Example: sum in range

Write a function that takes integers n and m as input (n <= m),
and returns the sum of all numbers from n to m.

```
def sum(n, m):
    if (n == m): return n
    else: return n + sum(n+1, m)
```

Note: objects with recursive structure

Lists

0	1	2	4	5	5	6	8	9	9

Strings (a list of characters)

“Dammit I’m mad”

Problems related to these objects often have very natural recursive solutions.

Example: sumList(L)

Write a function that takes a list of integers as input and returns the sum of all the elements in the list.

| 3 | 5 | 2 | 6 | 9 | 1 | 5 |

sum(| 3 | 5 | 2 | 6 | 9 | 1 | 5 |) =

3 + sum(| 5 | 2 | 6 | 9 | 1 | 5 |)

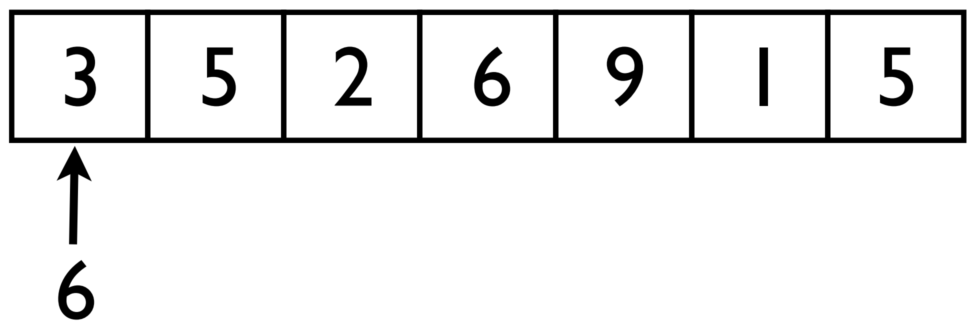
Example: sumList(L)

Write a function that takes a list of integers as input and returns the sum of all the elements in the list.

```
def sum(L):
    if (len(L) == 0): return 0
    else: return L[0] + sum(L[1:])
```

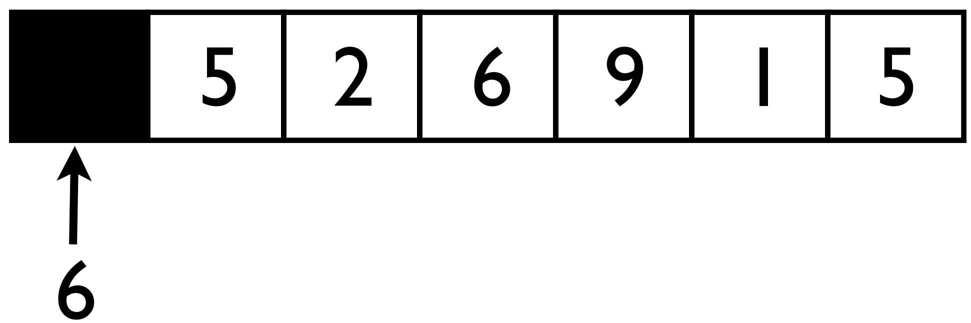
Example: isElement(L, e)

Write a function that checks if a given element is in a given list.



Example: isElement(L, e)

Write a function that checks if a given element is in a given list.



Example: isElement(L, e)

Write a function that checks if a given element is in a given list.

```
def isElement(L, e):
    if (len(L) == 0): return False
    else:
        if (L[0] == e): return True
        else: return isElement(L[1:], e)
```

Example: isPalindrome(s)

Write a function that checks if a given string is a palindrome.

h a n n a h

Example: isPalindrome(s)

Write a function that checks if a given string is a palindrome.

should be palindrome

h a n n a h

Example: isPalindrome(s)

Write a function that checks if a given string is a palindrome.

```
def isPalindrome(s):
    if (len(s) <= 1): return True
    else:
        return (s[0] == s[len(s)-1]  and  isPalindrome(s[1:len(s)-1]))
```

Example: isPrime(n)

Tricky: Doesn't seem like calling isPrime(n) for smaller n would be useful.

Idea:

Think of another function such that:

- its solution can be used to solve isPrime(n)
- it has a recursive structure

Example: isPrime(n)

2	3	4	5	6	7	…	n**0.5

Want to check if one of these numbers is a factor:

- check if 2 is a factor
- if not, check (recursively) if one of the remaining numbers is a factor.

Example: isPrime(n)

2	3	4	5	6	7	…	n**0.5

hasNoFactorStartingFrom(n, m)

return True if n has no factors between m and n**0.5

```
def hasNoFactorStartingFrom(n, m):
    if (m*m > n): return True
    return (n%m != 0) and hasNoFactorStartingFrom(n, m+1)

def isPrime(n):
    if (n < 2): return False
    return hasNoFactorStartingFrom(n, 2)
```

Example: isPrime(n)

2	3	4	5	6	7	…	n**0.5

```
def isPrime(n, m):
    if (n < 2): return False
    if (m*m > n): return True
    return (n%m != 0) and isPrime(n, m+1)
```

Example: isPrime(n)

2 3 4 5 6 7 … n**0.5

```
def isPrime(n, m=2):
    if (n < 2): return False
    if (m*m > n): return True
    return (n%m != 0) and isPrime(n, m+1)
```

Example: nthPrime(n)

```
def nthPrime(n):
    if (n == 0): return 2
    m = nthPrime(n-1) + 1
    while(True):
        if (isPrime(m)): return m
        m += 1
```

Can we do it **without** using a loop?

Example: nthPrime(n)

```
def nthPrime(n, start):
    # return the nth prime starting from the integer start
    if (n == 0 and isPrime(start)): return start
    elif (isPrime(start)):
        return nthPrime(n-1, start+1)
    else:
        return nthPrime(n, start+1)

# printing the 10th prime number
print(nthPrime(10, 2))
```

Example: nthPrime(n)

```
def nthPrime(n, start=2):
    # return the nth prime starting from the integer start
    if (n == 0 and isPrime(start)): return start
    elif (isPrime(start)):
        return nthPrime(n-1, start+1)
    else:
        return nthPrime(n, start+1)

# printing the 10th prime number
print(nthPrime(10))
```