**Solutions for Problems 3.13, 3.18, and 3.19**

3.13. For this manipulator, we can easily find the coordinates of the end-effector position $o = [o_x, o_y, o_z]^T$ as:

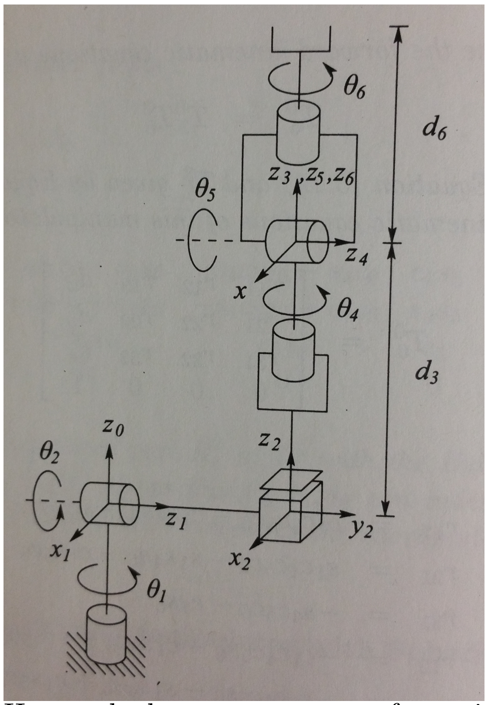
$$o_x = (d_3 + 1)\cos(\theta_1) \quad ; \quad o_y = (d_3 + 1)\sin(\theta_1) \quad ; \quad o_z = d_2 + 1. \tag{1}$$

Hence, given the desired end-effector position as $d = [d_x, d_y, d_z]^T$, we can solve for the joint variables $(\theta_1, d_2, d_3)$ as:

$$\theta_1 = \text{Atan2}(d_x, d_y) \quad ; \quad d_2 = d_z - 1 \quad ; \quad d_3 = -1 + \sqrt{d_x^2 + d_y^2} \tag{2}$$

where Atan2 denotes the two-argument arc-tangent function wherein Atan2$(x, y)$ is essentially $\tan^{-1}(y/x)$, but with the signs of $x$ and $y$ also taken into account.

3.18. The choice of the D-H coordinate frames and the corresponding D-H table are shown below.




| | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| Link 1 | 0 | -90$^o$ | 0 | $\theta_1^*$ |
| Link 2 | 0 | 90$^o$ | $d_2$ | $\theta_2^*$ |
| Link 3 | 0 | 0 | $d_3^*$ | 0 |
| Link 4 | 0 | -90$^o$ | 0 | $\theta_4^*$ |
| Link 5 | 0 | 90$^o$ | 0 | $\theta_5^*$ |
| Link 6 | 0 | 0 | $d_6$ | $\theta_6^*$ |

Hence, the homogeneous transformation matrices $A_1, \ldots, A_6$ and the homogeneous transformation matrix $H_6^0$ are given as:

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; ; \; A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; ; \; A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; ; \; A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; ; \; A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$H_6^0 = A_1 A_2 A_3 A_4 A_5 A_6 \tag{5}$$

To solve the inverse kinematics for this manipulator given the desired end-effector position $d$ and desired orientation $R$, the desired coordinates of the wrist center $o_c^0$ are found as

$$o_c^0 = d - d_6 R[0, 0, 1]^T \tag{6}$$

Here, $d_6$ is the distance from the wrist center to the origin of the end-effector coordinate frame. Computing $A_1 A_2 A_3$ and $A_4 A_5 A_6$, we get

$$H_3^0 = A_1A_2A_3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2 - d_2s_1 \\ c_2s_1 & c_1 & s_1s_2 & d_2c_1 + d_3s_1s_2 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

$$H_6^3 = A_4A_5A_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_6s_4 - c_4c_5s_6 & c_4s_5 & d_6c_4s_5 \\ c_4s_6 + c_5c_6s_4 & c_4c_6 - c_5s_4s_6 & s_4s_5 & d_6s_4s_5 \\ -c_6s_5 & s_5s_6 & c_5 & d_6c_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (8)$$

Using the concept of kinematic decoupling, we will solve for the first three joint variables ($\theta_1$, $\theta_2$, $d_3$) to place the wrist center at the desired location and then solve for the last three joint variables ($\theta_4$, $\theta_5$, $\theta_6$) to set the desired angular orientation of the end-effector. With $o$ being the end-effector position, it can be seen that $o - d_6R_6^0[0,0,1]^T = [d_3c_1s_2 - d_2s_1, d_2c_1 + d_3s_1s_2, d_3c_2]^T$. Hence, if the desired location of the wrist center is found using (6) as $o_c^0 = [x_c, y_c, z_c]^T$, then to find $\theta_1$, $\theta_2$, and $d_3$, we need to solve the equations

$$x_c = d_3c_1s_2 - d_2s_1 \quad (9)$$

$$y_c = d_2c_1 + d_3s_1s_2 \quad (10)$$

$$z_c = d_3c_2. \quad (11)$$

Hence,

$$\theta_2 = \cos^{-1}\left(\frac{z_c}{d_3}\right) \quad (12)$$

Also, $x_c^2 + y_c^2 + z_c^2 = d_3^2 + d_2^2$. Therefore,

$$d_3 = \sqrt{x_c^2 + y_c^2 + z_c^2 - d_2^2}. \quad (13)$$

From the solutions for $\theta_2$ and $d_3$ in equations (12) and (13), we can then find $\theta_1$ using equations (9) and (10) which are of the form $Ac_1 + Bs_1 = x_c$ and $As_1 - Bc_1 = y_c$ where $A = d_3s_2$ and $B = -d_2$. This pair of equations for $\theta_1$ is identical in form to the equations in the solution of Problem 3.11 in Homework 5. Hence, using the same solution method as in that problem, we see that $\theta_1 = \text{Atan2}(x_c, y_c) - \text{Atan2}(d_3s_2, d_2)$ where Atan2 denotes the two-argument arc-tangent function wherein $\text{Atan2}(x, y)$ is essentially $\tan^{-1}(y/x)$, but with the signs of $x$ and $y$ also taken into account. From equation (12), we get two solutions for $\theta_2$. If $d_3$ is constrained to be positive, (13) gives us a unique solution for $d_3$. Once $\theta_2$ and $d_3$ are specified, we get a unique solution for $\theta_1$ as described above. Hence, in general, we get two solutions for the inverse position kinematics given the desired position of the wrist center.

Next, to find $\theta_4$, $\theta_5$, and $\theta_6$, note that the rotation matrix part of $H_6^3 = A_4A_5A_6$ in (8) is exactly of the form of the Euler angle $(Z - Y - Z)$ rotation matrix representation. Hence, $\theta_4$, $\theta_5$, and $\theta_6$ are the Euler angles $(\phi, \theta, \psi)$ corresponding to the desired rotation matrix $R_6^3$, which is $(R_3^0)^T R$ where $R$ is the given rotation matrix corresponding to the desired end-effector orientation and $R_3^0$ is the rotation matrix corresponding to the first three joint variables ($\theta_1$, $\theta_2$, $d_3$). We know the rotation matrix $R_3^0$ from (7). Hence, from the solutions that we found above for the first three joint variables ($\theta_1$, $\theta_2$, $d_3$), we find $R_3^0$ and therefore find the desired value of $R_6^3$ as $(R_3^0)^T R$. Then, from this desired $R_6^3$ matrix, we can find $\theta_4$, $\theta_5$, and $\theta_6$ using the standard way of finding Euler angles corresponding to a given rotation matrix, i.e., denoting the $(i,j)^{th}$ element of this desired $R_6^3$ matrix by $\tilde{r}_{ij}$, we have $\theta_5 = \cos^{-1}(\tilde{r}_{33})$, which gives us two possible values of $\theta_5$, one of which has a positive value of $s_5$ and the other a negative value of $s_5$. Then, we have $\theta_6 = \text{Atan2}(-\tilde{r}_{31}, \tilde{r}_{32})$ for the case of positive $s_5$ and $\theta_6 = \text{Atan2}(\tilde{r}_{31}, -\tilde{r}_{32})$ for the case of negative $s_5$. Similarly, using the $(1,3)$ and $(2,3)$ elements of the desired $R_6^3$ matrix, we see that $c_4s_5 = \tilde{r}_{13}$ and $s_4s_5 = \tilde{r}_{23}$. Hence, $\theta_4 = \text{Atan2}(\tilde{r}_{13}, \tilde{r}_{23})$ for the case of positive $s_5$ and $\theta_4 = \text{Atan2}(-\tilde{r}_{13}, -\tilde{r}_{23})$ for the case of negative $s_5$. In the special case that $\tilde{r}_{33} = 1$, we get $\theta_5 = 0$ and we can only find $\theta_4 + \theta_6$, but not the individual angles $\theta_4$ and $\theta_6$ uniquely. Similarly, in the special case that $\tilde{r}_{33} = -1$, we get $\theta_5 = \pi$ and we can only find $\theta_4 - \theta_6$, but not the individual angles $\theta_4$ and $\theta_6$ uniquely.

3.19. The forward kinematics of the PUMA 260 manipulator was written in Problem 3.10 in Homework 5. To solve the inverse kinematics for this manipulator given the desired end-effector position

$d$ and desired orientation $R$, the desired coordinates of the wrist center $o_c^0$ are

$$o_c^0 = d - d_6 R[0,0,1]^T \tag{14}$$

Here, $d_6$ is the distance from the wrist center to the origin of the end-effector coordinate frame. From the forward kinematics we found in Problem 3.10 in Homework 5, we find that with $o$ being the end-effector position,

$$o - d_6 R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d_4(c_1c_2s_3 + c_1c_3s_2) + d_2s_1 + a_2c_1c_2 \\ d_4(c_2s_1s_3 + c_3s_1s_2) - d_2c_1 + a_2c_2s_1 \\ d_1 - d_4(c_2c_3 - s_2s_3) + a_2s_2 \end{bmatrix} \tag{15}$$

Hence, if the desired location of the wrist center is found using (14) as $o_c^0 = [x_c, y_c, z_c]^T$, then we see from (15) that to find $\theta_1$, $\theta_2$, and $\theta_3$, we need to solve the equations

$$x_c = d_4c_1s_{23} + d_2s_1 + a_2c_1c_2 \tag{16}$$

$$y_c = d_4s_1s_{23} - d_2c_1 + a_2c_2s_1 \tag{17}$$

$$z_c = d_1 - d_4c_{23} + a_2s_2. \tag{18}$$

Hence, $x_c^2+y_c^2+(z_c-d_1)^2 = d_4^2+d_2^2+a_2^2+2a_2d_4(c_2s_{23}-s_2c_{23}) = d_4^2+d_2^2+a_2^2+2a_2d_4s_3$. Therefore,

$$\theta_3 = \sin^{-1}\left(\frac{x_c^2+y_c^2+(z_c-d_1)^2-d_4^2-d_2^2-a_2^2}{2a_2d_4}\right) \tag{19}$$

Once the solution for $\theta_3$ has been found from (19), we can use equation (18) to find $\theta_2$ since this equation is equivalent to $(a_2+d_4s_3)s_2-d_4c_3c_2 = z_c-d_1$, i.e., an equation of the form $Ac_2+Bs_2 = C$, the solution for which can be written as

$$\theta_2 = -\text{Atan2}(B,A) + \sin^{-1}\left(\frac{C}{\sqrt{A^2+B^2}}\right). \tag{20}$$

Then, with $\theta_2$ and $\theta_3$ known from equations (20) and (19), we can find $\theta_1$ using equations (16) and (17) since these equations are of the form $Ac_1+Bs_1 = x_c$ and $As_1-Bc_1 = y_c$ where $A = d_4s_{23}+a_2c_2$ and $B = d_2$, the solution for which can be written as $\theta_1 = \text{Atan2}(x_c, y_c) + \text{Atan2}(d_4s_{23}+a_2c_2, d_2)$. From equation (19), we get two possible solutions for $\theta_3$ and from (20), we get two possible solutions for $\theta_2$. Once $\theta_2$ and $\theta_3$ are specified, we get a unique solution for $\theta_1$. Hence, in general, we get four solutions for the inverse position kinematics given the desired position of the wrist center.

Using the concept of kinematic decoupling, we can solve for the first three joint variables ($\theta_1$, $\theta_2$, $\theta_3$) to place the wrist center at the desired location and then solve for the last three joint variables ($\theta_4$, $\theta_5$, $\theta_6$) to set the desired angular orientation of the end-effector. This part is identical to the calculation of $\theta_4$, $\theta_5$, and $\theta_6$ in the solution of Problem 3.18 above.

For this manipulator, we have

$$H_3^0 = A_1A_2A_3 = \begin{bmatrix} c_1c_2c_3 - c_1s_2s_3 & s_1 & c_1c_2s_3 + c_1c_3s_2 & d_2s_1 + a_2c_1c_2 \\ c_2c_3s_1 - s_1s_2s_3 & -c_1 & c_2s_1s_3 + c_3s_1s_2 & a_2c_2s_1 - d_2c_1 \\ c_2s_3 + c_3s_2 & 0 & s_2s_3 - c_2c_3 & d_1 + a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

$$H_6^3 = A_4A_5A_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_6s_4 - c_4c_5s_6 & c_4s_5 & d_6c_4s_5 \\ c_4s_6 + c_5c_6s_4 & c_4c_6 - c_5s_4s_6 & s_4s_5 & d_6s_4s_5 \\ -c_6s_5 & s_5s_6 & c_5 & d_4 + d_6c_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

Note that the rotation matrix part of $H_6^3 = A_4A_5A_6$ in (22) is exactly of the form of the Euler angle $(Z-Y-Z)$ rotation matrix representation. Hence, $\theta_4$, $\theta_5$, and $\theta_6$ are the Euler angles $(\phi, \theta, \psi)$ corresponding to the desired rotation matrix $R_6^3$, which is $(R_3^0)^TR$ where $R$ is the given rotation matrix corresponding to the desired end-effector orientation and $R_3^0$ is the rotation matrix corresponding to the first three joint variables ($\theta_1$, $\theta_2$, $\theta_3$). We know the rotation matrix $R_3^0$ from (21) as a function of ($\theta_1$, $\theta_2$, $\theta_3$). Hence, from the solutions that we found above for the first three joint variables ($\theta_1$, $\theta_2$, $\theta_3$), we find $R_3^0$ and therefore the desired value of $R_6^3$ as $(R_3^0)^TR$. Then, from this desired $R_6^3$ matrix, we can find $\theta_4$, $\theta_5$, and $\theta_6$ using the standard way of finding Euler angles corresponding to a given rotation matrix, exactly as in the solution for Problem 3.18 above.