Representations of Rotation Matrices

A rotation matrix R in 3D space is given by a 3×3 matrix. Hence, R has 9 elements. However, these elements are not all independent numerical quantities since there is a set of properties that any rotation matrix must satisfy. Specifically, in any rotation matrix R, each row (and each column) must be a unit vector. Also, any pair of two different rows must be mutually orthogonal and any pair of two different columns must be mutually orthogonal, i.e., denoting the $(i,j)^{th}$ element of R by r_{ij}, we have the equations:

$$\sum_{j=1}^{3} r_{ij}^2 = 1 \quad \text{for any } i = 1,2,3 \tag{1}$$

$$\sum_{k=1}^{3} r_{ik} r_{jk} = 0 \quad \text{for any } i = 1,2,3 \text{ and } j = 1,2,3 \text{ such that } i \neq j. \tag{2}$$

The equations (1) and (2) represent, respectively, the properties that each row of R is a unit vector and that any two different rows are mutually orthogonal. The equivalent properties about columns can be shown to follow directly from equations (1) and (2). Hence, the set of equations in (1) and (2) represent the set of constraints on the 9 elements of R. The total number of equations from (1) and (2) is 6. Hence, there are a total of 6 constraints on the 9 elements of R, implying that there are effectively 3 independent variables in R, i.e., any rotation matrix R can be written as a function of 3 independent variables. The choice of these 3 independent variables to define R is not unique and many different combinations of variables can be chosen in terms of which the set of all rotation matrices can be parameterized. Some of these choices of parameterizations of rotation matrices are discussed below.

Parameterization of rotation matrices utilizing three elementary rotation matrices with one repeated axis: Since a rotation matrix can be, as discussed above, characterized in terms of three independent variables, one way to write a parameterized form of rotation matrices is through a decomposition into elementary rotation matrices. There are many ways to do this. For example, the product $R_{x,\phi} R_{y,\theta} R_{x,\psi}$ is a rotation matrix where the notations $R_{x,\phi}$, $R_{y,\theta}$, etc., denote the elementary rotation matrices (e.g., $R_{x,\phi}$ is a rotation around the X axis by angle ϕ). This product of three elementary rotation matrices can be physically visualized as the composite rotation matrix obtained for the following transformation: rotate around the X axis by angle ϕ, then rotate around the new Y axis by angle θ, then rotate around the new X axis by angle ψ. Since each successive rotation in this formulation is defined in terms of the new (i.e., the current, and not the original) frame, the corresponding rotation matrices are combined by post-multiplying.

Note that the same product $R_{x,\phi} R_{y,\theta} R_{x,\psi}$ of three elementary matrices can also be equivalently physically visualized as the following transformation: rotate around the X axis by angle ψ, then rotate around the original Y axis by angle θ, then rotate around the original X axis by angle ϕ. Since each successive rotation in this formulation is defined in terms of the original (i.e., not the new or current) frame, the corresponding rotation matrices are combined by pre-multiplying.

Instead of a composition of three elementary matrices as $R_{x,\phi} R_{y,\theta} R_{x,\psi}$, i.e., as an $X-Y-X$ combination, we could use other combinations such as $X-Z-X$, $Z-X-Z$, etc. In general, there are six possible combinations: $X-Y-X$, $X-Z-X$, $Y-X-Y$, $Y-Z-Y$, $Z-X-Z$, and $Z-Y-Z$.

For example, considering the $Z-Y-Z$ combination, which is one of the more commonly used combinations, $R = R_{z,\phi} R_{y,\theta} R_{z.\psi}$, i.e.,

$$R = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} \tag{3}$$

where $c_\phi = \cos(\phi)$, $s_\phi = \sin(\phi)$ $c_\psi = \cos(\psi)$, $s_\psi = \sin(\psi)$ $c_\theta = \cos(\theta)$, $s_\theta = \sin(\theta)$. Given a set of three numbers ϕ, θ, and ψ denoting the Z-Y-Z Euler angles, the corresponding rotation matrix can be found directly from (3).

Also, given a rotation matrix R with its elements denoted as r_{ij} (i.e., the $(i,j)^{th}$ element of R is denoted as r_{ij}), the corresponding Z-Y-Z Euler angles can be found by solving for ϕ, θ, and ψ using equation (3). For example, using the $(3,3)$ element of R, we have $r_{33} = c_\theta$. Hence, $\theta = \cos^{-1}(r_{33})$. Note that the function $\cos^{-1}$ is not unique, but has, in general, two possible values (within a 2π angular range). Hence, there are two possible values of θ; for each of these values of θ, the choices of ϕ and ψ are unique as defined below. Hence, given a rotation matrix, there are, typically, two possible solutions for the corresponding Z-Y-Z Euler angles. In the special case that the $(3,3)$ element of R is 1, we get $\theta = 0$. In this case, the Euler angle parameterization of the rotation matrix reduces to $R = R_{z,\phi} R_{z,\psi} = R_{z,\phi+\psi}$; hence, the sum $\phi + \psi$ can be found from the given

rotation matrix R but not the individual angles ϕ and ψ (i.e., an infinite number of possible solutions for ϕ and ψ). Similarly, if the $(3,3)$ element of a given rotation matrix R is -1, then we get $\theta = \pi$ and it can be shown that the difference $\phi - \psi$ can be found but not the individual angles ϕ and ψ. Except for the special cases that the $(3,3)$ element of R is +1 or -1, the angle θ has two possible values as discussed above; one of these values will have a positive s_θ and the other value will have a negative s_θ. Using the $(3,1)$ and $(3,2)$ elements of R, we see that $s_\theta c_\psi = -r_{31}$ and $s_\theta s_\psi = r_{32}$. Hence, $\psi = \text{Atan2}(-r_{31}, r_{32})$ for the case of positive s_θ and $\psi = \text{Atan2}(r_{31}, -r_{32})$ for the case of negative s_θ. Similarly, using the $(1,3)$ and $(2,3)$ elements of R, we see that $c_\phi s_\theta = r_{13}$ and $s_\phi s_\theta = r_{23}$. Hence, $\phi = \text{Atan2}(r_{13}, r_{23})$ for the case of positive s_θ and $\phi = \text{Atan2}(-r_{13}, -r_{23})$ for the case of negative s_θ.
Note on Atan2 function: Here, Atan2 denotes the two-argument arc-tangent function, i.e., Atan2(x, y) is essentially $\tan^{-1}(y/x)$, but also takes into account the signs of x and y. Hence, Atan2(x, y) is the angle α such that $\cos(\alpha) = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin(\alpha) = \frac{y}{\sqrt{x^2+y^2}}$. Note that the order of the arguments in this definition of the Atan2 function is opposite to the C/C++ atan2 function (i.e., atan2(y,x) using the atan2 function in C/C++ is equivalent to Atan2(x, y) defined above).

Roll-Pitch-Yaw angles: Another way to parameterize rotation matrices is using three independent variables representing rotations around X, Y, and Z axes. These rotation angles are called roll (around X axis), pitch (around Y axis), and yaw (around Z axis). Since multiplication of rotation matrices is not commutative, different orders of rotations around X, Y, and Z axes would yield physically different composite rotations, i.e., for example, $R_{x,\theta_x} R_{y,\theta_y} R_{z,\theta_z}$ is, in general, not equal to $R_{z,\theta_z} R_{y,\theta_y} R_{x,\theta_x}$, i.e., the order of the elementary rotation matrices is important to specify. One commonly used convention is $R_{z,\theta_z} R_{y,\theta_y} R_{x,\theta_x}$. This product of three elementary rotation matrices can be physically visualized as the composite rotation matrix obtained for the following transformation: rotate around the Z axis by angle θ_z, then rotate around the new Y axis by angle θ_y, then rotate around the new X axis by angle θ_x. Since each successive rotation in this formulation is defined in terms of the new (i.e., the current, and not the original) frame, the corresponding rotation matrices are combined by post-multiplying. Note that the same product $R_{z,\theta_z} R_{y,\theta_y} R_{x,\theta_x}$ of three elementary matrices can also be equivalently physically visualized as the following transformation: rotate around the X axis by angle θ_x, then rotate around the original Y axis by angle θ_y, then rotate around the original z axis by angle θ_z; since each successive rotation in this formulation is defined in terms of the original (i.e., not the new or current) frame, the corresponding rotation matrices are combined by pre-multiplying.

With $R = R_{z,\theta_z} R_{y,\theta_y} R_{x,\theta_x}$, we get

$$R = \begin{bmatrix} c_z & -s_z & 0 \\ s_z & c_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_y & 0 & s_y \\ 0 & 1 & 0 \\ -s_y & 0 & c_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_x & -s_x \\ 0 & s_x & c_x \end{bmatrix} = \begin{bmatrix} c_z c_y & -s_z c_x + c_z s_y s_x & s_z s_x + c_z s_y c_x \\ s_z c_y & c_z c_x + s_z s_y s_x & -c_z s_x + s_z s_y c_x \\ -s_y & c_y s_x & c_y c_x \end{bmatrix} \tag{4}$$

where $c_x = \cos(\theta_x)$, $s_x = \sin(\theta_x)$, $c_y = \cos(\theta_y)$, $s_y = \sin(\theta_y)$, $c_z = \cos(\theta_z)$, and $s_z = \sin(\theta_z)$. Given a set of three numbers θ_x, θ_y, and θ_z denoting the roll, pitch, and yaw angles, the corresponding rotation matrix can be found directly from (4).

Also, given a rotation matrix R with its elements denoted as r_{ij} (i.e., the $(i,j)^{th}$ element of R is denoted as r_{ij}), the corresponding roll, pitch, and yaw angles can be found by solving for θ_x, θ_y, and θ_z using equation (4). For example, using the $(3,1)$ element of R, we have $r_{31} = -s_y$. Hence, $\theta_y = \sin^{-1}(-r_{31})$. Note that the function $\sin^{-1}$ is not unique, but has, in general, two possible values (within a 2π angular range). Hence, there are two possible values of θ_y; one of these two values will correspond to a positive c_y and the other value will correspond to a negative c_y. For each of these values of θ_y, the choices of θ_x and θ_z are unique. Using the $(1,1)$ and $(2,1)$ elements of R, we see that $c_z c_y = r_{11}$ and $s_z c_y = r_{21}$. Hence, $\theta_z = \text{Atan2}(r_{11}, r_{21})$ for the case of positive c_y and $\theta_z = \text{Atan2}(-r_{11}, -r_{21})$ for the case of negative c_y. Similarly, using the $(3,2)$ and $(3,3)$ elements of R, we see that $c_y s_x = r_{32}$ and $c_y c_x = r_{33}$. Hence, $\theta_x = \text{Atan2}(r_{33}, r_{32})$ for the case of positive c_y and $\theta_x = \text{Atan2}(-r_{33}, -r_{32})$ for the case of negative c_y. Hence, given a rotation matrix, there are, typically, two possible solutions for the corresponding roll, pitch, and yaw angles.

In the special case that the $(3,1)$ element of R is 1 or -1, we get $\theta_y = \frac{3\pi}{2}$ or $\frac{\pi}{2}$, respectively. In this case, the r_{11}, r_{21}, r_{32}, and r_{33} elements are all zero. Hence, θ_x and θ_z would need to be solved for using the expressions in the $(1,2)$, $(1,3)$, $(2,2)$, and $(2,3)$ elements. Except in this special case, it is seen above that θ_x can be found from the $(3,2)$ and $(3,3)$ elements and θ_z can be found from the $(1,1)$ and $(2,1)$ elements.

Axis-angle representation: Instead of parameterizing a rotation matrix as a product of three elementary rotation matrices (as in the Euler angles or in the roll-pitch-yaw parameterizations defined above), an alternative parameterization is to characterize a rotation matrix as a single rotation around an appropriate axis. In this axis-angle representation, a rotation matrix is characterized as a rotation by an angle θ around an axis direction (a unit vector) k, and is denoted as $R_{k,\theta}$. This combination of a unit vector k and an angle θ corresponds to three independent variables (as expected since, as discussed above, a rotation matrix can be parameterized in terms of three independent variables) since the constraint that k is a unit vector (and therefore has unit magnitude) reduces the number of effectively independent variables in k to 2.

Given $k = [k_x, k_y, k_z]^T$ with $k_x^2 + k_y^2 + k_z^2 = 1$, the rotation matrix $R_{k,\theta}$ corresponding to a rotation around axis k by an angle θ can be found using a few different ways as described below.

One way to find $R_{k,\theta}$ is to utilize the formula that we had found earlier for a rotation performed around a different frame (i.e., if a rotation from a frame 1 to a frame 2 is defined numerically as R in terms of a frame c, then the rotation matrix R_2^1 is given by $R_2^1 = (R_1^c)^T R(R_1^c)$). For example, if we consider a coordinate frame c in which the Z axis is along the direction of k, then in this frame c, the rotation around the axis k is simply an elementary rotation around the Z axis. Hence, in frame c, the rotation matrix is defined numerically as $R_{z,\theta}$. With frame 1 being the original coordinate frame and frame 2 being the final coordinate frame after the rotation around axis k by angle θ, the rotation matrix R_2^1 can therefore be found as $R_{k,\theta} = R_2^1 = (R_1^c)^T R_{z,\theta}(R_1^c)$. To find the rotation matrix R_1^c, first note that the exact definition of the frame c is not important as long as its Z axis is chosen along direction k, i.e., there are no specific constraints on the X and Y axes in the frame c. The condition that the Z axis of the frame c is along the direction of the specified vector k can be characterized as the requirement that if a point p at unit distance along the direction of the vector k is considered, its coordinates in the frame c are of the form $[0, 0, 1]^T$; also, the coordinates of the point p in the frame 1 are $[k_x, k_y, k_z]^T$ since k is a unit vector that is numerically defined relative to the original frame (i.e., frame 1). Hence,

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_1^c \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}. \tag{5}$$

Since $(R_1^c)^{-1} = (R_1^c)^T$, we get

$$\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = (R_1^c)^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{6}$$

To find a choice of a rotation matrix R_1^c that satisfies this condition, we can use, for example, the roll-pitch-yaw parameterization, i.e., $R_1^c = R_{z,\theta_z} R_{y,\theta_y} R_{x,\theta_x}$. Using (4), this results in the equations $-s_y = k_x$, $c_y s_x = k_y$, and $c_y c_x = k_z$, where $c_x = \cos(\theta_x)$, $s_x = \sin(\theta_x)$, $c_y = \cos(\theta_y)$, and $s_y = \sin(\theta_y)$. Hence, we get

$$s_y = -k_x \quad ; \quad c_y = \sqrt{k_y^2 + k_z^2} \quad ; \quad s_x = \frac{k_y}{\sqrt{k_y^2 + k_z^2}} \quad ; \quad c_x = \frac{k_z}{\sqrt{k_y^2 + k_z^2}}. \tag{7}$$

Since there is no explicit condition on θ_z, we can pick $\theta_z = 0$, i.e., $R_1^c = R_{y,\theta_y} R_{x,\theta_x}$ with θ_x and θ_y defined by the conditions in (7). Finally, we get $R_{k,\theta} = R_2^1 = (R_1^c)^T R_{z,\theta}(R_1^c) = R_{x,-\theta_x} R_{y,-\theta_y} R_{z,\theta} R_{y,\theta_y} R_{x,\theta_x}$. Expanding out this matrix product and doing some algebraic simplifications, we get

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix} \tag{8}$$

where $c_\theta = \cos(\theta)$, $s_\theta = \sin(\theta)$, and $v_\theta = \mathrm{vers}(\theta) = 1 - \cos(\theta)$.

Note that in the derivation of the axis-angle representation above, the choice of R_1^c is not unique since the only condition on this matrix is that the Z axis in frame c should correspond with the vector k in the original coordinate frame. Hence, we could, for example, have found the rotation matrix R_1^c using the Euler angle representation above instead of roll-pitch-yaw representation; the final rotation matrix $R_{k,\theta}$ would be identical to (8) using any such R_1^c.

Another way to derive the axis-angle representation shown in (8) is using a geometrical analysis from the definition of rotation around an axis vector k. For this purpose, consider a point p in 3D space. Recall that one interpretation of a rotation matrix is as a physical rotation/transformation of a rigid body, e.g., as a mapping from a point in 3D space to a new point. In this interpretation of rotation matrices, a rotation matrix R_2^1 transforms a point p to a new point $p_{new} = R_2^1 p$. Given an axis vector k, the point p can be written as a combination of two parts, a part that is along the direction k and a part that is perpendicular to the direction k, i.e., as $p = p_a + p_p$, where p_a is along the direction k and p_p is perpendicular to the direction k. The part along the direction k is, by definition, $p_a = (p.k)k$. Also, $p_p = p - p_a$. When applying a rotation around the vector k, the part along k does not change and the part perpendicular to k will be rotated by the angle θ, i.e., $p_{new} = p_a + \tilde{p}_p$ where $\tilde{p}_p$ denotes a rotated version of p_p. To find $\tilde{p}_p$, note that if p_p is rotated by 90 degrees around k, then we would get $k \times p_p$ (this is, by its definition, the vector perpendicular to both k and p_p); the rotation of p_p is in the plane perpendicular to k; hence, in general, if p_p is rotated by angle θ around k, then we get $\tilde{p}_p = p_p \cos(\theta) + (k \times p_p)\sin(\theta)$. Note that since $k \times p_a = 0$, we get $k \times p_p = k \times p$. Therefore,

$$p_{new} = p_a + p_p c_\theta + (k \times p)s_\theta \ = p_a + (p - p_a)c_\theta + (k \times p)s_\theta \ = pc_\theta + p_a v_\theta + (k \times p)s_\theta \ = pc_\theta + (p.k)kv_\theta + (k \times p)s_\theta.$$

We can write $p = Ip$ where I is the 3×3 identity matrix. Also, $(p.k)k$ can be expanded as $(p.k)k = Pp$ where the matrix P is defined as

$$P = \begin{bmatrix} k_x^2 & k_x k_y & k_x k_z \\ k_x k_y & k_y^2 & k_y k_z \\ k_x k_z & k_y k_z & k_z^2 \end{bmatrix} p. \tag{9}$$

Also, by the definition of the cross product, $k \times p = S(k)p$ where $S(k)$ is the skew-symmetric matrix defined as

$$S(k) = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}. \tag{10}$$

Hence, $p_{new} = (Ic_\theta + Pv_\theta + S(k)s_\theta)p$. Therefore, $R_2^1 = Ic_\theta + Pv_\theta + S(k)s_\theta$. Expanding out the elements of this matrix using (9) and (10), we get an identical matrix to (8).

Given a unit vector k and an angle θ, the rotation matrix $R_{k,\theta}$ can be found directly from (8). Also, given a rotation matrix R with its elements denoted as r_{ij} (i.e., the $(i,j)^{th}$ element of R is denoted as r_{ij}), the corresponding axis vector k and angle θ can be found by solving for k_x, k_y, k_z, and θ using equation (8). Using $r_{11} = k_x^2 v_\theta + c_\theta$, $r_{22} = k_y^2 v_\theta + c_\theta$, and $r_{33} = k_z^2 v_\theta + c_\theta$, and using the property $k_x^2 + k_y^2 + k_z^2 = 1$, we get $v_\theta + 3c_\theta = r_{11} + r_{22} + r_{33}$. Hence, since $v_\theta = (1 - c_\theta)$, we get $1 + 2c_\theta = r_{11} + r_{22} + r_{33}$. Therefore, $\theta = \cos^{-1}(0.5[r_{11} + r_{22} + r_{33} - 1])$. Also, using the off-diagonal elements of $R_{k,\theta}$ from (8), we get

$$k_x = \frac{r_{32} - r_{32}}{2s_\theta} \quad ; \quad k_y = \frac{r_{13} - r_{31}}{2s_\theta} \quad ; \quad k_z = \frac{r_{21} - r_{12}}{2s_\theta} \tag{11}$$

where s_θ is found as $\sqrt{1 - c_\theta^2}$ from the definition of c_θ given above.

Note on cross product as skew-symmetric matrix: Given vectors $k_1 = [k_{1x}, k_{1y}, k_{1z}]^T$ and $k_2 = [k_{2x}, k_{2y}, k_{2z}]^T$, the cross product of these vectors can be found as follows: denoting the unit vectors in the X, Y, and Z axis directions by i, j, and k, respectively, we have $k_1 = k_{1x}i + k_{1y}j + k_{1z}k$ and $k_2 = k_{2x}i + k_{2y}j + k_{2z}k$. From the definition of cross product, we have $i \times j = k$, $j \times i = -k$, $i \times k = -j$, $k \times i = j$, $j \times k = i$, and $k \times j = -i$. Hence, expanding, $k_1 \times k_2$, we can write $k_1 \times k_2$ in the form $S(k_1)k_2$ where $S(k_1)$ is the skew-symmetric matrix

$$S(k_1) = \begin{bmatrix} 0 & -k_{1z} & k_{1y} \\ k_{1z} & 0 & -k_{1x} \\ -k_{1y} & k_{1x} & 0 \end{bmatrix}. \tag{12}$$

Quaternion representation: A quaternion is a 4×1 vector denoted as $q = [q_1, q_2, q_3, q_4]^T$. A unit quaternion is a quaternion with unit magnitude, i.e., $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$. The quaternion representation can be defined in terms of the axis-angle representation as follows: A rotation matrix given in axis-angle representation as $R_{k,\theta}$ can be represented as a unit quaternion $q = [q_1, q_2, q_3, q_4]^T$ with elements

$$q_1 = \cos\left(\frac{\theta}{2}\right) \quad ; q_2 = k_x \sin\left(\frac{\theta}{2}\right) \quad ; q_3 = k_y \sin\left(\frac{\theta}{2}\right) \quad ; q_4 = k_z \sin\left(\frac{\theta}{2}\right). \tag{13}$$

Using the trigonometric identities that $s_\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$, $c_\theta = \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2}) = 1 - 2\sin^2(\frac{\theta}{2}) = 2\cos^2(\frac{\theta}{2}) - 1$, the rotation matrix $R_{k,\theta}$ corresponding to the equivalent quaternion q in (13) can be written as

$$R_{k,\theta} = \begin{bmatrix} 1 - 2q_3^2 - 2q_4^2 & 2q_2q_3 - 2q_4q_1 & 2q_2q_4 + 2q_3q_1 \\ 2q_2q_3 + 2q_4q_1 & 1 - 2q_2^2 - 2q_4^2 & 2q_3q_4 - 2q_2q_1 \\ 2q_2q_4 - 2q_3q_1 & 2q_3q_4 + 2q_2q_1 & 1 - 2q_2^2 - 2q_3^2 \end{bmatrix}. \tag{14}$$

Given a quaternion $q = [q_1, q_2, q_3, q_4]^T$, the corresponding rotation matrix can be found directly from (14). Also, given a rotation matrix R with its elements denoted as r_{ij} (i.e., the $(i,j)^{th}$ element of R is denoted as r_{ij}), the corresponding quaternion can be found by first finding the equivalent axis vector k and angle θ and then finding the corresponding quaternion using (13).