b) By the associative property of matrix multiplication,

$$(x_1, x_2)x_3 \;=\; x_1(x_2x_3).$$

for $x_1, x_2, x_3 \in SO(n)$

c) The $n \times n$ identity matrix satisfies the third property.

d) Since $x^T x = xx^T = I$, it follows that $x^T = x^{-1}$

2-8 For a rotation of θ about the x axis we have

$$\begin{aligned} x_0 \cdot x_1 &= 1 \\ y_0 \cdot y_1 &= \cos\theta \\ z_0 \cdot z_1 &= \cos\theta \\ z_0 \cdot y_1 &= \sin\theta \\ y_0 \cdot z_1 &= -\sin\theta \end{aligned}$$

and all other dot products are zero. Substituting into the rotation matrix in Section 2.2.2 gives

$$R_0^1 \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

For a rotation of θ about the y axis we have

$$\begin{aligned} y_0 \cdot y_1 &= 1 \\ x_0 \cdot x_1 &= \cos\theta \\ z_0 \cdot z_i &= \cos\theta \\ z_0 \cdot x_1 &= -\sin\theta \\ x_0 \cdot z_1 &= \sin\theta \end{aligned}$$

and all other dot products are zero. Again using the rotation matrix gives

$$R \;=\; \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

2-9 Let

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2).$$

From Cramer's rule and the fact that $A \in SO(3)$ we have

$$A^{-1} \;=\; \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

which implies that $a = d$ and $b = -c$. Thus

$$A \;=\; \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$$

with $\det A = 1 = a^2 + c^2$. Define $\theta = \tan^{-1}(c/a)$. Then $\cos\theta = a$ and $\sin\theta = c$.

2-10

$$R = R_{y,\psi}R_{x,\phi}R_{z,\theta}$$

2-11

$$R = R_{z,\theta}R_{x,\phi}R_{x,\psi}$$

2-12

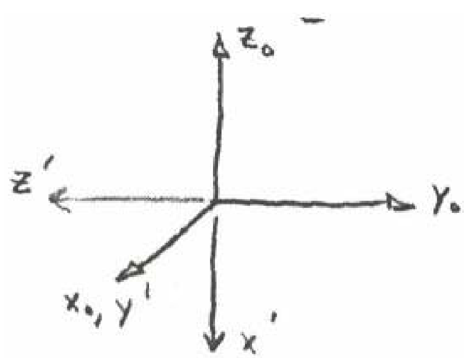
$$R = R_{z,\alpha}R_{x,\phi}R_{z,\theta}R_{x,\psi}$$

2-13

$$R = R_{z,\alpha}R_{z,\theta}R_{x,\phi}R_{x,\psi}$$

2-14

$$R = R_{y,\frac{\pi}{2}}R_{x,\frac{\pi}{2}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$



2-15

$$R_3^2 = R_1^2 R_3^1 \quad \text{where} \quad R_1^2 = (R_2^1)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Therefore,

$$R_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \end{bmatrix}$$

2-16 If r_{11}, r_{21} are not both zero, then

- $c_\theta \neq 0$ and $r_{31} = -s_\theta \neq \pm 1$
- r_{32}, r_{33} are not both zero.

so, $c_\theta = \pm\sqrt{1 - r_{31}^2}$ and $\theta = $Atan2$\big(\pm\sqrt{1-r_{31}^2}, r_{31}\big)$.
Follow a development similar to that provided for the Euler angles to find ϕ, θ, and ψ.

2-17 Straightforward; follow directions given in sentence preceding the equation.

2-18 Straightforward. Substitute for r_{ij} in Equation (2.45) using the matrix elements given in Equation (2.43).

2-19 If λ is an eigenvalue of R and k is a unit eigenvector corresponding to λ then, $Rk = \lambda k$. Since R is a rotation $\|Rk\| = \|k\|$. This implies that $|\lambda| = 1$, i.e., the eigenvalues of R are on the unit circle in the complex plane. Since the characteristic polynomial of R is of degree three at least one eigenvalue of R must be real. Hence $+1$ or -1 is an eigenvalue of R. Now, since $+1 = \det R = \lambda_1\lambda_2\lambda_3$ where $\{\lambda_1, \lambda_2, \lambda_3\}$ is the set of eigenvalues of R, it is easy to see that if -1 is an eigenvalue then $\{\lambda_1, \lambda_2, \lambda_3\} = \{-1, -1, +1\}$. In any case $+1$ is always an eigenvalue of R.

The vector k defines the axis of rotation in the angle/axis representation of R.

2-20

$$R_{k,\theta} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} + \frac{1}{\sqrt{3}} \\ \frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{1}{3} - \frac{1}{\sqrt{3}} \\ \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} \end{bmatrix}$$

2-21 Straightforward.

2-22

$$\begin{aligned} & R_{x,\theta} R_{y,\phi} R_{z,\pi} R_{y,-\phi} R_{x,-\theta} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} -\cos(2\phi) & -2\cos(\phi)\sin(\phi)\sin(\theta) & \cos(\theta)\sin(2\phi) \\ -2\cos(\phi)\sin(\phi)\sin(\theta) & -\cos(\theta)^2 - \cos(2\phi)\sin(\theta)^2 & -\cos(\phi)^2\sin(2\theta) \\ \cos(\theta)\sin(2\phi) & -\cos(\phi)^2\sin(2\theta) & \cos(\phi)^2\cos(\theta)^2 - \cos(\theta)^2\sin(\phi)^2 - \sin(\theta)^2 \end{bmatrix} \end{aligned}$$

2-23

$$R = R_{y,90} R_{z,45} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right) = \cos^{1}\left(\frac{\frac{\sqrt{2}}{2} - 1}{2}\right) = 98.42^\circ$$