Solutions for Problems 2.17 and 2.24

2.17.

Given the rotation axis $k = [k_x, k_y, k_z]^T$ with $k_x^2 + k_y^2 + k_z^2 = 1$, one way to find the rotation matrix $R_{k,\theta}$ corresponding to a rotation around axis k by an angle θ is to utilize the formula for a rotation performed around a different frame (i.e., if a rotation from a frame 1 to a frame 2 is defined numerically as R in terms of a frame c, then the rotation matrix R_2^1 is given by $R_2^1 = (R_1^c)^T R (R_1^c))$. For example, if we consider a coordinate frame c in which the Z axis is along the direction of k, then in this frame c, the rotation around the axis k is simply an elementary rotation around the Z axis. Hence, in frame c, the rotation matrix is defined numerically as $R_{z,\theta}$. With frame 1 being the original coordinate frame and frame 2 being the final coordinate frame after the rotation around axis k by angle θ, the rotation matrix R_2^1 can therefore be found as $R_{k,\theta} = R_2^1 = (R_1^c)^T R_{z,\theta}(R_1^c)$. To find the rotation matrix R_1^c, first note that the exact definition of the frame c is not important as long as its Z axis is chosen along direction k, i.e., there are no specific constraints on the X and Y axes in the frame c. The condition that the Z axis of the frame c is along the direction of the specified vector k can be characterized as the requirement that if a point p at unit distance along the direction of the vector k is considered, its coordinates in the frame c are of the form $[0, 0, 1]^T$; also, the coordinates of the point p in the frame 1 are $[k_x, k_y, k_z]^T$ since k is a unit vector that is numerically defined relative to the original frame (i.e., frame 1). Hence,

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_1^c \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}. \tag{1}$$

Since $(R_1^c)^{-1} = (R_1^c)^T$, we get

$$\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = (R_1^c)^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2}$$

To find a choice of a rotation matrix R_1^c that satisfies this condition, we can use, for example, the roll-pitch-yaw parameterization, i.e., $R_1^c = R_{z,\theta_z} R_{y,\theta_y} R_{x,\theta_x}$. Using the equations for the roll-pitch-yaw parameterization, this results in the equations $-s_y = k_x$, $c_y s_x = k_y$, and $c_y c_x = k_z$, where $c_x = \cos(\theta_x)$, $s_x = \sin(\theta_x)$, $c_y = \cos(\theta_y)$, and $s_y = \sin(\theta_y)$. Hence, we get

$$s_y = -k_x \quad ; \quad c_y = \sqrt{k_y^2 + k_z^2} \quad ; \quad s_x = \frac{k_y}{\sqrt{k_y^2 + k_z^2}} \quad ; \quad c_x = \frac{k_z}{\sqrt{k_y^2 + k_z^2}}. \tag{3}$$

Since there is no explicit condition on θ_z, we can pick $\theta_z = 0$, i.e., $R_1^c = R_{y,\theta_y} R_{x,\theta_x}$ with θ_x and θ_y defined by the conditions in (3). Finally, we get $R_{k,\theta} = R_2^1 = (R_1^c)^T R_{z,\theta}(R_1^c) = R_{x,-\theta_x} R_{y,-\theta_y} R_{z,\theta} R_{y,\theta_y} R_{x,\theta_x}$, i.e.,

$$R_1^c = R_{y,\theta_y} R_{x,\theta_x} = \begin{bmatrix} c_y & 0 & s_y \\ 0 & 1 & 0 \\ -s_y & 0 & c_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_x & -s_x \\ 0 & s_x & c_x \end{bmatrix} \tag{4}$$

$$R_{k,\theta} = (R_1^c)^T \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1^c \tag{5}$$

where s_x, c_x, s_y, and c_y are as found before in (3) and s_θ and c_θ denote $\sin(\theta)$ and $\cos(\theta)$, respectively.

Expanding out the matrix products above and doing some algebraic simplifications, we get

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix} \tag{6}$$

where $v_\theta = \mathrm{vers}(\theta) = 1 - \cos(\theta)$.

A symbolic code in MATLAB to compute and algebraically simplify the matrix products above is available at: `http://crrl.poly.edu/5223/axis_angle.m`.

2.24. The rotation matrix corresponding to the given Euler angles ($\phi = \frac{\pi}{2}$, $\theta = 0$, $\psi = \frac{\pi}{4}$) is

$$R_1^0 = R_{z,\phi} R_{y,\theta} R_{z,\psi} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

Hence, the x_1 axis relative to the base frame is $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]^T$.