Euler-Lagrange formulation for dynamics of an n-link manipulator

In the Euler-Lagrange dynamics formulation, the dynamics of an n-link manipulator are written as:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \quad , i = 1, \ldots, n \tag{1}$$

where the Lagrangian $\mathcal{L}$ is defined as $\mathcal{L} = K - P$ with K being the kinetic energy of the system and P being the potential energy of the system. τ_i is the force/torque corresponding to the i^{th} joint of the manipulator.

Given an n-link manipulator, the kinetic energy of the manipulator can be written as:

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q} \tag{2}$$

with $D(q)$ defined as

$$D(q) = \sum_{i=1}^{n} \left\{ m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i^0 I_i (R_i^0)^T J_{\omega_i} \right\}. \tag{3}$$

Here, m_i denotes the mass of the i^{th} link, I_i denotes the inertia matrix in the link-fixed frame with its origin at the center of mass of the link, J_{v_i} denotes the velocity Jacobian for the center of mass of link i, and J_{ω_i} denotes the angular velocity Jacobian for link i, i.e., the velocity (written relative to frame 0) of the center of mass of link i is written as $v_i^{(0)} = J_{v_i}(q)\dot{q}$ and the angular velocity (written relative to frame 0) of link i is written as $\omega_i^{(0)} = J_{\omega_i}(q)\dot{q}$. Note that $D(q)$ as defined in equation (3) is a symmetric matrix.

Inertia matrix: Note that since I_i is the inertia matrix written relative to the link-fixed frame, $R_i^0 I_i (R_i^0)^T$ is the inertia matrix written relative to an inertial frame (with the origin of the frame at the center of mass of the link). The inertia matrix is typically a constant matrix when written in the link-fixed frame. The inertia matrix I_i is a 3×3 symmetric matrix whose elements can be found by a volume integration, i.e.,

$$I_i = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{4}$$

where $I_{xx} = \int\int\int (y^2 + z^2)\rho(x, y, z)dx\,dy\,dz$, $I_{yy} = \int\int\int (x^2 + z^2)\rho(x, y, z)dx\,dy\,dz$,
$I_{zz} = \int\int\int (x^2 + y^2)\rho(x, y, z)dx\,dy\,dz$, $I_{xy} = I_{yx} = -\int\int\int xy\rho(x, y, z)dx\,dy\,dz$,
$I_{xz} = I_{zx} = -\int\int\int xz\rho(x, y, z)dx\,dy\,dz$, and $I_{yz} = I_{zy} = -\int\int\int yz\rho(x, y, z)dx\,dy\,dz$.
$\rho(x, y, z)$ denotes the mass density of the rigid body at the position (x, y, z). The integrals in the expressions for I_{xx}, I_{xy}, etc., are computed over the entire volume of the rigid body.

The potential energy of the n-link manipulator can be written as

$$P = \sum_{i=1}^{n} m_i g^T r_{ci} \tag{5}$$

where g has magnitude equal to acceleration due to gravity (≈ 9.81 m/s^2) and is along the direction opposite to gravity (written relative to frame 0) and r_{ci} is the position of the center of mass of link i (again, written relative to frame 0). The vector g is defined to be in the direction opposite to gravity since the potential energy should increase with increasing distance from the earth's surface. Alternatively, g can be defined to be in the direction of gravity and a negative sign can be added on the right hand side of equation (5) (i.e., $P = -\sum_{i=1}^{n} m_i g^T r_{ci}$).

If the kinetic energy and potential energy functions that were found as in equations (2) and (5) are algebraically simple, then it is easy to simply substitute $\mathcal{L} = K - P$ into the Euler-Lagrange equation (1) to find the dynamics equations. Alternatively, a more formal procedure is to use the Christoffel symbols defined below.

From the matrix $D(q)$ that was found in equation (3), the Christoffel symbols c_{ijk} are found as:

$$c_{ijk} = \frac{1}{2}\left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \tag{6}$$

where expressions such as d_{ij} denote the $(i, j)^{th}$ element, etc., of the matrix $D(q)$. The Christoffel symbols need to be found for all i, j, k in $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, n\}$. In writing the Christoffel symbols, we can use the property that $c_{ijk} = c_{jik}$ to reduce the number of Christoffel symbols that need to be explicitly calculated by around a half.

From the potential energy (5), define the functions

$$g_k(q) = \frac{\partial P}{\partial q_k} \quad k = 1, \ldots, n. \tag{7}$$

The Euler-Lagrange dynamics equations can be written as:

$$\sum_{i=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k \quad k = 1, \ldots, n. \tag{8}$$

From the Christoffel symbols, define a matrix $C(q, \dot{q})$ to be the $n \times n$ matrix that has its $(k, j)^{th}$ element to be

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i. \tag{9}$$

Then, the Euler-Lagrange dynamics equations from (8) can be written in a matrix form as

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \tag{10}$$

where $g(q) = [g_1(q), \ldots, g_n(q)]^T$ and $\tau = [\tau_1, \ldots, \tau_n]^T$.

Some properties of the D and C matrices:

- The $D(q)$ matrix is symmetric and positive-definite.
- The matrix $N(q, \dot{q})$ defined as $N(q, \dot{q}) = \dot{D}(q) - 2C(q, \dot{q})$ is skew symmetric, i.e., $[N(q, \dot{q})]^T = -N(q, \dot{q})$.

Derivation of equation (8): Denoting the $(i, j)^{th}$ element of the matrix $D(q)$ by d_{ij}, the kinetic energy of the manipulator is seen from equation (2) to be of the form

$$K = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j. \tag{11}$$

The potential energy P depends only on q and does not depend on $\dot{q}$. Hence, we see that for any k in $1, \ldots, n$:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j. \tag{12}$$

Hence,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{j=1}^{n}\left\{\frac{d}{dt}d_{kj}(q)\right\}\dot{q}_j \tag{13}$$

$$= \sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j. \tag{14}$$

Also, note that

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}. \tag{15}$$

Hence, the Euler-Lagrange equation (1) can be written as:

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k. \tag{16}$$

By interchanging the dummy variables of summation, we can write $\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \sum_{j=1}^{n}\sum_{i=1}^{n}\left\{\frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_j\dot{q}_i = \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j$. Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j. \tag{17}$$

Therefore, from (16), we get

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k. \tag{18}$$

Hence, from the definition of the Christoffel symbols from (6), we get the dynamics equations shown in equation (8).