Solutions for Problems 4.13 and 4.15

4.13. A rotation matrix is given as $R = R_{z,\psi}R_{y,\theta}R_{z,\phi}$. Since $\dot{R}_{z,\psi} = S(\dot{\psi}k)R_{z,\psi}$, $\dot{R}_{y,\theta} = S(\dot{\theta}j)R_{y,\theta}$, and $\dot{R}_{z,\phi} = S(\dot{\phi}k)R_{z,\phi}$, where $j = [0,1,0]^T$ and $k = [0,0,1]^T$, we have

$$
\begin{aligned}
\dot{R} &= \dot{R}_{z,\psi}R_{y,\theta}R_{z,\phi} + R_{z,\psi}\dot{R}_{y,\theta}R_{z,\phi} + R_{z,\psi}R_{y,\theta}\dot{R}_{z,\phi} \\
&= S(\dot{\psi}k)R_{z,\psi}R_{y,\theta}R_{z,\phi} + R_{z,\psi}S(\dot{\theta}j)R_{y,\theta}R_{z,\phi} + R_{z,\psi}R_{y,\theta}S(\dot{\phi}k)R_{z,\phi} \\
&= S(\dot{\psi}k)R_{z,\psi}R_{y,\theta}R_{z,\phi} + R_{z,\psi}S(\dot{\theta}j)R_{z,\psi}^T R_{z,\psi}R_{y,\theta}R_{z,\phi} + R_{z,\psi}R_{y,\theta}S(\dot{\phi}k)R_{y,\theta}^T R_{z,\psi}^T R_{z,\psi}R_{y,\theta}R_{z,\phi} \\
&= S(\dot{\psi}k)R_{z,\psi}R_{y,\theta}R_{z,\phi} + S(R_{z,\psi}\dot{\theta}j)R_{z,\psi}R_{y,\theta}R_{z,\phi} + S(R_{z,\psi}R_{y,\theta}\dot{\phi}k)R_{z,\psi}R_{y,\theta}R_{z,\phi} \\
&= S(\dot{\psi}k + R_{z,\psi}\dot{\theta}j + R_{z,\psi}R_{y,\theta}\dot{\phi}k)R \qquad (1)
\end{aligned}
$$

Hence, $\dot{R} = S(\omega)R$ where

$$
\begin{aligned}
\omega &= \dot{\psi}k + R_{z,\psi}\dot{\theta}j + R_{z,\psi}R_{y,\theta}\dot{\phi}k \\
&= \{c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta}\}i + \{s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta}\}j + \{\dot{\psi} + c_\theta \dot{\phi}\}k \qquad (2)
\end{aligned}
$$

4.15. We are given two coordinate frames, 0 and 1, and the homogeneous transformation $H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$ between them. Denoting the coordinates of a point as p^0 and p^1 in the two coordinate frames, we have $p^0 = R_1^0 p^1 + d_1^0$. Hence, if R_1^0 and d_1^0 are constants, we have $\dot{p}^0 = R_1^0 \dot{p}^1$. Hence, the velocity of the given point in the 0-frame is $R_1^0[3,1,0]^T = [-1,3,0]^T$.