Example of dynamics computation (Euler-Lagrange and Newton-Euler formulations): Prismatic-Prismatic-Prismatic (PPP) manipulator (3-link Cartesian manipulator) – Solution to Problem 7.7

Consider a Prismatic-Prismatic-Prismatic (PPP) manipulator (3-link Cartesian manipulator). The dynamics of the manipulator will depend on the arrangement of the joints and links (i.e., which direction the first joint actuates in, which direction the second joint actuates in, etc.). Let the first joint actuate in the vertical direction (i.e., up and down in the direction of gravity). Let the coordinate system of the base frame (frame 0) be such that the z_0 axis is along the translation axis of the first joint, the y_0 axis is parallel to the translation axis of the second joint, and the x_0 axis is parallel to the translation axis of the third joint. Also, let gravity be along $-z_0$. The joint variables are $q_1 = d_1$, $q_2 = d_2$, and $q_3 = d_3$. Let the masses of the links be m_1, m_2, and m_3.

Since this manipulator has only prismatic joints, the inertia matrices of the links do not need to be calculated. However, since the part (a) of Problem 7.7 asks to compute the inertia matrices of the links, we can find the elements of the inertia matrices by the volume integrals for I_{xx}, I_{xy}, etc. The inertia matrix for each link will depend on the angular orientation of the link relative to the corresponding link-fixed frame. For example, if, according to the Denavit-Hartenberg (D-H) convention, we pick z_1 along the actuation axis of the second joint and pick x_1 to be perpendicular to both z_0 and z_1, then we would have x_1 to be parallel to x_0 and y_1 to be along $-z_0$. Then, y_1 is parallel to the edge of the link which is of length 1 and x_1 and z_1 are parallel to the edges which are of length $\frac{1}{4}$. Then, if the origin of the 1-frame is at the center of link 1 (which would be the case according to the D-H convention if the actuation axis of the second joint is connected centered with the center of the first link), we would have: $I_{xx} = \frac{m}{12}(L^2 + H^2) = \frac{17}{192}$, $I_{yy} = \frac{m}{12}(W^2 + H^2) = \frac{1}{96}$, $I_{zz} = \frac{m}{12}(W^2 + L^2) = \frac{17}{192}$. By symmetry, the cross products of inertia (I_{xy}, I_{xz}, etc.) are all zero. Therefore, the 3×3 inertia matrix for link 1 would be $I_1 = \begin{bmatrix} \frac{17}{192} & 0 & 0 \\ 0 & \frac{1}{96} & 0 \\ 0 & 0 & \frac{17}{192} \end{bmatrix}$. However, if we were to pick the 1-frame such that the origin of the 1-frame is at the end of the link, i.e., offset along y_1 axis by distance $\frac{1}{2}$, then, an additional $m_1\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ would be added to I_{xx} and I_{zz}; thus, the 3×3 inertia matrix for link 1 would be $I_{1,e} = \begin{bmatrix} \frac{65}{192} & 0 & 0 \\ 0 & \frac{1}{96} & 0 \\ 0 & 0 & \frac{65}{192} \end{bmatrix}$. The inertia matrices for links 2 and 3 can be found similarly given the angular orientations of these links relative to their link-fixed coordinate frames and the origins of the coordinate frames.

Some simplifying assumptions: Let the joint variable d_1 be defined to be the vertical component of the vector from the origin of frame 0 to the center of mass of link 1 (i.e., the potential energy of link 1 is $m_1 g d_1$). Also, assume that the joints and links are attached such that the centers of mass of the three links are in the same horizontal plane, i.e., the vertical elevations of the centers of mass of link 2 and link 3 are also d_1. Hence, the potential energies of the second and third links are $m_2 g d_1$ and $m_3 g d_1$. Also, assume that the linkage between links 2 and 3 is such that when joint 2 actuates, it shifts the center of mass of link 3 by distance q_2. Mathematically, since this is a purely prismatic manipulator (i.e., inertia matrices are not relevant), this is equivalent to assuming that the mass of each link is at the end of the link.

Euler-Lagrange formulation to find the dynamics: Since this manipulator has only prismatic joints, the angular velocity Jacobian matrices corresponding to the three links are

$$J_{\omega_1} = J_{\omega_2} = J_{\omega_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

We find the linear velocity Jacobian matrices of the three links to be:

$$J_{v_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad ; \quad J_{v_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad ; \quad J_{v_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2}$$

Hence, the matrix $D(q)$ is given by:

$$D(q) = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + m_3 J_{v_3}^T J_{v_3} + J_{\omega_1}^T R_1^0 I_1 (R_1^0)^T J_{\omega_1} + J_{\omega_2}^T R_2^0 I_2 (R_2^0)^T J_{\omega_2} + J_{\omega_3}^T R_3^0 I_3 (R_3^0)^T J_{\omega_3} \quad (3)$$

$$= \begin{bmatrix} m_1 + m_2 + m_3 & 0 & 0 \\ 0 & m_2 + m_3 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad (4)$$

Since $D(q)$ shown in equation (4) is a constant matrix, all the Christoffel symbols c_{ijk} are zero for this manipulator.

The potential energy of the manipulator is given by:

$$P = m_1 g q_1 + m_2 g q_1 + m_3 g q_1. \quad (5)$$

Hence,

$$g(q) = \begin{bmatrix} \frac{\partial P}{\partial q_1} \\ \frac{\partial P}{\partial q_2} \\ \frac{\partial P}{\partial q_3} \end{bmatrix} = \begin{bmatrix} (m_1 + m_2 + m_3) g \\ 0 \\ 0 \end{bmatrix}. \quad (6)$$

Hence, the dynamical equations of the manipulator are given by:

$$D(q) \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + g(q) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \quad (7)$$

where f_1, f_2, and f_3 are the applied forces at the three prismatic joints. Hence, the dynamical equations can be written as:

$$\begin{aligned} (m_1 + m_2 + m_3)\ddot{q}_1 + (m_1 + m_2 + m_3) g &= f_1 \\ (m_2 + m_3)\ddot{q}_2 &= f_2 \\ m_3 \ddot{q}_3 &= f_3. \end{aligned} \quad (8)$$

Newton-Euler formulation to find the dynamics: Since this manipulator has only prismatic joints, the torque equations in the Newton-Euler formulation do not need to be written and the force equations can be written in either body-fixed frame or inertial frame.

Using body-fixed frame:

- Forward recursion: The angular velocity of link 1 is $\omega_1 = [0, 0, 0]^T$. To assign the Denavit-Hartenberg coordinate frames, pick z_0 upward and x_0 pointing out of the page (i.e., y_0 to the right); pick z_1 pointing to the right (actuation axis of the second joint) and x_1 pointing out of the page (i.e., x_1 is parallel to x_0); pick z_2 pointing out of the page (actuation axis of the third joint) and x_2 pointing down (i.e., x_2 is parallel to $-z_0$). Pick z_3 along same direction as z_2 (with origin of frame 3 at the end-effector) and x_3 parallel to x_2. Then, we have y_1 parallel to $-z_0$, z_1 parallel to y_0, y_2 parallel to y_0, z_2 parallel to x_0. The Denavit-Hartenberg table for this manipulator is:

Link	a_i	α_i	d_i	θ_i
1	0	-90^o	q_1	0
2	0	90^o	q_2	90^o
3	0	0	q_3	0

  By the simplifying assumptions mentioned on page 1, the linear accelerations of the center of mass of link 1 and of the end of link 1 are both $a_{c,1} = a_{e,1} = R_0^1 \ddot{q}_1 z_0 = -\ddot{q}_1 \vec{j}$. Note that we can, in general, define the end of the link to be at any convenient location (fixed to the link) as long as we keep the equations consistent. Here, the notations $\vec{i}$, $\vec{j}$, and $\vec{k}$ denote the 3×1 unit vectors, i.e., $\vec{i} = [1, 0, 0]^T$, $\vec{j} = [0, 1, 0]^T$, $\vec{k} = [0, 0, 1]^T$. The gravity vector written in frame 1 is given by $g_1 = g\vec{j}$. The angular velocity of link 2 is $\omega_2 = 0$. Again, by the simplifying assumptions listed on page 1, the linear acceleration of the center of mass of link 2 and the end of link 2 are: $a_{c,2} = a_{e,2} = R_1^2 a_{e,1} + R_0^2 \ddot{q}_2 y_0 = -R_1^2 \ddot{q}_1 \vec{j} + \ddot{q}_2 \vec{j} = -\ddot{q}_1 \vec{i} + \ddot{q}_2 \vec{j}$. The gravity vector written in frame 2 is also $g_2 = g\vec{i}$. The angular velocity of link 3 is $\omega_3 = 0$. Again, by the simplifying assumptions listed on page 1, the linear acceleration of the center of mass of link 3 is: $a_{c,3} = a_{e,3} = R_2^3 a_{e,2} + R_0^3 \ddot{q}_3 x_0 = R_2^3(-\ddot{q}_1 \vec{i} + \ddot{q}_2 \vec{j}) + \ddot{q}_3 \vec{k} = -\ddot{q}_1 \vec{i} + \ddot{q}_2 \vec{j} + \ddot{q}_3 \vec{k}$. The gravity vector written in frame 3 is also $g_3 = g\vec{i}$.

- Backward recursion: The torque equations do not need to be written since this manipulator has only prismatic joints. Start with $f_4 = 0$. Then, $f_3 = m_3(a_{c,3} - g_3) = m_3[-(\ddot{q}_1 + g)\vec{i} + \ddot{q}_2\vec{j} + \ddot{q}_3\vec{k}]$. Then, $f_2 = R_3^2 f_3 + m_2(a_{c,2} - g_2) = m_3[-(\ddot{q}_1 + g)\vec{i} + \ddot{q}_2\vec{j} + \ddot{q}_3\vec{k}] + m_2[-(\ddot{q}_1 + g)\vec{i} + \ddot{q}_2\vec{j}]$. Note that R_3^2 is the identity matrix. Then,

$$f_1 = R_2^1 f_2 + m_1(a_{c,1} - g_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -m_3(\ddot{q}_1 + g) - m_2(\ddot{q}_1 + g) \\ m_3\ddot{q}_2 + m_2\ddot{q}_2 \\ m_3\ddot{q}_3 \end{bmatrix} - m_1(\ddot{q}_1 + g)\vec{j} \tag{9}$$

$$= \begin{bmatrix} m_3\ddot{q}_3 \\ -m_3(\ddot{q}_1 + g) - m_2(\ddot{q}_1 + g) - m_1(\ddot{q}_1 + g) \\ m_3\ddot{q}_2 + m_2\ddot{q}_2 \end{bmatrix}. \tag{10}$$

  Since the actuation of the first joint is along $z_0 = -y_1$, the actuated joint force for the first joint is $-f_{1,y}$. Since the actuation of the second joint is along $z_1 = y_2$, the actuated joint force for the second joint is $f_{2,y}$. Since the actuation of the third joint is along $z_2 = z_3$, the actuated joint force for the second joint is $f_{3,z}$. Another way to see which joint forces are the externally actuated forces is to look at $f_i^T z_{i-1}^{(i)}$ where $z_{i-1}^{(i)}$ denotes the axis z_{i-1} written relative to frame i; here, $f_1^T z_0^{(1)} = -f_{1,y}$, $f_2^T z_1^{(2)} = f_{2,y}$, and $f_3^T z_2^{(3)} = f_{3,z}$. We see that $-f_{1,y} = m_3(\ddot{q}_1 + g) + m_2(\ddot{q}_1 + g) + m_1(\ddot{q}_1 + g)$, $f_{2,y} = m_3\ddot{q}_2 + m_2\ddot{q}_2$, and and $f_{3,z} = m_3\ddot{q}_3$. Therefore, the dynamics equations are:

$$\begin{aligned} (m_1 + m_2 + m_3)\ddot{q}_1 + (m_1 + m_2 + m_3)g &= u_1 \\ (m_2 + m_3)\ddot{q}_2 &= u_2 \\ m_3\ddot{q}_3 &= u_3 \end{aligned} \tag{11}$$

  where u_1, u_2, and u_3 are the actuated forces for the first, second, and third joints, respectively.

Using inertial frame:

- Forward recursion: The angular velocity of link 1 is $\omega_1 = [0, 0, 0]^T$. By the simplifying assumptions mentioned on page 1, the linear accelerations of the center of mass of link 1 and of the end of link 1 are $a_{c,1} = a_{e,1} = \ddot{q}_1\vec{k}$. Since we are using inertial frame, the gravity vectors are given by $g_1 = g_2 = g_3 = -g\vec{k}$. The angular velocity of link 2 is $\omega_2 = 0$. Again, by the simplifying assumptions listed on page 1, the linear acceleration of the center of mass of link 2 and end of link 2 are: $a_{c,2} = a_{e,2} = \ddot{q}_1\vec{k} + \ddot{q}_2\vec{j}$. The angular velocity of link 3 is $\omega_3 = 0$. Again, by the simplifying assumptions listed on page 1, the linear acceleration of the center of mass of link 3 is: $a_{c,3} = \ddot{q}_1\vec{k} + \ddot{q}_2\vec{j} + \ddot{q}_3\vec{i}$.

- Backward recursion: The torque equations do not need to be written since this manipulator has only prismatic joints. Start with $f_4 = 0$. Then, $f_3 = m_3(a_{c,3} - g_3) = m_3[(\ddot{q}_1 + g)\vec{k} + \ddot{q}_2\vec{j} + \ddot{q}_3\vec{i}]$. Then, $f_2 = f_3 + m_2(a_{c,2} - g_2) = m_3[(\ddot{q}_1 + g)\vec{k} + \ddot{q}_2\vec{j} + \ddot{q}_3\vec{i}] + m_2[(\ddot{q}_1 + g)\vec{k} + \ddot{q}_2\vec{j}]$. Note that if we are doing the calculations in inertial frame, then all the forces and torques are being written in the same coordinate frame and hence, we do not need to multiply with any rotation matrices during the recursive calculations of the forces and torques. Then, $f_1 = f_2 + m_1(a_{c,1} - g_1) = m_3[(\ddot{q}_1 + g)\vec{k} + \ddot{q}_2\vec{j} + \ddot{q}_3\vec{i}] + m_2[(\ddot{q}_1 + g)\vec{k} + \ddot{q}_2\vec{j}] + m_1[(\ddot{q}_1 + g)\vec{k}]$. If we are writing the equations in inertial frame, the actuated joint forces for this manipulator are the z component of f_1, i.e., $f_{1,z}$, the y component of f_2, i.e., $f_{2,y}$, and the x component of f_3, i.e., $f_{3,x}$. We see that $f_{1,z} = (m_1 + m_2 + m_3)\ddot{q}_1 + (m_1 + m_2 + m_3)g$, $f_{2,y} = (m_2 + m_3)\ddot{q}_2$, and $f_{3,x} = m_3\ddot{q}_3$. Therefore, the dynamics equations are:

$$\begin{aligned} (m_1 + m_2 + m_3)\ddot{q}_1 + (m_1 + m_2 + m_3)g &= u_1 \\ (m_2 + m_3)\ddot{q}_2 &= u_2 \\ m_3\ddot{q}_3 &= u_3 \end{aligned} \tag{12}$$

  where u_1, u_2, and u_3 are the actuated forces for the first, second, and third joints, respectively.