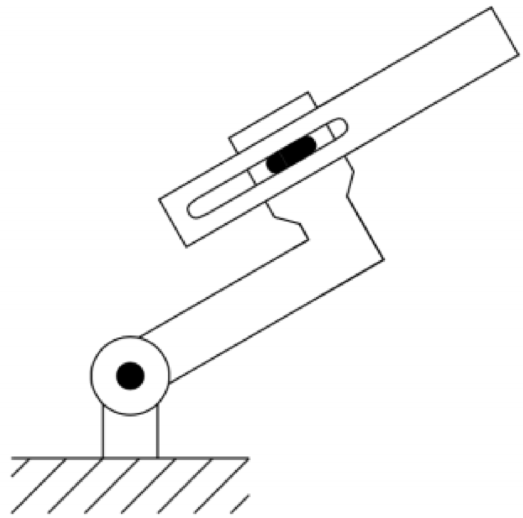
Example of dynamics computation (Euler-Lagrange and Newton-Euler formulations): Revolute-Prismatic (RP) manipulator (Problem 7.8)

Consider the Revolute-Prismatic (RP) manipulator shown in the figure below.



Let the coordinate system of the base frame (frame 0) be such that z_0 is pointing out of the page and x_0 is pointing to the right. Then, y_0 is pointing towards top in the figure. The joint variables are $q_1 = \theta_1$ and $q_2 = d_2$. Let the masses of the two links be m_1 and m_2. Since this is a planar manipulator and rotation is only around the z_0 axis, only the inertia around the z axis is relevant; let $I_{1,z}$ and $I_{2,z}$ denote the moments of inertia of links 1 and 2, respectively, around the axis pointing out of the page ($I_{1,z}$ and $I_{2,z}$ are both defined relative to a coodinate frame with origin at the center of mass of the link).

If the planar motion of the manipulator is in the horizontal plane, then gravity terms are not relevant. If the planar motion of the manipulator is in the vertical plane, then gravity terms need to be considered.

Let gravity be in the downward direction in the figure (i.e., in the $-y_0$ direction). As shown in the figure, the first link has an L-shape; let the longer length of this link be denoted by l and the shorter length (the offset) by a. Let l_{c1} denote the distance from the base (origin of frame 0) to the center of mass of link 1. Also, assume that the linkage between links 1 and 2 is such that when joint 2 actuates, it shifts the center of mass of link 2 by distance q_2.

Euler-Lagrange formulation to find the dynamics: The angular velocity Jacobian matrix corresponding to the first link is:

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{1}$$

Since the second joint is prismatic, the angular velocity Jacobian matrix corresponding to the second link is also:

$$J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

We find the linear velocity Jacobian matrices of the first and second link to be:

$$J_{v_1} = \begin{bmatrix} -l_{c1}s_1 & 0 \\ l_{c1}c_1 & 0 \\ 0 & 0 \end{bmatrix} \quad ; \quad J_{v_2} = \begin{bmatrix} -l_1 s_1 - ac_1 - q_2 s_1 & c_1 \\ l_1 c_1 - as_1 + q_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}. \tag{3}$$

where $s_1 = \sin(q_1)$ and $c_1 = \cos(q_1)$. Hence, the matrix $D(q)$ is given by:

$$D(q) = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_1}^T R_1^0 I_1 (R_1^0)^T J_{\omega_1} + J_{\omega_2}^T R_2^0 I_2 (R_2^0)^T J_{\omega_2} \tag{4}$$

$$= \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \tag{5}$$

where

$$d_{11} = m_2 a^2 + m_2 l_1^2 + 2m_2 l_1 q_2 + m_1 l_{c1}^2 + m_2 q_2^2 + I_{1,z} + I_{2,z} \tag{6}$$

$$d_{12} = d_{21} = -a m_2 \tag{7}$$

$$d_{22} = m_2 \tag{8}$$

As described above, since the rotation of both links is only about the z_0 axis, only the moments of inertia about the axis pointing out of the page are relevant (i.e., $I_{1,z}$ and $I_{2,z}$).

The Christoffel symbols for the manipulator are found as follows:

$$\begin{aligned}
c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0 \\
c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -l_1 m_2 - m_2 q_2 \\
c_{121} = c_{211} &= \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = l_1 m_2 + m_2 q_2 \\
c_{122} = c_{212} &= \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\
c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\
c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0
\end{aligned} \tag{9}$$

The potential energy of the manipulator is given by:

$$P = m_1 g l_{c1} s_1 + m_2 g (l_1 s_1 + a c_1 + q_2 s_1). \tag{10}$$

Hence,

$$g(q) = \begin{bmatrix} \frac{\partial P}{\partial q_1} \\ \frac{\partial P}{\partial q_2} \end{bmatrix} = \begin{bmatrix} m_1 g l_{c1} c_1 + m_2 g (l_1 c_1 - a s_1 + q_2 c_1) \\ m_2 g s_1 \end{bmatrix}. \tag{11}$$

We find the $(k,j)^{th}$ elements of the matrix $C(q,\dot{q})$ to be

$$\begin{aligned}
c_{11} &= c_{211}\dot{q}_2 \\
c_{12} &= c_{121}\dot{q}_1 \\
c_{21} &= c_{112}\dot{q}_1 \\
c_{22} &= 0
\end{aligned} \tag{12}$$

The dynamical equations of the manipulator are given by:

$$D(q)\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + C(q,\dot{q})\dot{q} + g(q) = \begin{bmatrix} \tau_1 \\ f_2 \end{bmatrix} \tag{13}$$

where τ_1 is the applied torque at the first joint (revolute) and f_2 is the applied force at the second joint (prismatic). Hence, the dynamical equations can be written as:

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{121}\dot{q}_1\dot{q}_2 + m_1 g l_{c1} c_1 + m_2 g (l_1 c_1 - a s_1 + q_2 c_1) = \tau_1 \tag{14}$$

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + m_2 g s_1 = f_2. \tag{15}$$

Newton-Euler formulation to find the dynamics:

- Forward recursion: To assign the Denavit-Hartenberg coordinate frames, pick z_0 pointing out of the page and x_0 to the right in the figure (i.e., y_0 pointing towards top in the figure); pick z_1 pointing along the actuation axis of the second joint; x_1 should be picked such that it intersects both z_0 and z_1 and is perpendicular to both z_0 and z_1. Hence, x_1 is in the plane of the page. Then, y_1 can point out of the page. Pick z_2 along the same direction as z_1 (with origin of frame 2 at the end-effector) and x_2 parallel to x_1. Then, we have y_1 and y_2 parallel to z_0. The Denavit-Hartenberg table for this manipulator is:

Link	a_i	α_i	d_i	θ_i
1	a	90^o	0	$90^o + q_1$
2	0	0	$l_1 + q_2$	0

Since the rotation is physically around the z_0 axis, and the axes y_1 and y_2 are parallel to z_0, the angular velocities of both link 1 and link 2 (written relative to the link-fixed frames) are $\dot{q}_1\vec{j}$, i.e., $\omega_1 = \omega_2 = \dot{q}_1\vec{j}$. Here, the notations $\vec{i}$, $\vec{j}$, and $\vec{k}$ denote the 3×1 unit vectors, i.e., $\vec{i} = [1,0,0]^T$, $\vec{j} = [0,1,0]^T$, $\vec{k} = [0,0,1]^T$. Since the base frame is stationary, we have $a_{c,0} = a_{e,0} = 0$. Looking at the orientation of frame 1, we have $r_{1,c1} = l_{c1}\vec{k}$. Hence, the linear acceleration of the center of mass of link 1 is:

$$a_{c,1} = R_0^1 a_{e,0} + \dot{\omega}_1 \times r_{1,c1} + \omega_1 \times (\omega_1 \times r_{1,c1}) = l_{c1}\ddot{q}_1\vec{i} - l_{c1}\dot{q}_1^2\vec{k}. \tag{16}$$

Taking the end of link 1 to be the location where it meets the next link (i.e., the tip of the shorter end of the L shape of link 1; note that the end of the link can be defined to be at any convenient location fixed to the link as long as we keep the equations consistent), we can write $r_{1,2} = l_1\vec{k} + a\vec{i}$. Hence, $a_{e,1}$ is

$$a_{e,1} = R_0^1 a_{e,0} + \dot{\omega}_1 \times r_{1,2} + \omega_1 \times (\omega_1 \times r_{1,2}) = l_1\ddot{q}_1\vec{i} - a\ddot{q}_1\vec{k} - l_1\dot{q}_1^2\vec{k} - a\dot{q}_1^2\vec{i}. \tag{17}$$

The gravity vector written in frame 1 is given by $g_1 = -R_0^1 g\vec{j} = -g[c_1, 0, s_1]^T$. The gravity vector written in frame 2 is also given by $g_2 = -R_0^2 g\vec{j} = -g[c_1, 0, s_1]^T$.

The linear acceleration of the center of mass of link 2 is:
$a_{c,2} = R_1^2 a_{e,1} + \dot{\omega}_2 \times r_{2,c2} + \omega_2 \times (\omega_2 \times r_{2,c2}) + R_0^2\ddot{q}_2 z_1^{(0)} + [(R_1^2\omega_1) \times (R_0^2 z_1^{(0)})]\dot{q}_2$. Note that $R_0^2 z_1^{(0)} = \vec{k}$, i.e., the z_1 axis (actuation axis of joint 2) is along the z_2 direction when seen relative to frame 2. Also, $r_{2,c2} = q_2\vec{k}$. Hence,

$$a_{c,2} = (l_1 + q_2)\ddot{q}_1\vec{i} - a\ddot{q}_1\vec{k} - (l_1 + q_2)\dot{q}_1^2\vec{k} - a\dot{q}_1^2\vec{i} + \ddot{q}_2\vec{k} + \dot{q}_1\dot{q}_2\vec{i}. \tag{18}$$

- Backward recursion: Start with $f_3 = \tau_3 = 0$. Then,

$$f_2 = m_2(a_{c,2} - g_2) = m_2[(l_1 + q_2)\ddot{q}_1\vec{i} - a\ddot{q}_1\vec{k} - (l_1 + q_2)\dot{q}_1^2\vec{k} - a\dot{q}_1^2\vec{i} + \ddot{q}_2\vec{k} + \dot{q}_1\dot{q}_2\vec{i} + gc_1\vec{i} + gs_1\vec{k}]. \tag{19}$$

$$\tau_2 = -f_2 \times r_{2,c2} + I_2\dot{\omega}_2 + \omega_2 \times (I_2\omega_2) = \{m_2[q_2(l_1 + q_2)\ddot{q}_1 - q_2 a\dot{q}_1^2 + q_2\dot{q}_1\dot{q}_2 + q_2 gc_1] + I_{2,z}\ddot{q}_1\}\vec{j} \tag{20}$$

Then,

$$\begin{aligned} f_1 &= R_2^1 f_2 + m_1(a_{c,1} - g_1) \\ &= m_2[(l_1 + q_2)\ddot{q}_1\vec{i} - a\ddot{q}_1\vec{k} - (l_1 + q_2)\dot{q}_1^2\vec{k} - a\dot{q}_1^2\vec{i} + \ddot{q}_2\vec{k} + \dot{q}_1\dot{q}_2\vec{i} + gc_1\vec{i} + gs_1\vec{k}] \\ &\quad + m_1(l_{c1}\ddot{q}_1\vec{i} - l_{c1}\dot{q}_1^2\vec{k} + gc_1\vec{i} + gs_1\vec{k}). \end{aligned} \tag{21}$$

Also, since $r_{2,c1} = (l_{c1} - l_1)\vec{k} - a\vec{i}$,

$$\begin{aligned} \tau_1 &= R_2^1\tau_2 - f_1 \times r_{1,c1} + (R_2^1 f_2) \times r_{2,c1} + I_1\dot{\omega}_1 + \omega_1 \times (I_1\omega_1) \\ &= \{m_2[q_2(l_1 + q_2)\ddot{q}_1 - q_2 a\dot{q}_1^2 + q_2\dot{q}_1\dot{q}_2 + q_2 gc_1] + I_{2,z}\ddot{q}_1 + l_{c1}f_{1,x} - (l_{c1} - l_1)f_{2,x} - af_{2,z} + I_{1,z}\ddot{q}_1\}\vec{j} \end{aligned} \tag{22}$$

where

$$f_{1,x} = m_2[(l_1 + q_2)\ddot{q}_1 - a\dot{q}_1^2 + \dot{q}_1\dot{q}_2 + gc_1] + m_1(l_{c1}\ddot{q}_1 + gc_1) \tag{23}$$

$$f_{2,x} = m_2[(l_1 + q_2)\ddot{q}_1 - a\dot{q}_1^2 + \dot{q}_1\dot{q}_2 + gc_1] \tag{24}$$

$$f_{2,z} = m_2[-a\ddot{q}_1 - (l_1 + q_2)\dot{q}_1^2 + \ddot{q}_2 + gs_1] \tag{25}$$

Since the actuation of the first joint is along $z_0 = y_1$, the actuated joint torque for the first joint (revolute) is $\tau_{1,y}$. Since the actuation of the second joint is along $z_1 = z_2$, the actuated joint force for the second joint is $f_{2,z}$. Another way to see which joint forces are the externally actuated forces is to look at $f_i^T z_{i-1}^{(i)}$ or $\tau_i^T z_{i-1}^{(i)}$ (depending on whether joint i is prismatic or revolute) where $z_{i-1}^{(i)}$ denotes the axis z_{i-1} written relative to frame i; here, $\tau_1^T z_0^{(1)} = \tau_{1,y}$ and $f_2^T z_1^{(2)} = f_{2,z}$.

Therefore, the dynamics equations are:

$$m_2[q_2(l_1 + q_2)\ddot{q}_1 - q_2 a\dot{q}_1^2 + q_2\dot{q}_1\dot{q}_2 + q_2 gc_1] + l_{c1}f_{1,x} - (l_{c1} - l_1)f_{2,x} - af_{2,z} + (I_{1,z} + I_{2,z})\ddot{q}_1 = u_1 \tag{26}$$

$$m_2[-a\ddot{q}_1 - (l_1 + q_2)\dot{q}_1^2 + \ddot{q}_2 + gs_1] = u_2 \tag{27}$$

where u_1 is the actuated torque for the first joint and u_2 is the actuated force for the second joint, respectively.

Substituting (23)-(25) into (26), we get the dynamics equation (14). Also, we see that (27) is the same as the dynamics equation (15).