Kinematics (Forward Kinematics, Inverse Kinematics, and Velocity Kinematics)

Forward Kinematics: Given a robotic manipulator, the forward kinematics problem is to find the position and angular orientation of the end-effector given the joint variables (i.e., the angles for the revolute joints and the displacements for the prismatic joints in the manipulator). Given a robotic manipulator with n joints, we number the robot's links as 0 to n starting from the base of the robot and we number the robot's joints as 1 to n, i.e., the i^{th} joint connects links $(i-1)$ and i. Hence, by this numbering of the joints and links, we see that the link i moves when joint i is actuated. Denote the joint variables by $q_1, \ldots, q_n$ where q_i is the joint angle if the i^{th} joint is revolute and q_i is the joint displacement if the i^{th} joint is prismatic, i.e.,

$$q_i = \begin{cases} \theta_i & \text{if the } i^{th} \text{ joint is revolute} \\ d_i & \text{if the } i^{th} \text{ joint is prismatic} \end{cases} \tag{1}$$

We attach a coordinate frame to each link of the robot. The coordinate frames are also numbered from 0 to n, i.e., the i^{th} coordinate frame is attached to the i^{th} link of the robot. The i^{th} coordinate frame is denoted as $o_i x_i y_i z_i$ where o_i is the origin of the i^{th} coordinate frame and x_i, y_i, z_i denote the X, Y, and Z direction unit vectors of the i^{th} coordinate frame. The relation between the $(i-1)^{th}$ coordinate frame and the i^{th} coordinate frame can be written as a homogeneous transformation matrix denoted as H_i^{i-1} or simply as A_i where this homogeneous transformation matrix is of the form

$$A_i = H_i^{i-1} = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix}. \tag{2}$$

R_i^{i-1} is the 3×3 rotation matrix between frames $(i-1)$ and i. o_i^{i-1} is the 3×1 position offset between the origins of the frames $(i-1)$ and i. Since the frame $(i-1)$ is attached to the $(i-1)^{th}$ link and the frame i is attached to the i^{th} link, and since the links $(i-1)$ and i are connected by the i^{th} joint, we see that each A_i is only a function of the i^{th} joint variable, i.e., of q_i. Hence, we denote A_i as $A_i(q_i)$.

The transformation matrix between any two frames i and j attached to the robot manipulator can be written in terms of the A_i matrices. Denoting the transformation matrix between the i^{th} frame and the j^{th} frame by H_j^i or T_j^i, we have

$$H_j^i = T_j^i = \begin{cases} A_{i+1} A_{i+2} \ldots A_j & \text{if } i < j \\ I & \text{if } i = j \\ (T_i^j)^{-1} & \text{if } i > j \end{cases} \tag{3}$$

For example, $T_3^0 = A_1 A_2 A_3$, $T_4^2 = A_3 A_4$, and $T_2^4 = (T_4^2)^{-1} = (A_3 A_4)^{-1}$.

The homogeneous transformation matrix between the base frame (frame 0) and the end-effector frame (frame n) is given by $T_n^0 = A_1 \ldots A_n$. Since each A_i is a function of the corresponding joint variable q_i, T_n^0 is of the form $A_1(q_1) \ldots A_n(q_n)$.

The rotational and translational parts of the homogeneous transformation matrix T_j^i are denoted as R_j^i and o_j^i, i.e., T_j^i is of the form

$$T_j^i = \begin{bmatrix} R_j^i & o_j^i \\ 0 & 1 \end{bmatrix}. \tag{4}$$

Since $T_j^i = T_{j-1}^i T_j^{j-1}$, we can write the position offset between frames i and j in the form $o_j^i = o_{j-1}^i + R_{j-1}^i o_j^{j-1}$.

In general, the homogeneous transformation matrix $A_i = H_i^{i-1}$ can be parameterized in terms of 6 numbers (three angular parameters and three position parameters), e.g., the Euler angles for the rotation part of the homogeneous transformation matrix and three position coordinates for the translational part of the homogeneous transformation matrix. However, by choosing the coordinate frames $(i-1)$ and i in a specific way (the Denavit-Hartenberg convention), we can reduce the number of required parameters to 4. Furthermore, typically, one of these four parameters would be the actuated joint variable (i.e., the joint angle for a revolute joint or the joint displacement for a prismatic joint) and the other three parameters are dependent on the robot geometry (e.g., link length, etc.). The Denavit-Hartenberg (or D-H) convention specifies that the coordinate frames are chosen such that

1. The axis x_i is perpendicular to the axis z_{i-1}.
2. The axis x_i intersects the axis z_{i-1}.

Furthermore, since the axis x_i forms part of a coordinate frame along with axis z_i, the axis x_i should definitely be perpendicular with axis z_i and should intersect axis z_i (the point of intersection between the axes x_i and z_i

is the origin o_i of the i^{th} coordinate frame). If the D-H convention properties listed above are satisfied, then we see that since $x_i.z_{i-1}$ is 0, the $(3,1)$ element (element in the third row and first column) of the rotation matrix R_i^{i-1} should be 0. Also, since each row and each column of a rotation matrix should be unit vectors, we see that the rotation matrix R_i^{i-1} should be of the form

$$R_i^{i-1} = \begin{bmatrix} c_\theta & * & * \\ s_\theta & * & * \\ 0 & s_\alpha & c_\alpha \end{bmatrix} \tag{5}$$

with θ and α being some angles and the $*$ symbols in equation (5) denoting elements still to be found in the rotation matrix R_i^{i-1}. Comparing the form of the rotation matrix R_i^{i-1} in equation (5) with the roll-pitch-yaw representation, for example, we can show that the remaining elements in the rotation matrix R_i^{i-1} in equation (5) should be as follows:

$$R_i^{i-1} = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}. \tag{6}$$

Hence, comparing this rotation matrix with the roll-pitch-yaw representation, we see that the rotation matrix R_i^{i-1} is of the form $R_{z,\theta}R_{x,\alpha}$.

The form of the translation part o_i^{i-1} of the homogeneous transformation matrix T_i^{i-1} can also be written using the D-H convention properties listed above. Since axis x_i intersects axis z_{i-1} by the D-H convention properties, we see that the position offset between the origins of the $(i-1)^{th}$ frame and the i^{th} frame can be written as a combination of some distance along z_{i-1} and some distance along x_i. Denoting these distances along z_{i-1} and along x_i by d and a, respectively, we see that the position offset between the origins of the $(i-1)^{th}$ frame and the i^{th} frame can be written in the $(i-1)^{th}$ frame as

$$dz_{i-1}^{i-1} + ax_i^{i-1} = d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a\begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix} = \begin{bmatrix} ac_\theta \\ as_\theta \\ d \end{bmatrix}. \tag{7}$$

From the equations (6) and (7), we see that the homogeneous transformation matrix $A_i = T_i^{i-1}$ is of the form

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

$$= \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i}. \tag{9}$$

Here, the four parameters a_i, α_i, d_i, and θ_i are referred to as the link length, link twist, link offset, and joint angle, respectively, and can be found geometrically as follows:

- a_i : distance between the axes z_{i-1} and z_i, measured along the axis x_i
- α_i : angle between the axes z_{i-1} and z_i, measured in a plane normal to x_i
- d_i : distance from origin o_{i-1} to the intersection of axis x_i with axis z_{i-1}, measured along z_{i-1}
- θ_i : angle between the axes x_{i-1} and x_i, measured in a plane normal to z_{i-1}.

By picking the z axes of the coordinate frames to be along the axes of actuation of the corresponding joints, we can make the joint variable effectively one of these four parameters (i.e., θ_i for a revolute joint or d_i for a prismatic joint). The other three of the four parameters described above would be constants that depend on the robot geometry (e.g., link length, etc.). The axis of actuation is the axis of rotation for a revolute joint and is the axis of translation for a prismatic joint. Hence, the sequence of steps to pick the coordinate frames according to the D-H convention is as follows:

- First pick the z axes to be along the axes of actuation for the joints, i.e., z_0 is picked along the axis of actuation of the first joint, z_1 is picked along the axis of actuation of the second joint, etc., z_{n-1} is picked along the axis of actuation of the n^{th} joint. The axis z_n which corresponds to the end-effector can be picked to be along z_{n-1} or any other convenient direction depending on the end-effector.

- Then pick the x axes according to the D-H convention properties.
- Finally, the y axes are assigned to form a right-handed coordinate frame along with the corresponding x and z axes.

The choice of the x_i axes according to the D-H convention properties depends on the z_{i-1} and z_i axes. The various possibilities of the z_{i-1} and z_i axes are summarized below:

- The axes z_{i-1} and z_i are not coplanar. In this case, the x_i axis is picked to be along the unique shortest line segment from the z_{i-1} line to the z_i line that is perpendicular to both z_{i-1} and z_i. The x_i axis is along this unique shortest line segment; there are two possibilities for the direction of the x_i axis (the two directions along this line segment), either of which can be used (one direction is simply the opposite of the other). The intersection between the x_i axis and the z_i axis is defined to be the origin o_i of the i^{th} coordinate frame.
- The axes z_{i-1} and z_i are parallel. In this case, x_i is defined to be in the same plane as z_{i-1} and z_i and is perpendicular to both z_{i-1} and z_i. There are infinitely many possibilities for x_i, any of which can be picked; these infinite number of possibilities for x_i correspond to sliding the origin o_i along the z_i axis. One possibility is to choose the perpendicular direction that passes through o_{i-1} as the x_i axis. Then, d_i is seen to be zero since the vector between the origins o_{i-1} and o_i is along the x_i axis. Also, since the axes z_{i-1} and z_i are parallel, the angle α_i is also zero in this case.
- The axes z_{i-1} and z_i intersect. In this case, the axis x_i is picked to be normal (i.e., perpendicular) to the plane formed by the axes z_{i-1} and z_i. The origin o_i is the point of intersection of the axes z_{i-1} and z_i. In this case, since the axes z_{i-1} and z_i intersect, the distance a_i is zero.
- The axes z_{i-1} and z_i are identical (along the same direction). In this case, the axis x_i can be picked to be any vector perpendicular to z_{i-1}, which is the same as z_i. The point of intersection of x_i and z_i is defined to be the origin o_i. In this case, the parameters α_i and a_i are zero.

After the coordinate frames $0, \ldots, n$ are picked as described above, the geometric definitions of the parameters a_i, α_i, d_i, and θ_i that were described above are utilized to form a table as follows:

	a_i	α_i	d_i	θ_i
Link 1				
Link 2				
Link 3				

This table will have n rows of parameters where n is the number of joints in the robotic manipulator. From this table, the matrices $A_1, \ldots, A_n$ are formed using equation (8). Using these matrices, the homogeneous transformation matrix T_j^i can be found for any i and j using equation (3). In particular, the homogeneous transformation matrix T_n^0 between the base frame and the end-effector frame is the product $A_1 A_2 \ldots A_n$.

Inverse Kinematics: Given a robotic manipulator, the inverse kinematics problem is to find the joint variables $q_1, \ldots, q_n$ given the end-effector position or orientation (or both). As shown in equation (1), the joint variable q_i is the joint angle θ_i if the i^{th} joint is revolute and is the joint displacement d_i if the i^{th} joint is prismatic. As described above, the homogeneous transformation matrix between the base frame and the end-effector frame is given by $T_n^0 = A_1 A_2 \ldots A_n$. This homogeneous transformation matrix is of the form

$$T_n^0 = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}. \tag{10}$$

Since each A_i is a function of the corresponding joint variable q_i, the homogeneous transformation matrix T_n^0 is a function of $q_1, \ldots, q_n$.

In the *inverse position kinematics* problem, the position offset o_n^0 is given as a numerical quantity o (which represents the desired end-effector position) and the joint variables $q_1, \ldots, q_n$ are to be solved for. Since o_n^0 is a function of $q_1, \ldots, q_n$, this forms essentially a nonlinear equation $o_n^0(q_1, \ldots, q_n) = o$.

In the general *inverse kinematics problem*, both the desired position and the desired orientation of the end-effector are specified as a desired position offset o and a desired rotation matrix R. Hence, this forms a set of nonlinear equations of the form $o_n^0(q_1, \ldots, q_n) = o$; $R_n^0(q_1, \ldots, q_n) = R$.

In general, for both the inverse position kinematics problem and for the general inverse kinematics problem, the solution could be non-unique (there could be multiple solutions; there could even be an infinite number of solutions). Also, it is possible that given a specific desired position/orientation of the end-effector, there could be no solution to the inverse kinematics problem if the desired position/orientation of the end-effector is outside the feasible workspace of the robot (e.g., due to the lengths of the links, constraints on the joint angles, etc.).

In the particular case when the last three joints of the robot manipulator form a spherical wrist (or more generally, when the last three joint axes intersect at a point), then the general inverse kinematics problem can be simplified to two smaller inverse kinematics problems by considering the center of the spherical wrist. The z_{n-1} and z_n axes can be taken to be along the same direction. Also, by the definition of the rotation matrix, the z_n axis is specified (in terms of coordinate frame 0) by the last column of the rotation matrix R. Hence, if the desired position of the end-effector is given as o, then the corresponding desired position of the wrist center (in terms of the coordinate frame 0) is given by

$$(o_c^0)_{\text{desired}} = o - d_n R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{11}$$

Next, note that the position of the wrist center does not depend on the wrist angles (the last three revolute joints of the robot manipulator). Hence, the position of the wrist center is a function of $q_1, \ldots, q_{n-3}$. Thus, given the desired position of the wrist center from equation (11), the first $(n-3)$ joint variables $q_1, \ldots, q_{n-3}$, can be found using the equation $o_c^0(q_1, \ldots, q_{n-3}) = (o_c^0)_{\text{desired}}$, i.e., an inverse position kinematics problem for the first $(n-3)$ joint variables. For the specific case of a six degree of freedom manipulator, this corresponds to the first 3 joint variables. Once $q_1, \ldots, q_{n-3}$ are solved for as described above based on the desired position of the wrist center, the matrices $A_1, \ldots, A_{n-3}$ are known since these are functions of $q_1, \ldots, q_{n-3}$, respectively. Using the rotation parts $R_1^0, \ldots, R_{n-3}^{n-4}$ of these matrices, we get $R_{n-3}^0 = R_1^0 \ldots R_{n-3}^{n-4}$. Since the desired orientation of the end-effector is specified by a rotation matrix R, we should have $R_{n-3}^0 R_n^{n-3} = R$. Hence, we get

$$R_n^{n-3} = (R_{n-3}^0)^T R. \tag{12}$$

The right hand side of this equation is fully known since R is a given rotation matrix (the desired orientation of the end-effector) and R_{n-3}^0 is a function of the joint variables $q_1, \ldots, q_{n-3}$ that we solved for above. Hence, the desired value of the rotation matrix R_n^{n-3} is known by equation (12). The rotation matrix R_n^{n-3} is a function of only the the joint variables q_{n-2}, q_{n-1}, and q_n. Hence, we can solve for these three joint variables from the desired value of R_n^{n-3}. For a spherical wrist, the structure of the rotation matrix R_n^{n-3} is identical to the Euler angle representation. Hence, the three joint angles q_{n-2}, q_{n-1}, and q_n can be found simply as the Euler angles ϕ, θ, and ψ of the Euler angle representation corresponding to the rotation matrix $(R_{n-3}^0)^T R$.

Velocity Kinematics: Given a robotic manipulator, the velocity kinematics problem is to find the relation between the joint variable rates (i.e., $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$) and the end-effector velocity (i.e., linear velocity and angular velocity of the end-effector).

To write the velocity kinematics, the representation of a vector cross-product in terms of a skew-symmetric matrix will be useful. Given two vectors k and p, the cross product of these vectors can be written as a product with a skew-symmetric matrix, i.e., the cross product $k \times p$ is equal to $S(k)p$ where $S(k)$ is the skew-symmetric matrix defined as

$$S(k) = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}. \tag{13}$$

A matrix S is said to be skew-symmetric if $S + S^T = 0$. Hence, the diagonal elements of a skew-symmetric matrix are zero and the off-diagonal elements satisfy the relation $s_{ij} = -s_{ji}$ where s_{ij} is the (i, j) element of S and s_{ji} is the (j, i) element.

Given a 3×1 vector a, the matrix $S(a)$ is a 3×3 skew-symmetric matrix. If R is any rotation matrix, then it can be shown that $RS(a)R^T = S(Ra)$ where a is any 3×1 vector. To show this property of skew-symmetric matrices, note that for any two vectors a and b, since the physical vector corresponding to the cross-product of these two vectors is not dependent on a specific coordinate transformation, we have $R[a \times b] = (Ra) \times (Rb)$. Hence, $RS(a)R^T b = R[a \times R^T b] = (Ra) \times (RR^T b) = (Ra) \times b = S(Ra)b$. Hence, $RS(a)R^T = S(Ra)$.

The skew-symmetric matrix function $S()$ is a linear function in the sense that given any two 3×1 vectors a and b and any two scalars α and β, the following equation is satisfied: $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$.

Given any 3×1 vectors a and x, we can show that $x^T S(a)x = 0$. To see this, note that $x^T S(a)x$ is a scalar. Hence, $x^T S(a)x = [x^T S(a)x]^T$. Therefore, $x^T S(a)x = [x^T S(a)x]^T = x^T S^T(a)x = -x^T S(a)x$. Hence, $x^T S(a)x = -x^T S(a)x$. Therefore, $x^T S(a)x = 0$.

Consider any rotation matrix R that is a function of a parameter θ (that could be, for instance, an angle or could be the time variable). Then, since $R(\theta)R^T(\theta) = I$, we see that

$$\frac{dR}{d\theta}R^T(\theta) + R(\theta)\frac{dR^T(\theta)}{d\theta} = 0. \tag{14}$$

Defining $S = \frac{dR}{d\theta}R^T(\theta)$, we see from (14) that $S + S^T = 0$. Hence, this matrix S is a skew-symmetric matrix. Therefore, $\frac{dR}{d\theta} = SR(\theta)$ with S being a skew-symmetric matrix.

If $R(\theta) = R_{x,\theta}$, then, by a direct calculation, we see that the corresponding matrix $S = \frac{dR}{d\theta}R^T(\theta)$ is $S(i)$ where i is the X-axis unit vector. Similarly, if $R(\theta) = R_{y,\theta}$, then the corresponding matrix $S = \frac{dR}{d\theta}R^T(\theta)$ is $S(j)$ where j is the Y-axis unit vector. If $R(\theta) = R_{z,\theta}$, then the corresponding matrix $S = \frac{dR}{d\theta}R^T(\theta)$ is $S(k)$ where k is the Z-axis unit vector. In general, if $R(\theta)$ is the rotation around an axis k by an angle θ, i.e., if $R(\theta) = R_{k,\theta}$, then the corresponding matrix $S = \frac{dR}{d\theta}R^T(\theta)$ is $S(k)$. Also, note that if θ is an angle, then the derivative of the rotation matrix with respect to time t can be written as $\dot{R} = \dot{\theta}SR(\theta)$ where $S = \frac{dR}{d\theta}R^T(\theta)$. Hence, if R is written as an axis-angle representation as $R(\theta) = R_{k,\theta}$, then we have $\dot{R} = S(\dot{\theta}k)R$. The product $\dot{\theta}k$ is the angular rate $\dot{\theta}$ multiplied by the axis of rotation k; this product $\dot{\theta}k$ intuitively corresponds to the angular velocity ω, i.e., $\omega = \dot{\theta}k$. Hence, we can write $\dot{R} = S(\omega(t))R$ where $\omega(t)$ is the angular velocity at time t.

Considering a point p that is stationary with respect to a coordinate frame 1, and considering a rotation matrix R_1^0 between coordinate frames 0 and 1 (and taking the origins of the frames 0 and 1 to be the same), we can write $\dot{p}^0 = \dot{R}_1^0 p^1$ where p^0 and p^1 are the 3×1 representations of the point p relative to frames 0 and 1, respectively. Hence, $\dot{p}^0 = S(\omega(t))R_1^0 p^1 = \omega(t) \times [R_1^0 p^1] = \omega(t) \times p^0$ where $\omega(t)$ is the angular velocity (at time t) of the frame 1 with respect to the frame 0. If the frame 1 is moving relative to frame 0, then we can write $p^0 = R_1^0 p^1 + o$ where o is the position offset between the origins of frames 0 and 1 (written relative to frame 0). Hence, $\dot{p}^0 = \dot{R}_1^0 p^1 + v = S(\omega(t))r + v$ where v denotes $\dot{o}$ (i.e., the rate of change of the position offset o, written relative to frame 0) and r denotes $R_1^0 p^1$ (i.e., the vector from the origin of frame 1 to the point p, written relative to frame 0). In general, if the point p is also moving relative to the frame 1 with some rate of change $\dot{p}^1$, then $\dot{p}^0 = \dot{R}_1^0 p^1 + R_1^0 \dot{p}^1 + v = S(\omega(t))r + R_1^0 \dot{p}^1 + v$ where v denotes $\dot{o}$ and r denotes $R_1^0 p^1$.

Considering coordinate frames 0, 1, and 2, we can write $R_2^0 = R_1^0 R_2^1$. Hence,

$$\dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1 \quad = \quad S(\omega_{0,1}^0)R_1^0 R_2^1 + R_1^0 S(\omega_{1,2}^1)R_2^1 \tag{15}$$

where $\omega_{a,b}^r$ denotes the angular velocity (written relative to frame r) of frame b with respect to frame a. Hence, $\omega_{1,2}^0 = R_1^0 \omega_{1,2}^1$. Therefore, from equation (15), we see that

$$\begin{aligned}\dot{R}_2^0 &= S(\omega_{0,1}^0)R_1^0 R_2^1 + R_1^0 S(\omega_{1,2}^1)[R_1^0]^T R_1^0 R_2^1 \\ &= S(\omega_{0,1}^0)R_1^0 R_2^1 + S(R_1^0 \omega_{1,2}^1)R_1^0 R_2^1 \\ &= S(\omega_{0,1}^0 + R_1^0 \omega_{1,2}^1)R_1^0 R_2^1 \quad = \quad S(\omega_{0,1}^0 + \omega_{1,2}^0)R_2^0.\end{aligned} \tag{16}$$

Since $\dot{R}_2^0$ should be equal to $S(\omega_{0,2}^0)R_2^0$, we see from equation (16) that

$$\omega_{0,2}^0 = \omega_{0,1}^0 + \omega_{1,2}^0. \tag{17}$$

In general, if we have coordinate frames $0, 1, \ldots, n$, then

$$\begin{aligned}\omega_{0,n}^0 &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + \ldots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \\ &= \omega_{0,1}^0 + \omega_{1,2}^0 + \ldots + \omega_{n-1,n}^0.\end{aligned} \tag{18}$$

Consider a robotic manipulator with n joints with corresponding joint variables $q_1, \ldots, q_n$. Denote $q = [q_1, \ldots, q_n]^T$. The homogeneous transformation matrix between the base frame and the end-effector frame is a function of q and can be written in the form

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}. \tag{19}$$

Then, $\dot{o}_n^0$ is the linear velocity (written relative to the base frame, i.e., frame 0) of the end-effector; denote this linear velocity by v_n^0. Also, the time derivative R_n^0 is of the form $S(\omega_n^0)R_n^0$ where ω_n^0 is the angular velocity (written relative to frame 0) of the end-effector.

Noting that o_n^0 is a function of $q = [q_1, \ldots, q_n]^T$, we can write $\dot{o}_n^0$ in the form $\dot{o}_n^0 = \sum_{i=1}^n \frac{\partial o_n^0}{\partial q_i}\dot{q}_i$. Hence, we can write $v_n^0 = J_v \dot{q}$ where

$$J_v = \begin{bmatrix} \frac{\partial o_n^0}{\partial q_1} & \frac{\partial o_n^0}{\partial q_2} & \ldots & \frac{\partial o_n^0}{\partial q_n} \end{bmatrix}. \tag{20}$$

J_v is a $3 \times n$ matrix that is called the linear velocity Jacobian.

Note that the actuation of a prismatic joint (joint i for example) would have the effect of moving the end-effector along the direction z_{i-1} (since by the D-H convention, the axis z_{i-1} is picked to be along the axis of actuation for the i^{th} joint). Hence, if joint i is prismatic, then the contribution to the linear velocity v_n^0 is simply $\dot{d}_i z_{i-1}^0$. Since, for a prismatic joint, we have $q_i = d_i$, we have $\dot{d}_i z_{i-1}^0 = \dot{q}_i z_{i-1}^0$. Hence, denoting the i^{th} column of the matrix J_v by J_{v_i}, we see that if the i^{th} joint is prismatic, then $J_{v_i} = z_{i-1}^0$.

If the i^{th} joint is revolute, then its actuation would have the effect of rotating the end-effector around an arc that has a radial vector $(o_n^0 - o_{i-1}^0)$ where o_{i-1}^0 denotes the origin of the $(i-1)^{th}$ coordinate frame (written relative to frame 0). Since the axis of rotation of this joint is the axis z_{i-1} by the D-H convention, we see that, if the i^{th} joint is revolute, then the linear velocity imparted to the end-effector by actuation of the i^{th} joint is given by $(\dot{\theta}_i z_{i-1}^0) \times (o_n^0 - o_{i-1}^0)$. Hence, if the i^{th} joint is revolute, then the i^{th} column of the matrix J_v is $J_{v_i} = z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)$.

Therefore, the linear velocity Jacobian can be written as $J_v = [J_{v_1} \ \ldots \ J_{v_n}]$ where the i^{th} column J_{v_i} is given by

$$J_{v_i} = \begin{cases} z_{i-1}^0 \text{ if the } i^{th} \text{ joint is prismatic} \\ z_{i-1}^0 \times (o_n^0 - o_{i-1}^0) \text{ if the } i^{th} \text{ joint is revolute} \end{cases} \tag{21}$$

The angular velocity ω_n^0 can be written as $\omega_n^0 = J_\omega \dot{q}$ where J_ω is a $3 \times n$ matrix that is called the angular velocity Jacobian. The actuation of a prismatic joint does not contribute to the angular velocity of the end-effector. Hence, if the i^{th} joint is prismatic, then the i^{th} column of the matrix J_ω is $J_{\omega_i} = 0$. If the i^{th} joint is revolute, then since, by the D-H convention, the corresponding axis of rotation is picked to be the z_{i-1} axis, the effect of actuation of the i^{th} joint would be to impart an angular velocity of $\dot{\theta}_i z_{i-1}$ to the end-effector. Therefore, if the i^{th} joint is revolute, then the i^{th} column of the matrix J_ω is $J_{\omega_i} = z_{i-1}^0$. Therefore, the angular velocity Jacobian can be written as $J_\omega = [J_{\omega_1} \ \ldots \ J_{\omega_n}]$ where the i^{th} column J_{ω_i} is given by

$$J_{\omega_i} = \rho_i z_{i-1}^0 \tag{22}$$

where

$$\rho_i = \begin{cases} 0 \text{ if the } i^{th} \text{ joint is prismatic} \\ 1 \text{ if the } i^{th} \text{ joint is revolute} \end{cases} \tag{23}$$

Denoting $\xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix}$, we can write

$$\dot{\xi} = J\dot{q} \qquad \text{where} \quad J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}. \tag{24}$$

Hence, given the joint variable rates (i.e., the time derivatives of the joint variables $q_1, \ldots, q_n$), the linear and angular velocities of the end-effector can be found from (24). The inverse velocity kinematics problem is to find $\dot{q}$ given a desired ξ (i.e., a desired set of linear and angular velocities of the end-effector). From (24), we can solve for $\dot{q}$ as $\dot{q} = J^{-1}\xi$ if J is a square and invertible matrix. In more generality, defining the pseudo-inverse $J^+ = J^T(JJ^T)^{-1}$, we can write $\dot{q} = J^+\xi + (I - J^+J)b$ with b being any $n \times 1$ vector.

Relationship between the Joint Forces/Torques and the End-Effector Forces/Torques: Denoting the vector of forces and torques at the end-effector by a 6×1 vector F and the vector of joint forces/torques (forces for prismatic joints, torques for revolute joints) by an $n \times 1$ vector τ, we can write the physical work performed as either $F^T \delta X$ or as $\tau^T \delta q$. Here, δX is the infinitesimal displacement in the task space (i.e., infinitesimal change in the end-effector position and orientation) and δq is the corresponding infinitesimal change in the joint variables. Note that δX and δq are related by $\delta X = J(q)\delta q$. Using the principle of virtual work, the difference between $F^T \delta X$ and $\tau^T \delta q$ should be zero. Hence, $F^T J(q) = \tau^T$. Therefore, $\tau = J^T(q)F$.