Homework 8 Solutions

1. (Problem 5.20 from textbook)

   We want to find a LSPB trajectory $q(t)$ to satisfy the general requirements $q(0) = q_0$, $\dot{q}(0) = 0$, $q(t_f) = q_1$, and $\dot{q}(t_f) = v_f$. In this problem, we are also given that $t_f = 2$ and $v_f = 1$. The LSPB trajectory has an initial parabolic segment (until a blend time), a constant-velocity segment, and a final parabolic segment. Note that, in general, the blend times at the start and end of the motion can be different (e.g., t_{b1} and t_{b2}). The LSPB trajectory can be written as a combination of three segments.

$$q(t) = \begin{cases} a_0 + a_1 t + a_2 t^2 \text{ for } t \in [0, t_{b1}] \\ c_0 + Vt \text{ for } t \in [t_{b1}, t_f - t_{b2}] \\ b_0 + b_1 t + b_2 t^2 \text{ for } t \in [t_f - t_{b2}, t_f] \end{cases} \tag{1}$$

   where $a_0, a_1, a_2, c_0, V, b_0, b_1$, and b_2 are constants. V is the velocity during the constant-velocity segment. To satisfy the given requirements, we should have

$$a_0 = q_0 \tag{2}$$
$$a_1 = 0 \tag{3}$$
$$b_0 + b_1 t_f + b_2 t_f^2 = q_1 \tag{4}$$
$$b_1 + 2b_2 t_f = v_f. \tag{5}$$

   Also, the position and velocity of the initial parabolic segment and the constant-velocity segment must match at time t_{b1}. Hence, we also get the constraints

$$a_0 + a_1 t_{b1} + a_2 t_{b1}^2 = c_0 + V t_{b1} \tag{6}$$
$$a_1 + 2a_2 t_{b1} = V \tag{7}$$

   Also, the position and velocity of the constant-velocity segment and the final parabolic segment must match at time $t_f - t_{b2}$. This results in the equations

$$b_0 + b_1(t_f - t_{b2}) + b_2(t_f - t_{b2})^2 = c_0 + V(t_f - t_{b2}) \tag{8}$$
$$b_1 + 2b_2(t_f - t_{b2}) = V. \tag{9}$$

   Therefore, from (2)-(9), we get a total of 8 linear equations in the variables $a_0, a_1, a_2, c_0, V, b_0, b_1$, and b_2. The equations (2) and (3) directly give us a_0 and a_1. From equations (6) and (7), we get

$$a_2 = \frac{V}{2t_{b1}} \tag{10}$$
$$c_0 = q_0 - \frac{V t_{b1}}{2}. \tag{11}$$

   Using (2), (3), (10), and (11), and writing the equations (4), (5), (8), (9) in matrix form, we get

$$\begin{bmatrix} 0 & 1 & t_f & t_f^2 \\ 0 & 0 & 1 & 2t_f \\ \frac{t_{b1}}{2} - t_f + t_{b2} & 1 & (t_f - t_{b2}) & (t_f - t_{b2})^2 \\ -1 & 0 & 1 & 2(t_f - t_{b2}) \end{bmatrix} \begin{bmatrix} V \\ b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ v_f \\ q_0 \\ 0 \end{bmatrix}. \tag{12}$$

   Picking values for t_{b1} and t_{b2}, we can find V, b_0, b_1, and b_2 from (12). For example, picking $t_{b1} = 0.1$, $t_{b2} = 0.1$, the LSPB trajectory from (1) for $q_0 = 0$ and $q_1 = 5$ is shown in Figure 1 below.

2. (Problem 5.22 from textbook) A Matlab program to generate an LSPB trajectory, given the parameters q_0, q_f, t_f, v_f, t_{b1}, and t_{b2}, as defined in the solution to Problem 5.20 above, is at: `http://crrl.poly.edu/EL5223/lspb.m`.

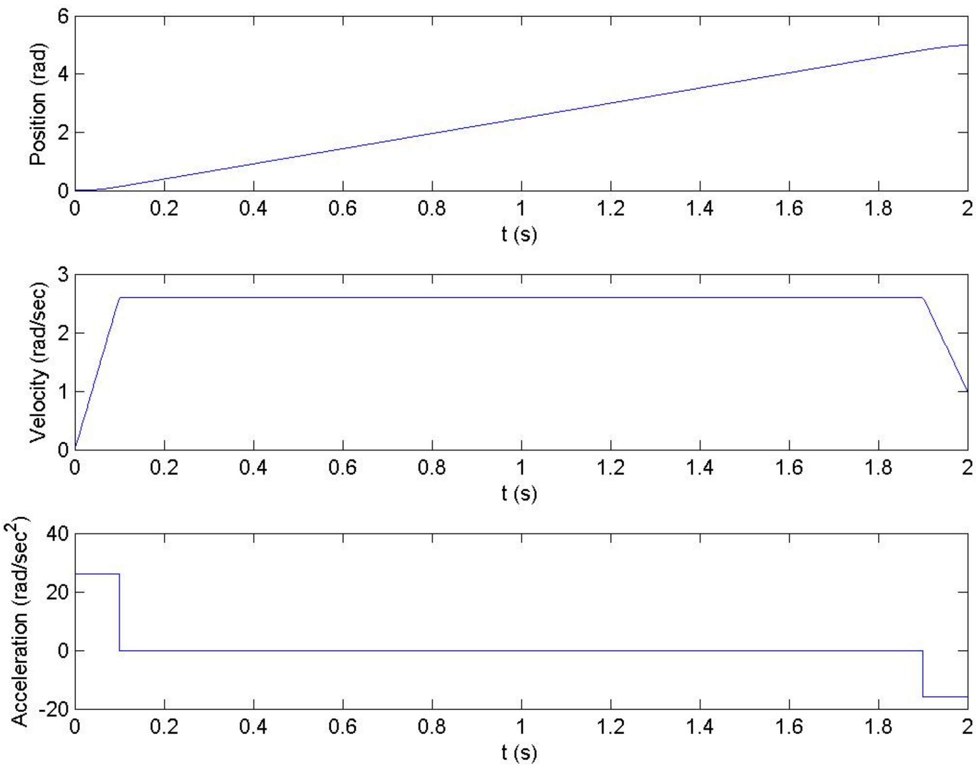

Figure 1: LSPB trajectory: position, velocity, and acceleration profiles.

3. The position of the center of mass of the cart is given by $[q_1, 0, 0]^T$. Here, for simplicity, we can consider the cart to be simply a point (without any height) since the height of the cart will only add a constant term to the potential energy and will not affect the resulting dynamics equations. The velocity of the center of mass of the cart is $[\dot{q}_1, 0, 0]^T$. The position of the center of mass of the pole is $[q_1 + l_2 \sin(q_2), 0, l_2 \cos(q_2)]^T$ and the velocity of the center of mass of the pole is $[\dot{q}_1 + l_2 \cos(q_2)\dot{q}_2, 0, -l_2 \sin(q_2)\dot{q}_2]^T$. The angular velocity of the pole is $[0, \dot{q}_2, 0]^T$. Hence, the kinetic energy and potential energy of the system are given by $K = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2[\dot{q}_1^2 + l_2^2 \dot{q}_2^2 + 2 l_2 \cos(q_2)\dot{q}_1 \dot{q}_2] + \frac{1}{2} I_2 \dot{q}_2^2$ and $P = m_2 g l_2 \cos(q_2)$. Defining the Lagrangian $\mathcal{L} = K - P$, the dynamics of the system are given by the Euler-Lagrange dynamics equations as:

$$\begin{aligned}
F_1 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} && (13)\\
&= \frac{d}{dt}\left[(m_1 + m_2)\dot{q}_1 + m_2 l_2 \cos(q_2)\dot{q}_2\right] && (14)\\
&= (m_1 + m_2)\ddot{q}_1 + m_2 l_2 \cos(q_2)\ddot{q}_2 - m_2 l_2 \sin(q_2)\dot{q}_2^2 && (15)\\
\tau_2 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2} && (16)\\
&= \frac{d}{dt}\left[(m_2 l_2^2 + I_2)\dot{q}_2 + m_2 l_2 \cos(q_2)\dot{q}_1\right] - (-m_2 l_2 \sin(q_2)\dot{q}_1\dot{q}_2 + m_2 g l_2 \sin(q_2)) && (17)\\
&= (m_2 l_2^2 + I_2)\ddot{q}_2 + m_2 l_2 \cos(q_2)\ddot{q}_1 - m_2 l_2 \sin(q_2)\dot{q}_1\dot{q}_2 + m_2 l_2 \sin(q_2)\dot{q}_1\dot{q}_2 - m_2 g l_2 \sin(q_2) && (18)\\
&= (m_2 l_2^2 + I_2)\ddot{q}_2 + m_2 l_2 \cos(q_2)\ddot{q}_1 - m_2 g l_2 \sin(q_2) && (19)
\end{aligned}$$

where F_1 is the force at the prismatic joint (joint 1) and τ_2 is the torque at the revolute joint (joint 2). If the revolute joint is not actively controlled (i.e., if the only actuation is the force on the cart), then τ_2 is simply zero.

4. (Problem 7.3 from textbook) We are asked to find the moments of inertia and cross products of inertia of a uniform rectangular solid of sides a, b, c with respect to a coordinate system with origin at one corner and axes along the edges of the solid. With ρ denoting the mass density of the rigid body, we have $m = \rho abc$. Then,

$$\begin{aligned}
I_{xx} &= \int_0^c \int_0^b \int_0^a (y^2 + z^2)\rho(x,y,z)dx\,dy\,dz &= \frac{m}{3}(b^2 + c^2) && (20)\\
I_{yy} &= \int_0^c \int_0^b \int_0^a (x^2 + z^2)\rho(x,y,z)dx\,dy\,dz &= \frac{m}{3}(a^2 + c^2) && (21)\\
I_{zz} &= \int_0^c \int_0^b \int_0^a (x^2 + y^2)\rho(x,y,z)dx\,dy\,dz &= \frac{m}{3}(a^2 + b^2) && (22)\\
I_{xy} = I_{yx} &= -\int_0^c \int_0^b \int_0^a xy\rho(x,y,z)dx\,dy\,dz &= -\frac{m}{4}ab && (23)\\
I_{xz} = I_{zx} &= -\int_0^c \int_0^b \int_0^a xz\rho(x,y,z)dx\,dy\,dz &= -\frac{m}{4}ac && (24)\\
I_{yz} = I_{zy} &= -\int_0^c \int_0^b \int_0^a yz\rho(x,y,z)dx\,dy\,dz &= -\frac{m}{4}bc. && (25)
\end{aligned}$$