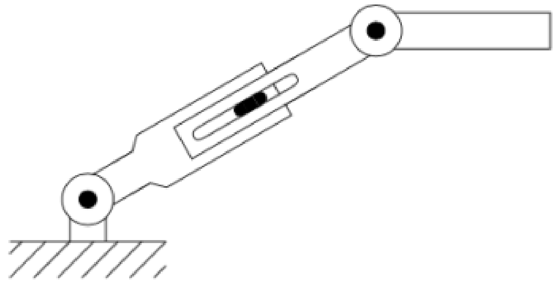
Example of dynamics computation (Euler-Lagrange and Newton-Euler formulations): Revolute-Prismatic-Revolute (RPR) manipulator (Problem 7.9)

Consider the Revolute-Prismatic-Revolute (RPR) manipulator shown in the figure below.



Let the coordinate system of the base frame (frame 0) be such that z_0 is pointing out of the page and x_0 is pointing to the right. Then, y_0 is pointing towards top in the figure. The joint variables are $q_1 = \theta_1$, $q_2 = d_2$, and $q_3 = \theta_3$. Let the masses of the three links be m_1, m_2, and m_3. Since this is a planar manipulator and rotation is only around the z_0 axis, only the inertia around the vertical axis is relevant; let $I_{1,z}$, $I_{2,z}$, and $I_{3,z}$ denote the moments of inertia of links 1, 2, and 3, respectively, around the axis pointing out of the page (for each link, the moments of inertia are defined relative to a coodinate frame with origin at the center of mass of the link).

If the planar motion of the manipulator is in the horizontal plane, then gravity terms are not relevant. If the planar motion of the manipulator is in the vertical plane, then gravity terms need to be considered.

Let gravity be in the downward direction in the figure (i.e., in the $-y_0$ direction). Let l_{c1} denote the distance from the base (origin of frame 0) to the center of mass of link 1. Let l_1 be the length of link 1. Then, the combined length of links 1 and 2 is $l_1 + q_2$. Also, assume that the linkage between links 1 and 2 is such that when joint 2 actuates, it shifts the center of mass of link 2 by distance q_2. Let the distance from the point where links 2 and 3 meet to the center of mass of link 3 be l_{c3}.

Euler-Lagrange formulation to find the dynamics: The angular velocity Jacobian matrices for the three links are:

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad ; \quad J_{\omega_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad ; \quad J_{\omega_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{1}$$

The linear velocity Jacobian matrices for the three links are:

$$J_{v_1} = \begin{bmatrix} -l_{c1}s_1 & 0 & 0 \\ l_{c1}c_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad J_{v_2} = \begin{bmatrix} -(l_1+q_2)s_1 & c_1 & 0 \\ (l_1+q_2)c_1 & s_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad J_{v_3} = \begin{bmatrix} -(l_1+q_2)s_1 - l_{c3}s_{13} & c_1 & -l_{c3}s_{13} \\ (l_1+q_2)c_1 + l_{c3}c_{13} & s_1 & l_{c3}c_{13} \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

where $s_1 = \sin(q_1)$, $c_1 = \cos(q_1)$, $s_{13} = \sin(q_1+q_3)$, and $c_{13} = \cos(q_1+q_3)$. Hence, the matrix $D(q)$ is given by:

$$D(q) = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + m_3 J_{v_3}^T J_{v_3} + J_{\omega_1}^T R_1^0 I_1 (R_1^0)^T J_{\omega_1} + J_{\omega_2}^T R_2^0 I_2 (R_2^0)^T J_{\omega_2} + J_{\omega_3}^T R_3^0 I_3 (R_3^0)^T J_{\omega_3} \tag{3}$$

$$= \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \tag{4}$$

where

$$d_{11} = I_{1,z} + I_{2,z} + I_{3,z} + l_1^2 m_2 + l_1^2 m_3 + l_{c1}^2 m_1 + l_{c3}^2 m_3 + m_2 q_2^2 + m_3 q_2^2 + 2 l_1 m_2 q_2 + 2 l_1 m_3 q_2 + 2 l_1 l_{c3} m_3 \cos(q_3) + 2 l_{c3} m_3 q_2 \cos(q_3) \tag{5}$$

$$d_{12} = d_{21} = -l_{c3} m_3 \sin(q_3) \tag{6}$$

$$d_{13} = d_{31} = I_{3,z} + l_{c3}^2 m_3 + l_1 l_{c3} m_3 \cos(q_3) + l_{c3} m_3 q_2 \cos(q_3) \tag{7}$$

$$d_{22} = m_2 + m_3 \tag{8}$$

$$d_{23} = d_{32} = -l_{c3} m_3 \sin(q_3) \tag{9}$$

$$d_{33} = I_{3,z} + l_{c3}^2 m_3. \tag{10}$$

As described above, since the rotation of all the links is only about the z_0 axis, only the moments of inertia about the axis pointing out of the page are relevant (i.e., $I_{1,z}$, $I_{2,z}$, $I_{3,z}$).

Finding the Christoffel symbols c_{ijk} as

$$c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\} \tag{11}$$

for $i = 1, 2, 3$; $j = 1, 2, 3$; $k = 1, 2, 3$, and writing the matrix $C(q, \dot{q})$ with its its $(k, j)^{th}$ element being

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i, \tag{12}$$

we get

$$C(q,\dot{q}) = \begin{bmatrix} C_{11}(q,\dot{q}) & C_{12}(q,\dot{q}) & C_{13}(q,\dot{q}) \\ C_{21}(q,\dot{q}) & C_{22}(q,\dot{q}) & C_{23}(q,\dot{q}) \\ C_{31}(q,\dot{q}) & C_{32}(q,\dot{q}) & C_{33}(q,\dot{q}) \end{bmatrix} \tag{13}$$

where

$$C_{11}(q,\dot{q}) = \dot{q}_2(l_1 m_2 + l_1 m_3 + m_2 q_2 + m_3 q_2 + l_{c3} m_3 \cos(q_3)) - l_{c3} m_3 \dot{q}_3 \sin(q_3)(l_1 + q_2) \tag{14}$$
$$C_{12}(q,\dot{q}) = \dot{q}_1(l_1 m_2 + l_1 m_3 + m_2 q_2 + m_3 q_2 + l_{c3} m_3 \cos(q_3)) \tag{15}$$
$$C_{13}(q,\dot{q}) = -l_{c3} m_3 \sin(q_3)(l_1 + q_2)(\dot{q}_1 + \dot{q}_3) \tag{16}$$
$$C_{21}(q,\dot{q}) = -\dot{q}_1(l_1 m_2 + l_1 m_3 + m_2 q_2 + m_3 q_2 + l_{c3} m_3 \cos(q_3)) - l_{c3} m_3 \dot{q}_3 \cos(q_3) \tag{17}$$
$$C_{22}(q,\dot{q}) = 0 \tag{18}$$
$$C_{23}(q,\dot{q}) = -l_{c3} m_3 \cos(q_3)(\dot{q}_1 + \dot{q}_3) \tag{19}$$
$$C_{31}(q,\dot{q}) = l_{c3} m_3 \dot{q}_2 \cos(q_3) + l_{c3} m_3 \dot{q}_1 \sin(q_3)(l_1 + q_2) \tag{20}$$
$$C_{32}(q,\dot{q}) = l_{c3} m_3 \dot{q}_1 \cos(q_3) \tag{21}$$
$$C_{33}(q,\dot{q}) = 0 \tag{22}$$

The potential energy of the manipulator is given by:

$$P = m_1 g l_{c1} \sin(q_1) + m_2 g(l_1 + q_2)\sin(q_1) + m_3 g(\sin(q_1)(l_1 + q_2) + l_{c3} \sin(q_1 + q_3)). \tag{23}$$

Hence,

$$g(q) = \begin{bmatrix} \frac{\partial P}{\partial q_1} \\ \frac{\partial P}{\partial q_2} \\ \frac{\partial P}{\partial q_3} \end{bmatrix} = \begin{bmatrix} m_1 g l_{c1} \cos(q_1) + m_2 g(l_1 + q_2)\cos(q_1) + m_3 g(\cos(q_1)(l_1 + q_2) + l_{c3}\cos(q_1 + q_3)) \\ m_2 g \sin(q_1) + m_3 g \sin(q_1) \\ m_3 g l_{c3} \cos(q_1 + q_3) \end{bmatrix}. \tag{24}$$

The dynamical equations of the manipulator are given by:

$$D(q)\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + C(q,\dot{q})\dot{q} + g(q) = \begin{bmatrix} \tau_1 \\ f_2 \\ \tau_3 \end{bmatrix} \tag{25}$$

where τ_1 is the applied torque at the first joint (revolute), f_2 is the applied force at the second joint (prismatic), and τ_3 is the applied torque at the third joint (revolute).

Newton-Euler formulation to find the dynamics:

- Forward recursion: To assign the Denavit-Hartenberg coordinate frames, pick z_0 pointing out of the page and x_0 to the right in the figure (i.e., y_0 pointing towards top in the figure); pick z_1 pointing along the actuation axis of the second joint; x_1 should be picked such that it intersects both z_0 and z_1 and is perpendicular to both z_0 and z_1. Hence, x_1 is in the plane of the page and the origins of frames 0 and 1 are at the same location. Then, y_1 can point out of the page. Pick z_2 pointing out of the page (rotation axis of the third joint). x_2 has to be picked such that it intersects both z_1 and z_2 and is perpendicular to both z_1 and z_2. Hence, x_2 is in the plane of the page. y_2 is also in the plane of the page. y_2 is in the same direction as $-z_1$. Pick z_3 parallel to z_2 (with origin of frame 3 at the end-effector). Then, x_3 can be picked in the plane of the page pointing forward from the end-effector. The Denavit-Hartenberg table for this manipulator is:

Link	a_i	α_i	d_i	θ_i
1	0	90^o	0	$90^o + q_1$
2	0	-90^o	$l_1 + q_2$	0
3	l_3	0	0	$-90^o + q_3$

We have

$$R_1^0 = \begin{bmatrix} -s_1 & 0 & c_1 \\ c_1 & 0 & s_1 \\ 0 & 1 & 0 \end{bmatrix} \;\; ; \;\; R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \;\; ; \;\; R_3^2 = \begin{bmatrix} s_3 & c_3 & 0 \\ -c_3 & s_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{26}$$

The angular velocities of the links (written relative to the link-fixed frames) are $\omega_1 = \dot{q}_1\vec{j}$, $\omega_2 = \dot{q}_1\vec{k}$, and $\omega_3 = (\dot{q}_1+\dot{q}_3)\vec{k}$. Here, the notations $\vec{i}$, $\vec{j}$, and $\vec{k}$ denote the 3×1 unit vectors, i.e., $\vec{i} = [1,0,0]^T$, $\vec{j} = [0,1,0]^T$, $\vec{k} = [0,0,1]^T$. The gravity vector can be written in the link-fixed frames as: $g_1 = g[-c_1, 0, -s_1]^T$, $g_2 = g[-c_1, s_1, 0]^T$, and $g_3 = g[-s_{13}, -c_{13}, 0]^T$.

Since the base frame is stationary, we have $a_{c,0} = a_{e,0} = 0$. Looking at the orientation of frame 1, we have $r_{1,c1} = l_{c1}\vec{k}$. Hence, the linear acceleration of the center of mass of link 1 is:

$$a_{c,1} = R_0^1 a_{e,0} + \dot{\omega}_1 \times r_{1,c1} + \omega_1 \times (\omega_1 \times r_{1,c1}) = l_{c1}\ddot{q}_1\vec{i} - l_{c1}\dot{q}_1^2\vec{k}. \tag{27}$$

Since the origins of frames 0 and 1 are at the same point, it is convenient to take the end of link 1 to be at the origin of frames 0 and 1. Note that the end of the link can be defined to be at any convenient location fixed to the link as long as we keep the equations consistent. If we pick the end of link 1 to be at the fixed base (origin of frames 0 and 1), then $a_{e,1} = 0$. We can write $r_{2,c2} = -(l_1+q_2)\vec{j}$. Also, $r_{2,c1} = r_{1,c1}$. The linear acceleration $a_{c,2} = R_0^2 \dot{v}_{c,2}^{(0)}$ of the center of mass of link 2 can be found to be:
$a_{c,2} = (\ddot{q}_1(l_1+q_2) + 2\dot{q}_1\dot{q}_2)\vec{i} + ((l_1+q_2)\dot{q}_1^2 - \ddot{q}_2)\vec{j}$. Also, by the assumption mentioned near the top of page that the linkage between links 1 and 2 is such that when joint 2 actuates, it shifts the center of mass of link 2 by distance q_2, we can write $a_{e,2} = a_{c,2}$. Note that $r_{3,c3} = l_{c3}\vec{i}$ and $r_{3,c2} = [0,0,0]^T$. We can write

$$\begin{aligned} a_{c,3} = {} & (c_3((-l_1-q_2)\dot{q}_1^2 + \ddot{q}_2) - l_{c3}(\dot{q}_1+\dot{q}_3)^2 + s_3(\ddot{q}_1(l_1+q_2) + 2\dot{q}_1\dot{q}_2))\vec{i} \\ & + (l_{c3}(\ddot{q}_1+\ddot{q}_3) - s_3((-l_1-q_2)\dot{q}_1^2 + \ddot{q}_2) + c_3(\ddot{q}_1(l_1+q_2) + 2\dot{q}_1\dot{q}_2))\vec{j}. \end{aligned} \tag{28}$$

- Backward recursion: Start with $f_4 = \tau_4 = 0$. Then,

$$f_3 = m_3(a_{c,3} - g_3) = f_{3,x}\vec{i} + f_{3,y}\vec{j} \tag{29}$$

$$\tau_3 = -f_3 \times r_{3,c3} + I_3\dot{\omega}_3 + \omega_3 \times (I_3\omega_3) = \tau_{3,z}\vec{k} \tag{30}$$

where

$$f_{3,x} = m_3(c_3((-l_1-q_2)\dot{q}_1^2 + \ddot{q}_2) - l_{c3}(\dot{q}_1+\dot{q}_3)^2 + gs_{13} + s_3(\ddot{q}_1(l_1+q_2) + 2\dot{q}_1\dot{q}_2)) \tag{31}$$

$$f_{3,y} = m_3(l_{c3}(\ddot{q}_1+\ddot{q}_3) + gc_{13} - s_3((-l_1-q_2)\dot{q}_1^2 + \ddot{q}_2) + c_3(\ddot{q}_1(l_1+q_2) + 2\dot{q}_1\dot{q}_2)) \tag{32}$$

$$\tau_{3,z} = I_{3,z}(\ddot{q}_1+\ddot{q}_3) + l_{c3}m_3(l_{c3}(\ddot{q}_1+\ddot{q}_3) + gc_{13} - s_3((-l_1-q_2)\dot{q}_1^2 + \ddot{q}_2) + c_3(\ddot{q}_1(l_1+q_2) + 2\dot{q}_1\dot{q}_2)). \tag{33}$$

Then,

$$f_2 = R_3^2 f_3 + m_2(a_{c,2} - g_2) = f_{2.x}\vec{i} + f_{2.y}\vec{j} \tag{34}$$

$$\tau_2 = R_3^2\tau_3 - f_2 \times r_{2,c2} + (R_3^2 f_3) \times r_{3,c2} + I_2\dot{\omega}_2 + \omega_2 \times (I_2\omega_2) = \tau_{2,z}\vec{k} \tag{35}$$

where

$$\begin{aligned} f_{2,x} = {} & g(m_2+m_3)c_1 + l_1(m_2+m_3)\ddot{q}_1 + (m_2+m_3)q_2\ddot{q}_1 + 2(m_2+m_3)\dot{q}_1\dot{q}_2 + l_{c3}m_3(\ddot{q}_1+\ddot{q}_3)c_3 \\ & - l_{c3}m_3(\dot{q}_1^2+\dot{q}_3^2)s_3 - 2l_{c3}m_3\dot{q}_1\dot{q}_3 s_3 \end{aligned} \tag{36}$$

$$\begin{aligned} f_{2,y} = {} & l_1 m_2\dot{q}_1^2 - (m_2+m_3)\ddot{q}_2 + l_1 m_3\dot{q}_1^2 + (m_2+m_3)q_2\dot{q}_1^2 - g(m_2+m_3)s_1 + l_{c3}m_3(\ddot{q}_1+\ddot{q}_3)s_3 \\ & + l_{c3}m_3(\dot{q}_1^2+\dot{q}_3^2)c_3 + 2l_{c3}m_3\dot{q}_1\dot{q}_3 c_3 \end{aligned} \tag{37}$$

$$\begin{aligned} \tau_{2,z} = {} & I_{2,z}\ddot{q}_1 + (l_1+q_2)(m_2(\tilde{a}_1 + gc_1) + m_3 s_3(c_3((-l_1-q_2)\dot{q}_1^2 + \ddot{q}_2) - l_{c3}(\dot{q}_1+\dot{q}_3)^2 + gs_{13} + s_3\tilde{a}_1) \\ & + m_3 c_3(l_{c3}(\ddot{q}_1+\ddot{q}_3) + gc_{13} - s_3((-l_1-q_2)\dot{q}_1^2 + \ddot{q}_2) + c_3\tilde{a}_1)) + I_{3,z}(\ddot{q}_1+\ddot{q}_3) \\ & + l_{c3}m_3(l_{c3}(\ddot{q}_1+\ddot{q}_3) + gc_{13} - s_3((-l_1-q_2)\dot{q}_1^2 + \ddot{q}_2) + c_3\tilde{a}_1) \end{aligned} \tag{38}$$

where $\tilde{a}_1 = (\ddot{q}_1(l_1 + q_2) + 2\dot{q}_1\dot{q}_2)$. Then,

$$f_1 = R_2^1 f_2 + m_1(a_{c,1} - g_1) = f_{1,x}\vec{i} + f_{1,z}\vec{k} \tag{39}$$

$$\tau_1 = R_2^1 \tau_2 - f_1 \times r_{1,c1} + (R_2^1 f_2) \times r_{2,c1} + I_1\dot{\omega}_1 + \omega_1 \times (I_1\omega_1) = \tau_{1,y}\vec{j} \tag{40}$$

where

$$\begin{aligned} f_{1,x} = {} & g(m_1 + m_2 + m_3)c_1 + l_1(m_2 + m_3)\ddot{q}_1 + l_{c1}m_1\ddot{q}_1 + (m_2 + m_3)q_2\ddot{q}_1 + 2m_2\dot{q}_1\dot{q}_2 + 2m_3\dot{q}_1\dot{q}_2 \\ & + l_{c3}m_3(\ddot{q}_1 + \ddot{q}_3)c_3 - l_{c3}m_3(\dot{q}_1^2 + \dot{q}_3^2)s_3 - 2l_{c3}m_3\dot{q}_1\dot{q}_3 s_3 \end{aligned} \tag{41}$$

$$\begin{aligned} f_{1,z} = {} & (m_2 + m_3)\ddot{q}_2 - l_1(m_2 + m_3)\dot{q}_1^2 - l_{c1}m_1\dot{q}_1^2 - (m_2 + m_3)q_2\dot{q}_1^2 + g(m_1 + m_2 + m_3)s_1 \\ & - l_{c3}m_3(\ddot{q}_1 + \ddot{q}_3)s_3 - l_{c3}m_3(\dot{q}_1^2 + \dot{q}_3^2)c_3 - 2l_{c3}m_3\dot{q}_1\dot{q}_3 c_3 \end{aligned} \tag{42}$$

$$\begin{aligned} \tau_{1,y} = {} & (I_{1,z} + I_{2,z} + I_{3,z})\ddot{q}_1 + I_{3,z}\ddot{q}_3 + (m_2 + m_3)\{l_1^2\ddot{q}_1 + q_2^2\ddot{q}_1 + 2l_1q_2\ddot{q}_1 + 2l_1\dot{q}_1\dot{q}_2 + 2q_2\dot{q}_1\dot{q}_2 + gl_1c_1 + gq_2c_1\} \\ & + l_{c1}^2m_1\ddot{q}_1 + l_{c3}^2m_3(\ddot{q}_1 + \ddot{q}_3) + gl_{c3}m_3c_{13} + gl_{c1}m_1c_1 - l_{c3}m_3\ddot{q}_2s_3 - (l_1 + q_2)l_{c3}m_3\dot{q}_3^2s_3 \\ & + l_1l_{c3}m_3(2\ddot{q}_1 + \ddot{q}_3)c_3 + l_{c3}m_3q_2(2\ddot{q}_1 + \ddot{q}_3)c_3 + 2l_{c3}m_3\dot{q}_1\dot{q}_2c_3 - 2(l_1 + q_2)l_{c3}m_3\dot{q}_1\dot{q}_3s_3 \end{aligned} \tag{43}$$

Since the actuation of the first joint is along $z_0 = y_1$, the actuated joint torque for the first joint (revolute) is $\tau_{1,y}$. Since the actuation of the second joint is along $z_1 = -y_2$, the actuated joint force for the second joint is $-f_{2,y}$. Since the actuation of the third joint is along $z_2 = z_3$, the actuated joint torque for the third joint is $\tau_{3,z}$. Another way to see which joint forces/torques are the externally actuated forces/torques is to look at $f_i^T z_{i-1}^{(i)}$ or $\tau_i^T z_{i-1}^{(i)}$ (depending on whether joint i is prismatic or revolute) where $z_{i-1}^{(i)}$ denotes the axis z_{i-1} written relative to frame i; here, $\tau_1^T z_0^{(1)} = \tau_{1,y}$, $f_2^T z_1^{(2)} = -f_{2,y}$, and $\tau_3^T z_2^{(3)} = \tau_{3,z}$.

Therefore, the dynamics equations are:

$$\begin{aligned} u_1 = {} & (I_{1,z} + I_{2,z} + I_{3,z})\ddot{q}_1 + I_{3,z}\ddot{q}_3 + (m_2 + m_3)\{l_1^2\ddot{q}_1 + q_2^2\ddot{q}_1 + 2l_1q_2\ddot{q}_1 + 2l_1\dot{q}_1\dot{q}_2 + 2q_2\dot{q}_1\dot{q}_2 + gl_1c_1 + gq_2c_1\} \\ & + l_{c1}^2m_1\ddot{q}_1 + l_{c3}^2m_3(\ddot{q}_1 + \ddot{q}_3) + gl_{c3}m_3c_{13} + gl_{c1}m_1c_1 - l_{c3}m_3\ddot{q}_2s_3 - (l_1 + q_2)l_{c3}m_3\dot{q}_3^2s_3 \\ & + l_1l_{c3}m_3(2\ddot{q}_1 + \ddot{q}_3)c_3 + l_{c3}m_3q_2(2\ddot{q}_1 + \ddot{q}_3)c_3 + 2l_{c3}m_3\dot{q}_1\dot{q}_2c_3 - 2(l_1 + q_2)l_{c3}m_3\dot{q}_1\dot{q}_3s_3 \end{aligned} \tag{44}$$

$$\begin{aligned} u_2 = {} & -\big\{l_1m_2\dot{q}_1^2 - (m_2 + m_3)\ddot{q}_2 + l_1m_3\dot{q}_1^2 + (m_2 + m_3)q_2\dot{q}_1^2 - g(m_2 + m_3)s_1 + l_{c3}m_3(\ddot{q}_1 + \ddot{q}_3)s_3 \\ & + l_{c3}m_3(\dot{q}_1^2 + \dot{q}_3^2)c_3 + 2l_{c3}m_3\dot{q}_1\dot{q}_3c_3\big\} \end{aligned} \tag{45}$$

$$u_3 = I_{3,z}(\ddot{q}_1 + \ddot{q}_3) + l_{c3}m_3(l_{c3}(\ddot{q}_1 + \ddot{q}_3) + gc_{13} - s_3((-l_1 - q_2)\dot{q}_1^2 + \ddot{q}_2) + c_3(\ddot{q}_1(l_1 + q_2) + 2\dot{q}_1\dot{q}_2)) \tag{46}$$

where u_1 is the actuated torque for the first joint, u_2 is the actuated force for the second joint, and u_3 is the actuated torque for the third joint.