Newton-Euler formulation for dynamics of an n-link manipulator

In the Newton-Euler formulation, the force and torque equations for each link are written and then all the forces and torques (including both the externally applied forces/torques and the constraint forces/torques) are found through a recursive procedure.

For a rigid body (with constant mass m), the force equation can be written as $f = ma$ where f is the total force acting on the body and a is the acceleration of the center of mass of the body. The quantities f and a can be written in either the inertial frame or the body-fixed frame; the equation $f = ma$ retains the same form in both cases. The torque equation for a rigid body is given by

$$\frac{d}{dt}(I_0\omega_0) = \tau_0 \tag{1}$$

where I_0 is the moment of inertia (i.e., inertia matrix) relative to an inertial frame whose origin is placed at the center of mass of the body, ω_0 is the angular velocity of the body (relative to the inertial frame), and τ_0 is the total torque acting on the body (again, written relative to the inertial frame). In general, the inertia matrix I_0 changes with changing angular orientation of the rigid body. The inertia matrix I_0 can be written as $I_0 = R_b^0 I (R_b^0)^T$ where R_b^0 is the rotation matrix that transforms coordinates from the body-fixed frame to the inertial frame and I is the inertia matrix written relative to the body-fixed frame. To see this expression for the inertia matrix, consider the rotational kinetic energy which can be written in the inertial frame as $\frac{1}{2}\omega_0^T I_0 \omega_0$ and in the body-fixed frame as $\frac{1}{2}\omega^T I\omega$ where ω is the angular velocity of the body written relative to the body-fixed frame. Since $\omega = (R_b^0)^T\omega_0$ and the rotational kinetic energy (which is a scalar quantity) should be the same whether written using the body-fixed frame or the inertial frame, we see that $\omega_0^T I_0\omega_0 = \omega^T I\omega = \omega_0^T R_b^0 I (R_b^0)^T\omega_0$ and therefore $I_0 = R_b^0 I (R_b^0)^T$. Hence, from equation (1), we get

$$\frac{d}{dt}(R_b^0 I (R_b^0)^T R_b^0\omega) = \tau_0. \tag{2}$$

Note that I, which is the inertia matrix written relative to the body-fixed frame is typically a constant matrix. Therefore,

$$\tau_0 = \frac{d}{dt}(R_b^0 I\omega) \;=\; R_b^0 I\dot{\omega} + \frac{d}{dt}(R_b^0)I\omega \;=\; R_b^0 I\dot{\omega} + S(\omega_0)R_b^0 I\omega. \tag{3}$$

Multiplying both sides of the equation (3) by $(R_b^0)^T$, and using the property that $RS(a)R^T = S(Ra)$ for any vector a and rotation matrix R, we get

$$\tau = (R_b^0)^T R_b^0 I\dot{\omega} + (R_b^0)^T S(\omega_0)R_b^0 I\omega \;=\; I\dot{\omega} + S((R_b^0)^T\omega_0)I\omega \;=\; I\dot{\omega} + S(\omega)I\omega \;=\; I\dot{\omega} + \omega\times(I\omega) \tag{4}$$

where τ denotes the total torque on the body written relative to the body-fixed frame.

To write the Newton-Euler formulation of the manipulator dynamics, define the following quantities, all of which are written relative to frame i:

- $a_{c,i}$: acceleration of the center of mass of link i
- $a_{e,i}$: acceleration of the end of link i (i.e., the origin of frame $i+1$)
- ω_i : angular velocity of frame i relative to frame 0
- g_i : acceleration due to gravity written in terms of frame i
- f_i : force exerted by link $i-1$ on link i
- τ_i : torque exerted by link $i-1$ on link i
- m_i : mass of link i
- I_i : inertia matrix of link i about a body-fixed frame that has its origin at the center of mass of link i and is parallel to frame i
- $r_{i,ci}$: vector from start of link i (i.e., end of link $i-1$, i.e., origin of frame i) to center of mass of link i
- $r_{i+1,ci}$: vector from end of link i (i.e., origin of frame $i+1$) to center of mass of link i
- $r_{i,i+1}$: vector from start of link i (i.e., origin of frame i) to end of link i (i.e., origin of frame $i+1$)

Note that all the quantities above are written relative to frame i.

The total force acting on link i, when written in the body-fixed frame (frame i) is $f_i - R^i_{i+1} f_{i+1} + m_i g_i$. Hence, the force equation for link i is:

$$f_i - R^i_{i+1} f_{i+1} + m_i g_i = m_i a_{c,i}. \tag{5}$$

The total torque acting on link i, when written in the body-fixed frame (frame i) is:
$\tau_i - R^i_{i+1}\tau_{i+1} + f_i \times r_{i,ci} - (R^i_{i+1} f_{i+1}) \times r_{i+1,ci}$.
Hence, the torque equation for link i is:

$$\tau_i - R^i_{i+1}\tau_{i+1} + f_i \times r_{i,ci} - (R^i_{i+1} f_{i+1}) \times r_{i+1,ci} = I_i \dot{\omega}_i + \omega_i \times (I_i \omega_i). \tag{6}$$

Note that there is no torque corresponding to the gravity force since the torque equation is being written relative to the center of mass of the link.

To find the quantities ω_i and $a_{c,i}$, a recursive procedure can be used. If the i^{th} joint is revolute, we can write:

$$\omega_i^{(0)} = \omega_{i-1}^{(0)} + z_{i-1}\dot{q}_i \tag{7}$$

where $\omega_{i-1}^{(0)}$ and $\omega_i^{(0)}$ denote the angular velocities of links $(i-1)$ and i, respectively, written relative to the frame 0. z_{i-1} denotes the rotation axis of the i^{th} joint (revolute joint), written relative to frame 0. Hence, multiplying both sides of (7) by R^i_0, we get

$$\omega_i = R^i_0 \omega_{i-1}^{(0)} + R^i_0 z_{i-1}\dot{q}_i \quad = \quad R^i_{i-1}\omega_{i-1} + R^i_0 z_{i-1}\dot{q}_i. \tag{8}$$

Denote $v_{c,i}$ and $v_{c,i}^{(0)}$ to be velocity of the center of mass of link i written relative to the body-fixed frame (frame i) and the base frame (frame 0), respectively. Denote $v_{e,i}$ and $v_{e,i}^{(0)}$ to be velocity of the end of link i written relative to the body-fixed frame (frame i) and the base frame (frame 0), respectively. If the i^{th} joint is revolute, we can write:

$$v_{c,i}^{(0)} = v_{e,i-1}^{(0)} + \omega_i^{(0)} \times r_{i,ci}^{(0)}. \tag{9}$$

Differentiating both sides of equation (9) with respect to time, we get

$$a_{c,i}^{(0)} = \dot{v}_{c,i}^{(0)} \quad = \quad a_{e,i-1}^{(0)} + \dot{\omega}_i^{(0)} \times r_{i,ci}^{(0)} + \omega_i^{(0)} \times (\omega_i^{(0)} \times r_{i,ci}^{(0)}). \tag{10}$$

Multiplying both sides of equation (10) by R^i_0, and using the property that $R(v_1 \times v_2) = (Rv_1) \times (Rv_2)$ for any vectors v_1 and v_2 and any rotation matrix R, we get:

$$a_{c,i} = R^i_{i-1} a_{e,i-1} + \dot{\omega}_i \times r_{i,ci} + \omega_i \times (\omega_i \times r_{i,ci}) \tag{11}$$

Also, we can write $a_{e,i}$ similarly as

$$a_{e,i} = R^i_{i-1} a_{e,i-1} + \dot{\omega}_i \times r_{i,i+1} + \omega_i \times (\omega_i \times r_{i,i+1}) \tag{12}$$

If the i^{th} joint is prismatic, the angular velocity of link i is physically equal to the angular velocity of link $i-1$ (there is no additional rotational motion as in the revolute joint). Hence, $\omega_i^{(0)} = \omega_{i-1}^{(0)}$ implying that

$$\omega_i = R^i_{i-1}\omega_{i-1}. \tag{13}$$

However, if joint i is a prismatic joint, the velocity of link i will have an additional component due to the translational motion of joint i. Hence, we can write

$$v_{c,i}^{(0)} = v_{e,i-1}^{(0)} + \omega_i^{(0)} \times r_{i,ci}^{(0)} + z_{i-1}\dot{q}_i. \tag{14}$$

Therefore, differentiating equation (14), we get

$$a_{c,i}^{(0)} = \dot{v}_{c,i}^{(0)} \quad = \quad a_{e,i-1}^{(0)} + \dot{\omega}_i^{(0)} \times r_{i,ci}^{(0)} + \omega_i^{(0)} \times (\omega_i^{(0)} \times r_{i,ci}^{(0)}) + \dot{z}_{i-1}\dot{q}_i + z_{i-1}\ddot{q}_i. \tag{15}$$

z_{i-1} denotes the translation axis of the i^{th} joint (prismatic joint), written relative to frame 0. Since the vector z_{i-1} is attached to the link $(i-1)$ which has an angular velocity ω_{i-1}, we can write $\dot{z}_{i-1} = \omega_{i-1}^{(0)} \times z_{i-1}$. Hence, multiplying both sides of equation (15) by R^i_0, and using the property that $R(v_1 \times v_2) = (Rv_1) \times (Rv_2)$ for any vectors v_1 and v_2 and any rotation matrix R, we get:

$$\begin{aligned} a_{c,i} &= R^i_{i-1} a_{e,i-1} + \dot{\omega}_i \times r_{i,ci} + \omega_i \times (\omega_i \times r_{i,ci}) + R^i_0[\omega_{i-1}^{(0)} \times z_{i-1}]\dot{q}_i + R^i_0 z_{i-1}\ddot{q}_i & (16) \\ &= R^i_{i-1} a_{e,i-1} + \dot{\omega}_i \times r_{i,ci} + \omega_i \times (\omega_i \times r_{i,ci}) + [(R^i_{i-1}\omega_{i-1}) \times (R^i_0 z_{i-1})]\dot{q}_i + R^i_0 z_{i-1}\ddot{q}_i & (17) \end{aligned}$$

Similarly, we can write $a_{e,i}$ as

$$a_{e,i} = R^i_{i-1} a_{e,i-1} + \dot{\omega}_i \times r_{i,i+1} + \omega_i \times (\omega_i \times r_{i,i+1}) + [(R^i_{i-1}\omega_{i-1}) \times (R^i_0 z_{i-1})]\dot{q}_i + R^i_0 z_{i-1}\ddot{q}_i. \tag{18}$$

Also, the quantities g_i are given by $g_i = R_0^i g_0$ where g_0 is the acceleration due to gravity written in terms of the base frame (i.e., frame 0).

Hence, the Newton-Euler method to write the dynamics can be summarized as follows:

- *Forward recursion:* Start with $\omega_0 = 0$, $a_{c,0} = 0$, $a_{e,0} = 0$. Recursively find $\omega_i, a_{c,i}$, and $a_{e,i}$ for $i = 1, \ldots, n$ using the equations in (19) given below if the i^{th} joint is revolute or the equations in (20) if the i^{th} joint is prismatic.

  Forward recursion equations for revolute joint:

$$\left.\begin{aligned} \omega_i &= R_{i-1}^i \omega_{i-1} + R_0^i z_{i-1} \dot{q}_i \\ a_{c,i} &= R_{i-1}^i a_{e,i-1} + \dot{\omega}_i \times r_{i,ci} + \omega_i \times (\omega_i \times r_{i,ci}) \\ a_{e,i} &= R_{i-1}^i a_{e,i-1} + \dot{\omega}_i \times r_{i,i+1} + \omega_i \times (\omega_i \times r_{i,i+1}) \end{aligned}\right\} \tag{19}$$

  Forward recursion equations for prismatic joint:

$$\left.\begin{aligned} \omega_i &= R_{i-1}^i \omega_{i-1} \\ a_{c,i} &= R_{i-1}^i a_{e,i-1} + \dot{\omega}_i \times r_{i,ci} + \omega_i \times (\omega_i \times r_{i,ci}) + [(R_{i-1}^i \omega_{i-1}) \times (R_0^i z_{i-1})]\dot{q}_i + R_0^i z_{i-1} \ddot{q}_i \\ a_{e,i} &= R_{i-1}^i a_{e,i-1} + \dot{\omega}_i \times r_{i,i+1} + \omega_i \times (\omega_i \times r_{i,i+1}) + [(R_{i-1}^i \omega_{i-1}) \times (R_0^i z_{i-1})]\dot{q}_i + R_0^i z_{i-1} \ddot{q}_i. \end{aligned}\right\} \tag{20}$$

  Find the quantities g_i as $g_i = R_0^i g_0$ or equivalently using the recursive equation $g_i = R_{i-1}^i g_{i-1}$. This equation is the same for both revolute and prismatic joints.

- *Backward recursion:* Start with $f_{n+1} = 0$, $\tau_{n+1} = 0$. Recursively find f_i and τ_i for $i = n, \ldots, 1$, using the equations:

$$\left.\begin{aligned} f_i &= R_{i+1}^i f_{i+1} - m_i g_i + m_i a_{c,i} \\ \tau_i &= R_{i+1}^i \tau_{i+1} - f_i \times r_{i,ci} + (R_{i+1}^i f_{i+1}) \times r_{i+1,ci} + I_i \dot{\omega}_i + \omega_i \times (I_i \omega_i). \end{aligned}\right\} \tag{21}$$

Once the forward and backward recursions are completed, we get expressions for all the force and torque quantities $f_n, \tau_n, f_{n-1}, \tau_{n-1}, \ldots, f_1, \tau_1$. This includes the external applied forces/torques (the actuator forces/torques on the joints) and the constraint forces/torques. To get the dynamical equations of the manipulator, we need to pick the equations that correspond to the external applied forces/torques. In general, since the actuation direction of the i^{th} joint is defined to be z_{i-1} in the Denavit-Hartenberg convention, the i^{th} generalized force will correspond to $f_i^T z_{i-1}^{(i)}$ if the i^{th} joint is a prismatic joint and to $\tau_i^T z_{i-1}^{(i)}$ if the i^{th} joint is a revolute joint. Here, $z_{i-1}^{(i)}$ denotes the axis z_{i-1} written relative to frame i, i.e., $z_{i-1}^{(i)} = R_0^i z_{i-1}$. Thus, to get the final set of dynamics equations for the manipulator from the set of expressions for $f_n, \tau_n, f_{n-1}, \tau_{n-1}, \ldots, f_1, \tau_1$ obtained from (21), we do the following: for each i from 1 to n, write the expression for $f_i^T z_{i-1}^{(i)}$ if the i^{th} joint is prismatic and write the expression for $\tau_i^T z_{i-1}^{(i)}$ if the i^{th} joint is revolute. Thus, we get a set of n scalar equations (each of which is a second-order differential equation). Once this final set of equations is written, it will be in the general form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau. \tag{22}$$

$D(q)$ is an $n \times n$ matrix function of q. $C(q, \dot{q})$ is an $n \times n$ matrix function of q and $\dot{q}$. $g(q)$ is an $n \times 1$ matrix function of q. τ is the $n \times 1$ vector of generalized forces (i.e., the forces/torques applied to the n joints).

Note: The primary reason for writing the force and torque equations in the body-fixed frame is that the inertia matrix written relative to the body-fixed frame is a constant matrix. In the special case in which all the joints in the manipulator are prismatic joints, then the angular velocities ω_i are all zero and therefore the torque equations are not relevant; in this case, we can, as well, write the force equations in the inertial frame and write the quantities $a_{c,i}$, $a_{e,i}$, and g_i relative to the inertial frame.

Additional force/torque on a link in the manipulator: If there is an additional external applied force or torque on a link in the manipulator, it can be easily taken into account in the Newton-Euler formulation by simply including the additional force or torque in the appropriate force equation (5) or torque equation (6).

An external applied force/torque on the end-effector can be modeled similarly by setting f_{n+1} and τ_{n+1} appropriately to include this external applied force/torque. However, a simpler approach is to use the property that we know from the principle of virtual work that if F_e is a 6×1 vector of forces and torques on the end-effector, then the resulting vector of generalized forces on the manipulator joints is given by $J^T(q)F_e$ where $J(q)$ is the manipulator Jacobian matrix (of dimension $6 \times n$). Therefore, the dynamics equations with the external applied force/torque on the end-effector is:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + J^T(q)F_e = \tau \tag{23}$$