Hamiltonian Formulation for Dynamics

The Hamiltonian formulation provides a method to write the dynamics of a robotic manipulator as a set of first-order differential equations instead of a set of second-order differential equations as in the Euler-Lagrange method.

Given the Lagrangian $\mathcal{L} = K - P$ with K being the kinetic energy of the system, P being the potential energy of the system, and $q_1, \ldots, q_n$ being the generalized coordinates of the system, define the generalized momentum p_k as

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}. \tag{1}$$

For an n-link robotic manipulator, we know that the kinetic energy K is given by $K = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j$ where d_{ij} denotes the $(i,j)^{th}$ element of the matrix $D(q)$. Hence, we have $p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j$. Therefore,

$$\sum_{k=1}^{n} \dot{q}_k p_k = \sum_{k=1}^{n} \dot{q}_k \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j = \sum_{k=1}^{n}\sum_{j=1}^{n} d_{kj}(q)\dot{q}_k\dot{q}_j = 2K. \tag{2}$$

Define the Hamiltonian H as $H = \sum_{k=1}^{n} \dot{q}_k p_k - \mathcal{L} = 2K - (K - P) = K + P$ where K is the kinetic energy and P is the potential energy. $\mathcal{L}$ is a function of q and $\dot{q}$, i.e., $\mathcal{L}(q, \dot{q})$. From Euler-Lagrange's equations, we know that $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i$. Since $p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$, we have from Euler-Lagrange's equations that $\frac{\partial \mathcal{L}}{\partial q_k} = \dot{p}_k - \tau_k$. From the equation that $H = \sum_{k=1}^{n} \dot{q}_k p_k - \mathcal{L}(q, \dot{q})$, we have $\frac{\partial H}{\partial p_k} = \dot{q}_k$. Also, $\frac{\partial H}{\partial q_k} = -\frac{\partial \mathcal{L}}{\partial q_k} = -\dot{p}_k + \tau_k$. Hence, we get

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} + \tau_k. \end{aligned} \tag{3}$$

Note that this is a set of first-order differential equations.

Since H is a function of $q_k, k = 1, \ldots, n$, and $p_k, k = 1, \ldots, n$, we have

$$\begin{aligned} \frac{dH}{dt} &= \sum_{k=1}^{n} \Big(\frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k \Big) \\ &= \sum_{k=1}^{n} \Big((-\dot{p}_k + \tau_k)\dot{q}_k + (\dot{q}_k)\dot{p}_k \Big) = \sum_{k=1}^{n} \tau_k \dot{q}_k = \dot{q}^T \tau. \end{aligned} \tag{4}$$

Since, as shown above, H is the total energy ($H = K + P$), the rate of change of H is the power injected into the system (assuming that there are no friction losses, etc.). Since power for a prismatic joint is the product of the joint's linear velocity and force and power for a revolute joint is the product of the joint's angular velocity and torque, we see that the total power is $\dot{q}^T\tau = \dot{q}_1^T\tau_1 + \ldots + \dot{q}_n^T\tau_n$ where each τ_i is the force or torque of the i^{th} joint (force for a prismatic joint and torque for a revolute joint).