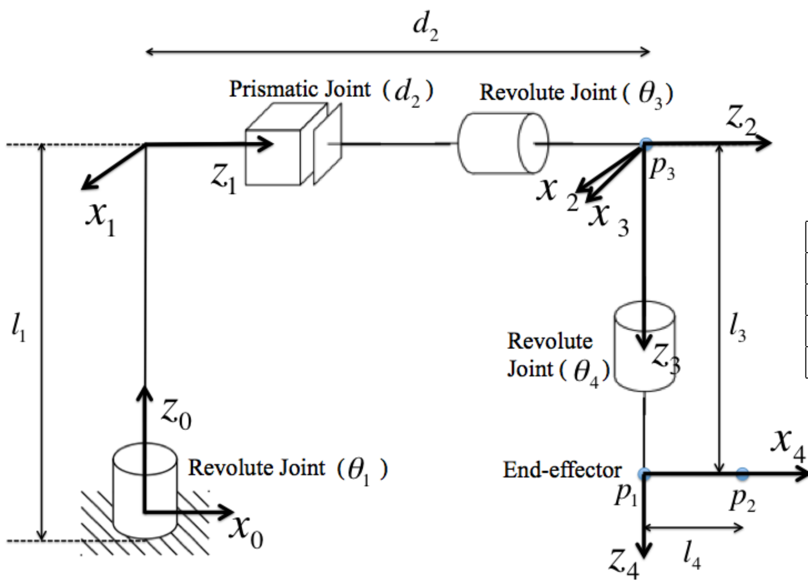
Outline of Solutions for Additional Problems for Practice

1. We are given a RPRR (revolute-prismatic-revolute-revolute) robot manipulator. The choice of the D-H coordinate frames and the corresponding D-H table are shown below. Here, the point p_1 is given to be the end-effector location.



	a_i	α_i	d_i	θ_i
Link 1	0	-90^o	l_1	θ_1^*
Link 2	0	0	d_2^*	0
Link 3	0	-90^o	0	θ_3^*
Link 4	0	0	l_3	θ_4^*

The homogeneous transformations A_1, A_2, A_3, and A_4 for this manipulator are given below.

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$A_3 = \begin{bmatrix} c_3 & 0 & -s_3 & 0 \\ s_3 & 0 & c_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$H = A_1 A_2 A_3 A_4 \tag{3}$$

We are asked to find the Jacobian for this robot manipulator (with p_1 being the end-effector position). Using the matrices A_1, A_2, etc., from the forward kinematics found above, we find

$$o_0 = [0,0,0]^T \;;\; o_1 = [0,0,l_1]^T \;;\; o_2 = [-d_2 s_1, d_2 c_1, l_1]^T \;;\; o_3 = o_2$$

$$o_4 = [-d_2 s_1 - l_3 c_1 s_3, d_2 c_1 - l_3 s_1 s_3, l_1 - l_3 c_3]^T \tag{4}$$

$$z_0 = [0,0,1]^T \;;\; z_1 = [-s_1, c_1, 0]^T \;;\; z_2 = z_1 \;;\; z_3 = [-c_1 s_3, -s_1 s_3, -c_3]^T \tag{5}$$

This manipulator is RPRR (first, third, and fourth joints revolute; second joint prismatic). Hence, the linear velocity Jacobian J_v and the angular velocity Jacobian J_ω are given by

$$\begin{aligned} J_v &= \begin{bmatrix} z_0 \times (o_4 - o_0) & z_1 & z_2 \times (o_4 - o_2) & z_3 \times (o_4 - o_3) \end{bmatrix} \\ &= \begin{bmatrix} l_3 s_1 s_3 - d_2 c_1 & -s_1 & -l_3 c_1 c_3 & 0 \\ -d_2 s_1 - l_3 c_1 s_3 & c_1 & -l_3 c_3 s_1 & 0 \\ 0 & 0 & l_3 s_3 c_1^2 + l_3 s_3 s_1^2 & 0 \end{bmatrix} \end{aligned} \tag{6}$$

$$J_\omega = \begin{bmatrix} z_0 & 0 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -s_1 & -c_1 s_3 \\ 0 & 0 & c_1 & -s_1 s_3 \\ 1 & 0 & 0 & -c_3 \end{bmatrix}. \tag{7}$$

The overall Jacobian matrix is $J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$.

2. For the given manipulator, we have

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$A = A_1 A_2 = \begin{bmatrix} c_1 c_2 & -c_1 s_2 & s_1 & c_1 l_2 c_2 \\ s_1 c_2 & -s_1 s_2 & -c_1 & s_1 l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since both joints are revolute, the linear velocity Jacobian J_v and the angular velocity Jacobian J_ω) are given by:

$$\begin{aligned} J_v &= \begin{bmatrix} z_0 \times (o_c - o_0) & z_1 \times (o_c - o_1) \end{bmatrix} \\ J_\omega &= \begin{bmatrix} z_0 & z_1 \end{bmatrix}. \end{aligned} \tag{8}$$

For the given manipulator, we have: $o_0 = [0,0,0]^T$, $o_1 = [0,0,d_1]^T$, $o_c = [c_1 l_2 c_2, s_1 l_2 c_2, l_2 s_2 + d_1]^T$, $z_0 = [0,0,1]^T$, $z_1 = [s_1, -c_1, 0]^T$. Hence,

$$\begin{aligned} J_v &= \begin{bmatrix} -s_1 l_2 c_2 & -c_1 l_2 s_2 \\ c_1 l_2 c_2 & -s_1 l_2 s_2 \\ 0 & l_2 c_2 \end{bmatrix} \\ J_\omega &= \begin{bmatrix} 0 & s_1 \\ 0 & -c_1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{9}$$

3. We are considering again the RR manipulator from the previous question.

   (a) Since each link is a point mass, the inertia matrices I_1 and I_2 are zero. Also, for this robotic manipulator, the mass corresponding to the first link does not move; hence, $J_{v_1} = 0$. Also, for this robotic manipulator, the mass of the first link does not move up and down; hence, potential energy of first link can be ignored. Define $q_1 = \theta_1$, $q_2 = \theta_2$, $q = [q_1, q_2]^T$.

   - Kinetic energy $K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$ with

$$D(q) = m_2 J_v^T J_v = \begin{bmatrix} m_2 l_2^2 c_2^2 & 0 \\ 0 & m_2 l_2^2 \end{bmatrix} \tag{10}$$

   - Assuming that z_0 is along the direction of gravity (but pointing upward), the potential energy is given by $P = m_2 g (l_2 s_2 + d_1)$ where $g = 9.81$ m/s^2 is the magnitude of acceleration due to gravity. As noted above, the potential energy of the first link can be ignored for this manipulator since the center of mass of the first link does not move up and down (i.e., potential energy of the first link is constant).

Hence, we have the Lagrangian $\mathcal{L} = K - P = \frac{1}{2}m_2 l_2^2 c_2^2 \dot{q}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{q}_2^2 - m_2 g(l_2 s_2 + d_1)$. Therefore,

$$\begin{aligned}
\frac{\partial L}{\partial q_1} &= 0 \\
\frac{\partial L}{\partial q_2} &= -m_2 l_2^2 c_2 s_2 \dot{q}_1^2 - m_2 g l_2 c_2 \\
\frac{\partial L}{\partial \dot{q}_1} &= m_2 l_2^2 c_2^2 \dot{q}_1 \\
\frac{\partial L}{\partial \dot{q}_2} &= m_2 l_2^2 \dot{q}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} &= m_2 l_2^2 c_2^2 \ddot{q}_1 - 2m_2 l_2^2 c_2 s_2 \dot{q}_2 \dot{q}_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} &= m_2 l_2^2 \ddot{q}_2
\end{aligned} \tag{11}$$

Hence, the dynamics of the robot manipulator are:

$$\begin{aligned}
m_2 l_2^2 c_2^2 \ddot{q}_1 - 2m_2 l_2^2 c_2 s_2 \dot{q}_2 \dot{q}_1 &= \tau_1 \\
m_2 l_2^2 \ddot{q}_2 + m_2 l_2^2 c_2 s_2 \dot{q}_1^2 + m_2 g l_2 c_2 &= \tau_2
\end{aligned} \tag{12}$$

Alternatively, we could use the Christoffel symbols: $c_{(1,1,1)} = 0$, $c_{(1,1,2)} = m_2 l_2^2 s_2 c_2$, $c_{(1,2,1)} = -m_2 l_2^2 s_2 c_2$, $c_{(1,2,2)} = 0$, $c_{(2,1,1)} = -m_2 l_2^2 s_2 c_2$, $c_{(2,1,2)} = 0$, $c_{(2,2,1)} = 0$, $c_{(2,2,2)} = 0$. Hence,

$$C(q,\dot{q}) = \begin{bmatrix} -m_2 l_2^2 s_2 c_2 \dot{q}_2 & -m_2 l_2^2 s_2 c_2 \dot{q}_1 \\ m_2 l_2^2 s_2 c_2 \dot{q}_1 & 0 \end{bmatrix}. \tag{13}$$

Also, $g(q) = [g_1(q), g_2(q)]^T$ with

$$\begin{aligned}
g_1(q) &= \frac{\partial P}{\partial q_1} = 0 \\
g_2(q) &= \frac{\partial P}{\partial q_2} = m_2 l_2 g c_2
\end{aligned} \tag{14}$$

The dynamics of the robotic manipulator can be written as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau. \tag{15}$$

(b) If the end-effector has a general force/torque F_e on it (from the environment) written as a 6×1 vector, the dynamics including this force/torque are

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + J^T(q)F_e = \tau. \tag{16}$$

4. To write the dynamics of the manipulator using the Newton-Euler method, we do a forward recursion to find the kinematic quantities of the links and a backward recursion to write all the forces and torques at the joints.

   Forward recursion: We start with $\omega_0 = 0$, $a_{c,0} = 0$, and $a_{e,0} = 0$. Then, we get $\omega_1 = R_0^1 \omega_0 + R_0^1 z_0 \dot{q}_1$. From the homogeneous transformation A_1 that we had found in Question 2, we see that $R_0^1 z_0 = \vec{j}$. Hence, $\omega_1 = \dot{q}_1 \vec{j}$. Here, the notations $\vec{i}$, $\vec{j}$, and $\vec{k}$ denote the 3×1 unit vectors, i.e., $\vec{i} = [1,0,0]^T$, $\vec{j} = [0,1,0]^T$, $\vec{k} = [0,0,1]^T$. Also, $\omega_2 = R_1^2 \omega_1 + R_0^2 z_1 \dot{q}_2 = s_2 \dot{q}_1 \vec{i} + c_2 \dot{q}_1 \vec{j} + \dot{q}_2 \vec{k}$. Therefore, $\dot{\omega}_1 = \ddot{q}_1 \vec{j}$ and $\dot{\omega}_2 = (s_2 \ddot{q}_1 + c_2 \dot{q}_2 \dot{q}_1)\vec{i} + (c_2 \ddot{q}_1 - s_2 \dot{q}_2 \dot{q}_1)\vec{j} + \ddot{q}_2 \vec{k}$

   The gravity vector can be written in the link-fixed frames as: $g_1 = g[0,-1,0]^T$ and $g_2 = g[-s_2, -c_2, 0]^T$. Since the base frame is stationary, we have $a_{c,0} = a_{e,0} = 0$. From A_1, we see that $r_{1,c1}$ is $[0,0,d_1]^T$ when written in frame 0; hence, in frame 1, $r_{1,c1} = R_0^1[0,0,d_1]^T = [0,d_1,0]^T = d_1 \vec{j}$. Also, $r_{1,2} = r_{1,c1}$ and $r_{2,c1} = [0,0,0]^T$. Similarly, $r_{2,c2} = [l_2,0,0]^T = l_2 \vec{i}$. Hence, the linear acceleration of the center of mass of link 1 is:

$$a_{c,1} = R_0^1 a_{e,0} + \dot{\omega}_1 \times r_{1,c1} + \omega_1 \times (\omega_1 \times r_{1,c1}) = [0,0,0]^T. \tag{17}$$

Also, the linear acceleration of the end of link 1 is $a_{e,1} = [0, 0, 0]^T$. The linear acceleration of the center of mass of link 2 can be found to be:

$$\begin{aligned} a_{c,2} &= R_1^2 a_{e,1} + \dot{\omega}_2 \times r_{2,c2} + \omega_2 \times (\omega_2 \times r_{2,c2}) \\ &= -l_2(c_2^2\dot{q}_1^2 + \dot{q}_2^2)\vec{i} + l_2(\ddot{q}_2 + s_2c_2\dot{q}_1^2)\vec{j} + l_2(-c_2\ddot{q}_1 + 2s_2\dot{q}_2\dot{q}_1)\vec{k}. \end{aligned} \tag{18}$$

Backward recursion: Note that for this manipulator, the inertia matrices I_1 and I_2 are zero since each link is assumed to be a point mass at the end of the link. Start with $f_3 = \tau_3 = 0$. Then, we can find f_2 and τ_2 and then f_1 and τ_1. From these quantities, we need to extract the equations corresponding to the axes of actuation (represented in the corresponding link-attached coordinate frames). The axis of actuation of joint 1 is $z_0 = [0, 0, 1]^T$ when written in frame 0 and is therefore $R_0^1 z_0 = [0, 1, 0]^T$ when written in frame 1. Therefore, for joint 1, we need $\tau_{1,y}$ for the dynamics. The axis of actuation of joint 2 is $z_1 = [s_1, -c_1, 0]^T$ when written in frame 0 and is therefore $R_0^2 z_1 = [0, 0, 1]^T$ when written in frame 2. Therefore, we need $\tau_{2,z}$ for the dynamics.

$$f_2 = m_2(a_{c,2} - g_2) = f_{2,x}\vec{i} + f_{2,y}\vec{j} + f_{2,z}\vec{k} \tag{19}$$

$$\tau_2 = -f_2 \times r_{2,c2} + I_2\dot{\omega}_2 + \omega_2 \times (I_2\omega_2) = \tau_{2,y}\vec{j} + \tau_{2,z}\vec{k} \tag{20}$$

where

$$f_{2,x} = m_2(gs_2 - l_2c_2^2\dot{q}_1^2 - l_2\dot{q}_2^2) \tag{21}$$

$$f_{2,y} = m_2(gc_2 + l_2\ddot{q}_2 + l_2s_2c_2\dot{q}_1^2) \tag{22}$$

$$f_{2,z} = m_2l_2(-c_2\ddot{q}_1 + 2s_2\dot{q}_2\dot{q}_1) \tag{23}$$

$$\tau_{2,y} = -m_2l_2^2(-c_2\ddot{q}_1 + 2s_2\dot{q}_2\dot{q}_1) \tag{24}$$

$$\tau 2, z = m_2l_2(gc_2 + l_2\ddot{q}_2 + l_2s_2c_2\dot{q}_1^2). \tag{25}$$

Then,

$$f_1 = R_2^1 f_2 + m_1(a_{c,1} - g_1) = f_{1,x}\vec{i} + f_{1,y}\vec{j} + f_{1,z}\vec{k} \tag{26}$$

$$\tau_1 = R_2^1\tau_2 - f_1 \times r_{1,c1} + (R_2^1 f_2) \times r_{2,c1} + I_1\dot{\omega}_1 + \omega_1 \times (I_1\omega_1) = \tau_{1,x}\vec{i} + \tau_{1,y}\vec{j} + \tau_{1,z}\vec{k}. \tag{27}$$

As discussed above, we need $\tau_{1,y}$ to write the dynamics. Doing the calculations from above, we get

$$\tau_{1,y} = m_2l_2^2c_2(c_2\ddot{q}_1 - 2s_2\dot{q}_2\dot{q}_1). \tag{28}$$

As discussed above, the actuated forces/torques in the link-attached coordinate frames are given by $\tau_{1,y}$ and $\tau_{2,z}$. Therefore, the dynamics equations are:

$$u_1 = m_2l_2^2c_2(c_2\ddot{q}_1 - 2s_2\dot{q}_2\dot{q}_1) \tag{29}$$

$$u_2 = m_2l_2(gc_2 + l_2\ddot{q}_2 + l_2s_2c_2\dot{q}_1^2) \tag{30}$$

where $u_1 = \tau_{1,y}$ is the actuated torque for the first joint and $u_2 = \tau_{2,z}$ is the actuated torque for the second joint. We see that the dynamics obtained above are identical to the dynamics obtained using the Euler-Lagrange method in the previous question.

5. In the previous questions (using either the Euler-Lagrange or the Newton-Euler), the dynamics of the manipulator were written in the form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau. \tag{31}$$

Now, we want to design controllers to make the joints of the manipulator track desired trajectories $q_d = [q_{1,d}, q_{2,d}]$.

(a) A proportional-derivative (PD) controller (independent joint control) for the manipulator can be written as: $\tau = -K_p\tilde{q} - K_d\dot{\tilde{q}}$ where $\tilde{q} = q - q_d$. K_p and K_d are diagonal matrices of the form $K_p = \text{diag}(K_{p,1}, K_{p,2})$ and $K_d = \text{diag}(K_{d,1}, K_{d,2})$ with $K_{p,1}$, $K_{p,2}$, $K_{d,1}$, and $K_{d,2}$ being positive constants. Hence, the independent joint controller comprises of two separate controllers, one for each joint in the manipulator, given by: $\tau_1 = -K_{p,1}\tilde{q}_1 - K_{d,1}\dot{\tilde{q}}_1$ and $\tau_2 = -K_{p,2}\tilde{q}_2 - K_{d,2}\dot{\tilde{q}}_2$ where $\tilde{q}_i = q_i - q_{i,d}$, $i = 1, 2$.

(b) The inverse dynamics controller for the manipulator is given by:

$$a_q = \ddot{q}_d - K_p\tilde{q} - K_d\dot{\tilde{q}} \tag{32}$$
$$\tau = D(q)a_q + C(q,\dot{q})\dot{q} + g(q) \tag{33}$$

with K_p and K_d being diagonal matrices (proportional and derivative gains).

6. In the given Euler angle representation, the rotation matrix is given as $R = R_{z,\phi}R_{y,\theta}R_{z,\psi}$.

(a) Hence, we have

$$\dot{R} = S(\dot{\phi}\vec{k})R_{z,\phi}R_{y,\theta}R_{z,\psi} + R_{z,\phi}S(\dot{\theta}\vec{j})R_{y,\theta}R_{z,\psi} + R_{z,\phi}R_{y,\theta}S(\dot{\psi}\vec{k})R_{z,\psi} \tag{34}$$
$$= S(\dot{\phi}\vec{k})R_{z,\phi}R_{y,\theta}R_{z,\psi} + S(R_{z,\phi}\dot{\theta}\vec{j})R_{z,\phi}R_{y,\theta}R_{z,\psi} + S(R_{z,\phi}R_{y,\theta}\dot{\psi}\vec{k})R_{z,\phi}R_{y,\theta}R_{z,\psi} \tag{35}$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ represent the unit vectors in the x, y, and z directions, respectively, i.e., $\vec{i} = [1,0,0]^T$, $\vec{j} = [0,1,0]^T$, and $\vec{k} = [0,0,1]^T$. From (35), since $\dot{R} = S(\omega^i)R$ where ω^i is the angular velocity expressed in the inertial frame, we can write ω^i as

$$\omega^i = (-s_\phi\dot{\theta} + c_\phi s_\theta\dot{\psi})\vec{i} + (c_\phi\dot{\theta} + s_\phi s_\theta\dot{\psi})\vec{j} + (\dot{\phi} + c_\theta\dot{\psi})\vec{k} \tag{36}$$
$$= \begin{bmatrix} -s_\phi\dot{\theta} + c_\phi s_\theta\dot{\psi} \\ c_\phi\dot{\theta} + s_\phi s_\theta\dot{\psi} \\ \dot{\phi} + c_\theta\dot{\psi} \end{bmatrix} \tag{37}$$

where $s_\phi = \sin(\phi)$, $c_\phi = \cos(\phi)$, etc. Hence, $\omega^i = J^i(q)\dot{q}$ where

$$J^i(q) = \begin{bmatrix} 0 & -s_\phi & c_\phi s_\theta \\ 0 & c_\phi & s_\phi s_\theta \\ 1 & 0 & c_\theta \end{bmatrix}. \tag{38}$$

Here, we want to write the kinetic energy in the form $K = \frac{1}{2}\omega^T I\omega$. Hence, we want ω in the body-fixed frame. This can be obtained from ω^i using $\omega = R^T\omega^i$ as

$$\omega = \begin{bmatrix} \dot{\theta}s_\psi - \dot{\phi}c_\psi s_\theta \\ \dot{\theta}c_\psi + \dot{\phi}s_\psi s_\theta \\ \dot{\psi} + \dot{\phi}c_\theta \end{bmatrix} \tag{39}$$

Equivalently, since when we consider ω in body-fixed frame, we should have $\dot{R} = RS(\omega)$, we can obtain ω directly from the expression for R as $R = R_{z,\phi}R_{y,\theta}R_{z,\psi}$ by writing $\dot{R}$ as

$$\dot{R} = S(\dot{\phi}\vec{k})R_{z,\phi}R_{y,\theta}R_{z,\psi} + R_{z,\phi}S(\dot{\theta}\vec{j})R_{y,\theta}R_{z,\psi} + R_{z,\phi}R_{y,\theta}S(\dot{\psi}\vec{k})R_{z,\psi} \tag{40}$$
$$= R_{z,\phi}R_{y,\theta}R_{z,\psi}S([R_{z,\phi}R_{y,\theta}R_{z,\psi}]^T\dot{\phi}\vec{k}) + R_{z,\phi}R_{y,\theta}R_{z,\psi}S([R_{y,\theta}R_{z,\psi}]^T\dot{\theta}\vec{j}) + R_{z,\phi}R_{y,\theta}R_{z,\psi}S([R_{z,\psi}]^T\dot{\psi}\vec{k}) \tag{41}$$

and obtaining $\omega = [R_{z,\phi}R_{y,\theta}R_{z,\psi}]^T\dot{\phi}\vec{k} + [R_{y,\theta}R_{z,\psi}]^T\dot{\theta}\vec{j} + [R_{z,\psi}]^T\dot{\psi}\vec{k}$.
Hence, $\omega = J(q)\dot{q}$ where

$$J(q) = \begin{bmatrix} -c_\psi s_\theta & s_\psi & 0 \\ s_\psi s_\theta & c_\psi & 0 \\ c_\theta & 0 & 1 \end{bmatrix}. \tag{42}$$

Equivalently, equation (42) can be obtained from equation (38) using $J = R^T J_i$.

(b) The kinetic energy of the system is given by $K = \frac{1}{2}\omega^T I \omega = \frac{1}{2}\dot{q}^T J^T(q) I J(q)\dot{q}$ where I is a diagonal matrix with diagonal elements I_{xx}, I_{yy}, and I_{zz}. $q = [\phi, \theta, \psi]^T$. Since there is no external force (in particular, no gravity), the potential energy is zero.

(c) The dynamics of the system can be written by applying the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad , i = 1, \ldots, 3 \tag{43}$$

with $\mathcal{L} = K = \frac{1}{2}\omega^T I \omega = \frac{1}{2}\dot{q}^T J^T(q) I J(q)\dot{q}$. Note that $\frac{\partial \mathcal{L}}{\partial \omega} = \omega^T I$ and $\frac{\partial \omega}{\partial \dot{q}} = J(q)$. Hence, we get $\frac{\partial \mathcal{L}}{\partial q} = \omega^T I \frac{\partial \omega}{\partial q}$ and $\frac{\partial \mathcal{L}}{\partial \dot{q}} = \omega^T I J$. Therefore, from equation (43), we get

$$\frac{d}{dt}\left(\omega^T I J\right) - \omega^T I \frac{\partial \omega}{\partial q} = 0. \tag{44}$$

Expanding $\frac{d}{dt}(\omega^T I J)$, we get $\frac{d}{dt}(\omega^T I J) = \dot{\omega}^T I J + \omega^T I \dot{J}$. Substituting this into equation (44), taking transpose throughout, and simplifying, we can write

$$I\dot{\omega} + \left\{\left(\dot{J} - \frac{\partial \omega}{\partial q}\right) J^{-1}\right\}^T I\omega = 0. \tag{45}$$

Using equations (39) and (42), we can write

$$\dot{J} = \begin{bmatrix} \dot{\psi} s_\psi s_\theta - \dot{\theta} c_\psi c_\theta & \dot{\psi} c_\psi & 0 \\ \dot{\psi} c_\psi s_\theta + \dot{\theta} c_\theta s_\psi & -\dot{\psi} s_\psi & 0 \\ -\dot{\theta} s_\theta & 0 & 0 \end{bmatrix} \tag{46}$$

$$\frac{\partial \omega}{\partial q} = \begin{bmatrix} 0 & -\dot{\phi} c_\psi c_\theta & \dot{\theta} c_\psi + \dot{\phi} s_\psi s_\theta \\ 0 & \dot{\phi} c_\theta s_\psi & -\dot{\theta} s_\psi + \dot{\phi} c_\psi s_\theta \\ 0 & -\dot{\phi} s_\theta & 0 \end{bmatrix} \tag{47}$$

Hence, from equations (42), (46), and (47), we get

$$\left(\dot{J} - \frac{\partial \omega}{\partial q}\right) J^{-1} = \begin{bmatrix} 0 & \dot{\psi} + \dot{\phi} c_\theta & -\dot{\theta} c_\psi - \dot{\phi} s_\psi s_\theta \\ -\dot{\psi} - \dot{\phi} c_\theta & 0 & \dot{\theta} s_\psi - \dot{\phi} c_\psi s_\theta \\ \dot{\theta} c_\psi + \dot{\phi} s_\psi s_\theta & \dot{\phi} c_\psi s_\theta - \dot{\theta} s_\psi & 0 \end{bmatrix} \tag{48}$$

Comparing with (39), we see that the right hand side of equation (48) is simply $-S(\omega)$. Hence,

$$\left(\dot{J} - \frac{\partial \omega}{\partial q}\right) J^{-1} = -S(\omega). \tag{49}$$

Therefore, since $[-S(\omega)]^T = S(\omega)$, equation (45) simplifies to

$$I\dot{\omega} + S(\omega) I \omega = 0. \tag{50}$$

Expanding the equation (50), we immediately obtain:

$$\begin{aligned} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y \omega_z &= 0 \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_z \omega_x &= 0 \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x \omega_y &= 0. \end{aligned} \tag{51}$$

7. We are given that a a controller has been implemented for the manipulator to make its effective dynamics have a stiffness (in task space, i.e., in terms of end effector generalized position) of $K_p = 50$ N/m. Also, we are given that the wall has a stiffness of $K_e = 1000$ N/m and that we want to apply a force $f_{des} = 25$ N on the wall. Hence, the virtual trajectory should be chosen such that it is inside the wall by distance $\frac{K_p + K_e}{K_p K_e} f_{des} = \frac{1050}{50000} 25 = 0.525$ m.