**Solutions for Problems 3.9, 3.10, 3.11, and 3.12**

3.9. Here, we are given a three-link Cartesian manipulator with a spherical wrist. We had considered a three-link Cartesian manipulator in Problem 3.7 in Homework 4. Using the D-H coordinate frames defined in the solution of Problem 3.7 in Homework 4 and adding the standard coordinate frames for the spherical wrist, we get the D-H table shown below:

| | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| Link 1 | 0 | -90$^o$ | $d_1^*$ | 0 |
| Link 2 | 0 | -90$^o$ | $d_2^*$ | 90$^o$ |
| Link 3 | 0 | 0 | $d_3^*$ | 0 |
| Link 4 | 0 | -90$^o$ | 0 | $\theta_4^*$ |
| Link 5 | 0 | 90$^o$ | 0 | $\theta_5^*$ |
| Link 6 | 0 | 0 | $d_6$ | $\theta_6^*$ |

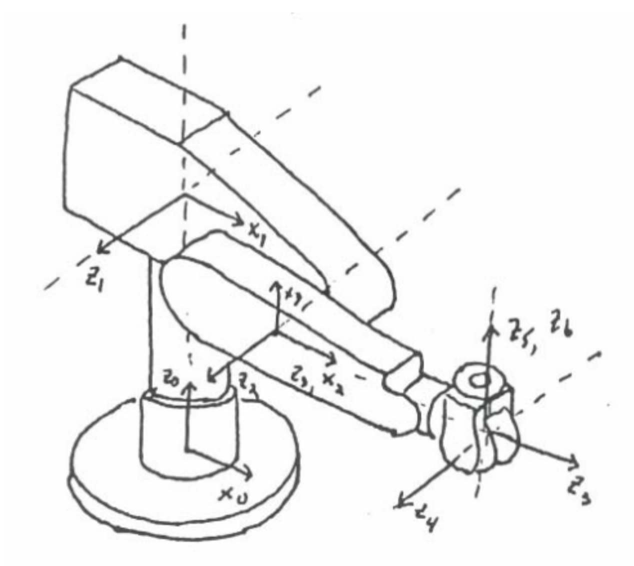
Hence, the homogeneous transformation matrices $A_1, \ldots, A_6$ and the homogeneous transformation matrix $H_6^0$ are given as:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$H_6^0 = A_1A_2A_3A_4A_5A_6 = \begin{bmatrix} c_6s_5 & -s_5s_6 & -c_5 & -d_3 - d_6c_5 \\ -c_4s_6 - c_5c_6s_4 & c_5s_4s_6 - c_4c_6 & -s_4s_5 & d_2 - d_6s_4s_5 \\ s_4s_6 - c_4c_5c_6 & c_6s_4 + c_4c_5s_6 & -c_4s_5 & d_1 - d_6c_4s_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

3.10 The choice of the D-H coordinate frames is shown in the figure below.




The corresponding D-H table is:

| | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| Link 1 | 0 | $90^o$ | $d_1$ | $\theta_1^*$ |
| Link 2 | $a_2$ | 0 | $d_2$ | $\theta_2^*$ |
| Link 3 | 0 | $90^o$ | 0 | $\theta_3^*$ |
| Link 4 | 0 | $-90^o$ | $d_4$ | $\theta_4^*$ |
| Link 5 | 0 | $90^o$ | 0 | $\theta_5^*$ |
| Link 6 | 0 | 0 | $d_6$ | $\theta_6^*$ |

Here, $a_2 = d_4 = 8$ inches and $d_1 = 13$ inches. In the figure, there is an offset between the 1-frame and the 2-frame; this offset would correspond to $d_2$. $d_6$ is the distance from the wrist center to the end-effector tip.

Hence, the homogeneous transformation matrices $A_1, \ldots, A_6$ and the homogeneous transformation matrix $H_6^0$ are given as:

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_3 = \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ s_3 & 0 & -c_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$H_6^0 = A_1A_2A_3A_4A_5A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

where

$r_{11} = s_6(c_4s_1 + s_4(c_1s_2s_3 - c_1c_2c_3)) + c_6(c_5(s_1s_4 - c_4(c_1s_2s_3 - c_1c_2c_3)) - s_5(c_1c_2s_3 + c_1c_3s_2))$

$r_{12} = c_6(c_4s_1 + s_4(c_1s_2s_3 - c_1c_2c_3)) - s_6(c_5(s_1s_4 - c_4(c_1s_2s_3 - c_1c_2c_3)) - s_5(c_1c_2s_3 + c_1c_3s_2))$

$r_{13} = s_5(s_1s_4 - c_4(c_1s_2s_3 - c_1c_2c_3)) + c_5(c_1c_2s_3 + c_1c_3s_2)$

$r_{14} = d_6(s_5(s_1s_4 - c_4(c_1s_2s_3 - c_1c_2c_3)) + c_5(c_1c_2s_3 + c_1c_3s_2)) + d_4(c_1c_2s_3 + c_1c_3s_2) + d_2s_1 + a_2c_1c_2$

$r_{21} = -s_6(c_1c_4 - s_4(s_1s_2s_3 - c_2c_3s_1)) - c_6(c_5(c_1s_4 + c_4(s_1s_2s_3 - c_2c_3s_1)) + s_5(c_2s_1s_3 + c_3s_1s_2))$

$r_{22} = s_6(c_5(c_1s_4 + c_4(s_1s_2s_3 - c_2c_3s_1)) + s_5(c_2s_1s_3 + c_3s_1s_2)) - c_6(c_1c_4 - s_4(s_1s_2s_3 - c_2c_3s_1))$

$r_{23} = c_5(c_2s_1s_3 + c_3s_1s_2) - s_5(c_1s_4 + c_4(s_1s_2s_3 - c_2c_3s_1))$

$r_{24} = d_4(c_2s_1s_3 + c_3s_1s_2) - d_6(s_5(c_1s_4 + c_4(s_1s_2s_3 - c_2c_3s_1)) - c_5(c_2s_1s_3 + c_3s_1s_2)) - d_2c_1 + a_2c_2s_1$

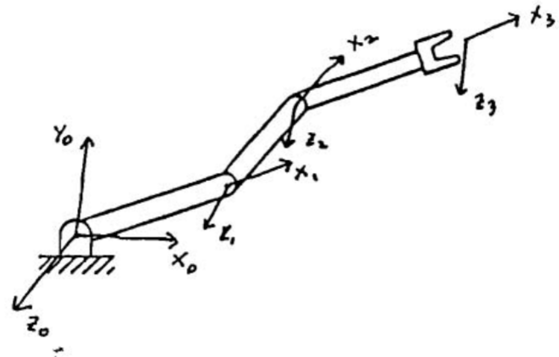
$r_{31} = c_6(s_5(c_2c_3 - s_2s_3) + c_4c_5(c_2s_3 + c_3s_2)) - s_4s_6(c_2s_3 + c_3s_2)$

$r_{32} = -s_6(s_5(c_2c_3 - s_2s_3) + c_4c_5(c_2s_3 + c_3s_2)) - c_6s_4(c_2s_3 + c_3s_2)$

$r_{33} = c_4s_5(c_2s_3 + c_3s_2) - c_5(c_2c_3 - s_2s_3)$

$r_{34} = d_1 - d_4(c_2c_3 - s_2s_3) + a_2s_2 - d_6(c_5(c_2c_3 - s_2s_3) - c_4s_5(c_2s_3 + c_3s_2))$

3.11 We are given a three-link planar arm. The choice of the D-H coordinate frames is shown in the figure below.




The corresponding D-H table is:

| | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| Link 1 | $a_1$ | 0 | 0 | $\theta_1^*$ |
| Link 2 | $a_2$ | 0 | 0 | $\theta_2^*$ |
| Link 3 | $a_3$ | 0 | 0 | $\theta_3^*$ |

Hence, the homogeneous transformation matrices $A_1, A_2, A_3$ and the homogeneous transformation matrix $H_3^0$ are given as:

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$$H_3^0 = A_1A_2A_3 \tag{8}$$

The translational part (the first three elements of the last column) of $H_3^0$ can be found to be

$$o_6^0 = \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix} \tag{9}$$

where $c_{12} = \cos(\theta_1 + \theta_2)$, etc. Hence, if only the desired position is given as $d = [d_x, d_y, 0]^T$, we have three unknown variables ($\theta_1$, $\theta_2$, $\theta_3$) to be found from the two equations:

$$d_x = a_1c_1 + a_2c_{12} + a_3c_{123} \tag{10}$$

$$d_y = a_1s_1 + a_2s_{12} + a_3s_{123}. \tag{11}$$

Hence, we will, in general, have infinitely many solutions to the inverse position kinematics problem. More specifically, the number of solutions will be

- infinitely many solutions if $d$ is inside the workspace (the set of points that the end-effector can possibly reach given all possible joint configurations of the robotic manipulator)
- 1 solution if $d$ is on the workspace boundary
- 0 solutions if $d$ is outside the workspace

Since we have three unknown variables and two equations to solve for, we get infinitely many solutions in general. One way to find all solutions is to take one of the variables as specified (e.g., $\theta_3$ or, more generally, some combination of $\theta_1$, $\theta_2$, and $\theta_3$) and then find the other two variables in terms of this specified variable. Then, we will get all the infinitely many solutions as parameterized in terms of this single variable, which can then take any value in the feasible range for that joint variable. This is equivalent to specifying one additional quantity, e.g., the desired angular orientation. Hence, the set of all possible inverse position kinematics solutions can be thought of as the union of the solutions for the inverse kinematics problem (with specified desired position and orientation) for all possible desired orientations, i.e., for all possible $\theta_d$ in the inverse kinematics computed below.

If the desired orientation $\theta_d$ of the end-effector is also specified, then, for this planar manipulator, we should have

$$\theta_1 + \theta_2 + \theta_3 = \theta_d. \tag{12}$$

Since, we know $\theta_1 + \theta_2 + \theta_3$, we can simplify the equations (10) and (11) as

$$d_x - a_3 \cos(\theta_d) = a_1 c_1 + a_2 c_{12} \tag{13}$$

$$d_y - a_3 \sin(\theta_d) = a_1 s_1 + a_2 s_{12}. \tag{14}$$

The equations (13) and (14) can also be thought of as corresponding to a *wrist center* wherein we consider the last joint to be a one degree-of-freedom wrist and then find the coordinates of the point before this last joint as $[d_x - a_3\cos(\theta_d), d_y - a_3\sin(\theta_d), 0]^T$. This is analogous to the method of finding the wrist center for a spherical wrist.

Denote $w_x = d_x - a_3\cos(\theta_d)$ and $w_y = d_y - a_3\sin(\theta_d)$. Taking the squares of (13) and (14) and adding, we get:

$$w_x^2 + w_y^2 = a_1^2 + a_2^2 + 2a_1a_2c_2. \tag{15}$$

Hence,

$$\theta_2 = \cos^{-1}\left(\frac{w_x^2 + w_y^2 - a_1^2 - a_2^2}{2a_1a_2}\right). \tag{16}$$

This gives us two possible values for $\theta_2$. Now, with either of these possible values of $\theta_2$, we can solve for $\theta_1$ utilizing, for example, equation (13), which when expanded gives us an equation of form

$$Ac_1 + Bs_1 = w_x \tag{17}$$

where $A = a_1 + a_2c_2$ and $B = -a_2s_2$. From (14), we similarly get the equation

$$As_1 - Bc_1 = w_y \tag{18}$$

Define $\alpha$ to be the angle Atan2$(A, B)$ where Atan2 denotes the two-argument arc-tangent function wherein Atan2$(x, y)$ is essentially $\tan^{-1}(y/x)$, but with the signs of $x$ and $y$ also taken into account.

Hence, $\sin(\alpha) = \frac{B}{\sqrt{A^2+B^2}}$ and $\cos(\alpha) = \frac{A}{\sqrt{A^2+B^2}}$. Therefore, from (17) and (18), we get

$$c_\alpha c_1 + s_\alpha s_1 = \frac{w_x}{\sqrt{A^2+B^2}} \tag{19}$$

$$c_\alpha s_1 - s_\alpha c_1 = \frac{w_y}{\sqrt{A^2+B^2}} \tag{20}$$

Hence, $\cos(\theta_1 - \alpha) = \frac{w_x}{\sqrt{A^2+B^2}}$ and $\sin(\theta_1 - \alpha) = \frac{w_y}{\sqrt{A^2+B^2}}$. Therefore,

$$\theta_1 = \alpha + \mathrm{Atan2}(w_x, w_y) = \mathrm{Atan2}(w_x, w_y) - \mathrm{Atan2}(a_1 + a_2c_2, a_2s_2) \tag{21}$$

Hence, given a solution for $\theta_2$, we can find $\theta_1$ using (21). Once solutions for $\theta_1$ and $\theta_2$ have been found, $\theta_3$ is found simply as $\theta_3 = \theta_d - \theta_1 - \theta_2$. Since we find, in general, two possible values of $\theta_2$ from (16), we see that when the desired position and orientation are both given, the number of solutions will be

- 2 solutions if the *wrist center* $(w_x, w_y)$ is inside the workspace of the two-link manipulator formed by only considering the first two links of the given manipulator
- 1 solution if the point $(w_x, w_y)$ is on the boundary of the workspace of the two-link manipulator
- 0 solutions if the point $(w_x, w_y)$ is outside the workspace of the two-link manipulator
- infinitely many solutions if the point $(w_x, w_y)$ is the origin and this point is inside the workspace of the two-link manipulator; in this case, we will have $a_1 = a_2$ and the solution for $\theta_2$ to place $(w_x, w_y)$ at the origin has to be $\theta_2 = \pi$. $\theta_1$ can then be any angle and therefore, we get infinitely many solutions.

3.12 We are given a three-link planar manipulator with first joint revolute, second joint prismatic, and third joint revolute. The direct kinematics for this manipulator was written in Homework 3.5 in

Homework 4. For the inverse position kinematics, if only the desired position of the end-effector is given as $d = [d_x, d_y, 0]^T$, we have three unknown variables $(\theta_1, d_2, \theta_3)$ to be found from the two equations:

$$d_x = d_2 s_1 + a_3 c_{13} \tag{22}$$

$$d_y = -d_2 c_1 + a_3 s_{13}. \tag{23}$$

Hence, we will, in general, have infinitely many solutions to the inverse position kinematics problem. More specifically, the number of solutions will be

- infinitely many solutions if $d$ is inside the workspace (the set of points that the end-effector can possibly reach given all possible joint configurations of the robotic manipulator)
- 1 solution if $d$ is on the workspace boundary
- 0 solutions if $d$ is outside the workspace

Since we have three unknown variables and two equations to solve for, we get infinitely many solutions in general. One way to find all solutions is to take one of the variables as specified (e.g., $\theta_3$ or, more generally, some combination of $\theta_1$, $d_2$, and $\theta_3$) and then find the other two variables in terms of this specified variable. Then, we will get all the infinitely many solutions as parameterized in terms of this single variable, which can then take any value in the feasible range for that joint variable. This is equivalent to specifying one additional quantity, e.g., the desired angular orientation. Hence, the set of all possible inverse position kinematics solutions can be thought of as the union of the solutions for the inverse kinematics problem (with specified desired position and orientation) for all possible desired orientations, i.e., for all possible $\theta_d$ in the inverse kinematics computed below.

If the desired orientation $\theta_d$ of the end-effector is also specified, then, for this planar manipulator, we should have

$$\theta_1 + \theta_3 = \theta_d. \tag{24}$$

Since, we know $\theta_1 + \theta_3$, we can simplify the equations (22) and (23) as

$$d_x - a_3 \cos(\theta_d) = d_2 s_1 \tag{25}$$

$$d_y - a_3 \sin(\theta_d) = -d_2 c_1 \tag{26}$$

The equations (25) and (26) can also be thought of as corresponding to a *wrist center* wherein we consider the last joint to be a one degree-of-freedom wrist and then find the coordinates of the point before this last joint as $[d_x - a_3 \cos(\theta_d), d_y - a_3 \sin(\theta_d), 0]^T$. This is analogous to the method of finding the wrist center for a spherical wrist.

Denote $w_x = d_x - a_3 \cos(\theta_d)$ and $w_y = d_y - a_3 \sin(\theta_d)$. From (25) and (26), we get:

$$\theta_1 = \mathrm{Atan2}(-w_y, w_x) \tag{27}$$

$$d_2 = \sqrt{w_x^2 + w_y^2} \tag{28}$$

where Atan2 denotes the two-argument arc-tangent function wherein $\mathrm{Atan2}(x, y)$ is essentially $\tan^{-1}(y/x)$, but with the signs of $x$ and $y$ also taken into account. Once the solution for $\theta_1$ has been found, $\theta_3$ is found simply as $\theta_3 = \theta_d - \theta_1$. Hence, we see that when the desired position and orientation are both given, the number of solutions will be

- 1 solution if the *wrist center* $(w_x, w_y)$ is on or inside the workspace of the two-link manipulator formed by only considering the first two links of the given manipulator
- 0 solutions if the point $(w_x, w_y)$ is outside the workspace of the two-link manipulator
- infinitely many solutions if the point $(w_x, w_y)$ is the origin and this point is inside the workspace of the two-link manipulator; in this case, we will have $d_2 = 0$ and $\theta_1$ can be any angle; therefore, we get infinitely many solutions.