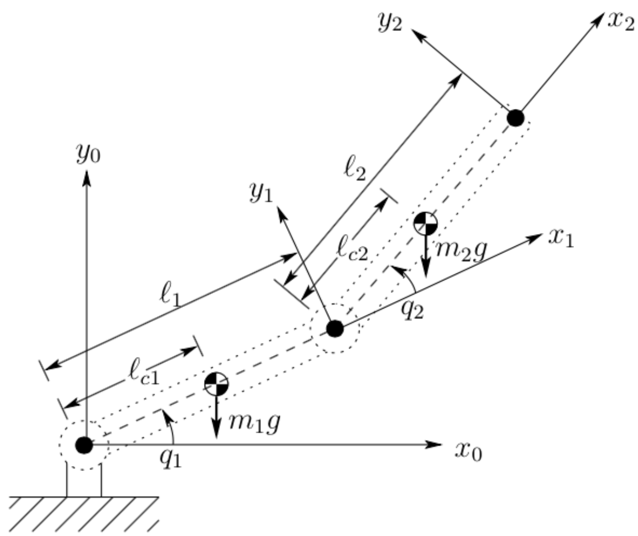
Example of dynamics computation (Euler-Lagrange and Newton-Euler formulations): Revolute-Revolute (RR) manipulator

Consider the Revolute-Revolute (RR) manipulator shown in the figure below.



The coordinate frames 0, 1, and 2 are shown in the figure. The joint variables are $q_1 = \theta_1$ and $q_2 = \theta_2$. Let the masses of the links be m_1 and m_2. Since this is a planar manipulator and rotation is only around the z_0 axis, only the inertia around the vertical axis is relevant; let $I_{1,z}$ and $I_{2,z}$ denote the moments of inertia of links 1 and 2, respectively, around the axis pointing out of the page (for each link, the moments of inertia are defined relative to a coodinate frame with origin at the center of mass of the link).

If the planar motion of the manipulator is in the horizontal plane, then gravity terms are not relevant. If the planar motion of the manipulator is in the vertical plane (as shown in the figure), then gravity terms need to be considered. In the figure, gravity is shown as being in the downward direction (i.e., in the $-y_0$ direction). Let l_{c1} denote the distance from the base (origin of frame 0) to the center of mass of link 1. Let l_1 be the length of link 1. Let the distance from the point where links 1 and 2 meet to the center of mass of link 2 be l_{c2}. Let l_2 be the length of link 2.

Euler-Lagrange formulation to find the dynamics: The angular velocity Jacobian matrices for the two links are:

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad ; \quad J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{1}$$

The linear velocity Jacobian matrices for the two links are:

$$J_{v_1} = \begin{bmatrix} -l_{c1}s_1 & 0 \\ l_{c1}c_1 & 0 \\ 0 & 0 \end{bmatrix} \quad ; \quad J_{v_2} = \begin{bmatrix} -l_{c2}s_{12} - l_1 s_1 & -l_{c2}s_{12} \\ l_{c2}c_{12} + l_1 c_1 & l_{c2}c_{12} \\ 0 & 0 \end{bmatrix} \tag{2}$$

where $s_1 = \sin(q_1)$, $c_1 = \cos(q_1)$, $s_{12} = \sin(q_1 + q_2)$, and $c_{12} = \cos(q_1 + q_2)$. Hence, the matrix $D(q)$ is given by:

$$D(q) = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_1}^T R_1^0 I_1 (R_1^0)^T J_{\omega_1} + J_{\omega_2}^T R_2^0 I_2 (R_2^0)^T J_{\omega_2} \tag{3}$$

$$= \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \tag{4}$$

where

$$d_{11} = m_2 l_1^2 + 2m_2 c_2 l_1 l_{c2} + m_1 l_{c1}^2 + m_2 l_{c2}^2 + I_{1z} + I_{2z} \tag{5}$$

$$d_{12} = d_{21} = m_2 l_{c2}^2 + l_1 m_2 c_2 l_{c2} + I_{2z} \tag{6}$$

$$d_{22} = m_2 l_{c2}^2 + I_{2z}. \tag{7}$$

As described above, since the rotation of all the links is only about the z_0 axis, only the moments of inertia about the axis pointing out of the page are relevant (i.e., $I_{1,z}$ and $I_{2,z}$).

Finding the Christoffel symbols c_{ijk} as

$$c_{ijk} = \frac{1}{2}\left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \tag{8}$$

for $i = 1, 2$; $j = 1, 2$; $k = 1, 2$, we get

$$c_{111} = c_{122} = c_{212} = c_{222} = 0 \; ; \; c_{112} = l_1 l_{c2} m_2 s_2 \; ; \; c_{121} = c_{211} = c_{221} = -l_1 l_{c2} m_2 s_2 \tag{9}$$

Writing the matrix $C(q, \dot{q})$ with its its $(k, j)^{th}$ element being $c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$, we get

$$C(q, \dot{q}) = \begin{bmatrix} C_{11}(q, \dot{q}) & C_{12}(q, \dot{q}) \\ C_{21}(q, \dot{q}) & C_{22}(q, \dot{q}) \end{bmatrix} \tag{10}$$

where

$$C_{11}(q, \dot{q}) = -l_1 l_{c2} m_2 \dot{q}_2 s_2 \tag{11}$$

$$C_{12}(q, \dot{q}) = -l_1 l_{c2} m_2 \dot{q}_1 s_2 - l_1 l_{c2} m_2 \dot{q}_2 s_2 \tag{12}$$

$$C_{21}(q, \dot{q}) = l_1 l_{c2} m_2 \dot{q}_1 s_2 \tag{13}$$

$$C_{22}(q, \dot{q}) = 0 \tag{14}$$

The potential energy of the manipulator is given by:

$$P = g m_2 (l_{c2} s_{12} + l_1 s_1) + g l_{c1} m_1 s_1. \tag{15}$$

Hence,

$$g(q) = \begin{bmatrix} \frac{\partial P}{\partial q_1} \\ \frac{\partial P}{\partial q_2} \end{bmatrix} = \begin{bmatrix} g m_2 (l_{c2} c_{12} + l_1 c_1) + g l_{c1} m_1 c_1 \\ g l_{c2} m_2 c_{12} \end{bmatrix}. \tag{16}$$

The dynamical equations of the manipulator are given by:

$$D(q) \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + C(q, \dot{q})\dot{q} + g(q) = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \tag{17}$$

where τ_1 is the applied torque at the first joint (revolute) and τ_2 is the applied torque at the second joint (revolute).

Newton-Euler formulation to find the dynamics:

- Forward recursion: The Denavit-Hartenberg coordinate frames are shown in the figure. The Denavit-Hartenberg table for this manipulator is:

Link	a_i	α_i	d_i	θ_i
1	l_1	0	0	q_1
2	l_2	0	0	q_2

We have

$$R_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; R_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

The angular velocities of the links (written relative to the link-fixed frames) are $\omega_1 = \dot{q}_1 \vec{k}$ and $\omega_2 = (\dot{q}_1 + \dot{q}_2)\vec{k}$. Here, the notations $\vec{i}$, $\vec{j}$, and $\vec{k}$ denote the 3×1 unit vectors, i.e., $\vec{i} = [1, 0, 0]^T$, $\vec{j} = [0, 1, 0]^T$, $\vec{k} = [0, 0, 1]^T$. The gravity vector can be written in the link-fixed frames as: $g_1 = g[-s_1, -c_1, 0]^T$ and $g_2 = g[-s_{12}, -c_{12}, 0]^T$.

Since the base frame is stationary, we have $a_{c,0} = a_{e,0} = 0$. Looking at the orientation of frame 1, we have $r_{1,c1} = l_{c1}\vec{i}$, $r_{1,2} = l_1\vec{i}$, and $r_{2,c1} = (l_{c1} - l_1)\vec{i}$. Hence, the linear acceleration of the center of mass of link 1 is:

$$a_{c,1} = R_0^1 a_{e,0} + \dot{\omega}_1 \times r_{1,c1} + \omega_1 \times (\omega_1 \times r_{1,c1}) = -l_{c1}\dot{q}_1^2\vec{i} + l_{c1}\ddot{q}_1\vec{j}. \tag{19}$$

Also, the linear acceleration of the end of link 1 is:

$$a_{e,1} = R_0^1 a_{e,0} + \dot{\omega}_1 \times r_{1,2} + \omega_1 \times (\omega_1 \times r_{1,2}) = -l_1\dot{q}_1^2\vec{i} + l_1\ddot{q}_1\vec{j}. \tag{20}$$

We can write $r_{2,c2} = l_{c2}\vec{i}$. The linear acceleration $a_{c,2} = R_0^2 \dot{v}_{c,2}^{(0)}$ of the center of mass of link 2 can be found to be:

$$\begin{aligned} a_{c,2} &= R_1^2 a_{e,1} + \dot{\omega}_2 \times r_{2,c2} + \omega_2 \times (\omega_2 \times r_{2,c2}) \\ &= (l_1\ddot{q}_1 s_2 - l_{c2}(\dot{q}_1 + \dot{q}_2)^2 - l_1\dot{q}_1^2 c_2)\vec{i} + (l_1 s_2 \dot{q}_1^2 + l_{c2}(\ddot{q}_1 + \ddot{q}_2) + l_1\ddot{q}_1 c_2)\vec{j}. \end{aligned} \tag{21}$$

- Backward recursion: Start with $f_3 = \tau_3 = 0$. Then,

$$f_2 = m_2(a_{c,2} - g_2) = f_{2,x}\vec{i} + f_{2,y}\vec{j} \tag{22}$$

$$\tau_2 = -f_2 \times r_{2,c2} + I_2\dot{\omega}_2 + \omega_2 \times (I_2\omega_2) = \tau_{2,z}\vec{k} \tag{23}$$

where

$$f_{2,x} = m_2(g s_{12} - l_{c2}(\dot{q}_1 + \dot{q}_2)^2 + l_1\ddot{q}_1 s_2 - l_1\dot{q}_1^2 c_2) \tag{24}$$

$$f_{2,y} = m_2(l_1 s_2 \dot{q}_1^2 + g c_{12} + l_{c2}(\ddot{q}_1 + \ddot{q}_2) + l_1\ddot{q}_1 c_2) \tag{25}$$

$$\tau_{2,z} = I_{2,z}(\ddot{q}_1 + \ddot{q}_2) + l_{c2}m_2(l_1 s_2\dot{q}_1^2 + g c_{12} + l_{c2}(\ddot{q}_1 + \ddot{q}_2) + l_1\ddot{q}_1 c_2). \tag{26}$$

Then,

$$f_1 = R_2^1 f_2 + m_1(a_{c,1} - g_1) = f_{1,x}\vec{i} + f_{1,y}\vec{j} \tag{27}$$

$$\tau_1 = R_2^1\tau_2 - f_1 \times r_{1,c1} + (R_2^1 f_2) \times r_{2,c1} + I_1\dot{\omega}_1 + \omega_1 \times (I_1\omega_1) = \tau_{1,z}\vec{k} \tag{28}$$

where

$$f_{1,x} = g(m_1 + m_2)s_1 - l_1 m_2\dot{q}_1^2 - l_{c1}m_1\dot{q}_1^2 - l_{c2}m_2(\ddot{q}_1 + \ddot{q}_2)s_2 - l_{c2}m_2(\dot{q}_1^2 + \dot{q}_2^2)c_2 - 2l_{c2}m_2\dot{q}_1\dot{q}_2 c_2 \tag{29}$$

$$\begin{aligned} f_{1,y} = &-l_{c2}m_2 s_2\dot{q}_1^2 - 2l_{c2}m_2 s_2\dot{q}_1\dot{q}_2 - l_{c2}m_2 s_2\dot{q}_2^2 + g m_1 c_1 + g m_2 c_1 + l_1 m_2\ddot{q}_1 + l_{c1}m_1\ddot{q}_1 \\ &+ l_{c2}m_2\ddot{q}_1 c_2 + l_{c2}m_2\ddot{q}_2 c_2 \end{aligned} \tag{30}$$

$$\begin{aligned} \tau_{1,z} = &I_{1,z}\ddot{q}_1 + I_{2,z}\ddot{q}_1 + I_{2,z}\ddot{q}_2 + l_1^2 m_2\ddot{q}_1 + l_{c1}^2 m_1\ddot{q}_1 + l_{c2}^2 m_2\ddot{q}_1 + l_{c2}^2 m_2\ddot{q}_2 + g l_1 m_2 c_1 + g l_{c1} m_1 c_1 \\ &+ g l_{c2} m_2 c_{12} - l_1 l_{c2} m_2\dot{q}_2^2 s_2 + 2 l_1 l_{c2} m_2\ddot{q}_1 c_2 + l_1 l_{c2} m_2\ddot{q}_2 c_2 - 2 l_1 l_{c2} m_2\dot{q}_1\dot{q}_2 s_2 \end{aligned} \tag{31}$$

Since the actuation of the first joint is along $z_0 = z_1$, the actuated joint torque for the first joint (revolute) is $\tau_{1,z}$. Since the actuation of the second joint is along $z_1 = z_2$, the actuated joint force for the second joint (revolute) is $\tau_{2,z}$. Another way to see which joint forces/torques are the externally actuated forces/torques is to look at $f_i^T z_{i-1}^{(i)}$ or $\tau_i^T z_{i-1}^{(i)}$ (depending on whether joint i is prismatic or revolute) where $z_{i-1}^{(i)}$ denotes the axis z_{i-1} written relative to frame i; here, $\tau_1^T z_0^{(1)} = \tau_{1,z}$ and $\tau_2^T z_1^{(2)} = \tau_{2,z}$.

Therefore, the dynamics equations are:

$$\begin{aligned} u_1 = &I_{1,z}\ddot{q}_1 + I_{2,z}\ddot{q}_1 + I_{2,z}\ddot{q}_2 + l_1^2 m_2\ddot{q}_1 + l_{c1}^2 m_1\ddot{q}_1 + l_{c2}^2 m_2\ddot{q}_1 + l_{c2}^2 m_2\ddot{q}_2 + g l_1 m_2 c_1 + g l_{c1} m_1 c_1 \\ &+ g l_{c2} m_2 c_{12} - l_1 l_{c2} m_2\dot{q}_2^2 s_2 + 2 l_1 l_{c2} m_2\ddot{q}_1 c_2 + l_1 l_{c2} m_2\ddot{q}_2 c_2 - 2 l_1 l_{c2} m_2\dot{q}_1\dot{q}_2 s_2 \end{aligned} \tag{32}$$

$$u_2 = I_{2,z}(\ddot{q}_1 + \ddot{q}_2) + l_{c2}m_2(l_1 s_2\dot{q}_1^2 + g c_{12} + l_{c2}(\ddot{q}_1 + \ddot{q}_2) + l_1\ddot{q}_1 c_2) \tag{33}$$

where u_1 is the actuated torque for the first joint and u_2 is the actuated torque for the second joint.