Math Insight

Ordinary differential equation examples

Example 1

Solve the ordinary differential equation (ODE)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 3$$

for $x(t)$.

Solution: Using the shortcut method outlined in the introduction to ODEs, we multiply through by dt and divide through by $5x-3$:

$$\frac{dx}{5x-3} = dt.$$

We integrate both sides

$$\begin{aligned} \int \frac{dx}{5x-3} &= \int dt \\ \frac{1}{5}\log|5x-3| &= t + C_1 \\ 5x-3 &= \pm\exp(5t+5C_1) \\ x &= \pm\frac{1}{5}\exp(5t+5C_1) + 3/5. \end{aligned}$$

Letting $C = \frac{1}{5}\exp(5C_1)$, we can write the solution as

$$x(t) = Ce^{5t} + \frac{3}{5}.$$

We check to see that $x(t)$ satisfies the ODE:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 5Ce^{5t} \\ 5x-3 = 5Ce^{5t} + 3 - 3 &= 5Ce^{5t}. \end{aligned}$$

Both expressions are equal, verifying our solution.

Example 2

Solve the ODE combined with initial condition:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 5x - 3 \\ x(2) &= 1. \end{aligned}$$

Solution: This is the same ODE as example 1, with solution

$$x(t) = Ce^{5t} + \frac{3}{5}.$$

We just need to use the initial condition $x(2) = 1$ to determine C.

C must satisfy

$$1 = Ce^{5\cdot 2} + \frac{3}{5},$$

so it must be

$$C = \frac{2}{5}e^{-10}.$$

Our solution is

$$x(t) = \frac{2}{5}e^{5(t-2)} + \frac{3}{5}.$$

You can verify that $x(2) = 1$.

Example 3

Solve the ODE with initial condition:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 7y^2x^3\\ y(2) &= 3.\end{aligned}$$

Solution: We multiply both sides of the ODE by dx, divide both sides by y^2, and integrate:

$$\begin{aligned}\int y^{-2}dy &= \int 7x^3\,dx\\ -y^{-1} &= \frac{7}{4}x^4 + C\\ y &= \frac{-1}{\frac{7}{4}x^4 + C}.\end{aligned}$$

The general solution is

$$y(x) = \frac{-1}{\frac{7}{4}x^4 + C}.$$

Verify the solution:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{-1}{\frac{7}{4}x^4 + C}\right)\\ &= \frac{7x^3}{(\frac{7}{4}x^4 + C)^2}.\end{aligned}$$

Given our solution for y, we know that

$$y(x)^2 = \left(\frac{-1}{\frac{7}{4}x^4 + C}\right)^2 = \frac{1}{(\frac{7}{4}x^4 + C)^2}.$$

Therefore, we see that indeed

$$\frac{dy}{dx} = \frac{7x^3}{(\frac{7}{4}x^4 + C)^2} = 7x^3y^2.$$

The solution satisfies the ODE.

To determine the constant C, we plug the solution into the equation for the initial conditions $y(2) = 3$:

$$3 = \frac{-1}{\frac{7}{4}2^4 + C}.$$

The constant C is

$$C = -28\frac{1}{3} = -\frac{85}{3},$$

and the final solution is

$$y(x) = \frac{-1}{\frac{7}{4}x^4 - \frac{85}{3}}.$$

See also

An introduction to ordinary differential equations
Solving linear ordinary differential equations using an integrating factor

Cite this as

Nykamp DQ, "Ordinary differential equation examples." From *Math Insight*. http://mathinsight.org/ordinary_differential_equation_introduction_examples

Keywords: differential equation, ordinary differential equation