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**KINEMATIC ANALYSIS AND SYNTHESIS OF  
FOUR-BAR MECHANISMS FOR  
STRAIGHT LINE COUPLER CURVES**

by

**Arun K. Natesan**

A Thesis Submitted

in

Partial Fulfillment

of the

Requirements for the Degree of

**MASTER OF SCIENCE**

in

**Mechanical Engineering**

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May, 1994

Title of Thesis: "Kinematic Analysis and Synthesis of Four Bar Mechanisms for  
Straight-Line Coupler Curves"

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Date: 5/10/94

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## **ABSTRACT**

Mechanisms are means of power transmission as well as motion transformers. A four-bar mechanism consists mainly of four planar links connected with four revolute joints. The input is usually given as rotary motion of a link and output can be obtained from the motion of another link or a coupler point. Straight line motion from a four bar linkages has been used in several ways as in a dwell mechanism and as a linkage to vehicle suspension.

This paper studies the straight line motion obtained from planar four-bar mechanisms and optimizes the design to produce the maximized straight line portion of the coupler-point curve. The equations of motion for four different four-bar mechanisms will be derived and dimensional requirements for these mechanisms will be obtained in order to produce the straight line motion. A numerical procedure will be studied and computer codes that generate the coupler curves will be presented. Following the numerical results study, a synthesis procedure will be given to help a designer in selecting the optimized straight line motion based on design criteria.

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# 1. INTRODUCTION

One of the main objects of designing a mechanism is to develop a system that transforms motion in a specific way to provide mechanical advantage. A typical problem in mechanism design is coordinating the input and output motions. A mechanism designed to produce a specified output as a function of input is called a *function generator*. Such a function generator which is capable of producing a straight line output has found a wide variety of applications.

A system that transmits forces in a predetermined manner to accomplish specific work may be considered a *machine*. A *mechanism* is the heart of a machine. It is a device that transforms one motion, for example the rotation of a driving shaft, into another, such as the rotation of the output shaft or the oscillation of a rocker arm. A mechanism consists of a series of connected moving parts which provide the specific motions and forces to do the work for which the machine is designed. A machine is usually driven by a motor which supplies constant speed and power. It is the mechanism which transforms this applied motion into the form demanded to perform the required task. The study of mechanisms is very important. With the tremendous advantages made in the design of instruments, automatic controls, and automated equipment, the study of mechanisms takes on new significance.

Once a need for a machine or mechanism with given characteristics is identified, the design process begins. Detailed analysis of displacements, velocities and accelerations is usually required. Kinematics is the study of motion. The study of motions in machines may be considered from the two different points of view generally identified as kinematic analysis and kinematic synthesis. Kinematic analysis is the determination of motion inherent in a given mechanism. Kinematic synthesis is the

reverse problem: it is the determination of mechanisms that are to fulfill certain motion specifications.

## **1. 1 A Brief History of the Development of the Kinematics of Mechanisms**

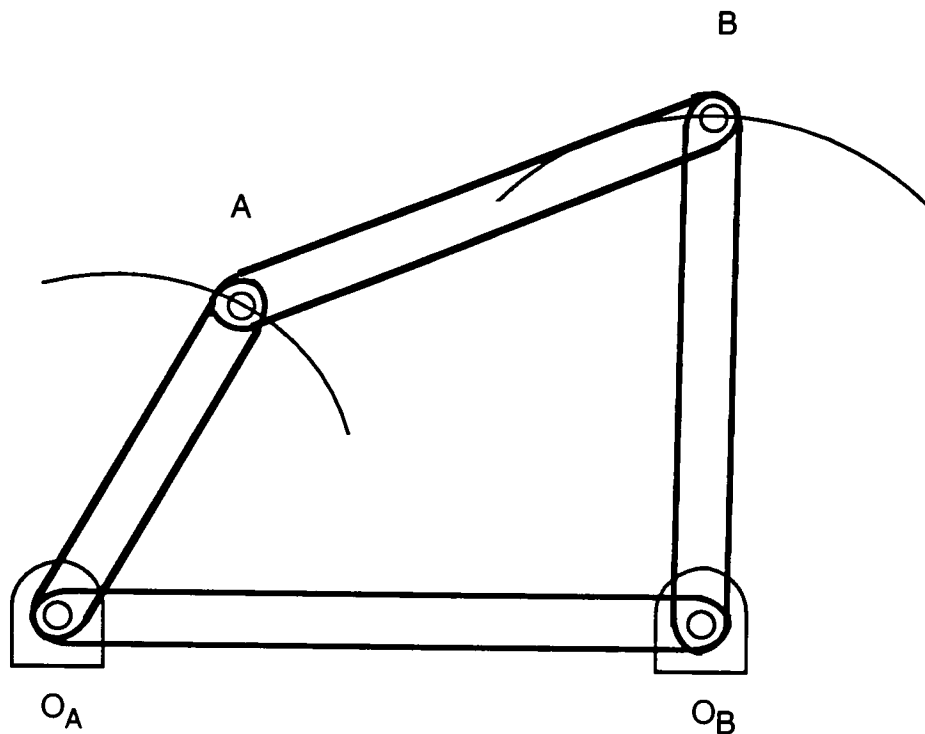
The history of kinematics, the story of the development of the geometry of motion, is composed of evolution in machines, mechanisms and mathematics. The recent investigation of mechanism design by mathematicians and engineers have been stimulated in part by the increase in operating speeds of machines and in part by the expectation of evolving more logical approaches to the development of mechanisms.

Franz Reuleaux was the first scientist who systematically analyzed mechanisms, devised machine elements, studied their combinations, and discovered those laws of operation which constituted the early science of machine kinematics. His now classical “Theoretische Kinematik” of 1875 presented many views finding general acceptance then that are current still and his second book, “Lehrbuch der Kinematik” (1900), consolidated and extended earlier notions. Reuleaux’s comprehensive and orderly views mark a high point in the development of kinematics. He devoted most of his work to the analysis of machine elements.

In the one hundred years that followed Reuleaux, the contributions of such scientists as W. Hartmann, H. Alt, F. Wittenbauer and L. Burmester developed the science of constructing mechanisms to satisfy specific motions, namely, kinematic synthesis. The techniques they used were based on mechanics and geometry.

It was not until 1940 that Svaoboda developed numerical methods to design a simple but versatile mechanism known as *four-bar linkage* (Fig. 1.1) to generate a desired function with sufficient accuracy for engineering purposes. The input crank is  $O_AA$  and





|           |   |          |
|-----------|---|----------|
| $O_A O_B$ | – | Frame    |
| $O_A A$   | – | Rocker   |
| $A B$     | – | Coupler  |
| $O_B B$   | – | Follower |

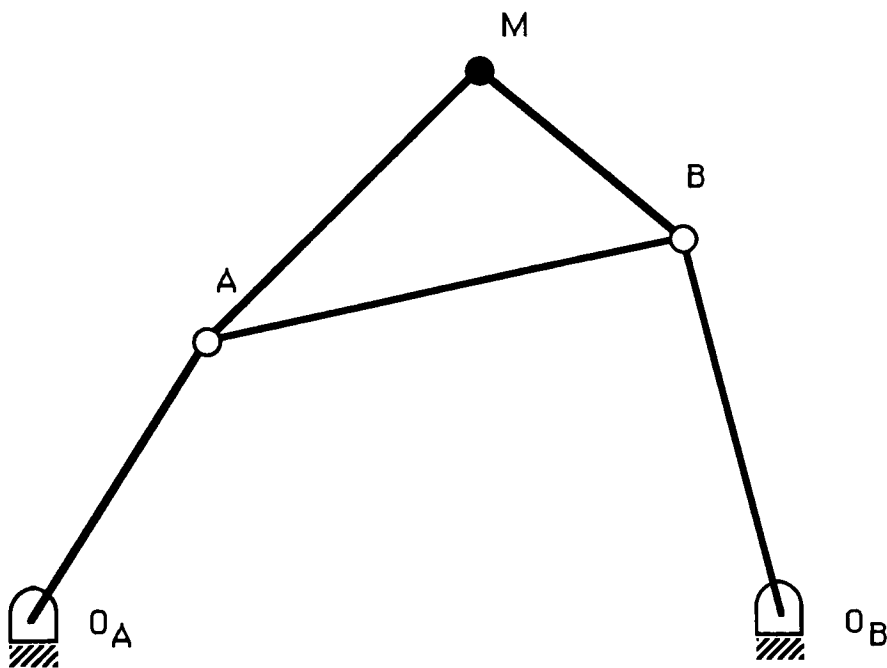
**Figure 1.1 A Basic Four-Bar Mechanism**

the output crank is  $O_B B$ . The scale to input crank indicates the values of the parameter of a function, and that on the output crank indicates the result of the function. Naturally, this four-bar linkage can generate only a limited number of functions because of the nature of the linkage itself. In 1951, the publication by Hrones and Nelson of an “atlas” containing approximately 10,000 coupler curves offered a very practical approach for the design engineers. The Kinematics of mechanisms has gradually become a popular field for scholarly and engineering investigation.

## 1. 2 Four-Bar Mechanism

A four-bar linkage is a versatile mechanism that is widely used in machines to transmit motion or to provide mechanical advantage. Four-bar linkages can also be used as function generators. Their low friction, higher capacity to carry load, ease of manufacturing, and reliability of performance in spite of manufacturing tolerances make them preferable over other mechanisms in certain applications. It is also the most fundamental linkage mechanism, and many more complex mechanisms contain the four-bar linkage as elements. Therefore, a basic understanding of its characteristics is essential.

A four-bar mechanism (Figure 1.2) consists of four rigid members: the *frame* or fixed member, to which pivoted the *crank* and *follower*, whose intermediary is aptly termed *coupler*. These members are connected by four *revolute* pairs. A point on the coupler



**Figure 1.2 A Four-Bar Linkage With Coupler Point on AB**

is called the *coupler point*, and its path when the crank is rotated is known as a *coupler point curve* or *coupler curve* and the number of such curves are infinite. By proper choice of link proportions and coupler point locations useful curves may be found. A curve's usefulness depends on the particular shape of a segment, for example, an approximate straight line or a circular arc, or on a peculiar shape of either the whole curve or parts of it. The coupler point because of its motion characteristic, is now the output of the linkage.

Four-bar mechanism has wide range of applications such as in the pantograph, universal drafting machine, Boehm's coupling, Poppet-valve gear, Whitworth quick-return mechanism and Corliss Valve-gear. A straight line output from a four-bar mechanism has been used in several ways and a few such applications are linkage for vehicle suspension, linkage for posthole borer, in textile industries and in material handling devices.

This work studies mechanisms and, in particular, the four-bar mechanisms. Four popular planar four-bar mechanisms that are capable of generating straight line motion will be analyzed. The equation of the coupler curve for these four-bar mechanisms will be derived and dimensional requirements for these mechanisms will be obtained in order to produce the straight line motion.

Kinematic analysis will not be complete without graphical tools for the designer to examine the output of the mechanisms. A numerical procedure will be studied and computer codes that generate the coupler curves will be presented. Following the numerical results study, a synthesis procedure will be given to help the designer in selecting the optimized straight line motion based on design criteria.

## 2. MECHANISM AND ITS COMPONENTS

Configurations of mechanisms have been incorporated into machines for centuries. In the last forty years, the kinematics of mechanisms has emerged as an engineering science and consistent terminology and definitions were necessitated to assist research and communication. A mechanism has been defined by Reuleux as a combination of rigid or resistant bodies so formed and connected that they move upon each other with definite relative motion. It is the device that transforms one motion, for example the rotation of a driving shaft, into another, such as the rotation of the output shaft or the oscillation of a rocker arm. A *linkage* or *linkwork* might be called the universal mechanism, since almost any conceivable motion can be produced by this device. A linkage, as applied to mechanisms, means a combination of a number of pairs of elements, such as levers, cranks, slides, etc., connected by rigid pieces or *links*, all the parts being connected by pin or pivoted joints allowing relative motion between the parts. All the parts must be so connected that when any one part is moved, definite motion is imparted to all the other parts. A few terms of particular interest to the study of kinematics and dynamics of mechanisms are defined below.

### **Link**

A *link* is one of the rigid bodies or members joined together. The term rigid link or sometimes simply link is an idealization used in the study of mechanisms that does not consider small deflections due to strains in machine members. A perfectly rigid or inextensible link can exist only as a textbook type of model of a real machine member. For typical machine parts, maximum dimension changes are of the order of only a one-thousandth of the part length. It is justified to neglect this small motion when considering the much greater motion characteristic of most mechanisms. The word link is used in a general sense to include cams, gears, and other machine members in

addition to cranks, connecting rods, and other pin-connected components.

## **Frame**

The fixed or stationary link in a mechanism is called the *frame*. When there is no link that is actually fixed, one link may be considered as being fixed and determine the motion of the other links relative to it. A frame is the reference from which all motions of the mechanisms are accounted for. In an automotive engine, the engine block is considered the frame, even though the automobile may be moving.

## **Joint or Kinematic Pair**

The connections between links that permit relative motion are called *joints*. An unconstrained rigid body has a mobility of six degrees of freedom. Each joint reduces the mobility of a system. The joint between a crank and connecting rod, for instance, is called a revolute joint or pin joint. The revolute joint has one degree-of-freedom in that if one element is fixed, the revolute joint allows the other only to rotate in a plane. A sphere joint has three degrees-of-freedom. Some of the practical joints are made up of several elements. Examples include universal joint; ball and roller bearings that are represented by the revolute joint; ball slides represented by the cylindrical joint; and ball screws represented by the helix.

## **Lower and Higher Pairs**

Connections between rigid bodies can be categorized as lower and higher pairs of elements. The two elements of a lower pair have theoretical surface contact with one another, while the two elements in the higher pair have theoretical point or line contact (if we disregard deflections). Lower pairs include revolute or pin connections - for example, a shaft in a bearing or the wrist pin joining a piston and connecting rod. Examples for higher pair include a pair of gears or a disk cam and a follower.

## Kinematic Chain

A *kinematic chain* is an assembly of links and joints. In a *closed kinematic chain*, each link is connected to two or more other links.

## Mechanism

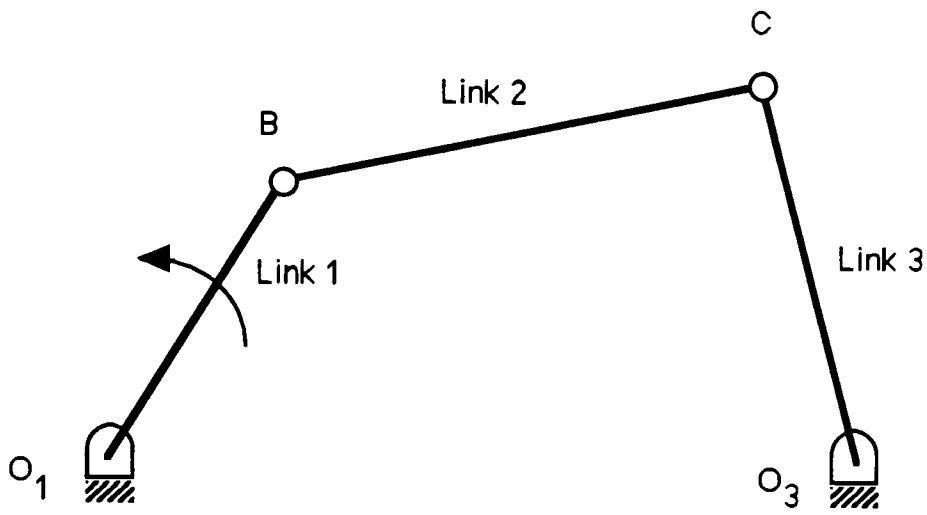
A *mechanism* is a kinematic chain in which one link is considered fixed for the purpose of analysis, but motion is possible in other links. As noted above, the link designated as the fixed link need not actually be stationary relative to the surface of the earth. A kinematic chain is usually identified as a mechanism if its primary purpose is modification or transmission of motion.

## Linkage

If kinematic chains are needed to be examined without regard to its ultimate use, an assemblage of rigid bodies connected by kinematic joints of lower pairs are identified as a *linkage*. Thus, both mechanisms and machines may be considered linkages. However in general, the term linkage is restricted to kinematic chains made up of lower pairs.

## Planar Motion and Planar Linkages

If all points in a system move in parallel planes, then that system undergoes *planar motion*. All the links in a *planar linkage* have planar motion. This work is concerned only with planar linkages. A *skeleton diagram* of a planar linkage (e.g. Fig. 2.1) is formed by connecting the pin centers by straight lines and projecting these centerlines on one of the planes of motion. The linkages in which motion can be described as taking place in parallel planes are called *spatial* or *three-dimensional* (3D) *linkages*.



**Fig 2.1 A Skeleton Diagram of a Planar Linkage**

### **Cycle and Period**

A *cycle* represents the complete sequence of positions of the links in a mechanism (all points attained between two identical positions). In a four-stroke-cycle engine, one thermodynamic cycle corresponds to two revolutions or cycles of the crankshaft but one revolution of the camshaft and, thus, one cycle of motion of the cam followers and valves. The time required to complete a cycle of motion is called the *period*.

### 3. FOUR-BAR MECHANISM AND ITS CLASSIFICATIONS

An important property of a classification system would be the aid it could furnish a designer in finding the forms and arrangements best suited to satisfying certain conditions. The planar four-bar mechanism which consists of four pin-connected rigid links gains its importance as a basic mechanism because it is one of the simplest of all mechanisms to produce. The *four-bar linkage* derives its renown from the fact that the members of a three bar linkage are incapable of relative motion and a linkage composed of more than four bars has indeterminate motion with a single input. Though it may assume many forms, often with little resemblance to the usual representation, a four-bar linkage consists of two members in pure rotation about fixed axes, called the *driving* and *follower* crank; a *coupler* in combined motion, which joins the moving ends of the cranks; and a fixed frame, which establishes the relative position of the stationary crank centers.

#### 3. 1. Classifications and the Grashof Criteria

There are two main classes of four bar mechanisms based on the rotational and dimensional limitations of its links called Grashof's criterion, which are:

1. Grashof mechanisms, which is comprised of:

- crank rocker mechanism

- drag link mechanism

- double rocker mechanism

- crossover-position or change point mechanism

2. Non-Grashof Mechanisms, which includes

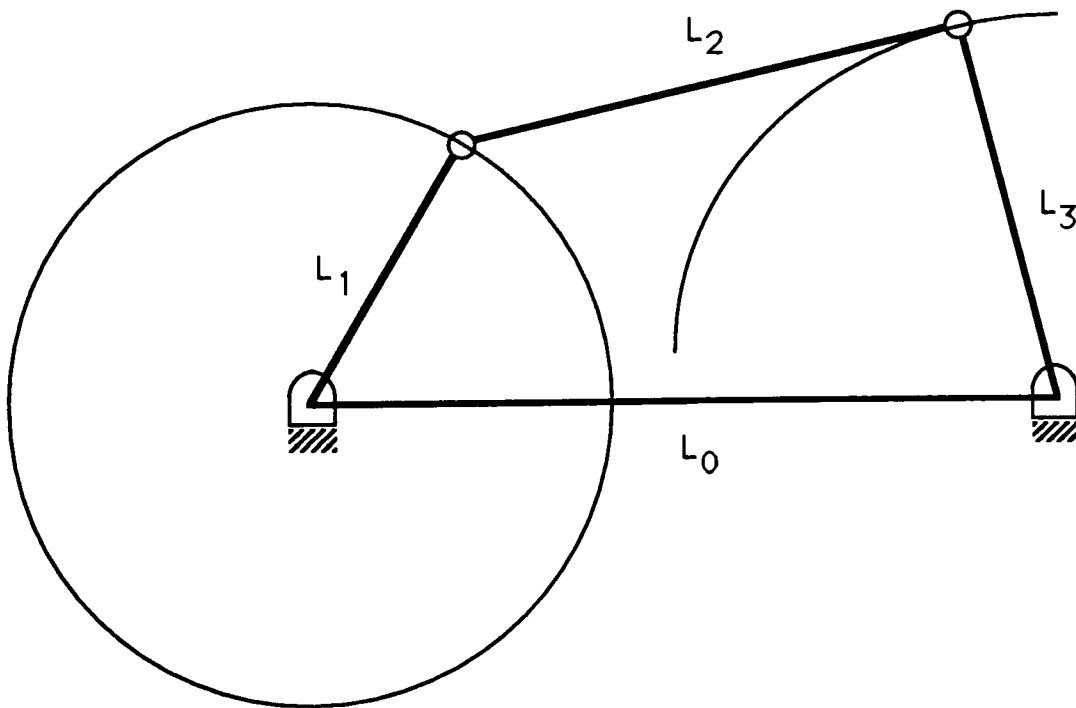
- double rocker mechanisms of the second kind or triple rocker mechanisms.



A Grashof mechanism is a four bar linkage in which one link can perform a complete rotation relative to the other three. This criterion would be considered if we plan to drive a linkage with a continuously rotating motor. It will be shown that the Grashof criterion is met if:

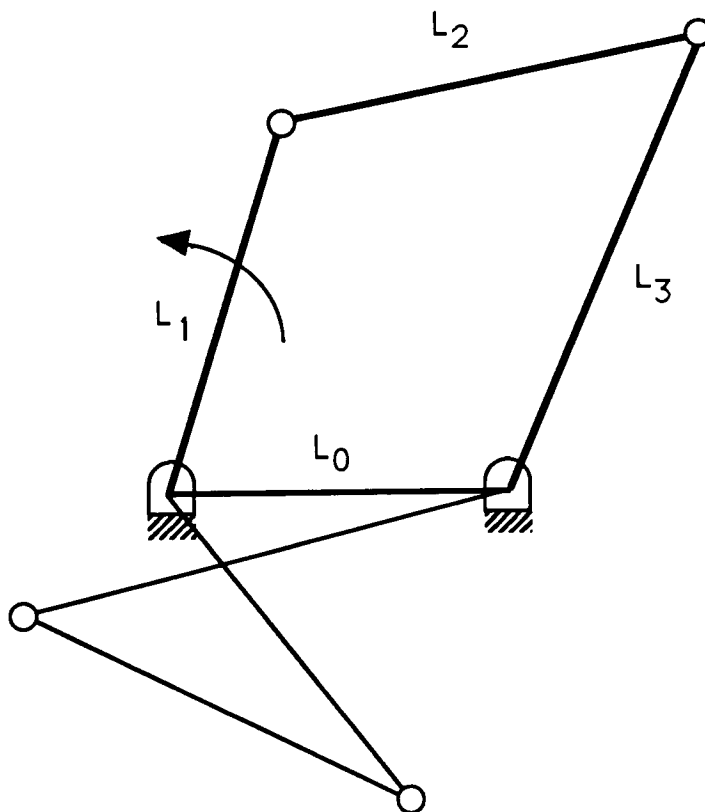
$$L_{\max} + L_{\min} \leq L_a + L_b \quad (3.1)$$

where link lengths are measured between bearing centers,  $L_{\max}$  is the length of the longest link,  $L_{\min}$  that of the shortest link,  $L_a$  and  $L_b$  are the lengths of the remaining links.



**Fig. 3.1 Crank Rocker Mechanism**

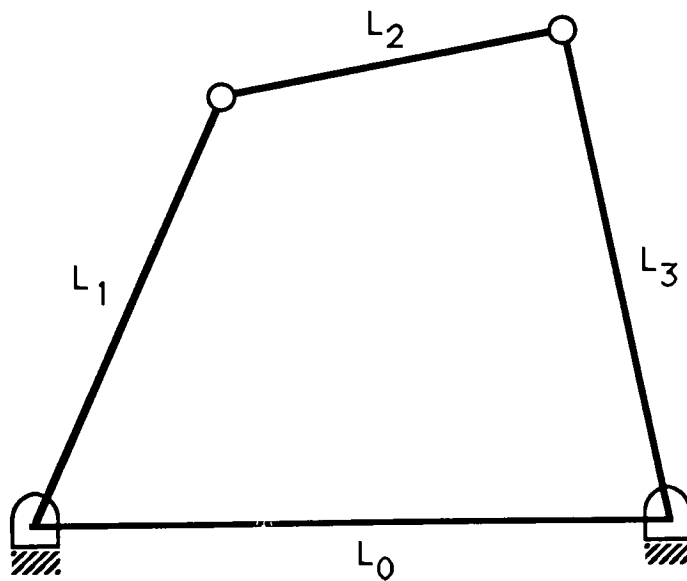
A Grashof mechanism in which the drive crank is the shortest (and  $L_{\max} + L_{\min} < L_a + L_b$ ) will act as a crank rocker mechanism. In Figure 3.1, the skeleton diagram represents a crank rocker mechanism where link 0 represents the frame, links 1 and 3 are the side links and link 2 is the coupler. The smallest side link, link 1, often acts as a driving crank. The rocker link (link 3) will oscillate while the crank (link 1) is rotated continuously in one direction.



**Fig. 3.2 Drag Link Mechanism**

A Grashof mechanism in which the fixed link is the shortest ( and  $L_{\max} + L_{\min} < L_a + L_b$ ) will act as a drag link mechanism, (see Figure 3.2). The drive crank will rotate through  $360^\circ$  along with the coupler and follower crank.

A Grashof mechanism in which coupler, the link opposite to the fixed link, is shortest and  $L_{\max} + L_{\min} < L_a + L_b$  will act as a double rocker mechanism, (see Figure 3.3). This is sometimes called double rocker of the first kind. Although the coupler can rotate  $360^\circ$ , neither crank can rotate through  $360^\circ$ . In a linkage of this type, the coupler can be used as the drive member.



**Figure 3.3 A Double Rocker Mechanism**

A Grashof mechanism in which  $L_{\max} + L_{\min} = L_a + L_b$  may be considered a crossover-position or change point mechanism. Relative motion of a crossover position may depend on inertia, spring forces, or other forces when links become collinear.

Any of the above classes of linkages may be driven by rotation of the coupler (the link opposite the fixed link), although the range of coupler in some cases may be very limited. The coupler effectively provides a hinge with a moving center. The coupler-driven linkages may be called polycentric.

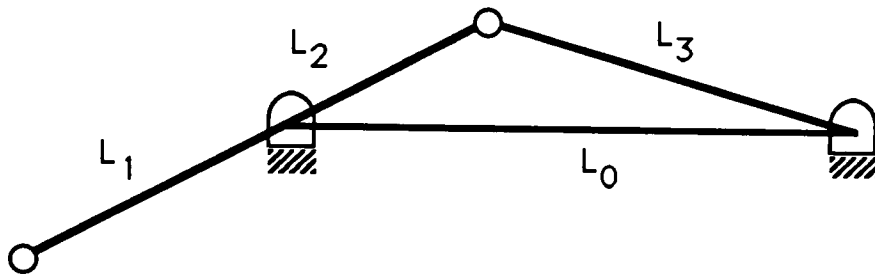
Four-bar mechanisms that do not satisfy the Grashof criterion,  $L_{\max} + L_{\min} \leq L_a + L_b$  are called double rocker mechanisms of the second kind or triple rocker mechanisms. In this case no link can rotate through  $360^\circ$ .

### **3. 2. Proof of Grashof Criteria**

To show the validity of the Grashof criteria, we may begin by examining a crank rocker mechanism. Referring to Figure 3.1 , we observe that the range of motion of link 3 is limited. The limiting positions of link 3 occur when links 1 and link 2 are collinear. The linkage arranges itself in the form of a triangle. Using the triangle inequality, we obtain the required relationships between lengths of the crank rocker mechanism.

First, using Figure 3.4, we have an inequality relating the length of the fixed link to the others:

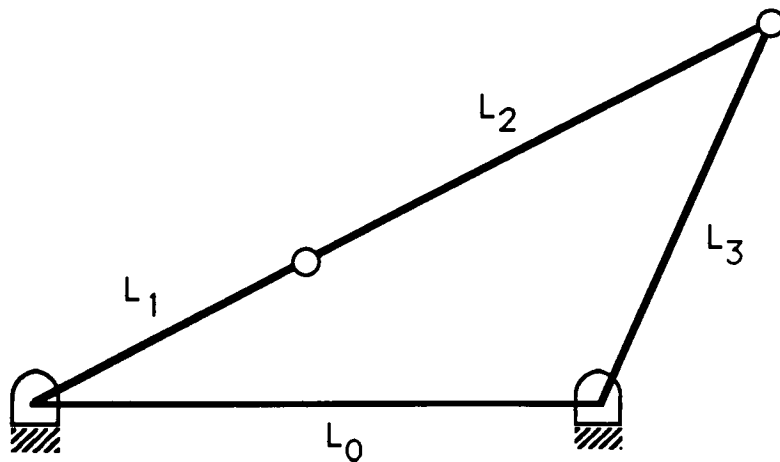
$$L_0 < L_2 - L_1 + L_3 \quad (3.2)$$



**Figure 3.4 A Limiting Position of Crank Rocker Mechanism**

Next, a similar expression is obtained for follower crank 3:

$$L_3 < L_2 - L_1 + L_0 \quad (3.3)$$



**Figure 3.5 Another Limiting Position of Crank Rocker Mechanism**

From Figure 3.5, The length of link 1 added to link 2 is related to the others to form inequality:

$$L_1 + L_2 < L_0 + L_3 \quad (3.4)$$

Combining inequalities 3.2, 3.3 and 3.4, we have

$$L_1 + |L_2 - L_3| < L_0 < L_1 + L_3 \quad (3.5)$$

Actually, there are three additional possible inequalities based on the triangle formed in Figure 3.5, but these three inequalities are redundant.

Inequalities 3.2 and 3.3, respectively, may be written in the following form:

$$L_1 < -L_0 + L_2 + L_3 \quad (3.6)$$

$$L_1 < L_0 + L_2 - L_3 \quad (3.7)$$

In this form, the inequalities may be added to obtain

$$2L_1 < 2L_2 \quad \text{or} \quad L_1 < L_2 \quad (3.8)$$

Similarly using inequality 3.2 and 3.4, we have

$$L_1 < L_3 \quad (3.9)$$

Using 3.3 with 3.4, we have

$$L_1 < L_0 \quad (3.10)$$

Thus, if the driver crank (which we label link 1) is the shortest link in a four-bar mechanism, we may have a crank rocker mechanism. If inequalities 3.2, 3.3 and 3.4 are satisfied for the given mechanism, the identification of the mechanism as a crank rocker mechanism is then positive; link 3 will oscillate as link 1 rotates continuously.

Substituting  $L_{\min}$  for  $L_1$  and  $L_{\max}$  for each of  $L_0$ ,  $L_2$ , and  $L_3$  in turn, in inequality 3.5, we see that it is identical to the more concise requirements for a crank rocker mechanism: (a)  $L_{\max} + L_{\min} < L_a + L_b$  and (b) the crank is the shortest link. If link 0, the fixed link, is shortest, as in the drag link mechanism, we may substitute  $L_0$  for  $L_1$ ,  $L_1$  for  $L_2$ ,  $L_2$  for  $L_3$  and  $L_3$  for  $L_0$  in inequality 3.2 to obtain

$$L_0 + |L_1 - L_2| < L_3 < L_1 - L_0 + L_2 \quad (3.11)$$

If link 2, the coupler link, is shortest, as in a double rocker mechanism, by similar permutation, we obtain

$$L_2 + |L_3 - L_0| < L_1 < L_3 - L_2 + L_0 \quad (3.12)$$

Substituting as we did in inequality 3.5 we see that in equalities 3.11 and 3.12 satisfy the Grashof inequality:

$$L_{\max} + L_{\min} < L_a + L_b \quad (3.13)$$

Each of these mechanisms may be considered as inversion of the others. Four-bar

linkages that violate the Grashof criteria are triple rocker mechanisms. Each Grashof mechanism has two assembly configurations. The positions attainable in one assembly configuration are not attainable in the other. A summary of the results is given in Table 3.1.

**Table 3. 1. Summary of the Criteria of Motion for Each Class of Four-Bar Linkages**

| $L_{min}$ :        | shortest link;              |                                    |
|--------------------|-----------------------------|------------------------------------|
| $L_{max}$ :        | longest link;               |                                    |
| $L_a$ and $L_b$ :  | links of intermediate links |                                    |
| Type of Mechanism  | Shortest Link               | Relationship Between Links         |
| GRASHOF            | Any                         | $L_{max} + L_{min} \leq L_a + L_b$ |
| Crank rocker       | Driver crank                | $L_{max} + L_{min} < L_a + L_b$    |
| Drag link          | Fixed link                  | $L_{max} + L_{min} < L_a + L_b$    |
| Double-rocker      | Coupler                     | $L_{max} + L_{min} < L_a + L_b$    |
| Crossover-position | Any                         | $L_{max} + L_{min} = L_a + L_b$    |
| NON-GRASHOF        |                             |                                    |
| Triple-rocker      | Any                         | $L_{max} + L_{min} > L_a + L_b$    |



## 4. SPECIAL FOUR-BAR MECHANISMS FOR APPROXIMATE STRAIGHT LINE OUTPUT

One of the special applications of four-bar linkages is as function generators. The atlas, *Analysis of the four-bar linkage*, by Hrones and Nelson, contains a few coupler curves with approximate straight lines that have been useful for practical design problems. Four well-known four-bar linkages which are capable of generating straight line motion will be introduced in this chapter. However, those mechanisms given by the atlas are inflexible in design, and only occasionally they fit the problems the designers face in practice; in most cases to suit particular needs mechanisms for straight line motions needed to be developed. In this chapter, these four mechanisms will be categorically defined and their mobility will be investigated.

### 4. 1. Evans Linkage

Evans Linkage is a crank rocker mechanism in which the crank  $L_1$  rotates through a

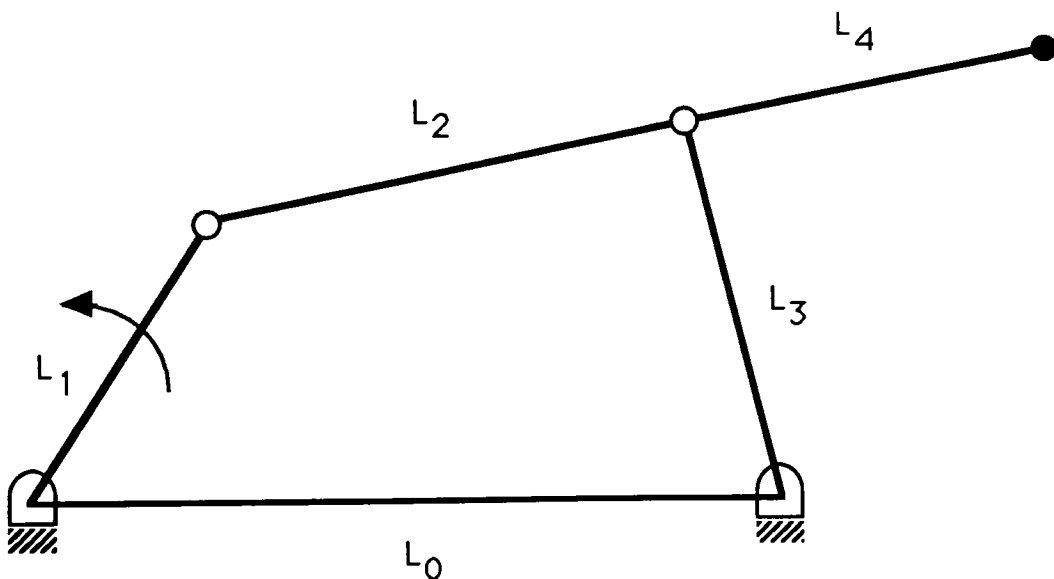
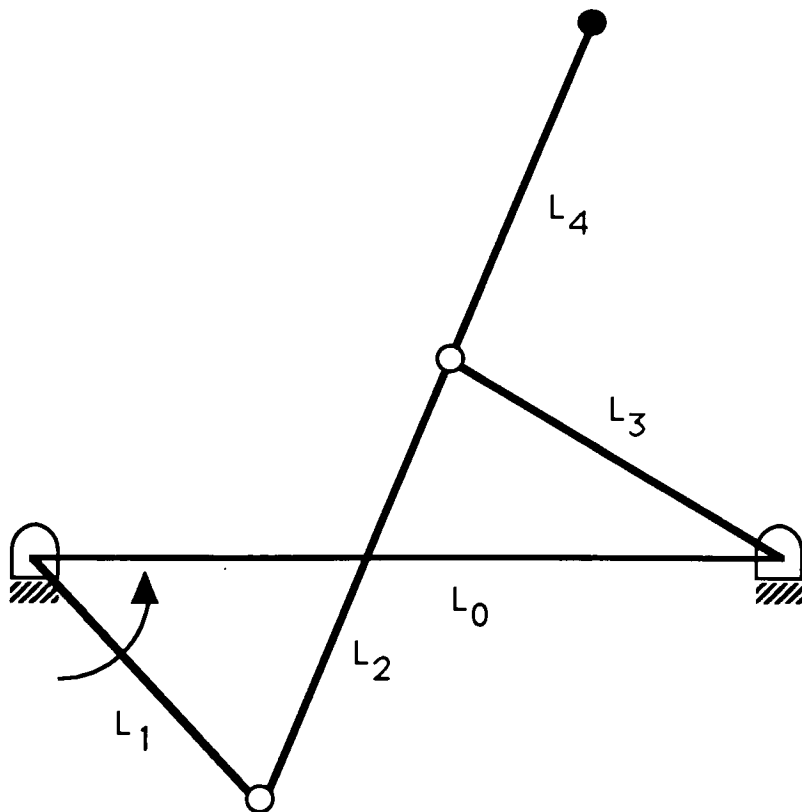


Fig 4.1 Evans Linkage

complete rotation and thus used as the input link (See Figures 4.1 and 4.2). The output motion is obtained from the point M which is located on the extension of the coupler link  $L_2$ . Here  $L_0$  is the fixed link or frame. The Grashof condition for this linkage can be stated as

$$L_{\max} + L_{\min} < L_a + L_b \quad (4.1)$$

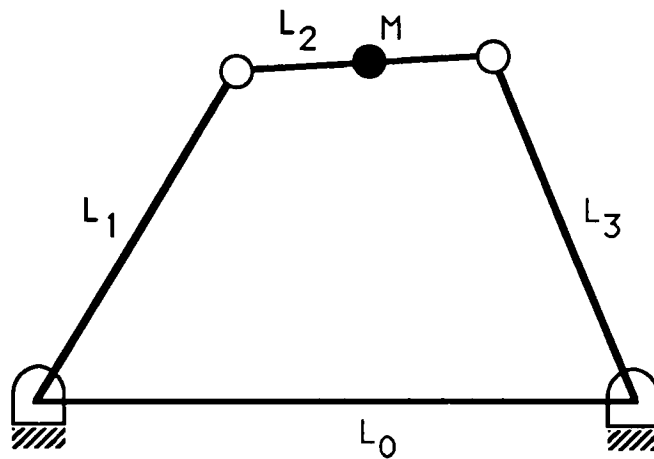
and the crank should be the smallest link. Since this is a crank rocker mechanism, the crank is free to rotate through a complete rotation with respect to the frame.



**Figure 4.2 Another Position of Evans Linkage**

## 4. 2. Chebyshev Linkage

Chebyshev Linkage is a four-bar double rocker mechanism where the coupler rotates through  $360^\circ$ . Figure 4.3 shows a schematic diagram of the Chebyshev linkage. In Chebyshev's Linkage either the crank or the coupler can be used as the input link. The output is obtained from the point M located at the middle of the coupler. The limiting positions of the crank in relation to the frame for this mechanism are shown in Figures 4.4 and 4.5.



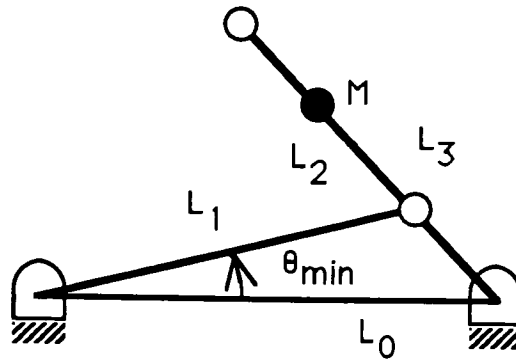
**Figure 4.3 Chebyshev Linkage**

The conditions for Chebyshev's Linkage is same as the Grashof's criteria:

$$L_{\max} + L_{\min} < L_a + L_b \quad (4.2)$$

and the smallest link is the coupler  $L_2$ .

The limiting angles between the crank and the frame for Chebyshev's Linkage are calculated using cosine law (refer Figure 4.4).

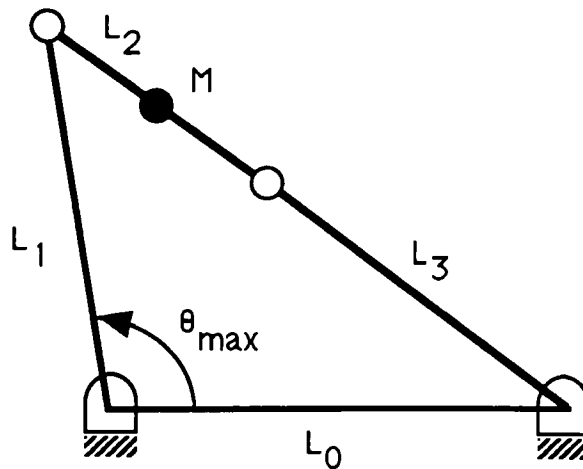


**Figure 4.4 Limiting Angle of Chebyshev Linkage**

By cosine law:

$$(L_3 - L_2)^2 = L_0^2 + L_1^2 - 2 L_0 L_1 \cos \theta_{\min}$$

$$\Rightarrow \theta_{\min} = \cos^{-1} \{ [L_0^2 + L_1^2 - (L_3 - L_2)^2] / 2 L_0 L_1 \} \quad (4.3)$$



**Figure 4.5. Another Limiting Angle of Chebyshev Linkage**

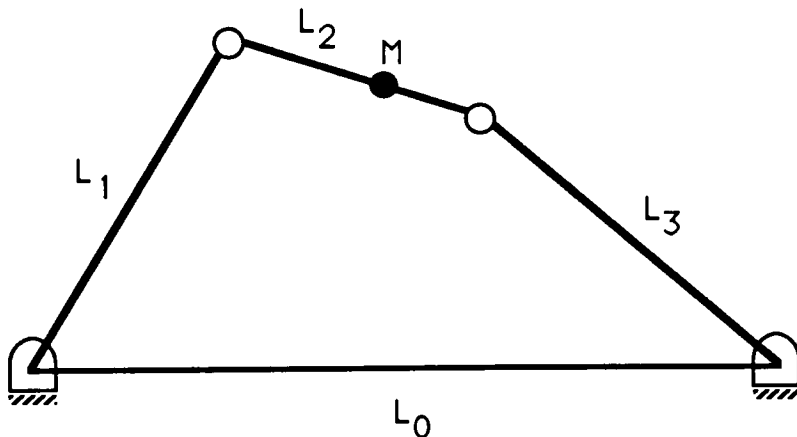
Similarly to calculate  $\theta_{\max}$  from Figure 4.5,

$$(L_3 + L_2)^2 = L_0^2 + L_1^2 - 2 L_0 L_1 \cos \theta_{\max}$$

$$\Rightarrow \theta_{\max} = \cos^{-1} \{ [L_0^2 + L_1^2 - (L_3 + L_2)^2] / 2 L_0 L_1 \} \quad (4.4)$$

### 4. 3. Watts Linkage

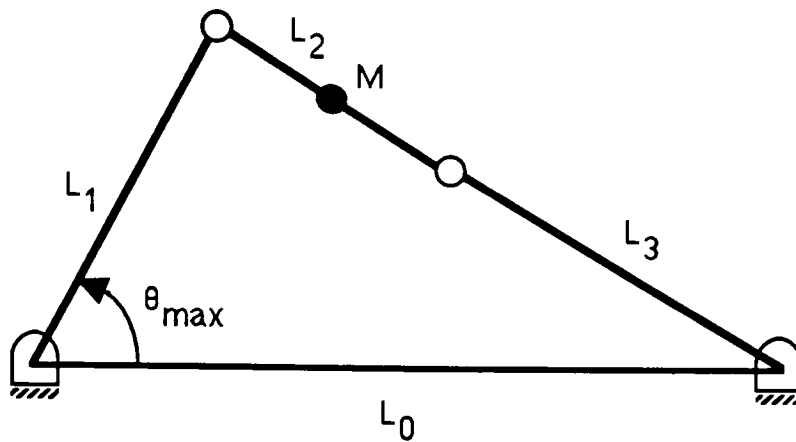
A pictorial representation of Watts Linkage (refer Figure 4.6) resembles that of the Chebyshev's. But it differs from Chebyshev's by the fact that it is not a double rocker mechanism which means no link in this mechanism rotates through 360°. The limiting positions of the crank for this mechanism are shown in Figures 4.7 and 4.8.



**Figure 4.6 Watts Linkage**

Being a non-Grashof mechanism, Watts Linkage does not satisfy the Grashof Criteria. The condition for this kind of link mechanism is,

$$L_{\max} + L_{\min} > L_a + L_b \quad (4.5)$$



**Figure 4.7 Limiting Position of Watts Linkage**

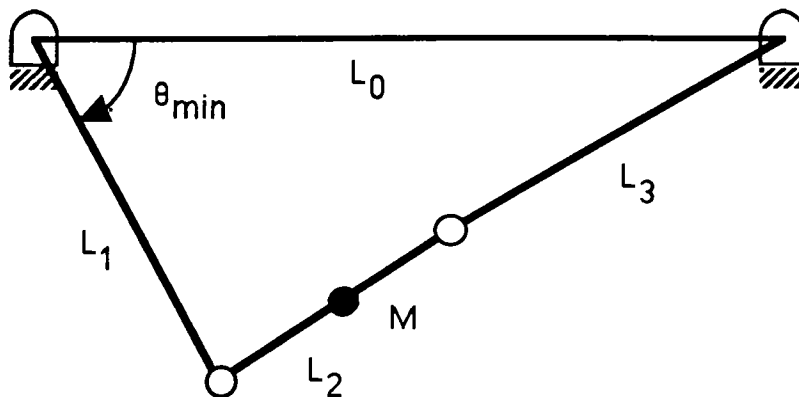
Figure 4.7 will show that the limiting angle  $\theta_{\max}$  for Watts Linkage can be calculated as same way as that of Chebyshev's.

$$(L_3 + L_2)^2 = L_0^2 + L_1^2 - 2 L_0 L_1 \cos \theta_{\max}$$

$$\Rightarrow \theta_{\max} = \cos^{-1} \{ [L_0^2 + L_1^2 - (L_3 + L_2)^2] / 2 L_0 L_1 \} \quad (4.6)$$

and in this case,

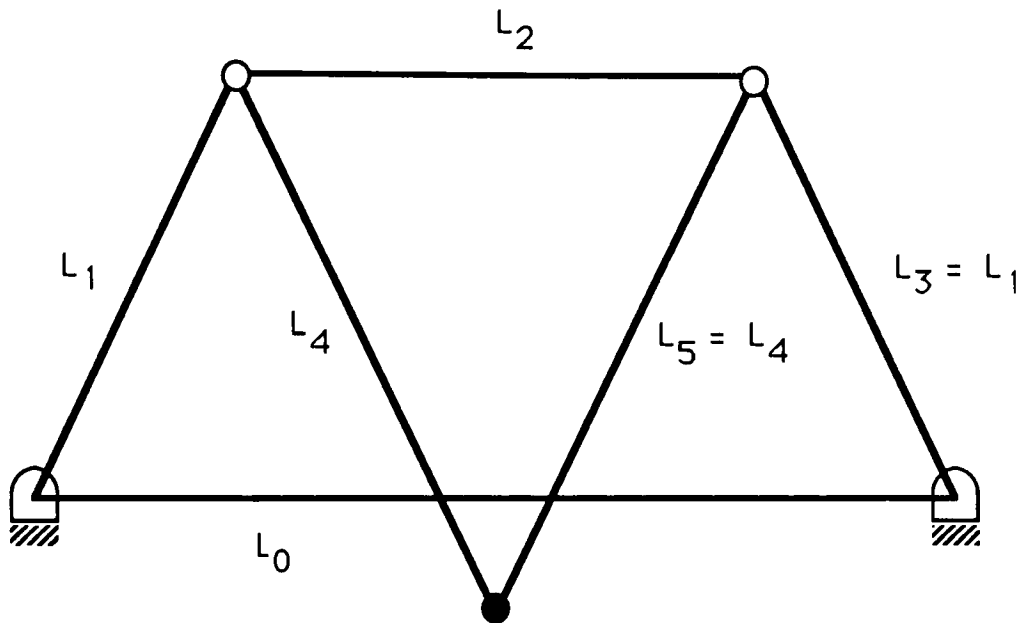
$$\theta_{\min} = -\theta_{\max} \quad (4.7)$$



**Figure 4.8 The Other Limiting Position of Watts Linkage**

#### 4. 4. Roberts Linkage

Roberts Linkage is a mechanism of triple rocker kind in which none of the links rotate through  $360^\circ$ . Figure 4.9 shows a Roberts mechanism.

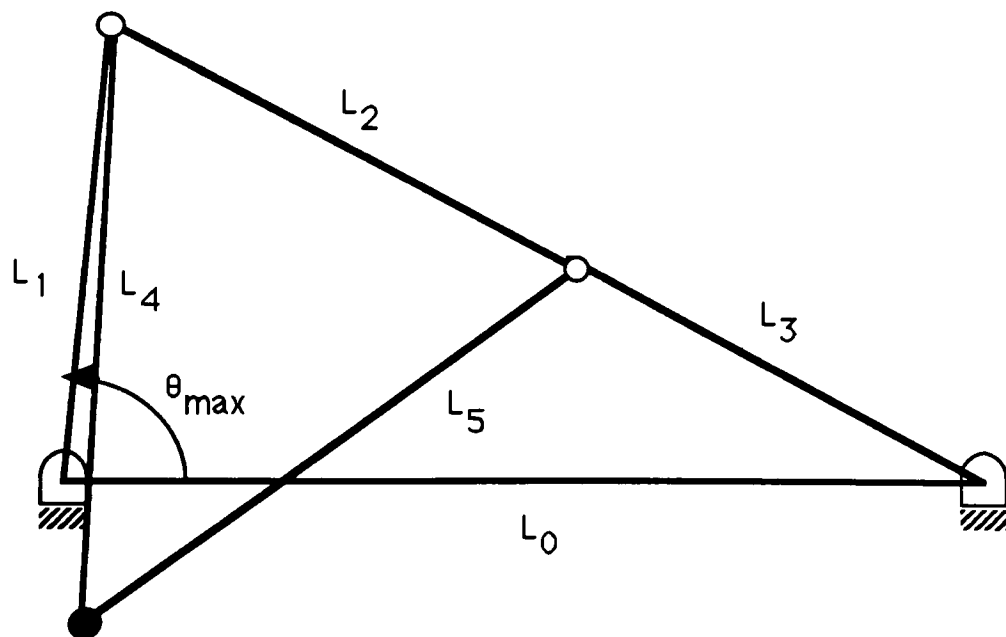
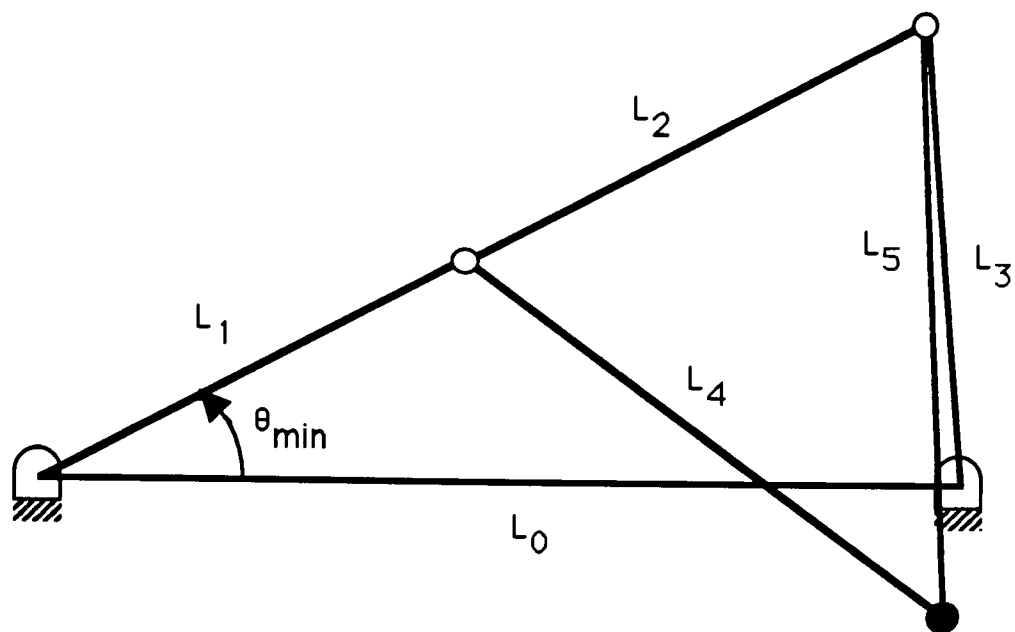


**Figure 4.9 Roberts Linkage**

From the limiting positions illustrated by Figure 4.10, by cosine law,

$$L_3^2 = L_0^2 + (L_1 + L_2)^2 - 2 L_0 (L_1 + L_2) \cos \theta_{\min}$$

$$\Rightarrow \theta_{\min} = \cos^{-1} \{ [ L_0^2 + (L_1 + L_2)^2 - L_3^2 ] / 2 L_0 (L_1 + L_2) \} \quad (4.8)$$



**Figure 4.11 The Limiting Positions of Roberts Linkage**



and  $\theta_{\max}$  is calculated as,

$$\begin{aligned} (L_2 + L_3)^2 &= L_0^2 + L_1^2 - 2 L_0 L_1 \cos \theta_{\max} \\ \Rightarrow \theta_{\max} &= \cos^{-1} \{ [ L_0^2 + L_1^2 - (L_2 + L_3)^2 ] / 2 L_0 L_1 \} \end{aligned} \quad (4.9)$$

## 5. POSITION ANALYSIS OF A BASIC FOUR-BAR MECHANISM

When mechanisms are analyzed both graphical and analytical methods can be useful. When position of a point or set of points are to be determined for a single linkage position, graphical methods are usually more convenient. Analytical methods are more practical when a sequence of positions of a mechanism must be analyzed. The use of a computer permits a detailed study of a full cycle of motion. Once the initial programming is completed, little effort is required to examine the effect of design changes. On the other hand, if we were to use graphical methods, each linkage position would require a separate plot and each change in length of a link would require a new sequence of plots.

This chapter deals with the analytical method of determining the positions of the links relative to one another. Methods of vector analysis are important tools, which could be used for mechanism analysis and synthesis.

### 5.1 Vectors

*Vectors* provide graphical and analytical means to represent motion. A quantity described by its *magnitude* and *direction* can be considered a vector and can be graphically represented by an arrow. The length of the arrow is proportional to the magnitude of the vector quantity and the direction of the arrow is the direction of the vector quantity. Graphical and analytical vector methods can be applied to linear displacements, velocities, accelerations and forces, and to torques and angular velocities and accelerations. Vectors will be identified in this study by **boldface** type to distinguish from scalar quantities.

In general, a vector of unit magnitude can be called a unit vector. Thus,  $\mathbf{A}^u = \mathbf{A} / A$  is a unit vector in the direction of  $\mathbf{A}$ , where  $A = |\mathbf{A}|$  is the magnitude of vector  $\mathbf{A}$ . A coordinate system in which the axes are mutually perpendicular is called a rectangular coordinate system. Unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  parallel to the  $x, y, z$  coordinate axes, respectively, are particularly useful, since we are going to use only rectangular coordinate system throughout this work. These unit vectors are also called rectangular unit vectors. A vector may be described in terms of its components along each coordinate axis.

When we use vectors to describe the motion of a linkage, it is advisable to make a sketch of the linkage adjacent to vector diagrams so that vector directions can be referred to linkage orientation.

### 5. 1. 1 Solution Of Planar Vector Equations

Consider the planar vector equation

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$$

or in terms of unit vectors ( $\mathbf{A}^u$  etc.) and magnitudes ( $A$  etc.),

$$\mathbf{A}^u A + \mathbf{B}^u B + \mathbf{C}^u C = \mathbf{0}$$

If the magnitude and direction of the same vector are unknown, then the solution is easily obtained. If  $C$  is unknown, we use

$$\mathbf{C} = -(\mathbf{A}_x + \mathbf{B}_x)\mathbf{i} - (\mathbf{A}_y + \mathbf{B}_y)\mathbf{j}$$

or

$$\mathbf{C} = -(\mathbf{A} \cdot \mathbf{i} + \mathbf{B} \cdot \mathbf{i})\mathbf{i} - (\mathbf{A} \cdot \mathbf{j} + \mathbf{B} \cdot \mathbf{j})\mathbf{j}$$

If the magnitudes of two different vectors are unknown, a vector elimination method may be used. Suppose, for example, magnitudes  $A = |\mathbf{A}|$  and  $B = |\mathbf{B}|$  are unknown in

the vector equation  $\mathbf{A}^u \mathbf{A} + \mathbf{B}^u \mathbf{B} + \mathbf{C}^u \mathbf{C} = \mathbf{0}$ . We take the dot product of each term with  $\mathbf{B}^u \times \mathbf{k}$  noting that  $\mathbf{B}^u \cdot (\mathbf{B}^u \times \mathbf{k}) = 0$  since vector  $\mathbf{B}^u$  is perpendicular to vector  $\mathbf{B}^u \times \mathbf{k}$ . Thus, we obtain

$$\mathbf{A}^u \mathbf{A} \cdot (\mathbf{B}^u \times \mathbf{k}) + \mathbf{C} \cdot (\mathbf{B}^u \times \mathbf{k}) = 0$$

from which the magnitude of vector  $\mathbf{A}$  is given by

$$A = \frac{-\mathbf{C} \cdot (\mathbf{B}^u \times \mathbf{k})}{\mathbf{A}^u \cdot (\mathbf{B}^u \times \mathbf{k})} \quad (5.1)$$

Similarly, the magnitude of  $\mathbf{B}$  is given by

$$B = \frac{-\mathbf{C} \cdot (\mathbf{A}^u \times \mathbf{k})}{\mathbf{B}^u \cdot (\mathbf{A}^u \times \mathbf{k})} \quad (5.2)$$

If the vector directions  $\mathbf{A}^u$  and  $\mathbf{B}^u$  are unknown but all vector magnitudes are known, the solution to the equation  $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$  is more difficult. The results in this case, as given in Reference 10, are

$$\begin{aligned} \mathbf{A} = & -\{ B^2 - [(C^2 + B^2 - A^2) / 2C]^2 \}^{(1/2)} (\mathbf{C}^u \times \mathbf{k}) \\ & + \{ [(C^2 + B^2 - A^2) / 2C] - C \} \mathbf{C}^u \end{aligned} \quad (5.3a)$$

$$\begin{aligned} \text{or } \mathbf{A} = & +\{ B^2 - [(C^2 + B^2 - A^2) / 2C]^2 \}^{(1/2)} (\mathbf{C}^u \times \mathbf{k}) \\ & + \{ [(C^2 + B^2 - A^2) / 2C] - C \} \mathbf{C}^u \end{aligned} \quad (5.3b)$$

$$\begin{aligned} \text{and } \mathbf{B} = & +\{ B^2 - [(C^2 + B^2 - A^2) / 2C]^2 \}^{(1/2)} (\mathbf{C}^u \times \mathbf{k}) \\ & + [(C^2 + B^2 - A^2) / 2C] \mathbf{C}^u \end{aligned} \quad (5.4a)$$

$$\begin{aligned} \text{or } \mathbf{B} = & -\{ B^2 - [(C^2 + B^2 - A^2) / 2C]^2 \}^{(1/2)} (\mathbf{C}^u \times \mathbf{k}) \\ & + [(C^2 + B^2 - A^2) / 2C] \mathbf{C}^u \end{aligned} \quad (5.4b)$$

When the magnitude of A and the direction of B are unknown, A and B may be found by the following equations:

$$\mathbf{A} = \{ -\mathbf{C} \cdot \mathbf{A}^u \pm \sqrt{B^2 - [\mathbf{C} \cdot (\mathbf{A}^u \times \mathbf{k})]^2} \} \mathbf{A}^u \quad (5.5)$$

$$\mathbf{B} = -[\mathbf{C} \cdot (\mathbf{A}^u \times \mathbf{k})] (\mathbf{A}^u \times \mathbf{k}) \pm \sqrt{B^2 - [\mathbf{C} \cdot (\mathbf{A}^u \times \mathbf{k})]^2} \} \mathbf{A}^u \quad (5.6)$$

The above approach uses vector notation throughout, unlike alternate methods that use vector analysis to derive scalar equations. When the above method is used for computer-aided analysis and design of mechanisms, computer subroutines will be incorporated to handle the conversion from vector to scalars. The above equation will be applied to the analysis of planar linkages.

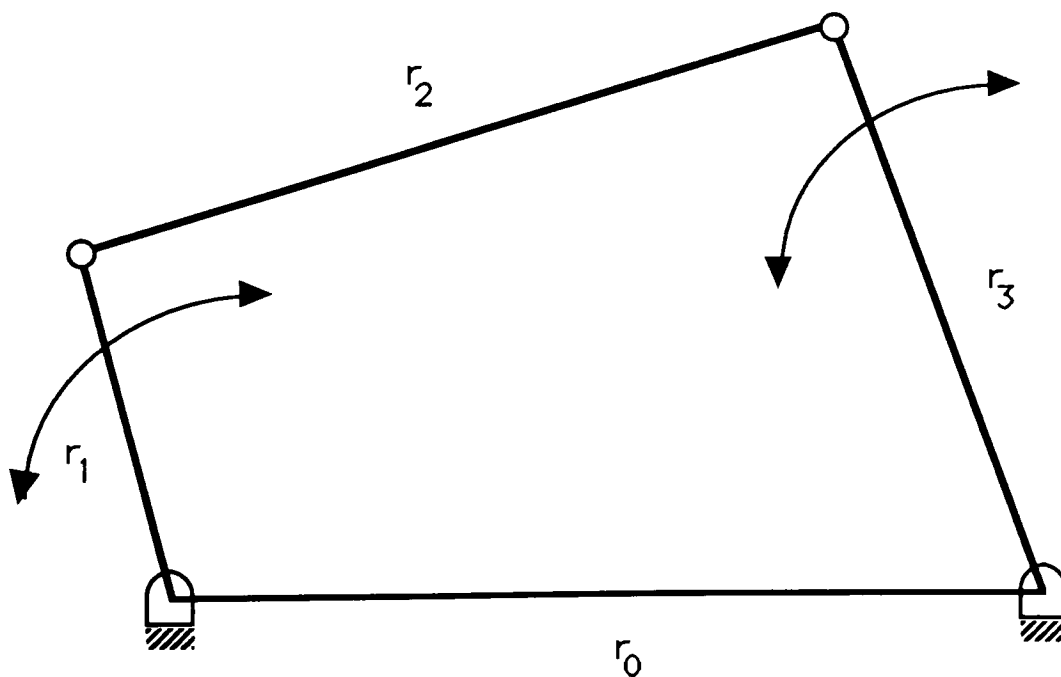
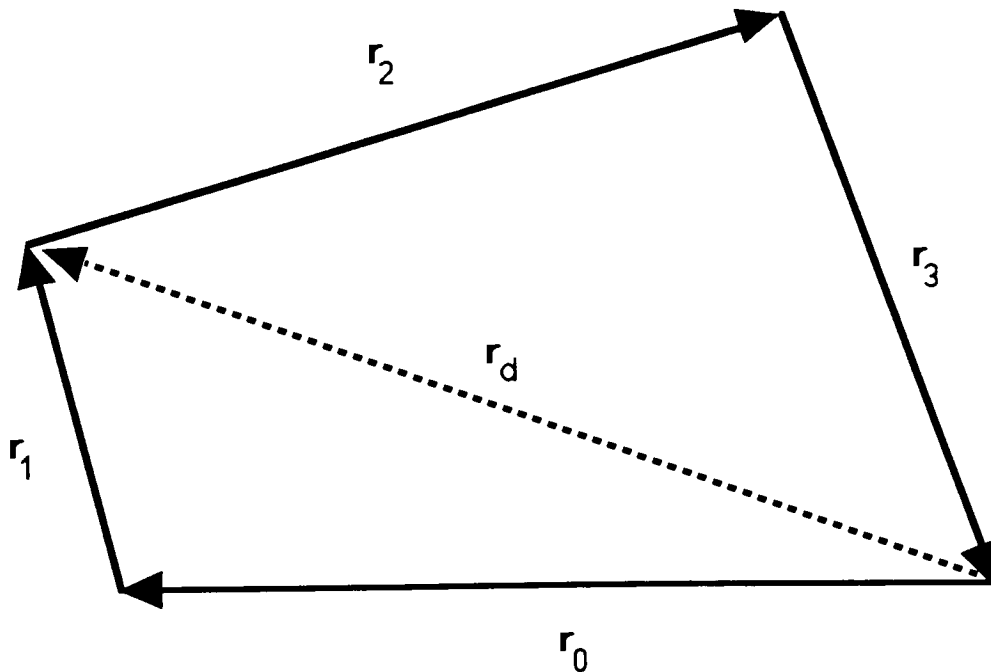


Figure 5.1 A Basic Four-Bar Planar Linkage

## 5. 2 The Four Bar Linkage

A graphical layout of a four bar linkage can be easily constructed. We only require to know the position of one link be given in relative to the frame and the link lengths. One such layout for the simplest four bar mechanism is given in Fig. 5.1. Analytical formulas are to be developed to determine all the link positions needed to write a computer program.

The following analysis provides an analytical solution for a simple mechanism shown as vector notations in Fig. 5.2 and this can be modified to suit the different kinds of four bar mechanisms in the later parts of this chapter. Note that there are two different modes of assembly possible for a non-Grashof mechanism. (Refer Chapter 3).



**Figure 5.2 Vector Representation of a Planar Four-Bar Linkage**

### 5. 2. 1 Position Analysis Using Vector Cross Product

Equations 5.3 and 5.4 may be used to find linkage displacements. These equations apply when directions of vectors **A** and **B** are unknown. The four bar planar linkage of Fig. 5.2 is described by the vector equation

$$\mathbf{r}_0 + \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = \mathbf{0}$$

or

$$\mathbf{r}_d = -(\mathbf{r}_2 + \mathbf{r}_3)$$

where the diagonal is given by

$$\mathbf{r}_d = \mathbf{r}_0 + \mathbf{r}_1$$

If the lengths of the links are specified and orientation of link 1 is given, then the following substitution may be made in equations 5.3 and 5.4:

$$\mathbf{A} = \mathbf{r}_2$$

$$\mathbf{B} = \mathbf{r}_3$$

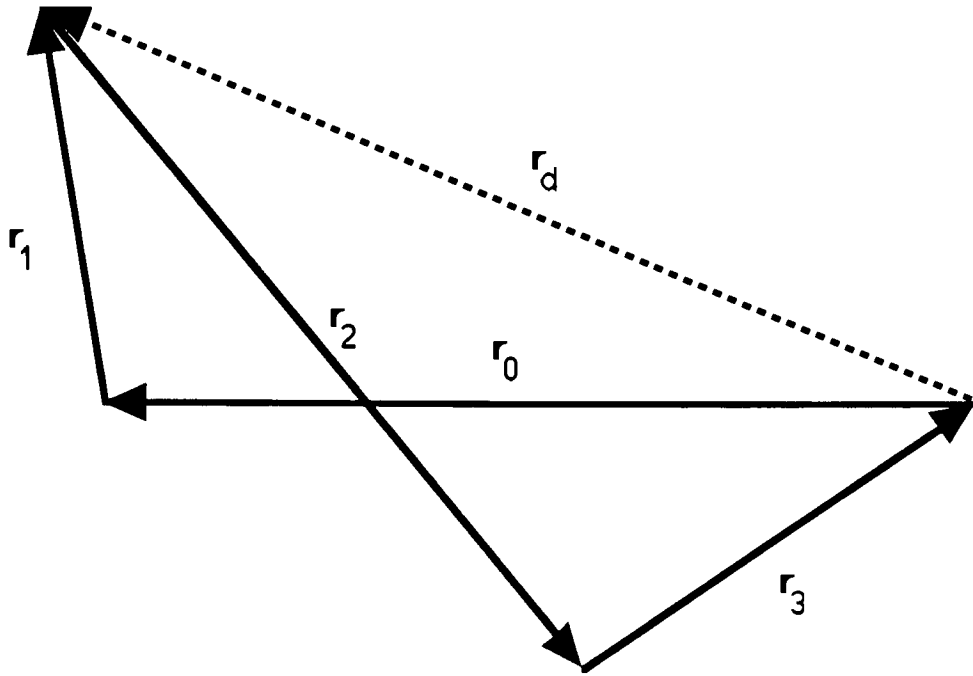
$$\mathbf{C} = \mathbf{r}_d = \mathbf{r}_0 + \mathbf{r}_1$$

yielding, if the linkage is assembled so that the vector loop  $\mathbf{r}_2\mathbf{r}_3\mathbf{r}_d$  is clockwise as in Fig. 5.2,

$$\begin{aligned} \mathbf{r}_2 = & -\{ r_3^2 - [(r_3^2 - r_2^2 + r_d^2) / (2 r_d)]^2 \}^{1/2} (\mathbf{r}_d^u \times \mathbf{k}) \\ & + [(r_3^2 - r_2^2 + r_d^2) / (2 r_d) - r_d] \mathbf{r}_d^u \end{aligned} \quad (5.7)$$

$$\begin{aligned} \mathbf{r}_3 = & +\{ r_3^2 - [(r_3^2 - r_2^2 + r_d^2) / (2 r_d)]^2 \}^{1/2} (\mathbf{r}_d^u \times \mathbf{k}) \\ & - [(r_3^2 - r_2^2 + r_d^2) / (2 r_d) - r_d] \mathbf{r}_d^u \end{aligned} \quad (5.8)$$

and if the vector loop  $\mathbf{r}_2\mathbf{r}_3\mathbf{r}_d$  is counterclockwise as in Fig. 5.3,



**Figure 5.3 Vector Representation of a Four-Bar Linkage  
(Alternative Mode of Assembly)**

$$\begin{aligned} \mathbf{r}_2 = & + \{ r_3^2 - [ ( r_3^2 - r_2^2 + r_d^2 ) / ( 2 r_d ) ]^2 \}^{1/2} ( \mathbf{r}_d^u \times \mathbf{k} ) \\ & + [ ( r_3^2 - r_2^2 + r_d^2 ) / ( 2 r_d ) - r_d ] \mathbf{r}_d^u \end{aligned} \quad (5.9)$$

$$\begin{aligned} \mathbf{r}_3 = & - \{ r_3^2 - [ ( r_3^2 - r_2^2 + r_d^2 ) / ( 2 r_d ) ]^2 \}^{1/2} ( \mathbf{r}_d^u \times \mathbf{k} ) \\ & - [ ( r_3^2 - r_2^2 + r_d^2 ) / ( 2 r_d ) - r_d ] \mathbf{r}_d^u \end{aligned} \quad (5.10)$$

In order to be used in a computer program, analytical formulas are to be developed for different link positions. These formulas can be developed using the concepts of vector operations and incorporating them.



## 6. EQUATION OF COUPLER CURVE OF A GENERIC FOUR BAR LINKAGE

In order to investigate algebraically the function generated by a coupler point, a generic equation of this coupler curve should be obtained. The equation of the coupler point curve for a four-bar linkage was first derived by Samuel Roberts by using analytic geometry. The equation will be written in Cartesian coordinates, with x axis along the line of centers  $O_A O_B$  and the y axis perpendicular to that line at  $O_A$  (See Fig. 6.1). Let  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x, y)$  be, respectively, the coordinates of points A, B and coupler point M; then

$$\begin{aligned} x_1 &= x - b \cos \theta & y_1 &= y - b \sin \theta \\ \text{and} \quad x_2 &= x - a \cos (\theta + \gamma) & y_2 &= y - a \sin (\theta + \gamma) \end{aligned} \quad (6.1)$$

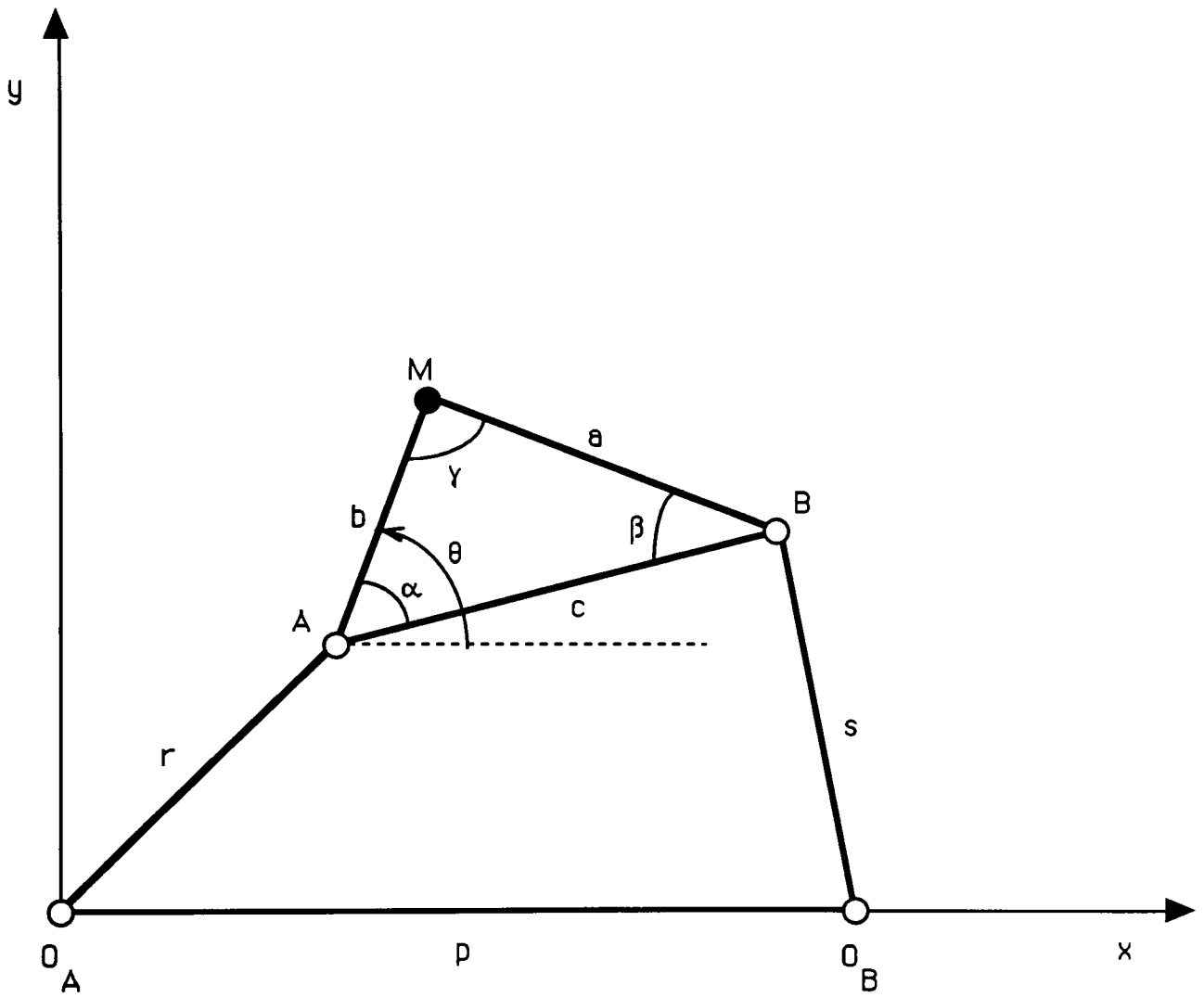
Since A and B describe circles (or arcs of circles) about centers  $O_A$  and  $O_B$ , respectively,

$$x_1^2 + y_1^2 = r^2 \quad \text{and} \quad (x_2 - p)^2 + y_2^2 = s^2 \quad (6.2)$$

Substituting the values of  $x_1, y_1$  and  $x_2, y_2$  into the last two equations yields,

$$\begin{aligned} (x - b \cos \theta)^2 + (y - b \sin \theta)^2 &= r^2 \\ \text{and} \quad [x - a \cos (\theta + \gamma) - p]^2 + [a \sin (\theta + \gamma)]^2 &= s^2 \end{aligned} \quad (6.3)$$

which, by application of trigonometric identities, ordering of terms and simplification, become:



**Figure 6.1 A Graphical Layout of a Generic Four-Bar Linkage**

$$x \cos \theta + y \sin \theta = (x^2 + y^2 + b^2 - r^2) / 2 b$$

$$\begin{aligned} \text{and} \quad & [ (x - p) \cos \gamma + y \sin \gamma ] \cos \theta - [ (x - p) \sin \gamma - y \cos \gamma ] \sin \theta \\ & = [(x - p)^2 + y^2 + a^2 - s^2] / 2 a \end{aligned} \quad (6.4)$$

The equation of the coupler-point curve may now be obtained by elimination of  $\theta$  between the last two equations. Solving these equations for  $\cos \theta$  and  $\sin \theta$  and substituting the values obtained into identity  $\cos^2 \theta + \sin^2 \theta = 1$  yields the general four-bar coupler curve equation:

$$\begin{aligned} & \{ \sin \alpha [(x - p) \sin \gamma - y \cos \gamma] (x^2 + y^2 + b^2 - r^2) \\ & \quad + y \sin \beta [(x - p)^2 + y^2 + a^2 - s^2] \}^2 \\ & + \{ \sin \alpha [(x - p) \cos \gamma + y \sin \gamma] (x^2 + y^2 + b^2 - r^2) \\ & \quad - x \sin \beta [(x - p)^2 + y^2 + a^2 - s^2] \}^2 \\ & = 4 k^2 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma [x(x - p) - y^2 - p y \cot \gamma]^2 \end{aligned} \quad (6.5)$$

In this,  $k$  is the constant of the sine law applied to the triangle ABM,

$$k = \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

This equation is of the sixth degree because one of its property is that a straight line will intersect it in no more than six points. In the following sections the equation of motion of specific four-bar mechanisms will be derived in a similar approach.

## **7. FOUR-BAR MECHANISMS THAT GENERATE SYMMETRICAL COUPLER CURVES**

Symmetrical curves generated by a four-bar mechanism have received a great deal of attention due to their wide applications. In this chapter conditions for a four-bar mechanism to produce a symmetrical coupler curve will be discussed. The following theorem and proof were presented by Berzak (Reference 8 and 9).

### **Symmetrical Coupler Curves**

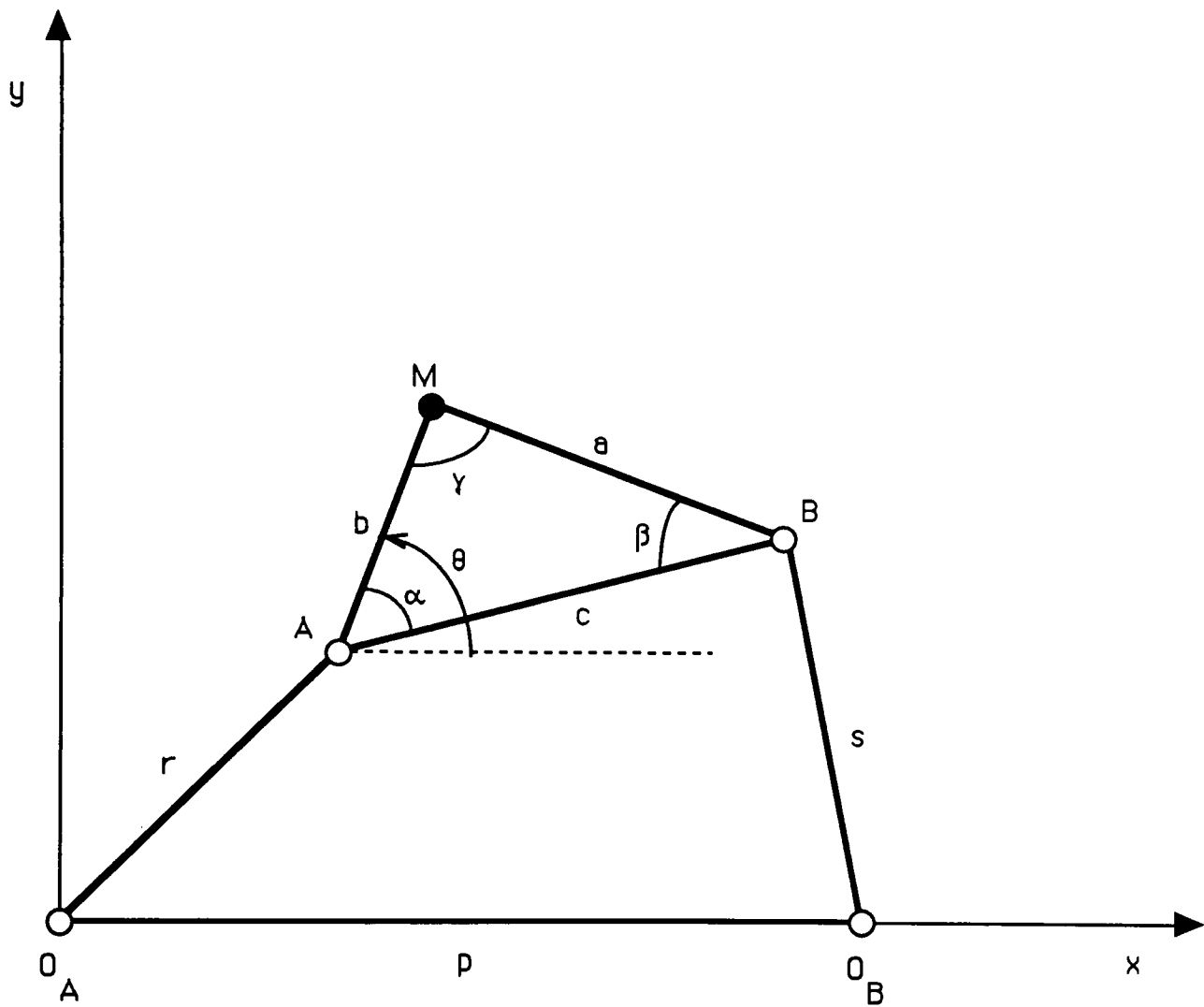
Let a symmetrical four-bar mechanism be defined as a four-bar mechanism for which the length of the crank and the length of the rocker are equal.

**Theorem 1:** In a symmetrical four-bar mechanism, any point in the coupler plane which lies on the perpendicular bisector of the coupler, generates a symmetrical curve, with the perpendicular bisector of the frame as a line of symmetry.

### **Proof of Theorem 1 Using Analytical Geometry**

The equation of the coupler curve generated by point M of the four bar mechanism shown in Fig. 7.1 is given by the equation 7.1. This equation was first derived by Samuel Roberts and is presented with slight modifications. The derivation of this equation is explained in Chapter 6.

The equation is written in Cartesian coordinates with origin at the center of rotation of the crank  $O_A$ , and x-axis along  $O_AO_B$ , where  $O_B$  is the center of rotation of rocker.



**Figure 7.1 Graphical Layout of a Four-Bar Mechanism**

$$\begin{aligned}
& \{ \sin \alpha [(x - p) \sin \gamma - y \cos \gamma] (x^2 + y^2 + b^2 - r^2) \\
& + y \sin \beta [(x - p)^2 + y^2 + a^2 - s^2] \}^2 \\
& + \{ \sin \alpha [(x - p) \cos \gamma + y \sin \gamma] (x^2 + y^2 + b^2 - r^2) \\
& - x \sin \beta [(x - p)^2 + y^2 + a^2 - s^2] \}^2 \\
& = 4 k^2 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma [x (x - p) - y^2 - p y \cot \gamma]^2
\end{aligned} \tag{7.1}$$

In this,  $k$  is the constant of the sine law applied to the triangle ABM,

$$k = \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

All angles in this are positive when measured in the counter clockwise direction. A special four-bar mechanism is shown in Fig. 7.2. For this mechanism:

$$r = s \tag{7.2}$$

In view of the choice of point M, triangle ABM is isosceles. Therefore:

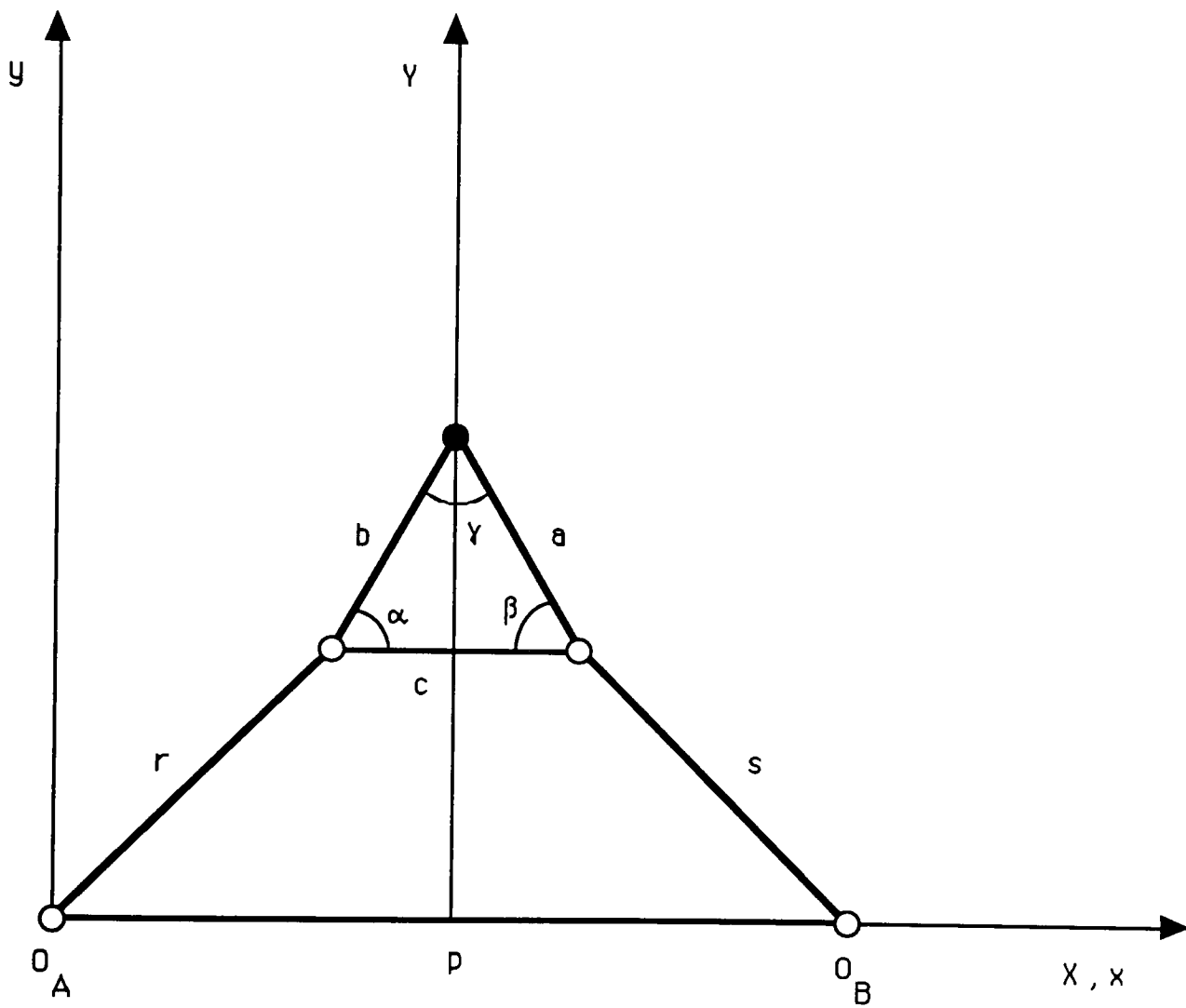
$$\begin{aligned}
& a = b \\
& \text{and } \alpha = \beta
\end{aligned} \tag{7.3}$$

From the condition that the sum of the angles in a triangle is  $180^\circ$  together with equation 7.3 the following conditions are obtained.

$$\sin \alpha = \sin 2\beta \tag{7.4a}$$

$$\cos \alpha = \cos 2\beta \tag{7.4b}$$

Substituting these into equation 7.1, we get



**Figure 7.2 A Symmetric Linkage**

$$\begin{aligned}
& \{ \sin \beta [ (x - p) \sin 2\beta + y \cos 2\beta ] (x^2 + y^2 + b^2 - r^2) \\
& + y \sin \beta [ (x - p)^2 + y^2 + b^2 - r^2 ]^2 \\
& + \{ \sin \beta [ (x - p) (-\cos 2\beta) + y \sin 2\beta ] (x^2 + y^2 + b^2 - r^2) \\
& - x \sin \beta [ (x - p)^2 + y^2 + b^2 - r^2 ]^2 \\
& = 4k^2 \sin^4 \beta \sin^2 2\beta [ x(x - p) - y^2 + p y \cot 2\beta ]^2
\end{aligned} \tag{7.5}$$

Now if we will transform the equation to a new X,Y coordinate system. The origin of this coordinate system is at the center of the frame. The X-axis coincides with the x-axis, and X,Y coordinate system is also right-handed.

The transformation equations are defined by the equations:

$$\begin{aligned}
x &= X + p/2 \\
\text{and } y &= Y
\end{aligned} \tag{7.6}$$

Substitution of these transformation into equation 7.5 results in,

$$\begin{aligned}
& X^6 \sin^2 2\beta + X^4 [ p^2 \sin^2 \beta (1 - 3 \cos^2 \beta) + 2 \sin^2 2\beta (b^2 - r^2) \\
& - 4 b^2 \sin^2 2\beta \sin^2 \beta ] - 2 X^4 Y p \sin^2 \beta \sin 2\beta + 3 X^4 Y^2 \sin^2 2\beta \\
& + X^2 [ (p^4 / 4) \sin^2 \beta (1 - 3 \sin^2 \beta) - 2 p^2 (b^2 - r^2) \sin^2 \beta + (b^2 - r^2)^2 \sin^2 2\beta \\
& + 2 b^2 p^2 \sin^2 \beta \sin^2 2\beta ] + X^2 Y \{ p \sin 2\beta [ \sin^2 \beta (p^2 - 4 b^2 + 4 r^2) \\
& - 8 b^2 \cos 2\beta \sin^2 \beta ] \} + X^2 Y^2 [ 2 p^2 \sin^4 \beta \\
& - 8 b^2 \sin^2 \beta \sin^2 2\beta + 4 (b^2 - r^2) \sin^2 2\beta ] - 4 X^2 Y^3 p \sin 2\beta \sin^2 \beta \\
& + 3 X^2 Y^4 \sin^2 2\beta + Y^6 \sin^2 2\beta - 2 Y^5 p \sin 2\beta \sin^2 \beta \\
& + Y^4 [ 2 \sin^2 2\beta (b^2 - r^2) + p^2 \sin^2 \beta (1 + \cos^2 \beta) - 4 b^2 \sin^2 \beta \sin^2 2\beta ] \\
& + Y^3 [ p \sin 2\beta \sin^2 \beta (-p^2 - 4 b^2 + 4 r^2) - 8 b^2 p \sin 2\beta \sin^2 \beta \cos 2\beta ]
\end{aligned}$$



$$\begin{aligned}
& + Y^2 [ ( p^4 / 4 ) \sin^2\beta ( 1 + \sin^2\beta + ( b^2 - r^2 )^2 \sin^2 2\beta + 2 p^2 ( b^2 - r^2 ) \sin^2\beta \\
& - 4 b^2 p^2 \sin^2\beta ( 1 - 6 \cos^2\beta + 6 \cos^4\beta ) ] \\
& + Y \{ ( p / 2 ) \sin 2\beta \sin^2\beta [ - p^4 / 4 - 4 ( b^2 - r^2 )^2 - 2 p^2 ( b^2 - r^2 ) ] \\
& + 2 b^2 p^3 \sin 2\beta \cos 2\beta \sin^2\beta \} \\
& + p^2 \sin^4\beta [ ( p^4 / 16 ) + ( b^2 - r^2 )^2 + ( p^2 / 2 ) ( b^2 - r^2 ) ] \\
& - ( 1 / 4 ) b^2 p^4 \sin^2 2\beta \sin^2\beta = 0
\end{aligned} \tag{7.7}$$

A close examination of the above equation will show that all coefficients of odd powers of  $X$  are zero, which follows that the  $Y$ -axis coincides with the line of symmetry of the coupler curve.

The following theorem and proof are presented by Hartenberg and Denavit, in Reference 1.

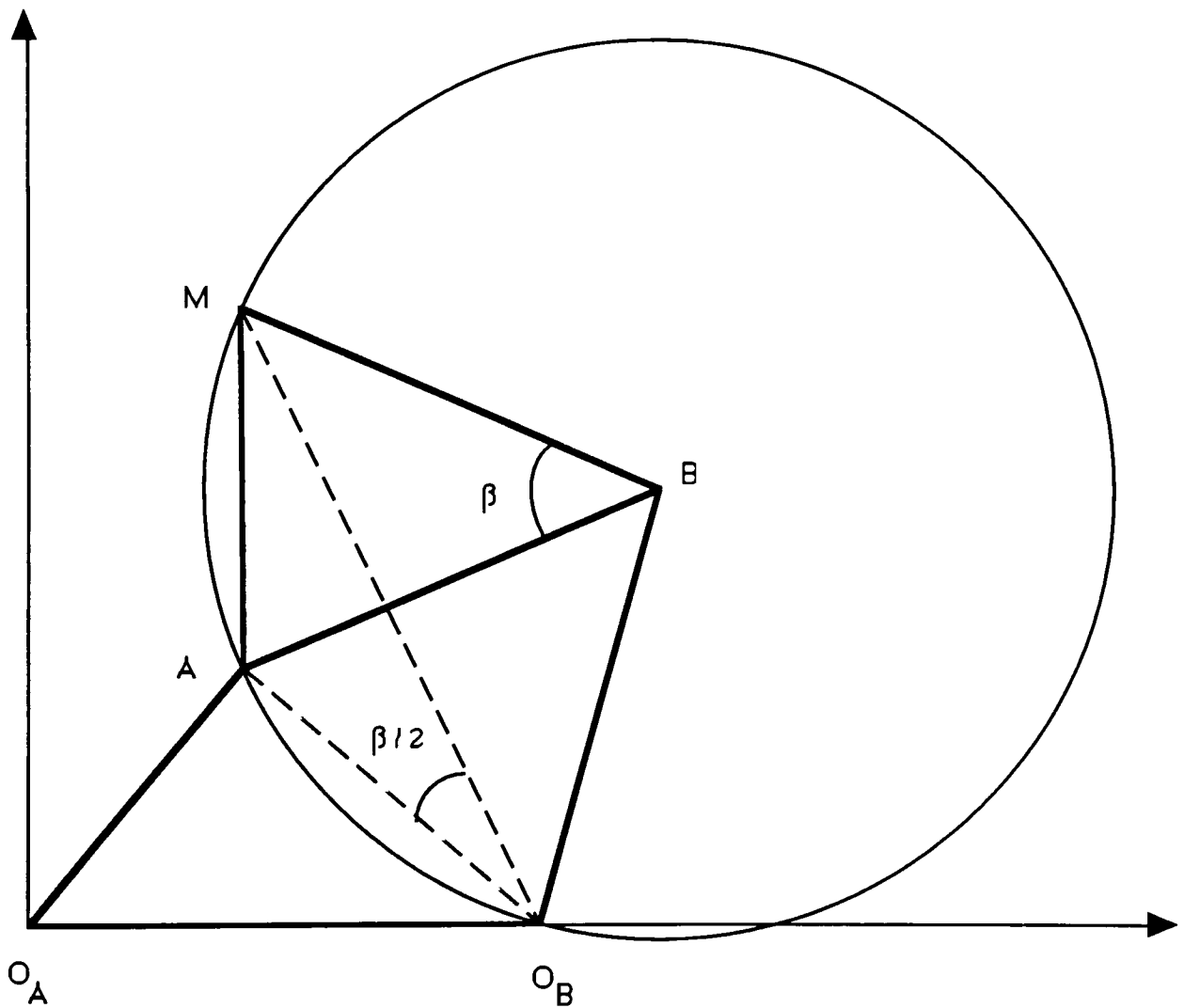
**Theorem 2: Coupler curves that are symmetrical about an axis may be generated by a four bar linkage with a coupler base  $AB$  and follower  $O_B B$  of equal length. The coupler point must lie anywhere on the circle centered at  $B$  and passing through  $A$  (refer figure 7.3).**

Since  $BO_B = BA = BM$ , the above circle also passes through  $O_B$  and the inscribed angle  $AO_B M$  satisfies the relation

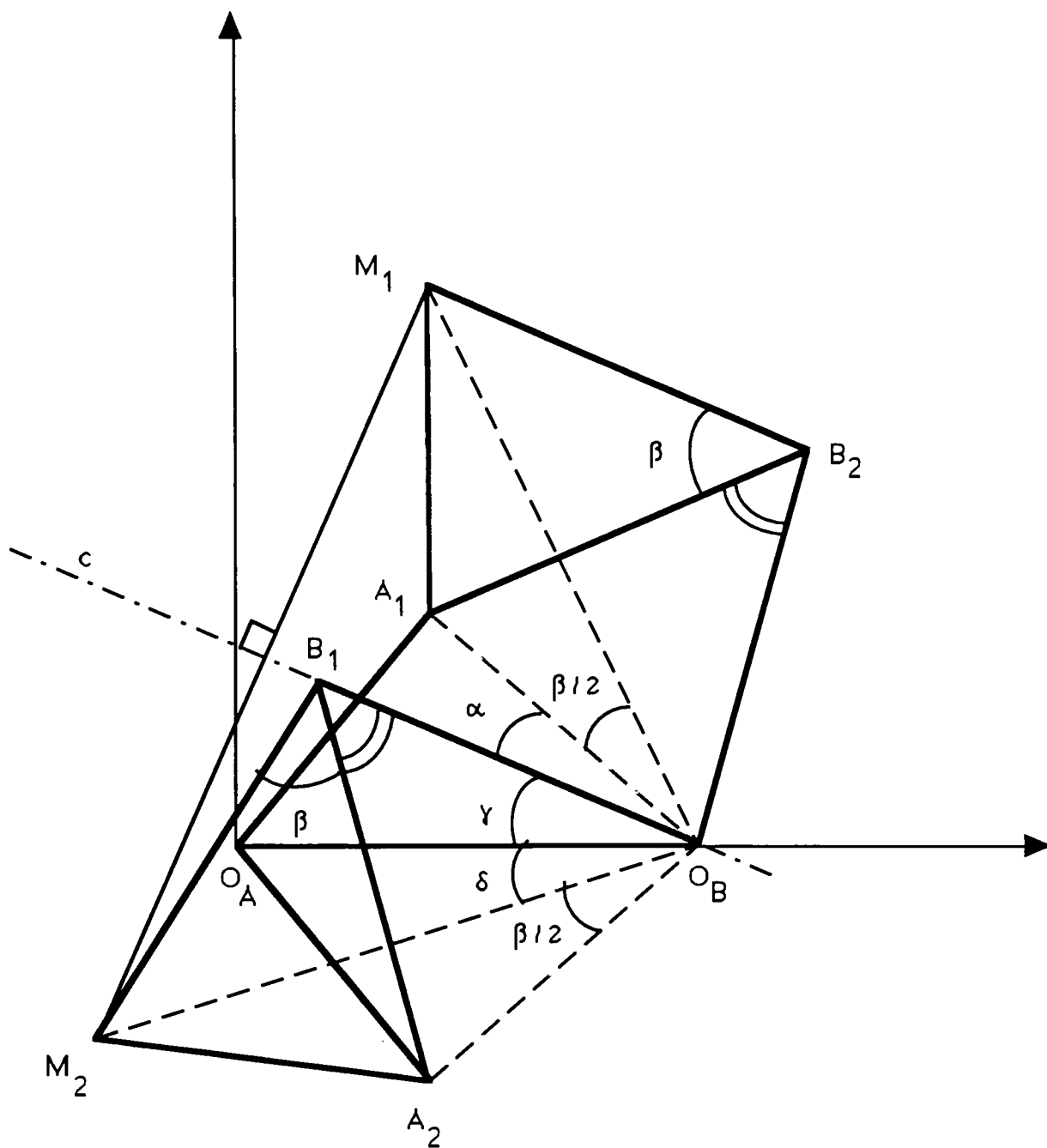
$$\angle AO_B M = \angle ABM / 2 = \beta / 2 = \text{constant.} \tag{7.8}$$

Consider now the linkage in two positions  $O_A A_1 B_1 O_B$  and  $O_A A_2 B_2 O_B$  for which points  $A_1$  and  $A_2$  are symmetrical with respect to the line of fixed centers  $O_A O_B$  (Figure 7.4). For these positions, triangles  $O_B A_1 B_1$  and  $O_B A_2 B_2$  are equal, since

corresponding sides are equal, whence  $\beta_1 = \beta_2$ .



**Figure 7.3 A Four-Bar Mechanism That Generates Symmetrical Coupler Curves**



**Figure 7.4 Two Positions of a Four-Bar Corresponding to Symmetrical Points  $M_1$  and  $M_2$  on the Coupler Curve**

Now, the isosceles triangles  $O_BB_1M_1$  and  $O_BB_2M_2$  are equal,

$$O_B M_1 = O_B M_2 \quad (7.9)$$

and the midnormal  $c$  to  $M_1M_2$  passes through  $O_B$  and bisects the angle  $M_1O_B M_2$ , whence

$$\delta + \gamma = \alpha + \beta / 2 \quad (7.10)$$

Since the angles  $O_A O_B A_1$  and  $O_A O_B A_2$  are also equal ( $A_1$  symmetric to  $A_2$  with respect to  $O_A O_B$ ),

$$\gamma + \alpha = \beta / 2 + \delta \quad (7.11)$$

Adding the last two equations yields

$$\gamma = \beta / 2 \quad (7.12)$$

The midnormal  $c$  to  $M_1M_2$ , therefore makes a constant angle with the line of fixed centers  $O_A O_B$ , whence it is an axis of symmetry for the coupler curve generated by point  $M$ . It should further be noted that symmetric points on the coupler curve corresponds to symmetric positions of the crank with respect to the line of fixed centers  $O_A O_B$ .

## 8. ANALYSIS OF FOUR-BAR LINKAGES GENERATING STRAIGHT LINE COUPLER CURVE

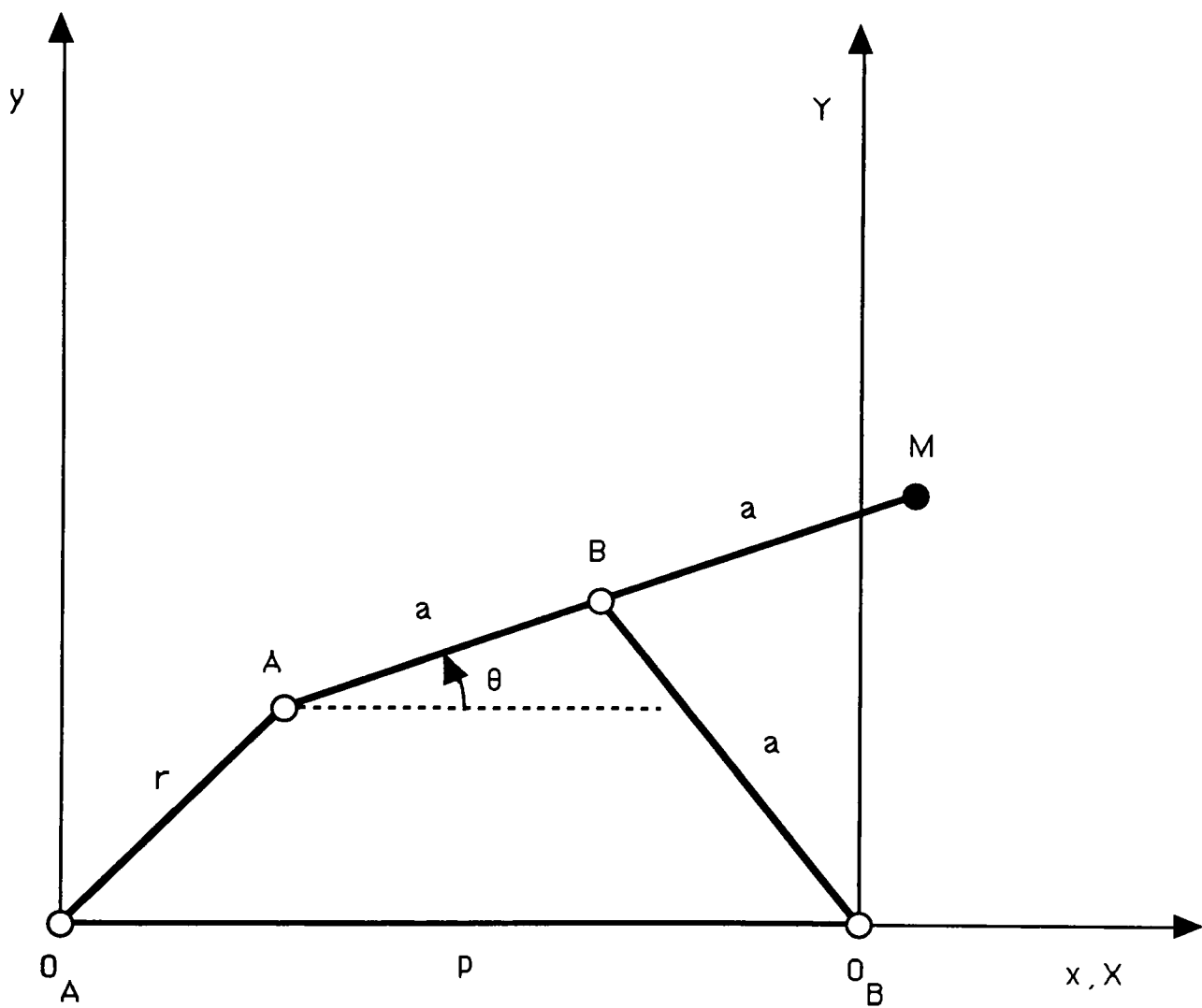
When James Watt built a steam engine in 1769, he had to design a linkage to guide a pin along a straight line path, since there were no machine tools at that time capable of producing straight metal slides of sufficient precision. Since then straight line mechanisms have found numerous applications.

Straight line mechanisms are not of historic interest alone, however, since four bar mechanisms have their own advantages over the conventional slides. Many four-bar mechanisms that generate approximate straight line motion, have been designed in the past. Although they have proven to be very useful for practical design problems, these “ready-made” straight line motions only occasionally fit the design problems in practice; in most cases the designer must develop straight line motion to suit his/her particular needs.

This chapter derives the necessary condition to obtain a straight line output from four bar mechanisms, in order to aid the designer in designing his/her mechanism for straight line motion to fit his/her needs.

### 8. 1 Evans Linkage

A schematic layout of symmetrical Evans linkage is shown in Fig. 8.1. From Chapter 7, we are now aware that for a symmetric coupler curve in the standard Evans linkage (refer Fig. 4.1) the values of  $s$ ,  $a$  and  $b$  must be equal and the Fig. 8.1. is shown as such. The x-axis is drawn along the frame  $O_A O_B$  and the y-axis perpendicular at  $O_A$ . Let  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x, y)$  be respectively, the coordinates of points A, B and the coupler point M; then



**Figure 8.1 Graphical Layout of Symmetrical Evans Linkage**

$$\begin{aligned} x_1 &= x - 2a \cos \theta & y_1 &= y - 2a \sin \theta \\ \text{and } x_2 &= x - a \cos \theta & y_2 &= y - a \sin \theta \end{aligned}$$

Since A and B describe circles (or arcs of circles) about centers  $O_A$  and  $O_B$ , respectively,

$$x_1^2 + y_1^2 = r^2 \quad \text{and} \quad (x_2 - p)^2 + y_2^2 = a^2$$

Substituting the values of  $x_1, y_1$  and  $x_2, y_2$  in these two equations gives,

$$\begin{aligned} (x - 2a \cos \theta)^2 + (y - 2a \sin \theta)^2 &= r^2 \\ \text{and } (x - a \cos \theta - p)^2 + (y - a \sin \theta)^2 &= a^2 \end{aligned}$$

Expanding and solving these two equations for  $\cos \theta$  and  $\sin \theta$  yields,

$$\begin{aligned} \cos \theta &= \frac{4a^2 - r^2 + x^2 + y^2}{4ap} - \frac{p^2 - 2px + x^2 + y^2}{2ap} \\ \sin \theta &= \frac{(p - x)(4a^2 - r^2 + x^2 + y^2)}{4apy} - \frac{x(p^2 - 2px + x^2 + y^2)}{2apy} \end{aligned}$$

Substituting these values in the trigonometric identity  $\cos^2 \theta + \sin^2 \theta = 1$ , and simplifying we get the equation of the coupler curve:

$$\begin{aligned} 16a^4p^2 - 8a^2p^2r^2 + p^2r^4 - 32a^4px + 16a^2p^3x + 16a^2pr^2x - \\ 4p^3r^2x - 2p^4x + 16a^4x^2 - 40a^2p^2x^2 + 4p^4x^2 - 8a^2r^2x^2 + \end{aligned}$$

$$\begin{aligned}
& 10 p^2 r^2 x^2 + r^4 x^2 + 32 a^2 p x^3 - 12 p^3 x^3 - 8 p r^2 x^3 - 8 a^2 x^4 + 13 p^2 x^4 + \\
& 2 r^2 x^4 - 6 p x^5 + x^6 + 16 a^4 y^2 - 24 a^2 p^2 y^2 + 4 p^4 y^2 - 8 a^2 r^2 y^2 + \\
& 2 p^2 r^2 y^2 + r^4 y^2 + 32 a^2 p x y^2 - 12 p^3 x y^2 - 8 p r^2 x y^2 - 16 a^2 x^2 y^2 + \\
& 18 p^2 x^2 y^2 + 4 r^2 x^2 y^2 - 12 p x^3 y^2 + 3 x^4 y^2 - 8 a^2 y^4 + 5 p^2 y^4 + 2 r^2 y^4 - \\
& 6 p x y^4 + 3 x^2 y^4 + y^6 = 0
\end{aligned} \tag{8.1}$$

Now, to find the coordinates of the points where the output curve would meet the Y axis, substituting  $x = p$  equation 8.1 and simplifying results in,

$$y^2 (-4 a^2 + p^2 - 2 p r + r^2 + y^2) (-4 a^2 + p^2 + 2 p r + r^2 + y^2) = 0 \tag{8.2}$$

whose non-zero, non-negative solutions of  $y$  are,

$$y = \{ (4 a^2 - p^2 + 2 p r - r^2)^{1/2}, (4 a^2 - p^2 - 2 p r - r^2)^{1/2} \} \tag{8.3}$$

Differentiating the equation of coupler curve with respect to  $x$  yields ( $y' = dy/dx$ ),

$$\begin{aligned}
& -32 a^4 p + 16 a^2 p^3 + 16 a^2 p r^2 - 4 p^3 r^2 - 2 p r^4 + 32 a^4 x - 80 a^2 p^2 x + \\
& 8 p^4 x - 16 a^2 r^2 x + 20 p^2 r^2 x + 2 r^4 x + 96 a^2 p x^2 - 36 p^3 x^2 - 24 p r^2 x^2 - \\
& 32 a^2 x^3 + 52 p^2 x^3 + 8 r^2 x^3 - 30 p x^4 + 6 x^5 + 32 a^2 p y^2 - 12 p^3 y^2 - \\
& 8 p r^2 y^2 - 32 a^2 x y^2 + 36 p^2 x y^2 + 8 r^2 x y^2 - 36 p x y^2 + 12 x^3 y^2 - 6 p y^4 + \\
& 6 x y^4 + y' (32 a^4 y - 48 a^2 p^2 y + 8 p^4 y - 16 a^2 r^2 y + 4 p^2 r^2 y + 2 r^4 y + \\
& 64 a^2 p x y - 24 p^3 x y - 16 p r^2 x y - 32 a^2 x^2 y + 36 p^2 x^2 y + 8 r^2 x^2 y - \\
& 24 p x^3 y + 6 x^4 y - 32 a^2 y^3 + 20 p^2 y^3 + 8 r^2 y^3 - 24 p x y^3 + 12 x^2 y^3 + \\
& 6 y^5) = 0
\end{aligned} \tag{8.4}$$



Substituting the two values of  $y$  from equation 8.3 in equation 8.4 successively and solving for  $dy/dx$  results in,

$$dy/dx = 0, \quad (8.5)$$

which means the slope of the coupler curve at both the points it crosses the axis of symmetry (Y axis) is zero. From calculus, we know that the formula for radius of curvature of the curve  $f(x, y) = 0$  at any point is,

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} \quad (8.6)$$

Therefore, for a straight line the radius of curvature must be infinity which will happen only if the second derivative of  $y$  ( $d^2y/dx^2$ ) is equal to zero at that point. Differentiating equation (8.4) again with respect to  $x$  gives ( $y'' = d^2y/dx^2$ ),

$$\begin{aligned} & 32 a^4 - 80 a^2 p^2 + 8 p^4 - 16 a^2 r^2 + 20 p^2 r^2 + 2 r^4 + 192 a^2 p x - 72 p^3 x - \\ & 48 p r^2 x - 96 a^2 x^2 + 156 p^2 x^2 + 24 r^2 x^2 - 120 p x^3 + 30 x^4 - 32 a^2 y^2 + \\ & 36 p^2 y^2 + 8 r^2 y^2 - 72 p x y^2 + 36 x^2 y^2 + 6 y^4 + 2 y' ( 64 a^2 p y - 24 p^3 y - \\ & 16 p r^2 y - 64 a^2 x y + 72 p^2 x y + 16 r^2 x y - 72 p x^2 y + 24 x^3 y - 24 p y^3 + \\ & 24 x y^3 ) + y'^2 ( 32 a^4 - 48 a^2 p^2 + 8 p^4 - 16 a^2 r^2 + 4 p^2 r^2 + 2 r^4 + \\ & 64 a^2 p x - 24 p^3 x - 16 p r^2 x - 32 a^2 x^2 + 36 p^2 x^2 + 8 r^2 x^2 - 24 p x^3 + \\ & 6 x^4 - 96 a^2 y^2 + 60 p^2 y^2 + 24 r^2 y^2 - 72 p x y^2 + 36 x^2 y^2 + 30 y^4 ) + \\ & y'' ( 32 a^4 y - 48 a^2 p^2 y + 8 p^4 y - 16 a^2 r^2 y + 4 p^2 r^2 y + 2 r^4 y + \\ & 64 a^2 p x y - 24 p^3 x y - 16 p r^2 x y - 32 a^2 x^2 y + 36 p^2 x^2 y + 8 r^2 x^2 y - \\ & 24 p x^3 y + 6 x^4 y - 32 a^2 y^3 + 20 p^2 y^3 + 8 r^2 y^3 - 24 p x y^3 + 12 x^2 y^3 + 6 y^5 ) \\ & = 0 \end{aligned} \quad (8.7)$$

So for us to get a condition for a straight line coupler curve, substituting the values of x and y in the equation 8.7 and setting  $d^2y/dx^2 = 0$  and simplifying we get,

$$\begin{aligned} 8p(p^3 - 4a^2r + 3p^2r + 3pr^2 + r^3) &= 0 \\ \Rightarrow (p^3 - 4a^2r + 3p^2r + 3pr^2 + r^3) &= 0 \end{aligned} \quad (8.8)$$

since p cannot be equal to zero. These dimensions are unique only to their ratio. So let us set the value of r as 1 and solve for a,

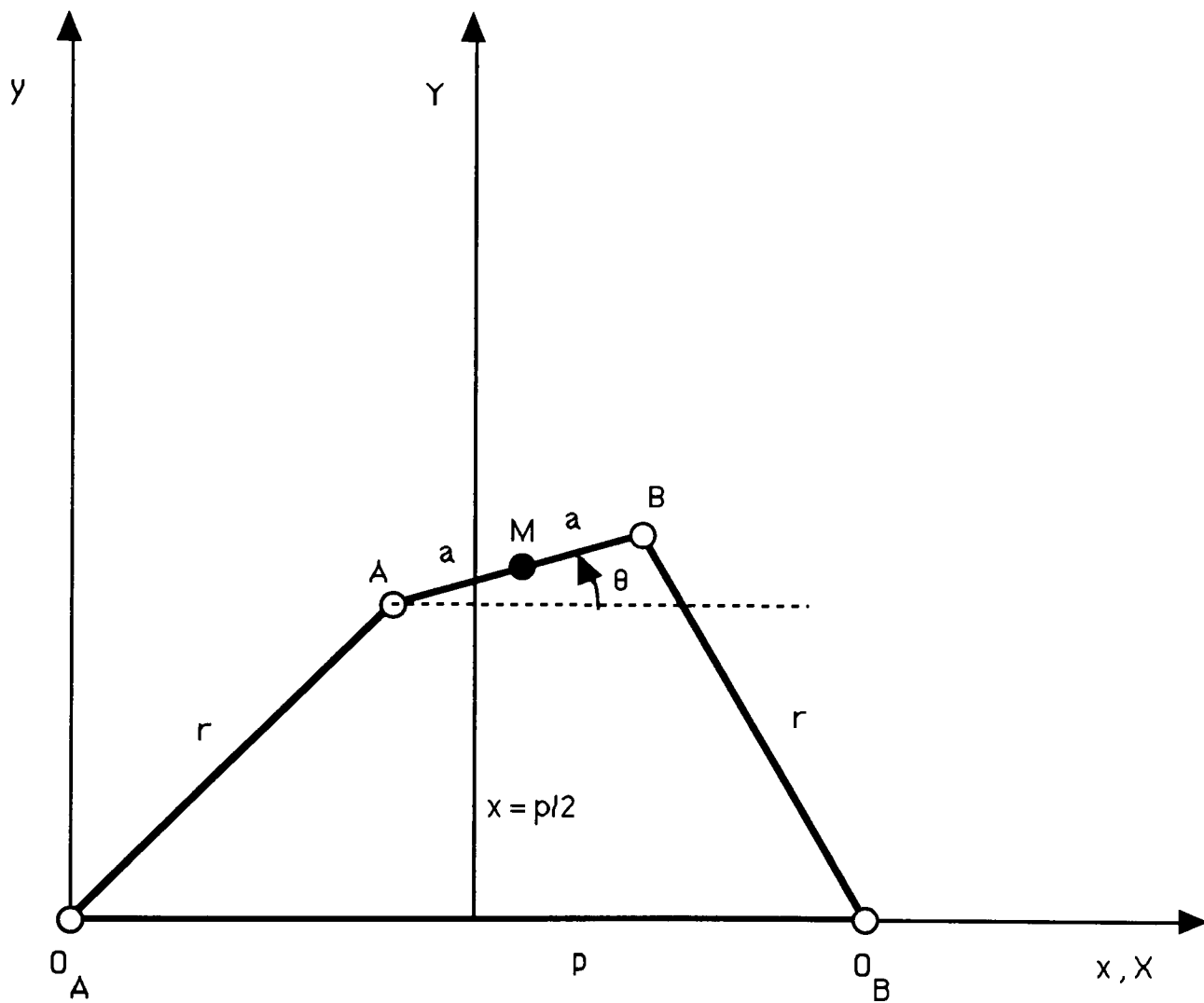
$$a = \pm \frac{[1 + 3p + 3p^2 + p^3]^{1/2}}{2} \quad (8.9)$$

Disregarding the negative value, and simplifying the condition for a straight line coupler curve for a symmetrical Evans Linkage is given by,

$$a = \frac{(1 + p)^{3/2}}{2} \quad (8.10)$$

## 8. 2 Chebyshev's Linkage

Although Chebyshev and Watts linkages may fall into different categories, their modes of assembly are identical (refer chapter 4) which shows their equation of the coupler curve would be the same. Fig. 8.2 shows a generic symmetric Chebyshev and Watts Linkage. Note that these are symmetric about a vertical line,  $x = p/2$  (Y axis) which is essential to produce a symmetric coupler curve as explained in Chapter 7. This line would be the line of symmetry for the coupler curve as well.



**Figure 8.2 Graphical Layout of Symmetrical Chebyshev Linkage**

The equation of the coupler curve is obtained as follows: From Fig. 8.2 the values of coordinates of points A (  $x_1$  ,  $y_1$  ) and B (  $x_2$  ,  $y_2$  ) from M (  $x$  ,  $y$  ) can be written as

$$\begin{aligned}x_1 &= x - a \cos \theta & y_1 &= y - a \sin \theta \\x_2 &= x + a \cos \theta & y_2 &= y + a \sin \theta\end{aligned}$$

The points A and B follows in a circle around centers  $O_A$  and  $O_B$ , respectively, which means

$$x_1^2 + y_1^2 = r^2 \quad \text{and} \quad (x_2 - p)^2 + y_2^2 = r^2$$

From the values of  $x_1, y_1$  and  $x_2, y_2$  in terms of  $x$  and  $\theta$ , these equations become,

$$\begin{aligned}(x - a \cos \theta)^2 + (y - a \sin \theta)^2 &= r^2 \\(x + a \cos \theta - p)^2 + (y + a \sin \theta)^2 &= r^2\end{aligned}$$

From these the solutions for  $\cos \theta$  and  $\sin \theta$  are found to be:

$$\begin{aligned}\cos \theta &= \frac{4 a^2 - r^2 + x^2 + y^2}{4 a p} - \frac{p^2 - 2 p x + x^2 + y^2}{4 a p} \\ \sin \theta &= \frac{(p - x) (4 a^2 - r^2 + x^2 + y^2)}{4 a p y} - \frac{x (p^2 - 2 p x + x^2 + y^2)}{2 a p y}\end{aligned}$$

Substituting these in the trigonometric identity  $\cos^2 \theta + \sin^2 \theta = 1$  as before, we obtain the coupler curve equation,

$$\begin{aligned}
& a^4 p^2 - 2 a^2 p^2 r^2 + p^2 r^4 - 4 a^4 p x - 2 a^2 p^3 x + 8 a^2 p r^2 x + 2 p^3 r^2 x - \\
& 4 p r^4 x + 4 a^4 x^2 + 10 a^2 p^2 x^2 + p^4 x^2 - 8 a^2 r^2 x^2 - 10 p^2 r^2 x^2 + 4 r^4 x^2 - \\
& 16 a^2 p x^3 - 6 p^3 x^3 + 16 p r^2 x^3 + 8 a^2 x^4 + 13 p^2 x^4 - 8 r^2 x^4 - 12 p x^5 + \\
& 4 x^6 + 4 a^4 y^2 + 2 a^2 p^2 y^2 + p^4 y^2 - 8 a^2 r^2 y^2 - 6 p^2 r^2 y^2 + 4 r^4 y^2 - \\
& 16 a^2 p x y^2 - 6 p^3 x y^2 + 16 p r^2 x y^2 + 16 a^2 x^2 y^2 + 18 p^2 x^2 y^2 - \\
& 16 r^2 x^2 y^2 - 24 p x^3 y^2 + 12 x^4 y^2 + 8 a^2 y^4 + 5 p^2 y^4 - 8 r^2 y^4 - 12 p x y^4 + \\
& 12 x^2 y^4 + 4 y^6 = 0
\end{aligned} \tag{8.11}$$

Equation 8.11 is the equation of the coupler point curve for symmetric Chebyshev and Watts linkages. From this equation the non-zero, non negative values of  $y$  at  $x = p/2$  are,

$$y = \left\{ \frac{\sqrt{-4 a^2 + 4 a p - p^2 + 4 r^2}}{2}, \frac{\sqrt{-4 a^2 - 4 a p - p^2 + 4 r^2}}{2} \right\} \tag{8.12}$$

Differentiating the equation of the coupler curve 8.11 with respect to  $x$  yields (  $dy/dx$  is denoted as  $y'$  ),

$$\begin{aligned}
& -4 a^4 p - 2 a^2 p^3 + 8 a^2 p r^2 + 2 p^3 r^2 - 4 p r^4 + 8 a^4 x + 20 a^2 p^2 x + 2 p^4 x - \\
& 16 a^2 r^2 x - 20 p^2 r^2 x + 8 r^4 x - 48 a^2 p x^2 - 18 p^3 x^2 + 48 p r^2 x^2 + 32 a^2 x^3 + \\
& 52 p^2 x^3 - 32 r^2 x^3 - 60 p x^4 + 24 x^5 - 16 a^2 p y^2 - 6 p^3 y^2 + 16 p r^2 y^2 + \\
& 32 a^2 x y^2 + 36 p^2 x y^2 - 32 r^2 x y^2 - 72 p x^2 y^2 + 48 x^3 y^2 - 12 p y^4 + \\
& 24 x y^4 + y' ( 8 a^4 y + 4 a^2 p^2 y + 2 p^4 y - 16 a^2 r^2 y - 12 p^2 r^2 y + 8 r^4 y - \\
& 32 a^2 p x y - 12 p^3 x y + 32 p r^2 x y + 32 a^2 x^2 y + 36 p^2 x^2 y - 32 r^2 x^2 y - \\
& 48 p x^3 y + 24 x^4 y + 32 a^2 y^3 + 20 p^2 y^3 - 32 r^2 y^3 - 48 p x y^3 + 48 x^2 y^3 + \\
& 24 y^5 ) = 0
\end{aligned} \tag{8.13}$$

Substituting the two sets of values of  $x$  and  $y$  from equation 8.12 in equation 8.13 gives the solution to  $dy/dx$  as,

$$dy/dx = 0 \quad (8.14)$$

for both points, which means the slope of the curve at both of these points are zero.

For the coupler point curve to be a straight line at any point, the radius of curvature at that point is to be infinity, which is possible only when  $d^2y/dx^2 = 0$ . When differentiated again equation 8.13 becomes (  $d^2y/dx^2 = y''$  ),

$$\begin{aligned} & 8a^4 + 20a^2p^2 + 2p^4 - 16a^2r^2 - 20p^2r^2 + 8r^4 - 96a^2px - 36p^3x + \\ & 96pr^2x + 96a^2x^2 + 156p^2x^2 - 96r^2x^2 - 240px^3 + 120x^4 + 32a^2y^2 + \\ & 36p^2y^2 - 32r^2y^2 - 144pxy^2 + 144x^2y^2 + 24y^4 + 2y' ( -32a^2py - \\ & 12p^3y + 32pr^2y + 64a^2xy + 72p^2xy - 64r^2xy - 144px^2y + 96x^3y - \\ & 48py^3 + 96xy^3 ) + y'^2 ( 8a^4 + 4a^2p^2 + 2p^4 - 16a^2r^2 - 12p^2r^2 + 8r^4 - \\ & 32a^2px - 12p^3x + 32pr^2x + 32a^2x^2 + 36p^2x^2 - 32r^2x^2 - 48px^3 + \\ & 24x^4 + 96a^2y^2 + 60p^2y^2 - 96r^2y^2 - 144pxy^2 + 144x^2y^2 + 120y^4 ) + \\ & y'' ( 8a^4y + 4a^2p^2y + 2p^4y - 16a^2r^2y - 12p^2r^2y + 8r^4y - 32a^2pxy - \\ & 24x^4y + 32a^2y^3 + 20p^2y^3 - 32r^2y^3 - 48pxy^3 + 48x^2y^3 + 24y^5 ) = 0 \end{aligned} \quad (8.14)$$

In the above equation, setting  $d^2y/dx^2 = 0$  at the points described by equation 8.12, results in

$$\begin{aligned} & 2p ( 8a^3 + 12a^2p + 6ap^2 + p^3 - 8ar^2 ) = 0 \\ \text{or} \quad & ( 8a^3 + 12a^2p + 6ap^2 + p^3 - 8ar^2 ) = 0 \end{aligned} \quad (8.16)$$

In order to make these dimensions true to its ratio, we set the length of the coupler (which from the Fig. 8.2 is equal to '2a') as 1, which is done by giving the value of 'a' as 1/2.

$$1 + 3p + 3p^2 + p^3 - 4r^2 = 0 \quad (8.17)$$

From which the condition for a symmetric Chebyshev or Watts linkage for a straight line coupler curve can be stated as:

$$r = \frac{\sqrt{1 + 3p + 3p^2 + p^3}}{2} \quad (8.18)$$

which can be rewritten as,

$$r = \frac{(1 + p)^{3/2}}{2} \quad (8.19)$$

Note that this condition is identical to that for the Evans linkage.

## 9. NUMERICAL GENERATION OF COUPLER POINT CURVES

If a mechanism is to be examined for its output, it is necessary to obtain all the positions of the links to assure proper design, for which graphical methods of analysis are convenient. Computers are excellent tools for graphical examination. This chapter will discuss methods to generate coupler point curves for different types of mechanisms.

Chapter 5 has given the solutions to the vector equations to the basic four bar equations. If these equations to be used in a computer program to generate the coupler curve, subroutines are necessary to convert the vector quantities to scalar using the concept of vector operations. From these equations for the simple four-bar mechanism, algorithms for specific mechanisms can be derived by making some modifications.

### 9. 1 Conversion of Vector Solutions to Scalar

The vector representation of a simple four-bar mechanism is shown in Fig. 9.1. The equations for the value of  $r_2$  obtained from the vector analysis in section 5.2.1 (Equation 5.7) can be decomposed into x and y coordinates as

$$r_{2x} = (r_3^2 - a^2)^{1/2} r_{d2} + (a - r_d) r_{d1} \quad (9.1a)$$

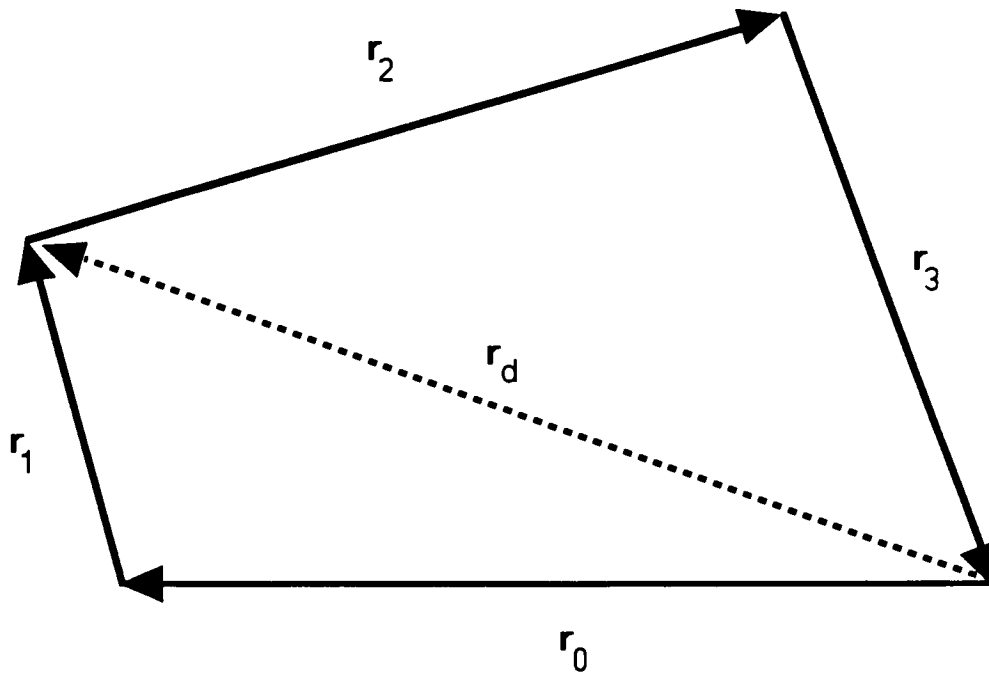
$$r_{2y} = - (r_3^2 - a^2)^{1/2} r_{d1} + (a - r_d) r_{d2} \quad (9.1b)$$

$$\text{where} \quad a = (r_3^2 - r_2^2 + r_d^2) / 2 r_d \quad (9.1c)$$

$$r_{d1} = (r_1 \cos \theta - r_0) / r_d \quad (9.1d)$$

$$\text{and} \quad r_{d2} = (r_1 \sin \theta) / r_d \quad (9.1e)$$





**Figure 9.1 Vector Representation of a Planar Four-Bar Linkage**

Again if the vector loop  $r_2 r_3 r_d$  is counterclockwise as shown in Fig. 9.2, the x and y components of the solution to vector  $r_2$  given by equation 5.9 are,

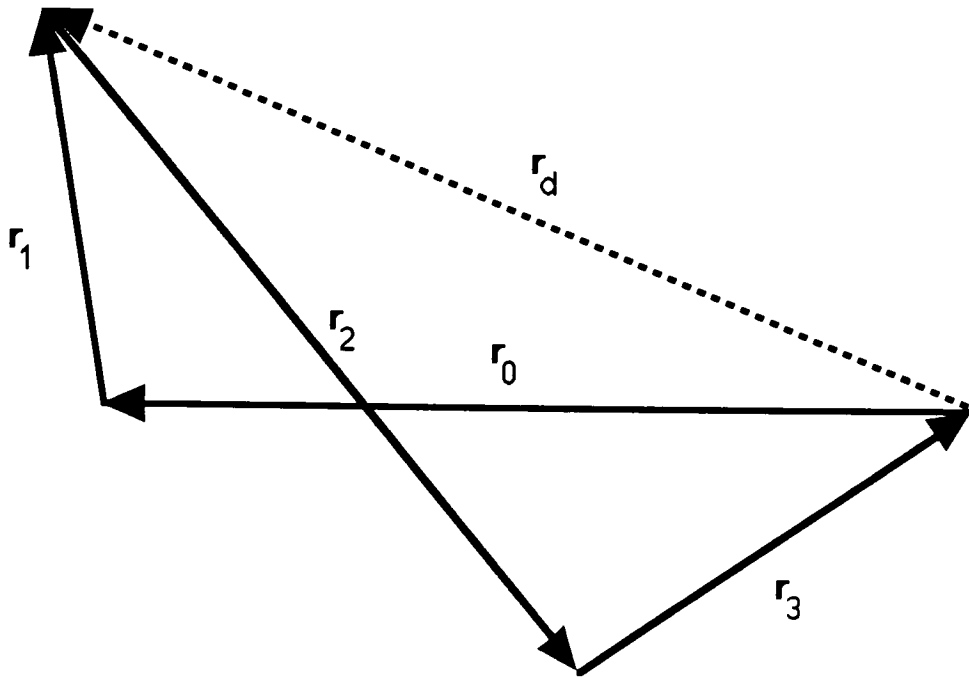
$$r_{2x} = - (r_3^2 - a^2)^{1/2} r_{d2} + (a - r_d) r_{d1} \quad (9.2a)$$

$$r_{2y} = (r_3^2 - a^2)^{1/2} r_{d1} + (a - r_d) r_{d2} \quad (9.2b)$$

where 
$$a = (r_3^2 - r_2^2 + r_d^2) / 2 r_d \quad (9.2c)$$

$$r_{d1} = (r_1 \cos \theta - r_0) / r_d \quad (9.2d)$$

and 
$$r_{d2} = (r_1 \sin \theta) / r_d \quad (9.2e)$$



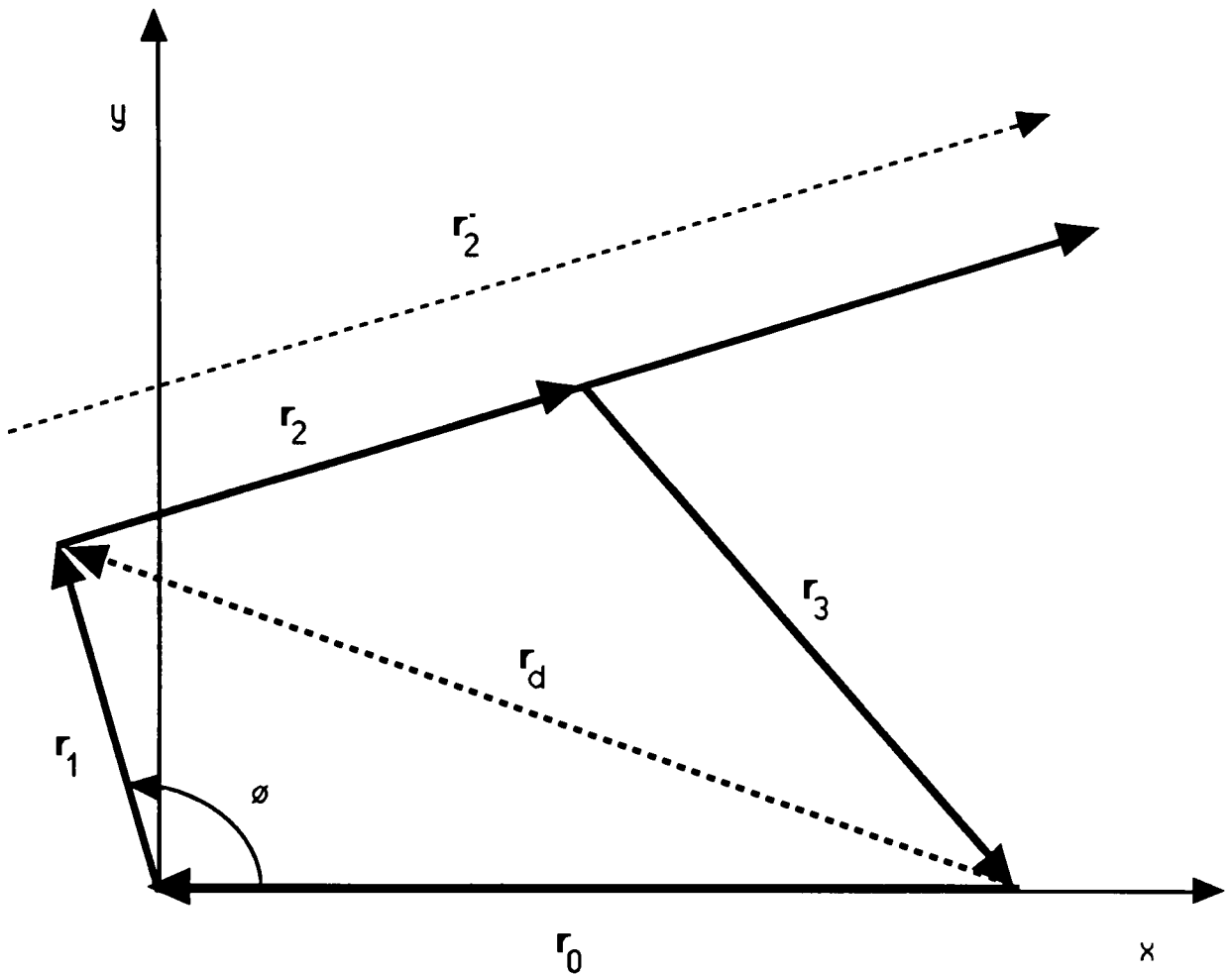
**Figure 9.2 Vector Representation of a Four-Bar Linkage  
(Alternative Mode of Assembly)**

### 9. 2. 1 Evans Linkage

A vector representation of the Evans Linkage is shown in Fig. 9.3. Evans linkage is obtained from the simple four-bar mechanism by simply extending the coupler (represented by the vector  $r_2$ ). Therefore the formula for the components of  $r'_2$  in figure can be derived from equations 9.1 as

$$r'_{2x} = (r'_2 / r_2) [ (r_3^2 - a^2)^{1/2} r_{d2} + (a - r_d) r_{d1} ] \quad (9.3a)$$

$$r'_{2y} = (r'_2 / r_2) [ - (r_3^2 - a^2)^{1/2} r_{d1} + (a - r_d) r_{d2} ] \quad (9.3b)$$



**Figure 9.3 Vector Representation of Evans Linkage**

From which the coordinates of the output point M (x,y) are obtained as

$$x = r'_2x + r_1 \cos \phi \quad (9.4a)$$

and  $y = r'_2y + r_1 \sin \phi \quad (9.4b)$

The coupler point curve for Evans linkage is obtained by substituting the value of  $\phi$  from  $0^\circ$  to  $360^\circ$  in small increments and plotting the positions of the coupler point successively. Appendix B gives the the computer program that utilizes this algorithm

and plots the output curve for a given set of dimensions of the links. Appendix A gives some samples of plots thus obtained.

### 9. 2. 2 Chebyshev Linkage

A Chebyshev linkage is a crank-rocker mechanism in which the coupler rotates through  $360^\circ$ . A schematic representation of the Chebyshev's mechanism is shown in Fig. 9.4. Since the output point M may lie anywhere on the coupler, the value of the

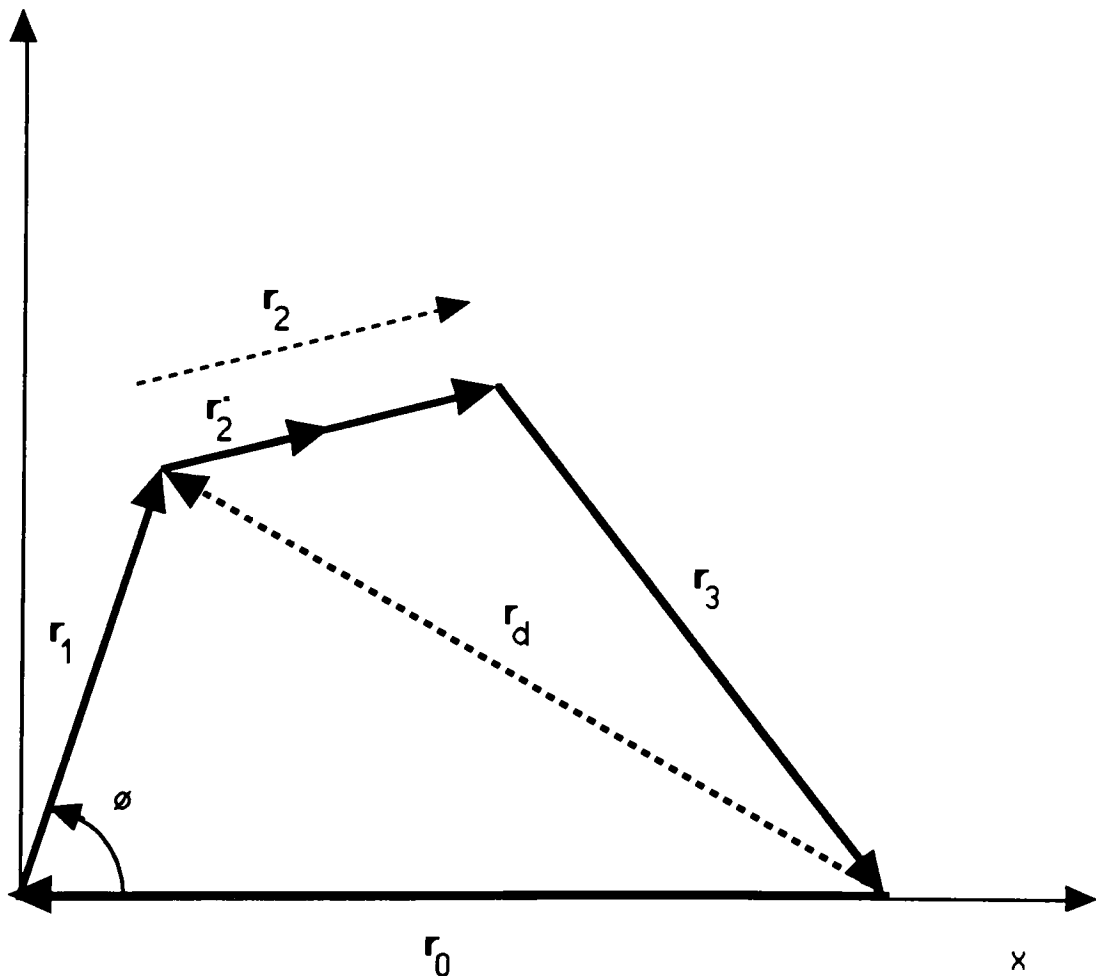


Figure 9.4 Vector Representation of Chebyshev Linkage

vector  $\mathbf{r}'_2$  is unchanged from that of  $\mathbf{r}_2$  given by equation 9.11 except for its magnitude. The components of vector  $\mathbf{r}'_2$  is calculated as follows:

$$r'_{2x} = (r'_2 / r_2) [ (r_3^2 - a^2)^{1/2} r_{d2} + (a - r_d) r_d ] \quad (9.5a)$$

$$r'_{2y} = (r'_2 / r_2) [ - (r_3^2 - a^2)^{1/2} r_{d1} + (a - r_d) r_{d2} ] \quad (9.5b)$$

The position vector of the point can be obtained by the vector addition of vectors  $\mathbf{r}_1$  and  $\mathbf{r}'_2$ .

$$x = r'_{2x} + r_1 \cos \varnothing \quad (9.6a)$$

$$\text{and } y = r'_{2y} + r_1 \sin \varnothing \quad (9.6b)$$

The output curve can be obtained by finding the position for small increments of  $\varnothing$  between  $\varnothing_{\max}$  and  $\varnothing_{\min}$  using. And in the same way the positions are calculated for the second half rotation of the coupler using,

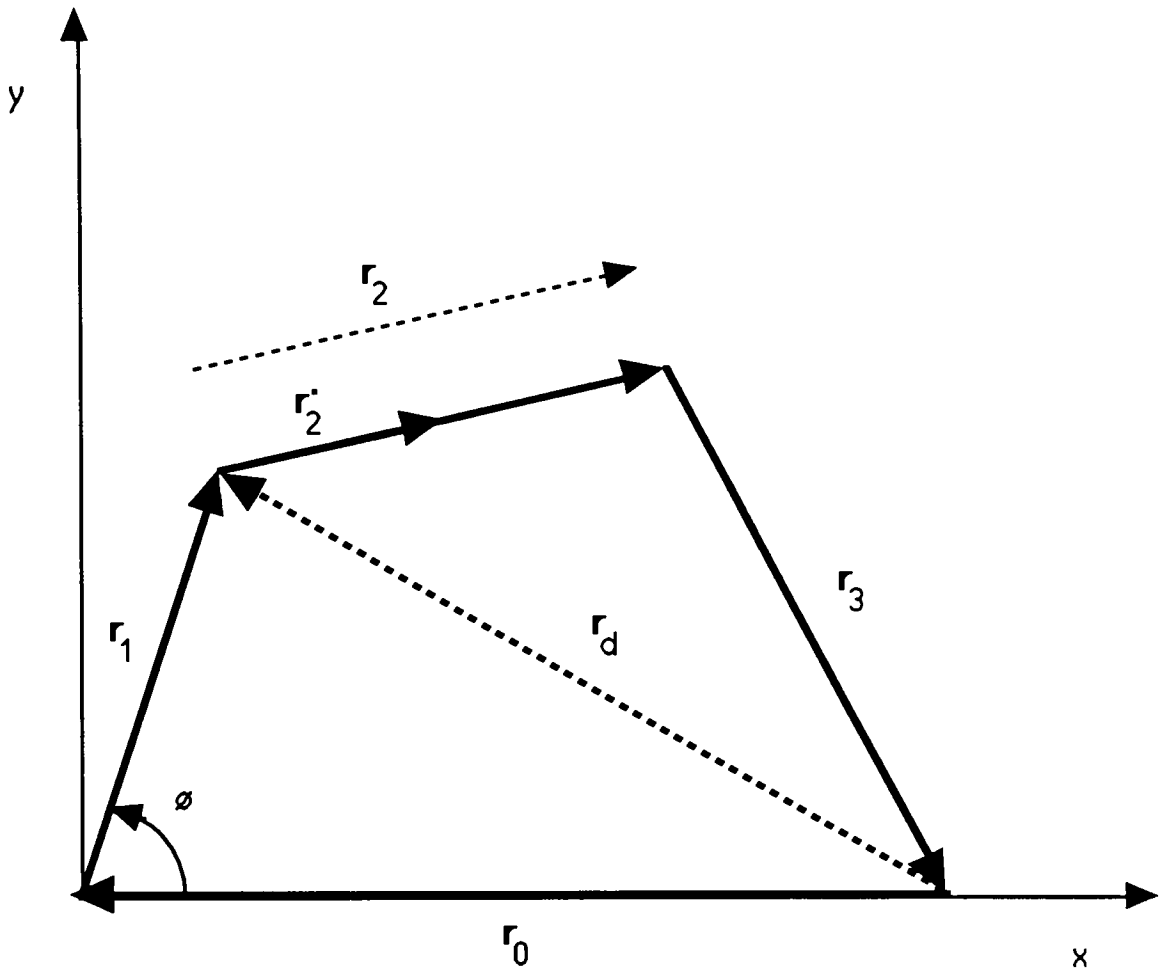
$$r'_{2x} = (r'_2 / r_2) [ - (r_3^2 - a^2)^{1/2} r_{d2} + (a - r_d) r_{d1} ] \quad (9.7a)$$

$$r'_{2y} = (r'_2 / r_2) [ (r_3^2 - a^2)^{1/2} r_{d1} + (a - r_d) r_{d2} ] \quad (9.7b)$$

Chapter 4 explains these limiting angles  $\varnothing_{\max}$  and  $\varnothing_{\min}$ , and how to calculate them. A computer code developed using the above algorithm to plot the coupler point curves for Chebyshev Linkage is given in Appendix B and Appendix A contains the plots obtained from the program.

### 9. 2. 3 Watts Linkage

Although Watts linkage looks similar to Chebyshev Linkage, it differs from it in the sense that none of the links rotates through  $360^\circ$ . The vector representation of the watts linkage is shown in Fig. 9.5. The procedure for generation of the coupler curve



**Figure 9.5 Vector Representation of Watts Linkage**

for Watts linkage is done the same way as that of Chebyshev's except that its limiting angles are different. These limiting angles are explained in Chapter 4.

For the first half of the coupler curve, the positions for every small increments of angles between  $\phi_{\max}$  and  $\phi_{\min}$  are calculated using,

$$r'_{2x} = (r'_2 / r_2) [ (r_3^2 - a^2)^{1/2} r_{d2} + (a - r_d) r_{d1} ] \quad (9.8a)$$

$$r'_{2y} = (r'_2 / r_2) [ - (r_3^2 - a^2)^{1/2} r_{d1} + (a - r_d) r_{d2} ] \quad (9.8b)$$

And for the second half using

$$r'_{2x} = (r'_2 / r_2) [ - (r_3^2 - a^2)^{1/2} r_{d2} + (a - r_d) r_{d1} ] \quad (9.9a)$$

$$r'_{2y} = (r'_2 / r_2) [ (r_3^2 - a^2)^{1/2} r_{d1} + (a - r_d) r_{d2} ] \quad (9.9b)$$

where  $a$ ,  $r_{d1}$  and  $r_{d2}$  are calculated as before and the position vector of the point can be obtained by the vector addition of vectors  $\mathbf{r}_1$  and  $\mathbf{r}'_2$ .

$$x = r'_{2x} + r_1 \cos \varnothing \quad (9.10a)$$

$$\text{and } y = r'_{2y} + r_1 \sin \varnothing \quad (9.10b)$$

Appendix A contains some samples of the plots generated using computers and the codes required to generate those plots can be found in Appendix B.

#### 9. 2. 4 Roberts Linkage

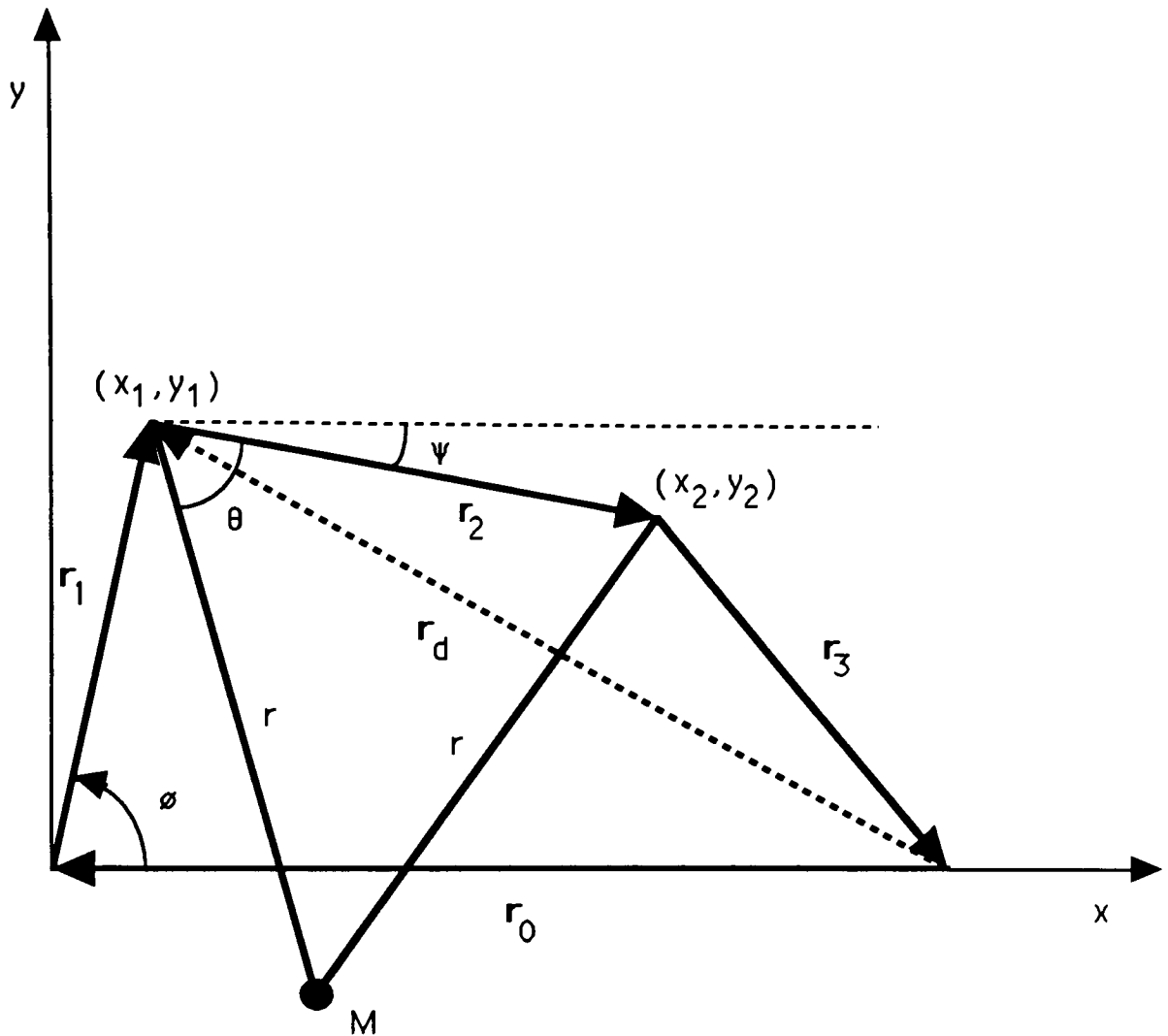
The vector representation of a symmetrical Roberts Linkage is shown in Fig. 9.6. As we have seen already, the  $x$  and  $y$  components of the vector  $\mathbf{r}_2$  can be calculated from equations 9.1a through 9.1e. The coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$  are calculated as follows,

$$x_1 = r_1 \cos \varnothing \quad (9.11a)$$

$$y_1 = r_1 \sin \varnothing \quad (9.11b)$$

$$x_2 = x_1 + r'_{2x} \quad (9.12a)$$

$$y_2 = y_1 + r'_{2y} \quad (9.12b)$$



**Figure 9.6 Vector Layout of Roberts Linkage**

To find the coordinates of the point  $M(x,y)$ , from Figure 9.6, by the cosine law,

$$r^2 = r_2^2 + r^2 - 2 r r_2 \cos \theta$$

$$\Rightarrow \theta = \cos^{-1} [r_2 / 2 r]$$
(9.13)

and the angle

$$\psi = \sin^{-1} [(y_1 - y_2) / r_2]$$
(9.14)



Finally the coordinates of the output point M are obtained by,

$$x = x_1 + r \cos ( \theta + \psi ) \quad (9.15a)$$

$$\text{and } y = y_1 - r \sin ( \theta + \psi ) \quad (9.15b)$$

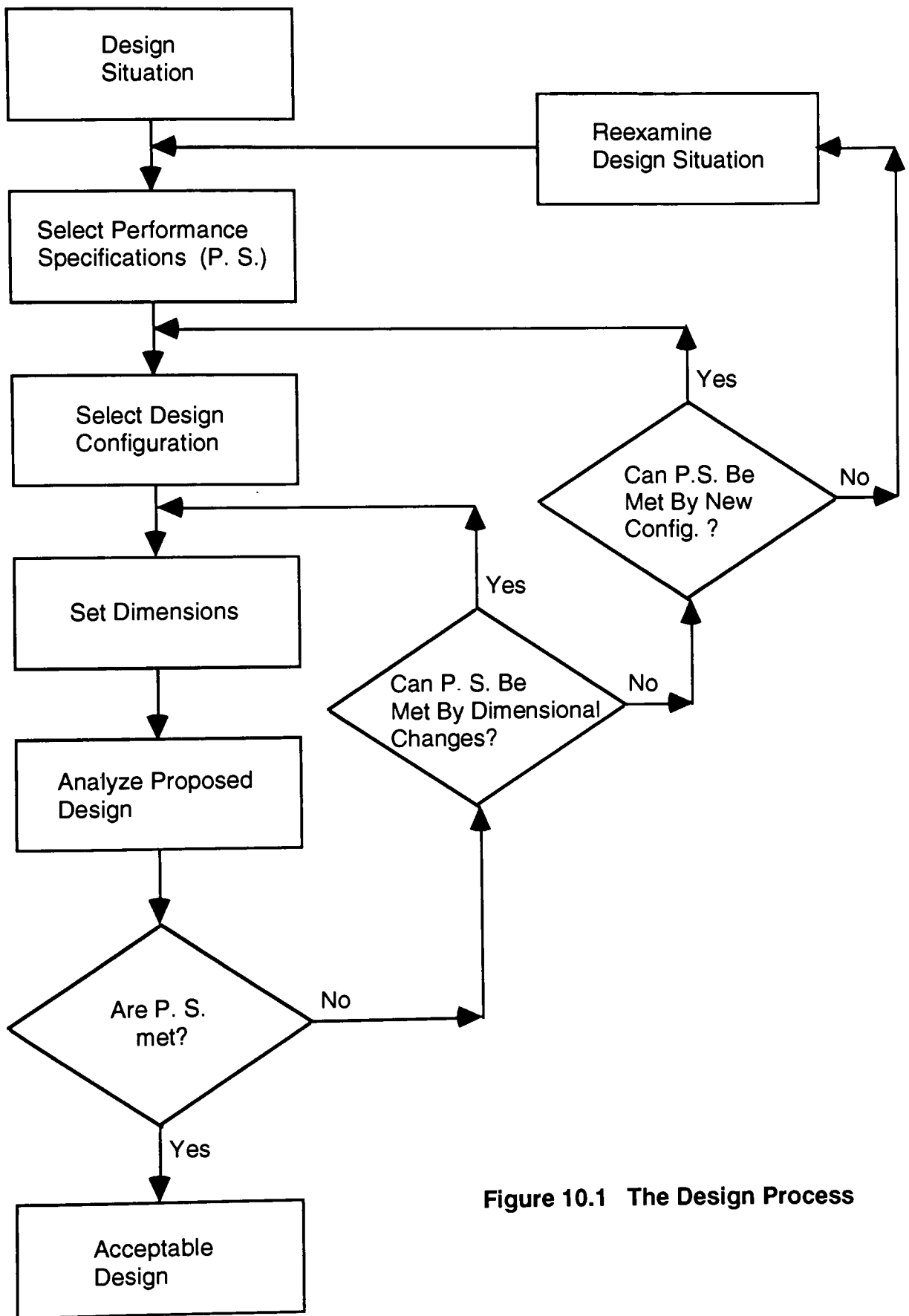
These coordinates of the coupler point can be found for small intervals of  $\theta$  between  $\theta_{\min}$  and  $\theta_{\max}$  (see Chapter 4 on how to calculate them). The computer generated plots and codes for Roberts Linkage are given Appendix A and Appendix B respectively.

## 10. SYNTHESIS PROCEDURE FOR DESIGNING A FOUR-BAR LINKAGE TO GENERATE STRAIGHT LINE COUPLER CURVE

In the linkages studied, conditions have been developed for a mechanism to generate a straight line and numerical and geometrical methods to study these outputs are also discussed in detail. However it is a different task to start with a required straight line motion and to try to proportion a mechanism to produce this motion. This procedure is known as the *synthesis of mechanisms*. In the application of synthesis to design of a mechanism , the problem divides itself into two parts (a) the type of mechanism to be used, (b) the proportions and lengths of the links necessary.

Analysis, as applied to mechanisms, implies that given the dimensions of a linkage it is required to find its motion characteristics. Synthesis will be the exact opposite; the motion characteristics are specified and it is needed to find the linkage that will produce the specified motion. These design of linkages for specific applications relies heavily on human judgment and ingenuity. This design process may be illustrated by the flow chart given in Figure 10.1. This process of human interaction includes creativity and possibly mathematical analysis and computation for which the dimensional requirements that have been derived so far and the computer codes that have been developed will be of immense help. While it is unlikely that human creativity can be completely replaced, these tools and knowledge can be employed to relieve the designer of many of the routine processes that would be otherwise necessary.

This chapter gives the designer a synthesis procedure in order to help him/her to select and design a four-bar linkage that will meet the design criteria and will provide means to check its performance, and save some laborious trial and error processes.



**Figure 10.1 The Design Process**

## **10. 1 Kinematic Synthesis**

As it was said earlier the synthesis of a four-bar linkage requires the designer to reach decisions on (1) the form or type of mechanism, (2) the proportions (lengths) of the links necessary to accomplish the specified motion transformation. The first phase is called the *type synthesis*. Here the choice of the linkage mechanism is determined. The other phase is *dimensional synthesis*, presents challenging problems, to which the greater part of this work is addressed.

### **10. 1. 1 Type synthesis**

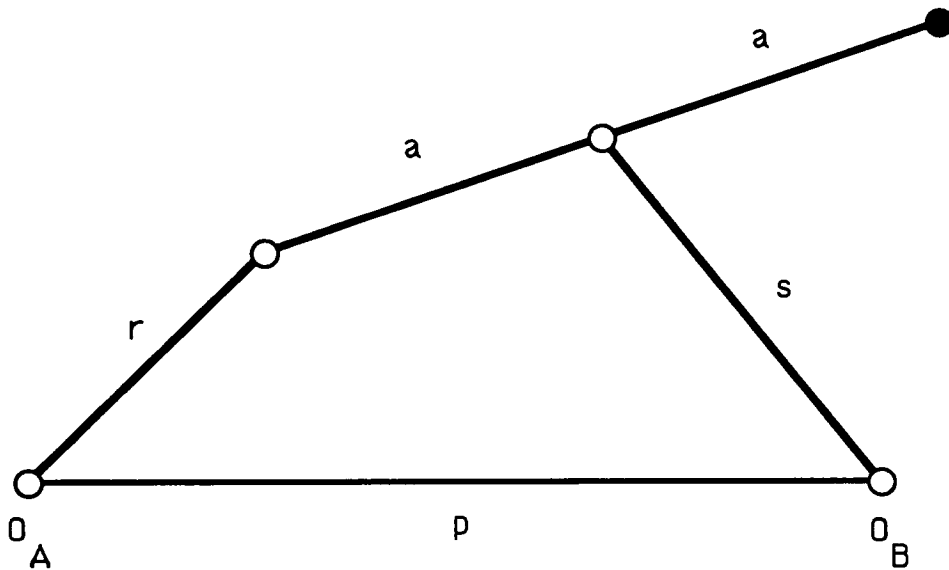
The selection of the type of four-bar linkage needed to accomplish a purpose depends to a great extent on consideration of usage, the coordination of input and output and availability. If the mode of input is a motor, which often is, the choice of a crank rocker mechanism (such as Evans linkage and Chebyshev linkage) would be appropriate, since in this type of mechanism the input link is capable of complete revolutions. On the other hand, if the input is from a rocker as in steam engine configuration, Watts linkage or Roberts linkage could be useful. As the coupler curve plots in Appendix A show that the coupler curve for Watts linkage has a double point on its output, it could be used as the return mechanism as well.

With so many factors, there can be no scheme whereby a mechanism may be uniquely determined on naming desired motion specifications. It will be necessary to consider a line-up of possible combinations that could do the job, from which the “best one” for the particular application in view is chosen.

### **10. 1. 2 Dimensional Synthesis**

By dimensional synthesis, it is meant determination of the dimensions of parts necessary to create a mechanism that will effect a desired motion transformation. In

considering the dimensional synthesis, it recognized that it has two aspects, called approximate and exact. For approximate synthesis, Chapter 8 provides the dimensional conditions for a four-bar linkage to produce a straight line coupler curve as follows:

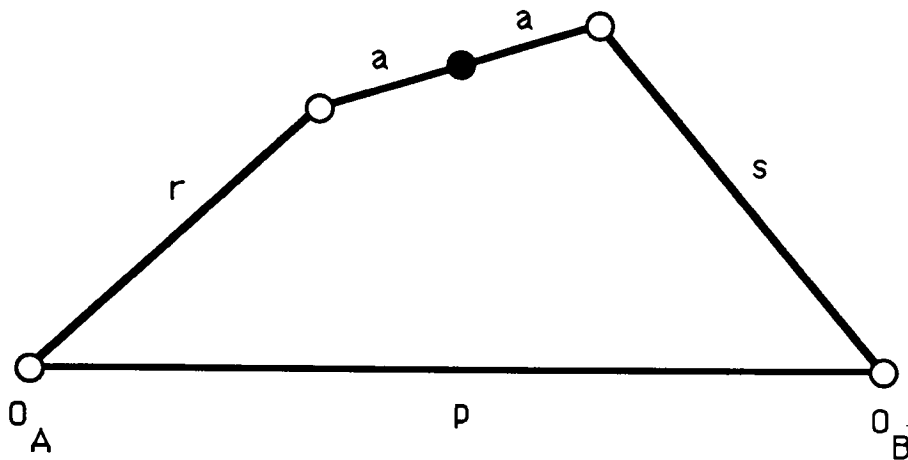


**Figure 10.2      Evans Linkage for Symmetrical Coupler Curves**

For a symmetrical Evans Linkage shown in Figure 10.2,

$$a = \frac{(1 + p)^{3/2}}{2} \quad (10.1)$$

where the length of the crank is considered as unity. The above condition is true to the dimensional ratio of the links. This means any multiple of these dimensions will still generate a straight line. It can be seen from the straight line plots given in Appendix A, that the range x to y ration decreases as p increases. The designer may choose a p value that suits his design and performance restrictions.



**Figure 10.3 A Symmetrical Chebyshev Linkage**

In case of a symmetrical Chebyshev linkage as the one in Figure 10.3,

$$r = \frac{(1 + p)^{3/2}}{2} \quad (10.2)$$

here the length of the coupler which can rotate completely is considered as one. As in Evans Linkage, here any multiple of these dimensions will produce a straight line coupler curve. The value of  $p$  is chosen to suit the needs then other dimensions are determined as required by the condition 10.2.

Geometric methods can furnish, with superb accuracy, quick and dependable solution to the problem of exact synthesis. Chapters 9 is dedicated to this method and computer codes are given in Appendix B that are created to provide the designer the graphical means. They give direct feeling for mechanical details which will be very important in reducing a given solution to hardware and which may be obtained directly on the drawing board without making use of sometimes unfamiliar or unavailable analytical techniques.

## 11. CONCLUSION

The objective of this work is to aid a designer with tools and knowledge to design a four-bar mechanism to produce a straight line coupler curve that would meet his/her design criteria.

In this work, mechanisms and in particular four bar linkages have been studied and their components and classifications have been discussed. Four special four-linkages that are capable of producing a straight line output are introduced and their mobilities have been explained in detail.

The two conditions that are required for a four-bar mechanism to deliver a symmetrical output have been provided along with their proofs. Two crank rocker mechanisms that generate straight line coupler curve have been analyzed and the dimensional requirements in order for them to generate straight lines have been derived.

Position analyses of all the links in a four-bar mechanism have been done using vector algebra and based on this, separate algorithms to numerically generate the coupler curves have been arrived at. Computer codes and the plots thus obtained are added in appendices.

Finally, synthesis procedure for the designer to design a mechanism that would meet his/her needs and ways to make finer adjustments to meet the performance criteria are provided.

It has thus been shown that it is possible to synthesize a four-bar mechanism analytically that could generate a straight line as part of its coupler curve. This work

will help the designer to synthesize analytically, a four-bar mechanism with proper selection of dimensions thereby saving the laborious task of choosing, by trial and error, a mechanism that had already been designed which would rarely coincide the requirements. With the help of the analytical tools presented the designer could also make finer modifications to the design to meet exact design requirements. Finally, to examine the output of the mechanism designed the graphical methods presented would be an excellent tool.



## **References**

1. Hartenberg, R. S. and J. Denavit, Kinematic Synthesis of Linkages. New York. McGraw-Hill Book Company, 1964.
2. Wilson, C. E., Sadler, J. P. and Michels, W. J., Kinematics and Dynamics of Machinery. Harper & Row Publishers, New York, 1983.
3. Mabie, H. H. and Ocvirk, F. W, Mechanisms and Dynamics of Machinery, Third Edition. John Wiley & Sons, 1975.
4. Tao, D. C., Applied Linkage Synthesis, Addison - Wesley Publishing Company, Inc.1964.
5. Lent, D., Analysis and Design of Mechanisms, Second Edition. Prentice-Hall, Inc., New Jersey, 1970.
6. Vallence, A. and Farris, M. E., Principles of Mechanism. The Macmillan Company, New York, 1933.
7. S. Molian, The Design of Cam Mechanisms and Linkages. American Elsevier Publishing Company, Inc., New York, 1968.
8. N. Berzak, Direct Derivation of the Symmetrical Coupler Curves of the Planar Four-Bar Linkage.

9. N. Berzak, Symmetrical Coupler Curves from Symmetrical Four-Bar Mechanisms.
10. M. A. Chase, Vector Analysis of Linkages, Transactions of American Society of Mechanical Engineers, Journal of Engineering for Industries, Series B55, No. 3 (August 1963): pages 289 - 297
11. E. A. Dilksman, A Modified Design for I-Type Double-Cranks to Approximate a Straight Line, Transactions of the ASME, Vol. 110, December 1988, pages 446-451.
12. J. A. Hrones and G. L. Nelson, Analysis of Four-Bar Linkage; Its Application To The Synthesis of Mechanisms, Wiley, New York, 1951.

# APPENDIX A

## PLOTS OF STRAIGHT LINE COUPLER CURVES

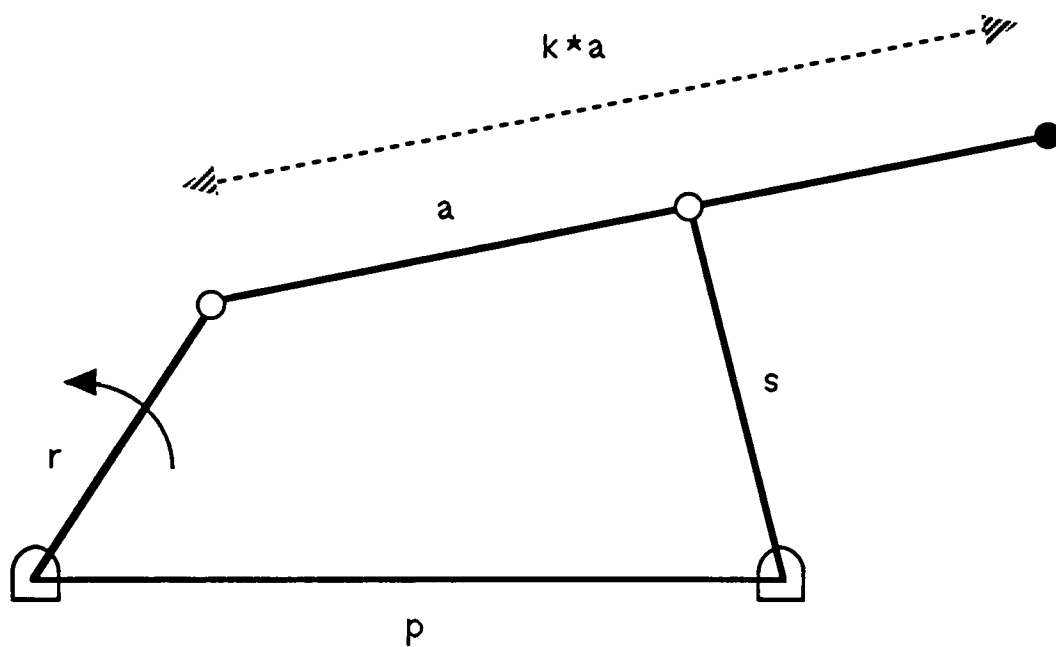


Fig. A.1 Evans Linkage



$$p = 2.00, r = 1.00, s = 2.59, a = 2.59$$

k is kept as 2.00

EVANS LINKAGE



$$p = 2.00, r = 1.00, s = 2.50, a = 2.50$$

k is kept as 2.00

EVANS LINKAGE



$$p = 3.00, r = 1.00, s = 4.00, a = 4.00$$

k is kept as 2.00

EVANS LINKAGE



$$p = 3.00, r = 1.00, s = 4.10, a = 4.10$$

k is kept as 2.00

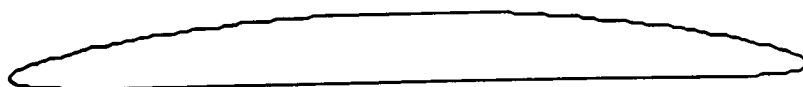
EVANS LINKAGE



$p = 4.00$ ,  $r = 1.00$ ,  $s = 5.59$ ,  $a = 5.59$

$k$  is kept as 2.00

EVANS LINKAGE



$p = 4.00$ ,  $r = 1.00$ ,  $s = 5.90$ ,  $a = 5.90$

$k$  is kept as 2.00

EVANS LINKAGE



$p = 5.00, r = 1.00, s = 7.35, a = 7.35$

k is kept as 2.00

EVANS LINKAGE



$p = 5.00, r = 1.00, s = 7.95, a = 7.95$

k is kept as 2.00

EVANS LINKAGE

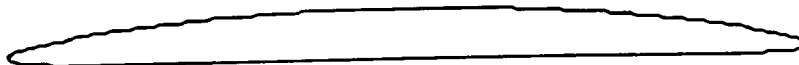




$p = 6.00, r = 1.00, s = 9.26, a = 9.26$

$k$  is kept as 2.00

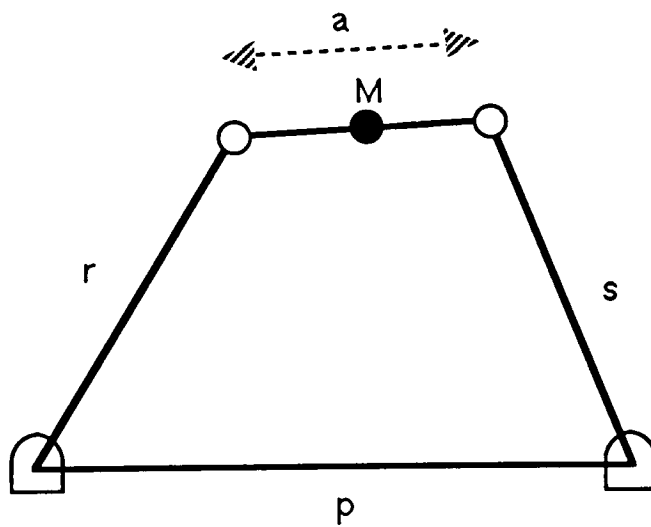
EVANS LINKAGE



$p = 6.00, r = 1.00, s = 10.20, a = 10.20$

$k$  is kept as 2.00

EVANS LINKAGE



**Figure A.2 Chebyshev Linkage**



$p = 2.00, r = 2.59, s = 2.59, a = 1.00$

CHEBYSHEV LINKAGE



$p = 2.00, r = 2.50, s = 2.50, a = 1.00$

CHEBYSHEV LINKAGE



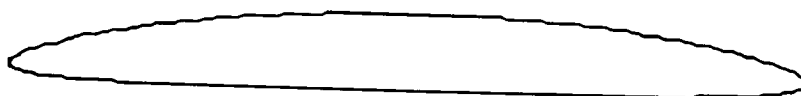
$p = 3.00, r = 4.00, s = 4.00, a = 1.00$

CHEBYSHEV LINKAGE



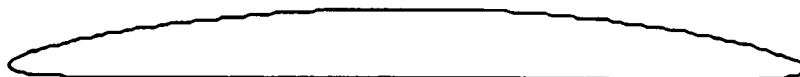
$p = 3.00, r = 4.10, s = 4.10, a = 1.00$

CHEBYSHEV LINKAGE



$p = 4.00, r = 5.59, s = 5.59, a = 1.00$

CHEBYSHEV LINKAGE



$p = 4.00, r = 5.90, s = 5.90, a = 1.00$

CHEBYSHEV LINKAGE



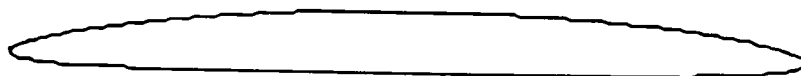
$p = 5.00, r = 7.35, s = 7.35, a = 1.00$

CHEBYSHEV LINKAGE



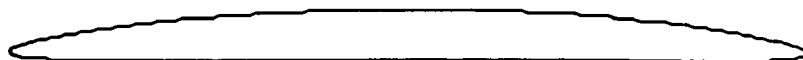
$p = 5.00, r = 7.95, s = 7.95, a = 1.00$

CHEBYSHEV LINKAGE



$$p = 6.00, r = 9.26, s = 9.26, a = 1.00$$

CHEBYSHEV LINKAGE



$$p = 6.00, r = 10.20, s = 10.20, a = 1.00$$

CHEBYSHEV LINKAGE

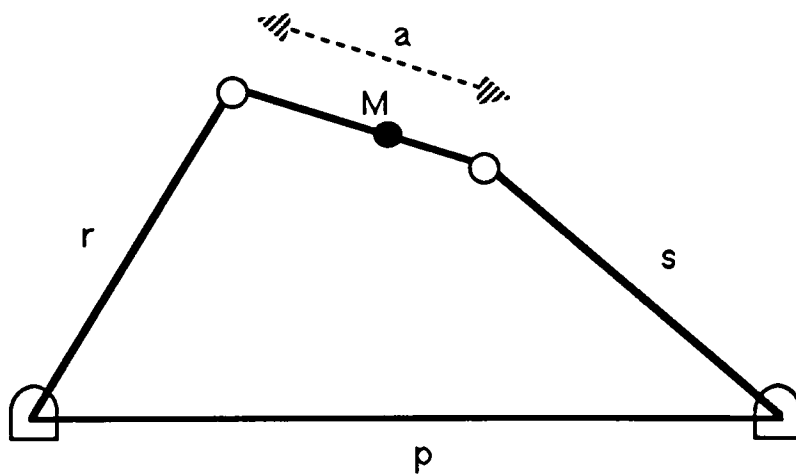
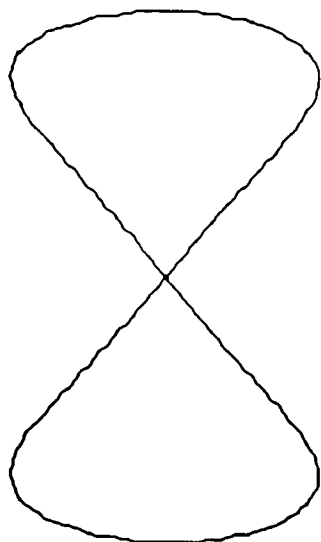


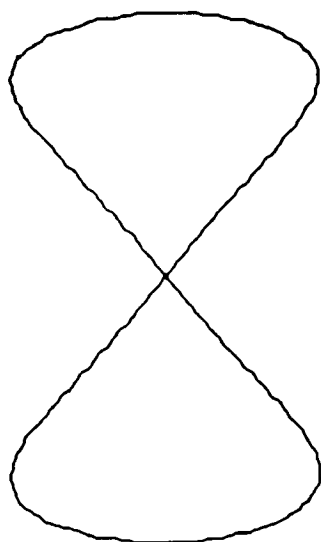
Figure A.3 Watts Linkage





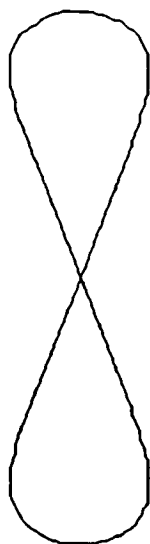
$p = 2.00, r = 1.00, s = 1.00, a = 1.00$

WATTS LINKAGE



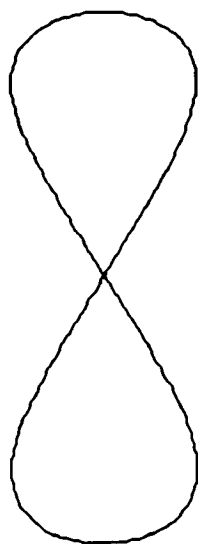
$p = 4.00, r = 2.00, s = 2.00, a = 2.00$

WATTS LINKAGE



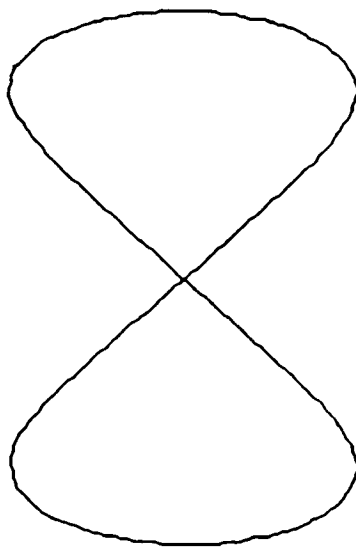
$p = 4.00, r = 2.00, s = 2.00, a = 1.00$

WATTS LINKAGE



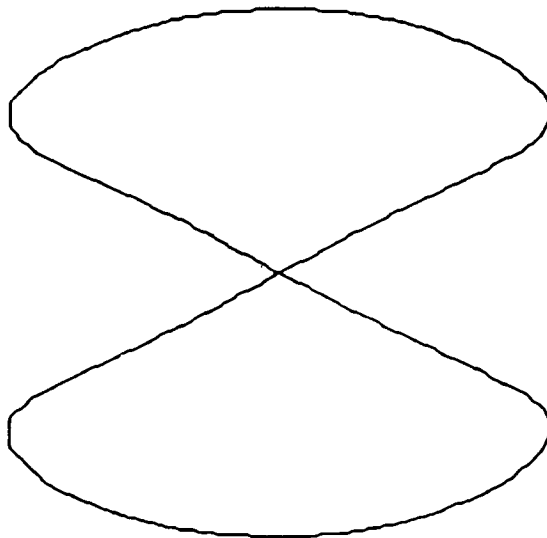
$p = 5.00, r = 2.00, s = 2.00, a = 2.00$

WATTS LINKAGE



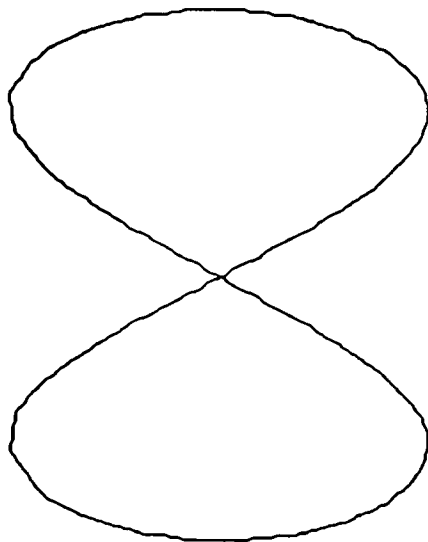
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WATTS LINKAGE



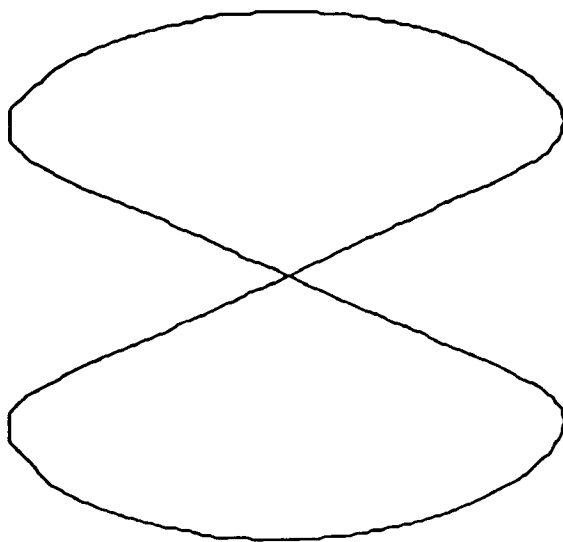
$p = 5.00, r = 2.00, s = 2.00, a = 4.00$

WATTS LINKAGE



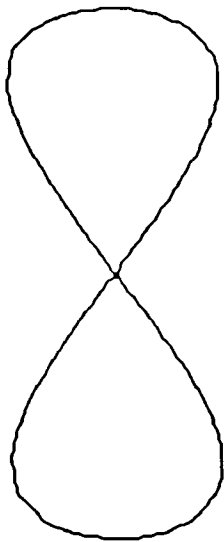
$p = 6.00, r = 1.00, s = 1.00, a = 5.00$

WATTS LINKAGE



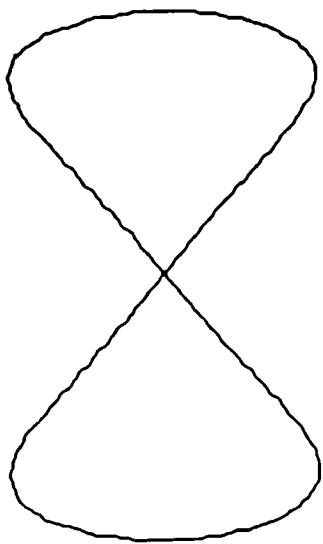
$p = 6.00, r = 2.00, s = 2.00, a = 5.00$

WATTS LINKAGE



$p = 6.00, r = 2.00, s = 2.00, a = 3.00$

WATTS LINKAGE



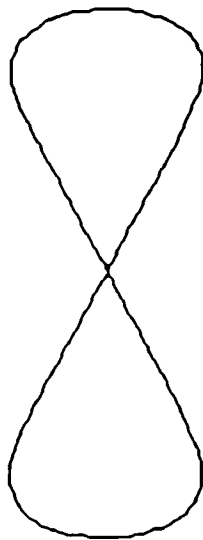
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WATTS LINKAGE



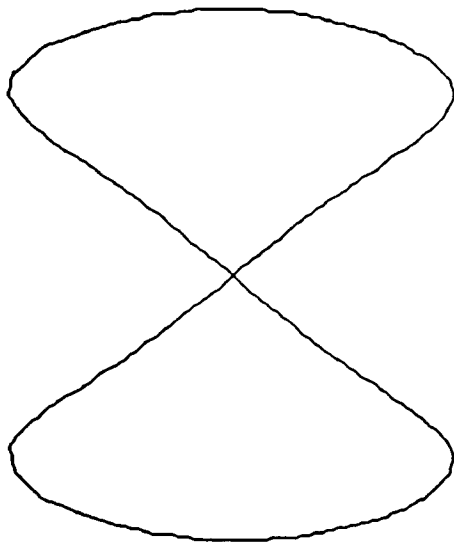
$$p = 6.00, r = 3.00, s = 3.00, a = 1.00$$

WATTS LINKAGE



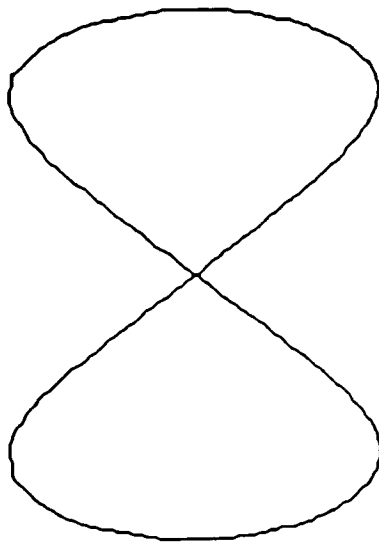
$$p = 6.00, r = 3.00, s = 3.00, a = 2.00$$

WATTS LINKAGE



$p = 6.00, r = 3.00, s = 3.00, a = 4.00$

WATTS LINKAGE



$p = 6.00, r = 2.00, s = 2.00, a = 4.00$

WATTS LINKAGE



### Figure A.4 Roberts Linkage



---

$$p = 2.00, r = 2.00, s = 2.00, a = 1.00$$

$$b = 2.00$$

ROBERTS LINKAGE

---

$$p = 3.00, r = 2.00, s = 2.00, a = 1.00$$

$$b = 1.00$$

ROBERTS LINKAGE

---

$$p = 3.00, r = 3.00, s = 3.00, a = 1.00$$

$$b = 1.00$$

ROBERTS LINKAGE

---

$$p = 4.00, r = 3.00, s = 3.00, a = 1.00$$

$$b = 1.00$$

ROBERTS LINKAGE

---

$$p = 4.00, r = 2.00, s = 2.00, a = 2.00$$

$$b = 3.00$$

ROBERTS LINKAGE

---

$$p = 4.00, r = 3.00, s = 3.00, a = 2.00$$

$$b = 3.00$$

ROBERTS LINKAGE

## APPENDIX B

CODE TO GENERATE  
COUPLER POINT CURVES

---

**program** Evans; {A program that plots the coupler curve for any Evans Mechanism}

---

**var**  
  xl, yl: **array**[0..200] **of** real;  
  r0, r1, r2, r3, k, w: real;  
  x, y, xmax, xmin, ymax, ymin, range: real;  
  i, xpos, ypos, cen1, cen2, cen3: integer;  
  crank: boolean;  
  cap1, cap2, cap3: **string**;  
  data: text;  
  window, trect: rect;

**procedure** initialize (window: rect); {initializes the windows}  
**begin**  
  setrect(window, 0, 20, screenbits.bounds.right, screenbits.bounds.bottom);  
  setdrawingrect(window);  
  showdrawing;  
**end**;**{initialize}**

**procedure** readdata (**var** r0, r1, r2, r3, k: real); {reads data}  
**begin**  
  readln(data, r0, r1, r2, r3, k);  
**end**;**{readdata}**

**procedure** verify (**var** crank: boolean; r0, r1, r2, r3: real); {verifies if the data complies to be a crank  
rocker mechanism}  
  **var**  
    lmax, lt, la, lb: real;  
  **begin**  
    **if** (r1 < r0) **and** (r1 < r2) **and** (r1 < r3) **then**  
      crank := true  
    **else**  
      crank := false;  
    **if** crank **then**  
      **begin**  
        lmax := r0;  
        **if** (r1 > lmax) **then**  
          lmax := r1;  
        **if** (r2 > lmax) **then**  
          lmax := r2;  
        **if** (r3 > lmax) **then**  
          lmax := r3;  
        **if** (r0 + r1 + r2 + r3 - 2 \* (lmax + r1)) > 0 **then**  
          crank := true  
        **else**  
          crank := false;  
      **end**;**{if}**  
    **end**;**{verify}**

---

**procedure** iteration (var x, y: real; r0, r1, r2, r3, k, w: real);

**var**

rd, rd1, rd2, a, r2x, r2y: real;

**begin**

rd := sqrt(sqr(r0) + sqr(r1) - 2 \* r0 \* r1 \* cos(w));

rd1 := (r1 \* cos(w) - r0) / rd;

rd2 := r1 \* sin(w) / rd;

a := (sqr(r3) - sqr(r2) + sqr(rd)) / (2 \* rd);

r2x := k \* (sqrt(sqr(r3) - sqr(a)) \* rd2 + (a - rd) \* rd1);

r2y := k \* (-sqrt(sqr(r3) - sqr(a)) \* rd1 + (a - rd) \* rd2);

x := r2x + r1 \* cos(w);

y := r2y + r1 \* sin(w);

**end;**{iteration}

**begin** {main program}

initialize(window);

reset(data, 'data');

readdata(r0, r1, r2, r3, k);

verify(crank, r0, r1, r2, r3);

**if** crank **then**

**begin**

**for** i := 0 to 180 **do**

**begin**

w := i \* 0.034906585;

iteration(x, y, r0, r1, r2, r3, k, w);

xl[i] := x;

yl[i] := y;

**end;**{for}

xmax := xl[1];

ymin := yl[1];

xmin := xl[1];

ymin := yl[1];

**for** i := 1 to 180 **do**

**begin**

**if** xl[i] > xmax **then**

xmax := xl[i];

**if** yl[i] > ymax **then**

ymax := yl[i];

**if** xl[i] < xmin **then**

xmin := xl[i];

**if** yl[i] < ymin **then**

ymin := yl[i];

**end;**{for}

cap1 := stringof('p = ', r0 : 1 : 2, ', r = ', r1 : 1 : 2, ', s = ', r3 : 1 : 2, ', a = ', r2 : 1 : 2);

cap2 := Stringof(' k is kept as ', k : 1 : 2);

cen1 := 300 - stringwidth(cap1) div 2;

cen2 := 300 - stringwidth(cap2) div 2;

moveto(cen1, 300);

drawstring(cap1);

---

```
moveto(cen2, 325);
drawstring(cap2);
cap3 := stringof('EVANS LINKAGE');
cen3 := 300 - stringwidth(cap3) div 2;
moveto(cen3, 400);
drawstring(cap3);
range := xmax - xmin;
if (ymax - ymin > range) then
  range := ymax - ymin;
xpos := 175 + round(250 * (xl[0] - xmin) / range);
ypos := 225 - round(250 * (yl[0] - ymin) / range);
moveto(xpos, ypos);
for i := 1 to 180 do
  begin
    xpos := 175 + round(250 * (xl[i] - xmin) / range);
    ypos := 225 - round(250 * (yl[i] - ymin) / range);
    lineto(xpos, ypos);
  end;{for}
end
else
  begin
    setrect(trect, 100, 125, 400, 175);
    settextrtrect(trect);
    showtext;
    writeln;
    writeln('Sorry! This is not a crank rocker mechanism!!');
  end;{else}
end.{evans}
```

---

**program** Chebyshev; {A program that plots the coupler curve for any Chebyshev Mechanism}

```
var
  xl, yl: array[1..200] of real;
  r0, r1, r2, r3, w: real;
  x, y, xmax, xmin, ymax, ymin, range: real;
  i, xpos, ypos, cen1, cen2: integer;
  crank: boolean;
  cap1, cap2: string;
  data: text;
  window, trect: rect;

procedure initialize (window: rect); {initializes the windows}
begin
  setrect(window, 0, 20, screenbits.bounds.right, screenbits.bounds.bottom);
  setdrawingrect(window);
  showdrawing;
end;{initialize}

procedure readdata (var r0, r1, r2, r3: real); {reads data}
begin
  readln(data, r0, r1, r2, r3);
end;{readdata}

procedure verify (var crank: boolean; r0, r1, r2, r3: real);
{verifies if the data complies to be a double rocker mechanism}
var
  lmax, lt, la, lb: real;
begin
  if (r1 < r0) and (r1 < r2) and (r1 < r3) then
    crank := true
  else
    crank := false;
  if crank then
    begin
      lmax := r0;
      if (r1 > lmax) then
        lmax := r1;
      if (r2 > lmax) then
        lmax := r2;
      if (r3 > lmax) then
        lmax := r3;
      if (r0 + r1 + r2 + r3 - 2 * (lmax + r1)) > 0 then
        crank := true
      else
        crank := false;
    end;{if}
  end;{verify}
```



```
procedure iteration (var x, y: real; r0, r1, r2, r3, w: real);
var
  rd, rd1, rd2, a, r2x, r2y: real;
  r, x1, y1, x2, y2, t, p: real;
begin
  rd := sqrt(sqr(r0) + sqr(r1) - 2 * r0 * r1 * cos(w));
  rd1 := (r1 * cos(w) - r0) / rd;
  rd2 := r1 * sin(w) / rd;
  a := (sqr(r3) - sqr(r2) + sqr(rd)) / (2 * rd);
  r2x := sqrt(sqr(r3) - sqr(a)) * rd2 + (a - rd) * rd1;
  r2y := -sqrt(sqr(r3) - sqr(a)) * rd1 + (a - rd) * rd2;
  x1 := r2x + r1 * cos(w);
  y1 := r2y + r1 * sin(w);
  x2 := (r1 / 2) * cos(w);
  y2 := (r1 / 2) * sin(w);
  x := x2 - r0;
  y := y2;
  r := sqrt(sqr(x) + sqr(y));
  t := arcsin(y2 / r);
  p := arcsin((r0 - x1) / r3);
  x := r * sin(t + p);
  y := r * cos(t + p);
end;{iteration}
```

```
begin {Main Program}
  initialize(window);
  reset(data, 'data');
  readdata(r0, r1, r2, r3);
  verify(crank, r0, r1, r2, r3);
  if crank then
    begin
      for i := 1 to 181 do
        begin
          w := i * 0.034906585;
          iteration(x, y, r0, r1, r2, r3, w);
          xl[i] := x;
          yl[i] := y;
        end;{for}
      xmax := xl[1];
      ymax := yl[1];
      xmin := xl[1];
      ymin := yl[1];
      for i := 2 to 181 do
        begin
          if xl[i] > xmax then
            xmax := xl[i];
          if yl[i] > ymax then
            ymax := yl[i];
          if xl[i] < xmin then
```

---

```
xmin := xl[i];
if yl[i] < ymin then
  ymin := yl[i];
end;{for}
cap1 := stringof('p = ', r3 : 1 : 2, ', r = ', r0 : 1 : 2, ', s = ', r2 : 1 : 2, ', a = ', r1 : 1 : 2);
cen1 := 300 - stringwidth(cap1) div 2;
moveto(cen1, 300);
drawstring(cap1);
cap2 := stringof('CHEBYSHEV LINKAGE');
cen2 := 300 - stringwidth(cap2) div 2;
moveto(cen2, 350);
drawstring(cap2);
range := xmax - xmin;
if (ymax - ymin > range) then
  range := ymax - ymin;
xpos := 175 + round(250 * (xl[1] - xmin) / range);
ypos := 225 - round(250 * (yl[1] - ymin) / range);
moveto(xpos, ypos);
for i := 2 to 181 do
  begin
    xpos := 175 + round(250 * (xl[i] - xmin) / range);
    ypos := 225 - round(250 * (yl[i] - ymin) / range);
    lineto(xpos, ypos);
  end;{for}
end
else
  begin
    setrect(trect, 100, 125, 400, 175);
    settextrect(trect);
    showtext;
    writeln;
    writeln('Sorry! This is not a double rocker mechanism!!');
  end;{else}
end.{evens}
```

---

program Watts;

**var**

xl, yl: **array**[1..200] **of** real;  
r0, r1, r2, r3, w, d, x, y: real;  
wmax, wmin, xmax, ymax, xmin, ymin, range: real;  
i, xpos, ypos, cen1, cen2: integer;  
crank: **boolean**;  
cap1, cap2: **string**;  
data: text;  
window, trect: rect;

**procedure** initialize (window: rect);

**begin**

setrect(window, 0, 20, screenbits.bounds.right, screenbits.bounds.bottom);  
setdrawingrect(window);  
showdrawing;

**end**;**{initialize}**

**procedure** readdata (var r0, r1, r2, r3: real);

**begin**

readln(data, r0, r1, r2, r3);

**end**;**{readdata}**

**procedure** verify (var crank: **boolean**; r0, r1, r2, r3: real);

**begin**

**if** (r1 + r2 + r3) > r0 **then**

crank := true

**else**

crank := false;

**if** (r0 + r2 + r3) > r1 **then**

crank := true

**else**

crank := false;

**if** (r0 + r1 + r3) > r2 **then**

crank := true

**else**

crank := false;

**if** (r0 + r1 + r2) > r3 **then**

crank := true

**else**

crank := false;

**end**;**{verify}**

**procedure** iteration1 (var x, y: real; r0, r1, r2, r3, w: real);

**var**

rd, rd1, rd2, a, r2x, r2y: real;

**begin**

rd := sqrt(sqr(r0) + sqr(r1) - 2 \* r0 \* r1 \* cos(w));

```

rd1 := (r1 * cos(w) - r0) / rd;
rd2 := r1 * sin(w) / rd;
a := (sqr(r3) - sqr(r2) + sqr(rd)) / (2 * rd);
r2x := sqrt(sqr(r3) - sqr(a)) * rd2 + (a - rd) * rd1;
r2y := -sqrt(sqr(r3) - sqr(a)) * rd1 + (a - rd) * rd2;
x := (r2x / 2) + r1 * cos(w);
y := (r2y / 2) + r1 * sin(w);
end;{iteration}

```

```

procedure iteration2 (var x, y: real; r0, r1, r2, r3, w: real);
  var
    rd, rd1, rd2, a, r2x, r2y: real;
begin
  rd := sqrt(sqr(r0) + sqr(r1) - 2 * r0 * r1 * cos(w));
  rd1 := (r1 * cos(w) - r0) / rd;
  rd2 := r1 * sin(w) / rd;
  a := (sqr(r3) - sqr(r2) + sqr(rd)) / (2 * rd);
  r2x := -sqrt(sqr(r3) - sqr(a)) * rd2 + (a - rd) * rd1;
  r2y := sqrt(sqr(r3) - sqr(a)) * rd1 + (a - rd) * rd2;
  x := (r2x / 2) + r1 * cos(w);
  y := (r2y / 2) + r1 * sin(w);
end;{iteration}

```

```

begin
  initialize(window);
  reset(data, 'data');
  readdata(r0, r1, r2, r3);
  verify(crank, r0, r1, r2, r3);
  wmin := -arccos((sqr(r0) + sqr(r1) - sqr(r3 + r2)) / (2 * r0 * r1)) + 0.0000001;
  wmax := arccos((sqr(r0) + sqr(r1) - sqr(r3 + r2)) / (2 * r0 * r1)) - 0.0000001;
  if crank then
    begin
      d := (wmax - wmin) / 90;
      for i := 1 to 91 do
        begin
          w := wmin + (i - 1) * d;
          iteration1(x, y, r0, r1, r2, r3, w);
          xl[i] := x;
          yl[i] := y;
        end;{for}
      for i := 92 to 182 do
        begin
          w := wmin + (182 - i) * d;
          iteration2(x, y, r0, r1, r2, r3, w);
          xl[i] := x;
          yl[i] := y;
        end;{for}
      xmax := xl[1];
      ymax := yl[1];
    end;
  end;

```

---

```
xmin := xl[1];
ymin := yl[1];
for i := 2 to 182 do
  begin
    if xl[i] > xmax then
      xmax := xl[i];
    if yl[i] > ymax then
      ymax := yl[i];
    if xl[i] < xmin then
      xmin := xl[i];
    if yl[i] < ymin then
      ymin := yl[i];
  end;{for}
range := xmax - xmin;
if (ymax - ymin) > range then
  range := ymax - ymin;
cap1 := stringof('p = ', r0 : 1 : 2, ', r = ', r1 : 1 : 2, ', s = ', r3 : 1 : 2, ', a = ', r2 : 1 : 2);
cap2 := Stringof('WATTS LINKAGE');
cen1 := 300 - stringwidth(cap1) div 2;
cen2 := 300 - stringwidth(cap2) div 2;
moveto(cen1, 300);
drawstring(cap1);
moveto(cen2, 350);
drawstring(cap2);
xpos := 175 + round(250 * (xl[1] - xmin) / range);
ypos := 225 - round(250 * (yl[1] - ymin) / range);
moveto(xpos, ypos);
for i := 2 to 182 do
  begin
    xpos := 175 + round(250 * (xl[i] - xmin) / range);
    ypos := 225 - round(250 * (yl[i] - ymin) / range);
    lineto(xpos, ypos);
  end;{for}
xpos := 175 + round(250 * (xl[1] - xmin) / range);
ypos := 225 - round(250 * (yl[1] - ymin) / range);
lineto(xpos, ypos);
end{if crank}
else
  begin
    setrect(trect, 100, 125, 320, 175);
    settextrect(trect);
    showtext;
    writeln;
    writeln('Uh-huh! This is not gonna work!!');
  end;{else}
end.{watts}
```

---

program Roberts;

**var**

xl, yl: **array**[1..200] **of** real;  
r0, r1, r2, r3, r, t, w, d, x, y: real;  
wmin, wmax, xmax, xmin, ymax, ymin, range: real;  
i, xpos, ypos, cen1, cen2, cen3: integer;  
crank: boolean;  
cap1, cap2, cap3: **string**;  
data: text;  
window, trect: rect;

**procedure** initialize (window: rect);

**begin**

  setrect(window, 0, 20, screenbits.bounds.right, screenbits.bounds.bottom);  
  setdrawingrect(window);  
  showdrawing;

**end**;  
{initialize}

**procedure** readdata (var r0, r1, r2, r3, r: real);

**begin**

  readln(data, r0, r1, r2, r3, r);

**end**;  
{readdata}

**procedure** verify (var crank: boolean; r0, r1, r2, r3: real);

**var**

  lmax, lt, la, lb: real;

**begin**

**if** (r0 > r1) **and** (r0 > r2) **and** (r0 > r3) **then**

    crank := true

**else**

    crank := true;

**if** crank **then**

**if** (r1 + r2 + r3) > r0 **then**

      crank := true

**else**

      crank := false;

**end**;  
{verify}

**procedure** iteration (var x, y: real; r0, r1, r2, r3, r, w: real);

**var**

  rd, rd1, rd2, a, r2x, r2y, x1, y1, x2, y2, p: real;

**begin**

  rd := sqrt(sqrt(r0) + sqrt(r1) - 2 \* r0 \* r1 \* cos(w));

  rd1 := (r1 \* cos(w) - r0) / rd;

  rd2 := r1 \* sin(w) / rd;

  a := (sqrt(r3) - sqrt(r2) + sqrt(rd)) / (2 \* rd);

  r2x := sqrt(sqrt(r3) - sqrt(a)) \* rd2 + (a - rd) \* rd1;

  r2y := -sqrt(sqrt(r3) - sqrt(a)) \* rd1 + (a - rd) \* rd2;

```
x1 := r1 * cos(w);
y1 := r1 * sin(w);
x2 := r2x + x1;
y2 := r2y + y1;
p := arcsin((y2 - y1) / r2);
t := arccos(r2 / (2 * r));
x := x1 + r * cos(p - t);
y := y1 + r * sin(p - t);
end;{iteration}
```

```
begin
initialize(window);
reset(data, 'data');
readdata(r0, r1, r2, r3, r);
verify(crank, r0, r1, r2, r3);
wmin := arccos((sqr(r0) + sqr(r1 + r2) - sqr(r3)) / (2 * r0 * (r1 + r2))) + 0.0000001;
wmax := arccos((sqr(r0) + sqr(r1) - sqr(r3 + r2)) / (2 * r0 * r1)) - 0.0000001;
if crank then
begin
d := (wmax - wmin) / 180;
for i := 1 to 181 do
begin
w := wmin + (i - 1) * d;
iteration(x, y, r0, r1, r2, r3, r, w);
xl[i] := x;
yl[i] := y;
end;{for}
xmax := xl[1];
ymax := yl[1];
xmin := xl[1];
ymin := yl[1];
for i := 2 to 181 do
begin
if xl[i] > xmax then
xmax := xl[i];
if yl[i] > ymax then
ymax := yl[i];
if xl[i] < xmin then
xmin := xl[i];
if yl[i] < ymin then
ymin := yl[i];
end;{for}
cap1 := stringof('p = ', r0 : 1 : 2, ', r = ', r1 : 1 : 2, ', s = ', r3 : 1 : 2, ', a = ', r2 : 1 : 2);
cap2 := Stringof('b = ', r : 1 : 2);
cen1 := 300 - stringwidth(cap1) div 2;
cen2 := 300 - stringwidth(cap2) div 2;
moveto(cen1, 300);
drawstring(cap1);
moveto(cen2, 325);
```

---

```
drawstring(cap2);
cap3 := stringof('ROBERTS LINKAGE');
cen3 := 300 - stringwidth(cap3) div 2;
moveto(cen3, 400);
drawstring(cap3);
range := xmax - xmin;
if (ymax - ymin > range) then
    range := ymax - ymin;
xpos := 175 + round(250 * (xl[1] - xmin) / range);
ypos := 225 - round(250 * (yl[1] - ymin) / range);
moveto(xpos, ypos);
for i := 2 to 181 do
    begin
        xpos := 175 + round(250 * (xl[i] - xmin) / range);
        ypos := 225 - round(250 * (yl[i] - ymin) / range);
        lineto(xpos, ypos);
    end;{for}
end {if crank}
else
    begin
        setrect(trect, 100, 125, 400, 175);
        settextrrect(trect);
        showtext;
        writeln;
        writeln('Uh-huh! This is not gonna work!!');
    end;{else}
end.{evans}
```