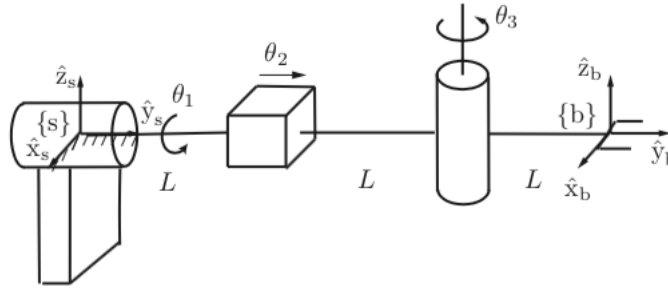


1. We deal in this note with the example shown in the Figure below



It has three joints, RPR joints, but note that the end-effector frame is *not* collocated with a joint.

2. Denote by $T_{sb}(\theta)$ the position of the end-effector frame with respect to the inertial frame s . We have seen how to express T_{sb} using the product of exponential formula in two different ways:

$$T_{sb} = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M$$

or

$$T_{sb} = M e^{[B_1]\theta_1} e^{[B_2]\theta_2} e^{[B_3]\theta_3}.$$

For the first formula, M is the end-effector frame, expressed in s -frame, when all joint angles are zero, and S_i is the screw vector of joint i , assuming all joints are at zero. For the second formula, M is the same as before, but B_i is the screw of joint i in the body frame, when all joint angles are zero.

3. For the mechanism addressed here, the screw vectors are $S_1 = (0, 1, 0, 0, 0, 0)^\top$, $S_2 = (0, 0, 0, 0, 1, 0)^\top$ and $S_3 = (0, 0, 1, 2L, 0, 0)^\top$. The first two are obvious, being a revolute joint collocated with the reference frame for 1 and a prismatic joint along \hat{y} for 2. For the third one, the first three entries are the coordinates of the axis of rotation in s frame: here, it is \hat{z}_s , and hence we get $(0, 0, 1)$. For the last three entries, recall that you need to express in the s frame the instantaneous velocity of the origin of the s frame when joint 3 is activated. It is easy to see that this velocity is $(2L, 0, 0)$. We can also use the formula $v_{s3} = -\omega_{s3} \times q_{s3}$ where q_{s3} is any vector joining the origin of frame s to the axis of rotation; for example we can take $q_{s3} = (0, 2L, 0)$.

We now compute

$$e^{[S_1]\theta_1} = \begin{bmatrix} \text{Rot}(\hat{y}, \theta_1) & 0 \\ 0 & 1 \end{bmatrix}, e^{[S_2]\theta_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, e^{[S_3]\theta_3} = \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & 2L \sin \theta_3 \\ \sin \theta_3 & \cos \theta_3 & 0 & 2L - 2L \cos \theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that we can also find a “physical” interpretation for these matrix exponentials. To evaluate $e^{[S_i]\theta_i}$, you can picture that the whole space, and thus the reference frame s , moves according to the twist described by S_i . Call the transformed s frame the s' frame. Then the rotation matrix part (R part) of $e^{[S_i]\theta_i}$ is simply obtained by writing in columns the coordinates the vectors of the s' -frame in the s frame, and the translation part (p part) of $e^{[S_i]\theta_i}$ is the position of the origin of the s' frame in the s frame. Let us apply this to S_2 . The whole space moves along with joint 2, it is easy to see that s' is a translation of s ; hence the basis vector of the s' frame are the same as the one of the s frame, but the center is now at position $(0, \theta_2, 0)$, expressed in the s frame. We thus get that R is the identity and $p = (0, \theta_2, 0)$ as verified above. Using the same approach, you can deduce the value of $e^{[S_3]\theta_3}$ as well. It is also always safe to simply compute the exponential.

4. For the matrix M , we have

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Putting everything together, we get

$$T_{sb}(\theta) = \begin{bmatrix} \cos \theta_1 \cos \theta_3 & -\cos \theta_1 \sin \theta_3 & \sin \theta_1 & -L \cos \theta_1 \sin \theta_3 \\ \sin \theta_3 & \cos \theta_3 & 0 & 2L + \theta_2 + L \cos \theta_3 \\ -\cos \theta_3 \sin \theta_1 & \sin \theta_1 \sin \theta_3 & \cos \theta_1 & L \sin \theta_1 \sin \theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. Recall that the Jacobian relates the velocity of the joints (the $\dot{\theta}_s$'s) to a twist of the end-effector, according to

$$\mathcal{V}_s = J_s \dot{\theta} \text{ and } \mathcal{V}_b = J_b \dot{\theta},$$

where $J_b(\theta)$ and $J_s(\theta)$ are the body and space Jacobians respectively. Hence the body Jacobian relates the joint velocities to a body twist of the end-effector, and the space Jacobian relates the same joint velocities to the space twist of the end-effector. In terms of dimensions, J has 6 rows and as many columns as there are joints. The vector θ has one column and as many rows as there are joints. Here, there are 3 joints, generalizing to n joints is obvious.

6. We now rederive the formula for the Jacobian. We have

$$\mathcal{V}_s = \dot{T}_{sb} T_{sb}^{-1}$$

and

$$\begin{aligned} \dot{T}_{sb} &= \frac{d}{dt} (e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M) \\ &= [S_1] \dot{\theta}_1 e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M + e^{[S_1]\theta_1} [S_2] \dot{\theta}_2 e^{[S_2]\theta_2} e^{[S_3]\theta_3} M + e^{[S_1]\theta_1} e^{[S_2]\theta_2} [S_3] \dot{\theta}_3 e^{[S_3]\theta_3} M. \end{aligned}$$

Furthermore,

$$T_{sb}^{-1} = M^{-1} e^{-[S_3]\theta_3} e^{-[S_2]\theta_2} e^{-[S_1]\theta_1}.$$

Now recall that

$$\text{Ad}_T[\mathcal{S}] = T[\mathcal{S}]T^{-1}$$

in general for an arbitrary screw matrix $[\mathcal{S}]$. The same transformation is applied to the vector form of the screw \mathcal{S} by defining the matrix Ad_T (we overload the notation):

$$\text{Ad}_T = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

where $T = (R, p)$. The definitions are such that

$$[\text{Ad}_T \mathcal{S}] = \text{Ad}_T[\mathcal{S}].$$

Note that, for the above equation, on the left, this is the 6×6 matrix, and on the right, this is the operator that multiplies the matrix $[\mathcal{S}]$ by T and T^{-1} on the left and on the right respectively.

With this notation, we have after simplifications

$$[\mathcal{V}_s] = [\mathcal{S}_1]\dot{\theta}_1 + \text{Ad}_{e^{[\mathcal{S}_1]\theta_1}}[\mathcal{S}_2]\dot{\theta}_2 + \text{Ad}_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}[\mathcal{S}_3]\dot{\theta}_3.$$

We can also write this equation as

$$\mathcal{V}_s = \underbrace{\begin{bmatrix} \mathcal{S}_1 & \text{Ad}_{e^{[\mathcal{S}_1]\theta_1}} \mathcal{S}_2 & \text{Ad}_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}} \mathcal{S}_3 \end{bmatrix}}_{J_s} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix},$$

from which we deduce the form of the Jacobian is J_s .

7. In words, for the space Jacobian, its i th row is the twist of the i th link for a motion for the i th joint, when joint angles θ_1 to θ_{i-1} are set to arbitrary values. Indeed, we see that the first column in \mathcal{S}_1 , which is the twist of the first joint. The second column is $\text{Ad}_{e^{[\mathcal{S}_1]\theta_1}} \mathcal{S}_2$; recall that the Ad operator is used to change frames; here, \mathcal{S}_2 is the twist of the second joint when all variables are zero, and $e^{[\mathcal{S}_1]\theta_1}$ is the transformation that moves joint 1 by the value θ_1 . Thus $\text{Ad}_{e^{[\mathcal{S}_1]\theta_1}} \mathcal{S}_2$ is simply expressing twist 2 when the joint 1 is at θ_1 .
8. We now evaluate the space Jacobian of the mechanism shown above, using the formulas and using our interpretation of them. We use the notation $(\bar{\omega}_{si}, \bar{v}_{si})$ for the top three and bottom the entries of the i th column of J_s respectively. We keep ω_{si} and v_{si} for the entries of $\mathcal{S}_i = (\omega_{si}, v_{si})^\top$.

First column For the first column, there is no preceding joint in the chain. We thus need to write the twist of the first link when the first joint moves, expressed in the s -frame. Here, the first joint is a revolute joint with axes \hat{y}_s , and the axis goes through the origin of the s -frame. Hence $\bar{\omega}_{s1} = \omega_{s1} = (0, 1, 0)^\top$ and $\bar{v}_{s1} = v_{s1} = 0$ and

$$J_{s1} = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)^\top$$

Second column For the second column, we need to write the twist of link 2 when joint 2 moves, assuming that joint 1 is at an arbitrary position θ_1 . Here, joint 2 is a prismatic joint in direction \hat{y}_s when $\theta_1 = 0$. We see that when θ_1 is arbitrary, the axis of translation does not change, and is still \hat{y}_s . Hence

$$J_{s2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^\top.$$

To see this using the formula, it suffices to write $\text{Ad}_{e^{[S_1]\theta_1}}$, where

$$e^{[S_1]\theta_1} = \begin{bmatrix} \text{Rot}(\hat{y}, \theta_1) & 0 \\ 0 & 1 \end{bmatrix}.$$

Because a rotation around \hat{y} leaves \hat{y} fixed, and because $S_2 = (0, 0, 0, 0, 1, 0)$, we obtain the same result as above for J_{s2} .

Third column For the last column, we need to write the twist of the end-effector frame, when the last joint moves and assuming θ_1 and θ_2 are at arbitrary positions, in the s -frame. The last joint is a revolute joint around \hat{z}_s when $\theta = 0$. When θ_1 and θ_2 are non-zero, this axis is not \hat{z}_s anymore, but is rotated around \hat{y}_s by θ_1 . The translation does not change ω . Hence we have

$$\bar{\omega}_{s3} = \text{Rot}(y, \theta_1)(0, 0, 1)^\top = \begin{bmatrix} \cos \theta_1 & 0 & \sin \theta_1 \\ 0 & 1 & 0 \\ -\sin \theta_1 & 0 & \cos \theta_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin \theta_1 \\ 0 \\ \cos \theta_1 \end{bmatrix}.$$

Now for \bar{v}_{s3} , we need the velocity of the center of the s -frame, expressed in the s frame, when we assume that the whole space moves according to joint 3. To obtain it, we can use the formula $\bar{v}_{s3} = -\bar{\omega}_{s3} \times q_{s3}$ where q_{s3} is a vector joining the origin of the s -frame to any point on the axis of rotation. When computing q_{s3} , *do not forget that θ_1 and θ_2 have arbitrary values*. We can take for example $q_{s3} = (0, 2L + \theta_2, 0)^\top$. This yields

$$\bar{v}_{s3} = -\bar{\omega}_{s3} \times q_{s3} = \begin{bmatrix} \sin \theta_1 \\ 0 \\ \cos \theta_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2L + \theta_2 \\ 0 \end{bmatrix} = \begin{bmatrix} (2L + \theta_2) \cos \theta_1 \\ 0 \\ -(2L + \theta_2) \sin \theta_1 \end{bmatrix}$$

Finally, we get the space Jacobian

$$J_s = \begin{bmatrix} 0 & 0 & \sin \theta_1 \\ 1 & 0 & 0 \\ 0 & 0 & \cos \theta_1 \\ 0 & 0 & (2L + \theta_2) \cos \theta_1 \\ 0 & 1 & 0 \\ 0 & 0 & -(2L + \theta_2) \sin \theta_1 \end{bmatrix}$$

9. Now let us derive the formula for the body Jacobian. The body Jacobian satisfies

$$\mathcal{V}_b = J_b \dot{\theta}.$$

It is obtained similarly to the space Jacobian, but we start with the body frame expression of T_{sb} , namely

$$T_{sb} = M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_3]\theta_3}.$$

Differentiating, we get

$$\begin{aligned}\dot{T}_{sb} &= \frac{d}{dt} M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_3]\theta_3} \\ &= M e^{[\mathcal{B}_1]\theta_1} [\mathcal{B}_1] \dot{\theta}_1 e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_3]\theta_3} + M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} [\mathcal{B}_2] \dot{\theta}_2 e^{[\mathcal{B}_3]\theta_3} + M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_3]\theta_3} [\mathcal{B}_3] \dot{\theta}_3.\end{aligned}$$

Multiplying on the left by T_{sb} , we get

$$\mathcal{V}_b = T_{sb}^{-1} \dot{T}_{sb} = \text{Ad}_{e^{-[\mathcal{B}_3]\theta_3} e^{-[\mathcal{B}_2]\theta_2}} [\mathcal{B}_1] \dot{\theta}_1 + \text{Ad}_{e^{-[\mathcal{B}_3]\theta_3}} [\mathcal{B}_2] \dot{\theta}_2 + [\mathcal{B}_3] \dot{\theta}_3.$$

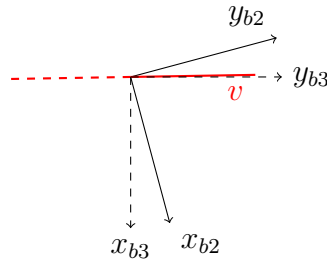
Note the minus sign in the exponentials! We conclude that the i th column of the body Jacobian is

$$J_{bi} = \text{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \dots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}} [\mathcal{B}_i].$$

10. We now derive the body Jacobian of the mechanism depicted above. The i th column of the body Jacobian is the twist induced by a motion of the i th joint, expressed in the end-effector (body) frame, when we assume that $\theta_{i+1}, \dots, \theta_n$ are at arbitrary positions. It is a bit more challenging to evaluate than J_s in that as we compute the columns, our reference frame b keeps on changing, whereas when computing the space Jacobian, the s frame is fixed. In terms of amount of computation, they are however equivalent.

Third column The easiest column to compute is the last column, corresponding to a motion of the last joint. There are no ulterior joints, so it is simply the body twist of the last joint (twist in body frame). Here, we see that $\bar{\omega}_{b3} = \omega_{b3} = (0, 0, 1)^\top$ and $\bar{v}_{b3} = v_{b3} = (-L, 0, 0)^\top$, where we recall that v_{b3} is the velocity of the origin of the b -frame, when joint 3 rotate, expressed in the b frame. We can also obtain it as $v_{b3} = -\omega_{b3} \times q_{b3}$ with $q_{b3} = (0, -L, 0)^\top$.

Second column We need the body twist of joint 2, in b frame, when θ_3 is arbitrary. The second joint being prismatic, we know that $\bar{\omega}_{b2} = \omega_{b2} = (0, 0, 0)^\top$. Now assuming we have rotated around ω_{b3} by θ_3 , we have (see from the top).



From the drawing, we see that $\bar{v}_{b2} = (\sin \theta_3, \cos \theta_3, 0)$.

We now explain how to obtain it without drawing. Feel free to skip this part, until the **** below. Recall that v_{b2} is the translation vector in the b frame before the latter rotates by θ_3 : $v_{b2} = (0, 1, 0)$ and denote by R_{b3} the orientation of the b frame when all angles are zero, expressed in the b

frame. Then R_{b3} is simply the identity. (Recall, we need to express the columns of the basis b in the basis $b!$). Now denote by R_{b2} the matrix whose columns are the vectors of the end-effector frame *after* the rotation by θ_3 . We have

$$R_{b2} = \text{Rot}(\hat{z}, \theta_3) R_{b3}.$$

The vectors x_{b2} and y_{b2} in the figure above are the vectors in the first and second columns of R_{b2} . Now we write the equality equation for the translation vector v , expressed in the bases $b3$ (all joints at zero) and $b2$ (all joints at zero except 3). In the basis $b3$, $v = y_{b3} = R_{b3}v_{b2}$, and in the basis $b2$, it is equal to $v = x_{b2}\bar{v}_{b2,1} + y_{b2}\bar{v}_{b2,2} + z_{b2}\bar{v}_{b2,3} = R_{b2}\bar{v}_{b2}$, where x_{b2} , y_{b2} and z_{b2} are the columns of R_{b2} and we seek to determine \bar{v}_{b2} . Hence we have

$$R_{b3}v_{b2} = R_{b2}\bar{v}_{b2} = \text{Rot}(\hat{z}, \theta_3)\bar{v}_{b2} \Rightarrow \bar{v}_{b2} = \text{Rot}(\hat{z}, -\theta_3)v_{b2}.$$

We obtain as above

$$v_{b2} = \begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin \theta_3 \\ \cos \theta_3 \\ 0 \end{bmatrix}.$$

****This reasoning may seem long and complicated, but it achieves a simple thing: start from the end-effector, and apply the *inverse of the transformations* of the joints. This is opposite, in a sense, to what we did for the space Jacobian, where we applied the transformations of the joints. You can see this appearing already in the formulas for the Jacobian, where J_s has a plus sign in the exponentials, but J_b a minus sign.

We now compare this with the formula for the second column of the body Jacobian: $J_{b2} = \text{Ad}_{e^{-[\mathcal{B}_3]\theta_3}}[\mathcal{B}_2]$. Given that $\mathcal{B}_3 = (0, 0, 1, -L, 0, 0)^\top$, we obtain

$$e^{-[\mathcal{B}_3]\theta_3} = \begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 & L \sin \theta_3 \\ -\sin \theta_3 & \cos \theta_3 & 0 & L \cos \theta_3 - L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A quick computation shows that $\text{Ad}_{e^{-[\mathcal{B}_3]\theta_3}}(0, 0, 0, 0, 1, 0)^\top$ yields the same value for the second column of the Jacobian.

First column This is the body twist of joint 1 in b frame when θ_2 and θ_3 are arbitrary. From looking at the mechanism, we see that the axis of translation of joint 2 is the same as the axis of rotation of joint 1, hence we immediately have $\bar{\omega}_{b1} = (\sin \theta_3, \cos \theta_3, 0)^\top$. We could also draw a picture, but it would be similar to the one above since the axes are the same. Now for \bar{v}_{b1} , we need to express q_{b1} , a vector from the center of the b frame, with θ_2, θ_3 arbitrary, to the axis of rotation. This vector is also expressed in the b frame, hence we can take $q_{b1} = (0, -L, 0)^\top$, from which we obtain

$$\bar{v}_{b1} = -\bar{\omega}_{b1} \times q_{b1} = - \begin{bmatrix} \sin \theta_3 \\ \cos \theta_3 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -L \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L \sin \theta_3 \end{bmatrix}.$$

This can also be verified by inspection of the figure.

Let us also with the derivation using the general formula. We have that $\mathcal{B}_1 = (0, 1, 0, 0, 0, 0)^\top$, since when all joints angles are zero, joint one is a rotation around an axis that goes through the origin of the b frame. We have,

$$J_{b1} = \text{Ad}_{e^{-[\mathcal{B}_3]\theta_3}} \text{Ad}_{e^{-[\mathcal{B}_2]\theta_2}} \mathcal{B}_1.$$

We know that $e^{-[\mathcal{B}_2]\theta_2}$ is a translation of distance $-\theta_2$ along the \hat{y} axis:

$$e^{-[\mathcal{B}_2]\theta_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $e^{-[\mathcal{B}_3]\theta_3}$ has been calculated above. We get after computing the product of Ad's the expression

$$J_{b1} = \begin{bmatrix} \cos(\theta_3) & \sin(\theta_3) & 0 & 0 & 0 & 0 \\ -\sin(\theta_3) & \cos(\theta_3) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & L \cos(\theta_3) - L - \theta_2 \cos(\theta_3) & \cos(\theta_3) & \sin(\theta_3) & 0 \\ 0 & 0 & -\sin(\theta_3)(L - \theta_2) & -\sin(\theta_3) & \cos(\theta_3) & 0 \\ \theta_2 - L + L \cos(\theta_3) & L \sin(\theta_3) & 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{B}_1 = \begin{bmatrix} \sin \theta_3 \\ \cos \theta_3 \\ 0 \\ 0 \\ 0 \\ L \sin \theta_3 \end{bmatrix}.$$

This matches our previous answer.

Summarizing, we obtain the body Jacobian

$$J_b = \begin{bmatrix} \sin \theta_3 & 0 & 0 \\ \cos \theta_3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \sin \theta_3 & -L \\ 0 & \cos \theta_3 & 0 \\ L \sin \theta_3 & 0 & 0 \end{bmatrix}$$

- II. Recall that if \mathcal{V}_b and \mathcal{V}_s are the same twists (e.g. twist of the end-effector), expressed in two different frames (space and body frame), then they are related according to

$$\mathcal{V}_s = \text{Ad}_{T_{sb}} \mathcal{V}_b.$$

Recall that if $T = (R, p)$, then

$$\text{Ad}_T = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}.$$

From there, we can also deduce the rule for obtaining J_s from J_b and vice-versa. We have

$$\mathcal{V}_s = J_s \dot{\theta} = \text{Ad}_{T_{sb}} \mathcal{V}_b = \text{Ad}_{T_{sb}} J_b \dot{\theta}.$$

Looking at the second and last terms in the above chain of equalities, and since this is true for any θ , we get

$$J_s = \text{Ad}_{T_{sb}} J_b.$$

Hence Jacobian and twists obey the same rule when changing the frame with respect to which we write them.

12. We can verify that

$$T_{sb}^{-1} = \begin{bmatrix} \cos(\theta_1) \cos(\theta_3) & \sin(\theta_3) & -\cos(\theta_3) \sin(\theta_1) & -\sin(\theta_3) (2L + \theta_2) \\ -\cos(\theta_1) \sin(\theta_3) & \cos(\theta_3) & \sin(\theta_1) \sin(\theta_3) & -L - 2L \cos(\theta_3) - \theta_2 \cos(\theta_3) \\ \sin(\theta_1) & 0 & \cos(t_1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\text{Ad}_{T_{sb}^{-1}} J_s = J_b.$$