REVIEW OVER VECTORS

I. Scalars & Vectors:

- A scalar is a quantity that is defined by its value only. This value can be positive, negative or zero

  Example

$$mass = 5\ kg$$

- A vector is a quantity that can be described by a value -called magnitude and always positive- and a direction.

  Example A car moving north with a speed of 60 mph

- Vectors are always represented by symbols with an overhead arrows.

  Example

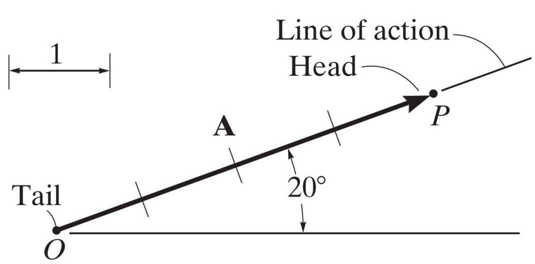
$$\vec{V} : 60\ mph,\ north$$

  here the magnitude of vector $\vec{V}$ is 60 mph (positive), also represented by $|\vec{V}|$ or V , its direction is north.

- Graphically a vector is represented by an arrow where:
  - The direction of the arrow is the direction of the vector.
  - The length is the magnitude and may be drawn to scale.

- We usually identify the direction of a vector by the angle it makes with a certain reference axis like E, N, S or W.

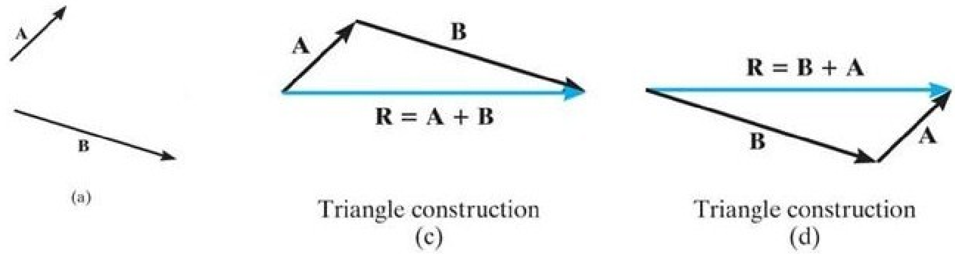
  Example: $\vec{A} : 4\ units, 20^o\ north\ of\ east$



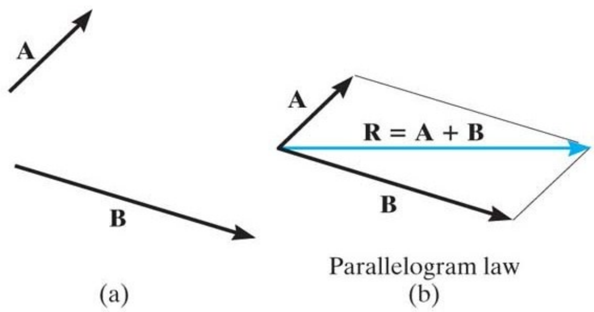
I-1. Adding vectors graphically:

There are two major ways of adding vectors, The first method is called *parallelogram construction* and the second called the *triangular construction*. The vector sum is called **resultant** and represented by $\vec{R}$. Foe example the resultant of vectors $\vec{A}$ and $\vec{B}$ is $\vec{R} = \vec{A} + \vec{B}$

- To add two or more vectors using the triangular construction we simply connect the arrows head to tail. The vector sum is the arrow starting at the tail of the first vector and ending at the head of the last vector.



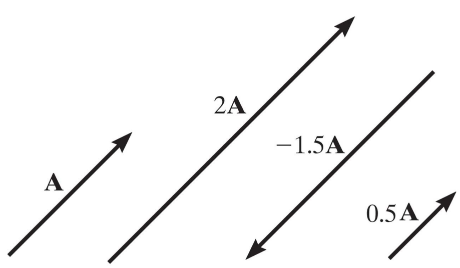
- To add two vectors using the parallelogram construction you first join the vectors at their tails then draw parallel lines from the head of each vector forming the adjacent sides of a parallelogram. The diagonal is the resultant of the two vectors



- You can measure graphically the magnitude and direction of the vector resultant $\vec{R}$

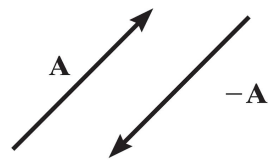
I-2. Multiplication by a factor:

- The resultant of $\alpha \cdot \vec{A}$ where α is a scalar is a vector:
  - The magnitude is equal to $\mid \alpha \mid \cdot A$
  - The direction is:
    - The same as the direction of $\vec{A}$ if α positive.
    - Opposite to the direction of $\vec{A}$ if α is negative.



Note:

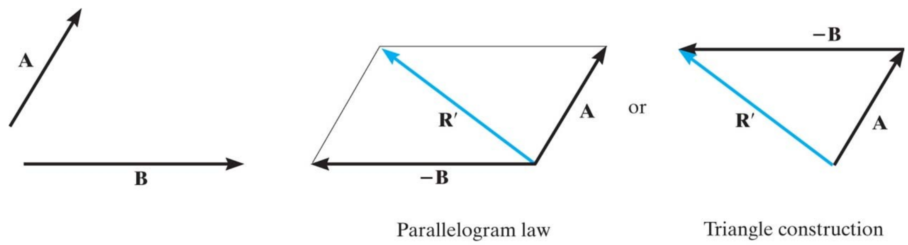
$(-\vec{A})$ is the vector opposite to $\vec{A}$. It has the same magnitude as $\vec{A}$, but opposite direction.



I-3. Subtraction of vectors:

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

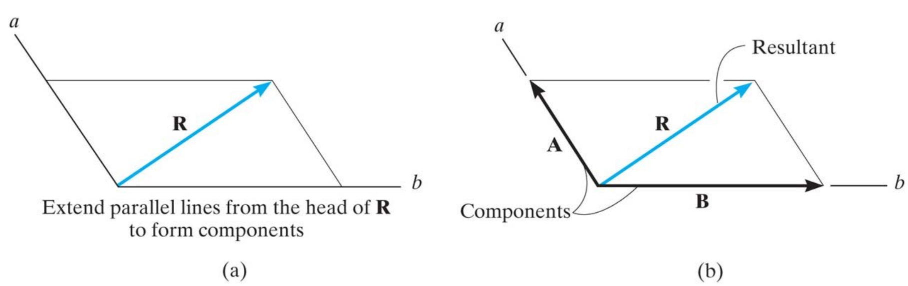
Which is the same as the resultant of $\vec{A}$ and the opposite of $\vec{B}$



II. Components of a vector:

A vector can always be represented as the sum of two components having known lines of action.

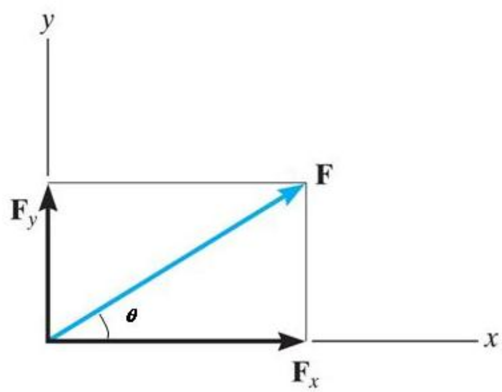
Example:



In this example the lines of action are (a) and (b). One can extend parallel lines to both (a) and (b) from the head of the vector then figure out the components $\vec{A}$ and $\vec{B}$. notice also that $\vec{R} = \vec{A} + \vec{B}$.

Note:

When the lines of action are perpendicular to each other the vector components are known as the rectangular components of the vector. In the example below the lines of action are the x and y axes. The rectangular components of vector $\vec{F}$ are labeled respectively $\vec{F}_x$ for the component along the x-axis (also refereed to as x-component of $\vec{F}$) and $\vec{F}_y$ for the component along the y-axis (also refereed to as y-component of $\vec{F}$).



$$\vec{F} = \vec{F}_x + \vec{F}_y$$

The magnitude of the x and y component of the vector can be explicitly determined from the magnitude and direction of the original vector

$$|\vec{F}_x| = F \times cos(\theta)$$
$$|\vec{F}_y| = F \times sin(\theta)$$

where F is the magnitude of vector $\vec{F}$.

Note:

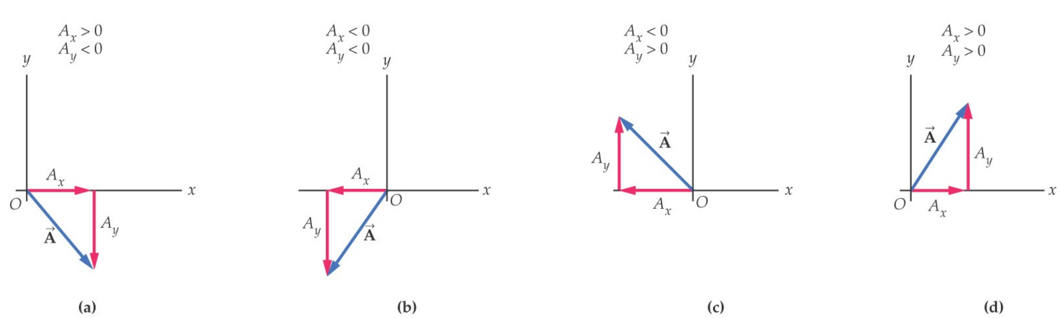
1. $\vec{F}_x$ can only point to the right or left along the x-axis or precisely in the positive or negative direction of the x-axis. Similarly $\vec{F}_y$ can only point up or down on the y-axis or in the positive or negative direction of the y-axis. alternatively one can use the following representation:

$$F_x = \pm \mid \vec{F}_x \mid$$

$$F_y = \pm \mid \vec{F}_y \mid$$

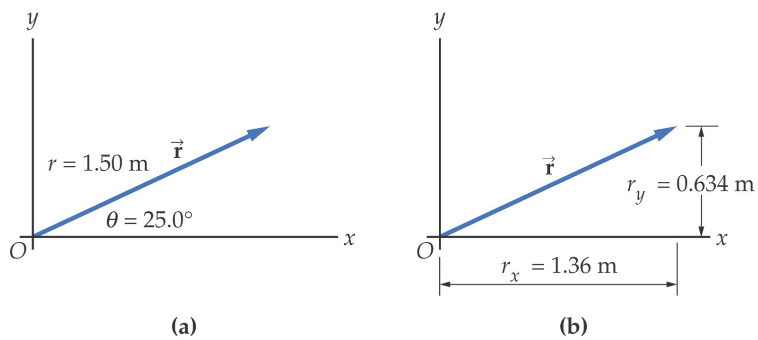
Where the + sign is used if $\vec{A}_x$ or $\vec{F}_y$ points in the positive direction and the $-$ sign is used if $\vec{F}_x$ or $\vec{F}_y$ points in the negative direction. The same vector can now be represented in the following notation: $\vec{F}$: F_x , F_y also known as the component representation of vector $\vec{F}$.

Example 1:



2. We can go from one representation to another using the adequate relations.

<u>*Example 1:*</u> $\vec{r}: 1.5\ m,\ 25^o\ north\ of\ east$



$$|\ \vec{r}_x\ |= (1.5\ m) \times cos(25) = 1.36\ m$$

$$|\ \vec{r}_y\ |= (1.5\ m) \times sin(25) = 0.634\ m$$

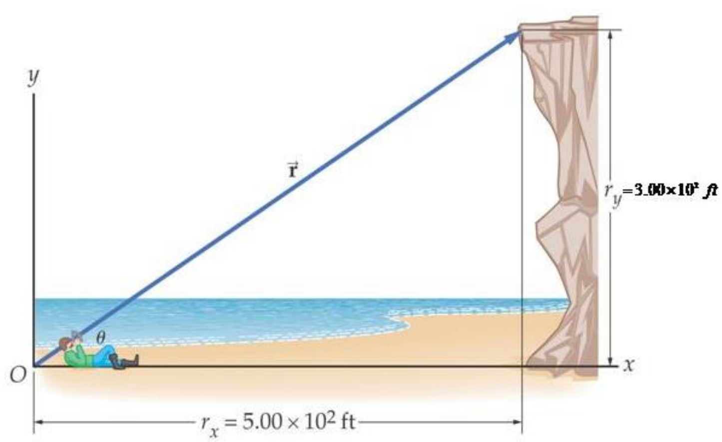
Both components point in the positive direction:

$$r_x = +1.7\ m$$

$$r_y = +1.0\ m$$

$$\boxed{\vec{r}: +1.36\ m\ ,\ \ +0.634\ m}$$

<u>*Example 2:*</u> $\vec{r}: +5.00 \times 10^2\ ft\ ,\ \ +3.00 \times 10^2\ ft$

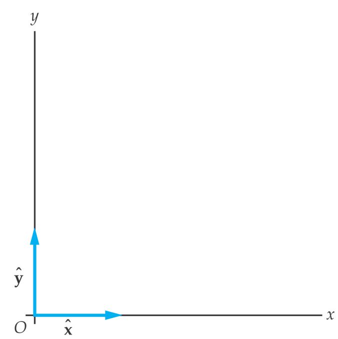


$$|\ \vec{r}\ |= \sqrt{r_x^2 + r_y^2} = \sqrt{(+5.00 \times 10^2\ ft)^2 + (+3.00 \times 10^2\ ft)^2} = 5.83 \times 10^2\ f$$

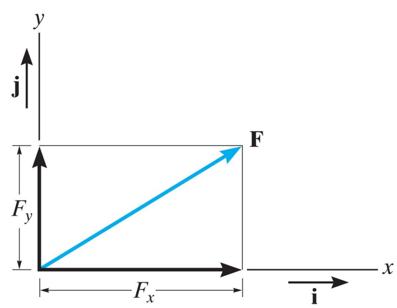
$$\theta = tan^{-1}(\frac{3.00 \times 10^2\ ft}{5.00 \times 10^2\ ft}) = 31.0^o$$

Note: Unit vectors

A unit vector is a vector that has a magnitude of exactly 1 and points in a particular direction. It lacks both dimension and unit. Its sole purpose is to point, that is, to specify a direction. The unit vectors in the positive directions of the x, y axes are labeled $\hat{x}$, $\hat{y}$, where the hat is used instead of an overhead arrow as for other vectors. Sometimes the symbols $\hat{i}$, $\hat{j}$ are used respectively instead of $\hat{x}$, and $\hat{y}$.



Unit vectors are very useful for expressing other vectors. For example, a vector that is expressed in the components notation as $\vec{F} : F_x\ ,\ F_y$, can also be written using the unit vectors along the x and y axes as $\vec{F} = F_x\ \hat{x} + F_y\ \hat{y}$ or $\vec{F} = F_x\ \hat{i} + F_y\ \hat{j}$



II-1. Addition and Subtraction of vector Components in 2D:

- Multiplying a vector by a scalar is equivalent to multiplying the components by the same scalar

  Example:

  If $\vec{A} = A_x\ \hat{i} + A_y\ \hat{j}$ then $\alpha\ \vec{A} = \alpha\ A_x\ \hat{i} + \alpha\ A_y\ \hat{j}$ where α is any scalar. particularly if $\alpha = -1$ the opposite to $\vec{A}$, $-\vec{A} = -A_x\ \hat{i} - A_y\ \hat{j}$

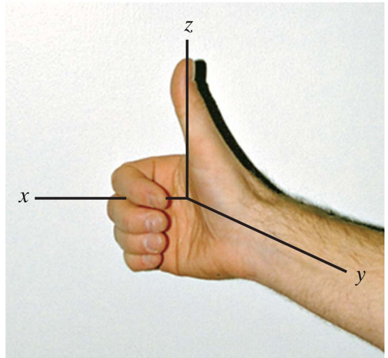
- In general adding or subtracting vector is equivalent to adding or subtracting there components.

  Example:

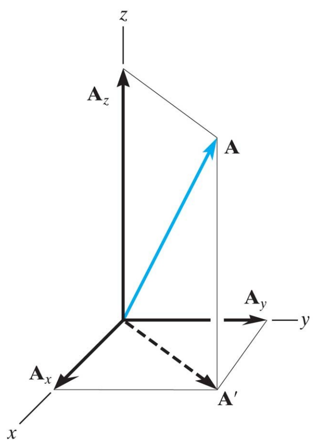
  If $\vec{A} = A_x\ \hat{i} + A_y\ \hat{j}$ and $\vec{B} = B_x\ \hat{i} + B_y\ \hat{j}$ then $\alpha\ \vec{A} + \beta\ \vec{B} = (\alpha\ A_x + \beta\ B_x)\ \hat{i} + (\alpha\ A_y + \beta\ B_y)\ \hat{j}$ where α and β are any scalars.

II-2. Vectors in space:

Everything considered up to this point involved only two dimensions, where all vectors belonged to the same plane also known as coplanar vectors. In the following we will discuss problems involving the three dimensions if space. A cartesian coordinate system is often used to solve problems in three dimensions. The coordinate system is right handed which means that the thumb of the right hand points in the direction of the positive z-axis when the right hand fingers are curled about the axis and directed from the positive x toward the positive y-axis.



Consider a vector acting at the origin of the system of rectangular coordinates x, y, z as shown in the figure below.



Vector $\vec{A}$ is the sum of an horizontal vector $\vec{A}'$, and a vertical vector $\vec{A}_z$ along the z-axis.

$$\vec{A} = \vec{A}' + \vec{A}_z$$

$\vec{A}'$ lies in the x-y plane and can then be decomposed into two components one along each axis such that:

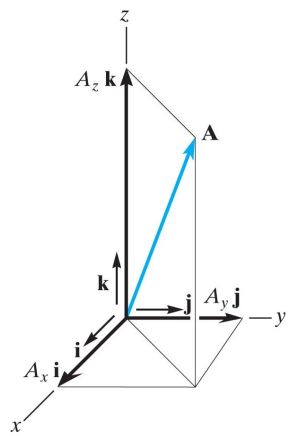
$$\vec{A}' = \vec{A}_x + \vec{A}_y$$

consequently vector $\vec{A}$ is written as the sum of three components one along each axis:

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$$

Introducing the unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, directed respectively along the x, y, z axes as shown in the figure below, we can express $\vec{A}$ in the form, also known as the cartesian representation of vector $\vec{A}$

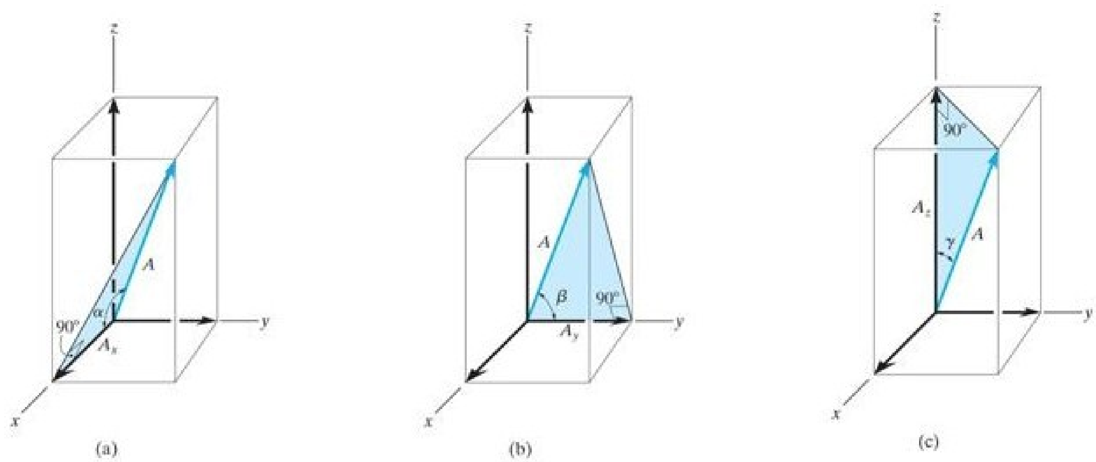
$$\vec{A} = A_x\ \hat{\imath} + A_y\ \hat{\jmath} + A_z\ \hat{k}$$



- The magnitude of $\vec{A}$ is expressed as

$$\mid \vec{A} \mid = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

- The direction of vector $\vec{A}$ is defined by the angles α, β, and γ measured between the tail of $\vec{A}$ and the positive x, y, z axes located at the tail of $\vec{A}$.
- The angles α, β, and γ are found from their direction cosines as shown in the figure below.



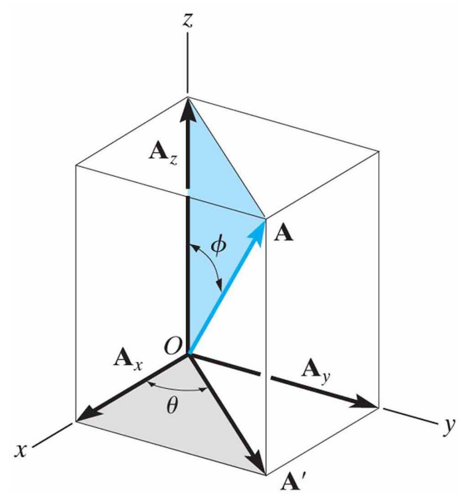
$$\cos\alpha = \frac{A_x}{A}, \quad \cos\beta = \frac{A_y}{A}, \quad \cos\gamma = \frac{A_z}{A}$$

- Notice also that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This means that only two of the angles have to be specified the third can be found from the previous relation.
- An easy way of obtaining these direction cosines is to form a unit vector $\hat{u}_A$ in the direction of $\vec{A}$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = A\ \cos(\alpha)\hat{i} + A\ \cos(\beta)\hat{j} + A\ \cos(\gamma)\hat{k}$$
$$\hat{u}_A = \frac{\vec{A}}{A} = \cos(\alpha)\hat{i} + \cos(\beta)\hat{j} + \cos(\gamma)\hat{k}$$

$\hat{u}_A$ is a unit vector (magnitude 1) of the same direction as $\vec{A}$

- Sometimes the direction of $\vec{A}$ is specified by two angles instead of three, this times the angles θ and ϕ are defined as shown below



we have the following

$$A_z = A\ \cos(\phi)$$

$$A_x = A\ \sin(\phi)\ \cos(\theta)$$

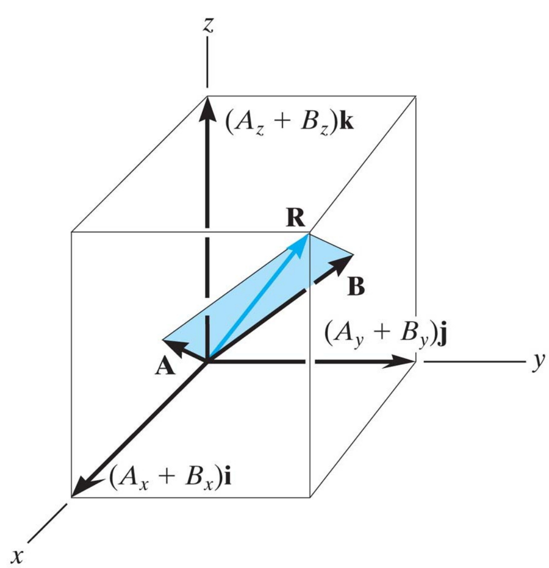
$$A_x = A\ \sin(\phi)\ \sin(\theta)$$

$$\vec{A} = A\ \sin(\phi)\ \cos(\theta)\hat{\imath} + A\ \sin(\phi)\ \sin(\theta)\hat{\jmath} + A\ \cos(\phi)hatk$$

II-2-1. Addition of cartesian vectors:

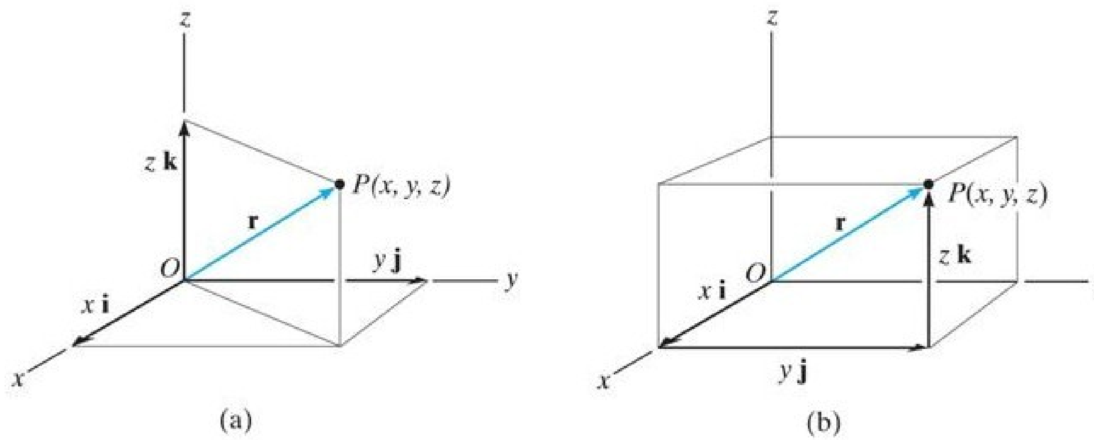
To find the resultant of set of vectors $\vec{A}$, and $\vec{B}$, express each force as a cartesian vector and add the $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ components of all the vectors in the system.

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\ \hat{\imath} + (A_y + B_y)\ \hat{\jmath} + (A_z + B_z)\ \hat{k}$$



III. Position vectors:

- The position vector $\vec{r}$ is defined as a fixed vector which locates a point in space relative to another point. For example, from the origin of coordinates O, the point in space P(x, y, z) has position vector $\vec{r} = x\ \hat{i} + y\ \hat{j} + z\ \hat{k}$



- More generally, the position vector may be directed from point A to point B in space. In this case, the position vector is again denoted by $\vec{r}$ (or sometimes $\vec{r}_{AB}$) and given by

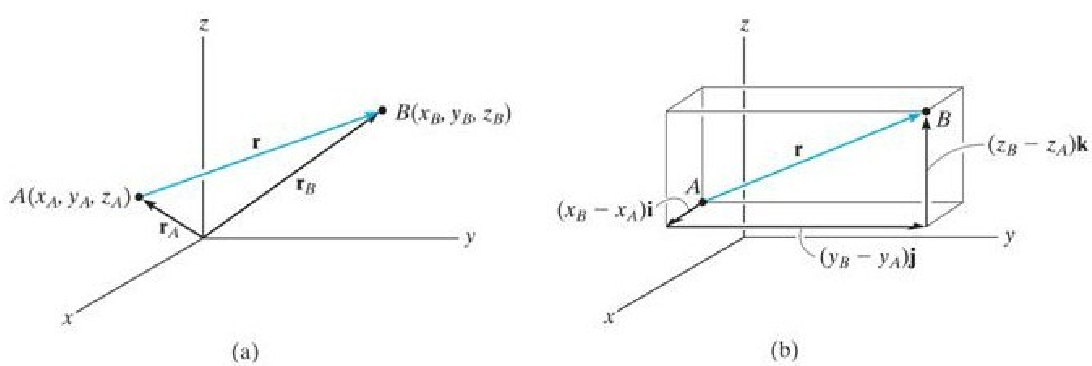
$$\vec{r}_{AB} = \vec{r}_B - \vec{r}_A$$

where $\vec{r}_A$ and $\vec{r}_B$ are the position vectors of A and B from the origin of coordinate O. For example if $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ then

$$\vec{r}_A = x_A\ \hat{i} + y_A\ \hat{j} + z_A\ \hat{k}$$

$$\vec{r}_B = x_B\ \hat{i} + y_B\ \hat{j} + z_B\ \hat{k}$$

$$\vec{r}_{AB} = (x_B - x_A)\ \hat{i} + (y_B - y_A)\ \hat{j} + (z_B - z_A)\ \hat{k}$$



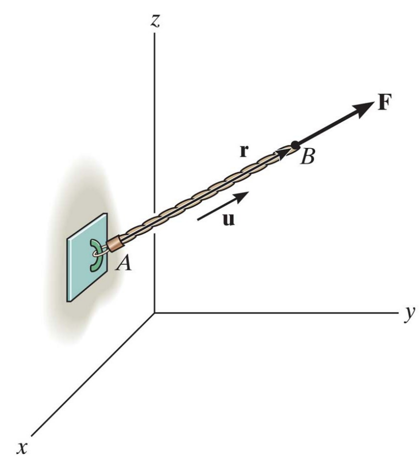
IV. Force vector directed along a line:

A force $\vec{F}$ (with magnitude F) acting in the direction of a line represented by a position vector $\vec{r}$ can be written in the form

$$\vec{F} = F\left(\frac{\vec{r}}{r}\right) = F\hat{u}$$

where $\hat{u} = \dfrac{\vec{r}}{r}$ is a unit vector representing the direction of the line. For example for a position vector $\vec{r}$ directed from $A(x_A, y_A, z_A)$ to $B(x_B, y_B, z_B)$ as shown in the figure below

$$\vec{F} = F\left(\frac{\vec{r}}{r}\right) = F\hat{u} = F\left(\frac{(x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$



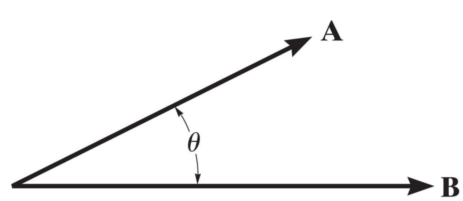
V. Dot product

- The dot product is used to determine
  - the angle between two vectors
  - The projection of a vector in a specified direction
- The dot product of two vector $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \cdot \vec{B} = A\ B\ \cos(\theta)$$

where A and B are the magnitudes of $\vec{A}$ and $\vec{B}$, respectively, and θ is the angles between the tails of $\vec{A}$ and $\vec{B}$.

$$\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{A\ B}\right)$$



- The dot products of two vectors is a scalar quantity
- We have the following results:

$$\hat{x} \cdot \hat{x} = 1$$

$$\hat{y} \cdot \hat{y} = 1$$

$$\hat{z} \cdot \hat{z} = 1$$

$$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{x} \cdot \hat{z} = 0$$

- Using the above results we have the following:

$$\vec{A} \cdot \vec{B} = (A_x\ \hat{x} + A_y\ \hat{y} + A_z\ \hat{z}) \cdot (B_x\ \hat{x} + B_y\ \hat{y} + B_z\ \hat{z}) = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z$$

- The dot product is commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. and distributive $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C})$. Also if a is a real number then $a(\vec{A} \cdot \vec{B}) = (a\vec{A}) \cdot \vec{B} = \vec{A} \cdot (a\vec{B}) = (\vec{A} \cdot \vec{B})a$
- In some engineering applications, you must resolve a vector int o components which are parallel and perpendicular to a given line. The component of vector $\vec{A}$ in the direction specified by the unit vector $\hat{u}$ is given by

$$A_{\|} = A \cdot \hat{u} = A\cos(\theta)$$

  This component is also referred to a the scalar projection of $\vec{A}$ onto the line with direction $\hat{u}$ or the component of vector $\vec{A}$ parallel to the line with direction $\hat{u}$. Clearly $\vec{A}_{\|}$ is defined by $\vec{A}_{\|} = A_{\|}\ \hat{u}$
- Once the parallel component has been determined, we can determine the component of $\vec{A}$ perpendicular to a line with direction $\hat{u}$ by

$$A_{\perp} = \sqrt{A^2 - A_{\|}^2}$$

- Clearly, in terms of vectros

$$\vec{A} = \vec{A}_{\|} + \vec{A}_{\perp}$$