Newton-Euler Equations

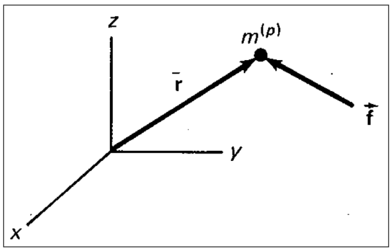
Reading material:
Chapter 8 Basic Concepts in Dynamics

Dynamics of a Particle

$$\vec{f} = m^{(p)}\vec{a}$$

Or in vector form:

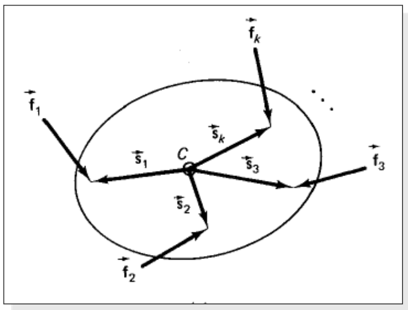
$$\mathbf{f} = m^{(p)}\ddot{\mathbf{r}}$$



f = force (*N*)
r = x ***i*** + y ***j*** + z ***k*** = [x, y, z]^T (*m*)
a = d^2**r**/dt^2 = acceleration (*m/s^2*)
$m^{(p)}$ = mass of the particle P (*kg*)

$$\mathbf{f} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ f_{(z)} \end{bmatrix}_{3\times1} = m^{(p)}\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}_{3\times1} = \begin{bmatrix} m^{(p)} & 0 & 0 \\ 0 & m^{(p)} & 0 \\ 0 & 0 & m^{(p)} \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}_{3\times1}$$

Equivalent Systems of a Rigid Body



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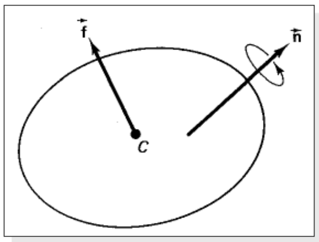


將所有外力轉換爲作用於質心之力和力矩, 以便作系統化的處理. (由於力矩爲自由向量, 故不必轉換)

$$\mathbf{f} = \sum_{i=1}^{k} \mathbf{f}_i$$

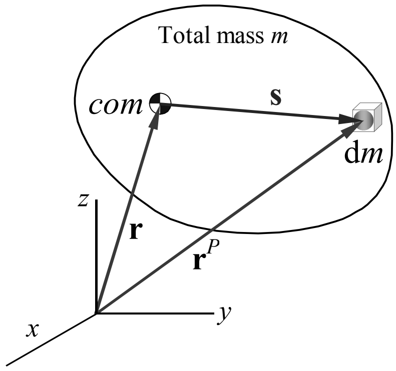
$$\mathbf{n} = \sum_{i=1}^{k} \mathbf{n}_i = \sum_{i=1}^{k} \tilde{\mathbf{s}}_i \mathbf{f}_i^{(e)}$$

Rigid Body's Center of Mass (com)

$$\mathbf{r} = \frac{1}{m}\int_{(v)} \mathbf{r}^P \, dm$$

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}$$

$$\cancel{\mathbf{r}} = \frac{1}{m}\int_{(v)} (\mathbf{r} + \mathbf{s}) \, dm = \cancel{\mathbf{r}} + \frac{1}{m}\int_{(v)} \mathbf{s} \, dm$$



$$\int_{(v)} \mathbf{s} dm = 0 \Rightarrow \int_{(v)} \dot{\mathbf{s}} dm = 0, \quad \int_{(v)} \ddot{\mathbf{s}} dm = 0$$

剛體上, 各質點相對於質心之運動所造成之動量和爲零

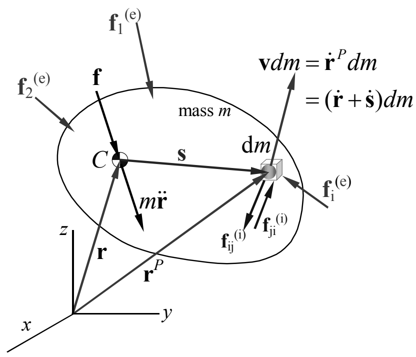
A. Newton's Equation of Motion
Momentum & Translational Equation of Motion

$$\int_{(v)} \dot{\mathbf{s}}\,dm = 0$$

- Momentum **P** ≡ ∫**v**d*m* = ∫d(**r**+**s**)/d*t* d*m* = ∫(d**r**/d*t*) d*m* = (d**r**/d*t*) *m*
  - 即整個剛體的動量 = [剛體之質量] [質心之速度]
- Newton equation: The rate of change of the linear momentum equals the total force applied to the body; i.e.,

$$\dot{\mathbf{P}} = \frac{d\mathbf{P}}{dt} = \frac{d(m\dot{\mathbf{r}})}{dt} = m\ddot{\mathbf{r}} = \sum \mathbf{f}_i^{(e)} + \sum \mathbf{f}_{ij}^{(i)} = \mathbf{f}$$

Internal forces cancel out in pairs



f = the resultant of external forces (*N*), **act at the com**
r = x ***i*** + y ***j*** + z ***k*** = [x, y, z]^T (*m*) =the vector from origin to com
a = d^2**r**/dt^2 = acceleration of the com(*m/s*^{2})
m = mass of the rigid body (*kg*)

$$\mathbf{f} = m\ddot{\mathbf{r}}$$

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B. Euler's Equation of Motion
Angular Momentum & Rotational Equation of Motion

$$\dot{\mathbf{H}} = \frac{d\mathbf{H}}{dt} = \mathbf{n}$$

$$\boxed{\mathbf{n} = \mathbf{J}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{J}\boldsymbol{\omega}}$$

where

$$\mathbf{J} = -\int_{(v)} \tilde{\mathbf{s}}\tilde{\mathbf{s}}\,dm = \text{global inertia tensor}$$

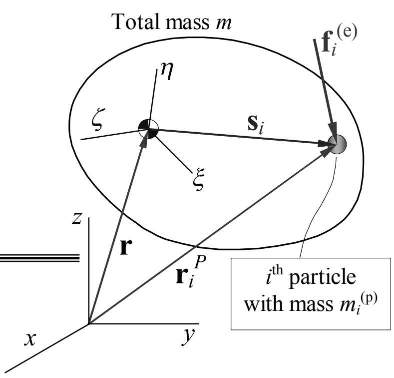
$$\mathbf{n} = \sum_{i=1}^{k} \mathbf{n}_i = \sum_{i=1}^{k} \tilde{\mathbf{s}}_i \mathbf{f}_i^{(e)}$$

$$\boxed{\mathbf{n}' = \mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}'}$$

where

$$\mathbf{J}' = -\int_{(v)} \tilde{\mathbf{s}}'\,\tilde{\mathbf{s}}'\,dm = \text{local inertia tensor}$$

$$\mathbf{n}' = \sum_{i=1}^{k} \mathbf{n}_i' = \sum_{i=1}^{k} \tilde{\mathbf{s}}_i'\,\mathbf{f}_i^{(e)}$$



Note: <u>**J**' *is a constant matrix*</u>, while <u>**J** *is a function of the angular orientation of the body*</u>

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B.1 dH_G/dt = n = total torque

$$\int_{(v)} \dot{\mathbf{s}}\, dm = 0$$

$$\mathbf{H}_G \equiv \int_v \mathbf{s} \times (\dot{\mathbf{r}}^P dm)$$

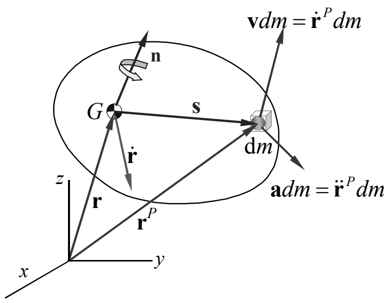
$$\dot{\mathbf{H}}_G = \frac{d}{dt}\int_v \mathbf{s} \times (\dot{\mathbf{r}}^P dm) = \int_v \frac{d}{dt}\left(\mathbf{s} \times \dot{\mathbf{r}}^P dm\right)$$

$$= \int \dot{\mathbf{s}} \times \dot{\mathbf{r}}^P dm + \int \mathbf{s} \times \ddot{\mathbf{r}}^P dm$$

$$= \int \dot{\mathbf{s}} \times (\dot{\mathbf{r}} + \dot{\mathbf{s}}) dm + \int \mathbf{s} \times (\mathbf{f}^{(e)} + \mathbf{f}^{(i)})$$

$$= \int \dot{\mathbf{s}} \times \dot{\mathbf{r}} dm + \cancel{\int \dot{\mathbf{s}} \times \dot{\mathbf{s}} dm} + \int \mathbf{s} \times \mathbf{f}^{(e)} + \cancel{\int \mathbf{s} \times \mathbf{f}^{(i)}}$$

Internal forces cancel out in pairs



$$\dot{\mathbf{H}}_G = (\cancel{\int \dot{\mathbf{s}} dm}) \times \dot{\mathbf{r}} + \mathbf{n} = \mathbf{n}$$

Euler equation: The rate of change of the angular momentum equals the total torque applied to the body

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B.2 Angular momentum H_G= J ω

$$\int_{(v)} \mathbf{s}\, dm = 0$$

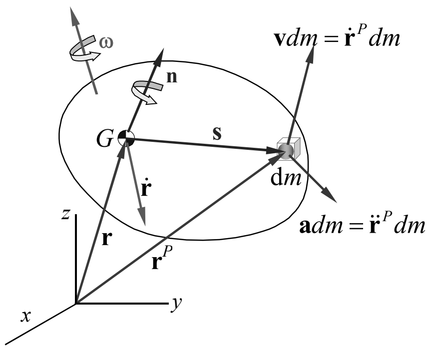
$$\mathbf{H}_G \equiv \int_v \mathbf{s} \times (\dot{\mathbf{r}}^P dm) = \int \mathbf{s} \times (\dot{\mathbf{r}} + \dot{\mathbf{s}}) dm$$

$$= \int \mathbf{s} \times \dot{\mathbf{r}} dm + \int \mathbf{s} \times \dot{\mathbf{s}} dm$$

$$= (\int \mathbf{s} dm) \times \dot{\mathbf{r}} + \int \mathbf{s} \times (\boldsymbol{\omega} \times \mathbf{s}) dm$$

$$= \int \mathbf{s} \times (\boldsymbol{\omega} \times \mathbf{s}) dm$$

$$= -\int \mathbf{s} \times (\mathbf{s} \times \boldsymbol{\omega}) dm$$



$$\mathbf{H}_G = -\int \tilde{\mathbf{s}}\tilde{\mathbf{s}} \boldsymbol{\omega} dm = -(\int \tilde{\mathbf{s}}\tilde{\mathbf{s}} dm)\boldsymbol{\omega}$$

Inertia tensor w.r.t inertia frame

$$\mathbf{J} \equiv -\int_{(v)} \tilde{\mathbf{s}}\tilde{\mathbf{s}}\, dm$$

$$\mathbf{H}_G = \mathbf{J}\boldsymbol{\omega}$$

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B.3 Inertia Tensor w.r.t body frame

$$\mathbf{J}' \equiv -\int_{(v)} \tilde{\mathbf{s}}'\tilde{\mathbf{s}}' dm = \text{local inertia tensor}$$

$$\mathbf{J}' = -\int_{(v)} \begin{bmatrix} s_{(\eta)}^2 + s_{(\zeta)}^2 & -s_{(\xi)}s_{(\eta)} & -s_{(\xi)}s_{(\zeta)} \\ -s_{(\xi)}s_{(\eta)} & s_{(\zeta)}^2 + s_{(\xi)}^2 & -s_{(\eta)}s_{(\zeta)} \\ -s_{(\xi)}s_{(\zeta)} & -s_{(\eta)}s_{(\zeta)} & s_{(\xi)}^2 + s_{(\eta)}^2 \end{bmatrix} dm = \begin{bmatrix} j_{\xi\xi} & j_{\xi\eta} & j_{\xi\zeta} \\ j_{\xi\eta} & j_{\eta\eta} & j_{\eta\zeta} \\ j_{\xi\zeta} & j_{\eta\zeta} & j_{\zeta\zeta} \end{bmatrix}$$

- A new array of moments and products of inertia would be obtained if a different system of axes were used.
- If the axes are chosen to coincide with principal axes, the inertia matrix takes the diagonal form:

$$\mathbf{J}' = \begin{bmatrix} J_{\xi\xi} & 0 & 0 \\ 0 & J_{\eta\eta} & 0 \\ 0 & 0 & J_{\zeta\zeta} \end{bmatrix}$$

- How to do this?

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B.4 The Relation Between (Local) Inertia Tensor and Global Inertia Tensor

$$\mathbf{J} = -\int_{(v)} \tilde{\mathbf{s}}\tilde{\mathbf{s}} dm$$

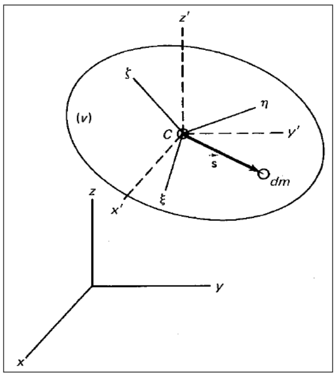
$$\tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T$$

$$\mathbf{A}^{-1} = \mathbf{A}^T \Rightarrow \mathbf{A}\mathbf{A}^T = \mathbf{I}$$

$$= -\int_{(v)} (\mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T)(\mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T) dm$$

$$= \mathbf{A}\left(-\int_{(v)} \tilde{\mathbf{s}}'\tilde{\mathbf{s}}' dm\right)\mathbf{A}^T$$

$$= \mathbf{A}\mathbf{J}'\mathbf{A}^T$$



- Which is used to transform the inertia tensor from <u>local coordinates</u> to <u>global coordinates</u>

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B.5 Derive the Euler Equation in Terms of Inertia Frame

- **J** = **AJ**'**A**^{T}, where **J**' is a constant matrix
- Angular momentum **H** = ∫**s**×(ω×**s**)dm = -∫**s**×(**s**×ω)dm = **J** ω
- Euler equation: d**H**/dt = **n**

$$\mathbf{H} = \mathbf{J}\boldsymbol{\omega} = (\mathbf{A}\mathbf{J}'\mathbf{A}^T)(\mathbf{A}\boldsymbol{\omega}') = \mathbf{A}\mathbf{J}'\boldsymbol{\omega}'$$

$$\dot{\mathbf{H}} = \dot{\mathbf{A}}\mathbf{J}'\boldsymbol{\omega}' + \mathbf{A}\mathbf{J}'\dot{\boldsymbol{\omega}}' = (\tilde{\boldsymbol{\omega}}\mathbf{A})\mathbf{J}'(\mathbf{A}^T\boldsymbol{\omega}) + \mathbf{A}\mathbf{J}'(\mathbf{A}^T\dot{\boldsymbol{\omega}}) = \tilde{\boldsymbol{\omega}}\mathbf{J}\boldsymbol{\omega} + \mathbf{J}\dot{\boldsymbol{\omega}}$$

$$\dot{\mathbf{H}} = \frac{d\mathbf{H}}{dt} = \mathbf{n}$$

$$\mathbf{n} = \mathbf{J}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{J}\boldsymbol{\omega}$$

$$\mathbf{n}' = \mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}'$$

$$\boldsymbol{\omega} = \mathbf{A}\boldsymbol{\omega}' \Leftrightarrow \boldsymbol{\omega}' = \mathbf{A}^T\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\omega}} = \mathbf{A}\dot{\boldsymbol{\omega}}' \Leftrightarrow \dot{\boldsymbol{\omega}}' = \mathbf{A}^T\dot{\boldsymbol{\omega}}$$

因爲**J**'爲固定矩陣, 故可先將 **J** 轉換爲**J**', 使易於求時間導數

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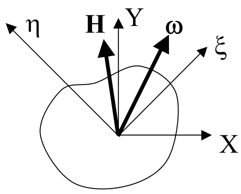
Principal Axes and Principal Moments of Inertia

$$\mathbf{J}' = -\int_{(v)} \tilde{\mathbf{s}}'\tilde{\mathbf{s}}'\,dm$$

- Angular momentum **H**=∫**s**×(ω×**s**)dm=-∫**s**×(**s**×ω)dm=**J**ω=(**AJ**'**A**^{T})ω
- If the body coordinate axes are oriented such that they become principal axes of inertia of the body, we have



$$\begin{bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{bmatrix} = \begin{bmatrix} J_{\xi\xi} & 0 & 0 \\ 0 & J_{\eta\eta} & 0 \\ 0 & 0 & J_{\zeta\zeta} \end{bmatrix} \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}$$

- In such a case, if the body is spinning about any of its principal axes, the resulting angular momentum **H** must be parallel to the angular velocity ω. For instance, letting ω =[ω_ξ, 0,0] yields

$$\begin{bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{bmatrix} = \begin{bmatrix} J_{\xi\xi} & 0 & 0 \\ 0 & J_{\eta\eta} & 0 \\ 0 & 0 & J_{\zeta\zeta} \end{bmatrix} \begin{bmatrix} \omega_\xi \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} J_{\xi\xi}\omega_\xi \\ 0 \\ 0 \end{bmatrix} = J_{\xi\xi} \begin{bmatrix} \omega_\xi \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{H} = J_{\xi\xi}\boldsymbol{\omega}$$

- Now we know that **H** =λω (λ is a scalar) if ω parallel to a principal axis.
- Conversely, we may choose ω as unknown, and solve the eigenvalue problem **H**≡**J**ω=λω for ω. The resulting eigenvector ω must be coincident with a principal axis, and the λ the principal moment of inertia corresponding to that axis.

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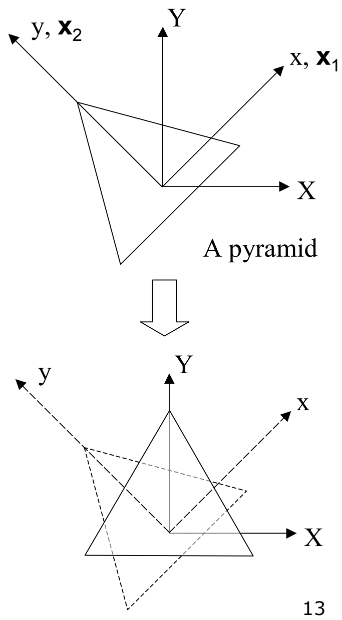
Principal Axes and Principal Moments of Inertia (cont)

$H = J\omega = \lambda\omega$

- Find the eigenvectors $\mathbf{x}_i$ (i=1,2,3) of a known inertia tensor **J**
- Make a matrix **R** using the eigenvectors as columns
  - $R=[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$
  - The eigenvectors are the unit vectors of the principal axes
- Note that **R** can be thought of as a rotation matrix that rotate body from an orientation X-Y-Z to another orientation x-y-z.



- If one make a matrix A with the eigenvectors as rows ($A=\mathbf{R}^T=\mathbf{R}^{-1}$) , and then transform the vertices of a body. One will see the new moment of inertia matrix for the transformed body is diagonal.

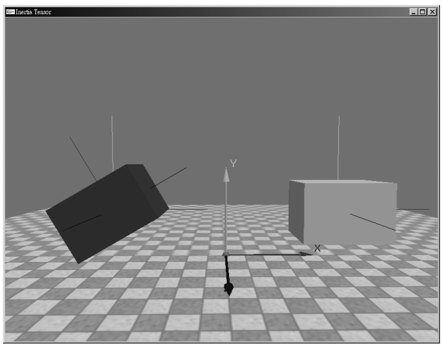
Example

- Reading in 8 vertices (original cuboid, V=3x2x2=12 m^3)
- Reading in 12 faces
- mass = +11.999995
- center of mass: (+0.000000, -0.000000, +0.000000)
- inertia tensor with origin at c.o.m. :
- +9.249993 -2.165063 +0.000000
- -2.165063 +11.749994 +0.000000
- +0.000000 +0.000000 +12.999991

- Eigenvectors (in column) :
- +0.866025 -0.500000 +0.000000
- +0.500000 +0.866025 +0.000000
- +0.000000 +0.000000 +1.000000

- Eigenvalue :
- +7.999994 +12.999994 +12.999991

- Writing out 8 vertices (modified cuboid)
- writing out 12 faces



- Reading in 8 vertices (modified cuboid)
- Reading in 12 faces
- mass = +11.999994 (true value=12.0)
- center of mass: (+0.000000, +0.000000, +0.000000)
- inertia tensor with origin at c.o.m. : (I_{xx}=8, I_{yy}=I_{zz}=13.0)
- +7.999992 -0.000003 +0.000000
- -0.000003 +12.999993 +0.000000
- +0.000000 +0.000000 +12.999990

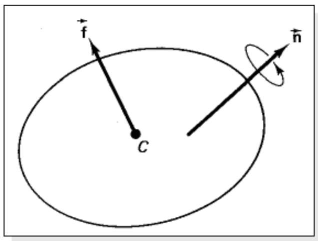
Motion Equations of An Unconstraint Body

$$\boxed{m_i \ddot{\mathbf{r}}_i = \mathbf{f}_i}$$
$$\boxed{\mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' = \mathbf{n}'}$$

$$\Rightarrow \boxed{\begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}' \end{bmatrix}_i \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix}_i + \begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \end{bmatrix}_i = \begin{bmatrix} \mathbf{f} \\ \mathbf{n}' \end{bmatrix}_i}$$

where $\mathbf{N}_\mathbf{i} = diag[m, m, m]_\mathbf{i}$

6×6	$\mathbf{M}_i = \begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}' \end{bmatrix}_i$
6×1	$\mathbf{h}_i = \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix}_i$
6×1	$\mathbf{b}_i = \begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \end{bmatrix}_i$
6×1	$\mathbf{g}_i = \begin{bmatrix} \mathbf{f} \\ \mathbf{n}' \end{bmatrix}_i$



$$\boxed{\mathbf{M}_\mathbf{i}\dot{\mathbf{h}}_\mathbf{i} + \mathbf{b}_\mathbf{i} = \mathbf{g}_\mathbf{i}}$$

Motion Equations of A System of Unconstrained Bodies

- For the *i*th body of a system, we have

$$\boxed{\mathbf{M}_\mathbf{i}\dot{\mathbf{h}}_\mathbf{i} + \mathbf{b}_\mathbf{i} = \mathbf{g}_\mathbf{i}} \quad \mathbf{M}_i = \begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}' \end{bmatrix}_i \quad \mathbf{h}_i = \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix}_i \quad \mathbf{b}_i = \begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \end{bmatrix}_i \quad \mathbf{g}_i = \begin{bmatrix} \mathbf{f} \\ \mathbf{n}' \end{bmatrix}_i$$

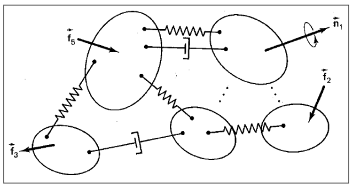
- Repeating this equation for all the *b* bodies of a system yields

$$\begin{bmatrix} \mathbf{M}_1 & & & \\ & \mathbf{M}_2 & & \\ & & \ddots & \\ & & & \mathbf{M}_b \end{bmatrix} \begin{bmatrix} \dot{\mathbf{h}}_1 \\ \dot{\mathbf{h}}_2 \\ \\ \dot{\mathbf{h}}_b \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_b \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_b \end{bmatrix}$$

6b×6b 6b×1 6b×1 6b×1

or

$$\boxed{\mathbf{M}\dot{\mathbf{h}} + \mathbf{b} = \mathbf{g}}$$



Motion Equations of A System of Constrained Bodies

$$\boxed{\mathbf{\Phi} \equiv \mathbf{\Phi}(\mathbf{q}) = \mathbf{0}}$$

$$\boxed{\mathbf{M}\dot{\mathbf{h}} + \mathbf{b} = \mathbf{g} + \mathbf{g}^{(c)}} \longleftarrow \text{Unconstrained } \boxed{\mathbf{M}\dot{\mathbf{h}} + \mathbf{b} = \mathbf{g}}$$

where $g^{(c)}$ is the vector of joint reaction forces of constraint forces

$$\mathbf{g}^{(c)} = \begin{bmatrix} \mathbf{g}_1^{(c)} \\ \mathbf{g}_2^{(c)} \\ \vdots \\ \mathbf{g}_b^{(c)} \end{bmatrix}_{6b\times 1} = \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}$$