Planar Dynamics
(Newton-Euler Method)

Reading material: Chapter 9

Equations of Motion for an Unconstrained Body

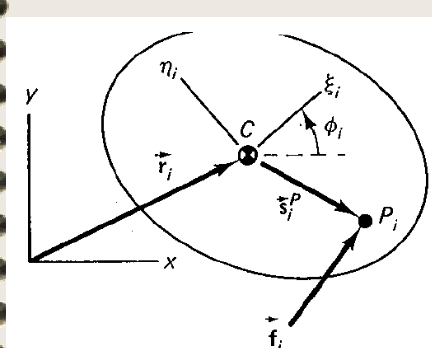
$$m_i \ddot{\mathbf{r}}_i = \mathbf{f}_i$$

$$\mathbf{J}_i' \dot{\boldsymbol{\omega}}_i' + \tilde{\boldsymbol{\omega}}_i' \mathbf{J}_i' \boldsymbol{\omega}_i' = \mathbf{n}_i'$$

$$\begin{aligned} m_i \ddot{x}_i &= f_{(x)_i} \\ m_i \ddot{y}_i &= f_{(y)_i} \\ j_{\zeta\zeta_i} \ddot{\phi}_i &= n_i \end{aligned}$$

$$\begin{bmatrix} m & & \\ & m & \\ & & \mu \end{bmatrix}_i \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i$$

$$\boxed{\mathbf{M}_i \ddot{\mathbf{q}}_i = \mathbf{g}_i}$$



Equations of Motion for a System of Unconstrained Bodies

$$\begin{bmatrix} m & & \\ & m & \\ & & \mu \end{bmatrix}_i \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i$$

$$\boxed{\mathbf{M}_i \ddot{\mathbf{q}}_i = \mathbf{g}_i}$$

For a system of b unconstrained bodies, this equation is repeated b times to yield

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} = \mathbf{g}}$$

3b×3b

$$\begin{bmatrix} \mathbf{M}_1 & & & \\ & \mathbf{M}_2 & & \\ & & \ddots & \\ & & & \mathbf{M}_b \end{bmatrix}$$

3b×1

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_b \end{bmatrix}_{3b\times 1} = [q_1 \quad q_2 \quad \cdots \quad q_n]^T$$

3b×1

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_b \end{bmatrix}$$

Equations of Motion for a System of Constrained Bodies

For b bodies with m independent constraints, we have

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{g} + \mathbf{g}^{(c)}$$

$$\mathbf{g}^{(c)} = \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{g}}$$

$n(=3b)$ equations with $n+m$ unknowns

For the m independent constraints $\Phi=0$, we have

$$\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -[(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + 2\mathbf{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} + \mathbf{\Phi}_{tt}] \equiv \boldsymbol{\gamma}$$

$$\boxed{\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{bmatrix}}$$

Ax = c

(n+m)x(n+m) (n+m)x1 (n+m)x1

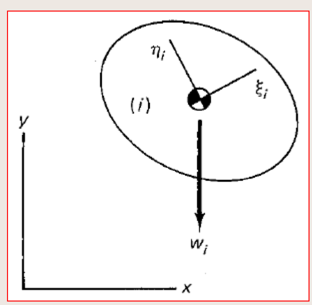
Forces Vector g

- Gravitational Forces
- Single Force or Moment
- Translational Actuators
- Translational Springs
- Translational Dampers
- Rotational Springs
- Joint Reaction Forces
  - Revolute Joint
  - Translational Joint

Gravitational Forces

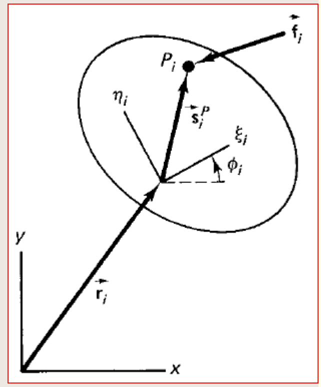
- For each body i with a mass of m_i,

$$\mathbf{g}_i^{(gravity)} = \begin{bmatrix} 0 \\ -m_i G \\ 0 \end{bmatrix}_i$$



A Body Acted Upon by a Single Force

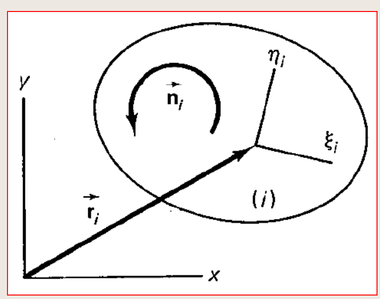
$$\mathbf{g}_i^{(\text{single } f)} = \begin{bmatrix} f_x \\ f_y \\ n \end{bmatrix}_i$$



$$\begin{aligned} n_i &= (\tilde{\mathbf{s}}_i^P \mathbf{f}_i)_{(z)} \\ &= s_{(y)_i}^P f_{(x)_i} + s_{(x)_i}^P f_{(y)_i} \\ &= -(\xi_i^P \sin \phi_i + \eta_i^P \cos \phi_i) f_{(x)_i} + (\xi_i^P \cos \phi_i - \eta_i^P \sin \phi_i) f_{(y)_i} \end{aligned}$$

A Body Acted Upon by a Single Moment

$$\mathbf{g}_i^{(\text{single } n)} = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}_i$$



Example: Body Acted Upon by a Force and a Moment

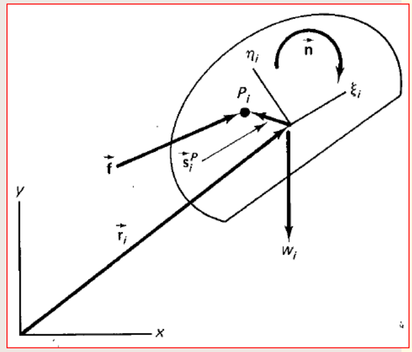
Body i, with a mass of 2, is acted upon by gravity, a constant force, and a pure moment, as shown in the illustration. The constant force has the components $\mathbf{f} = [1.2, 0.5]^T$, and the magnitude of the pure moment is $\boxed{0.6}$. Determine the vector of force for this body if $\mathbf{s}_i'^P = [-0.2, 0.3]^T$, $\phi_i = 30°$, and $\mathbf{r}_i = [2.1, 1.6]^T$.

$$w_i = 2 \times 9.81 = 19.62$$

$$n_i = (\breve{\mathbf{s}}_i^P \mathbf{f}_i)_{(z)}$$

$$n_i = -0.35$$

$$\mathbf{g}_i = \begin{bmatrix} 1.2 \\ 0.5 - 19.62 \\ -0.35 - \boxed{0.6} \end{bmatrix} = \begin{bmatrix} 1.2 \\ -19.12 \\ -0.95 \end{bmatrix}$$

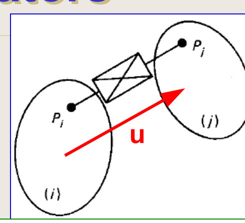


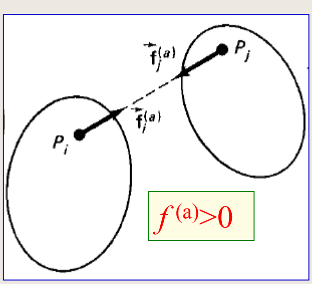
Translational Actuators

$$\boldsymbol{l} = \mathbf{r}_j + \mathbf{A}_j \mathbf{s}_j'^P - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}_i'^P$$

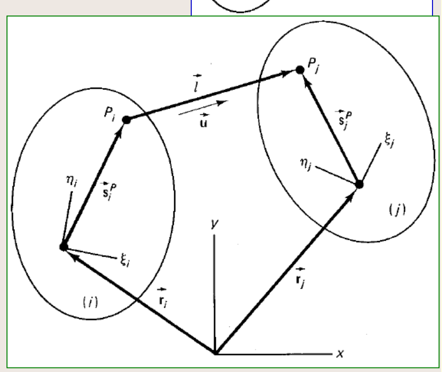
$$\mathbf{u} = \frac{\boldsymbol{l}}{l} \qquad l = (\boldsymbol{l}^T \boldsymbol{l})^{1/2}$$

$$\mathbf{f}_i^{(a)} = f^{(a)} \mathbf{u}$$





$f^{(a)} > 0$



Translational Springs

$$\mathbf{f}_i^{(s)} = f^{(s)}\mathbf{u}$$

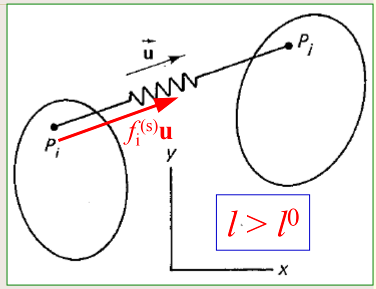
where

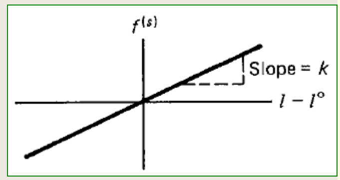
$$f^{(s)} = k(l - l^0)$$

$$\boldsymbol{l} = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^P$$

$$l = (\boldsymbol{l}^T\boldsymbol{l})^{1/2}$$

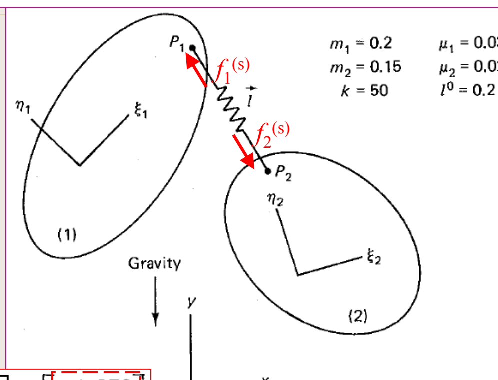
$$\mathbf{u} = \frac{\boldsymbol{l}}{l}$$





Example

Two bodies are connected by a translational spring, where $\mathbf{s}_1'^P = [0.15, 0]^T$ and $\mathbf{s}_2'^P = [0, 0.1]^T$ (see the illustration). Write the equations of motion when $\mathbf{q}_1 = [-0.1, 0.2, 0.785]^T$ and $\mathbf{q}_2 = [0.1, 0.1, 0.262]^T$, and then calculate the accelerations.



$l = 0.129$

$\mathbf{u} = [0.528, -0.849]^T$

$f^{(s)} = 50(0.129 - 0.2) = -3.558$

$\mathbf{f}_1^{(s)} = [-1.878, 3.022]^T$

$\mathbf{f}_2^{(s)} = [1.878, -3.022]^T$

$$\begin{bmatrix} 0.2 & & & & & \\ & 0.2 & & & & \\ & & 0.03 & & & \\ & & & 0.15 & & \\ & & & & 0.15 & \\ & & & & & 0.02 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\phi}_2 \end{bmatrix} = \begin{bmatrix} -1.878 \\ 1.060 \\ 0.520 \\ 1.878 \\ -4.494 \\ -0.103 \end{bmatrix}$$

Translational Dampers

$$\mathbf{f}_i^{(d)} = f^{(d)}\mathbf{u}$$

where

$$f^{(d)} = d\dot{l}$$

Damping coefficient

$$l = (\boldsymbol{l}^T\boldsymbol{l})^{1/2}$$

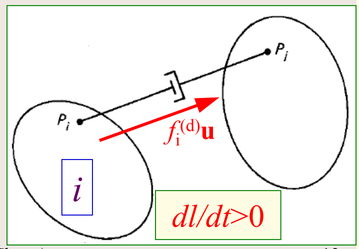
$$\dot{l} = \frac{1}{2}(\boldsymbol{l}^T\boldsymbol{l})^{-\frac{1}{2}}\frac{d}{dt}(\boldsymbol{l}^T\boldsymbol{l}) = \frac{1}{2l}(\dot{\boldsymbol{l}}^T\boldsymbol{l}+\boldsymbol{l}^T\dot{\boldsymbol{l}}) = \frac{\dot{\boldsymbol{l}}^T\boldsymbol{l}}{l}$$

Velocity component in the direction of $\boldsymbol{l}$

$$\boldsymbol{l} = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^P$$

$$\dot{\boldsymbol{l}} = \dot{\mathbf{r}}_j + \dot{\phi}_j\mathbf{B}_j\mathbf{s}_j'^P - \dot{\mathbf{r}}_i - \dot{\phi}_i\mathbf{B}_i\mathbf{s}_i'^P$$

$$\mathbf{B}_k = \begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix}_k \qquad k = i, j$$

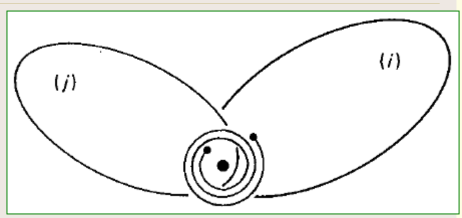


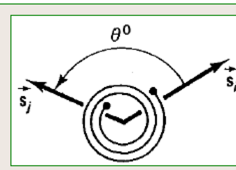
Rotational Springs

$$n^{(r\text{-}s)} = k(\theta - \theta^0)$$

Deformed angle

Undeformed angle

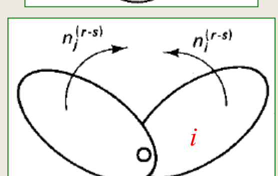




$$n_i^{(r\text{-}s)} = n^{(r\text{-}s)}$$

$$n_j^{(r\text{-}s)} = -n^{(r\text{-}s)}$$

$$\theta > \theta^0$$

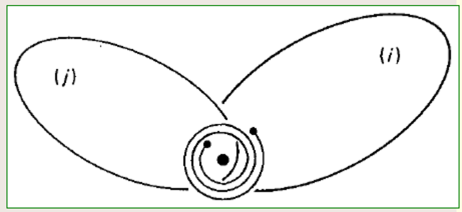


Rotational Dampers

$$n^{(r\text{-}d)} = d\dot{\theta}$$

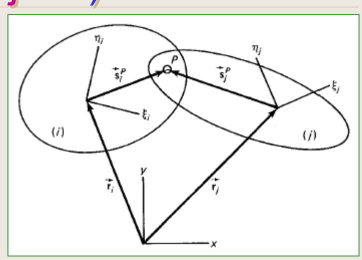
Time rate of change of the element angle

Damping coefficient



Constraint Reaction Forces
(Revolute joint)

$$\mathbf{g}^{(c)} = \mathbf{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda}$$



For object i :

$$\mathbf{M}_i\ddot{\mathbf{q}}_i - \mathbf{\Phi}_{\mathbf{q}_i}^{T}\boldsymbol{\lambda} = \mathbf{g}_i$$

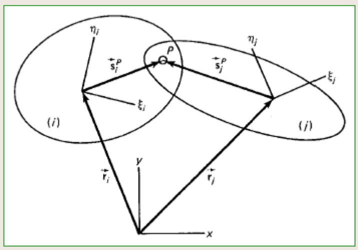
$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix}_i \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(y_i^P - y_i) & (x_i^P - x_i) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i$$

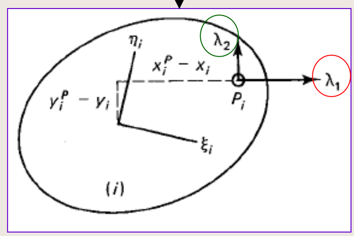
	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\Phi^{(r,2)}$	1	0	$-(y_i^P - y_i)$	-1	0	$(y_j^P - y_j)$
	0	1	$(x_i^P - x_i)$	0	-1	$-(x_j^P - x_j)$

Revolute joint Object *i*

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix}_i \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(y_i^P - y_i) & (x_i^P - x_i) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i$$

$$m_i\ddot{x}_i = f_{(x)_i} + \lambda_1$$
$$m_i\ddot{y}_i = f_{(y)_i} + \lambda_2$$
$$\mu_i\ddot{\phi}_i = n_i - (y_i^P - y_i)\lambda_1 + (x_i^P - x_i)\lambda_2$$

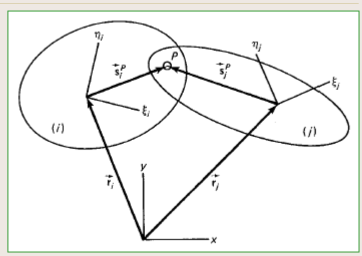




Revolute joint Object *j*

For object *j* :

$$\mathbf{M}_j\ddot{\mathbf{q}}_j - \mathbf{\Phi}_{\mathbf{q}_j}^T\boldsymbol{\lambda} = \mathbf{g}_j$$



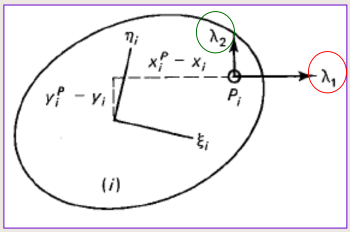
$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix}_j \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_j - \begin{bmatrix} \boxed{-1 \quad\quad 0} \\ 0 \quad\quad -1 \\ (y_j^P - y_j) \quad -(x_j^P - x_j) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{(x)} \\ \mathbf{f}_{(y)} \\ n \end{bmatrix}_j$$

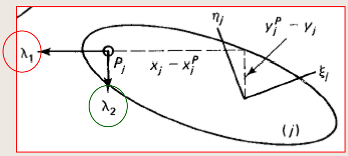
	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\mathbf{\Phi}^{(r,2)}$	1	0	$-(y_i^P - y_i)$	-1	0	$(y_j^P - y_j)$
	0	1	$(x_i^P - x_i)$	0	-1	$-(x_j^P - x_j)$

Revolute joint Object j

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix}_j \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_j - \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ (y_j^P - y_j) & -(x_j^P - x_j) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{(x)} \\ \mathbf{f}_{(y)} \\ n \end{bmatrix}_j$$

$$\begin{aligned} m_j \ddot{x}_j &= f_{(x)_j} - \lambda_1 \\ m_j \ddot{y}_j &= f_{(y)_j} - \lambda_2 \\ \mu_j \ddot{\phi}_j &= n_j + (y_j^P - y_j)\lambda_1 - (x_j^P - x_j)\lambda_2 \end{aligned}$$





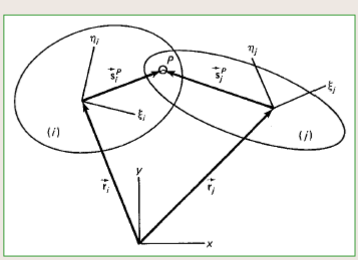
Example

Consider a system of two bodies connected by a revolute joint as shown in Fig. 9.9(a). The external forces acting on the system are gravity, a constant force of 10 N acting on body i in the negative x direction, and a constant force of 10 N acting on body j in the positive x direction. Calculate the joint reaction forces at the instant for which

$$\mathbf{q}_i = [1.58, 1.59, 0.6]^T, \qquad \mathbf{q}_j = [3.4, 1.96, 0.2]^T$$

$$\dot{\mathbf{q}}_i = [1.1, 0.2, -0.02]^T, \qquad \dot{\mathbf{q}}_j = [1.14, 0.24, 0.03]^T$$

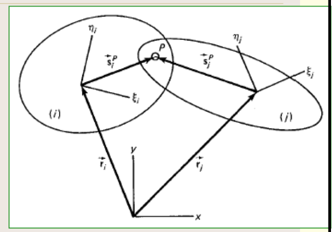
The constant quantities for this system are: $m_i = 1.2, m_j = 2, \mu_i = 2.5, \mu_j = 4$, $\mathbf{s}_i'^P = [0.9, 0.7]^T$, and $\mathbf{s}_j'^P = [-1.3, 1]^T$.



Example (cont.)

The constraint equations for this revolute joint are

$$x_i + 0.9\cos\phi_i - 0.7\sin\phi_i - x_j + 1.3\cos\phi_j + \sin\phi_j = 0$$
$$y_i + 0.9\sin\phi_i + 0.7\cos\phi_i - y_j + 1.3\sin\phi_j - \cos\phi_j = 0$$



The Jacobian matrix for these constraints is

$$\mathbf{\Phi_q} = \begin{bmatrix} 1 & 0 & -1.09 & -1 & 0 & 0.72 \\ 0 & 1 & 0.35 & 0 & -1 & 1.47 \end{bmatrix}$$

the equations of motion for body i are

$$\mathbf{M}_i\ddot{\mathbf{q}}_i - \mathbf{\Phi}_{\mathbf{q}_i}^T\boldsymbol{\lambda} = \mathbf{g}_i \longrightarrow \begin{aligned} &1.2\ddot{x}_i - \lambda_1 = -10 \\ &1.2\ddot{y}_i - \lambda_2 = -11.77 \\ &2.5\ddot{\phi}_i + 1.09\lambda_1 - 0.35\lambda_2 = 0 \end{aligned}$$

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\Phi^{(r,2)}$	1	0	$-(y_i^P - y_i)$	-1	0	$(y_j^P - y_j)$
	0	1	$(x_i^P - x_i)$	0	-1	$-(x_j^P - x_j)$

Example (cont.)

Similarly, the equations of motion for body j are :

$$\mathbf{M}_j\ddot{\mathbf{q}}_j - \mathbf{\Phi}_{\mathbf{q}_j}^T\boldsymbol{\lambda} = \mathbf{g}_j$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix}_j \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_j - \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ (y_j^P - y_j) & -(x_j^P - x_j) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{(x)} \\ \mathbf{f}_{(y)} \\ n \end{bmatrix}_j$$

$$2\ddot{x}_j + \lambda_1 = 10$$
$$2\ddot{y}_j + \lambda_2 = -19.62$$
$$4\ddot{\phi}_j - 0.72\lambda_1 - 1.47\lambda_2 = 0$$

Example (cont.)

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{bmatrix}$$

(6+2)x(6+2) (6+2)x1 (6+2)x1

Ax = c

$$1.2\ddot{x}_i - \lambda_1 = -10$$
$$1.2\ddot{y}_i - \lambda_2 = -11.77$$
$$2.5\ddot{\phi}_i + 1.09\lambda_1 - 0.35\lambda_2 = 0$$
$$2\ddot{x}_j + \lambda_1 = 10$$
$$2\ddot{y}_j + \lambda_2 = -19.62$$
$$4\ddot{\phi}_j - 0.72\lambda_1 - 1.47\lambda_2 = 0$$

$$\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{g}$$

There are six equations in 8 unknowns, and hence two more equations are needed. The two additional equations are the acceleration equations :

$$\ddot{x}_i - 1.09\ddot{\phi}_i - \ddot{x}_j + 0.72\ddot{\phi}_j = 0$$
$$\ddot{y}_i + 0.35\ddot{\phi}_i - \ddot{y}_j + 1.47\ddot{\phi}_j = 0$$

$$\mathbf{\Phi}_q \ddot{\mathbf{q}} = \gamma$$

Example (cont.)

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{bmatrix}$$

(6+2)x(6+2) (6+2)x1 (6+2)x1

$$\ddot{\mathbf{q}}_i = [-2.571, -10.154, -3.061]^T$$
$$\ddot{\mathbf{q}}_j = [1.543, -9.604, 1.096]^T$$
$$\boldsymbol{\lambda} = [6.915, -0.413]^T$$

Constraint forces :

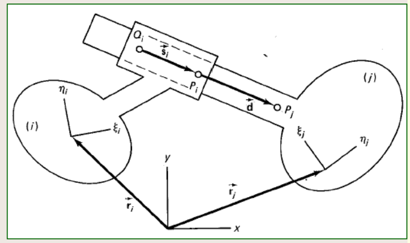
$$\mathbf{g}^{(c)} = \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1.09 & 0.35 \\ -1 & 0 \\ 0 & -1 \\ 0.72 & 1.47 \end{bmatrix} \begin{bmatrix} 6.915 \\ -0.413 \end{bmatrix} = \begin{bmatrix} 6.915 \\ -0.413 \\ -7.6819 \\ -6.915 \\ 0.413 \\ 4.3717 \end{bmatrix} = \begin{bmatrix} f_{ix} \\ f_{iy} \\ n_i \\ f_{jy} \\ f_{jy} \\ n_j \end{bmatrix}$$

拘束對質心之作用力

拘束力對質心之扭矩

Constraint Reaction Forces
(Translational joint)

$$\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{g}$$



For object i :

$$m_i\ddot{x}_i = f_{(x)_i} + (y_i^P - y_i^Q)\lambda_1$$

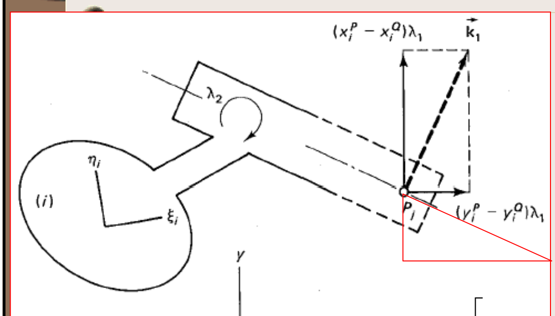
$$m_i\ddot{y}_i = f_{(y)_i} - (x_i^P - x_i^Q)\lambda_1$$

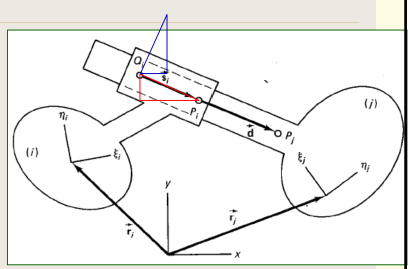
$$\mu_i\ddot{\phi}_i = n_i - [(x_j^P - x_i)(x_i^P - x_i^Q) + (y_j^P - y_i)(y_i^P - y_i^Q)]\lambda_1 + \lambda_2$$

$$\mathbf{\Phi}^{(t,2)} = \begin{bmatrix} (x_i^P - x_i^Q)(y_j^P - y_i^P) - (y_i^P - y_i^Q)(x_j^P - x_i^P) \\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\Phi^{(t,2)}$	$(y_i^P - y_i^Q)$	$-(x_i^P - x_i^Q)$	$-(x_j^P - x_i)(x_i^P - x_i^Q) - (y_j^P - y_i)(y_i^P - y_i^Q)$	$-(y_i^P - y_i^Q)$	$(x_i^P - x_i^Q)$	$(x_j^P - x_j)(x_i^P - x_i^Q) + (y_j^P - y_j)(y_i^P - y_i^Q)$
	0	0	1	0	0	-1

Reaction force Acting on the COM: $g^{(c)}$





$$\mathbf{g}^{(c)} = \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \begin{bmatrix} y_i^P - y_i^Q & 0 \\ -(x_i^P - x_i^Q) & 0 \\ -(x_j^P - x_i)(x_i^P - x_i^Q) - (y_j^P - y_i)(y_i^P - y_i^Q) & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$m_i\ddot{x}_i = f_{(x)_i} + (y_i^P - y_i^Q)\lambda_1$$

$$m_i\ddot{y}_i = f_{(y)_i} - (x_i^P - x_i^Q)\lambda_1$$

k$_1$

$$\mu_i\ddot{\phi}_i = n_i - [(x_j^P - x_i)(x_i^P - x_i^Q) + (y_j^P - y_i)(y_i^P - y_i^Q)]\lambda_1 + \lambda_2$$

Determine Static Forces in a Structure

- A mechanical system becomes a **structure** when *the number of constraint equations* **m =** *the number of coordinates in the system* **n**
- For a structure, we have

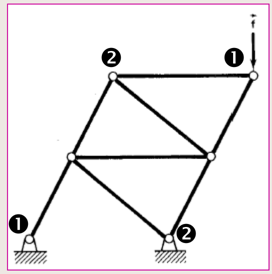
$$\boxed{\dot{\mathbf{q}} = \ddot{\mathbf{q}} = 0}$$

$$\downarrow$$

$$\boxed{\mathbf{M}\cancel{\ddot{\mathbf{q}}} - \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{g}}$$

$$\downarrow$$

$$\boxed{\boldsymbol{\lambda} = -(\mathbf{\Phi}_q^T)^{-1}\mathbf{g}}$$



$$m = 24 + 3 = 27$$

$$n = (8 + 1) \times 3 = 27$$

Dof = n-m=0

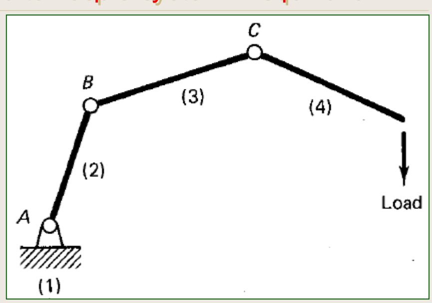
Then, the static (constraint) forces associated with the constraints can be determined from

$$\boxed{\mathbf{g}^{(c)} = \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}}$$

Static Balance Forces

Determine the forces needed to keep a system in equilibrium

- Q: What moments must the motors at A, B, and C apply on the bodies in order to keep the system in equilibrium, in the configuration shown?



$$\mathbf{M}\cancel{\ddot{\mathbf{q}}} - \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{g}^{(k)} + \mathbf{g}^{(u)}$$

In equilibrium

The vector of unknown forces

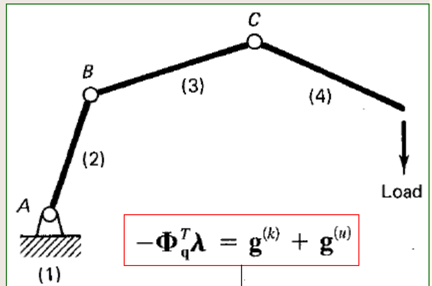
The vector of known forces

$$\boxed{-\mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{g}^{(k)} + \mathbf{g}^{(u)}}$$

Static Balance Forces

- The system has 3 dof (three unknown moments)
- We must add three artificial constraint equations to the system :

$$\Phi_1^* \equiv \phi_2 - \phi_1 - c_1 = 0$$
$$\Phi_2^* \equiv \phi_3 - \phi_2 - c_2 = 0$$
$$\Phi_3^* \equiv \phi_4 - \phi_3 - c_3 = 0$$



$$-\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{g}^{(k)} + \mathbf{g}^{(u)}$$

$$\mathbf{g}^{(u)} = \mathbf{\Phi}_q^{*T}\boldsymbol{\lambda}^*$$

$$-\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} - \mathbf{\Phi}_{\mathbf{q}}^{*T}\boldsymbol{\lambda}^* = \mathbf{g}^{(k)}$$

$$\begin{bmatrix}\mathbf{\Phi}_q^T & \mathbf{\Phi}_q^{*T}\end{bmatrix}\begin{bmatrix}-\boldsymbol{\lambda} \\ -\boldsymbol{\lambda}^*\end{bmatrix} = \mathbf{g}^{(k)}$$

The static balance moments $g^{(u)}$ can be determined after solving the linear equation

$\mathbf{A}\mathbf{x} = \mathbf{c}$

Example

- Replace the actuator in Fig. a with a rigid bar in order to determine the static balance force, that the actuator should supply to keep the mechanism in equilibrium.

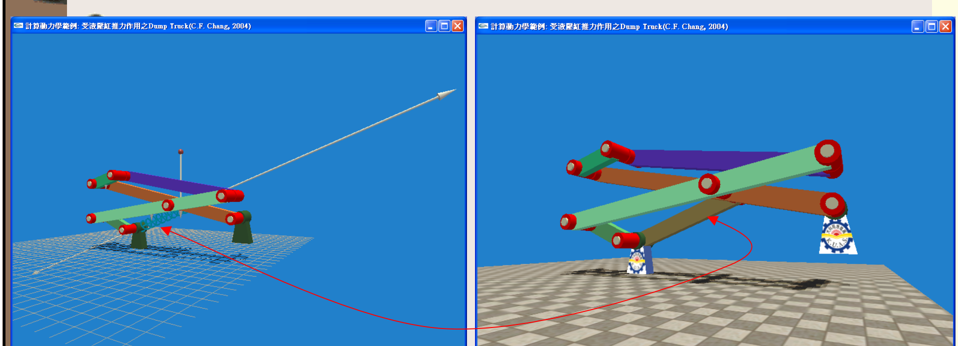


Fig. a Fig. b

Kinetostatic Analysis / Inverse Dynamic Analysis

- Forward dynamic analysis:
  - Determine the motion of a system with known external forces by solving the equations of motion
- Inverse dynamic analysis:
  - determine the forces that must act on a system to produce the desired motion

Example of Inverse Dynamic Analysis

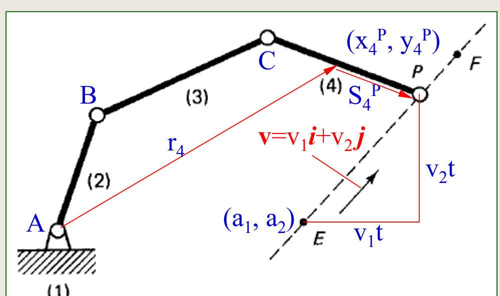
- Find the torque that actuators A, B, and C must supply, as a function of time, in order to produce the following motion:
  - Point P must move along the known path EF
  - Point P keep a constant velocity
  - The angle of body 4 remain unchanged w.r.t the line EF

The required motion can be expressed as the following constraints:

$$\Phi_1^* \equiv x_4^P - a_1 - v_1 t = 0$$
$$\Phi_2^* \equiv y_4^P - a_2 - v_2 t = 0$$
$$\Phi_3^* \equiv \phi_4 - c_1 = 0$$

$$\Updownarrow$$

$$\boldsymbol{\Phi}^*(\mathbf{q}, t) = \boldsymbol{0}$$



$$\mathbf{P}_4 = \mathbf{r}_4 + \mathbf{A}_4 \mathbf{S}_4^{\prime P}$$

Example of Inverse Dynamic Analysis (cont)

1. determine position, velocity, and acceleration

For kinematic constraints (3 revolute joints+1 fixed link) , we have m=2x3+3=9 equations $\boxed{\boldsymbol{\Phi}(\mathbf{q})=0}$

For driving constraints, we have 3 equations: (k=dof=n-m=3x4-9=3)

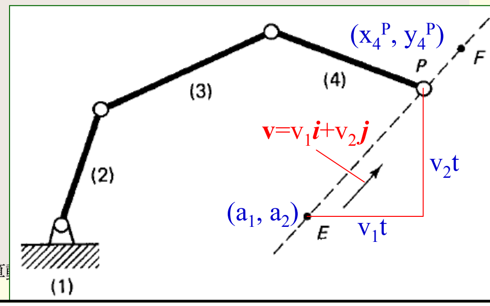
$$\boxed{\boldsymbol{\Phi}^{*}(\mathbf{q},t)=0}$$

$$\boxed{\mathbf{P}_4=\mathbf{r}_4+\mathbf{A}_4\mathbf{S}_4'^{P}}$$

There are 9+3=12 equations in n=3x4=12 unknowns for the system

Thus, we can solve these equations to determine

$\boxed{\mathbf{q},\ \dot{\mathbf{q}},\ \text{and}\ \ddot{\mathbf{q}}}$



(review)

-Position Analysis-

- Providing that there are <u>*m* constraints</u>
- When appending <u>*k* driving constraints</u> to the system, it has $n = m + k$ equations in n variables:

$\Phi = \Phi(\mathbf{q}) = 0$ *m* kinematic constraints

$\Phi^{(d)} = \Phi(\mathbf{q},\mathbf{t}) = 0$ *k* driving constraints

- In general, these equations are also nonlinear
- So, the configuration of the system may be determined by using iterative numerical method

(review)
-Velocity Analysis-

For kinematic constraints : $\boxed{\Phi = \Phi(\mathbf{q})}$

$$\boxed{\dot{\mathbf{\Phi}} = \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = 0}$$

For driving constraints: $\boxed{\Phi = \Phi(\mathbf{q}\,,\,t)}$

$$\boxed{\dot{\mathbf{\Phi}}^{(d)} = \mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}} + \mathbf{\Phi}_{t}^{(d)} = \mathbf{0}}$$

$$\underbrace{\begin{bmatrix} \mathbf{\Phi}_{\mathbf{q}} \\ \mathbf{\Phi}_{\mathbf{q}}^{(d)} \end{bmatrix}}_{\mathbf{A}_{m\times m}} \underbrace{\dot{\mathbf{q}}}_{\mathbf{x}_{m\times 1}} = \underbrace{\begin{bmatrix} 0 \\ -\mathbf{\Phi}_{t}^{(d)} \end{bmatrix}}_{\mathbf{c}_{m\times 1}}$$

$$\mathbf{A}_{m\times m}\,\mathbf{x}_{m\times 1} = \mathbf{c}_{m\times 1}$$

velocities can be determined by solving Ax=c

(review)
-Acceleration Analysis-

For kinematic constraints : $\boxed{\Phi = \Phi(\mathbf{q})}$ $\boxed{\dot{\mathbf{\Phi}} = \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = 0}$

$$\boxed{\ddot{\mathbf{\Phi}} = (\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = 0}$$

For driving constraints: $\boxed{\Phi = \Phi(\mathbf{q}\,,\,t)}$ $\boxed{\dot{\mathbf{\Phi}}^{(d)} = \mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}} + \mathbf{\Phi}_{t}^{(d)} = \mathbf{0}}$

$$\begin{aligned} \ddot{\mathbf{\Phi}}^{(d)} &= \frac{d}{dt}\dot{\mathbf{\Phi}}_{\mathbf{q}}^{(d)} = \frac{d}{dt}(\mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}} + \mathbf{\Phi}_{t}^{(d)}) \\ &= (\mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + (\mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}})_{t} + \mathbf{\Phi}_{tq}^{(d)}\dot{\mathbf{q}} + \mathbf{\Phi}_{tt}^{(d)} \\ &= (\mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}t}^{(d)}\dot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{(d)}\ddot{\mathbf{q}} + \mathbf{\Phi}_{tq}^{(d)}\dot{\mathbf{q}} + \mathbf{\Phi}_{tt}^{(d)} = 0 \end{aligned}$$

$$\boxed{\therefore (\mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + 2\mathbf{\Phi}_{\mathbf{q}t}^{(d)}\dot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{(d)}\ddot{\mathbf{q}} + \mathbf{\Phi}_{tt}^{(d)} = 0}$$

(review)
-Acceleration Analysis (cont.)

$$\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} \equiv \boldsymbol{\gamma}$$

Kinematic constraints

$$\mathbf{\Phi}_{\mathbf{q}}^{(d)}\ddot{\mathbf{q}} = -[(\mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + 2\mathbf{\Phi}_{\mathbf{q}t}^{(d)}\dot{\mathbf{q}} + \mathbf{\Phi}_{tt}^{(d)}] \equiv \boldsymbol{\gamma}^{(d)}$$

Driving constraints

$$\underbrace{\begin{bmatrix} \mathbf{\Phi}_{\mathbf{q}} \\ \mathbf{\Phi}_{\mathbf{q}}^{(d)} \end{bmatrix}}_{\mathbf{A}_{m\times m}} \underbrace{\ddot{\mathbf{q}}}_{\mathbf{x}_{m\times 1}} = \underbrace{\begin{bmatrix} -(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} \\ -(\mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t}^{(d)}\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}^{(d)} \end{bmatrix}}_{\mathbf{c}_{m\times 1}}$$

$$\mathbf{A}_{m\times m}\,\mathbf{x}_{m\times 1} = \mathbf{c}_{m\times 1}$$

accelerations can be determined by solving Ax=c

Example of Inverse Dynamic Analysis (cont)

derive velocity and acceleration Equations for Driving Constraints

$$\mathbf{P}_4 = \mathbf{r}_4 + \mathbf{A}_4\mathbf{S}'^{P}_4$$

$$\begin{aligned} \Phi_1^* &\equiv x_4^P - a_1 - v_1 t = 0 \\ \Phi_2^* &\equiv y_4^P - a_2 - v_2 t = 0 \\ \Phi_3^* &\equiv \phi_4 - c_1 = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{\Phi}_1^* &= (x_4 + \xi_4^P \cos\phi_4 - \eta_4^P \sin\phi_4) - a_1 - v_1 t = 0 \\ \mathbf{\Phi}_2^* &= (y_4 + \xi_4^P \sin\phi_4 + \eta_4^P \cos\phi_4) - a_2 - v_2 t = 0 \\ \mathbf{\Phi}_3^* &= \phi_4 - c_1 = 0 \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{\Phi}}_1^* &= \dot{x}_4 - v_1 = 0 \\ \dot{\mathbf{\Phi}}_2^* &= \dot{y}_4 - v_2 = 0 \\ \dot{\mathbf{\Phi}}_3^* &= \dot{\phi}_4 = 0 \end{aligned}$$

$$\mathbf{\Phi}_{\mathbf{q}}^{(d)}\dot{\mathbf{q}} = -\mathbf{\Phi}_t^{(d)}$$

$$\mathbf{\Phi}_q^{(d)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \dot{\mathbf{q}} = \begin{bmatrix} \dot{x}_4 \\ \dot{y}_4 \\ \dot{\phi}_4 \end{bmatrix} \qquad -\mathbf{\Phi}_t^{(d)} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \ddot{\mathbf{\Phi}}_1^* &= \ddot{x}_4 = 0 \\ \ddot{\mathbf{\Phi}}_2^* &= \ddot{y}_4 = 0 \\ \ddot{\mathbf{\Phi}}_3^* &= \ddot{\phi}_4 = 0 \end{aligned}$$

$$\mathbf{\Phi}_{\mathbf{q}}^{(d)}\ddot{\mathbf{q}} = \boldsymbol{\gamma}^{(d)}$$

$$\boldsymbol{\gamma}^{(d)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Another method (Link4)

$\Phi_q^{(d)}$	x_4	y_4	ϕ_4
$\partial\Phi_1^*/\partial$	1	0	0
$\partial\Phi_2^*/\partial$	0	1	0
$\partial\Phi_3^*/\partial$	0	0	1

Example of Inverse Dynamic Analysis (cont)

2. determine forces by solving equations of motion

$$\mathbf{M}\ddot{\mathbf{q}} - \boldsymbol{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda} = \mathbf{g}^{(k)} + \mathbf{g}^{(u)}$$

The vector of unknown forces

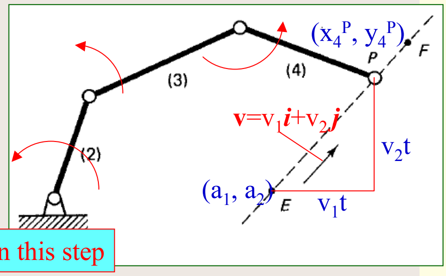
The vector of known forces

$$\boldsymbol{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda} + \mathbf{g}^{(u)} = \mathbf{M}\ddot{\mathbf{q}} - \mathbf{g}^{(k)}$$

$\mathbf{M}\ddot{\mathbf{q}}$, $\boldsymbol{\Phi}_{\mathbf{q}}$, and $\mathbf{g}^{(k)}$ are known

$$\begin{array}{cc} \left[\begin{array}{cc} \boldsymbol{\Phi}_{q}^{T} & \mathbf{I}^{*} \end{array}\right] & \left[\begin{array}{c} \boldsymbol{\lambda} \\ \mathbf{g}^{(u)} \end{array}\right] = \mathbf{M}\ddot{\mathbf{q}} - \mathbf{g}^{(k)} \\ 12\times 9 \quad 12\times 3 & \end{array}$$

$$\mathbf{A}_{12\times 12}\,\mathbf{x}_{(9+3)\times 1} = \mathbf{c}_{12\times 1}$$



Note: Driving constraints are removed in this step

These equations can be solved for the driving forces $\mathbf{g}^{(u)}$ of the actuators A, B, and C and the joint reaction forces λ

Constraint Violation Stabilization Method

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{\Phi}_{\mathbf{q}}^{T} \\ \boldsymbol{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \boldsymbol{\gamma} \end{bmatrix}$$

Goal: Suppress the growth of error and achieve a stable response

Denote $\mathbf{q}^*, \dot{\mathbf{q}}^*$, and $\ddot{\mathbf{q}}^*$ as computed values of $\mathbf{q}, \dot{\mathbf{q}}$, and $\ddot{\mathbf{q}}$, respectively

If there are violations in the position and velocity constraints such that

$$\mathbf{q}^* - \mathbf{q} = \Delta\mathbf{q}$$

$$\dot{\mathbf{q}}^* - \dot{\mathbf{q}} = \Delta\dot{\mathbf{q}}$$

The acceleration vector $d\mathbf{q}^*/dt^2$ determined from the equations of motion may different from the correct $d\mathbf{q}/dt^2$, but satisfies

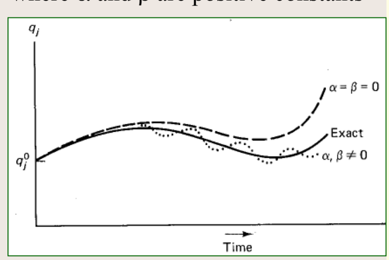
$$\ddot{\boldsymbol{\Phi}} = \boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}}^* - \boldsymbol{\gamma}^* = \mathbf{0} \qquad \ddot{\mathbf{q}}^* - \ddot{\mathbf{q}} = \Delta\ddot{\mathbf{q}}$$

After integration the error of acceleration may lead to additional errors in position and velocity in the next step.

In order to have $\Delta\ddot{\mathbf{q}} = 0$ in the next few steps, a closed-loop system with feedback is introduced as follows

$$\Delta\ddot{\mathbf{q}} + 2\alpha\Delta\dot{\mathbf{q}} + \beta^2\Delta\mathbf{q} = 0$$

where α and β are positive constants



Constraint Violation Stabilization Method (cont)

$$\Delta\ddot{\mathbf{q}} + 2\alpha\Delta\dot{\mathbf{q}} + \beta^2\Delta\mathbf{q} = 0$$

$$\ddot{\mathbf{q}}^* - \ddot{\mathbf{q}} = \Delta\ddot{\mathbf{q}}$$

$$\dot{\Phi} = \Phi_q\dot{\mathbf{q}}^* = \cancel{\Phi_q\dot{\mathbf{q}}} + \Phi_q\Delta\dot{\mathbf{q}} = \Phi_q\Delta\dot{\mathbf{q}}$$

$$\Phi(\mathbf{q}^*) = \Phi(\mathbf{q}+\Delta\mathbf{q}) = \cancel{\Phi(\mathbf{q})} + \Phi_q\Delta\mathbf{q} = \Phi_q\Delta\mathbf{q}$$

$$\Phi_q\Delta\ddot{\mathbf{q}} + 2\alpha\Phi_q\Delta\dot{\mathbf{q}} + \beta^2\Phi_q\Delta\mathbf{q} = 0$$

$$\Phi_q\ddot{\mathbf{q}}^* - \Phi_q\ddot{\mathbf{q}} + 2\alpha\dot{\Phi} + \beta^2\Phi = 0$$

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \boldsymbol{\gamma}$$

$$\Phi_q\ddot{\mathbf{q}}^* = \boldsymbol{\gamma} - 2\alpha\dot{\Phi} - \beta^2\Phi$$

$$\mathbf{M}\ddot{\mathbf{q}}^* - \boldsymbol{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{Q}$$

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{\Phi}_{\mathbf{q}}^T \\ \boldsymbol{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^* \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \boldsymbol{\gamma} - 2\alpha\dot{\boldsymbol{\Phi}} - \beta^2\boldsymbol{\Phi} \end{bmatrix}$$