Spatial Kinematics

Reading Material: Chapter 7

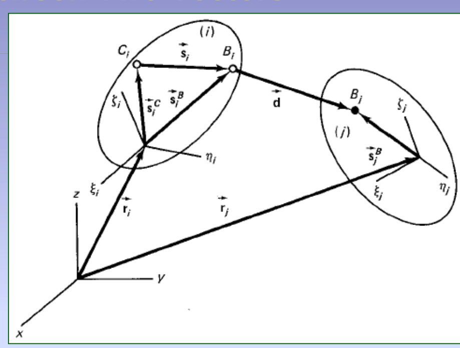
Relative Constraints Between Two Vectors

Two Perpendicular Vectors

- For two vectors that are embedded in different bodies, the constraint equation imposing perpendicularity is

Normal type 1 constraint having 1 equation

$$\Phi^{(n1,1)} \equiv \mathbf{s}_i^T \mathbf{s}_j = \mathbf{s}_i'^T \mathbf{A}_i^T \mathbf{A}_j \mathbf{s}_j' = 0$$



- If a vector $\mathbf{d}$, which is connected between bodies i and j, has to remain perpendicular to $\mathbf{s}_i$, the constraint is:

$$\mathbf{s}_i^T = (\mathbf{A}_i \mathbf{s}_i')^T = \mathbf{s}_i'^T \mathbf{A}_i^T$$

Normal type 2 constraint having 1 equation

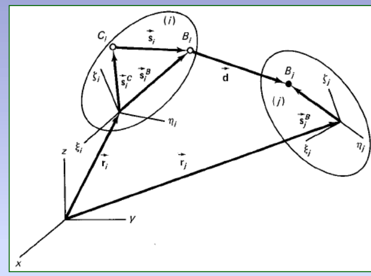
$$\Phi^{(n2,1)} \equiv \mathbf{s}_i^T \mathbf{d} = \mathbf{s}_i'^T \mathbf{A}_i^T (\mathbf{r}_j + \mathbf{A}_j \mathbf{s}_j'^B - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}_i'^B) = 0$$

Relative Constraints Between Two Vectors

Two Parallel Vectors

$$\mathbf{s}_i = \mathbf{A}_i \mathbf{s}_i'$$
$$\tilde{\mathbf{s}}_i = \mathbf{A}_i \tilde{\mathbf{s}}_i' \mathbf{A}_i^T$$

- For two vectors that embedded in different bodies, the constraints imposing parallelism are



Parallel type 1 constraint having **2** equation

$$\boldsymbol{\Phi}^{(p1,2)} \equiv \tilde{\mathbf{s}}_i \mathbf{s}_j = \mathbf{A}_i \tilde{\mathbf{s}}_i' \mathbf{A}_i^T \mathbf{A}_j \mathbf{s}_j' = \mathit{0}$$

- If the vector **d**, which is connected between bodies i and j, has to remain parallel to $\mathbf{s}_i$, the constraint is:

Parallel type 2 constraint having **2** equation

$$\boldsymbol{\Phi}^{(p2,2)} \equiv \tilde{\mathbf{s}}_i \mathbf{d} = \mathbf{A}_i \tilde{\mathbf{s}}_i' \mathbf{A}_i^T (\mathbf{r}_j + \mathbf{A}_j \mathbf{s}_j'^B - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}_i'^B) = \mathit{0}$$

$\Phi^{(p1,2)}$ and $\Phi^{(p2,2)}$ provide three equations each. However, only two of them are independent

Q/A

- $\mathbf{a}\times\mathbf{b}=0$ ➔
  - $a_y b_z - a_z b_y = 0$ (1)
  - $a_x b_z - a_z b_x = 0$ (2)
  - $a_x b_y - a_y b_x = 0$ (3)
- 若滿足前兩式, 則可得$a_y = a_z b_y / b_z$ 和 $a_x = a_z b_x / b_z$
- 將此二關係代入第3式, 可發現恒滿足第3式
- 換言之, 只須滿足(1)(2)兩式, 自然會滿足(3)式, 故三式中只有兩式爲線性獨立

$\Phi^{(p1,2)}$ and $\Phi^{(p2,2)}$ provide three equations each. However, only two of them are independent

Remark on Two Parallel Vectors

How to select the two independent equations

Ans: Compare the absolute values of $s_{(x)i}$, $s_{(y)i}$, and $s_{(z)i}$, and select the two equations having the largest terms.

For instance,

$$\Phi^{(p1,2)} \equiv \tilde{\mathbf{s}}_i \mathbf{s}_j$$

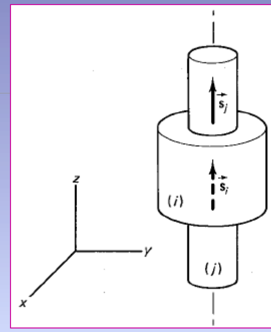
$$\mathbf{s}_i \equiv [s_{(x)}, s_{(y)}, s_{(z)}]_i^T$$

$$\mathbf{s}_j \equiv [s_{(x)}, s_{(y)}, s_{(z)}]_j^T$$

$$-s_{(z)i}s_{(y)j} + s_{(y)i}s_{(z)j} = 0 \quad (a)$$

$$s_{(z)i}s_{(x)j} - s_{(x)i}s_{(z)j} = 0 \quad (b)$$

$$-s_{(y)i}s_{(x)j} + s_{(x)i}s_{(y)j} = 0 \quad (c)$$



From the figure, we have

$$s_{(x)i} = s_{(y)i} = 0$$

$$s_{(x)j} = s_{(y)j} = 0$$

$$\partial(\text{Eq. } c)/\partial \cdots \begin{array}{c} \begin{array}{ccc} & \mathbf{p}_i & \quad\quad \mathbf{p}_j \end{array} \\ \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & ① & \cdots\cdots & ② & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \end{array}$$

$$① = -\frac{\partial s_{(y)i}}{\partial \mathbf{p}_i} s_{(x)j} + \frac{\partial s_{(x)i}}{\partial \mathbf{p}_i} s_{(y)j}$$

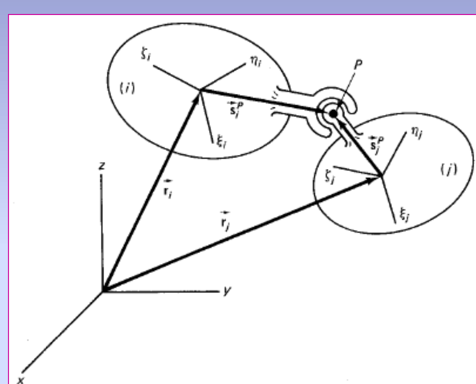
$$② = -\frac{\partial s_{(x)j}}{\partial \mathbf{p}_j} s_{(y)i} + \frac{\partial s_{(y)j}}{\partial \mathbf{p}_j} s_{(x)i}$$

Hence, Eq. c leads to a reduction in the row rank of the Jacobian matrix and to numerical difficulties. Therefore Eqs. a and b are selected because they have nonzero s_z.

Constraints for Spherical Joints

$$\vec{r}_i + \vec{s}_i^{\,P} - \vec{s}_j^{\,P} - \vec{r}_j = \vec{0}$$

$$\Phi^{(s,3)} \equiv \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i'^{P} - \mathbf{A}_j \mathbf{s}_j'^{P} - \mathbf{r}_j = \mathbf{0}$$



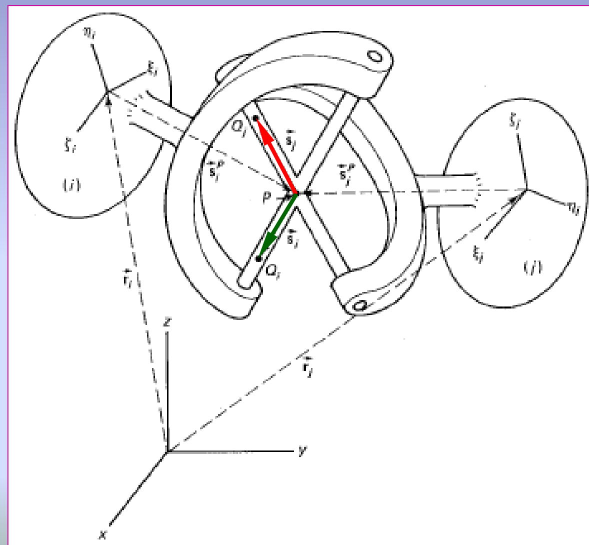
Since a spherical joint imposes three constraint equations on the two bodies connected by it, the two bodies have three relative degrees of freedom

Constraints for Universal Joints

$$\Phi^{(s,3)} = \mathbf{r}_i^P - \mathbf{r}_j^P = 0$$

$$\Phi^{(n1,1)} \equiv \mathbf{s}_i^T \mathbf{s}_j = 0$$

A universal joint imposes **four** constraint equations on the two bodies connected by it. Thus, the two bodies have **two** relative degrees of freedom

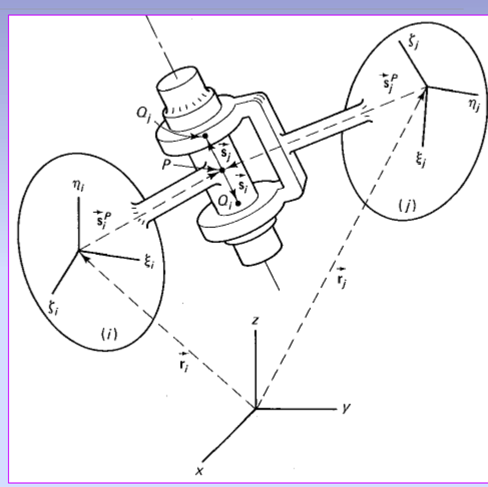


Constraints for Revolute Joints

$$\Phi^{(s,3)} = \mathbf{r}_i^P - \mathbf{r}_j^P = 0$$

$$\Phi^{(p1,2)} \equiv \tilde{\mathbf{s}}_i \mathbf{s}_j = 0$$

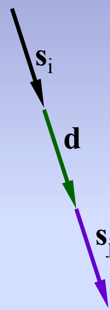
A revolute joint imposes **five** constraint equations on the two bodies connected by it. Thus, the two bodies have **one** relative degrees of freedom

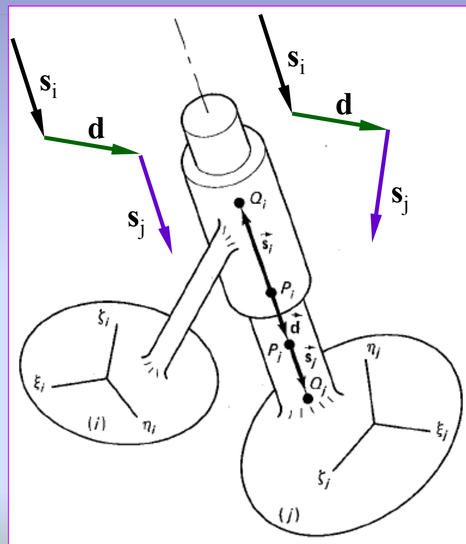


Constraints for Cylindrical Joints

$$\mathbf{\Phi}^{(p1,2)} \equiv \tilde{\mathbf{s}}_i \, \mathbf{s}_j = 0$$

$$\mathbf{\Phi}^{(p2,2)} \equiv \tilde{\mathbf{s}}_i \, \mathbf{d} = 0$$





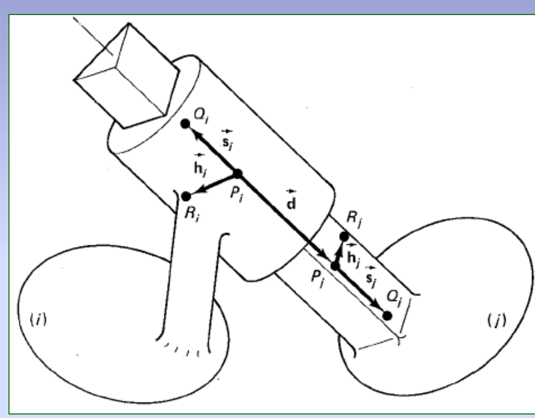
The two bodies have **two** relative degrees of freedom

Constraints for Prismatic Joints

$$\mathbf{\Phi}^{(p1,2)} \equiv \tilde{\mathbf{s}}_i \mathbf{s}_j = 0$$

$$\mathbf{\Phi}^{(p2,2)} \equiv \tilde{\mathbf{s}}_i \mathbf{d} = 0$$

$$\mathbf{\Phi}^{(n1,1)} \equiv \mathbf{h}_i^T \mathbf{h}_j = 0$$



The two bodies have **one** relative degrees of freedom

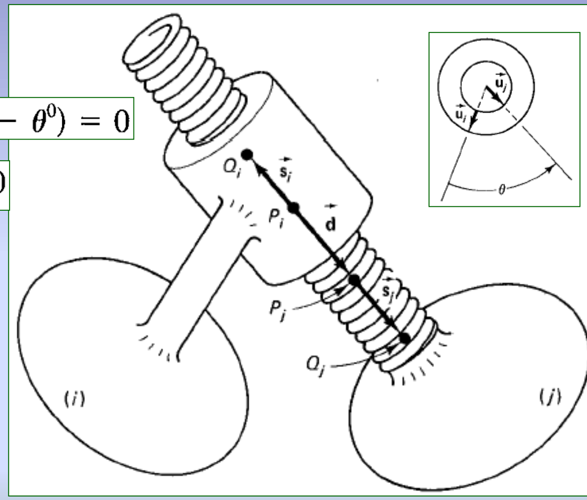
Constraints for Screw Joints

$$\Phi^{(p1,2)} \equiv \tilde{\mathbf{s}}_i \mathbf{s}_j = \mathbf{0}$$

$$\Phi^{(p2,2)} \equiv \tilde{\mathbf{s}}_i \mathbf{d} = \mathbf{0}$$

$$\Phi^{(r,1)} \equiv (d - d^0) - \alpha(\theta - \theta^0) = 0$$

$$\Phi^{(\theta,1)} \equiv \mathbf{u}_i^T \mathbf{u}_j - \cos\theta = 0$$



$\alpha = \Delta d / \Delta\theta$ =pitch/2π

θ is an artificial coordinate. It should be included in vector **q**

θ^0 Is the initial angle between the two unit vector $\mathbf{u}_i$ and $\mathbf{u}_j$ embedded in bodies i and j, respectively

Relative DOF =n-m= 1
(n=6+1=7, m=6)

State Variables of Spatial Systems

$$\mathbf{q}_i = \begin{bmatrix} x \\ y \\ z \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}_{7\times1} = \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}_i$$

e_0 : angle of rotation
e : an unit vector along the axis of rotation

$$\mathbf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}_{4\times1} = \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix}$$

Different Expressions for Velocity and Acceleration (ref: chap. 6)

$$\mathbf{q}_i = \begin{bmatrix}\mathbf{r}\\ \mathbf{p}\end{bmatrix}_i \rightarrow \dot{\mathbf{q}}_i = \begin{bmatrix}\dot{\mathbf{r}}\\ \dot{\mathbf{p}}\end{bmatrix}^i_{7\times1} = \begin{bmatrix}\dot{\mathbf{r}}\\ \boldsymbol{\omega}\end{bmatrix}^i_{6\times1} = \begin{bmatrix}\dot{\mathbf{r}}\\ \boldsymbol{\omega}'\end{bmatrix}^i_{6\times1} \rightarrow \ddot{\mathbf{q}}_i = \begin{bmatrix}\ddot{\mathbf{r}}\\ \ddot{\mathbf{p}}\end{bmatrix}^i_{7\times1} = \begin{bmatrix}\ddot{\mathbf{r}}\\ \dot{\boldsymbol{\omega}}\end{bmatrix}^i_{6\times1} = \begin{bmatrix}\ddot{\mathbf{r}}\\ \dot{\boldsymbol{\omega}}'\end{bmatrix}^i_{6\times1}$$

$$\dot{\mathbf{r}} = \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{3\times1}$$

$$\boldsymbol{\omega} = \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}_{3\times1} = \begin{bmatrix}\omega_x'\\ \omega_y'\\ \omega_z'\end{bmatrix}_{3\times1}$$

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^T\boldsymbol{\omega} = \frac{1}{2}\begin{bmatrix}-e_1 & -e_2 & -e_3\\ e_0 & e_3 & -e_2\\ -e_3 & e_0 & e_1\\ e_2 & -e_1 & e_0\end{bmatrix}_{4\times3}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}_{3\times1}$$

$$\boldsymbol{\omega} = 2\mathbf{G}\dot{\mathbf{p}}$$

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{L}^T\boldsymbol{\omega}' = \frac{1}{2}\begin{bmatrix}-e_1 & -e_2 & -e_3\\ e_0 & -e_3 & e_2\\ e_3 & e_0 & -e_1\\ -e_2 & e_1 & e_0\end{bmatrix}_{4\times3}\begin{bmatrix}\omega_\xi\\ \omega_\eta\\ \omega_\zeta\end{bmatrix}_{3\times1}$$

$$\boldsymbol{\omega}' = 2\mathbf{L}\dot{\mathbf{p}}$$

$$\ddot{\mathbf{r}} = \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{3\times1}$$

$$\ddot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\dot{\boldsymbol{\omega}} - \tfrac{1}{4}(\boldsymbol{\omega}^T\boldsymbol{\omega})\mathbf{p}$$

$$\dot{\boldsymbol{\omega}} = 2\mathbf{G}\ddot{\mathbf{p}}$$

$$\ddot{\mathbf{p}} = \tfrac{1}{2}\mathbf{L}^T\dot{\boldsymbol{\omega}}' - \tfrac{1}{4}(\boldsymbol{\omega}'^T\boldsymbol{\omega}')\mathbf{p}$$

$$\dot{\boldsymbol{\omega}}' = 2\mathbf{L}\ddot{\mathbf{p}}$$

Jacobian Entries for n1 (Normal Type 1) constraint (ref pp.298-299)

If we choose
$$\dot{\mathbf{q}}_i = \begin{bmatrix}\dot{\mathbf{r}}\\ \boldsymbol{\omega}'\end{bmatrix}^i_{6\times1} \quad \ddot{\mathbf{q}}_i = \begin{bmatrix}\ddot{\mathbf{r}}\\ \dot{\boldsymbol{\omega}}'\end{bmatrix}^i_{6\times1}$$

$$\begin{aligned}\Phi^{(n1,1)} &\equiv \mathbf{s}_i^T\mathbf{s}_j\\ &= \mathbf{s}_i'^T\mathbf{A}_i^T\mathbf{A}_j\mathbf{s}_j' = 0\end{aligned}$$

$$\mathbf{s} = \mathbf{A}\mathbf{s}' \Rightarrow \dot{\mathbf{s}} = \dot{\mathbf{A}}\mathbf{s}'$$

$$\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A} = \mathbf{A}\tilde{\boldsymbol{\omega}}'$$

$$\tilde{\boldsymbol{\omega}}'\mathbf{s}' = -\tilde{\mathbf{s}}'\boldsymbol{\omega}'$$

$$\begin{aligned}\dot{\Phi}^{(n1,1)} &\equiv \mathbf{s}_i^T\dot{\mathbf{s}}_j + \mathbf{s}_j^T\dot{\mathbf{s}}_i\\ &= \mathbf{s}_i^T\dot{\mathbf{A}}_j\mathbf{s}_j' + \mathbf{s}_j^T\dot{\mathbf{A}}_i\mathbf{s}_i'\\ &= \mathbf{s}_i^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}_j'\mathbf{s}_j' + \mathbf{s}_j^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i'\\ &= -\mathbf{s}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\boldsymbol{\omega}_j' - \mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i'\boldsymbol{\omega}_i'\\ &= \left[-\mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i',\ -\mathbf{s}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\right]\begin{bmatrix}\boldsymbol{\omega}_i'\\ \boldsymbol{\omega}_j'\end{bmatrix}\end{aligned} = \begin{bmatrix}0 & -\mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i' & 0 & -\mathbf{s}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\end{bmatrix}\begin{bmatrix}\dot{\mathbf{r}}_i\\ \boldsymbol{\omega}_i'\\ \dot{\mathbf{r}}_j\\ \boldsymbol{\omega}_j'\end{bmatrix} = \boldsymbol{\Phi}_q\dot{\mathbf{q}} = 0$$

	$\frac{\partial\Phi}{\partial\mathbf{r}_i}$	$\frac{\partial\Phi}{\partial\boldsymbol{\omega}_i'}$	$\frac{\partial\Phi}{\partial\mathbf{r}_j}$	$\frac{\partial\Phi}{\partial\boldsymbol{\omega}_j'}$	$\gamma^{\#}$
$\Phi^{(n1,1)}$	0^T	$-\mathbf{s}_j^T\tilde{\mathbf{s}}_i\mathbf{A}_i$	0^T	$-\mathbf{s}_i^T\tilde{\mathbf{s}}_j\mathbf{A}_j$	$-2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j + \dot{\mathbf{s}}_i^T\tilde{\boldsymbol{\omega}}_i\mathbf{s}_j + \dot{\mathbf{s}}_j^T\tilde{\boldsymbol{\omega}}_j\mathbf{s}_i$

The γ for n1 (Normal Type 1) constraint (ref pp.298-299)

$$\tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T$$

$$\mathbf{A}\tilde{\mathbf{s}}' = \tilde{\mathbf{s}}\mathbf{A}$$

$$\dot{\Phi} = \begin{bmatrix} 0 & -\mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i' & 0 & -\mathbf{s}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j' \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i' \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}_j' \end{bmatrix} = 0$$

$$\ddot{\Phi}^{(n1,1)} \equiv [-\mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i', -\mathbf{s}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'] \begin{bmatrix} \dot{\boldsymbol{\omega}}_i' \\ \dot{\boldsymbol{\omega}}_j' \end{bmatrix} + (\dot{\mathbf{s}}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i' - \mathbf{s}_j^T\dot{\mathbf{A}}_i\tilde{\mathbf{s}}_i')\boldsymbol{\omega}_i' + (-\dot{\mathbf{s}}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j' - \mathbf{s}_i^T\dot{\mathbf{A}}_j\tilde{\mathbf{s}}_j')\boldsymbol{\omega}_j' = 0$$

$$\ddot{\Phi} = \Phi_q\ddot{\mathbf{q}} - \gamma^{\#} = 0$$

$$\begin{aligned} \gamma^{\#(n1,1)} &\equiv \dot{\mathbf{s}}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i'\boldsymbol{\omega}_i' + \mathbf{s}_j^T\dot{\mathbf{A}}_i\tilde{\mathbf{s}}_i'\boldsymbol{\omega}_i' + \dot{\mathbf{s}}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\boldsymbol{\omega}_j' + \mathbf{s}_i^T\dot{\mathbf{A}}_j^T\tilde{\mathbf{s}}_j'\boldsymbol{\omega}_j \\ &= \dot{\mathbf{s}}_j^T\tilde{\mathbf{s}}_i\boldsymbol{\omega}_i + \mathbf{s}_j^T\tilde{\boldsymbol{\omega}}_i\tilde{\mathbf{s}}_i\boldsymbol{\omega}_i + \dot{\mathbf{s}}_i^T\tilde{\mathbf{s}}_j\boldsymbol{\omega}_j + \mathbf{s}_i^T\tilde{\boldsymbol{\omega}}_j\tilde{\mathbf{s}}_j\boldsymbol{\omega}_j \\ &= -2\dot{\mathbf{s}}_j^T\dot{\mathbf{s}}_i - \mathbf{s}_j^T\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i - \mathbf{s}_i^T\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_j \\ &= -2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j + \dot{\mathbf{s}}_i^T\tilde{\boldsymbol{\omega}}_i\mathbf{s}_j + \dot{\mathbf{s}}_j^T\tilde{\boldsymbol{\omega}}_j\mathbf{s}_i \end{aligned}$$

	$\frac{\partial\Phi}{\partial\dot{\mathbf{r}}_i}$	$\frac{\partial\Phi}{\partial\boldsymbol{\omega}_i'}$	$\frac{\partial\Phi}{\partial\dot{\mathbf{r}}_j}$	$\frac{\partial\Phi}{\partial\boldsymbol{\omega}_j'}$	$\gamma^{\#}$
$\Phi^{(n1,1)}$	0^T	$-\mathbf{s}_j^T\tilde{\mathbf{s}}_i\mathbf{A}_i$	0^T	$-\mathbf{s}_i^T\tilde{\mathbf{s}}_j\mathbf{A}_j$	$-2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j + \dot{\mathbf{s}}_i^T\tilde{\boldsymbol{\omega}}_i\mathbf{s}_j + \dot{\mathbf{s}}_j^T\tilde{\boldsymbol{\omega}}_j\mathbf{s}_i$

Jacobian Entries for Spherical Joint

$$\tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T$$

$$\mathbf{A}\tilde{\mathbf{s}}' = \tilde{\mathbf{s}}\mathbf{A}$$

Position Constraints: $\Phi^{(s,3)} = \mathbf{r}_i^P - \mathbf{r}_j^P = (\mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i') - (\mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j') = 0$

Velocity Constraints: $\dot{\Phi}^{(s,3)} = \dot{\mathbf{r}}_i^P - \dot{\mathbf{r}}_j^P = (\dot{\mathbf{r}}_i + \dot{\mathbf{A}}_i\mathbf{s}_i') - (\dot{\mathbf{r}}_j + \dot{\mathbf{A}}_j\mathbf{s}_j') = 0$

Acceleration Constraints: $\ddot{\Phi}^{(s,3)} = \ddot{\mathbf{r}}_i^P - \ddot{\mathbf{r}}_j^P = 0$

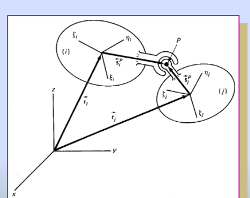
$$\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A} = \mathbf{A}\tilde{\boldsymbol{\omega}}' \quad 6.97$$

$$\dot{\mathbf{A}}\mathbf{s}' = \mathbf{A}\tilde{\boldsymbol{\omega}}'\mathbf{s}' = -\mathbf{A}\tilde{\mathbf{s}}'\boldsymbol{\omega}'$$

If we choose $\dot{\mathbf{q}}_i = \begin{bmatrix} \dot{\mathbf{r}} \\ \boldsymbol{\omega}' \end{bmatrix}_{6\times1}^i$ $\ddot{\mathbf{q}}_i = \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix}_{6\times1}^i$

$$\dot{\Phi}^{(s,3)} = (\dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\mathbf{s}}_i'\boldsymbol{\omega}_i') - (\dot{\mathbf{r}}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j'\boldsymbol{\omega}_j')$$

$$= \begin{bmatrix} \mathbf{I} & -\mathbf{A}_i\tilde{\mathbf{s}}_i' & -\mathbf{I} & \mathbf{A}_j\tilde{\mathbf{s}}_j' \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i' \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}_j' \end{bmatrix} = \Phi_q\dot{\mathbf{q}} = 0$$



$$\Phi_q = \begin{bmatrix} \mathbf{I} & -\mathbf{A}_i\tilde{\mathbf{s}}_i' & -\mathbf{I} & \mathbf{A}_j\tilde{\mathbf{s}}_j' \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\tilde{\mathbf{s}}_i\mathbf{A}_i & -\mathbf{I} & \tilde{\mathbf{s}}_j\mathbf{A}_j \end{bmatrix}$$

The $\gamma^{\#}$ for Spherical Joint (cont.)

$$\dot{\mathbf{\Phi}}^{(s,3)} = \begin{bmatrix} \mathbf{I} & -\mathbf{A}_i\tilde{\mathbf{s}}_i' & -\mathbf{I} & \mathbf{A}_j\tilde{\mathbf{s}}_j' \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i' \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}_j' \end{bmatrix} = 0$$

$$\ddot{\mathbf{\Phi}}^{(s,3)} = \begin{bmatrix} \mathbf{I} & -\mathbf{A}_i\tilde{\mathbf{s}}_i' & -\mathbf{I} & \mathbf{A}_j\tilde{\mathbf{s}}_j' \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i' \\ \ddot{\mathbf{r}}_j \\ \dot{\boldsymbol{\omega}}_j' \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\mathbf{A}}_i\tilde{\mathbf{s}}_i' & 0 & \dot{\mathbf{A}}_j\tilde{\mathbf{s}}_j' \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i' \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}_j' \end{bmatrix} = \mathbf{\Phi}_q\ddot{\mathbf{q}} - \boldsymbol{\gamma}^{\#} = 0$$

$$\boldsymbol{\gamma}^{\#} = \begin{bmatrix} 0 & \dot{\mathbf{A}}_i\tilde{\mathbf{s}}_i' & 0 & -\dot{\mathbf{A}}_j\tilde{\mathbf{s}}_j' \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i' \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}_j' \end{bmatrix} = -\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i + \tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j$$

$$\tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T$$

$$\mathbf{A}\tilde{\mathbf{s}}' = \tilde{\mathbf{s}}\mathbf{A}$$

$$\dot{\mathbf{A}}\tilde{\mathbf{s}}'\boldsymbol{\omega}' = \tilde{\boldsymbol{\omega}}\mathbf{A}\tilde{\mathbf{s}}'\boldsymbol{\omega}' = \tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}\mathbf{A}\boldsymbol{\omega}' = \tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}\boldsymbol{\omega} = -\tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{s} = -\tilde{\boldsymbol{\omega}}\dot{\mathbf{s}}$$

Velocity Equations & Conversion

$$\mathbf{\Phi} = 0$$

$$\dot{\mathbf{\Phi}} = \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}$$

$$= [\mathbf{\Phi}_{\mathbf{r}_1}, \mathbf{\Phi}_{\mathbf{p}_1}, \ldots, \mathbf{\Phi}_{\mathbf{r}_b}, \mathbf{\Phi}_{\mathbf{p}_b}] \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \dot{\mathbf{p}}_1 \\ \vdots \\ \dot{\mathbf{r}}_b \\ \dot{\mathbf{p}}_b \end{bmatrix}$$

$$= \left[\mathbf{\Phi}_{\mathbf{r}_1}, \tfrac{1}{2}\mathbf{\Phi}_{\mathbf{p}_1}\mathbf{L}_1^T, \ldots, \mathbf{\Phi}_{\mathbf{r}_b}, \tfrac{1}{2}\mathbf{\Phi}_{\mathbf{p}_b}\mathbf{L}_b^T\right] \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \boldsymbol{\omega}_1' \\ \vdots \\ \dot{\mathbf{r}}_b \\ \boldsymbol{\omega}_b' \end{bmatrix} = 0$$

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{L}^T\boldsymbol{\omega}' = \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{bmatrix}_{4\times 3} \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}_{3\times 1}$$

$$\mathbf{\Phi}_p\dot{\mathbf{p}} = \frac{1}{2}\mathbf{\Phi}_p\mathbf{L}^T\boldsymbol{\omega}'$$

Acceleration Equations & Conversion

$$\boldsymbol{\Phi} = 0$$

$$\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = -\boldsymbol{\Phi}_t$$

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \boldsymbol{\gamma}$$

$$\mathbf{q}_i = \begin{bmatrix} x \\ y \\ z \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}_{7\times 1} = \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}_i$$

$$\ddot{\mathbf{p}}_i = \tfrac{1}{2}\mathbf{L}_i^T\dot{\boldsymbol{\omega}}_i' - \tfrac{1}{4}\omega^2\mathbf{p}_i$$

$$\boldsymbol{\Phi}_{\mathbf{p}_i}\ddot{\mathbf{p}}_i = \boldsymbol{\Phi}_{\mathbf{p}_i}(\tfrac{1}{2}\mathbf{L}_i^T\dot{\boldsymbol{\omega}}_i' - \tfrac{1}{4}\omega_i^2\mathbf{p}_i) = \tfrac{1}{2}\boldsymbol{\Phi}_{\mathbf{p}_i}\mathbf{L}_i^T\dot{\boldsymbol{\omega}}_i' - \tfrac{1}{4}\omega_i^2\boldsymbol{\Phi}_{\mathbf{p}_i}\mathbf{p}_i$$

$$[\boldsymbol{\Phi}_{\mathbf{r}_1}, \boldsymbol{\Phi}_{\mathbf{p}_1}, \ldots, \boldsymbol{\Phi}_{\mathbf{r}_b}, \boldsymbol{\Phi}_{\mathbf{p}_b}] \begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \ddot{\mathbf{p}}_1 \\ \vdots \\ \ddot{\mathbf{r}}_b \\ \ddot{\mathbf{p}}_b \end{bmatrix} = \boldsymbol{\gamma}$$

$$[\boldsymbol{\Phi}_{\mathbf{r}_1}, \tfrac{1}{2}\boldsymbol{\Phi}_{\mathbf{p}_1}\mathbf{L}_1^T, \ldots, \boldsymbol{\Phi}_{\mathbf{r}_b}, \tfrac{1}{2}\boldsymbol{\Phi}_{\mathbf{p}_b}\mathbf{L}_b^T] \begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \dot{\boldsymbol{\omega}}_1' \\ \vdots \\ \ddot{\mathbf{r}}_b \\ \dot{\boldsymbol{\omega}}_b' \end{bmatrix} = \boldsymbol{\gamma}^{\#}$$

Components in the Expansion of the Most Common Constraints (ref: p.299)

If we chose $\dot{\mathbf{q}}_i = \begin{bmatrix} \dot{\mathbf{r}} \\ \boldsymbol{\omega}' \end{bmatrix}^i_{6\times 1}$ $\ddot{\mathbf{q}}_i = \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix}^i_{6\times 1}$

	$\frac{\partial\Phi}{\partial\dot{\mathbf{r}}_i}$	$\frac{\partial\Phi}{\partial\boldsymbol{\omega}'_i}$	$\frac{\partial\Phi}{\partial\dot{\mathbf{r}}_j}$	$\frac{\partial\Phi}{\partial\boldsymbol{\omega}'_j}$	$\boldsymbol{\gamma}^{\#}$
$\boldsymbol{\Phi}$	$\boldsymbol{\Phi}_{\mathbf{r}_i}^{(m)}$	$\frac{1}{2}\boldsymbol{\Phi}_{\mathbf{p}_i}^{(m)}\mathbf{L}_i^T$	$\boldsymbol{\Phi}_{\mathbf{r}_j}^{(m)}$	$\frac{1}{2}\boldsymbol{\Phi}_{\mathbf{p}_j}^{(m)}\mathbf{L}_j^T$	$\boldsymbol{\gamma}^{\#}$
$\boldsymbol{\Phi}^{(n1,1)}$	$\mathbf{0}^T$	$-\mathbf{s}_j^T\tilde{\mathbf{s}}_i\mathbf{A}_i$	$\mathbf{0}^T$	$-\mathbf{s}_i^T\tilde{\mathbf{s}}_j\mathbf{A}_j$	$-2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j + \dot{\mathbf{s}}_i^T\tilde{\boldsymbol{\omega}}_i\mathbf{s}_j + \dot{\mathbf{s}}_j^T\tilde{\boldsymbol{\omega}}_j\mathbf{s}_i$
$\boldsymbol{\Phi}^{(n2,1)}$	$-\mathbf{s}_i^T$	$-(\mathbf{d} + \mathbf{s}_i^B)^T\tilde{\mathbf{s}}_i\mathbf{A}_i$	$\mathbf{s}_i^T$	$-\mathbf{s}_i^T\tilde{\mathbf{s}}_j^B\mathbf{A}_j$	$-2\dot{\mathbf{d}}^T\dot{\mathbf{s}}_i - \mathbf{d}^T\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i + \mathbf{s}_i^T(\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i^B - \tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j^B)$
$\boldsymbol{\Phi}^{(p1,2)}$	$\mathbf{0}$	$\tilde{\mathbf{s}}_j\tilde{\mathbf{s}}_i\mathbf{A}_i$	$\mathbf{0}$	$-\tilde{\mathbf{s}}_i\tilde{\mathbf{s}}_j\mathbf{A}_j$	$-2\tilde{\dot{\mathbf{s}}}_i\dot{\mathbf{s}}_j + \tilde{\mathbf{s}}_j\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i - \tilde{\mathbf{s}}_i\tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j$
$\boldsymbol{\Phi}^{(p2,2)}$	$-\tilde{\mathbf{s}}_i$	$(\tilde{\mathbf{s}}_i\tilde{\mathbf{s}}_i^B + \tilde{\mathbf{d}}\tilde{\mathbf{s}}_i)\mathbf{A}_i$	$\tilde{\mathbf{s}}_i$	$-\tilde{\mathbf{s}}_i\tilde{\mathbf{s}}_j^B\mathbf{A}_j$	$-2\tilde{\dot{\mathbf{s}}}_i\dot{\mathbf{d}} + \tilde{\mathbf{s}}_i(\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i^B - \tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j^B) + \tilde{\dot{\mathbf{d}}}\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i$
$\boldsymbol{\Phi}^{(s,3)}$	$\mathbf{I}$	$-\tilde{\mathbf{s}}_i^P\mathbf{A}_i$	$-\mathbf{I}$	$\tilde{\mathbf{s}}_j^P\mathbf{A}_j$	$-\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i^P + \tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j^P$

	$\frac{\partial\Phi}{\partial\dot{\mathbf{r}}_i}$	$\frac{\partial\Phi}{\partial\boldsymbol{\omega}'_i}$	$\frac{\partial\Phi}{\partial\dot{\mathbf{r}}_j}$	$\frac{\partial\Phi}{\partial\boldsymbol{\omega}'_j}$	$\boldsymbol{\gamma}^{\#}$
$\boldsymbol{\Phi}^{(n1,1)}$	$\mathbf{0}^T$	$-\mathbf{s}_j^T\tilde{\mathbf{s}}_i\mathbf{A}_i$	$\mathbf{0}^T$	$-\mathbf{s}_i^T\tilde{\mathbf{s}}_j\mathbf{A}_j$	$-2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j + \dot{\mathbf{s}}_i^T\tilde{\boldsymbol{\omega}}_i\mathbf{s}_j + \dot{\mathbf{s}}_j^T\tilde{\boldsymbol{\omega}}_j\mathbf{s}_i$

Components of the Jacobin Matrix of the Most common Constraints ref: p. 201

If we chose $\dot{\mathbf{q}}_i = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{p}} \end{bmatrix}^i_{7\times1}$ $\ddot{\mathbf{q}}_i = \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{p}} \end{bmatrix}^i_{7\times1}$

$\mathbf{\Phi}$	$\mathbf{\Phi}_{\mathbf{r}_i}$	$\mathbf{\Phi}_{\mathbf{p}_i}$	$\mathbf{\Phi}_{\mathbf{r}_j}$	$\mathbf{\Phi}_{\mathbf{p}_j}$
$\Phi^{(n1,1)}$	$\mathbf{0}^T$	$\mathbf{s}_j^T\mathbf{C}_i$	$\mathbf{0}^T$	$\mathbf{s}_i^T\mathbf{C}_j$
$\Phi^{(n2,1)}$	$-\mathbf{s}_i^T$	$-\mathbf{s}_i^T\mathbf{B}_i + \mathbf{d}^T\mathbf{C}_i$	$\mathbf{s}_i^T$	$\mathbf{s}_i^T\mathbf{B}_j$
$\mathbf{\Phi}^{(p1,2)}$	$\mathbf{0}$	$-\tilde{\mathbf{s}}_j\mathbf{C}_i$	$\mathbf{0}$	$\tilde{\mathbf{s}}_i\mathbf{C}_j$
$\mathbf{\Phi}^{(p2,2)}$	$-\tilde{\mathbf{s}}_i$	$-\tilde{\mathbf{s}}_i\mathbf{B}_i - \tilde{\mathbf{d}}\mathbf{C}_i$	$\tilde{\mathbf{s}}_i$	$\tilde{\mathbf{s}}_i\mathbf{B}_j$
$\mathbf{\Phi}^{(s,3)}$	$\mathbf{I}$	$\mathbf{C}_i$	$-\mathbf{I}$	$-\mathbf{C}_j$

$$\mathbf{B}_k = 2(\mathbf{G}_k\bar{\mathbf{s}}_k'^B + \mathbf{s}_k'^B\mathbf{p}_k^T)$$

$$\mathbf{C}_k = 2(\mathbf{G}_k\bar{\mathbf{s}}_k' + \mathbf{s}_k'\mathbf{p}_k^T)$$

$$k = i, j$$

Modified Jacobian Matrix and Modified Vector γ (ref: 203)

If eliminating some equivalent terms from both sides of the velocity and acceleration equations, one have

Velocity equation: $\mathbf{\Phi}_\mathbf{q}^{(m)}\dot{\mathbf{q}} = -\mathbf{\Phi}_t = \begin{bmatrix} 0 \\ -\mathbf{\Phi}_t^{(d)} \end{bmatrix}$

Acceleration equation: $\mathbf{\Phi}_\mathbf{q}^{(m)}\ddot{\mathbf{q}} = \gamma^{(m)}$

$$\mathbf{\Phi}_\mathbf{q}^{(m)}\ddot{\mathbf{q}} = \begin{bmatrix} \cdots & \mathbf{\Phi}_{\mathbf{r}_i}^{(m)} & \mathbf{\Phi}_{\mathbf{p}_i}^{(m)} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \ddot{\mathbf{r}}_i \\ \ddot{\mathbf{p}}_i \\ \vdots \end{bmatrix} = \gamma^{(m)}$$

$\mathbf{\Phi}$	$\mathbf{\Phi}_{\mathbf{r}_i}^{(m)}$	$\mathbf{\Phi}_{\mathbf{p}_i}^{(m)}$	$\mathbf{\Phi}_{\mathbf{r}_j}^{(m)}$	$\mathbf{\Phi}_{\mathbf{p}_j}^{(m)}$	$\gamma^{(m)}$
$\Phi^{(n1,1)}$	$\mathbf{0}^T$	$2\mathbf{s}_j^T\mathbf{G}_i\bar{\mathbf{s}}_i'$	$\mathbf{0}^T$	$2\mathbf{s}_i^T\mathbf{G}_j\bar{\mathbf{s}}_j'$	$\mathbf{s}_i^T\mathbf{h}_j + \mathbf{s}_j^T\mathbf{h}_i - 2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j$
$\Phi^{(n2,1)}$	$-\mathbf{s}_i^T$	$-2\mathbf{s}_i^T\mathbf{G}_i\bar{\mathbf{s}}_i'^B + 2\mathbf{d}^T\mathbf{G}_i\bar{\mathbf{s}}_i'$	$\mathbf{s}_i^T$	$2\mathbf{s}_i^T\mathbf{G}_j\mathbf{s}_j'^B$	$+\mathbf{s}_i^T(\mathbf{h}_i^B - \mathbf{h}_j^B) + \mathbf{d}^T\mathbf{h}_i - 2\dot{\mathbf{s}}_i^T\dot{\mathbf{d}}$
$\mathbf{\Phi}^{(p1,2)}$	$\mathbf{0}$	$-2\tilde{\mathbf{s}}_j\mathbf{G}_i\bar{\mathbf{s}}_i'$	$\mathbf{0}$	$2\tilde{\mathbf{s}}_i\mathbf{G}_j\bar{\mathbf{s}}_j'$	$\tilde{\mathbf{s}}_i\mathbf{h}_j - \tilde{\mathbf{s}}_j\mathbf{h}_i - 2\tilde{\dot{\mathbf{s}}}_i\dot{\mathbf{s}}_j$
$\mathbf{\Phi}^{(p2,2)}$	$-\tilde{\mathbf{s}}_i$	$-2\tilde{\mathbf{s}}_i\mathbf{G}_i\bar{\mathbf{s}}_i'^B - 2\tilde{\mathbf{d}}\mathbf{G}_i\bar{s}_i'$	$\tilde{\mathbf{s}}_i$	$2\tilde{\mathbf{s}}_i\mathbf{G}_j\bar{\mathbf{s}}_j'^B$	$+\tilde{\mathbf{s}}_i(\mathbf{h}_j^B - \mathbf{h}_i^B) - \tilde{\mathbf{d}}\mathbf{h}_i - 2\tilde{\dot{\mathbf{s}}}_i\dot{\mathbf{d}}$
$\mathbf{\Phi}^{(s,3)}$	$\mathbf{I}$	$2\mathbf{G}_i\bar{\mathbf{s}}_i'^P$	$-\mathbf{I}$	$-2\mathbf{G}_j\bar{\mathbf{s}}_j'^P$	$\mathbf{h}_i^P - \mathbf{h}_j^P$

$\mathbf{h}_k = -2\dot{\mathbf{G}}_k\dot{\mathbf{L}}_k^T\mathbf{s}_k'$ $\mathbf{h}_k^B = -2\dot{\mathbf{G}}_k\dot{\mathbf{L}}_k^T\mathbf{s}_k'^B$ $\mathbf{h}_k^P = -2\dot{\mathbf{G}}_k\dot{\mathbf{L}}_k^T\mathbf{s}_k'^P$ $k = i, j$

Position Analysis & Newton-Raphson Iteration

- Determine **q** to satisfy Φ(**q**)=0 $\boxed{\mathbf{p}^T\mathbf{p} - 1 = 0}$

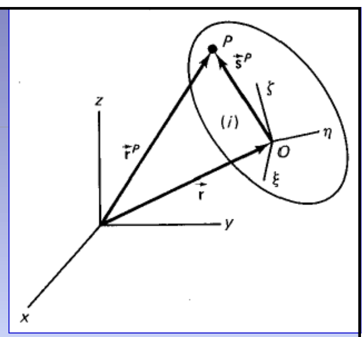
$$\boxed{\mathbf{q}^{j+1} = \mathbf{q}^j + \Delta\mathbf{q}} \qquad \boxed{\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}^j)\Delta\mathbf{q} = -\mathbf{\Phi}(\mathbf{q}^j)}$$

$$\mathbf{\Phi}_{\mathbf{q}} = \begin{bmatrix} \cdots & \boxed{\mathbf{\Phi}_{\mathbf{r}_i} \quad \mathbf{\Phi}_{\mathbf{p}_i}} & \cdots \end{bmatrix}_{m\times 7b} \qquad \mathbf{q} = \begin{bmatrix} \vdots \\ \mathbf{r}_i \\ \mathbf{p}_i \\ \vdots \end{bmatrix}_{7b\times 1}$$

A Kind of Integration & Conversion

Express angular motions in terms of local coordinates

(ref: Chap. 13, p.317-)



$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \boldsymbol{\gamma}^{\#} \end{bmatrix} \Rightarrow \ddot{\mathbf{q}} = \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix}$$

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \boldsymbol{\omega}' \\ \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix} \rightarrow \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{p}} \\ \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix} \xrightarrow{\text{integrate}} \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \\ \dot{\mathbf{r}} \\ \boldsymbol{\omega}' \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{r} \\ \mathbf{A} \\ \dot{\mathbf{r}} \\ \boldsymbol{\omega}' \end{bmatrix} \rightarrow \boxed{\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P}$$

A: Rotation Matrix

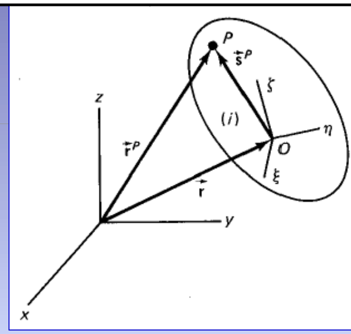
$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{L}^T\boldsymbol{\omega}'$$

$$= \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{bmatrix}_{4\times 3} \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}_{3\times 1}$$

$$\mathbf{A} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

Another Kind of Integration & Conversion

Express angular motions in terms of global coordinates



$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \boldsymbol{\gamma}^{\#} \end{bmatrix} \Rightarrow \ddot{\mathbf{q}} = \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \boldsymbol{\omega} \\ \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} \rightarrow \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{p}} \\ \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} \xrightarrow{\text{integrate}} \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \\ \dot{\mathbf{r}} \\ \boldsymbol{\omega} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{r} \\ \mathbf{A} \\ \dot{\mathbf{r}} \\ \boldsymbol{\omega} \end{bmatrix} \rightarrow \mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P$$

A: Rotation Matrix

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^T\boldsymbol{\omega}$$

$$= \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & e_3 & -e_2 \\ -e_3 & e_0 & e_1 \\ e_2 & -e_1 & e_0 \end{bmatrix}_{4\times 3} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{3\times 1}$$

$$\mathbf{A} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$