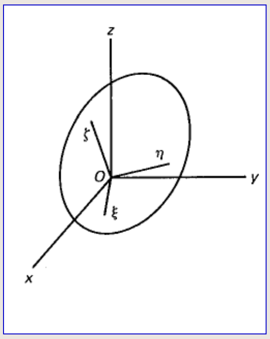
Planar Kinematics

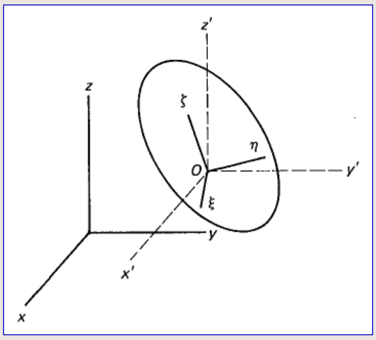
Reading material: Chapter 4

2004/7/8 計算動力學(C.F. Chang) 1

Transformation Between Body Coordinates and Global Coordinates (p153. 6.1)

- The configuration of the $\xi\eta\zeta$ axes w.r.t the xyz axes can be considered as a rotation and a translation (Chasles theorem).





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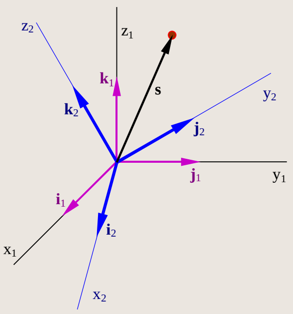
The configuration of the body axes w.r.t the global axes

$\mathbf{P} = x_1\,\mathbf{i}_1 + y_1\,\mathbf{j}_1 + z_1\,\mathbf{k}_1,$ in CS1 Eq.(a)

$\mathbf{P} = x_2\,\mathbf{i}_2 + y_2\,\mathbf{j}_2 + z_2\,\mathbf{k}_2,$ in CS2 Eq.(b)



$$\mathbf{i}_2 = (\mathbf{i}_2\cdot\mathbf{i}_1)\,\mathbf{i}_1 + (\mathbf{i}_2\cdot\mathbf{j}_1)\,\mathbf{j}_1 + (\mathbf{i}_2\cdot\mathbf{k}_1)\,\mathbf{k}_1$$
$$\mathbf{j}_2 = (\mathbf{j}_2\cdot\mathbf{i}_1)\,\mathbf{i}_1 + (\mathbf{j}_2\cdot\mathbf{j}_1)\,\mathbf{j}_1 + (\mathbf{j}_2\cdot\mathbf{k}_1)\,\mathbf{k}_1$$
$$\mathbf{k}_2 = (\mathbf{k}_2\cdot\mathbf{i}_1)\,\mathbf{i}_1 + (\mathbf{k}_2\cdot\mathbf{j}_1)\,\mathbf{j}_1 + (\mathbf{k}_2\cdot\mathbf{k}_1)\,\mathbf{k}_1$$

$$\begin{aligned}\therefore\ \mathbf{P} = &\ [\,x_2(\mathbf{i}_2\cdot\mathbf{i}_1) + y_2(\mathbf{j}_2\cdot\mathbf{i}_1) + z_2(\mathbf{k}_2\cdot\mathbf{i}_1)\,]\ \mathbf{i}_1\\ &+[\,x_2(\mathbf{i}_2\cdot\mathbf{j}_1) + y_2(\mathbf{j}_2\cdot\mathbf{j}_1) + z_2(\mathbf{k}_2\cdot\mathbf{j}_1)\,]\ \mathbf{j}_1\\ &+[\,x_2(\mathbf{i}_2\cdot\mathbf{k}_1) + y_2(\mathbf{j}_2\cdot\mathbf{k}_1) + z_2(\mathbf{k}_2\cdot\mathbf{k}_1)\,]\ \mathbf{k}_1\end{aligned}$$

…….Eq.(c)

Comparing Eqs. (c) and (a) leads to

$$\begin{aligned}x_1 &= x_2(\mathbf{i}_2\cdot\mathbf{i}_1) + y_2(\mathbf{j}_2\cdot\mathbf{i}_1) + z_2(\mathbf{k}_2\cdot\mathbf{i}_1)\\ y_1 &= x_2(\mathbf{i}_2\cdot\mathbf{j}_1) + y_2(\mathbf{j}_2\cdot\mathbf{j}_1) + z_2(\mathbf{k}_2\cdot\mathbf{j}_1)\\ z_1 &= x_2(\mathbf{i}_2\cdot\mathbf{k}_1) + y_2(\mathbf{j}_2\cdot\mathbf{k}_1) + z_2(\mathbf{k}_2\cdot\mathbf{k}_1)\end{aligned}$$

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The configuration of the body axes w.r.t the global axes (pure rotation)

$$\begin{aligned}x_1 &= x_2(\mathbf{i}_2\cdot\mathbf{i}_1) + y_2(\mathbf{j}_2\cdot\mathbf{i}_1) + z_2(\mathbf{k}_2\cdot\mathbf{i}_1)\\ y_1 &= x_2(\mathbf{i}_2\cdot\mathbf{j}_1) + y_2(\mathbf{j}_2\cdot\mathbf{j}_1) + z_2(\mathbf{k}_2\cdot\mathbf{j}_1)\\ z_1 &= x_2(\mathbf{i}_2\cdot\mathbf{k}_1) + y_2(\mathbf{j}_2\cdot\mathbf{k}_1) + z_2(\mathbf{k}_2\cdot\mathbf{k}_1)\end{aligned} \Rightarrow \boxed{\mathbf{s} = \mathbf{A}\mathbf{s}'}$$

s= $[x_1,y_1,z_1]^T$ =*coordinate vector in terms of world coordinates*

s' = $[x_2,y_2,z_2]^T$ = *coordinate vector in terms of body coordinates*

A: *a rotation matrix describing the orientation of the moving body*

$$\mathbf{A} = \begin{bmatrix}(\mathbf{i}_2\bullet\mathbf{i}_1) & (\mathbf{j}_2\bullet\mathbf{i}_1) & (\mathbf{k}_2\bullet\mathbf{i}_1)\\ (\mathbf{i}_2\bullet\mathbf{j}_1) & (\mathbf{j}_2\bullet\mathbf{j}_1) & (\mathbf{k}_2\bullet\mathbf{j}_1)\\ (\mathbf{i}_2\bullet\mathbf{k}_1) & (\mathbf{j}_2\bullet\mathbf{k}_1) & (\mathbf{k}_2\bullet\mathbf{k}_1)\end{bmatrix}$$

- Each column of A is a unit vector, and the unit vectors are orthogonal to each other.
- Similarly, it can be easily derived that $\mathbf{s}'=\mathbf{A}^T\mathbf{s}$
- Thus, **A** is a orthonormal matrix, i.e., $A^{-1}=A^T$

$$\boxed{\mathbf{s}' = \mathbf{A}^{-1}\mathbf{s} = \mathbf{A}^T\mathbf{s}}$$

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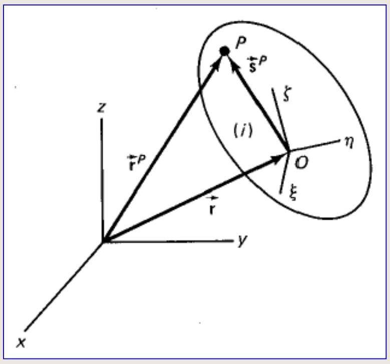
The configuration of the body axes w.r.t the global axes (rotation + translation)

- If the origins of the body and global coordinate systems do not coincide, an additional translation is necessary. Thus,



$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P$$

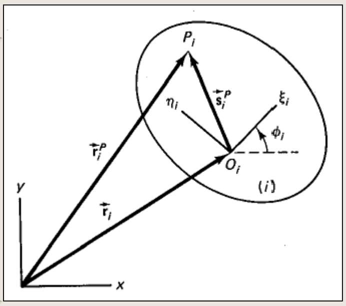
- A vector with a superscript prime is one that contains local components
- A vector without of a superscript prime is one that contains global components

Locating Points Relative to the Body-Fixed and Global Coordinate Systems (Planar Motion)

For a point P_i on body i,

$$\mathbf{s}_i^P = \mathbf{A}_i \mathbf{s}_i'^P$$

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i'^P$$



$$\mathbf{A}_i = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}_i$$

A_i = *a rotation matrix representing the* *orientation* *of the moving body*

$\mathbf{s}_i'^P = [\xi^P, \eta^P]_i^T$

= *coordinate of point P_i w.r.t body-fixed coordinate system $\xi_i \eta_i$*

This is a constant vector

$\mathbf{r}_i = [x, y]_i^T$

= *the origin of the body-fixed coordinate system*

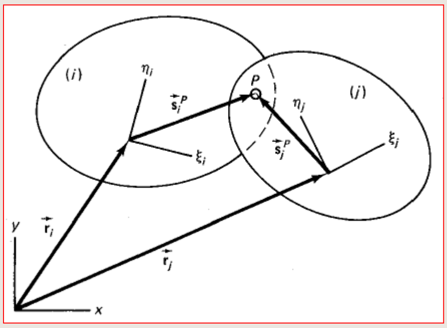
$\mathbf{r}_i^P = [x^P, y^P]_i^T$

= *global coordinates of point P_i*

Function of (x_i, y_i, ϕ_i)

Kinematic Constraint—Revolute Joints (p. 81)

- The constraint equations for the revolute joint can be obtained from the vector loop equation:



$$\vec{r}_i + \vec{s}_i^P - \vec{s}_j^P - \vec{r}_j = \vec{0}$$

$$\mathbf{r}_i + \mathbf{s}_i^P - \mathbf{r}_j - \mathbf{s}_j^P = \mathbf{0}$$

$$\Phi^{(r,2)} \equiv \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i'^P - \mathbf{r}_j - \mathbf{A}_j\mathbf{s}_j'^P = \mathbf{0}$$

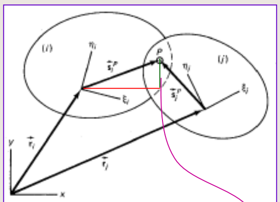
- The superscripts of Φ indicating
  1. *the constraint type*
  2. *the number of algebraic equations*
- $\Phi^{(r,2)}$ denotes the *revolute joint* constraint which contains two equations

Kinematic Constraint—Revolute Joints (pp.103-106, Table 4.2)

$$\vec{r}_i + \vec{s}_i^P - \vec{s}_j^P - \vec{r}_j = \vec{0}$$

$$\mathbf{r}_i + \mathbf{s}_i^P - \mathbf{r}_j - \mathbf{s}_j^P = \mathbf{0}$$

$$\Phi^{(r,2)} \equiv \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i'^P - \mathbf{r}_j - \mathbf{A}_j\mathbf{s}_j'^P = \mathbf{0}$$



$$\Phi^{(r,2)} \equiv \begin{bmatrix} x_i + \xi_i^P\cos\phi_i - \eta_i^P\sin\phi_i - x_j - \xi_j^P\cos\phi_j + \eta_j^P\sin\phi_j \\ y_i + \xi_i^P\sin\phi_i + \eta_i^P\cos\phi_i - y_j - \xi_j^P\sin\phi_j - \eta_j^P\cos\phi_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial\Phi^{(r,1st)}}{\partial\phi_i} = -\xi_i^P\sin\phi_i - \eta_i^P\cos\phi_i = -(y_i^P - y_i)$$

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\Phi^{(r,2)}$	1	0	$-(y_i^P - y_i)$	-1	0	$(y_j^P - y_j)$
	0	1	$(x_i^P - x_i)$	0	-1	$-(x_j^P - x_j)$

The Entries of the Jacobian Matrix for Revolute Joints (pp.103-106, Table 4.2)

$$\boxed{\mathbf{\Phi_q \dot{q}} = -\mathbf{\Phi}_t \qquad \text{Eq.(a)}}$$

- Taking the time derivative of the constraint equations yields

$$\boxed{\begin{aligned} &\dot{x}_i - (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i - \dot{x}_j + (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_j = 0 \\ &\dot{y}_i + (\xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i)\dot{\phi}_i - \dot{y}_j - (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\dot{\phi}_j = 0 \end{aligned}}$$

- The entries of the Jacobian Matrix for revolute joints can be determined by expressing the two equations in the following form and comparing with velocity equation

$$\begin{array}{c} \\ \partial\Phi^{(r,1st)}/\partial\ldots \\ \partial\Phi^{(r,2nd)}/\partial\ldots \end{array} \begin{array}{c} \begin{array}{cccccc} x_i & y_i & \phi_i & x_j & y_j & \phi_j \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 5 & 6 & 0 & 7 & 8 \end{bmatrix} \end{array} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\phi}_i \\ \dot{x}_j \\ \dot{y}_j \\ \dot{\phi}_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \boxed{\mathbf{\Phi}_t = 0}$$

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\Phi^{(r,2)}$	1	0	$-(y_i^P - y_i)$	-1	0	$(y_j^P - y_j)$
	0	1	$(x_i^P - x_i)$	0	-1	$-(x_j^P - x_j)$

The RHS of Acceleration Equation for Revolute Joints (P.107, Table 4.3)

$$\boxed{\mathbf{\Phi_q \ddot{q}} = -[(\mathbf{\Phi_q \dot{q}})_\mathbf{q} \mathbf{\dot{q}} + 2\mathbf{\Phi}_{\mathbf{q}t}\mathbf{\dot{q}} + \mathbf{\Phi}_{tt}] \equiv \boldsymbol{\gamma} \qquad \text{Eq(b)}}$$

- Differentiating twice the constraint equation $\Phi^{(r,2)}$ w.r.t. time yields

$$\boxed{\begin{aligned} &\ddot{x}_i - (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\ddot{\phi}_i - (\xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i)\dot{\phi}_i^2 - \ddot{x}_j \\ &\qquad + (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\ddot{\phi}_j + (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\dot{\phi}_j^2 = 0 \\ &\ddot{y}_i + (\xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i)\ddot{\phi}_i - (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i^2 - \ddot{y}_j \\ &\qquad - (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\ddot{\phi}_j + (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_j^2 = 0 \end{aligned}}$$

- These two equations can be written in the form of Eq (b) as follows

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 5 & 6 & 0 & 7 & 8 \end{bmatrix} \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \\ \ddot{x}_j \\ \ddot{y}_j \\ \ddot{\phi}_j \end{bmatrix} = \begin{bmatrix} (\xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i)\dot{\phi}_i^2 - (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\dot{\phi}_j^2 \\ (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i^2 - (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_i^2 \end{bmatrix}$$

The RHS of Acceleration Equation for Revolute Joints (P.107, Table 4.3)

$$
\begin{bmatrix} ① & 0 & ② & ③ & 0 & ④ \\ 0 & ⑤ & ⑥ & 0 & ⑦ & ⑧ \end{bmatrix}
\begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \\ \ddot{x}_j \\ \ddot{y}_j \\ \ddot{\phi}_j \end{bmatrix}
$$

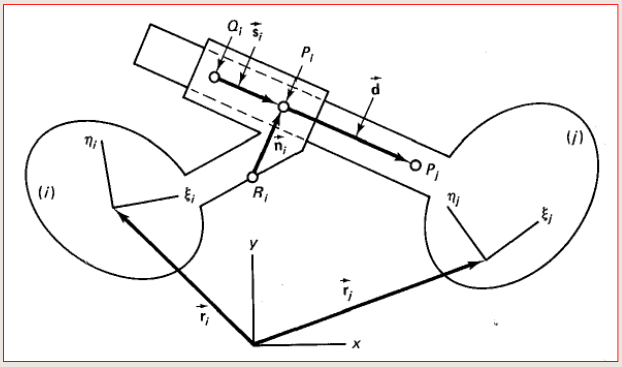
$$
= \begin{bmatrix} (\xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i)\dot{\phi}_i^2 - (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\dot{\phi}_j^2 \\ (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i^2 - (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_i^2 \end{bmatrix}
$$

$$
= \boxed{\begin{bmatrix} (x_i^P - x_i)\dot{\phi}_i^2 - (x_j^P - x_j)\dot{\phi}_j^2 \\ (y_i^P - y_i)\dot{\phi}_i^2 - (y_j^P - y_j)\dot{\phi}_j^2 \end{bmatrix}}
$$

$$
= \mathbf{s}_i^P\dot{\phi}_i^2 - \mathbf{s}_j^P\dot{\phi}_j^2
$$

	$\boldsymbol{\gamma}$
$\Phi^{(r,2)}$	$(x_i^P - x_i)\dot{\phi}_i^2 - (x_j^P - x_j)\dot{\phi}_j^2$
	$(y_i^P - y_i)\dot{\phi}_i^2 - (y_j^P - y_j)\dot{\phi}_j^2$

Kinematic Constraint—Translational Joints (pp.82-84)



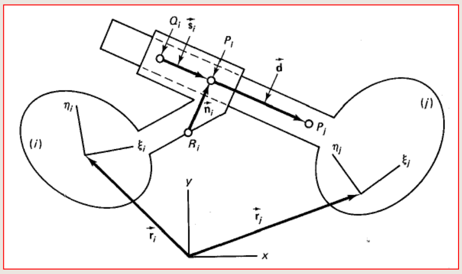
- There is no relative rotation between the two bodies i and j
  - $\phi_i - \phi_j = \text{constant} = \phi_i^0 - \phi_j^0$
- The two vector $\mathbf{s}_i$ and $\mathbf{d}$ must remain parallel all the time (in other words, Q_i, P_i on body i and P_j on body j must be on a line)

Kinematic Constraint—Translational Joints (pp.82-84)

- If we rotate vector $\mathbf{s}_i$ by 90 degree counterclockwise to determine $\mathbf{n}_i$
- The collinear condition for Q_i, P_i and P_j can be rewritten as $\mathbf{n}_i^T\mathbf{d} = 0$, i.e.,

$$\left(\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix}\begin{bmatrix}x_i^P - x_i^Q\\ y_i^P - y_i^Q\end{bmatrix}\right)^T \begin{bmatrix}x_j^P - x_i^P\\ y_j^P - y_i^P\end{bmatrix} = \left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_i^P - x_i^Q\\ y_i^P - y_i^Q\end{bmatrix}\right)^T \begin{bmatrix}x_j^P - x_i^P\\ y_j^P - y_i^P\end{bmatrix}$$

$$= \begin{bmatrix}-(y_i^P - y_i^Q)\\ x_i^P - x_i^Q\end{bmatrix}^T \begin{bmatrix}x_j^P - x_i^P\\ y_j^P - y_i^P\end{bmatrix} = -(y_i^P - y_i^Q)(x_j^P - x_i^P) + (x_i^P - x_i^Q)(y_j^P - y_i^P) = 0$$



Kinematic Constraint—Translational Joints (pp.82-84)

$$-(y_i^P - y_i^Q)(x_j^P - x_i^P) + (x_i^P - x_i^Q)(y_j^P - y_i^P) = 0$$

$$\phi_i - \phi_j = \phi_i^0 - \phi_j^0$$

Thus, the constraint equations for a translational joint can be expressed as

$$\boxed{\mathbf{\Phi}^{(t,2)} = \begin{bmatrix}(x_i^P - x_i^Q)(y_j^P - y_i^P) - (y_i^P - y_i^Q)(x_j^P - x_i^P)\\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}}$$

The Entries of the Jacobian Matrix for Trnaslational Joints (Table 4.2)

$$\boxed{\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = -\mathbf{\Phi}_t \qquad \text{Eq.(a)}}$$

$$\mathbf{\Phi}^{(t,2)} = \begin{bmatrix} (x_i^P - x_i^Q)(y_j^P - y_i^P) - (y_i^P - y_i^Q)(x_j^P - x_i^P) \\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Taking the time derivative of the constraint equations yields

$$\mathbf{\Phi}_{\mathbf{q}} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\phi}_i \\ \dot{x}_j \\ \dot{y}_j \\ \dot{\phi}_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \boxed{\mathbf{\Phi}_t = 0}$$

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\Phi^{(t,2)}$	$(y_i^P - y_i^Q)$	$-(x_i^P - x_i^Q)$	$-(x_j^P - x_i)(x_i^P - x_i^Q) - (y_j^P - y_i)(y_i^P - y_i^Q)$	$-(y_i^P - y_i^Q)$	$(x_i^P - x_i^Q)$	$(x_j^P - x_j)(x_i^P - x_i^Q) + (y_j^P - y_j)(y_i^P - y_i^Q)$
	0	0	1	0	0	-1

The RHS of Acceleration Equation for Translational Joints (P.107, Table 4.3)

$$\boxed{\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -[(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + 2\mathbf{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} + \mathbf{\Phi}_{tt}] \equiv \boldsymbol{\gamma} \qquad \text{Eq(b)}}$$

- Differentiating twice the constraint equation $\Phi^{(t,2)}$ w.r.t. time yields

$$\mathbf{\Phi}_{\mathbf{q}} \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \\ \ddot{x}_j \\ \ddot{y}_j \\ \ddot{\phi}_j \end{bmatrix} = \overbrace{\begin{pmatrix} -2[(x_i^P - x_i^Q)(\dot{x}_i - \dot{x}_j) + (y_i^P - y_i^Q)(\dot{y}_i - \dot{y}_j)]\dot{\phi}_i \\ - [(x_i^P - x_i^Q)(y_i - y_j) - (y_i^P - y_i^Q)(x_i - x_j)]\dot{\phi}_i^2 \\ \hdashline 0 \end{pmatrix}}^{\boxed{\gamma}}$$

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
$\Phi^{(t,2)}$	$(y_i^P - y_i^Q)$	$-(x_i^P - x_i^Q)$	$-(x_j^P - x_i)(x_i^P - x_i^Q) - (y_j^P - y_i)(y_i^P - y_i^Q)$	$-(y_i^P - y_i^Q)$	$(x_i^P - x_i^Q)$	$(x_j^P - x_j)(x_i^P - x_i^Q) + (y_j^P - y_j)(y_i^P - y_i^Q)$
	0	0	1	0	0	-1

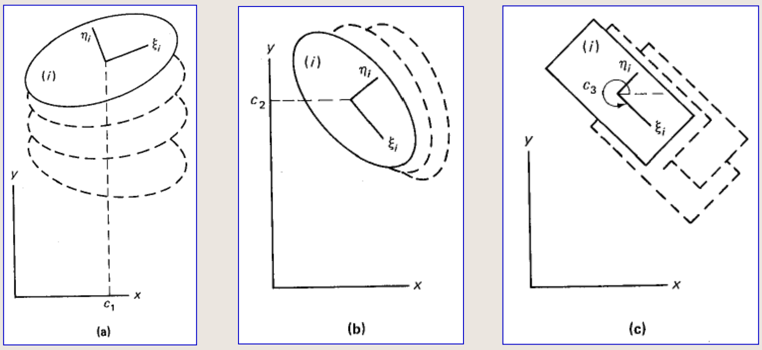
Simplified Constraints

a. Body can move with *constant* x_i, $\Phi = x_i - c_1 = 0$

b. Body can move with *constant* y_i, $\Phi = y_i - c_2 = 0$

c. Body can move with *constant* ϕ_i, $\Phi = \phi_i - c_3 = 0$

※A nonmoving body is subjected to the three simplified constraints



Driving Constraints-1

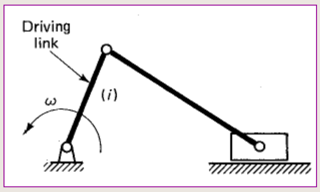
1. Rotate a body with specified function:

- $\Phi = \phi_i - d_1(t) = 0$
- For instance, driving the body with initial angle ϕ_i^0, initial angular velocity ω_i^0, and specific angular acceleration α_i yields

$$\boxed{d_1(t) = \phi_i^0 + \omega_i^0 t + \tfrac{1}{2}\alpha_i t^2 = \text{rotation angle at time t}}$$

- If the body is driven with constant angular velocity ω_i, then

$$\boxed{d_1(t) = \phi_i^0 + \omega_i t}$$



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Driving Constraints-2

2. Translate a body with specified function:

$\Phi = x_i - d_2(t) = 0$

and/or

$\Phi = y_i - d_3(t) = 0$

- If driving the body with initial position (x_i^0, y_i^0), initial velocity $\mathbf{v}_i^0$, and specific angular acceleration $\mathbf{a}_i$ yields

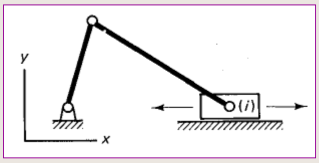
$$\boxed{d_2(t) = x_i^0 + v_{ix}^0 t + \tfrac{1}{2} a_{ix} t^2 = \text{x - coordinate at time t}}$$

$$\boxed{d_3(t) = y_i^0 + v_{iy}^0 t + \tfrac{1}{2} a_{iy} t^2 = \text{y - coordinate at time t}}$$

- If the body is driven with constant velocity v_i, then

$$\boxed{d_2(t) = x_i^0 + v_{ix} t}$$

$$\boxed{d_3(t) = y_i^0 + v_{iy} t}$$



Driving Constraints-3

2. Translate a point P with specified function:

$\Phi = x_i^P - d_4(t) = 0$

and/or

$\Phi = y_i^P - d_5(t) = 0$

- If driving the body with initial position (x_i^{P0}, y_i^{P0}), initial velocity $\mathbf{v}_i^{P0}$, and specific angular acceleration $\mathbf{a}_i^P$ yields

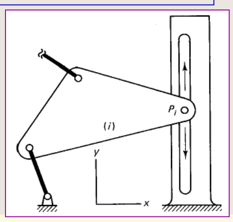
$$\boxed{d_4(t) = x_i^{P0} + v_{ix}^{P0} t + \tfrac{1}{2} a_{ix}^P t^2 = \textit{specified } x_i^P \textit{ at time t}}$$

$$\boxed{d_5(t) = y_i^{P0} + v_{iy}^{P0} t + \tfrac{1}{2} a_{iy}^P t^2 = \textit{specified } y_i^P \textit{ at time t}}$$

- If the body is driven with constant velocity v_i^P, then

$$\boxed{d_4(t) = x_i^{P0} + v_{ix}^P t}$$

$$\boxed{d_5(t) = y_i^{P0} + v_{iy}^P t}$$



$-\Phi_t$ and γ

- What are the rhs of velocity & acceleration equations for those constraints other than revolute & translational joints?
- For simplified constraints:
  - $\Phi_t = 0$
  - $\gamma = 0$

$$\boxed{\Phi = q_i - c_i = 0}$$

- For driving constraint that specifies ω and α :
  - $\Phi_t = \omega + \alpha t$
  - $\gamma = \alpha$

$$\boxed{\Phi = \phi_i - \phi_i^0 + \omega_i t + \tfrac{1}{2}\alpha_i t^2 = 0}$$

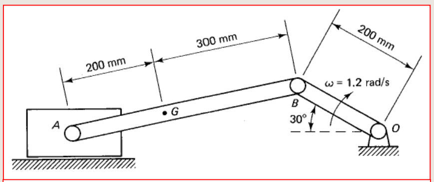
- For driving constraint that specifies v and a :
  - $\Phi_t = v + a t$
  - $\gamma = a$

$$\boxed{\Phi = x_i - x_i^0 + v_{ix} t + \tfrac{1}{2} a_{ix} t^2 = 0}$$

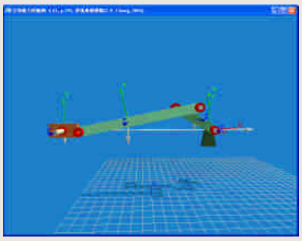
$$\boxed{\Phi = y_i - y_i^0 + v_{iy} t + \tfrac{1}{2} a_{iy} t^2 = 0}$$

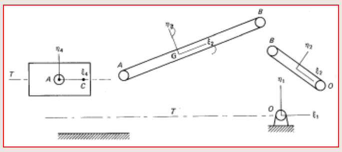
An Example of Kinematic Modeling & Kinematic Analysis

- n = 3 x (4 links) = 12
- m = 1fixed link + three rev joints + 1 trans joint + 1 driving joint
  = 1x3+3x2+1x2+1=12
- Determine the Jacobian matrix and the global coordinates of the bodies, **q**, when ϕ_2=330°



x_1 = 0.0,	y_1 = 0.0,	ϕ_1 = 0.0
x_2 = −86.6,	y_2 = 50.0,	ϕ_2 = 5.76
x_3 = −380.5,	y_3 = 40.0,	ϕ_3 = 0.2
x_4 = −663.1,	y_4 = 0.0,	ϕ_4 = 0.0

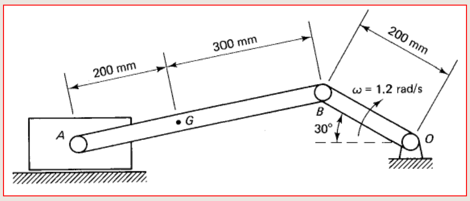




Set Up Constraints – nonmoving body

- In this mechanism, there is only one nonmoving body, link 1.
- For the fixed link 1, the following three simplified constraints can be imposed on the system :

$$\begin{aligned} \Phi_1 &\equiv x_1 &= 0.0 \\ \Phi_2 &\equiv y_1 &= 0.0 \\ \Phi_3 &\equiv \Phi_1 &= 0.0 \end{aligned}$$



$$\begin{array}{c|cccccccccccc} & x_1 & y_1 & \phi_1 & x_2 & y_2 & \phi_2 & x_3 & y_3 & \phi_3 & x_4 & y_4 & \phi_4 \\ \hline \partial\Phi_1/\partial\ldots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial\Phi_2/\partial\ldots & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial\Phi_3/\partial\ldots & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

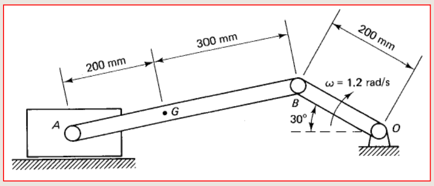
Set Up Constraints – Revolute Joints

- Data for the three revolute joints:

$\xi_4^A = 0.0,$	$\eta_4^A = 0.0,$	$\xi_3^A = -200.0,$	$\eta_3^A = 0.0$	Joint A
$\xi_3^B = 300.0,$	$\eta_3^B = 0.0,$	$\xi_2^B = -100.0,$	$\eta_2^B = 0.0$	Joint B
$\xi_2^0 = 100.0$	$\eta_2^0 = 0.0,$	$\xi_1^0 = 0.0,$	$\eta_1^0 = 0.0$	Joint O

$$\boldsymbol{\Phi}^{(r,2)} \equiv \begin{bmatrix} x_i + \xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i - x_j - \xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j \\ y_i + \xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i - y_j - \xi_j^P \sin\phi_j - \eta_j^P \cos\phi_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{lll} x_1 = 0.0, & y_1 = 0.0, & \phi_1 = 0.0 \\ x_2 = -86.6, & y_2 = 50.0, & \phi_2 = 5.76 \\ x_3 = -380.5, & y_3 = 40.0, & \phi_3 = 0.2 \\ x_4 = -663.1, & y_4 = 0.0, & \phi_4 = 0.0 \end{array}$$



Joint A

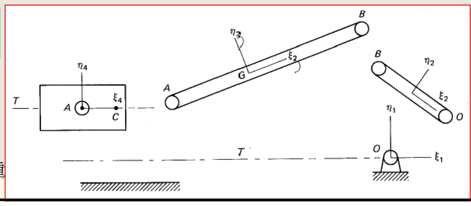
$$\begin{aligned} \Phi_4 &\equiv x_4 - x_3 + 200\cos\phi_3 = 0 \\ \Phi_5 &\equiv y_4 - y_3 + 200\sin\phi_3 = 0 \end{aligned}$$

Joint B

$$\begin{aligned} \Phi_6 &\equiv x_3 + 300\cos\phi_3 - x_2 + 100\cos\phi_2 = 0 \\ \Phi_7 &\equiv y_3 + 300\sin\phi_3 - y_2 + 100\sin\phi_2 = 0 \end{aligned}$$

Joint O

$$\begin{aligned} \Phi_8 &\equiv x_2 + 100\cos\phi_2 - x_1 = 0 \\ \Phi_9 &\equiv y_2 + 100\sin\phi_2 - y_1 = 0 \end{aligned}$$



Set Up Constraints – Revolute Joints (cont.)

Joint A

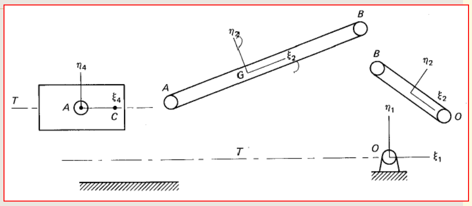
$\Phi_4 \equiv x_4 - x_3 + 200 \cos \phi_3 = 0$
$\Phi_5 \equiv y_4 - y_3 + 200 \sin \phi_3 = 0$

Joint B

$\Phi_6 \equiv x_3 + 300 \cos \phi_3 - x_2 + 100 \cos \phi_2 = 0$
$\Phi_7 \equiv y_3 + 300 \sin \phi_3 - y_2 + 100 \sin \phi_2 = 0$

Joint O

$\Phi_8 \equiv x_2 + 100 \cos \phi_2 - x_1 = 0$
$\Phi_9 \equiv y_2 + 100 \sin \phi_2 - y_1 = 0$

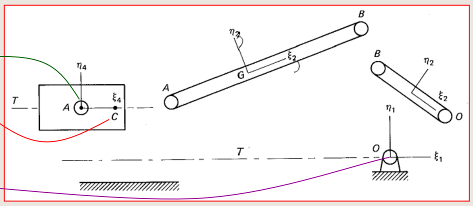


	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial \phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial \phi_j$
$\Phi^{(r,2)}$	1	0	$-(y_i^P - y_i)$	-1	0	$(y_j^P - y_j)$
	0	1	$(x_i^P - x_i)$	0	-1	$-(x_j^P - x_j)$

	Link 1			Link 2			Link 3			Link 4		
	x_1	y_1	ϕ_1	x_2	y_2	ϕ_2	x_3	y_3	ϕ_3	x_4	y_4	ϕ_4
$\partial\Phi_4/\partial\ldots$	0	0	0	0	0	0	4	0	5	6	0	7
$\partial\Phi_5/\partial\ldots$	0	0	0	0	0	0	0	8	9	0	10	11
$\partial\Phi_6/\partial\ldots$	0	0	0	12	0	13	14	0	15	0	0	0
$\partial\Phi_7/\partial\ldots$	0	0	0	0	16	17	0	18	19	0	0	0
$\partial\Phi_8/\partial\ldots$	20	0	21	22	0	23	0	0	0	0	0	0
$\partial\Phi_9/\partial\ldots$	0	24	25	0	26	27	0	0	0	0	0	0

Set Up Constraints – Translational Joints

Two points on link 4:
$\xi_4^A = 0.0, \quad \eta_4^A = 0.0$
$\xi_4^C = 100.0, \quad \eta_4^C = 0.0$
A points on link 1:
$\xi_1^0 = 0.0, \quad \eta_1^0 = 0.0$



$$\Phi^{(t,2)} = \begin{bmatrix} (x_i^P - x_i^Q)(y_j^P - y_i^P) - (y_i^P - y_i^Q)(x_j^P - x_i^P) \\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Phi_{10} \equiv (-100 \cos \phi_4)(y_1 - 100 \sin \phi_1 - y_4) - (x_1 - 100 \cos \phi_1 - x_4)(-100 \sin \phi_4) = 0$
$\Phi_{11} \equiv \phi_4 - \phi_1 = 0$

Set Up Constraints – Translational Joints (cont.)

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial \phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial \phi_j$
$\Phi^{(t,2)}$	$(y_i^P - y_i^Q)$	$-(x_i^P - x_i^Q)$	$-(x_j^P - x_i)(x_i^P - x_i^Q) - (y_j^P - y_i)(y_i^P - y_i^Q)$	$-(y_i^P - y_i^Q)$	$(x_i^P - x_i^Q)$	$(x_j^P - x_j)(x_i^P - x_i^Q) + (y_j^P - y_j)(y_i^P - y_i^Q)$
	0	0	1	0	0	−1

Link 1 | Link 4

	x_1	y_1	ϕ_1	x_2	y_2	ϕ_2	x_3	y_3	ϕ_3	x_4	y_4	ϕ_4
$\partial\Phi_{10}/\partial\ldots$	28	29	30	0	0	0	0	0	0	31	32	33
$\partial\Phi_{11}/\partial\ldots$	0	0	34	0	0	0	0	0	0	0	0	35

(28) $= 100 \sin \phi_4$

(29) $= -100 \cos \phi_4$

(30) $= 10{,}000(\cos\phi_1 \cos\phi_4 + \sin\phi_1 \sin\phi_4)$

(31) $= -$(28)

(32) $= -$(29)

(33) $= 100[\sin\phi_4(y_1 - y_4) + \cos\phi_4(x_1 - x_4)]$

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Set Up Constraints – driving constraint

- Given:
  - initial angle $\phi_2^0 = 330° = 5.76$ rad
  - angular velocity $\omega_2 = \omega_2^0 = -1.2$ rad/sec (cw)
  - angular acceleration $\alpha_2 = 0$ rad/sec^2

$$\Phi = \phi_i - (\phi_i^0 + \omega_i^0 t + \tfrac{1}{2}\alpha_i t^2) = 0$$

$$\Phi = \phi_2 - 5.76 + 1.2t = 0 \quad \Longrightarrow \quad \Phi_t = 1.2$$

Link 1 | Link 2 | Link 3 | Link 4

	x_1	y_1	ϕ_1	x_2	y_2	ϕ_2	x_3	y_3	ϕ_3	x_4	y_4	ϕ_4
$\partial\Phi_{12}/\partial\ldots$	0	0	0	0	0	36	0	0	0	0	0	0

$\partial\Phi/\partial\phi_2 = 1$

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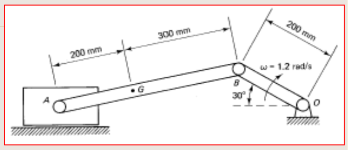
Position Analysis (Assembly)

- Determine **q** to satisfy $\Phi(\mathbf{q})=0$

$$\boxed{\mathbf{q}^{j+1}=\mathbf{q}^{j}+\Delta\mathbf{q}} \quad \boxed{\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}^{j})\Delta\mathbf{q}=-\mathbf{\Phi}(\mathbf{q}^{j})}$$



•The Jacobian matrix for the slider-crank mechanism:

		Link 1			Link 2			Link 3			Link 4		
		x_1	y_1	ϕ_1	x_2	y_2	ϕ_2	x_3	y_3	ϕ_3	x_4	y_4	ϕ_4
Fixed link	$\partial\Phi_1/\partial\ldots$	1	0	0	0	0	0	0	0	0	0	0	0
	$\partial\Phi_2/\partial\ldots$	0	2	0	0	0	0	0	0	0	0	0	0
	$\partial\Phi_3/\partial\ldots$	0	0	3	0	0	0	0	0	0	0	0	0
Revolute	$\partial\Phi_4/\partial\ldots$	0	0	0	0	0	0	4	0	5	6	0	7
	$\partial\Phi_5/\partial\ldots$	0	0	0	0	0	0	0	8	9	0	10	11
Revolute	$\partial\Phi_6/\partial\ldots$	0	0	0	12	0	13	14	0	15	0	0	0
	$\partial\Phi_7/\partial\ldots$	0	0	0	0	16	17	0	18	19	0	0	0
Revolute	$\partial\Phi_8/\partial\ldots$	20	0	21	22	0	23	0	0	0	0	0	0
	$\partial\Phi_9/\partial\ldots$	0	24	25	0	26	27	0	0	0	0	0	0
Translational	$\partial\Phi_{10}/\partial\ldots$	28	29	30	0	0	0	0	0	0	31	32	33
	$\partial\Phi_{11}/\partial\ldots$	0	0	34	0	0	0	0	0	0	0	0	35
Driving	$\partial\Phi_{12}/\partial\ldots$	0	0	0	0	0	36	0	0	0	0	0	0

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Velocity Analysis

$$\boxed{\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}=-\mathbf{\Phi}_{t}}$$

- Construct Jacobian matrix $\Phi_{\mathbf{q}}$
- Determine the rhs of velocity equation - $\Phi_{\mathbf{t}}$
- Solve velocity equation for d**q**/dt

		Link 1			Link 2			Link 3			Link 4		
		x_1	y_1	ϕ_1	x_2	y_2	ϕ_2	x_3	y_3	ϕ_3	x_4	y_4	ϕ_4
Fixed link	$\partial\Phi_1/\partial\ldots$	1	0	0	0	0	0	0	0	0	0	0	0
	$\partial\Phi_2/\partial\ldots$	0	2	0	0	0	0	0	0	0	0	0	0
	$\partial\Phi_3/\partial\ldots$	0	0	3	0	0	0	0	0	0	0	0	0
Revolute	$\partial\Phi_4/\partial\ldots$	0	0	0	0	0	0	4	0	5	6	0	7
	$\partial\Phi_5/\partial\ldots$	0	0	0	0	0	0	0	8	9	0	10	11
Revolute	$\partial\Phi_6/\partial\ldots$	0	0	0	12	0	13	14	0	15	0	0	0
	$\partial\Phi_7/\partial\ldots$	0	0	0	0	16	17	0	18	19	0	0	0
Revolute	$\partial\Phi_8/\partial\ldots$	20	0	21	22	0	23	0	0	0	0	0	0
	$\partial\Phi_9/\partial\ldots$	0	24	25	0	26	27	0	0	0	0	0	0
Translational	$\partial\Phi_{10}/\partial\ldots$	28	29	30	0	0	0	0	0	0	31	32	33
	$\partial\Phi_{11}/\partial\ldots$	0	0	34	0	0	0	0	0	0	0	0	35
Driving	$\partial\Phi_{12}/\partial\ldots$	0	0	0	0	0	36	0	0	0	0	0	0

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Acceleration Analysis

$$\boxed{\Phi_{\mathbf{q}}\ddot{\mathbf{q}} = -[(\Phi_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + 2\Phi_{\mathbf{q}t}\dot{\mathbf{q}} + \Phi_{tt}] \equiv \gamma}$$

- Construct Jacobian matrix $\Phi_{\mathbf{q}}$ (this matrix is the same as that for velocity analysis)
- Determine the rhs of acceleration equation γ
- Solve acceleration equation for $d^2\mathbf{q}/dt^2$

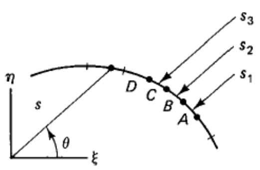
$$\Phi_{\mathbf{q}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\phi}_2 \\ \vdots \\ \ddot{x}_4 \\ \ddot{y}_4 \\ \ddot{\phi}_4 \end{bmatrix}_{(12)\times 1} = \begin{pmatrix} \\ \\ \boxed{\gamma} \\ \\ \\ \end{pmatrix}_{12\text{x}1}$$

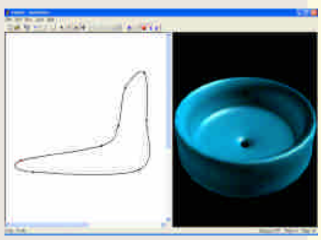
Curve Representation

- For any point on a planar curve, its coordinate can be written as

  $\mathbf{s'}_j^P = [\xi_j, \eta_j]^T = [\, s\cos\theta, s\sin\theta\,]^T$ (Local coordinates)

Point	θ (rad)	s (cm)
.	.	.
.	.	.
.	.	.
A	0.2	3.75
B	0.3	3.57
C	0.4	3.35
D	0.5	3.10
.	.	.
.	.	.
.	.	.



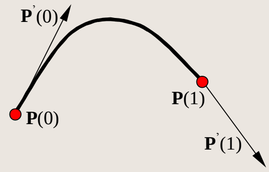


- In order to determine a smooth curve passing through all the specified discrete points, lots of Parametric Curves had been suggested, e.g., Biarc, Hermite, Bezier, B-Spline, NURBS

Cubic Spline

- $s(\theta) = a_3\theta^3 + a_2\theta^2 + a_1\theta + a_0 = [\theta^3 \;\; \theta^2 \;\; \theta \;\; 1]\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \mathbf{UA}, \;\; \mathbf{A}=\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$
- $ds/d\theta = 3a_3\theta^2 + 2a_2\theta + a_1 = [3\theta^2 \;\; 2\theta \;\; 1 \;\; 0]\,\mathbf{A}$
- For each segment of the curve, we need four equations to solve for the four coefficients $a_0, \ldots, a_3$, maxtrix **A**
- For instance, Hermite curves are defined by two points and two tangent vectors

s(0)	=	[0 0 0 1]	A
s(1)	=	[1 1 1 1]	A
s'(0)	=	[0 0 1 0]	A
s'(1)	=	[3 2 1 0]	A



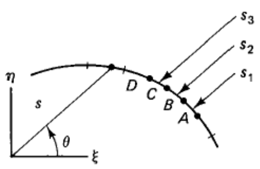
$$\begin{bmatrix} s(0) \\ s(1) \\ s'(0) \\ s'(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} A$$

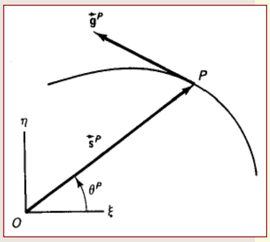
➔ S=UA ➔ $A=U^{-1}S$

The vector tangent to a curve

- The curve representation is determined when coefficient matrix A for each segment is found.
- Then, the coordinates of points on the curve can be evaluated by specifying θ because $\mathbf{s'}_j^P = [\xi_j, \eta_j]^T = [s\cos\theta, s\sin\theta]^T$

Point	θ (rad)	s (cm)
.	.	.
A	0.2	3.75
B	0.3	3.57
C	0.4	3.35
D	0.5	3.10
.	.	.



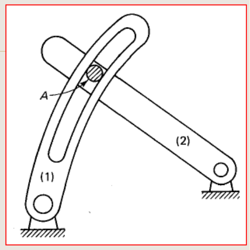


- And the vector tangent to the curve at P is determined by

$$g'^P = \frac{ds'^P}{d\theta} = \frac{d}{d\theta}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \dfrac{d\xi}{d\theta} \\ \dfrac{d\eta}{d\theta} \end{bmatrix} = \begin{bmatrix} \mu \\ \nu \end{bmatrix}$$

(Local coordinate)

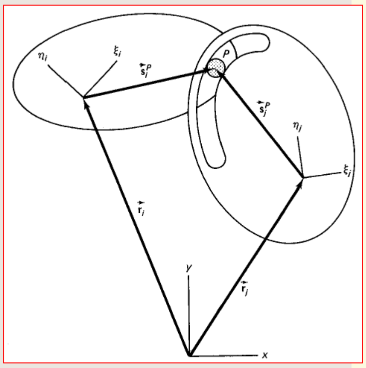
Point-Follower (pp. 97-98)



- Provided that a pin is attached rigidly to body *i* and can move in the slot on body *j*, the constraint equations are

$$\Phi = \underbrace{\mathbf{r}_j+\mathbf{s}_i^P}\ -\mathbf{r}_j\ -\mathbf{s}_j^P=0$$

$$\Phi^{(p\text{-}f,2)} \equiv \begin{bmatrix} \boxed{x_i^P} - \boxed{(x_j} + \boxed{\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)} \\ \boxed{y_i^P} - \boxed{(y_j} + \boxed{\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)} \end{bmatrix} = \boldsymbol{0}$$



- Note that $\mathbf{s}_j^P = \mathbf{A}\mathbf{s}'^P_j$
- $\mathbf{s}'^P_j = [\xi_j, \eta_j]^T = [s\cos\theta, s\sin\theta]^T$ represents a curve while $\mathbf{s}'^P_i = [\xi_i, \eta_i]^T$ is a constant vector

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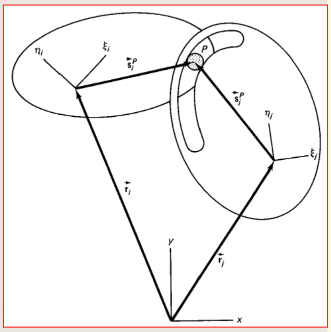
Point-Follower (pp. 97-98)

$$\Phi = \underbrace{\mathbf{r}_j+\mathbf{s}_i^P}\ -\mathbf{r}_j\ -\mathbf{s}_j^P=0$$

$$\Phi^{(p\text{-}f,2)} \equiv \begin{bmatrix} \boxed{x_i^P} - \boxed{(x_j} + \boxed{\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)} \\ \boxed{y_i^P} - \boxed{(y_j} + \boxed{\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)} \end{bmatrix} = \boldsymbol{0}$$



- $\mathbf{s}_j^P = \mathbf{A}\mathbf{s}'^P_j$
- $\mathbf{s}'^P_j = [\xi_j, \eta_j]^T = [s\cos\theta, s\sin\theta]^T$ represents a curve
- Since we need θ to identify the contact point on body j, an artificial coordinate θ must be added to the vector of coordinates $\mathbf{q}_j$. i.e.,

$$\mathbf{q}_i = \begin{bmatrix} x_i \\ y_i \\ \phi_i \end{bmatrix}, \mathbf{q}_j = \begin{bmatrix} x_j \\ y_j \\ \phi_j \\ \theta \end{bmatrix}$$

•$-\Phi_t$ = ? γ = ? [Assignment #3a]

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Cam-Followers (knife edge or point follower)

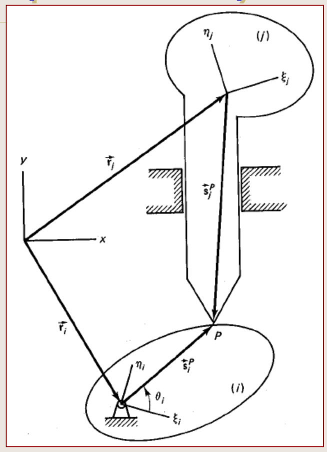
$$\mathbf{q}_j = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_j, \mathbf{q}_i = \begin{bmatrix} x \\ y \\ \phi \\ \theta \end{bmatrix}_i$$

$$\boldsymbol{\Phi}^{(c\text{-}f\text{-}1,2)} \equiv \mathbf{r}_i^P - \mathbf{r}_j^P = \mathbf{0}$$

where

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i \mathbf{s}'^P_i$$

$$\mathbf{r}_j^P = \mathbf{r}_j + \mathbf{A}_j \mathbf{s}'^P_j$$



- Or expressed in expended form:

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}_i + \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}_i \begin{bmatrix} s\cos\theta \\ s\sin\theta \end{bmatrix}_i}_{r_i^P} - \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}_j - \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}_j \begin{bmatrix} \xi^P \\ \eta^P \end{bmatrix}_j}_{r_j^P} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

function of θ

constant

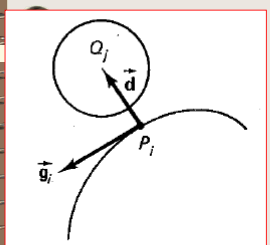
Cam-Followers (offset roller follower)

$$\boldsymbol{\Phi}^{(c\text{-}f\text{-}2,2)} = \begin{bmatrix} \mathbf{g}_i^T \mathbf{d} \\ \mathbf{d}^T\mathbf{d} - \rho^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

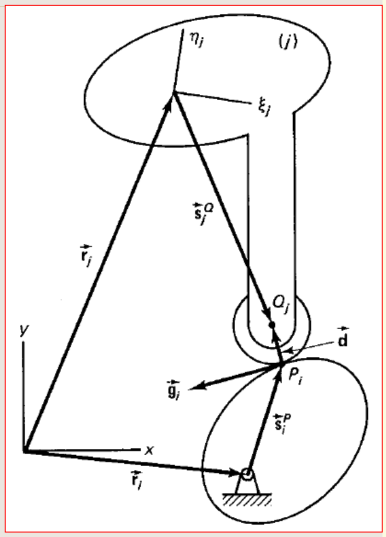
where

ρ=radius of the roller

$$\mathbf{d} = \mathbf{r}_j + \mathbf{A}_j \mathbf{s}'^Q_j - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}'^P_i$$



$$\mathbf{q}_j = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_j, \mathbf{q}_i = \begin{bmatrix} x \\ y \\ \phi \\ \theta \end{bmatrix}_i$$



Cam-Followers (flat-faced follower)

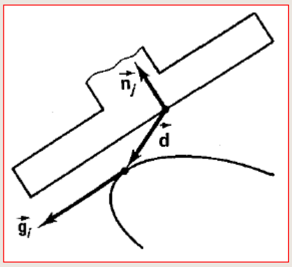
$$\boldsymbol{\Phi}^{(c\text{-}f\text{-}3,2)} = \begin{bmatrix} \mathbf{g}_i^T \mathbf{n}_j \\ \mathbf{d}^T \mathbf{n}_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

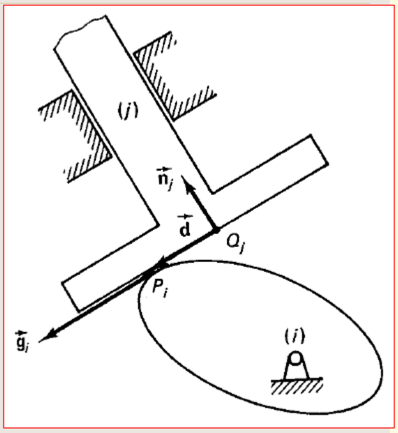
where

$\mathbf{g}_i$=tangent vector at P_i

$\mathbf{n}_j$=normal vector at Q_j

$\mathbf{d}=\mathbf{r}_i^P-\mathbf{r}_j^Q$





$$\mathbf{q}_j = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_j, \mathbf{q}_i = \begin{bmatrix} x \\ y \\ \phi \\ \theta \end{bmatrix}_i$$

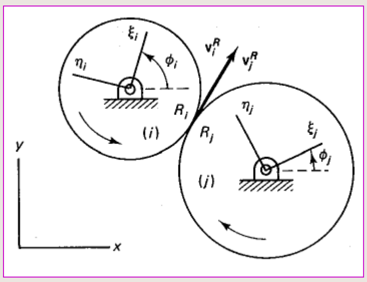
Gears/Pure Rolling: case 1

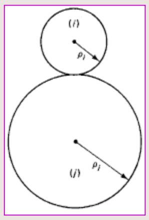
- Constraint: no slippage between the two pitch circles

$$v_i^R = v_j^R$$

$$\rho_i \dot{\phi}_i = -\rho_j \dot{\phi}_j$$

$$\Phi^{(g\text{-}1,1)} \equiv (\phi_i - \phi_i^0)\rho_i + (\phi_j - \phi_j^0)\rho_j = 0$$



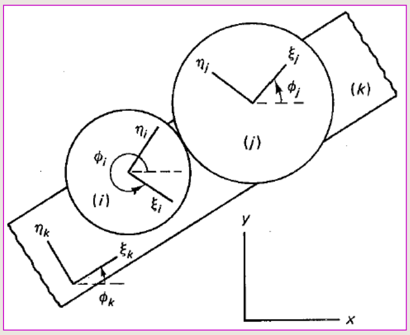


Gears/Pure Rolling: case 2

- Constraint: no slippage between the two pitch circles

$$v^R_{i/k} = v^R_{j/k}$$

$$(\dot{\phi}_i - \dot{\phi}_k)\rho_i = -(\dot{\phi}_j - \dot{\phi}_k)\rho_j$$



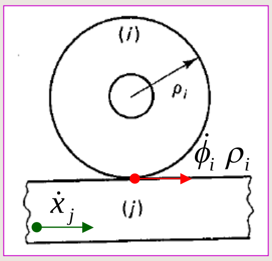
$$\Phi^{(g\text{-}2,1)} \equiv [(\phi_i - \phi_i^0) - (\phi_k - \phi_k^0)]\rho_i + [(\phi_j - \phi_j^0) - (\phi_k - \phi_k^0)]\rho_j = 0$$

Rack & Pinion/Pure Rolling: case 3

- Pinion i rotates about a fixed point
- Rack j translate parallel to the x-axis

$$\dot{\phi}_i\rho_i = \dot{x}_j$$

$$\Phi^{(r\text{-}p\text{-}1,1)} \equiv (\phi_i - \phi_i^0)\rho_i - (x_j - x_j^0) = 0$$



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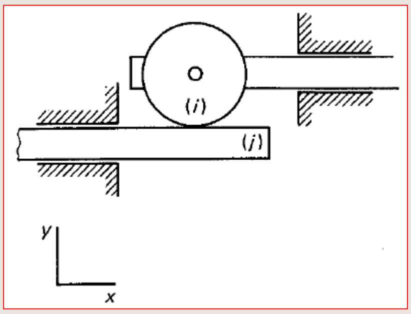
Department of Mechanical Engineering

Rack & Pinion/Pure Rolling: case 4



$$\dot{x}_i + \dot{\phi}_i \rho_i = \dot{x}_j$$

$$\Phi^{(r\text{-}p\text{-}2,\,1)} \equiv (x_i - x_i^0) + (\phi_i - \phi_i^0)\rho_i - (x_j - x_j^0) = 0$$

- End of chapter 4