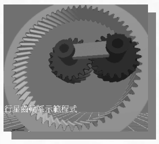
Introduction to Other Formulations

1. Joint Coordinate Method (Chapter 13)
2. Featherstone’s Formulation



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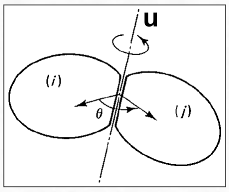
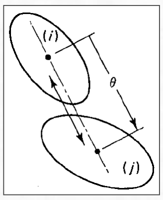
Joint Coordinates

- If bodies i and j are connected by a kinematic joint having relative (joint) coordinate, θ_{ij}, then the coordinates $\mathbf{q}_j$ can be determined from $\mathbf{q}_i$ and θ_{ij}





$$\boxed{\mathbf{q}_j = f^{(c)}(\mathbf{q}_i, \boldsymbol{\theta}_{ij})}$$

$$\boxed{\dot{\mathbf{q}}_j = f^{(v)}(\dot{\mathbf{q}}_i, \dot{\boldsymbol{\theta}}_{ij})} \quad \boxed{Ex: \boldsymbol{\omega}_j = \boldsymbol{\omega}_i + \dot{\theta}\mathbf{u}}$$

$$\boxed{\ddot{\mathbf{q}}_j = f^{(a)}(\dot{\mathbf{q}}_i, \ddot{\mathbf{q}}_i, \dot{\boldsymbol{\theta}}_{ij}, \ddot{\boldsymbol{\theta}}_{ij})} \quad \boxed{Ex: \dot{\boldsymbol{\omega}}_j = \dot{\boldsymbol{\omega}}_i + \ddot{\theta}\mathbf{u} + \dot{\theta}\dot{\mathbf{u}} = \dot{\boldsymbol{\omega}}_i + \ddot{\theta}\mathbf{u} + \dot{\theta}\tilde{\boldsymbol{\omega}}_i\mathbf{u}}$$

$$\updownarrow$$

$$\boxed{\ddot{\boldsymbol{\theta}}_{ij} = f^{(a)}(\ddot{\mathbf{q}}_i, \ddot{\mathbf{q}}_j)}$$

注意: 爲了可進行加減運算, 故各向量必須使用同一座標系加以表示, 通常先將各向量轉換至機架(座)之座標系才作加減.

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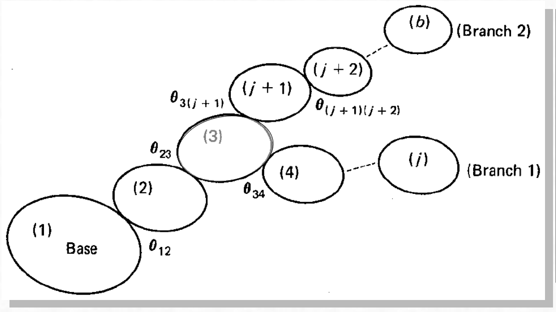
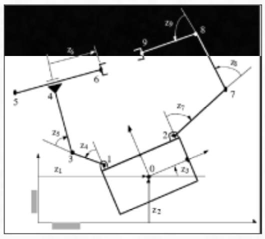
Open-Chain Systems: System Topology



Body i	Body j	
1 (base)	2	
2	3	branch 1
3	4	
4	. . .	
. . .	j	
3	$j+1$	branch 2
$j+1$	$j+2$	
$j+2$	. . .	
. . .	b	

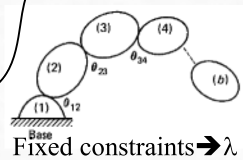


Open-Chain Systems: Solving Procedures

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_\mathbf{q}^T \\ \mathbf{\Phi}_\mathbf{q} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} \mathbf{q}_1 \\ \boldsymbol{\theta} \\ \dot{\mathbf{q}}_1 \\ \dot{\boldsymbol{\theta}} \end{bmatrix} \quad \dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\boldsymbol{\theta}} \\ \ddot{\mathbf{q}}_1 \\ \ddot{\boldsymbol{\theta}} \end{bmatrix}$$

(a) Main routine
- **(a.1)** Specify initial conditions for $\mathbf{q}$ and $\dot{\mathbf{q}}$.
- **(a.2)** Specify (or automatically determine) the topology of the system.
- **(a.3)** Compute initial conditions for $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$.
- **(a.4)** Transfer the initial values to $\mathbf{y}$ (Eq. 13.39 or 13.40).

(b) Numerical integration routing



Fixed constraints➔λ

(c) DIFEQN routine
- **(c.1)** Transfer the contents of $\mathbf{y}$ to $\mathbf{q}_{\text{base}}$, $\dot{\mathbf{q}}_{\text{base}}$ (if there is a floating base body), $\boldsymbol{\theta}$, and $\dot{\boldsymbol{\theta}}$.
- **(c.2)** Compute $\mathbf{q}$ and $\dot{\mathbf{q}}$ for all of the bodies (Eqs. 13.36 and 13.37).
- **(c.3)** Evaluate $\mathbf{M}$, $\mathbf{\Phi}_\mathbf{q}$, $\mathbf{g}$, and $\boldsymbol{\gamma}$.
- **(c.4)** Solve Eq. 13.16 for $\ddot{\mathbf{q}}$ and $\boldsymbol{\lambda}$.
- **(c.5)** Compute $\ddot{\boldsymbol{\theta}}$ (Eq. 13.38).
- **(c.6)** Transfer $\dot{\mathbf{q}}_{\text{base}}$, $\ddot{\mathbf{q}}_{\text{base}}$ (if there is a floating base body), $\dot{\boldsymbol{\theta}}$, and $\ddot{\boldsymbol{\theta}}$ to $\dot{\mathbf{y}}$.
- **(c.7)** Return.

$$\mathbf{q}_j = f^{(c)}(\mathbf{q}_i, \boldsymbol{\theta}_{ij})$$

$$\dot{\mathbf{q}}_j = f^{(v)}(\dot{\mathbf{q}}_i, \dot{\boldsymbol{\theta}}_{ij})$$

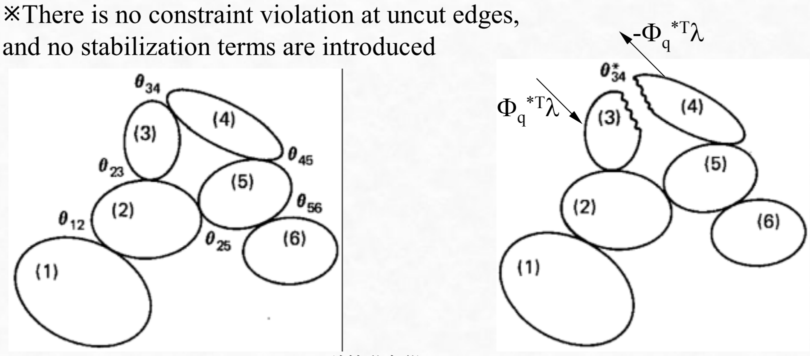
$$\ddot{\boldsymbol{\theta}}_{ij} = f^{(a)}(\ddot{\mathbf{q}}_i, \ddot{\mathbf{q}}_j)$$

Close-Chain Systems: Cut-Joint method

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_\mathbf{q}^T \\ \mathbf{\Phi}_\mathbf{q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_\mathbf{q}^T & \mathbf{\Phi}_\mathbf{q}^{*T} \\ \mathbf{\Phi}_\mathbf{q} & \mathbf{0} & \mathbf{0} \\ \mathbf{\Phi}_\mathbf{q}^* & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \\ -\boldsymbol{\lambda}^* \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^* - 2\alpha\dot{\mathbf{\Phi}}^* - \beta^2\mathbf{\Phi}^* \end{bmatrix} \Leftarrow \begin{cases} \mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_\mathbf{q}^T\boldsymbol{\lambda} - \mathbf{\Phi}_\mathbf{q}^{*T}\boldsymbol{\lambda}^* = \mathbf{g} \\ \mathbf{\Phi}_\mathbf{q}\ddot{\mathbf{q}} = \boldsymbol{\gamma} \\ \mathbf{\Phi}_\mathbf{q}^*\ddot{\mathbf{q}} = \boldsymbol{\gamma}^* - 2\alpha\dot{\mathbf{\Phi}}^* - \beta^2\mathbf{\Phi}^* \end{cases}$$

$\Phi(q)$: constraint equations for uncut edges
$\Phi^*(q)$: constraint equations for cut edges
※There is no constraint violation at uncut edges, and no stabilization terms are introduced



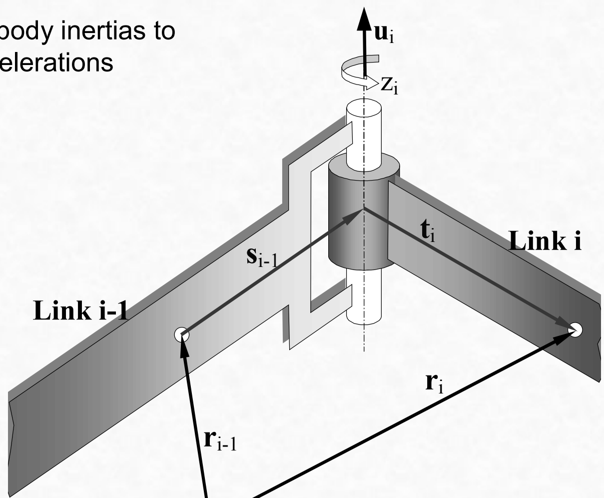
Basic Concept for Featherstone's Formulation

- Calculating a series of articulated-body inertias, one for each link
- Using the articulated-body inertias to calculate the joint accelerations

首先, 由內而外求各桿件之位置和速度.

其次, 由外而內求複合桿件之相關性質, 例如慣性矩陣和作用力;

最後, 由基座開始逐一向外計算各接頭之相對加速度



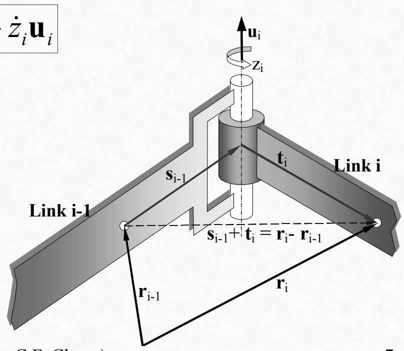
Featherstone's Formulation

Ref: R. Featherstone, 1987, Robot Dynamics Algorithms, Kluwer Academic Publishers

- Here we adopt the notation introduced by Bae and Haug (1987) instead of the notation of spatial vector (**Bae, D.-S. and Haug, E.j., "A Recursive Formulation for Constrained Mechanical System Dynamics. Part I: Open-Loop Systems, Part II: Closed-Loop Systems," Mechanics of Structures and Machines, Vol. 15, pp. 359-382, pp. 481-506.**)
- In this method, the type of joint is arbitrary. The relative positions of adjacent links can be described by using the four parameters introduced by Denavit & Hartenberg
- **For simplicity we will assume that <u>all the joints are of revolute type</u>**

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_{i-1} + \tilde{\boldsymbol{\omega}}_{i-1}\mathbf{s}_{i-1} + \tilde{\boldsymbol{\omega}}_i\mathbf{t}_i \qquad \boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \dot{z}_i\mathbf{u}_i$$

$$\begin{aligned}\dot{\mathbf{r}}_i &= \dot{\mathbf{r}}_{i-1} + \tilde{\boldsymbol{\omega}}_{i-1}\mathbf{s}_{i-1} + (\tilde{\boldsymbol{\omega}}_{i-1} + \dot{z}_i\tilde{\mathbf{u}}_i)\mathbf{t}_i \\ &= \dot{\mathbf{r}}_{i-1} + \tilde{\boldsymbol{\omega}}_{i-1}(\mathbf{s}_{i-1} + \mathbf{t}_i) + \dot{z}_i\tilde{\mathbf{u}}_i\mathbf{t}_i \\ &= \dot{\mathbf{r}}_{i-1} + \tilde{\boldsymbol{\omega}}_{i-1}(\mathbf{r}_i - \mathbf{r}_{i-1}) + \dot{z}_i\tilde{\mathbf{u}}_i\mathbf{t}_i\end{aligned}$$



各向量必須使用同一座標系加以表示才能相加減
此處使用慣性座標系(全域座標系)
dz_i/dt為桿i相對於桿i-1之角速度

Recursive Relation Between the Velocities of Two Consecutive Bodies

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \dot{z}_i\mathbf{u}_i \qquad \dot{\mathbf{r}}_i = \dot{\mathbf{r}}_{i-1} + \tilde{\boldsymbol{\omega}}_{i-1}(\mathbf{r}_i - \mathbf{r}_{i-1}) + \dot{z}_i\tilde{\mathbf{u}}_i\mathbf{t}_i$$

$$\mathbf{Y}_i \equiv \begin{bmatrix}\dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \tilde{\mathbf{r}}_{i-1} - \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{r}}_{i-1} \\ \boldsymbol{\omega}_{i-1}\end{bmatrix} + \begin{bmatrix}\tilde{\mathbf{u}}_i\mathbf{t}_i \\ \mathbf{u}_i\end{bmatrix}\dot{z}_i \qquad \mathbf{B}_i \equiv \begin{bmatrix}\mathbf{I} & \tilde{\mathbf{r}}_{i-1} - \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I}\end{bmatrix}$$

$$\mathbf{Y}_\mathbf{i} = \mathbf{B}_i\mathbf{Y}_{\mathbf{i}-1} + \mathbf{b}_i\dot{z}_\mathbf{i} \qquad \mathbf{b}_i \equiv \begin{bmatrix}\tilde{\mathbf{u}}_i\mathbf{t}_i \\ \mathbf{u}_i\end{bmatrix}$$

$$\dot{\mathbf{Y}}_\mathbf{i} = \begin{bmatrix}\ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i\end{bmatrix} = \dot{\mathbf{B}}_i\mathbf{Y}_{\mathbf{i}-1} + \mathbf{B}_i\dot{\mathbf{Y}}_{\mathbf{i}-1} + \dot{\mathbf{b}}_i\dot{z}_\mathbf{i} + \mathbf{b}_i\ddot{z}_\mathbf{i}$$

$$\dot{\mathbf{Y}}_\mathbf{i} = \mathbf{B}_i\dot{\mathbf{Y}}_{\mathbf{i}-1} + \mathbf{b}_i\ddot{z}_\mathbf{i} + \mathbf{d}_\mathbf{i} \qquad \mathbf{d}_\mathbf{i} = \dot{\mathbf{B}}_i\mathbf{Y}_{\mathbf{i}-1} + \dot{\mathbf{b}}_i\dot{z}_\mathbf{i}$$

Thus, one can compute recursively forward the velocity and acceleration of links from i=1 to i=N if knowing dz/dt, d^2z/dt^2, as well as <u>the velocity and acceleration of the **base body**</u>. Note: Accelerations yield inertia forces and inertia torques

Relative Angular Acceleration at the Last Joint

$$\mathbf{Y_i} \equiv \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i \end{bmatrix} = \mathbf{B}_i \mathbf{Y_{i-1}} + \mathbf{b}_i \dot{z}_\mathbf{i}$$

$$\mathbf{B}_i \equiv \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_{i-1} - \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \qquad \mathbf{b}_i \equiv \begin{bmatrix} \tilde{\mathbf{u}}_i \mathbf{t}_i \\ \mathbf{u}_i \end{bmatrix}$$

$$\mathbf{d_i} = \dot{\mathbf{B}}_i \mathbf{Y_{i-1}} + \dot{\mathbf{b}}_i \dot{z}_\mathbf{i}$$

$$\dot{\mathbf{Y}}_\mathbf{i} = \mathbf{B}_i \dot{\mathbf{Y}}_\mathbf{i-1} + \mathbf{b}_i \ddot{z}_\mathbf{i} + \mathbf{d_i}$$

- The principle of virtual power(虛功率原理) for an *N*-link serial manipulator can be expressed as (作用於剛體上之外力和慣性力的合力)

$$W^* \equiv \sum_{i=1}^{N} \dot{\mathbf{q}}_i^{*T} \mathbf{F}_i = \sum_{i=1}^{N} \mathbf{Y_i^{*T}} \left(\mathbf{M_i} \dot{\mathbf{Y}}_\mathbf{i} - \mathbf{Q_i} \right) = 0$$

$$\mathbf{M}_i = \begin{bmatrix} m_i \mathbf{I}_i & 0 \\ 0 & \mathbf{J}_i \end{bmatrix} \qquad \dot{\mathbf{Y}}_i = \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i \end{bmatrix} \qquad \mathbf{Q}_i = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{n}_i - \tilde{\boldsymbol{\omega}}_i \mathbf{J}_i \boldsymbol{\omega}_i \end{bmatrix}$$

$$\sum_{\mathbf{i=1}}^{\mathbf{N-2}} \mathbf{Y_i^{*T}} \left(\mathbf{M_i} \dot{\mathbf{Y}}_\mathbf{i} - \mathbf{Q_i} \right) + \mathbf{Y}_{N-1}^{*\mathbf{T}} \left(\mathbf{M}_{N-1} \dot{\mathbf{Y}}_{N-1} - \mathbf{Q}_{N-1} \right) + \mathbf{Y}_N^{*\mathbf{T}} \left(\mathbf{M}_N \dot{\mathbf{Y}}_N - \mathbf{Q}_N \right) = 0$$

Substituting Y_N^* and dY_N/dt

$$\sum_{\mathbf{i=1}}^{\mathbf{N-2}} \mathbf{Y_i^{*T}} \left(\mathbf{M_i} \dot{\mathbf{Y}}_\mathbf{i} - \mathbf{Q_i} \right) + \mathbf{Y}_{N-1}^{*\mathbf{T}} \left(\mathbf{M}_{N-1} \dot{\mathbf{Y}}_{N-1} - \mathbf{Q}_{N-1} \right) + \left(\mathbf{Y}_{N-1}^{*\mathbf{T}} \mathbf{B}_N^\mathbf{T} + \mathbf{b}_N^\mathbf{T} \dot{z}_N^* \right) \left\{ \mathbf{M}_N (\mathbf{B}_N \dot{\mathbf{Y}}_{N-1} + \mathbf{b}_N \ddot{z}_N + \mathbf{d}_N) - \mathbf{Q}_N \right\} = 0$$

$$\sum_{\mathbf{i=1}}^{\mathbf{N-2}} \mathbf{Y_i^{*T}} \left(\mathbf{M_i} \dot{\mathbf{Y}}_\mathbf{i} - \mathbf{Q_i} \right) + \mathbf{Y}_{N-1}^{*\mathbf{T}} \left\{ \left(\mathbf{M}_{N-1} + \mathbf{B}_N^\mathbf{T} \mathbf{M}_N \mathbf{B}_N \right) \dot{\mathbf{Y}}_{N-1} - \left(\mathbf{Q}_{N-1} + \mathbf{B}_N^\mathbf{T} \mathbf{Q}_N - \mathbf{B}_N^\mathbf{T} \mathbf{M}_N \mathbf{d}_N \right) + \mathbf{B}_N^\mathbf{T} \mathbf{M}_N \mathbf{b}_N \ddot{z}_N \right\}$$

$$+ \dot{z}_N^* \left[\mathbf{b}_N^\mathbf{T} \mathbf{M}_N \left(\mathbf{B}_N \dot{\mathbf{Y}}_{N-1} + \mathbf{b}_N \ddot{z}_N + \mathbf{d}_N \right) - \mathbf{b}_N^\mathbf{T} \mathbf{Q}_N \right] = 0 \longrightarrow \boxed{= 0}$$

Note that the value of an independent virtual velocity can be arbitrarily specified. If choosing $dz_N/dt=0$ the sum of all terms but the last one equals to zero. Consequently, the last term should be zero

$$\ddot{z}_N = (\mathbf{b}_N^T \mathbf{M}_N \mathbf{b}_N)^{-1} \left\{ \mathbf{b}_N^T \mathbf{Q}_N - \mathbf{b}_N^T \mathbf{M}_N \mathbf{B}_N \dot{\mathbf{Y}}_{N-1} - \mathbf{b}_N^T \mathbf{M}_N \mathbf{d}_N \right\}$$

This equation can be used to determine the relative angular acceleration at joint N if $\mathbf{M}_\mathrm{N}$ *and* $\mathbf{Q}_\mathrm{N}$ *are known*

Relative Angular Acceleration at Joint *i*

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{n}_i - \tilde{\boldsymbol{\omega}}_i \mathbf{J}_i \boldsymbol{\omega}_i \end{bmatrix} \qquad \mathbf{B}_i \equiv \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_{i-1} - \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{M}_i = \begin{bmatrix} m_i \mathbf{I}_i & 0 \\ 0 & \mathbf{J}_i \end{bmatrix} \qquad \mathbf{b}_i \equiv \begin{bmatrix} \tilde{\mathbf{u}}_i \mathbf{t}_i \\ \mathbf{u}_i \end{bmatrix} \qquad \mathbf{d_i} = \dot{\mathbf{B}}_i \mathbf{Y_{i-1}} + \dot{\mathbf{b}}_i \dot{z}_\mathbf{i} \qquad \mathbf{Y}_i = \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i \end{bmatrix}$$

$$\ddot{z}_N = (\mathbf{b}_N^T \mathbf{M}_N \mathbf{b}_N)^{-1} \left\{ \mathbf{b}_N^T \mathbf{Q}_N - \mathbf{b}_N^T \mathbf{M}_N \mathbf{B}_N \dot{\mathbf{Y}}_{N-1} - \mathbf{b}_N^T \mathbf{M}_N \mathbf{d}_N \right\}$$

$$\sum_{\mathbf{i=1}}^{\mathbf{N-2}} \mathbf{Y_i^{*T}} \left(\mathbf{M_i} \dot{\mathbf{Y}}_\mathbf{i} - \mathbf{Q_i} \right) +$$
$$\mathbf{Y}_{N-1}^{*\mathbf{T}} \left\{ \left(\mathbf{M}_{N-1} + \mathbf{B}_N^\mathbf{T} \mathbf{M}_N \mathbf{B}_N \right) \dot{\mathbf{Y}}_{N-1} - \left(\mathbf{Q}_{N-1} + \mathbf{B}_N^\mathbf{T} \mathbf{Q}_N - \mathbf{B}_N^\mathbf{T} \mathbf{M}_N \mathbf{d}_N \right) + \mathbf{B}_N^\mathbf{T} \mathbf{M}_N \mathbf{b}_N \ddot{z}_N \right\} = 0$$

$d^2 z_i/dt^2$爲桿*i*相對於桿*i*-1之角速度

Define

$$\overline{\mathbf{M}}_\mathbf{N-1} \equiv \mathbf{M_{N-1}} + \mathbf{B_N^T} \left(\mathbf{M_N} - \mathbf{M_N} \mathbf{b_N} (\mathbf{b_N^T} \mathbf{M_N} \mathbf{b_N})^{-1} \mathbf{b_N^T} \mathbf{M_N} \right) \mathbf{B_N}$$

$$\overline{\mathbf{Q}}_\mathbf{N-1} \equiv \mathbf{Q_{N-1}} + \left(\mathbf{B_N^T} - \mathbf{B_N^T} \mathbf{M_N} \mathbf{b_N} (\mathbf{b_N^T} \mathbf{M_N} \mathbf{b_N})^{-1} \mathbf{b_N^T} \right) (\mathbf{Q_N} - \mathbf{M_N} \mathbf{d_N})$$

$$\sum_{\mathrm{i=1}}^{\mathrm{N-2}} \mathbf{Y_i^{*T}} \left(\mathbf{M_i} \dot{\mathbf{Y}}_\mathbf{i} - \mathbf{Q_i} \right) + \mathbf{Y_{N-1}^{*T}} \left(\overline{\mathbf{M}}_\mathbf{N-1} \dot{\mathbf{Y}}_\mathbf{N-1} - \overline{\mathbf{Q}}_\mathbf{N} \right) = 0$$

Repeat the same process of eliminating the last two terms yields

$$\overline{\mathbf{M}}_\mathbf{i-1} \equiv \mathbf{M_{i-1}} + \mathbf{B_i^T} \left(\overline{\mathbf{M}}_\mathbf{i} - \overline{\mathbf{M}}_\mathbf{i} \mathbf{b_i} \left(\mathbf{b_i^T} \overline{\mathbf{M}}_\mathbf{i} \mathbf{b_i} \right)^{-1} \mathbf{b_i^T} \overline{\mathbf{M}}_\mathbf{i} \right) \mathbf{B_i}$$

$$\overline{\mathbf{Q}}_\mathbf{i-1} \equiv \mathbf{Q_{i-1}} + \left(\mathbf{B_i^T} - \mathbf{B_i^T} \overline{\mathbf{M}}_\mathbf{i} \mathbf{b_i} \left(\mathbf{b_i^T} \overline{\mathbf{M}}_\mathbf{i} \mathbf{b_i} \right)^{-1} \mathbf{b_i^T} \right) \left(\overline{\mathbf{Q}}_\mathbf{i} - \overline{\mathbf{M}}_\mathbf{i} \mathbf{d_i} \right)$$

$i = N, N\text{-}1, \ldots, 2, 1$

$$\ddot{z}_\mathrm{i} = \left(\mathbf{b_i^T} \overline{\mathbf{M}}_\mathbf{i} \mathbf{b_i} \right)^{-1} \left(\mathbf{b_i^T} \overline{\mathbf{Q}}_\mathbf{i} - \mathbf{b_i^T} \overline{\mathbf{M}}_\mathbf{i} \mathbf{B_i} \dot{\mathbf{Y}}_\mathbf{i-1} - \mathbf{b_i^T} \overline{\mathbf{M}}_\mathbf{i} \mathbf{d_i} \right)$$

For base 1: $\mathbf{Y_1^{*T}} \left(\overline{\mathbf{M}}_1 \dot{\mathbf{Y}}_1 - \overline{\mathbf{Q}}_1 \right) = 0$

$$\dot{\mathbf{Y}}_1 = \overline{\mathbf{M}}_1^{-1} \overline{\mathbf{Q}}_1$$

物理意義: 將桿*i*及其後之桿件(i+1~N)視爲一支剛桿再求解此桿與前一桿(i-1)之共同接頭的相對角加速度

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{n}_i - \tilde{\boldsymbol{\omega}}_i \mathbf{J}_i \boldsymbol{\omega}_i \end{bmatrix} \quad \mathbf{M}_i = \begin{bmatrix} m_i \mathbf{I}_i & 0 \\ 0 & \mathbf{J}_i \end{bmatrix} \quad \mathbf{Y}_i = \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i \end{bmatrix}$$

Solving Procedures

$$\mathbf{B}_i \equiv \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_{i-1} - \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \mathbf{b}_i \equiv \begin{bmatrix} \tilde{\mathbf{u}}_i \mathbf{t}_i \\ \mathbf{u}_i \end{bmatrix} \quad \mathbf{d}_\mathrm{i} = \dot{\mathbf{B}}_i \mathbf{Y}_{\mathrm{i}-1} + \dot{\mathbf{b}}_i \dot{z}_\mathrm{i}$$

1. Knowing the position and velocity of the base body and the joint relative positions z and velocity dz/dt, one can compute recursively forward the Cartesian position and velocity of all the links from i=1 to i=N.

   Step 1 根據各接頭之相對角位置和相對角速度, 由基座開始逐一求各桿件質心之位置, 速度和角速度(Y_i).

   $$\mathbf{Y}_\mathrm{i} = \mathbf{B}_\mathrm{i}\, \mathbf{Y}_{\mathrm{i}-1} + \mathbf{b}_\mathrm{i}\, \dot{z}_i$$

2. The articulated inertias $\overline{\mathbf{M}}$, the forces $\overline{\mathbf{Q}}$, and the coefficients for determining $d^2 z_i/dt^2$ are then computed recursively backwards from i=N to i=1, using the following equations

   $$\overline{\mathbf{M}}_{\mathrm{i}-1} \equiv \mathbf{M}_{\mathrm{i}-1} + \mathbf{B}_\mathrm{i}^\mathrm{T} \left(\overline{\mathbf{M}}_\mathrm{i} - \overline{\mathbf{M}}_\mathrm{i}\, \mathbf{b}_\mathrm{i} \left(\mathbf{b}_\mathrm{i}^\mathrm{T}\, \overline{\mathbf{M}}_\mathrm{i}\, \mathbf{b}_\mathrm{i}\right)^{-1} \mathbf{b}_\mathrm{i}^\mathrm{T}\, \overline{\mathbf{M}}_\mathrm{i}\right) \mathbf{B}_\mathrm{i}$$

   $$\overline{\mathbf{Q}}_{\mathrm{i}-1} \equiv \mathbf{Q}_{\mathrm{i}-1} + \left(\mathbf{B}_\mathrm{i}^\mathrm{T} - \mathbf{B}_\mathrm{i}^\mathrm{T}\, \overline{\mathbf{M}}_\mathrm{i}\, \mathbf{b}_\mathrm{i} \left(\mathbf{b}_\mathrm{i}^\mathrm{T}\, \overline{\mathbf{M}}_\mathrm{i}\, \mathbf{b}_\mathrm{i}\right)^{-1} \mathbf{b}_\mathrm{i}^\mathrm{T}\right) \left(\overline{\mathbf{Q}}_\mathrm{i} - \overline{\mathbf{M}}_\mathrm{i}\, \mathbf{d}_\mathrm{i}\right)$$

   Step 2 由最末端之桿件開始, 計算下一步驟所需之參數, 例如複合桿件之慣性矩陣M和作用力Q.

3. Finally, the acceleration of the base body (dY_1/dt) is computed and then the relative accelerations $d^2 z_i/dt^2$ are determined recursively forward from i=1 to i=N, using equation

   $$\dot{\mathbf{Y}}_1 = \overline{\mathbf{M}}_1^{-1}\, \overline{\mathbf{Q}}_1 \qquad \dot{\mathbf{Y}}_\mathrm{i} = \mathbf{B}_\mathrm{i}\, \dot{\mathbf{Y}}_{\mathrm{i}-1} + \mathbf{b}_\mathrm{i}\, \ddot{z}_i + \mathbf{d}_\mathrm{i}$$

   $$\ddot{z}_\mathrm{i} = \left(\mathbf{b}_\mathrm{i}^\mathrm{T}\, \overline{\mathbf{M}}_\mathrm{i}\, \mathbf{b}_\mathrm{i}\right)^{-1} \left(\mathbf{b}_\mathrm{i}^\mathrm{T}\, \overline{\mathbf{Q}}_\mathrm{i} - \mathbf{b}_\mathrm{i}^\mathrm{T}\, \overline{\mathbf{M}}_\mathrm{i}\, \mathbf{B}_\mathrm{i}\, \dot{\mathbf{Y}}_{\mathrm{i}-1} - \mathbf{b}_\mathrm{i}^\mathrm{T}\, \overline{\mathbf{M}}_\mathrm{i}\, \mathbf{d}_\mathrm{i}\right)$$

   Step 3 由基座開始, 先求加速度dY_i/dt, 再根據前一步驟已求得之M_i和Q_i求桿件受力後各接頭之相對角加速度$d^2 z_i/dt^2$.
   <<將目前之角加速度和角速度積分後可得新角速度和角位置, 以確定各桿之新狀態>>