Analytical Dynamics

Ref:

1. Classical Mechanics, Ed.2, Herbert Goldstein, Addison-Wesley, London,1980.
2. J. García de Jalón and E. Bayo, *Kinematic and Dynamic Simulation of Multibody Systems--The Real-Time Challenge*, Springer-Verlag, 1994.

Virtual Displacements

- A virtual displacement is defined as an infinitesimal imaginary change of configuration of a system at a stationary time that is consistent with its boundary and constraint conditions.
  1. They are infinitesimal changes in the coordinates
  2. They must be consistent with the constraints of the system, but they are otherwise arbitrary
  3. They are taking place without any change in time, so that the forces and constraints do not change during the process. In other words, the time variable is consider fixed.
  4. They are customarily denoted by δq

dr vs δr

- Given: $\boxed{\mathbf{r} = f(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n, t)}$

$$
\begin{aligned}
d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial \mathbf{q}_1} d\mathbf{q}_1 + \frac{\partial \mathbf{r}}{\partial \mathbf{q}_2} d\mathbf{q}_2 + \ldots + \frac{\partial \mathbf{r}}{\partial \mathbf{q}_n} d\mathbf{q}_n + \frac{\partial \mathbf{r}}{\partial t} dt \\
&= \sum_{i=1}^{n} \frac{\partial \mathbf{r}}{\partial \mathbf{q}_i} d\mathbf{q}_i + \frac{\partial \mathbf{r}_j}{\partial t} dt
\end{aligned}
$$

$$
d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} d\mathbf{q} + \frac{\partial \mathbf{r}}{\partial t} dt
$$

(3×1) (3×n) (n × 1) (3×1)

$$
\begin{aligned}
\delta\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial \mathbf{q}_1} \delta\mathbf{q}_1 + \frac{\partial \mathbf{r}}{\partial \mathbf{q}_2} \delta\mathbf{q}_2 + \ldots + \frac{\partial \mathbf{r}}{\partial \mathbf{q}_n} \delta\mathbf{q}_n \\
&= \sum_{i=1}^{n} \frac{\partial \mathbf{r}}{\partial \mathbf{q}_i} \delta\mathbf{q}_i
\end{aligned}
$$

$$
\delta\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \delta\mathbf{q}
$$

- The main difference between dr and δr is that virtual displacements are taking place without any change in time; i.e. the time variable is consider fixed.

Virtual Displacements of a Single Pendulum

It can be seen that the following condition must be satisfied by the virtual displacements δx and δy

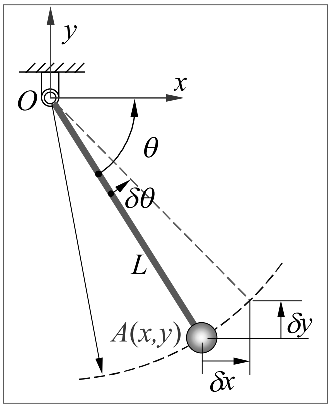
$$\Phi = x^2 + y^2 - L^2 = 0$$

$$\frac{d\Phi}{dt} = \left(\frac{d\Phi}{d\mathbf{q}}\right)\left(\frac{d\mathbf{q}}{dt}\right) = \Phi_{\mathbf{q}}\left(\frac{d\mathbf{q}}{dt}\right)$$

$$d\Phi = \Phi_{\mathbf{q}} d\mathbf{q}$$

$$\delta\Phi = \Phi_{\mathbf{q}} \delta\mathbf{q} = \begin{bmatrix} 2x & 2y \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = 0$$

δq: Virtual displacements
q : Generalized coordinates

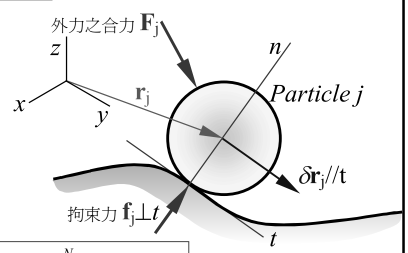


Dependent variables: (x, y)
Constraint: 1
Dof = 1

Note: the virtual displacement at point *A* is perpendicular to the rod *OA*.

The Principle of Virtual Work

- The principle of virtual work is a powerful principle which is suitable for the dynamic analysis of connected rigid and flexible multibody systems
- For a system of particles in static equilibrium, the resultant force $\mathbf{R}_j$ on each particle is zero, so that the associated virtual work $\delta W=\sum \mathbf{R}_j \delta \mathbf{r}_j=0$.



$$\delta W=\sum_{j}^{N}\mathbf{R}_j\bullet\delta\mathbf{r}_j=\sum_{j}^{N}\left(\mathbf{F}_j+\mathbf{f}_j\right)\bullet\delta\mathbf{r}_j=\sum_{j}^{N}\mathbf{F}_j\bullet\delta\mathbf{r}_j+\sum_{j}^{N}\mathbf{f}_j\bullet\delta\mathbf{r}_j=0$$

$$\because \mathbf{f}_j\bullet\delta\mathbf{r}_j=0,\quad j=1,\cdots N$$

$$\delta W=\sum_{j}^{N}\mathbf{F}_j\bullet\delta\mathbf{r}_j=0$$

注意: $\mathbf{F}_j$爲外加之作用力(applied force),
但不含拘束所造成之拘束力

- This is the mathematical expression of the principle of virtual work.
- Since rigid bodies can be regarded as systems of particles with constraints, this principle is also valid for rigid bodies.
- Constrain force is workless force. (拘束力不作功)

Virtual Work and Generalized Forces

$$\delta\mathbf{r}_j=\sum_{i=1}^{n}\frac{\partial\mathbf{r}_j}{\partial\mathbf{q}_i}\delta\mathbf{q}_i$$

- Generalized principle of virtual work

$$\begin{aligned}\delta W&=\sum_{j}^{N}\mathbf{F}_j\bullet\delta\mathbf{r}_j=\sum_{j}^{N}\mathbf{F}_j\bullet\Big(\sum_{i=1}^{n}\frac{\partial\mathbf{r}_j}{\partial\mathbf{q}_i}\delta\mathbf{q}_i\Big)=\sum_{j}^{N}\Big(\sum_{i=1}^{n}\mathbf{F}_j\bullet\frac{\partial\mathbf{r}_j}{\partial\mathbf{q}_i}\delta\mathbf{q}_i\Big)\\&=\sum_{i}^{n}\Big(\sum_{j}^{N}\mathbf{F}_j\bullet\frac{\partial\mathbf{r}_j}{\partial\mathbf{q}_i}\delta\mathbf{q}_i\Big)=\sum_{i}^{n}\Big(\sum_{j}^{N}\mathbf{F}_j\bullet\frac{\partial\mathbf{r}_j}{\partial\mathbf{q}_i}\Big)\delta\mathbf{q}_i=\sum_{i}^{n}\mathbf{Q}_i\bullet\delta\mathbf{q}_i=0\end{aligned}$$

where $\mathbf{Q}_i\equiv\sum_{j}^{N}\mathbf{F}_j\bullet\dfrac{\partial\mathbf{r}_j}{\partial\mathbf{q}_i}=\text{Generalized force}$

- It can be written in a compact form as

$$\delta W=\sum_{i}^{n}\mathbf{Q}_i\bullet\delta\mathbf{q}_i=\left[\mathbf{Q}_1\ \cdots\ \mathbf{Q}_n\right]\begin{bmatrix}\delta\mathbf{q}_1\\\vdots\\\delta\mathbf{q}_n\end{bmatrix}=\mathbf{Q}^T\delta\mathbf{q}=\delta\mathbf{q}^T\mathbf{Q}=0$$

$$\begin{aligned}\delta W&=\delta\mathbf{r}^T\mathbf{F}=\left(\frac{\partial\mathbf{r}}{\partial\mathbf{q}}\delta\mathbf{q}\right)^T\mathbf{F}\\&=\delta\mathbf{q}^T\left\{\left(\frac{\partial\mathbf{r}}{\partial\mathbf{q}}\right)^T\mathbf{F}\right\}\\&=\delta\mathbf{q}^T\mathbf{Q}=0\end{aligned}$$

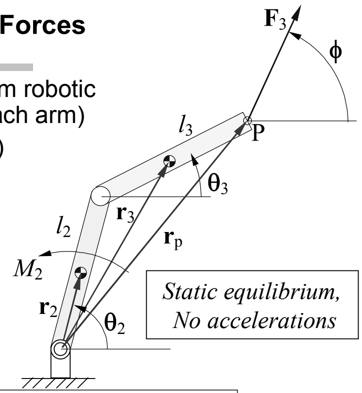
- Since q_i is independent, we can choose $\delta q_i=0$ for all $i\neq k$ to yield $\delta W=Q_k\delta q_k$
  Thus, Q_k is the force associated with δq_k, and $Q_k=\delta W/\delta q_k$

注意: 廣義力**Q**爲來自**F**, 它可以是力或力偶, 但不含不作功之拘束力和力偶.

Example: **Virtual Work and Generalized Forces**

Determine the generalized forces of the two-arm robotic manipulator. (com is located at the center of each arm)

Sol: Dof=2 (independent coordinates: θ_2 and θ_3)



$m_2\mathbf{g}\bullet\delta\mathbf{r}_2 = -m_2 g\delta r_{2y}$ $\quad \mathbf{F}_3\bullet\delta\mathbf{r}_P = \mathbf{F}_3^T\delta\mathbf{r}_P$

$$\boxed{\delta W = M_2\delta\theta_2 - m_2 g\delta r_{2y} - m_3 g\delta r_{3y} + \mathbf{F}_3^T\delta\mathbf{r}_p}$$

$$r_{2y} = (l_2/2)\sin\theta_2$$
$$r_{3y} = l_2\sin\theta_2 + (l_3/2)\sin\theta_3$$
$$\mathbf{r}_p = \begin{bmatrix} l_2\cos\theta_2 + l_3\cos\theta_3 \\ l_2\sin\theta_2 + l_3\sin\theta_3 \end{bmatrix}$$

$\Rightarrow$

$$\delta r_{2y} = (l_2/2)\cos\theta_2\delta\theta_2$$
$$\delta r_{3y} = l_2\cos\theta_2\delta\theta_2 + (l_3/2)\cos\theta_3\delta\theta_3$$
$$\delta\mathbf{r}_p = \begin{bmatrix} -l_2\sin\theta_2\delta\theta_2 - l_3\sin\theta_3\delta\theta_3 \\ l_2\cos\theta_2\delta\theta_2 + l_3\cos\theta_3\delta\theta_3 \end{bmatrix}$$

$$\delta W = M_2\delta\theta_2 - m_2 g(l_2/2)\cos\theta_2\delta\theta_2 - m_3 g\{l_2\cos\theta_2\delta\theta_2 + (l_3/2)\cos\theta_3\delta\theta_3\} + \begin{bmatrix} F_3\cos\phi & F_3\sin\phi \end{bmatrix}\begin{bmatrix} -l_2\sin\theta_2\delta\theta_2 - l_3\sin\theta_3\delta\theta_3 \\ l_2\cos\theta_2\delta\theta_2 + l_3\cos\theta_3\delta\theta_3 \end{bmatrix}$$

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Example: **Virtual Work and Generalized Forces** (cont'd)

$$\delta W = M_2\delta\theta_2 - m_2 g(l_2/2)\cos\theta_2\delta\theta_2 - m_3 g\{l_2\cos\theta_2\delta\theta_2 + (l_3/2)\cos\theta_3\delta\theta_3\} + F_3\cos\phi\{-l_2\sin\theta_2\delta\theta_2 - l_3\sin\theta_3\delta\theta_3\} + F_3\sin\phi\{l_2\cos\theta_2\delta\theta_2 + l_3\cos\theta_3\delta\theta_3\}$$

$$\delta W = \{M_2 - m_2 g(l_2/2)\cos\theta_2 - m_3 g l_2\cos\theta_2 - (F_3\cos\phi)(l_2\sin\theta_2) + (F_3\sin\phi)(l_2\cos\theta_2)\}\delta\theta_2 + \{-m_3 g(l_3/2)\cos\theta_3 - (F_3\cos\phi)(l_3\sin\theta_3) + (F_3\sin\phi)(l_3\cos\theta_3)\}\delta\theta_3$$

$$\because \delta W = \mathbf{Q}^T\delta\mathbf{q} = \begin{bmatrix} \mathbf{Q}_{\theta_2} & \mathbf{Q}_{\theta_3} \end{bmatrix}\begin{bmatrix} \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} = \mathbf{Q}_{\theta_2}\delta\theta_2 + \mathbf{Q}_{\theta_3}\delta\theta_3$$

$\therefore$ Comparing the two equations yields generalized forces :

$$\begin{aligned}\mathbf{Q}_{\theta_2} &= M_2 - m_2 g(l_2/2)\cos\theta_2 - m_3 g l_2\cos\theta_2 - (F_3\cos\phi)(l_2\sin\theta_2) + (F_3\sin\phi)(l_2\cos\theta_2) \\ &= M_2 - m_2 g(l_2/2)\cos\theta_2 - m_3 g l_2\cos\theta_2 + F_3 l_2\sin(\phi - \theta_2)\end{aligned}$$

$$\begin{aligned}\mathbf{Q}_{\theta_3} &= -m_3 g(l_3/2)\cos\theta_3 - (F_3\cos\phi)(l_3\sin\theta_3) + (F_3\sin\phi)(l_3\cos\theta_3) \\ &= -m_3 g(l_3/2)\cos\theta_3 + F_3 l_3\sin(\phi - \theta_3)\end{aligned}$$

因爲兩個廣義力皆對應於旋轉角, 故皆爲力偶型態

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D'Alembert's principle & Kane Equation

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} d\mathbf{q} + \frac{\partial \mathbf{r}}{\partial t} dt$$

$$\delta \mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \delta \mathbf{q}$$

- D'Alembert's principle
  - Newton's 2nd law: resultant external force $\mathbf{F}^e=\mathbf{M}(d^2\mathbf{r}/dt)$
  - This implies $-\mathbf{M}(d^2\mathbf{r}/dt)$ can be regarded as a force—Inertia force
  - Thus, the sum of external forces $\mathbf{F}^e$ and the inertia forces $-\mathbf{M}(d^2\mathbf{r}/dt)$ of the system must satisfy
    $\mathbf{F}= \mathbf{F}^e +[-\mathbf{M}(d^2\mathbf{r}/dt)] =0$ for systems in dynamic equilibrium
- Kane Equation: By using the principle of virtual work, we have

$$\delta W = \sum_j^N \mathbf{F}_j \bullet \delta \mathbf{r}_j = (\delta \mathbf{r})^T \mathbf{F} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}} \delta \mathbf{q}\right)^T (\mathbf{F}^e - \mathbf{M}\ddot{\mathbf{r}}) = \delta \mathbf{q}^T \left\{ \left(\frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}}\right)^T (\mathbf{F}^e - \mathbf{M}\ddot{\mathbf{r}}) \right\} = 0$$

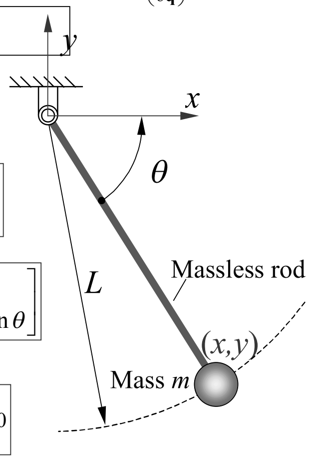
$$\therefore \underbrace{\left(\frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}}\right)^T}_{\text{Partial Velocity (部份速度/偏速度)}} (\underbrace{\mathbf{F}^e}_{\text{Resultant external forces (外力之合力)}} \underbrace{- \mathbf{M}\ddot{\mathbf{r}}}_{\text{Inertia forces (慣性力)}}) = \sum \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\mathbf{q}}_j} \bullet (\mathbf{F}_i^e - \mathbf{M}_i \ddot{\mathbf{r}}_i) = 0$$

注意: **F, F**^{e}不含拘束所造成之拘束力

Example

$$\left(\frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}}\right)^T (\mathbf{Q} - \mathbf{M}\ddot{\mathbf{r}}) = 0$$

Derive the equation of motion using Kane equation



$$\mathbf{r} = \begin{bmatrix} L\cos\theta \\ -L\sin\theta \end{bmatrix} \qquad \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$\dot{\mathbf{r}} = \dot{\theta} \begin{bmatrix} -L\sin\theta \\ -L\cos\theta \end{bmatrix} \rightarrow \ddot{\mathbf{r}} = \ddot{\theta} \begin{bmatrix} -L\sin\theta \\ -L\cos\theta \end{bmatrix} + \dot{\theta}^2 \begin{bmatrix} -L\cos\theta \\ L\sin\theta \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}} = \frac{\partial \dot{\mathbf{r}}}{\partial \dot{\theta}} = \begin{bmatrix} -L\sin\theta \\ -L\cos\theta \end{bmatrix} \qquad \mathbf{F}^e - \mathbf{M}\ddot{\mathbf{r}} = m \begin{bmatrix} \ddot{\theta} L\sin\theta + \dot{\theta}^2 L\cos\theta \\ -g - \ddot{\theta} L\cos\theta + \dot{\theta}^2 L\sin\theta \end{bmatrix}$$

$$\left(\frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}}\right)^T (\mathbf{F}^e - \mathbf{M}\ddot{\mathbf{r}}) = m[-L\sin\theta \quad -L\cos\theta] \begin{bmatrix} -\ddot{\theta} L\sin\theta - \dot{\theta}^2 L\cos\theta \\ -g - \ddot{\theta} L\cos\theta + \dot{\theta}^2 L\sin\theta \end{bmatrix} = 0$$

$$\begin{aligned} &\Rightarrow L\sin\theta(\ddot{\theta} L\sin\theta + \dot{\theta}^2 L\cos\theta) + L\cos\theta(g + \ddot{\theta} L\cos\theta - \dot{\theta}^2 L\sin\theta) \\ &= \boxed{\ddot{\theta}(L\sin\theta)^2} + \dot{\theta}^2 L^2 \cancel{\sin}\theta\cos\theta + gL\cos\theta + \boxed{\ddot{\theta}(L\cos\theta)^2} - \dot{\theta}^2 L^2 \cancel{\sin}\theta\cos\theta \\ &= \ddot{\theta} L^2 + gL\cos\theta = 0 \end{aligned} \rightarrow \therefore \ddot{\theta} + \frac{g}{L}\cos\theta = 0$$

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Lagrange's Equation

- Lagrange (1736-1813) created the powerful tool – *the principle of virtual work*, and used it as the starting point to formulate *Lagrange's equation of motion*
- Using the principle of virtual work in dynamics, one concludes that
  - if the n generalized coordinates are independent, the system equations of motions can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

j =1, 2, …, n.

T : system's kinetic energy
q_j : the independent generalized coordinates
Q_j: the generalized forces associated with the coordinate q_j

注意: 此處之廣義力**Q**將由虛功原理決定, 故它可以是力或力偶, 但不含不作功之拘束力和力偶.

Kinetic Energy for Translational Motion

$$\mathbf{r}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}_{3\times1}$$

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{\mathbf{r}}_1 \bullet \dot{\mathbf{r}}_1 + \cdots + \frac{1}{2}m_N\dot{\mathbf{r}}_N \bullet \dot{\mathbf{r}}_N \\
&= \frac{1}{2}m_1\dot{\mathbf{r}}_1^T\dot{\mathbf{r}}_1 + \cdots + \frac{1}{2}m_N\dot{\mathbf{r}}_N^T\dot{\mathbf{r}}_N \\
&= \frac{1}{2}\begin{bmatrix} \dot{\mathbf{r}}_1^T & \cdots & \dot{\mathbf{r}}_N^T \end{bmatrix}_{1\times3N} \begin{bmatrix} m_1\dot{\mathbf{r}}_1 \\ \vdots \\ m_N\dot{\mathbf{r}}_N \end{bmatrix}_{3N\times1} \\
&= \frac{1}{2}\begin{bmatrix} \dot{\mathbf{r}}_1^T & \cdots & \dot{\mathbf{r}}_N^T \end{bmatrix}_{1\times3N} \begin{bmatrix} m_1\mathbf{I}_3 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & m_N\mathbf{I}_3 \end{bmatrix}_{3N\times3N} \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \vdots \\ \dot{\mathbf{r}}_N \end{bmatrix}_{3N\times1} \\
&= \frac{1}{2}\dot{\mathbf{r}}^T\mathbf{M}\dot{\mathbf{r}}
\end{aligned}$$

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{r}_\mathbf{q}\dot{\mathbf{q}}$$

$$\begin{aligned}
T &= \frac{1}{2}\dot{\mathbf{r}}^T\mathbf{M}\dot{\mathbf{r}} \\
&= \frac{1}{2}\left(\mathbf{r}_\mathbf{q}\dot{\mathbf{q}}\right)^T\mathbf{M}\left(\mathbf{r}_\mathbf{q}\dot{\mathbf{q}}\right) \\
&= \frac{1}{2}\dot{\mathbf{q}}^T\left(\mathbf{r}_\mathbf{q}^T\mathbf{M}\mathbf{r}_\mathbf{q}\right)\dot{\mathbf{q}} \\
&= \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}
\end{aligned}$$

$$\therefore T = \sum_i^N \frac{1}{2}m_i\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i = \frac{1}{2}\sum_i^N m_i\dot{\mathbf{r}}_i^T\dot{\mathbf{r}}_i = \frac{1}{2}\dot{\mathbf{r}}^T\mathbf{M}(\mathbf{r})\dot{\mathbf{r}} = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$

Derivation of Lagrange's Equation

$$T=\sum_i^N \frac{1}{2} m_i \dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i = \frac{1}{2}\sum_i^N m_i \dot{\mathbf{r}}_i^T \dot{\mathbf{r}}_i = \frac{1}{2}\dot{\mathbf{r}}^T \mathbf{M}\dot{\mathbf{r}} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Why? Proof it!

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i^N \frac{\partial}{\partial \dot{q}_j}\left(\frac{m_i}{2}\dot{\mathbf{r}}_i^T \dot{\mathbf{r}}_i\right) = \sum_i^N \frac{m_i}{2}\frac{\partial}{\partial \dot{q}_j}\left(\dot{\mathbf{r}}_i^T \dot{\mathbf{r}}_i\right) = \sum_i^N m_i \dot{\mathbf{r}}_i^T \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \sum_i^N m_i \dot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\sum_i^N m_i \dot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \sum_i^N m_i \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} + \sum_i^N m_i \dot{\mathbf{r}}_i^T \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}$$

$$\mathbf{Q}_j \equiv \sum_i^N \mathbf{F}_i \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} = \sum_i^N \mathbf{F}_i^T \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} = \sum_i^N m_i \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j}$$

$$\frac{\partial T}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\sum_i^N \frac{m_i}{2}\dot{\mathbf{r}}_i^T \dot{\mathbf{r}}_i\right) = \sum_i^N \frac{\partial}{\partial q_j}\left(\frac{m_i}{2}\dot{\mathbf{r}}_i^T \dot{\mathbf{r}}_i\right) = \sum_i^N m_i \dot{\mathbf{r}}_i^T \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \mathbf{Q}_j$$

Example:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j}\right) = Q_j$$

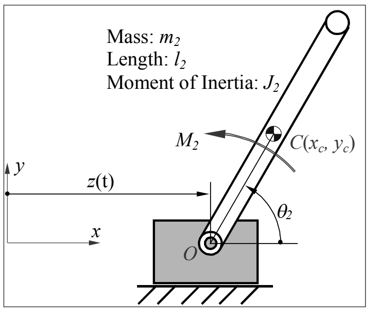
- Derive the equation of motion of the system provided that z(t) is specified.
- Sol: DOF=1, θ_2 is the only independent coordinate

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial T}{\partial \theta_2}\right) = Q_{\theta_2}$$

$$x_c = z(t) + (l_2/2)\cos\theta_2 \qquad y_c = (l_2/2)\sin\theta_2$$

$$\Rightarrow \quad \dot{x}_c = \dot{z}(t) - (l_2/2)\dot{\theta}_2 \sin\theta_2 \qquad \dot{y}_c = (l_2/2)\dot{\theta}_2\cos\theta_2$$

$$T = \frac{1}{2}m_2(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2}J_2\dot{\theta}_2^2 = \frac{1}{2}m_2\left[\dot{z}^2 - \dot{z}l_2\dot{\theta}_2\sin\theta_2 + \left(l_2\dot{\theta}_2/2\right)^2\right] + \frac{1}{2}J_2\dot{\theta}_2^2$$



$$\frac{\partial T}{\partial \theta_2} = -\frac{1}{2}m_2\dot{z}l_2\dot{\theta}_2\cos\theta_2$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = -\frac{1}{2}m_2\dot{z}l_2\sin\theta_2 + \left[J_2 + \frac{m_2 l_2^2}{4}\right]\dot{\theta}_2$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = -\frac{1}{2}m_2\ddot{z}l_2\sin\theta_2 - \frac{1}{2}m_2\dot{z}l_2\dot{\theta}_2\cos\theta_2 + \left[J_2 + \frac{m_2 l_2^2}{4}\right]\ddot{\theta}_2$$

$$J_O \equiv J_2 + m_2\left(\frac{l_2}{2}\right)^2$$

Example (cont'd)

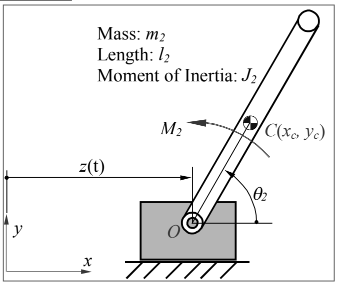
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = -\frac{1}{2}m_2\ddot{z}l_2\sin\theta_2 - \frac{1}{2}m_2\dot{z}l_2\dot{\theta}_2\cos\theta_2 + J_O\ddot{\theta}_2 \qquad J_O \equiv J_2 + m_2\left(\frac{l_2}{2}\right)^2$$

$$\frac{\partial T}{\partial \theta_2} = -\frac{1}{2}m_2\dot{z}l_2\dot{\theta}_2\cos\theta_2$$

$$\begin{aligned}\delta W &= -m_2 g\delta y_c + M_2\delta\theta_2 \\ &= -m_2 g(l_2/2)\cos\theta_2\delta\theta_2 + M_2\delta\theta_2 \\ &= Q_{\theta_2}\delta\theta_2\end{aligned}$$

$$\therefore Q_{\theta_2} = -m_2 g(l_2/2)\cos\theta_2 + M_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial T}{\partial \theta_2}\right) = Q_{\theta_2}$$



$\therefore$ The equation of motion of the system is

$$J_O\ddot{\theta}_2 + m_2 g\left(\frac{l_2}{2}\right)\cos\theta_2 = M_2 + \frac{1}{2}m_2\ddot{z}l_2\sin\theta_2$$

Another Form of Lagrange's Equation

$$T = T(\mathbf{q}, \dot{\mathbf{q}})$$

$$V = V(\mathbf{q})$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

Define a new function called ***Lagrangian*** *L*=*T*-*V*

註: 此式與前式之不同處在於: 將保守力所作之功轉爲勢能V

$$\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial (L+V)}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \cancelto{0}{\frac{\partial V(\mathbf{q})}{\partial \dot{q}_j}} = \frac{\partial L}{\partial \dot{q}_j}$$

$$Q_j = Q_j^c + Q_j^{nc}$$

$$Q_j^c = \sum_j^N \mathbf{F}_j^c \bullet \frac{\partial \mathbf{r}_j}{\partial \mathbf{q}_i} = \sum_j^N \left(-\nabla_i V_i\right) \bullet \frac{\partial \mathbf{r}_j}{\partial \mathbf{q}_i} = -\frac{\partial V}{\partial q_j}$$

$$\frac{\partial T}{\partial q_j} = \frac{\partial (L+V)}{\partial q_j} = \frac{\partial L}{\partial q_j} + \frac{\partial V(\mathbf{q})}{\partial q_j} = \frac{\partial L}{\partial q_j} - Q_j^c$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial \dot{q}_j} + \cancel{Q_j^c} = \cancel{Q_j^c} + Q_j^{nc} \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial \dot{q}_j} = Q_j^{nc}$$

Summary: Various Forms of Lagrange's Equation

If the coordinates **q** are independent

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\left(\frac{\partial T}{\partial q_j}\right)=Q_j$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\left(\frac{\partial L}{\partial q_j}\right)=Q_j^{nc}$$

$$\text{where } L \equiv T-V$$

$$Q_j^{nc} = \text{Nonconservative forces}$$

Q_j: the generalized forces associated with the coordinate q_j

If the coordinates **q** are not independent (變數非獨立故須加入拘束條件和拘束力)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\left(\frac{\partial L}{\partial q_j}\right)+\Phi_q^T\lambda=Q_j^{nc}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)-\left(\frac{\partial L}{\partial \mathbf{q}}\right)+\mathbf{\Phi}_q^T\boldsymbol{\lambda}=\mathbf{Q}^{nc}$$ Matrix form

- where λ are the so called Lagrange multipliers, they are selected to make the equality of the two sides of the equation
- In fact, $\Phi_q^T\lambda$ are the constraint forces. Columns of Φ_q^T (=rows of Φ_q) give the direction of constraint forces, and λ is the vector of their magnitudes)

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Example 1: **Lagrange Equation ⇒ Equation of Motion**

- Use the Lagrange's equation to write the equations of motion of a mechanical system with kinetic energy T, potential energy V, external forces Q_{ex}, and whose constraint conditions are $\Phi(\mathbf{q})=0$

$$L=T-V$$

$$T=T(\mathbf{q},\dot{\mathbf{q}})=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$

$$V=V(\mathbf{q})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)-\left(\frac{\partial L}{\partial \mathbf{q}}\right)+\Phi_{\mathbf{q}}^T\boldsymbol{\lambda}=\mathbf{Q}^{nc}$$

$$\frac{\partial L}{\partial \dot{\mathbf{q}}}=\frac{\partial T}{\partial \dot{\mathbf{q}}}=\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)=\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}}+\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}$$

$$\frac{\partial L}{\partial \mathbf{q}}=\frac{\partial T}{\partial \mathbf{q}}-\frac{\partial V}{\partial \mathbf{q}}=T_q-V_q$$

$$\therefore \mathbf{M}\ddot{\mathbf{q}}+\Phi_{\mathbf{q}}^T\boldsymbol{\lambda}=\mathbf{Q}_{nc}-\dot{\mathbf{M}}\dot{\mathbf{q}}+T_q-V_q$$

$\mathbf{Q}_{nc}$ 外力但不含重力和彈簧力等保守力

$\dot{\mathbf{M}}\dot{\mathbf{q}}, T_q$ 含科氏力和離心力

V_q 含重力和彈簧力等保守力

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Example 2: Using Independent Coordinates

Find the equation of motion of the system subject to gravity, using the two independent coordinates q_1 and q_2. Dof=2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)-\left(\frac{\partial L}{\partial \mathbf{q}}\right)+\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}=\mathbf{Q}_{nc}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=\begin{bmatrix} q_1\cos q_2 \\ q_1\sin q_2 \end{bmatrix} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix}=\begin{bmatrix} \dot{q}_1\cos q_2 - q_1(\sin q_2)\dot{q}_2 \\ \dot{q}_1\sin q_2 + q_1(\cos q_2)\dot{q}_2 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}=\begin{bmatrix} L\cos q_2 \\ L\sin q_2 \end{bmatrix} \qquad \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix}=\begin{bmatrix} -L(\sin q_2)\dot{q}_2 \\ L(\cos q_2)\dot{q}_2 \end{bmatrix}$$

$$T=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$$

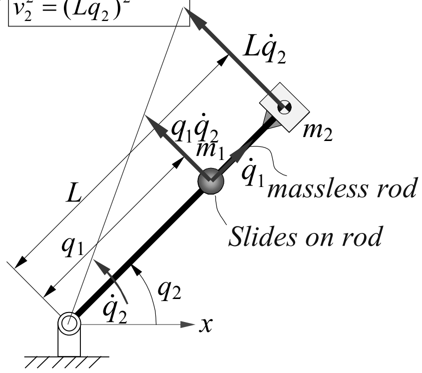
$$T=\frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2)+\frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)$$

$$v_1^2=\dot{q}_1^2+(q_1\dot{q}_2)^2$$

$$v_2^2=(L\dot{q}_2)^2$$

$$T=\frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2L^2+m_1q_1^2 \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$$

$$T=T(\mathbf{q},\dot{\mathbf{q}})=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$



Example 2: Using Independent Coordinates (cont'd)

$$T=\frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2L^2+m_1q_1^2 \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$$

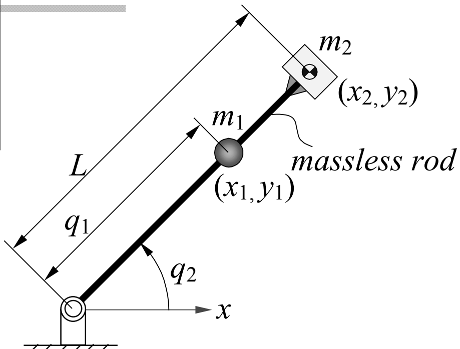
$$=\frac{1}{2}\left\{m_1\dot{q}_1^2+(m_2L^2+m_1q_1^2)\dot{q}_2^2\right\}$$

$$V=(m_1g)(q_1\sin q_2)+(m_2g)(L\sin q_2)$$

$$=(m_1q_1+m_2L)g\sin q_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)-\left(\frac{\partial L}{\partial \mathbf{q}}\right)+\cancel{\Phi_{\mathbf{q}}^T\boldsymbol{\lambda}}=\cancel{\mathbf{Q}_{nc}}$$

$$L=T-V$$



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2L^2+m_1q_1^2 \end{bmatrix}\begin{bmatrix}\ddot{q}_1 \\ \ddot{q}_2\end{bmatrix}=\underbrace{\begin{bmatrix} 0 \\ -2m_1q_1\dot{q}_1\dot{q}_2 \end{bmatrix}}_{\text{科氏力}}+\overbrace{\begin{bmatrix} m_1q_1\dot{q}_2^2 \\ 0 \end{bmatrix}}^{\text{離心力}}-\begin{bmatrix} m_1g\sin q_2 \\ (m_1q_1+m_2L)g\cos q_2 \end{bmatrix}\begin{matrix}\text{力} \\ \text{力矩}\end{matrix}$$

$$\dot{\mathbf{M}}\dot{\mathbf{q}}=\begin{bmatrix} 0 & 0 \\ 0 & 2m_1q_1\dot{q}_1 \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}=\begin{bmatrix} 0 \\ 2m_1q_1\dot{q}_1\dot{q}_2 \end{bmatrix}$$

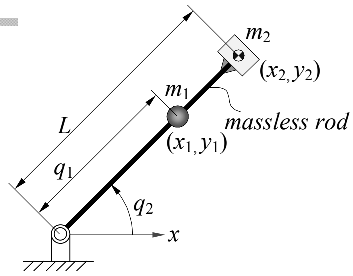
$$T_q=\begin{bmatrix} \dfrac{\partial T}{\partial q_1} \\ \dfrac{\partial T}{\partial q_2} \end{bmatrix}=\begin{bmatrix} m_1q_1\dot{q}_2^2 \\ 0 \end{bmatrix} \qquad V_q=\begin{bmatrix} \dfrac{\partial V}{\partial q_1} \\ \dfrac{\partial V}{\partial q_2} \end{bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{q}}+\cancel{\Phi_{\mathbf{q}}^T\boldsymbol{\lambda}}=\cancel{\mathbf{Q}_{nc}}-\dot{\mathbf{M}}\dot{\mathbf{q}}+T_q-V_q$$

Example 3: Repeat example 2 using the Cartesian coordinates of the two masses

$$T=\frac{1}{2}\{\dot{x}_1\dot{y}_1\dot{x}_2\dot{y}_2\}\underbrace{\begin{bmatrix} m_1 & 0 & 0 & 0\\ 0 & m_1 & 0 & 0\\ 0 & 0 & m_2 & 0\\ 0 & 0 & 0 & m_2\end{bmatrix}}_{\mathbf{M}}\begin{Bmatrix}\dot{x}_1\\ \dot{y}_1\\ \dot{x}_2\\ \dot{y}_2\end{Bmatrix}$$

$$V=m_1\,g\,y_1+m_2\,g\,y_2$$



❑ Since the coordinates are not independent, we need constraint equations:

$$x_2^2+y_2^2-L^2=0$$
$$x_1\,y_2-x_2\,y_1=0$$

$y_2/x_2 = y_1/x_1$ = slope

$$\begin{bmatrix} m_1 & 0 & 0 & 0\\ 0 & m_1 & 0 & 0\\ 0 & 0 & m_2 & 0\\ 0 & 0 & 0 & m_2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{y}_1\\ \ddot{x}_2\\ \ddot{y}_2\end{Bmatrix}+\begin{bmatrix} 0 & y_2\\ 0 & -x_2\\ 2x_2 & -y_1\\ 2y_2 & x_1\end{bmatrix}\begin{Bmatrix}\lambda_1\\ \lambda_2\end{Bmatrix}=\begin{Bmatrix}0\\ -m_1g\\ 0\\ -m_2g\end{Bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{q}}+\Phi_{\mathbf{q}}^T\boldsymbol{\lambda}=\mathbf{Q}_{nc}-\dot{\mathbf{M}}\dot{\mathbf{q}}+T_q-V_q$$

$$V_q=\begin{bmatrix}\frac{\partial V}{\partial x_1}\\ \frac{\partial V}{\partial y_1}\\ \frac{\partial V}{\partial x_2}\\ \frac{\partial V}{\partial y_2}\end{bmatrix}$$

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Discussion

- If using dependent coordinates (Example 3),
  - the number of equations is increased when compared to the results of using independent coordinates(Example 2)
  - the mass matrix M and gravity forces V_q are constant
  - The degree of non-linearity is decreased, since there are neither velocity dependent terms nor transcendental functions (e.g., cosq_2)
  - The Jacobian matrix of the constraints is linear in q

$$\begin{bmatrix} m_1 & 0 & 0 & 0\\ 0 & m_1 & 0 & 0\\ 0 & 0 & m_2 & 0\\ 0 & 0 & 0 & m_2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{y}_1\\ \ddot{x}_2\\ \ddot{y}_2\end{Bmatrix}+\begin{bmatrix} 0 & y_2\\ 0 & -x_2\\ 2x_2 & -y_1\\ 2y_2 & x_1\end{bmatrix}\begin{Bmatrix}\lambda_1\\ \lambda_2\end{Bmatrix}=\begin{Bmatrix}0\\ -m_1g\\ 0\\ -m_2g\end{Bmatrix}$$

$$\begin{bmatrix} m_1 & 0\\ 0 & m_2L^2+m_1q_1^2\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix}=\begin{bmatrix}0\\ -2m_1q_1\dot{q}_1\dot{q}_2\end{bmatrix}+\begin{bmatrix}m_1q_1\dot{q}_2^2\\ 0\end{bmatrix}-\begin{bmatrix}m_1g\sin q_2\\ (m_1q_1+m_2L)g\cos q_2\end{bmatrix}$$

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Numerical Implementation
Formulation in Independent Coordinates

- Define: $\mathbf{y} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \Longrightarrow \dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix}$

- Start at a time t in which the <u>positions</u> and <u>velocities</u> are known

- Determine <u>accelerations</u> at time t

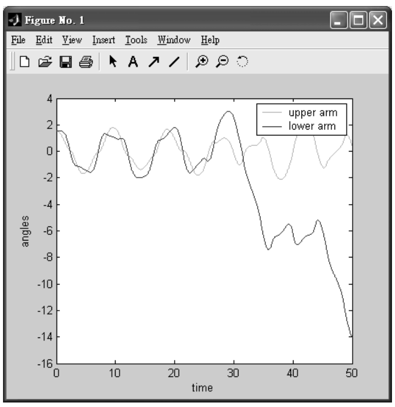
If using independent coordinates

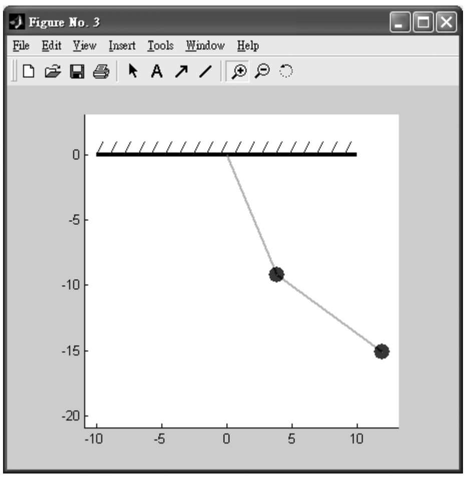
$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \cancel{\Phi_{\mathbf{q}}^{T}\boldsymbol{\lambda}} = \mathbf{Q}_{nc} - \dot{\mathbf{M}}\dot{\mathbf{q}} + T_q - V_q}$$

- Integrate d**y**(t)/dt, current velocities and accelerations, to obtain state variables **y**(t+Δt), new positions and velocities. That is,

$$\dot{\mathbf{y}}_t = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix}_t \xrightarrow{\text{Numerical integration}} \mathbf{y}_{t+\Delta t} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}_{t+\Delta t}$$

Numerical Example: Double Pendulum





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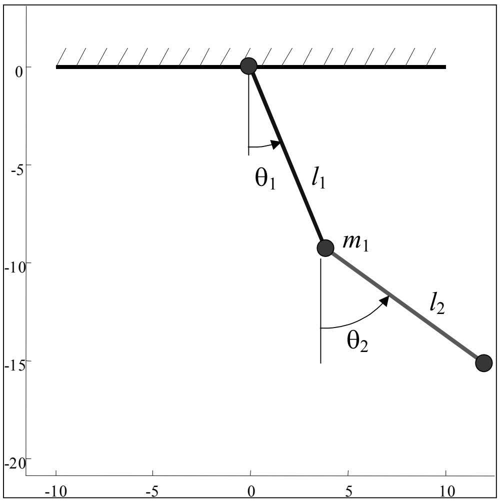
Numerical Example: Double Pendulum



$$\mathbf{y} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ \ddot{\theta}_1 \\ y_4 \\ \ddot{\theta}_2 \end{bmatrix}$$

Numerical Example: Double Pendulum

```
function ydot = f(t,y,l,m)
g=9.8;
a = y(3)-y(1);
sa = sin(a);
ca = cos(a);
temp = (l(1)*m(2)*y(2)*y(2)*sa*ca + m(2)*g*sin(y(3))*ca +
    m(2)*l(2)*y(4)*y(4)*sa - (m(1)+m(2))*g*sin(y(1))) / (l(1)*(m(1)+m(2))-
    l(1)*m(2)*ca*ca);
ydot = [  y(2);
        temp;
        y(4);
        -l(1)/l(2)*( temp*ca+y(2)*y(2)*sa )-g/l(2)*sin(y(3)) ];
% y= [q1, q1p, q2, q2p] = [y(1), y(2), y(3), y(4)]
% ydot = [q1p, q1pp, q2p, q2pp] = [y(2),q1pp,y(4),q2pp];
```

$$\mathbf{y} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ \ddot{\theta}_1 \\ y_4 \\ \ddot{\theta}_2 \end{bmatrix}$$

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Dept. of Mechanical Engineering

Numerical Example : Double Pendulum

```
function double_pendulum
  l = [10, 10]; % link length
  m = [2, 2]; % mass
  [t,Q]=ode45( @f,...
        [0:0.01:50],... % t
        [pi/2,0.0,pi/2,0.0],... %y
        [],... % option
        l,... % my extra argument 1
        m... % my extra argument 2
        );
  % ode solver returns new state variable in Q
  % Q = [q1, q1d, q2, q2d] = [theta1, omega1, theta2, omega2]
  y1=-l(1)*cos(Q(:,1));% position of sphere1
  x1=l(1)*sin(Q(:,1));
  y2=y1-l(2)*cos(Q(:,3));% position of sphere2
  x2=x1+l(2)*sin(Q(:,3));
```


MATLAB Functions: ODE Solvers
Solve initial value problems for ordinary differential equations

Solver	Problem Type	Order of Accuracy	When to Use
ode45	Nonstiff	Medium	Most of the time. This should be the first solver you try
ode23	Nonstiff	Low	If using crude error tolerances or solving moderately stiff problems
ode113	Nonstiff	Low to high	If using stringent error tolerances or solving a computationally intensive ODE file
ode15s	Stiff	Low to medium	If ode45 is slow because the problem is stiff
ode23s	Stiff	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant
ode23t	Moderately Stiff	Low	If the problem is only moderately stiff and you need a solution without numerical damping
ode23tb	Stiff	Low	If using crude error tolerances to solve stiff systems

What Is A Stiff System

- A stiff system is referred to as any initial-value problem in which the complete solution consists of fast and slow components.
- Technically, when the eigenvalues are widely spread, the system is said to be stiff.
- A stiff solver is the one that is well-suited for solving "stiff" equations
- A "stiff differential equation" is the one whose response changes rapidly over a time scale that is short compared to the time scale over which we are interested in the solution.
- A stable differential equation is called **stiff** when it has a decaying exponential particular solution with a ***time constant*** which is very small relative to the interval over which it is being solved.
- The ***time constant*** of a solution to a differential equation is the time it takes to decay by a factor of 1/e.
- For instance, the equation y'=-ky has the solution ce^{-ky}. This solution decays by a factor of 1/e in time 1/k. So, the time constant is 1/k.

Syntax for Calling ODE Solver

- Syntax:
  - [T,Y,TE,YE]=solver(odefun,tspan,y0,options, p1,p2...)
  - p1,p2…are optional parameters that the solver passes to odefun and all the functions specified in options..
- For instance,
  - y0 = [20; 0];
  - options = odeset('Events',@events);
  - [t,y,te,ye] = ode23(@f,[tstart:0.01:tfinal],y0,options,R);
  - Prarmeter R will be passed to @f and @events

Virtual Velocity

$$\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = -\boldsymbol{\Phi}_t$$

- A virtual velocity vector is defined as <u>a set of imaginary velocities at a stationary time</u> that is consistent with the homogeneous form of the velocity constraint conditions, i.e., having no RHS term Φ_t

  For instance,

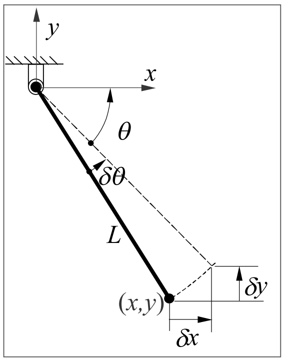
$$\Phi = x^2 + y^2 - L^2 = 0$$

$$\delta\Phi = \Phi_{\mathbf{q}}\delta\mathbf{q} = \begin{bmatrix} 2x & 2y \end{bmatrix}\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = 0$$

$$\dot{\Phi} = \Phi_{\mathbf{q}}\dot{\mathbf{q}}^* = \begin{bmatrix} 2x & 2y \end{bmatrix}\begin{bmatrix} \dot{x}^* \\ \dot{y}^* \end{bmatrix} = 0$$

velocity constraint equation

Virtual velocity vector $\dot{\mathbf{q}}^*$



Virtual Velocity (虛速度) vs. Variation of Velocity (速度之變分)

- Virtual velocities need not be infinitesimal
- A virtual velocity (finite) is a virtual displacement (infinitesimal) divided by δt (infinitesimal)

$$\dot{\mathbf{q}}^* = \frac{\delta\mathbf{q}}{\delta t} \neq \delta\dot{\mathbf{q}}$$

Variation of velocity

Virtual velocity

- Since a virtual velocity is derived from a virtual displacement, it satisfies the kinematic constraints, and thus is referred to as a <u>kinematically admissible virtual velocity</u>(虛速度必須滿足運動拘束).

The Principle of Virtual Power (虛功率原理)

- D'Alembert's principle:
  F= $\mathbf{Q}_e$+[-**M**(d²**q**/dt)] =0 when the system is dynamically balanced
  ($\mathbf{Q}_e$: external forces)
- Principle of Virtual work: $\delta W = \sum_j^N \mathbf{F}_j \bullet \delta \mathbf{r}_j = \mathbf{Q}^T \delta \mathbf{q} = \delta \mathbf{q}^T \mathbf{Q} = 0$
- The principle of virtual power

$$\boxed{\delta W = \delta \mathbf{q}^T \mathbf{F} = 0} \longrightarrow \boxed{\frac{\delta W}{\delta t} = \left(\frac{\delta \mathbf{q}}{\delta t}\right)^T \mathbf{F} = \dot{\mathbf{q}}^{*T} \mathbf{F} = 0}$$

(作用於剛體上之外力和慣性力的合力)

$$\boxed{\frac{\delta W}{\delta t} \equiv \sum \dot{\mathbf{q}}_i^* \bullet \mathbf{F}_i = \underset{\text{1xn}}{\dot{\mathbf{q}}^{*T}} \underset{\text{nx1}}{\mathbf{F}} = 0}$$

where $\boxed{\underset{\text{nx1}}{\mathbf{F}} = \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}_e}$

External forces

Inertia forces

The external forces $\mathbf{Q}_e$ may simply be written as **Q** hereafter.

Virtual Power & Formulation in Dependent Coordinates

$$\boxed{W^* \equiv \sum \dot{\mathbf{q}}_i^* \bullet \mathbf{F}_i = \dot{\mathbf{q}}^{*T} \mathbf{F} = 0} \quad \text{——— a} \qquad \boxed{\mathbf{F} = \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}}$$

- Recall that $\Phi_q^T\lambda$ are the constraint forces. Columns of Φ_q^T or rows of Φq give the direction of constraint forces, and λ is the vector of their magnitudes
- The constraint forces are in the directions of the constraint violations
- That is, each column of $\Phi_q^T\lambda$ is normal to the contact surface or the corresponding virtual velocity, and thus

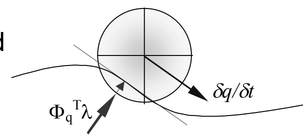
$$\boxed{\dot{\mathbf{q}}^{*T} \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = 0} \quad \text{——— b}$$

- Equations (a) and (b) can be added to yield

$$\boxed{W^* = \dot{\mathbf{q}}^{*T}(\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q} + \mathbf{\Phi}_q^T \boldsymbol{\lambda}) = 0}$$

$$\Downarrow$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_q^T \boldsymbol{\lambda} = \mathbf{Q}} \quad \xLeftrightarrow{\text{analogous}} \quad \boxed{\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{Q}_{nc} - \dot{\mathbf{M}}\dot{\mathbf{q}} + T_q - V_q}$$



Methods for Solving Equations of Motion with Lagrange's Multipliers

- Method 1

$$\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{Q}_{nc} - \dot{\mathbf{M}}\dot{\mathbf{q}} + T_q - V_q$$

$$\mathbf{M}\ \ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T\ \boldsymbol{\lambda} = \mathbf{Q}$$

nxn nx1 nxm mx1 nx1

- n = number of coordinates
- m = number of λ or constraint equations
- There are n equations in n+m unknowns ($d^2\mathbf{q}/dt^2$ and λ), so we need m constraints for providing m additional equations:

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -[(\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} + 2\boldsymbol{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} + \boldsymbol{\Phi}_{tt}] \equiv \boldsymbol{\gamma} \Leftarrow \boldsymbol{\Phi} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{\Phi}_{\mathbf{q}}^T \\ \boldsymbol{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \boldsymbol{\gamma} \end{bmatrix}$$

(n+m)x(n+m) (n+m)x1 (n+m)x1

Ax = b

Methods for Solving Equations of Motion with Lagrange's Multipliers (cont'd)

- Method 2

$$\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{Q}$$

$$\Downarrow$$

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{Q} - \Phi_{\mathbf{q}}^T \boldsymbol{\lambda})$$

Premultiply both sides by $\Phi_{\mathbf{q}}$

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \boldsymbol{\gamma}$$

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \boldsymbol{\Phi}_{\mathbf{q}}\mathbf{M}^{-1}(\mathbf{Q} - \boldsymbol{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}) = \boldsymbol{\gamma}$$

$$\Downarrow$$

$$(\boldsymbol{\Phi}_{\mathbf{q}}\mathbf{M}^{-1}\boldsymbol{\Phi}_{\mathbf{q}}^T)\boldsymbol{\lambda} = \boldsymbol{\Phi}_{\mathbf{q}}\mathbf{M}^{-1}\mathbf{Q} - \boldsymbol{\gamma}$$

Ax = b

From which Lagrange multiplier λ can be found, and the accelerations can be determined from the follow equation:

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{Q} - \Phi_{\mathbf{q}}^T \boldsymbol{\lambda})$$

Note: Matrix **M** should be nonsingular.

Example 4: Double Pendulum with Moment of Inertia (ref: E. J. Haug, 1992)

- Given: Each bar is 2 unit long with the centroid located at its center.
- Derive the EOM of the planar double pendulum.
- Sol: *Since* q_1 and q_2 are independent coordinates, and there are no non-conservative forces. Thus,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)-\left(\frac{\partial L}{\partial \mathbf{q}}\right)+\Phi_{\mathbf{q}}^{T}\boldsymbol{\lambda}=\mathbf{Q}^{nc}}$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}}+\cancel{\Phi_{\mathbf{q}}^{T}\boldsymbol{\lambda}}=\cancel{\mathbf{Q}_{nc}}-\dot{\mathbf{M}}\dot{\mathbf{q}}+T_q-V_q}$$

$$\boxed{T=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}=\frac{1}{2}\sum_{i=1}^{2}\left(m_i\dot{r}_i^2+J_i\omega_i^2\right)}$$

$J_i = m_i(2)^2/12 = m_i/3$

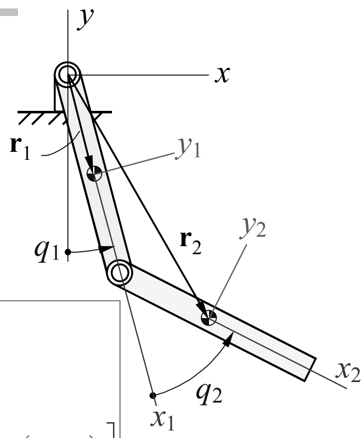
$$\boxed{V=\sum_{i=1}^{2}m_i g r_{iy}}$$

$$\mathbf{r}_1=\begin{bmatrix}\frac{l_1}{2}\sin q_1\\ -\frac{l_1}{2}\cos q_1\end{bmatrix}$$

$$\mathbf{r}_2=\begin{bmatrix}l_1\sin q_1+\frac{l_2}{2}\sin(q_1+q_2)\\ -l_1\cos q_1-\frac{l_2}{2}\cos(q_1+q_2)\end{bmatrix}$$

$\omega_1=\dot{q}_1$

$\omega_2=\dot{q}_1+\dot{q}_2$



Note: relative coordinates are used in this example.

Example 4 (cont'd)

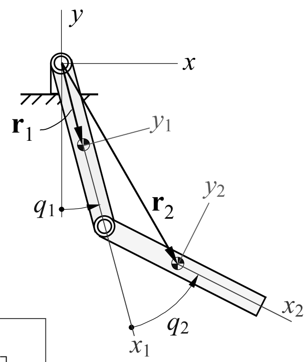
$$\boxed{\mathbf{M}\ddot{\mathbf{q}}+\cancel{\Phi_{\mathbf{q}}^{T}\boldsymbol{\lambda}}=\cancel{\mathbf{Q}_{nc}}-\dot{\mathbf{M}}\dot{\mathbf{q}}+T_q-V_q}$$

$$T=\frac{1}{2}\sum_{i=1}^{2}\left(m_i\dot{r}_i^2+J_i\omega_i^2\right)=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}$$

$$\dot{\mathbf{r}}_1=(\mathbf{r}_1)_{\mathbf{q}}\dot{\mathbf{q}}=\begin{bmatrix}\frac{l_1}{2}\cos q_1 & 0\\ \frac{l_1}{2}\sin q_1 & 0\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

$$V=\sum_{i=1}^{2}m_i g r_{iy}$$

$$\dot{\mathbf{r}}_2=(\mathbf{r}_2)_{\mathbf{q}}\dot{\mathbf{q}}=\begin{bmatrix}l_1\cos q_1+\frac{l_2}{2}\cos(q_1+q_2) & \frac{l_2}{2}\cos(q_1+q_2)\\ l_1\sin q_1+\frac{l_2}{2}\sin(q_1+q_2) & \frac{l_2}{2}\sin(q_1+q_2)\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$



$$\mathbf{r}_1=\begin{bmatrix}\frac{l_1}{2}\sin q_1\\ -\frac{l_1}{2}\cos q_1\end{bmatrix}$$

$$\mathbf{r}_2=\begin{bmatrix}l_1\sin q_1+\frac{l_2}{2}\sin(q_1+q_2)\\ -l_1\cos q_1-\frac{l_2}{2}\cos(q_1+q_2)\end{bmatrix}$$

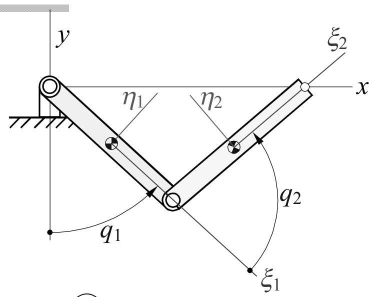
$\omega_1=\dot{q}_1$

$\omega_2=\dot{q}_1+\dot{q}_2$

$$\boxed{\begin{bmatrix}\frac{4m_1}{3}+m_1\left(\frac{16}{3}+4\cos q_2\right) & m_2\left(\frac{4}{3}+2\cos q_2\right)\\ m_2\left(\frac{4}{3}+2\cos q_2\right) & \frac{4m_2}{3}\end{bmatrix}\ddot{\mathbf{q}}=-\begin{bmatrix}-2m_2\sin q_2\left(\dot{q}_2^2+2\dot{q}_1\dot{q}_2\right)+m_2g\left(2\sin q_1+\sin(q_1+q_2)\right)+m_1g\sin q_1\\ 2m_2\sin q_2\dot{q}_1^2+m_2g\sin(q_1+q_2)\end{bmatrix}}$$

Example 5: Double Pendulum➔Slider Crank Mechanism

- Given: The endpoint of body 2 in Ex. 4 slides along the x-axis.
- Derive the EOM.
- Sol: In this case the coordinates are not independent, and we have to consider the following constraint equation



$$\boxed{\Phi = -2\cos q_1 - 2\cos(q_1+q_2) = 0}$$

$$\boxed{\Phi_q = \begin{bmatrix} 2\sin q_1 + 2\sin(q_1+q_2) & 2\sin(q_1+q_2) \end{bmatrix}}$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T\boldsymbol{\lambda} = \cancel{\mathbf{Q}}_{nc} - \dot{\mathbf{M}}\dot{\mathbf{q}} + T_q - V_q}$$

$$\begin{bmatrix} \frac{4m_1}{3} + m_1\left(\frac{16}{3} + 4\cos q_2\right) & m_2\left(\frac{4}{3} + 2\cos q_2\right) \\ m_2\left(\frac{4}{3} + 2\cos q_2\right) & \frac{4m_2}{3} \end{bmatrix}\ddot{\mathbf{q}} + \begin{bmatrix} 2\sin q_1 + 2\sin(q_1+q_2) \\ 2\sin(q_1+q_2) \end{bmatrix}\lambda$$
$$= -\begin{bmatrix} -2m_2\sin q_2\left(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right) + m_2 g\left(2\sin q_1 + \sin(q_1+q_2)\right) + m_1 g\sin q_1 \\ 2m_2\sin q_2\dot{q}_1^2 + m_2 g\sin(q_1+q_2) \end{bmatrix}$$

2 equations in 3 unknowns, q_1, q_2, λ.

Example 5 (cont'd)

$$\boxed{\Phi_q = \begin{bmatrix} 2\sin q_1 + 2\sin(q_1+q_2) & 2\sin(q_1+q_2) \end{bmatrix}}$$

$$\boxed{\Phi = -2\cos q_1 - 2\cos(q_1+q_2) = 0}$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T\boldsymbol{\lambda} = \cancel{\mathbf{Q}}_{nc} - \dot{\mathbf{M}}\dot{\mathbf{q}} + T_q - V_q}$$

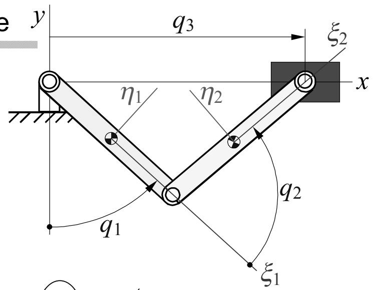
$$\boxed{\Phi_q\ddot{\mathbf{q}} = -2\left(\cos q_1 + \cos(q_1+q_2)\right)\dot{q}_1^2 - 2\cos(q_1+q_2)\left(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2\right) \equiv \gamma}$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T\lambda = \mathbf{Q}}$$

$$\boxed{\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \gamma \end{bmatrix}}$$

$$\left[\begin{array}{cc:c} \frac{4m_1}{3} + m_1\left(\frac{16}{3} + 4\cos q_2\right) & m_2\left(\frac{4}{3} + 2\cos q_2\right) & 2\sin q_1 + 2\sin(q_1+q_2) \\ m_2\left(\frac{4}{3} + 2\cos q_2\right) & \frac{4m_2}{3} & 2\sin(q_1+q_2) \\ \hdashline 2\sin q_1 + 2\sin(q_1+q_2) & 2\sin(q_1+q_2) & 0 \end{array}\right]\left[\begin{array}{c} \ddot{q}_1 \\ \ddot{q}_2 \\ \hdashline \lambda \end{array}\right]$$
$$= \left[\begin{array}{c} -2m_2\sin q_2\left(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right) + m_2 g\left(2\sin q_1 + \sin(q_1+q_2)\right) + m_1 g\sin q_1 \\ 2m_2\sin q_2\dot{q}_1^2 + m_2 g\sin(q_1+q_2) \\ \hdashline 2\left(\cos q_1 + \cos(q_1+q_2)\right)\dot{q}_1^2 + 2\cos(q_1+q_2)\left(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2\right) \end{array}\right]$$

Example 6: Slider's Mass Is Not Negligible



- Given: The mass of the slider in Ex. 5 plays an important role in dynamics.
- Derive the EOM.

$$\mathbf{\Phi}=\begin{bmatrix}2\sin q_1+2\sin(q_1+q_2)-q_3\\-2\cos q_1-2\cos(q_1+q_2)\end{bmatrix}=\mathbf{0}$$

$$\mathbf{\Phi_q}=\begin{bmatrix}2\cos q_1+2\cos(q_1+q_2) & 2\cos(q_1+q_2) & -1\\2\sin q_1+2\sin(q_1+q_2) & 2\sin(q_1+q_2) & 0\end{bmatrix}$$

$$\mathbf{M\ddot{q}}+\Phi_{\mathbf{q}}^T\boldsymbol{\lambda}=\mathbf{Q}_{nc}-\mathbf{\dot{M}\dot{q}}+T_q-V_q$$

$$\left[\begin{array}{cc|c}\frac{4m_1}{3}+m_1\left(\frac{16}{3}+4\cos q_2\right) & m_2\left(\frac{4}{3}+2\cos q_2\right) & 0\\ m_2\left(\frac{4}{3}+2\cos q_2\right) & \frac{4m_2}{3} & 0\\ \hline 0 & 0 & m_3\end{array}\right]\mathbf{\ddot{q}}+\left[\begin{array}{cc}2\cos q_1+2\cos(q_1+q_2) & 2\sin q_1+2\sin(q_1+q_2)\\ 2\cos(q_1+q_2) & 2\sin(q_1+q_2)\\ \hline -1 & 0\end{array}\right]\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}$$

$$=-\left[\begin{array}{c}-2m_2\sin q_2\left(\dot{q}_2^2+2\dot{q}_1\dot{q}_2\right)+m_2g\left(2\sin q_1+\sin(q_1+q_2)\right)+m_1g\sin q_1\\ 2m_2\sin q_2\dot{q}_1^2+m_2g\sin(q_1+q_2)\\ \hline 0\end{array}\right]$$

3 equations in 5 unknowns, q_1, q_2, q_3, λ_1, λ_2

Example 5 (cont'd)

$$\Phi_q=\left[2\sin q_1+2\sin(q_1+q_2)\quad 2\sin(q_1+q_2)\right]$$

$$\mathbf{\Phi}=\begin{bmatrix}2\sin q_1+2\sin(q_1+q_2)-q_3\\-2\cos q_1-2\cos(q_1+q_2)\end{bmatrix}=\mathbf{0}$$

$$\mathbf{M\ddot{q}}+\Phi_{\mathbf{q}}^T\boldsymbol{\lambda}=\mathbf{Q}_{nc}-\mathbf{\dot{M}\dot{q}}+T_q-V_q$$

$$\Phi_q\mathbf{\ddot{q}}=\begin{bmatrix}2\left(\sin q_1+\sin(q_1+q_2)\right)\dot{q}_1^2+2\sin(q_1+q_2)\left(2\dot{q}_1\dot{q}_2+\dot{q}_2^2\right)\\-2\left(\cos q_1+\cos(q_1+q_2)\right)\dot{q}_1^2-2\cos(q_1+q_2)\left(2\dot{q}_1\dot{q}_2+\dot{q}_2^2\right)\end{bmatrix}\equiv\boldsymbol{\gamma}_{2\times1}$$

$$\mathbf{M\ddot{q}}+\Phi_{\mathbf{q}}^T\lambda=\mathbf{Q}$$

$$\begin{bmatrix}\mathbf{M}_{3\times3} & \mathbf{\Phi}_{\mathbf{q}}^T\\ \mathbf{\Phi_q} & \mathbf{0}_{2\times2}\end{bmatrix}_{5\times5}\begin{bmatrix}\mathbf{\ddot{q}}_{3\times1}\\ \boldsymbol{\lambda}_{2\times1}\end{bmatrix}_{5\times1}=\begin{bmatrix}\mathbf{Q}_{3\times1}\\ \boldsymbol{\gamma}_{2\times1}\end{bmatrix}_{5\times1}$$

$$\mathbf{M}_{3\times3}=\begin{bmatrix}\frac{4m_1}{3}+m_1\left(\frac{16}{3}+4\cos q_2\right) & m_2\left(\frac{4}{3}+2\cos q_2\right) & 0\\ m_2\left(\frac{4}{3}+2\cos q_2\right) & \frac{4m_2}{3} & 0\\ 0 & 0 & m_3\end{bmatrix}$$

$$\mathbf{Q}_{3\times1}=-\begin{bmatrix}-2m_2\sin q_2\left(\dot{q}_2^2+2\dot{q}_1\dot{q}_2\right)+m_2g\left(2\sin q_1+\sin(q_1+q_2)\right)+m_1g\sin q_1\\ 2m_2\sin q_2\dot{q}_1^2+m_2g\sin(q_1+q_2)\\ 0\end{bmatrix}$$

$$\mathbf{\Phi_q}=\begin{bmatrix}2\cos q_1+2\cos(q_1+q_2) & 2\cos(q_1+q_2) & -1\\2\sin q_1+2\sin(q_1+q_2) & 2\sin(q_1+q_2) & 0\end{bmatrix}_{2\times3}$$

$$\begin{bmatrix}\mathbf{\ddot{q}}\\ \boldsymbol{\lambda}\end{bmatrix}_{5\times1}=\begin{bmatrix}\ddot{q}_1\\\ddot{q}_2\\\ddot{q}_3\\\lambda_1\\\lambda_2\end{bmatrix}$$

Solving Method Based on the Projection Matrix R
Review SVD in Basic Concepts

- Let $\mathbf{r}_i$ be a set of *k=n-m* linearly independent vectors that constitute a basis for the null space of Φ_q. Any velocity vector dq/dt can be expressed as a linear combination of this basis as follows:

$$\begin{aligned}\dot{\mathbf{q}}_{n\times 1} &= \mathbf{r}_1\dot{z}_1 + \mathbf{r}_1\dot{z}_2 + \cdots + \mathbf{r}_k\dot{z}_k \\ &= \begin{bmatrix}\mathbf{r}_1 & \cdots & \mathbf{r}_k\end{bmatrix}_{n\times k}\begin{bmatrix}\dot{z}_1\\ \vdots \\ \dot{z}_k\end{bmatrix}_{k\times 1} \\ &= \mathbf{R}_{n\times k}\dot{\mathbf{z}}_{k\times 1}\end{aligned}$$

- Matrix R plays an important role in some of the most efficient formulation for dynamic analysis.
- From the previous discussion, we know that

$$\boldsymbol{\Phi}_{\mathbf{q}}\mathbf{r}_i = \mathbf{0} \quad (i=1,\ldots,k) \longrightarrow \boldsymbol{\Phi}_{\mathbf{q}}\begin{bmatrix}\mathbf{r}_1 & \cdots & \mathbf{r}_k\end{bmatrix} = \boldsymbol{\Phi}_{\mathbf{q}}\mathbf{R} = \mathbf{0}$$

- Matrix R can be determined by calculating the SVD of Φq^T
- Let $A=\Phi_q^T=\mathbf{USV}^T \Rightarrow A^T\mathbf{U}_i=\Phi_q\mathbf{U}_i=0$ (Note: we had proven in chapter 3 that $\mathbf{A}^T\mathbf{U}_i=\mathbf{U}_i^T\mathbf{A}=\mathbf{0}$) $\Rightarrow$ $\mathbf{R}=\mathbf{U}_i$

Solving Method Based on the Projection Matrix R (cont'd)

$$\mathbf{M}\ \ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T\ \boldsymbol{\lambda} = \mathbf{Q}$$

Premultiply each term by R^T

$$\mathbf{R}_{k\times n}^T\mathbf{M}_{n\times n}\ddot{\mathbf{q}}_{n\times 1} + \mathbf{R}_{k\times n}^T\Phi_{\mathbf{q}}^T\boldsymbol{\lambda}_{m\times 1} = \mathbf{R}_{k\times n}^T\mathbf{Q}_{n\times 1}$$

k=n-m

$$\boldsymbol{\Phi}_{\mathbf{q}}\mathbf{R} = \mathbf{0}$$

n-m equations in *n* unknowns

$$\mathbf{R}_{k\times n}^T\mathbf{M}_{n\times n}\ddot{\mathbf{q}}_{n\times 1} = \mathbf{R}_{k\times n}^T\mathbf{Q}_{n\times 1}$$

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}}_{n\times 1} = \boldsymbol{\gamma}_{m\times 1}$$

$$\begin{bmatrix}\boldsymbol{\Phi}_{\mathbf{q}}\\ \mathbf{R}_{k\times n}^T\mathbf{M}_{n\times n}\end{bmatrix}_{n\times n}\ddot{\mathbf{q}}_{n\times 1} = \begin{bmatrix}\boldsymbol{\gamma}_{m\times 1}\\ \mathbf{R}_{k\times n}^T\mathbf{Q}_{n\times 1}\end{bmatrix}_{n\times 1}$$

$$\boxed{\mathbf{Ax} = \mathbf{b}}$$

Merit: Coordinate partitioning is not necessary.
(優點爲不必以手動方式分離自變數和因變數. 因爲求**R**時已自動將其分離)

Solving Method Based on the Projection Matrix R (cont'd)

- Derive previous equation using virtual power and virtual velocity

$$W^* \equiv \sum_{i=1}^{n} F_i \dot{q}^* = \dot{\mathbf{q}}^{*T}\mathbf{F} = \dot{\mathbf{q}}^{*T}(\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}) = 0$$

$$\dot{\mathbf{q}}_{n\times 1} = \mathbf{r}_1\dot{z}_1 + \mathbf{r}_1\dot{z}_2 + \cdots + \mathbf{r}_k\dot{z}_k = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_k]_{n\times k}\begin{bmatrix}\dot{z}_1\\ \vdots \\ \dot{z}_k\end{bmatrix}_{k\times 1} = \mathbf{R}_{n\times k}\dot{\mathbf{z}}_{k\times 1}$$

$$\dot{\mathbf{z}}^{*T}_{1\times k}\mathbf{R}^T_{k\times n}(\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}) = 0$$

(dz^*/dt) is independent

$$\mathbf{R}^T_{k\times n}(\mathbf{M}_{n\times n}\ddot{\mathbf{q}}_{n\times 1} - \mathbf{Q}_{n\times 1}) = 0$$

n-m equations in n unknowns

$$\mathbf{R}^T_{k\times n}\mathbf{M}_{n\times n}\ddot{\mathbf{q}}_{n\times 1} = \mathbf{R}^T_{k\times n}\mathbf{Q}_{n\times 1}$$

$$\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \gamma$$

$$\begin{bmatrix}\mathbf{\Phi}_{\mathbf{q}} \\ \mathbf{R}^T_{k\times n}\mathbf{M}_{n\times n}\end{bmatrix}_{n\times n}\ddot{\mathbf{q}}_{n\times 1} = \begin{bmatrix}\gamma_{m\times 1} \\ \mathbf{R}^T_{k\times n}\mathbf{Q}_{n\times 1}\end{bmatrix}_{n\times 1}$$

Numerical Implementation
Solving Method Based on the Projection Matrix R

- Start at a time t in which the positions and velocities are known

Define $\mathbf{y} = \begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} \Longrightarrow \dot{\mathbf{y}} = \begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix}$

- Determine matrix **R** by calculating the null space of of Φ_q^T.
- Solving the the following equation for the dependent accelerations

$$\begin{bmatrix}\mathbf{\Phi}_{\mathbf{q}} \\ \mathbf{R}^T_{k\times n}\mathbf{M}_{n\times n}\end{bmatrix}_{n\times n}\ddot{\mathbf{q}}_{n\times 1} = \begin{bmatrix}\gamma_{m\times 1} \\ \mathbf{R}^T_{k\times n}\mathbf{Q}_{n\times 1}\end{bmatrix}_{n\times 1}$$

- Integrate d**y**(t)/dt to obtain the new state variables **y**(t+Δt). That is,

$$\dot{\mathbf{y}}_t = \begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix}_t \xrightarrow{\text{Numerical integration}} \mathbf{y}_{t+\Delta t} = \begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix}_{t+\Delta t}$$

Hamiltonian Formulation--Velocity-Based Formulation

- The preceding equations of motion are a set of 2nd order differential equations. The technique is called acceleration-based formulations since the accelerations of the bodies are involved
- Hamilton introduced a transformation that leads to a set of first order differential equations
- The derived equations are called the canonical or Hamilton's equations

Generalized Momentum (廣義動量)

Define **generalized momenta** $\boxed{\mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial (T-V)}{\partial \dot{\mathbf{q}}} = \frac{\partial T}{\partial \dot{\mathbf{q}}} = \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}}$

$$\boxed{T = T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}} \quad \boxed{V = V(\mathbf{q})}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{Q}_{ex}}$$

$$\mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{q}}}$$

$$\boxed{\dot{\mathbf{p}} = \mathbf{Q}_{ex} + \frac{\partial L}{\partial \mathbf{q}} - \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}}$$

$$\boxed{\frac{\partial L}{\partial \mathbf{q}} = \dot{\mathbf{p}} - \mathbf{Q}_{ex} + \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}}$$

$$\because L = L(\dot{\mathbf{q}}, \mathbf{q}, t)$$

$$\therefore dL = d\dot{\mathbf{q}}^T \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) + d\mathbf{q}^T \left(\frac{\partial L}{\partial \mathbf{q}}\right) + dt\left(\frac{\partial L}{\partial t}\right)$$

$$\boxed{dL = d\dot{\mathbf{q}}^T \mathbf{p} + d\mathbf{q}^T \left(\dot{\mathbf{p}} - \mathbf{Q}_{ex} + \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}\right) + dt\left(\frac{\partial L}{\partial t}\right)}$$

$$\because d(\mathbf{p}^T \dot{\mathbf{q}}) = d\mathbf{p}^T \dot{\mathbf{q}} + \mathbf{p}^T d\dot{\mathbf{q}}$$

$$\therefore d\dot{\mathbf{q}}^T \mathbf{p} = \mathbf{p}^T d\dot{\mathbf{q}} = d(\mathbf{p}^T \dot{\mathbf{q}}) - d\mathbf{p}^T \dot{\mathbf{q}}$$

$$\boxed{d(\mathbf{p}^T \dot{\mathbf{q}} - L) = d\mathbf{p}^T \dot{\mathbf{q}} + d\mathbf{q}^T \left(\mathbf{Q}_{ex} - \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} - \dot{\mathbf{p}}\right) - dt\left(\frac{\partial L}{\partial t}\right)}$$

Hamiltonian Function & Canonical Equations

- Define ***Hamiltonian function*** $\boxed{H \equiv \mathbf{p}^T \dot{\mathbf{q}} - L}$

$$\because H = H(\mathbf{p}, \mathbf{q}, t)$$

$$\therefore dH = d\mathbf{p}^T \left(\frac{\partial H}{\partial \mathbf{p}}\right) + d\mathbf{q}^T \left(\frac{\partial H}{\partial \mathbf{q}}\right) + dt \left(\frac{\partial H}{\partial t}\right)$$

$$d(\mathbf{p}^T \dot{\mathbf{q}} - L) = d\mathbf{p}^T \dot{\mathbf{q}} + d\mathbf{q}^T \left(\mathbf{Q}_{ex} - \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} - \dot{\mathbf{p}}\right) - dt \left(\frac{\partial L}{\partial t}\right)$$

$$\boxed{\frac{\partial H}{\partial \mathbf{p}} = \dot{\mathbf{q}}} \qquad \boxed{\frac{\partial H}{\partial \mathbf{q}} = \mathbf{Q}_{ex} - \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} - \dot{\mathbf{p}}}$$

- These equations are called the ***canonical equations of Hamilton***.
- These ***2n first-order differential equations*** can replace the **n second-order differential equations**

$$\boxed{\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}} \qquad \boxed{\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{Q}_{ex} - \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}}$$

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Procedure for Using the Canonical Equations (Formulation in Independent Coordinates)

$$\mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = 0$$

1. Define the Lagrangian *L* as a function of the coordinates and velocities using *L=T-V*
2. Define the vector of generalized momenta using Eq. (a)
3. Define the Hamiltonian H using Eq. (b)
4. Obtain the first-order differential equations of the system by substituting the H into Eq. (c)

$$\mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial T}{\partial \dot{\mathbf{q}}} \quad \text{(a)}$$

$$H \equiv \mathbf{p}^T \dot{\mathbf{q}} - L \quad \text{(b)}$$

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{Q}_{ex} - \cancel{\mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}} \quad \text{(c)}$$

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Example (Using Independent Coordinates)

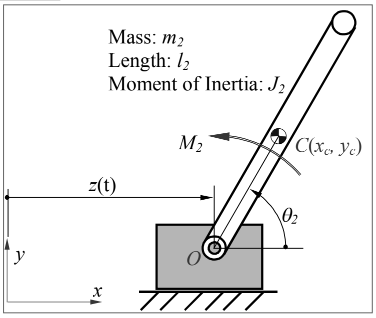
1. Define the Lagrangian L as a function of the coordinates and velocities

$$T=\frac{1}{2}m_2(\dot{x}_c^2+\dot{y}_c^2)+\frac{1}{2}J_2\dot{\theta}_2^2$$
$$=\frac{1}{2}m_2\left[\dot{z}^2-\dot{z}l_2\dot{\theta}_2\sin\theta_2+\left(l_2\dot{\theta}_2/2\right)^2\right]+\frac{1}{2}J_2\dot{\theta}_2^2$$

$$V=m_2gy_c=m_2g\frac{l_2}{2}\sin\theta_2$$

$$L=T-V$$
$$=\frac{1}{2}m_2\left[\dot{z}^2-\dot{z}l_2\dot{\theta}_2\sin\theta_2\right]+\frac{1}{2}J_O\dot{\theta}_2^2-m_2g\frac{l_2}{2}\sin\theta_2$$

where $J_O\equiv J_2+m_2\left(\frac{l_2}{2}\right)^2$



2. Define the generalized momentum

$$P=\frac{\partial L}{\partial\dot{\theta}_2}=\frac{\partial T}{\partial\dot{\theta}_2}=-\frac{1}{2}m_2\dot{z}l_2\sin\theta_2+J_O\dot{\theta}_2$$

$$\Downarrow$$

$$\dot{\theta}_2=\frac{1}{J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]$$

Example (cont'd)

$$L=T-V$$
$$=\frac{1}{2}m_2\left[\dot{z}^2-\dot{z}l_2\dot{\theta}_2\sin\theta_2\right]+\frac{1}{2}J_O\dot{\theta}_2^2-m_2g\frac{l_2}{2}\sin\theta_2$$

3. Define the Hamiltonian $H\equiv\mathbf{p}^T\dot{\mathbf{q}}-L$

$$H=P\dot{\theta}_2-L$$
$$=P\dot{\theta}_2-\frac{1}{2}m_2\left[\dot{z}^2-\dot{z}l_2\dot{\theta}_2\sin\theta_2\right]-\frac{1}{2}J_O\dot{\theta}_2^2+m_2g\frac{l_2}{2}\sin\theta_2$$

$$\dot{\theta}_2=\frac{1}{J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]$$

$$\Downarrow$$

$$H=\frac{P}{J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]$$
$$-\frac{1}{2}m_2\left\{\dot{z}^2-\frac{\dot{z}l_2}{J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]\sin\theta_2\right\}$$
$$-\frac{1}{2J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]^2$$
$$+m_2g\frac{l_2}{2}\sin\theta_2$$

Example (cont'd)

$$H=\frac{P}{J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]$$
$$-\frac{1}{2}m_2\left\{\dot{z}^2-\frac{\dot{z}l_2}{J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]\sin\theta_2\right\}$$
$$-\frac{1}{2J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]^2$$
$$+m_2g\frac{l_2}{2}\sin\theta_2$$

$$\dot{\mathbf{q}}=\frac{\partial H}{\partial \mathbf{p}}$$
$$\dot{\mathbf{p}}=-\frac{\partial H}{\partial \mathbf{q}}+\mathbf{Q}_{nc}-\cancel{\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}} \quad \text{(c)}$$

4. Obtain the first-order differential equations using Eq. (c)

$$\dot{p}=-\frac{\partial H}{\partial\theta_2}+M_2$$
$$=-\frac{P}{2J_O}m_2\dot{z}l_2\cos\theta_2-\frac{1}{4J_O}[m_2\dot{z}l_2]^2\sin\theta_2\cos\theta_2-m_2g\frac{l_2}{2}\cos\theta_2+M_2$$

$$\dot{\theta}_2=\frac{\partial H}{\partial P}=\frac{1}{J_O}\left[P+\frac{1}{2}m_2\dot{z}l_2\sin\theta_2\right]$$

1. Start at time t when *p* and $q(\theta_2)$ are known
2. Determine $d\theta_2/dt$ and dp/dt
3. Integrate to determine θ_2 and p at time $t+\Delta t$
4. Update state and go to step 2

Another Form of Canonical Equations of Hamilton

$$T=T(\mathbf{q},\dot{\mathbf{q}})=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \qquad V=V(\mathbf{q})$$

$$\dot{\mathbf{q}}=\frac{\partial H}{\partial \mathbf{p}}$$
$$\dot{\mathbf{p}}=-\frac{\partial H}{\partial \mathbf{q}}+\mathbf{Q}_{ex}-\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}$$

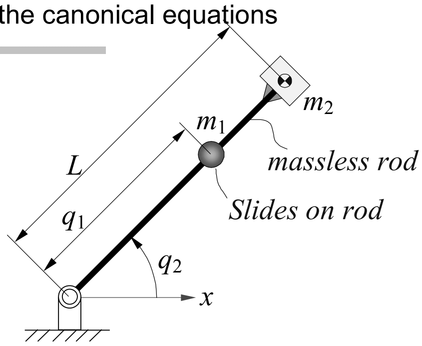
$$L=T-V=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}-V(\mathbf{q})$$
$$\mathbf{p}\equiv\frac{\partial L}{\partial \dot{\mathbf{q}}}$$
$$\boxed{\mathbf{p}=\mathbf{M}\dot{\mathbf{q}}}$$

$$H\equiv\mathbf{p}^T\dot{\mathbf{q}}-L$$
$$\Downarrow$$
$$\frac{\partial H}{\partial \mathbf{q}}=\frac{\partial}{\partial \mathbf{q}}(\mathbf{p}^T\dot{\mathbf{q}}-L)=-\frac{\partial L}{\partial \mathbf{q}}=-L_{\mathbf{q}}$$
$$\dot{\mathbf{p}}=-\frac{\partial H}{\partial \mathbf{q}}+\mathbf{Q}_{ex}-\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}$$
$$\boxed{\dot{\mathbf{p}}=L_{\mathbf{q}}+\mathbf{Q}_{ex}-\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}}$$

Example: Repeat Ex. 2 using the canonical equations

$$T=\frac{1}{2}\begin{bmatrix}\dot q_1 & \dot q_2\end{bmatrix}\begin{bmatrix}m_1 & 0\\ 0 & m_2L^2+m_1q_1^2\end{bmatrix}\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix}$$

$$V=(m_1q_1+m_2L)g\sin q_2$$



$$\boxed{\mathbf{p}=\mathbf{M}\dot{\mathbf{q}}}$$

$$\begin{bmatrix}p_1\\ p_2\end{bmatrix}=\begin{bmatrix}m_1 & 0\\ 0 & m_2L^2+m_1q_1^2\end{bmatrix}\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix}$$

Using independent coordinates

$$\boxed{\dot{\mathbf{p}}=L_q+\cancel{\mathbf{Q}_{ex}}-\cancel{\mathbf{\Phi}_q^T\boldsymbol{\lambda}}}$$

$$T_q=\begin{bmatrix}\frac{\partial T}{\partial q_1}\\ \frac{\partial T}{\partial q_2}\end{bmatrix}=\begin{bmatrix}m_1q_1\dot q_2^2\\ 0\end{bmatrix}$$

$$V_q=\begin{bmatrix}\frac{\partial V}{\partial q_1}\\ \frac{\partial V}{\partial q_2}\end{bmatrix}$$

$$\begin{bmatrix}\dot p_1\\ \dot p_2\end{bmatrix}=T_q-V_q=\underbrace{\begin{bmatrix}m_1q_1\dot q_2^2\\ 0\end{bmatrix}}_{T_{q1},\ T_{q2}}-\underbrace{\begin{bmatrix}m_1g\sin q_2\\ (m_1q_1+m_2L)g\cos q_2\end{bmatrix}}_{V_{q1},\ V_{q2}}$$

Comparison

$$\boxed{\mathbf{p}=\mathbf{M}\dot{\mathbf{q}}}\qquad \boxed{\dot{\mathbf{p}}=L_q+\cancel{\mathbf{Q}_{ex}}-\cancel{\mathbf{\Phi}_q^T\boldsymbol{\lambda}}}$$

Canonical Equations of Hamilton
Two 1st order differential equations

$$\begin{bmatrix}p_1\\ p_2\end{bmatrix}=\begin{bmatrix}m_1 & 0\\ 0 & m_2L^2+m_1q_1^2\end{bmatrix}\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix}$$

$$\begin{bmatrix}\dot p_1\\ \dot p_2\end{bmatrix}=T_q-V_q=\begin{bmatrix}m_1q_1\dot q_2^2\\ 0\end{bmatrix}-\begin{bmatrix}m_1g\sin q_2\\ (m_1q_1+m_2L)g\cos q_2\end{bmatrix}$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}}+\cancel{\Phi_q^T\lambda}=\cancel{\mathbf{Q}_{nc}}-\dot{\mathbf{M}}\dot{\mathbf{q}}+T_q-V_q}$$

Lagrange Equation
One 2nd order differential equation

$$\begin{bmatrix}m_1 & 0\\ 0 & m_2L^2+m_1q_1^2\end{bmatrix}\begin{bmatrix}\ddot q_1\\ \ddot q_2\end{bmatrix}=\begin{bmatrix}0\\ -2m_1q_1\dot q_1\dot q_2\end{bmatrix}+\begin{bmatrix}m_1q_1\dot q_2^2\\ 0\end{bmatrix}-\begin{bmatrix}m_1g\sin q_2\\ (m_1q_1+m_2L)g\cos q_2\end{bmatrix}$$

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Conservative Systems + Independent Coordinates

$$T=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \qquad V=V(\mathbf{q})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)-\left(\frac{\partial L}{\partial \mathbf{q}}\right)+\cancel{\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}}=\cancel{\mathbf{Q}_{nc}}$$

$$L=T-V=f(\dot{\mathbf{q}},\mathbf{q},t)$$

$$\frac{dL}{dt}=\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\frac{d\dot{\mathbf{q}}}{dt}+\left(\frac{\partial L}{\partial \mathbf{q}}\right)\frac{d\mathbf{q}}{dt}$$

$$=\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\frac{d\dot{\mathbf{q}}}{dt}+\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\right\}\dot{\mathbf{q}}$$

$$=\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}}\right\}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)=\left(\frac{\partial L}{\partial \mathbf{q}}\right)$$

$$H\equiv\mathbf{p}^T\dot{\mathbf{q}}-L$$

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}}-L\right\}=0 \longleftarrow \mathbf{p}=\frac{\partial L}{\partial \dot{\mathbf{q}}}$$

$$\frac{d}{dt}\left\{\mathbf{p}^T\dot{\mathbf{q}}-L\right\}=0$$

$$\frac{dH}{dt}=0 \longrightarrow H=const.$$

- Thus, in the case of a conservative system, the Hamiltonian is a constant

The Hamiltonian for Conservative Systems

$$\mathbf{p}=\mathbf{M}\dot{\mathbf{q}} \qquad T=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}$$

$$H\equiv\mathbf{p}^T\dot{\mathbf{q}}-L=const$$

$$\mathbf{p}^T\dot{\mathbf{q}}=\dot{\mathbf{q}}^T\mathbf{p}=\dot{\mathbf{q}}^T(\mathbf{M}\dot{\mathbf{q}})=2T$$

$$\boxed{H=2T-(T-V)=T+V}$$

- This implies that, for a conservative system, the ***Hamiltonian*** is the sum of the kinetic and potential energies of the system and ***it remains constant throughout the system motion***

Example

Figure 24 shows a homogeneous circular cylinder of radius r, mass m, and mass moment of inertia J about its center of mass, where $J = m(r)^2/2$. The cylinder rolls without slipping on a curved surface of radius R. Use the principle of conservation of energy to derive the equation of motion of the cylinder.

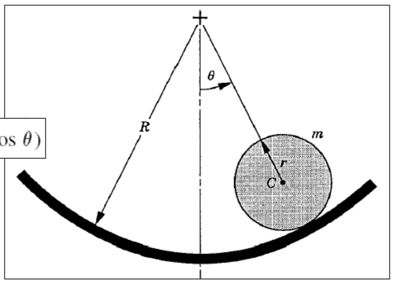
$$\boxed{v_c = (R-r)\dot{\theta}} \quad \boxed{\omega = \frac{v_c}{r} = \frac{(R-r)\dot{\theta}}{r}}$$

$$\boxed{T = \tfrac{1}{2}m(v_c)^2 + \tfrac{1}{2}J(\omega)^2} \boxed{= \tfrac{3}{4}\, m(R-r)^2(\dot{\theta})^2}$$

$$\boxed{V = mg(R-r)(1-\cos\theta)}$$

$$\boxed{H = T + V = \tfrac{3}{4}\, m(R-r)^2(\dot{\theta})^2 + mg(R-r)(1-\cos\theta)}$$

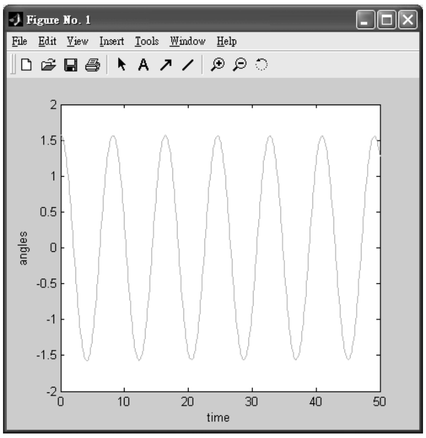
$$\boxed{\frac{dH}{dt} = \frac{3}{2}\, m(R-r)^2\dot{\theta}\ddot{\theta} + mg(R-r)\dot{\theta}\,\sin\theta = 0}$$

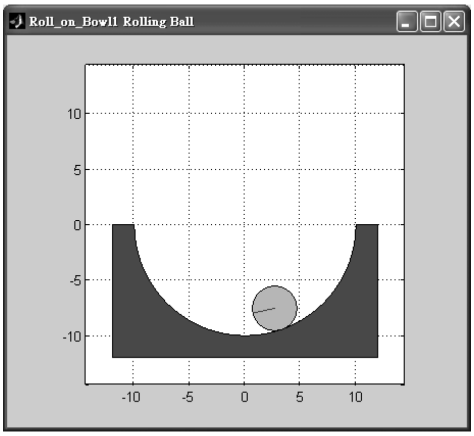


∴The equation of motion of the cylinder is

$$\boxed{\tfrac{3}{2}(R-r)\ddot{\theta} + g\,\sin\theta = 0}$$

Implementation with Matlab





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Formulation Based on the Canonical Equations (Formulation in dependent Coordinates)

$\boxed{\mathbf{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda} \neq 0}$

$\boxed{T = 1/2\ \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\ \dot{\mathbf{q}}}$ $\boxed{V = V(\mathbf{q})}$

$$\left.\begin{array}{l} \boxed{L^{*} = T - V + \dot{\Phi}^{T}\sigma = \frac{1}{2}\dot{\mathbf{q}}^{T}\mathbf{M}\dot{\mathbf{q}} - V(\mathbf{q}) + \dot{\Phi}^{T}\sigma} \\ \boxed{\mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{q}}}} \end{array}\right\} \boxed{\mathbf{p} = \mathbf{M}\dot{\mathbf{q}} + \Phi_{\mathbf{q}}^{T}\boldsymbol{\sigma}}$$

$$\boxed{H = \mathbf{p}^{T}\dot{\mathbf{q}} - L}$$

$$\Downarrow$$

$$\left.\begin{array}{l} \boxed{\frac{\partial H}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}}(\mathbf{p}^{T}\dot{\mathbf{q}} - L) = -\frac{\partial L}{\partial \mathbf{q}} = -L_{\mathbf{q}}} \\ \boxed{\frac{\partial H}{\partial \mathbf{q}} = \mathbf{Q}_{ex} - \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} - \dot{\mathbf{p}}} \end{array}\right\} \boxed{\dot{\mathbf{p}} = L_{\mathbf{q}} + \mathbf{Q}_{ex} + \dot{\mathbf{\Phi}}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\sigma}}$$

Solving Procedures

1. Start at time t when **p** and **q** are known
2. Solve for dq/dt and σ at time t

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}} \\ \mathbf{\Phi}_{\mathbf{q}} & 0 \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}} \\ -\boldsymbol{\sigma} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p} \\ 0 \end{Bmatrix}$$

$$\boxed{\mathbf{p} = \mathbf{M}\dot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\sigma}}$$

$$\boxed{\dot{\mathbf{\Phi}} = \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}}$$

3. Compute $\boxed{\dot{\mathbf{p}} = L_{\mathbf{q}} + \mathbf{Q}_{ex} + \dot{\mathbf{\Phi}}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\sigma}}$

4. Obtain the vectors **p** and **q** at time t+Δt by the numerical integration:
5. Update the time variable and go to step 2

Runge-Kutta Algorithm

$$\dot{y} = g \Rightarrow dy = gdt$$

$$Euler: y^{i+1} - y^{i+1} \cong g\Delta t$$

$$y^{i+1} = y^i + hg$$

$$g = \tfrac{1}{6}(f_1 + 2f_2 + 2f_3 + f_4)$$

$$f_1 = f(y^i, t^i)$$

$$f_2 = f\left(y^i + \frac{h}{2}f_1, t^i + \frac{h}{2}\right)$$

$$f_3 = f\left(y^i + \frac{h}{2}f_2, t^i + \frac{h}{2}\right)$$

$$f_4 = f\left(y^i + hf_3, t^i + h\right)$$

Subroutine RUNGK4. The argument parameters in this subroutine are:

H	Time step
NSTEP	Number of time steps
N	Number of dependent variables (same as the number of differential equations)
Y	An N-vector of dependent variables **y**
F	An N-vector which upon return will contain $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$
F1, F2, F3, F4, YY	N-vectors of working arrays

Runge-Kutta Algorithm (cont.)

```
      SUBROUTINE RUNGK4 (T,H,NSTEP,N,Y,F,F1,F2,F3,F4,YY)
      DIMENSION Y(N),F(N),F1(N),F2(N),F3(N),F4(N),YY(N)
      HH=0.5*H
      TS=T
      WRITE (1,200)
      DO 100 I=1,NSTEP
        WRITE (1,210) T,(Y(J),J=1,N)
        CALL DIFEQN (T,N,Y,F)              求dY/dt (return in F)
        DO 10 J=1.N
10        F1(J)=H*F(J)
        TT=T+HH
        DO 20 J=1,N
20        YY(J)=Y(J)+0.5*F1(J)
        CALL DIFEQN (TT,N,YY,F)            求dY/dt
        DO 30 J=1,N
          F2(J)=H*F(J)
30        YY(J)=Y(J)+0.5*F2(J)
        CALL DIFEQN (TT,N,YY,F)            求dY/dt
        TT=T+H
        DO 40 J=1,N
          F3(J)=H*F(J)
40        YY(J)=Y(J)+F3(J)
        CALL DIFEQN (TT,N,YY,F)            求dY/dt
        T=TS+H*FLOAT(I)
        DO 50 J=1,N
          F4(J)=H*F(J)
50        Y(J)=Y(J)+(F1(J)+2.0*F2(J)+2.0*F3(J)+F4(J))/6.0
100   CONTINUE
200   FORMAT (5X,'    TIME     Y')      設定列印格式
210   FORMAT (5X,4F10.6)
      RETURN
```

$$g = \tfrac{1}{6}(f_1 + 2f_2 + 2f_3 + f_4) \qquad y^{i+1} = y^i + hg$$

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