Euler Parameters

Reading material: Chapter 6

What is Euler Parameters?

- Euler parameters can be considered as an example of the quaternions (四元數)
- Euler parameters are a normalized form of parameters known as quaternions (p. 153)
- Euler parameters are widely used in general purpose multibody computer programs
  - *to avoid the singularities associated with Euler angles*
  - *to reduce memory in representing rotations ,*
  - *to reduce computation time in actually rotating or updating the equations of motion*

$$\mathbf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}_{4\times 1} = \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix}$$

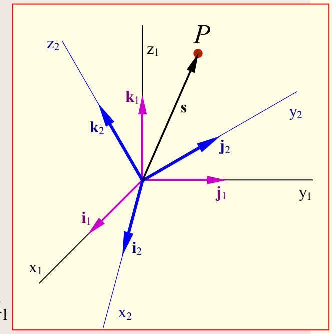
e_0 : angle of rotation
e : an unit vector along the axis of rotation

(Review)
The configuration of the body axes w.r.t the global axes

$\mathbf{P} = x_1\,\mathbf{i}_1 + y_1\,\mathbf{j}_1 + z_1\,\mathbf{k}_1,$ in CS1 Eq.(a)
$\mathbf{P} = x_2\,\mathbf{i}_2 + y_2\,\mathbf{j}_2 + z_2\,\mathbf{k}_2,$ in CS2 Eq.(b)

$$\mathbf{i}_2 = (\mathbf{i}_2 \cdot \mathbf{i}_1)\,\mathbf{i}_1 + (\mathbf{i}_2 \cdot \mathbf{j}_1)\,\mathbf{j}_1 + (\mathbf{i}_2 \cdot \mathbf{k}_1)\,\mathbf{k}_1$$
$$\mathbf{j}_2 = (\mathbf{j}_2 \cdot \mathbf{i}_1)\,\mathbf{i}_1 + (\mathbf{j}_2 \cdot \mathbf{j}_1)\,\mathbf{j}_1 + (\mathbf{j}_2 \cdot \mathbf{k}_1)\,\mathbf{k}_1$$
$$\mathbf{j}_3 = (\mathbf{k}_2 \cdot \mathbf{i}_1)\,\mathbf{i}_1 + (\mathbf{k}_2 \cdot \mathbf{j}_1)\,\mathbf{j}_1 + (\mathbf{k}_2 \cdot \mathbf{k}_1)\,\mathbf{k}_1$$

$$\begin{aligned}\therefore \mathbf{P} = \; & [\,x_2\,(\mathbf{i}_2 \cdot \mathbf{i}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{i}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{i}_1)\,]\;\mathbf{i}_1\\ & +[\,x_2\,(\mathbf{i}_2 \cdot \mathbf{j}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{j}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{j}_1)\,]\;\mathbf{j}_1\\ & +[\,x_2\,(\mathbf{i}_2 \cdot \mathbf{k}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{k}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{k}_1)\,]\;\mathbf{k}_1\end{aligned}$$
…….Eq.(c)



Comparing Eqs. (c) and (a) leads to

$$x_1 = (\mathbf{i}_2 \cdot \mathbf{i}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{i}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{i}_1)$$
$$y_1 = (\mathbf{i}_2 \cdot \mathbf{j}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{j}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{j}_1)$$
$$z_1 = (\mathbf{i}_2 \cdot \mathbf{k}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{k}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{k}_1)$$

(Review)
The configuration of the body axes w.r.t the global axes (pure rotation)

$$x_1 = (\mathbf{i}_2 \cdot \mathbf{i}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{i}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{i}_1)$$
$$y_1 = (\mathbf{i}_2 \cdot \mathbf{j}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{j}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{j}_1)$$
$$z_1 = (\mathbf{i}_2 \cdot \mathbf{k}_1) + y_2\,(\mathbf{j}_2 \cdot \mathbf{k}_1) + z_2(\mathbf{k}_2 \cdot \mathbf{k}_1)$$

$$\boxed{\mathbf{s} = \mathbf{A}\mathbf{s}'}$$

s= $[x_1,y_1,z_1]^T$ =*coordinate vector in terms of world coordinates*
s' = $[x_2,y_2,z_2]^T$ = *coordinate vector in terms of body coordinates*
A: *a rotation matrix representing the orientation of the moving body*

$$\mathbf{A} = \begin{bmatrix} (\mathbf{i}_2 \bullet \mathbf{i}_1) & (\mathbf{j}_2 \bullet \mathbf{i}_1) & (\mathbf{k}_2 \bullet \mathbf{i}_1) \\ (\mathbf{i}_2 \bullet \mathbf{j}_1) & (\mathbf{j}_2 \bullet \mathbf{j}_1) & (\mathbf{k}_2 \bullet \mathbf{j}_1) \\ (\mathbf{i}_2 \bullet \mathbf{k}_1) & (\mathbf{j}_2 \bullet \mathbf{k}_1) & (\mathbf{k}_2 \bullet \mathbf{k}_1) \end{bmatrix}$$

- Each column of A is a unit vector, and the unit vectors are orthogonal to each other.
- Similarly, it can be easily derived that $\mathbf{s}' = \mathbf{A}^T\mathbf{s}$
- Thus, **A** is a orthonormal matrix, i.e., $A^{-1}=A^T$

$$\boxed{\mathbf{s}' = \mathbf{A}^{-1}\mathbf{s} = \mathbf{A}^T\mathbf{s}}$$

Practical Consideration

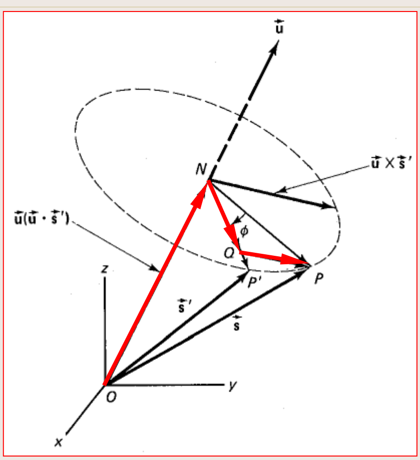
- It is seen that the nine direction cosines in matrix **A** define the orientation of a body relative to the global coordinate system.
- **A** must be a orthogonal matrix, i.e., $A^{-1}=A^T$, or $AA^T=I$

$$\mathbf{M}=\mathbf{A}\mathbf{A}^T=\begin{bmatrix}(\mathbf{i}_2\bullet\mathbf{i}_1) & (\mathbf{j}_2\bullet\mathbf{i}_1) & (\mathbf{k}_2\bullet\mathbf{i}_1)\\ (\mathbf{i}_2\bullet\mathbf{j}_1) & (\mathbf{j}_2\bullet\mathbf{j}_1) & (\mathbf{k}_2\bullet\mathbf{j}_1)\\ (\mathbf{i}_2\bullet\mathbf{k}_1) & (\mathbf{j}_2\bullet\mathbf{k}_1) & (\mathbf{k}_2\bullet\mathbf{k}_1)\end{bmatrix}\begin{bmatrix}(\mathbf{i}_2\bullet\mathbf{i}_1) & (\mathbf{i}_2\bullet\mathbf{j}_1) & (\mathbf{i}_2\bullet\mathbf{k}_1)\\ (\mathbf{j}_2\bullet\mathbf{i}_1) & (\mathbf{j}_2\bullet\mathbf{j}_1) & (\mathbf{j}_2\bullet\mathbf{k}_1)\\ (\mathbf{k}_2\bullet\mathbf{i}_1) & (\mathbf{k}_2\bullet\mathbf{j}_1) & (\mathbf{k}_2\bullet\mathbf{k}_1)\end{bmatrix}=\mathbf{I}$$

$$\Rightarrow\left\{\begin{array}{l}\mathbf{M}_{11}=1\\ \mathbf{M}_{22}=1\\ \mathbf{M}_{33}=1\\ \mathbf{M}_{12}=\mathbf{M}_{21}=(\mathbf{i}_2\bullet\mathbf{i}_1)(\mathbf{i}_2\bullet\mathbf{j}_1)+(\mathbf{j}_2\bullet\mathbf{i}_1)(\mathbf{j}_2\bullet\mathbf{j}_1)+(\mathbf{k}_2\bullet\mathbf{i}_1)(\mathbf{k}_2\bullet\mathbf{j}_1)=0\\ \mathbf{M}_{13}=\mathbf{M}_{31}=0\\ \mathbf{M}_{23}=\mathbf{M}_{32}=0\end{array}\right\}$$

- It is impractical to specify the nine direction cosines to satisfy the six constraints
- Only three direction cosines are independent

Express Matrix A in terms of Euler Parameters



$$\mathbf{p}=\begin{bmatrix}e_0\\ e_1\\ e_2\\ e_3\end{bmatrix}_{4\times 1}=\begin{bmatrix}e_0\\ \mathbf{e}\end{bmatrix}$$

$$\vec{s}=\overrightarrow{ON}+\overrightarrow{NQ}+\overrightarrow{QP}$$

Express Matrix A in terms of Euler Parameters

$$\boxed{\vec{s} = \overrightarrow{ON} + \overrightarrow{NQ} + \overrightarrow{QP}}$$

$$\overrightarrow{ON} = \boxed{\vec{u}(\vec{u}\cdot\vec{s}')}$$

$$\overrightarrow{NQ} = [\vec{s}' - \vec{u}(\vec{u}\cdot\vec{s}')]\boxed{\cos\phi}$$

$$\overrightarrow{QP} = \boxed{\vec{u}\times\vec{s}'\sin\phi}$$

$$\boxed{\overrightarrow{NQ} = m\cos\phi\frac{\overrightarrow{NP'}}{\overline{NP'}} = \overrightarrow{NP'}\cos\phi}$$

$$\boxed{\overrightarrow{QP} = m\sin\phi\frac{\vec{u}\times\vec{s}'}{\left|\vec{u}\times\vec{s}'\right|} = \vec{u}\times\vec{s}'\sin\phi}$$

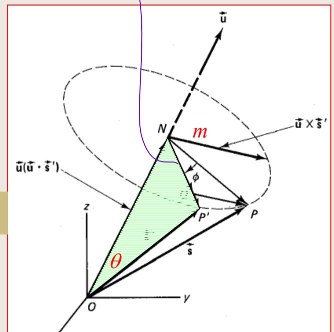
$$\vec{s} = \vec{s}'\boxed{\cos\phi} + \vec{u}(\vec{u}\cdot\vec{s}')(\boxed{1} - \boxed{\cos\phi}) + \boxed{\vec{u}\times\vec{s}'\sin\phi}$$

$$\cos\phi = 2\cos^2\frac{\phi}{2} - 1$$

$$\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$$

$$1 - \cos\phi = 2\sin^2\frac{\phi}{2}$$

$$\boxed{e_0 = \cos\frac{\phi}{2} \qquad \vec{e} = \vec{u}\sin\frac{\phi}{2}}$$



radius $m = s'\sin\theta = |\mathbf{u}\times\mathbf{s}'| = NP'$

幾何向量

$$\vec{s} = (2e_0^2 - 1)\vec{s}' + 2\vec{e}(\vec{e}\cdot\vec{s}') + 2e_0\vec{e}\times\vec{s}'$$

代數矩陣

$$\mathbf{s} = (2e_0^2 - 1)\mathbf{s}' + 2\mathbf{e}(\mathbf{e}^T\mathbf{s}') + 2e_0(\tilde{\mathbf{e}}\mathbf{s}')$$

$$\mathbf{s} = [(2e_0^2 - 1)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0\tilde{\mathbf{e}}]\mathbf{s}'$$

Transform Euler Parameters to Matrix A

$$\left.\begin{array}{l}\mathbf{s} = [(2e_0^2 - 1)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0\tilde{\mathbf{e}}]\mathbf{s}' \\ \mathbf{s} = \mathbf{A}\mathbf{s}'\end{array}\right\} \quad \mathbf{A} = (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})$$

- **Thus, matrix A can be easily determined from specified Quaternion/Euler parameters**

$$\mathbf{A} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

$$\mathbf{A}^T = (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T - e_0\tilde{\mathbf{e}})$$

$$\mathbf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}_{4\times1} = \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \cos\frac{\phi}{2} \\ \mathbf{u}\sin\frac{\phi}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\phi}{2} \\ u_x\sin\frac{\phi}{2} \\ u_y\sin\frac{\phi}{2} \\ u_z\sin\frac{\phi}{2} \end{bmatrix}$$

Some Properties about Euler Parameters

- The four Euler parameters are not independent, because

$$e_0^2 + \mathbf{e}^T\mathbf{e} = \cos^2(\frac{\phi}{2}) + \sin^2(\frac{\phi}{2}) = 1 \Rightarrow \mathbf{e}^T\mathbf{e} = 1 - e_0^2$$

$$e_0 = \cos\frac{\phi}{2}, \quad \vec{e} = \vec{u}\sin\frac{\phi}{2}$$

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

- Any vector lying along the rotation axis must have the same components in both initial and final coordinate systems (**e** lies along the rotation axis)

$$\mathbf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}_{4\times 1} = \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix}$$

$$\begin{aligned} \mathbf{e}' &= \mathbf{A}^T\mathbf{e} \\ &= (2e_0^2 - 1)\mathbf{e} + 2(\mathbf{e}\mathbf{e}^T - e_0\tilde{\mathbf{e}})\mathbf{e} \\ &= (2e_0^2 - 1)\mathbf{e} + 2\mathbf{e}(1 - e_0^2) \\ &= (2e_0^2 - 1)\mathbf{e} + 2(1 - e_0^2)\mathbf{e} \\ &= (2e_0^2 - 1 + 2 - 2e_0^2)\mathbf{e} \\ &= \mathbf{e} \end{aligned}$$

$$\mathbf{A} = (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})$$

$$\mathbf{A}^T = (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T - e_0\tilde{\mathbf{e}})$$

Determine Euler Parameters from Rotation Matrix A

- Assume that the nine direction cosines of a transformation matrix are given as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{A} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

- Define:

$$\begin{aligned} \operatorname{tr}\mathbf{A} &= a_{11} + a_{22} + a_{33} \\ &= 2(3e_0^2 + e_1^2 + e_2^2 + e_3^2) - 3 \\ &= 2(2e_0^2 + 1) - 3 \\ &= 4e_0^2 - 1 \end{aligned}$$

$$\therefore \quad e_0^2 = \frac{\operatorname{tr}\mathbf{A} + 1}{4}$$

Determine Euler Parameters from Rotation Matrix A ($e_0 \neq 0$)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} a_{32} - a_{23} &= 4e_0e_1 \\ a_{13} - a_{31} &= 4e_0e_2 \\ a_{21} - a_{12} &= 4e_0e_3 \end{aligned}$$

$$e_0^2 = \frac{\operatorname{tr}\mathbf{A} + 1}{4}$$

- If $e_0 \neq 0$, then

$$e_1 = \frac{a_{32} - a_{23}}{4e_0}$$

$$e_2 = \frac{a_{13} - a_{31}}{4e_0}$$

$$e_3 = \frac{a_{21} - a_{12}}{4e_0}$$

- The sign of e_0 may be selected as positive or negative
- Changing the sign of e_0 does not influence the rotation matrix

Example: Determine Euler Parameters from Rotation Matrix A ($e_0 \neq 0$)

$$\mathbf{A} = \begin{bmatrix} 0.5449 & -0.5549 & 0.6285 \\ 0.3111 & 0.8299 & 0.4629 \\ -0.7785 & -0.0567 & 0.6249 \end{bmatrix}$$

Determine the four Euler parameters corresponding to this transformation

Solution The trace of **A** is calculated from Eq. 6.24:

$$\operatorname{tr}\mathbf{A} = 0.5449 + 0.8299 + 0.6249 = 1.9997$$

Then, Eq. 6.25 yields $e_0^2 = 0.7499$. Selecting the positive sign for e_0, we find that $e_0 = 0.866$. From Eq. 6.27,

$$e_1 = \frac{-0.0567 - 0.4629}{4.0(0.866)} = -0.15$$

$$e_2 = \frac{0.6285 + 0.7785}{4.0(0.866)} = 0.406$$

$$e_3 = \frac{0.3111 + 0.5549}{4.0(0.866)} = 0.25$$

A test can be performed to check that the four parameters satisfy the constraint of Eq. 6.21. Either the four parameters are $\mathbf{p} = [0.866, -0.15, 0.406, 0.25]^T$, or, if the sign of e_0 is changed, the four parameters become $\mathbf{p} = [-0.866, 0.15, -0.406, -0.25]^T$.

Determine Euler Parameters from Rotation Matrix A (e_0 =0)

$$a_{11} = 2(e_0^2 + e_1^2) - 1 = 2\left(\frac{\mathrm{tr}\,\mathbf{A} + 1}{4} + e_1^2\right) - 1 \quad\Rightarrow\quad e_1^2 = \frac{1 + 2a_{11} - \mathrm{tr}\,\mathbf{A}}{4}$$

$$e_0^2 = \frac{\mathrm{tr}\,\mathbf{A} + 1}{4}$$

similarly,

$$e_2^2 = \frac{1 + 2a_{22} - \mathrm{tr}\,\mathbf{A}}{4} \qquad e_3^2 = \frac{1 + 2a_{33} - \mathrm{tr}\,\mathbf{A}}{4}$$

- Since $e_0 = \cos(\phi/2) = 0$ ➔ $\phi = \pm\pi, \pm 3\pi$
- Suppose e_1, e_2, or e_3 is nonzero (determined from one of the above equations), its sign may be selected as positive or negative
- Then, the other two parameters can be determined by the following equations:

$$a_{21} + a_{12} = 4e_1e_2$$
$$a_{31} + a_{13} = 4e_1e_3$$
$$a_{32} + a_{23} = 4e_2e_3$$

Example: Determine Euler Parameters from Rotation Matrix A (e_0 =0)

Determine the four Euler parameters for transformation matrix

$$\mathbf{A} = \begin{bmatrix} -0.280 & -0.600 & -0.749 \\ -0.600 & -0.500 & 0.625 \\ -0.749 & 0.625 & -0.220 \end{bmatrix}$$

Solution The trace of **A** is found from Eq. 6.24:

$$\mathrm{tr}\,\mathbf{A} = -0.280 - 0.500 - 0.220 = -1.0$$

Then, Eq. 6.25 yields $e_0 = 0.0$. From Eq. 6.26 it is found that

$$e_1^2 = \frac{1.0 + 2.0(-0.28) + 1.0}{4.0} = 0.36$$

Therefore, $e_1 = \pm 0.6$. If the positive sign is selected for e_1, then, Eq. 6.28 yields

$$e_2 = \frac{-0.6 - 0.6}{4.0(0.6)} = -0.5$$

$$e_3 = \frac{-0.749 - 0.749}{4.0(0.6)} = -0.624$$

The vector of the Euler parameters is $\mathbf{p} = [0.0, 0.6, -0.5, -0.624]^T$ or $\mathbf{p} = [0.0, -0.6, 0.5, 0.624]^T$.

Identities with Euler Parameters

- Define

$$\mathbf{G} = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} = [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}]$$

$$\mathbf{L} = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix} = [-\mathbf{e}, -\tilde{\mathbf{e}} + e_0\mathbf{I}]$$

- Each row of G and L is orthogonal to p=[e_0,**e**]^T; i.e.,

$$\mathbf{Gp} = [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}]\begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix} = [-e_0\mathbf{e} + \tilde{\mathbf{e}}\mathbf{e} + e_0\mathbf{e}] = \mathbf{0}$$

$$\mathbf{Lp} = \mathbf{0}$$

$\mathbf{Gp} = \mathbf{0} = \mathbf{p}^T\mathbf{G}^T$ (6.40)

$\mathbf{G}^T\mathbf{G}=\mathbf{L}^T\mathbf{L}=-\mathbf{p}\mathbf{p}^T+\mathbf{I}^*$ (6.46)

$\mathbf{A}=\mathbf{G}\mathbf{L}^T$ (6.49)

$d\mathbf{A}/dt=2(d\mathbf{G}/dt)\mathbf{L}^T$ (6.56)

$\widetilde{\mathbf{G}\dot{\mathbf{p}}} = -\mathbf{G}\dot{\mathbf{G}}^T$ (6.57)

$\mathbf{G}\dot{\mathbf{G}}^T = -\dot{\mathbf{G}}\mathbf{G}^T$ (6.59)

Identities with Arbitrary Vector s

$$\boxed{\mathbf{A}\tilde{\mathbf{s}}' = \tilde{\mathbf{s}}\mathbf{A}}$$

$$\boxed{\tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T}$$

$$\overset{+}{\mathbf{a}} \equiv \begin{bmatrix} 0 & -\mathbf{a}^T \\ \mathbf{a} & \tilde{\mathbf{a}} \end{bmatrix}$$

$$\overset{-}{\mathbf{a}} \equiv \begin{bmatrix} 0 & -\mathbf{a}^T \\ \mathbf{a} & -\tilde{\mathbf{a}} \end{bmatrix}$$

Time Derivative of a Vector Attached to a Moving Body (P.172)

- The global location of a point *P* that is fixed in the local coordinate system is given by

$$\boxed{\mathbf{s} = \mathbf{A}\mathbf{s}'}$$

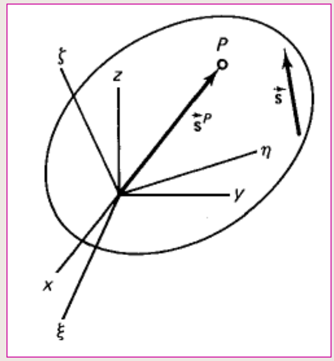
$$\downarrow$$

$$\boxed{\dot{\mathbf{s}}^P = \dot{\mathbf{A}}\mathbf{s}'^P + \mathbf{A}\cancelto{0}{\dot{\mathbf{s}}'^P}}$$

$$\downarrow$$

$$\boxed{\dot{\mathbf{s}}^P = \dot{\mathbf{A}}\mathbf{s}'^P}$$

$$\boxed{\dot{\mathbf{A}} = ?}$$



Time Derivative of Rotation Matrix A (pp.173-174)

Let $\boxed{\dot{\mathbf{A}} = \mathbf{\Omega}\mathbf{A}}$ $\boxed{\mathbf{\Omega}\text{ Is an unknown matrix}}$

$$\boxed{\because \mathbf{A}^T\mathbf{A} = \mathbf{I}}$$

$$\downarrow$$

$$\boxed{\dot{\mathbf{A}}^T\mathbf{A} + \mathbf{A}^T\dot{\mathbf{A}} = \mathbf{0}}$$

$$\downarrow$$

$$\boxed{\mathbf{A}^T\mathbf{\Omega}^T\mathbf{A} + \mathbf{A}^T\mathbf{\Omega}\mathbf{A} = \mathbf{0}}$$

$$\downarrow$$

$$\boxed{\mathbf{A}(\mathbf{A}^T\mathbf{\Omega}^T\mathbf{A} + \mathbf{A}^T\mathbf{\Omega}\mathbf{A})\mathbf{A}^T = \mathbf{0}}$$

$$\downarrow$$

$$\boxed{\mathbf{\Omega}^T + \mathbf{\Omega} = \mathbf{0}}$$

$$\downarrow$$

$$\boxed{\mathbf{\Omega}^T = -\mathbf{\Omega}}$$

$\therefore \mathbf{\Omega}$ is a skew-symmetric matrix, and we can assume that

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \tilde{\boldsymbol{\omega}}$$

$$\boxed{\therefore \dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A}}$$

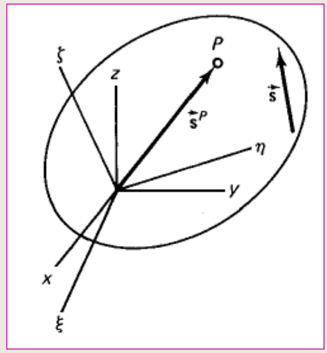
What is the physical meaning of ω ?

Time Derivative of Any Vector s Attached to a Moving Body (pp.173-174)

$$\dot{\mathbf{s}}^P = \dot{\mathbf{A}}\mathbf{s}'^P \qquad \dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A}$$

$$\begin{aligned}\dot{\mathbf{s}}^P &= \tilde{\omega}\mathbf{A}\mathbf{s}'^P \\ &= \tilde{\omega}\mathbf{s}^P\end{aligned}$$

$$\mathbf{s}^P = \mathbf{A}\mathbf{s}'^P$$



∴Any vector s attached to the local coordinate system can be written as

$$\boxed{\dot{\mathbf{s}} = \tilde{\boldsymbol{\omega}}\mathbf{s} \Leftrightarrow \dot{\vec{\mathbf{s}}} = \boldsymbol{\omega} \times \vec{\mathbf{s}}}$$

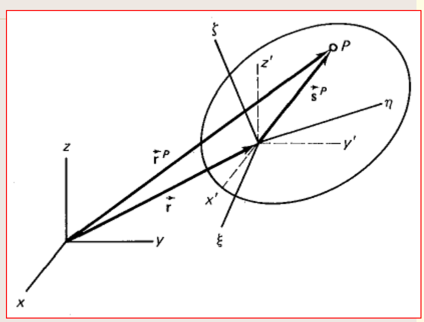
Thus, ω is the angular velocity of the moving body

Velocity of a Point on Moving Body

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P$$

$$\begin{aligned}\dot{\mathbf{r}}^P &= \dot{\mathbf{r}} + \dot{\mathbf{s}}^P \\ &= \dot{\mathbf{r}} + \tilde{\omega}\mathbf{s}^P\end{aligned}$$

$$\dot{\mathbf{s}}^P = \tilde{\boldsymbol{\omega}}\mathbf{s}^P$$



Express Angular Velocity in terms of Quaternion and Its Time Derivative (P.174)

$$\boxed{\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A}}$$

$$\boxed{\dot{\mathbf{A}}\mathbf{A}^T = \tilde{\boldsymbol{\omega}}\mathbf{A}\mathbf{A}^T = \tilde{\boldsymbol{\omega}}}$$

$$\dot{\mathbf{A}} = \dot{\mathbf{G}}\mathbf{L}^T + \mathbf{G}\dot{\mathbf{L}}^T = 2\mathbf{G}\dot{\mathbf{L}}^T = 2\dot{\mathbf{G}}\mathbf{L}^T \qquad (6.56)$$

$$\mathbf{A} = \mathbf{G}\mathbf{L}^T \qquad (6.49)$$

$$\boxed{2\dot{\mathbf{G}}\mathbf{L}^T\mathbf{L}\mathbf{G}^T = \tilde{\boldsymbol{\omega}}}$$

$$\mathbf{L}^T\mathbf{L} = -\mathbf{p}\mathbf{p}^T + \mathbf{I}^* \qquad (6.46)$$

$$\boxed{2\dot{\mathbf{G}}(-\mathbf{p}\mathbf{p}^T + \mathbf{I})\mathbf{G}^T = \tilde{\boldsymbol{\omega}}}$$

$$\mathbf{G}\mathbf{p} = \mathbf{0} = \mathbf{p}^T\mathbf{G}^T \qquad (6.40)$$

$$\boxed{2\dot{\mathbf{G}}\mathbf{G}^T = \tilde{\boldsymbol{\omega}}}$$

$$\mathbf{G}\dot{\mathbf{G}}^T = -\dot{\mathbf{G}}\mathbf{G}^T \qquad (6.59)$$

$$\widetilde{\mathbf{G}\dot{\mathbf{p}}} = -\mathbf{G}\dot{\mathbf{G}}^T \qquad (6.57)$$

$$\boxed{2\widetilde{\mathbf{G}\dot{\mathbf{p}}} = \tilde{\boldsymbol{\omega}}}$$

$$\boxed{\boldsymbol{\omega} = 2\mathbf{G}\dot{\mathbf{p}}} \Longleftrightarrow \begin{bmatrix}\omega_{(x)}\\ \omega_{(y)}\\ \omega_{(z)}\end{bmatrix} = 2\begin{bmatrix}-e_1 & e_0 & -e_3 & e_2\\ -e_2 & e_3 & e_0 & -e_1\\ -e_3 & -e_2 & e_1 & e_0\end{bmatrix}\begin{bmatrix}\dot{e}_0\\ \dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix}$$

Express Angular Velocity in terms of Quaternion and Its Time Derivative (P.175)

$$\boxed{\boldsymbol{\omega} = 2\mathbf{G}\dot{\mathbf{p}}}$$

$$\begin{bmatrix}\omega_{(x)}\\ \omega_{(y)}\\ \omega_{(z)}\end{bmatrix} = 2\begin{bmatrix}-e_1 & e_0 & -e_3 & e_2\\ -e_2 & e_3 & e_0 & -e_1\\ -e_3 & -e_2 & e_1 & e_0\end{bmatrix}\begin{bmatrix}\dot{e}_0\\ \dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix}$$

global coordinates

$$\boxed{\boldsymbol{\omega}' = 2\mathbf{L}\dot{\mathbf{p}}}$$

$$\begin{bmatrix}\omega_{(\xi)}\\ \omega_{(\eta)}\\ \omega_{(\zeta)}\end{bmatrix} = 2\begin{bmatrix}-e_1 & e_0 & e_3 & -e_2\\ -e_2 & -e_3 & e_0 & e_1\\ -e_3 & e_2 & -e_1 & e_0\end{bmatrix}\begin{bmatrix}\dot{e}_0\\ \dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix}$$

local coordinates

Express Time Derivative of Quaternion in terms Angular Velocity (P.175)

$$\boxed{\dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\boldsymbol{\omega}}$$

$$\boxed{\dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{L}^T\boldsymbol{\omega}'}$$

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^T\boldsymbol{\omega} = \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & e_3 & -e_2 \\ -e_3 & e_0 & e_1 \\ e_2 & -e_1 & e_0 \end{bmatrix}_{4\times 3}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{3\times 1}$$

global coordinates

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{L}^T\boldsymbol{\omega}' = \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{bmatrix}_{4\times 3}\begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}_{3\times 1}$$

local coordinates