Bivariate Option Pricing Using Dynamic Copula Models

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Abstract

This paper examines the behavior of bivariate option prices in the presence of association between the underlying assets. Parametric families of copulas offering various alternatives to the normal dependence structure are used to model this association, which is explicitly assumed to vary over time as a function of the volatilities of the assets. These dynamic copula models are applied to better-of-two-markets and worse-of-two-markets options on the S&P500 and Nasdaq indexes. Results show that option prices implied by dynamic copula models differ substantially from prices implied by models that fix the dependence between the underlyings, particularly in times of high volatilities. Furthermore, the normal copula produces option prices that differ significantly from non-normal copula prices, irrespective of initial volatility levels. Within the class of non-normal copula families considered, option prices are robust with respect to the copula choice.

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1 Introduction

In today's economy, multivariate (or rainbow) options are viewed as excellent tools for hedging the risk of multiple assets. These options, which are written on two or more underlying securities or indexes, usually take the form of calls (or puts) that give the right to buy (or sell) the best or worst performer of a number of underlying assets. Other examples include forward contracts whose payoff is equal to that of the best or worst performer of its underlyings, and spread options on the difference between the prices of two assets.

One of the key determinants in the valuation of multivariate options is the dependence between the underlying assets. Consider for instance a bivariate call-on-max option, namely a contract that gives the holder the right to purchase the more valuable of two underlying assets for a prespecified strike price. Intuitively, the value of such an option should be smaller if the underlyings tend to move together than when they move in opposite directions. More generally, the dependence between the underlyings could change over time. Accounting for time variation in the dependence structure between assets should prove helpful in providing a more realistic valuation of multivariate options.

Over the years, various generalizations of the Black–Scholes (1973) Brownian motion framework have been used to model multivariate option prices. Examples include Margrabe (1978), Stulz (1982), Johnson (1987), Reiner (1992), and Shimko (1994). In these papers, the dependence between assets is modelled by their correlation. However, unless asset returns are well represented by a multivariate normal distribution, correlation is often an unsatisfactory measure of dependence; see, for instance, Embrechts, McNeil and Straumann (2002). Furthermore, it is a stylized fact of financial markets that correlations observed under ordinary market conditions differ substantially from correlations observed in hectic periods. In particular, asset prices have a greater tendency to move together in bad states of the economy than in quiet periods; see, for instance, Boyer, Gibson and Loretan (1999) and Patton (2003, 2004) and references therein. These "correlation breakdowns," associated with economic downturns, suggest a dynamic model of the dependence structure of asset returns.

In this paper, the relation between bivariate option prices and the dependence structure of the underlying financial assets is modelled dynamically through copulas. A copula is a multivariate distribution function each of whose marginals is uniform on the unit interval. It has been known since the work of Sklar (1959) that any multivariate continuous distribution function can be uniquely factored into its marginals and a copula. Thus while correlation measures dependence through a single number, the dependence between multiple assets is fully captured by the copula. From a practical point of view, the advantage of the copula-based approach to modelling is that appropriate marginal distributions for the components of a multivariate system can be selected by any desired method, and then linked through a copula or family of copulas suitably chosen to represent the dependence prevailing between the components.

The use of copulas to price multivariate options is not new. For example, in Rosenberg (1999), univariate options data are used to estimate marginal risk-neutral densities, which are linked with a Plackett copula to obtain a bivariate risk-neutral density from which bivariate claims are valuated. This semiparametric procedure uses a particular identifying assumption on the risk-neutral correlation to fix the copula parameter. Cherubini and Luciano (2002) extend Rosenberg's work by considering other families of copulas. In Rosenberg (2003), a risk-neutral bivariate distribution is estimated from nonparametric estimates of the marginal distributions and a nonparametric estimate of the copula.

An innovating feature of the present paper, however, is that, contrary to earlier works on multivariate option pricing, the dependence structure of the underlying assets is not treated as fixed, but rather as possibly varying over time. Taking into account this time variation is important because it may influence option prices. This paper proposes a model for the time variation of the dependence structure, in which a parametric copula is specified whose dependence parameter is allowed to change with the volatilities of the underlying assets. A distinct advantage of the parametric approach is that while the model may be misspecified, the robustness of the conclusions can easily be verified by repeating the analysis for as many different copula families as desired.

A similar dynamic-copula approach has already been used in the foreign exchange market literature by Patton (2003), who found time variation to be significant in a copula model for asymmetric dependence between two exchange rates where the dependence parameter followed a ARMA-type process. While Patton's goal was to study the effect of asymmetric dependence on portfolio returns, the objective of the present paper is very different. The main focus here is on the effect of time variation in the underlying dependence structure on the price of multivariate options.

In the empirical study presented herein, multivariate options on two important American equity index returns are considered: the S&P500 and the Nasdaq. An analysis of the results suggests that allowing for time variation in the dependence structure of the underlyings produces substantially different option prices than under constant dependence, particularly in times of increased volatility. Moreover, option prices implied by a normal dynamic dependence structure differ significantly from option prices implied by non-normal dynamic dependence structures. These findings suggest that unless the dependence between the S&P500 and Nasdaq stock indexes is well described by a normal copula, alternative copula families should be considered. Option prices turned out to be robust among the alternative—i.e., non-normal—copula models considered in this study.

The remainder of this paper is organized as follows. Section 2 describes the payoff structure of better-of-two-markets and worse-of-two-markets claims, and explains in detail the proposed dynamic-dependence option valuation scheme. The empirical results are presented in Section 3, and conclusions are given in Section 4.

2 Option valuation with time-varying dependence

Multivariate options come in a wide variety of payoff schemes. The most commonly traded options of this kind are basket options on a portfolio of assets, such as index options. Other examples include spread options, some of which are traded on commodity exchanges (see, for example, Rosenberg (1998)), or dual-strike and multivariate-digital options.

This paper concentrates on European-type options on the best (worst) performer of several assets, sometimes referred to as outperformance (underperformance) options. As these are typically traded over the counter, price data are not available. Therefore, valuation models cannot be tested empirically. However, a robustness study comparing models with different assumptions remains feasible, and this is the objective pursued herein. While the study described in the sequel is restricted to options on better- and worse-of-two-markets claims, the technique is sufficiently general to analyze the aforementioned alternative multivariate options as well, and may thus be of wider interest.

One can distinguish four types of better-of-two-markets or worse-of-two-markets claims: call options on the better performer, put options on the worse performer, call options on the worse performer, and put options on the better performer. These may be referred to as call-on-max, put-on-min, call-on-min, and put-on-max options, respectively. The expiration payoffs of these claims may be defined in terms of the underlying asset prices. For instance, the payoff of a call-on-max option would be $\max\{\max(S_1, S_2) - K, 0\}$, with S_i being the price at maturity of asset $i \in \{1, 2\}$ and K the strike price. Of course, initial asset prices need to be close for the option to make sense. For expository reasons, it is assumed here that they are exactly equal to an amount S, say, and the option premium is expressed as a percentage (in basis points, or 0.01%) of this common S. This, by linearity, can be done by valuing an option on S_1/S and S_2/S, with strike K/S. Effectively, all initial prices are normalized to unity. Using this convention, the expiration payoffs of the four types of contracts are given by:

$$\begin{array}{lcl} \text{call on max} & : & \max\{\max(R_1, R_2) - E, 0\}, \\ \text{put on min} & : & \max\{E - \min(R_1, R_2), 0\}, \\ \text{call on min} & : & \max\{\min(R_1, R_2) - E, 0\}, \\ \text{put on max} & : & \max\{E - \max(R_1, R_2), 0\}, \end{array}$$

where $R_i = S_i/S$ is the gross return at maturity on underlying $i \in \{1, 2\}$, and $E = K/S$ denotes the normalized exercise price of the option.

The proposed scheme for valuating these options is as follows. Let $r_{i,t+1} = \log R_{i,t+1}$ be the log return on index $i \in \{1, 2\}$ from time t to time $t + 1$, and let $\mathcal{I}_t = \sigma((r_{1,s}, r_{2,s}) : s \leq t)$ denote all return information available at time t. First, the objective bivariate distribution of the log returns $(r_{1,t+1}, r_{2,t+1})$ is specified conditional on past information $\mathcal{I}_t$. It is assumed that this conditional distribution has Gaussian margins and a certain conditional copula C_t. The model allows volatilities and dependence to be time varying in a non-deterministic way; the volatilities of the conditional joint distribution are modelled as functions of past squared return innovations, while the conditional copula is assumed to depend on the past via Kendall's tau as a function of past volatilities. The model is stationary; see Comte and Lieberman (2003) for a discussion.

The next step in the valuation scheme is the derivation of the joint risk-neutral return process from the objective bivariate distribution. The specification of the objective marginals, in conjunction with the assumption that the objective and the risk-neutral conditional copulas are the same, allows for a particularly convenient transformation to risk neutrality; instead of deriving the bivariate risk-neutral distribution directly, it is found by transforming each of the marginal processes (and the copula) separately. The fair value of the option is then determined by taking the discounted expected value of the option's payoff under the risk-neutral distribution.

The specification chosen for the objective marginal distributions is from Duan (1995). It is general enough to capture volatility clustering, a stylized fact of equity returns for which there is overwhelming empirical evidence at the daily frequency, while still providing a relatively easy transformation to risk-neutral distributions. Each of the objective marginal distributions of the index returns is modelled by a GARCH(1,1) process with Gaussian innovations. It is repeated here for the sake of completeness; see Bollerslev (1986). For $i \in \{1, 2\}$,

$$\begin{aligned} r_{i,t+1} &= \mu_i + \eta_{i,t+1} \,, \\ h_{i,t+1} &= \omega_i + \beta_i h_{i,t} + \alpha_i \eta_{i,t+1}^2 \,, \\ \mathcal{L}_P(\eta_{i,t+1}|\mathcal{I}_t) &= \mathcal{N}(0, h_{i,t}), \end{aligned}$$

where $\omega_i > 0$, $\beta_i > 0$, and $\alpha_i > 0$, and $\mathcal{L}_P(\cdot|\mathcal{I}_t)$ denotes the objective probability law conditional on the information set $\mathcal{I}_t$, which includes all realized returns on both indexes. The marginal distributions are specified conditional on this common information set, so that copula theory can be used to construct a joint conditional distribution. Failure to use the same conditioning information for the margins and the copula will, in general, lead to invalid joint density models. This point is emphasized by Patton (2003). The GARCH parameters are estimated by maximum likelihood, using the unconditional variance level $\omega_i/(1 - \beta_i - \alpha_i)$ as starting value $h_{i,0}$.

It must be stressed that, in the light of Sklar's theorem, in principle *any* choice for the marginal distributions is consistent with the copula approach. The vast collection of alternatives that have been used by other authors to model univariate index return distributions includes (variants of) continuous-time geometric Brownian motion of Black and Scholes (1973), and the discrete-time binomial model of Cox, Ross and Rubinstein (1979). Again, the GARCH specification that is employed here is appealing as it allows for an easy change of measure in addition to being able to capture volatility clustering. In particular, Duan (1995) shows that, under certain conditions, the change of measure comes down to a change in the drift. The law of the returns under the risk-neutral probability measure (Q) is given by:

$$\begin{aligned} r_{i,t+1} &= r_f - \tfrac{1}{2} h_{i,t} + \eta_{i,t+1}^* \,, \\ h_{i,t+1} &= \omega_i + \beta_i h_{i,t} + \alpha_i (r_{i,t+1} - \mu_i)^2 \,, \\ \mathcal{L}_Q(\eta_{i,t+1}^*|\mathcal{I}_t) &= \mathcal{N}(0, h_{i,t}), \end{aligned}$$

where r_f is the risk-free rate, which is assumed to be constant. Recall that under the risk-neutral measure, actualized prices are martingales.

It is important to note here that the specified marginal distributions are conditional on the common information set $\mathcal{I}_t$, but that it is assumed that both conditional margins only depend on their own past, i.e.,

$$\mathcal{L}_P(r_{i,t+1}|\mathcal{I}_t) = \mathcal{L}_P(r_{i,t+1}|\mathcal{I}_{i,t}),$$

where $\mathcal{I}_{i,t} = \sigma(r_{i,s} : s \leq t)$ denotes the information on index i available at time t. What this means in particular is that return spillovers or volatility spillovers from one index to the other are excluded. This restriction is necessary for the application of Duan's change of measure.

An alternative, nonparametric approach is to use univariate option price data to obtain arbitrage-free estimates of the marginal risk-neutral densities, as in Ait-Sahalia and Lo (1998). This route is taken by Rosenberg (2003). Clearly, an advantage of this approach is that it does not impose restrictions on the asset return processes or on the functional form of the risk-neutral densities. However, this flexibility comes at the cost of imprecise estimates, especially if the distributions are time-varying.

The description of the joint distribution of the index returns under the objective probability measure is completed by fixing the conditional copula. A set of well-known one-parameter copula families is considered for this purpose. They are the Frank, Gumbel–Hougaard, Plackett, Galambos, and normal families. Their cumulative distribution functions are given in Appendix A. For all of these copulas, there is a one-to-one relation between the dependence parameter—denoted θ—and Kendall's nonparametric measure of association. For any copula C_θ, Kendall's tau is related to θ in the following way:

$$\tau(\theta) = 4\mathrm{E}C_\theta(U, V) - 1, \tag{1}$$

where (U, V) is distributed as C_θ, and E denotes the expectation operator with respect to U and V. Appendix B displays closed-form formulas for the population value of Kendall's tau for some of the copula models under consideration.

This relation suggests a natural way to estimate the copula. An estimate of θ is readily obtained by computing the sample version of tau on a (sub)sample of paired index-return observations, inverting Relation (1), and plugging in the sample tau. See Appendix C for a definition of the sample version of Kendall's tau. This method-of-moment type procedure yields a rank-based estimate of the association parameter which is consistent, under the assumption that the selected family of copulas describes accurately the dependence structure of the equity indexes. Other methods could be used without fundamentally altering this approach, e.g., inversion of Spearman's rho, or the maximum pseudo-likelihood method.

The proposed technique assumes that the objective and risk-neutral copulas are identical, so that the objective joint returns process is easily transformed into its risk-neutral counterpart, using the risk-neutral marginals and the copula. Rosenberg (2003) makes this assumption as well. If bivariate option price data were available, equality of the objective and risk-neutral copulas could be tested or the appropriate risk-neutral copula could be estimated. Only data on prices of bivariate claims would reveal information about the risk-neutral dependence structure. Information about the risk-neutral dependence structure can never be extracted from univariate option prices—which *are* available—as these only bear relevance to the marginal risk-neutral processes, and not to the joint risk-neutral process. Identification of the bivariate density requires knowledge of both the marginal densities and the dependence function that links them together.

Variation of the dependence structure through time can be modelled by means of the conditional copula, which was introduced by Patton (2003, 2004) and recently extended by Fermanian and Wegkamp (2004). In the present paper, time variation in the copula is modelled by allowing the dependence parameter to evolve through time according to a particular equation. The forcing variables in this equation are the conditional volatilities of the underlying assets. These are also the forcing variables that are typically chosen to model time-varying correlations; see, e.g., the BEKK model introduced by Engle and Kroner (1995). Additional motivation is provided by the evidence on correlation breakdowns, which suggests that financial markets exhibit high dependence in periods of high volatility. Patton (2003) proposes an ARMA-type process linking the dependence parameter to absolute differences in return innovations, which is another way to capture the same idea.

To be more specific, let τ_t be Kendall's measure of association at time t, and let $h_{i,t}$ be the objective conditional variance estimate at time t of underlying index return $i \in \{1, 2\}$ implied by Duan's GARCH option pricing model. It is assumed that

$$\tau_t = \gamma(h_{1,t}, h_{2,t}) \tag{2}$$

for some function $\gamma(\cdot, \cdot)$ to be specified later. This conditional measure of association governs the degree of dependence for the risk-neutral copula under consideration.

The proposed valuation scheme is implemented using Monte Carlo simulations. Pairs of random variates are drawn from the copula implied by the estimated conditional risk-neutral measure of association, which are then transformed to return innovations using Duan's GARCH model. Subsequently, the payoffs implied by these innovations are averaged and discounted at the risk-free rate. The result then constitutes the fair value of the option. Algorithms for random variable generation from the non-normal copulas are given in Genest and MacKay (1986), Genest (1987), Ghoudi, Khoudraji and Rivest (1998), and Nelsen (1999). For the normal copula, a straightforward Cholesky decomposition may be used.

3 Pricing options on two equity indexes

The dynamic-dependence valuation scheme outlined in Section 2 is applied to better-of-two-markets and worse-of-two-markets options on the S&P500 and Nasdaq indexes. A sample consisting of pairs of daily returns on the S&P500 and Nasdaq from January 1, 1993 to August 30, 2002 was obtained from Datastream. The sample size is $T = 2422$. The maximum likelihood estimates of the GARCH parameters for the marginal index return processes may be found in Table I. These estimates are based upon an initial variance level $h_{i,0} = \omega_i/(1 - \beta_i - \alpha_i)$. The values for α and β nearly add up to one. These estimates are in line with previously reported values.

Figure 1 depicts the time series of the estimated standardized GARCH innovations $(\eta_{1,t+1}/\sqrt{h_{1,t}}, \eta_{2,t+1}/\sqrt{h_{2,t}})$ for the last 250 trading days in the sample. (For clarity, the picture is restricted to a subsample; other episodes show a similar pattern.) Note that outliers typically occur simultaneously and in the same direction. This positive dependence between the two series is even more apparent from Figure 2, which displays the support set of the empirical copula of the standardized return innovations. This scatter plot consists of the observed pairs of ranks (divided by $T + 1$) for the estimated standardized GARCH innovations of the two markets. In case of static dependence, the empirical copula estimates the copula function; under regularity conditions, the empirical copula function converges to the true (here, objective) copula function, see Van der Vaart and Wellner (1996). In case of time-varying dependence, it is an estimate of the copula of the unconditional distribution. Notice the pronounced positive dependence, particularly in the tails. The sample version of Kendall's tau for the entire sample amounts to 0.60, confirming positive dependence. Figure 3 gives an impression of how this dependence measure of the standardized return innovations evolves over time. It shows rolling-window estimates of Kendall's tau using window sizes of two months, i.e., Kendall's tau at day t is computed using the 20 trading days prior to day t, day t itself, and the 20 trading days after day t. While the estimates show considerable variation, a slight upward trend over the sample period is discernable.

The time variation in the copula is governed by Equation (2). It models the dependence measure as a function of the conditional volatilities of the index returns. The following specification of this function is proposed:

$$\gamma(h_1, h_2) = \gamma_0 + \gamma_1 \log \max(h_1, h_2). \tag{3}$$

To motivate this specification, recall that the evidence on correlation breakdowns suggests that increased dependence occurs in hectic periods. Hence, theory predicts a positive value of γ_1. The maximum operator reflects that hectic periods in either market may cause dependence to go up. Since volatility in both markets is highly dependent, the actual specification is likely not to affect the results in the present section too much. The parameters γ_0 and γ_1 were estimated by regressing the non-parametric (rolling-window) estimates of Kendall's tau from Figure 3 on the estimated log maximum conditional volatility. The regression is illustrated in Figure 4. The slope coefficient, γ_1, was estimated at 0.063; positive, as expected. The estimated dependence measure implied by these parameter estimates,

$$\gamma(h_{1,t}, h_{2,t}) = \gamma_0 + \gamma_1 \log \max(h_{1,t}, h_{2,t}),$$

was then used to fix the conditional risk-neutral copula at time t.

Return innovations were sampled from this conditional copula to compute the price of the option. In total, the Monte Carlo study was based on $100,000$ replications, leading to simulation errors in the order of magnitude of 1 basis point for one-month maturity claims. The option prices turned out to be robust with respect to the choice of the window used to calculate Kendall's tau. Halving or doubling the window size only led to slight price changes in the order of magnitude of the simulation errors.

Clearly, the option price depends on the initial levels of volatility of the underlyings. Prices for three levels of initial volatility were computed: low, medium, and high volatility, where medium volatility is defined as the estimated unconditional variance $\omega/(1 - \beta - \alpha)$, and low and high volatility are one-fourth of and four times this amount, respectively. Furthermore, different maturities were considered, ranging from one day to one month (i.e., 20 trading days). The strike price was set at levels between 0.98 and 1.02. Finally, the risk-free rate was assumed to be 4 percent per annum.

The results show that allowing for time varying dependence leads to different option prices than under static dependence, in particular in times of high volatility. This is illustrated in Figure 5 which displays, for various copula parametrizations, the price (measured in basis points) of a one-month put-on-max option as a function of the exercise price implied by dynamic dependence, and compares it to the option price under three levels of static dependence: low, medium, and high static dependence. The medium level of dependence is equal to the average measure of dependence found in the sample, 0.60; the low and high levels are 0.40 and 0.80, respectively. Note that a static model for the dependence structure, which uses the sample measure of dependence of 0.60, underestimates the option price generated by the dynamic model considerably for all copula parametrizations and over the entire range of strike prices considered. The difference is significant since the 95% confidence intervals of the price estimates do not overlap. In the interest of clarity, confidence intervals are not displayed here, but available from the authors upon request. Note that the prices implied by dynamic copulas are between the high and the medium static-dependence prices, suggesting that the dynamic model implies a dependence that is on average stronger than in the medium static-dependence case. Interestingly, price differences between the dynamic and static model vanish as initial volatilities are at a medium level; see Figure 6. The same holds for low initial volatilities (not shown), again, across a broad range of copula families and strike prices.

It is also interesting to compare option prices produced by different dynamic copula families. It turns out that prices implied by the normal copula deviate substantially from prices implied by the other copula families. Outside the normal class, the copula choice appears to be irrelevant. This suggests that unless the dependence between index returns can be described by a normal model, alternative specifications should be considered. These findings are illustrated in Figures 7 and 8 which depict dynamic-dependence one-month call-on-max and put-on-min option prices respectively, as a function of their strike under medium initial volatilities. The prices implied by the normal copula are significantly lower than the prices implied by the other copulas across the whole range of strike prices. The effect is there at other maturities as well. The difference between normal and non-normal prices is also found for high and low initial volatility levels. The differences are less significant for call-on-min and put-on-max options.

4 Conclusions

This paper studies the relation between multivariate options prices and the dependence structure of the underlying assets. A copula-based model was proposed for the valuation of claims on multiple assets. A novel feature of the proposed model is that, contrary to earlier works on multivariate option pricing, the dependence structure is not taken as fixed, but rather as potentially varying with time. The time variation in the dependence structure was modelled using various parametric copulas by letting the copula parameter depend on the conditional volatilities of the underlyings.

This dynamic copula model was applied to better- and worse-of-two-markets options on the S&P500 and Nasdaq indexes for a variety of copula parametrizations. Option prices implied by the dynamic model turned out to differ substantially from prices implied by a model that fixes the dependence between the underlying indexes, especially in high-volatility market conditions. Hence, the application suggests that time variation in the dependence between the S&P500 and the Nasdaq is important for the price of options on these indexes. A comparison of option prices computed from different copula families shows that the normal family produces prices that differ significantly from the ones implied by the non-normal alternatives. These findings suggests that if the dependence between the index returns is not well represented by a normal copula, alternative copulas need to be considered. The empirical relevance of such alternatives is apparent given the evidence of non-normality in financial markets.

A One-Parameter Copula Families

The table below displays several one-parameter copula families.

Frank	$C_\theta(u,v) = \frac{1}{\theta}\log\left\{1 + \frac{(e^{\theta u}-1)(e^{\theta v}-1)}{e^{\theta}-1}\right\}$
Gumbel–Hougaard	$C_\theta(u,v) = \exp\left\{-\left(\lvert\log u\rvert^\theta + \lvert\log v\rvert^\theta\right)^{\frac{1}{\theta}}\right\}$
Plackett	$C_\theta(u,v) = \frac{1+(\theta-1)(u+v)-\sqrt{[1+(\theta-1)(u+v)]^2-4uv\theta(\theta-1)}}{2(\theta-1)}$
Galambos	$C_\theta(u,v) = uv\exp\left\{\left(\lvert\log u\rvert^\theta + \lvert\log v\rvert^\theta\right)^{\frac{1}{\theta}}\right\}$
Normal	$C_\theta(u,v) = N_\theta(\Phi^{-1}(u), \Phi^{-1}(v))$

Note: Φ is the standard (univariate) normal distribution function, and N_θ denotes the standard bivariate normal distribution function with correlation coefficient θ.

B Kendall's tau

The table below provides expressions—closed-form if available—of the relation between Kendall's tau and the dependence parameter for the copula families considered in Appendix A.

Frank	$\tau(\theta) = 1 - 4\left\{D_1(-\theta) - 1\right\}/\theta$
Gumbel-Hougaard	$\tau(\theta) = 1 - 1/\theta$
Plackett	$\tau(\theta) = 4\int_0^1\int_0^1 C_\theta(u,v)dC_\theta(u,v) - 1$
Galambos	$\tau(\theta) = \frac{\theta+1}{\theta}\int_0^1\left(\frac{1}{t^{1/\theta}} + \frac{1}{(1-t)^{1/\theta}} - 1\right)^{-1} dt$
Normal	$\tau(\theta) = \frac{2}{\pi}\arcsin\theta$

Note: D_1 denote the first-order Debye function, $D_1(-\theta) = \frac{1}{\theta}\int_0^\theta \frac{t}{e^t-1}dt + \frac{\theta}{2}$.

C Sample version of Kendall's tau

Let $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ be a random sample of n observations from a vector (X, Y) of continuous random variables. Two distinct pairs (X_i, Y_i) and (X_j, Y_j) are said to be concordant if $(X_i - X_j)(Y_i - Y_j) > 0$, and discordant if $(X_i - X_j)(Y_i - Y_j) < 0$. Kendall's tau for the sample is then defined as $t = (c - d)/(c + d)$, where c denotes the number of concordant pairs, and d is the number of discordant pairs.

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Table I: Maximum likelihood estimates of the GARCH parameters for the marginal index return processes. Figures in brackets are robust quasi-maximum likelihood standard errors.

Parameter	S&P500		Nasdaq	
$\mu \times 10^2$	0.0674	(0.0168)	0.0812	(0.0246)
$\omega \times 10^5$	0.0680	(0.0398)	0.1895	(0.0987)
β	0.9258	(0.0220)	0.8906	(0.0309)
α	0.0680	(0.0198)	0.1015	(0.0288)

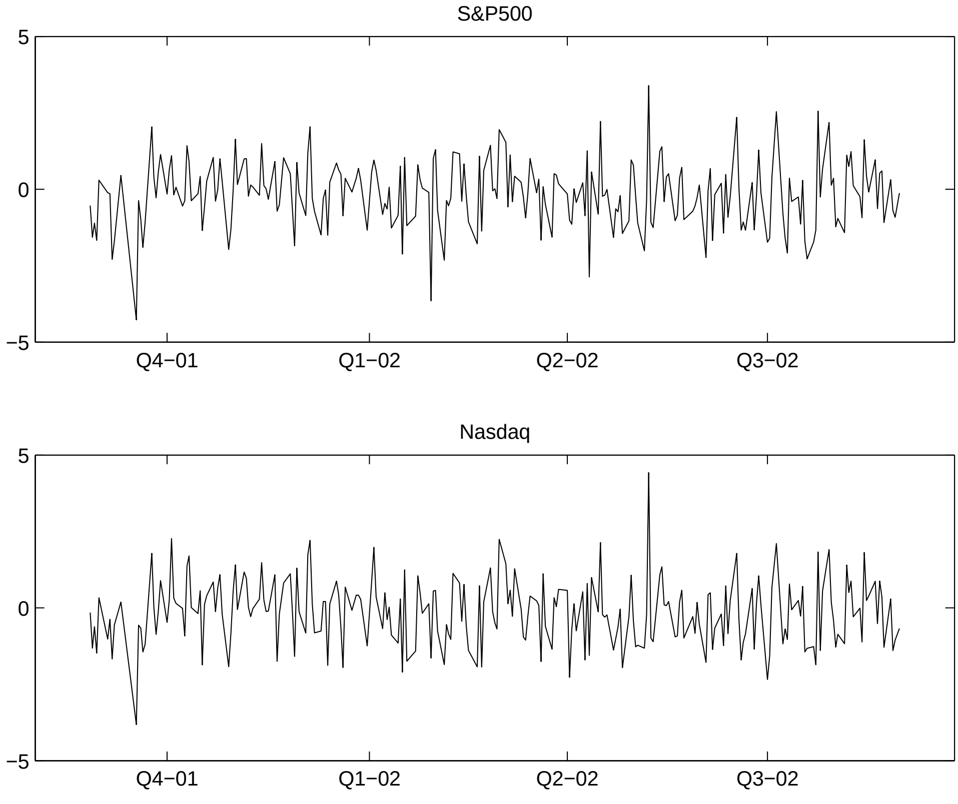

Figure 1: Daily standardized GARCH innovations for S&P500 and Nasdaq for the last 250 trading days in the sample.

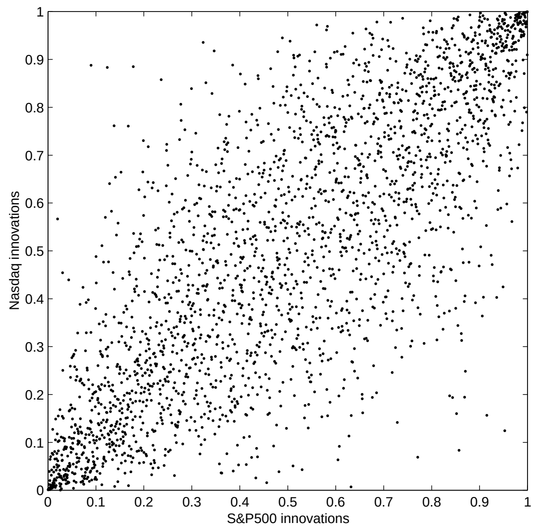

Figure 2: Support set of the empirical copula of the standardized GARCH innovations.

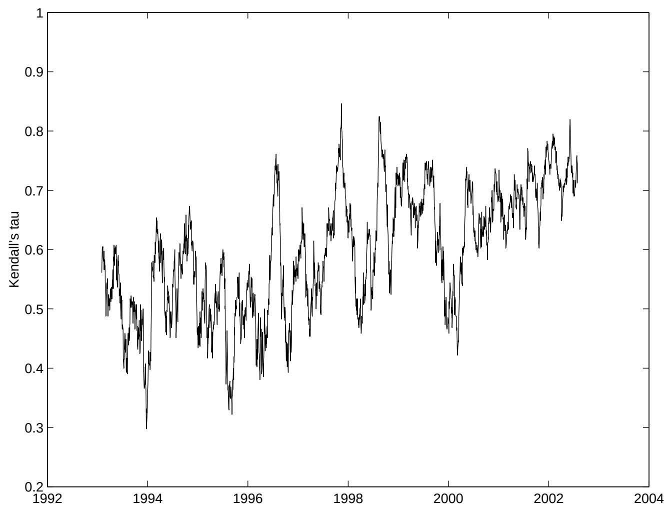

Figure 3: Rolling-window estimates of Kendall's tau for the standardized return innovations using a window size of 41 trading days.

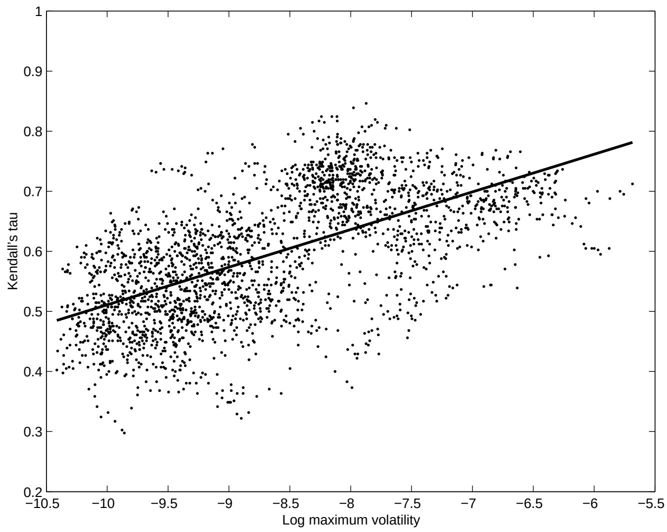

Figure 4: Regression of rolling-window estimates of Kendall's tau for the standardized return innovations on the logarithm of the maximum return volatility.

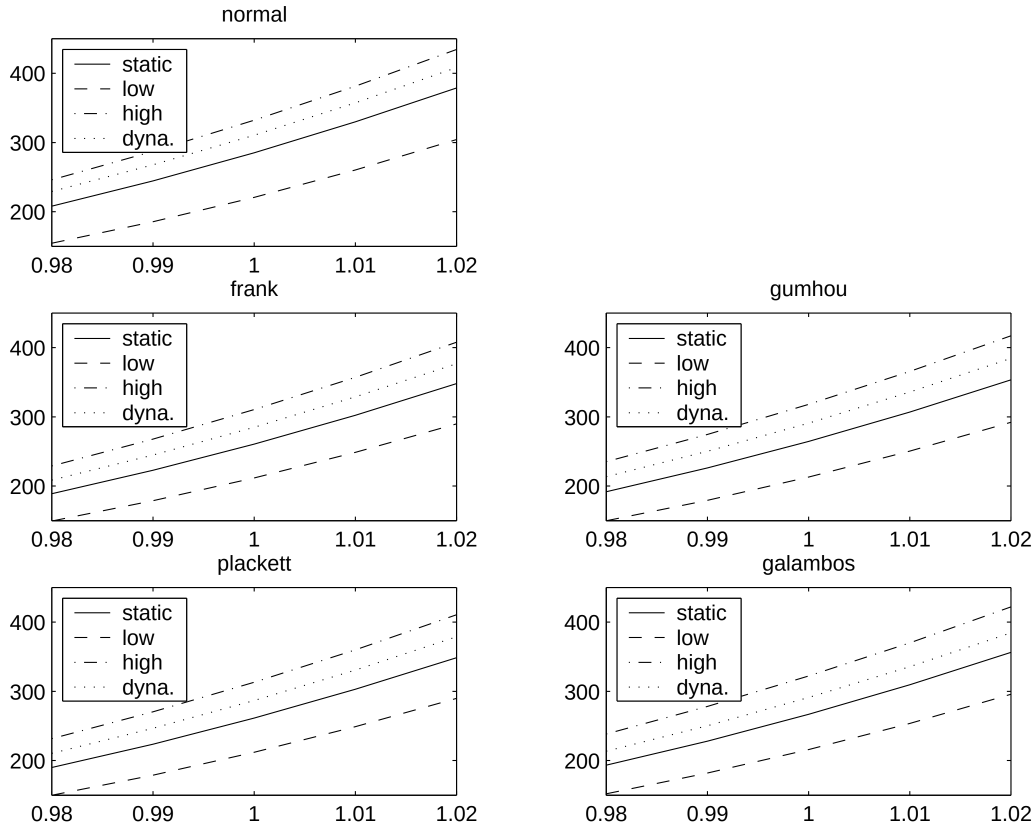

Figure 5: One-month maturity put-on-max prices in basis points as a function of the strike under high initial volatilities for dynamic dependence and for low, medium, and high static dependence for various copulas.

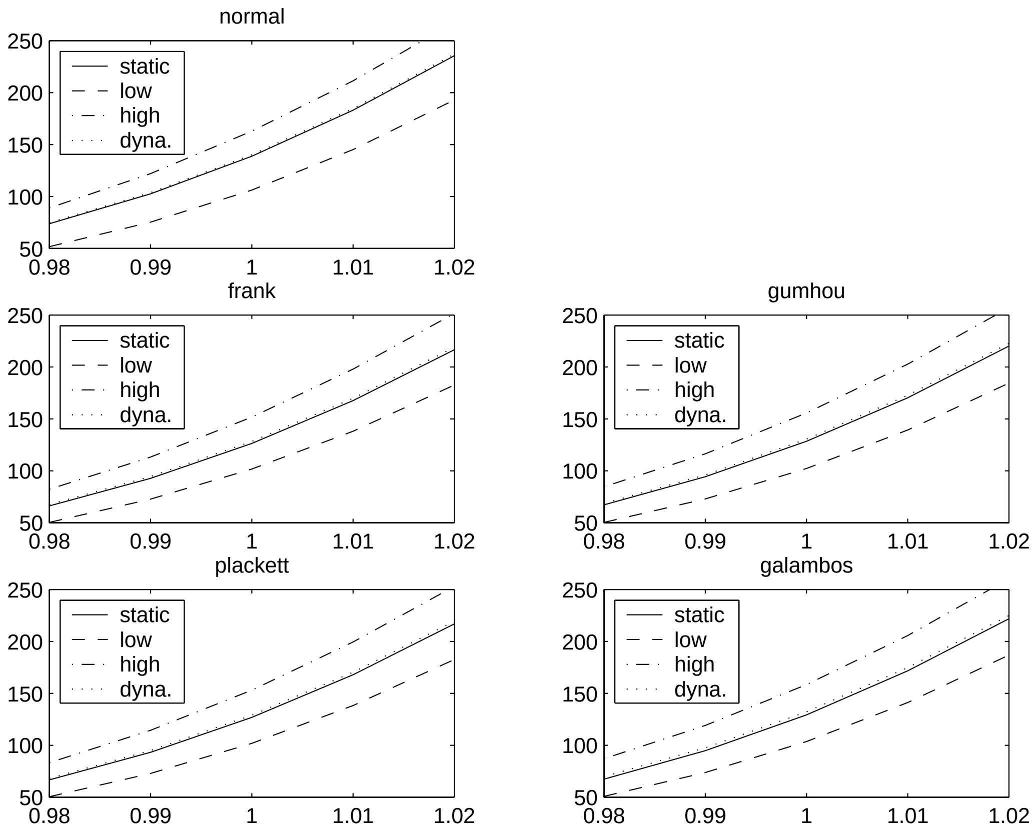

Figure 6: One-month maturity put-on-max prices in basis points as a function of the strike under medium initial volatilities for dynamic dependence and for low, medium, and high static dependence for various copulas.

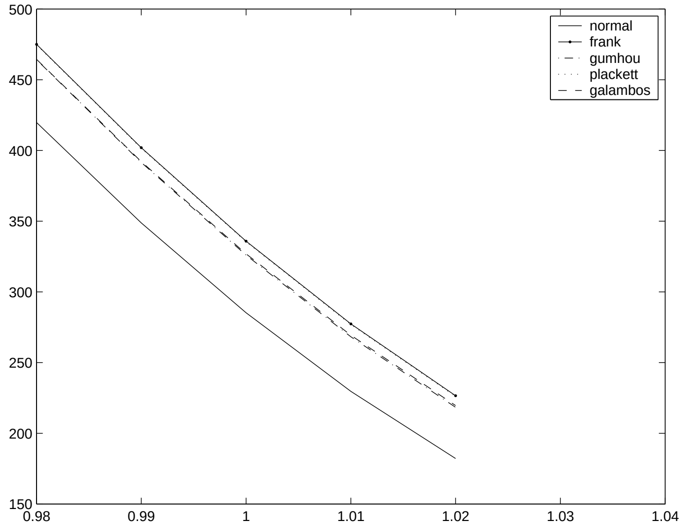

Figure 7: One-month call-on-max prices in basis points as a function of the strike under dynamic dependence and medium initial volatilities for various copula models.

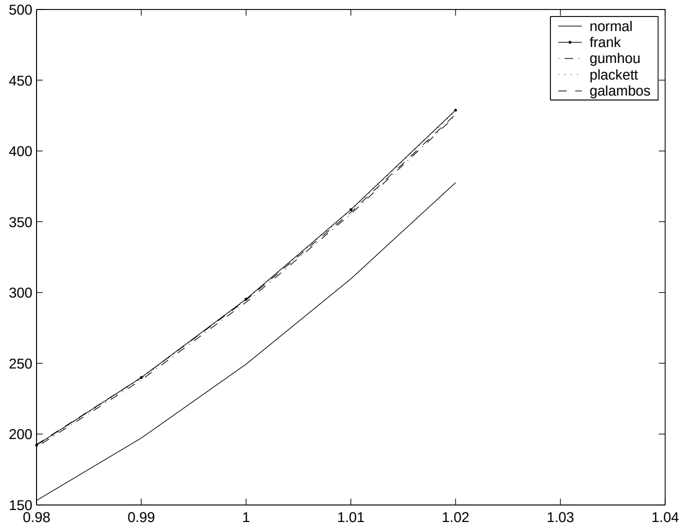

Figure 8: One-month put-on-min prices in basis points as a function of the strike under dynamic dependence and medium initial volatilities for various copula models.