A Note on Standard No-Arbitrage Conditions

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Abstract We discuss standard no-arbitrage conditions for a plain vanilla call price function. Furthermore, a new no-arbitrage condition is discussed which is derived from an observation by Gatheral (2000).

1 Introduction

Option markets underly certain economic relationships which have to hold always to avoid arbitrage opportunities. Arbitrage in this context is the possibility to set up a trading strategy without net investment of capital which yields a profit with positive probability without taking any risk, see Delbaen and Schachermayer (2006), Carr and Madan (2005). The economic relationships can be derived from theoretical considerations and have to be respected by market quotes independent of the model behind these quotes. Restrictions on the form of theoretical economic objects are known from other areas of economic research: Utility functions are increasing and concave, production and cost functions are monotonic (see Ait-Sahalia and Jefferson (2003)). In option markets, theoretical considerations can be used to significantly reduce the class of potential function candidates to represent a price function: The function which maps the strike against the call price is known to be positive, monotonically decreasing and convex. Furthermore, the behavior of the function and its derivatives for large and small strikes is known. The theory is even developed enough to state an explicit value of the price function at a certain point (zero strike) and the value of its derivative at the same point,

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independent of the model[1]. The arbitrage question is relevant if one considers an option market where call prices and strikes for different maturities are quoted. Our research question is how to find an arbitrage-free function $C(K,t)$ which is consistent with the quotes and returns a call price for any strike and maturity. As stated before, financial theory imposes shape restrictions on $C(K,t)$ and its derivatives. If one moves to the implied volatility space, the same restrictions have to hold. These restrictions are presented below.

2 Standard No-Arbitrage Conditions

Let $C(K)$ be a call price function for a fixed time to maturity $\tau = T - t$. Let r_d be the domestic rate and r_f the foreign rate (or dividend yield), both rates are assumed to be continuously compounded. The variable S denotes the spot at time t and K the strike. Assume that the price function fulfills the standard regularity conditions such as differentiability and smoothness. The forward rate is given as

$$f = Se^{(r_d - r_f)\tau} \tag{1}$$

while the value of the forward contract is

$$v_f = Se^{-r_f\tau} - Ke^{-r_d\tau}. \tag{2}$$

The standard Black-Scholes formula is

$$\begin{aligned} v(S_t, K, \sigma, \phi) &= v(S, r_d, r_f, K, \sigma, t, T, \phi) \\ &= \phi[e^{-r_f\tau}\, S\, N(\phi d_+) \;-\; e^{-r_d\tau}\, K\, N(\phi d_-)] \\ &= \phi e^{-r_d\tau}\, [f\, N(\phi d_+) \;-\; K\, N(\phi d_-)], \end{aligned} \tag{3}$$

where

$$d_\pm = \frac{\ln\left(\frac{f}{K}\right) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

K : strike of the FX option,

σ : volatility,

$\phi = +1$ for a call , $\phi = -1$ for a put,

$N(x)$: cumulative normal distribution function.

[1] In this section the focus is put on call price functions. The put price functions can be incorporated by transforming them to call price functions with the put-call parity.

2.1 Conditions on a Call Price Function for a Fixed Maturity

The commonly stated no-arbitrage conditions on the call price function $C(K)$ with maturity τ (see Merton (1973), Fengler (2009), Birke and Pilz (2009)) are

$$\max(Se^{-r_f\tau} - Ke^{-r_d\tau}, 0) \leq C(K) \leq Se^{-r_f\tau} \tag{4}$$

$$-e^{-r_d\tau} \leq C'(K) \leq 0 \tag{5}$$

$$C''(K) \geq 0 \tag{6}$$

Condition (4) shows that the call price function should always be at least as much as the forward contract, since the call pays more than the forward contract at time T in any case. Furthermore, a call can only have a positive payment at time T which restricts the call price to be a positive number. A call pays always less than a position in a foreign currency, which is the intuition behind the right inequality in (4). A higher strike leads to a lower payoff, which shows that the call price is monotonically decreasing in the strike. This is represented on the right hand side of Inequality (5). The left hand side of Inequality (5) and the intuition behind inequality (6) can be explained by viewing the call price as an expectation under the risk-neutral measure

$$C(K) = e^{-r_d\tau} \int_K^\infty (S - K)\phi(S)dS, \tag{7}$$

with $\phi(S)$ being the risk-neutral density. From this equation, it follows that

$$\frac{\partial C}{\partial K} = C'(K) = e^{-r_d\tau}[\Phi(K) - 1], \tag{8}$$

with $\Phi(K)$ being the risk-neutral c.d.f.. Inequality (5) follows from the fact that a regular c.d.f. can only attain values within $[0, 1]$. Differentiating again with respect to the strike yields

$$\frac{\partial^2 C}{\partial K^2} = C''(K) = e^{-r_d\tau}\phi(K). \tag{9}$$

This is the well known result by Breeden and Litzenberger (1978). Since the density must be positive, Inequality (6) must hold. From a different point of view this inequality can be seen as the requirement that a butterfly position has a positive value due to the positive payoff of this position at time T. From representations (7) and (8) and under the assumption of a regular underlying process which is absorbed at zero one can deduce (see Carr (2004))

$$
\begin{aligned}
C(0) &= Se^{-r_f\tau}, \\
C'(0) &= -e^{-r_d\tau}, \\
C''(0) &= 0.
\end{aligned} \tag{10}
$$

The first condition states the model independent price of a zero strike call. Equivalently, it states that the forward rate has to be the one in Equation (1), independent of the model. The third condition is stated in Carr (2004). The conditions above can be found in Merton (1973), Fengler (2009), Brunner and Hafner (2003), Härdle and Hlávka (2009),Wang et al. (2004), Ait-Sahalia and Jefferson (2003), Birke and Pilz (2009). Simple no-arbitrage considerations lead to a remarkably concrete shape requirement for the call price function.

2.2 Conditions on Call Price Functions with Different Times to Maturity

Fengler (2009) states an arbitrage condition for call options with different times to maturity which he accredits to Reiner (2000). Let two maturities $T_1 < T_2$ be given and two strikes K_1, K_2 which are related by the forward-moneyness

$$\frac{K_1}{f(t,T_1)} = \frac{K_2}{f(t,T_2)}$$

with f being the forward rate in Equation (1). Assume time dependent deterministic interest rates $r_d(t), r_f(t)$. There is no calendar arbitrage if

$$C(K_2,T_2) \geq e^{-\int_{T_1}^{T_2} r_f(s)\mathrm{d}s} C(K_1,T_1). \tag{11}$$

For $T_2 \to T_1$ this results in the following partial differential inequality (given in Carr (2004))

$$\frac{\partial C}{\partial T}(K,T) + (r_d(T) - r_f(T))K\frac{\partial C}{\partial K}(K,T) + r_f(T)C(K,T) \geq 0 \qquad \forall (K,T)$$

with T being the time to maturity. Note that this inequality can be reformulated using homogeneity assumptions, see Reiss and Wystup (2001). Such a transformation shows that the inequality is not very strict and almost always fulfilled - as long as the call price function increases with the time to maturity. This is also apparent in the inequality above by setting $r_d(T) \approx r_f(T)$.

2.3 Conditions on the Implied Volatility Function for a Fix Maturity

Assume that an implied volatility function $\sigma(K)$ is considered instead of a call price function. Plugging $\sigma(K)$ in the standard Black-Scholes function $C_{BS}(K,\sigma)$ yields a market consistent price. The function $C_{BS}(K,\sigma(K))$ has to fulfill all arbitrage conditions stated before as it is a standard call function, only expressed in the Black-Scholes framework. Given that

$$\frac{\partial C}{\partial K} \leq 0$$

yields with standard calculus rules (see Carr (2004)):

$$\frac{\partial C}{\partial K} = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \sigma}\frac{\partial \sigma}{\partial K} \leq 0 \Leftrightarrow \frac{\partial \sigma}{\partial K} \leq \frac{N(d_-)}{Kn(d_-)\sqrt{\tau}} \tag{12}$$

using the same notation as in Equation (3). An additional condition given by Gatheral (2000) will play a crucial role in the later section. At this stage, we simply state that the condition puts a lower bound on the smile slope (see Lee (2005)) such that

$$-\frac{N(-d_+)}{Kn(d_+)\sqrt{\tau}} \leq \frac{\partial \sigma}{\partial K}$$

This condition yields a stricter lower bound than the simple condition implied by the restriction

$$\frac{\partial P}{\partial K} \geq 0$$

with P being the put price. The derivations above show that one has an upper and lower bound on the implied volatility smile slope to avoid arbitrage. Finally, convexity arguments imply

$$\frac{\partial^2 C}{\partial K^2} = \frac{\partial}{\partial K}\Big(\frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \sigma}\frac{\partial \sigma}{\partial K}\Big) \geq 0 \tag{13}$$

$$\tag{14}$$

This is equivalent to

$$\frac{\partial^2 \sigma}{\partial K^2} \geq -\frac{1}{K^2\sigma(K)\tau} - 2\frac{d_+}{\sigma(K)K\sqrt{\tau}}\frac{\partial \sigma}{\partial K} - \frac{d_+ d_-}{\sigma(K)}\Big(\frac{\partial \sigma}{\partial K}\Big)^2. \tag{15}$$

Equation (15) shows that a lower bound on the convexity of the implied volatility is available. Note that this lower bound is not necessarily positive: An implied volatility smile can be concave.

3 A Stricter Condition

Consider again the standard arbitrage conditions commonly stated in the literature:

$$\begin{aligned}\max(Se^{-r_f\tau} - Ke^{-r_d\tau}, 0) \leq\ & C(K)\ \leq Se^{-r_f\tau},\\ -e^{-r_d\tau} \leq\ & C'(K)\ \leq 0,\\ & C''(K) \geq 0.\end{aligned}$$

In this section, we will consider whether some of the conditions above are redundant. In the second step, the question will be raised, whether it is sufficient to impose the conditions above on a call price interpolation on an interval $[K_1, K_n]$ to avoid arbitrage. An example will be shown which shows that this is not the case. This example is constructed by deriving a stricter necessary condition which is motivated by a statement by Gatheral (2000).

3.1 Gatheral's Condition

Gatheral mentions an additional arbitrage condition in Gatheral (2000), which is

$$\frac{\partial(\frac{P(K)}{K})}{\partial K} \geq 0. \tag{16}$$

This condition is motivated by the payoff of this position at time T which is

$$\frac{1}{K}\max(K - S, 0).$$

The time T payoff is plotted in Figure 1. One can observe that the time T payoff is increasing in strike. The consequence is that the time t value should also increase in strike which is equivalent to a monotonically increasing value function which is the intuition behind (16)[2]. Gatheral's condition can be reformulated in call price space. From put-call parity, we know

$$P(K) = C(K) + Ke^{-r_d\tau} - Se^{-r_f\tau}$$

and consequently for $K > 0$

$$\frac{P(K)}{K} = \frac{C(K)}{K} + e^{-r_d\tau} - e^{-r_f\tau}\frac{S}{K}.$$

Taking the derivative with respect to the strike yields, with Equation (16),

[2] To the knowledge of the author, the only source mentioning this condition - apart from the original source - is Lee (2005).

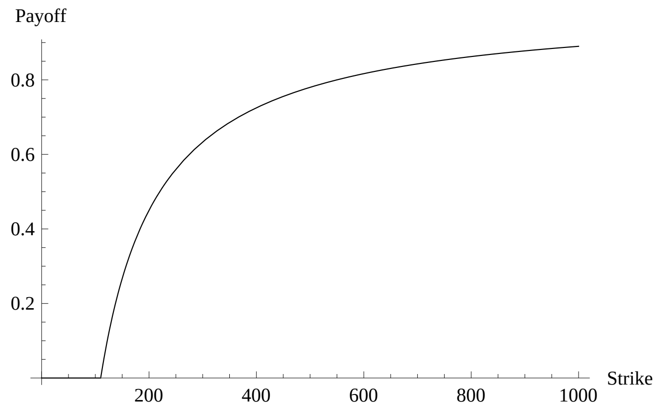

Fig. 1: Chart of $\frac{1}{K}\max(K-S,0)$ with $S=120$

$$\frac{\partial\left(\frac{P(K)}{K}\right)}{\partial K} = \frac{1}{K}\frac{\partial C(K)}{\partial K} - \frac{C(K)}{K^2} + e^{-r_f\tau}\frac{S_t}{K^2} \geq 0.$$

This implies that

$$\frac{\partial C(K)}{\partial K} \geq \frac{C(K)}{K} - e^{-r_f\tau}\frac{S}{K} = \frac{C(K)-e^{-r_f\tau}S}{K}. \tag{17}$$

If the call price function is above the forward price (Inequality (4) holds), the lower slope bound (17) is stricter than the lower bound in Equation (5) due to

$$\frac{C(K)-e^{-r_f\tau}S}{K} \underset{(4)}{\geq} \frac{Se^{-r_f\tau}-e^{-r_d\tau}K-Se^{-r_f\tau}}{K} = -e^{-r_d\tau}. \tag{18}$$

3.2 Intuition behind Gatheral's Condition

The condition (17) is intuitive, if one considers the condition in the framework of a digital cash or nothing call. Note, that a digital cash or nothing call $C_D(K)$ paying out 1 unit of the domestic currency if $S_T \geq K$ can be represented as the negative of the vanilla call function derivative:

$$C_D(K) = -C'(K).$$

This is a consequence of the representation of the digital as a call spread limit

$$C_D(K) = \lim_{h\to 0}\frac{C(K)-C(K+h)}{h} = -\lim_{h\to 0}\frac{C(K+h)-C(K)}{h} = -C'(K),$$

where again all necessary assumptions are assumed to hold (i.e. no free lunch with vanishing risk, see (Joshi, 2003, Page 122)). The result can also be found in Jeanblanc et al. (2009). The condition in Inequality (5)

$$-e^{-r_d\tau} \leq C'(K) \Leftrightarrow e^{-r_d\tau} \geq -C'(K) = C_D(K)$$

can be interpreted intuitively: As the digital call never pays out more than 1, it should be worth less than a zero bond with value $e^{-r_d\tau}$ which pays out 1 domestic currency unit in any case. Also, the digital should always have a positive value, which is implied by the right hand side of Equation (5). In this context, the zero coupon bond can be considered as a super-replication for the digital cash or nothing call. Alternatively, $e^{-r_d\tau}$ can be seen as the price of a zero strike digital call. Condition (17) describes another super-replication strategy, which is stricter than the zero coupon bond super-replication and consequently imposes a stricter bound on the digital call. Consider the negative of the portfolio of Equation (17): a long position in $e^{-r_f\tau}/K$ underlying units and a short position in $1/K$ vanilla calls with strike K. The value of this position at time t is:

$$\frac{Se^{-r_f\tau} - C(K)}{K}.$$

The payoff of this portfolio at time T is

$$\begin{cases} \frac{S_T-(S_T-K)}{K} = 1 \text{ if } S_T \geq K, \\ \frac{S_T}{K} \text{ if } S_T < K. \end{cases} \tag{19}$$

The portfolio pays always more than the digital and always less than the zero bond with a notional of 1. Condition (17) can be seen as

$$C'(K) \geq \frac{C(K) - e^{-r_f\tau}S}{K} \Leftrightarrow C_D(K) = -C'(K) \leq \frac{Se^{-r_f\tau} - C(K)}{K}$$

which should always be true due to the argument above (the payoff of the digital is always less). This super-replication explains condition (17). The old and new upper digital call bounds are shown graphically in Figure (2).

3.3 Simplifying the Standard Conditions

Consider again the conditions stated in Equation (10):

$$\begin{aligned} C(0) &= Se^{-r_f\tau}, \\ C'(0) &= -e^{-r_d\tau}. \end{aligned} \tag{20}$$

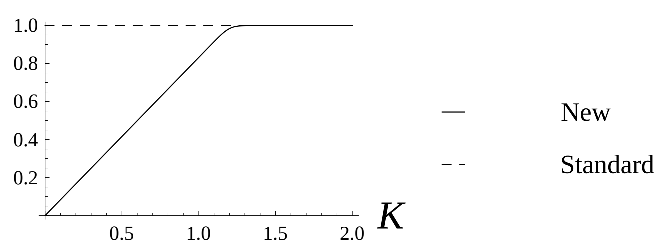

Fig. 2: Time t digital call price bounds for a digital with strike 1.2

Let $C(K)$ be the call price for strike K. We will proof that a positive, monotonically decreasing convex function that fulfills the conditions above automatically fulfills Inequality (4), the left hand side of Inequality (5), and Inequality (17). Birke and Pilz (2009) cover parts of the following results by claiming that conditions (5), (6), the positiveness and the zero strike call matching already imply condition (4). The result stated below takes the zero strike slope and Gatheral's condition into account. The following definition of convexity, which can be found in standard textbooks, will be useful in all proofs: A function C is convex in $S \subset \mathbb{R}$ if and only if for any $x, y \in S$ such that C is differentiable in x we have

$$C(y) \geq C(x) + C'(x)(y - x).$$

Lemma 1. *Let the conditions in Equation (20) be fulfilled by a positive, monotonically decreasing, convex call interpolation function $C(K)$ which is differentiable in $[0, \infty)$. Then the following conditions hold:*

$$C(K) \leq Se^{-r_f \tau} \quad \forall K \in [0, \infty) \tag{21}$$

$$C(K) \geq \max(Se^{-r_f \tau} - Ke^{-r_d \tau}, 0) \quad \forall K \in [0, \infty) \tag{22}$$

$$C'(K) \geq \frac{C(K) - e^{-r_f \tau} S}{K} \geq -e^{-r_d \tau} \quad \forall K \in (0, \infty). \tag{23}$$

Proof. The first condition can be proved easily, since $C(0) = Se^{-r_f \tau}$ and a monotonically decreasing function is considered. Consider the previously mentioned convexity condition

$$C(y) \geq C(x) + C'(x)(y - x).$$

Set $y = K$ and $x = 0$. Then we have

$$C(K) \geq C(0) + C'(0)K = Se^{-r_f \tau} - Ke^{-r_d \tau}. \tag{24}$$

This condition holds for all strikes $K \geq 0$. This, and the positiveness of the function $C(K)$ implies Equation (22). For $y = 0$ and $x = K$ we have

$$C(0) - C(K) \geq -C'(K)K,$$

which is equivalent to

$$C'(K) \geq \frac{C(K) - C(0)}{K} = \frac{C(K)}{K} - \frac{S}{K}e^{-r_f\tau} \tag{25}$$

which shows the middle part of Equation (23). Using Inequality (24) yields that, for $K > 0$,

$$C'(K) \geq \frac{C(K) - Se^{-r_f\tau}}{K} \geq \frac{Se^{-r_f\tau} - Ke^{-r_d\tau} - Se^{-r_f\tau}}{K} = -e^{-r_d\tau}$$

This proves the right inequality in Equation (23). □

The benefit of this result is that the standard arbitrage conditions can be simplified by letting the call function C start with the correct value and slope at strike zero.

3.4 Interpolation on an Interval $[K_1, K_n]$

What, if we do not consider the zero strike call and its derivative? The typical case would be an interpolation withing a strike range $[K_1, K_n]$, where market option prices for a discrete strike grid are available. Then, some of the conditions will not be fulfilled by default and need to be considered explicitly. Subsequently, we will derive a simple condition on the call price function $C(K)$ which is equivalent to Equation (17). Consider for this case the interval $[K_1, K_n]$ and the following portfolio value (the one from Equation (17)):

$$\frac{Se^{-r_f\tau} - C(K)}{K} \tag{26}$$

with $K \in [K_1, K_n]$. The payoff of the corresponding portfolio at time T is

$$\begin{cases} \frac{S_T - (S_T - K)}{K} = 1 \text{ if } S_T \geq K \\ \frac{S_T}{K} \text{ if } S_T < K \end{cases} \tag{27}$$

Consider for $y, K \in [K_1, K_n]$ with $y < K$ the following portfolio value

$$\frac{C(y) - C(K)}{K - y}. \tag{28}$$

The payoff of the corresponding portfolio at time T is

$$\begin{cases} \frac{S_T - y - (S_T - K)}{K - y} = 1 \text{ if } S_T \geq K \\ \frac{S_T - y}{K - y} \text{ if } y \leq S_T < K \\ 0 \text{ if } S_T < y \end{cases} \tag{29}$$

For the situation $y \leq S_T < K$ the first portfolio (26) always pays off more than the second (28):

$$\begin{aligned} \frac{S_T}{K} - \frac{S_T - y}{K - y} &= \frac{S_T(K - y) - (S_T - y)K}{K(K - y)} = \frac{S_T K - S_T y - K S_T + K y}{K(K - y)} \\ &= \frac{y(K - S_T)}{K(K - y)} > 0 \end{aligned}$$

Looking at the other scenarios for S at time T shows that the first portfolio corresponding to the value in Equation (26) pays more than the second one in Equation (28) irrespective of the final value S_T. By arbitrage this implies that, at time t:

$$\frac{Se^{-r_f \tau} - C(K)}{K} \geq \frac{C(y) - C(K)}{K - y} \tag{30}$$

must hold for any $y < K$ with $K, y \in [K_1, K_n]$. This no-arbitrage condition is the finite difference equivalent to Inequality (17). We will first show a simple restriction on the call price function $C(K)$ which guarantees that condition (30) is fulfilled. In the next step, the equivalence to Inequality (17) will be shown. The first result follows from the following Lemma.

Lemma 2. *Let $C(K)$ be convex on $[K_1, K_n]$. If for all $K \in [K_1, K_n]$*

$$C(K) \geq \frac{K[C(K_1) - Se^{-r_f \tau}]}{K_1} + Se^{-r_f \tau} \tag{31}$$

holds, then

$$\frac{Se^{-r_f \tau} - C(K)}{K} \geq \frac{C(y) - C(K)}{K - y} \tag{32}$$

holds for all $y < K$ and $y \in [K_1, K_n], K \in (K_1, K_n]$.

Proof. In the first step, we show that the function

$$h(y) = \frac{C(y) - C(K)}{K - y}$$

is decreasing for $y < K$ and $y \in [K_1, K_n], K \in (K_1, K_n]$. Taking the derivative of $h(y)$ yields

$$h'(y) = \frac{\partial}{\partial y}\Big(\frac{C(y) - C(K)}{K - y}\Big) = \frac{C'(y)(K - y) + C(y) - C(K)}{(K - y)^2}. \tag{33}$$

For any K, y we have from convexity

$$C(K) \geq C(y) + C'(y)(K-y) \Leftrightarrow 0 \geq C(y) + C'(y)(K-y) - C(K).$$

This shows that the numerator in Equation (33) is negative, implying that $h(y)$ has a negative derivative. Consequently, $h(y)$ is a decreasing function whose largest value is

$$\Big(\frac{C(K_1) - C(K)}{K - K_1}\Big)$$

For each K we thus have to ensure

$$\frac{Se^{-r_f\tau} - C(K)}{K} \geq \Big(\frac{C(K_1) - C(K)}{K - K_1}\Big)$$

to show that (32) holds. Reformulating yields

$$\begin{aligned}
\frac{Se^{-r_f\tau} - C(K)}{K} &\geq \Big(\frac{C(K_1) - C(K)}{K - K_1}\Big) \\
\Leftrightarrow C(K) &\geq \frac{KC(K_1) - Se^{-r_f\tau}(K - K_1)}{K_1} \\
&= \frac{K[C(K_1) - Se^{-r_f\tau}] + Se^{-r_f\tau}K_1}{K_1} \\
&= \frac{K[C(K_1) - Se^{-r_f\tau}]}{K_1} + Se^{-r_f\tau}
\end{aligned}$$

This completes the proof. □

Note, that the lower bound in Inequality (31) is given by a linear function g in K with $g(0) = Se^{-r_f\tau}$ and $g(K_1) = C(K_1)$. Thus, we can check for arbitrage graphically by drawing this line and checking if the call function falls below the line at any $K \in [K_1, K_n]$. Compare the inequality

$$C(K) \geq \frac{K[C(K_1) - Se^{-r_f\tau}]}{K_1} + Se^{-r_f\tau}$$

to the standard lower bound representing the forward from Equation (2),

$$C(K) \geq -Ke^{-r_d\tau} + Se^{-r_f\tau}. \tag{34}$$

If

$$\frac{C(K_1) - Se^{-r_f\tau}}{K_1} \geq -e^{-r_d\tau}$$

holds, then we have a stricter lower bound since both lower bound functions are linear in K with the same value at $K = 0$, but the first one has a larger slope. It can be shown that this is indeed a stricter lower bound if one can ensure that the call price at strike K_1 is above the corresponding forward:

$$C(K_1) \geq Se^{-r_f\tau} - K_1e^{-r_d\tau}. \tag{35}$$

The following can then be derived

$$\frac{C(K_1) - Se^{-r_f\tau}}{K_1} \underset{(35)}{\geq} \frac{Se^{-r_f\tau} - K_1 e^{-r_d\tau} - Se^{-r_f\tau}}{K_1} = -e^{-r_d\tau}.$$

Thus, we have found a stricter lower bound on the call price function which needs to be respected to avoid arbitrage. This condition is important since the interpolation function $C(K)$ is defined on $[K_1, K_n]$ and does not include the zero strike call and its derivative as stated in Lemma (1). This implies that the conditions from Lemma (1) - including condition (17) - are not automatically fulfilled. What about inequality (17) which is a lower bound on the call derivative? Remember that this inequality can be transformed into an upper bound on the digital call. Do we have to include this condition in addition to condition (31)? The following Lemma shows that Inequality (31) and Inequality (17) are equivalent.

Lemma 3. *Let the call function $C(K)$ be convex on $[K_1, K_n]$. Inequality*

$$C(K) \geq \frac{K[C(K_1) - Se^{-r_f\tau}]}{K_1} + Se^{-r_f\tau} \tag{36}$$

holds if and only if

$$C'(K) \geq \frac{C(K) - Se^{-r_f\tau}}{K} \tag{37}$$

holds for all $K \in [K_1, K_n]$.

Proof. Let (36) hold. From convexity we have

$$C(y) \geq C(x) + C'(x)(y - x)$$

for all $x, y \in [K_1, K_n]$. Let $K \in (K_1, K_n]$. Then we deduce

$$C(K_1) \geq C(K) + C'(K)(K_1 - K) \Leftrightarrow \frac{C(K_1) - C(K)}{K_1 - K} \leq C'(K)$$

From Lemma 2 we know that if

$$C(K) \geq \frac{K[C(K_1) - Se^{-r_f\tau}]}{K_1} + Se^{-r_f\tau}$$

holds, then

$$\frac{Se^{-r_f\tau} - C(K)}{K} \geq \frac{C(K_1) - C(K)}{K - K_1} \tag{38}$$

holds, which leads to

$$\frac{C(K_1) - C(K)}{K - K_1} \leq \frac{Se^{-r_f\tau} - C(K)}{K} \Leftrightarrow C'(K) \geq \frac{C(K_1) - C(K)}{K_1 - K} \geq \frac{C(K) - Se^{-r_f\tau}}{K}.$$

Now consider the finite difference approximation of $C'(K_1)$. We have for $h > 0$

$$\begin{aligned}\frac{C(K_1+h)-C(K_1)}{h} &\underset{(36)}{\geq} \frac{1}{h}\Big[(K_1+h)\frac{C(K_1)-Se^{-r_f\tau}}{K_1}+Se^{-r_f\tau}-C(K_1)\Big]\\ &= \frac{C(K_1)-Se^{-r_f\tau}}{K_1}\end{aligned}$$

If $g(x) \geq f(x)$ for all x, then their limit has the same order. Consequently, we have

$$C'(K_1)=\lim_{h\to 0^+}\frac{C(K_1+h)-C(K_1)}{h}\geq \frac{C(K_1)-Se^{-r_f\tau}}{K_1}.$$

These results confirm Inequality (37). The reverse conclusion follows. From convexity we have

$$\begin{aligned}C(K) &\geq C(K_1)+C'(K_1)(K-K_1)\\ &\underset{(37)}{\geq} C(K_1)+\frac{C(K_1)-Se^{-r_f\tau}}{K_1}(K-K_1)\\ &= K\Big[\frac{C(K_1)-Se^{-r_f\tau}}{K_1}\Big]+Se^{-r_f\tau}.\end{aligned}$$

Which proves Inequality (36). □

3.5 Arbitrage Example

In Figure 3 a call function is plotted which fulfills the standard conditions but not the new condition (31), which is represented by the dashed straight line. It is obvious, that the line (31) is above the call price function below the strike of 66. The call price function represents a cubic between the following strikes and call prices

$$K_1=60, C(K_1)=19.5706, K_2=75, C(K_2)=11.699$$

The call prices are Black-Scholes prices with the following parameters:

$$\sigma(K_1)=0.29, \sigma(K_2)=0.26, r_d=0.031, r_f=0.012, \tau=3.0, S=70.$$

As stated before, the call price function is given by

$$C(K)=a+b(K-K_1)+c(K-K_1)^2+d(K-K_1)^3 \tag{39}$$

with

$$a = 19.5706, b = -0.900513, c = 0.0151978, d = 0.000656768.$$

In Figure 4, the first and second derivatives of the cubic are shown. The

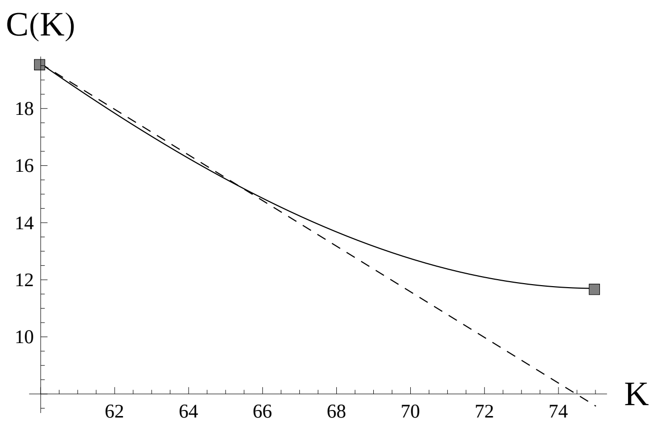

Fig. 3: Arbitrage Violation Chart. Dashed line represents condition (31). Black line is the call price function (39).

standard arbitrage bounds on the first derivative are represented by horizontal dashed lines. One can see that both derivatives fulfill the standard conditions. Finally, one can show that the call price function is also above

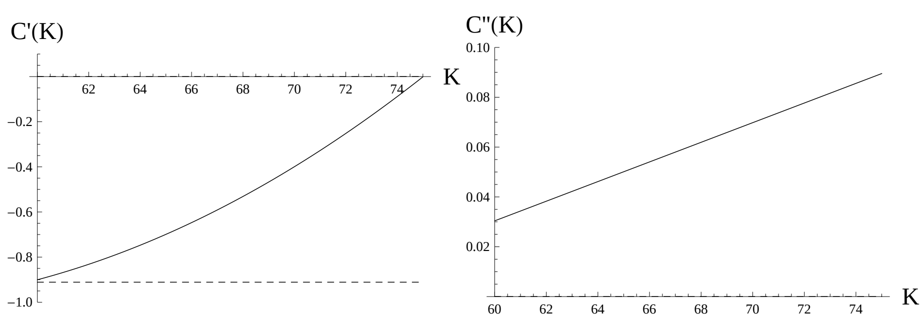

Fig. 4: First (left panel) and second (right panel) derivative of the call price function. Bounds are represented with dashed lines.

the forward. The violation of Inequality (31) can be exploited as follows:

- Choose the strike $\overline{K} = 62.8 \in [K_1, K_2]$.
- Buy the portfolio corresponding to the value

$$\frac{e^{-r_f \tau} S - C(\overline{K})}{\overline{K}}.$$

The price of this position in our example is 0.801625.

- On the other hand, sell the following portfolio value

$$\frac{C(K_1) - C(\overline{K})}{\overline{K} - K_1}.$$

The price of this position in our example is 0.85281.

- The initial profit is 0.051185. In addition, the first portfolio has a payoff which is strictly larger than the one of the short portfolio.

4 Summary

We have stated the common no-arbitrage conditions for a fixed time to maturity which appear in the literature

$$\begin{aligned} \max(Se^{-r_f\tau} - Ke^{-r_d\tau}, 0) \leq\ & C(K) \ \leq Se^{-r_f\tau}, \\ -e^{-r_d\tau} \leq\ & C'(K) \ \leq 0, \\ & C''(K) \geq 0. \end{aligned}$$

We have then derived the following lower bound on the call strike derivative which has to hold in addition to the conditions above:

$$\frac{\partial C(K)}{\partial K} \geq \frac{C(K) - e^{-r_f\tau}S}{K}.$$

To guarantee that all conditions are fulfilled on a strike interval $[K_1, K_n]$, one can choose two alternative sets of conditions which are implied by Lemma (1) and Lemma (3) respectively:

Condition Set 1:

$$\begin{aligned} C(0) &= Se^{-r_f\tau} \\ C'(0) &= -e^{-r_d\tau} \\ C(K) &\geq 0 \quad \forall K \in [0, K_n] \\ C'(K) &\leq 0 \quad \forall K \in [0, K_n] \\ C''(K) &\geq 0 \quad \forall K \in [0, K_n] \end{aligned}$$

In this case, one has to explicitly consider the extrapolation despite the focus on the interval $[K_1, K_n]$.

Condition Set 2:

$$
\begin{aligned}
C(K_1) &\geq Se^{-r_f\tau} - K_1e^{-r_d\tau} \\
C(K) &\geq \frac{K[C(K_1) - Se^{-r_f\tau}]}{K_1} + Se^{-r_f\tau} \quad \forall K \in [K_1, K_n] \\
C(K) &\geq 0 \quad \forall K \in [K_1, K_n] \\
C'(K) &\leq 0 \quad \forall K \in [K_1, K_n] \\
C''(K) &\geq 0 \quad \forall K \in [K_1, K_n]
\end{aligned}
$$

References

Ait-Sahalia, Y. and D. Jefferson, "Nonparametric option pricing under shape restrictions," *Journal of Econometrics*, 2003, *116*, 9–47.

Birke, M. and K.F. Pilz, "Nonparametric option pricing with no-arbitrage constraints," *Journal of Financial Econometrics*, 2009, *7* (2), 53–76.

Breeden, D. T. and R.H. Litzenberger, "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business*, 1978, *51* (4), 621–651.

Brunner, B. and R. Hafner, "Arbitrage-free estimation of the risk-neutral density from the implied volatility smile," *Journal of Computational Finance*, 2003, *7* (1), 75–106.

Carr, P., "Implied Vol Constraints," November 2004. Bloomberg Working Paper.

_ **and D.B. Madan**, "A note on sufficient conditions for no arbitrage," *Finance Research Letters*, 2005, *2* (3), 125–130.

Delbaen, F. and W. Schachermayer, *The Mathematics of Arbitrage*, Springer, 2006.

Fengler, M.R., "Arbitrage-free smoothing of the implied volatility surface," *Quantitative Finance*, 2009, *9* (4), 417–428.

Gatheral, J., "Rational shapes of the volatility surface," 2000. Merrill Lynch Presentation.

Härdle, W. and Z. Hlávka, "Dynamics of state price densities," *Journal of Econometrics*, 2009, *150* (1), 1–15.

Jeanblanc, M., M. Yor, and M. Chesney, *Mathematical methods for financial markets*, Springer Verlag, 2009.

Joshi, M.S., *The concepts and practice of mathematical finance*, Cambridge University Press, 2003.

Lee, R.W., "Implied volatility: Statics, dynamics, and probabilistic interpretation," *Recent Advances in Applied Probability*, 2005, pp. 241–268.

Merton, R.C., "Theory of rational option pricing," *The Bell Journal of Economics and Management Science*, 1973, *4* (1), 141–183.

Reiner, E., "Calendar Spreads, characteristic functions, and variance interpolation," 2000. Mimeo.

Reiss, O. and U. Wystup, "Computing option price sensitivities using homogeneity and other tricks," *The Journal of Derivatives*, 2001, *9* (2), 41–53.

Wang, Y., H. Yin, and L. Qi, "No-arbitrage interpolation of the option price function and its reformulation," *Journal of Optimization Theory and Applications*, 2004, *120* (3), 627–649.