Department of Mathematics Uppsala University
Seminar on Brownian Motion Autumn 2010
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Topic: Continuity properties and non-differentiability of Brownian motion

Notes on the sections 1.2 and 1.3 from the book Brownian motion by P. Mörters and Y. Peres.

1 Continuity properties of Brownian motion

In this section we describe continuity properties of Brownian motion using the modulus of continuity. This will lead to a proof of local α-Hölder continuity of Brownian motion.

1.1 Modulus of continuity

A modulus of continuity of the function $B : [0,1] \to \mathbb{R}$ is a function $\varphi : [0,\infty) \to [0,\infty)$ with $\lim_{h\downarrow 0} \varphi(h) = 0$ such that

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \frac{|B(t+h)-B(t)|}{\varphi(h)} \le 1.$$

In the book it is shown that there exists a constant $C > 0$ such that, almost surely, the function $C \cdot \sqrt{h\log(1/h)}$ is a modulus of continuity of Brownian motion.

Remark (on the proof of Theorem 1.12). If we are given a Brownian motion path, we can find a sequence of piecewise linear functions that forms Lévy's construction of that Brownian motion path. For almost every such path we can find N such that $\|F_n\|_\infty < c\sqrt{n}2^{-n/2}$ for $n > N$.

For large n, the function F_n has 2^n pieces. Hence, the image of F_n' consists of many values. However, the supremum norm of F_n' is not getting smaller with larger n. Therefore, in the third display on page 15, we use mean-value theorem to replace only the first l terms with the derivatives.

We need to find an upper bound on the sum

$$h\sum_{n=0}^{N} \|F_n'\|_\infty + 2ch \sum_{n=N+1}^{l} \sqrt{n}2^{n/2} + 2c \sum_{n=l+1}^{\infty} \sqrt{n}2^{-n/2}.$$

We can notice that there exists h small enough that the first sum is smaller than $\sqrt{h\log(1/h)}$. Now we can choose l such that $2^{-(l+1)} < h \le 2^{-l}$. The second and third term can be bounded by some constants multiples of $\sqrt{h\log(1/h)}$, where the constants don't depend on the choice of Brownian motion path. □

Later on, we can show that for $c < \sqrt{2}$, almost surely, in every neighbourhood of 0, there exist some h and $t \in [0,1-h]$ such that

$$|B(t+h)-B(t)| \ge c\sqrt{h\log(1/h)}.$$

This we can write also as, for any $c < \sqrt{2}$,

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \frac{|B(t+h)-B(t)|}{c\sqrt{h\log(1/h)}} \ge 1.$$

Remark (on the proof of Theorem 1.13, third display). For the events $A_{k,n}$ we also have that $\mathbb{P}(A_{k,n}) = \mathbb{P}(B(1) > c\sqrt{n}) \to 0$ as $n \to \infty$. Now, we can write

$$\mathbb{P}\left(\bigcap_{k=0}^{\lfloor e^n - 1\rfloor} A_{k,n}^c\right) \le (1-\mathbb{P}(A_{0,n}))^{e^n-2} \le \exp(-(e^n-2)\mathbb{P}(A_{0,n})) \to 0 \quad \text{as} \quad n\to\infty.$$

□

1.2 α-Hölder continuity

If $\alpha < 1/2$, then, almost surely, Brownian motion is everywhere locally α-Hölder continuous.

Remark (on the proof of Corollary 1.20). We would like to show that for every $t > 0$ there exist $\varepsilon > 0$ and $c > 0$ such that

$$|B(t+h)-B(t)| < ch^\alpha$$

whenever $|h| < \varepsilon$. First, note that there exist $h_1 > 0$ such that

$$h^\alpha > \sqrt{h\log(1/h)} \quad \text{for} \quad h < h_1.$$

Thus, if we show that, almost surely,

$$|B(t+h)-B(t)| < c\sqrt{h\log(1/h)}$$

then we are done. This claim is similar to the claim of Theorem 1.12, but there t was in the interval $[0,1]$ and h was positive. For any t we can find integer k such that $k \le t < k+1$. If we now restrict our Brownian motion to the interval $[k,k+1]$ and take the value $B(k)$ as the origin, we can apply Theorem 1.12 to the new Brownian motion $\{\hat{B}(\hat{t}) = B(k+\hat{t}) - B(k) : \hat{t} \in [0,1]\}$. For this Brownian motion we can find $h_2(t) > 0$ such that $h_2(t) < k+1-t$ and that for all $h < h_2(t)$, almost surely,

$$|B(t+h)-B(t)| = |\hat{B}(t-k+h) - \hat{B}(t-k)| \le c\sqrt{h\log(1/h)}.$$

Also, to prove the result for negative value of h, we can take the same restricted Brownian motion, but now with the origin in $B(k+1)$ and going backwards, $\{\tilde{B}(\tilde{t}) = B(k+1-\tilde{t}) - B(k+1) : \tilde{t} \in [0,1]\}$. Now, we can find $h_3(t) > 0$ such that $h_3(t) < t-k$ and that for all $0 < h < h_3(t)$, almost surely,

$$|B(t-h)-B(t)| = |\tilde{B}(k+1-t+h) - \tilde{B}(k+1-t)| \le c\sqrt{h\log(1/h)}.$$

Thus, for any t we can find $h_0(t) = \min\{h_1, h_2(t), h_3(t)\}$, such that almost surely,

$$|B(t+h)-B(t)| < ch^\alpha \text{ for } |h| < h_0(t).$$

Therefore Brownian motion is locally α-Hölder continuous. □

Remark. Brownian motion defined on any closed interval (or any compact set) is α-Hölder continuous. □

2 Non-differentiability of Brownian motion

The increment $B(t+h)-B(t)$ is a normal random variable with mean 0, and variance h, thus $h^{-1}(B(t+h)-B(t))$ is a normal random variable with mean 0 and variance h^{-1}. When h goes to 0, the variance of that random varible goes to infinity. Therefore, we cannot expect that the upper and lower right derivatives are finite. Here we have two theorems that confirm our intuition.

Theorem (1.27)**.** *Fix $t \geq 0$. Then, almost surely, Brownian motion is not differentiable at t. Moreover, the upper right derivative $D^*B(t) = +\infty$ and the lower right derivative $D_*(t) = -\infty$.*

Theorem (1.30)**.** *Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely, for all t, either $D^*B(t) = +\infty$ or $D_*(t) = -\infty$ or both.*

Here is another formulation of Theorem 1.30:

Almost all paths of Brownian motion are nowhere differentiable, since

$$\mathbb{P}\left(\forall t \geq 0 : \limsup_{h\downarrow 0} \frac{|B(t+h)-B(t)|}{h} = +\infty\right) = 1.$$

To prove Theorem 1.30, we need to show that for any M,

$$\mathbb{P}\left(\exists t \in [0,1] \text{ s.t. } \limsup_{h\downarrow 0} \frac{|B(t+h)-B(t)|}{h} \leq M\right) = 0.$$

The statements of Theorems 1.27 and 1.30 might seem quite similar, but one theorem doesn't imply the other. From Theorem 1.27 we know that the measure of the set of paths that are non-differentiable at t is 1, but we cannot conclude that the intersection over all t of these sets of paths has measure 1 itself as it is an uncountable intersection. Theorem 1.30 claims that the measure of the set of all paths that are non-differentiable at t is 1, but it does not prove that *both* upper and lower right derivatives do not exist.