Lecture 1

We should start with administrative stuff from the syllabus. Instructor, grader, undergrad tutors, solicitation for office hour times, text, homework due on Tuesday in class. Grade breakdown.

Linear algebra is the study of linear functions. In $\mathbb{R}^n$ these are functions f satisfying

$$f(x+y) = f(x) + f(y) \text{ and } f(cx) = cf(x), \; x, y \in \mathbb{R}^n, \; c \in \mathbb{R}.$$

We will generalize this immediately, taking from R^n only what we absolutely need. We start by looking at the c value above: it is called a scalar. Generally scalars do not need to come from $\mathbb{R}$. There is only some amount of structure we need for the set of scalars.

Definition 0.1. *A set $\mathbb{F}$ is called a field if for each $a, b \in \mathbb{F}$, there is an element $ab \in \mathbb{F}$ and another $a + b \in \mathbb{F}$ such that*

1. *for all $a, b, c \in \mathbb{F}$, $(ab)c = a(bc)$ and $(a + b) + c = a + (b + c)$,*
2. *for all $a, b \in \mathbb{F}$, $ab = ba$ and $a + b = b + a$,*
3. *there exist element $0, 1 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$, $a + 0 = a$ and $1a = a$,*
4. *for all $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$ and if $a \neq 0$ there is an element $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$ and*
5. *for all $a, b, c \in \mathbb{F}$, $a(b + c) = ab + ac$.*

This is our generalization of $\mathbb{R}$. Note one interesting point: there is nothing that asserts that $\mathbb{F}$ must be infinite, and indeed there are finite fields. Take any prime p and consider the set $\mathbb{Z}_p$ given by

$$\mathbb{Z}_p = \{0, \ldots, p-1\} \text{ with modular arithmetic .}$$

That is, $a + b$ is defined as $(a + b) \mod p$ (for instance $(2 + 5) \mod 3 = 1$). Then this is a field. You will verify this in the exercises. Another neat fact: if $\mathbb{F}$ is a finite field then it must have p^n elements for some prime p and $n \in \mathbb{N}$. You will prove this too.

Other examples are $\mathbb{R}$ and $\mathbb{C}$.

Given our field of scalars we are ready to generalize the idea of $\mathbb{R}^n$; we will call this a vector space.

Definition 0.2. *A collection $(V, \mathbb{F})$ of a set V and a field $\mathbb{F}$ is called a vector space (the elements of V called vectors and those of $\mathbb{F}$ called scalars) if the following hold. For each $v, w \in V$ there is a vector sum $v + w \in V$ such that*

1. *there is one (and only one) vector called $\vec{0}$ such that $v + \vec{0} = v$ for all $v \in V$,*
2. *for each $v \in V$ there is one (and only one) vector $-v$ such that $v + (-v) = \vec{0}$,*

3. *for all* $v, w \in V$, $v + w = w + v$,
4. *for all* $v, w, z \in V$, $v + (w + z) = (v + w) + z$.

Furthermore for all $v \in V$ *and* $c \in \mathbb{F}$ *there is a scalar product* $cv \in V$ *such that*

1. *for all* $v \in V$, $1v = v$,
2. *for all* $v \in V$ *and* $c, d \in \mathbb{F}$, $(cd)v = c(dv)$,
3. *for all* $v, w \in V$ *and* $c \in \mathbb{F}$, $c(v + w) = cv + cw$ *and*
4. *for all* $v \in V$ *and* $c, d \in \mathbb{F}$, $(c + d)v = cv + dv$.

This is really a ton of rules but they have to be verified! In case $(V, \mathbb{F})$ is a vector space, we will typically say V is a vector space over $\mathbb{F}$ or V is an $\mathbb{F}$-vector space. Let's look at some examples.

1. Take $V = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$. We define addition as you would imagine:
$$(v_1, \ldots, v_n) + (w_1, \ldots, w_n) = (v_1 + w_1, \ldots, v_n + w_n)$$
and scalar multiplication by
$$c(v_1, \ldots, v_n) = (cv_1, \ldots, cv_n) \ .$$
2. Let $\mathbb{F}$ be any field with $n \in \mathbb{N}$ and write
$$\mathbb{F}^n = \{(a_1, \ldots, a_n) : a_i \in \mathbb{F} \text{ for } i = 1, \ldots, n\}$$
and define addition and scalar multiplication as above. This is a vector space. Note in particular that $\mathbb{F}$ is a vector space over itself.
3. If $\mathbb{F}_1 \subset \mathbb{F}_2$ are fields (with the same $0, 1$ and operations) then $\mathbb{F}_2$ is a vector space over $\mathbb{F}_1$. This situation is called a *field extension*.
4. Let S be any nonempty set and $\mathbb{F}$ a field. Then define
$$V = \{f : S \to \mathbb{F} : f \text{ a function}\} \ .$$
Then V is an $\mathbb{F}$-vector space using the operations
$$(f_1 + f_2)(s) = f_1(s) + f_2(s) \text{ and } (cf_1)(s) = c(f_1(s)) \ .$$

Facts everyone should see once.

1. For all $c \in \mathbb{F}$, $c\vec{0} = \vec{0}$.

Proof.
$$c\vec{0} = c(\vec{0} + \vec{0}) = c\vec{0} + c\vec{0}$$

$$\begin{aligned}\vec{0} = c\vec{0} + (-c\vec{0}) &= (c\vec{0} + c\vec{0}) + (-c\vec{0}) \\ &= (c\vec{0}) + (c\vec{0} + (-c\vec{0})) = c\vec{0}\ .\end{aligned}$$

□

2. For all $v \in V$, $0v = \vec{0}$.

   Proof. $0v = (0 + 0)v = 0v + 0v$. Adding $-(0v)$ to both sides gives the result. □

3. For all $v \in V$, $(-1)v = -v$.

   Proof.
   $$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = \vec{0}\ .$$
   □

Subspaces

Definition 0.3. *Let V be a vector space over $\mathbb{F}$. Then $W \subset V$ is called a subspace of V if W is a vector space over $\mathbb{F}$ using the same operations as in V.*

Suppose we are given a vector space V. To check that $W \subset V$ is a subspace we need to verify eight properties! Do not worry – many of them follow immediately, by "inheritance." That is, they are true simply because they were true in V. For example if V is a vector space over $\mathbb{F}$ and $v, w \in W$ then clearly $v + w = w + v$, since these are also vectors in V and addition is commutative in V.

We only need to check the following.

1. $\vec{0} \in W$.

2. (closed under addition) For all $v, w \in W$, $v + w \in W$.

3. (closed under scalar multiplication) For all $v \in W$ and $c \in \mathbb{F}$, $cv \in W$.

4. (closed under inverses) For all $v \in W$, $-v \in W$.

Proposition 0.4. *Let $(V, \mathbb{F})$ be a vector space. Then $W \subset V$ is a subspace if and only if it is nonempty and for all $v, w \in W$ and $c \in \mathbb{F}$, $cv + w \in W$.*

Proof. Suppose that W satisfies the property in the proposition. Then let $v \in W$. Taking $v = w$ and $c = -1$, we get $\vec{0} = v + (-1)v \in W$. Next, if $c \in \mathbb{F}$ then $cv = \vec{0} + cv \in W$. If $w \in W$ then $v + (1)w = v + w \in W$, giving W as a subspace. Conversely, if W is a subspace then for all $v \in W$ and $c \in \mathbb{F}$, $cv \in W$, so if $w \in W$, we get $cv + w \in W$. Furthermore W is nonempty since it contains $\vec{0}$. □

If V is a vector space over $\mathbb{F}$ with W_1, W_2 subspaces we can generate a new space. We define

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\} \ .$$

Generally we define

$$W_1 + \cdots + W_n = (W_1 + \cdots + W_{n-1}) + W_n \ .$$

Claim 0.5. *$W_1 + W_2$ is a subspace.*

Proof. First it is nonempty. Next if $v, w \in W_1 + W_2$ and $c \in \mathbb{F}$, we can write $v = w_1 + w_2$ and $w = w_1' + w_2'$ for $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$. Then

$$cv + w = c(w_1 + w_2) + (w_1' + w_2') = (cw_1 + w_1') + (cw_2 + w_2') \ .$$

Since W_1 and W_2 are subspaces, the first element is in W_1 and the second in W_2, giving $cv + w \in W_1 + W_2$, so it is a subspace. □

Question from class. If V is a vector space over $\mathbb{F}$ and W is a subset of V that is a vector space using the same operations of addition and scalar multiplication, can the zero element of W be different from the zero element of V? No. Let $\vec{0}_W$ be the zero element from W. Then $\vec{0}_W + \vec{0}_W = \vec{0}_W$. However denoting by v the additive inverse element of $\vec{0}_W$ from V, we have

$$\vec{0} = \vec{0}_W + v = (\vec{0}_W + \vec{0}_W) + v = \vec{0}_W + (\vec{0}_W + v) = \vec{0}_W \ .$$