Lecture 9

Dual maps

Given $T : V \to W$ that is linear, we will define a corresponding transformation T^t on the dual spaces, but it will act in the other direction. We will have $T^t : W^* \to V^*$.

Definition 0.1. *If $T : V \to W$ is linear, we define the function $T^t : W^* \to V^*$ by the following. Given $g \in W^*$, set $T^t(g) \in V^*$ as the linear functional such that*

$$(T^t(g))(v) = g(T(v)) \text{ for all } v \in V \ .$$

T^t is called the transpose of T.

Note that the definition here is $T^t(g) = g \circ T$. Since both maps on the right are linear, so is their composition. So $T^t(g)$ is in fact a linear functional (it is in V^*).

Proposition 0.2. *If $T : V \to W$ is linear then $T^t : W^* \to V^*$ is linear.*

Proof. Let $g_1, g_2 \in W^*$ and $c \in \mathbb{F}$. We want to show that $T^t(cg_1 + g_2) = cT^t(g_1) + T^t(g_2)$ and these are both elements of V^*, so we want to show they act the same on each element of V. So let $v \in V$ and compute

$$\begin{aligned}(T^t(cg_1 + g_2))(v) = (cg_1 + g_2)(T(v)) = cg_1(T(v)) + g_2(T(v)) &= c(T^t(g_1))(v) + (T^t(g_2))(v) \\ &= (cT^t(g_1) + T^t(g_2))(v) \ .\end{aligned}$$

□

The matrix of T^t can be written in a very simple way using dual bases.

Theorem 0.3. *Let $T : V \to W$ be linear and B, C bases for V and W. Writing B^* and C^* for the dual bases,*

$$[T^t]_{B^*}^{C^*} = \left([T]_C^B\right)^t \ .$$

The matrix on the right the transpose matrix; that is, if A is a matrix then the transposed matrix A^t is defined by $(A^t)_{i,j} = A_{j,i}$.

Proof. Let $B = \{v_1, \ldots, v_n\}$ and $C = \{w_1, \ldots, w_m\}$. When we build the matrix $[T]_C^B$, we make the j-th column by expanding $T(v_j)$ in terms of the basis C. So our matrix can be rewritten as

$$[T]_C^B = \begin{pmatrix} w_1^*(T(v_1)) & w_1^*(T(v_2)) & \cdots & w_1^*(T(v_n)) \\ w_2^*(T(v_1)) & w_2^*(T(v_2)) & \cdots & w_2^*(T(v_n)) \\ & \cdots & \cdots & \\ w_m^*(T(v_1)) & w_m^*(T(v_2)) & \cdots & w_m^*(T(v_n)) \end{pmatrix} \ .$$

To build the matrix $[T^t]_{B^*}^{C^*}$, we begin with the first vector of C^* and express it in terms of B^*. We write

$$T^t(w_1^*) = a_1 v_1^* + \cdots + a_n v_n^* \ .$$

The coefficients have a simple form:

$$\begin{aligned} T^t(w_1^*) &= (T^t(w_1^*))(v_1)v_1^* + \cdots + (T^t(w_1^*))(v_n)v_n^* \\ &= w_1^*(T(v_1))v_1^* + \cdots + w_1^*(T(v_n))v_n^* \ . \end{aligned}$$

This means the first column of our matrix is

$$\begin{pmatrix} w_1^*(T(v_1)) \\ \cdots \\ w_1^*(T(v_n)) \end{pmatrix} \ .$$

This is just the first row of $[T]_C^B$. Similarly, the j-th column is the j-th row of $[T]_C^B$ and this completes the proof. $\square$

Proposition 0.4. *Let $T : V \to W$ be linear with V, W finite-dimensional. Then*

1. $N(T^t) = R(T)^\perp$,
2. $R(T^t) = N(T)^\perp$,
3. *rank*$(T^t) =$ *rank*(T) *and nullity*$(T^t) =$ *nullity*(T).

Proof. For the first item, let $g \in R(T)^\perp$. Then we would like to show that $g \in N(T^t)$, or that $T^t(g) = 0$. Since $T^t(g) \in V^*$ this amounts to showing that $(T^t(g))(v) = 0$ for all $v \in V$. So let $v \in V$ and compute

$$(T^t(g))(v) = g(T(v)) = 0 \ ,$$

since g annihilates the range of T. This shows that $R(T)^\perp \subset N(T^t)$. For the other direction, let $g \in N(T^t)$ and $w \in R(T)$. Then we can find $v \in V$ such that $w = T(v)$ and so

$$g(w) = g(T(v)) = (T^t(g))(v) = 0 \ ,$$

since $T^t(g) = 0$. This completes the proof of the first item.

Next if $f \in R(T^t)$ we can find $g \in W^*$ such that $f = T^t(g)$. If $v \in N(T)$ then

$$f(v) = (T^t(g))(v) = g(T(v)) = g(0) = 0 \ ,$$

so $f \in N(T)^\perp$. This shows that $R(T^t) \subset N(T)^\perp$. To show the other direction, we count dimensions.

$$\begin{aligned} \dim R(T^t) &= \dim W - \dim N(T^t) \\ &= \dim W - \dim R(T)^\perp \\ &= \dim R(T) \\ &= \dim W - \dim N(T) \\ &= \dim N(T)^\perp \ . \end{aligned}$$

Since these spaces have the same dimension and one is contained in the other, they must be equal.

The last item follows from dimension counting as well.

$\square$

Double dual

We now move one level up, to look at the dual of the dual, the double dual.

Definition 0.5. *If V is a vector space, we define the double dual V^{**} as the dual of V^*. It is the space $L(V^*, \mathbb{F})$ of linear functionals on V^*.*

As before, when V is finite-dimensional, since $\dim L(V^*, \mathbb{F}) = \dim(V^*) \dim(\mathbb{F})$, we find

$$\dim V^{**} = \dim V \text{ when } \dim V < \infty .$$

Exercise. Show this is true even if $\dim V = \infty$.

There are some simple elements of V^{**}, the evaluation maps. given $v \in V$ we set

$$eval_v : V^* \to \mathbb{F} \text{ by } eval_v(f) = f(v) .$$

- $eval_v$ is a linear functional on V^*. To see this, note first that it certainly maps V^* to $\mathbb{F}$, so we must only show it is linear. The proof is the same as in the previous homework: let $f, g \in V^*$ and $c \in \mathbb{F}$. Then

$$eval_v(cf + g) = (cf + g)(v) = cf(v) + g(v) = c\ eval_v(f) + eval_v(g) .$$

- In fact the map $\Phi : V \to V^{**}$ given by $\Phi(v) = eval_v$ is an isomorphism when $\dim V < \infty$. It is called the *natural isomorphism*. We first show it is linear, so let $v_1, v_2 \in V$ and $c \in \mathbb{F}$. If $f \in V^*$ then

$$\begin{aligned}(\Phi(cv_1 + v_2))(f) = eval_{cv_1+v_2}(f) &= f(cv_1 + v_2) = cf(v_1) + f(v_2) \\ &= c\ eval_{v_1}(f) + eval_{v_2}(f) = c(\Phi(v_1))(f) + (\Phi(v_2))(f) \\ &= (c\Phi(v_1) + \Phi(v_2))(f) .\end{aligned}$$

  Since this is true for all f, we get $\Phi(cv_1 + v_2) = c\Phi(v_1) + \Phi(v_2)$.

  Now to prove that Φ is an isomorphism we only need to show injective (since V and V^{**} have the same dimension). So assume that $\Phi(v) = 0$ (the zero element of V^{**}). Then for all $f \in V^*$, we have $f(v) = eval_v(f) = 0$. We now use a lemma to finish the proof.

  Lemma 0.6. *If $v \in V$ is nonzero, there exists $f \in V^*$ such that $f(v) \neq 0$.*

  Proof. Let $v \in V$ be nonzero and extend it to a basis B for v. Then the element v^* in the dual basis B^* has $v^*(v) = 1$. So set $f = v^*$. □

  Because $f(v) = 0$ for all $f \in V^*$, the lemma says $v = \vec{0}$. Therefore $N(\Phi) = \{\vec{0}\}$ and Φ is injective, implying it is an isomorphism.