Lecture 7

Last time we saw that if V and W have dimension n and m and we fix bases B of V and C of W then there is an isomorphism $\Phi : L(V, W) \to M_{m,n}(\mathbb{F})$ given by

$$\Phi(T) = [T]_C^B .$$

A simple corollary of this follows. Because Φ of any basis is a basis, these spaces have the same dimension:

Corollary 0.1. *The dimension of $L(V, W)$ is mn, where V has dimension n and W has dimension m. Given bases B of V and C of W, a basis of $L(V, W)$ is given by the set of size mn*

$$\{T_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq m\} ,$$

where $T_{i,j}$ is the unique linear transformation sending v_i to w_j and all other elements of B to $\vec{0}$.

Proof. Since $L(V, W)$ and $M_{m,n}(\mathbb{F})$ are isomorphic, they have the same dimension, which in the latter case is mn (that was a homework problem). Further the basis of $M_{m,n}(\mathbb{F})$ of size mn given by the matrices with a 1 in the (i, j)-th entry and 0 everywhere else map by Φ^{-1} to a basis for $L(V, W)$, and it is exactly the set listed in the corollary. □

We can now give many nice properties of the matrix representation.

1. Let $T : V \to W$ and $U : W \to Z$ be linear with B, C, D bases for V, W, Z. For any $v \in V$,
$$[(U \circ T)v]_D = [U(T(v))]_D = [U]_D^C [T(v)]_C = [U]_D^C [T]_C^B [v]_B .$$
However $[U \circ T]_D^B$ is the unique matrix with this property, so we find
$$[U \circ T]_D^B = [U]_D^C [T]_C^B .$$
In other words, transformation composition corresponds to matrix multiplication. A good way to remember this is that the C's "cancel out" on the right.

2. If $T : V \to W$ is an isomorphism, setting $Id_V : V \to V$ as the identity map and $Id_W : W \to W$ as the identity map and I as the identity matrix,
$$I = [Id_V]_B^B = [T]_C^B [T^{-1}]_B^C$$
$$I = [Id_W]_C^C = [T^{-1}]_B^C [T]_C^B .$$
In other words, $[T]_C^B$ is an invertible matrix.

Definition 0.2. *We say that $A \in M_{n,n}(\mathbb{F})$ is invertible if there is a $B \in M_{n,n}(\mathbb{F})$ such that $AB = BA = I$.*

You will show in the homework if A is invertible, there is exactly one (invertible) B that satisfies $AB = BA = I$. Therefore we write $A^{-1} = B$. This gives

$$\left([T]_C^B\right)^{-1} = [T^{-1}]_B^C \ .$$

Exercise: if A is an invertible $n \times n$ matrix and B is a basis for V then there is an isomorphism $T : V \to V$ such that $[T]_B^B = A$.

We summarize the relation between linear transformations and matrices using the following table. Fix V, W, $T : V \to W$ and bases B, C of V, W.

Linear transformations	Matrices
$v \in V$	the $n \times 1$ column vector $[v]_B$
$w \in W$	the $m \times 1$ column vector $[w]_C$
T	the $m \times n$ matrix $[T]_C^B$
$U \circ T$ (composition)	$[U]_D^C [T]_C^B$ (matrix multiplication)
isomorphisms	invertible matrices

3. **Change of basis.** Suppose we have $T : V \to W$ with B, C bases of V, W. We would like to relate $[T]_C^B$ to $[T]_{C'}^{B'}$, the matrix relative to other bases B', C' of V, W. How do we do this? Consider the matrices $[Id_V]_B^{B'}$ and $[Id_W]_{C'}^C$:

$$[Id_W]_{C'}^C [T]_C^B [Id_V]_B^{B'} = [Id_W \circ T \circ Id_V]_{C'}^{B'} = [T]_{C'}^{B'} \ .$$

Note that $[Id_W]_{C'}^C$ and $[Id_V]_B^{B'}$ are invertible. Therefore:

- If $T : V \to W$ is linear and B, B' are bases of V with C, C' bases of W, there exist invertible matrices $P = [Id_W]_{C'}^C \in M_{m,m}(\mathbb{F})$ and $Q = [Id_V]_B^{B'} \in M_{n,n}(\mathbb{F})$ such that

$$[T]_{C'}^{B'} = P[T]_C^B Q \ .$$

- Not only is each $[Id_V]_B^{B'}$ invertible, each invertible matrix can be seen as a change of basis matrix: given a basis B of V and an invertible matrix $P \in M_{n,n}(\mathbb{F})$, there exists a basis B' of V such that $P = [Id_V]_B^{B'}$.

  Proof. By the exercise above, there is an isomorphism $T_P : V \to V$ such that $[T_P]_B^B = P$. Writing $B = \{v_1, \ldots, v_n\}$, define $B' = \{T(v_1), \ldots, T(v_n)\}$. Then the j-th column of $[Id_V]_B^{B'}$ is computed by evaluating

$$[Id_V(T(v_j))]_B = [T(v_j)]_B = j\text{-th column of } [T_P]_B^B = P \ .$$

  So $[Id_V]_B^{B'}$ and P have the same columns and are thus equal. □

- In one case we have a simpler form for P and Q. Suppose that $T : V \to V$ is linear and B, B' are bases for V. Then

$$[T]_{B'}^{B'} = [Id_V]_{B'}^B [T]_B^B [Id_V]_B^{B'} \ .$$

  That is, we have $[T]_{B'}^{B'} = P^{-1}[T]_B^B P$, where P is an invertible $n \times n$ matrix. This motivates the definition

Definition 0.3. *Two $n \times n$ matrices A and B are said to be similar if there is an invertible $n \times n$ matrix P such that $B = P^{-1}AP$.*

The message is that similar matrices represent the same transformation but relative to a different basis. Therefore if there is some property of matrices that is the same for all matrices that are similar, we are right to say it is a property of the underlying transformation. For instance we define the trace of an $n \times n$ matrix A by

$$Tr(A) = \sum_{i=1}^{n} A_{i,i} \ .$$

We can show easily that $Tr(AB) = Tr(BA)$:

$$\begin{aligned} Tr(AB) &= \sum_{i=1}^{n} (AB)_{i,i} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{i,k} B_{k,i} \\ &= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{k,i} A_{i,k} = \sum_{k=1}^{n} (BA)_{k,k} = Tr(BA) \ . \end{aligned}$$

Therefore if P is invertible, $Tr(P^{-1}AP) = Tr(APP^{-1}) = Tr(A)$. This means that if $T : V \to V$ is linear, we can define its trace as $Tr(T) = Tr([T]_B^B)$ for any basis of B (and it will not depend on our choice of B!).