Today we will prove the existence and uniqueness of the determinant.

Theorem 0.1. *Let V be an $\mathbb{F}$-vector space and $\{e_1, \ldots, e_n\}$ a basis. There exists a unique n-linear alternating function f on V such that $f(e_1, \ldots, e_n) = 1$.*

Proof. We will first prove uniqueness, so assume that f is an n-linear alternating function on V such that $f(v_1, \ldots, v_n) = 1$. We will show that f must have a certain form. Let $v_1, \ldots, v_n \in V$ and write then as

$$v_k = a_{1,k}e_1 + \cdots + a_{n,k}e_n \ .$$

We can then expand using n-linearity:

$$\begin{aligned} f(v_1, \ldots, v_n) = f(a_{1,1}e_1 + \cdots + a_{n,1}e_n, v_2, \ldots, v_n) &= \sum_{i_1=1}^{n} a_{i,1} f(e_i, v_2, \ldots, v_n) \\ &= \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} a_{i_1,1} \cdots a_{i_n,n} f(e_{i_1}, \ldots, e_{i_n}) \\ &= \sum_{i_1,\ldots,i_n} a_{i_1,1} \cdots a_{i_n,n} f(e_{i_1}, \ldots, e_{i_n}) \ . \end{aligned}$$

Since f is alternating, all choices of $i_1, \ldots, i_n$ that are not distinct have $f(e_{i_1}, \ldots, e_{i_n}) = 0$. So we can write this as

$$\sum_{i_1,\ldots,i_n \text{ distinct}} a_{i_1,1} \cdots a_{i_n,n} f(e_{i_1}, \ldots, e_{i,n}) \ .$$

The choices of distinct $i_1, \ldots, i_n$ can be made using permutations. Each permutation $\pi \in S_n$ gives exactly one such choice. So we can yet again write as

$$\sum_{\pi \in S_n} a_{\pi(1),1} \cdots a_{\pi(n),n} f(e_{\pi(1)}, \ldots, e_{\pi(n)}) \ .$$

Using the lemma from last time, $f(e_{\pi(1)}, \ldots, e_{\pi(n)}) = \text{sgn}(\pi) f(e_1, \ldots, e_n) = \text{sgn}(\pi)$, so

$$f(v_1, \ldots, v_n) = \sum_{\pi \in S_n} \text{sgn}(\pi) a_{\pi(1),1} \cdots a_{\pi(n),n} \ .$$

If g is any other n-linear alternating function with $g(e_1, \ldots, e_n) = 1$ then the same computation as above gives the same formula for $g(v_1, \ldots, v_n)$, so $f = g$. This shows uniqueness.

For existence, we need to show that the formula above actually gives an n-linear alternating function with $f(e_1, \ldots, e_n) = 1$.

1. We first show $f(e_1, \ldots, e_n) = 1$. To do this, we write
$$e_k = a_{1,k}e_1 + \cdots + a_{n,k}e_n \ ,$$
where $a_{k,j} = 0$ unless $j = k$, in which case it is 1. If $\pi \in S_n$ is not the identity, we can find $k \neq j$ such that $\pi(k) = j$. This means that $a_{\pi(k),k} = a_{j,k} = 0$ and so $\text{sgn}(\pi)a_{\pi(1),1} \cdots a_{\pi(n),n} = 0$. Therefore
$$f(e_1, \ldots, e_n) = \sum_{\pi \in S_n} \text{sgn}(\pi)a_{\pi(1),1} \cdots a_{\pi(n),n} = \text{sgn}(id)a_{1,1} \cdots a_{n,n} = 1 \ .$$

2. Next we show alternating. Suppose that $v_i = v_j$ for some $i \neq j$ and let $\tau_{i,j}$ be the transposition (ij). Split all permutations into A, those which invert i and j and $S_n \setminus A$, those which do not. Then if $v_k = a_{1,k}e_1 + \cdots + a_{n,k}e_n$, we can write
$$\begin{aligned} f(v_1, \ldots, v_n) &= \sum_{\pi \in S_n} \text{sgn}(\pi)a_{\pi(1),1} \cdots a_{\pi(n),n} \\ &= \sum_{\pi \in A} \text{sgn}(\pi)a_{\pi(1),1} \cdots a_{\pi(n),n} + \sum_{\pi \in S_n \setminus A} \text{sgn}(\pi\tau_{i,j})a_{\pi\tau_{i,j}(1),1} \cdots a_{\pi\tau_{i,j}(n),n} \\ &= \sum_{\pi \in A} \text{sgn}(\pi)[a_{\pi(1),1} \cdots a_{\pi(n),n} - a_{\pi\tau_{i,j}(1),1} \cdots a_{\pi\tau_{i,j}(n),n}] \ . \end{aligned}$$
Note however that $a_{\pi\tau_{i,j}(i),i} = a_{\pi(j),i} = a_{\pi(j),j}$, since $v_i = v_j$. Similarly $a_{\pi\tau_{i,j}(j),j} = a_{\pi(i),i}$. Therefore
$$a_{\pi(1),1} \cdots a_{\pi(n),n} = a_{\pi\tau_{i,j}(1),1} \cdots a_{\pi\tau_{i,j}(n),n} \ .$$
So the above sum is zero and we are done.

3. For n-linearity, we will just show it in the first coordinate. So let $v, v_1, \ldots, v_n \in V$ and $c \in \mathbb{F}$. Writing
$$v_k = a_{1,k}e_1 + \cdots + a_{n,k}e_n \text{ and } v = a_1e_1 + \cdots + a_ne_n \ ,$$
then $cv + v_1 = (ca_1 + a_{1,1})e_1 + \cdots + (ca_n + a_{n,1})e_n$, so
$$\begin{aligned} f(cv + v_1, v_2, \ldots, v_n) &= \sum_{\pi \in S_n} \text{sgn}(\pi)(ca_{\pi(1)} + a_{\pi(1),1})a_{\pi(2),2} \cdots a_{\pi(n),n} \\ &= c\sum_{\pi \in S_n} \text{sgn}(\pi)a_{\pi(1)}a_{\pi(2),2} \cdots a_{\pi(n),n} + \sum_{\pi \in S_n} \text{sgn}(\pi)a_{\pi(1),1} \cdots a_{\pi(n),n} \\ &= cf(v, v_2, \ldots, v_n) + f(v_1, \ldots, v_n) \ . \end{aligned}$$

□

One nice property of n-linear alternating functions is that they can determine when vectors are linearly independent.

Theorem 0.2. *Let V be an n-dimensional $\mathbb{F}$-vector space and f a nonzero n-linear alternating function on V. Then $\{v_1, \ldots, v_n\}$ is linearly independent if and only if $f(v_1, \ldots, v_n) \neq 0$.*

Proof. If $n = 1$ then the proof is an exercise, so take $n \geq 2$ and first assume that the vectors are linearly dependent. Then we can write one as a linear combination of the others. Suppose for example that $v_1 = b_2 v_2 + \cdots + b_n v_n$. Then

$$f(v_1, \ldots, v_n) = b_2 f(v_2, v_2, \ldots, v_n) + \cdots + b_n f(v_n, v_2, \ldots, v_n) = 0 \ .$$

Here we have used that f is alternating.

Conversely suppose that $\{v_1, \ldots, v_n\}$ is linearly independent. Then it must be a basis. We can then proceed exactly along the development given above and, if $u_1, \ldots, u_n$ are vectors written as

$$u_k = a_{1,k} v_1 + \cdots + a_{n,k} v_n \ ,$$

then if $f(v_1, \ldots, v_n) = 0$, we find

$$f(u_1, \ldots, u_n) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{\pi(1),1} \cdots a_{\pi(n),n} f(v_1, \ldots, v_n) = 0 \ .$$

Therefore f is zero. This is a contradiction, so $f(v_1, \ldots v_n) \neq 0$. □

Definition 0.3. *Choosing $V = \mathbb{F}^n$ and $e_1, \ldots, e_n$ the standard basis, we write* det *(the determinant) for the unique n-linear alternating function such that* $\det(e_1, \ldots, e_n) = 1$. *If $A \in M_{n,n}(\mathbb{F})$ we define* $\det(A) = \det(\vec{a}_1, \ldots, \vec{a}_n)$, *where $\vec{a}_i$ is the i-th column of A.*

Corollary 0.4. *Let $A \in M_{n,n}(\mathbb{F})$. Then* $\det(A) \neq 0$ *if and only if A is invertible.*

Proof. By the previous theorem, $\det(A) \neq 0$ if and only if the columns of A are linearly independent. This is equivalent to saying that the column rank of A is n, or that A is invertible. □

One of the most important properties of the determinant is that it factors through products (compositions).

Theorem 0.5. *Let $A, B \in M_{n,n}(\mathbb{F})$. We have the following factorization:*

$$\det AB = \det A \cdot \det B \ .$$

Proof. First if $\det A = 0$ the matrix A cannot be invertible and therefore neither is AB, so $\det AB = 0$, proving the formula in that case. Otherwise we have $\det A \neq 0$. In this case we will use a method of proof that is very common when dealing with determinants. We will define a function on matrices that is n-linear and alternating as a function of the columns, mapping the identity to 1, and use the uniqueness of the determinant to see that it is just the determinant function. So define $f : M_{n,n}(\mathbb{F}) \to \mathbb{F}$ by

$$f(B) = \frac{\det AB}{\det A} \ .$$

First note that if I_n is the $n \times n$ identity matrix, $f(I_n) = (\det AI_n)/\det A = 1$. Next, if B has two equal columns, its column rank is strictly less than n and so is the column rank of AB, meaning that AB is non-invertible. This gives $f(B) = 0/\det A = 0$.

Last to show n-linearity of f as a function of the columns of B, write B in terms of its columns as $(\vec{b}_1, \ldots, \vec{b}_n)$. Note that if e_i is the i-th standard basis vector, then we can write $\vec{b}_i = Be_i$. Therefore the i-th column of AB is $(AB)e_i = A\vec{b}_i$ and $AB = (A\vec{b}_1, \ldots, A\vec{b}_n)$. Thus if $\vec{b}_1, \vec{b}_1'$ are column vectors and $c \in \mathbb{F}$,

$$\begin{aligned}
\det \left[A(c\vec{b}_1 + \vec{b}_1', \vec{b}_2, \ldots, \vec{b}_n)\right] &= \det(A(c\vec{b}_1 + \vec{b}_1'), A\vec{b}_2, \ldots, A\vec{b}_n) \\
&= \det(cA\vec{b}_1 + A\vec{b}_1', A\vec{b}_2, \ldots, A\vec{b}_n) \\
&= c\det(A\vec{b}_1, A\vec{b}_2, \ldots, A\vec{b}_n) + \det(A\vec{b}_1', A\vec{b}_2, \ldots, A\vec{b}_n) \ .
\end{aligned}$$

This means that $\det AB$ is n-linear (at least in the first column – the same argument works for all columns), and so is f.

There is exactly one n-linear alternating function f with $f(I_n) = 1$, so $f(B) = \det B$. □

Here are some consequences.

- Similar matrices have the same determinant.

  Proof.

  $$\det P^{-1}AP = \det P^{-1} \det A \det P = \det A \det P^{-1} \det P = \det A \det I_n = \det A \ .$$

  □

- If A is invertible then $\det(A^{-1}) = \frac{1}{\det A}$.