Bivariate option pricing with copulas

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Abstract

In this paper we suggest the adoption of copula functions in order to price bivariate contingent claims. Copulas enable us to imbed the marginal distributions extracted from vertical spreads in the options markets in a multivariate pricing kernel. We prove that such kernel is a copula function, and that its super-replication strategy is represented by the Fréchet bounds. As applications, we provide prices for binary digital options, options on the minimum and options to exchange one asset for another. For each of these products, we provide no-arbitrage pricing bounds, as well as the values consistent with independence of the underlying assets. As a final reference value, we use a copula function calibrated on historical data.

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1. Introduction

One of the open issues in contingent claim pricing is how to extend, in full generality, the well-known risk-neutral valuation technique to the multivariate case, that is the case of contingent claims written on "baskets" of several underlying assets, rather than on a single security.

As the existing literature demonstrates, even the pricing problem of bivariate contingent claims is far from elementary, whenever the simplifying – and sometimes unrealistic – hypothesis of independence or perfect correlation between the underlying risks is abandoned. It becomes even more involved when another questionable assumption is dropped, that of joint normality of returns. In this paper we describe and implement some new results concerning bivariate contingent claims, which hold in a very general setting.

Options on the maximum, minimum or options to exchange one asset for another are "traditional" examples of bivariate claims. Recently, bivariate digital options have come into fashion, particularly as building blocks, i.e. coupons, of structured debts products. More generally, one can also envisage a bivariate feature in every OTC contract, due to the joint riskiness of the underlying and the counterpart business (Klein, 1996).

In this paper we abandon both the linear correlation and gaussian assumptions in order to price bivariate claims. We explore copula functions as pricing devices. Copulas allow us not only to separate the impact on the joint distribution of the marginals and the association structure, but also to exploit non-parametric measures of the latter.

The paper is structured as follows: section 2 presents the multivariate pricing problem in detail. Section 3 introduces the notion of copula, reviews some of its properties and introduces the relationship with well-known non-parametric association measures. Section 4 explains how to use copulas to represent the bivariate digital option price, i.e. the pricing kernel of the bivariate economy. Section 5 extends the results to other bivariate claims, namely options on the minimum and options to exchange one asset for another. Section 6 presents an extensive application to four market indices (Mib30, S&P500, FTSE, DAX), for which we price bivariate digital options, options on the minimum and options to exchange. Section 7 concludes.

2. Multivariate Contingent Claims

In mathematical terms, the multivariate feature of a contingent claim shows up in a pay-off, which in general can be written as

$$g(f(S_i(T), T; i = 1, 2...n))$$

where $g(.)$ is a univariate pay-off function which identifies the derivative contract, $f(.)$ is a multivariate function which describes how the n underlying securities determine the final cash-flow, S_i denotes the price of the $i-th$ underlying security and T is the contract maturity. In what follows, for notational convenience, we skip the reference to T whenever it is not needed.

As an example, in the case of rainbow call options g is the familiar payoff function

$$g(f) = \max(f - K, 0)$$

where K is the strike price, while $f(\cdot)$ may select the minimum (or maximum or some kind of average) of n assets:

$$f(S_i(T), T; i = 1, 2...n) = \min(S_i(T); i = 1, 2...n)$$

The option on the minimum (or maximum) was first studied by Stulz (1982) in the lognormal case. In the same setting, Margrabe (1978) had already studied the most specific instance of the option to exchange.

Another example, which is easier to address, is the multivariate digital option. In this case the g function is simply a multiplicative constant; f spots the event that each underlying be greater than or equal to some corresponding threshold K_i

$$f(S_i(T), T; i = 1, 2...n) = I_{\{(S_1(T) \geq K_1) \cap .. \cap (S_n(T) \geq K_n)\}}$$

where I is the indicator function.

The multi-asset feature has a long standing history in fairly standard and mature derivative markets. The most standard case is offered by the market of options on futures: as typically the standardization feature of futures contracts entails a delivery grade option (or quality option), the option on the futures contract may be seen as an option written on a basket of deliverable products. The quality option issue, however, has not posed much of a problem as for the pricing techniques: in fact, the standardization feature of futures contracts also implies that most of the products eligible for delivery were chosen in such a way as to grant a high degree of similarity. In other words, the products chosen as deliverable under the contract are typically strongly correlated and considering them perfectly correlated is not much of a mistake. This is what is done when using a one factor model of the term structure in order to evaluate the quality option on long term interest futures and options .

It cannot be ignored that the perfect correlation assumption always leads to an approximation to the problem. Even in the case of the quality option in interest rate futures we know that this approach is not able to take into account some market anomalies, such as coupon or seasoning effects, that may play a relevant role in the determination of the cheapest asset to deliver. Besides this, there are also cases in which the imperfect correlation issue cannot be avoided, as it represents the main motivation which inspires the product. In fact, the pricing task gets more involved for products in which the multiasset feature is meant to provide diversification. The most straightforward example is offered by call options written on the minimum or maximum among some market indices. In these cases, ignoring imperfect dependence among the markets may lead to substantial mispricing of the products, as well as to inaccurate hedging policies and unreliable risk evaluations.

While the multivariate – or simply bivariate – pricing problem may be already complex in a standard gaussian world, the evaluation task is compounded by the well known evidence of departures from normality. Following the stock market crash in October 1987, departures from normality have shown up in the effects of smile and term structure of volatility and have become common ground of work for both academics and traders.

Of course, jointly taking into account non-normality of yields and their dependence structure makes the two problems far more involved. As a simple example of this complexity, just consider the fact that the linear correlation figure cannot be safely used in a non-normal world, as it may not be able to take values over the whole range from -1 to +1 (see for instance Embrechts, McNeil and Straumann,1999a, 1999b). In a non-gaussian world we may well observe a correlation figure lower than 1 in a case in which there is perfect dependence between the variables. So, the correlation figure may lead to mis-representation of the degree of diversification in a portfolio. By the same token, in a non-gaussian world a trader may pursue the task of modifying the correlation figure of his portfolio to a value which is simply impossible to reach under the real bivariate distribution in the data.

A possible strategy to address the problem of dependency under non-normality is to separate the two issues, i.e. working with non-gaussian marginal probability distributions and using some techniques to combine these distributions in a bivariate setting. This is the approach followed by Rosenberg (1999), who uses the set of Plackett distributions to price bivariate contingent claims consistently with given marginals. In the sequel we generalize his approach using copula functions, of which the Plackett family is only a specific case. The main advantage of the copula approach to pricing is to write the bivariate pricing kernel as a function of univariate pricing functions. This enables us to carry out sensitivity analysis with respect to the dependence structure of the underlying assets, separately from that on univariate prices.

3. Mathematical background

In what follows we give the definition of copula function and some of its basic properties, while we refer the reader interested in a more detailed treatment to Nelsen (1999) and Joe (1997). Since in the sequel we are going to price bivariate claims, here we stick to the bivariate copula: nonetheless, most of the results carry over to the general multivariate setting.

Definition 3.1. *A two-dimensional copula C is a real function defined on $I^2 \stackrel{d}{=} [0,1] \times [0,1]$, with range $I \stackrel{d}{=} [0,1]$, such that*
for every (v,z) of $I^2, C(v,0) = 0 = C(0,z)$, $C(v,1) = v, C(1,z) = z$;
for every rectangle $[v_1, v_2] \times [z_1, z_2]$ in I^2,with $v_1 \leq v_2$ and $z_1 \leq z_2$, $C(v_2, z_2) - C(v_2, z_1) - C(v_1, z_2) + C(v_1, z_1) \geq 0$

As such, it can represent the joint distribution function of two standard uniform random variables U_1, U_2:

$$C(u_1, u_2) = \Pr(U_1 \leq u_1, U_2 \leq u_2)$$

We can use this feature in order to re-write via copulas the joint distribution function of two (even non-uniform) random variables. The most interesting fact about copulas in this sense is Sklar's theorem:

Theorem 3.2 (Sklar (1959)). *Let $F(x,y)$ be a joint distribution function with continuous marginals $F_1(x)$ and $F_2(y)$. Then there exists a unique copula such that*

$$F(x,y) = C(F_1(x), F_2(y)) \tag{3.1}$$

Conversely, if C is a copula and $F_1(x)$, $F_2(y)$ are continuous univariate distributions, $F(x,y) = C(F_1(x), F_2(y))$ is a joint distribution function with marginals $F_1(x)$, $F_2(y)$.

The theorem suggests then to represent the multiplicity of joint distributions consistent with given marginals through copulas.

Three specific copulas are worth mentioning: the *product* copula, the *minimum* and the *maximum* copulas. Families of copulas which encompass all of these copulas are called *comprehensive*. As for the first, the copula representation of a distribution F degenerates into the so-called product copula, $C(v,z) = v \cdot z$, if and only if X and Y are independent. As for the others, they derive from the well-known Fréchet-Hoeffding result in probability theory, stating that every joint distribution function is constrained between the bounds

$$\max(F_1(x) + F_2(y) - 1, 0) \leq F(x,y) \leq \min(F_1(x), F_2(y)) \tag{3.2}$$

As a consequence of Sklar's theorem, the Fréchet-Hoeffding bounds exist for copulas too:

$$\max(v + z - 1, 0) \leq C(v,z) \leq \min(v,z)$$

In correspondence of the extreme copula bounds, there is perfect positive and negative dependence between the variables, and every variable can be obtained as a deterministic function of the other (see Embrechts, McNeil and Straumann,1999 for a proof). Let us define the generalized inverse of a distribution function $y = F_2(x)$, as

$$F_2^{-1}(y) = \inf \{t \in R : F_2(t) \geq y, 0 < y < 1\}$$

We can state that

Theorem 3.3 (Hoeffding (1940), Fréchet (1957)). *If the continuous random variables X and Y have the copula $\min(v,z)$, then there exists a monotonically increasing function U such that*

$$Y = U(X) \qquad U = F_2^{-1}(F_1)$$

where F_2^{-1} is the generalized inverse of F_2. If instead they have the copula $\max(v + z - 1, 0)$, then there exists a monotonically decreasing function L such that

$$Y = L(X) \qquad L = F_2^{-1}(1 - F_1)$$

The converse of the previous results holds too.

In the first case, X and Y are called *comonotonic*, while in the second they are deemed *countermonotonic*.

Copulas are linked to non-parametric association measures by useful relationships. As an example, Kendall's τ may be proved to be

$$\tau = 4 \int \int_{I^2} C(v,z) dC(v,z) - 1$$

while Spearman's ρ measure is given by

$$\rho = 12 \int \int_{I^2} C(v,z) dv dz - 3 \tag{3.3}$$

As a consequence of these results, it can be proved that the bounds of τ and ρ are always -1 and +1. This is not true for the linear correlation coefficient, whose value depends on the specific shape of the marginal distribution functions, as can be glanced from the relationship

$$cov(X,Y) = \int \int_D (F(x,y) - F_1(x)F_2(y))\, dx dy \tag{3.4}$$

where D is the cartesian product of X and Y's domains.

A particular class of copulas, the so-called Archimedean, is particularly easy to handle and will be used in this paper (see Genest and MacKay, 1986, Genest and Rivest, 1993).

Archimedean copulas may be constructed using a generating function $\phi : [0,1] \rightarrow [0,\infty]$ continuous, strictly decreasing, convex and such that $\phi(1) = 0$. Given such a function ϕ a copula may be generated computing

$$C(v,z) = \phi^{[-1]}\ (\phi(v) + \phi(z)) \tag{3.5}$$

where $\phi^{[-1]}$ is the pseudo-inverse of ϕ, defined as

$$\phi^{[-1]}(u) = \begin{cases} \phi^{-1}(u) & 0 \leq u \leq \phi(0) \\ 0 & \phi(0) \leq u \leq \infty \end{cases}$$

Among Archimedean copulas, we are going to consider only the one-parameter ones, which are constructed using a generator $\varphi_\alpha(t)$, indexed by the parameter α. The table below describes well known Archimedean copulas and their generators (for an extensive list see Nelsen, 1999):

Family	$\phi_\alpha(t)$	range for α	$C(v,z)$
Gumbel (1960)	$(-\ln t)^\alpha$	$[1,+\infty)$	$\exp\left\{-\left[(-\ln v)^\alpha+(-\ln z)^\alpha\right]^{1/\alpha}\right\}$
Clayton (1978)	$\frac{1}{\alpha}\left(t^{-\alpha}-1\right)$	$[-1,0)\cup(0,+\infty)$	$\max\left[\left(v^{-\alpha}+z^{-\alpha}-1\right)^{-1/\alpha},0\right]$
Frank (1979)	$-\ln\frac{\exp(-\alpha t)-1}{\exp(-\alpha)-1}$	$(-\infty,0)\cup(0,+\infty)$	$-\frac{1}{\alpha}\ln\left(1+\frac{(\exp(-\alpha v)-1)(\exp(-\alpha z)-1)}{\exp(-\alpha)-1}\right)$

Table 3: some Archimedean copulas

The second and third are particularly interesting since they are *comprehensive* according to the definition above. The Gumbel family gives the product copula if $\alpha=1$ and the upper Fréchet bound $\min(v,z)$ for $\alpha\to+\infty$: it describes positive association only. The Clayton family gives the product copula if $\alpha\to0$, the lower Fréchet bound $\max(v+z-1,0)$ when $\alpha=-1$, and $\min(v,z)$ for $\alpha\to+\infty$. To end up with, the Frank's family, which is discussed at length in Genest (1987), reduces to the product copula if $\alpha\to0$, and reaches the lower and upper Fréchet bounds for $\alpha\to-\infty$ and $\alpha\to+\infty$, respectively.

4. Pricing bivariate digital options

As a first step to price bivariate contingent claims with copulas we may focus on the case of bivariate digital options. Every practitioner would agree that this case is not simply of an academic interest, as this kind of derivative is present in some widely known structured finance products such as bivariate digital notes, that is debt instruments promising to pay a fixed coupon if the prices of two assets are above some predefined strike levels at some future date and zero otherwise. In order to price and hedge products like these one has to find a replicating strategy for the bivariate digital option. As there is not a market for them, this task would lead us straight into the problem of market incompleteness. One could even argue that the problem is further complicated by the fact that replicating a digital note written on a single underlying asset would also be involved, as a market of digital options is not available for each and every strike: while this is true, we may assume that products like these may be satisfactorily approximated using vertical spreads, as suggested in the seminal work by Breeden and Litzenberger (1978) and in all the more recent literature drawing from that idea (Shimko 1993, Rubinstein 1994, Derman and Kani 1994a,b to quote a few).

In order to focus on the bivariate feature of the pricing problem, we assume that we may replicate and price two single digital options with the same exercise date T written on the underlying markets S_1 and S_2 for strikes K_1 and K_2 respectively. Our problem is then to use these products to replicate a bivariate option which pays 1 if $S_1 \geq K_1$ and $S_2 \geq K_2$ and zero otherwise. Let us first break the sample space, which coincides with the positive quadrant, in the four relevant regions

	State H	State L
State H	$S_1 \geq K_1, S_2 \geq K_2$	$S_1 \geq K_1, S_2 < K_2$
State L	$S_1 < K_1, S_2 \geq K_2$	$S_1 < K_1, S_2 < K_2$

Table 1: breaking down the sample space for the digital option.

The bivariate digital option pays 1 unit only if both of the assets are in state H, that is in the upper left cell of the table. The single digital options written on assets 1 and 2 pay in the first row and the first column respectively. In table 2 below we sum up the payoffs of these different assets and we make clear which prices are observed in the market. We denote with P_1, P_2 and B the prices of the single digital options and the risk-free asset respectively.

	Price	HH	HL	LH	LL
Digital option asset 1	P_1	1	1	0	0
Digital option asset 2	P_2	1	0	1	0
Risk free asset	B	1	1	1	1
Bivariate digital option	?	1	0	0	0

Table 2: Prices and payoffs for digital options

Our problem is to use no-arbitrage arguments to recover the price of the bivariate digital option. Some interesting no arbitrage implications can be easily obtained by comparing its pay-off with that of portfolios of the single digital options and the risk free asset. Furthermore, as we face a pricing problem in an incomplete market, we may expect to find super-replication strategies leading to pricing bounds for the bivariate asset. The following proposition states such bounds for the price.

Proposition 4.1. *The no-arbitrage price $P(S_1 \geq K_1, S_2 \geq K_2)$ of a bivariate digital option is bounded by the inequality*

$$\max(P_1 + P_2 - B, 0) \leq P(S_1 \geq K_1, S_2 \geq K_2) \leq \min(P_1, P_2) \qquad (4.1)$$

Proof: assume first that the right side of the inequality is violated: say that, without loss of generality, it is $P(S_1 \geq K_1, S_2 \geq K_2) > P_1$; in this case selling the bivariate digital option and buying the single digital option would allow a free lunch in the state $[S_1 \geq K_1, S_2 < K_2]$. As for the left side of the inequality, it is straightforward to see that P must be non-negative. There is also a bound $P_1 + P_2 - B$: assume in fact that $P_1 + P_2 - B > P(S_1 \geq K_1, S_2 \geq K_2)$; in this case buying the bivariate digital option and a risk-free asset and selling the two single digital options would allow a free lunch in the current date with non-negative pay-off in the future (actually, the pay-off could even be positive if state $[S_1 < K_1, S_2 < K_2]$ occurred) $\square$.

The proposition exploits a static super-replication strategy for the bivariate digital option: the lower and upper bounds have a direct financial meaning, as they describe the pricing bounds for long and short positions in the bivariate options. The result may sound even more suggestive if we use forward prices. As we know, the forward prices are defined as $P(S_1 \geq K_1, S_2 \geq K_2)/B$, P_1/B and P_2/B for the double and single digital options respectively. We have then

$$\max\left(\frac{P_1}{B} + \frac{P_2}{B} - 1, 0\right) \leq \frac{P(S_1 \geq K_1, S_2 \geq K_2)}{B} \leq \min\left(\frac{P_1}{B}, \frac{P_2}{B}\right) \qquad (4.2)$$

and it is easy to recognize that the two bounds constraining the forward price of the double digital option are the Fréchet bounds taking the forward prices of the single digital options as arguments. Let us observe and stress that these bounds emerged from no-arbitrage considerations only. Furthermore, it must be reminded that the Fréchet bounds fulfill the conditions defining copula functions, suggesting that this result could hide a more general finding. We may then take one step further and investigate the features of the forward price of the double digital option. Such price obviously represents the pricing kernel for the bivariate economy. In particular, we are lead to conjecture that such bivariate kernel be a function of the kind $P(S_1 \geq K_1, S_2 \geq K_2)/B = C(P_1/B, P_2/B)$. The following proposition proves that this conjecture turns out to be verified.

Proposition 4.2. *The bivariate pricing kernel is a function $C(v, z)$ taking the univariate pricing kernels as arguments. In order to rule out arbitrage opportunities the function must fulfill the following requirements:*

- it is defined in $I^2 = [0,1] \times [0,1]$ and takes values in $I = [0,1]$;
- for every v and z of $I^2, C(v,0) = 0 = C(0,z)$, $C(v,1) = v, C(1,z) = z$;
- for every rectangle $[v_1, v_2] \times [z_1, z_2]$ in I^2,with $v_1 \leq v_2$ and $z_1 \leq z_2$,

$$C(v_2, z_2) - C(v_2, z_1) - C(v_1, z_2) + C(v_1, z_1) \geq 0$$

Proof: The first condition is trivial: the prices of the digital options cannot be higher than the risk-free asset B, implying than the forward prices of both the univariate and bivariate digital are bounded in the unit interval. As for the second condition, it follows directly from the no-arbitrage inequality (4.2), by substituting the values 0 and 1 for $v = P_1/B$ or $z = P_2/B$. As for the last requirement, consider taking two different strike prices $K_{11} > K_{12}$ for the first security, and $K_{21} > K_{22}$ for the second. Denote with v_1 the forward price of the first digital corresponding to the strike K_{11}; with v_2 that of the first digital for the strike K_{12} and use an analogous notation for the second security. Then, the third condition above can be re-written as

$$P(S_1 \geq K_{12}, S_2 \geq K_{22}) - P(S_1 \geq K_{12}, S_2 \geq K_{21}) +$$

$$-P(S_1 \geq K_{11}, S_2 \geq K_{22}) + P(S_1 \geq K_{11}, S_2 \geq K_{21}) \geq 0$$

As such, it implies that a spread position in bivariate options paying one unit if the two underlying assets end in the region $[K_{12}, K_{11}] \times [K_{22}, K_{21}]$ cannot have negative value[1]$\square$.

Matching the two propositions above with the mathematical definitions given in the previous paragraph we may restate the main results of our analysis as

Proposition 4.3. *The arbitrage-free pricing kernel of a bivariate contingent claim is a copula function taking the univariate pricing kernels as arguments $P(S_1 \geq K_1, S_2 \geq K_2)/B = C(P_1/B, P_2/B)$, and the corresponding super-replication strategies are represented by the Fréchet bounds:*

$$\max\left(\frac{P_1}{B} + \frac{P_2}{B} - 1, 0\right) \leq C\left(\frac{P_1}{B}, \frac{P_2}{B}\right) \leq \min\left(\frac{P_1}{B}, \frac{P_2}{B}\right)$$

[1]This property is akin to the requirement, in the one dimensional pricing problem, that the option price be decreasing and convex in the strike.

It must be stressed that in order to prove the result we did not rely on any assumption concerning the probabilistic nature of the arguments of the pricing function: these are only required to be no-arbitrage prices of single digital options. In this respect, our results carry over to more general incomplete market pricing models that may use capacities, i.e. non-additive functionals, rather than probability measures, as pricing kernels. In the case in which the market is complete, so that these options can be exactly replicated and the forward prices are the risk-neutral probabilities

$$\frac{P_1}{B} = \Pr\left(S_1 \geq K_1\right), \frac{P_2}{B} = \Pr\left(S_2 \geq K_2\right)$$

we are allowed to give a straightforward probabilistic interpretation of our results, directly resorting to Sklar's theorem. In a complete market, in fact, the bivariate pricing kernel is a bivariate probability measure and Sklar's theorem ensures that it could be written as a copula function taking the marginals as arguments. We may then write

$$\frac{P(S_1 \geq K_1, S_2 \geq K_2)}{B} = \Pr(S_1 \geq K_1, S_2 \geq K_2) = C\left(\frac{P_1}{B}, \frac{P_2}{B}\right) \tag{4.3}$$

The strength of these results is that they enable us to break the bivariate pricing kernel of the economy into a function of marginal univariate kernels: we may then separately extract marginal pricing kernels from vertical spreads and the bivariate pricing kernel of the economy from the dependence structure in the data.

A final word is in order to clarify the financial meaning the two no-arbitrage bounds in (4.2), corresponding to the minimal and maximal copula functions. Looking over the result in theorem 3.3, these bounds associate our super-replication strategies to the cases of perfect negative and positive dependence between the two underlying assets. An interesting question is how wide the range between the two pricing bounds can be. It may be the case that assuming perfect dependence leads us to overlook the most relevant feature of a bivariate contingent claim. In this case, relying on some figure representing the dependence structure in the data may help to yield a more precise evaluation of the contingent claim. Indeed, knowing the dependence structure in the data may help to characterize the copula function used in order to price the bivariate contingent claim, as we are going to show over the next sections.

5. Pricing other bivariate claims

5.1. Options on the minimum of two assets

We turn now to the pricing of the option on the minimum between two risky assets, which has been priced, in the Black-Scholes (lognormal) framework, by Stulz (1982). We assume deterministic, non zero interest rates.

The payoff of the call option on the minimum, with maturity T, is

$$\max(\min(S_1(T), S_2(T)) - K, 0)$$

where K is the strike price. Provided only that interest rates are non-stochastic, its price at time t is

$$B(T-t)\left[\int_K^{+\infty} qg(q)dq - K(1 - G(K))\right] \tag{5.1}$$

where $B(T-t)$ is the value at time t of the zero-coupon bond with maturity T, $g(q)$ is the risk-neutral density of the minimum, while $G(K)$ is the corresponding distribution function evaluated at K. In turn, since G can be computed to be

$$G(q) = F_{S_1}(q) + F_{S_2}(q) - c(F_{S_1}(q), F_{S_2}(q))$$

where c is the copula density, the density $g(q)$ is

$$g(q) = f_{S_1}(q) + f_{S_2}(q) - c_1(F_{S_1}(q), F_{S_2}(q))f_{S_1}(q) - c_2(F_{S_1}(q), F_{S_2}(q))f_{S_2}(q) \tag{5.2}$$

where c_1 and c_2 are the partial derivatives of c with respect to its arguments, while f_{S_i}, $i = 1, 2$, are the densities corresponding to F_{S_i}.

5.2. Options to exchange

The price of the option to exchange one asset for another, originally derived – for lognormal distributions – by Margrabe (1978), can be obtained as a portfolio of one underlying and a zero-strike option on the minimum. In fact, the payoff of the exchange option is

$$\max(S_1(T) - S_2(T), 0)$$

which can be rewritten as

$$S_1 - \max(\min(S_1, S_2), 0)$$

Recalling that the risk-neutral expected value of the underlying at maturity is the forward price, it follows that its price is the current value of the first underlying minus the price of the option on the minimum, with strike equal to zero. Using this device, under the assumption of deterministic, non zero interest rates, we obtain the exchange option value at time t as

$$S_1(t) - B(T-t)\int_0^{+\infty} qg(q)dq \tag{5.3}$$

where the function $g(q)$ is as defined above.

6. Empirical applications

In this section we apply copulas to the pricing problem of bivariate derivatives written on four indices: MIB30, S&P500, FTSE, DAX.

We first derive the risk-neutral marginal density function of each index, using the technique proposed by Shimko (1993)[2]. We then obtain the lower and upper Fréchet bounds for their joint distribution functions, which are the pricing kernel bounds. In order to pick out a single value between these bounds, one would need to observe at least one price of a bivariate claim. Since typically these products are not traded on organized markets, we present a sensitivity analysis of the bivariate option values with respect to the dependence structure of the underlying assets. As a reference case, we choose a price consistent with the dependence statistics estimated on historical data using a non parametric procedure[3].

6.1. Descriptive statistics and implied marginal densities

The data used for implied volatilities, time to maturity, dividend points and risk-free rate corresponding to the different markets were downloaded from Bloomberg on March 27, 2000 and refer to European calls closing prices with June expiration and different strikes. As for the non-parametric dependence estimation, we used the time series of the corresponding indices, from January 2, 1999 to March 27, 2000.

[2]The estimation technique for the marginals can be changed without modifying the core of our procedure.

[3]Sufficient conditions for the latter to be coincident with the no-arbitrage one, in spite of having been estimated on historical data, are provided in Rosenberg (2000).

We estimate the risk-neutral marginal distribution of each index (MIB30, S&P500, FTSE, DAX) on the cross section data of European calls. As a result, given a quadratic smile function $\sigma(K) = A_0 + A_1 K + A_2 K^2$, where K is the strike price, we recover the risk-neutral distribution function F_S of the underlying S:

$$F_S(s) = 1 + sn(D_2(s))\sqrt{T-t}(A_1 + 2A_2 s) - N(D_2(s)) \tag{6.1}$$

$$D_2(s) = \frac{\ln \frac{S(t)}{B(T-t)s}}{\sigma(s)\sqrt{T-t}} - \frac{1}{2}\sigma(s)\sqrt{T-t}$$

where $T-t$ is the time to maturity, $n(\cdot)$ and $N(\cdot)$ are respectively the density and the distribution of the standard normal, $S(t)$ is the current value of the underlying, less the discounted dividends, and $B(T-t)$ is the risk-free discount factor for the same maturity of the option. Figure 6.1. reports the marginal risk-neutral density functions obtained from the data.

Insert here figure 6.1

Using the four marginals so obtained, we numerically compute the lower and upper Fréchet bounds for the joint distributions. Substituting them in (3.3) we recover the correlation bounds in figure 6.2 below.

Insert here figure 6.2

The Fréchet bounds could be exploited for super-replication pricing of bivariate claims. Before doing that, we present the results of copula estimation using historical data, that will serve as a reference point for pricing: the copula estimation procedure is described in Frees and Valdez (1998) and based on Genest and Rivest (1993).

In figure 6.3 we report the estimated parameters α for the families of Archimedean copulas described in table 3, as well as the standard non parametric statistics (Kendall's τ and Spearman's ρ).

Insert here figure 6.3

In what follows we will use Frank's copula, both on the ground that it turned out to provide a better fit in all cases - as measured visually and by the mean square error - than the Clayton, and on the fact that the Gumbel copula does not allow for negative dependence.

6.2. Option pricing

Using the Frank's copula and the estimation of the marginal densities described above we get the following joint distribution for each couple of indices

$$F(s_1, s_2) = -\frac{1}{\alpha} ln \left(1 + \frac{(\exp(-\alpha F_{S_1}(s_1)) - 1)(\exp(-\alpha F_{S_2}(s_2)) - 1)}{\exp(-\alpha) - 1} \right) \quad (6.2)$$

where $F_{S_1}(s_1)$, $F_{S_2}(s_2)$ are defined according to (6.1). This is the basic tool for option pricing, once the proper parameter estimates are plugged in. As an example, in figure 6.4. below we present the joint distribution (6.2) for the DAX/FTSE case, together with the corresponding level curves[4]. The distribution was computed using the historical estimate of α.

Insert here figure 6.4

6.2.1. Bivariate digital option

First of all, using the joint distribution (5.2) we are able to price bivariate digital options. In order to stress the immediate relationship between copulas and bivariate digital prices, we assume a zero interest rate and use proposition 4.3. above. In order to account for a deterministic, non-zero interest rate, it would be sufficient to scale the prices presented below with the discount factor $B(T-t)$.

Figure 6.5. below presents some examples, again with α estimated from historical data, for the cases in which the corresponding single digital options are out-of-the-money (OTM), nearly at-the money (ATM) and in-the money (ITM). The bivariate digital options in the figure are then evaluated under different moneyness combinations for each underlying. The maturity is that of the corresponding marginals, i.e. three months.

Incidentally, the reader can easily notice that the familiar behavior of prices with respect to moneyness is respected: prices are decreasing, for every couple of

[4] Level curves can be defined in terms of risk-neutral joint distributions or in terms of copulas. With respect to the latter they are defined as usual as

$$\left\{(v, z) \in I^2 \mid C(v, z) = t\right\}$$

In the Archimedean case they become

$$z = \varphi^{[-1]} (\varphi(t) - \varphi(v))$$

indices, from top left towards bottom right and, ceteris paribus, from left to right and from top to bottom.

Insert here figure 6.5

By letting the strikes vary on a finer grid, we obtain the whole pricing surface for the three-month digital option on DAX and FTSE, which corresponds to the risk-neutral probability that both underlying assets be above the corresponding strikes. In figure 6.6 below we present the pricing surface corresponding to the historically calibrated copula and to the product copula.

Insert here figure 6.6

In figure 6.7 we present a numerical example of the sensitivity analysis shown graphically above. The example refers to the case in which both of the strikes are either ITM or nearly ATM. We analyze how the value of the option changes with respect to the parameter α, as we move from perfect dependence to independence of the underlying assets. By letting α reach its bounds the bivariate digital option prices converge to the Fréchet bounds, while the independence case ($\alpha = 0$) is simply the product copula.

Insert here figure 6.7

6.2.2. Option on the minimum of two assets

In order to implement formula (5.1), which gives the no-arbitrage price of the option on the minimum between two assets in the non-normal framework, one needs first the densities of the single underlying assets, f_{S_i}. For the cases described in the previous sections, i.e. in the risk-neutral sense, and in particular given the expression (6.1) for the marginal distribution functions, these densities turn out to be

$$f_{S_i}(q) = n(D_{2i}(q)) \left[\mathcal{D}_{2i}(q) - (A_{1i} + 2A_{2i}q)\sqrt{T-t}(1 - D_{2i}(q)\mathcal{D}_{2i}(q)) - 2A_{2i}q\sqrt{T-t} \right] \tag{6.3}$$

where

$$\mathcal{D}_{2i}(q) = -\frac{1}{q\sigma_i(q)\sqrt{T-t}} - \frac{D_{1i}(q)(A_{1i} + 2A_{2i}q)}{\sigma_i(q)}$$

$$D_{1i}(q) = D_{2i}(q) + \sigma_i(q)\sqrt{T-t}$$

and the functions $D_{2i}(q)$ and $\sigma_i(q)$ are defined as in section 5.1.

Figure 6.8 presents the prices for ITM, ATM, OTM three-months call options on the minimum between DAX and FTSE, computed according to (5.1) and using our estimate for the risk-neutral joint distribution function.

Insert here figure 6.8

In the first column we present the strike prices, corresponding to ITM (5653), ATM (6653) and OTM (7653) options. Moneyness is defined with respect to the FTSE index, whose current value on March 27, 2000, was 6653. From left to right, we present the prices corresponding to the historical association between the two indices ($\alpha = 4.469$), to an hypothetical quasi perfect positive association ($\alpha = 100$), to independency ($\alpha = .0001$) and to quasi perfect negative association ($\alpha = -100$). The reader can easily notice that, ceteris paribus, option prices are increasing with the association between the underlying assets, as the prices get closer to the super-replication values. They are decreasing with the strike, as usual.

6.2.3. Option to exchange one asset for another

The last line of the table in figure 6.8 presents the price of the option to exchange DAX for FTSE, corresponding to different α values. In order to understand the pricing technique used, let us recall from section 5 that the value at t of the option to exchange S_1 for S_2 at time T is

$$S_1(t) - B(T-t)\int_0^{+\infty} qg(q)dq$$

where $g(q)$ is defined in (5.2) and is implemented knowing $f_{S_i}(q), i = 1, 2$, reported in the previous section.

7. Conclusions

In this paper we suggest a strategy to address the joint issues of non-normality of returns and dependence in the bivariate contingent pricing problem. It is well known that under non-normality of returns the linear correlation is not an useful indicator of dependence, as it may not cover the whole unit range. We suggest to resort to the concept of copula in order to account for this problem: using copula functions enables to "decouple" the pricing problem, addressing the specification of marginal distributions and the dependence problem separately. A relevant advantage of this approach is that it is directly amenable to concrete applications: as a matter of fact, we have very well developed markets for many products, and the marginal distributions can be directly computed from the prices of vertical spreads of plain vanilla options. The use of copula functions enables us to link these marginal distributions in a bivariate pricing kernel.

As an example of the power of the approach, we present an application to four stock market indices: for each market we use vertical spreads and the interpolation technique due to Shimko to recover the implied marginal probability distributions from the market. We provide prices for bivariate digital options, which represent the basis of prices for all of the bivariate contingent claims in the economy, as well as for options on the minimum and options to exchange one asset for another. For each of these products, we provide super-replication prices, as well as the values consistent with independence of the underlying assets. As a final reference value, we show how to price the bivariate options using a copula function calibrated on historical data.

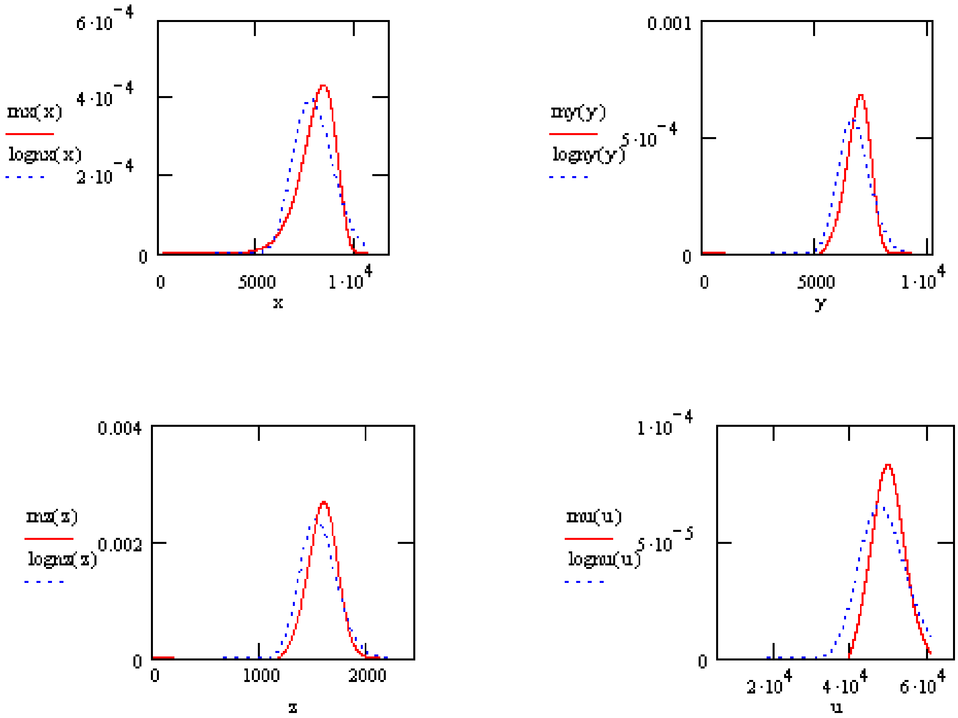

Figure 7.1: Solid lines: risk-neutral densities of DAX (x), FTSE (y), S&P500 (z), MIB30 (u), computed according to (5.4). Dotted lines: corresponding lognormal densities.

Mib-S&P	Mib-FTSE	Mib-DAX	S&P-FTSE	S&P-DAX	FTSE-DAX
-0.673	-0.642	-0.678	-0.868	-0.9	-0.865
0.73	0.732	0.735	0.997	0.985	0.984

Figure 7.2: Linear correlation bounds

	Sample Size	Kendall's tau	Spearman's rho	Gumbel's alpha	Clayton's alpha	Frank's alpha
Mib-S&P	306	0.372	0.548	1.593	1.185	3.789
Mib-FTSE	308	0.351	0.508	1.540	1.080	3.518
Mib-DAX	311	0.433	0.580	1.765	1.530	4.642
S&P-FTSE	304	0.581	0.772	2.387	2.774	7.445
S&P-DAX	306	0.646	0.846	2.828	3.657	9.317
FTSE-DAX	311	0.406	0.582	1.683	1.367	4.469

Figure 7.3: Estimated Kendall's τ, Spearman's R, and α for MIB30, S&P500, FTSE, DAX,1/2/99-3/27/00

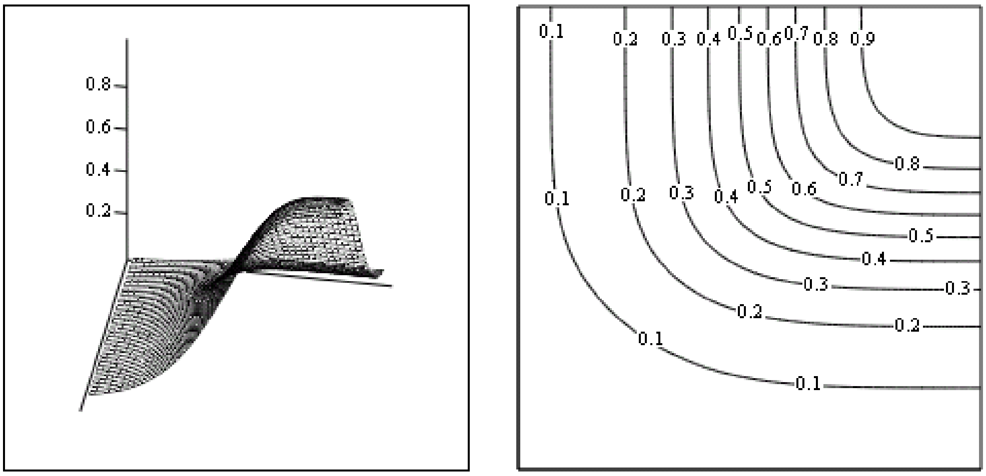

Figure 7.4: Joint distribution and level curves, DAX-FTSE

S&P-Mib		ITM	ATM	OTM
	Strike	38450	49702	60953
ITM	1218	0.9254	0.4344	0.0000
ATM	1649	0.2684	0.1999	0.0000
OTM	2079	0.0000	0.0000	0.0000

Mib-FTSE		ITM	ATM	OTM
	Strike	5322	6851	8379
ITM	38450	0.9268	0.4982	0.0000
ATM	49702	0.4341	0.3177	0.0000
OTM	60953	0.0000	0.0000	0.0000

DAX-Mib		ITM	ATM	OTM
	Strike	38450	49702	60953
ITM	6287	0.9010	0.4335	0.0000
ATM	8166	0.4630	0.3235	0.0000
OTM	10045	0.0000	0.0000	0.0000

S&P-FTSE		ITM	ATM	OTM
	Strike	5322	6851	8379
ITM	1218	0.9261	0.5028	0.0000
ATM	1649	0.2698	0.2515	0.0000
OTM	2079	0.0000	0.0000	0.0000

S&P-DAX		ITM	ATM	OTM
	Strike	6287	8166	10045
ITM	1218	0.9039	0.4656	0.0000
ATM	1649	0.2699	0.2553	0.0000
OTM	2079	0.0000	0.0000	0.0000

DAX-FTSE		ITM	ATM	OTM
	Strike	5322	6851	8379
ITM	6287	0.8977	0.4958	0.0000
ATM	8219	0.4398	0.3390	0.0000
OTM	10152	0.0000	0.0000	0.0000

Figure 7.5: Three-months binary digital option prices, selected strikes

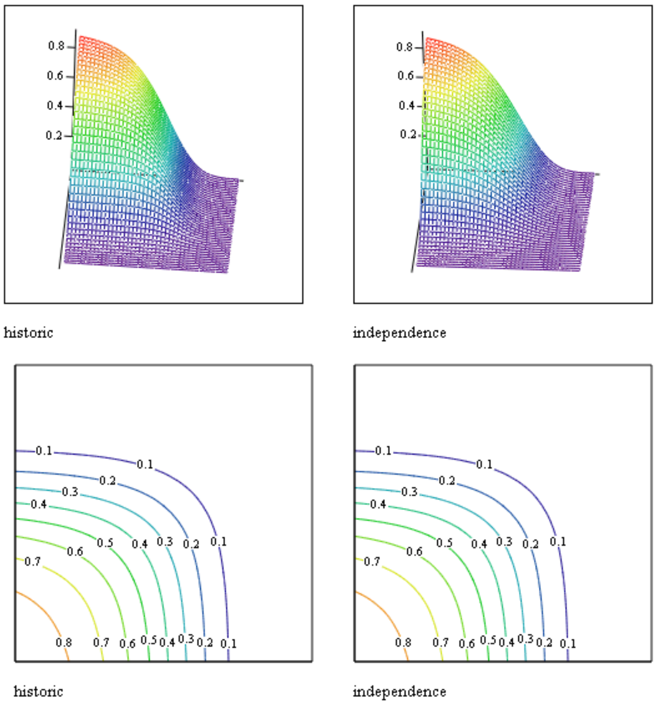

Figure 7.6: Binary digital option prices on DAX and FTSE, strikes 6300-10700, 5300-9300, computed with: the historically calibrated copula and the product one (independence).

DAX strike	FTSE strike	α -4.469	α --100	α -0.0001	α -100
6287	5322	0.8977	0.8864	0.8894	0.9278
6287	6851	0.4958	0.4313	0.4671	0.5034
8219	5322	0.4398	0.4016	0.4247	0.4430
8219	6851	0.3390	0.0000	0.2230	0.4430

Figure 7.7: Sensitivity analysis for the digital binary option, DAX-FTSE

K	$\alpha=4.469$	$\alpha=-100$	$\alpha=0.0001$	$\alpha=100$
5653	996.0	927.4	972.4	1021.0
6653	257.1	154.6	227.2	264.9
7653	6.6	0.0	4.4	6.9
opt.to exchange	1075.0	679.3	892.5	1187.0

Figure 7.8: Prices of the call option on the minimum and option to exchange DAX for FTSE

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