Valuation Theory: Stochastic Discount Factor

Economic Foundations for Computational Finance

MSCF, 45-905

Chris Telmer

Tepper School of Business, Carnegie Mellon University

Spring 2007

Carnegie Mellon Tepper SCHOOL OF BUSINESS

Stochastic Discount Factor

SDF

- Valuation theory: value *cash flows*, stochastic processes which assets are a claim to.
- $\{\delta_t\}_{t=1}^T \equiv$ *cash flow* or *dividend*
- $q_t \equiv$ time t, ex-dividend price of δ_t
- $m_{t,t+\tau} \equiv$ *stochastic discount factor* (SDF): a random variable that satisfies, for all t,

$$q_t = E_t \sum_{\tau=1}^{T-t} m_{t,t+\tau}\delta_{t+\tau} \quad , \tag{1}$$

  and $m_{t,t+\tau} > 0$. Conditional expectation is E_t.
- Existence of m_t is guaranteed by no-arbitrage.

Simplest Example

Discounting with constant interest rates:

- Suppose that $\delta_t = \delta$ for all t. Then, from intro finance:

$$\begin{aligned} q_t &= \sum_{\tau=1}^{T-t} \frac{\delta}{(1+r)^\tau} \\ &= \delta\left(\frac{1}{r} - \frac{1}{r(1+r)^{T-t}}\right) \end{aligned}$$

  For $T \to \infty$, $q_t = \delta/r$, the ‘dividend discount model.’
- What is the SDF for this case?

$$m_{t,t+\tau} = (1+r)^{-\tau} \quad .$$

- We’ll see that when the SDF, $m_{t,t+\tau}$, is constant for each maturity $t+\tau$, investors are *risk neutral*.

Recursive Representation

- By no-arbitrage (as we will see),

$$m_{t,t+2} = m_{t,t+1} m_{t+1,t+2} \quad ,$$

- More generally,

$$m_{t,t+\tau} = \prod_{k=1}^{\tau} m_{t+k-1,t+k} \quad .$$

- Therefore, by iterated expectations, equation (1) can be written recursively (show as an exercise):

$$q_t = E_t \, m_{t+1} (q_{t+1} + \delta_{t+1}) \quad , \tag{2}$$

where we denote $m_{t+1} \equiv m_{t,t+1}$.

Interpretation

Summarizing the two characterizations:

$$\begin{aligned} q_t &= E_t \sum_{\tau=1}^{T-t} m_{t,t+\tau} \delta_{t+\tau} && (3) \\ &= E_t \, m_{t+1} (q_{t+1} + \delta_{t+1}) && (4) \end{aligned}$$

Interpretation:

- m is a "*stochastic discount factor*" (SDF).
  - Equation (3): it 'discounts' multiperiod-ahead cash flows
  - Equation (4): synonymously, it discounts one-period-ahead payoffs.

Returns

- Repeating the recursive representation, equation (4):

$$q_t = E_t\, m_{t+1}(q_{t+1} + \delta_{t+1}) \tag{5}$$

- We call $(q_{t+1} + \delta_{t+1})$ a *payoff*
- We call $(q_{t+1} + \delta_{t+1})/q_t$ a *return*. Dividing (5) by q_t:

$$1 = E_t\, m_{t+1} \frac{q_{t+1} + \delta_{t+1}}{q_t} \tag{6}$$

$$1 = E_t\, m_{t+1}(1 + r_{t+1}) \tag{7}$$

- m, therefore, also 'discounts returns.' Returns are simply payoffs to unit-value portfolios (*e.g.*, the payoff on $1/q_t$ shares of stock *is* the return).

Excess Returns

- Suppose there are J assets, $j = 1, 2, ..., J$. The recursive, return representation for asset j:

$$E_t m_{t+1}(1 + r_{j,t+1}) = 1 \tag{8}$$

- Subtract the expression for asset k from asset j:

$$E_t m_{t+1}(r_{j,t+1} - r_{k,t+1}) = 0$$

- This says that *excess returns* must be orthogonal to the SDF.
- Special case: risk-free return is r_t (known at t):

$$E_t m_{t+1}(r_{j,t+1} - r_t) = 0$$

The *excess return over the risk-free rate* on asset j must also be orthogonal to the SDF.

Risk-Free Return

- Suppose that $\delta_{t+\tau} = 1$ for $\tau = 1$ and 0 otherwise. This is called a *one-period riskless bond*. Denote the bond's price as b_t^1. Apply the SDF valuation formula:

$$b_t^1 = E_t m_{t+1}$$

KEY: the conditional mean of the SDF *is* the one-period riskless bond price.

- The risk-free rate is *defined* as

$$(1 + r_t) \equiv 1/b_t^1$$

- Therefore,

$$\frac{1}{1 + r_t} = E_t m_{t+1}$$

- Similarly, if $\delta_{t+\tau} = 1$ for $\tau = n$ and 0 otherwise, this is an *n-period zero coupon bond*. It's price, b_t^n must satisfy:

$$b_t^n = E_t m_{t,t+\tau}$$

Summary

Stochastic Discount Factor
- SDF
- Simplest Example
- Recursive Representation
- Interpretation
- Returns
- Excess Returns
- Risk-Free Return
- Summary

Risk Premiums

Joint Lognormality

Lognormal Bond Pricing

The equation which the SDF must satisfy (by no-arbitrage) gets called the "fundamental pricing equation." It is helpful to remember that it can be written in (at least) 3 ways:

1. *Payoffs*:

2. *Returns*:

3. *Excess Returns*:

Risk Premiums

Carnegie Mellon Tepper SCHOOL OF BUSINESS

Stochastic Discount Factor

Risk Premiums
- Risk Premiums
- Covariance Risk
- Risk-Adjusted PDV
- Unconditional Risk
- Time Varying Riskless Rate
- β-representation (CAPM)
- Risk: price and quantity

Joint Lognormality

Lognormal Bond Pricing

Risk Premiums

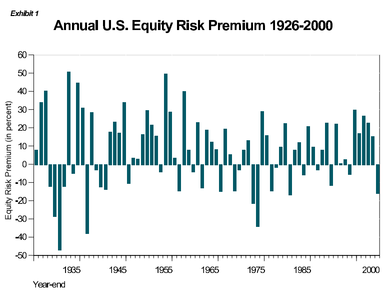
- Define the risk premium on asset j as the *expected* excess return over the risk-free rate:

$$\text{Risk Premium}_t = E_t(r_{j,t+1} - r_t)$$

- This is the *conditional* risk premium. The *unconditional* risk premium is (by iterated expectations):

$$\text{Risk Premium} = E\Big(E_t(r_{j,t+1} - r_t)\Big) = E(r_{j,t+1} - r_t)$$

For $j = S\&P500$, this is the object that one can estimate with:



Covariance Risk

- Which assets will pay risk premiums? Those with returns which covary *negatively* with the SDF:

$$\begin{aligned} E_t(m_{t+1}(r_{j,t+1} - r_t)) &= 0 \\ \Longrightarrow E_t m_{t+1} E_t(r_{j,t+1} - r_t) &= -\mathit{Cov}_t(m_{t+1}\,,\, r_{j,t+1}) \end{aligned}$$

  If the covariance $= 0$, then $E_t r_{j,t+1} = r_t$, so that the *conditional* risk premium is zero. By iterated expectations, so is the *unconditional* risk premium: $E\, r_{j,t+1} = r_t$.

- Assets with returns which covary positively with the SDF will pay a *negative* risk premium.

- Constant SDF is a special case. If m_{t+1} is a constant, then, trivially, the covariance $= 0$. Then the risk premium $= 0$. We say that constant SDF implies that investors are *risk neutral*.

Carnegie Mellon Tepper SCHOOL OF BUSINESS

Stochastic Discount Factor
Risk Premiums
- Risk Premiums
- Covariance Risk
- Risk-Adjusted PDV
- Unconditional Risk
- Time Varying Riskless Rate
- β-representation (CAPM)
- Risk: price and quantity
Joint Lognormality
Lognormal Bond Pricing

Risk-Adjusted PDV

You can do capital budgeting if you know the SDF:

- A project pays a random amount δ_{t+1} and costs q_t. Investment return is $(1 + r_{t+1}) = \delta_{t+1}/q_t$.

- By no-arbitrage

$$\begin{aligned} E_t[m_{t+1}(r_{t+1} - r_t)] &= 0 \\ \Longrightarrow E_t(r_{t+1}) &= r_t - \mathit{Cov}_t(m_{t+1}, r_{t+1})/E_t(m_{t+1}) \end{aligned}$$

- The expected return on investment is *defined* as $1 + E_t r_{t+1} \equiv E_t \delta_{t+1}/q_t$. Sub the above into this:

$$q_t = \frac{E_t(\delta_{t+1})}{1 + r_r - \mathit{Cov}_t(r_{t+1}, m_{t+1})/E_t(m_{t+1})}$$

- The project is priced as the 'expected, discounted present value.' The SDF tells us what 'discounting' means in this context.

Carnegie Mellon Tepper SCHOOL OF BUSINESS

Unconditional Risk

- *Conditional* moments such as $E_t r_{t+1}$ and $\mathit{Cov}_t(m_{t+1}, r_{t+1})$ are hard to measure.

- In many cases it is enough to work with the *unconditional* moments and risk premiums. These are derived using iterated expectations. To see this, recall equation (8), for all assets, j:

$$\begin{aligned} E_t\, m_{t+1}\,(1+r_{j,t+1}) &= 1 \qquad (9)\\ \Longrightarrow E\, m_{t+1}\,(1+r_{j,t+1}) &= 1\\ \Longrightarrow E\, m_{t+1}\,(r_{j,t+1}-r_t) &= 0\\ \Longrightarrow E\,(r_{j,t+1}-r_t) &= -\frac{\mathit{Cov}(m_{t+1}, r_{j,t+1})}{E\, m_{t+1}} \end{aligned}$$

- The *unconditional* risk premium is the term on the right.

Carnegie Mellon Tepper SCHOOL OF BUSINESS

Time Varying Riskless Rate

- Recall how we solve for the one-period riskless interest rate:

$$b_t^1 = \frac{1}{1+r_t} = E_t m_{t+1}$$

- This is the one-period *conditionally* riskless interest rate: one-period ahead we know the return, for sure: r_t.

- Unconditionally, there *can* be interest rate risk. The interest rate, r_t can vary over time. *Constant* interest rates are a special case.

- The unconditional mean and variance of the riskless rate are:

$$\begin{aligned} \text{Unconditional Mean} &= E(r_t)\\ \text{Unconditional Variance} &= \mathit{Var}(r_t) \end{aligned}$$

β-representation (CAPM)

- When working with unconditional moments, we don't need the time subscripts. So, we'll denote the typical risky asset return with r_j and the riskless return with r_f.

- Denote $1 + r^*$ as the return on some "benchmark" (risky) portfolio. Using equations (9),

$$\begin{aligned} \frac{E(r_j - r_f)}{E(r^* - r_f)} &= \frac{\mathit{Cov}(m, r_j)}{\mathit{Cov}(m, r^*)} \equiv \beta_j \\ \Longrightarrow E(r_j - r_f) &= \beta_j E(r^* - r_f) \ , \end{aligned}$$

  where β_j is the ratio of the two covariances.

- If $m = a + br^*$, then

$$\beta_j = \mathit{Cov}(r^*, r_j)/\mathit{Var}(r^*)$$

- If r^* is the market portfolio return, this is the CAPM

Risk: price and quantity

Consider the β-representation:

$$E\,r_j = r_f + \underbrace{\beta_j}_{quantity} \underbrace{E(r^* - r_f)}_{price} \ ,$$

- β_j measures the quantity of risk in asset j. It is specific to asset j. If $\mathit{Cov}(m, r_j) = 0$ — *i.e.*, if the return is uncorrelated with the SDF — then the asset is riskless.

- The excess return $E(r^* - r_f)$ is the market price of risk. It applies to the whole market (all traded assets). It is a measure how much the market rewards takers of one unit of 'β-risk.'

Joint Lognormality

Simplest Example

- A one-period asset pays off δ_{t+1} and costs q_t.
- Assume that m_{t+1} and δ_{t+1} are jointly lognormal. This can be written as

$$\begin{aligned} -\log m_{t+1} &\sim \mathrm{N}(\mu_m\,,\,\lambda^2) \\ \log \delta_{t+1} &\sim \mathrm{N}(\mu\,,\,\sigma^2) \\ \mathit{Cov}(-\log m_{t+1}\,,\,\log \delta_{t+1}) &= \sigma_{\delta,m} > 0 \end{aligned}$$

- Equivalently,

$$\begin{aligned} -\log m_{t+1} &= \mu_m + \lambda\varepsilon_{t+1} \\ \log \delta_{t+1} &= \mu + \sigma u_{t+1} \\ \begin{bmatrix} \varepsilon_{t+1} \\ u_{t+1} \end{bmatrix} &\sim \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \end{aligned}$$

Carnegie Mellon Tepper SCHOOL OF BUSINESS

Solution

- λ is the 'price of risk.' If $\rho > 0$, then the SDF covaries *negatively* with the return on the asset. The asset price, by no-arbitrage, is:

$$q_t = e^{-(r+\lambda\sigma\rho)} \; E_t \, \delta_{t+1} \quad .$$

- See the note on Blackboard which works out this solution and provides comments.

Carnegie Mellon Tepper SCHOOL OF BUSINESS

Lognormal Bond Pricing

Carnegie Mellon Tepper SCHOOL OF BUSINESS

Definitions

Define b_t^n as the price of an n-period zero-coupon bond. We'll use the following definitions:

- Continuously compounded *yield* or *spot interest rate*:

$$y_t^n = -n^{-1} \log b_t^n$$

- Forward interest rates:

$$f_t^n = \log(b_t^n / b_t^{n+1})$$

Define the *short rate* as r_t. By definition, $r_t = y_t^1 = f_t^0$. Also,

$$y_t^n = n^{-1} \sum_{i=0}^{n-1} f_t^i \quad .$$

Yields are averages of forward rates.

Carnegie Mellon Tepper SCHOOL OF BUSINESS

SDF

- Recursive representation of the SDF, reproduced from equation (5).

$$q_t = E_t \left(m_{t+1}(\delta_{t+1} + q_{t+1}) \right)$$

Or, in terms of returns,

$$1 = E_t \left(m_{t+1}(1 + r_{t+1}) \right) \tag{10}$$

- The date-t, one-period holding return on an $n+1$-period bond is

$$\frac{b_{t+1}^n}{b_t^{n+1}}$$

- Substituting this into equation (10) gives

$$E_t \, m_{t+1} \frac{b_{t+1}^n}{b_t^{n+1}} = 1 \tag{11}$$

- Or equivalently the "fundamental pricing equation" for risk-free bonds:

$$b_t^{n+1} = E_t \, m_{t+1} b_{t+1}^n \tag{12}$$

Model

- Repeating equation (12)

$$b_t^{n+1} = E_t\, m_{t+1} b_{t+1}^n \tag{13}$$

- The "Vasicek SDF" is:

$$-\log m_{t+1} \;=\; \delta + z_t + \lambda\varepsilon_{t+1}. \tag{14}$$

where

$$z_{t+1} \;=\; (1-\varphi)\theta + \varphi z_t + \sigma\varepsilon_{t+1} \quad , \tag{15}$$

with $\{\varepsilon_{t+1}\} \sim \mathrm{N}(0,1)$.

- For reasons soon to be clear, we'll set $\delta = \lambda^2/2$

Short rate

- Short rate is defined as $r_t = -\log b_t^1$. From (13)

$$b_t^1 = E_t\, m_{t+1}$$

- m_{t+1} is conditionally lognormal with,

$$\begin{aligned} E_t(\log m_{t+1}) &= -(\delta + z_t) \\ \mathit{Var}_t(\log m_{t+1}) &= \lambda^2 \end{aligned}$$

- Therefore,

$$E_t m_{t+1} = \exp(-\delta - z_t + \lambda^2/2) = b_t^1$$

and, since $r_t \equiv -\log b_t^1$ and $\delta = \lambda^2/2$,

$$r_t = z_t$$

Stochastic Discount Factor
Risk Premiums
Joint Lognormality
Lognormal Bond Pricing
- Definitions
- SDF
- Model
- Short rate
- Long rates
- ...continued
- Calibration

Long rates

- Guess that the solution is:

$$-\log b_t^n = A_n + B_n z_t \tag{16}$$

for some maturity-dependent coefficients, A_n, B_n.

- Since $b_t^0 = 1$ we know $A_0 = B_0 = 0$. Further, the short-rate solution gives us $A_1 = 0$ and $B_1 = 1$.

- We use valuation equation (13) to find b_t^{n+1}. The RHS involves,

$$\begin{aligned}\log m_{t+1} + \log b_{t+1}^n &= -\delta - z_t - \lambda\varepsilon_{t+1} - A_n - B_n z_{t+1} \\ &= -[A_n + \delta + B_n(1-\varphi)\theta] - (1 + B_n\varphi)z_t - (\lambda + B_n\sigma)\varepsilon_{t+1},\end{aligned}$$

- The conditional moments are

$$E_t(\log m_{t+1} + \log b_{t+1}^n) = -[A_n + \delta + B_n(1-\varphi)\theta] - (1 + B_n\varphi)z_t$$

and

$$\mathit{Var}_t(\log m_{t+1} + \log b_t^n) = (\lambda + B_n\sigma)^2.$$

...continued

Stochastic Discount Factor
Risk Premiums
Joint Lognormality
Lognormal Bond Pricing
- Definitions
- SDF
- Model
- Short rate
- Long rates
- ...continued
- Calibration

- The implied bond price is therefore

$$-\log b_t^{n+1} = A_n + \delta + B_n(1-\varphi)\theta - (\lambda + B_n\sigma)^2/2 + (1 + B_n\varphi)z_t.$$

- Lining up coefficients with (16) gives us the recursions

$$\begin{aligned}A_{n+1} &= A_n + \delta + B_n(1-\varphi)\theta - (\lambda + B_n\sigma)^2/2 \qquad (17)\\ B_{n+1} &= 1 + B_n\varphi. \qquad (18)\end{aligned}$$

- These equations look complicated, but given values for $(\theta, \varphi, \sigma, \lambda)$, we can easily evaluate them on a spreadsheet. They are a closed-form solution to the model, in the sense of being computable with a finite number of elementary operations.

Calibration

The model:

$$
\begin{aligned}
r_t &= z_t \\
z_{t+1} &= (1-\varphi)\theta + \varphi z_t + \sigma\varepsilon_{t+1} \\
ny_t^n &= A_n + B_n z_t \\
A_{n+1} &= A_n + \delta + B_n(1-\varphi)\theta - (\lambda + B_n\sigma)^2/2 \\
B_{n+1} &= 1 + B_n\varphi
\end{aligned}
$$

Calibration (or estimation):

- Choose θ, φ and σ to match the mean, variance and autocorrelation of the short rate
- Choose λ to match the *average* slope of the yield curve.
- Note that λ has the interpretation of the *price of risk*. If it is zero the yield curve is flat.