The Black-Scholes-Merton model and the Greeks

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2010–10–04

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1 Introduction

"The Greeks" is a name for a collection of quantities that can be obtained from differentiating for example the price of an option with respect to different variables. They represent the sensitivity of the price of derivatives to changes in the underlying parameters on which the derivative price processes are based. We will discuss some fundamental risk concepts where the use of derivatives come into play. A famous concept is "Δ-*hedging*", which uses the derivative with respect to the price of the underlying. The market model used in this paper will be the Black-Scholes-Merton model. We will show how Δ-hedging works and also give some intuition behind the other greeks. To have some intuition behind the Black-Scholes-Merton model we will do a short heuristic derivation of it. We will also give the formula for the price of a simple European call option but will not show how to obtain this solution as this would take too much space and use of advanced mathematics.

2 The Black-Scholes-Merton model

In the Black-Scholes-Merton market model quantities such as interest rate, volatility and mean rate-of-return are held constant. We introduce some notation for these quantities that will be used throughout this paper.

$$\begin{cases} r = & \text{interest rate} \\ \mu = & \text{mean rate-of-return of stock} \\ \sigma = & \text{volatility of stock} \end{cases}$$

Let now $x(t)$ denote the price of an asset. By the definition of interest rate we have

$$dx(t) = rx(t)dt.$$

The solution to this ordinary differential equation (ODE) is given by

$$x(t) = x_0 e^{rt}.$$

However, in the real world things are not always deterministic. By introducing some perturbation in the interest rate we can write the differential for $x(t)$ as

$$dX(t) = (r + \sigma\xi(t))X(t)dt = rX(t)dt + \sigma X(t)dW(t), \tag{1}$$

where $W(t)$[1] denotes a Brownian motion. The random variable $\xi(t)$ is commonly referred to as *white noise* and σ is the intensity of the white noise. In financial jargong σ is called volatility. This is a stochastic differential equation (SDE) for $X(t)$ with a drift term $rX(t)dt$ and a diffusion term $\sigma X(t)dW(t)$. Suppose $X(t)$ is the price of a stock at time t, then r can be interpreted as the

[1] The notation W comes from Wiener process, which has got its name from the mathematician Norbert Wiener.

mean rate-of-return of the stock and σ the volatility of the stock. The solution of this stochastic differential equation will be presented below. With risk of straying too far off here, we still wish to clarify some things here. Notice that we called μ the mean rate-of-return of the stock earlier, and here we have r. What is the difference? Well, stocks usually do not have the same rate-of-return as the market, i.e. $\mu \neq r$ and moreover, μ is impossible to statistically estimate. Actually, the SDE for $X(t)$ should read

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t).$$

One can, however, make a change of probability measure[2] using results from stochastic analysis to obtain the SDE given by (1). The fact that we do not need to know μ when pricing options is absolutely key to its success! With this said, we give the solution of (1) and make a change of notation from $X(t)$ to $S(t)$.

The solution to (1) is given by

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma W(t)},$$

where $S(0)$ denotes the stock price at time $t = 0$.

With our expression for the stockprice at time t we now face the challenge of pricing an option on this stock. We consider one of the simplest options, a European call option, which has payoff function $\max(S(T) - K, 0)$ at time of maturity T. Once again we do not prove the following result but merely present it. The price of a European call option with time of maturity T can be calculated as

$$c(S(t), t, K, T) = e^{-r(T-t)}\mathbb{E}\left[\max(S(T) - K, 0)|S(t)\right].$$

Here $S(t)$ denotes the price of the underlying stock at time t and K is the strike price. Omitting the calculations we get that

$$c(S(t), t, K, T) = S(t)\Phi\left(\frac{\log(\frac{S(t)}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{r(T-t)}\Phi\left(\frac{\log(\frac{S(t)}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right), \tag{2}$$

where $\Phi(x) = \int_{-\infty}^{x} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$.

For ease of notation we set

$$d_+ = \frac{\log(\frac{S(t)}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_- = \frac{\log(\frac{S(t)}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

3 Δ-hedging

Derivatives was invented with the purpose of letting for example a farmer insure his crops. That is, if he fears the harvest is going to be bad he can *hedge* away the risk by taking a long position

[2] The change of probability measure for stochastic processes is done using Girsanov's theorem.

in put options. Today investors can use derivatives to reduce risk in a portfolio. This will usually lower the overall portfolio return because they have to pay a premium for these derivatives, but the portfolio will be subject to less risk. When a bank issues a derivative they need to proctect themselves from potential losses due to the short position in the derivative, so banks hedge away the risks. We are now going to derive how one obtains the Δ-hedging rule. Let $c(x,t)$ denote the price a European call option if $S(t) = x$. In *no arbitrage pricing* one finds a hedging portfolio, i.e. a portfolio that replicates the derivative and by arbitrage reasoning finds the fair price of the derivative. Let $X(t)$ denote the hedging portfolio value process. A hedging portfolio starts with some initial value $X(0)$ and invests in the stock and the money market in such a way so that $c(S(t),t) = X(t)$ for all $t \in [0,T]$. One way to ensure this equality is to require

$$d\left(e^{-rt}X(t)\right) = d\left(e^{-rt}c(S(t),t)\right), \quad \text{for all } t \in [0,T) \tag{3}$$

and $X(0) = c(S(0),0)$. Integrating both sides from 0 to t gives

$$e^{-rt}X(t) - X(0) = e^{-rt}c(S(t),t) - c(S(0),0)$$

and we see that with $X(0) = c(S(0),0)$ these terms cancel leaving us with the desired equality. To see what must hold for (3) to hold we must first compute the differential of the discounted portfolio value process and the differential of the discounted option price process. These calculations make use of Itô's lemma[3], a fundamental result in stochastic calculus.

$$d\left(e^{-rt}c(S(t),t)\right) = e^{-rt}\left(-rc(S(t),t)dt + c_t(S(t),t)dt + c_x(S(t),t)dS(t) + \frac{1}{2}c_{xx}(S(t),t)(dS(t))^2\right) =$$

$$= e^{-rt}\left(-rc(S(t),t) + c_t(S(t),t) + \mu S(t)c_x(S(t),t) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(S(t),t)\right)dt + e^{-rt}\sigma S(t)c_x(S(t),t)dW(t).$$

Before we tackle the differential of the discounted portfolio value process we want to explain why the differential looks the way it does. Assume an investor starts with some initial capital $X(0)$ and at each time t holds $\Delta(t)$ shares of the stock, where $\Delta(t)$ is to be determined, and invests the remainder in the money market account. Then the differential of the portfolio value process becomes

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt =$$
$$= rX(t)dt + \Delta(t)(\mu - r)S(t)dt + \Delta(t)\sigma S(t)dW(t).$$

With this in mind we can find $d\left(e^{-rt}X(t)\right)$.

$$d\left(e^{-rt}X(t)\right) = e^{-rt}\left(-rX(t)dt + dX(t)\right) =$$

$$= \Delta(t)(\mu - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t).$$

We equate the differentials and get

$$\Delta(t)(\mu - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) =$$

$$= e^{-rt}\left(-rc(S(t),t) + c_t(S(t),t) + \mu S(t)c_x(S(t),t) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(S(t),t)\right)dt + e^{-rt}\sigma S(t)c_x(S(t),t)dW(t).$$

[3] $d(f(X(t))) = f_x(X(t))dX(t) + \frac{1}{2}f_{xx}(X(t))dX(t)dX(t)$.

First we cancel an e^{-rt} from each term and then set the $dW(t)$ terms equal. We see that we should choose $\Delta(t) = c_x(S(t), t)$, which is the famous Δ-hedging rule. With this choice of $\Delta(t)$ we now equate the dt terms and see that the terms containing μ cancels! We are left with

$$rc(S(t), t) = c_t(S(t), t) + rS(t)c(S(t), t) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(S(t), t),$$

for all $t \in [0, T)$. So to find the correct price one should seek a function $c(x, t)$ which solves the Black-Scholes-Merton partial differential equation

$$rc(x, t) = c_t(x, t) + rxc(x, t) + \frac{1}{2}\sigma^2 x^2 c_{xx}(x, t), \quad c(x, T) = \max(x - K, 0).$$

Anyway, we got that to in order to hedge a short position in an option one should hold $c_x(S(t), t)$ shares of the stock at each time t. Equation (2) gives an explicit formula for the price of a European call option, so to find $c_x(S(t), t)$ we simply differentiate this expression with respect to $S(t)$.

For this greek, the delta $= \frac{\partial c}{\partial S}$, we compute it explicitly where the financial instrument is a European call option. We first notice that $\frac{\partial d_+}{\partial S} = \frac{\partial d_-}{\partial S}$. With $\varphi(x) = \Phi'(x)$ and by using the chain rule we get

$$\frac{\partial c}{\partial S}(S(t), t) = \Phi(d_+) + S(t)\varphi(d_+)\frac{\partial d_+}{\partial S} - Ke^{-r(T-t)}\varphi(d_-)\frac{\partial d_-}{\partial S} =$$

$$= \Phi(d_+) + \frac{\partial d_-}{\partial S}\left(S(t)\varphi(d_+) - Ke^{-r(T-t)}\varphi(d_-)\right).$$

It can be proven that $S(t)\varphi(d_+) - Ke^{-r(T-t)}\varphi(d_-) = 0$, and we obtain that

$$\frac{\partial c}{\partial S}(S(t), t) = \Phi(d_+).$$

Because $0 \leq \Phi(x) \leq 1$ we see that one should always hold between 0 and 1 shares of the stock in order to hedge a short position in a European call option.

Assume now that we take a short position in a European call option. For our future obligation at time T we get some money $c(S(t), t)$. For this money we buy $\frac{\partial c}{\partial S}(S(t), t)$ shares of the stock and invest the rest in the money market account. The amount invested in the money market account becomes

$$c(S(t), t) - c_x(S(t), t)S(t) = -Ke^{-r(T-t)}\Phi(d_-) = M,$$

and we notice that $M < 0$. So to hedge a short position in a European call option one has to borrow money. By continuously rebalancing the hedging the portfolio all the way up to time of maturity T one can ensure that the value of the hedging portfolio agrees with the value of the European call option at time T, and the option has been succesfully hedged. Figure(1) graphically displays Δ-hedging.

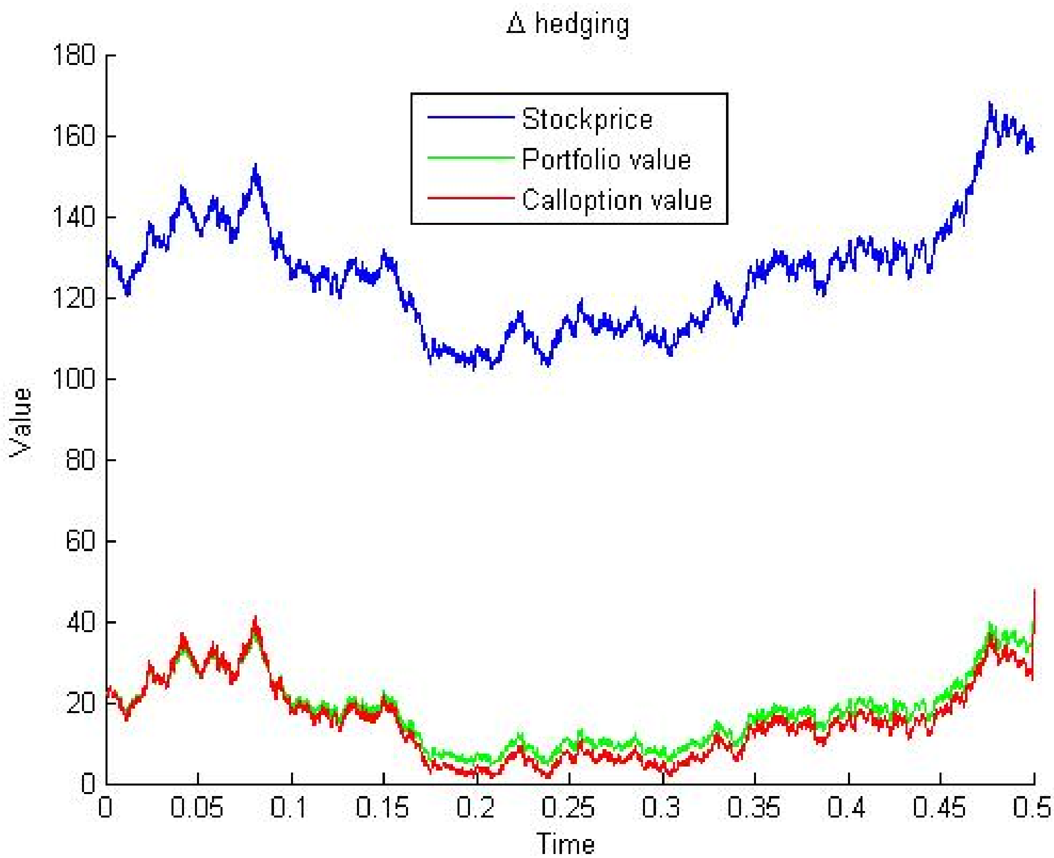

Figure 1: Δ-hedging, shows the value of the stock, the European call option and the hedging portfolio.

4 Other greeks

There are other greeks that are of great importance as well. We give a list with some of the most common togheter with their respective names. Let V denote the value of a financial derivative and let $\tau = T - t$.

$$\begin{cases} \frac{\partial V}{\partial S} &= \text{delta} \\ \frac{\partial V}{\partial \sigma} &= \text{vega} \\ \frac{\partial V}{\partial t} &= \text{theta} \end{cases}$$

One can show that

$$\begin{cases} \frac{\partial V}{\partial S} = \Phi(d_+) \\ \frac{\partial V}{\partial \sigma} = S(t)\Phi(d_+)\sqrt{T-t} \\ \frac{\partial V}{\partial \tau} = \frac{S(t)\Phi(d_+)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_-). \end{cases}$$

Delta we have already discussed.

The greek vega measures the price sensitivity of the derivative with respect to the underlying parameter σ, which is the volatility of the underlying stock. It is easy to see that vega is a positive quantity, so the price of the derivative increases with increasing volatility. Vega can be interpreted as the amount of money the price of the derivative increases with when the volatility of the underlying stock increases with 1%. One can put togheter so called delta-neutral and long-gamma portfolios that benefit in times of high volatility, so this quantity is useful for traders.

Theta is a negative quantity so the price of the derivative decreases as time t approaches time of maturity T. Consider a simple European call option. Even if the option is “out-of-the-money”, i.e. $S(t) < K$ there is still some chance that the option will expire in-the-money if the time until maturity is not very close. However, as time approaches maturity and the option is still out-of-the-money, the probability of the option expiring in-the-money decreases and therefore the price of the option decreases with time.

You can also take the derivative with respect to the strike price K, $\frac{\partial V}{\partial K}$. Again we notice that $\frac{\partial d_+}{\partial K} = \frac{\partial d_-}{\partial K}$. With a European call option as our example derivative we get

$$\frac{\partial c}{\partial K}(S(t), t) = S(t)\varphi(d_+)\frac{\partial d_+}{\partial K} - e^{-r\tau}\Phi(d_-) - Ke^{-r\tau}\varphi(d_-)\frac{\partial d_-}{\partial K} = -e^{-r\tau}\Phi(d_-).$$

This is a negative quantity, so the price of a European call option decreases with increasing strike K. An intuitive result one might say. With a higher strike K the chances of a higher payoff at maturity becomes smaller, so it feels reasonable that the price of the call option is lower for higher strikes.

5 Higher order greeks

There are other greeks which we have not mentioned. One could for example take the derivative (in calculus) of the derivative value process with respect to the price process of the underlying first, and then take the derivative with respect to time to obtain the “charm”, or “delta-decay”. In mathematical terms this becomes $\frac{\partial V}{\partial S \partial \tau}$. This is a measure of how much the delta of an option changes with time. With the short option hedging in mind we recall that one should hold $\frac{\partial V}{\partial S}$ shares of the underlying when hedging a short position in the derivative, so the charm simply becomes a measure on how much we need to change our hedging position with time. The charm can be an important quantity when monitoring a delta-hedging position for example over a weekend, because delta-hedging is a hedging strategy which needs to be paid attention to continuously. Of course, in practice it is impossible to rebalance all the time, but once or twice a day is usually sufficient.

Another important quantity, although it does not involve higher order derivatives, is the lambda, or elasticity. It is a measure of the change in percentage of the option price per percentage change in the underlying price. This is also called leverage. Lambda is defined by

$$\lambda = \frac{\partial V}{\partial S} \cdot \frac{S}{V}.$$

6 Conclusion

The Black-Scholes-Merton model is a basic model for asset prices and its creators have won the Nobel prize. Still, there are obvious drawbacks with this model. For the first, stock prices are not really geometric Brownian motions and the parameters r (interest rate) and σ (volatility) are not constant in the real world. To improve on this model one could try to find a better model for the interest rate. There are various such models, and some of them are used more than others. One such model is the Cox-Ingersoll-Ross (CIR) model,

$$dR(t) = b(a - R(t))dt + \sigma\sqrt{R(t)}dW(t).$$

This is a much more advanced model than simply letting the interest rate be a constant which is boring and not very realistic. The drawback is that this stochastic differential equation does not have an explicit solution and these kinds of more advanced models will make computations more challenging. Note that with this model for interest rate we are not in the Black-Scholes-Merton model anymore, because there we had a constant interest rate r.

7 Study guide

Interested readers are referred to Steven E. Shreve's book 'Stochastic Calculus for Finance II: Continuous time models' and also to Christer Borell's legendary introductory compendium 'Introduction to the Black-Scholes theory'. Many results in mathematical finance make heavy use of stochastic analysis and a good introductory book to this subject is Fima C. Klebaner's book 'Introduction to Stochastic Calculus with Applications'. It has two chapters entirely devoted to finance and they deal with stock- and FX options and also with bonds and interest rate models. Other recommended books are 'Options, Futures and Other Derivatives' by John C. Hull, which shows how academia and real-world practice come togheter, and also 'Paul Wilmott on Quantitative Finance' by Paul Wilmott.

References

[1] Steven E. Shreve, Stochastic Calculus for Finance II: Continuous time models

[2] Christer Borell, Introduction to the Black-Scholes theory

[3] Wikipedia, `http://www.wikipedia.com`

[4] Fima C. Klebaner, Introduction to Stochastic Calculus with Applications