Lecture 2

Examples.

1. For all $n_1 \leq n_2$, $\mathbb{C}^{n_1}$ is a subspace of $\mathbb{C}^{n_2}$ (as $\mathbb{C}$-vector spaces).
2. Given a vector space V over $\mathbb{F}$, $\{\vec{0}\}$ is a subspace.
3. In $\mathbb{R}^2$, any subspace is either (a) $\mathbb{R}^2$, (b) $\{\vec{0}\}$ or (c) a line through the origin. Why? If W is a subspace and contains some $w \neq \vec{0}$, it must contain the entire line spanned by w; that is, the set $\{cw : c \in \mathbb{R}\}$. This is a line through the origin. If it contains anything outside this line, we can use this new vector along with w to generate all of $\mathbb{R}^2$.
4. Generally in $\mathbb{R}^n$, any subspace is a hyperplane through the origin.

Last time we saw that if W_1 and W_2 are subspaces of a vector space V then

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1,\ w_2 \in W_2\}$$

is also a subspace. This is actually the smallest subspace containing both W_1 and W_2. You might think this would be $W_1 \cup W_2$, but generally, the union does not need to be a subspace. Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ and

$$W_1 = \{(x,0) : x \in \mathbb{R}\},\ W_2 = \{(0,y) : y \in \mathbb{R}\}\ .$$

Then both of these are subspaces but their union is not, since it is not closed under addition ($(1,1) = (1,0) + (0,1) \notin W_1 \cup W_2$).

In the case that $W_1 \cap W_2 = \{\vec{0}\}$, we say that $W_1 + W_2$ is a *direct sum* and we write it $W_1 \oplus W_2$.

Spanning

Given a subset S (not necessarily a subspace) of a vector space V we want to generate the smallest subspace containing S.

Definition 0.1. *Let V be a vector space and $S \subset V$. The span of S is defined*

$$Span(S) = \cap_{W \in \mathcal{C}_S} W\ ,$$

where $\mathcal{C}_S$ is the collection of subspaces of V containing S.

Note that the Span is the smallest subspace containing S in that if W is another subspace containing S then $\text{Span}(S) \subset W$. The fact that $\text{Span}(S)$ is a subspace follows from:

Proposition 0.2. *Let $\mathcal{C}$ be a collection of subspaces of a vector space V. Then $\cap_{W \in \mathcal{C}} W$ is a subspace.*

Proof. First each $W in \mathcal{C}$ contains $\vec{0}$, so $\cap_{W\in\mathcal{C}} W$ is nonempty. If $v, w \in \cap_{W\in\mathcal{C}} W$ and $c \in \mathbb{F}$ then $v, w \in W$ for all $W \in \mathcal{C}$. Since each W is a subspace, $cv + w \in W$ for all $W \in \mathcal{C}$, meaning that $cv + w \in \cap_{W\in\mathcal{C}} W$, completing the proof. □

Examples.

1. $\text{Span}(\emptyset) = \{\vec{0}\}$.
2. If W is a subspace of V then $\text{Span}(W) = W$.
3. $\text{Span}(\text{Span}(S)) = \text{Span}(S)$.
4. If $S \subset T \subset V$ then $\text{Span}(S) \subset \text{Span}(T)$.

There is a different way to generate the span of a set. We can imagine that our initial definition of span is from the "outside in." That is, we are intersecting spaces outside of S. The second will be from the "inside out": it builds the span from within, using the elements of S. To define it, we introduce some notation.

Definition 0.3. *If $S \subset V$ then $v \in V$ is said to be a* linear combination *of elements of S if there are finitely many elements $v_1, \ldots, v_n \in S$ and scalars $a_1, \ldots, a_n \in \mathbb{F}$ such that $v = a_1v_1 + \cdots + a_nv_n$.*

Theorem 0.4. *Let $S \subset V$ be nonempty. Then Span(S) is the set of all linear combinations of elements of S.*

Proof. Let $\hat{S}$ be the set of all linear combinations of elements of S. We first prove $\hat{S} \subset \text{Span}(S)$, so let $a_1v_1 + \cdots + a_nv_n \in \hat{S}$. Each of the v_i's is in S and therefore in $\text{Span}(S)$. By closure of $\text{Span}(S)$ under addition and scalar multiplication, we find $a_1v_1 + \cdots + a_nv_n \in \text{Span}(S)$.

To show that $\text{Span}(S) \subset \hat{S}$, it suffices to show that $\hat{S}$ is a subspace of V; then it is one of the spaces we are intersecting to get $\text{Span}(S)$ and we will be done. Because $S \neq \emptyset$ we can find $s \in S$ and then $1s$ is a linear combination of elements of S, making the Span nonempty. So let $v, w \in \text{Span}(S)$ and $c \in \mathbb{F}$. We can write $v = a_1v_1 + \cdots + a_nv_n$ and $w = b_1w_1 + \cdots + b_kw_k$ for $v_i, w_i \in S$. Then

$$cv + w = (ca_1)v_1 + \cdots + (ca_n)v_n + b_1w_1 + \cdots + b_kw_k \in \hat{S}\ .$$

□

Corollary 0.5. *If W_1, W_2 are subspaces of V then Span$(W_1 \cup W_2) = W_1 + W_2$.*

Proof. Because $\vec{0} \in W_1$ and in W_2, we have $W_1 + W_2 \supset (W_1 \cup W_2)$. Therefore $W_1 + W_2$ is one of the subspaces we intersect to get the span and $\text{Span}(W_1 \cup W_2) \subset W_1 + W_2$. Conversely, any element in $W_1 + W_2$ is in $\text{Span}(W_1 \cup W_2)$ as it is already a linear combination of elements of $W_1 \cup W_2$. □

Definition 0.6. *A vector space V is finitely generated if there is a finite set $S \subset V$ such that $V = Span(S)$. Such an S is called a generating set.*

- The space $\mathbb{R}^n$ is finitely generated: we can choose

$$S = \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\} .$$

- The space

$$\mathbb{R}^\infty_c = \{(x_1, x_2, \ldots) : x_i \in \mathbb{R}, \quad \text{finitely many nonzero terms}\}$$

  with coordinate-wise addition and scalar multiplication is not finitely generated.

Generating sets are closely linked to linear independence.

Linear independence

Definition 0.7. *A set $S \subset V$ is called linearly dependent if there exists $v \in S$ such that $v \in Span(S \setminus \{v\})$. We decree that $\emptyset$ is linearly independent; that is, not linearly dependent.*

The intuition is that a set is linearly dependent if there are unnecessary elements in it to span Span(S). Indeed, we can restate this condition for $S \neq \emptyset$ as

$$S \text{ linearly dependent iff } \exists v \in S \text{ such that } \mathrm{Span}(S) = \mathrm{Span}(S \setminus \{v\}) .$$

Exercise: prove this!

Examples.

1. $\{\vec{0}\}$ is linearly dependent in any vector space.
2. In $\mathbb{C}^2$, $\{(1,0),(0,1),(1,1)\}$ is linearly dependent, since $(1,1) \in \mathrm{Span}(\{(1,0),(0,1)\})$.
3. In $\mathbb{C}^n$,

$$\{(1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$$

   is linearly independent. Indeed, suppose we remove any element from this set. For simplicity let us take the first. Then every element in the span of the others must have zero first-coordinate, and cannot be $(1,0,\ldots,0)$.

There is a very simple condition we can check to see if a set is linearly independent.

Proposition 0.8. *Let V be a vector space and $S \subset V$. Then S is linearly independent if and only if whenever $a_1,\ldots,a_n \in \mathbb{F}$ and $v_1,\ldots,v_n \in S$ satisfy*

$$a_1 v_1 + \cdots + a_n v_n = \vec{0}$$

we must have $a_1 = \cdots = a_n = 0$.

Proof. If $S = \emptyset$ then S is linearly independent. Furthermore, it satisfies the condition of the proposition vacuously: it is true because we cannot ever find a linear combination of elements of S equal to $\vec{0}$.

Otherwise suppose that S is linearly dependent but $S \neq \emptyset$. Then we can find $v \in S$ such that $v \in \text{Span}(S \setminus \{v\})$. Therefore v is a linear combination of elements of $S \setminus \{v\}$: we can find $w_1, \ldots, w_n \in S \setminus \{v\}$ and scalars $a_1, \ldots, a_n$ such that $v = a_1 w_1 + \cdots + a_n w_n$. Then

$$(-a_1)w_1 + \cdots + (-a_n)w_n + v = \vec{0} \ .$$

This is a linear combination of elements of S equal to $\vec{0}$ with not all coefficients equal to 0, proving that if the condition of the proposition holds, then S must be linearly independent.

Conversely if S is linearly independent suppose that

$$a_1 v_1 + \cdots + a_n v_n = \vec{0}$$

for some $v_1, \ldots, v_n \in S$ and $a_1, \ldots, a_n \in \mathbb{F}$. If the coefficients are not all 0, we can find one, say a_1 which is nonzero. Then we solve:

$$v_1 = (-a_1^{-1}) \left[a_2 v_2 + \cdots + a_n v_n \right] \ ,$$

giving $v_1 \in \text{Span}(S \setminus \{v_1\})$. (Note here that a_1^{-1} is defined since $a_1 \neq 0$ and all nonzero field elements are invertible.) $\square$

Corollary 0.9. *Let $S_1 \subset S_2 \subset V$, an $\mathbb{F}$-vector space.*

1. *If S_1 is linearly dependent, so is S_2.*
2. *If S_2 is linearly independent, so is S_1.*

Proof. The first item follows from the second, so we prove the second. Suppose that S_2 is linearly independent and that $v_1, \ldots, v_n \in S_1$, $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$a_1 v_1 + \cdots + a_n v_n = \vec{0} \ .$$

Since these vectors are also in S_2 and S_2 is linearly independent, $a_1 = \cdots = a_n = 0$. Thus S_1 is linearly independent. $\square$