# Lecture 4

Now we have one of two main subspace theorems. It says we can extend a basis for a subspace to a basis for the full space.

**Theorem 0.1** (One subspace theorem)**.** *Let $W$ be a subspace of a finite-dimensional vector space $V$. If $B_W$ is a basis for $W$, there exists a basis $B$ of $V$ containing $B_W$.*

*Proof.* Consider all linearly independent subsets of $V$ that contain $B_W$ (there is at least one, $B_W$!) and choose one, $S$, of maximal size. We know that $\#S \leq \dim V$ and if $\#S = \dim V$ it must be a basis and we are done, so assume that $\#S = k < \dim V$. We must then have $\text{Span}(S) \neq V$ so choose a vector $v \in V \setminus \text{Span}(S)$. We claim that $S \cup \{v\}$ is linearly independent, contradicting maximality of $S$. To see this write $S = \{v_1, \ldots, v_k\}$ and

$$a_1 v_1 + \cdots + a_k v_k + bv = \vec{0} \ .$$

If $b \neq 0$ then we can solve for $v$, getting $v \in \text{Span}(S)$, a contradiction, so we must have $b = 0$. But then $a_1 v_1 + \cdots + a_k v_k = \vec{0}$ and linear independence of $S$ gives $a_i = 0$ for all $i$, a contradiction. $\square$

The second subspace theorem will follow from a dimension theorem.

**Theorem 0.2.** *Let $W_1, W_2$ be subspaces of $V$, a finite-dimensional vector space. Then*

$$dim(W_1 + W_2) + dim(W_1 \cap W_2) = dim(W_1) + dim(W_2) \ .$$

*Proof.* Let $\hat{B}$ be a basis for the intersection $W_1 \cap W_2$. By the one subspace theorem we can find bases $B_1$ and $B_2$ of $W_1$ and $W_2$ respectively that both contain $\hat{B}$. Write

$$\begin{aligned} \hat{B} &= \{v_1, \ldots, v_k\} \\ B_1 &= \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_l\} \\ B_2 &= \{v_1, \ldots, v_k, w_{k+1}, \ldots, w_m\} \ . \end{aligned}$$

We will now show that $B = B_1 \cup B_2$ is a basis for $W_1 + W_2$. This will prove the theorem, since then $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = k + (l + m - k) = l + m$.

To show that $B$ is a basis for $W_1 + W_2$ we first must prove $\text{Span}(B) = W_1 + W_2$. Since $B \subset W_1 + W_2$, we have $\text{Span}(B) \subset \text{Span}(W_1 + W_2) = W_1 + W_2$. On the other hand, each vector in $W_1 + W_2$ can be written as $w_1 + w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$. Because $B$ contains a basis for each of $W_1$ and $W_2$, these vectors $w_1$ and $w_2$ can be written in terms of vectors in $B$, so $w_1 + w_2 \in \text{Span}(B)$.

Next we show that $B$ is linearly independent. We set a linear combination equal to zero:

$$a_1 v_1 + \cdots + a_k v_k + a_{k+1} v_{k+1} + \cdots + a_l v_l + b_{k+1} w_{k+1} + \cdots + b_m w_m = \vec{0} \ . \tag{1}$$

By subtracting the $w$ terms to one side we find that $b_{k+1} w_{k+1} + \cdots + b_m w_m \in W_1$. But this sum is already in $W_2$, so it must be in the intersection. As $\hat{B}$ is a basis for the intersection we can write

$$b_{k+1} w_{k+1} + \cdots + b_m w_m = c_1 v_1 + \cdots + c_k v_k$$

for some $c_i$'s in $\mathbb{F}$. Subtracting the $w$'s to one side and using linear independence of $B_2$ gives $b_{k+1} = \cdots = b_m = 0$. Therefore (1) reads

$$a_1 v_1 + \cdots + a_l v_l = \vec{0} \ .$$

Using linear independence of $B_1$ gives $a_i = 0$ for all $i$ and thus $B$ is linearly independent. $\square$

The proof of this theorem gives:

**Theorem 0.3** (Two subspace theorem)**.** *If $W_1, W_2$ are subspaces of a finite-dimensional vector space $V$, there exists a basis of $V$ that contains bases of $W_1$ and $W_2$.*

*Proof.* Use the proof of the last theorem to get a basis for $W_1 + W_2$ containing bases of $W_1$ and $W_2$. Then use the one-subspace theorem to extend it to $V$. $\square$

Note the difference from the one subspace theorem. We are not claiming that you can extend any given bases of $W_1$ and $W_2$ to a basis of $V$. We are just claiming there exists at least one basis of $V$ such that part of this basis is a basis for $W_1$ and part is a basis for $W_2$.

In fact, given bases of $W_1$ and $W_2$ we cannot generally find a basis of $V$ containing these bases. Take

$$V = \mathbb{R}^3,\ W_1 = \{(x, y, 0) : x, y \in \mathbb{R}\},\ W_2 = \{(x, 0, z) : x, z \in \mathbb{R}\} \ .$$

If we take bases $B_1 = \{(1, 0, 0), (1, 1, 0)\}$ and $B_2 = \{(1, 0, 1), (0, 0, 1)\}$, there is no basis of $V = \mathbb{R}^3$ containing both $B_1$ and $B_2$ since $V$ is 3-dimensional.

We now move on to the main subject of the course, linear transformations.

## Linear transformations

**Definition 0.4.** *Let $V$ and $W$ be vector spaces over the same field $\mathbb{F}$. A function $T : V \to W$ is called a linear transformation if*

$$T(v_1 + v_2) = T(v_1) + T(v_2) \text{ and } T(cv_1) = cT(v_1) \text{ for all } v_1, v_2 \in V \text{ and } c \in \mathbb{F} \ .$$

As usual, we only need to check the condition

$$T(cv_1 + v_2) = cT(v_1) + T(v_2) \text{ for } v_1, v_2 \in V \text{ and } c \in \mathbb{F} \ .$$

**Examples**

1. Consider $\mathbb{C}$ as a vector space over itself. Then if $T : \mathbb{C} \to \mathbb{C}$ is linear, we can write

   $$T(z) = zT(1)$$

   so $T$ is completely determined by its value at 1.

2. Let $V$ be finite dimensional and $B = \{v_1, \ldots, v_n\}$ a basis for $V$. Each $v \in V$ can be written uniquely as
$$v = a_1 v_1 + \cdots + a_n v_n \text{ for } a_i \in \mathbb{F} .$$
So define $T : V \to \mathbb{F}^n$ by $T(v) = (a_1, \ldots, a_n)$. This is called the *coordinate map relative to $B$*. It is linear because if $v = a_1 v_1 + \cdots + a_n v_n$, $w = b_1 v_1 + \cdots + b_n v_n$ and $c \in \mathbb{F}$,
$$cv + w = (ca_1 + b_1)v_1 + \cdots + (ca_n + b_n)v_n$$
is one representation of $cv + w$ in terms of the basis. But this representation is unique, so we get
$$\begin{aligned} T(cv + w) &= (ca_1 + b_1, \ldots, ca_n + b_n) \\ &= c(a_1, \ldots, a_n) + (b_1, \ldots, b_n) \\ &= cT(v) + T(w) . \end{aligned}$$

3. Given any $m \times n$ matrix $A$ with entries from $\mathbb{F}$ (the notation from the homework is $A \in M_{m,n}(\mathbb{F})$, we can define a linear transformations $L_A : \mathbb{F}^n \to \mathbb{F}^m$ and $R_A : \mathbb{F}^m \to \mathbb{F}^n$ by
$$L_A(v) = A \cdot v \text{ and } R_A(v) = v \cdot A .$$
Here we are using matrix multiplication and in the first case, representing $v$ as a column vector. In the second, $v$ is a row vector.

4. In fact, the set of linear transformations from $V$ to $W$, written $L(V, W)$, forms a vector space! Since the space of functions from $V$ to $W$ is a vector space, it suffices to check that it is a subspace. So given $T, U \in L(V, W)$ and $c \in \mathbb{F}$, we must show that $cT + U$ is a linear transformation. So let $v_1, v_2 \in V$ and $c' \in \mathbb{F}$:
$$\begin{aligned} (cT + U)(c'v + w) &= (cT)(c'v + w) + U(c'v + w) \\ &= c(T(c'v + w)) + U(c'v + w) \\ &= c(c'T(v) + T(w)) + c'U(v) + U(w) \\ &= c'(cT(v) + U(v)) + cT(w) + U(w) \\ &= c'(cT + U)v + (cT + U)(w) . \end{aligned}$$