Now that we have done the primary decomposition of V into generalized eigenspaces, we will do a secondary decomposition, and break each space into smaller subspaces, each spanned by a *chain basis*. To do this we notice that if we consider $\lambda_i I - T$ on the generalized eigenspace $\hat{E}_{\lambda_i}$, then it is nilpotent.

Definition 0.1. *A linear $U : V \to V$ is called nilpotent if there is some $k \geq 1$ such that $U^k = 0$. The minimal k is called the degree of U.*

We will consider a nilpotent $U : V \to V$ and find a nice matrix representation for it. We will relate this basis back to T later. The nice representation will come from chains.

Definition 0.2. *A set $\{v, U(v), U^2(v), \ldots, U^l(v)\}$ is called a chain of length l for U if $U^s(v) \neq \vec{0}$ for $s \leq l$ but $U^{l+1}(v) = \vec{0}$.*

Theorem 0.3 (Structure theorem for nilpotent operators). *Let $U : V \to V$ be nilpotent and dim $V < \infty$. There exists a basis for V consisting of chains for U.*

Our main tool to prove the theorem will be linear independence mod a subspace. Recall that if W is a subspace of V then $v_1, \ldots, v_k$ are said to be *linearly independent mod W* if whenever

$$a_1 v_1 + \cdots + a_k v_k \in W \ ,$$

it follows that $v_1, \ldots, v_k \in W$. We will give some important lemmas about this concept. Many of these can be seen as statements we have derived at the beginning of the semester, but in the setting of quotient spaces, specifically in V/W. For example, the following is analogous to the one-subspace (basis extension) theorem:

Proposition 0.4. *Let $W_1 \subset W_2$ be subspaces of V. If dim $W_2 -$ dim $W_1 = m$ and $v_1, \ldots, v_l \in W_2$ are linearly independent mod W_1 we can find $m - l$ vectors $v_{l+1}, \ldots, v_m \in W_2 \setminus W_1$ such that $\{v_1, \ldots, v_m\}$ is linearly dependent mod W_1.*

Proof. You showed in homework that $\{v_1, \ldots, v_l\}$ is linearly independent mod W_1 if and only if $\{v_1 + W, \ldots, v_l + W\}$ is linearly independent in W_2/W_1. Further, you showed that the dimension of W_2/W_1 is dim $W_2 -$dim W_1. So we can use the one-subspace theorem in W_2/W_1 to extend $\{v_1 + W_1, \ldots, v_l + W_1\}$ to a basis of W_2/W_1, adding elements $C_{l+1}, \ldots, C_m \in W_2/W_1$. Each of these elements can be written as $v + W_1$ for some v, so we obtain a set

$$\{v_1 + W_1, \ldots, v_l + W_1, v_{l+1} + W_1, \ldots, v_m + W_1\}$$

which is a basis of W_2/W_1. By the equivalence above, $\{v_1, \ldots, v_m\}$ is then linearly independent mod W_1. □

Another similar statement is:

Proposition 0.5. *Let $W_1 \subset W_2$ be subspaces of V. If dim $W_2 -$ dim $W_1 = m$ and $\{v_1, \ldots, v_l\} \subset W_2$ is linearly independent mod W_1 then $l \leq m$.*

Proof. Since $\{v_1, \ldots, v_l\}$ is linearly independent mod W_1, $\{v_1 + W_2, \ldots, v_l + W_2\}$ is linearly independent in W_2/W_1. This space has dimension m, so by Steinitz, $l \leq m$. □

Now let's specialize to the case of subspaces associated to a nilpotent operator. Given a nilpotent $U : V \to V$ of degree k with V finite-dimensional, we construct the subspaces

$$N_0 = \{\vec{0}\},\ N_1 = N(U), \ldots, N_{k-1} = N(U^{k-1}),\ N_k = V\ .$$

Note that

$$N_0 \subset N_1 \subset \cdots \subset N_k \text{ and } N_{k-1} \neq V\ .$$

We will prove a couple of properties about this "tower" of subspaces.

1. If $v \in N_j \setminus N_{j-1}$ for $j = 2, \ldots, k$ then $U(v) \in N_{j-1} \setminus N_{j-2}$.

   Proof. If $v \in N_j \setminus N_{j-1}$ then $U^j(v) = \vec{0}$ but $U^{j-1}(v) \neq \vec{0}$. Thus

   $$U^{j-1}(U(v)) = \vec{0} \text{ but } U^{j-2}(U(v)) \neq \vec{0}\ ,$$

   meaning $U(v) \in N_{j-1} \setminus N_{j-2}$. □

2. If $\{v_1, \ldots, v_l\}$ is linearly independent mod N_j for $j \geq 1$ then $\{U(v_1), \ldots, U(v_l)\}$ is linearly independent mod N_{j-1}.

   Proof. Suppose that $\{v_1, \ldots, v_l\}$ is linearly independent mod N_j and $j \geq 1$. Then suppose

   $$a_1U(v_1) + \cdots + a_lU(v_l) \in N_{j-1} \text{ for some } a_1, \ldots, a_l \in \mathbb{F}\ .$$

   Then we can write $U(a_1v_1 + \cdots + a_lv_l) \in N_{j-1}$, meaning

   $$U^j(a_1v_1 + \cdots + a_lv_l) = U^{j-1}(U(a_1v_1 + \cdots + a_lv_l)) = \vec{0}\ .$$

   Thus $a_1v_1 + \cdots a_lv_l \in N_j$ and linear independence mod N_j gives $a_i = 0$ for all i. We conclude $\{U(v_1), \ldots, U(v_l)\}$ is linearly independent mod N_{j-1}. □

Finally we prove the structure theorem for nilpotent operators.

Proof of structure theorem. We will prove by induction on the degree of U. We will prove a slightly stronger statement: for any k, let S_k be the statement "whenever $U : V \to V$ is nilpotent of degree k, writing $m = \dim N_k - \dim N_{k-1}$, if $\{v_1, \ldots, v_m\}$ is linearly independent mod N_{k-1} then there is a basis of V consisting of chains for U such that $v_1, \ldots, v_m$ each begin a chain."

For $k = 1$, the statement is pretty easy. Let $U : V \to V$ be nilpotent of degree 1. Then $m = \dim N_1 - \dim N_0 = \dim V$. Also if $\{v_1, \ldots, v_m\}$ is linearly independent mod N_0, since $N_0 = \{\vec{0}\}$, this set is truly linearly independent and thus a basis. Now since $U(v) = \vec{0}$ for all v, each v_i starts a chain of length 1 and we are done.

Now let $U : V \to V$ be nilpotent of degree $k \geq 2$ and assume that the statement S_l holds for $l = k-1$. Suppose that $\{v_1, \ldots, v_{d_k}\}$ are given vectors that are linearly independent mod N_{k-1} and $d_k = \dim N_k - \dim N_{k-1}$. By the second property above,

$$\{U(v_1), \ldots, U(v_{d_k})\} \text{ is linearly independent mod } N_{k-2} \ ,$$

so by the first proposition again we may extend it to a set

$$\{U(v_1), \ldots, U(v_{d_k}), w_1, \ldots, w_{m-d_k}\} \text{ with } \dim N_{k-1} - \dim N_{k-2} = m \geq d_k$$

which is linearly independent mod N_{k-2}. Now we apply the statement S_{k-1} to this set to start chains. The space N_{k-1} is U-invariant, and so we can restrict U to it, defining the restricted operator U_{k-1}. It is not hard to check that it is nilpotent of degree $k-1$ and has tower of nullspaces equal to the set of first $k-1$ subspaces for U. That is, $N(U_{k-1}^j) = N_j$ for $j = 0, \ldots, k-1$. So the inductive hypothesis says that there is a basis B_{k-1} of N_{k-1} consisting of chains for U_{k-1} such that each of $U(v_1), \ldots, U(v_{d_k}), w_1, \ldots, w_{m-d_k}$ starts a chain. Now we may simply append v_i to the chain started by $U(v_i)$ for $i = 1, \ldots, d_k$ to get m chains with a total of $m + d_k$ elements. Since $m + d_k = \dim V$, we are left to just check that $\{v_1, \ldots, v_{d_k}\} \cup B_{k-1}$ is linearly independent; then it will be a basis for V consisting of chains for U (such that $v_1, \ldots, v_{d_k}$ each start a chain).

To prove that, note that B_{k-1} is linearly independent (and a subset of N_{k-1}) and $\{v_1, \ldots, v_{d_k}\}$ is linearly independent mod N_{k-1}. Thus if we have a linear combination

$$a_1 v_1 + \cdots + a_{d_k} v_{d_k} + \sum_{v \in B_{k-1}} b_v v = \vec{0} \ ,$$

then $a_1 v_1 + \cdots + a_{d_k} v_{d_k} \in N_{k-1}$ and linear independence mod N_{k-1} gives $a_1 = \cdots = a_{d_k} = 0$. Thus we have $\sum_{v \in B_{k-1}} b_v v = \vec{0}$ and linear independence of B_{k-1} gives $b_v = 0$ for all v. $\square$