MAT217 HW 3
DUE TUES. FEB. 26, 2013

1. Read Section 1.4 in the Hoffman-Kunze handout and do exercises 3, 7.

2. (From Axler) Give an example of a function $T : \mathbb{R}^2 \to \mathbb{R}$ such that $T(av) = aT(v)$ for all $a \in \mathbb{R}$ and $V \in \mathbb{R}^2$ but T is not linear.

3. (From Hoffman-Kunze) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) \ .$$
Show that T is linear. Is T invertible? If so, find a rule defining T^{-1} like the above.

4. (From Hoffman-Kunze) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$T(x_1, x_2) = (-x_2, x_1) \ .$$
   (a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$? In other words, find $[T]_B^B$ where $B = \{(1, 0), (0, 1)\}$.
   (b) What is $[T]_B^B$ with $B = (\alpha_1, \alpha_2)$, where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, -1)$?
   (c) Let T be the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ given by
$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_3, -x_1 + 2x_2 + 4x_3) \ .$$
   What is $[T]_B^B$ with $B = \{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$ and $\alpha_3 = (2, 1, 1)$?

5. Let $T : V \to V$ be linear with $\dim V < \infty$. Show that the following two statements are equivalent.
   (A) $V = R(T) \oplus N(T)$.
   (B) $N(T) = N(T^2)$, where $T^2 = T \circ T$.

6. Let $T : V \to W$ be linear with $\dim(V) = n$ and $\dim(W) = m$.
   (a) Prove that if $n > m$ then T cannot be injective.
   (b) Prove that if $n < m$ then T cannot be surjective.
   (c) Prove that if $n = m$ then T is injective if and only if it is surjective.

7. Let V, W and Z be finite-dimensional vector spaces over $\mathbb{F}$. If $T : V \to W$ and $U : W \to Z$ are linear, show that
$$\text{rank}(UT) \leq \min\{\text{rank}(U), \text{rank}(T)\} \ .$$
Prove also that if either of U or T is invertible, the rank of UT is equal to the rank of the other one. Deduce that if $P : V \to V$ and $Q : W \to W$ are isomorphisms then the rank of QTP equals the rank of T.

8. Given an angle $\theta \in [0, 2\pi)$, let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates a vector clockwise about the origin by an angle θ. Find $[T_\theta]_B^B$, where $B = \{(1,0),(0,1)\}$.

9. Let V and W be finite dimensional vector spaces over $\mathbb{F}$ and $T : V \to W$ linear. Show there exist ordered bases B of V and C of W such that

$$\left([T]_C^B\right)_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 0 \text{ or } 1 & \text{if } i = j \end{cases} .$$

10. Let $\mathbb{F}$ be a field and consider the vector space of polynomials of degree at most n:

$$\mathbb{F}_n[x] = \{a_n x^n + \cdots + a_0 : a_i \in \mathbb{F} \text{ for } i = 0, \ldots, n\} .$$

   (a) Show that $B = \{1, x, x^2, \ldots, x^n\}$ is a basis for this space.

   (b) Fix an element $b \in \mathbb{F}$ and define the evaluation map $T_b : \mathbb{F}_n[x] \to \mathbb{F}$ by $T_b(p) = p(b)$. Show this is linear. Find the range and nullspace of T_b.

   (c) Give the representation of T_b in terms of the basis B for $\mathbb{F}_n[x]$ and the basis $\{1\}$ for $\mathbb{F}$.

   (d) For distinct $b_1, \ldots, b_{n+2}$ in $\mathbb{F}$ show that the functions $T_{b_1}, \ldots, T_{b_{n+2}}$ are linearly dependent in $L(\mathbb{F}_n[x], \mathbb{F})$. Deduce that any polynomial p in $\mathbb{F}_n[x]$ with at least $n+1$ zeros must have $p(x) = 0$ for all $x \in \mathbb{F}$.

11. Here you will give an alternative proof of the rank-nullity theorem. Let $T : V \to W$ be linear and suppose that $\dim(V) < \infty$.

   (a) Consider the quotient space $V/N(T)$ and define a function $\hat{T} : V/N(T) \to W$ as follows. If $C \in V/N(T)$ is some element we may represent it as $v + N(T)$ for some $v \in V$. Select one such element v and define $\hat{T}(C) = T(v)$. Show that this definition does not depend on the choice of v, so long as $v + N(T) = C$; that is, that $\hat{T}$ as defined is a (well-defined) function.

   (b) Prove that $\hat{T}$ is an isomorphism from $V/N(T)$ to $R(T)$. (This is a version of the *first isomorphism theorem* when it is proved for groups.)

   (c) Deduce the rank-nullity theorem.