Lecture 19: Bilinear and symmetric bilinear forms

Here are some nice consequences of the equality $[f]_B^B = [L_f]_{B^*}^B$.

1. The definition of rank of f does not depend on the choice of basis B. Indeed, let C be another basis. Then

$$\text{rank}(f) = \text{rank}[f]_B^B = \text{rank}([L_f]_{B^*}^B) = \text{rank}([L_f]_{C^*}^C) = \text{rank}[f]_C^C \ .$$

   The third equality follows from the fact that the rank of the matrix of L_f does not depend on the bases used to represent it.

2. We can equally well define R_f to be the map from V to V^* by

$$R_f(v)(w) = f(w,v) \ .$$

   Then $[R_f]_{B^*}^B = [f]_B^B$. The reason is as follows. Define $g(v,w) = f(w,v)$. Then $L_g = R_f$. Now the matrix $[g]_B^B$ is easily seen to be the transpose of $[f]_B^B$ (its (i,j)-th entry is $g(v_j, v_i) = f(v_i, v_j)$). So

$$[f]_B^B = \left([g]_B^B\right)^t = [L_g]_{B^*}^B = [R_f]_{B^*}^B \ .$$

3. We say that a bilinear form is *degenerate* if its rank is not equal to $\dim V$. We can now state many equivalent conditions for this: the following are equivalent when $f \in Bil(V, \mathbb{F})$ and $\dim V = n < \infty$.

   (a) f is degenerate.

   (b) Define the nullspace of f to be

$$N(f) = \{v \in V : f(v,w) = 0 \text{ for all } w \in V\} \ .$$

   (This is also called the left nullspace.) Then $N(f) \neq \{\vec{0}\}$.

   (c) Defining the right nullspace by

$$N_R(f) = \{v \in V : f(w,v) = 0 \text{ for all } w \in V\} \ ,$$

   then $N_R(f) \neq \{\vec{0}\}$.

   Note here that $N(f) = N(L_f)$ and $N_R(f) = N(R_f)$. By this representation, we have

$$\text{rank}(f) + \dim N(f) = \dim V \ .$$

   This comes from the rank-nullity theorem applied to L_f.

Theorem 0.1. *Let V be finite-dimensional. The map $\Phi_L : Bil(V, \mathbb{F}) \to L(V, V^*)$ given by*

$$\Phi_L(f) = L_f$$

is an isomorphism.

Proof. First we show linearity. Given $f, g \in Bil(V, \mathbb{F})$ and $c \in \mathbb{F}$, we want to show that

$$\Phi_L(cf + g) = c\Phi_L(f) + \Phi_L(g) \ .$$

To do this, we need to show that when we apply each side to a vector $v \in V$, we get the same result. The result will be in the dual space, so we need to show this result, applied to a vector $w \in V$, is the same. Thus we compute

$$\left(\Phi_L(cf + g)(v)\right)(w) = (cf + g)(v, w) = cf(v, w) + g(v, w)$$

and the right side is

$$\begin{aligned} \left((c\Phi_L(f) + \Phi_L(g))(v)\right)(w) &= \left(c(\Phi_L(f)(v)) + \Phi_L(g)(v)\right)(w) \\ &= c\left((\Phi_L(f)(v))(w)\right) + (\Phi_L(g)(v))(w) \\ &= cf(v, w) + g(v, w) \ . \end{aligned}$$

To show bijectivity, we note that the dimension of $Bil(V, \mathbb{F})$ is n^2, the same as that of $L(V, V^*)$ (since the map sending a bilinear form to a matrix is an isomorphism). Thus we need only show one-to-one. If $\Phi_L(f) = 0$, then $\Phi_L(f)(v) = 0$ for all $v \in V$, meaning for all $w \in V$,

$$0 = (\Phi_L(f)(v))(w) = f(v, w) \ .$$

This being true for all v, w means f is zero, so Φ_L is injective. □

Now we move to changing coordinates. This is one big difference between the matrix of a linear transformation and the matrix of a bilinear form. Instead of conjugating by a change of basis matrix as before, we multiply on the right by the change of basis matrix and on the left by its transpose.

Proposition 0.2. *Let f be a bilinear form on V, a finite-dimensional vector space over $\mathbb{F}$. If B, B' are bases of V then*

$$[f]_{B'}^{B'} = \left([I]_B^{B'}\right)^t [f]_B^B [I]_B^{B'} \ .$$

Proof. For any $v, w \in V$,

$$\begin{aligned} [w]_{B'}^t \left(\left([I]_B^{B'}\right)^t [f]_B^B [I]_B^{B'}\right)[v]_{B'} &= \left([I]_B^{B'}[w]_{B'}\right)^t [f]_B^B \left([I]_B^{B'}[v]_{B'}\right) \\ &= [w]_B^t [f]_B^B [v]_B \\ &= f(v, w) \ . \end{aligned}$$

□

Another way to see the theorem is that if a matrix A represents a bilinear form in some basis and P is an invertible matrix, then $P^t A P$ represents the bilinear form in a different basis.

Definition 0.3. *A form $f \in Bil(V, \mathbb{F})$ is called symmetric if $f(v,w) = f(w,v)$ for all $v, w \in V$. The space of symmetric bilinear forms is denoted $Sym(V, \mathbb{F})$.*

Symmetric forms are represented by symmetric matrices. That is, if f is symmetric and B is a basis, then $[f]_B^B$ is equal to its transpose $([f]_B^B)^t$. One of the fundamental theorems about symmetric bilinear forms is that they can be diagonalized.

Definition 0.4. *A basis B of V is called orthogonal relative to $f \in Bil(V, \mathbb{F})$ if $f(v,w) = 0$ for all distinct $v, w \in B$.*

The basis B being orthogonal relative to f is equivalent to $[f]_B^B$ being a diagonal matrix.

Theorem 0.5 (Diagonalization of symmetric bilinear forms)**.** *Let V be a finite-dimensional vector space over $\mathbb{F}$, a field of characteristic not equal to 2. If $f \in Sym(V, \mathbb{F})$ then V has basis orthogonal relative to f.*

Remark. *Equivalently, if $A \in M_{n,n}(\mathbb{F})$ is symmetric, then there is an invertible $P \in M_{n,n}(\mathbb{F})$ such that $P^t AP$ is diagonal.*

Proof. First note that if f is the zero form then the theorem is trivially true. So assume $f \neq 0$.

We will argue by induction on the dimension of V. For the base case, if V has dimension 1, then any basis is orthogonal relative to f. If $\dim(V) = n > 1$ then we begin by finding a first element of our basis. To do this, let $v \in V$; we will need to make sure that v can be chosen such that $f(v,v) \neq 0$. For this, we use a lemma.

Lemma 0.6. *Let $f \in Sym(V, \mathbb{F})$ be nonzero. If $\mathbb{F}$ does not have characteristic two then*

$$f(v,v) = 0 \text{ for all } v \in V \Rightarrow f = 0 \ .$$

Proof. The idea of the proof is to develop a so-called "polarization identity." For $v, w \in V$,

$$f(v+w, v+w) - f(v-w, v-w) = 4f(v,w) \ ,$$

so

$$f(v,w) = \frac{1}{4}(f(v+w, v+w) - f(v-w, v-w)) \ .$$

Note that what we have written as $1/4$ is actually the inverse of 4 in $\mathbb{F}$. This exists because $\mathbb{F}$ does not have characteristic 2. Therefore if $f(z,z) = 0$ for all z, we apply this for $z = v + w$ and $z = v - w$ to find $f(v,w) = 0$. □

From the lemma, since $f \neq 0$, we can find $v \in V$ such that $f(v,v) \neq 0$. This implies that v itself is nonzero. Now consider the function $L_f(v)$. Since it is a nonzero linear functional (for instance $L_f(v)(v) \neq 0$) its nullspace must be of dimension $n - 1$. Define $\hat{f}$ to be f restricted to $N(L_f(v))$ and note that $\hat{f}$ is a symmetric bilinear form on $N(L_f(v))$. Since this has dimension strictly less than that of V, we use induction to find $\{v_1, \ldots, v_{n-1}\}$, a basis for $N(L_f(v))$ that is orthogonal relative to $\hat{f}$.

Since $v \notin N(L_f(v))$, it follows that $\{v_1, \ldots, v_{n-1}, v\}$ is a basis for V. It is also orthogonal because $f(v_i, v_j) = \hat{f}(v_i, v_j) = 0$ whenever $i, j \in \{1, \ldots, n-1\}$ and for $i = 1, \ldots, n-1$ we have

$$f(v_i, v) = f(v, v_i) = 0 \text{ since } v_i \in N(L_f(v)) \ .$$

This completes the proof. $\square$