MAT217 HW 2

Due Tues. Feb. 19, 2013

Notation.

Throughout, $(V, \mathbb{F})$ is a vector space. Recall that if W is a subspace of V, we write V/W for the set $\{v + W : v \in V\}$ and that this set has a vector space structure given by

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \text{ and } c(v + W) = (cv + W),\ v_1, v_2 \in V,\ c \in \mathbb{F}\ .$$

Exercises.

1. Read Section 1.3 in the Hoffman-Kunze handout and do exercises 1, 4, 5.
2. (From Hoffman-Kunze) Are the vectors

$$(1, 1, 2, 4), (2, -1, -5, 2), (1, -1, -4, 0) \text{ and } (2, 1, 1, 6)$$

linearly independent in $\mathbb{R}^4$? Find a basis for the subspace spanned by these vectors.

3. (From Hoffman-Kunze) If $\mathbb{F}$ is a field and $m, n \in \mathbb{N}$, we write $M_{m,n}(\mathbb{F})$ for the set of $m \times n$ matrices with entries from $\mathbb{F}$ and the (i, j)-th entry of a typical element A is written $A_{i,j}$. Addition and scalar multiplication is performed coordinate-wise: if $A, B \in M_{m,n}(\mathbb{F})$ and $c \in \mathbb{F}$, then $(cA + B)_{i,j} = cA_{i,j} + B_{i,j}$. Let $V = M_{2,2}(\mathbb{F})$,

$$W_1 = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} : x, y, z \in \mathbb{F} \right\} \text{ and } W_2 = \left\{ \begin{pmatrix} x & y \\ -x & z \end{pmatrix} : x, y, z \in \mathbb{F} \right\}\ .$$

Prove that these are subspaces of V and find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

4. (From Hoffman-Kunze) For $m, n \in \mathbb{N}$, find a basis for $M_{m,n}(\mathbb{F})$.
5. Show that if $S \subset V$ is a finite generating set then S contains a basis for V. Deduce that V is finitely generated if and only if it has a finite basis.
6. Let $S \subset V$.
   (a) Suppose that S generates V but no proper subset of S generates V (that is, S is a minimal spanning set). Show that S is a basis.
   (b) Suppose that S is linearly independent and is not a proper subset of any linearly independent set in V (that is, S is a maximal linearly independent set). Show that S is a basis.
7. Let W be a subspace of V.

(a) We say that $S \subset V$ is linearly independent modulo W if whenever $v_1, \ldots, v_k \in S$ and $a_1, \ldots, a_k \in \mathbb{F}$ are such that

$$a_1 v_1 + \cdots + a_k v_k \in W$$

then $a_1 = \cdots = a_k = 0$. Show that S is linearly independent modulo W if and only if the set $\{v + W : v \in S\}$ is linearly independent as a subset of V/W.

(b) Assume now that V has dimension $n < \infty$. If W has dimension m, show that V/W has dimension $n - m$.

Hint. Let B_W be a basis for W and use the one subspace theorem to extend it to a basis B for V. Show that $\{v + W : v \in B \text{ but } v \notin B_W\}$ is a basis for V/W.

(c) Let $W_1 \subset W_2 \subset V$ be subspaces. Show that

$$\dim W_2/W_1 + \dim V/W_2 = \dim V/W_1 \ .$$

8. If $W_1, \ldots, W_k$ are subspaces of V we write $W_1 \oplus \cdots \oplus W_k$ for the sum space $W_1 + \cdots + W_k$ if

$$W_j \cap [W_1 + \cdots + W_{j-1}] = \{\vec{0}\} \text{ for all } j = 2, \ldots, k \ .$$

In this case we say that the subspaces $W_1, \ldots, W_k$ are independent.

(a) For $k = 2$, this definition is what we gave in class: W_1 and W_2 are independent if and only if $W_1 \cap W_2 = \{\vec{0}\}$. Give an example to show that for $k > 2$ this is not true. That is, if $W_1, \ldots, W_k$ satisfy $W_i \cap W_j = \{\vec{0}\}$ for all $i \neq j$ then these spaces need not be independent.

(b) Prove that the following are equivalent.

1. $W_1, \ldots, W_k$ are independent.
2. Whenever $w_1 + \cdots + w_k = \vec{0}$ for $w_i \in W_i$ for all i then $w_i = \vec{0}$ for all i.
3. Whenever B_i is a basis for W_i for all i, the B_i's are disjoint and $B := \cup_{i=1}^k B_i$ is a basis for $W_1 + \cdots + W_k$.

9. Give an example to show that there is no "three subspace theorem." That is, if W_1, W_2, W_3 are subspaces of V then there need not exist a basis of V containing a basis for W_i for all $i = 1, 2, 3$.

10. Let F be a finite field. Define a sequence (s_n) of elements of F by $s_1 = 1$ and $s_{n+1} = s_n + 1$ for $n \in \mathbb{N}$. Last, define the *characteristic* of F as

$$\text{char}(F) = \min\{n \in \mathbb{N} : s_n = 0\} \ .$$

(If the set on the right is empty, we set $\text{char}(F) = 0$.)

(a) Show that because F is finite, its characteristic is a prime number p.

(b) Show that the set $\{0, s_1, \ldots, s_{p-1}\}$ with the same addition and multiplication as in $\mathbb{F}$ is itself a field, called the prime subfield of F.

(c) Using the fact that F can be viewed as a vector space over its prime subfield, show that F has p^n elements for some $n \in \mathbb{N}$.