MIDTERM SOLUTIONS

1. **Let $V, W$ and $Z$ be finite-dimensional vector spaces over $\mathbb{F}$ with $T : V \to W$ and $U : W \to Z$ linear. Show that if $U$ is invertible, then rank$(U \circ T) =$ rank$(T)$.**

   **Solution.** The rank of $T$ is defined as the dimension of the range of $T$. This is a subspace of $W$, a finite dimensional space, so the rank is finite and we can write $\{w_1, \ldots, w_k\}$ as a basis for the range $R(T)$. We claim that the elements $U(w_1), \ldots, U(w_k)$ are distinct and form a basis for the range $R(U \circ T)$. The rank of $U \circ T$ is the number of elements in a basis for $R(U \circ T)$, so this will complete the proof.

   If $U(w_i) = U(w_j)$ for some $i, j \in \{1, \ldots, k\}$ then since $U$ is invertible, it is in particular injective, giving $w_i = w_j$, so the elements $U(w_1), \ldots, U(w_k)$ are distinct. To show linear independence, suppose that

   $$a_1 U(w_1) + \cdots + a_k U(w_k) = \vec{0} \text{ for } a_i \in \mathbb{F} \ .$$

   Then using linearity we find $U(a_1 w_1 + \cdots + a_k w_k) = \vec{0}$, so since $U$ is injective, $a_1 w_1 + \cdots + a_k w_k = \vec{0}$. The $w_i$'s were chosen to be a basis for $R(T)$ so they are linearly independent and we conclude that $a_i = 0$ for all $i$. This means $\{w_1, \ldots, w_k\}$ is linearly independent.

   Last we prove spanning, so let $z \in R(U \circ T)$; this means there is a $v \in V$ such that $z = U(T(v))$. As $T(v) \in R(T)$ we can write it in terms of the basis: $T(v) = b_1 w_1 + \cdots + b_k w_k$ for some $b_i \in \mathbb{F}$. Last we find

   $$z = U(T(v)) = U(b_1 w_1 + \cdots + b_k w_k) = b_1 U(w_1) + \cdots + b_k U(w_k) \ ,$$

   giving $R(U \circ T) \subset \text{Span}(\{U(w_1), \ldots, U(w_k)\})$. For the other inclusion, each $w_i$ is in $R(T)$, so we can write $w_i = T(v_i)$ for $v_i \in V$. Thus $U(w_i) = (U \circ T)(v_i) \in R(U \circ T)$, and

   $$\text{Span}(\{U(w_1), \ldots, U(w_k)\}) \subset \text{Span}(R(U \circ T)) = R(U \circ T) \ .$$

2. **Let $V$ be an $\mathbb{F}$-vector space and $S$ be a linearly independent subset of $V$. Show that for each nonzero $v \in$ Span$(S)$ there exists exactly one choice of $v_1, \ldots, v_n \in S$ and nonzero coefficients $a_1, \ldots, a_n \in \mathbb{F}$ such that $v = a_1 v_1 + \cdots + a_n v_n$.**

   **Solution.** (From the notes.) Let $v \in \text{Span}(S)$ be nonzero. By characterization of the span as the set of linear combinations of elements of $S$, there is at least one representation as above. To show it is unique, suppose that $v = a_1 v_1 + \cdots + a_n v_n$ and $v = b_1 w_1 + \cdots + b_k w_k$ and write $S_1 = \{v_1, \ldots, v_n\}$, $S_2 = \{w_1, \ldots, w_k\}$. We can arrange the $S_i$'s so that the elements $v_1 = w_1, \ldots, v_m = w_m$ are the common ones; that is, the ones in $S_1 \cap S_2$. Then

   $$\vec{0} = v - v = \sum_{j=1}^{m} (a_j - b_j) v_j + \sum_{l=m+1}^{n} a_l v_l + \sum_{p=m+1}^{k} b_p w_p \ .$$

This is just a linear combination of elements of $S$, so by linear independence, all coefficients are zero, implying that $a_j = b_j$ for $j = 1, \ldots, m$, and all other $a_i$'s and $b_p$'s are zero. Thus all nonzero coefficients are the same in the linear combinations and we are done.

3. **Let $V$ be a finite dimensional $\mathbb{F}$-vector space and $W_1, W_2$ subspaces of $V$ such that $W_1^\perp = W_2^\perp$. Show that $W_1 = W_2$.**

   **Solution.** Define the linear map $\Phi : V \to V^{**}$ by $\Phi(v) = eval_v$. We saw in class that this is an isomorphism and further that for any subspace $W$ of $V$ we have $\Phi(W) = (W^\perp)^\perp$. Therefore
$$\Phi(W_1) = (W_1^\perp)^\perp = (W_2^\perp)^\perp = \Phi(W_2) \ .$$
   So for any $w_1 \in W_1$, we have $\Phi(w_1) \in \Phi(W_2)$, so there exists $w_2 \in W_2$ such that $\Phi(w_1) = \Phi(w_2)$. Since $\Phi$ is injective, $w_1 = w_2$ and therefore $w_1 \in W_2$. This proves that $W_1 \subset W_2$. The same argument, switching the roles of $W_1$ and $W_2$ shows that $W_2 \subset W_1$ also, and finishes the proof.

4. **Let $V$ and $W$ be finite-dimensional $\mathbb{F}$-vector spaces with $T_1 : V \to W$ and $T_2 : V \to W$ linear. Show that if $T_1$ and $T_2$ have the same nullspace, there exists an isomorphism $Q : W \to W$ such that $T_1 = Q \circ T_2$.**

   **Solution.** Let $\{v_1, \ldots, v_k\}$ be a basis for the nullspace of $T_1$ (and hence of $T_2$). We saw in the proof of the rank-nullity theorem that if we complete this to a basis $\{v_1, \ldots, v_n\}$ of $V$ then $\{T_1(v_{k+1}), \ldots, T_1(v_n)\}$ is a basis for $R(T_1)$. The same statement, applied to $T_2$ shows that $\{T_2(v_{k+1}), \ldots, T_2(v_n)\}$ is a basis for $R(T_2)$. Extend each of these to bases of $W$ (writing $m$ for the dimension of $W$):
$$\{w_1, \ldots, w_{m-k}, T_1(v_{k+1}), \ldots, T_1(v_n)\} \text{ and } \{w_1', \ldots, w_{m-k}', T_2(v_{k+1}), \ldots, T_2(v_n)\} \ .$$
   We now define the linear transformation $Q : W \to W$ (using the slogan) as the unique one that satisfies
$$Q(w_i') = w_i \text{ for } i = 1, \ldots, m-k \text{ and } Q(T_2(v_j)) = T_1(v_j)) \text{ for } j = k+1, \ldots, n \ .$$
   Since $Q$ maps one basis of $W$ to another basis of $W$, it is an isomorphism. A quick way to see this is that the range of $Q$ contains a linearly independent set of size $m$, so its dimension must be $m$, meaning that $Q$ is surjective. However any surjective linear transformation between spaces of the same dimension is an isomorphism.

   Furthermore $Q \circ T_2$ has the same action as $T_1$ on the basis $\{v_1, \ldots, v_n\}$: for $i = 1, \ldots, k$ we have $Q(T_2(v_i)) = Q(\vec{0}) = \vec{0} = T_1(v_i)$ and for $i = k+1, \ldots, n$ we have $Q(T_2(v_i)) = T_1(v_i))$. Since these linear transformations agree on a basis, the slogan says they are the same. Thus $Q \circ T_2 = T_1$.

5. **Let $V = \mathbb{R}^2$, viewed as a vector space over $\mathbb{R}$, and let $T : V \to V$ be the linear transformation specified by**
$$T((1,0)) = (2,2) \text{ and } T((1,1)) = (1,0) \ .$$

(a) **Find the matrix** $[T]_C^B$**, where** $B = \{(1,0),(0,1)\}$ **and** $C = \{(1,0),(1,1)\}$**.**

(b) **Find a matrix** $P \in M_{2,2}(\mathbb{R})$ **such that** $[T]_C^B = P[T]_B^B$**.**

**Solution.** For the first part, we start by writing $T((1,0))$ in terms of $C$:

$$T((1,0)) = (2,2) = 0 \cdot (1,0) + 2 \cdot (1,1)\ .$$

We then write $T((0,1))$ in terms of $C$:

$$T((0,1)) = T((1,1) - (1,0)) = (1,0) - (2,2) = (-1,-2) = 1 \cdot (1,0) - 2 \cdot (1,1)\ .$$

Putting these coefficients into the columns of our matrix, we find

$$[T]_C^B = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}\ .$$

For the second part, we recall our matrix multiplication rule:

$$[T]_C^B = [Id \circ T]_C^B = [Id]_C^B [T]_B^B\ ,$$

so we compute $[Id]_C^B$. The first basis vector in $B$ is transformed as thus:

$$Id((1,0)) = (1,0) = 1 \cdot (1,0) + 0 \cdot (1,1)\ .$$

As for the second vector,

$$Id((0,1)) = (0,1) = -1 \cdot (1,0) + 1 \cdot (1,1)\ .$$

Therefore

$$P = [Id]_C^B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\ .$$