# Lecture 8: Dual spaces

We have been talking about coordinates, so let's examine them more closely. Let $V$ be an $n$-dimensional vector space and fix a basis $B = \{v_1, \ldots, v_n\}$ of $V$. We can write any vector $V$ in coordinates relative to $B$ as

$$[v]_B = \begin{pmatrix} a_1 \\ \cdots \\ a_n \end{pmatrix} \text{, where } v = a_1 v_1 + \cdots + a_n v_n \ .$$

For any $i = 1, \ldots, n$ we can define the $i$-th coordinate map by $v_i^* : V \to \mathbb{F}$ given by $v_i^*(v) = a_i$, where $a_i$ is the $i$-th entry of $[v]_B$. These elements $v_i^*$ are linear and are thus in the space $L(V, \mathbb{F})$. This space comes up so much we give it a name:

**Definition 0.1.** *We write $V^* = L(V, \mathbb{F})$ and call it the dual space to $V$. Elements of $V^*$ will be written $f$ and called linear functionals.*

Given any basis $B = \{v_1, \ldots, v_n\}$ we call $B^* = \{v_1^*, \ldots, v_n^*\}$ the basis of $V^*$ dual to $B$.

**Proposition 0.2.** *If $B$ is a basis of $V$ then $B^*$ is a basis of $V^*$.*

*Proof.* The dimension of $V^*$ is $n$, the dimension of $V$, so we must show $B^*$ is linearly independent or spanning. We show linearly independent: suppose that

$$a_1 v_1^* + \cdots + a_n v_n^* = \vec{0} \ ,$$

where $\vec{0}$ on the right is the zero transformation from $V$ to $\mathbb{F}$. Apply both sides to $v_i$. For $i \neq j$ we get $v_j^*(v_i) = 0$, since the $j$-th coordinate of $v_i$ is 0. For $i = j$ we get $v_i^*(v_i) = 1$, so

$$a_i = (a_1 v_1^* + \cdots + a_n v_n^*)(v_i) = \vec{0}(v_i) = 0 \ .$$

This is true for all $i$ so $B^*$ is linearly independent and we are done. □

It is not surprising that $B^*$ is a basis of $V^*$. The reason is that each element $f \in V^*$ can be written in its matrix form using the basis $B$ of $V$ and $\{1\}$ of $\mathbb{F}$. Then then matrix for $v_i^*$ is

$$[v_i^*]_{\{1\}}^B = (0 \cdots 0 \ 1 \ 0 \cdots 0) \ ,$$

where the 1 is in the $i$-th spot. Clearly these form a basis for $M_{1,n}(\mathbb{F})$ and since the map sending linear transformations to their matrices relative to these bases is an isomorphisms, so should $B^*$ be a basis of $V^*$.

There is an alternate characterization: each $v_i^*$ is in $L(V, \mathbb{F})$ so can be identified by its action on the basis $B$:

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \ .$$

One nice thing about considering the dual basis $B^*$ is that we can write an arbitrary $f \in V^*$ in terms of the basis $B^*$ quite easily.

**Proposition 0.3.** *Let $B$ be a basis for $V$ and $B^*$ the dual basis for $V^*$. Then if $f \in V^*$,*

$$f = f(v_1)v_1^* + \cdots + f(v_n)v_n^* \ .$$

*Proof.* We simply need to check that both sides give the same answer when evaluated at the basis of $V$. So apply each to $v_i$: the left side gives $f(v_i)$ and the right gives

$$(f(v_1)v_1^* + \cdots + f(v_n)v_n^*)(v_i) = f(v_1)v_1^*(v_i) + \cdots + f(v_n)v_n^*(v_i) = f(v_i)v_i^*(v_i) = f(v_i) \ .$$

□

A nice way to think about linear functionals involves their nullspaces. By the rank-nullity theorem, if $f \in V^*$,

$$\dim(N(f)) + \dim(R(f)) = \dim(V) \ .$$

Because $R(f) \subset \mathbb{F}$, it is at most one-dimensional. Therefore $N(f) = V$ or $N(f)$ is $n-1$ dimensional, where $n = \dim(V)$. This gives

- If $f$ is not the zero functional, $\text{nullity}(f) = \dim(V) - 1$. A subspace of this dimension is called a *hyperspace*.

Because of the simple structure of the nullspace, we can characterize linear functionals easily.

**Proposition 0.4.** *Two nonzero elements $f, g \in V^*$ are equal if and only if they have the same nullspace $N = N(f) = N(g)$ and they agree at one vector outside $N$.*

*Proof.* One direction is clear, so suppose that $N = N(f) = N(g)$ and $v \in V \setminus N$ satisfies $f(v) = g(v)$. You can check that if $B_N$ is a basis for $N$ then $B_N \cup \{v\}$ is a basis for $V$. (The proof is similar to how we proved the one-subspace theorem.) But then $f$ and $g$ agree on $B_N \cup \{v\}$ and must agree everywhere, giving $f = g$. □

## Annihilators

As we have seen before, one useful tool for the study of linear transformations is the nullspace. We will consider the dual version of this now: given $S \subset V$, we give a name to those $f \in V^*$ such that $S \subset N(f)$.

**Definition 0.5.** *If $S \subset V$ then the annihilator of $S$ is*

$$S^\perp = \{f \in V^* : f(s) = 0 \text{ for all } s \in S\} \ .$$

Note that if $S \subset T$ then $S^\perp \supset T^\perp$.

**Proposition 0.6.** *Let $S \subset V$ (not necessarily a subspace).*

1. *$S^\perp$ is a subspace of $V^*$.*

*2.* $S^\perp = (Span(S))^\perp$.

*3. Let* $V$ *be finite-dimensional with* $U$ *a subspace. Let* $\{v_1, \ldots, v_k\}$ *be a basis for* $U$ *and extend it to a basis* $\{v_1, \ldots, v_n\}$ *for* $V$*. If* $\{v_1^*, \ldots, v_n^*\}$ *is the dual basis then* $\{v_{k+1}^*, \ldots, v_n^*\}$ *is a basis for* $U^\perp$*.*

*Proof.* For the first item, $S^\perp$ contains the zero linear functional, so it is nonempty. If $f, g \in S^\perp$ and $c \in \mathbb{F}$ then for any $s \in S$,

$$(cf + g)(s) = cf(s) + g(s) = c \cdot 0 + 0 = 0 \ ,$$

so $cf + g \in S^\perp$. Thus $S^\perp$ is a subspace of $V^*$.

Next since $S \subset \text{Span}(S)$, we have $S^\perp \supset (\text{Span}(S))^\perp$. Conversely, if $f(s) = 0$ for all $s \in S$ then let $a_1 s_1 + \cdots + a_k s_k \in \text{Span}(S)$. Then

$$f(a_1 s_1 + \cdots + a_k s_k) = a_1 f(s_1) + \cdots + a_k f(s_k) = 0 \ ,$$

so $f \in (\text{Span}(S))^\perp$.

For the third item, each of $v_{k+1}^*, \ldots, v_n^*$ annihilates $v_1, \ldots, v_k$, so they annihilate everything in the span, that is, $U$. In other words, they are in $U^\perp$, and we already know they are linearly independent since they are part of the dual basis. To show they span $U^\perp$, let $f \in U^\perp$ and write $f$ in terms of the dual basis using the previous proposition:

$$\begin{aligned} f = f(v_1)v_1^* + \cdots + f(v_k)v_k^* + f(v_{k+1})v_{k+1}^* + \cdots + f(v_n)v_n^* &= f(v_{k+1})v_{k+1}^* + \cdots + f(v_n)v_n^* \\ &\in \text{Span}(\{v_{k+1}^*, \ldots, v_n^*\}) \ . \end{aligned}$$

□

**Corollary 0.7.** *If* $V$ *is finite dimensional and* $W$ *is a subspace,*

$$dim(V) = dim(W) + dim(W^\perp) \ .$$

*Proof.* This follows from item 3 above. □