A PROOF OF CAUCHY-SCHWARZ

We will prove the Cauchy-Schwarz inequality. Let V be an inner product space over $\mathbb{C}$, and $u, v \in V$, then

$$|\langle u, v\rangle| \leq \|u\|\|v\|. \tag{1}$$

The idea is to start from a weaker inequality and upgrade it by exploiting some invariances of the quantities involved. The starting point is the positive-definiteness of the inner product:

$$0 \leq \langle u - v, u - v\rangle. \tag{2}$$

Expanding the inner product on the right, we find:

$$\begin{aligned} \langle u - v, u - v\rangle &= \langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle \\ &= \|u\|^2 - 2\Re\langle u, v\rangle + \|v\|^2. \end{aligned} \tag{3}$$

To pass to the second line, we have used the Hermitian property

$$\langle v, u\rangle = \overline{\langle u, v\rangle}$$

and the fact that for any complex number z,

$$z + \bar{z} = 2\Re z.$$

Combining (2) and , we find

$$2\Re\langle u, v\rangle \leq \|u\|^2 + \|v\|^2,$$

which we rewrite as

$$\Re\langle u, v\rangle \leq \frac{\|u\|^2}{2} + \frac{\|v\|^2}{2}. \tag{4}$$

This inequality is *weaker* than (1): the left side is in general smaller than $|\langle u, v\rangle|$, while the right side is larger than $\|u\|\|v\|$.

To improve the situation, notice that if $\lambda \in \mathbb{C}$ is such that $|\lambda| = 1$, then

$$\|\lambda v\|^2 = |\lambda|^2\|v\|^2 = \|v\|^2$$

while

$$\Re\langle u, \lambda v\rangle = \Re\bar{\lambda}\langle u, v\rangle.$$

So for *any* λ such that $|\lambda| = 1$:

$$\Re\bar{\lambda}\langle u, v\rangle \leq \frac{\|u\|^2}{2} + \frac{\|v\|^2}{2}. \tag{5}$$

This is one of those great situations where we have an inequality holding for every value of some parameter. The obvious next step is to optimize the choice of λ. To see how to do this in the present case, recall that any complex number z can be written as

$$z = |z|e^{i\theta},$$

where θ is real. Thus also

$$\langle u, v\rangle = e^{i\theta}|\langle u, v\rangle|$$

for some θ. Similarly write $\lambda = 1 \cdot e^{i\alpha}$ for some $\alpha > 0$. Then

$$\Re\bar{\lambda}\langle u, v\rangle = \Re e^{-i\alpha}e^{i\theta}|\langle u, v\rangle| = \cos(\theta - \alpha)|\langle u, v\rangle|,$$

recalling that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Using this in (5), we find

$$\cos(\theta - \alpha)|\langle u, v\rangle| \leq \frac{\|u\|^2}{2} + \frac{\|v\|^2}{2},$$

whether θ is determined by u and v, but α is at our disposal. We get the strongest statement by setting $\alpha = \theta$, improving (5) to

$$|\langle u, v\rangle| \leq \frac{\|u\|^2}{2} + \frac{\|v\|^2}{2}. \tag{6}$$

We can further improve (6) by introducing a new parameter $c > 0$. Notice that if we apply (6) with u replaced by $c \cdot u$ and v replaced by $c \cdot v$, then the left side is

$$|\langle c\dot{u}, \frac{1}{c} \cdot v\rangle| = \frac{c}{c}|\langle u, v\rangle| = |\langle u, v\rangle|.$$

On the other hand, the right side of (6) is now

$$c^2\frac{\|u\|^2}{2} + \frac{1}{c^2}\frac{\|v\|^2}{2},$$

so that for *any* $c > 0$, we have

$$|\langle u, v\rangle| \leq c^2\frac{\|u\|^2}{2} + \frac{1}{c^2}\frac{\|v\|^2}{2}. \tag{7}$$

Since the left side is independent of c, we should once again optimize in our free parameter c to get the best possible inequality. The inequality will obviously be strongest for the smallest possible value of the right side. The function

$$x \mapsto x\frac{\|u\|^2}{2} + \frac{1}{x}\frac{\|v\|^2}{2}$$

is differentiable for $x > 0$ and tends to infinity as $x \to 0^+$ and $x \to \infty$. It thus has a (global) minimum at the zero of its derivative, which occurs when

$$\frac{\|u\|^2}{2} - \frac{1}{x^2}\frac{\|v\|^2}{2} = 0,$$

or

$$x_0 = \frac{\|v\|^2}{\|u\|^2}.$$

This tells that the optimal choice of c in (7) for fixed $\|u\|$ and $\|v\|$ is $c = \sqrt{x_0}$, which leads to the right side being

$$\frac{\|v\|}{\|u\|}\frac{\|u\|^2}{2} + \frac{\|u\|}{\|v\|}\frac{\|v\|^2}{2} = \|u\|\|v\|.$$

This finishes the proof of (1).