MAT217 Spring 2013
Final exam Solutions

1. **Let $T : V \to V$ be linear with V a vector space over $\mathbb{F}$. If $v_1, \ldots, v_k$ are eigenvectors of T for eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively, with $\lambda_i \neq \lambda_j$ for $i \neq j$, show that $\{v_1, \ldots, v_k\}$ is linearly independent.**

   Solution. (From the notes.) As usual, suppose that

   $$a_1 v_1 + \cdots + a_k v_k = \vec{0} \ .$$

   Let's assume that we have already removed all vectors which have nonzero coefficients, and among all such linear combinations equal to zero, this is one with the least number of coefficients. We may assume that there are at least two coefficients, or else we would have $a_1 v_1 = \vec{0}$ and since $v_1 \neq 0$ we would have $a_1 = 0$, meaning all coefficients are zero and $\{v_1, \ldots, v_k\}$ is linearly independent.

   So apply T to both sides:

   $$a_1 T(v_1) + \cdots + a_k T(v_k) = \vec{0} \ .$$

   Since these are eigenvectors, we can rewrite as

   $$a_1 \lambda_1 v_1 + \cdots + a_k \lambda_k v_k = \vec{0} \ .$$

   However multiplying the linear combination by λ_1 we get

   $$a_1 \lambda_1 v_1 + \cdots + a_k \lambda_1 v_k = \vec{0} \ .$$

   Subtracting these two,

   $$a_2(\lambda_1 - \lambda_2) v_2 + \cdots + a_k(\lambda_1 - \lambda_k) v_k = \vec{0} \ .$$

   All λ_i's were distinct and all a_i's were nonzero, so this is a linear combination of the v_i's equal to zero with fewer nonzero coefficients than in the original one, a contradiction.

2. **Let V be a vector space over $\mathbb{F}$ and f a symmetric bilinear form on V. If W is a subspace of V such that $V = W \oplus N(f)$ show that f_W, the restriction of f to W, is non-degenerate.**

   Solution. To show f_W is non-degenerate we must show it has zero nullspace. Recalling the definition of nullspace of a bilinear form g on a vector space $\hat{V}$:

   $$N(g) = \{\hat{v} \in \hat{V} : g(\hat{v}, v') = 0 \text{ for all } v' \in \hat{V}\} \ ,$$

   we then have

   $$N(f_W) = \{w \in W : f_W(w, w') = 0 \text{ for all } w' \in W\} \ .$$

Since f_W behaves just as f on W,

$$N(f_W) = \{w \in W : f(w, w') = 0 \text{ for all } w' \in W\} \ .$$

Let $w \in N(f_W)$. If $v \in V$ we may write $v = w_1 + w_2$ for unique $w_1 \in W$ and $w_2 \in N(f)$. (Here we are using the assumption that $V = W \oplus N(f)$.) Since f is symmetric,

$$f(w, v) = f(w, w_1 + w_2) = f(w, w_1) + f(w, w_2) = f(w, w_1) + f(w_2, w) \ .$$

The vector $w_2 \in N(f)$, so $f(w_2, w) = 0$. However $w \in N(f_W)$ so $f(w, w_1) = 0$. This means $f(w, v) = 0$. As v was arbitrary, this means $w \in N(f)$. But $w \in W$ and $W \cap N(f) = \vec{0}$, so $w = \vec{0}$, implying $N(f_W) = \{\vec{0}\}$.

3. **Let V be a finite-dimensional complex inner product space and $T : V \to V$ be linear. Suppose that $T^* = T^2$.**

   (a) **Show that T is diagonalizable and find all possible eigenvalues of T.**

   Solution. Note that T is normal:

   $$TT^* = TT^2 = T^2T = T^*T \ .$$

   By a theorem in class, this means T is unitarily diagonalizable. So choose an orthonormal basis B such that $[T]_B^B$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. By another theorem in class, the matrix of $[T^*]_B^B$ (since B is orthonormal) is just the conjugate transpose of $[T]_B^B$, so because $[T^2]_B^B$ has diagonal entries $\lambda_1^2, \ldots, \lambda_n^2$ and $T^* = T^2$, we find

   $$\lambda_j^2 = \overline{\lambda}_j \text{ for all } j = 1, \ldots, n \ .$$

   If we take absolute value of both sides,

   $$|\lambda_j|^2 = |\lambda_j^2| = |\overline{\lambda}_j| = |\lambda_j| \ ,$$

   meaning $|\lambda_j| = 1$ or 0. If it is zero, then $\lambda_j = 0$. Otherwise, multiplying by λ_j on both sides,

   $$\lambda_j^3 = |\lambda_j|^2 = 1 \ .$$

   This implies that $\lambda_j = 1, \ e^{i2\pi/3}$ or $e^{i4\pi/3}$. Therefore each eigenvalue λ satisfies

   $$\lambda \in \{0, 1, e^{i2\pi/3}, e^{i4\pi/3}\} \ .$$

   We can see that all such λ's above can be eigenvalues of operators T with $T^* = T^2$. Consider any diagonal complex matrix A which contains all of the numbers above on the diagonal and let T be a linear transformation on a complex inner product space whose matrix relative to an orthonormal basis is A. Then the matrix of T^* relative to the same basis is just $\overline{A}^t$. T has eigenvalues equal to the diagonal elements and, indeed, since $A^2 = \overline{A}^t$, we have $T^2 = T^*$.

(b) **Find the possible Jordan forms for TT^*.**

Solution. Because $T^* = T^2$, $TT^* = T^3$. Therefore if we use the same orthonormal basis B that makes $[T]_B^B$ diagonal, we have $[TT^*]_B^B$ a diagonal matrix with entries $\lambda_1^3, \ldots, \lambda_n^3$. But as we saw above, for all j, either $\lambda_j^3 = 0$ or 1 so $[TT^*]_B^B$ is a diagonal matrix with entries 0 or 1 (this characterizes an orthogonal projection actually – we could have seen this by noting that TT^* is a self-adjoint projection). These matrices are in Jordan form, so each TT^* must have Jordan form equal to a diagonal matrix with 0's and 1's.

As before, each such matrix is a possible Jordan form for TT^*. Just take a diagonal matrix A with 0's and 1's and let T be a linear transformation on a complex inner product space whose matrix relative to some orthonormal basis B is A. Then $[T^*]_B^B$ is again just A and $[TT^*]_B^B = [T]_B^B[T]_B^B = A$. Since A is a Jordan matrix and Jordan form is unique (up to permutation of blocks), TT^* has Jordan form A.

4. **Let $A \in M_{n,n}(\mathbb{F})$ with $\mathbb{F}$ algebraically closed. Show there exists an invertible $P \in M_{n,n}(\mathbb{F})$ such that $A^t = P^{-1}AP$.**

Solution. Choose an invertible Q such that $Q^{-1}AQ = J$ is in Jordan form. Then

$$(Q^t)^{-1}A^tQ^t = (Q^{-1}AQ)^t = J^t \ ,$$

so A^t is similar to J^t. Because similarity is an equivalence relation it then suffices to show that J and J^t are similar. If J consists of only one Jordan block:

$$J = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \cdots & & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} ,$$

Then letting Q be the anti-diagonal matrix

$$Q = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

(which is its own inverse), we easily see that $J^t = QJQ = Q^{-1}JQ$, so that J is similar to J^t. In general, if J has Jordan blocks

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & J_3 & \\ & & & \cdots \end{pmatrix} ,$$

then we let Q equal the block matrix

$$Q = \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & Q_3 & \\ & & & \ldots \end{pmatrix} ,$$

where Q_i is an anti-diagonal matrix of the above form and dimension equal to that of J_i. Then $J^t = QJQ = Q^{-1}JQ$, so that J is similar to J^t.

5. **Find the characteristic polynomial of the following matrix:**

$$A = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix} .$$

What is the minimal polynomial of A? Prove your answer.

Solution. We first compute the characteristic polynomial:

$$c_A(x) = \det(xI - A) = \det \begin{pmatrix} x+1 & 0 & -1 \\ 3 & x-4 & -1 \\ 0 & 0 & x-2 \end{pmatrix} .$$

By using Laplace expansion on the last row, we find

$$c_A(x) = (x-2) \det \begin{pmatrix} x+1 & 0 \\ 3 & x-4 \end{pmatrix} = (x-2)(x+1)(x-4) .$$

The roots are $2, -1$ and 4 so these are the eigenvalues.

The minimal polynomial was shown in homework to be monic and to have roots equal to all the eigenvalues. By Cayley-Hamilton, it divides the characteristic polynomial, so has degree at most 3. The only polynomial satisfying these conditions is $(x-2)(x+1)(x-4)$ so this is also the minimal polynomial.

6. **Let $A \in M_{n,n}(\mathbb{C})$ and for $i, j \in \{1, \ldots, n\}$ recall the definition of $A(i|j)$, the (i,j)-th minor: it is the $(n-1) \times (n-1)$ matrix obtained from A by removing the i-th row and j-th column. Show that the rank of A is at least $n-1$ if and only if there exist $i, j \in \{1, \ldots, n\}$ such that $\det A(i|j) \neq 0$.**

Solution. Assume first that A has rank at least $n-1$. Then because rank equals column rank, we can find $n-1$ linearly independent columns. Writing A' for the $n \times (n-1)$ matrix whose columns are the above mentioned columns, A' has rank $n-1$, since it has $n-1$ linearly independent columns. Since column rank equals row rank, it must have $n-1$ linearly independent rows. Let A'' be the matrix whose rows are the above mentioned rows. Then A'' is $n-1$ dimensional and has $n-1$ linearly independent columns, so it has full rank. Therefore $\det A'' \neq 0$. But A'' is just the (i,j)-th minor of A for some $i, j \in \{1, \ldots, n\}$.

Conversely, assume that $\det A(i|j) \neq 0$ for some $i, j \in \{1, \ldots, n\}$. Then let A' be the matrix obtained from A by removing the i-th row. Since A' is just $A(i|j)$ but with another column, its column rank is at least equal to that of $A(i|j)$, which is $n-1$, since $A(i|j)$ has nonzero determinant and therefore has full rank. As column rank equals row rank, all $n-1$ rows of A' are linearly independent. However these are rows of A as well, so A has at least $n-1$ linearly independent rows, giving $\operatorname{rank}(A) \geq n-1$.