MAT217 HW 11
Due Tues. Apr. 30, 2013

1. Let V be a vector space and $f \in Sym(V, \mathbb{F})$. If W is a subspace of V such that $V = W \oplus N(f)$, show that f_W, the restriction of f to W, is non-degenerate.

2. Let V be a vector space over $\mathbb{F}$ with characteristic not equal to 2. Show that if V is finite dimensional and W is a subspace such that the restriction f_W of $f \in Sym(V, \mathbb{F})$ to W is non-degenerate, then $V = W \oplus W^{\perp_f}$. Here $W^{\perp_f}$ is defined as

$$W^{\perp_f} = \{v \in V : f(v, w) = 0 \text{ for all } w \in W\} .$$

   Hint. Use induction on $\dim W$.

3. Let V be a vector space of dimension $n < \infty$ and $f \in Sym(V, \mathbb{F})$ be non-degenerate.

   (a) Show that any orthogonal (relative to f) set of nonzero vectors

$$\{v_1, \ldots, v_n\} \subset V$$

   is a basis for V.

   (b) A linear $T : V \to V$ is called orthogonal relative to f if $f(T(v), T(w)) = f(v, w)$ for all $v, w \in V$. Show that if T is orthogonal then it is invertible.

   (c) For any $g \in Bil(V, \mathbb{F})$ and linear $U : V \to V$ we can define $g_U : V \times V \to \mathbb{F}$ by

$$g_U(v, w) = g(U(v), U(w)) .$$

   Show that $g_U \in Bil(V, \mathbb{F})$. Given a basis B of V, how do we express the matrix of g_U relative to that of g? Use this to find the determinant of any T that is orthogonal relative to f.

   (d) Show that the orthogonal group

$$O(f) = \{T \in L(V, V) : T \text{ is orthogonal relative to } f\}$$

   is, in fact, a group under composition.

4. Let V be a finite-dimensional $\mathbb{F}$-vector space such that $\text{char}(\mathbb{F}) \neq 2$. If f is a skew-symmetric bilinear form on V (that is, $f(v, w) = -f(w, v)$ for all $v, w \in V$) can one find a basis B of V such that $[f]_B^B$ is diagonal?

5. Let f be a symmetric bilinear form on $\mathbb{R}^n$.

   (a) Show that

$$f_H((v, w), (x, y)) := f(v, x) + f(w, y) - if(v, y) + if(w, x)$$

   defines a Hermitian form on $\mathbb{C}^n$. (Here we are writing (v, w) for the vector $v + iw$ as in last homework.)

(b) Show that $N(f_H) = \text{Span}(\iota(N(f)))$, where ι is the embedding $\iota(v) = (v, 0)$.

(c) Show that if f is an inner product then so is f_H.

6. For the matrix A below, find an invertible matrix S such that $S^t AS$ is diagonal:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} .$$

7. (From Hoffman-Kunze) Let V be a finite dimensional vector space over $\mathbb{C}$.

   (a) Show that $Sym(V, \mathbb{C})$ is a subspace of $Bil(V, \mathbb{C})$.

   (b) Find the dimension of $Sym(V, \mathbb{C})$.

8. Let A be a symmetric matrix in $M_{n,n}(\mathbb{R})$.

   (a) A is called positive-definite if $Av \cdot v > 0$ for all nonzero $v \in \mathbb{R}^n$. (Here $\cdot$ is the standard dot-product.) Show that A is positive-definite if and only if there exists an invertible $B \in M_{n,n}(\mathbb{R})$ such that $A = B^t B$.

   (b) A is called positive semi-definite if $Av \cdot v \geq 0$ for all $v \in \mathbb{R}^n$. Formulate a similar result to the above for such A.

9. (From Hoffman-Kunze) Let V be a finite-dimensional vector space over $\mathbb{C}$ with $f, g \in Sym(V, \mathbb{C})$. Show that there is an invertible $T : V \to V$ such that $f(T(v), T(w)) = g(v, w)$ for all $v, w \in V$ if and only if f and g have the same rank. Is the same statement true over $\mathbb{R}$?

10. Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space.

    (a) Define $\| \cdot \| : V \to \mathbb{R}$ by

    $$\|v\| = \sqrt{\langle v, v \rangle} .$$

    Show that for $v, w \in V$,

    $$|\langle v, w \rangle| \leq \|v\| \|w\| .$$

    (b) Show that $\| \cdot \|$ is a norm on V.

    (c) Show that there exists an orthonormal basis B of V.

11. (From Hoffman-Kunze) Let V be the vector space of all $n \times n$ matrices over $\mathbb{C}$, with the inner product $\langle A, B \rangle = Tr(AB^*)$. Find the orthogonal complement of the subspace of diagonal matrices. Here B^* is the conjugate transpose.