# Lecture 11

Last time we talked about permutations and saw that if $\tau$ is an adjacent transposition and $\pi \in S_n$ then the number of inversion pairs of $\tau\pi$ (written $N(\tau\pi)$) differs from $N(\pi)$ by exactly one. We can now iterate this result:

**Corollary 0.1.** *If $\pi \in S_n$ and $\tau$ is a transposition then*

$$N(\tau\pi) - N(\pi) \text{ is odd .}$$

*Proof.* Here just use the previous lemma and the fact that any transposition can be written as a product of an odd number of adjacent transpositions: if $a < b$ then

$$(ab) = (a\ a+1)(a+1\ a+2)\cdots(b-1\ b)(b-2\ b-1)\cdots(a\ a+1)\ .$$

□

We can then give the proof of the theorem on signature.

*Proof.* Let $\pi \in S_n$ and write

$$\tau_1 \cdots \tau_k = \pi = \hat{\tau}_1 \cdots \hat{\tau}_l$$

for transpositions $\tau_i, \hat{\tau}_i$. Then as a transposition is its own inverse,

$$Id = \hat{\tau}_1 \cdots \hat{\tau}_l \tau_k^{-1} \cdots \tau_1^{-1}\ .$$

The number of inversion pairs of $Id$ is 0. Counting the inversion pairs on the right side using the last corollary, each transposition contributes an odd number of pairs. Therefore we find that $l + k$ must be even. But since $l + k + (l - k) = 2l$ is even, we must have $l - k$ even as well. This completes the proof. □

Now that we have the signature theorem, we list a couple of facts:

- $\text{sgn}(Id) = 1$.
- If $\tau$ is a transposition, then $\text{sgn}(\tau) = -1$.
- If $\sigma_1, \sigma_2 \in S_n$ then $\text{sgn}(\sigma_1\sigma_2) = \text{sgn}(\sigma_1) \cdot \text{sgn}(\sigma_2)$.

Next begin determinants, motivated by the example of the volume of a parallelepiped. Look at Section 4.2 in the notes.