MAT217 HW 8
Due Tues. Apr. 9, 2013

1. Let $A \in M_{n\times n}(F)$ for some field F. Recall that if $1 \le i,j \le n$ then the (i,j)-th minor of A, written $A(i|j)$, is the $(n-1)\times(n-1)$ matrix obtained by removing the i-th row and j-th column from A. Define the *cofactor*

$$C_{i,j} = (-1)^{i+j} \det A(i|j) \ .$$

Note that the Laplace expansion for the determinant can be written

$$\det A = \sum_{i=1}^{n} A_{i,j} C_{i,j} \ .$$

   (a) Show that if $1 \le j,k \le n$ with $j \neq k$ then

$$\sum_{i=1}^{n} A_{i,k} C_{i,j} = 0 \ .$$

   (b) Define the *classical adjoint* of A, written adj A, by

$$(\text{adj } A)_{i,j} = C_{j,i} \ .$$

   Show that $(\text{adj } A)A = (\det A)I$.

   (c) Show that $A(\text{adj } A) = (\det A)I$ and deduce that if A is invertible then

$$A^{-1} = (\det A)^{-1} \text{adj } A \ .$$

   Hint: begin by applying the result of the previous part to A^t.

   (d) Use the formula in the last part to find the inverses of the following matrices:

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 6 & 0 & 1 & 1 \end{pmatrix} .$$

2. Consider a system of equations in n variables with coefficients from a field $\mathbb{F}$. We can write this as $AX = Y$ for an $n\times n$ matrix A, an $n\times 1$ matrix X (with entries $x_1,\ldots,x_n$) and an $n\times 1$ matrix Y (with entries $y_1,\ldots,y_n$). Given the matrices A and Y we would like to solve for X.

   (a) Show that

$$(\det A)x_j = \sum_{i=1}^{n} (-1)^{i+j} y_i \det A(i|j) \ .$$

(b) Show that if $\det A \neq 0$ then we have

$$x_j = (\det A)^{-1} \det B_j \ ,$$

where B_j is an $n \times n$ matrix obtained from A by replacing the j-th column of A by Y. This is known as *Cramer's rule.*

(c) Solve the following systems of equations using Cramer's rule.

$$\begin{cases} 2x - y + z & = 3 \\ 2y - z & = 1 \\ y - x & = 1 \end{cases} \qquad \begin{cases} 2x - y + z - 2t & = -5 \\ 2x + 2y - 3z + t & = -1 \\ -x + y - z & = -1 \\ 4x - 3y + 2z - 3t & = -8 \end{cases}$$

3. Let V be a vector space and $W_1, \ldots, W_k$ be subspaces. Show that the W_i's are independent if and only if for each $v \in W_1 + \cdots + W_k$, there exist unique $w_1 \in W_1, \ldots, w_k \in W_k$ such that $v = w_1 + \cdots + w_k$.

4. In this problem we will show that if $\mathbb{F}$ is algebraically closed then any linear $T : V \to V$ can be represented as an upper triangular matrix. This is a simpler result than (and is implied by) the Jordan Canonical form, which we will cover in class soon.

   We will argue by (strong) induction on the dimension of V. Clearly the result holds for $dim\ V = 1$. So suppose that for some $k \geq 1$ whenever $dim\ W \leq k$ and $U : W \to W$ is linear, we can find a basis of W relative to which the matrix of U is upper-triangular. Further, let V be a vector space of dimension $k + 1$ over $\mathbb{F}$ and $T : V \to V$ be linear.

   (a) Let λ be an eigenvalue of T. Show that the dimension of $R := R(T - \lambda I)$ is strictly less than $dim\ V$ and that R is T-invariant.

   (b) Apply the inductive hypothesis to $T|_R$ (the operator T restricted to R) to find a basis of R with respect to which $T|_R$ is upper-triangular. Extend this to a basis for V and complete the argument.

5. Let A be the matrix

$$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} .$$

   (a) Is A diagonalizable over $\mathbb{R}$? If so, find a basis for $\mathbb{R}^3$ of eigenvectors of A.

   (b) Is A diagonalizable over $\mathbb{C}$? If so, find a basis for $\mathbb{C}^3$ of eigenvectors of A.

6. Let A be the matrix

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} .$$

   Find A^n for all $n \geq 1$.

   Hint: first diagonalize A.

7. Let $A \in M_{n,n}(\mathbb{F})$ be upper-triangular. Show that the eigenvalues of A are the diagonal entries of A.

8. Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $T : V \to V$ be linear. Suppose that every subspace of V is T-invariant. What can you say about T?

9. A linear transformation $T : V \to V$ is called a *projection* if $T^2 = T$. Let $T : V \to V$ be a projection with $\dim V < \infty$.

   (a) Show that
   $$V = R(T) \oplus N(T)$$
   is a T-invariant direct sum.

   Hint. Use exercise 5, homework 3.

   (b) Show that there is a basis B of V such that $[T]_B^B$ is diagonal with entries equal to 1 or 0. How is this result different from that of exercise 9, homework 3?

   (c) Let $U : V \to V$ be linear such that $U^2 = I$. Derive a simple matrix representation for U.

   Hint. Consider $(1/2)(U + I)$.