IEOR E4703: Monte Carlo Simulation (Fall 2004)

Columbia University

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Assignment 1: Due Wednesday September 22

1. **(Very brief probability review)** Suppose the continuous random vector (X, Y) has the joint probability distribution

$$f(x,y) = \begin{cases} c(4x^2y + y^2), & x \in [0,1], y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

for some constant c.

   (a) Calculate $\mathrm{P}(X + Y > 1.5)$.

   (b) Calculate $\mathrm{E}[Y]$.

   (c) What is $\mathrm{Cov}(X, Y)$?

   (d) What is $f(x|y)$, the conditional probability density function of x given y?

   (e) Compute $\mathrm{E}[X|Y = .5]$.

2. **(Introduction to Matlab)** You do not need to submit this question. It only applies to those of you who plan to do the programming assignments in this course using Matlab and who have not had any prior exposure to Matlab.

   Whilst sitting at a computer with Matlab open, read and work through the examples of the first 15 pages of the Matlab primer by Kermit Sigmon. This primer can be downloaded by going to the website that is given in the first set of course lecture notes.

3. (Ross, Q3.3) Use simulation to estimate $\int_0^1 \exp(x)\ dx$. Compare your estimate with the exact answer if you can.

4. (Ross, Q3.6) Use simulation to estimate $\int_0^\infty x(1 + x^2)^{-2}\ dx$. Compare your estimate with the exact answer if you can.

5. (Ross, Q3.8) Use simulation to estimate $\int_0^1 \int_0^1 e^{(x+y)^2}\ dydx$. Compare your estimate with the exact answer if you can.

6. (Ross, Q3.10) Use simulation to approximate $\mathrm{Cov}(U, e^U)$, where U is uniform on $(0, 1)$. Compare your approximation with the exact answer.

7. (Law and Kelton, Q7.3) Without actually computing any Z_i's, determine which of the following mixed LCGs have full period:

   (a) $Z_i = (13Z_{i-1} + 13) \mod 16$

   (b) $Z_i = (12Z_{i-1} + 13) \mod 16$

   (c) $Z_i = (13Z_{i-1} + 12) \mod 16$

(d) $Z_i = (Z_{i-1} + 12) \mod 13$

8. (Law and Kelton, Q7.13). Suppose that $U_1, U_2, \ldots, U_k$ are IID $U(0,1)$ random variables. Show that the fractional part (i.e., ignoring anything to the left of the decimal point) of $U_1 + U_2 + \ldots + U_k$ is also uniformly distributed on $[0,1]$. (Hint: Use induction on k).

9. Recall the inventory example from Lecture 1. Estimate the optimal quantity, Q^*, that the retailer should order so as to maximize her expected profit. Do this by estimating the expected profit, $P(Q)$, when $Q = 90, 100, 110, 120, 130$ and 140. You may assume that the demand, D, is uniformly distributed on $(80, 140)$. Plot a graph of Q versus $P(Q)$ and use the graph to estimate Q^*.

10. Give the inverse transform algorithm to generate a random variate with the standard right-triangular distribution, i.e.,

$$f(x) = 2(1-x) \quad \text{for } 0 \leq x \leq 1$$

11. The double exponential distribution has density

$$f(x) = 0.5\ \mathbf{1}_{(-\infty,0)}(x)\exp(x) + 0.5\ \mathbf{1}_{[0,\infty)}(x)\exp(-x)$$

Show how to simulate a random variable with density $f(.)$ using the inverse transform method.