ADI finite difference schemes for the Heston model with correlation

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Contents

Option pricing in the Heston model

European call option gives the holder the right to buy a given asset at a prescribed *maturity* date T for a prescribed *strike* price K.

Let S_t denote the value of the asset at time $t \geq 0$.

The *payoff* of the call option is $\max(0, S_T - K)$.

For the evolution of S_t we consider the popular Heston stochastic volatility model (1993):

$$\begin{cases} dS_t &= r_{df}\, S_t\, dt + \sqrt{V_t}\, S_t\, dW_t^1\,, \\ dV_t &= \kappa(\eta - V_t)\, dt + \sigma\sqrt{V_t}\, dW_t^2 \end{cases}$$

with real parameters κ, η, σ, r_{df}.

W_t^1, W_t^2 are Brownian motions with correlation factor $\rho \in [-1, 1]$.

Let $u(s, v, t)$ be the fair price of the call option if $S_{T-t} = s$, $V_{T-t} = v$.

Financial option pricing theory yields that u satisfies a parabolic PDE,

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}s^2 v \frac{\partial^2 u}{\partial s^2} + \rho\sigma s v \frac{\partial^2 u}{\partial s \partial v} + \tfrac{1}{2}\sigma^2 v \frac{\partial^2 u}{\partial v^2} + r_{df} s \frac{\partial u}{\partial s} + \kappa(\eta - v)\frac{\partial u}{\partial v} - r_d u$$

for $0 < t \leq T$, $s > 0$, $v > 0$. We call this the Heston PDE.

The payoff gives the initial condition

$$u(s, v, 0) = \max(0, s - K).$$

Further, a boundary condition at $s = 0$ holds,

$$u(0, v, t) = 0.$$

The above equations constitute an initial-boundary value problem for a time-dependent convection-diffusion-reaction equation on an unbounded, two-dimensional spatial domain.

If the correlation $\rho \neq 0$, then there is a mixed-derivative term,

$$\frac{\partial^2 u}{\partial s\, \partial v}.$$

The Heston PDE forms a prototype for the many extensions of the well-known Black-Scholes PDE (1973) that arise in contemporary option pricing. These extensions are often comprised of parabolic PDEs in multiple space dimensions.

In finance there is a big demand for efficient, stable and robust codes for numerically solving such PDEs.

Semi-discretization Heston problem

To render the numerical solution of the Heston PDE feasible, first choose a bounded spatial domain $[0, S] \times [0, V]$ and appropriate additional boundary conditions.

We semi-discretize the PDE by replacing all spatial derivatives with suitable finite differences (FD).

Let $f : \mathbb{R} \to \mathbb{R}$ be any given function and $x_i = i \cdot \Delta x$ ($i \in \mathbb{Z}$), $\Delta x > 0$.

We deal with the following three FD formulas for the first derivative:

$$\begin{aligned}
f'(x_i) &\approx \left[\tfrac{1}{2}\, f_{i-2} - 2\, f_{i-1} + \tfrac{3}{2}\, f_i\right] / \Delta x, \\
f'(x_i) &\approx \left[-\tfrac{1}{2}\, f_{i-1} + \tfrac{1}{2}\, f_{i+1}\right] / \Delta x, \\
f'(x_i) &\approx \left[-\tfrac{3}{2}\, f_i + 2\, f_{i+1} - \tfrac{1}{2}\, f_{i+2}\right] / \Delta x.
\end{aligned}$$

These formulas are applied in the case of $\partial u / \partial s$ and $\partial u / \partial v$.

For the second derivative we take

$$f''(x_i) \;\approx\; [f_{i-1} - 2\, f_i + f_{i+1}]\,/(\Delta x)^2.$$

This FD formula is used for $\partial^2 u/\partial s^2$ and $\partial^2 u/\partial v^2$.

Next, suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $y_j = j \cdot \Delta y$ ($j \in \mathbb{Z}$), $\Delta y > 0$.

Then for the mixed derivative we consider

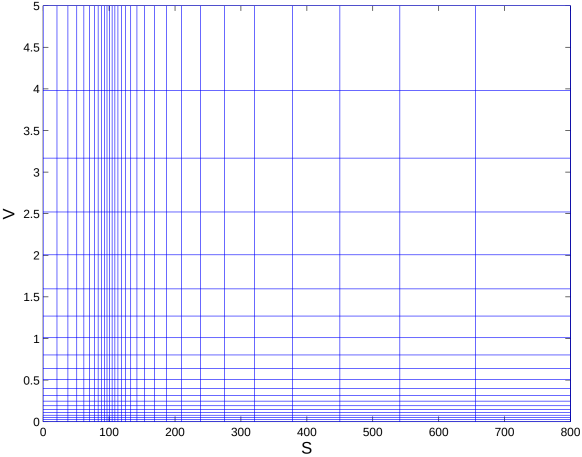
$$\frac{\partial^2 f}{\partial x \partial y}(x_i, y_j) \approx$$

$$\left[\tfrac{1}{4}\, f_{i-1,j-1} - \tfrac{1}{4}\, f_{i-1,j+1} - \tfrac{1}{4}\, f_{i+1,j-1} + \tfrac{1}{4}\, f_{i+1,j+1}\right]/(\Delta x \Delta y).$$

This FD formula is applied to $\partial^2 u/\partial s \partial v$.

All FD formulas above have a second-order truncation error.

In our actual application, we deal with a non-uniform grid in the (s, v)-domain such that many grid points lie near $(s, v) = (K, 0)$:

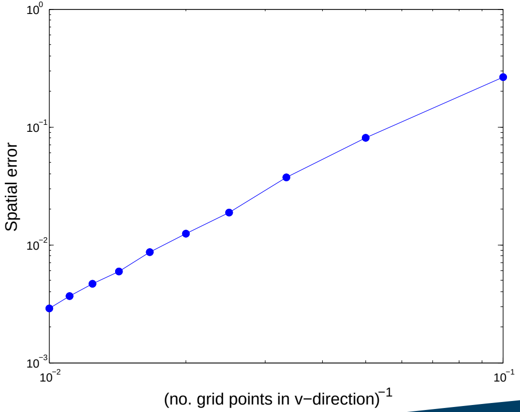


Semi-discretization error:

Comparison with code S. Foulon, KBC Bank. Data Bloomberg (2005):

$\kappa = 3, \eta = 0.12, \sigma = 0.04, \rho = 0.6, r_d = 0.01, r_f = 0.04, T = 1, K = 100.$

[no. of grid points s-direction] = $2\times$[no. of grid points v-direction]



Time integration semi-discrete Heston problem

FD discretization of the Heston problem yields an initial value problem for a large system of stiff ordinary differential equations (ODEs) of the form

$$U'(t) = A\,U(t) + b(t) \quad (0 < t \le T), \qquad U(0) = u_0$$

with given, fixed matrix A and vectors $b(t)$, u_0.

Standard implicit numerical methods such as the trapezoidal rule (Crank–Nicolson) are often not effective.

For the numerical time integration of the ODE system, we consider splitting schemes of the Alternating Direction Implicit (ADI) type.

Splitting:

$$A = A_0 + A_1 + A_2$$

where

- A_0 corresponds to $\partial^2 u/\partial s \partial v$ term (!)
- A_1 corresponds to $\partial u/\partial s$, $\partial^2 u/\partial s^2$ terms
- A_2 corresponds to $\partial u/\partial v$, $\partial^2 u/\partial v^2$ terms

Assume $b(t) \equiv 0$. Let $\Delta t > 0$ and grid points $t_n = n \cdot \Delta t$.

Four ADI schemes yielding $U_n \approx U(t_n)$ $(n = 1, 2, 3, \dots)$:

Douglas (Do) scheme

$$\begin{cases} Y_0 = U_{n-1} + \Delta t \, A \, U_{n-1} \, , \\ Y_j = Y_{j-1} + \theta \, \Delta t \, A_j \left(Y_j - U_{n-1} \right) \quad (j = 1, 2) \, , \\ U_n = Y_2 \, . \end{cases}$$

Parameter $\theta > 0$. Classical order (for general A_0, A_1, A_2) is 1.

Craig–Sneyd (CS) scheme

$$
\begin{cases}
Y_0 = U_{n-1} + \Delta t\, A\, U_{n-1}\,, \\[2ex]
Y_j = Y_{j-1} + \theta\, \Delta t\, A_j \left(Y_j - U_{n-1} \right) \quad (j = 1, 2)\,, \\[2ex]
\widetilde{Y}_0 = Y_0 + \frac{1}{2}\, \Delta t\, A_0 \left(Y_2 - U_{n-1} \right)\,, \\[2ex]
\widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta\, \Delta t\, A_j \left(\widetilde{Y}_j - U_{n-1} \right) \quad (j = 1, 2)\,, \\[2ex]
U_n = \widetilde{Y}_2\,.
\end{cases}
$$

Parameter $\theta > 0$.

Classical order is 2 iff $\theta = \frac{1}{2}$.

Modified Craig–Sneyd (MCS) scheme

$$\begin{cases} Y_0 = U_{n-1} + \Delta t\, A\, U_{n-1}\,, \\ Y_j = Y_{j-1} + \theta\,\Delta t\, A_j \left(Y_j - U_{n-1}\right) \quad (j = 1, 2)\,, \\ \widehat{Y}_0 = Y_0 + \theta\,\Delta t\, A_0\,(Y_2 - U_{n-1})\,, \\ \widetilde{Y}_0 = \widehat{Y}_0 + (\tfrac{1}{2} - \theta)\,\Delta t\, A\,(Y_2 - U_{n-1})\,, \\ \widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta\,\Delta t\, A_j\,(\widetilde{Y}_j - U_{n-1}) \quad (j = 1, 2)\,, \\ U_n = \widetilde{Y}_2\,. \end{cases}$$

Parameter $\theta > 0$.

Classical order is 2 for all θ.

Hundsdorfer–Verwer (HV) scheme

$$\begin{cases} Y_0 = U_{n-1} + \Delta t\, A\, U_{n-1}\,, \\[2ex] Y_j = Y_{j-1} + \theta\, \Delta t\, A_j \left(Y_j - U_{n-1} \right) \quad (j = 1,2)\,, \\[2ex] \widetilde{Y}_0 = Y_0 + \frac{1}{2}\, \Delta t\, A\,(Y_2 - U_{n-1})\,, \\[2ex] \widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta\, \Delta t\, A_j\, (\widetilde{Y}_j - Y_2) \quad (j = 1,2)\,, \\[2ex] U_n = \widetilde{Y}_2\,. \end{cases}$$

Parameter $\theta > 0$.

Classical order is 2 for all θ.

References

- Peaceman & Rachford (1955)
- Douglas & Rachford (1956)
- Brian (1961)
- Douglas (1962)
- McKee & Mitchell (1970)
- Van der Houwen & Verwer (1979)
- Craig & Sneyd (1988)
- McKee, Wall & Wilson (1996)
- Hundsdorfer (1999, 2002)
- Lanser, Blom & Verwer (2001)
- Hundsdorfer & Verwer (2003)
- In ’t Hout & Welfert (2007, 2009)
- In ’t Hout & Foulon (2010)

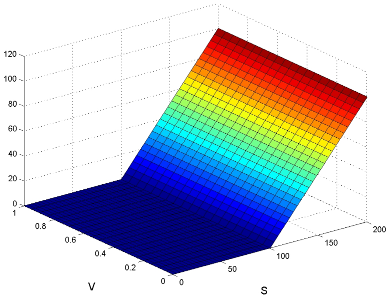
Unconditional stability results - in von Neumann sense - for ADI schemes when applied to FD discretizations of two-dimensional convection-diffusion problems with mixed derivative term:

- McKee, Wall & Wilson ('96):
  - Do scheme is stable if $\theta = \frac{1}{2}$
- Craig & Sneyd ('88):
  - CS scheme is stable if $\theta = \frac{1}{2}$ and no convection
- In 't Hout & Welfert ('07, '09):
  - CS scheme is stable if $\theta = \frac{1}{2}$
  - MCS scheme is stable if $\theta \geq \frac{1}{3}$ and no convection
  - HV scheme is stable if $\theta \geq 1 - \frac{1}{2}\sqrt{2}$ and no convection
  - HV scheme is stable if $\theta \geq \frac{1}{2} + \frac{1}{6}\sqrt{3}$: *conjecture*

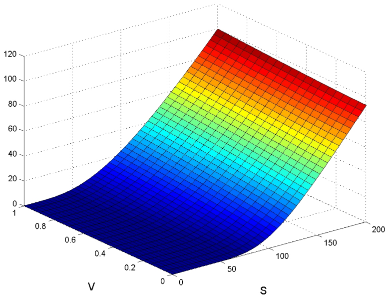
Numerical experiments

Bloomberg data

$\kappa = 3, \eta = 0.12, \sigma = 0.04, \rho = 0.6, r_d = 0.01, r_f = 0.04, T = 1, K = 100$



$t = 0$



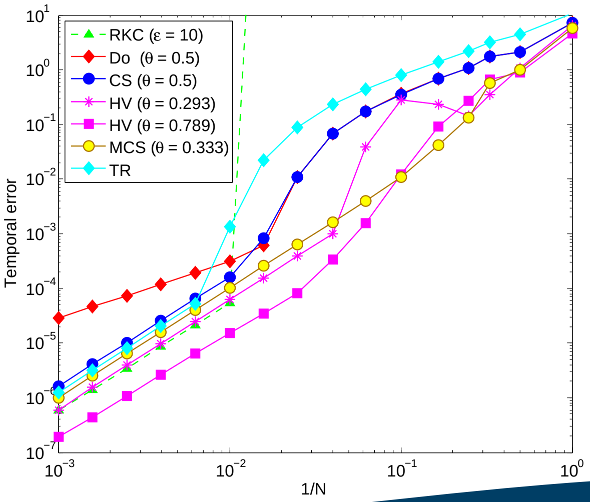
$t = T$

Numerical time-stepping schemes

- Do with $\theta = \frac{1}{2}$
- CS with $\theta = \frac{1}{2}$
- MCS with $\theta = \frac{1}{3}$
- HV with $\theta = 1 - \frac{1}{2}\sqrt{2}$ (HV1)
- HV with $\theta = \frac{1}{2} + \frac{1}{6}\sqrt{3}$ (HV2)
- Trapezoidal rule (TR)
- Runge–Kutta–Chebyshev (RKC)

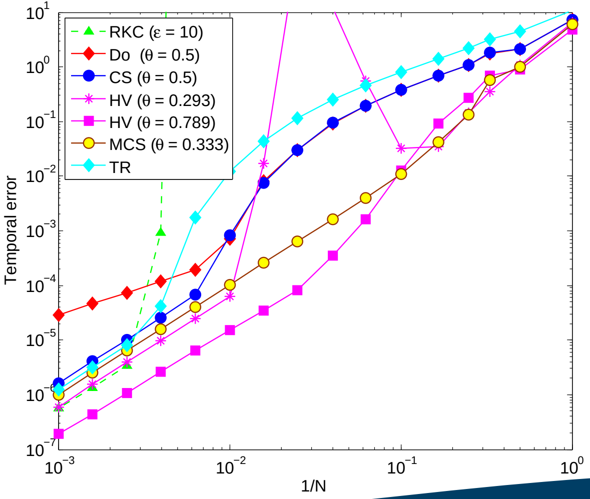
Number of grid points (s, v): 100×50.

Global temporal errors of ADI schemes w.r.t. ODE system:



Number of grid points (s, v): 200×100.

Global temporal errors of ADI schemes w.r.t. ODE system:



Observations

- TR: time-consuming.
- RKC, HV1: show instability, due to $\sigma \approx 0$.
- Do, CS, MCS, HV2, TR: show unconditional stability.
- Do, CS, TR: large errors for modest Δt, damping required.
- Do with damping: order of convergence 1.
- CS, TR with damping: order of convergence 2.
- MCS, HV2: order of convergence 2.

Conclusions and future research

Conclusions

- MCS with $\theta = \frac{1}{3}$ and HV with $\theta = \frac{1}{2} + \frac{1}{6}\sqrt{3}$ seem preferable.
- CS with $\theta = \frac{1}{2}$ and damping seems good second choice.

Current / future research

- Application of ADI FD approach to exotic options and higher-dimensional asset price models.
- Theoretical stability and convergence analysis.
- Practical implementation and experiments.
- Special features of option price models.
- Calibration, Greeks, ...