MAT217 HW 1

DUE TUES. FEB. 12, 2013

We will write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ for the natural numbers and integers, respectively. Let $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The rationals are $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$ and $\mathbb{R}$ stands for the real numbers.

1. Read Section 1.2 in the Hoffman-Kunze handout and do exercises 3, 4.

2. If $a, b \in \mathbb{N}$ we say that a divides b, written $a \mid b$, if there is another natural number c such that $b = ac$. Fix $m, n \in \mathbb{N}$ and define
$$S = \{mp + nq : p, q \in \mathbb{Z}\} \cap \mathbb{N} \ .$$
   (a) Let d be the smallest element of S. Show that $d \mid m$ and $d \mid n$.
   Hint. You can use the statement of the division algorithm without proof; that is, if $a, b \in \mathbb{N}$ then there exist $r, s \in \mathbb{N}^*$ such that $r < b$ and $a = bs + r$.
   (b) Show that if e is another element of $\mathbb{N}$ that divides both m and n then $e \mid d$. This number d is called the greatest common divisor of m and n, written $d = \gcd(m, n)$.
   (c) For any nonzero integers m, n define $\gcd(m, n) = \gcd(|m|, |n|)$. Show there exist $p, q \in \mathbb{Z}$ such that $mp + nq = \gcd(m, n)$.

3. Let p be a prime and $\mathbb{Z}_p$ be the set $\{0, \ldots, p-1\}$. Show that $\mathbb{Z}_p$ is a field using the operations
$$ab = (ab) \mod p \text{ and } a + b = (a + b) \mod p \ .$$
Here we have defined $a \mod p$ for $a \in \mathbb{N}^*$ as the unique $r \in \mathbb{N}^*$ with $r < p$ such that $a = ps + r$ for some $s \in \mathbb{N}^*$.

4. Let S be a nonempty set and $\mathbb{F}$ a field. Let V be the set of functions from S to $\mathbb{F}$ and define addition and scalar multiplication on $(V, \mathbb{F})$ by
$$(f + g)(s) = f(s) + g(s) \text{ and } (cf)(s) = c(f(s)) \ .$$
Show V is a vector space over $\mathbb{F}$.

5. (From Axler) Show that the union of two subspaces of a vector space V is a subspace if and only if one of the subspaces is contained in the other.

6. (From Axler) Let U_1, U_2, W be subspaces of a vector space V. Prove or give a counterexample:
   (a) If $U_1 + W = U_2 + W$ then $U_1 = U_2$.
   (b) If $V = U_1 \oplus W$ and $V = U_2 \oplus W$ then $U_1 = U_2$.

7. (From Axler) For each of the following subsets of $\mathbb{C}^3$, determine whether it is a subspace of $\mathbb{C}^3$:

(a) $\{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 0\}$.

(b) $\{(x_1, x_2, x_3) : x_1 x_2 x_3 = 0\}$.

8. Let W be a subspace of an $\mathbb{F}$-vector space V and define the set

$$V/W = \{v + W : v \in V\} \ .$$

Here the notation $v + W$ means the set $\{v + w : w \in W\}$, so V/W is a set whose elements are sets.

(a) Show that two elements $v_1 + W$ and $v_2 + W$ of V/W are equal if and only if $v_1 - v_2 \in W$. In this case we say that v_1 and v_2 are equivalent modulo W.

(b) Show that the elements of V/W form a partition of V. That is, their union is V and distinct elements must have empty intersection.

(c) In the case of $V = \mathbb{R}^2$ and $W = \{(x, y) : x + y = 0\}$, with $\mathbb{F} = \mathbb{R}$, give a geometric description of the elements of V/W.

(d) Define addition and scalar multiplication on V/W as follows. For $C_1, C_2 \in V/W$, select $v_1, v_2 \in V$ such that $C_1 = v_1 + W$ and $C_2 = v_2 + W$. Define

$$C_1 + C_2 = (v_1 + v_2) + W \text{ and } cC_1 = (cv_1) + W \text{ for } c \in \mathbb{F} \ .$$

Show that these definitions do not depend on the choice of v_1, v_2.

(e) Prove that the above operations turn V/W into an $\mathbb{F}$-vector space. It is called the *quotient space* of V over W.

9. If V is an $\mathbb{F}$-vector space, recall that V is *finitely generated* if there is a finite set $S \subset V$ such that $V = \mathrm{Span}(S)$.

(a) Is $\mathbb{R}$ finitely generated as a vector space over $\mathbb{Q}$?

(b) Is the space of functions from $\mathbb{R}$ to $\mathbb{R}$ finitely generated as a vector space over $\mathbb{R}$?