HW7

1. **Let V be an $\mathbb{F}$-vector space of dimension n and let f be a k-linear alternating function on V with $k > n$. Show that f is identically zero.**

   Solution. Let $v_1, \ldots, v_k \in V$; we will show that $f(v_1, \ldots, v_k) = 0$. First if $n = 0$ the vector space is just $\{\vec{0}\}$ and so f is clearly zero. Otherwise $n \geq 1$ and so $k \geq 2$. Since V has dimension n and $k > n$, the set $\{v_1, \ldots, v_k\}$ is linearly dependent. So we can find i such that $v_i \in \text{Span}(\cup_{j \neq i}\{v_j\})$. After reordering, we will assume that $i = 1$. Therefore we can find $a_2, \ldots, a_k \in \mathbb{F}$ such that

$$v_1 = a_2 v_2 + \cdots + a_k v_k \ .$$

   Now we plug into f and use the fact that it is alternating and k-linear:

$$\begin{aligned} f(v_1, \ldots, v_k) &= f(a_2 v_2 + \cdots + a_k v_k, v_2, \ldots, v_k) \\ &= a_2 f(v_2, v_2, \ldots, v_k) + \cdots + a_k f(v_k, v_2, \ldots, v_k) \\ &= 0 \ . \end{aligned}$$

9. **Let $T : V \to V$ be linear and B a finite basis for V. We define**

$$\det T = \det[T]_B^B \ .$$

   (a) **Show that the above definition does not depend on the choice of B.**
   Solution. Let B' be another basis of V. Then

$$\begin{aligned} \det[T]_{B'}^{B'} &= \det\left([I]_{B'}^B [T]_B^B [I]_B^{B'}\right) = \det[I]_{B'}^B \det[T]_B^B \det[I]_B^{B'} \\ &= \det[I]_{B'}^B \det[I]_B^{B'} \det[T]_B^B \\ &= \det\left([I]_{B'}^B [I]_B^{B'}\right) \det[T]_B^B \\ &= \det[T]_B^B \ . \end{aligned}$$

   (b) **Show that if f is any nonzero n-linear alternating function on V then**

$$\det T = \frac{f(T(v_1), \ldots, T(v_n))}{f(v_1, \ldots, v_n)} \ ,$$

   where we have written $B = \{v_1, \ldots, v_n\}$. (This is an alternate definition of $\det T$.)
   Solution. Let us define $g : V^n \to \mathbb{F}$ as

$$g(w_1, \ldots, w_n) = \frac{f(T(w_1), \ldots, T(w_n))}{f(v_1, \ldots, v_n)} \ .$$

   We claim that g is n-linear and alternating. Indeed, suppose that $w_i = w_j$ for some $i \neq j$. Then $T(w_i) = T(w_j)$ and so $f(T(w_1), \ldots, T(w_n)) = 0$, giving

$g(w_1, \ldots, w_n) = 0$. For n-linear, let i be some number between 1 and n and take $w_1, \ldots, w_n, w_i' \in V$ with $c \in \mathbb{F}$. Then

$$\begin{aligned}
&g(w_1, \ldots, cw_i + w_i', \ldots, w_n) \\
=&\frac{f(T(w_1), \ldots, T(cw_i + w_i'), \ldots, T(w_n))}{f(v_1, \ldots, v_n)} \\
=&\frac{f(T(w_1), \ldots, cT(w_i) + T(w_i'), \ldots, T(w_n))}{f(v_1, \ldots, v_n)} \\
=&\frac{cf(T(w_1), \ldots, T(w_i), \ldots, T(w_n)) + f(T(w_1), \ldots, T(w_i'), \ldots, T(w_n))}{f(v_1, \ldots, v_n)} \\
=&cg(w_1, \ldots, w_i, \ldots, w_n) + g(w_1, \ldots, w_i', \ldots, w_n) \ .
\end{aligned}$$

The right side of the equation in the problem is now equal to $g(v_1, \ldots, v_n)$. For the left side, define $h : V^n \to \mathbb{F}$ by

$$h(w_1, \ldots, w_n) = \det([T(w_1)]_B, \ldots, [T(w_n)]_B) \ .$$

Because det is an n-linear function on $\mathbb{F}^n$ and the function $w \mapsto [T(w)]_B$ is linear, the exact same computation as above shows that h is n-linear and alternating. Therefore to show that $h = g$ we must only show that they both take value 1 at a basis. To do this we split into two cases:

Case 1. If T is not invertible, then $\{T(v_1), \ldots, T(v_n)\}$ is linearly dependent. Then $[T]_B^B$ has linearly dependent columns and $\det T = 0$. Furthermore, f, being alternating, has $f(T(v_1), \ldots, T(v_n)) = 0$, so the equation in the problem holds.

Case 2. If T is invertible, then $\{T^{-1}(v_1), \ldots, T^{-1}(v_n)\}$ is a basis. Also,

$$g(T^{-1}(v_1), \ldots, T^{-1}(v_n)) = f(v_1, \ldots, v_n)/f(v_1, \ldots, v_n) = 1 \ ,$$

while

$$\begin{aligned}
h(T^{-1}(v_1), \ldots, T^{-1}(v_n)) &= \det([T(T^{-1}(v_1))]_B, \ldots, [T(T^{-1}(v_n))]_B) \\
&= \det([v_1]_B, \ldots, [v_n]_B) \\
&= \det(e_1, \ldots, e_n) = 1 \ ,
\end{aligned}$$

where $\{e_1, \ldots, e_n\}$ is the standard ordered basis. Therefore g and h give the value 1 at the same basis and by uniqueness of such n-linear alternating functions, they must be equal. Therefore

$$g(v_1, \ldots, v_n) = h(v_1, \ldots, v_n)$$

and we are done.