

LECTURE 3: STEINITZ EXCHANGE AND BASES

Recall the intuition that a set is linearly independent if each vector in it is truly needed to represent vectors in the span. Not only are they all needed, but linear independence implies that there is exactly one way to represent each vector of the span.

Proposition 0.1. *Let $S \subset V$ be linearly independent. Then for each nonzero vector $v \in \text{Span}(S)$ there exists exactly one choice of $v_1, \dots, v_n \in S$ and nonzero coefficients $a_1, \dots, a_n \in \mathbb{F}$ such that*

$$v = a_1 v_1 + \dots + a_n v_n .$$

Proof. Let $v \in \text{Span}(S)$ be nonzero. By characterization of the Span as the set of linear combinations of elements of S , there is at least one representation as above. To show it is unique, suppose that $v = a_1 v_1 + \dots + a_n v_n$ and $v = b_1 w_1 + \dots + b_k w_k$ and write $S_1 = \{v_1, \dots, v_n\}$, $S_2 = \{w_1, \dots, w_k\}$. We can rearrange the S_i 's so that the elements $v_1 = w_1, \dots, v_m = w_m$ are the common ones; that is, the ones in $S_1 \cap S_2$. Then

$$\vec{0} = v - v = \sum_{j=1}^m (a_j - b_j) v_j + \sum_{l=m+1}^n a_l v_l + \sum_{p=m+1}^k b_p w_p .$$

This is just a linear combination of elements of S , so by linear independence, all coefficients are zero, implying that $a_j = b_j$ for $j = 1, \dots, m$, and all other a_l 's and b_l 's are zero. Thus all nonzero coefficients are the same in the linear combinations and we are done. \square

We are now interested in maximal linearly independent sets. It turns out that these must generate V as well, and we will work toward proving that.

Definition 0.2. *Let V be an \mathbb{F} -vector space and $S \subset V$. If S generates V and is linearly independent then we call S a basis for V .*

Note that the above proposition says that any vector in V has a unique representation as a linear combination of elements from the basis.

We will soon see that any basis of V must have the same number of elements. To prove that, we need a famous lemma. It says that if we have a linearly independent T and a spanning set S , we can add $\#S - \#T$ vectors from S to T to make it spanning.

Theorem 0.3 (Steinitz exchange lemma). *Let $S = \{v_1, \dots, v_m\}$ satisfy $\text{Span}(S) = V$ and let $T = \{w_1, \dots, w_k\}$ be linearly independent. Then*

1. $k \leq m$ and
2. after possibly reordering the set S , we have

$$\text{Span}(\{w_1, \dots, w_k, v_{k+1}, \dots, v_m\}) = V .$$

Proof. The proof is by induction on k , the size of T . If $k = 0$ then T is empty and thus linearly independent. In this case, we do not exchange any elements of T with elements of S and the lemma simply states that $0 \leq m$ and $\text{Span}(S) = V$, which is true.

Suppose that for some $k \geq 0$ and all linearly independent sets T of size k , the lemma holds; we will prove it holds for $k + 1$, so let $T = \{w_1, \dots, w_{k+1}\}$ be a linearly independent set of size $k + 1$. By last lecture, $\{w_1, \dots, w_k\}$ is linearly independent and by induction, $k \leq m$ and we can reorder S so that

$$\text{Span}(\{w_1, \dots, w_k, v_{k+1}, \dots, v_m\}) = V .$$

Because of this we can find scalars a_1, \dots, a_m such that

$$a_1 w_1 + \dots + a_k w_k + a_{k+1} v_{k+1} + \dots + a_m v_m = w_{k+1} . \quad (1)$$

If $k = m$ or if $k \leq m - 1$ but all the coefficients a_{k+1}, \dots, a_m are zero, then we have $w_{k+1} \in \text{Span}(\{w_1, \dots, w_k\})$, a contradiction since T is linearly independent. Therefore we must have $k + 1 \leq m$ and at least one of a_{k+1}, \dots, a_m must be nonzero. Reorder the set S so that $a_{k+1} \neq 0$. Then we can solve for v_{k+1} in (1) to find

$$v_{k+1} \in \text{Span}(\{w_1, \dots, w_{k+1}, v_{k+2}, \dots, v_m\}) .$$

Therefore each element of $\{w_1, \dots, w_k, v_{k+1}, \dots, v_m\}$ can be represented as a linear combination of elements from $\{w_1, \dots, w_{k+1}, v_{k+2}, \dots, v_m\}$, and since the former set spans V , we see that

$$\text{Span}(\{w_1, \dots, w_{k+1}, v_{k+2}, \dots, v_m\}) = V .$$

This completes the proof. □

We can now give all the consequences of this theorem.

Corollary 0.4. *Let V be an \mathbb{F} -vector space. If B_1 and B_2 are both bases for V then they have the same number of elements.*

Proof. If B_1 is finite, with n elements, then suppose that B_2 has at least $n + 1$ elements. Choosing any such subset of size $n + 1$ as T and B_1 as the spanning set from the previous theorem, we see that $n + 1 \leq n$, a contradiction. This means $\#B_2 \leq \#B_1$. If on the other hand B_1 is infinite, then if B_2 were finite, we could reverse the roles of B_2 and B_1 , apply Steinitz again, and see $\#B_1 \leq \#B_2$, a contradiction. Therefore in all cases we have $\#B_2 \leq \#B_1$. Applying this same logic for B_1 and B_2 reversed, we get $\#B_1 \leq \#B_2$, proving the corollary. □

Definition 0.5. *A vector space with a basis of size n is called n -dimensional and we write $\dim(V) = n$. If this is true for some n we say the vector space is finite dimensional. Otherwise we say that V is infinite dimensional and write $\dim(V) = \infty$.*

Note that $\{\vec{0}\}$ is zero dimensional, since \emptyset is a basis for it.

Corollary 0.6. *Let V be an n -dimensional vector space ($n \geq 1$) and $S = \{v_1, \dots, v_m\}$.*

- 1. If $m < n$ then S cannot span V .*
- 2. If $m > n$ then S cannot be linearly independent.*
- 3. If $m = n$ then S is linearly independent if and only if it spans V .*

Proof. Let B be a basis for V . Then using Steinitz with B as the linearly independent set and S as the spanning set, we see that if S spans V then S has at least n elements, proving the first part. Similarly, using Steinitz with B as the spanning set and S as the linearly independent set, we get part two.

If $m = n$ and S is linearly independent then Steinitz implies that we can add 0 vectors from B to S to make S span V . This means S itself spans V . Conversely, if S spans V , then if it is not linearly independent, we can find $v \in S$ such that $v \in \text{Span}(S \setminus \{v\})$, or $V = \text{Span}(S) \subset \text{Span}(S \setminus \{v\})$. Therefore $S \setminus \{v\}$ is a smaller spanning set, contradicting the first part. \square

Corollary 0.7. *If W is a subspace of V then $\dim(W) \leq \dim(V)$. In particular, if V has a finite basis, so does W .*

Proof. If V is infinite dimensional there is nothing to prove, so let B be a finite basis for V of size n . Consider all subsets of W that are linearly independent. By the previous corollary, none of these have more than n elements (they cannot be infinite either since we could then extract a linearly independent subset of size $n + 1$). Choose any one with the largest number of elements and call it B_W . It must be a basis – the reason is that it is a maximal linearly independent subset of W (this is an exercise on this week's homework). Because it has no more than $\dim(V)$ number of elements, we are done. \square