## Lecture 6

We can give an alternative characterization of one-to-one and onto:

**Proposition 0.1.** *Let $T : V \to W$ be linear.*

1. *$T$ is injective if and only if it maps linearly independent sets of $V$ to linearly independent sets of $W$.*
2. *$T$ is surjective if and only if it maps spanning sets of $V$ to spanning sets of $W$.*
3. *$T$ is bijective if and only if it maps bases of $V$ to bases of $W$.*

*Proof.* The third part follows from the first two. For the first, assume that $T$ is injective and let $S \subset V$ be linearly independent. We will show that $T(S) = \{T(v) : v \in S\}$ is linearly independent. So let

$$a_1 T(v_1) + \cdots + a_n T(v_n) = \vec{0} \ .$$

This implies that $T(a_1 v_1 + \cdots + a_n v_n) = \vec{0}$, implying that $a_1 v_1 + \cdots + a_n v_n = \vec{0}$ by injectivity. But this is a linear combination of vectors in $S$, a linearly independent set, giving $a_i = 0$ for all $i$. Thus $T(S)$ is linearly independent.

Conversely suppose that $T$ maps linearly independent sets to linearly independent sets and let $v \in N(T)$. If $v \neq \vec{0}$ then $\{v\}$ is linearly independent, so $\{T(v)\}$ is linearly independent. But if $T(v) = \vec{0}$ this is impossible, since $\{\vec{0}\}$ is linearly dependent. Thus $v \neq \vec{0}$ and $N(T) = \{\vec{0}\}$, implying $T$ is injective.

For item two, suppose that $T$ is surjective and let $S$ be a spanning set for $V$. Then if $w \in W$ we can find $v \in V$ such that $T(v) = w$ and a linear combination of vectors of $S$ equal to $v$: $v = a_1 v_1 + \cdots + a_n v_n$ for $v_i \in S$. Therefore

$$w = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n) \ ,$$

meaning that we have $w \in \mathrm{Span}(T(S))$, so $T(S)$ spans $W$. Conversely if $T$ maps spanning sets to spanning sets, then $T(V) = R(T)$ must span $W$. But since $R(T)$ is a subspace of $W$, this means $R(T) = W$ and $T$ is onto. □

## Isomorphisms

**Definition 0.2.** *A linear transformation $T : V \to W$ that is bijective (that is, injective and surjective) is called an isomorphism.*

Generally speaking, we can view a bijection between sets $X$ and $Y$ as a relabeling of the elements of $X$ (to get those of $Y$). In the case of an isomorphism, this labeling also respects the vector space structure, being linear.

**Proposition 0.3.** *Let $T : V \to W$ be an isomorphism. Then $T^{-1} : W \to V$ is an isomorphism. Here, as always, the inverse function is defined by*

$$T^{-1}(w) = v \textit{ if and only if } T(v) = w \ .$$

*Proof.* It is an exercise to see that any bijection has a well-defined inverse function and that this inverse function is a bijection. (This was done, for example, in the 215 notes in the first chapter.) So we must only show that $T^{-1}$ is linear. To this end, let $w_1, w_2 \in W$ and $c \in \mathbb{F}$. Then

$$T(T^{-1}(cw_1 + w_2)) = cw_1 + w_2 \ ,$$

whereas

$$T(cT^{-1}(w_1) + T^{-1}(w_2)) = cT(T^{-1}(w_1)) + T(T^{-1}(w_2)) = cw_1 + w_2 \ .$$

Since $T$ is injective, this implies that $T^{-1}(cw_1 + w_2) = cT^{-1}(w_1) + T^{-1}(w_2)$. □

Using the notion of isomorphism, we can see that any $n$ dimensional vector space $V$ over $\mathbb{F}$ "is just" $\mathbb{F}^n$.

**Theorem 0.4.** *Let $V$ be an $n$-dimensional vector space over $\mathbb{F}$. Then $V$ is isomorphic to $\mathbb{F}^n$.*

*Proof.* Let $B = \{v_1, \ldots, v_n\}$ be a basis for $V$. We will think of $B$ as being ordered. Define the coordinate map $T_B : V \to \mathbb{F}^n$ as before as follows. Each $v \in V$ has a unique representation $v = a_1v_1 + \cdots + a_nv_n$. So set $T_B(v) = (a_1, \ldots, a_n)$. This was shown before to be a linear transformation. So we must just show it is an isomorphism.

Since the dimension of $V$ is equal to that of $\mathbb{F}^n$, we need only show that $T_B$ is onto. Then by the rank-nullity theorem, we will find

$$\dim N(T_B) = \dim(V) - \dim(R(T_B)) = \dim(V) - \dim(\mathbb{F}^n) = 0 \ ,$$

implying that $N(T_B) = \{\vec{0}\}$, and that $T_B$ is one-to-one. So to show onto, let $(a_1, \ldots, a_n) \in \mathbb{F}^n$. The element $v = a_1v_1 + \cdots + a_nv_n$ maps to it:

$$T_B(v) = T_B(a_1v_1 + \cdots + a_nv_n) = (a_1, \ldots, a_n) \ ,$$

so $T_B$ is an isomorphism. □

## Matrices and coordinates

We will now see that, just as $V$ with dimension $n$ "looks just like" $\mathbb{F}^n$, all linear maps from $V$ to $W$ look just like matrices with entries from $\mathbb{F}$.

Suppose that $T : V \to W$ is linear and these are finite dimensional vector spaces with dimension $n$ and $m$ respectively. Fix $B = \{v_1, \ldots, v_n\}$ and $C = \{w_1, \ldots, w_m\}$ to be bases of

$V$ and $W$ respectively. We know that $T$ is completely determined by its values on $B$, and each of these values lies in $W$, so we can write

$$T(v_1) = a_{1,1}w_1 + \cdots + a_{m,1}w_m$$
$$T(v_2) = a_{1,2}w_1 + \cdots + a_{m,2}w_m$$

and so on, up to

$$T(v_n) = a_{1,n}w_1 + \cdots + a_{m,n}w_m \ .$$

Now we take some arbitrary $v \in V$ and express it in terms of coordinates using $B$. This time we write it as a column vector and use the notation $[v]_B$ :

$$[v]_B = \begin{pmatrix} a_1 \\ \cdots \\ a_n \end{pmatrix}, \text{ where } v = a_1v_1 + \cdots + a_nv_n \ .$$

Let us compute $T(v)$ and write it in terms of $C$:

$$\begin{aligned} T(v) &= a_1T(v_1) + \cdots + a_nT(v_n) \\ &= a_1(a_{1,1}w_1 + \cdots + a_{m,1}w_m) + \cdots + a_n(a_{1,n}w_1 + \cdots + a_{m,n}w_m) \\ &= (a_1a_{1,1} + \cdots + a_na_{1,n})w_1 + \cdots + (a_1a_{m,1} + \cdots + a_na_{m,n})w_m \ . \end{aligned}$$

Therefore we can write $T(v)$ in coordinates using $C$ as

$$[T(v)]_C = \begin{pmatrix} a_1a_{1,1} + \cdots + a_na_{1,n} \\ \cdots \\ a_1a_{m,1} + \cdots + a_na_{m,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \cdots & \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \cdots \\ a_n \end{pmatrix} \ .$$

Therefore we have found on half of:

**Theorem 0.5** (Matrix representation)**.** *Let $T : V \to W$ be linear and $B = \{v_1, \ldots, v_n\}$ and $C = \{w_1, \ldots, w_m\}$ be (ordered) bases of $V$ and $W$ respectively. There exists a unique matrix, written $[T]_C^B$ such that for all $v \in V$,*

$$[T(v)]_C = [T]_C^B[v]_B \ .$$

*Proof.* We have already shown existence. To show uniqueness, suppose that $A$ is any $m \times n$ matrix with entries from $\mathbb{F}$ such that for all $v \in V$, $A[v]_B = [T(v)]_C$. Choose $v = v_i$ for some $i = 1, \ldots, n$ (one of the basis vectors in $B$). Then the coordinate representation of $v$ is $[v]_B = e_i$, the vector with all 0's but a 1 in the $i$-th spot. Now the product of matrices $A[v]_B$ actually gives the $i$-th column of $A$. We can see this by using the matrix multiplication formula: if $M$ is an $m \times n$ matrix and $N$ is an $n \times p$ matrix then the matrix $MN$ is $m \times p$ and its $(i, j)$-th coordinate is given by

$$(MN)_{i,j} = \sum_{k=1}^{n} M_{i,k}N_{k,j} \ .$$

Therefore as $A$ is $m \times n$ and $[v]_B$ is $n \times 1$, the matrix $A[v]_B$ is $m \times 1$ and its $(j,1)$-th coordinate is

$$(A[v]_B)_{j,1} = \sum_{k=1}^{n} A_{j,k} \left([v]_B\right)_{k,1} = \sum_{k=1}^{n} A_{j,k}(e_i)_{k,1} = A_{j,i} \ .$$

This means the entries of $A[v]_B$ are $A_{1,i}, A_{2,i}, \ldots, A_{m,i}$, the $i$-th column of $A$. However, this also equals $[T(e_i)]_C$, which is the $i$-th column of $[T]_C^B$ by construction. Thus $A$ and $[T]_C^B$ have the same columns and are thus equal. $\square$

In fact much more is true. What we have done so far is defined a mapping $\Phi : L(V,W) \to M_{m,n}(\mathbb{F})$ in the following manner. Given fixed bases $B$ and $C$ of sizes $n$ and $m$ respectively, we set

$$\Phi(T) = [T]_C^B \ .$$

This function is actually an isomorphism, meaning that the space of linear transformations is just a relabeling of the space of matrices (after choosing "coordinates" $B$ and $C$):

**Theorem 0.6.** *Given bases $B$ and $C$ of $V$ and $W$ of sizes $n$ and $m$, the spaces $L(V,W)$ and $M_{m,n}(\mathbb{F})$ are isomorphic via the mapping $\Phi$.*

*Proof.* We must show that $\Phi$ is a bijection and linear. First off, if $\Phi(T) = \Phi(U)$ then for all $v \in V$, we have

$$[T(v)]_C = \Phi(T)[v]_B = \Phi(U)[v]_B = [U(v)]_C \ .$$

But the map sending vectors in $W$ to their coordinates relative to $C$ is also a bijection, so $T(v) = U(v)$. Since this is true for all $v$, we get $T = U$, meaning $\Phi$ is injective. To show surjective, let $A$ be any $m \times n$ matrix with $(i,j)$-th entry $A_{i,j}$. Then we can define a linear transformation $T : V \to W$ by its action on the basis $B$: set

$$T(v_i) = A_{1,i}w_1 + \cdots + A_{m,i}w_m \ .$$

By the slogan, there is a unique linear transformation satisfying this and you can then check that $[T]_C^B = A$, meaning $\Phi$ is surjective and therefore a bijection.

To see that $\Phi$ is linear, let $T, U \in L(V,W)$ and $c \in \mathbb{F}$. Then the $i$-th column of $[cT+U]_C^B$ is simply the coefficients of $(cT+U)(v_i)$ expressed relative to the basis $C$. This coordinate map is linear, so

$$[(cT+U)(v_i)]_C = [cT(v_i) + U(v_i)]_C = c[T(v_i)]_C + [U(v_i)]_C \ ,$$

which is $c$ times the $i$-th column of $\Phi(T)$ plus the $i$-th column of $\Phi(U)$. Thus

$$[cT+U]_C^B = c[T]_C^B + [U]_C^B \ .$$

$\square$