MAT217 HW 9

DUE TUES. APR. 16, 2013

Notation. Let $T : V \to V$ be linear, where V is an $\mathbb{F}$-vector space. If $p \in \mathbb{F}[x]$ has the form $p(x) = a_n x^n + \cdots + a_1 x + a_0$ then we define the linear transformation $p(T) : V \to V$ by

$$p(T) = a_n T^n + \cdots + a_1 T + a_0 I \ .$$

Exercises.

1. Let V be a finite dimensional vector space with $T : V \to V$ linear and let

$$V = W_1 \oplus \cdots \oplus W_k$$

be a T-invariant direct sum decomposition. That is, the subspaces W_i are independent, sum to V and are each T-invariant. If $B_1, \ldots, B_k$ are bases for $W_1, \ldots, W_k$ respectively, let $B = \cup_{i=1}^k B_i$ and show that $[T]_B^B$ is a block diagonal matrix with k blocks of sizes $\#B_1, \ldots, \#B_k$.

2. Let $U : V \to V$ be nilpotent of degree k and $\dim V < \infty$. In this question we will sketch an approach to the structure theorem for nilpotent operators using quotient spaces.

   (a) For $i = 1, \ldots, k$, define $N_i = N(U^i)$ and $N_0 = \{\vec{0}\}$. Define the quotient spaces $\hat{N}_i = N_i / N_{i-1}$ and $\hat{N}_0 = \{\vec{0}\}$. Show that if $C \in \hat{N}_i$ for $i \geq 1$ then $U(C) \in \hat{N}_{i-1}$, where

   $$U(C) = \{U(v) : v \in C\} \ .$$

   (b) For $i = 1, \ldots, k$, define $U_i : \hat{N}_i \to \hat{N}_{i-1}$ by $U_i(C) = U(C)$. Show that U_i is linear.

   (c) For $i = 1, \ldots, k$, show that U_i is injective.

   (You do not need to do anything for this paragraph.) From this point on, the proof would proceed as follows. Let $C_1^{(k)}, \ldots, C_{l_k}^{(k)}$ be a basis of $\hat{N}_k$. By injectivity of U_k, $\{U_k(C_1^{(k)}), \ldots, U_k(C_{l_k}^{(k)})\}$ linearly independent. Extend it to a basis of $\hat{N}_{k-1}$ and call the elements of this basis $C_1^{(k-1)}, \ldots, C_{l_{k-1}}^{(k-1)}$, where $l_{k-1} \geq l_k$. Continue, and after constructing the basis of $\hat{N}_{k-r}$ whose elements are $C_1^{(k-r)}, \ldots, C_{l_{k-r}}^{(k-r)}$, extend their images under U_{k-r} to a basis of $\hat{N}_{k-r-1}$. In the end we get a family of bases of the spaces $\hat{N}_1, \ldots, \hat{N}_k$. This family is analogous to chain bases constructed in class, but now live in quotient spaces. At this point, one must just extract the chain bases, and you can think about how to do that.

3. **The minimal polynomial** Let V be a finite-dimensional $\mathbb{F}$-vector space and $T : V \to V$ linear.

   (a) Consider the subset $S \subset \mathbb{F}[x]$ defined by
   $$S = \{p \in \mathbb{F}[x] : p(T) = 0\} \ .$$
   Show that S contains a nonzero element.
   Hint. Let $\{v_1, \ldots, v_n\}$ be a basis for V and for each i, consider
   $$\{v_i, T(v_i), T^2(v_i), \ldots, T^n(v_i)\} \ .$$
   Show this set is linearly dependent and therefore there is a polynomial $p_i \in \mathbb{F}[x]$ such that $(p_i(T))(v_i) = \vec{0}$. Then define p as the product $p_1 \cdots p_n$.

   (b) Let $m_T \in S$ be a monic non-zero element of S of minimal degree. Show that m_T divides any other element of S. Conclude that m_T is unique. We call m_T the *minimal polynomial* of T.

   (c) Prove that the zeros of m_T are exactly the eigenvalues of T by completing the following steps.

      i. Suppose that $r \in \mathbb{F}$ is a zero of m_T. Show that
      $$m_T(x) = q(x)(x - r)^k$$
      for some $k \in \mathbb{N}$ and $q \in \mathbb{F}[x]$ such that $q(r) \neq 0$. Prove also that $q(T) \neq 0$.

      ii. Show that if $r \in \mathbb{F}$ is a zero of m_T then $rI - T$ is not invertible and so r is an eigenvalue of T.

      iii. Conversely, if λ is an eigenvalue of T, let v be a corresponding eigenvector. Show that if $p \in \mathbb{F}[x]$ then $(P(T))(v) = P(\lambda)v$. Conclude that λ is a zero of m_T.

4. Let $T : V \to V$ be linear and V a finite-dimensional vector space over $\mathbb{F}$. Let $p, q \in \mathbb{F}[x]$ be relatively prime (that is, their greatest common divisor is 1). We will show that
   $$N(p(T)q(T)) = N(p(T)) \oplus N(q(T)) \ .$$

   (a) Show that if $v \in N(p(T)) + N(q(T))$ then $v \in N(p(T)q(T))$.

   (b) Show that if $v \in N(p(T)q(T))$ then $v \in N(p(T)) + N(q(T))$.
   Hint. Since p, q are relatively prime, we may find $a, b \in \mathbb{F}[x]$ such that $ap+bq = 1$. Now apply this to v.

   (c) Show that $N(p(T)) \cap N(q(T)) = \{\vec{0}\}$.

5. Let $T : V \to V$ be linear and V a finite-dimensional vector space over an algebraically closed field $\mathbb{F}$. Show that if the minimal polynomial is factored as

$$m_T(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_k)^{n_k}$$

then $V = N(\lambda_1 I - T)^{n_1} \oplus \cdots \oplus N(\lambda_k I - T)^{n_k}$. How is this a different route to prove the primary decomposition theorem?

Hint. Use exercise 4.

6. Let $T : V \to V$ be linear and V a finite-dimensional vector space over $\mathbb{F}$. Show that T is diagonalizable if and only if there exist $\lambda_1, \ldots, \lambda_k$ in $\mathbb{F}$ such that

$$m_T(x) = (x - \lambda_1) \cdots (x - \lambda_k) \ .$$

Hint. Use exercise 4.

7. Let V be a finite dimensional vector space over $\mathbb{F}$, an algebraically closed field. If $T : V \to V$ is linear, show that the multiplicity of an eigenvalue λ in m_T is equal to the size of the largest Jordan block of T for λ. For which T is $c_T = m_T$?

8. Find the Jordan form for each of the following matrices over $\mathbb{C}$. Write the minimal polynomial and characteristic polynomial for each. To do this, first find the eigenvalues. Then, for each eigenvalue λ, find the dimensions of the nullspaces of $(A - \lambda I)^k$ for pertinent values of k (where A is the matrix in question). Use this information to deduce the block forms.

$$(a) \begin{pmatrix} -1 & 0 & 0 \\ 1 & 4 & -1 \\ -1 & 4 & 0 \end{pmatrix} \qquad (b) \begin{pmatrix} 2 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 2 \end{pmatrix} \qquad (c) \begin{pmatrix} 5 & 1 & 3 \\ 0 & 2 & 0 \\ -6 & -1 & -4 \end{pmatrix}$$

9. (a) The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 7 & 1 & 2 & 2 \\ 1 & 4 & -1 & -1 \\ -2 & 1 & 5 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$$

is $c(x) = (x - 6)^4$. Find an invertible matrix S such that $S^{-1}AS$ is in Jordan form.

(b) Find all complex matrices in Jordan form with characteristic polynomial

$$c(x) = (i - x)^3 (2 - x)^2 \ .$$

10. If $T : V \to V$ is linear and V is a finite-dimensional $\mathbb{F}$-vector space with $\mathbb{F}$ algebraically closed, we define the *algebraic multiplicity* of an eigenvalue λ to be $a(\lambda)$, the dimension of the generalized eigenspace $\hat{E}_\lambda$. The *geometric multiplicity* of λ is $g(\lambda)$, the dimension of the eigenspace E_λ. Finally, the index of λ is $i(\lambda)$, the length of the longest chain of generalized eigenvectors in $\hat{E}_\lambda$.

    Suppose that λ is an eigenvalue of T and $g = g(\lambda)$ and $i = i(\lambda)$ are given integers.

    (a) What is the minimal possible value for $a = a(\lambda)$?

    (b) What is the maximal possible for a?

    (c) Show that a can take any value between the answers for the above two questions.

    (d) What is the smallest dimension n of V for which there exist two linear transformations T and U from V to V with all of the following properties? (i) There exists $\lambda \in \mathbb{F}$ which is the only eigenvalue of either T or U, (ii) T and U are not similar transformations and (iii) the geometric multiplicity of λ for T equals that of U and similarly for the index.