MAT217 HW 10
Due Tues. Apr. 23, 2013

1. Let V be an $\mathbb{F}$-vector space and $\{v_1, \ldots, v_n\}$ a basis for V. Consider the dual basis $\{v_1^*, \ldots, v_n^*\}$ and for all pairs $i, j \in \{1, \ldots, n\}$ define $f_{i,j}(v, w) = v_i^*(v)v_j^*(w)$. Show that

$$\{f_{i,j} : i, j \in \{1, \ldots, n\}\}$$

is a basis for $Bil(V, \mathbb{F})$. Find the nullspace of each element in this basis.

2. (From Hoffman-Kunze) Let V be a finite dimensional $\mathbb{F}$-vector space and f a bilinear form on V of rank 1. Show that f can be written as $g_1 g_2$ for elements $g_1, g_2 \in V^*$.

3. (From Hoffman-Kunze) Which of the following functions f, defined on vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, are bilinear forms?
   (a) $f(x, y) = 1$.
   (b) $f(x, y) = (x_1 - y_1)^2 + x_2 y_2$.
   (c) $f(x, y) = (x_1 + y_1)^2 - (x_1 - y_1)^2$.
   (d) $f(x, y) = x_1 y_2 - x_2 y_1$.

4. (From Hoffman-Kunze) Let f be the bilinear form on $\mathbb{R}^2$ defined by

$$f((x_1, y_1), (x_2, y_2)) = x_1 x_2 + y_1 y_2 \ .$$

Find the matrix of f in each of the following bases:

$$\{(1, 0), (0, 1)\},\ \{(1, -1), (1, 1)\},\ \{(1, 2), (3, 4)\} \ .$$

5. Let $T : V \to V$ be linear on a finite-dimensional vector space. If W is a T-invariant subspace of V, define the restriction T_W of T to W.
   (a) Show the characteristic polynomial of T_W divides that of T. Show the minimal polynomial of T_W divides that of T.
   (b) Show that if T is diagonalizable then so is T_W.

6. Let $T, U : V \to V$ be linear on a finite-dimensional vector space. Assume that $TU = UT$ and that both T and U are diagonalizable. We will show that T and U are *simultaneously diagonalizable*; that is, there is a basis B such that both $[T]_B^B$ and $[U]_B^B$ are diagonal.
   (a) Show that each eigenspace E_λ^T of T is U-invariant.
   (b) Show that there is a basis of each E_λ^T consisting of eigenvectors for both T and U. Conclude that T and U are simultaneously diagonalizable.

7. In this problem we will inspect the interaction between $\mathbb{R}^n$ and $\mathbb{C}^n$. This will be used to establish the real Jordan form in the next problem.

   (a) Every vector in $\mathbb{C}^n$ can be written as $v+iw$ where $v, w \in \mathbb{R}^n$. Define the inclusion map $\iota : \mathbb{R}^n \to \mathbb{C}^n$ by $\iota(v) = v = v + i\vec{0}$. Show that ι is $\mathbb{R}$-linear; that is, $\iota(cv + w) = c\iota(v) + \iota(w)$ for $v, w \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

   (b) Define the complex conjugation map $\mathfrak{c} : \mathbb{C}^n \to \mathbb{C}^n$ by
   $$\mathfrak{c}(v + iw) = v - iw \ .$$
   Show that $\mathfrak{c}^2$ is the identity and $\mathfrak{c}$ is *anti-linear*; that is, $\mathfrak{c}$ is additive but $\mathfrak{c}(z(v + iw)) = \overline{z}\mathfrak{c}(v + iw)$. (Here $\overline{z}$ represents the complex conjugate of the number $z \in \mathbb{C}$.)

   (c) Prove that if W is a subspace of $\mathbb{R}^n$ then $\mathrm{Span}(\iota(W))$ is $\mathfrak{c}$-invariant. Conversely, if W' is a $\mathfrak{c}$-invariant subspace of $\mathbb{C}^n$, show that $W' = \mathrm{Span}(\iota(W))$ for some subspace W of $\mathbb{R}^n$.

8. In this problem we establish the real Jordan form. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. The *complexification* of T is defined as $T_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$ by
   $$T_{\mathbb{C}}(v + iw) = T(v) + iT(w) \ .$$

   (a) Show that $T_{\mathbb{C}}$ is a linear transformation on $\mathbb{C}^n$. If $\lambda \in \mathbb{C}$ is one of its eigenvalues and $\hat{E}_\lambda$ is the corresponding generalized eigenspace, show that $\mathfrak{c}(\hat{E}_\lambda) = \hat{E}_{\overline{\lambda}}$. (Here $\mathfrak{c}$ is the complex conjugation map from last problem.)

   (b) Show that the non-real eigenvalues of $T_{\mathbb{C}}$ come in pairs. In other words, show that we can list the distinct eigenvalues of $T_{\mathbb{C}}$ as
   $$\lambda_1, \ldots, \lambda_r, \sigma_1, \ldots, \sigma_{2m} \ ,$$
   where for each $j = 1, \ldots, r$, $\overline{\lambda_j} = \lambda_j$ and for each $i = 1, \ldots, m$, $\sigma_{2i-1} = \overline{\sigma_{2i}}$.

   (c) Because $\mathbb{C}$ is algebraically closed, the proof of Jordan form shows that
   $$\mathbb{C}^n = \hat{E}_{\lambda_1} \oplus \cdots \oplus \hat{E}_{\lambda_r} \oplus \hat{E}_{\sigma_1} \oplus \cdots \oplus \hat{E}_{\sigma_{2m}} \ .$$
   Using the previous two parts, show that for $j = 1, \ldots, r$ and $i = 1, \ldots, m$, the subspaces of $\mathbb{C}^n$
   $$\hat{E}_{\lambda_j} \text{ and } \hat{E}_{\sigma_{2i-1}} \oplus \hat{E}_{\sigma_{2i}}$$
   are $\mathfrak{c}$-invariant.

   (d) Deduce from the previous problem that there exist subspaces $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_m$ of $\mathbb{R}^n$ such that for each $j = 1, \ldots, r$ and $i = 1, \ldots, m$,
   $$\hat{E}_{\lambda_j} = \mathrm{Span}(\iota(X_j)) \text{ and } \hat{E}_{\sigma_{2i-1}} \oplus \hat{E}_{\sigma_{2i}} = \mathrm{Span}(\iota(Y_i)) \ .$$
   Show that $\mathbb{R}^n = X_1 \oplus \cdots \oplus X_r \oplus Y_1 \oplus \cdots \oplus Y_m$.

(e) Prove that for each $j = 1, \ldots, r$, the transformation $T - \lambda_j I$ restricted to X_j is nilpotent and thus we can find a basis B_j for X_j consisting entirely of chains for $T - \lambda_j I$.

(f) For each $k = 1, \ldots, m$, let

$$C_k = \{v_1^{(k)} + iw_1^{(k)}, \ldots, v_{n_k}^{(k)} + iw_{n_k}^{(k)}\}$$

be a basis of $\hat{E}_{\sigma_{2k-1}}$ consisting of chains for $T_{\mathbb{C}} - \sigma_{2k-1} I$. Prove that

$$\hat{C}_k = \{v_1^{(k)}, w_1^{(k)}, \ldots, v_{n_k}^{(k)}, w_{n_k}^{(k)}\}$$

is a basis for Y_k. Describe the form of the matrix representation of T restricted to Y_k, relative to the basis $\hat{C}_k$.

(g) Gathering the previous parts, state and prove a version of Jordan form for linear transformations on $\mathbb{R}^n$. Your version should be of the form "If $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear then there exists a basis B such that $[T]_B^B$ has the form ..."