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The coefficient of concordance for vague data

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Abstract

Kendall's coefficient of concordance is used traditionally in statistics for measuring agreement between k orderings ($k > 2$). A new measure of concordance which generalizes Kendall's coefficient is proposed. The suggested coefficient could be used in situations with missing information or noncomparable outputs.

Keywords: Coefficient of concordance; Measures of agreement; Ordering; Ranks; Fuzzy sets

1. Introduction

Suppose we observe k sets ($k > 2$) of rankings of n subjects and we want to decide whether there is any association or dependence between this rankings. If it is so, we are also interested in measuring the degree of agreements between these rankings (or orderings). A typical example is a situation with k judges or experts, called observers, each of whom is presented with the same set of n objects to be ranked. Then the desired measure of relationship will describe the total agreement or concordance within the group of observers in their judgements on the n objects under study.

The situation with $k > 2$ observers is a natural extension of the paired-sample problem where such well-known statistical tools as Kendall's τ or Spearman's rank correlation coefficient might be applied to measure the association between two orderings. Thus one possibility for measuring the agreement between k observers is to select one of the measures for the paired-sample problem and apply it to each of the $\binom{k}{2}$ sets of pairs of rankings of n objects. However, if we need a single measure of the overall association, instead of $\binom{k}{2}$ coefficients, then the so called, Kendall's coefficient of concordance should be used.

All statistical tools mentioned above deal with random but precise and unambiguous observations, even though they are orderings only. But in real life we often meet vague data and ambiguous answers which abound with missing information and hesitance. Such data exhibit uncertainty which has different sources than randomness. Thus traditional statistical tools cannot be directly applied there and new tools which admit uncertainty due to imprecision, vagueness and hesitance are strongly required.

The problem of the partial preorders comparison was considered, e.g. by Roy and Słowinski (1993). Hébert et al. (2003) extended Kendall's rank correlation to interval and fuzzy data. Kendall's τ and other rank-based nonparametric procedures for fuzzy data was also considered by Denœux et al. (2005). Their approach relies on the definition of a fuzzy

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partial order based on necessity index of strict dominance (see Dubois and Prade, 1980) and on a concept of fuzzy set of linear extensions of the fuzzy partial order. Unfortunately, their results—although mathematically elegant—are only applicable to very small problems, while for more objects Monte Carlo simulations have to be used. Another natural and simple mathematical tool based on IF-sets for dealing with partial ordering was considered by Grzegorzewski (2004b,c). He proposed generalized versions of Kendall's and Spearman's rank correlation coefficients for situations in which not all elements could be ordered. In the present paper we suggest how to measure a total association between k orderings which are not necessarily unambiguous and admit some hesitance. Our approach to partial orderings, contrary to that given by Denœux et al., seems to be much simpler and is not so sensitive to a sample size.

The paper is organized as follows. In Section 2 we introduce notation and recall basic information on the classical Kendall's coefficient of concordance. We also indicate there some motivations for this paper. Basic information on IF-sets are given in Section 3 while in Section 4 we propose a natural and simple method for modelling vague orderings, based on IF-sets. Then (Section 5) we show how to generalize the classical Kendall's coefficient of concordance to make it possible to measure association between rankings with missing information or noncomparable outputs. We also discuss basic properties of the generalized Kendall's coefficient of concordance and consider some examples.

2. The classical coefficient of concordance

Let $X=\{x_1,\ldots,x_n\}$ denote a finite universe of discourse, i.e. a finite set of considered objects. Suppose that elements (objects) $x_1,\ldots,x_n$ are ordered according to preferences of k observers $A_1,\ldots,A_k$. Then our data could be presented in the form of a two-way layout (or matrix) M of dimension $k\times n$ with row and column labels designating observers and objects, respectively. Thus the table entries R_{ij} (where $i=1,\ldots,k$ and $j=1,\ldots,n$) denote the rank given by the ith observer to the jth object, i.e.

$$M=\begin{bmatrix} R_{11} & R_{12} & \ldots & R_{1n} \\ R_{21} & R_{22} & \ldots & R_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ R_{k1} & R_{k2} & \ldots & R_{kn} \end{bmatrix}. \tag{1}$$

One may easily see that each row is a permutation of numbers $1,2,\ldots,n$. Moreover, in the jth column we have a collection of ranks given to object x_j by all observers. If, for example, x_j has the same preference relative to all other objects $x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n$ in the opinion of each of the k observers, then all ranks in the jth column will be identical. Therefore, the ranks in each column are indicative of the agreement among observers.

Let $R_1,\ldots,R_n$ denote column totals, i.e.

$$R_j=\sum_{i=1}^{k} R_{ij}. \tag{2}$$

Then the coefficient

$$W\left(A_1,\ldots,A_k\right)=\frac{12}{k^2 n\left(n^2-1\right)}\sum_{j=1}^{n}\left[R_j-\frac{k(n+1)}{2}\right]^2, \tag{3}$$

called Kendall's coefficient of concordance, provides a measure of total agreement within the group of k observers (see, e.g. Gibbons and Chakraborti, 2003).

Coefficient of concordance has various applications in psychology (see, e.g. Ferguson and Takane, 1989), social sciences, group decision making and so on.

Coefficient (3) ranges between 0 and 1, with 1 corresponding to perfect agreement (or concordance) and 0 indicating no agreement or independence of samples. As W increases, there is greater agreement among observers. Kendall's coefficient of concordance is also commutative and invariant under all order-preserving transformations.

Up to now we have assumed that there are no tied ranks in our data table. However, if tied observations also appear then the most common practice for dealing with them, as in most other nonparametric procedures, is to assign equal ranks to indistinguishable observations.

Let us now consider a following example:

Example 1. Four persons—John, Tom, Susan and Helen—were asked to rank the following films: *Ameli*, *Dogville*, *Leon*, *Matrix*, *Requiem for a dream*, *Stigmata*, *Talk to her* and *Titanic* from the most preferred to the less esteemed one. We just want to check if there is agreement within the group of these observers.

Suppose John likes *Leon* best. Next is *Ameli*, *Talk to her*, *Dogville*, *Stigmata*, *Requiem for a dream* and *Matrix* but he dislikes *Titanic*.

Tom also likes *Leon* best and then *Ameli* but next is *Dogville*, *Stigmata*, *Requiem for a dream*, *Matrix* and, as for John, *Titanic* is the worst. However, Tom has not seen *Talk to her* and thus he cannot classify this film.

Susan's favorite film is *Ameli*. Next is *Talk to her*, *Dogville*, *Stigmata*. Then is either *Leon* or *Requiem for a dream*, but Susan cannot decide which one of these two films is better. But she is sure that the worst is *Matrix* and the last but one is *Titanic*.

Finally, Helen's favorite film is *Talk to her*, then *Ameli*, *Leon*, *Dogville*, *Requiem for a dream*, *Titanic* and *Matrix*. However, Helen has not seen *Stigmata* and hence she cannot rank this film.

The matrix (2) which describes preferences of our four observers is as follows:

$$M_1 = \begin{bmatrix} 2 & 4 & 1 & 7 & 6 & 5 & 3 & 8 \\ 2 & 3 & 1 & 6 & 5 & 4 & (?) & 7 \\ 1 & 3 & 5 \text{ or } 6 & 8 & 5 \text{ or } 6 & 4 & 2 & 7 \\ 2 & 4 & 3 & 7 & 5 & (?) & 1 & 6 \end{bmatrix},$$

where(?) stands for the not ordered object.

It is evident that this kind of data require a coefficient of concordance to evaluate the association between rankings of our four observers. Unfortunately, Kendall's coefficient of concordance cannot be applied here. It is generally because not all elements have been ranked by each observer.

One way-out, sometimes applied in practice, is to remove and not to include into considerations all objects which are not ordered by all of the observers. However, this approach involves always a loss of information. Moreover, it may happen that if the number of ill-classified objects is large, eliminating them would be preclusive of applying Kendall's coefficient of concordance. Such a situation is given in a following example.

Example 2. Now, suppose that nine persons, and the matrix (2) which describes their preferences is given by

$$M_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (?) & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & (?) & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & (?) & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & (?) & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & (?) & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & (?) & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & (?) & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & (?) \end{bmatrix}, \tag{4}$$

where(?) stands, as before, for the not ordered object.

It is easily seen that if one is used to removing objects that are not univocally classified by all observers, he would have to eliminate here all the objects under study! Thus the association analysis using Kendall's coefficient would not be possible. On the other hand, just at a first glance on matrix M_2, it is evident that there exist an association between these nine observers.

Further on we suggest how to cope with problems like discussed above. In our approach we describe vagueness in rankings by the generalization of the standard fuzzy sets, suggested by Atanassov (1986).

3. IF-sets

Let X denote a universe of discourse. Then a fuzzy set C in X is defined as a set of ordered pairs

$$C = \left\{ \langle x, \mu_C(x) \rangle : x \in X \right\}, \tag{5}$$

where $\mu_C : X \to [0,1]$ is the membership function of C and $\mu_C(x)$ is the grade of belongingness of x into C (see Zadeh, 1965). Thus automatically the grade of nonbelongingness of x into C is equal to $1-\mu_C(x)$. However, in real life the linguistic negation not always identifies with logical negation. This situation is very common in natural language processing, computing with words, etc. Therefore Atanassov (1986, 1999) suggested a generalization of classical fuzzy set, called an intuitionistic fuzzy set. The name suggested by Atanassov is slightly misleading, because his sets have nothing in common with intuitionism known from logic. It seems that other name, e.g. incomplete fuzzy sets (which had the same abbreviation), would be even more adequate for the Atanassov sets. Thus finally, in order to avoid terminology problems, we call the Atanassov sets as IF-sets.

An IF-set C in X is given by a set of ordered triples

$$C = \left\{ \left\langle x, \mu_C(x), \nu_C(x) \right\rangle : x \in X \right\}, \tag{6}$$

where $\mu_C, \nu_C : X \to [0,1]$ are functions such that

$$0 \leqslant \mu_C(x) + \nu_C(x) \leqslant 1 \quad \forall x \in X. \tag{7}$$

For each x the numbers $\mu_C(x)$ and $\nu_C(x)$ represent the degree of membership and degree of nonmembership of the element $x \in X$ to $C \subset X$, respectively.

It is easily seen that an IF-set $\left\{ \left\langle x, \mu_C(x), 1-\mu_C(x) \right\rangle : x \in X \right\}$ is equivalent to (5), i.e. each fuzzy set is a particular case of the IF-set. We will denote a family of fuzzy sets in X by $FS(X)$, while $IFS(X)$ stands for the family of all IF-sets in X.

For each element $x \in X$ we can compute, so called, the IF-index of x in C defined as follows:

$$\pi_C(x) = 1 - \mu_C(x) - \nu_C(x), \tag{8}$$

which quantifies the amount of indeterminacy associated with x_i in C. It is seen immediately that $\pi_C(x) \in [0,1]$ $\forall x \in X$. If $C \in FS(X)$ then $\pi_C(x) = 0\ \forall x \in X$.

4. IF-sets in modelling rankings

In this section we will suggest how to apply IF-sets in modelling orderings or rankings. A method proposed below seems to be useful especially if not all elements under consideration could be ranked. In our approach we will attribute an IF-set to the ordering corresponding to each observer. For simplicity of notation we will identify orderings expressed by observers $A_1, \ldots, A_k$ with the corresponding IF-sets. Thus, for each $i = 1, \ldots, k$ let

$$A_i = \left\{ \left\langle x_j, \mu_{A_i}\left(x_j\right), \nu_{A_i}\left(x_j\right) \right\rangle : x_j \in X \right\} \tag{9}$$

denote an intuitionistic fuzzy subset of the universe of discourse $X = \{x_1, \ldots, x_n\}$, where membership function $\mu_{A_i}\left(x_j\right)$ indicates the degree to which x_j is the most preferred element by observer A_i, while nonmembership function $\nu_{A_i}\left(x_j\right)$ shows the degree to which x_j is the less preferred element by observer A_i.

Here a natural question arises: how to determine these membership and nonmembership functions. Let us recall that the only available information are orderings that admit ties and elements that cannot be ranked (i.e. we deal with orderings which are not necessarily linear orderings because there are elements which are noncomparable). Anyway, for each observer one can always specify two functions $w_{A_i}, b_{A_i} : X \to \{0, 1, \ldots, n-1\}$ defined as follows: for each given $x_j \in X$ let $w_{A_i}\left(x_j\right)$ denote the number of elements $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ surely worse than x_j, while $b_{A_i}\left(x_j\right)$ be equal to the number of elements $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ surely better than x_j in the ordering corresponding to the preferences expressed by observer A_i. Using functions $w_{A_i}\left(x_j\right)$ and $b_{A_i}\left(x_j\right)$ we may determine the requested membership and nonmembership functions as follows:

$$\mu_{A_i}\left(x_j\right) = \frac{w_{A_i}\left(x_j\right)}{n-1}, \tag{10}$$

$$\nu_{A_i}\left(x_j\right) = \frac{b_{A_i}\left(x_j\right)}{n-1}. \tag{11}$$

It is easily seen that $w_{A_i}\left(x_j\right), b_{A_i}\left(x_j\right) \in \{0, \ldots, n-1\}$ because we rank n elements and hence for each element $x_j \in X$ there exist no less than zero and no more than $n-1$ elements which are better (worse) than x_j. Moreover, we admit situations when the same rank is assigned to more than one element and elements that are not comparable with the others.

In such a way we get k well-defined IF-sets which describe nicely orderings corresponding to observers $A_1, \ldots, A_k$. It is seen that $\pi_{A_i}\left(x_j\right)=0$ for each $x_j \in X$ if and only if all elements are ranked by A_i and there are no ties. Conversely, if there exist such element $x_j \in X$ that $\pi_{A_i}\left(x_j\right)>0$ then it means that there are ties or noncomparable elements in the ordering made by A_i. Moreover, more ties or elements that are not comparable with the others are present, bigger values of the intuitionistic fuzzy index are observed. One may also notice that $\pi_{A_i}\left(x_j\right)=1$ if and only if element $x_j \in X$ is noncomparable with other element or all elements $x_1, \ldots, x_n$ have obtained the same rank in the ordering made by A_i.

Hence it is seen that IF-sets seem to be a natural and useful tool for modelling nonlinear orderings.

5. The generalized coefficient of concordance

According to (3) Kendall's coefficient of concordance might be expressed in a following way

$$W\left(A_1, \ldots, A_k\right)=\xi_{k,n} \times \rho\left(R, \overline{R^*}\right), \tag{12}$$

where $\xi_{k,n}$ is a constant coefficient (which depends on the number of observers k and observed objects n) and $\rho\left(R, \overline{R^*}\right)$ denotes a distance between the observed column totals $R=\left(R_1, \ldots, R_n\right)$ and the average column totals $\overline{R^*}$ obtained for perfect agreement between rankings. It can be shown that $\overline{R^*}=\left(\overline{R_1^*}, \ldots, \overline{R_n^*}\right)$ and $\overline{R_j^*}=k(n+1)/2$ for each $j=1, \ldots, n$. One may also find out that $\xi_{k,n}$ is the inverse of the maximum value of $\rho\left(R, \overline{R^*}\right)$ which is obtained when there is a perfect agreement between the observers.

Now to construct a straightforward generalization of Kendall's coefficient of concordance for orderings containing elements that cannot be ranked by all observers, we have to find counterparts of R and $\overline{R^*}$ and a suitable measure of distance between these two objects.

From now on we make the assumption that all objects under study are univocally ranked by at least one of the observers. Such set of rankings or orderings will be called *nondegenerated*. In other words, a set of rankings made by k observers is nondegenerated if and only if there exist at least one IF-set $A^{\#}$ among $A_1, \ldots, A_k$, corresponding to orderings produced by the observers, such that the intuitionistic index $\pi_{A^{\#}}\left(x_j\right)=0$ for each $x_j \in X$.

As we have suggested in the previous section, we would consider appropriate IF-sets instead of classical rankings. Thus instead of R will also consider an IF-set A, defined as follows:

$$A=\left\{\left\langle x_j, \mu_A\left(x_j\right), \nu_A\left(x_j\right)\right\rangle : x_j \in X\right\}, \tag{13}$$

where the membership and nonmembership functions μ_A and ν_A, respectively, are given by

$$\mu_A\left(x_j\right)=\frac{1}{k}\sum_{i=1}^{k} \mu_{A_i}\left(x_j\right), \tag{14}$$

$$\nu_A\left(x_j\right)=\frac{1}{k}\sum_{i=1}^{k} \nu_{A_i}\left(x_j\right). \tag{15}$$

If there is a perfect agreement within the group of observers and all objects are ranked without ties, then the resulting IF-set is of a form $A^*=\left\{\left\langle x_j, \mu_{A^*}\left(x_j\right), \nu_{A^*}\left(x_j\right)\right\rangle : x_j \in X\right\}$ such that the membership function is given by

$$\begin{aligned}
\mu_{A^*}(x_{j_1}) &= \frac{n-1}{n-1}=1,\\
\mu_{A^*}(x_{j_2}) &= \frac{n-2}{n-1},\\
\mu_{A^*}(x_{j_3}) &= \frac{n-3}{n-1},\\
&\vdots\\
\mu_{A^*}(x_{j_{n-1}}) &= \frac{1}{n-1},\\
\mu_{A^*}(x_{j_n}) &= 0,
\end{aligned} \tag{16}$$

where $x_{j_1}, \ldots, x_{j_n}$ is a permutation of elements $x_1, \ldots, x_n$ and the nonmembership function is

$$\nu_{A^*}\left(x_j\right) = 1 - \mu_{A^*}\left(x_j\right) \tag{17}$$

for each $j = 1, \ldots, n$. Therefore, for perfect agreement between rankings, instead of the average column totals $\overline{R^*}$ we obtain an IF-set $\overline{A^*} = \left\{\left\langle x_j, \mu_{\overline{A^*}}\left(x_j\right), \nu_{\overline{A^*}}\left(x_j\right)\right\rangle : x_j \in X\right\}$ such that

$$\mu_{\overline{A^*}}(x_1) = \cdots = \mu_{\overline{A^*}}(x_n) = \frac{1}{n} \cdot \frac{(0+1)n}{2} = \frac{1}{2}, \tag{18}$$

$$\nu_{\overline{A^*}}(x_1) = \cdots = \nu_{\overline{A^*}}(x_n) = \tfrac{1}{2}. \tag{19}$$

Now, after substituting R and $\overline{R^*}$ by IF-sets A and $\overline{A^*}$, respectively, we have to choose a suitable distance between these two IF-sets. Several measures of distance between IF-sets were considered in the literature (see, e.g. Grzegorzewski, 2004a). In this paper we will apply a distance proposed by Atanassov (1999), i.e. such function $d : IFS(X) \times IFS(X) \to R^+ \cup \{0\}$ which for any two IF-subsets $B = \left\{\left\langle x_j, \mu_B\left(x_j\right), \nu_B\left(x_j\right)\right\rangle : x_j \in X\right\}$ and $C = \left\{\left\langle x_j, \mu_C\left(x_j\right), \nu_C\left(x_j\right)\right\rangle : x_j \in X\right\}$ of the universe of discourse $X = \{x_1, \ldots, x_n\}$ is defined as

$$d(B, C) = \sum_{j=1}^{n} \left[\left(\mu_B\left(x_j\right) - \mu_C\left(x_j\right)\right)^2 + \left(\nu_B\left(x_j\right) - \nu_C\left(x_j\right)\right)^2\right]. \tag{20}$$

The use of the Atanassov distance can be motivated not only because (20) is a natural generalization of the distance used for fuzzy sets but it naturally leads to the same measure as used by Kendall.

For actual observed rankings, modelled by IF-sets $A_1, \ldots, A_k$, distance (20) between IF-set A obtained from (13)–(15) and $\overline{A^*}$ obtained from (18)–(19) is given by

$$d\left(A, \overline{A^*}\right) = \sum_{j=1}^{n} \left[\left(\mu_A\left(x_j\right) - \frac{1}{2}\right)^2 + \left(\nu_A\left(x_j\right) - \frac{1}{2}\right)^2\right]. \tag{21}$$

It can be shown that (21) attains the maximum value for the perfect agreement within the group of observers, i.e. for $A = A^*$. We are interested in this maximum value to obtain a counterpart of $\xi_{k,n}$ in (12), which in fact is a normalizing constant. Then, according to (16)–(17) we get

$$\begin{aligned}
d\left(A^*, \overline{A^*}\right) &= \sum_{j=1}^{n} \left[\left(\mu_{A^*}\left(x_j\right) - \frac{1}{2}\right)^2 + \left(\nu_{A^*}\left(x_j\right) - \frac{1}{2}\right)^2\right] \\
&= \sum_{j=1}^{n} \left[\left(\frac{n-j}{n-1} - \frac{1}{2}\right)^2 + \left(1 - \frac{n-j}{n-1} - \frac{1}{2}\right)^2\right] \\
&= 2\sum_{j=1}^{n} \left(\frac{n-j}{n-1} - \frac{1}{2}\right)^2 \\
&= \frac{1}{2(n-1)^2} \sum_{j=1}^{n} (n-2j+1)^2 \\
&= \frac{1}{2(n-1)^2} \frac{n(n^2-1)}{3} \\
&= \frac{n(n+1)}{6(n-1)}.
\end{aligned}$$

Hence now we can define a generalized version of Kendall's coefficient of concordance.

Definition. Let $A_i = \left\{\left\langle x_j, \mu_{A_i}\left(x_j\right), \nu_{A_i}\left(x_j\right)\right\rangle : x_j \in X\right\}$, $i = 1, \ldots, k$ denote intuitionistic fuzzy subsets of the universe of discourse $X = \{x_1, \ldots, x_n\}$ which correspond to nondegenerated orderings expressed by k observers. Then Kendall's coefficient of concordance $\widetilde{W}$ for this group of observers is given by

$$\widetilde{W}(A_1, \ldots, A_k) = \frac{6(n-1)}{n(n+1)} \sum_{j=1}^{n} \left[\left(\mu_A\left(x_j\right) - \frac{1}{2} \right)^2 + \left(\nu_A\left(x_j\right) - \frac{1}{2} \right)^2 \right]. \tag{22}$$

One may ask whether $\widetilde{W}$ satisfies the usual requirements of good coefficient of concordance. Moreover, is it really good generalization of the classical Kendall's coefficient of concordance, i.e. whether (22) reduces to (3) if all objects are univocally ranked by all observers $A_1, \ldots, A_k$? Basic properties of $\widetilde{W}$ are given in the following propositions that could be proved directly from the discussion given above.

Proposition 1. *Let $A_1, \ldots, A_k \in IFS(X)$ denote IF-subsets of the universe of discourse $X = \{x_1, \ldots, x_n\}$ which correspond to nondegenerated orderings expressed by k observers. Then $\widetilde{W}$ is commutative*, *i.e. for any permutation $A_{i_1}, \ldots, A_{i_k}$* of *$A_1, \ldots, A_k$ we have*

$$\widetilde{W}(A_1, \ldots, A_k) = \widetilde{W}(A_{i_1}, \ldots, A_{i_k}).$$

Proposition 2. *The generalized Kendall's coefficient of concordance $\widetilde{W}$* (22) *is invariant under all order-preserving transformations*.

Proposition 3. *Let $A_1, \ldots, A_k \in IFS(X)$ denote IF-subsets of the universe of discourse $X = \{x_1, \ldots, x_n\}$ which correspond to nondegenerated orderings expressed by k observers. Then we have*

$$0 \leqslant \widetilde{W}(A_1, \ldots, A_k) \leqslant 1.$$

Moreover, $\widetilde{W}(A_1, \ldots, A_n) = 1$ *if and only if there is a perfect agreement within the whole group of observers*.

From the discussion given above we may also conclude that as $\widetilde{W}$ increases, the set of ranks attributed to each object must become more and more similar. And conversely, if ranks given to objects by different observers vary then $\widetilde{W}$ decreases, because then (21) becomes smaller.

It is worth noting that in the case of more than two observers, discussed in this paper, there is no such thing as perfect disagreement between rankings. That is why Kendall's coefficient of concordance and its generalized version range from 0 to 1 (contrary to situations with two observers only, where correlation coefficients range from -1 to 1).

One can also prove that (22) is a natural and proper generalization of the classical statistical tool (3). Namely

Proposition 4. *If all n objects are univocally ranked* (*i.e. without ties*) *by all k observers then the generalized Kendall's coefficient of concordance $\widetilde{W}$* (22) *is equivalent to the classical Kendall's coefficient of concordance* W (3).

Example 1 (*continuation*). First of all let us construct IF-sets describing John's, Tom's, Susan's and Helen's preferences. We denote them by A_1, A_2, A_3 and A_4, respectively. According to (10)–(11) we get:

$$A_1 = \left\{ \left\langle Ameli, \tfrac{6}{7}, \tfrac{1}{7} \right\rangle, \left\langle Dogville, \tfrac{4}{7}, \tfrac{3}{7} \right\rangle, \langle Leon, 1, 0 \rangle, \right.$$
$$\left\langle Matrix, \tfrac{1}{7}, \tfrac{6}{7} \right\rangle, \left\langle Requiem\ for\ a\ dream, \tfrac{2}{7}, \tfrac{5}{7} \right\rangle,$$
$$\left. \left\langle Stigmata, \tfrac{3}{7}, \tfrac{4}{7} \right\rangle, \left\langle Talk\ to\ her, \tfrac{5}{7}, \tfrac{2}{7} \right\rangle, \langle Titanic, 0, 1 \rangle \right\},$$

$$A_2 = \left\{ \left\langle Ameli, \tfrac{5}{7}, \tfrac{1}{7} \right\rangle, \left\langle Dogville, \tfrac{4}{7}, \tfrac{2}{7} \right\rangle, \left\langle Leon, \tfrac{6}{7}, 0 \right\rangle, \right.$$
$$\left\langle Matrix, \tfrac{1}{7}, \tfrac{5}{7} \right\rangle, \left\langle Requiem\ for\ a\ dream, \tfrac{2}{7}, \tfrac{4}{7} \right\rangle,$$
$$\left. \left\langle Stigmata, \tfrac{3}{7}, \tfrac{3}{7} \right\rangle, \langle Talk\ to\ her, 0, 0 \rangle, \left\langle Titanic, 0, \tfrac{6}{7} \right\rangle \right\},$$

$$A_3 = \Big\{ \langle Ameli, 1, 0\rangle, \Big\langle Dogville, \tfrac{5}{7}, \tfrac{2}{7}\Big\rangle, \langle Leon, \tfrac{2}{7}, \tfrac{4}{7}\rangle, \langle Matrix, 0, 1\rangle, \langle Requiem\ for\ a\ dream, \tfrac{2}{7}, \tfrac{4}{7}\rangle, \langle Stigmata, \tfrac{4}{7}, \tfrac{3}{7}\rangle, \langle Talk\ to\ her, \tfrac{6}{7}, \tfrac{1}{7}\rangle, \langle Titanic, \tfrac{1}{7}, \tfrac{6}{7}\rangle \Big\},$$

$$A_4 = \Big\{ \Big\langle Ameli, \tfrac{5}{7}, \tfrac{1}{7}\Big\rangle, \langle Dogville, \tfrac{3}{7}, \tfrac{3}{7}\rangle, \langle Leon, \tfrac{4}{7}, \tfrac{2}{7}\rangle, \langle Matrix, 0, \tfrac{6}{7}\rangle, \langle Requiem\ for\ a\ dream, \tfrac{2}{7}, \tfrac{4}{7}\rangle, \langle Stigmata, 0, 0\rangle, \langle Talk\ to\ her, \tfrac{6}{7}, 0\rangle, \Big\langle Titanic, \tfrac{1}{7}, \tfrac{5}{7}\Big\rangle \Big\}.$$

In this example we have four observers (i.e. $k = 4$) and eight objects (i.e. $n = 8$). By (22) we get

$$\widetilde{W}(A_1, A_2, A_3, A_4) = 0.7485.$$

As it is seen there is no perfect agreement among our four observers, however the association is quite strong.

Example 2 (*continuation*). Using formulae (10) and (11) we may construct easily IF-sets $A_1, \ldots, A_9$ corresponding to rankings given in matrix (4) which express preferences of the nine observers. Thus we have

$$A_1 = \Big\{ \langle Ameli, 1, 0\rangle, \langle Dogville, \tfrac{6}{7}, \tfrac{1}{7}\rangle, \Big\langle Leon, \tfrac{5}{7}, \tfrac{2}{7}\Big\rangle, \langle Matrix, \tfrac{4}{7}, \tfrac{3}{7}\rangle, \langle Requiem\ for\ a\ dream, \tfrac{3}{7}, \tfrac{4}{7}\rangle, \Big\langle Stigmata, \tfrac{2}{7}, \tfrac{5}{7}\Big\rangle, \langle Talk\ to\ her, \tfrac{1}{7}, \tfrac{6}{7}\rangle, \langle Titanic, 0, 1\rangle \Big\},$$

$$A_2 = \Big\{ \langle Ameli, 0, 0\rangle, \langle Dogville, \tfrac{6}{7}, 0\rangle, \Big\langle Leon, \tfrac{5}{7}, \tfrac{1}{7}\Big\rangle, \langle Matrix, \tfrac{4}{7}, \tfrac{2}{7}\rangle, \langle Requiem\ for\ a\ dream, \tfrac{3}{7}, \tfrac{3}{7}\rangle, \langle Stigmata, \tfrac{2}{7}, \tfrac{4}{7}\rangle, \Big\langle Talk\ to\ her, \tfrac{1}{7}, \tfrac{5}{7}\Big\rangle, \langle Titanic, 0, \tfrac{6}{7}\rangle \Big\},$$

$$A_3 = \Big\{ \langle Ameli, \tfrac{6}{7}, 0\rangle, \langle Dogville, 0, 0\rangle, \Big\langle Leon, \tfrac{5}{7}, \tfrac{1}{7}\Big\rangle, \langle Matrix, \tfrac{4}{7}, \tfrac{2}{7}\rangle, \langle Requiem\ for\ a\ dream, \tfrac{3}{7}, \tfrac{3}{7}\rangle, \langle Stigmata, \tfrac{2}{7}, \tfrac{4}{7}\rangle, \Big\langle Talk\ to\ her, \tfrac{1}{7}, \tfrac{5}{7}\Big\rangle, \langle Titanic, 0, \tfrac{6}{7}\rangle \Big\},$$

$$\vdots$$

$$A_9 = \Big\{ \langle Ameli, \tfrac{6}{7}, 0\rangle, \Big\langle Dogville, \tfrac{5}{7}, \tfrac{1}{7}\Big\rangle, \langle Leon, \tfrac{4}{7}, \tfrac{2}{7}\rangle, \langle Matrix, \tfrac{3}{7}, \tfrac{3}{7}\rangle, \langle Requiem\ for\ a\ dream, \tfrac{2}{7}, \tfrac{4}{7}\rangle, \Big\langle Stigmata, \tfrac{1}{7}, \tfrac{5}{7}\Big\rangle, \langle Talk\ to\ her, 0, \tfrac{6}{7}\rangle, \langle Titanic, 0, 0\rangle \Big\}.$$

Then by (22) we get

$$\widetilde{W}(A_1, \ldots, A_9) = 0.72,$$

which confirms our suppositions that there is an association between these nine observers.

6. Conclusions

In the paper we have proposed how to generalize the well-known Kendall's coefficient of concordance to situations in which not all elements could be ordered. This coefficient possesses all basic requirements for such a measure of concordance. It seems that our coefficient might be useful in statistics, data mining and other situations where missing or noncomparable information occurs so often. In the suggested generalization we have assumed a nondegenerated set of rankings, which means that all objects under study have been univocally ranked by at least one of the observers. Thus further studies on the coefficient of concordance that would omit that assumption are still required.

References

Atanassov, K., 1986. Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20, 87–96.

Atanassov, K., 1999. Intuitionistic Fuzzy Sets: Theory and Applications. Physica-Verlag, Wurzburg.

Denœux, T., Masson, M.H., Hébert, P.A., 2005. Nonparametric rank-based statistics and significance tests for fuzzy data. Fuzzy Sets and Systems 153, 1–28.

Dubois, D., Prade, H., 1980. Fuzzy Sets and Systems: Theory and Applications. Academic Press, New York.

Ferguson, G.A., Takane, Y., 1989. Statistical Analysis in Psychology and Education. McGraw-Hill, New York.

Gibbons, J.D., Chakraborti, S., 2003. Nonparametric Statistical Inference. Marcel Dekker, New York.

Grzegorzewski, P., 2004a. Between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. Fuzzy Sets and Systems 148, 319–328.

Grzegorzewski, P., 2004b. On measuring association between preference systems. Proceedings of the 2004 IEEE International Conference on Fuzzy Systems, Budapest, Hungary, July 25–29, 2004, pp. 133–137.

Grzegorzewski, P., 2004c. The generalized Spearman's rank correlation coefficient. Proceedings of the Tenth International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, Perugia, Italy, July 4–9, 2004, pp. 1413–1418.

Hébert, P.A., Masson, M.H., Denœux, T., 2003. Fuzzy rank correlation between fuzzy numbers. In: Proceedings of the Tenth IFSA World Congress—IFSA 2003, Istanbul, Turkey, June 29—July 2, 2003, pp. 224–227.

Roy, B., Słowinski, R., 1993. Criterion of distance between technical programing and socio-economic priority. Oper. Res. 27, 45–60.

Zadeh, L.A., 1965. Fuzzy sets. Inform. and Control 8, 338–353.