MAT217 HW 7
DUE TUES. APR. 2, 2013

1. Let V be an $\mathbb{F}$-vector space of dimension n and let f be a k-linear alternating function on V with $k > n$. Show that f is identically zero.

2. Suppose that $A \in M_{n,n}(\mathbb{F})$ is *upper-triangular*; that is, $a_{i,j} = 0$ if $i > j$. Show that $\det A = a_{1,1}a_{2,2} \cdots a_{n,n}$. (Don't use the next exercise though!)

3. This exercise is a generalization of the previous one to block upper-triangular matrices. For $n \geq 2$ we say that $M \in M_{n,n}(\mathbb{F})$ is *block upper-triangular* if there exists k with $1 \leq k \leq n-1$ and matrices $A \in M_{k,k}(\mathbb{F}), B \in M_{k,n-k}(\mathbb{F})$ and $C \in M_{n-k,n-k}(\mathbb{F})$ such that M has the form
$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} .$$
That is, the elements of M are given by
$$M_{i,j} = \begin{cases} A_{i,j} & 1 \leq i \leq k,\ 1 \leq j \leq k \\ B_{i,j-k} & 1 \leq i \leq k,\ k < j \leq n \\ 0 & k < i \leq n,\ 1 \leq j \leq k \\ C_{i-k,j-k} & k < i \leq n,\ k < j \leq n \end{cases} .$$
We will show in this exercise that
$$\det M = \det A \cdot \det C .$$
   (a) Show that if $\det C = 0$ then the above formula holds.
   (b) Suppose that $\det C \neq 0$ and define a function $\phi : M_{k,k}(\mathbb{F}) \to \mathbb{F}$ by
$$\phi(\hat{A}) = [\det C]^{-1} \det \begin{pmatrix} \hat{A} & B \\ 0 & C \end{pmatrix} .$$
   That is, $\phi(\hat{A})$ is a scalar multiple of the determinant of the block upper-triangular matrix we get when we replace A by $\hat{A}$ and keep B and C fixed.
      i. Show that ϕ is k-linear as a function of the columns of $\hat{A}$.
      ii. Show that ϕ is alternating and satisfies $\phi(I_k) = 1$, where I_k is the $k \times k$ identity matrix.
      iii. Conclude that the above formula holds when $\det C \neq 0$.

4. Let $a_0, \ldots, a_n$ be distinct complex numbers. Write $M_n(a_0, \ldots, a_n)$ for the *Vandermonde* matrix
$$\begin{pmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ & & & \cdots & \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{pmatrix} .$$

The goal of this exercise is to prove the Vandermonde determinant formula

$$\det M_n(a_0, \ldots, a_n) = \prod_{0 \le i < j \le n} (a_j - a_i) \ .$$

We will argue by induction on n.

(a) Show that if $n = 2$ then the Vandermonde formula holds.

(b) Now suppose that $k \ge 2$ and that the formula holds for all $2 \le n \le k$. Show that it holds for $n = k + 1$ by completing the following outline.

   i. Define the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \det M_n(z, a_1, \ldots, a_n)$. Show that f is a polynomial of degree at most n.

   ii. Find all the zeros of f.
   Hint. Recall what was proved on a past homework: if a polynomial of degree n has at least $n + 1$ zeros then it must be identically zero.

   iii. Show that the coefficient of z^n is $(-1)^n \det M_{n-1}(a_1, \ldots, a_n)$.

   iv. Show that the Vandermonde formula holds for $n = k + 1$, completing the proof.

5. Show that if $A \in M_{n,n}(\mathbb{F})$ then $\det A = \det A^t$, the determinant of the transpose of A.

6. Let $A \in M_{7,7}(\mathbb{C})$ be anti-symmetric; that is, $A = -A^t$. What is $\det A$?

7. A field $\mathbb{F}$ is called *algebraically closed* if every $p \in \mathbb{F}[x]$ with $\deg(p) \ge 1$ has a zero in $\mathbb{F}$. Prove that if $\mathbb{F}$ is algebraically closed then for any nonzero $p \in \mathbb{F}[x]$, we can find $a, \lambda_1, \ldots, \lambda_k \in \mathbb{F}$ and natural numbers $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = \deg(p)$ such that

$$p(x) = a(x - \lambda_1)^{n_1} \cdots (x - \lambda_k)^{n_k} \ .$$

Here we say that $\lambda_1, \ldots, \lambda_k$ are the roots of p and $n_1, \ldots, n_k$ are their multiplicities.

Hint. Use induction on the degree of p.

8. Let $\mathbb{F}$ be algebraically closed. Show that for nonzero $p, q \in \mathbb{F}[x]$, the greatest common divisor of p and q is 1 if and only if p and q have no common root. Is this true for $\mathbb{F} = \mathbb{R}$?

9. Let $T : V \to V$ be linear and B a finite basis for V. We define

$$\det T = \det [T]_B^B \ .$$

(a) Show that the above definition does not depend on the choice of B.

(b) Show that if f is any nonzero n-linear alternating function on V then

$$\det T = \frac{f(T(v_1), \ldots, T(v_n))}{f(v_1, \ldots, v_n)} \ ,$$

where we have written $B = \{v_1, \ldots, v_n\}$. (This is an alternate definition of $\det T$.)