Lecture 5

We saw last time that we can add linear transformations and multiply them by scalars. These are just two ways to generate new linear transformations. Another obvious one is composition.

Proposition 0.1. *Let $T : V \to W$ and $U : W \to Z$ be linear (with all spaces over the same field $\mathbb{F}$). Then the composition $U \circ T$ is a linear transformation from V to Z.*

Proof. Let $v_1, v_2 \in V$ and $c \in \mathbb{F}$. Then

$$\begin{aligned}(U \circ T)(cv_1 + v_2) &= U(T(cv_1 + v_2)) = U(cT(v_1) + T(v_2)) \\ &= cU(T(v_1)) + U(T(v_2)) = c(U \circ T)(v_1) + (U \circ T)(v_2) \ .\end{aligned}$$

□

Recall that each $T : \mathbb{C} \to \mathbb{C}$ that is linear is completely determined by its value at 1. Note that $\{1\}$ is a basis. This fact holds true for all linear transformations and is one of the most important theorems of the course: in the words of Conway, each linear transformation is completely determined by its values on a basis, and any values will do!

Theorem 0.2 (The slogan)**.** *Let V and W be vector spaces over $\mathbb{F}$. If $\{v_1, \ldots, v_n\}$ is a basis for V and $w_1, \ldots, w_n$ are any vectors in W (with possible duplicates) then there is exactly one $T \in L(V, W)$ such that $T(v_i) = w_i$ for all $i = 1, \ldots, n$.*

Proof. This is an existence and uniqueness statement, so let's first prove uniqueness. Suppose that $T, U \in L(V, W)$ both map v_i to w_i for all i. Then write an arbitrary $v \in V$ uniquely as $v = a_1v_1 + \cdots + a_nv_n$. We have

$$\begin{aligned}T(v) = T(a_1v_1 + \cdots + a_nv_n) &= a_1T(v_1) + \cdots + a_nT(v_n) = a_1w_1 + \cdots + a_nw_n \\ &= a_1U(v_1) + \cdots + a_nU(v_n) = U(a_1v_1 + \cdots + a_nv_n) = U(v) \ .\end{aligned}$$

To prove existence we must construct one such linear map. Each $v \in V$ can be written uniquely as $v = a_1v_1 + \cdots + a_nv_n$, so define $T : V \to W$ by

$$T(v) = a_1w_1 + \cdots + a_nw_n \ .$$

The fact that T is a function (that is, for each $v \in V$ there is exactly one $w \in W$ such that $T(v) = w$) follows from uniqueness of the representation of v in terms of the basis. So we must show linearity. If $v, v' \in V$, write $v = a_1v_1 + \cdots + a_nv_n$ and $v' = b_1v_1 + \cdots + b_nv_n$. We have for $c \in \mathbb{F}$,

$$\begin{aligned}T(cv + v') &= T((ca_1 + b_1)v_1 + \cdots + (ca_n + b_n)v_n) \\ &= (ca_1 + b_1)w_1 + \cdots + (ca_n + b_n)w_n \\ &= c(a_1w_1 + \cdots + a_nw_n) + (b_1w_1 + \cdots + b_nw_n) \\ &= cT(v) + T(v') \ .\end{aligned}$$

□

Next we define two very important subspaces that are related to a linear transformation T.

Definition 0.3. *Let $T : V \to W$ be linear. The nullspace, or kernel, of T is the set $N(T) \subset V$ defined by*

$$N(T) = \{v \in V : T(v) = \vec{0}\} \ .$$

The range, or image, of T, is the set $R(T) \subset W$ defined by

$$R(T) = \{w \in W : T(v) = w \text{ for some } v \in V\} \ .$$

In the definition of $N(T)$ above, $\vec{0}$ is the zero vector in the space W.

Proposition 0.4. *Let $T : V \to W$ be linear. Then $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.*

Proof. First $N(T)$ is nonempty, since each linear transformation must map $\vec{0}$ to $\vec{0}$: $T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0}$. If $v_1, v_2 \in N(T)$ and $c \in \mathbb{F}$,

$$T(cv_1 + v_2) = cT(v_1) + T(v_2) = c\vec{0} + \vec{0} = \vec{0} \ ,$$

so $cv_1 + v_2 \in N(T)$, showing that $N(T)$ is a subspace of V. For $R(T)$, it is also non-empty, since $\vec{0}$ is mapped to by $\vec{0}$. If $w_1, w_2 \in R(T)$ and $c \in \mathbb{F}$, choose $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Then

$$cw_1 + w_2 = cT(v_1) + T(v_2) = T(cv_1 + v_2) \ ,$$

so $cw_1 + w_2$ is mapped to by $cv_1 + v_2$, a vector in V and we are done. $\square$

In the finite-dimensional case, the dimensions of these spaces are so important they get their own names: the *rank* of T is the dimension of $R(T)$ and the *nullity* of T is the dimension of $N(T)$. The next theorem relates these dimensions to each other.

Theorem 0.5 (Rank-nullity)**.** *Let $T : V \to W$ be linear and $dim(V) < \infty$. Then*

$$rank(T) + nullity(T) = dim(V) \ .$$

Proof. In a way, this theorem is best proved using quotient spaces, and you will do this in the homework. We will prove it the more standard way, by counting and using bases. Let $\{v_1, \ldots, v_k\}$ be a basis for the nullspace of T and extend it to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. We claim that $T(v_{k+1}), \ldots, T(v_n)$ are distinct and form a basis for $R(T)$; this will complete the proof. If $T(v_i) = T(v_j)$ for some $i, j \in \{k+1, \ldots, n\}$, we then have $T(v_i - v_j) = \vec{0}$, implying that $v_i - v_j \in N(T)$. But we have a basis for $N(T)$: we can write

$$v_i - v_j = a_1 v_1 + \cdots + a_k v_k$$

and subtracting $v_i - v_j$ to the other side, we have a linear combination of elements of a basis equal to zero with some nonzero coefficients, a contradiction.

Now we show $B = \{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for $R(T)$. They are clearly contained in the range, so $\mathrm{Span}(B) \subset R(T)$. Conversely, if $w \in R(T)$ we can write $w = T(v)$ for some $v \in V$ and using the basis, find coefficients such that b_i such that

$$w = T(v) = T(b_1 v_1 + \ldots + b_n v_n)\ .$$

Expanding the inside, we get $b_1 T(v_1) + \cdots + b_n T(v_n)$. The first k vectors are zero, since $v_1, \ldots, v_k \in N(T)$, so

$$w = b_{k+1} T(v_{k+1}) + \cdots + b_n T(v_n)\ ,$$

proving $w \in \mathrm{Span}(B)$ and therefore B spans $R(T)$.

For linear independence, let $b_{k+1} T(v_{k+1}) + \cdots + b_n T(v_n) = \vec{0}$. Then

$$\vec{0} = T(b_{k+1} v_{k+1} + \cdots + b_n v_n)\ ,$$

so $b_{k+1} v_{k+1} + \cdots + b_n v_n \in N(T)$. As before, we can then write these vectors in terms of $v_1, \ldots, v_k$, use linear independence of $\{v_1, \ldots, v_n\}$ to get $b_i = 0$ for all i. □

One reason the range and nullspace are important is that they tell us when a transformation is one-to-one (injective) or onto (surjective). Recall these definitions:

Definition 0.6. *If X and Y are sets and $f : X \to Y$ is a function then we say that f is one-to-one (injective) if f maps distinct points to distinct points; that is, if $x_1, x_2 \in X$ with $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. We say that f is onto (surjective) if each point of Y is mapped to by some x; that is, for each $y \in Y$ there exists $x \in X$ such that $f(x) = y$.*

Proposition 0.7. *Let $T : V \to W$ be linear. Then*

1. *T is injective if and only if $N(T) = \{\vec{0}\}$.*

2. *T is surjective if and only if $R(T) = W$.*

Proof. The second is just the definition of surjective, so we prove the first. Suppose that T is injective and let $v \in N(T)$. Then $T(v) = \vec{0} = T(\vec{0})$, but because T injective, $v = \vec{0}$, proving that $N(T) \subset \{\vec{0}\}$. As $N(T)$ is a subspace, we have $\{\vec{0}\} \subset N(T)$, giving equality.

Conversely suppose that $N(T) = \{\vec{0}\}$; we will prove that T is injective. So assume that $T(v_1) = T(v_2)$. By linearity, $T(v_1 - v_2) = \vec{0}$, so $v_1 - v_2 \in N(T)$. But he only vector in $N(T)$ is the zero vector, so $v_1 - v_2 = \vec{0}$, giving $v_1 = v_2$ and T is injective. □

In the previous proposition, the second part holds for all functions T, regardless of whether they are linear. The first, however, need not be true if T is not linear. (Think of an example!)

We can give an alternative characterization of one-to-one and onto:

Proposition 0.8. *Let $T : V \to W$ be linear.*

1. *T is injective if and only if it maps linearly independent sets of V to linearly independent sets of W.*

2. *T is surjective if and only if it maps spanning sets of V to spanning sets of W.*

3. *T is bijective if and only if it maps bases of V to bases of W.*

Proof. The third part follows from the first two. For the first, assume that T is injective and let $S \subset V$ be linearly independent. We will show that $T(S) = \{T(v) : v \in S\}$ is linearly independent. So let
$$a_1 T(v_1) + \cdots + a_n T(v_n) = \vec{0} \ .$$
This implies that $T(a_1 v_1 + \cdots + a_n v_n) = \vec{0}$, implying that $a_1 v_1 + \cdots + a_n v_n = \vec{0}$ by injectivity. But this is a linear combination of vectors in S, a linearly independent set, giving $a_i = 0$ for all i. Thus $T(S)$ is linearly independent.

Conversely suppose that T maps linearly independent sets to linearly independent sets and let $v \in N(T)$. If $v \neq \vec{0}$ then $\{v\}$ is linearly independent, so $\{T(v)\}$ is linearly independent. But if $T(v) = \vec{0}$ this is impossible, since $\{\vec{0}\}$ is linearly dependent. Thus $v \neq \vec{0}$ and $N(T) = \{\vec{0}\}$, implying T is injective.

For item two, suppose that T is surjective and let S be a spanning set for V. Then if $w \in W$ we can find $v \in V$ such that $T(v) = w$ and a linear combination of vectors of S equal to v: $v = a_1 v_1 + \cdots + a_n v_n$ for $v_i \in S$. Therefore
$$w = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n) \ ,$$
meaning that we have $w \in \text{Span}(T(S))$, so $T(S)$ spans W. Conversely if T maps spanning sets to spanning sets, then $T(V) = R(T)$ must span W. But since $R(T)$ is a subspace of W, this means $R(T) = W$ and T is onto. $\square$