MAT217 HW 4
Due Tues. Mar. 5, 2013

1. Read Section 1.5 in the Hoffman-Kunze handout and do exercises 4, 6.

2. Let $A \in M_{n,n}(\mathbb{F})$ be invertible and B be a basis for an n-dimensional $\mathbb{F}$-vector space V. Show there is an isomorphism $T : V \to V$ such that $[T]_B^B = A$.

3. (a) Let $A \in M_{n,n}(\mathbb{F})$ be invertible. Show that the inverse matrix is unique.

   (b) Let V be an n-dimensional $\mathbb{F}$-vector space and $T : V \to V$ and $U : V \to V$ be linear that satisfy
   $$(U \circ T)(v) = v \text{ for all } v \in V \ .$$
   Show that $(T \circ U)(v) = v$ for all $v \in V$.

   (c) Let $A, B \in M_{n,n}(\mathbb{F})$ satisfy $AB = I$. Show that $BA = I$.

4. If $A \in M_{m,n}(\mathbb{F})$ we define the *column rank* of A as the dimension of the span of the n different columns of A in $\mathbb{F}^m$. Similarly, we define the *row rank* of A as the dimension of the rows of A in $\mathbb{F}^n$.

   (a) Show that the column rank of A is equal to the rank of the linear transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$ defined by $L_A(v) = A \cdot v$, matrix multiplication of the column vector v by the matrix A on the left.

   (b) Use exercise 7 on the previous homework to show that if $P \in M_{n,n}(\mathbb{F})$ and $Q \in M_{m,m}(\mathbb{F})$ are both invertible then the column rank of A equals the column rank of QAP.

   (c) Show that the row rank of A is equal to the rank of the linear transformation $R_A : \mathbb{F}^m \to \mathbb{F}^n$ defined by $R_A(v) = v \cdot A$, viewing v as a row vector and multiplying by A on the right.

   (d) Show that if $P \in M_{n,n}(\mathbb{F})$ and $Q \in M_{m,m}(\mathbb{F})$ are both invertible then the row rank of A equals the row rank of QAP.

   (e) Use exercise 9 on the previous homework and parts (a) - (d) above to show that the row rank of A equals the column rank of A.

5. Given $m \in \mathbb{R}$ define the line
   $$L_m = \{(x, y) \in \mathbb{R}^2 : y = mx\} \ .$$

   (a) Let T_m be the function which maps a point in $\mathbb{R}^2$ to its closest point in L_m. Find the matrix of T_m relative to the standard basis.

   (b) Let R_m be the function which maps a point in $\mathbb{R}^2$ to the reflection of this point about the line L_m. Find the matrix of T_m relative to the standard basis.

   Hint for both. First find the matrix relative to a carefully chosen basis.

6. (From Hoffman-Kunze) Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbb{C}^3$ defined by $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$ and $\alpha_3 = (2, 2, 0)$. Find the dual basis B^*.

7. (From Hoffman-Kunze) Let W be the subspace of $\mathbb{R}^5$ spanned by the vectors $(1, 2, 1, 0, 0)$, $(0, 2, 3, 3, 1)$ and $(1, 4, 6, 4, 1)$. Find a basis for $W^\perp$.

8. (From Hoffman-Kunze) Prove that on the space $M_{n,n}(\mathbb{F})$, the trace function Tr is a linear functional. Show that, conversely, if some linear functional g on this space satisfies $g(AB) = g(BA)$ then g is a scalar multiple of the trace function.

9. Let V be an $\mathbb{F}$-vector space and C a basis of V^*. Show there is a basis B of V such that $B^* = C$.

10. Let V be an $\mathbb{F}$-vector space and $S' \subset V^*$. Define the lower annihilator

$$^\perp S' = \{v \in V : f(v) = 0 \text{ for all } f \in S'\} \ .$$

Show the following:

(a) $^\perp S' = {}^\perp\text{Span}(S')$.

(b) Assume that V is finite-dimensional and $U' \subset V^*$ is a subspace. Let $\{f_1, \ldots, f_n\}$ be a basis for V^* such that $\{f_1, \ldots, f_k\}$ is a basis for U'. If $\{v_1, \ldots, v_n\}$ is a basis for V such that $v_i^* = f_i$ for all i (given by exercise 9) then show that $\{v_{k+1}, \ldots, v_n\}$ is a basis for $^\perp U'$. In particular, deduce that $\dim(U') + \dim(^\perp U') = \dim(V)$.