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A Copula-based Approach to Option Pricing and Risk Assessment

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Abstract: Copulas are useful tools to study the relationship between random variables. In financial applications, they can separate the marginal distributions from the dynamic dependence of asset prices. The marginal distributions may assume some univariate volatility models whereas the dynamic dependence can be time-varying and depends on some explanatory variables. In this paper, we consider applications of copulas in finance. First, we combine the risk-neutral representation and copula-based models to price multivariate exotic derivatives. Second, we show that copula-based models can be used to assess value at risk of multiple assets. We demonstrate the applications using daily log returns of two market indices and compare the proposed method with others available in the literature.

Key words: Correlation, copula, derivative pricing, financial econometrics, value-at-risk.

1. Introduction

Most financial portfolios consist of multiple assets. To understand the time evolution of the value of a financial portfolio thus requires multivariate models that can handle the co-movements of the underlying price processes. The most commonly used statistical distribution for analyzing multiple asset returns is the multivariate normal distribution. However, a multivariate normal distribution restricts the association between margins to be linear as measured by the covariance. The real association between two asset returns is often much more complicated. In recent years, copulas are often used as an alternative to measure the association between assets. The basic idea of copulas is to separate the dependence between variables from their marginal distributions. The dependence structure can be flexible, including linear, nonlinear, or only tail dependent. The marginal distributions can be easily dealt with using the univariate volatility models available in the literature.

In this paper, we apply copulas to model multivariate asset returns with the aim of pricing derivatives and assessing value at risk. In recent years, financial derivatives contingent on multiple underlying assets have attracted much interest among researchers and practitioners, e.g., a digital option on two equity indices, which is an option promising to pay a fixed amount if the two indices are above some pre-specified levels (strike prices). The traditional approach to pricing such derivatives typically employs multivariate geometric Brownian motions. The copula approach, on the other hand, provides increased flexibility in modeling the marginal distributions without sacrificing the ability to compute option price. The first goal of this paper is to extend the univariate risk-neutral pricing of Duan (1995) to the multivariate case under the copula framework.

For risk management, copula-based models provide a general framework to measure tail dependence of asset returns and, hence, to assess more accurately the risk of a portfolio. Indeed, since the price paths of component assets can be characterized under the copula approach, the variation of the portfolio can be measured accordingly. The second goal of this paper is to show that copula-based methods can gain insights into value-at-risk (VaR) and other risk measurements.

This paper is organized as follows. Section 2 gives a brief review of the key concepts of copula. Section 3 presents an empirical application and demonstrates how the copula approach can be used in modeling bivariate return processes. Section 4 derives option pricing based on a GARCH-Copula model whereas Section 5 uses the new approach to derive VaR and demonstrates its application in risk management. Section 6 concludes.

2. Review of Copula Concept

The name "copula" was chosen to emphasize how a copula "couples" a joint distribution to its marginal distributions. The concept applies to high dimensional processes, but for simplicity we focus on the bivariate case in this paper. In what follows we briefly review the basic properties of a copula. Interested readers are referred to Nelsen (1999).

Definition 2.1. [Nelsen(1999), p.8] A two-dimensional copula C is a real function defined on $[0,1] \times [0,1]$ with range [0,1]. Furthermore, for every element (u,v) in the domain, $C(u,0) = C(0,v) = 0$, $C(u,1) = u$, and $C(1,v) = v$. For every rectangle $[u_1,u_2] \times [v_1,v_2]$ in the domain such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$, $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,u_1) \geq 0$.

Theorem 2.1. [Sklar's theorem, Nelsen(1999), p.15] Let $F(x,y)$ be a joint cumulative distribution function with marginal cumulative distributions $F_1(x)$ and $F_2(y)$. There exists a copula C such that for all real (x,y), $F(x,y) = C(F_1(x),F_2(y))$. If both F_1 and F_2 are continuous, then the copula is unique;

otherwise, C is uniquely determined on (range of F_1)$\times$ (range of F_2). Conversely, if C is a copula and F_1 and F_2 are cumulative distribution functions, then $F(x, y)$ defined above is a joint cumulative distribution function with F_1 and F_2 as its margins.

To describe the dependence structure of financial time series, we need some measures of association. The most widely used measure is the Pearson coefficient, which measures the linear association between two variables. However, without the normality assumption, Pearson coefficient, ρ, may be problematic. Indeed, as shown in Frechet (1957), ρ may not be bounded by 1 in absolute value and the bounds differ for different distributions. Thus, ρ is not a suitable measure of dependence when nonlinear relationship is of main interest. Some nonparametric measures of dependence are needed. Two such measures of dependence are often used, namely Kendall's tau and Spearman's rho. See, for example, Gibbons(1988).

Definition 2.2. [Nelsen (1999), Theorems 5.1.3 and 5.1.6] If (X, Y) forms a continuous, 2-dimensional random variable with copula C, then

$$\text{Kendall's tau} = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1, \tag{2.1}$$

$$\text{Spearman's rho} = 12\int_0^1\int_0^1 C(u,v)dudv - 3. \tag{2.2}$$

It can be shown that Kendall's tau and Spearman's rho are bounded by 1 in absolute value. In what follows, we apply these two nonparametric measures to describe the association between variables.

3. Financial Application of Copula Models

In this section, we employ univariate GARCH models to describe the marginal distributions of two asset returns and use copulas to model the dependence between the two assets.

3.1 The data

We use the daily log returns of Taiwan weighted stock index (TAIEX) and the New York Stock Exchange composite price index (NYSE) from January 1, 2001 to December 31, 2003 to demonstrate the application of copulas in finance. The data set obtained from DataStream consists of 782 observations. Some descriptive statistics and the scatter plot between the two return series are given in Table 1 and Figure 1, respectively. As expected, the mean returns of the two indices are close to zero and the returns have kurtosis greater than 3. On the other hand, the scatter plot is not informative.

Table 1: Summary statistics of the daily log returns, in percentages, of the NYSE composite index and Taiwan stock weighted index from January 1, 2001 to December 31, 2003. Data are obtained from DataStream.

	Min	1Q	median	3Q	Max	Mean	S.D.	Skewness	Kurtosis
TAIEX	-5.95	-0.92	0	0.95	5.61	0.03	1.67	0.18	3.71
NYSE	-4.70	-0.69	0	0.59	5.18	-0.00	1.18	0.09	4.67

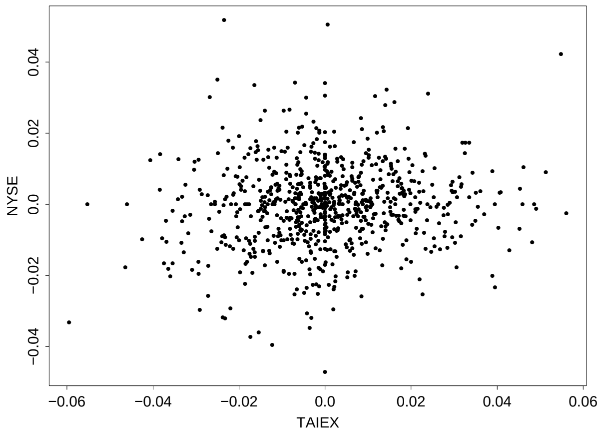

Figure 1: Scatter plot of the daily log returns of the NYSE index versus TAIEX index from 2001 to 2003.

3.2 An estimation procedure

We use the maximum likelihood (ML) method to estimate copula models for the two log-return series. For a single observation (x, y), the likelihood function under a copula model is

$$f(x, y) = f_1(x)f_2(y)C_{12}[F_1(x), F_2(y)], \tag{3.1}$$

where $f(x, y)$ is the joint density function of $F(x, y)$, $f_i(.)$ is the density function of $F_i(.)$, and C_{12} is defined as $\partial C(u, v)/\partial u \partial v$. Given the data, model parameters θ are estimated by maximizing the log likelihood function

$$\ell(\theta) = \sum_{t=1}^{n} \log(f_1(x_t)) + \log(f_2(y_t)) + \log\{C_{12}[F_1(x_t), F_2(y_t)]\} \tag{3.2}$$

Ideally, one would like to maximize the above likelihood function simultaneously over all parameters in the marginal distributions and the copula function. In practice, the joint estimation may be formidable, because the dimension of the parameters may be high and the dependence relation of the copula may involve a convoluted expression of the parameters. Therefore, a two-step estimation procedure is often used. The first step of the estimation is to estimate the GARCH models for the individual return series, i.e., the margins. In the second step, the estimated parameters of the margins are treated as known and used to evaluate the likelihood function in (3.2). Consequently, the objective function of the second step of estimation is

$$\sum_{t=1}^{n} \log\{C_{12}[\hat{F}_1(x_t), \hat{F}_2(y_t)]\}, \tag{3.3}$$

where the distribution functions $\hat{F}_1$ and $\hat{F}_2$ are evaluated using the estimated GARCH models. Patton (2006a) shows that such a two-step estimation procedure yields asymptotically efficient estimates. For simplicity, we adopt such a two-step estimation procedure in this paper.

3.3 Marginal models

Following the two-step estimation procedure, we employ the generalized autoregressive conditional heteroscedastic (GARCH) models for the individual return series of NYSE and TAIEX indices. A modeling procedure for univariate GARCH models can be found in Tsay (2005, ch. 3).

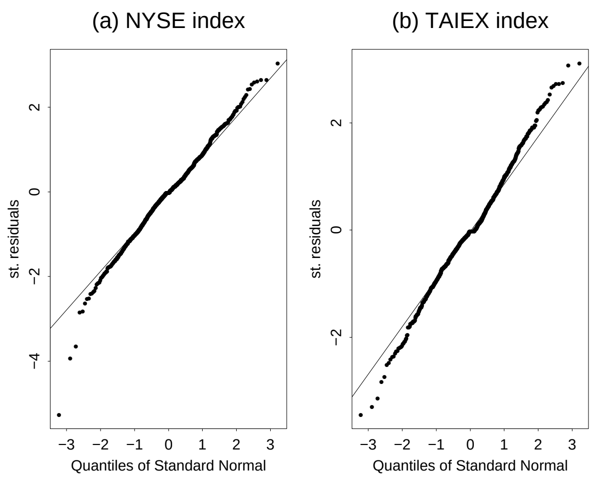

Figure 2: The normal quantile plots for the standardized residuals of a GARCH(1,1) model: (a) for the daily log returns of the NYSE index and (b) for the daily log returns of the TAIEX index from 2001 to 2003.

Table 2: Summary of parameter estimation for the marginal distributions of the daily log returns of the NYSE and TAIEX indices from 2001 to 2003. The numbers in parentheses are standard errors, and the asterisks indicates significance at the 5% level.

	$c \times 10^4$	$\alpha_0 \times 10^6$	$\alpha \times 10^2$	β	Innovation
NYSE	2.885	3.361	8.503	0.888	Gaussian
	(3.648)	(1.209)*	(1.809)*	(0.022)*	
TAIEX	3.895	0.094	2.823	0.971	Gaussian
	(5.634)	(1.072)	(0.845)*	(0.010)*	

For the daily log return series of the NYSE index, the autocorrelations and partial autocorrelations fail to show any significant serial dependence. Therefore, the mean equation for the return series is simply a constant. For the volatility equation, we choose a GARCH(1,1) model because the model has been widely used in the literature for daily asset returns. Consequently, the marginal model for the NYSE index returns is

$$r_t = c + a_t, \quad a_t = \sigma_t \varepsilon, \quad \varepsilon \sim N(0,1) \tag{3.4}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta \sigma_{t-1}^2, \tag{3.5}$$

where the parameter estimates are given in Table 2. The fitted model can be checked for adequacy in several ways; see, for instance, Zivot and Wang (2003) and Tsay (2005). In particular, the p-value of the Ljung-Box statistic $Q(12)$ is 0.7987 for the standardized residuals and 0.8148 for the squared standardized residuals. Thus, both the mean and volatility equations are adequate in describing the first two conditional moments of the data. In addition, the Lagrange multiplier test with 12 lags shows that no ARCH effects are left in the standardized residuals; the p-value of the test statistic is 0.7874. Finally, Figure 2(a) shows the normal quantile plot for the standardized residuals of the fitted GARCH(1,1) model. A straightline is imposed to aid interpretation. From the plot, except for some outlying data points, the Gaussian assumption is reasonable.

For the daily log returns of the TAIEX index, we also employ the same GARCH(1,1) model. Again, the parameter estimates are given in Table 2. The p-value of the Ljung-Box statistic $Q(12)$ is 0.746 for the standardized residuals, and 0.8331 for the squared standardized residuals. Thus, both the mean and volatility equations are adequate in describing the first two conditional moments of the data. Figure 2(b) shows the normal quantile plot for the standardized residuals of the fitted GARCH(1,1) model. In this particular case, the Gaussian assumption is not perfect, but acceptable. Indeed, the p-values of the Shapiro-Wilk test and the Jarque-Bera test for normality are 0.062 and 0.023, respectively. Note that strictly speaking, the normality assumption is not valid because there is a 7%

daily price limit on Taiwan Stock Exchange. However, the estimates shown in Table 2 are quasi-maximum likelihood estimates (QMLE), which are known to be consistent for GARCH models without the normality condition.

From Table 2, the fitted GARCH(1,1) model for the TAIEX index returns is essentially an IGARCH model because $\hat{\alpha} + \hat{\beta} \approx 1$ and $\hat{\alpha}_0 \approx 0$. While the IGARCH behavior is commonly seen in daily asset return series, it indicates that the series may contain some jumps in the volatility level. We intend to study the volatility jumps in future study.

3.4 Copula estimation

Empirical experience indicates that asset return series often exhibit certain pattern of co-movements. This is particularly so in recent years because the internet has greatly expedited the speed of information transmission and the economic globalization has increased the interdependence between financial markets. The extent of co-movement between asset returns, however, may be time-varying. To accommodate this characteristic of asset returns, the dependence parameters of a copula are often assumed to be functions of some time-dependent variables or explanatory variables.

Many copula functions are available in the literature. Following Rosenberg (1999), we choose the Plackett and Frank copulas because they enjoy some nice properties. First, the density functions of the Plackett and Frank copulas are flexible for the general marginal processes. Second, the association between the two margins of the copulas can be represented by a single parameter. Third, the two copulas are comprehensive. By varying the dependence parameter, both copulas are capable of covering a wide range of dependence between two processes. Table 3 summarizes the dependence structure of the two copulas.

Table 3: Dependence structure of Plackett and Frank copulas

Copula	Perfectly positive dependence	Perfectly negative dependence	Independence
Plackett	$\theta \to \infty$	$\theta \to 0$	$\theta = 1$
Frank	$\alpha \to \infty$	$\alpha \to -\infty$	$\alpha \to 0$

In what follows, we provide further details of Plackett and Frank copulas.

Plackett copula (with $\theta > 0$): Plackett copula is defined as

$$\begin{aligned} C(u,v|\theta) &= \frac{1}{2}(\theta-1)A - \sqrt{A^2 - 4uv(\theta-1)}\} \quad \text{if } \theta \neq 1, \\ &= uv \quad \text{if } \theta = 1, \end{aligned} \tag{3.6}$$

where $A = 1 + (\theta - 1)(u + v)$. Its second derivative can be shown as

$$C_{12}(u, v|\theta) = \frac{\theta[1 + (u - 2uv + v)(\theta - 1)]}{\{[1 + (\theta - 1)(u + v)]^2 - 4uv\theta(\theta - 1)\}^{3/2}}, \tag{3.7}$$

where θ is a positive parameter that characterizes the relationship between the two margins and is related to Spearman's rho as:

$$\begin{aligned} \text{Spearman's rho} &= \frac{\theta+1}{\theta-1} - \frac{2\theta}{(\theta-1)^2}\log(\theta) && \text{if } \theta \neq 1, \\ &= 0 && \text{if } \theta = 1. \end{aligned} \tag{3.8}$$

Many specifications are available to model the dependency parameter of a Plackett copula. To satisfy the constraint that θ is positive, structural forms are often imposed on $\log(\theta)$. We briefly review some of the specifications and estimation methods available in the literature. Gourieroux and Monfort (1992) and Rockinger and Jondeau (2005) decompose the unit square of the margins (hence, past observations) into a grid and entertain a parameter for each of the resulting regions. For instance, if the unit square is divided into 16 areas, their model becomes

$$\log(\theta_t)|I_{t-1} = \sum_{j=1}^{16} d_j I[(u_{t-1}, v_{t-1}) \in A_j], \tag{3.9}$$

where I_{t-1} denotes the information available at time $t-1$, A_j is the j-th area of the unit square and d_j is the unknown parameter associated with A_j.

Patton (2006b) uses autoregressive terms, consisting of lagged dependence parameter and lagged marginal outcomes, and some forcing variables to model the dependence parameters in Student-t and Joe-Clayton copulas. Finally, the dependence parameter can be a polynomial function of time such as

$$\log(\theta) = \sum_{j=1}^{J} d_j t^j, \tag{3.10}$$

where J is the order of the polynomial. In this paper, to stress that the dependence may vary with the volatility of marginal processes, we postulate that the dependence parameter can be written as

$$\log(\theta_t)|I_{t-1} = d_1 + d_2\sigma_t^{TAIEX} + d_3\sigma_t^{NYSE} + d_4\sqrt{\sigma_t^{TAIEX}\sigma_t^{NYSE}}, \tag{3.11}$$

where σ_t^j is the volatility of the j index return. Using the Plackett copula, we obtain the estimates [1.85, 215.04, 394.15, -702.15] of $[d_1, d_2, d_3, d_4]$ for the bivariate log return series of TAIEX and NYSE indices. The standard errors of the estimates are [0.75, 44.16, 30.30, 14.96].

Frank Copula Frank copula is a member of the Archimedean family and is defined as

$$C(u, v|\alpha) = \frac{1}{\alpha} \log \left[1 + \frac{(\exp(\alpha u) - 1)(\exp(\alpha v) - 1)}{\exp(\alpha) - 1} \right], \tag{3.12}$$

where α is a non-zero real number. The second derivative of the Frank copula is

$$C_{12}(u, v|\alpha) = \frac{\alpha \exp(\alpha u) \exp(\alpha v)}{[\exp(\alpha) - 1][1 + \frac{(\exp(\alpha u)-1)(\exp(\alpha v)-1)}{\exp(\alpha)-1}]^2}. \tag{3.13}$$

The dependence parameter α is related to Spearman's rho and Kendall's tau as (Frank, 1979)

$$\begin{aligned} \text{Spearman's rho} &= 1 - \frac{12}{\alpha}[D_2(-\alpha) - D_1(-\alpha)], & (3.14) \\ \text{Kendall's tau} &= 1 - \frac{4}{\alpha}[D_1(-\alpha) - 1], & (3.15) \end{aligned}$$

where D_K is the"Debye" function defined as $D_K(y) = \frac{K}{y^K} \int_0^y \frac{t^K}{\exp(t)-1} dt$. Similar to that of Plackett's copula, we employ the model

$$\alpha_t | I_{t-1} = d_1 + d_2 \sigma_t^{TAIEX} + d_3 \sigma_t^{NYSE} + d_4 \sqrt{\sigma_t^{TAIEX} \sigma_t^{NYSE}}, \tag{3.16}$$

for the dependence parameter. For the two return series considered, the estimates of $[d_1, d_2, d_3, d_4]$ are $[-3.4, -378.6, -700.5, 1249.8]$ for the Frank copula. The associated standard errors are [1.51, 91.86, 58.85, 27.76]. We remark that other explanatory variables such as the moving average of marginal volatilities or absolute log returns have been used to estimate the Plackett and Frank copulas. However, we did not find any significant improvement.

Figure 3 compares Spearman's rho and Kendall's tau for the two index return series under the Frank copula. As expected, the two measures of association exhibit similar patterns and show an increasing trend in recent years. Figure 4 shows Spearman's rho for the two index return series under the Plackett and Frank copulas. The dependence is rather similar, indicating that the measure is not sensitive to the choice of copulas. Most of the dependence measures are between 0 and 0.3.

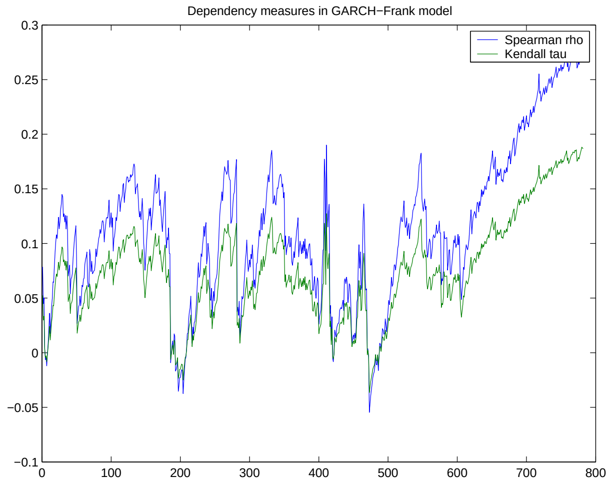

Figure 3: Dependence measures between the daily log returns of NYSE and TAIEX index under the Frank copula.

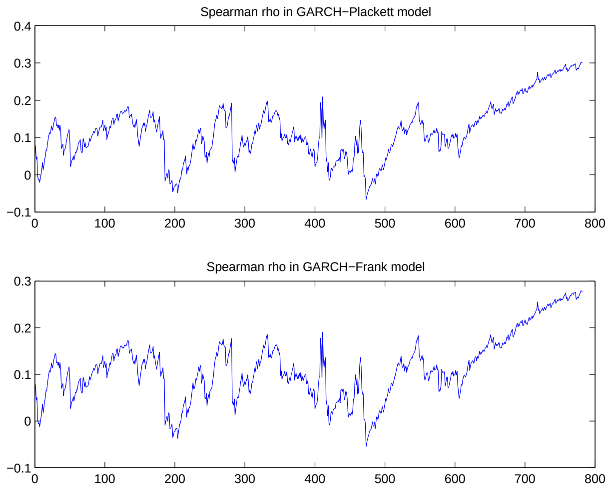

Figure 4: Spearman's rho between the daily log returns of the NYSE and TAIEX index under the Plackett and Frank copulas.

3.5 Comparison with bivariate GARCH models

For convenience, we refer to the model estimated in this paper as the GARCH-Copula model, which is different from the Copula-GARCH model of Jondeau and Rockinger (2002) because the latter uses Hansen's Student-t distribution as the innovations for the marginal GARCH models. Our goal here is to compare the GARCH-Copula model with the bivariate GARCH model commonly used in the literature.

Multivariate GARCH models are developed to model the cross-correlations between multiple asset returns. Several specifications are available, including the exponentially weighted covariance estimation, Diagonal VEC model, the BEKK model, and the dynamic correlation models. See Tsay (2005, Ch. 10) for descriptions. Among these models, the BEKK model of Engle and Kroner(1995) ensures that the resulting conditional covariance matrices are positive definite. We shall compare the GARCH-copula models with the BEKK models in modeling multivariate volatility.

Assume that the mean equation is zero so that $r_t = a_t = \Sigma_t^{1/2}\epsilon_t$, where $\{\epsilon_t\}$ is an *iid* sequence of random vectors with mean zero and identity covariance and Σ_t is the conditional covariance matrix of a_t given the past information at $t-1$. A 2-dimensional BEKK model of order (1,1) assumes the form

$$\Sigma_t = A_0 A_0' + A(a_{t-1}a_{t-1}')A' + B\Sigma_{t-1}B' \tag{3.17}$$

where Σ_t is the conditional covariance matrix of the bivariate process, A_0 is a lower triangular matrix, and A and B are square matrices.

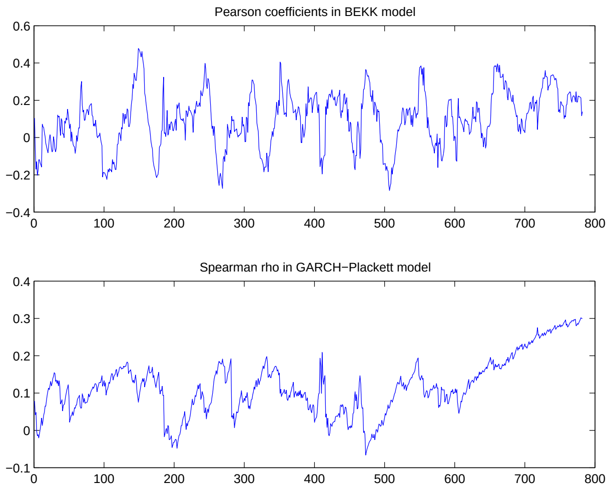

Figure 5: Dependence measures based on the BEKK(1,1) model and GARCH-Plackett model for the daily log return series of the NYSE and TAIEX indices.

The dependence measure of BEKK models is the Pearson coefficient, which is the commonly used linear correlation coefficient. Figure 5 compares the dependence measures of the BEKK and GARCH-Plackett models for the two log return series used in this paper. The figure shows clearly that there exist some differences in the dependence measure between the BEKK and GARCH-copula models. The correlation coefficients shown by the BEKK model are more volatile than the Spearman' rho of the GARCH-Plackett model. In addition, the BEKK model has 24% negative correlations whereas the GARCH-Plackett model has only 6% negative Spearman's rho. Thus, the BEKK model implies more negative association between the two return series.

In summary, there exist some fundamental differences between the BEKK and GARCH-Copula models. First, in the BEKK model, marginal distributions and the correlations between the two series are bounded together. In contrast, the margins and the association relationship are detachable under the GARCH-Copula model. Second, the correlation coefficients under the BEKK models measure the linear dependence between the two return series, but the dependence measures under GARCH-Copula models can be nonlinear.

4. Option Pricing with Copula-based Models

Partial differential equations and martingale pricing (Harrison and Kreps, 1979) are commonly used in option pricing. However, it is often difficult to derive a closed-form solution from a partial differential equation. Closed-form solutions like the Black-Scholes formula are rare and only applicable to vanilla European options. Closed-form solutions for options involving multiple underlying assets are in general not available. Therefore, the martingale pricing method is often used as an alternative. Martingale pricing is based on the risk-neutral measure and the derivative prices are often expressed in an expectation form, which can be solved by numerical simulation. In this paper, we adopt the martingale pricing.

Denote the n-variable risk-neutral density as $f^*(A_{1,t}, A_{2,t} \dots A_{n,t})$, where $A_{i,t}$ is the price of the ith underlying asset. The price of a path-independent option with $A_1, A_2, \dots$, and A_n as the underlying assets is

$$e^{-rT} E^*(g(A_{1,T}, A_{2,T} \dots A_{n,T})), \tag{4.1}$$

where g is the payoff function at maturity T and E^* is the expectation under the risk neutral probability. To price such multivariate contingent claims (MVCCs), several authors adopt the assumption that the underlying assets follow a multivariate geometric Brownian motion (MGBM). For example, Stulz (1982), Johnson (1987), Reiner (1992) and Shimko (1994) worked on continuous-time Brownian motions. Stapleton and Subrahmanyam (1984), Boyle(1988) and Rubinstein(1994) studied a discrete-time binomial tree model. However, the assumption of a lognormal dependence function is likely to generate biases in pricing. Rosenberg (2003) proposes a nonparametric method to price MVCCs. To estimate the marginal risk neutral density, he extends the Black-Scholes formula with the original constant volatility parameter replaced by a function of future price, exercise price, and option maturity. In addition, he uses nonparametric method to estimate the copula-based dependence function. A potential drawback of such an approach is that the methods require a large amount of trading data (futures or options) to perform the estimation.

In contrast to Rosenberg's approach, we employ GARCH models for the marginal processes and use conditional dependence to describe the co-movement of the underlying processes. GARCH models allow heterogeneous innovations and time-varying dependence to accommodate asymmetric correlations in the joint density. Because we work on discrete-time framework, it is more tractable and easier to price path-dependence MVCCs. In what follows, we demonstrate how to utilize copula methods to price bivariate exotic options.

Based on Sklar's Theorem, the joint density of the bivariate process (x, y) can be expressed via a copula function as

$$f(x,y) = f_1(x)f_2(y)C_{12}[F_1(x), F_2(y)], \tag{4.2}$$

where, as before, C_{12} is defined as $\partial C(u,v)/\partial u \partial v$ and f_i is the density function of F_i. Denote the joint risk-neutral density as

$$f^*(x,y) = f_1^*(x)f_2^*(y)C_{12}^*[F_1^*(x), F_2^*(y)]. \tag{4.3}$$

The task is then to find the risk neutral counterpart of the marginal process. In our case, the margins are GARCH processes. Duan (1995) develops a locally risk-neutral valuation relationship (LRNVR) for univariate GARCH processes. Let S_t be the asset price at date t and σ_t the conditional standard deviation of the log return series. The dynamic of the log return process is assumed to follow the model

$$\log(\frac{S_t}{S_{t-1}}) = r_f + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + a_t, \tag{4.4}$$

$$a_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0,1) \quad \text{under measure P (real world)}, \tag{4.5}$$

$$\sigma_t^2 = \alpha_0 + \alpha a_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{4.6}$$

where r_f is the riskfree interest rate and λ is the risk premium.

The LRNVR shows that, under measure Q (risk-neutral world), the dynamic becomes

$$\log(\frac{S_t}{S_{t-1}}) = r_f - \frac{1}{2}\sigma_t^2 + \xi_t, \tag{4.7}$$

$$\xi_t = \sigma_t \varepsilon_t^*, \quad \varepsilon_t^* \sim N(0,1) \quad \text{under measure Q (risk-neutral world)}, \tag{4.8}$$

$$\sigma_t^2 = \alpha_0 + \alpha(\varepsilon_{t-1}^* - \lambda)\sigma_{t-1}^2 + \beta\sigma_{t-1}^2. \tag{4.9}$$

Due to the complexity of the GARCH process, analytical solution for the GARCH-Copula option-pricing model is in general not available. Therefore, we use numerical methods to price the option.

The objective is to calculate the option price at maturity T, that is, the option expires at T days after December 31, 2003, which is the end of our estimation period. For demonstration purpose, we assume that the two returns follow the Plackett copula. This is a reasonable assumption because as shown in Figure 4 the dependence relationship between the NYSE and TAIEX index returns are similar under Frank and Plackett copulas. Indeed, interested readers can do the same exercises for the Frank copula.

The pricing procedure we use is as follows:

1. For each marginal process, MLE is used to estimate $(\alpha_0, \alpha, \beta, \lambda)$ in the real-world model. The data are the daily log returns from 1/1/2001 to 31/12/2003 with 782 observations. Based on the estimated results of the marginal processes, we estimate the parameters $[d_1, d_2, d_3, d_4]$ of the Plackett copula, i.e.,

$$\log \theta_t | I_{t-1} = d_1 + d_2 \sigma_t^{TAIEX} + d_3 \sigma_t^{NYSE} + d_4 \sqrt{\sigma_t^{TAIEX} \sigma_t^{NYSE}}.$$

2. Generate standard normal random variables (in risk-neutral world) for the NYSE index, say, $\varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_T^*$. For each t, use the risk neutral volatility equation to compute the conditional variance, σ_t^2, and set $u_t = \Phi(\varepsilon_t^*)$. The size of time increment is one day.

3. Given u_t, we follow the technique of Johnson (1987) to get v_t. See below for details. u_t and v_t will be distributed according to the Plackett copula with parameter $\hat{\theta}_t$, where $\hat{\theta}_t$ is the estimate of θ_t in Step 1.

4. Get $\varepsilon_{t,TAIEX}^* = \Phi^{-1}(v_t)$, the risk-neutral innovation of the TAIEX index return, and compute the conditional variance for the TAIEX index log returns.

5. From t =1, repeat Step 2 to Step 4 until t =T. We then obtain the innovation series (ε_t^*) and the conditional variance series (σ_t^2) for both index returns, and the asset price at maturity T is $S_T = S_0 \exp(r_f T - \frac{1}{2}\sum_{t=1}^{T} \sigma_t^2 + \sum_{t=1}^{T} \sigma_t \varepsilon_t^*)$. The initial price S_0 is 6440 for the NYSE index, which is the closing price at 12/31/2003, and 5891 for the TAIEX index, which is the closing price on 12/31/2003.

6. Repeat Step 2 to Step 5 for N runs (the number of simulation). Each run generates a pair of indices $(S_{T,i}^{NYSE}, S_{T,i}^{TAIEX})$ at the maturity T. Finally, we obtain the option price as $P = \exp(-r_f T) \frac{1}{N} \sum_{i=1}^{N} g(S_{T,i}^{NYSE}, S_{T,i}^{TAIEX})$, where $g(.)$ denotes the payoff function of the option.

For the generation of u and v under the Plackett copula (with dependence parameter θ), we adopt the procedure of Johnson (1987), which can be written as

1. $u = \Phi(\varepsilon^*)$(as mentioned in Step 2) is distributed uniformly on the interval $[0, 1]$ because ε^* is generated randomly, where the subscript t is omitted for simplicity.
2. Simulate another random variable z from uniform $[0, 1]$ that is independent of u.
3. Define $a = z(1 - z)$ and $b = \sqrt{\theta}\sqrt{\theta + 4au(1 - u)(1 - \theta)^2}$.
4. Compute $v = [2a(u\theta^2 + 1 - u) + \theta(1 - 2a) - (1 - 2z)b]/[2\theta + 2a(\theta - 1)^2]$.
5. u and v are distributed as a Plackett copula with parameter θ.

Table 4: Estimation results of the GARCH-Plackett copula, with risk premium, for the daily log returns of the NYSE and TAIEX indices. Innovations of the margins are Gaussian. The sample period is from January 1, 2001 to December 31, 2003 for 782 Observations.

	Marginal processes			
	α_0	α	β	λ
TAIEX	10^{-6}	0.035	0.96	0.00025
NYSE	3×10^{-6}	0.087	0.88	0.0002
	The dependence parameter under Plackett copula			
	d_1	d_2	d_3	d_4
Copula	0.915	31.885	52.47	-124.68

For other copula functions, there is a general procedure to simulate the (u, v) pairs; see Nelsen (1999). The procedure is as follows:

1. Generate two independent uniform [0,1] random variates u and z.
2. Set $v = C_u^{(-1)}(z)$, where $C_u = \partial C/\partial u$ and $C_u^{(-1)}$ is the inverse function of C_u.
3. (u, v) then follows the desired copula function.

In our example, Johnson's simulating method is preferred since it does not require the computation of the inverse copula function. The inverse copula functions often cannot be calculated analytically. One has to use a numerical algorithm to compute v, which in turn puts a heavy burden on computation, especially

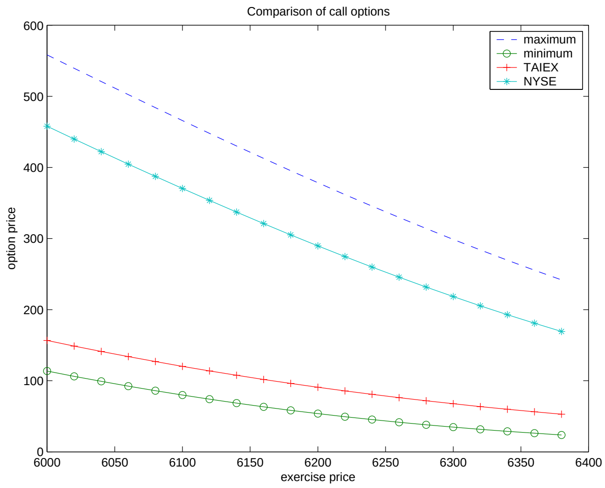

Figure 6: Comparisons of call prices for two types of options based on the daily log returns of the NYSE and TAIEX Indices. Bivariate options are based on a GARCH-copula model and univariate options are based on GARCH models.

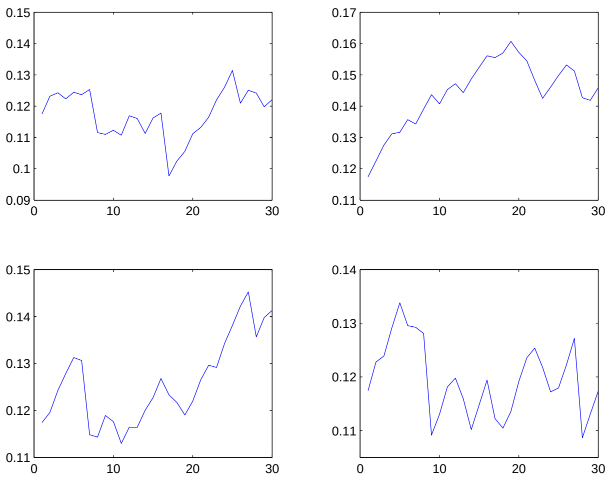

Figure 7: Samples of the evolution of Spearman's rho from the simulation of pricing call options.

when the number of simulation is large. For the two index return series considered, the estimated parameters for the bivariate process are summarized in Table 4, where the riskfree rate is fixed at 0%.

We are ready for option pricing. Figure 6 shows the prices of some call options with different exercise prices ranging from 6000 to 6400. In the plot, "maximum" stands for a call option on maximum, which has a payoff at the maturity date as $\max[\max(\text{NYSE, TAIEX}) - K, 0]$. Similarly, "minimum" stands for a call option on minimum, which has a payoff on maturity day max[min(NYSE, TAIEX) − $K, 0$]. The prices shown are obtained via simulation with 50,000 iterations. For comparison, we also show call option prices with a single underlying asset based on Duan's univariate risk neutral pricing method. The "TAIEX" and "NYSE" in the legend stand for the prices of standard call options with the underlying asset TAIEX and NYSE, respectively. Four randomly selected simulation results for the evolution of the Spearman's rho under the Plackett copula during the life of the option (i.e. 30 days) are shown in Figure 7. The Spearman's rho varies in the range [0.1, 0.16], which is reasonable compared with the estimation results in Section 3.

The effects of dependence between the two processes on option pricing are also of interest. Figure 8 shows the prices of a call option on the minimum under different dependence levels. In the plot, "positive" stands for strongly positive dependence with 0.99 for Spearman's rho (or $\theta = 10000$) and "negative" stands for strongly negative dependence with -0.99 for Spearman's rho (or $\theta = 0.0001$). Finally, "independent" means $\theta = 1$ in the Plackett copula and "Dynamic" stands for the dynamic relationship whose simulation results are showed above.

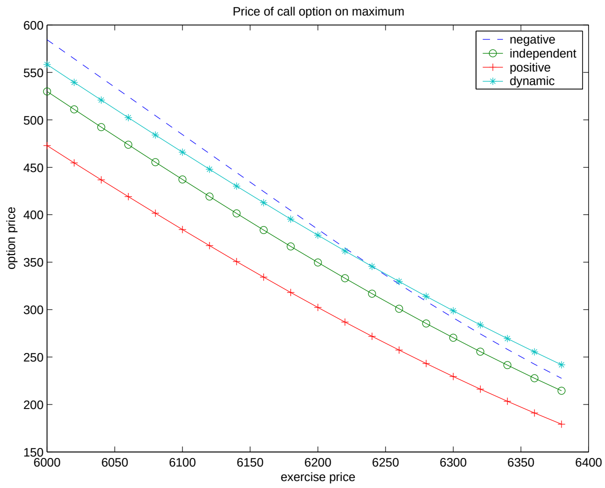

Figure 8: Effects of Dependence between Asset Returns on Call Option Prices.

For a call option on the minimum of the two indices, strongly negative dependence of the two processes yields the lowest option values. Indeed, when one index is at a high (low) level, the other one is likely to be at the low (high) level. In either case, the lower level index is treated as the final payoff asset. On the other hand, options with "strongly positive dependence" benefit from the co-movement of the two indices to high levels. The cases of independent and dynamic processes are simply in-between. From Figure 8, the price of a call option on the minimum

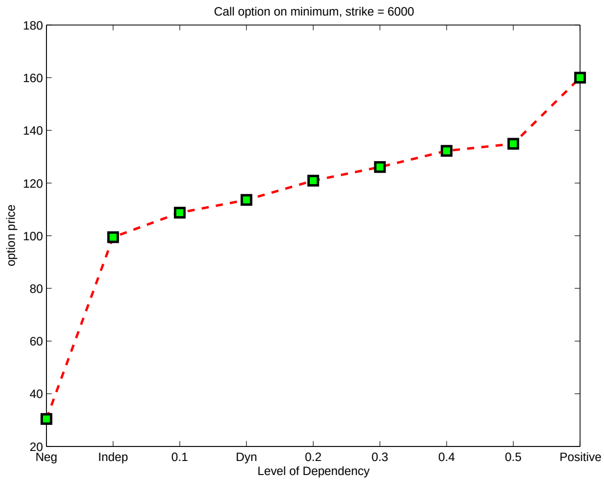

Figure 9: General pattern of price of a call option on minimum under various dependence levels.

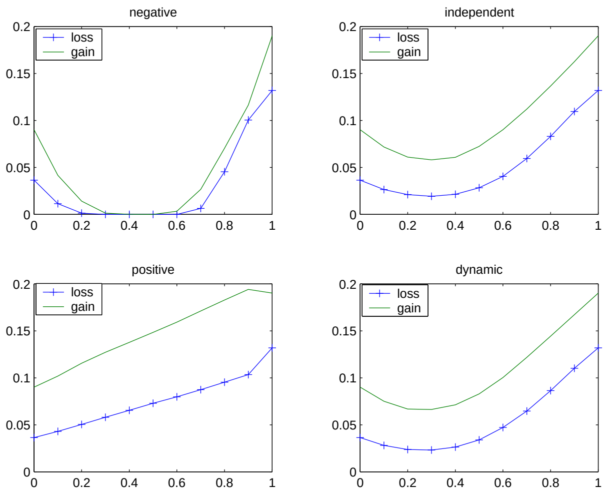

Figure 10: Chances of gaining or losing more than 5% for various portfolios consisting of the NYSE and TAIEX Indices.

seems to be an increasing functions of the dependence relationship. Figure 9 shows the price of a call option on the minimum (with strike = 6000) under various dependence levels. Calls with other strikes have a similar pattern. To gain a deeper understanding of the option behavior, we show the price of a call option on the minimum with various Spearman's rho in Table 5. The call price with dynamic dependence is bounded by call prices with Spearman's rho 0.1 and 0.2. Table 6 shows the corresponding standard deviations.

On the other hand, consider a call option on the maximum of the two market indices. That is to say, the payoff at the maturity day is max[max(NYSE, TAIEX)

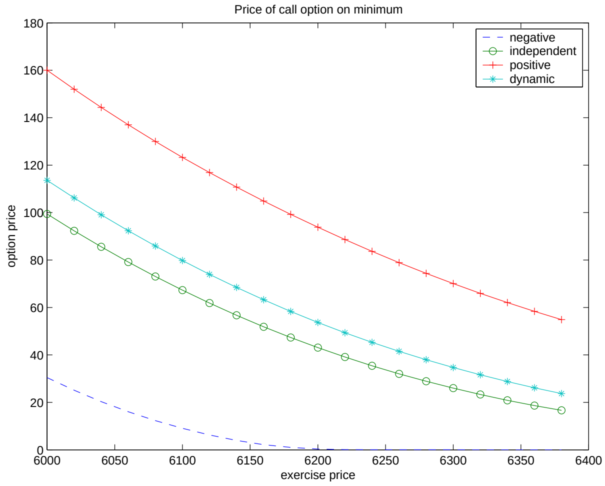

Figure 11: Effects of dependence between asset returns on call options when the payoR is based on the maximum.

Table 5: Prices of a call option on the minimum under various dependence. The leftmost column shows the strike and the top row shows the levels of Spearman's rho

	Neg.	Indep.	0.1	Dyn.	0.2	0.3	0.4	0.5	Positive
6000	30.42	99.43	108.75	113.60	120.90	126.09	132.21	134.90	160.01
6020	25.09	92.32	101.46	106.16	113.29	118.34	124.31	127.03	152.02
6040	20.28	85.56	94.51	99.08	106.01	110.92	116.74	119.48	144.35
6060	16.03	79.14	87.89	92.34	99.06	103.81	109.49	112.24	137.00
6080	12.29	73.06	81.60	85.90	92.42	97.04	102.55	105.30	129.98
6100	9.03	67.30	75.61	79.78	86.07	90.56	95.90	98.68	123.27
6120	6.24	61.83	69.94	73.96	80.02	84.40	89.54	92.34	116.86
6140	3.94	56.69	64.57	68.45	74.28	78.53	83.46	86.29	110.73
6160	2.16	51.86	59.48	63.25	68.82	72.95	77.67	80.53	104.86
6180	0.98	47.33	54.70	58.33	63.64	67.65	72.17	75.06	99.24
6200	0.30	43.08	50.21	53.69	58.75	62.64	66.97	69.87	93.83
6220	0.06	39.10	45.99	49.34	54.15	57.90	62.06	64.95	88.64
6240	0.01	35.42	42.04	45.29	49.84	53.44	57.43	60.29	83.66
6260	0.00	32.02	38.37	41.51	45.81	49.25	53.06	55.88	78.90
6280	0.00	28.88	34.95	37.98	42.02	45.30	48.94	51.71	74.36
6300	0.00	25.98	31.76	34.69	38.49	41.61	45.07	47.78	70.05
6320	0.00	23.30	28.80	31.63	35.19	38.15	41.44	44.09	65.96
6340	0.00	20.86	26.08	28.78	32.11	34.91	38.05	40.63	62.06
6360	0.00	18.64	23.57	26.14	29.24	31.90	34.88	37.36	58.35
6380	0.00	16.62	21.27	23.69	26.58	29.11	31.91	34.29	54.85

$-K, 0]$. Strongly negative dependence of the two processes can easily be in the money at the maturity. The option will suffer from the co-movements of the two indices to low levels under "strongly positive dependence". The case of "Independent" is bounded by the two extreme cases. We show their relation in Figure 10. Interestingly, the dynamic case, with slightly positive correlation on average, is not dominated by the strongly negative case. Tables 7 and 8 show the prices of a call option on the maximum with various Spearman's rho and their corresponding standard deviations. Intuitively, if the correlation between the two return series is higher, price of asset A tends to be higher when price of asset B goes up, which is good for the call option on the maximum. However, with higher correlation, price of asset A tends to be lower when price of asset B goes down, which is not desirable for a call option on the maximum. There is a tradeoff between the two situations. Figure 11 shows the price of a call option on the maximum (with strike = 6000) under various dependence levels. Calls with other strikes have a similar pattern.

Table 6: Prices of a call option on the maximum under various dependence. The leftmost column shows the strike and the top row shows the levels of Spearman's rho

	Neg.	Indep.	0.1	Dyn.	0.2	0.3	0.4	0.5	Positive
6000	584.42	529.89	548.97	558.32	561.72	549.54	528.54	508.57	472.82
6020	564.42	511.02	530.19	539.51	542.91	530.79	509.89	490.14	454.67
6040	544.42	492.30	511.59	520.86	524.27	512.21	491.41	471.90	436.75
6060	524.42	473.76	493.15	502.37	505.79	493.82	473.13	453.88	419.07
6080	504.42	455.40	474.87	484.03	487.47	475.58	455.03	436.06	401.62
6100	484.42	437.22	456.77	465.87	469.33	457.53	437.13	418.44	384.40
6120	464.42	419.21	438.88	447.89	451.34	439.64	419.40	401.00	367.40
6140	444.42	401.40	421.23	430.16	433.59	421.97	401.91	383.77	350.65
6160	424.43	383.85	403.82	412.64	416.06	404.54	384.67	366.81	334.18
6180	404.47	366.61	386.67	395.38	398.79	387.40	367.75	350.18	318.07
6200	384.56	349.65	369.83	378.43	381.80	370.54	351.13	333.85	302.28
6220	364.84	333.01	353.33	361.78	365.09	354.00	334.85	317.87	286.86
6240	345.54	316.75	337.21	345.50	348.74	337.85	318.94	302.29	271.88
6260	326.85	300.88	321.48	329.60	332.75	322.07	303.44	287.11	257.34
6280	308.74	285.40	306.07	314.00	317.05	306.60	288.28	272.31	243.20
6300	291.27	270.35	291.00	298.76	301.71	291.52	273.53	257.94	229.50
6320	274.40	255.68	276.33	283.92	286.77	276.84	259.21	243.96	216.21
6340	258.16	241.46	262.05	269.46	272.24	262.58	245.33	230.40	203.40
6360	242.51	227.70	248.21	255.43	258.15	248.76	231.88	217.29	191.06
6380	227.53	214.42	234.81	241.82	244.48	235.34	218.90	204.67	179.20

Table 7: Standard deviation of call price of an option on the maximum under various dependence. The leftmost column shows the strike and the top row shows the levels of Spearman's rho

	Neg.	Indep.	0.1	Dyn.	0.2	0.3	0.4	0.5	Positive
6000	2.54	1.90	1.89	2.10	2.11	4.64	2.80	2.86	2.01
6020	2.54	1.90	1.88	2.09	2.09	4.60	2.79	2.85	2.01
6040	2.54	1.89	1.88	2.08	2.08	4.56	2.78	2.85	2.00
6060	2.54	1.89	1.87	2.06	2.06	4.52	2.77	2.84	1.99
6080	2.54	1.89	1.86	2.05	2.04	4.48	2.76	2.83	1.99
6100	2.54	1.88	1.84	2.03	2.02	4.44	2.75	2.82	1.98
6120	2.54	1.88	1.83	2.01	2.00	4.39	2.75	2.82	1.97
6140	2.54	1.87	1.81	1.99	1.97	4.35	2.74	2.81	1.96
6160	2.54	1.87	1.79	1.96	1.94	4.29	2.73	2.80	1.95
6180	2.54	1.87	1.77	1.94	1.91	4.23	2.71	2.79	1.93
6200	2.54	1.86	1.75	1.91	1.89	4.17	2.70	2.77	1.92
6220	2.54	1.85	1.73	1.89	1.86	4.10	2.68	2.75	1.91
6240	2.53	1.84	1.70	1.86	1.83	4.03	2.65	2.73	1.89
6260	2.52	1.82	1.68	1.84	1.80	3.95	2.63	2.70	1.87
6280	2.49	1.81	1.66	1.81	1.77	3.85	2.59	2.67	1.85
6300	2.45	1.79	1.64	1.79	1.74	3.76	2.55	2.63	1.83
6320	2.41	1.77	1.61	1.76	1.71	3.66	2.51	2.59	1.81
6340	2.37	1.75	1.59	1.74	1.68	3.56	2.46	2.54	1.78
6360	2.32	1.73	1.56	1.71	1.65	3.46	2.42	2.49	1.75
6380	2.27	1.71	1.53	1.68	1.62	3.36	2.37	2.44	1.73

The above illustrations show how copula-based models can be used to price exotic options. Besides path-independent options, the model can also be used to price path-dependent options, such as barrier options, reset options, and look-back options. Since the dynamic (or the path) of bivariate processes can be simulated from the model, most options can be priced once their payoff functions are given.

4.1 Application in risk management

Market participants are interested in knowing the risk of their portfolios. For example, what is the probability of loss over 10 percent in the next ten trading days? What is the maximum loss in the following month? In this section, we show how the copula approach can be used in risk assessment. In particular, we consider an index portfolio: $w \times \text{TAIEX} + (1-w) \times \text{NYSE}$, where w is the weight on the TAIEX index.

In the literature, a commonly used method in risk measurement is value at risk (VaR). For a long position, the $q\%$ quantile of VaR (denoted by $\text{VaR}(q\%)$) is defined as: $Prob.[\text{loss} \leq \text{VaR}(q\%)] = q$. For example, say, $\text{VaR}(5\%) = 100$, it means the probability that the loss exceeds 100 is not larger than 5%. To calculate VaR, the first step is to figure out the conditional distribution of future returns and use the resulting predictive distribution to compute VaR. For example, consider the univariate GARCH (1,1) model:

$$\begin{aligned} r_t &= c + a_t, \quad a_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0,1) \\ \sigma_t^2 &= \alpha_0 + \alpha_1 a_{t-1}^2 + \beta \sigma_{t-1}^2. \end{aligned}$$

Let the current time (the forecast origin) be h. The p-period ahead distribution of r is Gaussian with mean cp and variance $\sigma_h^2(p)$, where $\sigma_h^2(p)$ can be computed recursively by

$$\sigma_h^2(i) = \alpha_0 + (\alpha_1 + \beta)\sigma_h^2(i-1), \quad i = 2, \ldots, p,$$

starting with $\sigma_h^2(1) = \alpha_0 + \alpha_1 a_h^2 + \beta \sigma_h^2$. If the market value of the portfolio is V, then VaR(5%)= $V \times \{cp - 1.65 \times [\sum_{i=1}^{p} \sigma_h(i)]^{1/2}\}$. See Tsay (2005, ch. 7) for further details.

However, using marginal GARCH models to calculate VaR for the portfolio, $w \times$ TAIEX $+ (1-w) \times$ NYSE, might encounter some difficulties. First, linear combinations of GARCH processes are in general not a GARCH process. In other words, if we model the portfolio returns as a GARCH process, we change the fundamental setting that each marginal return follows a GARCH model. Second, $\sigma^2(p)$ is not exactly known except for $p = 1$.

In this subsection, we consider another approach. Define the risk measurement as the probability that the portfolio will lose (or gain) more than some pre-specified level in certain period in the future. For example, suppose that we are interested in the probability of loss (or gain) more than 5 percent after ten trading days.

In Section 3, we have estimated the parameters of the marginal processes and the copula function. The estimated model can be used to generate sample paths of the NYSE and TAIEX indices for $t = h+1, \ldots, h+p$. For each sample path, the value of the portfolio at time $h+p$ can be calculate. These simulated portfolio values can then be used to compute VaR.

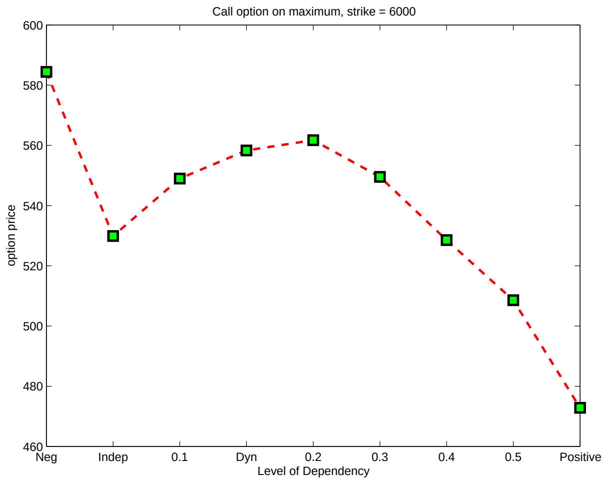

Figure 12: General pattern of the price of call options on the maximum under various dependence levels.

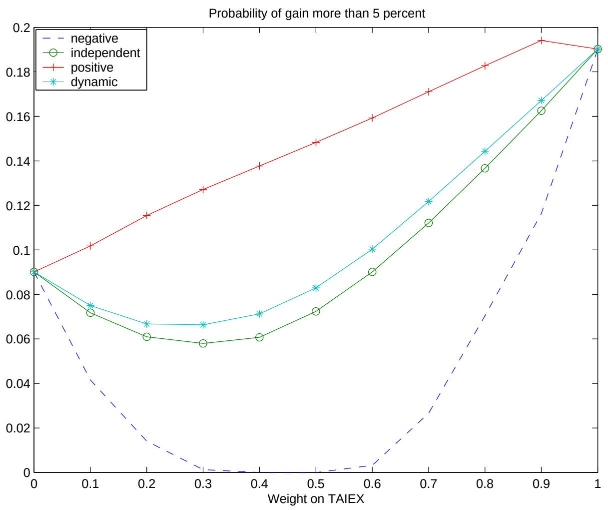

Figure 13: Probabilities of gain more than 5% for various portfolios consisting of the NYSE and TAIEX indices under different dependence between the two assets.

For the data considered in this paper, we use the fitted GARCH(1,1) marginal models and the Plackett copula to generate sample paths of NYSE and TAIEX indices. Specifically, 50,000 paths are generated and the probability of a loss (or gain) exceeding 5 percent of the original position is calculated. For comparison, we also compute the probabilities assuming different dependence structures. Figures 12 and 13 show the probabilities under different dependence relationships between NYSE and TAIEX. Tables 8 and 9 represent the same figures but with more dependence levels. In the plots, "dynamic" stands for time-varying dependence between the two processes. In particular, the dependence parameter, θ_t, of the Plackett copula satisfies Eq. (3.11). "Independent" means the two processes are independent ($\theta = 1$), "positive" means strongly positive dependence with Spearman's rho is 0.99 or $\theta = 10{,}000$, and "negative" means strongly negative dependence with Spearman's rho is -0.99 or $\theta = 0.0001$. For the case of "strongly positive dependence", the probability of a big loss tends to increase when the weight on TAIEX increases. Indeed, strong dependence makes diversification strategy ineffective. If the two indices are independent or negatively related, the investors can lower the chance of big loss via diversification. Specifically, zero probability of loss more than 5 percent can be achieved via diversification.

Figures 12 and 13 exhibit a similar pattern, indicating that the probabilities of loss and gain behave in the same manner. That is, a portfolio that has a higher probability of gaining more than 5% also has a higher probability of losing more than 5%. Note that at both ends of the curve, the portfolio consists of either NYSE index or TAIEX index only so that the loss or gain has nothing to do with the association between the two indexes.

Figure 14 shows combined plots of probabilities under different dependence relationships between the two indices. It is interesting to note that, no matter how NYSE and TAIEX are related, holding a portfolio of both indices always has a higher chance to gain over 5 percent than to lose more than 5 percent. It indicates that the returns of TAIEX and NYSE indices over the sample period have a positive drift. This is consistent with the estimation results shown in Table 2.

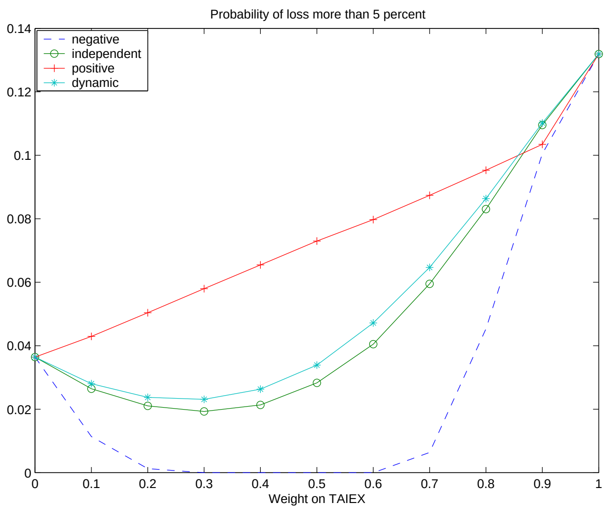

Figure 14: Probabilities of loss more than 5% for various portfolios consisting of the NYSE and TAIEX indices under different dependency between the two assets.

Table 8: Probability of loss more than 5% under various dependence relationships. The leftmost column is the weight of TAIEX and the top row shows the levels of Spearman's rho.

	Neg.	Indep.	0.1	Dyn.	0.2	0.3	0.4	0.5	Positive
0	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.036
0.1	0.011	0.026	0.028	0.028	0.029	0.031	0.033	0.034	0.043
0.2	0.001	0.021	0.024	0.024	0.026	0.029	0.032	0.035	0.050
0.3	0.000	0.019	0.023	0.023	0.026	0.029	0.034	0.038	0.058
0.4	0.000	0.021	0.026	0.026	0.030	0.034	0.038	0.044	0.065
0.5	0.000	0.028	0.033	0.034	0.037	0.042	0.047	0.052	0.073
0.6	0.000	0.040	0.046	0.047	0.049	0.053	0.058	0.063	0.080
0.7	0.006	0.059	0.063	0.065	0.066	0.068	0.072	0.077	0.087
0.8	0.045	0.083	0.085	0.086	0.087	0.087	0.090	0.092	0.095
0.9	0.100	0.110	0.110	0.110	0.110	0.107	0.109	0.109	0.103
1	0.132	0.132	0.132	0.132	0.132	0.132	0.132	0.132	0.132

Table 9: Probability of gain more than 5% under various dependence relationships. The leftmost column is the weight of TAIEX and the top row shows the levels of Spearman's rho

	Neg.	Indep.	0.1	Dyn.	0.2	0.3	0.4	0.5	Positive
0	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090	0.090
0.1	0.042	0.072	0.075	0.075	0.078	0.081	0.084	0.087	0.102
0.2	0.014	0.061	0.067	0.067	0.072	0.078	0.083	0.088	0.115
0.3	0.001	0.058	0.065	0.066	0.072	0.080	0.086	0.094	0.127
0.4	0.000	0.061	0.070	0.071	0.078	0.087	0.094	0.103	0.138
0.5	0.000	0.072	0.081	0.083	0.090	0.099	0.107	0.115	0.148
0.6	0.003	0.090	0.097	0.100	0.106	0.115	0.122	0.130	0.159
0.7	0.027	0.112	0.119	0.122	0.126	0.134	0.140	0.147	0.171
0.8	0.070	0.137	0.142	0.144	0.147	0.153	0.159	0.163	0.183
0.9	0.116	0.163	0.165	0.167	0.170	0.174	0.177	0.180	0.194
1	0.190	0.190	0.190	0.190	0.190	0.190	0.190	0.190	0.190

Finally, from the simulation results, we can construct the distribution of ΔV (the change in the portfolio value). This distribution enables us to compute VaR defined earlier. Some authors have used copula-based model to compute VaR. For instance, Embrechts, Hoing and Juri (2003) use copula to obtain bounds of VaR. Ivanov, Jordan, Panajotova and Schoess (2003) compare numerical results under different marginal settings. Micocci and Masala(2004) perform back-testing calculation to show that the copula approach gives more reliable results than do the traditional Monte Carlo methods based on the Gaussian assumption. Figures 15 and 16 show the VaR of the portfolio for 10-day horizon and 5% tail probability.

Note that, with different weights on TAIEX, the portfolios have different initial values so that V varies with the allocation on the TAIEX and NYSE indices.

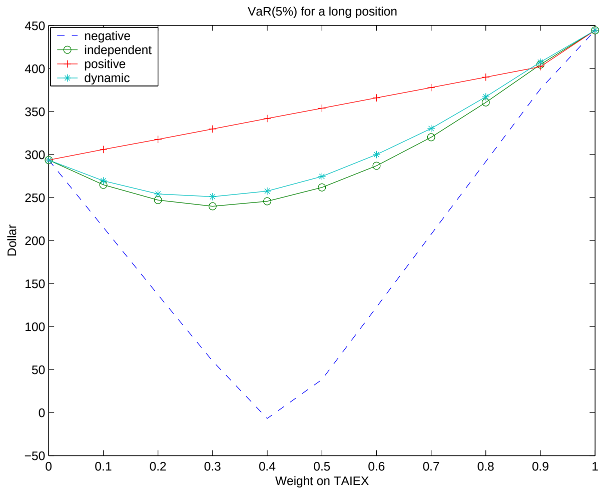

Figure 15: VaR for a long position of various portfolios. The tail probability is 5%.

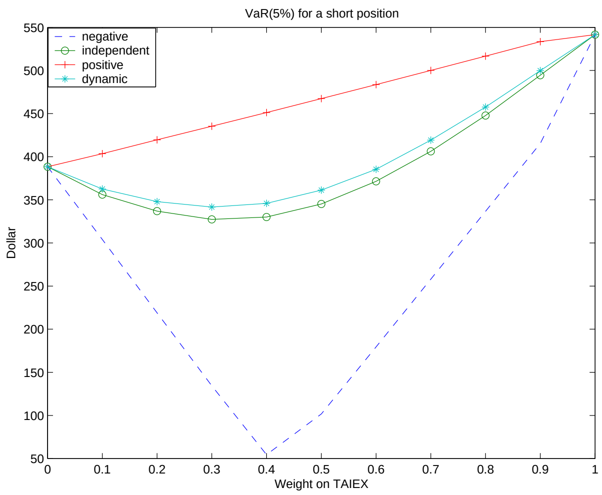

Figure 16: VaR for a short position of various portfolios. The tail probability is 5%.

4.2 Conclusion

This paper studied the dynamic of a bivariate financial process. We modeled the margins using the conventional time series models and linked the margins with a copula function. A two-step MLE procedure was used to estimate the parameters. Once the process was characterized, we used the fitted model to make inference concerning financial instruments contingent on the two underlying assets. In particular, we demonstrated how to calculate derivative prices with copula and the risk-neutral approach and how to assess risk of a portfolio via copula modeling.

Some concerns need to be addressed, however. Since option prices and VaR (or other risk measurements) are based on the MLE estimation of the model, we inevitably face the issue of model misspecification and parameter uncertainty. In particular, the validity of using the Plackett copula or the Frank copula needs further study. In recent years, this issue has started to attract some attention. For example, Fermanian and Scaillet (2003) propose a nonparametric estimation method for copula using a kernel-based approach. Chen and Fan (2006) use nonparametric marginal distributions and a parameterized copula to mitigate the inefficiency of the two-step estimation procedure. In addition, Chen, Fan and Patton (2003) develop two goodness-of-fit tests for copula models. To mitigate the problem of parameter uncertainty, one may use MCMC algorithms with some proper prior distributions.

Finally, the methodology considered in this paper has other applications. Since copula-based models describe a multivariate distribution by separating the marginal behavior from dynamic dependence, they substantially increase the flexibilities in modeling multivariate processes and are applicable to many scientific areas. For instance, copula-based models can be used to model multivariate extreme distributions, which are useful in the insurance industry. Also, the models can be used in modeling default risk and in pricing vulnerable credit derivatives. Interested readers are referred to Cherubini, Luciano and Vecchiato (2004) for more applications of copula methods in finance.

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