THE FUNDAMENTAL THEOREM OF ASSET PRICING: DISCRETE TIME

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Abstract. A simple statement and accessible proof of a new version of the Fundamental Theorem of Asset Pricing in discrete time are provided. Careful distinction is made between prices and cash flows in order to provide uniform treatment of all instruments. It is shown that there is no need for a "real-world" measure in order to specify a model for pricing derivative securities, as demonstrated by several examples.

1. Introduction

It is difficult to write a paper about the Fundamental Theorem of Asset Pricing that is longer than the bibliography required to do justice to the excellent work that has been done elucidating the key insight Fischer Black, Myron Scholes, and Robert Merton had in the early '70's. At that time, the Capital Asset Pricing Model and equilibrium reasoning dominated the theory of security valuation so the notion that the relatively weak assumption of no arbitrage could have such detailed implications about the price process was a huge leap forward.

One intriguing aspect of the development of the FTAP has been the technical difficulties involved in providing rigorous proofs and the related aspect of the increasingly convoluted statements of the theorem. The primary contributions of this paper are a statement of the fundamental theorem of asset pricing that is comprehensible to traders and risk managers and a proof that is accessible to students in a graduate level course in derivative securities. Emphasis is placed on distinguishing

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Peter Carr is entirely responsible for many enjoyable and instructive discussions on this topic. Walter Schachermeyer also provided valuable insights. I am entirely responsible for any omissions and errors.

between price and cash flows in order to give a unified treatment of all instruments. No artificial "real world" measures which are then changed to risk-neutral measures are required. (See also Biagini and Cont 2006.) One simply finds appropriate price deflators directly and gets down to the business of pricing derivative securities.

Section 2 gives a brief review of the history of the FTAP with an eye to demonstrating the increasingly esoteric mathematics involved. Section 3 states and proves the one period version and introduces a definition of arbitrage more closely suited to what practitioners would recognize. Several examples are presented to illustrate the usefulness of the theorem. In section 4 the general result for discrete time models is presented together with more examples. The last section finishes with some general remarks and a summary of the methodology proposed in this paper.

2. Review

From Merton's 1973 paper, "The manifest characteristic of (21) is the number of variables that it does *not* depend on" where (21) refers to the Black-Scholes 1973 option pricing formula for a call having strike E and expiration τ

$$f(S, \tau; E) = S\Phi(d_1) - Ee^{-rt}\Phi(d_1 - \sigma\sqrt{\tau}).$$

Here, Φ is the cumulative standard normal distribution, σ^2 is the instantaneous variance of the return on the stock and $d_1 = [\log(S/E) + (r + \frac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}$. In particular, the return on the stock does not make a showing, unlike in the Capital Asset Pricing Model where it shares center stage with covariance. This was the key insight in the connection between arbitrage-free models and martingales.

In the section immediately following Merton's claim he calls into question the rigor of Black and Scholes' proof and provides his own. His proof requires the bond process to have nonzero quadratic variation. Merton 1977 provides what is now considered to be the standard derivation.

A special case of the valuation formula that European option prices are the discounted expected value of the option payoff under the risk neutral measure makes its first appearance in the Cox and Ross 1976 paper. The first version of the FTAP in a form we would recognize today occurs in a Ross 1978 where it is called the Basic Valuation Theorem. The use of the Hahn-Banach theorem in the proof also makes its first appearance here, although it is not clear precisely what topological vector space is under consideration. The statement of the result is also couched in terms of market equilibrium, but that is not used in the proof. Only the lack of arbitrage in the model is required.

In Harrison and Kreps 1979, Harrison and Kreps provide the first rigorous proof of the one period FTAP (Theorem 1) in a Hilbert space setting. They are also the first to prove results for general diffusion processes with continuous, nonsingular coefficients and state “Theorem 3 can easily be extended to this larger class of processes, but one then needs quite a lot of measure theoretic notation to make a rigorous statement of the result.” This statement was premonitory.

The 1981 paper of Harrison and Pliska is primarily concerned with models in which markets are complete (Question 1.16), however they make the key observation, “Thus the parts of probability theory most relevant to the general question (1.16) are those results, usually abstract in appearance and French in origin, which are invariant under substitution of an equivalent measure.” This observation applies equally to incomplete market models and seems to have its genesis in the much earlier work of Kemeny 1955 and Shimony 1955 as pointed out by Schachermeyer.

D. Kreps 1981 was the first to replace the assumption of no arbitrage with that of no *free lunch*: “The financial market defined by (X, τ), M, and π admits a free lunch if there are nets $(m_\alpha)_{\alpha\in I} \in M_0$ and $(h_\alpha)_{\alpha\in I} \in X_+$ such that $\lim_{\alpha\in I}(m_\alpha - h_\alpha) = x$ for some $x \in X_+ \setminus \{0\}$.” It is safe to say the set of traders and risk managers that are able to comprehend this differs little from the empty set. It was a brilliant technical innovation in the theory but the problem with first assuming a measure for the paths instrument prices follow was that it made it difficult to apply the Hahn-Banach theorem. The dual of $L^\infty(\tau)$ under the norm topology is intractable.

The dual of $L^\infty(\tau)$ under the weak-star topology is $L^1(\tau)$, which by the Radon-Nikodym theorem can be identified with the set of measures that are absolutely continuous with respect to τ. This is what one wants when hunting for equivalent martingale measures, however one obstruction to the proof is that the positive functions in $L^\infty(\tau)$ do not form a weak-star open set. Krep's highly technical free lunch definition allowed him to use the full plate of open sets available in the norm topology that is required for a correct application of the Hahn-Banach theorem.

The escalation of technical machinery continues in Dalang, Morton and Willinger 1990. This paper gives a rigorous proof of the FTAP in discrete time for an arbitrary probability space and is closest to this paper in subject matter. They correctly point out an integrability condition on the price process is not economically meaningful since it is not invariant under change of measure. They give a proof that does not assume such a condition by invoking a nontrivial measurable selection theorem. They also mention, "However, if in addition the process were assumed to be bounded, ..." and point out how this assumption could simplify their proof. The robust arbitrage definition and the assumption of bounded prices is also used the original paper, Long Jr. 1990, on numeraire portfolios.

The pinnacle of abstraction comes in Delbaen and Schachermeyer 1994 where they state and prove the FTAP in the continuous time case. Theorem 1.1 states an equivalent martingale measure exists if and only if there is *no free lunch with vanishing risk*: "There should be no sequence of final payoffs of admissible integrands, $f_n = (H^n \cdot S)_\infty$, such that the negative parts f_n^- tend to zero uniformly and such that f_n tends almost surely to a $[0, \infty]$-valued function f_0 satisfying $P[f_0 > 0] > 0$." The authors were completely correct when they claim "The proof of Theorem 1.1 is quite technical..."

The fixation on change of measure and market completeness resulted in increasingly technical definitions and proofs. This paper presents a new version of the Fundamental Theorem of Asset Pricing in discrete time. No artificial probability measures are introduced and no "change of measure" is involved. The model allows for negative prices and for cash flows (e.g., dividends, coupons, rolls, etc.) to be associated with instruments. All instruments are treated on an equal basis and there is no need to assume the existence of a risk-free asset that can be used to fund trading strategies.

As is customary, perfect liquidity is assumed: every instrument can be instantaneously bought or sold in any quantity at the given price. What is not customary is that prices are bounded and there is no a priori measure on the space of possible outcomes. The algebras of sets that represent available information determine the price dynamics that are possible in an arbitrage-free model.

3. The One Period Model

The one period model is described by a vector, $x \in \mathbf{R}^m$, representing the prices of m instruments at the beginning of the period, a set Ω of all possible outcomes over the period, and a bounded function $X\colon \Omega \to \mathbf{R}^m$, representing the prices of the m instruments at the end of the period depending on the outcome, $\omega \in \Omega$.

Definition 3.1. *Arbitrage exists if there is a vector $\xi \in \mathbf{R}^m$ such that $\xi \cdot x < 0$ and $\xi \cdot X(\omega) \geq 0$ for all $\omega \in \Omega$.*

The cost of setting up the position ξ is $\xi \cdot x = \xi_1 x_1 + \cdots + \xi_m x_m$. This being negative means money is made by putting on the position. When the position is liquidated at the end of the period, the proceeds are $\xi \cdot X$. This being non-negative means no money is lost.

It is standard in the literature to introduce an arbitrary probability measure on Ω and use the conditions $\xi \cdot x = 0$ and $\xi \cdot X \geq 0$ with $E[\xi \cdot X] > 0$ to define an arbitrage opportunity, e.g., Shiryaev, Kabanov, Kramkov and Melnikov 1994 section 7.3, definition 1. Making nothing when setting up a position and having a nonzero probability of making a positive amount of money with no estimate of either the probability or amount of money to be made is not a realistic definition of an arbitrage opportunity. Traders want to know how much money they make up-front with no risk of loss after the trade is put on. This is what Garman 1985 calls *strong arbitrage*.

Define the *realized return* for a position, ξ, by $R_\xi = \xi \cdot X/\xi \cdot x$, whenever $\xi \cdot x \neq 0$. If there exists $\zeta \in \mathbf{R}^m$ with $\zeta \cdot X(\omega) = 1$ for $\omega \in \Omega$ (a zero coupon bond) then the price is $\zeta \cdot x = 1/R_\zeta$. Zero interest rates correspond to a realized return of 1.

Note that arbitrage is equivalent to the condition $R_\xi < 0$ on Ω for some $\xi \in \mathbf{R}^m$. In particular, negative interest rates do not necessarily imply arbitrage.

The set of all arbitrages form a cone since this set is closed under multiplication by a positive scalar and addition. The following version of the FTAP shows how to compute an arbitrage when it exists.

Theorem 3.1. *(One Period Fundamental Theorem of Asset Pricing) Arbitrage exists if and only if x does not belong to the smallest closed cone containing the range of X. If x^* is the nearest point in the cone to x, then $\xi = x^* - x$ is an arbitrage.*

Proof. If x belongs to the cone, it is arbitrarily close to a finite sum $\sum_j X(\omega_j)\pi_j$, where $\omega_j \in \Omega$ and $\pi_j > 0$ for all j. If $\xi \cdot X(\omega) \geq 0$ for all $\omega \in \Omega$ then $\xi \cdot \sum X(\omega_j)\pi_j \geq 0$, hence $\xi \cdot x$ cannot be negative. The other direction is a consequence of the following with $\mathcal{C}$ being the smallest closed cone containing $X(\Omega)$. $\square$

Lemma 3.2. *If $\mathcal{C} \subset \mathbf{R}^m$ is a closed cone and $x \notin \mathcal{C}$, then there exists $\xi \in \mathbf{R}^m$ such that $\xi \cdot x < 0$ and $\xi \cdot y \geq 0$ for all $y \in \mathcal{C}$.*

Proof. This result is well known, but we provide an elementary self-contained proof. Since $\mathcal{C}$ is closed and convex, there exists $x^* \in \mathcal{C}$ such that $\|x^* - x\| \leq \|y - x\|$ for all $y \in \mathcal{C}$. We have $\|x^* - x\| \leq \|tx^* - x\|$ for $t \geq 0$, so $0 \leq (t^2 - 1)\|x^*\|^2 - 2(t-1)x^* \cdot x = f(t)$. Because $f(t)$ is quadratic in t and vanishes at $t = 1$, we have $0 = f'(1) = 2\|x^*\|^2 - 2x^* \cdot x$, hence $\xi \cdot x^* = 0$. Now $0 < \|\xi\|^2 = \xi \cdot x^* - \xi \cdot x$, so $\xi \cdot x < 0$.

Since $\|x^* - x\| \leq \|ty + x^* - x\|$ for $t \geq 0$ and $y \in \mathcal{C}$, we have $0 \leq t^2\|y\|^2 + 2ty \cdot (x^* - x)$. Dividing by t and setting $t = 0$ shows $\xi \cdot y \geq 0$. $\square$

Let $B(\Omega)$ be the Banach algebra of bounded real-valued functions on Ω. Its dual, $B(\Omega)^* = ba(\Omega)$, is the space of finitely additive measures on Ω, e.g., Dunford and Schwartz 1954. If $\mathcal{P}$ is the set of non-negative measures in $ba(\Omega)$, then $\{\langle X, \Pi\rangle : \Pi \in \mathcal{P}\}$ is the smallest closed cone containing the range of X, where the angle brackets indicate the dual pairing. There is no arbitrage if and only if there exists a non-negative finitely additive measure, Π, on Ω such that $x = \langle X, \Pi\rangle$,

If $V \in B(\Omega)$ is the payoff function of an instrument and $V = \xi \cdot X$ for some $\xi \in \mathbf{R}^m$, then the cost of replicating the payoff is $\xi \cdot x = \langle \xi \cdot X, \Pi\rangle = \langle V, \Pi\rangle$. Of course the dimension of such perfectly replicating payoff functions can be at most m.

If a zero coupon bond, $\zeta \in \mathbf{R}^m$, exists then the risk-less realized return is $R = R_\zeta = 1/\Pi(\Omega)$. If we let $P = \Pi R$, then P is a probability measure and $x = \langle X/R, P\rangle = E[X]/R$.

3.1. **Examples.** This section illustrates the one period model with some examples.

Example 1. *(Forward) Let* $x = (1, s, 0)$*,* $\Omega = [0, \infty)$*, and* $X(\omega) = (R, \omega, \omega - f)$*.*

This models a bond with risk-less realized return R, a stock, and a forward contract on the stock with forward f. The smallest cone containing the range of X is spanned by $X(0) = (R, 0, -f)$ and $\lim_{\omega\to\infty} X(\omega)/\omega = (0, 1, 1)$. Solving $(1, s) = a(R, 0) + b(0, 1)$ gives $a = 1/R$ and $b = s$. This implies $0 = -f/R + s$ so $f = Rs$, the standard relationship between forward and spot prices in the one period case.

Example 2. *(Standard Binomial Model) Let* $x = (1, s, v)$*,* $\Omega = \{d, u\}$*,* $0 < d < u$*. and* $X(\omega) = (R, s\omega, V(s\omega))$*, where* V *is any given function.*

This is the usual parametrization for the one period binomial model with a risk-less bond having realized return R, and a stock having price s that can go to either sd or su. The smallest cone containing the range of X is spanned by $X(d)$ and $X(u)$. Solving $(1, s) = aX(d) + bX(u)$ for a and b yields $a = (u - R)/R(u - d)$

and $b = (R - d)/R(u - d)$. The condition that a and b are non-negative implies $d \le R \le u$. The no arbitrage condition on the third component implies

$$v = \frac{1}{R}\left(\frac{u-R}{u-d}V(sd) + \frac{R-d}{u-d}V(su)\right).$$

In a binomial model, the option is a linear combination of the bond and stock. Solving $V(sd) = mR+nsd$ and $V(su) = mR+nsu$ for n we see the number of shares of stock to purchase in order to replicate the option is $n = (V(su)-V(sd))/(su-sd)$. Note that if V is a call spread consisting of long one call with strike slightly greater than sd and short one call with strike slightly less than su, then $\partial v/\partial s = 0$ since $V'(sd) = 0 = V'(su)$.

Example 3. *(Binomial Model) Let* $x = (1, s, v)$, $\Omega = \{S^+, S^-\}$, *and* $X(\omega) = (R, \omega, V(\omega))$, *where* V *is any given function.*

As above we find

$$v = \frac{1}{R}\left(\frac{S^+ - Rs}{S^+ - S^-}V(S^-) + \frac{Rs - S^-}{S^+ - S^-}V(S^+)\right)$$

and the number of shares of stock required to replicate the option is $n = (V(S^+) - V(S^-)/(S^+ - S^-)$. Note $\partial v/\partial s = n$ indicates the number of stock shares to buy in order to replicate the option.

Example 4. *Let* $x = (1, 100, 6)$, $\Omega = [90, 110]$, *and* $X(\omega) = (1, \omega, \max\{\omega - 100, 0\})$.

This corresponds to zero interest rate, a stock having price 100 that will certainly end with a price in the range 90 to 110, and a call with strike 100. One might think the call could have any price between 0 are 10 without entailing arbitrage, but that is not the case.

This model is not arbitrage free. The smallest cone containing the range of X is spanned by $X(90)$, $X(100)$, and $X(110)$. It is easy to see that x does not belong to this cone since it lies above the plane determined by the origin, $X(90)$ and $X(110)$.

Using e_b, e_s, and e_c as unit vectors in the bond, stock, and call directions, $X(90) = e_b + 90e_s$ and $X(110) = e_b + 110e_s + 10e_c$. Grassmann calculus (Peano

1999) yields $X(110)\wedge X(90) = 90e_b\wedge e_s + 110e_s\wedge e_b + 10e_c\wedge e_b + 900e_c\wedge e_s = -900e_s\wedge e_c + 10e_c\wedge e_b - 20e_b\wedge e_s$. The vector perpendicular to this is $-900e_b + 10e_s - 20e_c$.

After dividing by 10, we can read off an arbitrage from this: borrow 90 using the bond, buy one share of stock, and sell two calls. The amount made by putting on this position is $-\xi\cdot x = 90 - 100 + 12 = 2$. At expiration the position will be liquidated to pay $\xi\cdot X(\omega) = -90 + \omega - 2\max\{\omega - 100, 0\} = 10 - |100-\omega| \geq 0$ for $90 \leq \omega \leq 110$.

Example 5. *Let* $x = (100, 9.1)$*,* $\Omega = [90, 110]$*, and* $X(\omega) = (\omega, \max\{\omega - 100, 0\})$*.*

Eliminating the bond does not imply the call can have any price between 0 and 10 without arbitrage. The position $\xi = (1, -11)$ is an arbitrage.

3.2. **An Alternate Proof.** The preceding proof of the fundamental theorem of asset pricing does not generalized to multiperiod models so we give a proof here that does.

Define $A\colon \mathbf{R}^m \to \mathbf{R}\oplus B(\Omega)$ by $A\xi = -\xi\cdot x \oplus \xi\cdot X$. This linear operator represents the account statements that would result from putting on the position ξ at the beginning of the period and taking it off at the end of the period. Define $\mathcal{P}$ to be the set of $p\oplus P$ where $p > 0$ is in $\mathbf{R}$ and $P \geq 0$ is in $B(\Omega)$. Arbitrage exists if and only if $\operatorname{ran} A = \{A\xi : \xi \in \mathbf{R}^m\}$ meets $\mathcal{P}$. If the intersection is empty, then by the Hahn-Banach theorem, Banach and Mazur 1933, there exists a hyperplane $\mathcal{H}$ containing $\operatorname{ran} A$ that does not intersect $\mathcal{P}$. Since we are working with the norm topology, clearly $1\oplus 1$ is the center of an open ball contained in $\mathcal{P}$, so the theorem applies. The hyperplane consist of all $y\oplus Y \in \mathbf{R}\oplus B(\Omega)$ such that $0 = y\pi + \langle Y, \Pi\rangle$ for some $\pi\oplus\Pi \in \mathbf{R}\oplus ba(\Omega)$.

First note that $\langle \mathcal{P}, \pi\oplus\Pi\rangle$ cannot contain both positive and negative values. If it did, the convexity of $\mathcal{P}$ would imply there is a point at which the dual pair is zero and thereby meet $\mathcal{H}$. We may assume that the dual pair is always positive and that $\pi = 1$. Since $0 = \langle A\xi, \pi\oplus\Pi\rangle = \langle -\xi\cdot x, \pi\rangle + \langle \xi\cdot X, \Pi\rangle$ for all $\xi \in \mathbf{R}^m$ it follows $x = \langle X, \Pi\rangle$ for the non-negative measure Π. This completes the alternate proof.

This proof does not yield the arbitrage vector when it exists, however it can be modified to do so. Define $\mathcal{P}^+ = \{\pi \oplus \Pi : \langle p \oplus P, \pi \oplus \Pi\rangle > 0, p \oplus P \in \mathcal{P}\}$. The Hahn-Banach theorem implies $\operatorname{ran} A \cap \mathcal{P} \neq \emptyset$ if and only if $\ker A^* \cap \mathcal{P}^+ = \emptyset$, where A^* is the adjoint of A and $\ker A^* = \{\pi \oplus \Pi : A^*(\pi \oplus \Pi) = 0\}$. If the later holds we know $0 < \inf_{\Pi \geq 0} \|-x + \langle X, \Pi\rangle\|$ since $A^*(\pi \oplus \Pi) = -x\pi + \langle X, \Pi\rangle$. The same technique as in the first proof can now be applied.

4. Multiperiod Model

The multiperiod model is specified by an increasing sequence of times $(t_j)_{0 \leq j \leq n}$ at which transactions can occur, a sequence of algebras $(\mathcal{F}_j)_{0 \leq j \leq n}$ on the set of possible outcomes Ω where $\mathcal{F}_j$ represents the information available at time t_j, a sequence of bounded $\mathbf{R}^m$ valued functions $(X_j)_{0 \leq j \leq n}$ with X_j being $\mathcal{F}_j$ measurable that represent the prices of m instruments, and a sequence of bounded $\mathbf{R}^m$ valued functions $(C_j)_{1 \leq j \leq n}$ with C_j being $\mathcal{F}_j$ measurable that represent the cash flows associated with holding one share of each instrument over the preceding time period. We further assume the cardinality of $\mathcal{F}_0$ is finite, but we do not assume the $\mathcal{F}_j$ are increasing.

Let $B(\Omega, \mathcal{F}, \mathbf{R}^m)$ denote the set of bounded $\mathcal{F}$ measurable functions on Ω taking values in $\mathbf{R}^m$.

Definition 4.1. *Arbitrage exists if there are $\Xi_i \in B(\Omega, \mathcal{F}_i, \mathbf{R}^m)$, $0 \leq i < n$, such that $-\Xi_0 \cdot X_0 > 0$, $\Xi_{i-1} \cdot (X_i + C_i) - \Xi_i \cdot X_i \geq 0$, $1 \leq i < n-1$, and $\Xi_{n-1} \cdot (X_n + C_n) \geq 0$.*

If we make the convention $\Xi_{-1} = 0 = \Xi_n$ then we can write the above as $\Xi_{i-1} \cdot (X_i + C_i) - \Xi_i \cdot X_i \geq 0$ for $0 \leq i \leq n$ with strict inequality at $i = 0$.

If we define $W_j = \sum_{i \leq j} \Xi_{i-1} \cdot (X_i + C_i) - \Xi_i \cdot X_i = -\Xi_j \cdot X_j + \sum_{i<j} \Xi_i \cdot (X_{i+1} + C_{i+1} - X_i)$ to be total wealth at t_j, then the arbitrage conditions are equivalent to $W_0 > 0$ and $(W_j)_{j=0}^n$ is non-decreasing. This will be the definition used in the follow-on paper dealing with continuous time.

Let $\mathcal{P} \subset \bigoplus_{j=0}^n B(\Omega, \mathcal{F}_j)$ be the cone of all $\oplus_j P_j$ such that $P_0 > 0$ and $P_j \geq 0$, $1 \leq j \leq n$. The dual cone, $\mathcal{P}^+$ is defined to be the set of all $\oplus_j \Pi_j$ in $\bigoplus_{j=0}^n ba(\Omega, \mathcal{F}_j)$ such that $\langle P, \Pi \rangle = \langle \oplus_j P_j, \oplus_j \Pi_j \rangle = \sum_j \langle P_j, \Pi_j \rangle > 0$.

Lemma 4.1. *The dual cone $\mathcal{P}^+$ consists of $\oplus_j \Pi_j$ such that $\Pi_0 > 0$, and $\Pi_j \geq 0$ and $1 \leq j \leq n$.*

Proof. Since $0 < \langle P_0, \Pi_0 \rangle$ for $P_0 > 0$ we have $\Pi_0(A) > 0$ for every atom of $\mathcal{F}_0$ so $\Pi_0 > 0$. For every $\epsilon > 0$ and any $j > 0$ we have $0 < \epsilon \Pi_0(\Omega) + \langle P_j, \Pi_j \rangle$ for every $P_j \geq 0$. This implies $\Pi_j \geq 0$. $\square$

If $\mathcal{F}$ is an algebra of sets on Ω, define $E_{\mathcal{F}} \colon ba(\Omega) \to ba(\Omega, \mathcal{F})$ by $E_{\mathcal{F}}(\Pi) = \Pi|_{\mathcal{F}}$, the restriction of Π to $\mathcal{F}$. Note this is a linear projection and $\Pi_{\mathcal{G}} \Pi_{\mathcal{F}} = \Pi_{\mathcal{G}}$ when $\mathcal{F} \supseteq \mathcal{G}$.

Theorem 4.2. *(Multiperiod Fundamental Theorem of Asset Pricing) There is no arbitrage if and only if there exists $\oplus_i \Pi_i \in \mathcal{P}^+$ such that*

$$X_i \Pi_i = E_{\mathcal{F}_i}[(X_{i+1} + C_{i+1}) \Pi_{i+1}], \quad 0 \leq i < n.$$

Proof. Define $A \colon \bigoplus_{i=0}^{n-1} B(\Omega, \mathcal{F}_i, \mathbf{R}^m) \to \bigoplus_{i=0}^n B(\Omega, \mathcal{F}_i)$ by

$$A(\bigoplus_{0 \leq i < n} \Xi_i) = \bigoplus_{0 \leq i \leq n} \Xi_{i-1} \cdot (X_i + C_i) - \Xi_i \cdot X_i,$$

where we use the convention $\Xi_{-1} = 0 = \Xi_n$. This represents the account statements that would occur at each transaction time from putting on position Ξ_j from t_j to t_{j+1}.

With $\mathcal{P}$ as above, no arbitrage is equivalent to $\operatorname{ran} A \cap \mathcal{P} = \emptyset$. Again, the norm topology ensures that $\mathcal{P}$ has an interior point so the Hahn-Banach theorem implies there exists a hyperplane $\mathcal{H} =^{\perp} \{\Pi\}$ for some $\Pi = \oplus_0^n \Pi_i$ containing $\operatorname{ran} A$ that does not meet $\mathcal{P}$. It is not possible that $\langle \mathcal{P}, \Pi \rangle$ takes on different signs. Otherwise the convexity of $\mathcal{P}$ would imply $0 = \langle P, \Pi \rangle$ for some $P \in \mathcal{P}$ so we may assume $\Pi \in \mathcal{P}^+$. Note

$$\begin{aligned} 0 &= \langle A(\oplus_i \Xi_i), \oplus_i \Pi_i \rangle \\ &= \sum_{i=0}^{n} \langle \Xi_{i-1} \cdot (X_i + C_i) - \Xi_i \cdot X_i, \Pi_i \rangle \\ &= \sum_{i=0}^{n-1} \langle \Xi_i, (X_{i+1} + C_{i+1})\Pi_{i+1} - X_i \Pi_i \rangle \end{aligned}$$

for all Ξ. This implies $0 = E_{\mathcal{F}_i}[(X_{i+1} + C_{i+1})\Pi_{i+1}] - X_i\Pi_i$ for $0 \le i < n$. $\square$

Define $E_j^k = E_j \cdots E_k$ where $E_i = E_{\mathcal{F}_i}$. The proof of the following corollary is trivial.

Corollary 4.3. *With notation as above,*

$$X_j\Pi_j = \sum_{j<i<k} E_j^i[C_i\Pi_i] + E_j^k[(X_k + C_k)\Pi_k], \quad j < k. \tag{1}$$

This corrects and generalizes formula (2) in chapter 2 of Duffie 1996. As we will see below, this corollary is the primary tool for constructing arbitrage free models. In the case of zero cash flows and increasing algebras, the no arbitrage condition is equivalent to $(X_j\Pi_j)_{j\ge 0}$ being a martingale, by a slight abuse of the word martingale.

In the one period case there is no need to distinguish between price and cash flows. In the multiperiod case one can account for the cash flows, as in Pliska 1997, by stipulating the price decreases by the amount of the cash flow. We find it advantageous to explicitly distinguish between prices and cash flows.

We say a strategy, Ξ, is *self-financing* if all but the first and last component of $A\Xi$ are zero. The cost at t_0 of creating the cash flow $\Xi_{n-1} \cdot (X_n + C_n)$ at t_n is $\Xi_0 \cdot X_0\Pi_0$.

If Ξ is self-financing we have

$$
\begin{aligned}
\Xi_0 \cdot X_0 \Pi_0 &= E_0[\Xi_0 \cdot (X_1 + C_1)\Pi_1] \\
&= E_0[\Xi_1 \cdot X_1 \Pi_1] \\
&= E_0[\Xi_1 \cdot E_1[(X_2 + C_2)\Pi_2]] \\
&= E_0[\Xi_1 \cdot (X_2 + C_2)\Pi_2] \\
&\dots \\
&= E_0[\Xi_{n-1} \cdot X_{n-1}\Pi_{n-1}] \\
&= E_0[\Xi_{n-1} \cdot E_{n-1}[(X_n + C_n)\Pi_n]] \\
&= E_0[\Xi_{n-1} \cdot (X_n + C_n)\Pi_n]
\end{aligned}
$$

If a European derivative has payoff $V\colon \Omega \to \mathbf{R}$ at t_n for which there exists a self-financing portfolio $(\Xi_j)_{0\le j<n}$ such that $\Xi_{n-1} \cdot (X_n + C_n) = V$, then the cost of a replicating strategy is $E_0[V\Pi_n]/\Pi_0$.

4.1. **Examples.**

Example 6. *(Short Rate Process) A short rate (realized return) process $(R_j)_{j\ge 0}$ is a scalar valued adapted process that defines instruments having price $X_j = 1$ and cash flow $C_j = 0$ at time t_j, and having price $X_{j+1} = 0$ and cash flow $C_{j+1} = R_j$ at time t_{j+1}.*

No arbitrage implies $\Pi_j = E_j[R_j \Pi_{j+1}]$ so $R_j = \Pi_j / E_j[\Pi_{j+1}]$. If the price deflators are predictable (Π_{j+1} is $\mathcal{F}_j$ measurable, $j \ge 0$) then $R_j = \Pi_j/\Pi_{j+1}$. In this case the short rate process determines the price deflators $\Pi_j = 1/(R_0 \cdots R_{j-1})$, $j > 0$. (We may assume $\Pi_0 = 1$ if $\mathcal{F}_0$ is trivial.)

Example 7. *(Zero Coupon Bonds) A zero coupon bond issued at time t_j and maturing at time t_k has price $X_j = D(j,k) = D_j(j,k)$ at t_j and one nonzero cash flow $C_k = 1$ at t_k.*

Note that a short rate process is equivalent to the collection of zero coupon bonds $D(j, j+1) = 1/R_j$. The arbitrage free prices $X_i = D_i(j,k)$, $j < i < k$ are determined by $D_i(j,k)\Pi_i = E_i[\Pi_k]$. Note $D_i(j,k) = D_i(i,k) = D(i,k) = E_i[\Pi_k]/\Pi_i$ for $j \le i \le k$.

Example 8. *(Forwards) A forward is a contract issued at time t_j and maturing at time t_k having price $X_j = 0$ at t_j and one nonzero cash flow $C_k = S_k - F(j,k)$ at time t_k, where S_k is the price at t_k of the underlying and $F(j,k) = F_j(j,k)$ is the forward rate that is specified at time t_j.*

No arbitrage implies $0 = E_j[(S_k - F(j,k))\Pi_k]/\Pi_j$ This implies $S_j = F(j,k)D(j,k)$ if the underlying pays no dividends. More generally, $\sum_{j<i<k} C_i D(j,i) + S_j = F(j,k)D(j,k)$. so $S_j = F(j,k)D(j,k) - \sum_{j<i<k} C_i D(j,i)$ if the stock pays fixed dividends C_i at t_i.

Example 9. *(Futures) Futures converge to an index, S_k, at expiration, t_k. Prior to that they are quoted as (Φ_j) according to what the market determines. The price of a futures is $X_j = 0$ for all j and has cash flows $C_j = \Phi_j - \Phi_{j-1}$, $j < k$, and $C_k = S_k - \Phi_{k-1}$.*

No arbitrage implies $0 = E_j[(\Phi_{j+1} - \Phi_j)\Pi_{j+1}]$ prior to expiration. Note (Φ_j) is a martingale if the deflators are predictable.

Example 10. *(Multiperiod Binomial Model) Fix the annualized realized return, R, the initial stock price, s, and the up and down increments, u, d. Let $\Omega = \{0,1\}^n$ and define $X_j = (R_j, S_j)$ by $R_j(\omega) = R^j$, where for ease of exposition we assume $t_j = j$. $S_j(\omega) = su^{W_j(\omega)}d^{j-W_j(\omega)}$, where $\omega = (\omega_1, ..., \omega_n)$ and $W(\omega) = \omega_1 + \cdots \omega_j$, $1 \le j \le n$.*

Many price deflators, (Π_j), $0 \le j \le n$ exist that make $(X_j\Pi_j)$ into a martingale. For $X \in B(\Omega)$ and $\Pi \in ba(\Omega)$ we define $X\Pi \in ba(\Omega)$ by $\langle Y, X\Pi\rangle = \langle YX, \Pi\rangle$ for $Y \in B(\Omega)$.

The canonical deflators are $\Pi_j(\omega) = R^{-j} p^{W_j(\omega)}(1-p)^{j-W_j(\omega)}$ where $p = (u - R)/(u-d)$. Note that if $p = 1/2$, then any u, d with $u+d = 2R$ yield an arbitrage-free model.

Define the measure $P \in ba(\Omega)$ by $P(\{\omega\}) = p^{W_n(\omega)}(1-p)^{n-W_n(\omega)}$ for some fixed $p \in [0,1]$. In general, $M_j = t^{W_j} c^{j-W_j}$ is a martingale under P if and only if $tp + c(1-p) = 1$.

If we are looking for price deflators of the form $\Pi_j = M_j$, then $R^j M_j$ is a martingale if and only if $Rtp + Rc(1-p) = 1$ and $S_j M_j$ is a martingale if and only if $utp + dc(1-p) = 1$. For any p between 0 and 1, we can find t and c that satisfy these conditions, viz. $t = (R-d)/Rp(u-d)$ and $c = (u-R)/R(1-p)(u-d)$.

Example 11. *(Multiperiod Geometric Brownian Motion) Fix the annualized return, $R > 0$, and the initial stock price, s, and fix the drift, μ, and volatility, σ. Let B_t be standard Brownian motion and define $X_j = (R^{t_j}, se^{(\mu-\sigma^2/2)t_j + \sigma B_{t_j}})$. We assume $t_0 = 0$.*

Again, many price deflators exist that make $X_j \Pi_j$ into a martingale. The canonical deflator is $\Pi_j = R^{-t_j} e^{-\lambda^2 t_j/2 - \lambda B_{t_j}} P$, where P is Wiener measure. In general $S_j \Pi_j$ is a martingale if and only if $r - \mu + \sigma^2/2 + \lambda^2/2 = (\sigma - \lambda)^2/2$, where $r = \log R$. This implies $\lambda = (\mu - r)/\sigma$.

The fact that S_j is not bounded has no material consequences when it comes to pricing. We may stop S_j at arbitrarily large values to fit the assumption that the prices are bounded. Since we can make the probability of stopping vanishingly small, calculation of option prices can be made arbitrarily close to those computed using the unbounded model.

5. Remarks

1) The proposed definition of arbitrage is not sufficient. Not only do traders want to know exactly how much they make upfront, they and their risk managers also want to hedge the subsequent gains they might make under favorable market conditions. (This has been empirically verified.)

2) As previously noted, $\partial v/\partial s \neq n$ in Example 2, however $\partial(Rv)/\partial R = ns$ for both Example 2 and 3. In words, the derivative of the future value of the option with respect to realized return is the dollar delta.

3) We don't assume algebras are increasing in order to model a recombining tree. In the standard binomial model the atoms of $\mathcal{F}_j$ are $\{W_j = j - 2k\}$, $0 \leq k \leq j$. This can be used to give a rigorous foundation to path bundling algorithms, e.g., Tilley 1993.

4) This theory only allows bounded functions as models of prices and positions. This corresponds to reality, but not to the classical Black-Merton-Scholes theory. As argued in the last example, this is not a real hindrance.

5) The examples show this theory has the same expressive power as the standard theory and illustrates the usefulness of distinguishing prices from cash flows. There is no need to cook up a "real world" measure. Not only does it ultimately get replaced, it adds considerable technical complications to the theory.

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