Lecture 23: Spectral theory in inner product spaces

Recall last time we defined $T : V \to V$ (in an inner product space V) to be unitary if T is invertible and $T^{-1} = T^*$. There are alternate characterizations of unitary operators.

Proposition 0.1. *The following are equivalent when V is a finite dimensional inner product space and $T : V \to V$ is linear.*

1. *T is unitary.*
2. *$\|T(v)\| = \|v\|$ for all $v \in V$.*
3. *$\langle T(v), T(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$.*
4. *$\{T(v_1), \dots, T(v_k)\}$ is orthonormal whenever $\{v_1, \dots, v_k\}$ is.*
5. *Whenever B is an orthonormal basis (for $\langle \cdot, \cdot \rangle$), the columns of $[T]_B^B$ are orthonormal (relative to the standard dot product).*

Proof. Suppose that T is unitary. Then if $v \in V$,

$$\|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle T^*Tv, v\rangle = \langle T^{-1}Tv, v\rangle = \langle v, v\rangle = \|v\|^2 \ .$$

So 1 implies 2.

Assume 2. Then using the polarization identity for Hermitian forms: for $v, w \in V$,

$$\begin{aligned} 4\langle T(u), T(v)\rangle &= \langle T(u+v), T(u+v)\rangle - \langle T(u-v), T(u-v)\rangle \\ &\quad + i\langle T(u+iv), T(u+iv)\rangle - i\langle T(u-iv), T(u-iv)\rangle \\ &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle + i\langle u+iv, u+iv\rangle - i\langle u-iv, u-iv\rangle \\ &= 4\langle u, v\rangle \ . \end{aligned}$$

So 2 implies 3.

Assuming 3, 4 follows immediately. If the vectors $\{v_1, \dots, v_k\}$ are orthonormal then

$$\langle v_i, v_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$

Then using $\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle$, this is still true for $\{T(v_1), \dots, T(v_k)\}$.

Assume 4. Then let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis . Then the matrix $[\langle \cdot, \cdot \rangle]_B^B$ is the identity. This means that for $u, v \in \mathbb{C}^n$,

$$\langle v, w\rangle = \overline{[w]_B}^t [\langle \cdot, \cdot \rangle]_B^B [v]_B = [v]_B \cdot [w]_B \ ,$$

where $\cdot$ is the standard dot product. Now the columns of $[T]_B^B$ are equal to $[T(v_1)]_B, \dots, [T(v_n])_B$ so we get by 4

$$[T(v_i)]_B \cdot [T(v_j)]_B = \langle T(v_i), T(v_j)\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$

This means 5 holds.

Last, assuming 5, we show that T is unitary. Taking B to be any orthonormal basis, the matrix $[T]_B^B$ has orthonormal (and thus linearly independent) columns, so it is invertible, giving T is invertible. Furthermore, the (i,j)-th entry of $[T^*T]_B^B$ is

$$e_j^t[T^*]_B^B[T]_B^B e_i = \left(\overline{[T]_B^B e_j}\right)^t [T]_B^B e_i \ ,$$

the standard dot product of the i-th column of $[T]_B^B$ with the j-th column, which is 1 if $i=j$ and zero otherwise. This means $[T^*T]_B^B$ is the identity matrix, giving $T^* = T^{-1}$. $\square$

Remark. *Given a matrix $A \in M_{n,n}(\mathbb{C})$ we say that A is unitary if A is invertible and $A^* = A^{-1}$. Here $A^* = \overline{A}^t$ is the conjugate transpose. Note that the (i,j)-th entry of A^*A is just the standard dot product of the i-th column of A with the j-th column of A. So A is unitary if and only if the columns of A are orthonormal. Thus the last part of the previous proposition says*

$$T \text{ unitary } \Leftrightarrow [T]_B^B \text{ unitary for any orthonormal basis } B \ .$$

We now study the eigenvalues of self-adjoint and unitary operators.

Theorem 0.2. *Let $T : V \to V$ be linear on a finite dimensional inner product space V and λ an eigenvalue of T.*

1. *If T is self-adjoint, $\lambda \in \mathbb{R}$.*
2. *If T is skew adjoint, $i\lambda \in \mathbb{R}$ (λ is imaginary).*
3. *If T is unitary, $|\lambda| = 1$.*

Proof. If T is self-adjoint, then if v is an eigenvector for eigenvalue λ,

$$\overline{\lambda}\langle v,v\rangle = \langle v, \lambda v\rangle = \langle v, T(v)\rangle = \langle T(v), v\rangle = \lambda\langle v,v\rangle \ .$$

Since $v \neq \vec{0}$, this means $\lambda = \overline{\lambda}$, or $\lambda \in \mathbb{R}$.

If T is skew-adjoint, then iT is self-adjoint:

$$(iT)^* = \bar{i}T^* = (-i)(-T) = iT \ .$$

So since the eigenvalues of iT are just $i\lambda$ for λ the eigenvalues of T, we see that all eigenvalues of T are imaginary.

If T is unitary with v an eigenvector for eigenvalue λ,

$$\lambda\overline{\lambda}\langle v,v\rangle = \langle T(v), T(v)\rangle = \langle v,v\rangle \ .$$

This means $\lambda\overline{\lambda} = 1$, or $|\lambda| = 1$ $\square$

The main theorem in spectral theory regards diagonalization of self-adjoint operators. In fact, they are more than diagonalizable; one can change the basis using a unitary transformation.

Definition 0.3. *A linear $T : V \to V$ is unitarily diagonalizable on an inner product space V if there exists an orthonormal basis of V consisting of eigenvectors for T. A matrix A is unitarily diagonalizable if there is a unitary matrix P such that*

$$P^{-1}AP \text{ is diagonal } .$$

The main theorem is the following. Recall that T is normal if $TT^* = T^*T$. Self-adjoint, skew-adjoint and unitary operators are normal.

Theorem 0.4. *Let $T : V \to V$ be linear and V a finite-dimensional inner product space. Then T is normal if and only if T is unitarily diagonalizable.*

Proof. One direction is easy. Suppose that T is unitarily diagonalizable. Then we can find an orthonormal basis B such that $[T]_B^B$ is diagonal. Then $[T^*]_B^B$ is also diagonal, since it is just the conjugate transpose of T. Any two diagonal matrices commute, so, in particular, these matrices commute. This means

$$[TT^*]_B^B = [T]_B^B[T^*]_B^B = [T^*]_B^B[T]_B^B = [T^*T]_B^B \ ,$$

giving $TT^* = T^*T$.

The other direction is more difficult. We will first show that if T is self-adjoint then T is unitarily diagonalizable. For that we need a lemma.

Lemma 0.5. *If $T : V \to V$ is linear then*

$$R(T)^\perp = N(T^*) \text{ and } N(T)^\perp = R(T^*) \ .$$

Proof. Let us assume that $v \in N(T^*)$. Then for any $w \in R(T)$ we can find $w' \in V$ such that $T(w') = w$. Then

$$\langle v, w\rangle = \langle v, T(w')\rangle = \langle T^*(v), w'\rangle = 0 \ .$$

So $v \in R(T)^\perp$. This means $N(T^*) \subset R(T)^\perp$. On the other hand, these spaces have the same dimension:

$$\dim\, R(T)^\perp = \dim\, V - \dim\, R(T) = \dim\, N(T) \ ,$$

but the matrix of T is just the conjugate transpose of that of T^* (in any orthonormal basis), so $\dim\, N(T) = \dim\, N(T^*)$, completing the proof of the first statement.

For the second, we apply the first statement to T^*:

$$R(T^*)^\perp = N(T)$$

and then perp both sides: $R(T^*) = N(T)^\perp$. □

Now we move to the proof.

Self-adjoint case. Assume that $T = T^*$; we will show that V has an orthonormal basis of eigenvectors for T by induction on dim V. First if dim $V = 1$ then any nonzero vector v is an eigenvector for T. Set our orthonormal basis to be $\{v/\|v\|\}$.

If dim $V > 1$ then by the fact that we are over $\mathbb{C}$, let v be any eigenvector for T^* (this is possible since $\mathbb{C}$ is algebraically closed) for eigenvalue λ. Then dim $N(T^* - \lambda I) > 0$ and thus

$$\begin{aligned} \dim\, R(T - \lambda I) &= \dim\, V - \dim\, N((T - \lambda I)^*) \\ &= \dim\, V - \dim\, N(T^* - \overline{\lambda} I)\ . \end{aligned}$$

However as T^* is self-adjoint, $\lambda \in \mathbb{R}$, so this is

$$\dim\, V - \dim\, N(T^* - \lambda I) < \dim\, V\ .$$

Furthermore if dim $R(T - \lambda I) = 0$ then we must have dim $N(T^* - \lambda I) = \dim\, V$, meaning the whole space is the eigenspace for T^*. In this case we just take any orthonormal basis B of V consisting of eigenvectors of T^* and write $[T^*]_B^B$ as a diagonal matrix in this basis. However as it is orthonormal, $[T]_B^B$ is just the conjugate transpose, and is thus diagonal, meaning B is an orthonormal basis of eigenvectors for T, completing the proof.

So we may assume that

$$0 < \dim\, R(T - \lambda I) < \dim\, V\ .$$

Now both $N = N(T^* - \lambda I)$ and $R = R(T - \lambda I)$ are T-invariant (check this!) so we can consider T_N and T_R, the restrictions of T to N and R. These are still self-adjoint, as for instance

$$\langle T_R v, w \rangle = \langle v, T_R w \rangle \text{ for all } v, w \in R\ ,$$

meaning $T_R^* = T_R$ (similarly for T_N). Therefore by induction we can find orthonormal bases

$$B_N = \{v_1, \ldots, v_l\} \text{ and } B_R = \{v_{l+1}, \ldots, v_n\}$$

of N and R consisting of eigenvectors for T_N and T_R (and thus of T). But N and R are perpendicular, meaning that $B = B_N \cup B_R$ is still orthonormal.

Normal case. Suppose now that T is normal and write T as a self-adjoint part and a skew-adjoint part:

$$T = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*) = T_1 + T_2\ .$$

Since T is normal, these two parts commute! Since self-adjoint operators are unitarily diagonalizable, so are skew adjoint ones (since whenever U is skew-adjoint, iU is self-adjoint). Therefore we have commuting unitarily diagonalizable transformations, they are simultaneously diagonalizable. This follows from a proof from the homework, where it was shown that commuting diagonalizable transformations are simultaneously diagonalizable. The transformations we consider are unitarily diagonalizable, but the same proof works here (check this!) So we can find an orthonormal basis B such that $[T_1]_B^B$ and $[T_2]_B^B$ are diagonal. Thus

$$[T]_B^B = [T_1]_B^B + [T_2]_B^B \text{ is diagonal}$$

and we are done. $\square$