

Dynamic Cointegrated Pairs Trading: Time-Consistent Mean-Variance Strategies

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Abstract

Cointegration is a useful econometric tool for identifying assets which share a common equilibrium. Cointegrated pairs trading is a trading strategy which attempts to take a profit when cointegrated assets depart from their equilibrium. This paper investigates the optimal dynamic trading of cointegrated assets using the classical mean-variance portfolio selection criterion. To ensure rational economic decisions, the optimal strategy is obtained over the set of time-consistent policies from which the optimization problem is enforced to obey the dynamic programming principle. We solve the optimal dynamic trading strategy in a closed-form explicit solution. This analytical tractability enables us to prove rigorously that cointegration ensures the existence of statistical arbitrage using a dynamic time-consistent mean-variance strategy. This provides the theoretical grounds for the market belief in cointegrated pairs trading. Comparison between time-consistent and pre-commitment trading strategies for cointegrated assets shows the former to be a persistent approach, whereas the latter makes it possible to generate infinite leverage once a cointegrating factor of the assets has a high mean reversion rate.

Keywords: Cointegration; Mean-Variance Portfolio Theory; Pair Trade; Time-consistency.

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1 Introduction

The concept of cointegration, which led to Nobel prize-winning work in 2003, originated with Granger (1981) and Engle and Granger's (1987) assertion that a linear combination of non-stationary economic series could be stationary, thus signifying the co-movement of economic series beyond the conventional correlation coefficient. Cointegration techniques have been applied to examining the co-movement of a variety of financial assets, including stocks (Cerchi and Havenner, 1988; Taylor and Tonks, 1989; Mylonidis and Kollias, 2010), exchange rates (Baillie and Bollerslev, 1989; Kellard et al., 2010) and commodities (Maslyuka and Smyth, 2009). They have also brought new insights into hedging (Alexander, 1999) and index tracking (Alexander and Dimitriu, 2005).

Cointegration is also an important tool in pairs trading (Vidyamurthy, 2004). As cointegrated stocks should theoretically have a narrow spread in long-term equilibrium, a hedge fund strategy is to bet on the spread by long-short or relative-value arbitrage. If the spread widens, then the strategy is to short the high stock and buy the low stock. As the spread narrows to some equilibrium value, unwinding the position results in a profit. Practitioners used to specify a threshold, which, when exceeded by the spread, would trigger a trade. The optimal choice of threshold, the amounts invested in the long and short positions, and the length of holding time are rules of thumb in practice. This relative-value arbitrage is, however, a suboptimal strategy. When two cointegrated stocks are undervalued relative to their long-term means, it could be better to long both rather than betting on their spread. Gatev et al. (2006) show that this simple relative-value arbitrage generates profits, but the contribution of mean reversion (or cointegration) is not significant. In addition, relative-value arbitrage is a static strategy that does not take advantage of dynamic trading.

The dynamic trading of cointegrated assets is an interesting topic in finance. Wachter (2002) considers the dynamic trading of a mean-reverting risky asset, a special case of cointegration, and a risk-free bond in a complete market. Jurek and Yang (2007) attempt to solve the dynamic trading of cointegrated assets using the utility function with constant risk aversion. Although they successfully solve the problem for the case of a single mean-reverting risky asset and a risk-free bond, their solution can be generalized only to *uncorrelated* cointegrated assets. Using the Markowitz (1952) mean-variance (MV) criterion, Chiu and Wong (2011) systematically investigate the optimal dynamic trading strategy of cointegrated assets with a general correlation structure. The latter's solution, however, is a time-inconsistent (or precommitment) policy, which violates the dynamic pro-

gramming principle of optimization. Such an obstacle is shared by most related work, such as that of Li and Ng (2000), Zhou and Li (2000), Bielecki et al. (2005), and many others due to the subtle non-separability of the MV objectives.

Investigation of time-consistent dynamic MV asset allocation has recently attracted a great deal of attention in the literature. In a discrete-time setting for example, Cui et al. (2010) show that it is possible to construct a trading strategy that always outperforms the optimal MV precommitment policy by taking advantage of time-inconsistency. In a continuous-time economy, Basak and Chabakauri (2010) propose a framework for deriving time-consistent optimal MV portfolio policies and apply it to the wealth allocation between a risky stock and a risk-free bond. Bjork et al. (2011) generalize their work to allow state-dependent risk aversion. Wang and Forsyth (2011) construct numerical schemes for the Hamilton-Jacobi-Bellman (HJB) equations associated with time-consistent MV portfolio problems subject to different market constraints.

In this paper, we investigate the dynamic trading of cointegrated assets and offer an optimal time-consistent asset allocation policy based on the MV criterion. The dynamics of the assets are the diffusion limit of the discrete-time error-correction model of cointegrated time series derived by Duan and Pliska (2004). Using the framework developed by Basak and Chabakauri (2010), time-consistent dynamic cointegrated pairs trading strategies are systematically derived in explicit and closed-form solutions.

In addition to its mathematical derivations, this paper contributes to the finance literature by providing insights into several important questions. For example, is there any relationship between cointegration and statistical arbitrage? Using the definition of statistical arbitrage in Hogan et al. (2004), we rigorously prove that the existence of cointegration ensures the existence of a statistical arbitrage strategy. In particular, an optimal time-consistent MV portfolio policy with an infinite investment time is precisely such a strategy. In a financial market in which non-stationary economic series cannot be error-corrected, time-consistent MV policies fail to deliver statistical arbitrage profits and are asymptotically no-arbitrage. In the Black-Scholes market, in which, the log-values of risky assets follow Brownian motions, statistical arbitrage asymptotically exists, but the long-term time-consistent MV policy suggests no investment.

Whereas the statistical arbitrage concept considered in Hogan et al. (2004) concerns long-term investments, practitioners are more concerned with finite-time, particularly short-term, investments. Does cointegration enhance profitability in finite time? By defining profitability as the trade-off between risk and return, our theory predicts that the time-consistent MV trading of cointegrated risky assets

generates a higher price of risk or Sharpe ratio than that of risky assets following the Black-Scholes model. We develop an index to measure the profitability of different cointegrated pairs.

As the relative-value arbitrage rule is a popular trading strategy, it is natural to ask whether our optimal trading strategy resembles a long-short trading rule. We provide a counter example that illustrates numerically a case of short-selling a pair of cointegrated assets for a risk-free bond. Such a situation occurs when the two assets are overvalued. Unlike the relative-value arbitrage rule reported in Gatev et al. (2006), the mean reversion rate contributes significantly to the profit of dynamic pairs trading with the MV objective.

What is the major advantage of the time-consistent policy over the precommitment policy? Basak and Chabakauri (2010) find that, in general, the time-consistent policy suggests less investment in risky assets than its precommitment counterpart, thus implying that the former is more prudent. Their game-theoretic argument further justifies the economic soundness of time-consistent MV policies. Using the Black-Scholes model of risky assets, Wang and Forsyth (2011) show numerically that the price of risk from the precommitment policy is greater than that from the time-consistent policy. We supplement their results by demonstrating that the precommitment policy for cointegrated assets suffers from the problem of a finite escape time at which investors are recommended to take infinite positions. Time-consistent policies do not share this problem. This shortcoming of the precommitment policy implies the inadequate robustness of the optimal trading rule to the estimated model parameters. The time-consistent policy, in contrast, is more robust, particularly to the mean reversion rate. The time-consistent constraint imposed on the portfolio problem confines us to rational decisions, which do not take full advantage of the assumed stochastic model, but render it more stable.

The rest of this paper is organized as follows. Section 2 introduces the cointegration dynamics and details the problem formulation. The time-consistent MV portfolio problem is completely solved in Section 3. The optimal policy and efficient frontier for trading cointegrated assets are derived in closed-form solutions. The analytical results are then applied to the pairs trading of cointegrated assets in Section 4. In particular, we investigate the arbitrage of dynamic cointegrated pairs trading using the MV portfolio theory. The derived time-consistent policy is contrasted with its precommitment policy. Numerical examples are provided to illustrate the application. Section 5 concludes the paper.

2 Problem formulation

2.1 Cointegration

Granger's representation theorem enables the cointegrated vector time series to be expressed as an error correction model. In discrete time, an error correction dynamic for the n -component asset price time series with k ($1 \leq k \leq n$) cointegrating factors is defined as follows.

$$\ln S_{i,t} - \ln S_{i,t-1} = \mu_i + \sum_{j=1}^k \delta_{ij} z_{j,t-1} + \sigma_{i,t} \epsilon_{i,t}, \quad i = 1, \dots, n,$$

$$z_{j,t} = a_j + b_j t + \sum_{i=1}^n c_{ij} \ln S_{i,t} \quad \text{for } j = 1, \dots, k,$$

where $S_{i,t}$ is the price of risky asset i at time t for $i = 1, \dots, n$; (c_{1j}, \dots, c_{nj}) are linearly independent vectors for $j = 1, \dots, k$; $\sigma_{i,t}$ is the volatility of asset i at time t ; and the random vector $[\epsilon_{1,t}, \dots, \epsilon_{n,t}]$ follows a multivariate normal distribution with mean zero and a constant correlation coefficient matrix. In the error correction model, the vector of k cointegrating factors, $[z_{1,t}, \dots, z_{k,t}]$, forms a weakly stationary time series such that each $z_{j,t}$ has a bounded variance at all time points for all $j = 1, 2, \dots, k$.

Our analysis is based on the diffusion limit of the discrete-time error correction model. Duan and Pliska (2004) derived the diffusion limit as

$$d \ln S_{i,t} = \left(\mu_i + \sum_{j=1}^k \delta_{ij} z_{j,t} \right) dt + \sigma_i d\widehat{W}_{i,t}, \quad i = 1, \dots, n, \quad (1)$$

$$z_{j,t} = a_j + b_j t + \sum_{i=1}^n c_{ij} \ln S_{i,t} \quad j = 1, \dots, k, \quad (2)$$

where $\widehat{W}_{i,t}$ are correlated Wiener processes.

2.2 The market

Using the concept of cointegration, we consider a financial market that contains $n + 1$ assets that are traded continuously within the time horizon $[0, T]$. These

assets are labeled by S_i for $i = 0, 1, 2, \dots, n$, with the 0-th asset being risk-free. The risk-free asset satisfies the following differential equation.

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt, \\ S_0(0) &= R_0 > 0, \end{aligned}$$

where $r(t)$ is the time deterministic risk-free rate. The risky assets are defined through their log-price processes, $X_1(t), \dots, X_n(t)$, where $X_j(t) = \ln S_j(t)$. By substituting (2) into (1), the vector of the log-prices of risky assets, $X(t)$, satisfies the stochastic differential equation (SDE),

$$dX(t) = [\theta(t) - \mathcal{A}X(t)]dt + \sigma(t)dW_t, \quad t \in [0, T], \quad (3)$$

where $W_t = (W_t^1, \dots, W_t^n)'$ is a standard \mathcal{F}_t -adapted n -dimensional Wiener process on a fixed filtered complete probability space $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{F}_t)$; W_t^i and W_t^j are mutually independent for all $i \neq j$; \mathcal{A} is an $n \times n$ constant matrix of cointegration coefficients; and $\sigma(t)\sigma(t)' = \Sigma(t)$ is the covariance matrix of an $\mathbb{R}^{n \times n}$ -valued continuous function on $[0, T]$. Therefore, risky assets are correlated with a time-varying covariance matrix in general. In line with the literature, we assume that the non-degeneracy condition of $\Sigma(t) \geq \delta I_n$ holds for all $t \in [0, T]$ and for some $\delta > 0$, whereas $\theta(t)$ and $\sigma(t)$ are continuous functions on $[0, T]$.

SDE (3) represents the diffusion limit of the first-order vector autoregressive (VAR(1)) model with Gaussian innovations. The error correction model is a special VAR model that requires all eigenvalues of \mathcal{A} to be nonnegative real numbers (or complex numbers with nonnegative real parts). For the time being, we impose no condition on the matrix \mathcal{A} to permit a general discussion. The concept of cointegration will be useful in our analysis of statistical arbitrage.

2.3 Mean-variance portfolio problem

Consider a mean-variance investor with an initial wealth of Y_0 in the specified financial market with cointegration. The investor seeks an admissible portfolio strategy so that the variance of the terminal wealth level is minimized and the expected final wealth equals \bar{Y} . Let $u_i(t)$ be the amount invested in asset i and $N_i(t)$ be the number of the i -th asset in the portfolio of the investor. The wealth of the investor at time t is then given by $Y(t) = \sum_{i=0}^n u_i(t) = \sum_{i=0}^n N_i(t)S_i(t)$. The portfolio

$$u(t) = (u_1(t), u_2(t), \dots, u_n(t))'$$

is said to be admissible if $u(t)$ is a non-anticipating and \mathcal{F}_t -adapted process such that

$$\int_0^T |u(\tau)|^2 d\tau < \infty \quad a.s.$$

In such a situation, we write $u \in \mathcal{L}_{\mathcal{F}_T}^2([0, T], \mathbb{R}^n)$.

Applying Itô's lemma to $Y(t)$ with respect to the cointegrating dynamics (3), the wealth process is given by

$$\begin{aligned} dY(t) &= [r(t)Y(t) + u(t)'\alpha(t)]dt + u(t)'\sigma(t)dW_t, \\ Y(0) &= Y_0, \end{aligned} \quad (4)$$

where $\mathbf{1}$ is the column vector with all elements being 1, and

$$\alpha(t) = \theta(t) - \mathcal{A}X(t) + \frac{1}{2}\mathcal{D}(\Sigma(t))\mathbf{1} - r(t)\mathbf{1}, \quad (5)$$

in which $\mathcal{D}(\Sigma(t))$ is the diagonal matrix with all diagonal elements equal to those of the covariance matrix $\Sigma(t) = \sigma(t)\sigma(t)'$. Note that $\alpha(t)$ is a random variable as it depends on $X(t)$ through (5). Therefore, our portfolio optimization problem involves a random parameter in the wealth process which complicates the solution derivation procedure.

Finding an optimal control $u(\cdot)$ with respect to a certain objective function is referred to as a portfolio selection problem in finance. The MV portfolio selection problem with cointegration is formulated as

$$\begin{aligned} \text{P(MV)}' : \quad & \min_u \quad \text{Var}(Y(T)) \\ \text{s.t.} \quad & \mathbb{E}[Y(T)] = \bar{Y}, \quad u \in \mathcal{L}_{\mathcal{F}_T}^2([0, T], \mathbb{R}^n), \quad (4), \end{aligned}$$

for a pre-specified expected final wealth \bar{Y} . A classical approach invokes the equivalent Lagrangian problem:

$$\begin{aligned} \text{P}(\lambda)' : \quad & \min_u \quad \text{Var}(Y(T)) - 2\lambda\mathbb{E}[Y(T)] \\ \text{s.t.} \quad & u \in \mathcal{L}_{\mathcal{F}_T}^2([0, T], \mathbb{R}^n), \quad (4). \end{aligned}$$

After solving $\text{P}(\lambda)'$ for any given $\lambda > 0$, the solution of $\text{P(MV)}'$ can be retrieved by matching \bar{Y} with a suitable value of λ .

Problems $\text{P(MV)}'$ and $\text{P}(\lambda)'$ have been systematically solved by Chiu and Wong (2011). However, the generic space $\mathcal{L}_{\mathcal{F}_T}^2([0, T], \mathbb{R}^n)$ of all possible asset

allocation policies does not guarantee that the solution follows the dynamic programming principle, which is a basic requirement of rational economic decisions (Strotz, 1956). In fact, Chiu and Wong (2011) obtained a precommitment policy.

For any given $\lambda > 0$, consider the utility function¹

$$U_t = \text{Var}(Y(T)|\mathcal{F}_t) - 2\lambda E[Y(T)|\mathcal{F}_t].$$

Using the law of iterated expectations, Basak and Chabakauri (2010) show that a portfolio policy satisfying

$$U_t = E[U_{t+\tau}|\mathcal{F}_t] + \text{Var}(E[Y(T)|\mathcal{F}_{t+\tau}]|\mathcal{F}_t), \quad t \geq 0, \tau > 0 \quad (6)$$

is a time-consistent policy that obeys the dynamic programming principle. Hence, we define the time-consistent solution space of the cointegrated assets as

$$\mathcal{U}(t, T) = \{u \in \mathcal{L}_{\mathcal{F}_T}^2([t, T], \mathbb{R}^n) : (4) \text{ and } (6) \text{ hold}\}. \quad (7)$$

The time-consistent versions of $P(\text{MV})'$ and $P(\lambda)'$ are respectively revised as

$$P(\text{MV}) : \quad \min_{u \in \mathcal{U}(0, T)} \text{Var}(Y(T)) \quad \text{s.t.} \quad E[Y(T)] = \bar{Y}$$

and

$$P(\lambda) : \quad \min_{u \in \mathcal{U}(0, T)} \text{Var}(Y(T)) - 2\lambda E[Y(T)].$$

3 The trading strategy and efficient frontier

3.1 Time-consistent recursive equation

To solve problem $P(\lambda)$, we need a time-consistent HJB equation for the value function,

$$J(t, Y(t), X(t), T) = \min_{u \in \mathcal{U}(t, T)} \text{Var}(Y(T)|\mathcal{F}_t) - 2\lambda E[Y(T)|\mathcal{F}_t]. \quad (8)$$

We denote $J^* := J(0, Y_0, X(0), T)$ as the initial value of the objective function in $P(\lambda)$ with the optimal policy adopted.

¹We use an alternative, but equivalent, utility function to Basak and Chabakauri (2010).

Using (4), an application of Itô's lemma shows that

$$d \left(e^{\int_t^T r(s)ds} Y(t) \right) = e^{\int_t^T r(s)ds} [\alpha(t)' u(t) dt + u(t)' \sigma(t) dW_t], \quad (9)$$

which implies

$$\mathbb{E}[Y(T)|\mathcal{F}_t] = e^{\int_t^T r(s)ds} Y(t) + \mathbb{E} \left[\int_t^T e^{\int_s^T r(\tau)d\tau} \alpha(s)' u(s) ds \middle| \mathcal{F}_t \right], \quad (10)$$

$$\begin{aligned} \text{Var}(Y(T)|\mathcal{F}_t) &= \text{Var} \left(\int_t^T e^{\int_s^T r(\tau)d\tau} \alpha(s)' u(s) ds \right. \\ &\quad \left. + \int_t^T e^{\int_s^T r(\tau)d\tau} u(s)' \sigma(s) dW_s \middle| \mathcal{F}_t \right). \end{aligned} \quad (11)$$

Substituting (10) and (11) into (8) results in

$$J(t, Y(t), X(t), T) = J_1(t, X, T) - 2\lambda e^{\int_t^T r(s)ds} Y(t), \quad (12)$$

where

$$J_1(t, X, T) = \min_{u \in \mathcal{U}(t, T)} \text{Var}(Y(T)|\mathcal{F}_t) - 2\lambda \mathbb{E} \left[\int_t^T e^{\int_s^T r(\tau)d\tau} \alpha(s)' u(s) ds \middle| \mathcal{F}_t \right]. \quad (13)$$

An interpretation of (12) is that the current wealth level, $Y(t)$, is linearly separated from the objective function. In other words, the optimal solution of $P(\lambda)$, $u^*(t, T)$, does not depend on the wealth level at time t as it suffices to operate the dynamic programming on the function J_1 which is independent of $Y(t)$; see (11). The aim of including T in the optimal solution is to highlight its dependence on the investment period specified by the investor. The Markovian nature of the economy further ensures that the optimal policy depends solely on $X(t)$ and t for any given constant λ . Hence, from now on, we designate the optimal policy as $u^*(t, X, T)$.

Lemma 3.1. The value function $J(t, Y(t), X(t), T)$ defined in (8) satisfies the following HJB equation.

$$\min_{u(t)} \left\{ \mathbb{E}[dJ|\mathcal{F}_t] + \text{Var}(d(e^{\int_t^T r(s)ds} Y) + d\Gamma|\mathcal{F}_t) \right\} = 0, \quad (14)$$

where

$$\Gamma(t, X(t), T) = \mathbb{E} \left[\int_t^T e^{\int_s^T r(\tau)d\tau} \alpha(s)' u^*(s) ds \middle| \mathcal{F}_t \right], \quad (15)$$

$$u^*(t, X(t), T) = e^{-\int_t^T r(s)ds} \left[\lambda \Sigma(t)^{-1} \alpha(t) - \frac{\partial \Gamma}{\partial X}(t, X(t)) \right], \quad (16)$$

$\alpha(t)$ is defined in (5), and $\frac{\partial}{\partial X}$ is the gradient operator with respect to the vector of log-asset values, X . Further, u^* in (16) is the optimal dynamic policy of $P(\lambda)$.

Although Lemma 3.1 is a multivariate generalization of Basak and Chabakauri's (2010) result, we do not regard it as an important contribution. The proof based on their approach is given in the appendix solely to produce a self-contained article. However, the major difficulty of our problem is the derivation of Γ in (15) and u^* in (16) in explicit closed-form solutions in a financial market with cointegrated assets. The key to deriving these solutions is the recognition of a quadratic form.

3.2 The optimal trading strategy

Lemma 3.2. The function $\Gamma(t, X(t))$ defined in (15) takes the form

$$\Gamma(t, X(t), T) = \lambda \hat{\mathcal{H}}(t, \alpha(t), T), \quad (17)$$

where $\alpha(t)$ is defined in (5), and

$$\hat{\mathcal{H}}(t, \alpha(t), T) = \hat{\mathbb{E}} \left[\int_t^T \alpha(s)' \Sigma(s)^{-1} \alpha(s) ds \middle| \mathcal{F}_t \right], \quad (18)$$

which uses the equivalent probability measure $\hat{\mathcal{P}}$:

$$\frac{d\hat{\mathcal{P}}}{d\mathcal{P}} = \exp \left\{ \int_t^T -\frac{1}{2} \alpha(s)' \Sigma(s)^{-1} \alpha(s) ds - \int_t^T \alpha(s)' \Sigma(s)^{-1} \sigma(s) dW_s \right\}.$$

Proof. By substituting (16) into (15), we arrive at

$$\Gamma = \mathbb{E} \left[\int_t^T \left(\lambda \alpha' \Sigma^{-1} \alpha - \left(\frac{\partial \Gamma}{\partial X} \right)' \alpha \right) ds \middle| \mathcal{F}_t \right]. \quad (19)$$

Alternatively, applying Itô's lemma to $\Gamma(t, X)$ with respect to X yields

$$- \Gamma = \mathbb{E} \left[\int_t^T \left(\frac{\partial \Gamma}{\partial t} + \left(\frac{\partial \Gamma}{\partial X} \right)' (\theta - \mathcal{A}X) + \frac{1}{2} \text{tr} \left(\sigma' \frac{\partial^2 \Gamma}{\partial X^2} \sigma \right) \right) ds \middle| \mathcal{F}_t \right], \quad (20)$$

in which we use the fact that $\Gamma(T, X) = 0$. Note that $\frac{\partial}{\partial X}$ is the gradient operator with respect to X , $\frac{\partial^2}{\partial X^2}$ the Hessian matrix operator, and $\text{tr}(\cdot)$ the trace of a

square matrix. Putting (19) and (20) together, we end up with a partial differential equation (PDE) of Γ :

$$\begin{aligned} \frac{\partial \Gamma}{\partial t} + \left(\frac{\partial \Gamma}{\partial X} \right)' (\theta - \mathcal{A}X - \alpha) + \frac{1}{2} \text{tr} \left(\sigma' \frac{\partial^2 \Gamma}{\partial X^2} \sigma \right) + \lambda \alpha' \Sigma^{-1} \alpha &= 0, \\ \Gamma(T, X, T) &= 0. \end{aligned} \quad (21)$$

As the coefficient of the gradient vector and the drift of X under \mathcal{P} are differed by a vector α , the result follows by the Feynman-Kač formula and the Girsanov theorem. \square

Proposition 3.1. The time-consistent dynamically optimal trading strategy of problem P(λ) is given by

$$u^*(t, X(t), T) = \lambda e^{-\int_t^T r(s)ds} \left[\left(\Sigma(t)^{-1} + \mathcal{A}' \widehat{K}(t, T) \right) \alpha(t) + \mathcal{A}' \widehat{N}(t, T) \right], \quad (22)$$

and the auxiliary function $\widehat{\mathcal{H}}$ appearing in Lemma 3.2 is

$$\widehat{\mathcal{H}}(t, \alpha, T) = \frac{1}{2} \alpha' \widehat{K}(t, T) \alpha + \widehat{N}(t, T)' \alpha + \widehat{M}(t, T), \quad (23)$$

where $\alpha(t)$ is defined in (5), $\mathcal{D} = \mathcal{D}(\Sigma(t))$,

$$\widehat{K}(t, T) = 2 \int_t^T \Sigma(s)^{-1} ds \in \mathbb{R}^{n \times n}; \quad (24)$$

$$\widehat{N}(t, T)' = \int_t^T \Theta(s)' \widehat{K}(s) ds \in \mathbb{R}^{1 \times n}; \quad (25)$$

$$\widehat{M}(t, T) = \int_t^T \left[\widehat{N}(s)' \Theta(s) + \frac{1}{2} \text{tr} \left(\sigma(s)' \mathcal{A}' \widehat{K}(s) \mathcal{A} \sigma(s) \right) \right] ds \in \mathbb{R}; \quad (26)$$

$$\Theta(t) = \dot{\theta} + \frac{1}{2} \left(\dot{\mathcal{D}} + \mathcal{A} \mathcal{D} \right) \mathbf{1} - (\dot{r} + \mathcal{A}r) \mathbf{1} \in \mathbb{R}^{1 \times n}. \quad (27)$$

Proof. We begin by deriving the explicit solution to function $\widehat{\mathcal{H}}$ in Lemma 3.2. By Itô's lemma and Girsanov's theorem, the dynamic of $\alpha(t)$ under the $\widehat{\mathcal{P}}$ -measure is given by

$$d\alpha(t) = \Theta dt - \mathcal{A} \sigma d\widehat{W}_t, \quad (28)$$

where Θ is given in (27) and \widehat{W}_t is the Wiener process under $\widehat{\mathcal{P}}$, such that

$$d\widehat{W}_t = dW_t - \sigma' \Sigma^{-1} \alpha dt.$$

Applying the Feynman-Kač formula to $\widehat{\mathcal{H}}(t, \alpha)$ with respect to (28), the PDE of $\widehat{\mathcal{H}}$ is obtained as

$$\frac{\partial \widehat{\mathcal{H}}}{\partial t} + \left(\frac{\partial \widehat{\mathcal{H}}}{\partial \alpha} \right)' \Theta + \frac{1}{2} \text{tr} \left(\sigma' \mathcal{A}' \frac{\partial^2 \widehat{\mathcal{H}}}{\partial \alpha^2} \mathcal{A} \sigma \right) + \alpha' \Sigma^{-1} \alpha = 0, \quad (29)$$

$$\widehat{\mathcal{H}}(T, \alpha, T) = 0. \quad (30)$$

Consider the quadratic form (23) where \widehat{K} , \widehat{N} and \widehat{M} satisfy (24), (25) and (26). Simple differentiations show that

$$\frac{\partial \widehat{\mathcal{H}}}{\partial t} = \frac{1}{2} \alpha' \dot{\widehat{K}} \alpha + \dot{\widehat{N}}' \alpha + \dot{\widehat{M}}, \quad \frac{\partial \widehat{\mathcal{H}}}{\partial \alpha} = \widehat{K} \alpha + \widehat{N}, \quad \frac{\partial^2 \widehat{\mathcal{H}}}{\partial \alpha^2} = \widehat{K}.$$

Substituting these formulas into (29) yields

$$\begin{aligned} & \frac{1}{2} \alpha' \dot{\widehat{K}} \alpha + \dot{\widehat{N}}' \alpha + \dot{\widehat{M}} + \left(\widehat{K} \alpha + \widehat{N} \right)' \Theta + \frac{1}{2} \text{tr} \left(\sigma' \mathcal{A}' \widehat{K} \mathcal{A} \sigma \right) + \alpha' \Sigma^{-1} \alpha \\ = & \alpha' \left(\frac{\dot{\widehat{K}}}{2} + \Sigma^{-1} \right) \alpha + \left(\dot{\widehat{N}}' + \Theta' \widehat{K} \right) \alpha + \dot{\widehat{M}} + \widehat{N}' \Theta + \frac{1}{2} \text{tr} \left(\sigma' \mathcal{A}' \widehat{K} \mathcal{A} \sigma \right) = 0. \end{aligned}$$

In other words, the proposed quadratic form satisfies the governing equation of (29). In addition, it is easy to check that $\widehat{\mathcal{H}}(T, \alpha, T) = 0$ using the quadratic form. Thus, it is the solution of PDE (29).

By Lemma 3.2, $\Gamma(t, X, T) = \lambda \widehat{\mathcal{H}}(t, \alpha, T)$. Simple differentiation shows that

$$\frac{\partial \Gamma}{\partial X} = -\lambda \mathcal{A}' \left(\widehat{K} \alpha + \widehat{N} \right).$$

Substituting this into the expression of u^* in Lemma 3.1 produces (22). \square

Proposition 3.2. The time-consistent dynamically optimal trading strategy of the MV portfolio problem P(MV) is given by

$$u^*(t, X(t), T) = \frac{\overline{Y} e^{-\int_t^T r(s) ds} - Y_0}{\widehat{\mathcal{H}}(0, \alpha(0), T)} \left[\left(\Sigma(t)^{-1} + \mathcal{A}' \widehat{K}(t, T) \right) \alpha(t) + \mathcal{A}' \widehat{N}(t, T) \right], \quad (31)$$

where $\widehat{\mathcal{H}}$, \widehat{K} , and \widehat{N} are obtained in Proposition 3.1 and $\alpha(t)$ is defined in (5).

Proof. Let Y^* be the wealth process with the optimal policy u^* adopted. By (10) and (15), it is clear that

$$\mathbb{E}[Y^*(T)] = e^{\int_0^T r(s)ds} Y(0) + \Gamma(0, X(0), T),$$

which is equivalent to

$$\mathbb{E}[Y^*(T)] = e^{\int_0^T r(s)ds} Y(0) + \lambda \widehat{\mathcal{H}}(0, \alpha(0), T). \quad (32)$$

In P(MV), the expected final wealth is pre-specified as $\mathbb{E}[Y^*(T)] = \bar{Y}$. Hence, we obtain the Lagrange multiplier as

$$\lambda = \frac{\bar{Y} - e^{\int_0^T r(s)ds} Y(0)}{\widehat{\mathcal{H}}(0, \alpha(0), T)}. \quad (33)$$

The result follows by substituting λ into the result in Proposition 3.1. \square

3.3 Mean-variance efficient frontier

An important topic in MV portfolio theory is the **efficient frontier**, which describes the relationship between the expected value and variance of the final wealth level which adopts the optimal trading strategy. For the MV portfolio selection problem, P(MV), the expected final wealth is pre-specified, and it suffices to calculate the variance of the optimal final wealth. Therefore, we focus on the variance of the terminal wealth.

Lemma 3.3. If the optimal time-consistent trading strategy $u^*(t, X(t), T)$ of P(λ) for $\lambda > 0$ shown in Proposition 3.1 is adopted, then the variance of the final wealth $Y^*(T)$ is given by

$$\text{Var}(Y^*(T)) = \lambda^2 \mathcal{H}(0, \alpha(0), T), \quad (34)$$

where

$$\mathcal{H}(t, \alpha(t), T) = \mathbb{E} \left[\int_t^T \alpha(s)' \Sigma(s)^{-1} \alpha(s) ds \middle| \mathcal{F}_t \right].$$

Proof. To calculate $\text{Var}(Y^*(T))$, we need the stochastic dynamics of Y^* . We first consider the process of Γ in (57):

$$d\Gamma = \left[\frac{\partial \Gamma}{\partial t} + \left(\frac{\partial \Gamma}{\partial X} \right)' (\theta - \mathcal{A}X) + \frac{1}{2} \text{tr} \left(\sigma' \frac{\partial^2 \Gamma}{\partial X^2} \sigma \right) \right] dt + \left(\frac{\partial \Gamma}{\partial X} \right)' \sigma dW_t.$$

Using (21), the foregoing process is simplified to

$$d\Gamma = \left[\left(\frac{\partial \Gamma}{\partial X} \right)' \alpha - \lambda \alpha' \Sigma^{-1} \alpha \right] dt + \left(\frac{\partial \Gamma}{\partial X} \right)' \sigma dW_t.$$

Equation (9) shows that

$$d(e^{\int_t^T r(s)ds} Y^*) = e^{\int_t^T r(s)ds} [\alpha' u^* dt + u^{*'} \sigma dW_t].$$

Applying (16) to the foregoing equation, and combining it with $d\Gamma$, we have

$$d(e^{\int_t^T r(s)ds} Y^*) + d\Gamma = [\lambda \alpha' \Sigma^{-1} \sigma] dW_t. \quad (35)$$

Integrating from 0 to T and taking the variance operator results in (34). \square

Proposition 3.3. The efficient frontier of the P(MV) is given by

$$\text{Var}(Y^*(T)) = \left(\frac{\bar{Y} - e^{\int_0^T r(s)ds} Y(0)}{\hat{\mathcal{H}}(0, \alpha(0), T)} \right)^2 \mathcal{H}(0, \alpha(0), T), \quad (36)$$

where $\hat{\mathcal{H}}$ has been obtained in (23) as

$$\mathcal{H}(t, \alpha(t), T) = \frac{1}{2} \alpha(t)' K(t, T) \alpha(t) + N(t, T)' \alpha(t) + M(t, T), \quad (37)$$

$$K(t, T) = 2 \int_t^T e^{\frac{1}{2} \mathcal{A}'(t-s) \Sigma(s)^{-1} e^{\frac{1}{2} \mathcal{A}(t-s)} ds, \quad (38)$$

$$N(t, T)' = \int_t^T \Theta(s)' K(s, T) e^{\mathcal{A}(t-s)} ds, \quad (39)$$

$$M(t, T) = \int_t^T N(s, T)' \Theta(s) + \frac{1}{2} \text{tr}(\sigma(s)' \mathcal{A}' K(s, T) \mathcal{A} \sigma(s)) ds, \quad (40)$$

Θ is defined in (27); and $e^{\mathcal{A}t}$ is the matrix exponential function of $\mathcal{A}t$.

Proof. Formula (36) immediately follows from (33) and Lemma 3.3. The remaining task is to show that the $\mathcal{H}(t, \alpha, T)$ defined in Lemma 3.3 really takes the form of (37).

By Itô's lemma, the dynamic of $\alpha(t)$ under the \mathcal{P} -measure is given by

$$d\alpha(t) = (\Theta - \mathcal{A}\alpha) dt - \mathcal{A}\sigma dW_t, \quad (41)$$

where Θ is given in (27). Applying the Feynman-Kač formula to $\mathcal{H}(t, \alpha, T)$ with respect to (41), the PDE governing \mathcal{H} is obtained as

$$\frac{\partial \mathcal{H}}{\partial t} + \left(\frac{\partial \mathcal{H}}{\partial \alpha} \right)' (\Theta - \mathcal{A}\alpha) + \frac{1}{2} \text{tr} \left(\sigma' \mathcal{A}' \frac{\partial^2 \mathcal{H}}{\partial \alpha^2} \mathcal{A} \sigma \right) + \alpha' \Sigma^{-1} \alpha = 0, \quad (42)$$

$$\mathcal{H}(T, \alpha, T) = 0. \quad (43)$$

Consider the quadratic form of (37)-(40) in which $K(t)$, $N(t)$ and $M(t)$ are solutions of the following matrix ordinary differential equations (ODEs).

$$\dot{K}(t, T) - K(t, T)\mathcal{A} - \mathcal{A}'K(t, T) + 2\Sigma(t)^{-1} = 0_{n \times n}, \quad K(T, T) = 0_{n \times n}, \quad (44)$$

$$\dot{N}(t, T)' - N(t, T)'\mathcal{A} + \Theta(t)'K(t, T) = 0, \quad N(T, T) = 0_{n \times 1}, \quad (45)$$

$$\dot{M}(t, T) + N(t, T)'\Theta(t) + \frac{1}{2} \text{tr} (\sigma(t)'\mathcal{A}'K(t, T)\mathcal{A}\sigma(t)) = 0, \quad M(T, T) = 0. \quad (46)$$

Simple differentiations show that

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{1}{2} \alpha' \dot{K} \alpha + \dot{N}' \alpha + \dot{M}, \quad \frac{\partial \mathcal{H}}{\partial \alpha} = K \alpha + N, \quad \frac{\partial^2 \mathcal{H}}{\partial \alpha^2} = K.$$

Substituting these formulas into (42) yields

$$\begin{aligned} & \frac{1}{2} \alpha' \left(\dot{K} - \mathcal{A}'K - K\mathcal{A} + 2\Sigma^{-1} \right) \alpha + \left(\dot{N}' - N'\mathcal{A} + \Theta'K \right) \alpha \\ & + \dot{M} + N'\Theta + \frac{1}{2} \text{tr} (\sigma' \mathcal{A}' K \mathcal{A} \sigma) = 0. \end{aligned}$$

This verifies that the quadratic form (37) satisfies the governing equation (42). In addition, the boundary conditions on K , N and M ensure that $\mathcal{H}(T, \alpha, T) = 0$. Hence, the quadratic form is the solution of the PDE (42). \square

The time-consistent efficient frontier in a financial market with cointegrated assets is developed in (36), which represents a curve depicted on the plane of mean and variance. However, the conventional approach is to use the plane of mean and standard deviation. Denote the standard deviation of the optimal terminal wealth as $\sigma_{Y^*(T)}$ so that $\text{Var}(Y^*(T)) = \sigma_{Y^*(T)}^2$. Applying the square root to both sides of (36), the efficient frontier can be alternatively expressed as

$$\overline{Y(T)} = \frac{\widehat{\mathcal{H}}(0, \alpha(0), T)}{\sqrt{\mathcal{H}(0, \alpha(0), T)}} \sigma_{Y^*(T)} + Y_0 e^{\int_0^T r(s) ds}, \quad (47)$$

which is a straight line with the slope being

$$\rho(\mathcal{A}, \alpha(0), T) = \frac{\hat{\mathcal{H}}(0, \alpha(0), T)}{\sqrt{\mathcal{H}(0, \alpha(0), T)}}. \quad (48)$$

This is the price of risk. When $\mathcal{A} \equiv 0$ and r , θ and σ are constant values, it becomes Black-Scholes asset dynamics, and the efficient frontier is reduced to

$$\bar{Y} = \sqrt{\alpha(0)' \Sigma^{-1} \alpha(0) T} \sigma_{Y^*(T)} + Y_0 e^{\int_0^T r(s) ds}, \quad (49)$$

because $\hat{K}(0, T) = K(0, T) = 2\Sigma^{-1}T$, $\hat{N} = N \equiv \vec{0}$ and $\hat{M} = M \equiv 0$.

4 Cointegration and statistical arbitrage

4.1 Statistical arbitrage possibility

Cointegrated pairs trading is usually referred to as a statistical arbitrage strategy in practice. In this section, we validate that the dynamically optimal MV portfolio allocation of cointegrated assets satisfies the four conditions for a statistical arbitrage strategy by Hogan et al. (2004): (i) the strategy is a zero initial cost ($Y_0 = 0$) self-financing trading strategy; (ii) in the limit of the infinite investment time, it has positive expected discounted profits; (iii) a probability of a loss converges to zero; and (iv) a time-averaged variance converges to zero if the probability of a loss does not become zero in finite time. We regard investors who consider only trading strategies that satisfy the aforementioned four conditions as arbitrageurs.

The first two conditions are clearly satisfied by the MV portfolio allocation problem because it considers time-consistent strategies, which are self-financing strategies, allows for zero initial wealth (cost) and seeks positive expected final wealth. Hence, it suffices for us to prove the remaining two conditions for the time-consistent MV optimal trading strategy as a statistical arbitrage one.

Proposition 4.1. Suppose that the market parameters, θ and σ , and the interest rate, r , are constant values. Consider the problem P(MV) in which $Y(0) = 0$ and $e^{-rT} \mathbb{E}[Y(T)] = \eta(T)$, for some pre-specified positive continuous function $\eta(T)$ such that, for some nonnegative constant \bar{C} ,

$$\lim_{T \rightarrow \infty} \frac{\eta(T)}{T} = \bar{C}. \quad (50)$$

If the eigenvalues of \mathcal{A} are nonnegative real numbers and $\mathcal{A} \neq 0$, then, using the optimal policy in Proposition 3.2, the optimal final wealth, $Y^*(T)$ satisfies the following limiting conditions.

$$\lim_{T \rightarrow \infty} \mathcal{P}(e^{-rT}Y^*(T) < 0) = 0; \quad (51)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{Var}(e^{-rT}Y^*(T)) = 0; \quad (52)$$

otherwise, if $\mathcal{A} = 0$, then (51) and (52) hold true if and only if $\overline{C} = 0$. Furthermore, if \mathcal{A} has a negative eigenvalue, then the limit in (52) tends to infinity.

When all of the eigenvalues of \mathcal{A} are nonnegative real numbers, the risky assets are cointegrated. An exceptional case is when $\mathcal{A} \equiv 0$, which reduces the cointegration dynamics to Black-Scholes dynamics. The log-returns of risky assets in the Black-Scholes economy are strictly stationary. When \mathcal{A} has a negative eigenvalue, some non-stationary time series exist and cannot be error-corrected. Proposition 4.1 asserts that there exists a statistical arbitrage strategy in the sense of Hogan et al. (2004) in the cointegration and Black-Scholes economies. However, the existence of a negative eigenvalue in \mathcal{A} prevents optimal MV portfolio policies from constituting statistical arbitrage because (52) is violated. In addition, a statistical arbitrage strategy, if it exists, can be constructed using the MV objective with zero cost and positive expected discounted wealth.

Condition (50) allows the discounted expected final wealth to grow, at most linearly, to infinity with the investment time. This condition is not restrictive. When $\eta(T) = \overline{Y}$ a constant value, condition (50) is clearly satisfied with \overline{C} being zero, and the expected final wealth increases with the risk-free interest rate. The condition is still satisfied when $\eta(T) = \overline{C}T$ for a positive \overline{C} . In this latter situation, the present value of the expected final wealth increases exactly linearly with the investment time. An investor who targets \$1 million present value in a year will target \$2 million present value in two years, and so on.

Remark: Proposition 4.1 can be generalized in a number of ways. The assumption on the constant parameters and an interest rate can be relaxed to time-varying parameters and interest rate with bounded first-order derivatives. When the condition that the eigenvalues of \mathcal{A} all be nonnegative real numbers is replaced by the condition that the eigenvalues of \mathcal{A} all have nonnegative real parts, (51) and (52) still hold true. In addition, (52) is violated once \mathcal{A} has an eigenvalue with a negative real part. Although these variations can be shown by modifying the proof in Appendix B, this version offers a friendly interpretation.

Proposition 4.2. Consider the conditions specified in Proposition 4.1. If all of the eigenvalues of \mathcal{A} are nonnegative real numbers, then the initial MV trading strategy for an infinite horizon investment is given by

$$\lim_{T \rightarrow \infty} u^*(0, T) = \begin{cases} 3\bar{C}\mathcal{A}'\Sigma^{-1}\Theta/(\Theta'\Sigma^{-1}\Theta), & \text{if } \mathcal{A} \neq 0 \\ 2\bar{C} \frac{\Sigma^{-1}[\theta + \frac{1}{2}\mathcal{D}(\Sigma)\mathbf{1} - r\mathbf{1}]}{[\theta + \frac{1}{2}\mathcal{D}(\Sigma)\mathbf{1} - r\mathbf{1}]'\Sigma^{-1}[\theta + \frac{1}{2}\mathcal{D}(\Sigma)\mathbf{1} - r\mathbf{1}]}, & \text{if } \mathcal{A} = 0, \end{cases} \quad (53)$$

where $\Sigma = \sigma\sigma'$ is the covariance matrix, Θ is defined in (27) and \bar{C} is defined in (50). Otherwise, the limit in (53) diverges.

Proof. The result immediately follows from the use of Proposition 3.2, (23) and (61). \square

Under the condition ensuring the existence of a statistical arbitrage opportunity, Proposition 4.2 explains the impact of the target expected long-term wealth and shows the importance of cointegration in statistical arbitrage. An investor whose expected long-term wealth satisfies the condition that $\eta(T)/T$ tends to zero for a large T ($\bar{C} = 0$) is too conservative. The strategy in (53) suggests not investing in risky assets ($u^* = 0$). This asymptotic strategy implies that conservative investors invest very little initially. An alternative investor with expected long-term wealth such that $\eta(T)/T$ diverges to infinity is overly aggressive because the optimal MV trading strategy suggests longing an infinite amount of undervalued assets and shorting an infinite amount of overvalued assets. This strategy cannot be implemented.

A reasonable and implementable infinite-horizon statistical arbitrage strategy is possible for an investor whose discounted expected wealth grows linearly with time, such as in the case of $\eta(T) = \bar{C}T$. In this situation, the strategy in (53) is a constant rebalancing policy. However, arbitrageurs do not invest in the Black-Scholes market ($\mathcal{A} = 0$). Proposition 4.1 suggests that arbitrageurs' expected long-term wealth levels should be very conservative in the Black-Scholes economy in the sense that $\bar{C} = 0$ in (50), which, by (53), results in no investment. The long-term strategy in (53) becomes practical in the cointegration economy ($\mathcal{A} \neq 0$). It suggests to arbitrageurs that they continuously rebalance the portfolio so as to keep constant values in long and short positions on risky assets. Using the MV paradigm, we now show that a practical statistical arbitrage is possible in a continuous-time cointegration economy, but not in the Black-Scholes economy or in a non-stationary economy, which cannot be error-corrected.

4.2 Statistical arbitrage profitability

Once an arbitrageur has identified several pairs of cointegrated assets, she may invest in only one of the pairs due to certain internal trading limits such as regulatory constraints. The conventional wisdom for selecting the best pair is to pick that which has the strongest mean-reversion in its cointegrating factor and which departs significantly from equilibrium. Chiu and Wong (2011) propose using the price of risk to quantify this wisdom because a higher price of risk implies a greater return for the same risk level.

We adopt Chiu and Wong's (2011) notion, but employ time consistent policies. The price of risk (48) depends on the value of $\alpha(0)$, which is given in (5) as

$$\alpha(0) = \left[\theta + \frac{1}{2} \mathcal{D}(\Sigma) \mathbf{1} - r \mathbf{1} \right] - \mathcal{A}X(0).$$

We interpret $\alpha(0)$ as the displacement of the initial log-asset values, $X(0)$, to equilibrium (the vector inside the square brackets). The cointegration coefficient matrix \mathcal{A} drives the speed of convergence or divergence to equilibrium.

The price of risk $\rho(\mathcal{A}, \alpha, T)$ in (48) can serve as an index to compare the profitability of various cointegrated pairs in terms of the trade-off between risk and return. When the index is zero, the efficient frontier is a horizontal line that indicates no (statistical) arbitrage. This is a risk-neutral market. When it is infinity, the efficient frontier becomes a vertical line, and it is possible to earn more than the risk-free rate by taking no risk. This is perfect arbitrage. In between, statistical arbitrage is possible. When \mathcal{A} is fixed, function $\rho(\mathcal{A}, \alpha, T)$ reflects the distance to equilibrium at the present moment. When α is fixed, the function $\rho(\mathcal{A}, \alpha, T)$ indicates the effect of mean-reversion generated by \mathcal{A} . This index thus simultaneously captures the aggregated effect of both mean-reversion and equilibrium departure.

The index $\rho(\mathcal{A}, \alpha, T)$ also provides an alternative angle with which to look at statistical arbitrage. By (48) and the calculations in Appendix B, the index tends to zero for a large T if \mathcal{A} has a negative eigenvalue. This situation refers to asymptotic no-arbitrage. If the eigenvalues of \mathcal{A} are nonnegative, then, for a large T ,

$$\rho(\mathcal{A}, \alpha, T) \sim \begin{cases} \mathcal{O}(T), & \text{if } \mathcal{A} = 0 \\ \mathcal{O}(T^{3/2}), & \text{if } \mathcal{A} \neq 0 \end{cases} \rightarrow \infty. \quad (54)$$

This is asymptotic arbitrage. The cointegration market, however, has an advantage over the Black-Scholes market because the divergent rate of the former is greater

than that of the latter. These factors echo Propositions 4.1 and 4.2, and further confirm the important role of cointegration in statistical arbitrage.

Our previous analysis of statistical arbitrage concentrates on long-term investment. For a finite-time investment, we now employ a numerical example to demonstrate the difference in the prices of risk between the cointegration and Black-Scholes markets. The parameters in Example 4.1 are taken from Chiu and Wong (2011). Later in this paper, we also use this example to compare the time-consistent policy with the precommitment policy of Chiu and Wong (2011).

Example 4.1. Consider two risky assets whose log-asset values at time t are $X(t) = [x_1(t) \ x_2(t)]'$ with constant parameters

$$\theta = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad \mathcal{A} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Clearly, $z(t) = x_2(t) + x_1(t)$ is a cointegrating factor that exhibits mean-reversion with a mean-reverting speed of 1. A risk-free asset is available in the market and the risk-free interest rate is 3%. We are interested in the corresponding MV problem with expected terminal wealth of $\bar{Y}(T)$ and an investment horizon of $T = 0.5$. These parameter values infer an equilibrium vector of $(0.09 \ 0.19)'$. Further suppose that $x_1(0) = \ln 1$ and $x_2(0) = \ln 2$. Then, the displacement to equilibrium is calculated as $\alpha(0) = (-0.2566 \ 0.1566)'$.

Our task is to compute the prices of risk of the optimal time-consistent MV policies in the cointegration market (3) and the Black-Scholes market with $\mathcal{A} = 0$, respectively. By Proposition 3.1, we have

$$\begin{aligned} \hat{K}(t; T) &= 2\Sigma^{-1}(T - t) = 50(T - t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \\ \hat{N}(t; T) &= \hat{K}(t; T)\Theta \frac{(T - t)}{2} = -0.25(T - t)^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \\ \hat{M}(t; T) &= \frac{(T - t)^2}{3}\Theta' \hat{N}(t, T) + \frac{(T - t)}{2} \text{tr} \left(\sigma' \mathcal{A}' \hat{K}(t; T) \mathcal{A} \sigma \right) \\ &= \frac{(T - t)^3}{600} + \frac{(T - t)^2}{2}; \\ \Theta &= \mathcal{A} \left(\frac{1}{2} \mathcal{D}(\Sigma) \mathbf{1} - r \mathbf{1} \right) = -0.01 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

By Proposition 3.3,

$$\begin{aligned}
K(t, T) &= 2 \int_t^T e^{\frac{1}{2}\mathcal{A}'(t-s)\Sigma^{-1}} e^{-\frac{1}{2}\mathcal{A}(t-s)} ds \\
&= 25(1 - e^{-(T-t)}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 25(T-t) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}; \\
N(t, T) &= \int_t^T e^{-\mathcal{A}'(t-s)} K(s, T) \Theta ds \\
&= \frac{(1 + T - t)e^{-(T-t)} - 1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \\
M(t, T) &= \int_t^T N(s, T)' \Theta + \frac{1}{2} \text{tr}(\sigma' \mathcal{A}' K(s, T) \mathcal{A} \sigma) ds \\
&= 1.01(T-t) - 1.02 + (1.02 + 0.01(T-t))e^{-(T-t)}.
\end{aligned}$$

Substituting all of these into (48), we obtain a price of risk of 1.2634. The solid line in Figure 1 is the efficient frontier of trading for the cointegrated pair of risky assets in Example 4.1 and the risk-free bond, whereas the dashed line is that for the trading of similar risky assets in the Black-Scholes market, i.e., $A \equiv 0$, and the risk-free bond. It can be seen that cointegration may generate a sizable degree of statistical arbitrage profits. The expected terminal wealth of trading the cointegrated pair is significantly larger than that of trading the pair of stocks that follow geometric Brownian motions with the same volatility and drifts equal to the initial drifts of the cointegrated assets. In fact, the price of risk for the assets following the Black-Scholes dynamics is only 0.7621.

Using the price of risk, we examine whether the statistical arbitrage profits increase with the mean-reverting speed of the cointegrating factor. We do so by perturbing the cointegration coefficient matrix, $\mathcal{A}'\epsilon = \epsilon\mathcal{A}$, where \mathcal{A} is presented in Example 4.1. To ensure fair comparison, we fix $\alpha(0)$ as in Example 4.1 for all ϵ . Thus, parameter θ should be adjusted accordingly here. In other words, the initial drifts of the processes of x_1 and x_2 maintain the same values for all $\epsilon \in (0, 1]$. Clearly, the cointegrating factor $z_t = x_1(t) + x_2(t)$ has a mean-reverting speed of ϵ . Figure 2 shows that the price of risk increases with ϵ , the mean-reverting rate. This is in a sharp contrast to the profits generated by the relative-value arbitrage rule reported by Gatev et al. (2006).

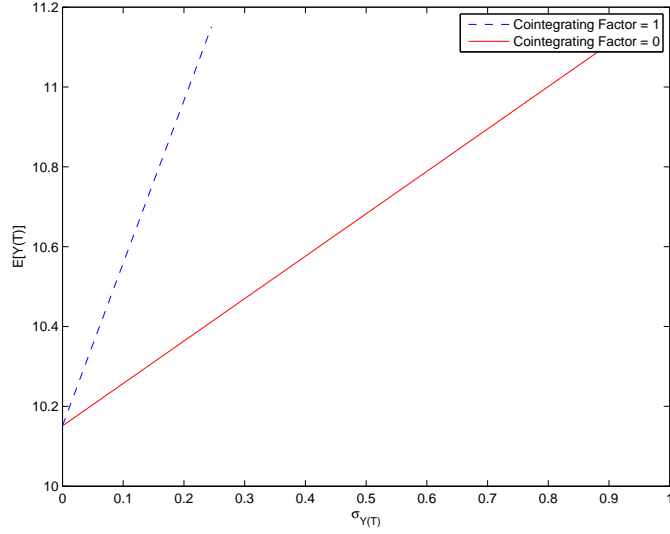


Figure 1: Efficient frontiers of Example 4.1.

4.3 Comparison with the precommitment policy

As this paper considers the time-consistent MV optimal trading strategy, a natural question is whether this strategy outperforms the precommitment policy obtained in the literature. More precisely, how can we fairly compare these two approaches for dynamic MV portfolio selection problems? Wang and Forsyth (2011) have already given a partial answer. Using a numerical scheme, they show numerically that the price of risk from the precommitment policy is always higher than that from the time-consistent policy in the Black-Scholes market. The fundamental reason is that the time-consistent policy observes an additional constraint associated with time-consistency (6). Wang and Forsyth (2011), however, argue that a time-consistent policy is still wealth considering as it makes economic sense.

The advantage of the time-consistent MV optimal policy can be articulated in the cointegration financial market via a quantitative approach. We further confirm that the price of risk from a precommitment policy is greater in the cointegration market. What makes the time-consistent policy appealing is its robustness and stability with respect to the estimated parameters. The precommitment policy in the cointegration market involves solving a matrix Riccati differential equation (MRDE), which may cause a finite escape time at which the solution blows up to infinity. This problem is hidden in the Black-Scholes market.

Consider the following initial optimal time-consistent and precommitment

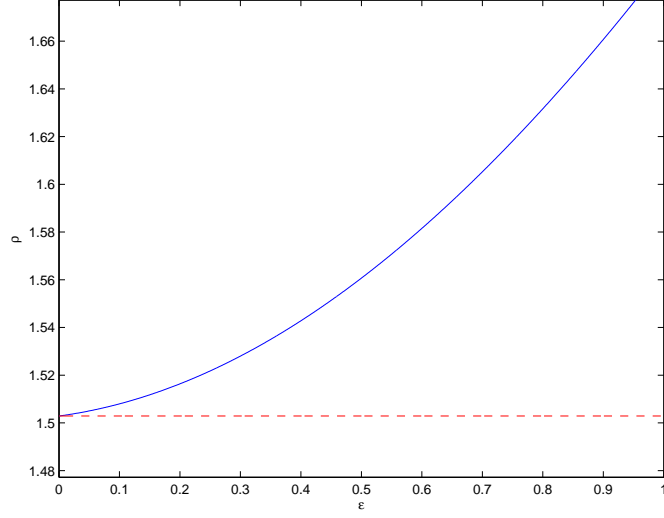


Figure 2: Statistical arbitrage profits and mean reverting rate.

policies.

$$u^*(0, X, T) = \frac{\bar{Y}_T e^{-rT} - Y_0}{\hat{\mathcal{H}}(0, \alpha, T)} \left[\left(\Sigma^{-1} + \mathcal{A}' \hat{K}(0, T) \right) \alpha + \mathcal{A}' \hat{N}(0, T) \right] \text{ and}$$

$$u_{\text{prec}}^*(0, X, T) = \frac{\bar{Y}_T e^{-rT} - Y_0}{1 - \mathcal{G}(0, \alpha, T)} \left[\left(\Sigma^{-1} - \mathcal{A}' K_{\mathcal{G}}(0, T) \right) \alpha - \mathcal{A}' N_{\mathcal{G}}(0, T) \right],$$

where $u_{\text{prec}}^*(t, X, T)$ denotes the precommitment policy derived by Chiu and Wong (2011),

$$\mathcal{G}(0, \alpha, T) = \exp \left\{ -\frac{1}{2} \alpha' K_{\mathcal{G}}(0, T) \alpha - N_{\mathcal{G}}'(0, T) \alpha - M_{\mathcal{G}}(0, T) \right\},$$

with $K_{\mathcal{G}}(t, T)$ being solved by a MRDE, $N_{\mathcal{G}}(t, T)$ by a system of ODEs and $M_{\mathcal{G}}(t, T)$ by a linear ODE. To illustrate our ideas and get rid of the subtle calculation associated with the MRDE, we employ Chiu and Wong's (2011) example and the parameters in Example 4.1 directly:

$$K_{\mathcal{G}}(t, T) = \frac{25 \sin(T-t)}{\cos(T-t) - \sin(T-t)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 25(T-t) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$N_{\mathcal{G}}(t, T) = \frac{1}{2} \tilde{N}(T-t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \tilde{N}(\tau) = \frac{\cos \tau - 1}{\cos \tau - \sin \tau},$$

$$M_{\mathcal{G}}(t, T) = \frac{(T-t)}{2} + \frac{\ln |\cos(T-t) - \sin(T-t)|}{2}$$

$$+ \frac{1}{100} \int_0^{T-t} \left(\tilde{N}(s)^2 - \tilde{N}(s) \right) ds.$$

It can be seen that $K_{\mathcal{G}}(0, T)$, $N_{\mathcal{G}}(0, T)$ and $M_{\mathcal{G}}(0, T)$ diverge at $T = \pi/4$, making $u^*(0, X, \pi/4)$ diverge as well.

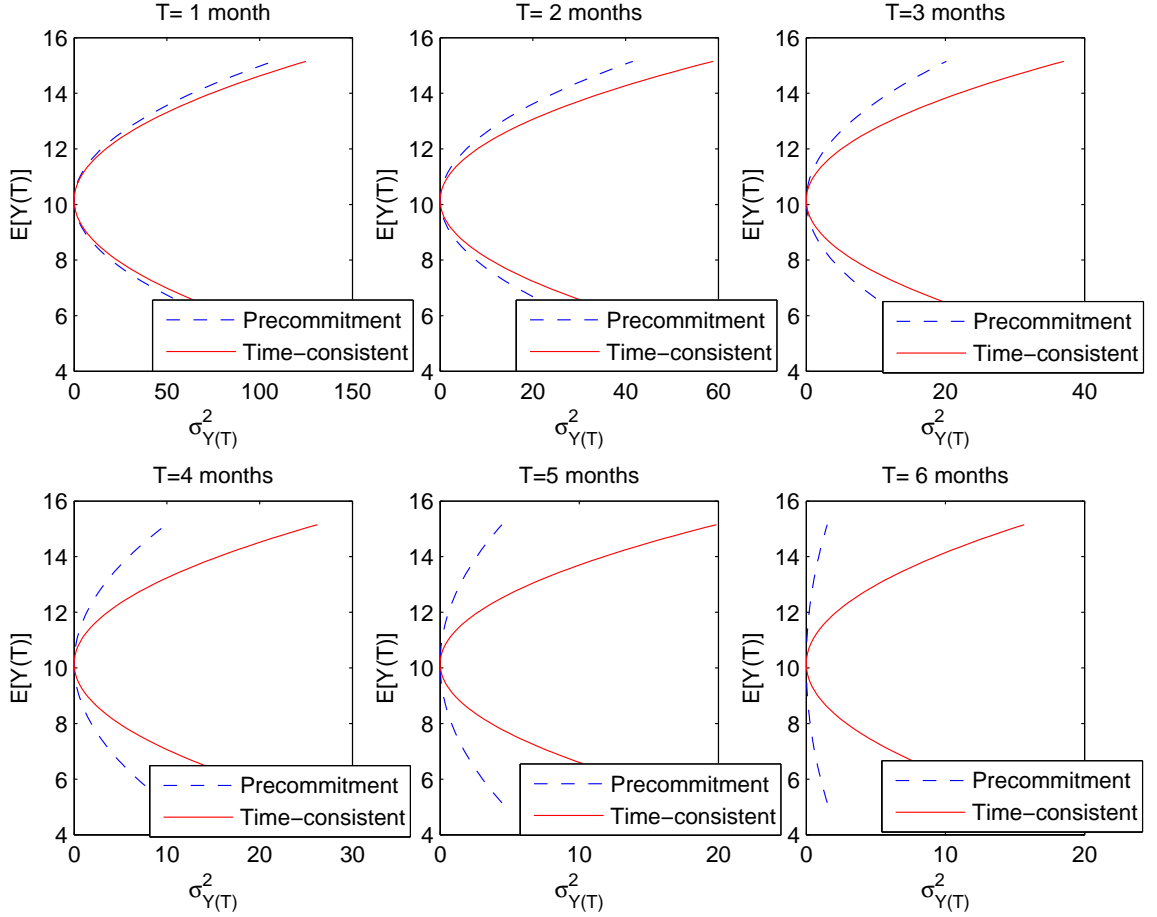


Figure 3: Time-consistent and precommitment efficient frontiers.

Figure 3 illustrates the efficient frontiers (on the mean-variance plane) of the time-consistent (the solid line) and precommitment (the dashed line) policies using the parameters in Example 4.1. By increasing the investment period from 1 month to 6 months, the price of risk from the efficient frontier of precommitment policies is clearly greater than that from time-consistent policies. A peculiarity is

that the efficient frontier of precommitment policies is an almost vertical line for 6-month investments, indicating almost perfect arbitrage. This is too good to be true. The time-consistent efficient frontier also has an increasing price of risk, but is much more persistent.

Important insights can be drawn by examining the optimal policies directly. Figure 4 shows the initial investment amounts in the risk-free asset (u_0^*), asset 1 (u_1^*) and asset 2 (u_2^*) under the assumption in Example 4.1 and the conditions that $\bar{Y}_T e^{-\int_0^T r(s)ds} = \20 and $Y_0 = \$10$. The investment period, however, varies from 0.001 to 0.76 years. For a short-term investment, the two policies are similar but the precommitment policy diverges when the investment period close to $0.76 \simeq \pi/4$. It can be shown by Lemma 4.2 in Chiu and Wong (2011) that the higher the mean reversion speed, the shorter the escape time. As the mean reversion speed in practice is usually of an order of 10^{-3} or even less, it is difficult to detect the problem. The implication of this example is essentially concerned with the sensitivity and robustness of the portfolio policy to the estimated parameters. In practice, the cointegration matrix \mathcal{A} and the other parameters are always estimated with errors. When the parameters are combined to estimate the mean reversion rate, the aggregate error has a greater impact on the optimal precommitment policy than the time-consistent policy. The aggregate estimation error is, however, difficult to control (Tang and Chen, 2009). The time-consistent MV policy is preferred for its robustness to the estimated parameters. Whereas the optimal precommitment policy can aggressively take full advantage of the assumed model, because it considers a generic set of trading policies, the optimal time-consistent policy is primarily concerned with rational decisions. The imprecise estimation of the assumed model thus becomes an additional factor to be considered in practice.

The policies shown in Figure 4 also demonstrate that the dynamic MV trading of cointegrated assets may resemble neither relative-value nor long-short arbitrage. Both precommitment and time-consistent policies typically leverage by shorting all risky assets for the risk-free bond for investment horizons less than 0.1 years. The precommitment policy alone becomes a long-short strategy for an investment horizon longer than 0.3 years.

5 Conclusion

Although the trading of cointegrated assets is believed to be able to generate statistical arbitrage profits, this belief has lacked rigorous theoretical support. By investigating time-consistent MV portfolio strategies for cointegrated assets in

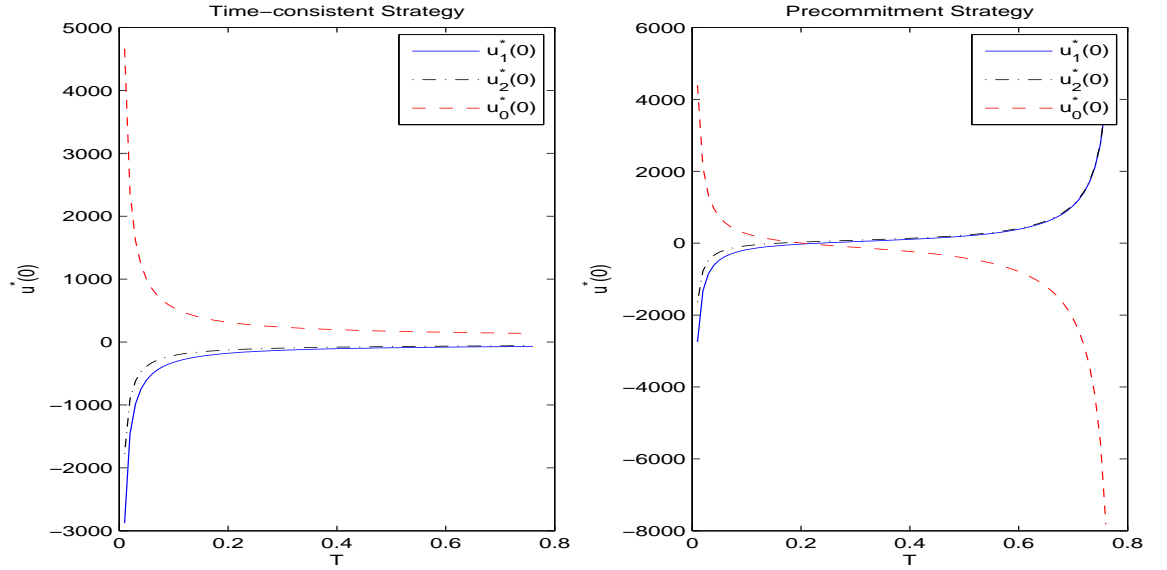


Figure 4: Investment amounts in time-consistent and precommitment policies.

a continuous-time economy, we derive the optimal trading strategy in a closed-form solution in this paper. This analytical solution allows us to prove rigorously the statistical arbitrage associated with cointegration dynamics. Our theory suggests that arbitrageurs do not invest when cointegration disappears in the market. Once cointegrated assets are identified, mean-variance arbitrageurs invest with discounted expected wealth that increases linearly with the investment horizon. The time-consistent optimal trading strategy does not necessarily resemble either relative-value or long-short strategies.

References

- [1] Alexander, C.O. 1999. Optimal hedging using cointegration. *Philosophical Transactions of the Royal Society A* 357, 2039-2058.
- [2] Alexander, C.O., A. Dimitriu. 2005. Indexing and statistical arbitrage: Tracking error or cointegration? *Journal of Portfolio Management* 32, 50-63.
- [3] Baillie, R., T. Bollerslev. 1989. Common stochastic trends in a system of exchange rates. *Journal of Finance* 44, 137-151.

- [4] Basak, S., G. Chabakauri. 2010. Dynamic mean-variance asset allocation. *Review of Financial Studies* 23, 2970-3016.
- [5] Bielecki, T., H. Jin, S. R. Pliska, X.Y. Zhou. 2005. Continuous-time mean-variance portfolio selection with bankruptcy prohibition. *Mathematical Finance* 15, 213-44.
- [6] Bjork, T., A. Murgoci, X.Y. Zhou. 2011. Mean-variance portfolio optimization with state dependent risk aversion. Working paper, Oxford University, Nomura Centre for Mathematical Finance.
- [7] Cerchi, M., A. Havenner. 1988. Cointegration and stock prices: The random walk on Wall Street revisited. *Journal of Economic Dynamics & Control* 12, 333-346.
- [8] Chiu, M.C., H.Y. Wong. 2011. Mean-variance portfolio selection of cointegrated assets. *Journal of Economic Dynamics & Control* 35, 1369-1385.
- [9] Cui, X.Y., D. Li, S.Y. Wang, S.S. Zhu. 2010. Better than dynamic mean-variance: Time inconsistency and free cash flow stream, *Mathematical Finance*, forthcoming.
- [10] Duan, J.C., S.R. Pliska. 2004. Option valuation with co-integrated asset prices. *Journal of Economic Dynamics & Control* 28, 727-754.
- [11] Engle, R., C. Granger. 1987. Co-integration and error correction: Representation, estimation and testing. *Econometrica* 55, 251-276.
- [12] Gatev, E., W.N. Goetzmann, K.G. Rouwenhorst. 2006. Pairs trading: Performance of a relative-value arbitrage rule. *Review of Financial Studies* 19, 797-827.
- [13] Granger, C. 1981. Some properties of time series data and their use in econometric model specification. *Journal of Econometrics* 23, 121-130.
- [14] Hogan, S., R. Jarrow, M. Teo, M. Warachkac. 2004. Testing market efficiency using statistical arbitrage with applications to momentum and value strategies. *Journal of Financial Economics* 73, 525-565.
- [15] Jurek, J.W., H. Yang. 2007. Dynamic portfolio selection in arbitrage. Working paper, Harvard Business School.

- [16] Li, D., W.L. Ng. 2000. Optimal dynamic portfolio selection: Multiperiod mean-variance formulation. *Mathematical Finance* 10, 387-406.
- [17] Liu, J., F.A. Longstaff. 2003. Losing money on arbitrage: Optimal dynamic portfolio choice in markets with arbitrage opportunities. *Review of Financial Studies* 17, 611-641.
- [18] Markowitz, H. 1952. Portfolio selection. *Journal of Finance* 7, 77-91.
- [19] Strotz, R. H. 1956. Myopia and inconsistency in dynamic utility maximization. *Review of Economic Studies* 23, 165-80.
- [20] Tang, C.Y., S.X. Chen. 2009. Parameter estimation and bias correction for diffusion processes. *Journal of Econometrics* 149, 65-81.
- [21] Taylor, M., I. Tonks. 1989. The internationalisation of stock markets and the abolition of U.K. exchange control. *Review of Economics and Statistics* 71, 332-336.
- [22] Vidyamurthy, G. 2004. *Pairs Trading – Quantitative Methods and Analysis*, Wiley: New York.
- [23] Wachter, J.A. 2002. Portfolio and consumption decisions under mean-reverting returns: An explicit solution for complete market. *Journal of Financial and Quantitative Analysis* 37, 63-91.
- [24] Wang, J., P.A. Forsyth. 2011. Continuous time mean variance asset allocation: A time consistent strategy. *European Journal of Operational Research* 209, 184-201.
- [25] Zhou X.Y., D. Li. 2000. Continuous-time mean-variance portfolio selection: A stochastic LQ framework. *Applied Mathematics & Optimization* 42, 19-33.

A Proof of Lemma 3.1

To simplify matters, we write $J(t) = J(t, Y(t), X(t))$. By the time-consistent property (6), for any $0 < \tau < T - t$,

$$J(t) = \min_{u \in \mathcal{U}(t, T)} \{ \text{Var}(Y(T) | \mathcal{F}_t) - 2\lambda \mathbb{E}[Y(T) | \mathcal{F}_t] \}$$

$$= \min_{u \in \mathcal{U}(t, t+\tau)} \mathbb{E}[J(t+\tau)|\mathcal{F}_t] + \text{Var}(\mathbb{E}[Y^*(T)|\mathcal{F}_{t+\tau}]|\mathcal{F}_t),$$

where $Y^*(T)$ is the terminal wealth with the optimal control adopted. Let u^* denote the optimal policy. By (10) and the definition of Γ in (15), we obtain an expression of $\mathbb{E}[Y^*(T)|\mathcal{F}_{t+\tau}]$ which brings us the following.

$$\begin{aligned} 0 &= \min_{u \in \mathcal{U}(t, t+\tau)} \mathbb{E}[J(t+\tau) - J(t)|\mathcal{F}_t] + \text{Var} \left(e^{\int_{t+\tau}^T r(s)ds} Y(t+\tau) + \Gamma(t+\tau) | \mathcal{F}_t \right) \\ &= \min_{u \in \mathcal{U}(t, t+\tau)} \left\{ \text{Var} \left(e^{\int_{t+\tau}^T r(s)ds} Y(t+\tau) - e^{\int_t^T r(s)ds} Y(t) + \Gamma(t+\tau) - \Gamma(t) \middle| \mathcal{F}_t \right) \right. \\ &\quad \left. + \mathbb{E}[J(t+\tau) - J(t)|\mathcal{F}_t] \right\}. \end{aligned} \quad (55)$$

Thus, (14) follows by letting $\tau \rightarrow 0$ in equation (55).

To derive an expression for the optimal time-consistent trading strategy $u^*(t)$, we solve equation (14). By (12), we have

$$\mathbb{E}[dJ|\mathcal{F}_t] = \mathbb{E} \left[-2\lambda e^{\int_t^T r(s)ds} \alpha(t)' u(t) dt + dJ_1 \middle| \mathcal{F}_t \right]. \quad (56)$$

Applying Itô's lemma to Γ yields

$$d\Gamma = \left[\frac{\partial \Gamma}{\partial t} + \left(\frac{\partial \Gamma}{\partial X} \right)' (\theta - \mathcal{A}X) + \frac{1}{2} tr \left(\sigma' \frac{\partial^2 \Gamma}{\partial X^2} \sigma \right) \right] dt + \left(\frac{\partial \Gamma}{\partial X} \right)' \sigma dW_t, \quad (57)$$

which implies that

$$\begin{aligned} &\text{Var} \left(d \left(e^{\int_t^T r(s)ds} Y(t) \right) + d\Gamma \middle| \mathcal{F}_t \right) \\ &= \mathbb{E} \left[e^{2 \int_t^T r(s)ds} u' \Sigma u + 2 e^{\int_t^T r(s)ds} u' \Sigma \frac{\partial \Gamma}{\partial X} + \frac{\partial \Gamma'}{\partial X} \Sigma \frac{\partial \Gamma}{\partial X} \middle| \mathcal{F}_t \right] dt. \end{aligned} \quad (58)$$

Substituting (13), (56) and (58) into (14) produces a straightforward quadratic minimization problem with respect to u . A simple completing square method gives the optimal policy in (16).

B Proof of Proposition 4.1

Substituting conditions $Y_0 = 0$ and $\mathbb{E}[Y_T] = \eta(T) \exp(\int_0^T r(s)ds)$ into (32) allows to solve for λ such that

$$\lambda = \frac{\eta(T) e^{\int_0^T r(s)ds}}{\widehat{\mathcal{H}}(0, \alpha(0), T)} > 0.$$

To prove (51), consider the dynamics of the optimal wealth level in (35):

$$\begin{aligned} Y_T^* &= \Gamma(0, X(0), T) + \lambda \int_0^T \alpha'(s) \Sigma^{-1} \sigma dW_s \\ &= \lambda \widehat{\mathcal{H}}(0, \alpha(0), T) + \lambda \int_0^T \alpha'(s) \Sigma^{-1} \sigma dW_s, \end{aligned}$$

where λ and $\widehat{\mathcal{H}}$ are positive. As $\mathcal{P}\{Y_T^* < 0\} = \mathcal{P}\{e^{\int_0^T r(s)ds} Y_T^* < 0\}$, we concentrate on the former probability.

$$\begin{aligned} \mathcal{P}\{Y_T^* < 0\} &= \mathcal{P}\left\{\lambda \widehat{\mathcal{H}}(0, \alpha(0)) + \lambda \int_0^T \alpha'(s) \Sigma^{-1} \sigma dW_s < 0\right\} \\ &= \mathcal{P}\left\{-\int_0^T \alpha'(s) \Sigma^{-1} \sigma dW_s > \widehat{\mathcal{H}}(0, \alpha(0))\right\} \\ &\leq \mathcal{P}\left\{\left|-\int_0^T \alpha'(s) \Sigma^{-1} \sigma dW_s\right| > \widehat{\mathcal{H}}(0, \alpha, T)\right\} \\ &\leq \frac{\text{Var}\left(\int_0^T \alpha'(s) \Sigma^{-1} \sigma dW_s\right)}{\widehat{\mathcal{H}}(0, \alpha, T)^2} \end{aligned} \tag{59}$$

$$= \frac{\mathbb{E}\left[\int_0^T \alpha'(s) \Sigma^{-1} \alpha(s) ds\right]}{\widehat{\mathcal{H}}(0, \alpha, T)^2} = \frac{\mathcal{H}(0, \alpha, T)}{\widehat{\mathcal{H}}(0, \alpha, T)^2}. \tag{60}$$

The Chebyshev inequality is applied in (59). It suffices for us to prove that the upper bound of the probability in (60) goes to zero when T tends to infinity. To do so, we investigate the asymptotic properties of $\widehat{\mathcal{H}}(0, \alpha, T)$ and $\mathcal{H}(0, \alpha, T)$ for a large T .

Let $\mathbb{P}_m(T)$ be the set of polynomials in T of an order no greater than m for some integer m . By Proposition 3.1, we have

$$\widehat{\mathcal{H}}(0, \alpha, T) = \frac{1}{2} \alpha' \widehat{K}(0, T) \alpha + \widehat{N}(0, T)' \alpha + \widehat{M}(0, T),$$

where, under the assumption of constant parameters,

$$\begin{aligned} \widehat{K}(0, T) &= 2\Sigma^{-1}T, \quad \widehat{N}(0, T)' = \Theta' \Sigma^{-1} T^2 \\ \widehat{M}(0, T) &= \Theta' \Sigma^{-1} \Theta \frac{T^3}{3} + \text{tr}(\sigma' \mathcal{A}' \Sigma^{-1} \mathcal{A} \sigma) \frac{T^2}{2}. \end{aligned} \tag{61}$$

Therefore, $\widehat{\mathcal{H}}(0, \alpha, T) \in \mathbb{P}_3(T)$. If $\mathcal{A} \neq 0$, then $\Theta' \Sigma^{-1} \Theta > 0$ and $\widehat{\mathcal{H}}(0, \alpha, T)$ diverges to infinity with the same order as T^3 . Hence, we write $\widehat{\mathcal{H}}(0, \alpha, T) \sim$

$\mathcal{O}(T^3)$ in this case. Alternatively, if $\mathcal{A} = 0$, then $\Theta = 0$ and $\widehat{\mathcal{H}}(0, \alpha, T) \in \mathbb{P}_1(T)$. We denote $\widehat{\mathcal{H}}(0, \alpha, T) \sim \mathcal{O}(T)$ in this latter case.

By Proposition 3.2, we have

$$\mathcal{H}(0, \alpha, T) = \frac{1}{2} \alpha' K(0, T) \alpha + N(0, T)' \alpha + M(0, T),$$

where

$$\begin{aligned} K(t, T) &= 2 \int_t^T e^{\frac{1}{2} \mathcal{A}'(t-s) \Sigma^{-1}} e^{\frac{1}{2} \mathcal{A}(t-s)} ds, \\ N(t, T)' &= \int_t^T \Theta' K(s, T) e^{\mathcal{A}(t-s)} ds, \\ M(t, T) &= \int_t^T N(s, T)' \Theta + \frac{1}{2} \text{tr}(\sigma' \mathcal{A}' K(s, T) \mathcal{A} \sigma) ds. \end{aligned}$$

The limiting property of $\mathcal{H}(0, \alpha, T)$ can then be assessed by investigating the limiting properties of K , M and N . It is obvious that $\mathcal{H}(0, \alpha, T) = 2\alpha' \Sigma^{-1} \alpha T$ and $\mathcal{H} \sim \mathcal{O}(T)$ once $\mathcal{A} = 0$.

We focus on the case in which $\mathcal{A} \neq 0$ and the eigenvalues are nonnegative. By the Cayley-Hamilton theorem on matrix exponential functions, for any $n \times n$ constant matrix \mathcal{A} ,

$$e^{-\mathcal{A}t} = \mathcal{B} + \sum_{i=0}^{n-1} c_i(t) \mathcal{A}^i, \quad (62)$$

where

$$c_i(t) = \sum_{m=1}^k f_{im}(t) e^{-\gamma_m t}, \quad (63)$$

$k \leq n$ is the number of non-zero eigenvalues of \mathcal{A} (or $\text{rank}(\mathcal{A})$); γ_m are the non-zero eigenvalues of \mathcal{A} for $m = 1, 2, \dots, k$; $f_{im}(t) \in \mathbb{P}_k(t)$ for all $i = 0, \dots, n-1$ and $m = 1, \dots, k$; and \mathcal{B} is an $n \times n$ constant matrix. The assumption on matrix \mathcal{A} implies that γ_m are positive for all $m = 1, 2, \dots, k$. For such a matrix \mathcal{A} , as $c_i(t)$ decays to zero in an exponential order of t for all $i = 0, \dots, n-1$, it is clear that, for any finite integers $n_0, \dots, n_{m'}$ and m' ,

$$\lim_{t \rightarrow \infty} \prod_{j=0}^{m'} \prod_{i=0}^{n-1} c_i(t)^{n_j} = 0,$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \prod_{j=0}^{m'} \prod_{i=0}^{n-1} c_i(t)^{n_j} dt = 0. \quad (64)$$

Consider the matrix-valued function $K(0, T)$ in (38), which diverges when T goes to infinity. By L'Hôpital's rule, we have

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{K(0, T)}{T} &= \lim_{T \rightarrow \infty} \frac{\partial}{\partial T} K(0, T) \\ &= \lim_{T \rightarrow \infty} 2e^{-\mathcal{A}'T/2} \Sigma^{-1} e^{-\mathcal{A}T/2} = 2\mathcal{B}'\Sigma^{-1}\mathcal{B}, \end{aligned} \quad (65)$$

where the last equality holds by (62). Hence, we write $K(0, T) \sim \mathcal{O}(T)$.

The vector-valued function $N(0, T)$ in (39) also diverges with T as it is an integration of $K(t, T)$. Thus, we apply L'Hôpital's rule to the following limit.

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{N(0, T)'}{T^2} &= \lim_{T \rightarrow \infty} \frac{\Theta'}{T^2} \int_0^T K(s, T) e^{-\mathcal{A}s} ds \\ &= \lim_{T \rightarrow \infty} \frac{\Theta'}{2T} \int_0^T \frac{\partial}{\partial T} K(s, T) e^{-\mathcal{A}s} ds \\ &= \lim_{T \rightarrow \infty} \frac{\Theta'}{T} \int_0^T e^{\frac{1}{2}\mathcal{A}'(s-T)} \Sigma^{-1} e^{\frac{1}{2}\mathcal{A}(s-T)} e^{-\mathcal{A}s} ds. \end{aligned}$$

By (62) and (64), we have

$$\lim_{T \rightarrow \infty} \frac{N(0, T)'}{T^2} = \Theta' \mathcal{B}' \Sigma^{-1} \mathcal{B}^2. \quad (66)$$

Hence, $N(0, T) \sim \mathcal{O}(T^2)$.

Applying similar tricks to $M(0, T)$ with some simple but tedious calculations shows that $M(0, T) \sim \mathcal{O}(T^3)$. Hence, $\mathcal{H}(0, \alpha, T) \sim \mathcal{O}(T^3)$.

Substituting all of these into (60), we conclude that

$$\mathcal{P}\{Y_T^* < 0\} \leq \frac{\mathcal{H}(0, \alpha, T)}{\widehat{\mathcal{H}}(0, \alpha, T)^2} \sim \begin{cases} \frac{\mathcal{O}(T)}{\mathcal{O}(T^2)}, & \text{if } \mathcal{A} = 0 \\ \frac{\mathcal{O}(T^3)}{\mathcal{O}(T^6)}, & \text{if } \mathcal{A} \neq 0 \end{cases} \rightarrow 0. \quad (67)$$

This completes the proof for (51).

We now prove (52). By Proposition 3.3 and (67), we have

$$\text{Var}(e^{-\int_0^T r(s)ds} Y_T^*) = \eta(T)^2 \frac{\mathcal{H}(0, \alpha, T)}{\widehat{\mathcal{H}}(0, \alpha, T)^2} \sim \begin{cases} \frac{\eta(T)^2}{\mathcal{O}(T)}, & \text{if } \mathcal{A} = 0 \\ \frac{\eta(T)^2}{\mathcal{O}(T^3)}, & \text{if } \mathcal{A} \neq 0 \end{cases}.$$

The condition (50) is sufficient to ensure that

$$\lim_{T \rightarrow \infty} \frac{\text{Var}(e^{-\int_0^T r(s)ds} Y_T^*)}{T} = 0$$

for $\mathcal{A} \neq 0$. Alternatively, we additionally require that $\overline{C} = 0$ in (50) to have the limit approaching zero for $\mathcal{A} = 0$.

Finally, consider the case in which \mathcal{A} has a negative eigenvalue. In (63), $c_i(t)$ diverges to infinity with an exponential order of t . It turns out that $\mathcal{H}(0, \alpha, T)$ tends to infinity with an exponential order of T and $\hat{\mathcal{H}}(0, \alpha, T)$ tends to infinity with the degree 3 polynomial order of T . By Proposition 3.3, the variance in (52) diverges.