Linear Algebra with Application

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Topic 1: Matrices and linear equations

1. Special types of matrices
2. Solution of a system of linear equations using matrix multiplication

Systems of Linear Equations

Probably the most important problem in mathematics is that of solving a system of linear equations. Well over 75 percent of all mathematical problems encountered in scientific or industrial applications involve solving a linear system at some stage.

We consider a number of important applications, including resource allocation, production scheduling, and economic planning.

Review :Systems of Linear Equations in Two Variables

let's consider the following simple example: If 2 adult tickets and 1 child ticket cost \$32, and if 1 adult ticket and 3 child tickets cost \$36, what is the price of each?

- Let: $x =$ price of adult ticket
  $y =$ price of child ticket
- Then: $2x + y = 32$
  $x + 3y = 36$

Contd.

- Definition Systems of Two Linear Equations in Two Variables
  Given the linear system

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

where a, b, c, d, h, and k are real constants, a pair of numbers $x = x_0$ and $y = y_0$ [also written as an ordered pair (x_0, y_0)] is a solution of this system if each equation is satisfied by the pair. The set of all such ordered pairs is called the solution set for the system. To solve a system is to find its solution set.

Contd.

- We will consider three methods of solving such systems: graphing, substitution, and elimination by addition. Each method has its advantages, depending on the situation.

Graphing

- Recall that the graph of a line is a graph of all the ordered pairs that satisfy the equation of the line. To solve the ticket problem by graphing, we graph both equations in the same coordinate system. The coordinates of any points that the graphs have in common must be solutions to the system since they satisfy both equations

Contd.

- **Example 1:** Solve the ticket problem by graphing:

$$2x + y = 32$$
$$x + 3y = 36$$

- **Solution** An easy way to find two distinct points on the first line is to find the x and y intercepts. Substitute $y = 0$ to find the x intercept ($2x = 32$, so $x = 16$), and substitute $x = 0$ to find the y intercept ($y = 32$). Then draw the line through (16, 0) and (0, 32). After graphing both lines in the same coordinate system (Fig. 1), estimate the coordinates of the intersection point:

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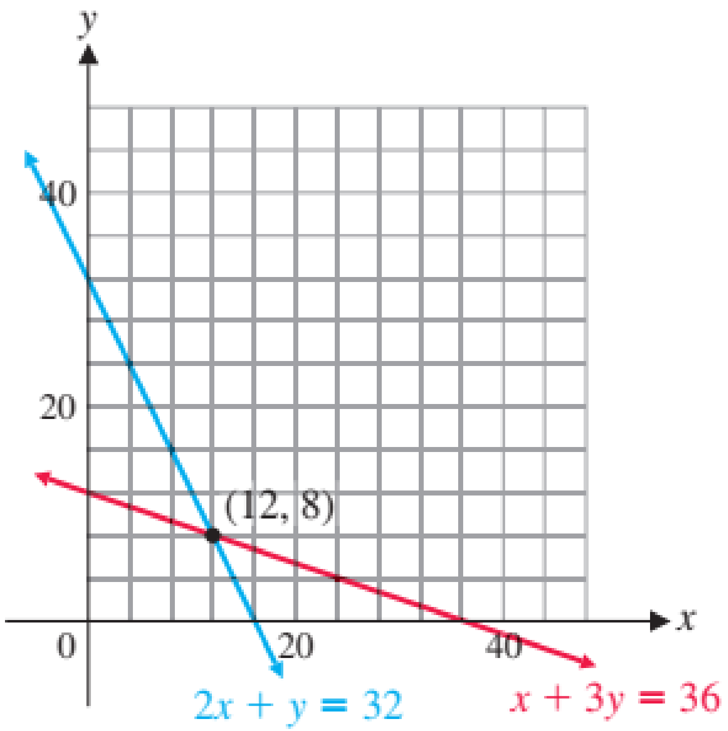


$x = \$12$ Adult ticket

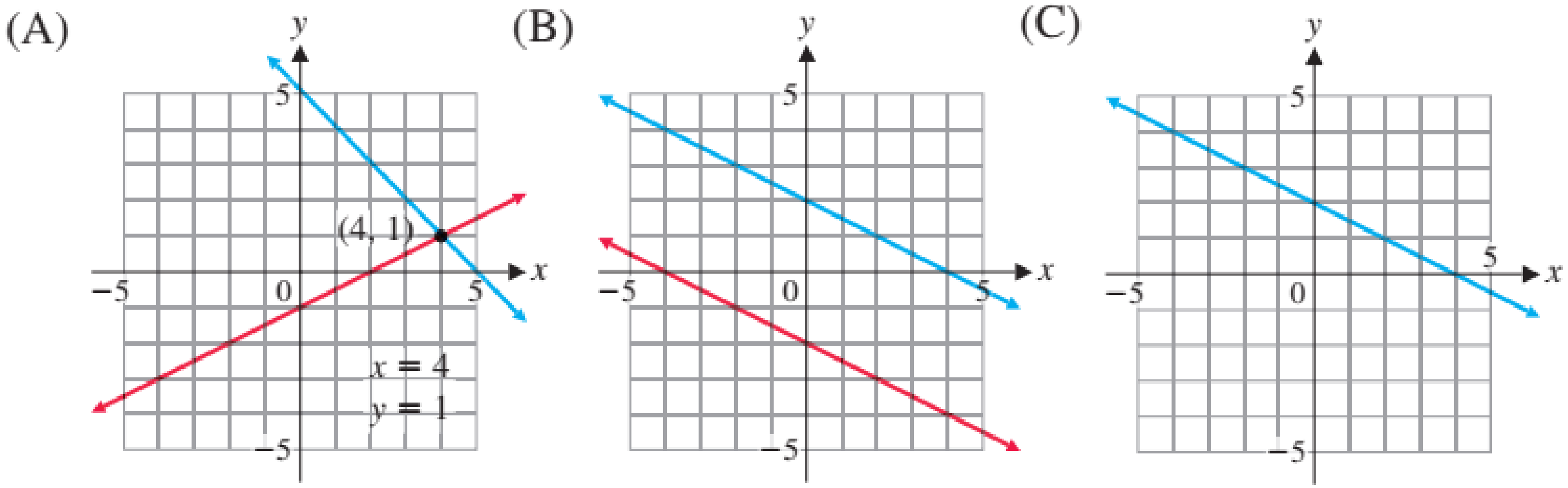
$y = \$8$ Child ticket

Contd.

- Example 2: Solve each of the following systems by graphing:

(A) $\begin{aligned} x - 2y &= 2 \\ x + y &= 5 \end{aligned}$ (B) $\begin{aligned} x + 2y &= -4 \\ 2x + 4y &= 8 \end{aligned}$ (C) $\begin{aligned} 2x + 4y &= 8 \\ x + 2y &= 4 \end{aligned}$

SOLUTION



(A) Intersection at one point only—exactly one solution

(B) Lines are parallel (each has slope $-\frac{1}{2}$)—no solutions

(C) Lines coincide—infinite number of solutions

Contd.

- Definition Systems of Linear Equations: Basic Terms
- A system of linear equations is **consistent** if it has one or more solutions and **inconsistent** if no solutions exist. Furthermore, a consistent system is said to be **independent** if it has exactly one solution (often referred to as the **unique solution**) and **dependent** if it has more than one solution. Two systems of equations are **equivalent** if they have the same solution set.

Contd.

- Referring to the three systems in Example 2, the system in part (A) is consistent and independent with the unique solution $x = 4$, $y = 1$. The system in part (B) is inconsistent. And the system in part (C) is consistent and dependent with an infinite number of solutions (all points on the two coinciding lines).

Contd.

THEOREM 1 Possible Solutions to a Linear System

The linear system

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

must have

(A) Exactly one solution — Consistent and independent

or

(B) No solution — Inconsistent

or

(C) Infinitely many solutions — Consistent and dependent

There are no other possibilities.

Contd.

Substitution

Now we review an algebraic method that is easy to use and provides exact solutions to a system of two equations in two variables, provided that solutions exist. In this method, first we choose one of two equations in a system and solve for one variable in terms of the other. (We make a choice that avoids fractions, if possible.) Then we **substitute** the result into the other equation and solve the resulting linear equation in one variable. Finally, we substitute this result back into the results of the first step to find the second variable.

Contd.

- Solve by substitution:

$$5x + y = 4$$
$$2x - 3y = 5$$

Solution: Solve either equation for one variable in terms of the other; then substitute into the remaining equation. In this problem, we avoid fractions by choosing the first equation and solving for y in terms of x:

Contd.

$5x + y = 4$ Solve the first equation for y in terms of x.

$y = \underbrace{4 - 5x}$ Substitute into the second equation.

$$\begin{aligned} 2x - 3y &= 5 && \text{Second equation} \\ 2x - 3(4 - 5x) &= 5 && \text{Solve for } x. \\ 2x - 12 + 15x &= 5 \\ 17x &= 17 \\ x &= 1 \end{aligned}$$

Now, replace x with 1 in $y = 4 - 5x$ to find y:

$$\begin{aligned} y &= 4 - 5x \\ y &= 4 - 5(1) \\ y &= -1 \end{aligned}$$

The solution is $x = 1, y = -1$ or $(1, -1)$.

Contd.

Elimination by Addition

- The methods of graphing and substitution both work well for systems involving two variables. However, neither is easily extended to larger systems. Now we turn to elimination by addition. This is probably the most important method of solution. It readily generalizes to larger systems and forms the basis for computer-based solution methods.

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THEOREM 2 Operations That Produce Equivalent Systems

A system of linear equations is transformed into an equivalent system if

(A) Two equations are interchanged.
(B) An equation is multiplied by a nonzero constant.
(C) A constant multiple of one equation is added to another equation.

- Any one of the three operations in Theorem 2 can be used to produce an equivalent system, but the operations in parts (B) and (C) will be of most use to us now. Part (A) becomes useful when we apply the theorem to larger systems.

Contd.

› Solve the following system using elimination by addition:

$$3x - 2y = 8$$
$$2x + 5y = -1$$

Solution: We use Theorem 2 to eliminate one of the variables, obtaining a system with an obvious solution:

Contd.

$$\begin{aligned} 3x - 2y &= 8 \\ 2x + 5y &= -1 \end{aligned}$$

Multiply the top equation by 5 and the bottom equation by 2 (Theorem 2B).

$$\boxed{\begin{aligned} 5(3x - 2y) &= 5(8) \\ 2(2x + 5y) &= 2(-1) \end{aligned}}$$

$$\begin{aligned} 15x - 10y &= 40 \\ 4x + 10y &= -2 \\ \hline 19x \qquad &= 38 \\ x &= 2 \end{aligned}$$

Add the top equation to the bottom equation (Theorem 2C), eliminating the y terms.

Divide both sides by 19, which is the same as multiplying the equation by $\frac{1}{19}$ (Theorem 2B).

This equation paired with either of the two original equations produces a system equivalent to the original system.

Contd.

Knowing that $x = 2$, we substitute this number back into either of the two original equations (we choose the second) to solve for y:

$$2(2) + 5y = -1$$

$$5y = -5$$

$$y = -1$$

The solution is $x = 2$, $y = -1$ or $(2, -1)$.

Contd.

Let's see what happens in the elimination process when a system has either no solution or infinitely many solutions. Consider the following system:

$$\begin{aligned} 2x + 6y &= -3 \\ x + 3y &= 2 \end{aligned}$$

Multiplying the second equation by -2 and adding, we obtain

$$\begin{array}{r} 2x + 6y = -3 \\ \underline{-2x - 6y = -4} \\ 0 \;\; = -7 \end{array} \qquad \text{Not possible}$$

We have obtained a contradiction. The assumption that the original system has solutions must be false. So the system has no solutions, and its solution set is the empty set. The graphs of the equations are parallel lines, and the system is inconsistent.

Contd.

Now consider the system

$$\begin{aligned} x - \tfrac{1}{2}y &= 4 \\ -2x + y &= -8 \end{aligned}$$

If we multiply the top equation by 2 and add the result to the bottom equation, we obtain

$$\begin{array}{r} 2x - y = 8 \\ \underline{-2x + y = -8} \\ 0 = 0 \end{array}$$

Contd.

Obtaining $0 = 0$ implies that the equations are equivalent; that is, their graphs coincide and the system is dependent. If we let $x = k$, where k is any real number, and solve either equation for y, we obtain $y = 2k - 8$. So $(k, 2k - 8)$ is a solution to this system for any real number k. The variable k is called a **parameter** and replacing k with a real number produces a **particular solution** to the system. For example, some particular solutions to this system are

$k = -1$	$k = 2$	$k = 5$	$k = 9.4$
$(-1, -10)$	$(2, -4)$	$(5, 2)$	$(9.4, 10.8)$

Contd.

›Applications

Many real-world problems are solved readily by constructing a mathematical model consisting of two linear equations in two variables and applying the solution methods that we have discussed.

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- Example of Diet Problem Jasmine wants to use milk and orange juice to increase the amount of calcium and vitamin A in her daily diet. An ounce of milk contains 37 milligrams of calcium and 57 micrograms* of vitamin A. An ounce of orange juice contains 5 milligrams of calcium and 65 micrograms of vitamin A. How many ounces of milk and orange juice should Jasmine drink each day to provide exactly 500 milligrams of calcium and 1,200 micrograms of vitamin A?

Contd.

- **Solution:** The first step in solving an application problem is to introduce the proper variables. Often, the question asked in the problem will guide you in this decision. Reading the last sentence in Example 6, we see that we must determine a certain number of ounces of milk and orange juice. So we introduce variables to represent these unknown quantities:

  x = number of ounces of milk

  y = number of ounces of orange juice

Contd.

- Next, we summarize the given information using a table. It is convenient to organize the table so that the quantities represented by the variables correspond to columns in the table (rather than to rows) as shown.

	Milk	Orange Juice	Total Needed
Calcium	37 mg/oz	5 mg/oz	500 mg
Vitamin A	57 μg/oz	65 μg/oz	1,200 μg

Contd.

Now we use the information in the table to form equations involving x and y:

$$\begin{pmatrix} \text{calcium in } x \text{ oz} \\ \text{of milk} \end{pmatrix} + \begin{pmatrix} \text{calcium in } y \text{ oz} \\ \text{of orange juice} \end{pmatrix} = \begin{pmatrix} \text{total calcium} \\ \text{needed (mg)} \end{pmatrix}$$

$$37x \quad + \quad 5y \quad = \quad 500$$

$$\begin{pmatrix} \text{vitamin A in } x \text{ oz} \\ \text{of milk} \end{pmatrix} + \begin{pmatrix} \text{vitamin A in } y \text{ oz} \\ \text{of orange juice} \end{pmatrix} = \begin{pmatrix} \text{total vitamin A} \\ \text{needed } (\mu\text{g}) \end{pmatrix}$$

$$57x \quad + \quad 65y \quad = \quad 1{,}200$$

Contd.

So we have the following model to solve:

$$\begin{aligned} 37x + 5y &= 500 \\ 57x + 65y &= 1{,}200 \end{aligned}$$

We can multiply the first equation by -13 and use elimination by addition:

$$\begin{array}{rcl} -481x - 65y &=& -6{,}500 \\ 57x + 65y &=& 1{,}200 \\ \hline -424x \qquad &=& -5{,}300 \\ x &=& \mathbf{12.5} \end{array} \qquad \begin{aligned} 37(\mathbf{12.5}) + 5y &= 500 \\ 5y &= 37.5 \\ y &= \mathbf{7.5} \end{aligned}$$

Drinking 12.5 ounces of milk and 7.5 ounces of orange juice each day will provide Jasmine with the required amounts of calcium and vitamin A.

System of linear Equations and Augmented Matrices

- A **matrix** is a rectangular array of numbers written within brackets. Two examples are

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 5 & 12 \\ 0 & 1 & 8 \\ -3 & 10 & 9 \\ -6 & 0 & -1 \end{bmatrix} \qquad (1)$$

- Each number in a matrix is called an **element** of the matrix. Matrix A has 6 elements arranged in 2 rows and 3 columns. Matrix B has 12 elements arranged in 4 rows and 3 columns. If a matrix has m rows and n columns, it is called an $m \times n$ **matrix** (read "m by n matrix").

Contd.

- The expression $m \times n$ is called the **size/order/shape** of the matrix, and the numbers m and n are called the **dimensions** of the matrix. It is important to note that the number of rows is always given first. Referring to equations (1), A is a 2×3 matrix and B is a 4×3 matrix. A matrix with n rows and n columns is called a **square matrix of order n**. A matrix with only 1 column is called a **column matrix**, and a matrix with only 1 row is called a **row matrix**.

Contd.

$$\begin{array}{ccc} 3 \times 3 & 4 \times 1 & 1 \times 4 \\ \begin{bmatrix} 0.5 & 0.2 & 1.0 \\ 0.0 & 0.3 & 0.5 \\ 0.7 & 0.0 & 0.2 \end{bmatrix} & \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 2 & \frac{1}{2} & 0 & -\frac{2}{3} \end{bmatrix} \end{array}$$

Square matrix of order 3 | Column matrix | Row matrix

- The **position** of an element in a matrix is given by the row and column containing the element. This is usually denoted using **double subscript notation** a_{ij}, where i is the row and j is the column containing the element a_{ij}, as illustrated below

Contd.

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad \begin{matrix} a_{11} = 1, & a_{12} = -4, & a_{13} = 5 \\ a_{21} = 7, & a_{22} = 0, & a_{23} = -2 \end{matrix}$$

- Note that a_{12} is read "*a* sub one two" (*not* "*a* sub twelve"). The elements $a_{11} = 1$ and $a_{22} = 0$ make up the *principal diagonal* of A. In general, the **principal diagonal** of a matrix A consists of the elements $a_{11}, a_{22}, a_{33}, \ldots\ldots..$

Contd.

- Matrices serve as a shorthand for solving systems of linear equations. Associated with the system

$$\begin{aligned} 2x - 3y &= 5 \\ x + 2y &= -3 \end{aligned} \qquad (2)$$

- are its **coefficient matrix, constant matrix,** and **augmented matrix:**

Coefficient matrix	Constant matrix	Augmented matrix
$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 5 \\ -3 \end{bmatrix}$	$\left[\begin{array}{cc\|c} 2 & -3 & 5 \\ 1 & 2 & -3 \end{array}\right]$

Contd.

- **Note that** the augmented matrix is just the coefficient matrix, augmented by the constant matrix. The vertical bar is included only as a visual aid to separate the coefficients from the constant terms. The augmented matrix contains all of the essential information about the linear system—everything but the names of the variables.

Contd.

- For ease of generalization to the larger systems in later sections, we will change the notation for the variables in system (2) to a subscript form. That is, in place of x and y, we use x_1 and x_2, respectively, and system (2) is rewritten as

$$\begin{aligned} 2x_1 - 3x_2 &= 5 \\ x_1 + 2x_2 &= -3 \end{aligned}$$

In general, associated with each linear system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= k_1 \\ a_{21}x_1 + a_{22}x_2 &= k_2 \end{aligned} \tag{3}$$

Contd.

where x_1 and x_2 are variables, is the *augmented matrix* of the system:

$$\begin{array}{l} \text{Column 1 } (C_1) \\ \text{Column 2 } (C_2) \\ \text{Column 3 } (C_3) \end{array}$$

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right] \begin{array}{l} \leftarrow \text{Row 1 } (R_1) \\ \leftarrow \text{Row 2 } (R_2) \end{array}$$

Contd.

- We say that two augmented matrices are row equivalent, denoted by the symbol $\sim$ placed between the two matrices, if they are augmented matrices of equivalent systems of equations. How do we transform augmented matrices into row-equivalent matrices?

Contd.

THEOREM 1 Operations That Produce Row-Equivalent Matrices

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following **row operations**:

(A) Two rows are interchanged $(R_i \leftrightarrow R_j)$.
(B) A row is multiplied by a nonzero constant $(kR_i \rightarrow R_i)$.
(C) A constant multiple of one row is added to another row $(kR_j + R_i \rightarrow R_i)$.

Note: The arrow $\rightarrow$ means "replaces."

Contd.

- Example: Solve using augmented matrix methods:

$$\begin{aligned} 3x_1 + 4x_2 &= 1 \\ x_1 - 2x_2 &= 7 \end{aligned} \qquad (4)$$

- Solution We start by writing the augmented matrix corresponding to system (4):

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \qquad (5)$$

Contd.

- Our objective is to use row operations from Theorem 1 to try to transform matrix (5) into the form

$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right] \tag{6}$$

- where m and n are real numbers. Then the solution to system (4) will be obvious, since matrix (6) will be the augmented matrix of the following system (a row in an augmented matrix always corresponds to an equation in a linear system):

Contd.

$$\begin{aligned} x_1 &= m \qquad\qquad & x_1 + 0x_2 &= m \\ x_2 &= n \qquad\qquad & 0x_1 + x_2 &= n \end{aligned}$$

- Now we use row operations to transform matrix (5) into form (6).
- Step 1 To get a 1 in the upper left corner, we interchange R_1 and R_2 (Theorem 1A):

$$\left[\begin{array}{rr|r} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

Contd.

- **Step 2** To get a 0 in the lower left corner, we multiply R_1 by (-3)and add to R_2 (Theorem 1C)—this changes R_2 but not R_1. Some people find it useful to write (-3R_1)outside the matrix to help reduce errors in arithmetic, as shown:

$$\begin{array}{c}\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 4 & 7 \end{array}\right] \\ \begin{array}{ccc} -3 & 6 & -21 \end{array}\end{array} \underbrace{(-3)R_1}_{} \overset{\sim}{+} R_2 \to R_2 \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

Contd.

- **Step 3** To get a 1 in the second row, second column, we multiply R_2 by $\frac{1}{10}$ (Theorem 1B):

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right] \tfrac{1}{10}R_2 \overset{\sim}{\rightarrow} R_2 \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

Contd.

› **Step 4** To get a 0 in the first row, second column, we multiply R_2 by 2 and add the result to R_1 (Theorem 1C)—this changes R_1 but not R_2:

$$\begin{array}{ccc} 0 & 2 & -4 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right] \underset{\sim}{\overbrace{2R_2} + R1 \to R1} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

Contd.

- We have accomplished our objective! The last matrix is the augmented matrix for the system

$$\begin{aligned} x_1 &= 3 \qquad & x_1 + 0x_2 &= 3 \\ x_2 &= -2 \qquad & 0x_1 + x_2 &= -2 \end{aligned} \tag{7}$$

- Since system (7) is equivalent to system (4), our starting system, we have solved system (4); that is, $x_1 = 3$ and $x_2 = -2$.

Contd.

- Example: Solve using augmented matrix methods:

$$\begin{aligned} 2x_1 - x_2 &= 4 \\ -6x_1 + 3x_2 &= -12 \end{aligned} \qquad (8)$$

- Solution:

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ -6 & 3 & -12 \end{array}\right] \quad \begin{array}{l} \frac{1}{2}R_1 \to R_1 \text{ (to get a 1 in the upper left corner)} \\ \frac{1}{3}R_2 \to R_2 \text{ (this simplifies } R_2\text{)} \end{array}$$

Contd.

$$\sim\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 2 \\ -2 & 1 & -4 \end{array}\right] \quad \underbrace{2R_1}_{} + R_2 \to R_2 \text{ (to get a 0 in the lower left corner)}$$

$$\begin{array}{ccc} 2 & -1 & 4 \end{array}$$

$$\sim\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Contd.

- The last matrix corresponds to the system

$$\begin{aligned} x_1 - \frac{1}{2}x_2 &= 2 \qquad & x_1 - \frac{1}{2}x_2 &= 2 \\ 0 &= 0 \qquad & 0x_1 + 0x_2 &= 0 \end{aligned} \tag{9}$$

- We start by solving $x_1 - \frac{1}{2}x_2 = 2$, the first equation in system (9), for either variable in terms of the other. We choose to solve for x_1 in terms of x_2 because it is easier:

Contd.

$$x_1 = \frac{1}{2}x_2 + 2 \qquad (10)$$

- we introduce a parameter t (we can use other letters, such as k, s, p, q, and so on, to represent a parameter also). If we let $x_2 = t$, then for any real number t,

Contd.

$$x_1 = \frac{1}{2}t + 2 \qquad (11)$$
$$x_2 = t$$

- represents a solution of system (8). Using ordered-pair notation, we write: For any real number t,

$$\left(\frac{1}{2}t + 2, t\right) \qquad (12)$$

- is a solution of system (8).

Contd.

- More formally, we write

$$\text{solution set} = \left\{ \left(\frac{1}{2}t + 2, t \right) \middle| t \in R \right\} \tag{13}$$

Contd.

- Example: Solve using augmented matrix methods

$$\begin{aligned} 2x_1 + 6x_2 &= -3 \\ x_1 + 3x_2 &= 2 \end{aligned}$$

- Solution

$$\left[\begin{array}{cc|c} 2 & 6 & -3 \\ 1 & 3 & 2 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

Contd.

$$\begin{array}{l}\sim\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 6 & -3 \end{array}\right] \qquad (-2)R_1 + R_2 \to R_2 \\ \quad\;\; -2 \quad -6 \qquad\quad -4 \end{array}$$

$$\sim\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & -7 \end{array}\right] \qquad R_2 \text{ implies the contradiction } 0 = -7.$$

Contd.

- This is the augmented matrix of the system

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ 0 &= -7 \end{aligned} \qquad \begin{aligned} x_1 + 3x_2 &= 2 \\ 0x_1 + 0x_2 &= -7 \end{aligned}$$

- The second equation is not satisfied by any ordered pair of real numbers.

Contd.

SUMMARY Possible Final Matrix Forms for a System of Two Linear Equations in Two Variables

Form 1: Exactly one solution (consistent and independent)	Form 2: Infinitely many solutions (consistent and dependent)	Form 3: No solution (inconsistent)
$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$	$\left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & p \end{array}\right]$

m, n, p are real numbers; $p \neq 0$