A + i

ST 122 : LINEAR ALGEBRA.

- MATRIX:

A rectangular array of maximum $m \times n$ numbers (real or complex) in the form of m-rows and n-columns is called a Matrix of order $m \times n$.

$$A_{m \times n} = \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}$$

Denoted as $A = [a_{ij}]_{m \times n}$: $i = 1, \ldots m; n = 1 \ldots$

- SPECIAL TYPES OF MATRICES.

i/ SYMMETRIC VS SKEW-SYMMETRIC MATRICES.

A matrix $A = [a_{ij}]_{m \times n}$ $i = 1 \ldots m$, $j = 1 \ldots n$ is to be symmetric if

$$a_{ij} = a_{ji} \; \forall \; i, j.$$

and it is skew-symmetric if

$$a_{ij} = -a_{ji} \; \forall \; i, j.$$

And A is a square matrix.

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ii) SQUARE MATRIX :

A matrix $A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$; i.e No. of rows = No. of columns.

iii) ROW AND COLUMN MATRICES :

$A = (a_{ij})_{m \times n}$ is

Row matrix if for a_{ij}, $i = 1$, $j = 1 \ldots n$; $a_{ij}\ \forall\ i, j$ but $m < n$ (Horizontal)

Column matrix if for a_{ij}, $i = 1 \ldots m$, $j = 1$; (Vertical) $a_{ij}\ \forall\ i, j$ but $m > n$ (

Rectangular $a_{ij}\ \forall\ ij$ such that $m > n$ or $m < n$;

iii) Square $a_{ij}\ \forall\ ij$ such that $m = n$.

iv) HORIZONTAL VS VERTICAL MATRICES ;

v) DIAGONAL, SCALAR or Identity MATRICES .

$$A = [a_{ij}]_{m \times n}.$$

... 2] - Scalar ;

1 = Identity ;

Real no .

... - Diagonal ;

$$a_{ij} \begin{cases} R\ \forall\ i = j ; & \text{- Diagonal .} \\ K\ \forall\ i = j . & \text{Scalar} \\ I\ \forall\ i = j & \text{- Identity ;} \\ a_{ij} = 0\ \forall\ i \neq j & \end{cases}$$

where K = Constant and I - Identity :

- Singular and Nonsingular Matrices

  Singular, $|A| = 0$, Has no soln.

  Non Singular $|A| \neq 0$, Has solutions.

- Upper and Lower triangular matrices.

  Upper triangular matrix, the upper entries ~~above~~ below the main diagonal are zero.

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & a_{24} \cdots & a_{2n} \\ 0 & 0 & 0 & & a_{mn} \end{bmatrix}$$

Lower triangular matrix - the entries above the main diagonal are zero

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots \\ a_{21} & a_{22} & 0 & \cdots & \\ a_{31} & a_{32} & a_{33} & \cdots & \end{bmatrix}$$

- Hermitian and skew-hermitian matrices.

LINEAR SYSTEMS:

It the system with m-linear equations which involves n-variables

- Types of solutions: - No solution.
  - Many solution.
  - One solution (Unique).

such as $a_{11}x_1 + a_{21}x_2 + \dots a_{n1}x_n = b_1$.

$a_{12}x_1 + a_{22}x_2 + \dots \dots a_{n2} = b_2$.

$a_{m1} + a_{n2}x_2 \dots + a_{mn}x_n = b_n$.

Approaches.

i) Graphing.
(ii) Substitution.
iii) Elimination.
(iv) Cramers Rule.
(v) Inverse Method

Advanced:

vi) GAUSS Elimination.
vii) GAUSS-JORDAN Elimination.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\underset{\sim}{x} = \underset{\sim}{b}.$$

$$\underset{\sim}{b} = \begin{cases} k; & \text{Heterogeneous system} \to \text{No soln.} \\ 0; & \text{Homogeneous system} \to \text{Unique, Many} \end{cases}$$

Upper triangular

Row echelon

GAUSS ELIMINATION (row reduction).

Is an algorithm in linear algebra for solving linear equations. Uses a sequence of elementary row operations to modify the matrix.

Types of elementary row operations.

i) swapping two rows.

ii) Multiplying a row by a constant (K) $K \neq 0$.

iii) Adding multiple of one row to another row.

- Matrix can be transformed into an upper triangular matrix (row echelon form).

Example;

$$x_1 - x_2 + x_3 = 3$$
$$-x_2 + x_1 - x_3 = 5$$

i) Augmented Matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 1 & -1 & -1 & 5 \end{array}\right]$$

Elementary row operations.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$R_1 - R_2 \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & -2 & 2 & -2 \end{array}\right]$$

Task 2:

$$x_1 - x_2 + 2x_3 = 4$$

$$x_1 + x_3 = 6$$

$$2x_1 - 3x_2 + 5x_3 = 4.$$

$$3x_1 + 2x_2 - x_3 = 1$$

Soln: $A\underset{\sim}{x} = b$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & -3 & 5 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

Augmented Matrix ~~[Ax~~ $[A : b]$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{array}\right] \xrightarrow[\substack{2R_2 - R_3 \\ 3R_1 - R_4}]{R_1 - R_2.}$$

Swapping of rows

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & -1 & 1 & -2 \\ 0 & 3 & -3 & 8 \\ 0 & -5 & 7 & 11 \end{array}\right] \xrightarrow[5R_2+R_4]{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 21 \end{array}\right]$$

$R_4 \leftrightarrow R_3$

$2X_3 = 21 \quad X_3 = 21/2$.

$-X_2 + X_3 = -2$

$-X_2 = -2 - 21/2 \quad X_2 = 25/2$.

$X_1 - X_2 + 2X_3 = 4$

$X_1 = 4 + 25/2 - 2(21/2)$

$X_1 = -9/2$.

$X_1 = -9/2, \; X_2 = 25/2, \; X_3 = 21/2$.

∴ The value of X_3 CANNOT be determined from $0X_3 = 2$

∴ There is No solution

Task 2: Find the solution.

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 9 \\ 3 & 3 & -1 & 1 & 1 & 5 \\ 2 & 2 & -1 & 1 & -2 & 1 \\ 4 & 4 & 1 & 0 & -3 & 4 \\ 8 & 5 & -2 & -1 & 2 & 3 \end{array}\right]$$

(iii) Row operations.

i/ Interchange the rows $R_i \leftrightarrow R_j$

ii/ Multiply a row

(d) i) Elementary Matrices

ii) An $m \times n$ matrix I_n is called E if it can be obtained from I_n by a single elementary row operations.

i] $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1}$ (ii) $A_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

elementary matrix

iii] $A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$ mR_4, $m \neq 0$.

3] $A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_3 + mR_2$

Definition:

Two matrices A and B are row equivalent if one matrix B can be obtained from another matrix A through a finite number of elementary matrices.

$E_1, E_2 \ldots E_n$

Such that $B = E_n \ldots E_2 E_1 A$.

k-matrices

$A\underline{x} = \underline{b}$;

$A^{-1}A\underline{x} = A^{-1}\underline{b}$

$x = A^{-1}\underline{b}$.

$B = E_3 E_2 E_1 A$.

$E_3^{-1} B = E_3^{-1} E_3 E_2 E_1 A$.

$E_3^{-1} B = \cancel{E_3^{-1}} \cancel{E_3} E_2 E_1 A$.

$E_2^{-1} E_3^{-1} B = E_1 A$

$E_1^{-1} E_2^{-1} E_3^{-1} B = E_1^{-1} E_1 A$.

$E_1^{-1} E_2^{-1} E_3^{-1} B = A$

$A = E_1^{-1} E_2^{-1} E_3^{-1} B$.

$\therefore A = E_1^{-1} E_2^{-1} \ldots E_k^{-1} B$

Inverse of E

E	E^{-1}
$R_i \Leftrightarrow R_j$	$R_i \Leftrightarrow R_j$
mR_i	$1/m R_i$; $m \neq 0$
$R_i + mR_j$	$R_i - mR_j$; $m \neq 0$

THE LU FACTORIZATION

MATRIX FACTORIZATION:

Matrix B is an inverse of A if after the row elementary operations the augmented matrix $[A \vdots I_n]$ is

$[I_n : B]$. Then $B = A^{-1}$ and $A^{-1}A = I_n$.

i.e $A\underset{\sim}{x} = \underset{\sim}{b}$.

$AA^{-1}\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$.

$I_n x = A^{-1}\underset{\sim}{b}$

$\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$.

From $A\underset{\sim}{x} = \underset{\sim}{b}$:. If A can be re-written as the prod of two matrix, L and U, then A is said to be factored into L and U.

$$A\underset{\sim}{x} = \underset{\sim}{b} \; = \; LU\underset{\sim}{x} = b$$

To solve $A\underset{\sim}{x} = b$ which is $LU\underset{\sim}{x} = \underset{\sim}{b}$.

1st let $\underset{\sim}{y} = u\underset{\sim}{x}$.

i.e $L\underset{\sim}{y} = \underset{\sim}{b}$.

determine y

2nd use y to determine x.

i.e $U\underset{\sim}{x} = y$

Topic.

(i) Matrix.

(ii) Special types.

(iii) Linear systems | 1-27

(iv) ~~Methods~~

(v) ~~Elementary mat~~ (87)

(vi) Inverse (76)

(vii) Determinants (128 – 1

(viii) LU Ratio (93)

In this approach

L – Lower triangular Matrix.

U – Upper triangular Matrix

$$U = E_K \cdots E_1 A.$$

$$A = E_1^{-1} \cdots E_K^{-1} U.$$

$$A = LU$$

where $L = E_1^{-1} \cdots E^{-K}$.

DETERMINANT :

If A is converted to Triangular Matrix, L or U then the the det(A) = products of diagonal entries of either L or U

$$\begin{bmatrix} a_{11} & & U \\ & a_{22} & \\ 0 & & a_{mn} \end{bmatrix}$$

Note:

i) $R_i \Leftrightarrow R_j$ – -ve multiplication to the product

ii) mR_i (m) x product

iii) $R_i + mR_j$ = Remains the same.

Show that the system

$2x_1 + 3x_2 = 0$

$4x_1 + 3x_2 - x_3 = 0$

$8x_1 + 3x_2 + 3x_3 = 0$

is (i) Homogeneous

(ii) It has unique soln $x = \underline{0}$.

Soln:

i) from $A\underset{\sim}{x} = \underset{\sim}{b}$, $\underset{\sim}{b} = \underset{\sim}{0}$, the the system is H.

ii) $A x = b$

$$\begin{pmatrix} 2 & 3 & 0 \\ 4 & 3 & -1 \\ 8 & 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gauss elimination.

$A\underset{\sim}{x} = \underset{\sim}{b}$.

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & 0 \\ 8 & 3 & 3 & 0 \end{array}\right] \quad \text{Targets } I_3 \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$-2R_1 + R_2$ $\quad$ $-2R_1 + R_2$.

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ & & & \end{array}\right] \quad E_3 \left[\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-4R_1+R_3 \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -9 & 3 & 0 \end{bmatrix}, \quad E_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$-3R_2+R_3 \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}, \quad E_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$R_i \pm mR_j \quad R_i \mp mR_j$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad E_3^{-1} \begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ 0 & 3 & 1 \end{bmatrix}$$

$U = E_3 E_2 E_1 A \qquad A = U E_3^{-1} E_2^{-1} E_1^{-1}$

$$U = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 6 \end{bmatrix}$$

$A = LU$

$L = E_1^{-1} E_2^{-1} E_3^{-1}$

$$-4R_1+R_3 \quad \left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -9 & 3 & 0 \end{array}\right], \qquad E_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$-3R_2+R_3 \quad \left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right], \qquad E_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$R_i \pm mR_j \quad R_i \mp mR_j$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad E_3^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

$U = E_3 E_2 E_1 A \qquad A = U E_3^{-1} E_2^{-1} E_1^{-1}$

$$U = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 6 \end{bmatrix}$$

$\therefore A = LU$

$L = E_1^{-1} E_2^{-1} E_3^{-1}$.

Given:

$2x_1 + 3x_2 = 0$

$4x_1 + 3x_2 - x_3 = 0$

$8x_1 + 3x_2 + 3x_3 = 0$

show that (i) if (i) Homogeneous

(1) $\underset{\sim}{x} = 0$

(i) Gauss Elimination
$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ +2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +3 & 1 \end{bmatrix}$$

$x_3 = 0, \; x_2 = 0, \; x_1 = 0$

(ii) Gauss-Jordan
$$\begin{array}{c} \tfrac{1}{2}R_1 \\ -\tfrac{1}{3} \\ \\ \end{array} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \qquad E_3^{-1} \quad E_2^{-1} \quad E_1^{-1}$$

(iii) LU-factorization: $A = LU$

But $U = E_3 E_2 E_1 A$.

$$\boxed{A = E_3^{-1} E_2^{-1} E_1^{-1} U}$$

But $A = LU$.

L = From the E_n^{-1}.

steps

(i) $\underset{\sim}{y} = U\underset{\sim}{x}$

(ii) $L\underset{\sim}{y} = \underset{\sim}{b}$

(iii) $U\underset{\sim}{x} = \underset{\sim}{y}$

Solve for x

(ii) $A\underset{\sim}{x} = \underset{\sim}{0}$

$LU\underset{\sim}{x} = \underset{\sim}{0}$.

Let $y = Ux$.

$\boxed{L}y = 0$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving $y_1 = 0, y_2 = 0, y_3\ 0$

$$U = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 6 \end{bmatrix}$$

Use different letter

But: $\underset{\sim}{y} = U\underset{\sim}{x}$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x_1

L can be obtained from the elementary matrices

$L = E_k^{-1} E_2^{-1} \dots E_k^{-1}$.

from: $A = LU$

But $A = E_3^{-1} E_2^{-1} E_1^{-1} U$.

If $[A \,\vdots\, I_n] \longrightarrow [I_n \,\vdots\, B]$.

Then $B = A^{-1}$

From: $A\underset{\sim}{x} = b$.

$$A A^{-1} x = b \cdot A^{-1}$$

$$\underset{\sim}{X} = A^{-1} \underset{\sim}{b}$$

$$A = \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 8 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \quad [A : I_n]$$

$$\xrightarrow{-3R_2 + R_3}_{R} \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|ccc} 2 & 0 & -1 & -1 & 1 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_3 + R_1}_{\frac{1}{6}R_3 + R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -\frac{2}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & -3 & 0 & -\frac{5}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right] \longrightarrow \begin{array}{c} \frac{1}{2}R_1 \\ -\frac{1}{3}R_2 \\ \frac{1}{6}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{[illegible]} & \frac{1}{[illegible]} & [illegible] \\ 0 & 1 & 0 & \frac{5}{[illegible]} & -\frac{1}{[illegible]} & [illegible] \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{array}\right]$$

$$[I_n : B]$$

$$B = \begin{bmatrix} -\frac{1}{3} & \frac{1}{4} & \frac{1}{12} \\ \frac{5}{9} & -\frac{1}{6} & -\frac{1}{18} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

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VECTORS

$\underset{\sim}{u} = (u_1, u_2 \ldots u_n)$

$\underset{\sim}{v} = (v_1, v_2, \ldots v_n)$

i) $\underset{\sim}{u} + \underset{\sim}{v} = u_1 + v_1 + u_2 + v_2 \ldots u_n + v_n$

ii) $c\underset{\sim}{u} = (cu_1, cu_2, cu_3 \ldots cu_n)$

iii) Zero vector $0 = (0, 0 \ldots 0)$.

Operations of Vectors

i) Summation.

ii) Substraction.

iii) Scalar multiplication.

Properties of Addition

i) $(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = u + (\underset{\sim}{v} + \underset{\sim}{w})$

ii) $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$.

iii) $\underset{\sim}{u} + (-\underset{\sim}{u}) = 0$

Multiplication:

i) $c\underset{\sim}{u}$.

ii) $c(\underset{\sim}{u} + \underset{\sim}{v}) = c\underset{\sim}{u} + c\underset{\sim}{v}$.

iii) $c(d\underset{\sim}{u}) = cd\underset{\sim}{u}$

collection of vectors and two operations $\oplus$ and $\odot$ such that the following hold for $u, v, w, \in V$

Vector Space: For one to be a subspace three conditions should be satisfied. (i) should contain 0 ~~(ii) Sum should be~~ positive. (iii)

1. Show that the set:

$W = (x_1, 0, x_3)$ such that x_1 and x_2 are real numbers, is a subspace of R^3 with the standard operations.

Let $\underset{\sim}{u} = (u_1, 0, u_3)$
$\underset{\sim}{v} = (v_1, 0, x_3)$.

i) $\underset{\sim}{0}$.
ii) $\underset{\sim}{u} + \underset{\sim}{v}$
iii) $c\underset{\sim}{u}$

(i) Does W contain 0?

Yes it does, since $u_1, v_1, u_3, v_3 \in R$

(ii) $\underset{\sim}{u} + \underset{\sim}{v} = (u_1 + v_1 + u_3 + v_3, 0)$
(iii) $c\underset{\sim}{u} = (cu_1, 0, cu_2)$.

1. Show that $w(x_1, x_2)$ such that $x_1 \geq 0$. and $x_2 \geq 0$ with the standard operations, is not a subspace of R^2

Soln:

Let $\underset{\sim}{u} = (u_1, u_2)$
$\underset{\sim}{v} = (v_1, v_2)$.
$c \in R$

(i) Does W contain 0?

Yes it does, since, $U_1, V_1, U_2, V_2 \in \mathbb{R} \geq 0$.

(ii) $\underline{u} + \underline{v} = (U_1 + V_1, U_2 + V_2)$. Hence $u+v = U$

since $u_1 + v_1 \geq 0$

$u_2 + v_2 \geq 0$

(iii) $c\underline{u} = cu_1, cu_2$.

$=$ ~~c(u_1,~~ since $c \in \mathbb{R}$, it can be -1, Hence does not satisfy.

* 2. Which of the two subsets is a subspace of $\mathbb{R}^2$.

a) The set of points on the line $x + 2y = 0$.

(b) The set of points on the line $x + 2y = 1$.

soln:

a) $x = -2y$.

$(-2y, y)$.

$W = \{(-2y, y), x, y \in \mathbb{R}\}$.

Let

$\underline{u} = (-2u_2, u_2)$.

$\underline{v} = (-2v_2, v_2)$.

Let $u_1 = (-2v_1, v_1)$

$u_2 = (-2v_2, v_2)$.

$u_1 + u_2 = (-2(v_1 + v_2), v_1 + v_2)$

Hence set is a subspace of $\mathbb{R}^2$.

b) Not subspace because every subspace contain zero vector $(0,0)$ is not on the line.

If every vector in vector space can be written as linear combination of vectors in a set

* **Spanning of Vectors:**

$US = V$.

$U_1, \ldots U_n, = V_n$, then S is called Spanning set of the vector space

$c_1 U_1 + c_2 U_2 + \ldots + c_n U_n = V$.

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & & u_{2m} \\ \vdots & & & \vdots \\ u_{n1} & \cdots & & u_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ v_n \end{bmatrix}$$

$u_1 = (-1, -1, 1)$

$u_2 = (0, 1, 1)$.

S spans V $v = (0, 1, 0)$

$$\begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

This can be solved by (i) Gauss
(ii) Gauss-Jordan
(iii) Inverse method
(iv) The $A = LU$

$c \neq R$

$UC = V$.

S can not be solved.

Example: $S\{v_1, v_2 \ldots v_n\}, V$.

in R^3, $v_1 = (1, 1, 1)$

$v_2 (-1, -1, 1)$

$S(v_1, v_2)$.

Let v^* be any vector which is in V.

$$c_1 \underset{\sim}{V_1} + c_2 \underset{\sim}{V_2} = v^*$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \cdot F$$

$$C = R.$$

There is $\underset{\sim}{c}$.

Linear Dependence and Linear Independence.

$$c_1 V_1 + c_2 V_2 + \dots + c_n V_n = V^*$$

$$c_1 V_1 + c_2 V_2 + \dots + c_n V_n = 0.$$

$$c = \begin{cases} \varphi & \text{Dependent} \\ 0 & \text{Independent (Homogeneous system)} \end{cases}$$

Basis and Dimension:

A set of vectors $S = \{V_1, V_2 \dots V_n\}$ in a vector space V is called a basis for V if the following conditions are true

(i) S spans V.

(ii) S is linearly independent.

vector in R^n

* Number of Vector Basis:

If a vector space V has one basis with n-vectors, then every basis for V has n vectors.

R^n, $S = \{n, \text{vectors}\}$

Eg: In R^3 - basis ~~set~~ with 3 vectors. Thus any basis will have 3 vectors.

* Defn of Dimension of Vector space.

If a vector space V has a basis consisting of n vectors, then the number n is called the dimension of V. Denoted as $\dim(V) = n$.

$R_n = n$ $M_{m,n} = (m \times n)$

~~$P_n = n+1$~~

* Uniqueness of the basis Representation.

* Rank of a Matrix

VECTOR: → any set that satisfies the axioms.

1. Vector space: n-space R^n, $n = 1, 2, 3 \ldots$
2. Vector Operations.
3. Subspace W.
   (i) $\underset{\sim}{0}$
   (ii) $\underset{\sim}{u} + \underset{\sim}{v}$
   (iii) $c\underset{\sim}{v}$.
4. Spanning set → $c_1 + c_2 + c_3 \neq 0$, (linearly dependent)
5. Dependence and Independence. $c_1 = c_2 = c_3 = 0$ (linearly independent)
6. Basis Dimension.
7. Rank of A
8. Row and column matrix.

(4) (i) S spans V → the vector space. $S = \{v_1, v_2, v_3 \ldots v_k\}$.
(ii) Span (s) → subspace

5. i) $u\underset{\sim}{c} = 0$
$c_1 + c_2 + c_3 \ldots c_k = 0$.
ii) $u \to u_T$

6. Basis. $S = \{v_1 \ldots v_k\}$.

Basis
i) S spans V.
ii) $\underset{\sim}{c} = \underset{\sim}{0}$.
iii) n-vectors in s.

4. Rank
(i) R^n
(ii) Matrix

Rank = dim(V)

5. Null place = dim(A).

Let V be a vector space of dimension n.

1. If $S = \{v_1, v_2 \ldots v_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.

2. If $S = \{v_1, v_2 \ldots v_n\}$ spans V, then S is a basis for V.

LINEAR TRANSFORMATION:

$T\ V \rightarrow W$

$T(v) \rightarrow W$

domain   Image (codomain)

(i) $T(u+v) = T(u) + T(v)$

(ii) $T(cu) = cT(u)$

(iii) $T(0) = 0$

(iv) $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$

$T(v) = c_1T(v_1) + c_2T(v_2) + \ldots c_nT(v_n)$

$T(1,0,0) = (2,2,2)$
$T(0,1,0) = (-1,-1,-1)$
$T(0,0,1) = (-1,2,1)$ $\quad T(v) = T(c_1v_1 + c_2v_2 + c_3v_3)$

$V = c_1v_1 + c_2v_2 + c_3v_3$ $\quad = c_1T(v_1) + c_2T(v_2) + c_3T(v_3)$

$$\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

$$= T\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 2T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= T\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + 3\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

$$= T\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = (3, -3, -1).$$

The function → Linear Transformation Defined by Matrix

$T: R^2 \to R^3$ as given below.

$$T(V) = AV = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

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Find w, $v = (2,-1)$.

Show T is L.

$v \in \mathbb{R}^2$.

(i) $T(\underline{0}) = A\underline{0} = 0$.

(ii) $T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v})$.

$T(v) = Av$.

$x = u + v$

$T(x) = A\underline{x}$.

$T(x) = A(\underline{u}+\underline{v})$.

$= A\underline{u} + A\underline{v}$.

$T(x) = T(u+v) = T(u) + T(v)$.

(iii) $T(cu) = cT(u)$.

let $\underline{y} = cu$.

$T(y) = A\underline{y}$.

$T(y) = A(c\underline{u})$.

$T(\underline{y}) = cA\underline{u}$.

$T(\underline{y}) = cT(\underline{u})$.

Since both satisfy it is L.

$T(v) = A \cdot v$, $A_{m \times n}$.

From; $T(v) = W$.

$W = A\underline{v}$.

Since $\underline{A_{m \times n}}$, check the given A; with $m \times n$ dimension. then for:

$T: V \Rightarrow W$

$V = R^n \quad W = R^m$.

Given; $T(v) = Av$

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}, \text{ Hence } v = (1, 2, 1).$$

determine v and W. $v = (1, 2, 1)$.

$A_{2 \times 4}$.

$V = R^4 \quad W = R^2$.

From $T(v) = Av$

$$T(v) = \underset{2 \times 4}{\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}},$$

Incompatible i.e $T(v)$ doesn't exist.

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Also can be given:

$T(v) = (v_1 - v_2,\ v_1 + 2v_2)$

To write into Av:

$$T(v) : \begin{bmatrix} v_1 & -v_2 \\ v_1 & 2v_2 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$A_{2\times2}$ $\quad 2\times1$

compatible.

$\therefore$ Transformation is possible.

$T\ V \to W$

$R^2 \to R^2$.

$T(x_1, x_2, x_3)$

$(2x_1 + x_2 - x_3,\ -x_1 + 3x_2 - 2x_3,\ 3x_2 + 4x_3)$.

$$T(x) = \begin{bmatrix} 2x_1 & x_2 & -x_3 \\ -x_1 & 3x_2 & -2x_3 \\ 0x_1 & 3x_2 & 4x_3 \end{bmatrix}$$

$T(x) = Ax$.

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$A_{3\times3}$ $\quad A_{3\times1}$

$T_x : \to W$ $\quad R^3 \to R^3$

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R^2 - Basis = 2, R^3 - 3.
$[1,0]$ $[0,1]$ $(1,0,0), (0,1,0)\ (0,0,1)$.

Basis:

$$B = [e_1, e_2 \ldots e_n]$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

If $T(v) = v'$

$T(v) = Av$.

(i) $A_{n\times n}$ - square matrix

(ii) A^{-1}, its inverse

(iii) $T = A^{-1}$, then Transformation = inverse.

$T(v) = A^{-1}v$

Eg. $T(x,y,z) = (x-y, x+y)$.

Find standard A.

$$T(x,y,z) = \begin{bmatrix} x & -y \\ x & y \end{bmatrix}.$$

$$T(x,y,z) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

$$T(x,y,z) = xT(e_1) + yT(e_2) + zT(e_3)$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}x + \begin{pmatrix} -1 \\ 1 \end{pmatrix}y + z\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$T(e_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$A \cdot (x, y, z)$

$$\underset{2\times 3}{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

Basis = $\{(-1,2,5), (0,6,18)\}$

Rank A = 2.

Basic column space

$C = \{(-1,3,5), (2,0,1)\}$.

Rank(A) = 2

Basic of Row space

$R = \{(1,3,3), (0,1,0,0), (0,0,0,1)\}$

Rank(A) = 3.

Basis of column space.

$C = \{(1,0,-3,3,2), (3,1,6,4,0), (2,0,-1,1,-2)\}$

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NULL SPACE

i, $A\underline{x} = 0$

$N(A) = \{ x \in R^n ; A\underline{x} = \underline{0} \}$

number of $\underline{x}$ is called nullity.

ii) $A\underline{x} = \underline{b}$

$\underline{x} = \underline{x}h + \underline{x}p$

$x_1 - 2x_3 + x_4 = 5$

$3x_1 + x_2 - 5x_3 = 8$

$x_1 + 2x_2 - 5x_4 = -9$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 1 & -5 & 0 \\ 1 & 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -9 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right]$$

$-3R_1 + R_2$

$-R_1 + R_2$

$-R_2 + R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

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~~Dimension~~ = Nullity + Rank.
No. of columns = Dimensions.

$x_1 - 2x_3 + x_4 = 5$

$x_2 + 11x_3 - 3x_4 = -7$.

Let $x_3 = s$;

$x_4 = t$.

$x_2 = -7 + 3t - s$

$x_1 = 5 - t + 2s$

Vector solution : $x = t \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ -7 \\ 0 \\ 0 \end{pmatrix}$.

Qn. $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 1 & -5 & 0 \\ 1 & 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -9 \end{bmatrix}$

i) Find (i) N(A) solution space (Homogeneous system).

(ii) Nullity of A (no. of vectors of that solution state - Dimension of null space).

(iii) solution of $Ax = b$.

(c) Basis of i) Row space of A.

(ii) ~~Column~~ space

(iii) Rank of A

(i) ~~A~~ ~~ज्ञान~~ spans R.

(ii) State Linearly dependent.

iii) Column space of A.

iv) R^4 and R^3

$$N(A) = \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 3 & 1 & -5 & 0 & 0 \\ 1 & 2 & 0 & -5 & 0 \end{array}\right]$$

U-triangular matrix.
$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

i.e $x_1 - 2x_3 + x_4 = 0$

$x_2 + x_3 - 3x_4 = 0$

Let $x_3 = s$, $x_4 = t$.

$x_1 = 2s - t$

$x_2 = 3t - s$

$x_3 = s$

$x_4 = t$

$$x = s\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$N(A) = \{x \in R^n : A\underline{x} = 0$

$N(A) = \{(2, -1, 1, 0), (-1, 3, 0, 1)\}$.

Row space of basis A, $R_b = \{(1, 0, -2, 1), (0, 1, 1, -2)\}$.

$A^T \rightarrow$ upper.

Column space of A $C_b = \{(1, 3, 2), (0, 1, 2)\}$.

D. = N + R
2 + 2 = 4.

Nullity of A = 2.

Rank of A = 2.

R^4 - cant be found because it does not have 4 vectors
R^3 - like wise.

=> To find basis column space; $A^T \rightarrow$ upper triangular matrix UTM, the row space in column form

=> Basis of column space:
Find the A^T and follow the normal space for basis of row space

chapter 5

TOPIC 3: ORTHOGONAL TRANSFORMATION

Definition:

Let $\underline{u}$ and $\underline{v} \in R^n$ (vectors in linear space (V))

1. The norm/length $\underline{u}$ is:

$$|u| = \sqrt{u_1^2 + u_2^2 + \dots u_n^2}$$

2. The distance between $\underline{u}$ and $\underline{v}$ is:

$$d(u,v) = |x| = \underset{\sim}{x} = \underline{u} - \underline{v}$$

3. The angle, θ between $\underline{u}$ and $\underline{v}$.

*Note:

$u \cdot v$ - dot (euclidean inner product for R^n) is

$\langle u,v \rangle$ - general inner product for vector space V (vector product).

$$\cos\theta = \frac{\langle u,v \rangle}{|u||v|}, \quad 0 \le \theta \le \pi$$

$$\therefore \theta = \cos^{-1}(P) \quad * \quad P = \frac{\langle u,v \rangle}{|\underset{\sim}{u}||v|}$$

4. $\underline{u}$ and $\underline{v}$ are orthogonal.

if $\langle u,v \rangle = 0$.

$$P = \frac{0}{|u||v|}, \quad P = \underline{\underline{0}}. \quad \theta = \cos^{-1}(0).$$

$$\therefore \theta = 90^\circ, \ \underline{\underline{\pi/2}}.$$

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5. Projections of $\underline{v}$ on u is

$$\text{Proj}_u\, v = \frac{v \cdot u}{\underline{u} \cdot \underline{u}} \cdot \underline{u}.$$

where: $\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + \dots u_nv_n$

$v \cdot v = v_1^2 + v_2^2 + \dots v_n^2$.

Projection of u on v, is.

$$\text{Proj}_V\, u = \frac{v \cdot u}{\underline{v} \cdot \underline{v}} \cdot \underline{v}$$

6. $\underline{u}$ is ortho~~gonal~~normal vector $u = (u_1, u_2, \dots, u_n)$,

iff $|u| = 1$. $v = (v_1, v_2, \dots, v_n)$.

$|u| = a$, $a \in R$ if $|u| \neq 1$, it corresponding,

$a \neq 1$; a ~~orthogonal~~ orthonormal vector $(\underline{w})$ is "

$$\underline{w} = \frac{1}{|u|} \cdot \underline{u}.$$

i. $$\underline{w} = \frac{u}{|u|}.$$

7. Orthogonal (Orthonormal) — Orthonormal.

1. $\langle v_i \cdot v_j \rangle = 0$, $i \neq j$.
2. ~~$|v_i| = 1$~~

when $i \neq j$

1. $\langle v_i, v_j \rangle = 0$.
2. $|v_i| = 1$

8. If $S = \{v_1, v_2, \ldots, v_n\}$ $v_i \in R^n$, such that v_i are orthogonal vectors then (i) v_i are linearly independent vectors. (ii) S is the basis set.

~~DEFINITIONS~~.

9. THE GRAM-SCHMIDT ORTHO~~GONAL~~NORMALIZATION PROCESS (GS-process).

Suppose, $S = \{v_1, \ldots v_n\}$ is Non-~~orthogonal~~ orthonormal set then $S \to Q$ via GS-process such that Q is ~~orthogonal~~ orthonormal set $\to$ i) $v_i v_j = 0$

(ii) $|v_i| = 1$

$S \to W \to Q$ — GS-process

W: $v_i v_j = 0$ — orthogonal

Q: $v_i v_j = 0$, $|v_i| = 1$ $\Big\} \to$ Orthonormal

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(i) Let $B = \{V_1, V_2 \ldots V_n\}$ be a basis for an I Ps -

(ii) ' $B' = \{W_1, W_2 \ldots W_n\}$ where W_i

(i) Let $B = \{V_1, V_2 \ldots V_n\}, \{W_1 \ldots, W_n\}, B = \{Q_1, Q_2 \ldots Q_n\}$.

GS - process.

$$W_1 = V_1$$

$$W_2 = V_2 - \text{Proj}_{W_1} V_2$$

$$W_3 = V_3 - \text{Proj}_{W_1} V_3 - \text{Proj}_{W_2} V_3$$

$W_i W_j = 0$, $W_n = V_n - \text{Proj}_{W_1} V_n - \text{Proj}_{W_2} V_n \ldots \text{Proj}_{W_{n-1}} V_n$

(iii) $B'' = q_i = \frac{1}{|W_i|} \cdot W_i, \; i = 1, 2 \ldots n$.

10. QR - Factorization :

$$A_{m \times n} = Q_{m \times n} \cdot R_{n \cdot n}$$

$$[V_1, V_2 \ldots V_n] = [q_1\, q_2 \ldots q_n] \begin{bmatrix} V_1 q_1 & V_2 q_2 & \ldots & V_n q_1 \\ 0 & & & \\ \vdots & & & \\ 0 & \ldots & & V_n q_n \end{bmatrix}$$

11. Fundamental Subspaces.

$$S = \left\{ \begin{array}{ccc} V_1 & V_2 & V_3 \\ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, & \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, & \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \end{array} \right\}$$

Find $A = QR$.

Soln:

The GS-process ← ~~To convert into orthogonal~~ Then to orthonormal.

$S \to W \to Q$. ⟶ i) $V_i V_j = 0 \ \forall i \neq j$

↓ (ii) $|V_i| = i = 1 \ldots n$

i) $V_i V_j = 0$.

Soln:

$V_1 \cdot V_2 = -1 + 0 - 1 = -2$.

$V_1 \cdot V_3 = 2 + 0 + 1 = 3$.

$V_2 \cdot V_3 = -2 + 0 - 1 = -3$

If they are orthogonal, the answer would be zero. Hence is it not W. So to convert $S \to W$ we use GS-process.

$W_1 = V_1 = (1, 0, 1)$.

$W_2 = V_2 - \text{Proj}_{W_1} V_2$

$= V_2 - \dfrac{(V_2 \cdot W_1)}{W_1 \cdot W_1} \cdot W_1$

$= (-1, 0, -1) - \dfrac{-2}{2}(1, 0, 1)$

$= (-1, 0, -1) + (1, 0, 1)$

$W_2 = (0, 0, 0)$

$$W_3 = V_3 - \underset{W_1}{\mathrm{Proj}}\, V_3 - \underset{W_2}{\mathrm{Proj}}\, V_3$$

$$W_3 = V_3 - \frac{V_3 \cdot W_1}{W_1 \cdot W_1} \cdot W_1 - \frac{V_3 \cdot W_2}{W_2 \cdot W_2} \cdot W_2 .$$

$$W_3 = (2,1,1) - \frac{3}{2}(1,0,1) - \frac{0}{\cancel{}}$$

$$W_3 = (2,1,1) + \left(\tfrac{3}{2}, 0, -\tfrac{3}{2}\right)$$

$$W_3 = \left(\tfrac{5}{2}, 1, -\tfrac{1}{2}\right).$$ correct

$$W_3 = \left(\tfrac{1}{2}, 1, -\tfrac{1}{2}\right).$$

Ignore the above

$$S = \underset{V_1}{\begin{pmatrix}1\\0\\1\end{pmatrix}}, \underset{V_2}{\begin{pmatrix}-2\\0\\-1\end{pmatrix}}, \underset{V_3}{\begin{pmatrix}1\\1\\1\end{pmatrix}}.$$

$$V_1 \cdot V_2 = -2+0-1 = -3$$

$$V_1 \cdot V_3 = 1+0+1 = 2 .$$

$$V_2 \cdot V_3 = -2+0-1 = -3$$

S is not W

$$S \to W$$

$$W_1 = V_1 = (1,0,1)$$

$$W_2 = V_2 - \underset{W_1}{\mathrm{Proj}}\, V_2 = (-2,0,-1) - \frac{(-3)}{2}(1,0,1)$$

$$= (-2,0,-1) + \left(\tfrac{3}{2}, 0, \tfrac{3}{2}\right)$$

$$W_2 = \left(-\tfrac{1}{2}, 0, \tfrac{1}{2}\right)$$

$$W_2 = \tfrac{1}{2}(-1,0,1)$$

-1+0+1

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$W_1 = (1, 0, 1)$

$W_2 = (-1, 0, 1)$ $\leftarrow$ not $\quad w_2 = (-1/2, 0, 1/2)$

$$W_3 = V_3 - \mathrm{Proj}_{W_1} V_3 - \mathrm{Proj}_{W_2} V_3$$

$$W_3 = (1,1,1) - \frac{2}{2}(1,0,1) - \frac{V_3 \cdot W_2}{W_2 \cdot W_2} \cdot W_2$$

$$W_3 = (1,1,1) - (1,0,1) - 0$$

$$W_3 = (0,1,0)$$

$W_1 = (1, 0, 1)$

$W_2 = (-1, 0, 1)$

$W_3 = (0, 1, 0)$

$$S \rightarrow W \rightarrow Q$$

To find Q

$|W_1| = \sqrt{2}$

$|W_2| = \sqrt{2}$

$|W_3| = 1$

$Q = \{q_1, q_2, \dots q_n\}$

$$Q_i = \frac{1}{||W_i||} \cdot W_i$$

$$Q_1 = \frac{1}{\sqrt{2}}(1,0,1) \quad Q_2 = \frac{1}{\sqrt{2}}(-1,0,1) \quad Q_3 = (0,1,0)$$

$Q = \{ (\sqrt{2}/2, 0\ \sqrt{2}/2), (-\sqrt{2}/2, 0, \sqrt{2}/2), (0, 1, 0) \}$.

$A = Q \cdot R$.

$$[v_1, v_2, v_3] = [q_1, q_2, q_3] \begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$$

$$\begin{bmatrix} v_1 q_1 & v_2 q_1 & v_3 q_1 \\ 0 & v_2 q_2 & v_3 q_2 \\ 0 & 0 & v_3 q_3 \end{bmatrix}$$

$v_1 q_1 = (1, 0, 1) \cdot (\sqrt{2}/2, 0, \sqrt{2}/2) = \sqrt{2}$.

$v_2 q_1 = (-2, 0, -1) \cdot (\sqrt{2}/2, 0, \sqrt{2}/2) =$

$v_2 \cdot q_3 = (-2, 0, -1) \cdot ($

$v_2 \cdot q_2 = (\quad)(\quad)$

$v_3 \cdot q_2 = (\quad)(\quad)$

$v_3 \cdot q_3 = (\quad)(\quad)$

EIGEN VALUES AND VECTORS.

i/ Values.

ii/ Vectors.

iii/ Application

Meaning. $A_{n\times n}$ $\underline{x}_{n\times 1}$

$$A\underline{x} = \lambda x$$

then λ = Eigen value

$\underline{x}$ = Eigen vector

$$A\underline{x} - \lambda\underline{x} = 0$$

$$\underline{x}(A - \lambda) = 0$$

(i) $(A_{n\times n} - \lambda I_n)\,\underline{x} = \underline{0}$ ... i

(ii) $(\lambda I_n - A)\,\underline{x} = \underline{0}$ ... (ii)

If (i) $|\lambda I_n - A|$. or

(ii) $|A - \lambda I_n| \neq 0$

Eigen vector "$\underline{x}$" Exists, then the eigen value exists

Qn: 1

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Determine x such that $A\underline{x} = \lambda x$.

Soln:

$A\underline{x} - \lambda x = 0$

$(A - \lambda I_n)\, \underline{X} = 0$.

$|A - \lambda I_n| = 0$ for λ.

$|A_{2\times 2} - \lambda I_{2\times 2}| = 0$.

$A - \lambda I_2$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}$$

$$A - I_2\lambda = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (3-\lambda) \end{vmatrix} = 0$$

$|A - \lambda I_2| = (1-\lambda)(3-\lambda) - 8 = 0$

$= 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

$= \lambda^2 - 4\lambda - 5 = 0$

$\lambda = 5$ or $\lambda = -1$

$\lambda_1 < \lambda_2 < \lambda \ldots \lambda_n$ $\qquad \lambda_1 = -1$

$\lambda_2 = 5$

$$[A - \lambda I_n] X = 0$$

For $\lambda = -1$ $\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} X = 0$

$R_1 - R_2$ $\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$2x_1 + 4x_2 = 0$

$2x_1 = -4x_2$

$x_1 = -2x_2$.

Let $t = x_2$.

$x_1 = -2t, \quad x_2 = t$

$x = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ for $\lambda = -1$

$x = (-2, 1), \lambda_1 = -1$

For $\lambda = 5$ $\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$

$R_1 + 2R_2$ $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$-4x_1 + 4x_2 = 0$

$-4x_1 = -4x_2$

$x_1 = x_2$

Let $x_1 = t$

$x_1 = t$

$x = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda_2 = 5$

$\lambda_2 = 5$

$x = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$x = (1, 1)$

Qn. 2. $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ Determine x, for which $Ax = \lambda x$.

Soln:

$Ax - \lambda x = 0$

$(A - \lambda I_3)\underline{x} = \underline{0}$.

$$A - \lambda I_3 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I_3 = \begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & -1 \\ -1 & -1 & -\lambda \end{bmatrix}.$$

$|A - \lambda I_3| = 0$.

To find the determinant through upper triangular matrix

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0.$$

To determine λ,

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \xrightarrow[\substack{(ii)\ -(1-\lambda)R_1 + R_2 \\ (iii)\ R_1 + R_2}]{(i)\ (-)\ R_1 \to R_2} \begin{bmatrix} 1 & 2-\lambda & 1 \\ 0 & 2-(\lambda-1)(\lambda-2) & (\lambda-2) \\ 0 & (1-\lambda) & (1-\lambda) \end{bmatrix}$$

$$-\begin{bmatrix} 1 & (2-\lambda) & 1 \\ 0 & 2-(1-\lambda)(2-\lambda) & (1-\lambda)-2 \\ 0 & (1-\lambda) & (1-\lambda) \end{bmatrix} \xrightarrow{\div(1-\lambda)} \begin{bmatrix} 1 & 2-\lambda & 1 \\ 0 & 2-(1-\lambda)(2-\lambda) & \lambda- \\ 0 & 1 & 1 \end{bmatrix}$$

$$(1-\lambda)\begin{bmatrix} 1 & 2-\lambda & \cancel{1} \quad 1 \\ 0 & 1 & \cancel{1} \quad 1 \\ 0 & 2-(1-\lambda)(2-\lambda) & \lambda-3 \end{bmatrix}$$

Let $p = (1-\lambda)(2-\lambda)$

$$(1-\lambda)\begin{bmatrix} 1 & 2-\lambda & 1 \\ 0 & 1 & 1 \\ 0 & 2-p & \lambda-3 \end{bmatrix} \xrightarrow{-(2-p)R_2+R_3}$$

$$\begin{bmatrix} 1 & 2-\lambda & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -(2-p)+(\lambda-3) \end{bmatrix}$$

3) $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

(b) show $\begin{vmatrix} (a+b) & a & a \\ a & (a+b) & a \\ a & a & (a+b) \end{vmatrix} = b^2(3a+b)$

(c) $A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$,

Determine λ such That $A \underset{n\times 1}{x} = \lambda \underset{n \times 1}{x}$..

$mR_i + R_j$

mR_i

$R_i \leftrightarrow R_j$.

$Det(A) = (-1)^n m \det(B)$.

PS: m is always multiplied to the reference row

$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

$|A - \lambda I_3| = 0$.

$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix}$ let $\alpha = 1-\lambda$.
$B = 2-\lambda$.

$$\begin{vmatrix} \alpha & 2 & -2 \\ 1 & B & 1 \\ -1 & -1 & \lambda \end{vmatrix}$$

$R_i \to R_j$ $$\begin{vmatrix} \alpha & 2 & -2 \\ 1 & B & 1 \\ -1 & -1 & \lambda \end{vmatrix}$$ de

$R_i \to R_j$ $$\begin{vmatrix} 1 & B & 1 \\ \alpha & 2 & -2 \\ -1 & -1 & \lambda \end{vmatrix}$$ $\det(A) = (-1)\det(B)$

$-\alpha R_1 + R_2$

$R_1 + R_3$ $$\begin{vmatrix} 1 & B & 1 \\ 0 & (2-\alpha B) & -(2+\alpha) \\ 0 & (B-1) & (1-\lambda) \end{vmatrix}$$

Let $t = 2 - \alpha B$

$w = -(2+\alpha)$

$r = B - 1$

$s = 1 - \lambda$

$$\begin{vmatrix} 1 & B & 1 \\ 0 & t & w \\ 0 & r & s \end{vmatrix}$$

$$-\frac{r}{t}R_2 + R_3 \quad \begin{vmatrix} 1 & k & 1 \\ 0 & t & w \\ 0 & 0 & s - rw/t \end{vmatrix}$$

$\det(A) = -\det B$

$\det(B) = 1 \times t \times s - \frac{rw}{t}$

$= t \times \frac{st - rw}{t}$

$\det(B) = st - rw$

$\det(B) = (1-\lambda)(2 - \alpha\beta) - (\beta - 1)(-(2+\alpha))$

$= (1-\lambda)(2-\alpha\beta) + (\beta - 1)(2+\alpha)$

$\alpha = 1 - \lambda, \quad \beta = 2 - \lambda$

$\alpha\beta = 2 - 3\lambda + \lambda^2$

$2 - \alpha\beta = 3\lambda - \lambda^2$

$\beta - 1 = 1 - \lambda$

$2 + \alpha = 3 - \lambda$

$\det(B) = [(1-\lambda)(3\lambda - \lambda^2)] + [(1-\lambda)(3-\lambda)]$

$= 1 - \lambda [(2\lambda - \lambda^2) + 3 - \lambda)]$

$= 1 - \lambda [3 + 2\lambda - \lambda^2]$

$= (1-\lambda)(3-\lambda)(\lambda+1)$

But $\det(A) = -\det(B)$

$\det(A) = -(1-\lambda)(3-\lambda)(\lambda+1)$

$|A - \lambda I_3| = 0$

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$$-(1-\lambda)(3-\lambda)(\lambda+1) = 0.$$

$$\lambda = 1, \quad \lambda = 3, \quad \lambda = -1$$

$$\lambda_1 = -1 \quad \lambda_2 = 1 \quad \lambda_3 = 3.$$

To Solve for x,

For $\lambda = -1$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & -1 \end{vmatrix}$$

For $\lambda = 1$

$$\begin{vmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix}$$

For $\lambda = 3$

$$\begin{vmatrix} -2 & 2 \\ 1 & -1 \\ -1 & -1 \end{vmatrix}$$

For each, find the upper triangular matrix.

For $\lambda = -1$

$$\left[\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right]$$

For $\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 2 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right]$$

For $\lambda = 3$

$$\left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & -1 & 3 & 0 \end{array}\right]$$

(b) $\begin{vmatrix} (a+b) & a & a \\ a & (a+b) & a \\ a & a & (a+b) \end{vmatrix} = b^2(3a+b)$

Let $\alpha = a+b$.

$$\begin{vmatrix} \alpha & a & a \\ a & \alpha & a \\ a & a & \alpha \end{vmatrix} \quad \begin{matrix} \\ -\frac{a}{\alpha}R_1 + R_2 \\ -\frac{a}{\alpha}R_1 + R_3 \end{matrix} \quad \begin{vmatrix} \alpha & a & a \\ 0 & \alpha - a^2/\alpha & a - a^2/\alpha \\ 0 & a - a^2/\alpha & \alpha - a^2/\alpha \end{vmatrix}$$

Let $t = \alpha - a^2/\alpha$.

$w = a - a^2/\alpha$.

$$\begin{vmatrix} \alpha & a & a \\ 0 & t & w \\ 0 & w & t \end{vmatrix}$$

$$-\frac{w}{t}R_2 + R_3 \quad \begin{vmatrix} \alpha & a & a \\ 0 & t & w \\ 0 & 0 & t - w^2/t \end{vmatrix}$$

$\det(A) = \det(B)$.

$= \alpha \times t \times \frac{t - w^2}{t}$

$= \alpha t \times \frac{t^2 - w^2}{t}$

$$= \alpha t^2 - \alpha w^2$$

$$= \alpha (t^2 - w^2)$$

$$= \alpha [(t+w)(t-w)]$$

$$= \alpha \left(\alpha - \frac{a^2}{\alpha} + a - \frac{a^2}{\alpha}\right)\left(\alpha - \frac{a^2}{\alpha} - a + \frac{a^2}{\alpha}\right)$$

$$= \alpha(\alpha - a)\left(\alpha + a - \frac{2a^2}{\alpha}\right)$$

$$= \alpha(\alpha - a)\left(\frac{\alpha^2 + a\alpha - 2a^2}{\alpha}\right)$$

$$= (\alpha - a)(\alpha^2 + \alpha a - 2a^2)$$

But $\alpha = a + b$

$$= (a+b-a)\left[(a+b)^2 + (a+b)a - 2a^2\right]$$

$$= b\left[(a+b)^2 + a^2 + ab - 2a^2\right]$$

$$= b\left[a^2 + 2ab + b^2 + a^2 + ab - 2a^2\right]$$

$$= b\left[3ab + b^2\right]$$

$$= b^2\left[3a + b\right]$$

$$\therefore \begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2[3a+b]$$

Hence shown

(1) ~~dos~~ no. of distribution
probability of survival
The no. of offsprings
in ~~future~~ future generation.

APPLICATION :

Model of POPULATION GROWTH :

1. Group the population into age classes of equal intervals, with the max. life span of a member being L-years, then the age intervals will be:

$$\left(0, \frac{L}{n}\right), \left(\frac{L}{n}, \frac{2L}{n}\right) \cdots \left(\frac{n-1}{n}L, L\right).$$

2. The number of the members in each age class Age distribution vector is.

$$\underline{x} = (x_1, x_2, \ldots x_n).$$

3. The probability of survival is $0 \le P_i \le 1$, $i = 1, 2 \ldots n$

4. The average no of off springs produced by a member in the i^{th} - Age class is b_i

Where

$b_i \geq 0$, $i = 1, 2, \ldots n$.

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Note:

i) $\underline{x} = (x_1, \dots x_n)$.

ii) $0 \leq P_i \leq 1, \quad i = 1, \dots, n-1$

iii) $b_i \geq 0, \quad i = 1 \dots n$.

$$
\begin{array}{cc}
 & \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{array} \\
A = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{array} & \begin{array}{cccccc} b_1 & b_i & b_3 & & b_{n-1} & b_n \\ P_1 & 0 & 0 & & 0 & 0 \\ 0 & P_2 & 0 & & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & P_{n-1} & 0 \end{array}
\end{array}
$$

1. Find distribution of x_{i+1}

$$\underline{x}_{i+1} = A x_i, \quad i = 1 \dots n-1$$

2. Find stable distribution

$$A\underline{x} = \lambda \underline{x} \qquad \lambda > 0$$