Linear Algebra with Applications

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Systems of Linear Equations

A linear equation in **n** unknowns is an equation of the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers and $x_1, x_2, \ldots, x_n$ are variables.

Contd.

- A linear system of **m** equations in **n** unknowns is then a system of the form:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

› is equivalent to a single matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

that is **Ax = b,** where $A = [a_{ij}]$ is the coefficient matrix of the system,

X= $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the vector of unknowns and b= $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is the vector of constants.

Contd.

- A **system of linear equations** (or a linear system) is a collection of one or more linear equations involving the same variables, $x_1,....,x_n$
- A **solution of the system** is a list of numbers, $s_1,...,s_n$, that makes each equation a true statement when the values $s_1,...,s_n$ are substituted for $x_1,...,x_n$ respectively.

Cont.

- **Solve a system** means "find all solutions to the system." The set of all possible solutions is called the **solution set of the linear system.**
- Two linear systems are called **equivalent** if they have the same solution set.

Contd.

- **Definition:** A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions. A system of linear equations is said to be **inconsistent** if it has no solution.

Contd.

- **Definition:** The process of applying the elementary row operations to transform the augmented matrix into row-echelon form is known as Gaussian Elimination.
- **Definition:** The process of applying the elementary row operations to transform the augmented matrix into reduced row-echelon form is known as Gauss-Jordan Elimination.

Contd.

- Definition: A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. **A pivot column** is a column of A that contains a pivot position.
- Definition: The **rank** of a matrix A, denoted rank A, is the number of pivot positions in any echelon matrix obtained from A by performing elementary row operations.

Contd.

- Fact: Whenever a **system is consistent**, the solution set can be described explicitly by solving the **reduced system of equations** for the basic variables in terms of the free variables. Each different choice of the free variable solution of the system, and every solution of the system is determined by a choice of the free variable.

Contd.

- **Definition:** **Solving a linear system** means to find a parametric description of the solution set or determine that the solution set is empty.
- **Fact:** Whenever **a system is consistent** and has free variables, the solution set has many parametric descriptions. When a **system is inconsistent**, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

Contd.

Theorem: Existence and Uniqueness Theorem

- A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—i.e., if and only if an echelon form of the augmented matrix has no row of the form [0 . . . 0 b] with b nonzero.

Contd.

Gaussian –Jordan Elimination Method.

- To solve a system of linear equations proceed as follows:

1. Carry the augmented matrix to a reduced echelon matrix using elementary row operations.

2. If a row of the form $[0 0 \cdots 0 | 1]$ occurs, the system is inconsistent.

Contd.

3. Otherwise assign the non-leading variables (if any) parameters and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters

Contd.

- **Fact:** If a linear system is consistent, then the solution set contains either

(i) a unique solution, when there are no free variables, or

(ii) infinitely many solutions, when there is at least one free variable.

Geometric interpretation

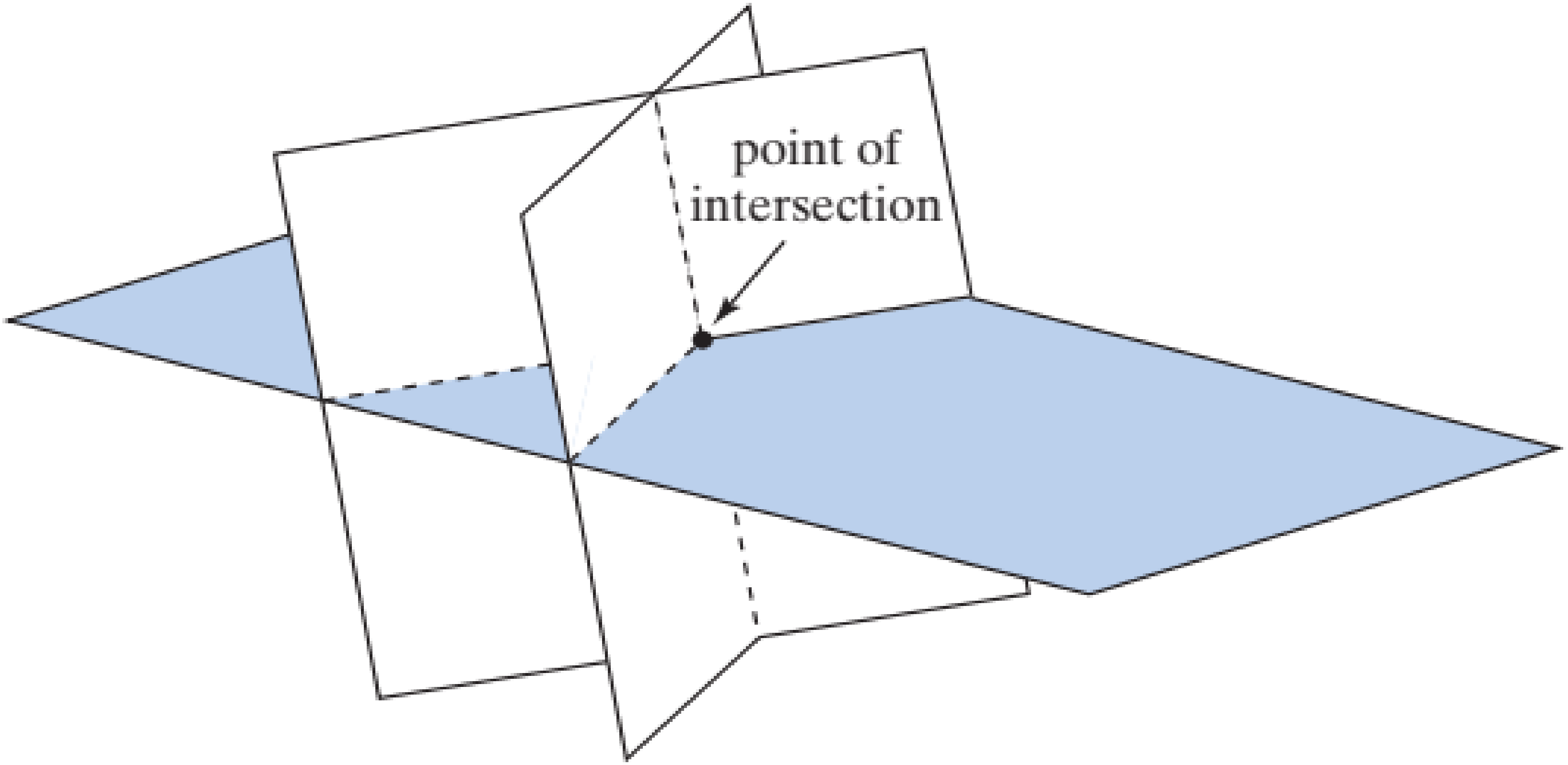

Figure 1 Three planes in space, intersecting at a point.

Contd.

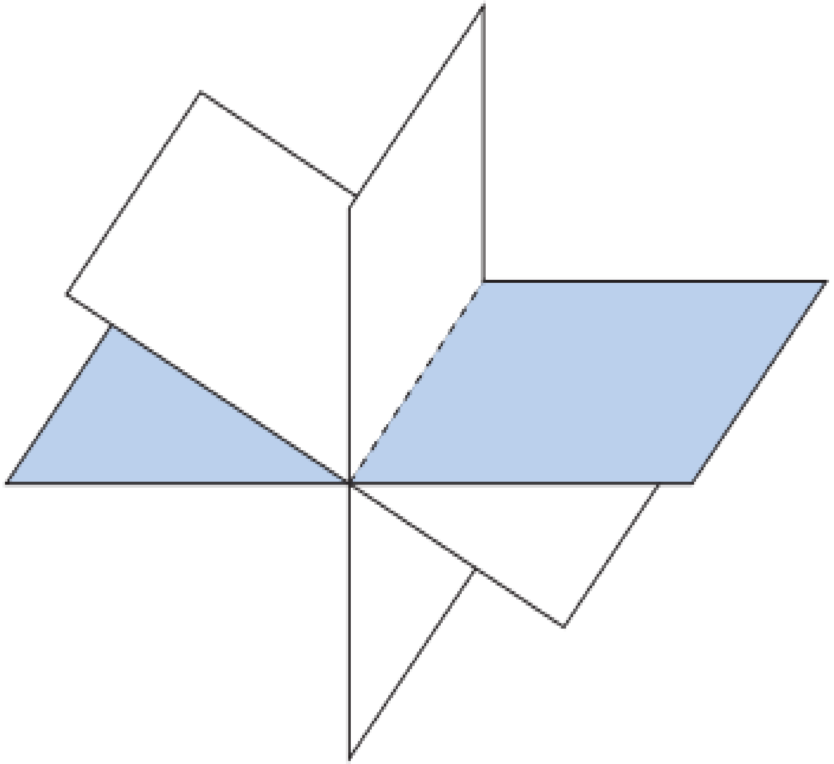

Figure 2(a) Three planes having a line in common.

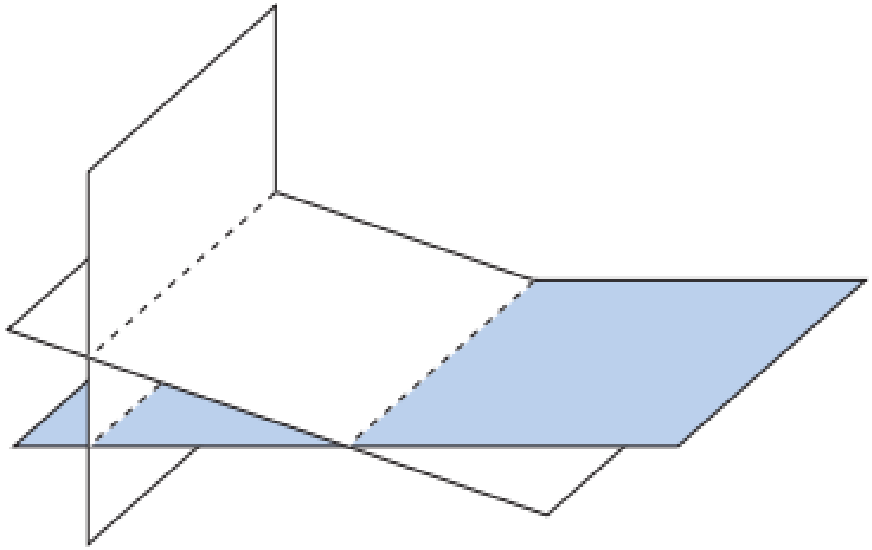

Figure 2(b) Three planes with no common intersection.

Contd.

- While three different planes in space usually intersect at a point, they may have a line in common (see Figure 2a) or may not have a common intersection at all, as shown in Figure 2b. Therefore, a system of three equations with three unknowns may have a unique solution, infinitely many solutions, or no solutions at all.

Solutions of Linear Systems

› **basic variable:**

any variable that corresponds to a pivot column in the augmented matrix of a system.

› **free variables:**

all non-basic variables.

Contd.

› Example 4

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \qquad \begin{aligned} x_1 + 6x_2 \qquad\quad + 3x_4 \qquad &= 0 \\ x_3 - 8x_4 \qquad &= 5 \\ x_5 &= 7 \end{aligned}$$

pivot columns: 1, 3, 5 basic variables: $x_1,\ x_3,\ x_5$

free variables: x_2 and x_4

Contd.

Final Step in Solving a Consistent Linear System

- After the augmented matrix is in reduced echelon form and the system is written down as a set of equations:
- Solve each equation for the basic variables in terms of the free variables (if any) in the equation.

Contd.

$$\begin{aligned} x_1 + 6x_2 \qquad\quad + 3x_4 \qquad\quad &= 0 \\ x_3 - 8x_4 \qquad\quad &= 5 \\ x_5 &= 7 \end{aligned} \qquad \begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

Contd.

- The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.)

Contd.

Definition: A homogeneous linear equation is one whose constant term is equal to zero. A system of linear equations is called homogeneous if each equation in the system is homogeneous. A homogeneous system has the form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

Note:

- $x_1=0,\ x_2=0,\cdots,x_n=0$ is always a solution to a homogeneous system of equations. We call this the trivial solution.
- The zero solution is usually called the trivial solution.
- Theorem: If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Contd.

- Theorem: A system of homogeneous equations has a nontrivial solution if and only if the equation has at least one free variable.

Inverse of a Matrix

- An $n \times n$ matrix A is invertible (or nonsingular) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

- where I_n is the identity matrix of order n. The matrix B is called the (multiplicative) inverse of A. A matrix that does not have an inverse is called noninvertible (or singular)

Contd.

Theorem 1. Let A and B be non singular n n matrices and let k be a nonzero scalar.

Then

- 1. $(AB)^{-1} = B^{-1}A^{-1}$.
- 2. $(A^{-1})^{-1} = A$.
- 3. $(kA)^{-1} = \frac{1}{k}A^{-1}$
- 4. $(A^2)^{-1} = (A^{-1})^2$ and in general, $(A^m)^{-1} = (A^{-1})^m$, for nonnegative integer m.

Step for Finding the Inverse of a Matrix by Gauss-Jordan Elimination

1) Write down the augmented matrix [A|I], i.e., the n × n matrix A with the n ×n identity matrix I at its side.
2) Perform row operations on [A|I] and reduce A to its reduced row echelon form.
3) If A can be reduced to the identity matrix I, then A^{-1} is the resulting matrix to the right of the vertical bar.
4) If A cannot be reduced to I, i.e., if a row of zeros appears to the left of the vertical bar, then A is not invertible.

Contd.

Example: Calculate the inverse of the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Contd.

- Step 1 Augment matrix A by the identity matrix.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Contd.

- **Step 2** Perform the three elementary row operations.

(1) $R_3 + (-1)R_1 \rightarrow R_3$

Multiply Row 1 by −1, add it to Row 3, and substitute the result for Row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right]$$

Contd.

- (2) $-\frac{1}{2}R_3 \rightarrow R_3$

Multiply Row 3 by $-\frac{1}{2}$ and substitute the result for Row 3.

$$\left[\begin{array}{ccc|ccr} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

Contd.

› (3) $R_1 + (-3)R_3 \rightarrow R_1$

Multiply Row 3 by –3, add it to Row 1, and substitute the result for Row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

Contd.

› (4) $R_2 + (-2)R_1 \rightarrow R_2$

Multiply Row 1 by −2, add it to Row 2, and substitute the result for Row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \\ 0 & -3 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

Contd.

- (5) $-\frac{1}{3}R_2 \rightarrow R_2$
- Multiply Row 2 by $-\frac{1}{3}$, and substitute the result for Row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

Contd.

- (6) $R_1 + (-2)R_2 \to R_1$
- Multiply Row 2 by -2, add it to Row 1, and substitute the result for Row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

Contd.

› The above matrix is equal to $[A^{-1}|I]$ and hence the inverse matrix of A is the right part of the matrix. That is,

$$A^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Using Inverse Methods to Solve Systems of Equations

- If the number of equations in a system equals the number of variables and the coefficient matrix has an inverse, then the system will always have a unique solution that can be found by using the inverse of the coefficient matrix to solve the corresponding matrix equation.

Matrix equation	Solution
$AX = B$	$X = A^{-1}B$

Contd.

- There are two cases where inverse methods will not work:

Case 1. The coefficient matrix is singular.

Case 2. The number of variables is not the same as the number of equations.

- In either case, use Gauss–Jordan elimination.