Linear Algebra with Applications

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Matrices

- A **matrix** is a rectangular arrangement of numbers (or, in some cases, symbols) whose position has significance. Typically, square brackets are used to denote a matrix.
- The standard mathematical convention to represent matrices in any one of the following three ways.

1. A matrix can be denoted by an uppercase letter such as A, B, C,……
2. A matrix can be denoted by a representative element enclosed in brackets, such as

   $[a_{ij}], [b_{ij}], [c_{ij}],$……
3. A matrix can be denoted by a rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Matrix Operations

- A general m × n matrix A may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Where the element in the i^{th} row and j^{th} column is denoted a_{ij}.

- **NB:** Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are equal if they are the same order (m × n)and $a_{ij} = b_{ij}$ for for and $1 \le i \le m$ and $1 \le j \le n$

Contd.

- **Addition and Subtraction**
- Two matrices may be added or subtracted if they are of the same order (have the same size/shape). Addition or subtraction is done element-wise. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 7 & 9 & 0 \\ 3 & 1 & 4 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 5 \\ 6 & 9 \end{bmatrix}$$

- Then

$$A + B = \begin{bmatrix} 8 & 11 & 3 \\ 7 & 6 & 10 \end{bmatrix}$$

Contd.

- And

$$A - B = \begin{bmatrix} -6 & -7 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$

- but A + C and A – C do not exist.

Contd.

- **Scalar Multiplication**
- Given a matrix A, we calculate kA by multiplying every element of A by the scalar k. For example,

if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Then

$$7A = \begin{bmatrix} 7 & 14 & 21 \\ 28 & 35 & 42 \end{bmatrix}$$

Contd.

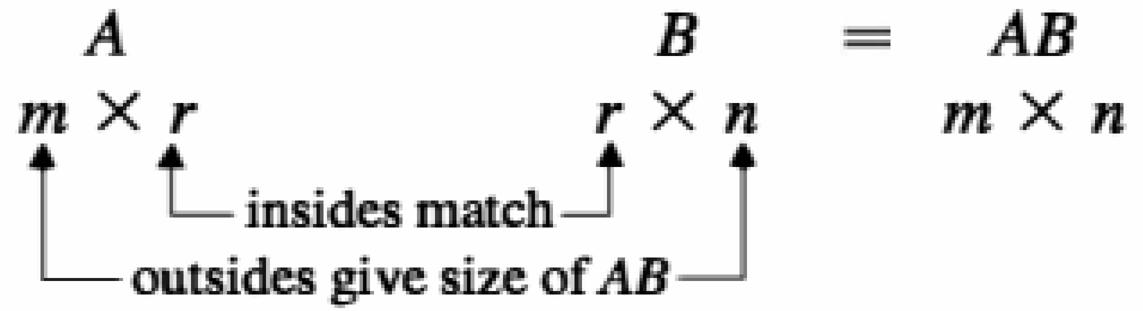
- **Multiplication of two Matrices**
- Not all matrices may be multiplied! Two matrices may be multiplied together only if they are compatible, that is, the number of columns of the first matrix must equal the number of rows of the second matrix. That is, if A is a $m \times n$ matrix and B is a $r \times p$ matrix then we can calculate AB only if $n = r$.

Contd.

- Size of a Product Matrix
- Let us now discuss the size of a product matrix. Let A be an $m \times r$ matrix and B be an $r \times n$ matrix. A has r columns and B has r rows. AB thus exists. The first row of AB is obtained by multiplying the first row of A by each column of B in turn. Thus the number of columns in AB is equal to the number of columns in B. The first column of AB results from multiplying each row of A in turn with the first column of B. Thus the number of rows in AB is equal to the number of rows in A. AB will be an $m \times n$ matrix.

Contd.

- If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.
- We can picture this result as follows:



Contd.

- The product $C = AB$ of an $m \times n$ matrix $A = [a_{ij}]$ and an $r \times p$ matrix $B = [b_{jk}]$ is defined only if and only if $r = n$ and then $C = [c_{ik}]$ is defined as the $m \times p$ with the entries

$$c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}$$

Contd.

- Example

$$\text{Given } A = \begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 3 & 8 \\ 9 & 2 \end{bmatrix}$$

- Find AB and BA.

Contd.

$$AB = \begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 8 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 45 & 33 \\ 51 & 32 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 3 & 8 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 5 \\ 51 & 25 & 52 \\ 21 & 31 & 46 \end{bmatrix}$$

Contd.

- The following properties hold for Matrix Multiplication

i. $AB \neq BA$ in general
ii. $AB = 0$ does not necessarily $A = 0$ or $B = 0$ or $BA = 0$.
iii. $kAB = k(AB) = A(kB)$
iv. $A(BC) = (AB)C$
v. $A(B + C) = AB + AC$
vi. $(A + B)C = AC + BC$

Contd.

- **Power of a Matrix**
- Integer powers of a matrix, e.g. A^2 , A^3 etc, are obtained by multiplying A by itself the required number of times. That is, $A^2 = AA$ and $A^3 = AAA = AA^2$ or $A^3 = A^2A$

Contd.

- **Transpose of a Matrix**
- The transpose of a matrix A, denoted A^T is the matrix obtained when the rows and columns of A are interchanged. For example, if $A = \begin{bmatrix} 6 & 3 & 5 \\ 4 & 7 & 2 \end{bmatrix}$
- Then $A^T = \begin{bmatrix} 6 & 4 \\ 3 & 7 \\ 5 & 2 \end{bmatrix}$
- Clearly $(A^T)^T = A$ and $(AB)^T = B^T A^T$

Contd.

Properties of Transpose

- Let A and B be matrices and c be a scalar. Assume that the sizes of the matrices are such that the operations can be performed.
  1. $(A + B)^T = A^T + B^T$ Transpose of a sum
  2. $(cA)^T = cA^T$ Transpose of a scalar multiple
  3. $(AB)^T = B^T A^T$ Transpose of a product
  4. $(A^T)^T = A$

Special Matrices

› **Zero Matrix** is a matrix of any size in which every element is zero.

$$O_{mn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Zero matrix O_{mn}

Contd.

- An **identity matrix** is a square matrix with zeros everywhere except along the **leading diagonal** where every element is a 1.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Identity matrix I_n

- For any matrix A there is an identity matrix I such that AI = A. There is also an identity matrix I such that IA = A. The identity matrix functions like the number 1 in real arithmetic.

Contd.

- A **Diagonal Matrix** is a square matrix with zeros everywhere except along the leading diagonal where there are some non-zero elements, e.g

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Diagonal matrix A

Contd.

- An **Upper Triangular Matrix** is a square matrix with zeros everywhere below the leading diagonal

$$\mathbf{U} = \begin{bmatrix} 6 & 4 & 2 & 1 \\ 0 & 6 & 4 & 2 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix},$$

Contd.

- A **Lower Triangular Matrix** is a square matrix with zeros everywhere above the leading diagonal

$$\mathbf{L} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 \\ -3 & 21 & 6 & 0 \\ -15 & -2 & 18 & 7 \end{bmatrix}.$$

- A **Symmetric Matrix A** is a square matrix with the property that $A^T = A$.
- The following are examples of symmetric matrices. Note the symmetry of these matrices about the main diagonal. All nondiagonal elements occur in pairs symmetrically located about the main diagonal.

match

$$\begin{bmatrix} 2 & 5 \\ 5 & -4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -4 \\ 1 & 7 & 8 \\ -4 & 8 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 7 & 3 & 9 \\ -2 & 3 & 2 & -3 \\ 4 & 9 & -3 & 6 \end{bmatrix}$$

match

Contd.

- A **Skew-Symmetrix Matrix** A is a square matrix with the property that $A^T = -A$.
- Also, for the matrix, $a_{ji} = -\ a_{ij}$(for all the values of i and j). The diagonal elements of a skew symmetric matrix are equal to zero. Some examples of skew symmetric matrices are:

$$P = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 2 & -7 \\ -2 & 0 & 3 \\ 7 & -3 & 0 \end{bmatrix}$$

Contd.

› **Echelon form (or row echelon form):**

1. All nonzero rows are above any row of zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero

› Example 1 Echelon form:

$$\text{(i)} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{(iii)} \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Contd.

- Reduced echelon form Add the conditions:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Contd.

- With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called Gauss-Jordan elimination after Carl Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a reduced row-echelon form is obtained.

The Row-Reduction Algorithm

- The row-reduced form of a matrix contains a great deal of information, both about the matrix itself and about systems of equations that may be associated with it.

Contd.

- **Notation for row operations.** Row reduction is made easier by having notation for various row operations. This is the notation that we will use here.

✓ Elementary transposition. $R \leftrightarrow R'$ means interchange rows R and R'.

Contd.

- **Elementary multiplication.** $R = cR$ means multiply the current row R by $c \neq 0$ and make the result the new row R.
- **Elementary modification.** $R = R + cR'$ means multiply R' by c, then add $cR0$ to R, and make the result the new row R.

Contd.

- *Row equivalence.* A matrix A is *row equivalent* to a matrix B if A can be transformed into B using a finite number of elementary row operations. Since such operations are reversible, B is also row-equivalent to A, and we simply say that A and B are row equivalent; we write A $\Leftrightarrow$B.

Contd.

- **Leading entry**. The *leading entry* in a row is the first non-zero entry in a row. The leading entries in each row of M are in boldface type.

$$M = \begin{pmatrix} 0 & \mathbf{1} & 3 & 2 \\ \mathbf{2} & 4 & 0 & -1 \\ 0 & 0 & \mathbf{6} & 5 \end{pmatrix}$$

Contd.

Theorem 1. Uniqueness Of The Reduced Echelon Form

- Each matrix is row-equivalent to one and only one reduced echelon matrix

pivot position

- a position of a leading entry in an echelon form of the matrix.

Contd.

pivot

- a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.

pivot column

- a column that contains a pivot position

Contd.

› Example 2 Find Row reduce to echelon form, and locate the pivot columns.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

$$\begin{array}{l} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Pivot column} \end{array}$$

Contd.

Pivot position

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

New pivot column

Possible pivots: 2, 5, –3

Contd.

Original matrix:

$$\begin{array}{c}\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \end{array}\end{array}$$

Pivot columns

Contd.

- **Example 3** Apply elementary row operations to transform the following matrix first into echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **Solution**
- **Step 1** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

- **Step 2** Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position
- Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\begin{array}{c} \begin{array}{l} \;\;\text{Pivot} \end{array} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

- **Step 3** Use row replacement operations to create zeros in all positions below the pivot.
- As a preliminary step, we could divide the top row by the pivot, 3. But with two 3's in column 1, it is just as easy to add -1 times row 1 to row 2.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

- **Step 4** Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.
- With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

- **For step 3**, we could insert an optional step of dividing the “top” row of the submatrix by the pivot, 2. Instead, we add $-\frac{3}{2}$ times the “top” row to the row below. This produces

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

Contd.

- Steps 1-3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

- **Step 5** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it by a scaling operation.
- The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

$$\begin{array}{l} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Row } 1 + (-6) \cdot \text{row } 3 \\ \leftarrow \text{Row } 2 + (-2) \cdot \text{row } 3 \\ \end{array}$$

Contd.

- The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \leftarrow \text{Row scaled by } \tfrac{1}{2}$$

Contd.

- Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row } 1 + (9) \cdot \text{row } 2$$

Contd.

› Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \leftarrow \text{Row scaled by } \tfrac{1}{3}$$