MT127 – Linear Algebra I
Lecture 1 – 2016/2017

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Matrices and properties

Matrices

An $m \times n$ **matrix** A is a rectangular array of mn real (or complex) numbers arranged in m horizontal **rows** and n vertical **columns**

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & \dots & a_{2n} \\ \vdots & \vdots & \dots & \dots & a_{ij} & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & \dots & a_{mn} \end{bmatrix} ; 1 \leq i \leq m, 1 \leq j \leq n$$

The ***i*th row** of A is $\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \quad (1 \leq i \leq m)$

The ***j*th column** of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad (1 \leq j \leq n)$

Matrices and properties cont.

Example (Motivation)

Road distances between the cities in km

	Arusha	**Dodoma**	**Mwanza**	**Lindi**
Arusha	0	450	650	1120
Dodoma	450	0	690	900
Mwanza	650	690	0	1590
Lindi	1120	900	1590	0

In matrix form

$$D = \begin{bmatrix} 0 & 450 & 650 & 1120 \\ 450 & 0 & 690 & 900 \\ 650 & 690 & 0 & 1590 \\ 1120 & 900 & 1590 & 0 \end{bmatrix}$$

What does d_{23} represent? what is its value, What is $d_{14} + d_{32}$?

Matrices and properties cont.

More concepts

- The entries from top left to bottom right form the main diagonal
- **Column matrix**; matrix with one column only
- **Row matrix**; matrix with one row only
- **Square matrix**: with $n = m$
- **Diagonal matrix**: Square matrix whose entries are all **zero** except those that form the main diagonal
  - Square matrix $[a_{ij}]$, where $a_{ij} = 0, \forall i \neq j$
  - Give an example
- **Scalar matrix**: diagonal matrix whose diagonal entries are equal
  - Scalar matrix $[a_{ij}]$, where $a_{ij} = k, \forall i = j \ , \quad k \in \Re$
  - Give an example of any order
- **Identity matrix** I: Scalar matrix whose main diagonal entries are 1
  - A scalar matrix $I = [a_{ij}]$, where $a_{ij} = 1, \forall i = j$
  - Give an example of Identity matrix of order 3, denoted as I_3

Matrices and properties cont.

True/False

- Identity matrix is a diagonal matrix
- Identity matrix is a square matrix
- Identity matrix is a scalar matrix

Zero matrix

A matrix A with all entries equal to zero; $a_{ij} = 0 \;, \quad \forall i \forall j$

Triangular matrix

- $A = [a_{ij}]$ is called **upper triangular** if $a_{ij} = 0 \;\; \forall i > j$
  $\Longrightarrow$ Elements below the main diagonal are zero
  - Give an example of upper triangular matrix
- $A = [a_{ij}]$ is called **lower triangular** if $a_{ij} = 0 \;\; \forall i < j$
  $\Longrightarrow$ Elements above the main diagonal are zero
  - Give an example of lower triangular matrix

Equal matrices

Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if and only if $a_{ij} = b_{ij}$, $1 \leq i \leq m, \quad 1 \leq j \leq n$

Transpose of a matrix

If $A = [a_{ij}]$ is an $m \times n$ matrix, then the $n \times m$ matrix $A^T = [a_{ij}^T]$, where $a_{ij}^T = a_{ji}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m$ is called the **transpose** of A

- Make your matrix and then compute its tranpose
- Compare the sizes of the two matrices

Symmetric matrix

Matrix $A = [a_{ij}]$ with real entries is called **symmetric** if $A = A^T$

- Give an example of a symmetric matrix
- It is Symmetric with respect to the main diagonal

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Matrix Operations

Matrix addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then the sum of A and B is the $m \times n$ matrix $C = [c_{ij}]$, defined by

$$c_{ij} = a_{ij} + b_{ij} \quad (1 \leq i \leq m, 1 \leq j \leq n)$$

Scalar multiplication

If $A = [a_{ij}]$ ia an $m \times n$ matrix and r is a real number, then the scalar multiple of A by r, rA, is the $m \times n$ matrix $B = [b_{ij}]$, where

$$b_{ij} = ra_{ij} \quad (1 \leq i \leq m, 1 \leq j \leq n)$$

Matrix subtraction

We can defind this using addition and scalar multiplication. If A and B are $m \times n$ matrices, we write $A + (-1)B$ as $A - B$ and call this the **difference** of A and B.

Example motivation

A manufacturer of a certain product makes three models, A, B and C. Each model is partially made in factory F_1 in Taiwan and then finished in factory F_2 in the United States. The total cost of each product consists of the manufacturing cost and the shipping cost are given in dollars.

$$\underbrace{\begin{array}{cc} \textbf{Man. cost} & \textbf{Ship. Cost} \\ 32 & 40 \\ 50 & 80 \\ 70 & 20 \end{array}}_{\textbf{Factory } F_1} \quad \underbrace{\begin{array}{cc} \textbf{Man. cost} & \textbf{Ship. Cost} \\ 40 & 60 \\ 50 & 50 \\ 130 & 20 \end{array}}_{\textbf{Factory } F_2} \quad \begin{array}{l} \\ \textbf{Model A} \\ \textbf{Model B} \\ \textbf{Model C} \end{array}$$

Write 3×2 matrices F_1 and F_2 which describe the costs at each factory. What is the total manufacturing and shipping costs of a model C?

Linear combination

If $A_1, A_2, A_3, \ldots, A_k$ are $m \times n$ matrices and $c_1, c_2, c_3, \ldots, c_k$ are real numbers, then the expression of the form

$$c_1 A_1 + c_2 A_2 + c_3 A_3 + \ldots + c_k A_k$$

is called a **linear combination** of $A_1, A_2, A_3, \ldots, A_k$, and $c_1, c_2, c_3, \ldots, c_k$ are called **coefficients**

Example

If $A = \begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix}$, then matrix $C = 3A - \frac{1}{2}B$ is a linear combination of A and B

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Dot Product

The **dot product** or **inner product** of the vectors $\underline{a} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}^T$ is defined as

$$\underline{a}.\underline{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = \sum_{i=i}^{n} a_i b_i$$

Example 1

The dot product of $\underline{u} = \begin{bmatrix} 1 & -2 & 3 & 4 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ is

$$\underline{u}.\underline{v} = 1 \times 2 + -2 \times 3 + 3 \times -2 + 4 \times 1 = -6$$

Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is an $p \times n$, then the product of A and B, denoted by AB, is the $m \times n$ matrix $C = [c_{ij}]$, defined by

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj} \\ &= \sum_{k=1}^{p} a_{ik}b_{kj} \qquad (1 \leq i \leq m, 1 \leq j \leq n) \end{aligned}$$

It says the i,j th element of AB is the dot product of the ith row, $row_i(A)$, and the jth column, $col_j(B)$ of B

Matrix Multiplication

$$
\begin{bmatrix}
a_{11} & a_{12} & \dots & a_{1p} \\
a_{21} & a_{22} & \dots & a_{2p} \\
\vdots & \vdots & & \vdots \\
\mathbf{a}_{i1} & \mathbf{a}_{i2} & \dots & \mathbf{a}_{ip} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mp}
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} & \dots & \mathbf{b}_{1j} & \dots & b_{1n} \\
b_{21} & b_{22} & \dots & \mathbf{b}_{2j} & \dots & b_{2n} \\
\vdots & \vdots & & \vdots & & \vdots \\
b_{p1} & b_{p2} & \dots & \mathbf{b}_{pj} & \dots & b_{pn}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
c_{11} & c_{12} & \dots & c_{1n} \\
c_{21} & c_{22} & \dots & c_{2n} \\
\vdots & \vdots & \mathbf{c}_{ij} & \vdots \\
c_{m1} & c_{m2} & \dots & c_{mn}
\end{bmatrix}
$$

$\Longrightarrow c_{ij} = \text{row}_i(A).\text{col}_j(B) = \sum_{k=1}^{p} a_{ik} b_{kj}$

Example 2

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}$. Find AB

Solution

$$\begin{aligned} AB &= \begin{bmatrix} \text{row}_1(A).\text{col}_1(B) & \text{row}_1(A).\text{col}_2(B) \\ \text{row}_2(A).\text{col}_1(B) & \text{row}_2(A).\text{col}_2(B) \end{bmatrix} \\ &= \begin{bmatrix} (1)(-2)+(2)(4)+(-1)(2) & (1)(5)+(2)(-3)+(-1)(1) \\ (3)(-2)+(1)(4)+(4)(2) & (3)(5)+(1)(-3)+(4)(1) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix} \end{aligned}$$

Example 3

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}$. Compute

1. the $(3, 2)$ entry of AB
2. the second column of AB.

Solution. Let $C = AB$, then

$$c_{32} = \text{row}_3(A).\text{col}_2(B) = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \mathbf{.} \begin{bmatrix} 4 & -1 & 2 \end{bmatrix}^T = -5$$

$$\text{col}_2(C) = A\text{col}_2(B) = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \\ -5 \end{bmatrix}$$

Example 4

Given any two non zero matrices whose product is a zero matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$$

Note: $AB = 0$ **does not** imply $A = 0$ or $B = 0$

Example 5

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$

Then $AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}$, but $B \neq C$

The matrix–vector product

Let the matrix A is $m \times n$ and c is an $n \times 1$ matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The matrix product Ac is $m \times 1$ matrix

$$Ac = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \text{row}_1(A).c \\ \text{row}_2(A).c \\ \vdots \\ \text{row}_m(A).c \end{bmatrix}$$

The matrix-vector product cont.

$$\begin{bmatrix} a_{11}c_1 + a_{12}c_2 + \ldots + a_{1n}c_n \\ a_{21}c_1 + a_{22}c_2 + \ldots + a_{2n}c_n \\ \vdots \\ a_{m1}c_1 + a_{m2}c_2 + \ldots + a_{mn}c_n \end{bmatrix} = c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= c_1 \mathrm{col}_1(A) + c_2 \mathrm{col}_2(A) + \ldots + c_n \mathrm{col}_n(A)$$

Linear combination!

Example 5

Repeat example 3 above to verify the results

$$\mathrm{col}_2(AB) = A\mathrm{col}_2(B) = 4 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \\ -5 \end{bmatrix}$$

The matrix-vector product cont.

Power of a matrix

Suppose that A is a square matrix. If p is a positive integer, then we define the powers of a matrix as follows

$$A^p = \underbrace{A.A.A\ldots A}_{\text{p factors}}$$

Power zero

If A is $n \times n$, we also define $A^0 = I_n$

Power rules

For nonnegative integers p and q

1. $A^p A^q = A^{p+q}$
2. $(A^p)^q = A^{pq}$
3. $(AB)^p \neq A^p B^p$ for square matrix in general

Matrix properties

Theorem (properties of matrix addition)

Let A, B, C and D be $m \times n$ matrices

1. $A + B = B + A$ (commutative property)
2. $A + (B + C) = (A + B) + C$ (associative property)
3. There is a unique $m \times n$ matrix $\mathbf{O}$ such that $A + \mathbf{O} = A$ for any $m \times n$ matrix A. The matrix $\mathbf{O}$ is called the $m \times n$ **additive identity** or **zero matrix**.
4. For each $m \times n$ matrix A, there is a unique matrix D such that

$$A + D = \mathbf{O}$$

we shall write D as $-A$, so that

$$A + (-A) = \mathbf{O}$$

The matrix $-A$ is called the **additive inverse** or negative of A.

Proof

We are going to prove part 1 and 3. The rest are left as exercise

Proof:

1) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, then

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}] = B + A$$

3) Let $A = [a_{ij}]$ and $U = [u_{ij}]$ be $m \times n$ matrices, then

$$A + U = A \Longleftrightarrow a_{ij} + u_{ij} = a_{ij}$$

which holds if and only if $u_{ij} = 0$. Thus U is the $m \times n$ matrix all of whose entries are zero; U is denoted by $\mathbf{O}$.

Matrix properties Cont.

Theorem (Matrix multiplication)

If A, B and C are of appropriate sizes, then

1. $A(BC) = (AB)C$ (associative)
2. $A(B + C) = AB + AC$ (distributive)
3. $(A + B)C = AC + BC$ (distributive)
4. If A is $m \times n$ matrix, then

$$I_m A = A I_n = A$$

Proof

Let us prove part 3. Let A and B be $m \times n$ matrices, and C be $n \times p$.
First observe that $(A+B)C$ and $AC+BC$ are both $m \times p$.
We only need to show equality of corresponding entries.
Let $Q = A + B$, where $Q = [q_{ij}]$,
$(A+B)C = QC$, the rsth entry of QC is given by

$$\sum_{k=1}^{n} q_{rk}c_{ks} = \sum_{k=1}^{n}(a_{rk}+b_{rk})c_{ks} = \sum_{k=1}^{n} a_{rk}c_{ks} + \sum_{k=1}^{n} b_{rk}c_{ks}$$
$$= AC + BC$$

Matrix properties Cont.

Theorem

If A and B are $m \times n$ matrices and C is an $n \times p$ matrix, then:

1. $(A + B)^T = A^T + B^T$
2. $(AC)^T = C^T A^T$
3. $(A^T)^T = A$

Proof

We prove part 2. First observe that both $(AC)^T$ and $C^T A^T$ are $p \times m$ matrices.
We then need to show equality of corresponding entries.
The ijth element of $(AC)^T$ is the jith element of AC, $(AC)^T$ is

$$\sum_{k=1}^{n} a_{jk} c_{ki}$$

Proof cont.

The ijth element of $C^T A^T$ is the dot product of

$$\begin{aligned}
&\mathrm{Row}_i(C^T)\mathrm{col}_j(A^T)\\
&= [c_{1i}, c_{2i}, \dots, c_{ni}].[a_{j1}, a_{j2}, \dots, a_{jn}]\\
&= c_{1i}a_{j1} + c_{2i}a_{21} + \dots + c_{ni}a_{jn}\\
&= \sum_{k=1}^{n} c_{ki}a_{jk}
\end{aligned}$$

Since $\sum_{k=1}^{n} a_{jk}c_{ki} = \sum_{k=1}^{n} c_{ki}a_{jk}$, it follows that

$$(AC)^T = C^T A^T$$

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Elementary Row operations

Elementary Row operations

An $m \times n$ matrix A is said to be in **reduced row echelon form (rref)** if it satisfies the following properties:

1. All zeros, if there are any, appear at the bottom of the matrix.
2. The first nonzero entry from the left of a nonzero row is a 1. This entry is called a **leading one** of its row.
3. For each nonzero row, the leading one appears to the right and below any leading one's in preceding rows.
4. If a column contains a leading one, then all other entry in that column are zero.

- A matrix in reduced row echelon form appears as a staircase ("echelon") pattern of leading ones descending from the upper left corner of the matrix.
- An $m \times n$ matrix satisfies 1, 2 and 3 is said to be in **row echelon form (ref)**.

Elementary Row operations

Example 1

The following matrices are in rref since they satisfy all conditions

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2

But the following are not in rref (why not?)

$$D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Elementary Row operations

Definition

An elementary row operations on an $m \times n$ matrix $A = [a_{ij}]$ is any of the following operations

1. Interchange two rows s and k of A; $R_s \longleftrightarrow R_k$
2. Multiply a row s by a constant $c \neq 0$; $cR_s \longrightarrow R_s$.
3. Add d times row s to row k of A; $dR_s + R_k \longrightarrow R_k$

Example 3

Let $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}$. Let us perform some row operations on A interchanging rows 1 and 3 of A

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix} \underset{\sim}{R_1 \longleftrightarrow R_3} \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Example 3 conti.

Multiplying the third row of A by $\frac{1}{3}$, we obtain

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix} \underset{\sim}{\frac{1}{3}R_1 \longrightarrow R_1} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

row 2 of A to row 3, we obtain

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix} \underset{\sim}{-2R_2 + R_3 \longrightarrow R_3} \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Elementary Row operations

Definition

An $m \times n$ matrix A is said to be **row equivalent** to an $m \times n$ matrix B if B can be obtained by applying a finite sequence of elementary row operations to A.

Facts

We can easily verify the following:

1. Every matrix is row equivalent to itself.
2. If A is row equivalent to B, then B is row equivalent to A; and
3. If A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

Theorem

Every $m \times n$ matrix is row equivalent to a matrix in row echelon form.

Example 4

Reduce the matrix $A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$

Identify a **pivotal column** and therefore pivots

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ \mathbf{2} & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \underset{\sim}{R_1 \longleftrightarrow R_3} \begin{bmatrix} \mathbf{2} & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

$$\underset{\sim}{\tfrac{1}{2}R_1 \longrightarrow R_1} \begin{bmatrix} \mathbf{1} & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \underset{\sim}{-2R_1 + R_4 \longrightarrow R_4} \begin{bmatrix} \mathbf{1} & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & \mathbf{2} & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

Elementary Row operations

Example 4 continue

$$R_2 \longleftrightarrow R_3 \underset{\sim}{} \begin{bmatrix} \mathbf{1} & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & \mathbf{2} & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \frac{1}{2}R_2 \longrightarrow R_2 \underset{\sim}{} \begin{bmatrix} \mathbf{1} & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & \mathbf{1} & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$2R_2 + R_3 \longrightarrow R_3 \underset{\sim}{} \begin{bmatrix} \mathbf{1} & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & \mathbf{1} & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & \mathbf{2} & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \frac{1}{2}R_3 \longrightarrow R_3 \underset{\sim}{} \begin{bmatrix} \mathbf{1} & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & \mathbf{1} & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & \mathbf{1} & \frac{3}{2} & 2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$-2R_3 + R_4 \longrightarrow R_4 \underset{\sim}{} \begin{bmatrix} \mathbf{1} & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & \mathbf{1} & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & \mathbf{1} & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Stop/continue?

Elementary Row operations

Theorem

Every $m \times n$ matrix $A = [a_{ij}]$ is row equivalent to a unique matrix in reduced row echelon form.

Example 5

Find the rref of the matrix A in example 4

$\longrightarrow$ Satisfy condition 4. Making zeros above the leading ones.

$$\begin{bmatrix} \mathbf{1} & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & \mathbf{1} & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & \mathbf{1} & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} -R_2 + R_1 \longrightarrow R_1 \\ \sim \\ -\frac{3}{2}R_3 + R_2 \longrightarrow R_2 \end{array} \begin{bmatrix} \mathbf{1} & 0 & -4 & 3 & \frac{3}{2} \\ 0 & \mathbf{1} & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & \mathbf{1} & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} -4R_3 + R_1 \longrightarrow R_1 \\ \sim \end{array} \begin{bmatrix} \mathbf{1} & 0 & 0 & 9 & \frac{19}{2} \\ 0 & \mathbf{1} & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & \mathbf{1} & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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System of Linear Equations

Theorem

Let $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ be two systems each of m equations in n unknowns. If the augmented matrix $\begin{bmatrix} A & | & b \end{bmatrix}$ and $\begin{bmatrix} C & | & d \end{bmatrix}$ of these systems are row equivalent, then both linear system have exactly the same solutions.

Corollary

If A and C are row equivalent $m \times n$ matrices, then the linear system $A\mathbf{x} = \mathbf{0}$ and $C\mathbf{x} = \mathbf{0}$ have exactly the same solutions.

Methods

We can find a row equivalent matrix by reducing to ref or rref

1. If $\begin{bmatrix} C & | & d \end{bmatrix}$ is in rref $\Longrightarrow$ **Gauss-Jordan reduction**
2. If $\begin{bmatrix} C & | & d \end{bmatrix}$ is in ref $\Longrightarrow$ **Gaussian elimination**

Method Algorithms

Gauss-Jordan reduction

Assume a linear system of the form $Ax = b$

1. Form the **augmented matrix** $\begin{bmatrix} A & \mid & b \end{bmatrix}$
2. Transform $\begin{bmatrix} A & \mid & b \end{bmatrix}$ to rref $\begin{bmatrix} C & \mid & d \end{bmatrix}$
3. For each non zero row of the matrix $\begin{bmatrix} C & \mid & d \end{bmatrix}$, solve the corresponding equation for the unknown associated with the leading one in that row.

Gaussian elimination

Assume a linear system of the form $Ax = b$

1. Form the **augmented matrix** $\begin{bmatrix} A & \mid & b \end{bmatrix}$
2. Transform $\begin{bmatrix} A & \mid & b \end{bmatrix}$ to some ref $\begin{bmatrix} C & \mid & d \end{bmatrix}$
3. Solve the linear system corresponding $\begin{bmatrix} C & \mid & d \end{bmatrix}$ by backward substitution

Example 1

Solve the linear system by Gauss-Jordan reduction

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8 \\ 3x - z &= 3 \end{aligned}$$

Solution: The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

Perform the row operations to rref, and you must obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example cont.

The linear system represented by the rref is

$$\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 3 \end{aligned}$$

So the **unique solution** to the given system is $x = 2, y = -1$ and $z = 3$

Example 2

Example: Solve the linear system

$$\begin{aligned} x + y + 2z - 5w &= 3 \\ 2x + 5y - z - 9w &= -3 \\ 2x + y - z + 3w &= -11 \\ x - 3y + 2z + 7w &= -5 \end{aligned}$$

by Gauss-Jordan reduction.

Solution

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & 1 & -1 & 3 & -11 \\ 1 & -3 & 2 & 7 & -5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & 1 & -1 & 3 & -11 \\ 1 & -3 & 2 & 7 & -5 \end{array}\right] \overset{\textbf{to rref}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The linear system represented by the rref is

$$\begin{cases} x + 2w & = -5 \\ y - 3w & = 2 \\ z - 2w & = 3 \end{cases} \Longrightarrow \begin{cases} x & = -5 - 2w \\ y & = 2 + 3w \\ z & = 3 + 2w \end{cases}$$

Example cont.

If we let $w = t$ **(the free variable)**, any real number, then a solution to the system is

$$\begin{cases} x &= -5 - 2t \\ y &= 2 + 3t \\ z &= 3 + 2t \\ w &= t \end{cases}$$

Example 3

Solve the linear system given in example 1 by Gaussian elimination.

Solution: We transform our augmented matrix to ref

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example cont.

This ref corresponds to the equivalent linear system

$$\begin{cases} x + 2y + 3z & = 9 \\ y + z & = 2 \\ z & = 3 \end{cases}$$

The process of back substitution starts with $z = 3$
we substitute in $y + z = 2 \Longrightarrow y = 2 - 3 = -1$ and
finally substitute these values in

$$x + 2y + 3z = 9 \Longrightarrow x = 9 - 2(-1) - 3(3) = 2$$

The unique solution is therefore $x = 2, y = -1$ and $z = 3$.

Example 4

Solve the linear system by Gauss-Jordan reduction

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ x_1 + 3x_2 + 5x_3 + 7x_4 &= 11 \\ x_1 - x_3 - 2x_4 &= -6 \end{aligned}$$

Solution: The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{array}\right]$$

You must be able to verify that

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Example conti.

The third row suggests $0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$, which has no solution for any x_i, consequently the given linear system has no solution.

Consistency

The linear system with at least one solution are called **consistent**, and linear systems with no solutions are called **inconsistent**

Homogeneous Equations

A linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned} \tag{1}$$

is called a **homogeneous equation**.

Hom eqns cont.

In matrix form we write it as $A\mathbf{x} = \mathbf{0}$

- Solution $x_1 = x_2 = x_3 = \ldots = x_n = 0$ of (1) is called the **trivial solution**
- Solution $x_1, x_2, x_3, \ldots, x_n$ in which not all the x_i are zero is called a **nontrivial solution**

Example 5

Consider the homogeneous equation

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = 0 \\ -x_1 + 3x_2 + 2x_3 & = 0 \\ 2x_1 + x_2 - 2x_3 & = 0 \end{cases}$$

Example 5 cont.

The Augmented matrix for this system is reduces as

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence the solution is

$$x_1 = x_2 = x_3 = 0$$

which means the system has only the trivial solution

Example 6

Consider the homogeneous equation

$$\begin{cases} x_1 + x_2 + x_3 + x_4 & = 0 \\ x_1 + x_4 & = 0 \\ x_1 + 2x_2 + x_3 & = 0 \end{cases}$$

Example 6 cont.

The Augmented matrix for this system reduces as

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Let $x_4 = t$ (the free variable), $t \in \Re$, then we have

$$\begin{aligned} x_3 + x_4 = 0 &\Longrightarrow x_3 = -t \\ x_2 - x_4 = 0 &\Longrightarrow x_2 = t \\ x_1 + x_4 = 0 &\Longrightarrow x_1 = -t \end{aligned}$$

The system has infinite many solutions $\mathbf{x} = (-t, t, -t, t)$, $t \in \Re$.

Note One solution for example, choose $t = 2 \Longrightarrow (-2, 2, -2, 2)$

Theorem

A homogeneous equation of m equations in n unknowns always has a nontrivial solution if $m < n$, that is, if the number of unknowns exceeds the number of equations.

Example 7

Recall the solution of example 2, the solution can be expressed as

$$x = \begin{bmatrix} -5-2t \\ 2+3t \\ 3+2t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} -5 \\ 2 \\ 3 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{\begin{bmatrix} -2t \\ 3t \\ 2t \\ t \end{bmatrix}}_{x_h}$$

then we can express $x = x_h + x_p$, where

- x_p is a particular solution to the given system and
- x_h is a solution corresponding to homogeneous system

We can easily verify that $Ax_p = b$ and $Ax_h = \mathbf{0}$

MT127 – Linear Algebra I
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LU Factorization/Decomposition

LU Decomposition

- A matrix A can be decomposed as a product of a lower triangular matrix and an upper triangular matrix; $A = LU$
- Can be used to solve linear system $Ax = b$ efficiently
- It provides cheapest way for solving linear system, easy to program!

Solution for $Ux = b$

Let matrix U is given by

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} & : & b_1 \\ 0 & u_{22} & u_{23} & \dots & u_{2n} & : & b_2 \\ 0 & 0 & u_{33} & \dots & u_{3n} & : & b_3 \\ \vdots & \vdots & \vdots & \dots & \vdots & : & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} & : & b_n \end{bmatrix}$$

Backward substitution Algorithm

The solution is obtained by the following algorithm

$$x_n = \frac{b_n}{u_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - u_{n-1}x_n}{u_{n-1\ n-1}}$$

$$\vdots$$

$$x_j = \frac{b_j - \sum_{k=n}^{j-1} u_{jk}x_k}{u_{jj}}, \quad j = n, n-1, \ldots, 2, 1$$

Forward substitution Algorithm

For lower triangular matrix L, the linear system is $Lx = b$, the augmented matrix is

$$L = \begin{bmatrix} \mathscr{L}_{11} & 0 & 0 & \dots & 0 & : & b_1 \\ \mathscr{L}_{21} & \mathscr{L}_{22} & 0 & \dots & 0 & : & b_2 \\ \mathscr{L}_{31} & \mathscr{L}_{31} & \mathscr{L}_{33} & \dots & 0 & : & b_3 \\ \vdots & \vdots & \vdots & \dots & \vdots & : & \vdots \\ \mathscr{L}_{n1} & \mathscr{L}_{n2} & \mathscr{L}_{n3} & \dots & \mathscr{L}_{nn} & : & b_n \end{bmatrix}$$

$$x_1 = \frac{b_1}{\mathscr{L}_{11}}$$

$$x_2 = \frac{b_2 - \mathscr{L}_{21}x_1}{\mathscr{L}_{22}}$$

$$\vdots$$

$$x_j = \frac{b_j - \sum_{k=1}^{j-1} \mathscr{L}_{jk}x_k}{\mathscr{L}_{jj}}, \quad j = 2, \dots, n$$

Procedure for using LU decomposition

Algorithm for solving linear system

Given a system of linear equations $Ax = b$, if we can decompose matrix A into LU, one can write the system as

$$(LU)x = b$$

Use associative property over matrix multiplication, one can write

$$L(Ux) = b$$

Create subproblem, Let $z = Ux$,

1. First solve for z using $Lz = b$ (forward substitution)
2. Then solve for x from $Ux = z$ (backward substitution)

Example

Use LU decomposition to solve the linear system

$$\begin{aligned} 6x_1 - 2x_2 - 4x_3 + 4x_4 &= 2 \\ 3x_1 - 3x_2 - 6x_3 + x_4 &= -4 \\ -12x_1 + 8x_2 + 21x_3 - 8x_4 &= 8 \\ -6x_1 - 10x_3 + 7x_4 &= -43 \end{aligned}$$

Solution

Let the coefficient matrix A and vector b be respectively

$$A = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ -4 \\ 8 \\ -43 \end{bmatrix}$$

Then the system of equations is $Ax = b$

Decomposition procedure step 1

Let $A = LU$, then we have to decompose the matrix A

- Build matrix U_1 by zero out below the first diagonal entry (6) of A $-\frac{1}{2}R_1 + R_2 \to R_2$, $2R_1 + R_3 \to R_3$ and $R_1 + R_4 \to R_4$
- At the same time build L_1 with 1's in the main diagonal and negative multipliers used in row operations in the first column of L_1

$$A \sim U_1 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & * & 1 & 0 \\ -1 & * & * & 1 \end{bmatrix}$$

Decomposition procedure step 2

- Zero out below the second diagonal entry of U_1 by the following operations $2R_2 + R_3 \rightarrow R_3$, $-R_2 + R_4 \rightarrow R_4$
- Keep the negative multipliers below the second diagonal entry of L_1

$$U_1 \sim U_2 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & * & 1 \end{bmatrix}$$

Decomposition procedure step 3

- Zero out below the third diagonal entry of U_2; $2R_3 + R_4 \rightarrow R_4$
- Keep the negative multipliers below the third diagonal entry of L_2

$$U_2 \sim U_3 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

Example cont., forward substitution

Now we have $(LU)x = b$, let $z = Ux$, solve $Lz = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \\ -43 \end{bmatrix}$$

By forward substitution we obtain

$$\implies \begin{cases} z_1 = 2 \\ z_2 = -4 - \frac{1}{2}z_1 = -5 \\ z_3 = 8 + 2z_1 + 2z_2 = 2 \\ z_4 = -43 + z_1 - z_2 + 2z_3 = -32 \end{cases}$$

The solution of the subproblem is therefore $z = \begin{bmatrix} 2 & -5 & 2 & -32 \end{bmatrix}^T$

Example cont., Back substitution

Now we solve $Ux = z$

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 2 \\ -32 \end{bmatrix}$$

Using backward substitution

$$\Longrightarrow \begin{cases} x_4 = \frac{-32}{8} = -4 \\ x_3 = \frac{2+2x_4}{5} = -\frac{6}{5} \\ x_2 = \frac{-5+4x_3+x_4}{-2} = \frac{69}{10} \\ x_1 = \frac{2+2x_2+4x_3-4x_4}{6} = \frac{9}{2} \end{cases}$$

The solution is therefore $x = \begin{bmatrix} -4 & -\frac{6}{5} & \frac{69}{10} & \frac{9}{2} \end{bmatrix}^T$

MT127 – Linear Algebra I
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The inverse of a matrix

The inverse

An $n \times n$ matrix A is called **nonsingular** (or **invertible**) if there exist an $n \times n$ matrix B such that

$$AB = BA = I_n$$

The matrix B is called the **inverse** of A. If there exist no such matrix B, then A is called **singular** (or *nonivertible*).

Example 1

Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$. Since $AB = BA = I_2$, we conclude that B is the inverse of A and that A is nonsingular.

Theorem

If a matrix has an inverse, then the inverse is unique

Proof

Let B and C are two inverses of A, then $BA = AC = I_n$, we have

$$B = BI_n = B(AC) = (BA)C = I_nC = C$$

Notation

If the inverse of A exist we shall denote is by A^{-1}. Thus

$$AA^{-1} = A^{-1}A = I_n$$

Theorem (Properties of the inverse)

1. If A is a nonsingular matrix, then A^{-1} is nonsingular and $(A^{-1})^{-1} = A$
2. If A and B are nonsingular matrices, then AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is nonsingular matrix, then $(A^T)^{-1} = (A^{-1})^T$

Proof

We prove 2 and the rest will be an exercise.
We have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$$

Therefore, AB is nonsingular. Since the inverse is unique, we conclude that

$$(AB)^{-1} = B^{-1}A^{-1}$$

Theorem

Suppose that A and B are $n \times n$ matrices, then $AB = I_n$ if and only if $BA = I_n$. **(Prove!)**

Practical method for finding an inverse

The practical procedure for computing the inverse $B = [b_{ij}]$ of $n \times n$ matrix $A = [a_{ij}]$ is obtained by solving a linear system with unknowns from the columns of matrix B

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

With **Gauss-Jordan reduction**, we can solve the system at once. Because the same coefficient matrix A applies for all columns of B. The right hand side is formed by the columns of the I_n

Algorithm

1. Form the $n \times 2n$ matrix $\begin{bmatrix} A & | & I_n \end{bmatrix}$ obtained by adjoining matrix I_n to matrix A.
2. Transform the resulted matrix to reduced row echelon form by using elementary row operations.
3. Suppose the resulted matrix in rref is $\begin{bmatrix} C & | & D \end{bmatrix}$
    1. If $C = I_n$, then $D = A^{-1}$.
    2. If $C \neq I_n$, then C has a row of zeros. In this case A is singular and has no inverse.

Example 2

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$.

Solution

The 3×6 $\begin{bmatrix} A & \mid & I_3 \end{bmatrix}$ is

$$\begin{bmatrix} A & \mid & I_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

transforming into rref, we have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] = \begin{bmatrix} I_3 & \mid & A^{-1} \end{bmatrix}$$

Example 2 cont.

$$\Longrightarrow A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}$$

Theorem

An $n \times n$ matrix is nonsingular if and only if it is row equivalent to I_n.

Linear systems and inverses

If A is an $n \times n$ matrix, the linear system $Ax = b$ is has n equations in n unknowns. Suppose A is nonsingular, then A^{-1} exists

$$\begin{aligned} A^{-1}(Ax) &= A^{-1}b \\ (\underbrace{A^{-1}A}_{I_n})x &= A^{-1}b \Longrightarrow x = A^{-1}b \end{aligned}$$

Thus if A is nonsingular, we have a unique solution.

Example 3

Find solution for linear system $Ax = b$, where A is the matrix in example 2 above and $b = \begin{bmatrix} 4 & 7 & 16 \end{bmatrix}^T$.

Solution

Using the inverse of A, we can define solution x as

$$x = A^{-1}b = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 16 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Example 4

Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$ if it exists.

Solution

No inverse. Why?

Row and column rank of a matrix

- The reduction of an $m \times n$ matrix A to ref/rref will produced a row of zeros whenever the row is a linear combination of some (or all) of the rows above
- If ref/rref has $r \leq m$ nonzero rows $\Longrightarrow$ r rows are linearly independent, and $m - r$ rows are linearly dependent on the first r rows.
- The number r is called the **row rank** of matrix A.
- If s of the n columns of an $m \times n$ matrix A are linearly independent, the number s is called the **column rank** of A.
- The rank of matrix A is denoted by rank(A)

Theorems

1. The row rank and column rank of a matrix A are equal.
2. Let A be any matrix, then $\text{rank}(A) = \text{rank}(A^T)$

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Permutation

Let $S = \{1, 2, 3, \ldots, n\}$ be the set of integers from 1 to n, arranged in ascending order. A rearrangement $j_1, j_2, j_3, \ldots, j_n$ of the element of S is called a **permutation** of S

Number of permutations

Let S_n be the set of all permutations in S, then there are $n!$ permutations is S.

Example

- S_1 consists of only 1 permutation of set $S = \{1\}$, namely 1
- S_2 consists of $2! = 2$ permutations of the set $S = \{1, 2\}$, namely 12 and 21
- S_3 consists of $3! = 6$ permutations of the set $S = \{1, 2, 3\}$, namely 123, 231, 312, 132, 213 and 321

Odd/even permutations

- A permutation $j_1, j_2, j_3, \dots, j_n$ of $S = \{1, 2, \dots, n\}$ is said to have inversion if a larger integer j_r precedes a smaller one j_s
- A permutation is called **even** or **odd** according to whether the total number of inversions in it is even or odd.

Example

Permutation 4132 of $S = \{1, 2, 3, 4\}$ has four inversions: $4 \longrightarrow 1$, $4 \longrightarrow 3$, $4 \longrightarrow 2$ and $3 \longrightarrow 2$. It is then an even permutation.

Note

If $n \geq 2$, S_n has $n!/2$ even permutations and an equal number of odd.

Examples

- In S_2, the permutation 21 is odd since it has one inversion and permutation 12 is even since it has no inversion
- In S_3 the even permutations are 123, 231 and 312, the odd permutations are 132, 213 and 321.

Determinant

Let $A = [a_{ij}]$ be $n \times n$ matrix. We define the determinant of A (written det(A) or $|A|$) by

$$\det(A) = |A| = \sum (\pm) a_{1j_1} a_{2j_2} a_{3j_3} \ldots a_{nj_n} \tag{1}$$

where the summation ranges over all permutations $j_1, j_2, j_3, \ldots, j_n$ of the set $S = \{1, 2, \ldots, n\}$.

- $+$ or $-$ according to whether the permutation is even or odd
- $\det(A)$ has $n!$ terms in the sum

Example

If $A = [a_{11}]$ is a 1×1 matrix, then S_1 has only one permutation in it; 1, which is even. Thus $\det(A) = a_{11}$

Example 2×2 matrix

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix, then to obtain det(A) we write down the terms

$$a_{1\_}a_{2\_} \qquad \text{and} \qquad a_{1\_}a_{2\_}$$

and fill in the blanks with all possible elements of S_2
The subscripts becomes 12 and 21. Since 12 is an even permutation, the term $a_{11}a_{22}$ has a $+$ sign. Since 21 is an odd permutation, the term $a_{12}a_{21}$ has a minus sign. Hence

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Example 3×3 matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3×3 matrix, then to obtain det(A) we write down the terms

$$a_{1-}a_{2-}a_{3-},\ a_{1-}a_{2-}a_{3-},\ a_{1-}a_{2-}a_{3-}$$
$$a_{1-}a_{2-}a_{3-},\ a_{1-}a_{2-}a_{3-},\ a_{1-}a_{2-}a_{3-}$$

Fill the elements of S_3 to fill the blanks, we obtain

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, verify that $\det(A) = 6$

Theorems: Properties of determinants

1. The determinant of a matrix and its transpose are equal, that is

$$\det(A) = \det(A^T)$$

2. If matrix B results from matrix A by interchanging two rows (columns) of A, then

$$\det(B) = -\det(A)$$

3. If two rows (columns) of A are equal, then

$$\det(A) = 0$$

4. If a row (column) of A consists of entirely of zeros, then

$$\det(A) = 0$$

5. If B is obtained from A by multiplying a row (column) of A by a real number k, then $\det(B) = k\det(A)$

Proof: Theorem 5

Suppose that the rth row of $A = [a_{ij}]$ is multiplied by k to obtain $B = [b_{ij}]$. Then $b_{ij} = a_{ij}$ if $i \neq r$ and $b_{rj} = ka_{rj}$. We obtain det(B) from definition

$$\begin{aligned}\det(B) &= \sum(\pm) b_{1j_1} b_{1j_1} \dots b_{rj_r} \dots b_{nj_n} \\ &= \sum(\pm) a_{1j_1} a_{1j_1} \dots (ka_{rj_r}) \dots a_{nj_n} \\ &= k \sum(\pm) a_{1j_1} a_{1j_1} \dots (a_{rj_r}) \dots a_{nj_n} = k\det(A)\end{aligned}$$

Example: $\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 2 \times 3 \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 6(4-1) = 18$

Theorem

If $B = [b_{ij}]$ is obtained from $A = [a_{ij}]$ by adding to each element of the rth row (column) of A a constant k times the corresponding element of the sth row (column) $r \neq s$, then $\det(B) = \det(A)$.

Example: $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}_{2r_2 + r_1 \longrightarrow r_1} = \begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}.$$

Theorem

If a matrix $A = [a_{ij}]$ is upper (lower) triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33}a_{44}\ldots a_{nn}$$

that is, the determinant of a triangular matrix is the product of the elements on the main diagonal.

Corollary: The determinant of a diagonal matrix is the product of the entries on its main diagonal.

Theorem

The determinant of a product of two matrices is the product of their determinants, that is

$$\det(AB) = \det(A)\det(B)$$

Example

$$\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{vmatrix}_{\frac{1}{2}r_3 \longrightarrow r_3} = 2\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{vmatrix}_{r_1 \longleftrightarrow r_3} = -2\begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{vmatrix}_{-3r_1+r_2 \longrightarrow r_2}$$

$$= -2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 4 & 3 & 2 \end{vmatrix}_{-4r_1+r_3 \longrightarrow r_3} = -2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{vmatrix}_{\frac{1}{4}r_2 \longrightarrow r_2}$$

$$= -2(4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & -5 & 10 \end{vmatrix}_{\frac{1}{5}r_3 \longrightarrow r_3} = -2(4)(5)\begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{vmatrix}_{-\frac{1}{2}r_2+r_3 \longrightarrow r_3}$$

Example conti

$$= -2(4)(5)\begin{vmatrix}1 & 2 & 3\\0 & -2 & -1\\0 & 0 & -\frac{3}{2}\end{vmatrix} = -2 \times 4 \times 5 \times 1 \times -2 \times -\frac{3}{2} = -\mathbf{120}$$

Remark

The method above is referred to as the computation via reduction to **triangular form**

MT127 – Linear Algebra I
Lecture 9 – 2016/2017

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Cofactor expansion and Applications

Let $A = [a_{ij}]$ be $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. The determinant $\det(M_{ij})$ is called the **minor** of a_{ij}. The **cofactor** A_{ij} of a_{ij} is defined as

$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$

where

$$(-1)^{i+j} = \begin{cases} -1 & \text{if } i+j \text{ is odd} \\ +1 & \text{if } i+j \text{ is even} \end{cases}$$

Example

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$, then

$$\det(M_{12}) = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 8 - 42 = -34$$

$$\implies A_{12} = (-1)^{1+2}\det(M_{12}) = - - 34 = 34$$

$$\det(M_{23}) = \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 3 + 7 = 10$$

$$\implies A_{23} = (-1)^{2+3}\det(M_{23}) = - - 10 = 10$$

$$\det(M_{31}) = \begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -6 - 10 = -16$$

$$\implies A_{31} = (-1)^{3+1}\det(M_{31}) = + - 16 = 16$$

Sign distribution

The sign distribution for cofactor elements form checkerboard pattern with $+$ in the $(1,1)$ position:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \\ \\ & n=3 & \end{matrix} \qquad \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \\ & n=4 & & \end{matrix}$$

Theorem

Let $A = [a_{ij}]$ be an $n \times n$ matrix, then for each $1 \leq i \leq n$

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} \qquad \text{(expansion along } i\text{th row)}$$

and for each $1 \leq j \leq n$

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} \qquad \text{(expansion along } j\text{th column)}$$

Case of 3×3 matrix

We consider the case $A = [a_{ij}]$ is 3×3 matrix, recall

$$\begin{aligned}
\det(A) &= a_{11}a_{22}a_{33} + a_{11}a_{22}a_{33} + a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} \\
&\quad - a_{11}a_{22}a_{33} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) \\
&\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \qquad \text{(expansion along } \mathit{1}\text{st row)}
\end{aligned}$$

Repeat the above proof with expansion along the 3rd row, 1st and 2nd column.

Example

Evaluate the determinant $\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$

Solution

The best option is to expand along the second column or the third row. Let us expand along the third row

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix} = (-1)^{3+1}(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + 0 + 0$$

$$+ (-1)^{3+4}(-3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

now, expanding along the first column

$$\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} = (-1)^{1+1}(2)\begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} + (-1)^{2+1}(2)\begin{vmatrix} -3 & 4 \\ -2 & 3 \end{vmatrix}$$

$$= 1(2)(9) + (-1)(2)(-1) = 20$$

Sol cont

expanding along the third row

$$\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix} = (-1)^{3+1}(2)\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} + (-1)^{3+3}(-2)\begin{vmatrix} 1 & 2 \\ -4 & 2 \end{vmatrix}$$

$$= 1(2)(8) + (1)(-2)(10) = -4$$

substituting these values, the value of the determinant is

$$(+1)(3)(20) + 0 + 0 + (-1)(-3)(-4) = 48$$

Example

Compute the determinant of the matrix in above example with row operations

Solution

$$
\begin{aligned}
\left|\begin{matrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{matrix}\right|_{c_4+c_1\longrightarrow c_4}
&= \left|\begin{matrix} 1 & 2 & -3 & 5 \\ -4 & 2 & 1 & -1 \\ 3 & 0 & 0 & 0 \\ 2 & 0 & -2 & 5 \end{matrix}\right| \\
&= (-1)^{1+3}(3)\left|\begin{matrix} 2 & -3 & 5 \\ 2 & 1 & -1 \\ 0 & -2 & 5 \end{matrix}\right|_{r_1-r_2\longrightarrow r_1} \\
&= (-1)^{4}(3)\left|\begin{matrix} 0 & -4 & 6 \\ 2 & 1 & -1 \\ 0 & -2 & 5 \end{matrix}\right| \\
&= (-1)^{4}(3)(-1)^{2+1}(2)\left|\begin{matrix} -4 & 6 \\ -2 & 5 \end{matrix}\right| \\
&= (+1)(3)(-1)(2)(-8) = 48
\end{aligned}
$$

The inverse of a Matrix

What is the value of

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \ldots + a_{in}A_{kn} \;, \qquad i \neq k$$

The inverse of a Matrix

Theorem: If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$\begin{aligned} a_{i1}A_{k1} + a_{i2}A_{k2} + \ldots + a_{in}A_{kn} &= 0 \;, \qquad \text{for } i \neq k \\ a_{1j}A_{1k} + a_{2j}A_{2k} + \ldots + a_{nj}A_{nk} &= 0 \;, \qquad \text{for } j \neq k \end{aligned}$$

Proof

We prove the first one: Consider a matrix B is obtained from A by replacing the kth row of A by its ith row. Thus B is a matrix having two identical rows. $\Longrightarrow \det(B) = 0$. The elements of the kth row are $a_{i1}, a_{i2}, \ldots, a_{in}$ and the corresponding cofactors are $A_{k1}, A_{k2}, \ldots, A_{kn}$.

$$\Longrightarrow 0 = \det(B) = a_{i1}A_{k1} + a_{i2}A_{k2} + \ldots + a_{in}A_{kn}$$

Ajdoint

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $n \times n$ matrix called the adjoint of A is the matrix whose **i,j**th element is the cofactor A_{ji} of a_{ji}. Thus

$$\text{ajd}A = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

Example

Compute adj A, if $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$

Solution

The cofactors of A are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18\ , \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6\ , \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10\ , \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1$$

$$A_{13} = -6, \quad A_{23} = -2, \quad A_{33} = 28$$

Then

$$\text{adj A} = [A_{ij}^T] = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

Theorem

If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$A(\text{ajd } A) = (\text{ajd } A)A = \det(A)I_n$$

Corollary If A is an $n \times n$ matrix and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)}(\text{adj } A) = \begin{bmatrix} \frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \cdots & \frac{A_{n1}}{\det(A)} \\ \frac{A_{12}}{\det(A)} & \frac{A_{22}}{\det(A)} & \cdots & \frac{A_{n2}}{\det(A)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A_{1n}}{\det(A)} & \frac{A_{2n}}{\det(A)} & \cdots & \frac{A_{nn}}{\det(A)} \end{bmatrix}$$

Example

Find the inverse of the matrix in above example.
The $\det(A) = -94$, then

$$A^{-1} = \frac{1}{\det(A)}(\text{adj } A) = \begin{bmatrix} \frac{18}{94} & \frac{6}{94} & \frac{10}{94} \\ -\frac{17}{94} & \frac{10}{94} & \frac{1}{94} \\ \frac{6}{94} & \frac{2}{94} & -\frac{28}{94} \end{bmatrix}$$

Theorem

Theorem: A matrix A is nonsingular if and only if $\det(A) \neq 0$.
(easy to prove)
Corollary: For an $n \times n$ matrix A, the homogeneous system $A\mathbf{x} = \mathbf{b}$ has a nontrivial solution if and only if $\det(A) \neq 0$.

Cramer's rule

Theorem (Cramer's rule)

Let

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= 0 \end{aligned}$$

be a linear system with coefficient matrix $A = [a_{ij}]$ st $A\mathbf{x} = \mathbf{b}$, where

$$b = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix}^T$$

If $\det(A) \neq 0$, then the system has the **unique solution**

$$x_1 = \frac{\det(A_1)}{\det(A)} \qquad x_2 = \frac{\det(A_2)}{\det(A)} \qquad \ldots, x_n = \frac{\det(A_n)}{\det(A)}$$

Proof

If $\det(A) \neq 0$, then A is nonsingular, hence

$$x = A^{-1}b = \begin{bmatrix} \frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \cdots & \frac{A_{n1}}{\det(A)} \\ \frac{A_{12}}{\det(A)} & \frac{A_{22}}{\det(A)} & \cdots & \frac{A_{n2}}{\det(A)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A_{1n}}{\det(A)} & \frac{A_{2n}}{\det(A)} & \cdots & \frac{A_{nn}}{\det(A)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Longrightarrow x_i = \frac{1}{\det(A)} \left[A_{1i}b_1 + A_{2i}b_2 + \ldots + A_{ni}b_n \right] \quad (1 \leq i \leq n)$$

Proof cont

now let

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1i-1} & b_1 & a_{1i+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2i-1} & b_2 & a_{2i+1} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{ni-1} & b_n & a_{ni+1} & \dots & a_{nn} \end{bmatrix}$$

If we evaluate $\det A_i$ by expanding along the **i**th column, we find that

$$\det(A_i) = A_{1i}b_1 + A_{2i}b_2 + \dots + A_{ni}b_n$$

$$\implies x_i = \frac{1}{\det(A)}\det(A_i) = \frac{\det(A_i)}{\det(A)}$$

Example

Solve the system by Cramer's rule

$$\begin{array}{rcrcrcr} -2x_1 & + & 3x_2 & - & x_3 & = & 1 \\ x_1 & + & 2x_2 & - & x_3 & = & 4 \\ -2x_1 & - & x_2 & + & x_3 & = & -3 \end{array}$$

Solution

Let the coefficient matrix is A, then

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2 \qquad |A_1| = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = -4$$

$$|A_2| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -6 \qquad |A_3| = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = -8$$

Solution cont

Using the Cramer's rule

$$x_i = \frac{|A_i|}{|A|}$$

$$\Longrightarrow x_1 = \frac{-4}{-2} = 2, \ x_2 = \frac{-6}{-2} = 3, \ x_3 = \frac{-8}{-2} = 4$$

Therefore the unique solution for the system is $x_1 = 2$, $x_2 = 3$ and $x_3 = 4$

MT127 – Linear Algebra I
Lecture 10 – 2016/2017

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Row and column rank of a matrix

- The reduction of an $m \times n$ matrix A to ref/rref will produced a row of zeros whenever the row is a linear combination of some (or all) of the rows above
- If ref/rref has $r \leq m$ nonzero rows $\Longrightarrow$ r rows are linearly independent, and $m - r$ rows are linearly dependent on the first r rows.
- The number r is called the **row rank** of matrix A.
- If s of the n columns of an $m \times n$ matrix A are linearly independent, the number s is called the **column rank** of A.
- The rank of matrix A is denoted by rank(A)

Theorems

1. The row rank and column rank of a matrix A are equal.
2. Let A be any matrix, then $\text{rank}(A) = \text{rank}(A^T)$

Proof

2) The columns of A are the rows of A^T, so the column rank of A is the row rank of A^T. Using the equality theorem, these two ranks are equal. i.e

$$\text{rank}(A) = \text{rank}(A^T)$$

Example 1

Let $A = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 & 0 \\ 2 & 1 & 7 & 0 & 10 & 1 \\ 1 & 0 & 3 & 2 & 6 & 4 \\ 1 & 0 & 3 & 0 & 4 & 0 \end{bmatrix}$

The row reduced echelon form of A is B ($A \sim B$)

$$B = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1 cont

Showing that the number of leading columns is 3, so the row rank of A is 3, and hence its **rank is 3**.

Basis

Three row vectors spanning a subspace of R^6, and so forming a **basis** for this subspace, these are $\underline{u}_1 = [1, 0, 3, 0, 4, 0]$, $\underline{u}_2 = [0, 1, 1, 0, 2, 1]$ and $\underline{u}_3 = [0, 0, 0, 1, 1, 2]$.

column rank

The row-reduced echelon form of A^T is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank and Singularity

Example cont

Number of leading column is 3, hence the row rank of $A^T = 3$
The three row vectors of A^T spanning a subspace of R^4, and so forming a basis for this subspace, these are

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Theorem

An $n \times n$ matrix A is nonsingular if and only if rank(A)$= n$

Proof

Suppose A is nonsingular, then $A \sim I_n \Longrightarrow \text{rank}(A) = n$.
Conversely, let $\text{rank}(A) = n$. Suppose $A \sim B$ in rref $\Longrightarrow \text{rank}(B) = n$,

Proof cont.

so the rows of B must be linearly independent. Hence B has no zero rows and since it is in rref, it must be I_n and so A is nonsingular.

Corollary 1

If A is an $n \times n$ matrix, then $\text{rank}(A) = n$ if and only if $\det(A) \neq 0$.

Corollary 2

Let A be an $n \times n$ matrix. The linear system $A\underline{x} = \underline{b}$ has unique solution for every $n \times 1$ matrix $\underline{b}$ if and only if $\text{rank}(A) = n$.

Corollary 3

The homogeneous system $A\underline{x} = \underline{0}$ of n linear equations in n unknowns has a nontrivial solution if and only if $\text{rank}(A) < n$.

Example

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$. If we transform A to rref matrix B, we find that $B = I_3$ (**verify!!**). Thus $\text{rank}(A) = 3$ and matrix A is nonsingular. Moreover the homogeneous system $A\underline{x} = \underline{0}$ has only the **trivial solution**.

Example

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -3 \\ 1 & 3 & 3 \end{bmatrix}$. If we transform A to rref matrix B, we find

$$B = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\text{rank}(A) = 2 < 3$, and A is singular. Moreover, $A\underline{x} = \underline{0}$ has a **nontrivial solution**.

Theorem

The linear system $A\underline{x} = \underline{b}$ has a solution if and only if $\text{rank}(A) = \text{rank}[A \mid \underline{b}]$; that is, if and only if the ranks of the coefficient and augmented matrices are equal.

Example

Consider the linear system

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Since $\text{rank}(A) = \text{rank}[A \mid \underline{b}] = 3$ **(verify!!)**, the linear system has a solution.

Example

Consider the linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & 4 \\ 2 & -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Since $\text{rank}(A) = 2 \neq \text{rank}[A \mid \underline{b}] = 3$ **(verify!!)**, the linear system has no solution.

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix. The real number λ is called an **eigenvalue** of A if there exists a **nonzero vector** $\underline{x}$ in $\Re^n$ such that

$$A\underline{x} = \lambda\underline{x} \tag{1}$$

A nonzero $\underline{x}$ satisfying (1) is an **eigenvector** of A associated with the eigenvalue λ.

Example

If $A = I_n$, then the only eigenvalue is $\lambda = 1$; every nonzero vector in $\Re^n$ is an eigenvector of A associated with the eigenvalue $\lambda = 1$.

$$I_n\underline{x} = 1\underline{x}$$

Example

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$, then

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Longrightarrow \underset{\sim}{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda_1 = \frac{1}{2}$,

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\Longrightarrow \underset{\sim}{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda_2 = -\frac{1}{2}$

Computing Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Let the eigenvector $\underline{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ is associated with the eigenvalue λ, then $\underline{x}$ must satisfy

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Longrightarrow \begin{array}{rcrcl} x_1 & + & x_2 & = & \lambda x_1 \\ -2x_1 & + & 4x_2 & = & \lambda x_2 \end{array}$$

$$\Longrightarrow \begin{cases} (\lambda - 1)x_1 - x_2 & = 0 \\ 2x_1 + (\lambda - 4)x_2 & = 0 \end{cases}$$

This is homogeneous equation, which has nontrivial solution iff the determinant of its coefficient is zero;

$$\begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = 0 \Longrightarrow (\lambda - 1)(\lambda - 4) + 2 = 0 \Longrightarrow \lambda_1 = 2, \lambda_2 = 3$$

To find the eigenvectors of A, we solve the linear system $A\underline{x} = \lambda\underline{x}$ when $\lambda = \lambda_1 = 2$, we have

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This gives

$$\begin{cases} x_1 + x_2 & = 2x_1 \\ -2x_1 + 4x_2 & = 2x_2 \end{cases} \Longrightarrow \begin{cases} x_1 - x_2 & = 0 \\ 2x_1 - 2x_2 & = 0 \end{cases} \Longrightarrow x_1 = x_2$$

Let $x_2 = r \in \Re$, then all eigenvectors associated with $\lambda_1 = 2$ are given by $\begin{bmatrix} r \\ r \end{bmatrix}$. In particular, choose $r = 1$;

$$\underline{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

when $\lambda = \lambda_2 = 3$, we have

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This gives

$$\begin{cases} x_1 + x_2 & = 3x_1 \\ -2x_1 + 4x_2 & = 3x_2 \end{cases} \Longrightarrow \begin{cases} 2x_1 - x_2 & = 0 \\ 2x_1 - x_2 & = 0 \end{cases} \Longrightarrow x_1 = \frac{1}{2}x_2$$

Let $x_2 = t \in \Re$, then all eigenvectors associated with $\lambda_2 = 3$ are given by $\begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix}$. In particular, choose $t = 2$;

$$\underline{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Practical method for finding λ and $\underline{x}$

We need to solve for $\underline{x}$ in for linear system $A\underline{x} = \lambda\underline{x}$, or

$$(\lambda I_n - A)\underline{x} = \underline{0}$$

This is a homogeneous system which has nontrivial solution iff

$$|\lambda I_n - A| = 0$$

Definition

Let A be an $n \times n$ matrix. The determinant $f(\lambda) = |\lambda I_n - A|$ is called the **characteristic polynomial** of A. The **characteristic equation** of A is

$$f(\lambda) = |\lambda I_n - A| = 0$$

Example

Find the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

Solution

The characteristic equation is

$$f(\lambda) = |\lambda I_n - A| = \begin{vmatrix} \lambda - 1 & -2 & 1 \\ -1 & \lambda - 0 & -1 \\ -4 & 4 & \lambda - 5 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Solution cont.

$$\Longrightarrow (\lambda-1)(\lambda-2)(\lambda-3)=0 \Longrightarrow \lambda_1=1, \lambda_2=2, \lambda_3=3$$

To find eigenvector we form a hom. system

$$(\lambda I_n - A)\mathbf{x} = \mathbf{0} \Longrightarrow \begin{bmatrix} \lambda-1 & -2 & 1 \\ -1 & \lambda & -1 \\ -4 & 4 & \lambda-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

when $\lambda = \lambda_1 = 1$, the augmented matrix with row equivalences

$$\left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ -4 & 4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -4 & 4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have x_3 as free variable, let $x_3 = t \in \Re$, then
$-2x_2 + x_3 = 0 \Longrightarrow x_2 = \frac{1}{2}t$ and
$x_1 - x_2 + x_3 = 0 \Longrightarrow x_1 = x_2 - x_3 = \frac{1}{2}t - t = -\frac{1}{2}t$

Solution cont.

The eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} -\frac{1}{2}t & \frac{1}{2}t & t \end{bmatrix}^T$. In particular, choose $t = 2$, we have

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

when $\lambda = \lambda_2 = 2$, the augmented matrix with row equivalences

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ -4 & 4 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have x_3 as free variable, let $x_3 = t \in \Re$, then
$-4x_2 + x_3 = 0 \Longrightarrow x_2 = \frac{1}{4}t$ and

$$x_1 - 2x_2 + x_3 = 0 \Longrightarrow x_1 = 2x_2 - x_3 = \frac{1}{2}t - t = -\frac{1}{2}t$$

Solution cont.

The eigenvectors are $\mathbf{x}_2 = \begin{bmatrix} -\frac{1}{2}t & \frac{1}{4}t & t \end{bmatrix}^T$. In particular, choose $t = 4$, we have

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

Similarly when $\lambda = \lambda_3 = 3$, we have $\mathbf{x}_3 = \begin{bmatrix} -\frac{1}{4}t & \frac{1}{4}t & t \end{bmatrix}^T$, $t \in \Re$.
Choose $t = 4$;

$$\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \qquad \textbf{(verify!!)}$$

Diagonalization

Similar matrices

A matrix B is said to be **similar** to a matrix A if there is a nonsingular matrix P such that

$$B = P^{-1}AP$$

Example

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Let $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Longrightarrow P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, and

$$B = P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Thus B is similar to A.

Properties

1. A is similar to A.
2. If B is similar to A, the A is similar to B.
3. If A is similar to B and B is similar to C, then A is similar to C.

Definition

A matrix A is said to be **diagonalizable** if it is **similar to a diagonal matrix**.

Example

If A and B are as in above example, then A is diagonalizable, since it is similar to diagonal matrix $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Theorem

Similar matrices have the same eigenvalues.

Proof

Let A and B be similar, then $B = P^{-1}AP$, for some nonsingular matrix P. The characteristic polynomial for B is

$$\begin{aligned} f_B(\lambda) &= \det(\lambda I_n - B) = \det(\lambda I_n - P^{-1}AP) \\ &= \det(P^{-1}\lambda I_n P - P^{-1}AP) = \det\big(P^{-1}(\lambda I_n - A)P\big) \\ &= \det(P^{-1})\det(\lambda I_n - A)\det(P) \\ &= \det(\lambda I_n - A) = f_A(\lambda) \end{aligned}$$

since $f_B(\lambda) = f_A(\lambda)$, it follows that A and B have the same eigenvalues.

Theorem

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Explanation

In this case A is similar to a diagonal matrix D, with $P^{-1}AP = D$, whose diagonal elements are the eigenvalues of A, while P is a matrix whose column are respectively the n linearly independent eigenvectors of A.

Example

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Find a diagonal matrix that is similar to A.

Solution

Find the eigenvalues and eigenvectors of A. Refer λ_i and $\underline{x}_i$ in Example 1

$$\lambda_1 = 2 \text{ with } \underline{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 3 \text{ with } \underline{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{then}$$

Solution cont.

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad , \text{ and}$$

$$P = \begin{bmatrix} [\underline{x}_1] & [\underline{x}_2] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

you can easily verify that $$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Example

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We find eigenvalues $\lambda_1 = \lambda_2 = 1$. Diagonalizable???

Theorem

A matrix A is diagonalizable if it has distinct real eigenvalues.

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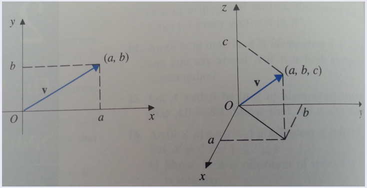
Geometric vectors

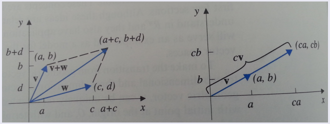
Geometric vectors can be represented graphically in R^2 or R^3.



- We will denote vector with underbar, vector x as $\underline{x}$
- In the figure, left, $\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ in R^2, right, $\underline{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in R^3

Vectors

Vector sums



- Vectors are added using parallelogram law, left figure
- Multiple of vector has same direction

Extension to non-geometric

- This study of vectors allow us to translate geometric properties into algebraic
- Extension of the concept is possible but with no geometrical representation. e.g elements in R^4

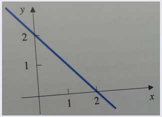
Example

Give a geometric interpretation of the subset W of R^2 defined by

$$W = \{\underline{x} : \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_1 + x_2 = 2\}$$

Solution

W is a line in the plane with equation $x + y = 2$



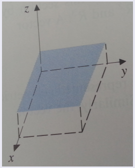
Example

Let W be the subset of R^3 defined by

$$W = \{\underline{x} : \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}, x_1, x_2 \in \Re\}$$

Solution

W can be viewed as the plane in three-space with equation $z = 1$



Vector Properties

R^n is the set of all n-dimensional vectors with real components:

$$R^n = \{\underline{x} : \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ x_1, x_2, \ldots, x_n \in \Re\}$$

If $\underline{x}$ and $\underline{y}$ are elements of R^n with $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, then vector

$\underline{x} + \underline{y}$ is defined by

Vectors space properties cont.

Vector Properties cont.

$$\underline{x} + \underline{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

If a is a real number, then the vector $a\underline{x}$ is defined to be

$$a\underline{x} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

Note: In the context of R^n, scalars are always real numbers

Theorem

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be any vectors in R^n, let c and d be any scalars. Then

(α) $\underline{u} + \underline{v}$ is a vector in R^n (i.e., R^n is closed under addition).

1. $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
2. $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$
3. There is a vector $\underline{0}$ in R^n such that $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$ for all $\underline{u}$ in R^n.
4. For each vector $\underline{u}$ in R^n, there is a vector $-\underline{u}$ in R^n such that $\underline{u} + (-\underline{u}) = \underline{0}$

(β) $c\underline{u}$ is a vector in R^n (i.e., R^n is closed under scalar multiplication)

1. $c(\underline{u} + \underline{v}) = c\underline{u} + c\underline{v}$
2. $(c + d)\underline{u} = c\underline{u} + c\underline{v}$
3. $c(d\underline{u}) = (cd)\underline{u}$
4. $1\underline{u} = \underline{u}$

The proof of the above properties is similar to what we did in matrices.

Vectors space properties cont.

Special vectors/operation

- $\underline{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is called **zero vector**.
- $-\underline{u}$ is called **negative** of $\underline{u}$, and that $-\underline{u} = (-1)\underline{u}$
- **Difference** $\underline{u} - \underline{v}$ can be regarded as a sum of

$$\underline{u} + (-\underline{v})$$

Magnitude/Norm of a vector

The **length** (or **magnitude** or **norm**) of a vector $\underline{u} = (u_1, u_2, \ldots, u_n)$ in R^n is defined by

$$||\underline{u}|| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Example

Let $\underline{u} = (2, 3, 2, -1)$ and $\underline{v} = (4, 2, 1, 3)$, then

$$||\underline{u}|| = \sqrt{2^2 + 3^2 + 2^2 + (-1)^2} = \sqrt{18}$$
$$||\underline{v}|| = \sqrt{4^2 + 2^2 + 1^2 + 3^2} = \sqrt{30}$$

The distance between the points $(2, 3, 2, -1)$ and $(4, 2, 1, 3)$ is the length of the vector $\underline{u} - \underline{v}$, thus

$$||\underline{u} - \underline{v}|| = \sqrt{(2-4)^2 + (3-2)^2 + (2-1)^2 + (-1-3)^2} = \sqrt{22}$$

Dot product

If $\underline{u} = (u_1, u_2, \ldots, u_n)$ and $\underline{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then their **dot product** is defined by

$$\underline{u}.\underline{v} = \sum_{k=1}^{n} u_k v_k$$

The dot product in R^n is also known as the **standard inner product**.

Dot product vs length of vector

If $\underline{u}$ is a vector in R^n, then we can use dot product definition to write

$$||\underline{u}|| = \sqrt{\underline{u}.\underline{u}}$$

Theorem /Dot product properties

If $\underline{u}$, $\underline{v}$ and $\underline{w}$ are vectors R^n and c is a scalar, then

1. $\underline{u}.\underline{u} \geq 0$; $\underline{u}.\underline{u} = 0$ if and only if $\underline{u} = \underline{0}$
2. $\underline{u}.\underline{v} = \underline{v}.\underline{u}$
3. $(\underline{u} + \underline{v}).\underline{w} = \underline{u}.\underline{w} + \underline{v}.\underline{w}$
4. $(c\underline{u}).\underline{v} = \underline{u}.(c\underline{v}) = c(\underline{u}.\underline{v})$

Cauchy-Schwarz Inequality

If $\underline{u}$ and $\underline{v}$ are vectors in R^n, then

$$|\underline{u}.\underline{v}| \leq ||\underline{u}||||\underline{v}||$$

Proof

If $\underline{u} = \underline{0}$ then $||\underline{v}|| = 0$ and $\underline{u}.\underline{v} = 0$, so CSE holds.
Now let r be a scalar and consider the vector $r\underline{u} + \underline{v}$

Proof cont.

By properties of dot product

$$
\begin{aligned}
0 \leq (r\underline{u} + \underline{v}).(r\underline{u} + \underline{v}) =& r^2\underline{u}.\underline{u} + 2r\underline{u}.\underline{v} + \underline{v}.\underline{v} \\
=& ar^2 + 2br + c
\end{aligned}
$$

where $a = \underline{u}.\underline{u}$, $b = \underline{u}.\underline{v}$ and $c = \underline{v} + \underline{v}$.
Now $p(r) = ar^2 + 2br + c$ is a quadratic polynomial in r, whose graph is opening upwards. Recall the roots r_i of the quadratic polynomial

$$
r_i = \frac{-2b \pm \sqrt{b^2 - 4ac}}{2a}
$$

since $p(r) \geq 0$, it means has no real roots or has only one real root.
$\Longrightarrow 4b^2 - 4ac \leq 0 \Longrightarrow b^2 \leq ac$
Then it follows, after taking the square root we have desired results

$$
(\underline{u}.\underline{v})^2 \leq (\underline{u}.\underline{u})(\underline{v}.\underline{v}) = ||\underline{u}||^2||\underline{v}||^2
$$

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Cauchy-Schwarz Inequality cont.

Example

Given $\underline{u} = (2, 3, 2, -1)$ and $\underline{v} = (4, 2, 1, 3)$. Then $\underline{u}.\underline{v} = 13$

$$|\underline{u}.\underline{v}| = 13 \le ||\underline{u}||||\underline{v}|| = \sqrt{18}\sqrt{30}$$

Angle between vectors

The angle $0 \le \theta \le \pi$ between vectors $\underline{u}$ and $\underline{v}$ is given by the use of Cauchy Schwarz Inequality as follows

$$\cos\theta = \frac{\underline{u}.\underline{v}}{||\underline{u}||||\underline{v}||}$$

Example

Given $\underline{u} = (1, 0, 0, 1)$ and $\underline{v} = (0, 1, 0, 1)$. Then $\underline{u}.\underline{v} = 1$

$$|\underline{u}|| = \sqrt{2}, \quad ||\underline{v}|| = \sqrt{2}, \quad \underline{u}.\underline{v} = 1$$

Example cont.

$$\cos\theta = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$

Orthogonal vectors

Two nonzero vectors $\underline{u}$ and $\underline{v}$ in R^n are said to be **orthogonal** if $\underline{u}.\underline{v} = 0$ $(\implies \cos\theta = 0)$

Parallel vectors

Two nonzero vectors $\underline{u}$ and $\underline{v}$ in R^n are said to be **parallel** if $|\underline{u}.\underline{v}| = ||\underline{u}||||\underline{v}||$ $(\implies \cos\theta = \pm 1)$.

Same direction

Two nonzero vectors $\underline{u}$ and $\underline{v}$ in R^n are said to be in **same direction** if $\underline{u}.\underline{v} = ||\underline{u}||||\underline{v}||$ $(\implies \cos\theta = 1)$.

Triangle inequality

Theorem

If $\underline{u}$ and $\underline{v}$ are vectors in R^n, then

$$||\underline{u} + \underline{v}|| \leq ||\underline{u}|| + ||\underline{v}||$$

Proof

$$\begin{aligned} ||\underline{u} + \underline{v}||^2 =& (\underline{u} + \underline{v}).(\underline{u} + \underline{v}) \\ =& \underline{u}.\underline{u} + 2(\underline{u}.\underline{v}) + \underline{v}.\underline{v} \\ =& ||\underline{u}||^2 + 2(\underline{u}.\underline{v}) + ||\underline{v}||^2 \end{aligned}$$

Using Cauchy-Schwarz Inequality we have

$$\begin{aligned} ||\underline{u}||^2 + 2(\underline{u}.\underline{v}) + ||\underline{v}||^2 \leq& ||\underline{u}||^2 + 2||\underline{u}||||\underline{v}|| + ||\underline{v}||^2 \\ =& (||\underline{u}|| + ||\underline{u}||)^2 \end{aligned}$$

Triangle inequality

Proof cont.

Taking square roots, we obtain desired results

Meaning of the triangle inequality

The triangle inequality in R^2 and R^3 states that the length of side of a triangle does not exceed the sum of the lengths of the other two sides.

Example

For $\underline{u}$ and $\underline{v}$ given in previous example, we have

$$||\underline{u}+\underline{v}|| = \sqrt{4} = 2 < \sqrt{2}+\sqrt{2} = ||\underline{u}|| + ||\underline{v}||$$

Pythagorean theorem

What happens when $\underline{u}.\underline{v} = 0$?

$$||\underline{u}+\underline{v}||^2 = ||\underline{u}||^2 + ||\underline{v}||^2$$

Unit vector

A **unit vector** $\hat{\underline{u}}$ in R^n is a vector of length 1. If $\underline{u}$ is a nonzero vector, then the vector

$$\hat{\underline{u}} = \frac{1}{||\underline{u}||}\underline{u}$$

is a unit vector in the direction of $\underline{u}$

Example

Given $\underline{u} = (1, 0, 0, 1)$, then $||\underline{u}|| = \sqrt{2}$, the vector $\hat{\underline{u}} = \frac{1}{\sqrt{2}}(1, 0, 0, 1)$ is a unit vector in the direction of $\underline{u}$.

Special unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$

For R^3, the unit vectors in the positive directions of $x-$, $y-$, and $z-$axes are denoted by $\underline{i} = (1, 0, 0)$, $\underline{j} = (0, 1, 0)$ and $\underline{k} = (0, 0, 1)$

General vector in R^3

If $\underline{u} = (x_1, y_1, z_1)$ is any vector in R^3, then we can write $\underline{u}$ as

$$\underline{i} = x_1\underline{i} + x_2\underline{j} + x_3\underline{k}$$

General vector in R^n

If $\underline{e}_1 = (1, 0, \dots, 0)$, $\underline{e}_2 = (0, 1, \dots, 0) \dots \underline{e}_n = (0, 0, \dots, 1)$ are the unit vectors in R^n that are mutually orthogonal. If $\underline{u} = (u_1, u_2, \dots, u_n)$ is any vector in R^n, then $\underline{u}$ can be written as a linear combination of e_i's as

$$\underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + \cdots + u_n\underline{e}_n$$

Note: The vector $\underline{e}_i$ can be viewed as the ith column of the identity matrix I_n.

Cross product

If $\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$ and $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$ are two vectors in R^3, then their **cross product** is the vector $\underline{u} \times \underline{v}$ defined by

$$\underline{u} \times \underline{v} = (u_2 v_3 - u_3 v_2)\underline{i} - (u_1 v_3 - u_3 v_1)\underline{j} + (u_1 v_2 - u_2 v_1)\underline{k}$$

Cross product as determinant

The cross product $\underline{u} \times \underline{v}$ can be written as a "determinant"

$$\begin{aligned} \underline{u} \times \underline{v} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \underline{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \underline{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \underline{k} \end{aligned}$$

Example

Let $\underline{u} = 2\underline{i} + \underline{j} + 2\underline{k}$ and $\underline{v} = 3\underline{i} - \underline{j} - 3\underline{k}$. Then the cross product

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix} = -\underline{i} + 12\underline{j} - 5\underline{k}$$

Properties of cross product

If $\underline{u}$, $\underline{v}$ and $\underline{w}$ are vectors in R^n and c is a scalar, then

1. $\underline{u} \times \underline{v} = -(\underline{v} \times \underline{u})$
2. $\underline{u} \times (\underline{v} + \underline{w}) = \underline{u} \times \underline{v} + \underline{u} \times \underline{w}$
3. $(\underline{u} + \underline{v}) \times \underline{w} = \underline{u} \times \underline{w} + \underline{v} \times \underline{w}$
4. $c(\underline{u} \times \underline{v}) = (c\underline{u}) \times \underline{v} = \underline{u} \times (c\underline{v})$
5. $\underline{u} \times \underline{u} = \underline{0}$
6. $\underline{0} \times \underline{u} = \underline{u} \times \underline{0} = \underline{0}$
7. $\underline{u} \times (\underline{v} \times \underline{w}) = (\underline{u} \cdot \underline{w})\underline{v} - (\underline{u} \cdot \underline{v})\underline{w}$
8. $(\underline{u} \times \underline{v}) \times \underline{w} = (\underline{w} \cdot \underline{u})\underline{v} - (\underline{w} \cdot \underline{v})\underline{u}$

Example

Using the definition of the cross product, we obtain the following

$$\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}$$

$$\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j} \qquad \text{clockwise product}$$

$$\underline{j} \times \underline{i} = -\underline{k}, \quad \underline{k} \times \underline{j} = -\underline{i}, \quad \underline{i} \times \underline{k} = -\underline{j} \qquad \text{counter-clockwise product}$$

Note

Many properties of real numbers hold for the cross product, however two important properties do not

1. Commutative property
2. Associative property

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More properties of cross product

More properties

The proof are left for exercise/tutorial 8

$$(\underline{u} \times \underline{v}) \cdot \underline{w} = \underline{u} \cdot (\underline{v} \times \underline{w})$$

$$(\underline{u} \times \underline{v}) \cdot \underline{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

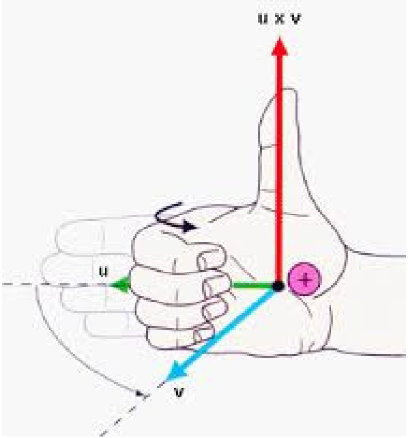
From the construction of $\underline{u} \times \underline{v}$, it follows that $\underline{u} \times \underline{v}$ is orthogonal to both $\underline{u}$ and $\underline{v}$; that is

$$(\underline{u} \times \underline{v}) \cdot \underline{u} = 0$$

$$(\underline{u} \times \underline{v}) \cdot \underline{v} = 0$$

- $\underline{u} \times \underline{v}$ is also orthogonal to the plane formed by $\underline{u}$ and $\underline{v}$

- The direction of $\underline{u} \times \underline{v}$ is given by the right hand rule. Given θ as the angle between $\underline{u}$ and $\underline{v}$, if we curl the fingers of the right hand in the direction of a rotation through the angle θ, then the thumb will point in the direction of $\underline{u} \times \underline{v}$.



Magnitude of $\underline{u} \times \underline{v}$

From the definition of the length of a vector, we have

$$\begin{aligned}
||\underline{u} \times \underline{v}||^2 =& (\underline{u} \times \underline{v}) \cdot (\underline{u} \times \underline{v}) \\
=& \underline{u} \cdot [\underline{v} \times (\underline{u} \times \underline{v})] \\
=& \underline{u} \cdot [(\underline{v} \cdot \underline{v})\underline{u} - (\underline{v} \cdot \underline{u})\underline{v}] \\
=& (\underline{u} \cdot \underline{u})(\underline{v} \cdot \underline{v}) - (\underline{v} \cdot \underline{u})(\underline{v} \cdot \underline{u}) \\
=& ||\underline{u}||^2||\underline{v}||^2 - (\underline{u} \cdot \underline{v})^2 \\
=& ||\underline{u}||^2||\underline{v}||^2 - ||\underline{u}||^2||\underline{v}||^2 \cos\theta \\
=& ||\underline{u}||^2||\underline{v}||^2(1 - \cos^2\theta)
\end{aligned}$$

More properties of cross product

magnitude conti.

Therefore

$$||\underline{u} \times \underline{v}||^2 = ||\underline{u}||^2||\underline{v}||^2 \sin^2\theta$$

Taking the square roots, we obtain

$$||\underline{u} \times \underline{v}|| = ||\underline{u}||||\underline{v}|| \sin\theta$$

where $\sin\theta \geq 0$ since $0 \leq \theta \leq \pi$

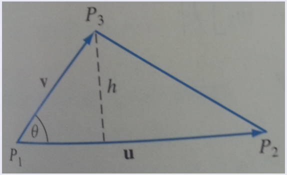
- It follows that $\underline{u}$ and $\underline{v}$ are parallel if and only if $\underline{u} \times \underline{v} = \underline{0}$

Area of a triangle

Consider the triangle with vertices P_1, P_2 and P_3, figure below. The area is

$$A = \frac{1}{2} bh$$

where b is the base and h is the height.



If we denote the vector formed by P_1 and P_2; $\overrightarrow{P_1P_2} = \underline{u} \Longrightarrow b = ||\underline{u}||$ and vector formed by P_1 and P_3; $\overrightarrow{P_1P_3} = \underline{v}$

we then find height h as

$$h = ||\underline{v}|| \sin\theta$$

Then the area is

$$A = \frac{1}{2}||\underline{u}||||\underline{v}|| \sin\theta = \frac{1}{2}||\underline{u} \times \underline{v}||$$

Example

Find the area of a triangle with vertices $P_1(2, 2, 4)$, $P_2(-1, 0, 5)$ and $P_3(3, 4, 3)$
We have

$$\underline{u} = \overrightarrow{P_1P_2} = -3\underline{i} - 2\underline{j} + \underline{k}$$
$$\underline{v} = \overrightarrow{P_1P_3} = \underline{i} + 2\underline{j} - \underline{k}$$

Applications of cross product

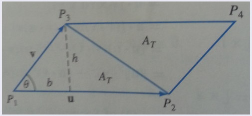
then

$$
\begin{aligned}
A =& \frac{1}{2}||\underline{u} \times \underline{v}|| \\
=& ||(-3\underline{i} - 2\underline{j} + \underline{k}) \times (\underline{i} + 2\underline{j} - \underline{k})|| \\
=& || - \underline{i} - 2\underline{j} - 2\underline{k}|| = \sqrt{5}
\end{aligned}
$$

Area of a parallelogram

The area of the parallelogram with adjacent sides $\underline{u}$ and $\underline{v}$ is twice that of a triangle with sama adjacent sides

$$A = ||\underline{u} \times \underline{v}||$$



Applications of cross product

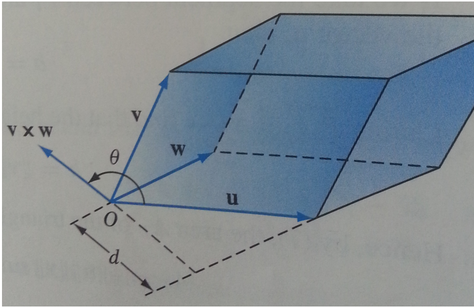
Consider the parallelepiped with a vertex at the origin and edges $\underline{u}$, $\underline{v}$ and $\underline{w}$. The volume V of the parallelepiped is

$$A = \text{Ar of face with } \underline{u} \text{ and } \underline{v} \times \text{ dist btn faces}$$

The distance d from this face to the face parallel to it is

$$d = ||\underline{u}|| \cos\theta$$

where θ is the angle between $\underline{u}$ and $\underline{v} \times \underline{w}$



Applications of cross product

Volume cont.

Hence,

$$V = ||\underline{v} \times \underline{w}||||\underline{u}|| \cos\theta| = |\underline{u} \cdot (\underline{v} \times \underline{w})|$$

using the property of cross products, we can also write

$$V = \left| \det\left(\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right) \right|$$

Example

Consider the parallelepiped with a vertex at the origin and edges
$\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$, $\underline{v} = \underline{i} + 3\underline{j} + \underline{k}$ and $\underline{w} = 2\underline{i} + \underline{j} + 2\underline{k}$.
Then $\underline{v} \times \underline{w} = 5\underline{i} - 5\underline{k}$
Hence $\underline{u} \cdot (\underline{v} \times \underline{w}) = -10$. Thus the volume V is

$$V = |\underline{u} \cdot (\underline{v} \times \underline{w})| = 10$$

Alternative 2

We can also compute the volume V as follows

$$V = \left|\det\left(\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right)\right|$$

$$= \left|\det\left(\begin{bmatrix} 1 & -2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}\right)\right| = |-10| = 10$$

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Lines in R^n

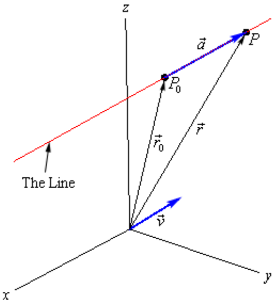
- Point P_0 given by $\underline{r}_0$ parallel to vector $\underline{v}$
- A general point P given by $\underline{r}$ on the line is given by **vector equation** of the line

$$\underline{r} = \underline{r}_0 + t\underline{v}$$

where t is a scalar

- **Parametric equation** are corresponding components equations

$$x_i = r_{0i} + tv_i, \;\; i = 1, \ldots, n$$



Vector $\underline{v}$ can be computed using any two points $\underline{a}_1$ and $\underline{a}_2$ on the line as

$$\underline{v} = \underline{a}_2 - \underline{a}_1$$

Lines in R^n

Example

Find the Parametric equations of the line in R^3 that passes through the points $\underline{a} = (1, 2, -1)$ and $\underline{b} = (3, 1, 4)$.

Solution

The parllel vector $\underline{d} = \underline{b} - \underline{a} = (2, -1, 5)$
Then, the vector equation is

$$\begin{aligned} \underline{x} =& x_0 + t\underline{d} \\ =& \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \end{aligned}$$

Parametric equations are therefore

$$x_1 = 1 + 2t, \quad x_2 = 2 - t, \quad x_3 = -1 + 5t$$

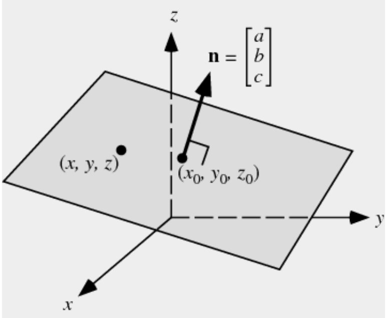
Planes

- A plane is a two-dimensional doubly ruled surface spanned by two linearly independent vectors.
- The generalization of the plane to higher dimensions is called a hyperplane.
- The **vector equation of the plane** with nonzero normal vector $\underline{n} = (n_1, n_2, n_3)$ through the point $\underline{x}_0 = (x_0, y_0, z_0)$ is

$$\underline{n} \cdot (\underline{x} - \underline{x}_0) = 0$$

- **Scalar equation** has the form

$$ax + by + cz + d = 0$$



Vector $\underline{n}$ can be computed as a cross product of any two vectors on the plane

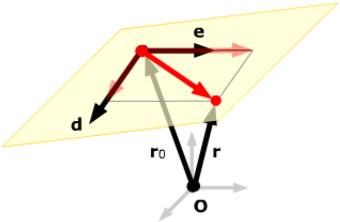
Alternative equation of a planes

- Let the two vectors $\underline{d}$ and $\underline{e}$ are linearly independent vectors on the plane S.
- Let P defined by vector $\underline{r}$ be a general point on the plane S
- Let P_0 defined by vector $\underline{r}_0$ be a given point on the plane S

The equation of the plane is then given by

$$\underline{x} = \underline{r}_0 + s\underline{d} + t\underline{e}$$

The **Parametric equations** can be extracted by equating components



Vectors $\underline{d}$ and $\underline{e}$ can be computed using any three points on the plane

Alternative equation of a planes

Example

Find the Parametric equations of the plane in R^4 passing through the points $\underline{a} = (1, 1, 1, 1)$, $\underline{b} = (2, 1, 1, 0)$ and $\underline{c} = (3, 2, 1, 0)$.
Vectors on the plane as $\underline{b} - \underline{a} = (1, 0, 0, -1)$, $\underline{c} - \underline{a} = (2, 1, 0, -1)$. Then the vector equation is given by

$$\begin{aligned}\underline{x} =& a + s(\underline{b} - \underline{a}) + t(\underline{c} - \underline{a}) \\ =& \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}\end{aligned}$$

The Parametric equations are therefore $\begin{cases} x_1 = 1 + s + 2t \\ x_2 = 1 + t \\ x_3 = 1, \quad x_4 = 1 - s - t \end{cases}$

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Vector spaces

A set of elements V is said to be a **vector space** over a **scalar field** S if an addition operation is defined between any two elements of V and a scalar multiplication operation is defined between any element of S and any vector in V. Moreover, if $\underline{u}$, $\underline{v}$ and $\underline{w}$ are vectors in V, and if a and b are any two scalars, then these 10 properties must hold

Closure properties:

c1) $\underline{u} + \underline{v} \in V$

c2) $a\underline{u} \in V$

Properties of addition

a1) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$

a2) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$

a3) $\exists \underline{0} \in V$ such that $\underline{v} + \underline{0} = \underline{v}$

a4) $\forall \underline{v} \in V$, $\exists (-\underline{v}) \in V$ such that $\underline{v} + (-\underline{v}) = \underline{0}$

Properties of scalar multiplication

m1) $a(b\underline{v}) = (ab)\underline{v}$

m2) $a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}$

m3) $(a + b)\underline{v} = a\underline{v} + b\underline{v}$

m4) $1\underline{v} = \underline{v}, \ \forall \underline{v} \in V$

Real vector space

- $\underline{0}$ is called **zero vector** (or **additive identity**)
- the vector $-\underline{v}$ is called **additive inverse** of $\underline{v}$
- if set of scalars S is $\Re$, then V is called a **real vector space**

Example 1

For any positive integer n, verify that R^n is a real vector space

Solution

We need to show that all 10 properties hold for R^n to be a vector space.
Let $\underline{u}, \underline{v}, \underline{w} \in R^n$ and $a, b \in R$

Closure properties

c1: $\underline{u} + \underline{v} \in R^n$, **c2:** $a\underline{v} \in R^n$

Addition properties

a1: The addition in R^n is commutative

a2: $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$, associative property of vector addition

a3: $\exists \underline{0} = [0\ 0\ 0\ 0\ \ \ldots\ \ 0]^T \in R^n$ such that $\underline{v} + \underline{0} = \underline{v}$

a4: $\forall \underline{v} \in R^n$, $\exists (-\underline{v}) \in R^n$ such that $\underline{v} + (-\underline{v}) = \underline{0}$

Solution cont.

Properties of scalar multiplication

m1) $a(b\underline{v}) = (ab)\underline{v}$, scalar multiplication property of vector

m2) $a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}$, distributive property

m3) $(a + b)\underline{v} = a\underline{v} + b\underline{v}$, scalar vector product property

m4) $1\underline{v} = \underline{v}, \ \forall \underline{v} \in R^n$

All 10 properties for vector space are satisfied and therefore R^n is a real vector space.

Example 2

Verify that the set of all 2×3 matrices with real entries is a real vector space

Solution

Let A, B and C be any 2×3 matrices and let λ, β be a scalars, then

Closure properties

c1: Addition of two 2×3 matrices is a 2×3 matrix

c2: $\lambda \underline{A}$ is also 2×3 matrix

Solution cont.

Addition properties

a1: The addition of matrices is commutative

a2: $A + (B + C) = (A + B) + C$, associative property for matrix addition

a3: $\exists$ 2×3 zero matrix $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ such that $A + O = A$

a4: For any 2×3 matrix A $\exists$ 2×3 matrix $(-A)$ such that $A + (-A) = O$

Properties of scalar multiplication

m1) $\lambda(\beta A) = (\lambda\beta)A$

m2) $\lambda(A + B) = \lambda A + \lambda B$

m3) $(\lambda + \beta)A = \lambda A + \beta A$

m4) $1A = A$, for any 2×3 matrix

All 10 properties for vector space are satisfied and therefore a set of 2×3 matrices with real entries is a vector space.

Example 3

Let P_2 denote the set of all real polynomials of degree 2 or less. Verify that P_2 is a real-vector space.

Solution

Let $p(x) = a_2x^2 + a_1x + a_0$, $q(x) = b_2x^2 + b_1x + b_0$, $r(x) = c_2x^2 + c_1x + c_0 \in P_2$ and $\lambda, \beta \in R$

Closure properties

c1:

$$\begin{aligned} p(x) + q(x) =& a_2x^2 + a_1x + a_0 + b_2x^2 + b_1x + b_0 \\ =& (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) \ \in P_2 \end{aligned}$$

c2: $\lambda p(x) = \lambda(a_2x^2 + a_1x + a_0) = (\lambda a_2)x^2 + (\lambda a_1)x + (\lambda a_0) \in P_2$

Solution cont.

Addition properties

a1:

$$\begin{aligned} p(x)+q(x) =&(a_2x^2+a_1x+a_0)+(b_2x^2+b_1x+b_0) \\ =&(b_2x^2+b_1x+b_0)+(a_2x^2+a_1x+a_0)=q(x)+p(x) \end{aligned}$$

a2:

$$\begin{aligned} &p(x)+(q(x)+r(x)) \\ &\quad = a_2x^2+a_1x+a_0+(b_2x^2+b_1x+b_0+c_2x^2+c_1x+c_0) \\ &\quad = \left((a_2x^2+a_1x+a_0)+(b_2x^2+b_1x+b_0)\right)+c_2x^2+c_1x+c_0 \\ &\quad = (p(x)+q(x))+r(x) \end{aligned}$$

a3: $\exists\ \theta(x)=0x^2+0x+0=0$ such that $p(x)+\theta(x)=p(x)$

a4: For any $p(x)\exists\ (-p(x))$ such that $p(x)+(-p(x))=0$

Solution cont.

Properties of scalar multiplication

m1) $\lambda(\beta p(x)) = (\lambda\beta)p(x)$ is true for any $p(x) \in P_2$, **verify!**

m2)

$$\begin{aligned}
\lambda(p(x)+q(x)) =& \lambda(a_2x^2 + a_1x + a_0 + b_2x^2 + b_1x + b_0) \\
=& \lambda(a_2x^2 + a_1x + a_0) + \lambda(b_2x^2 + b_1x + b_0) \\
=& \lambda p(x) + \lambda q(x)
\end{aligned}$$

m3)

$$\begin{aligned}
(\lambda+\beta)p(x) =& (\lambda+\beta)(a_2x^2 + a_1x + a_0) \\
=& (\lambda+\beta)(a_2)x^2 + (\lambda+\beta)(a_1)x + (\lambda+\beta)(a_0) \\
=& \lambda a_2x^2 + \beta a_2x^2 + \lambda a_1x + \beta a_1x + \lambda a_0 + \beta a_0 \\
=& \lambda(a_2x^2 + a_1x + a_0) + \beta(b_2x^2 + b_1x + b_0) \\
=& \lambda p(x) + \beta q(x)
\end{aligned}$$

Solution cont.

m4) $1p(x) = 1(a_2x^2 + a_1x + a_0) = a_2x^2 + a_1x + a_0 = p(x)$ All 10 properties of the vector product hold and therefore P_2 is a real vector space.

Take away

Let $C[a, b]$ be the set of functions defined by

$$C[a, b] = \{f(x) : f(x) \text{ is a real-valued continuous function, } a \leq x \leq b\}$$

Verify that $C[a, b]$ is real vector space.

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Lecture 16 – 2016/2017

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Subspaces

Let V be a vector space and W a nonempty subset of V. If W is a vector space with respect to the operations in V, then W is called a **Subspace** of V.

Example

Every vector space has at least two subspaces, which are they?

Solution

Itself and the subspace $\{0\}$. Verify that!

Theorem

Let V is a vector space with operations "$+$"and "$\times$"and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:

α) If $\underline{u}$ and $\underline{v}$ are any vectors in W, then $\underline{u} + \underline{v}$ is in W

β) If c is any real number and $\underline{u}$ is any vector in W, then $c\underline{u}$ is in W

Remarks

If a subset W of a vector space V does not contain the zero vector, then W is not a subspace of V.

Example

Consider the set W consisting of all 2×3 matrices of the form $\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$, where a, b, c and d are arbitrary real numbers. Show that W is subspace if M_{23}.

Solution

Let $\underline{u} = \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix}$. Then

$$\underline{u} + \underline{v} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & 0 \\ 0 & c_1 + c_2 & d_1 + d_2 \end{bmatrix} \quad \text{is in } W$$

so that (α) is satisfied.

$$k\underline{u} + \underline{v} = \begin{bmatrix} ka_1 & kb_1 & 0 \\ 0 & kc_1 & kd_1 \end{bmatrix} \quad \text{is in } W$$

so that (β) is satisfied.

Example

Which of the following subsets of R^2 with the usual operations of vector addition and scalar multiplications are subspaces?

1. W_1 is the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$.
2. W_2 is the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$, $y \geq 0$.
3. W_3 is the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x = 0$.

Solution

1. W_1 is the right half of the xy-plane. It is not a subspace of R^2 because if we take vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in W_1$, then the scalar multiple

$$-3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \end{bmatrix} \notin W_1$$

Example

2. The same vector in part (1), shows that W_2 is not a subspace
3. W_3 is the $y-$axis in the $xy-$plane. Proceed to check if W_3 is a subspace, Let $\underline{u} = \begin{bmatrix} 0 \\ b_1 \end{bmatrix}, \underline{v} = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \in W_3$, then

$$\underline{u} + \underline{v} = \begin{bmatrix} 0 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 + b_2 \end{bmatrix} \in W_3$$

so addition property holds,

$$c\underline{u} = c \begin{bmatrix} 0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ cb_1 \end{bmatrix} \in W_3$$

so scalar multiplication property hold as well. Therefore W_3 is a subspace of R^2

Null space

Consider the homogeneous system $A\underline{x} = \underline{0}$, where A is an $m \times n$ matrix. A solution contains of a vector $\underline{x}$ in R^n. Let W be the subset of R^n consisting of all solutions to the homogeneous system. Since $A\underline{0} = \underline{0}$, conclude that W is not empty. To check if W is a subspace, we verify the two properties of subspace.

Let $\underline{x}$ and $\underline{y}$ are solutions, then

$$A\underline{x} = \underline{0} \quad \text{and} \quad A\underline{y} = \underline{0} \quad , \text{ now}$$

$$A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y} = \underline{0} + \underline{0} = \underline{0}$$

so $\underline{x} + \underline{y}$ is a solution, also if c is a scalar, then

$$A(c\underline{x}) = c(A\underline{x}) = c\underline{0} = \underline{0}$$

so $c\underline{x}$ is a solution. Hence W is a subspace of R^n.

- The subspace W is called **solution space** or the **null space**.

Linear combination

Let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ be vectors in a vector space V. A vector $\underline{v} \in V$ is called a **linear combination** of $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ if

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_k\underline{v}_k$$

for some real numbers not all zeros $c_1, c_2, \ldots, c_k$.

Example

In R^3 let $\underline{v}_1 = (1, 2, 1)$, $\underline{v}_2 = (1, 0, 2)$ and $\underline{v}_3 = (1, 1, 0)$. The vector $\underline{v} = (2, 1, 5)$ is a linear combination of $\underline{v}_1, \underline{v}_1$ and $\underline{v}_3$ if we can find real numbers c_1, c_2 and c_3 so that

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3, \quad \text{then}$$

$$(2, 1, 5) = c_1(1, 2, 1) + c_2(1, 0, 2) + c_3(1, 1, 0)$$

Solution cont.

$$(2,1,5) = c_1(1,2,1) + c_2(1,0,2) + c_3(1,1,0)$$

Combining terms on the right and equating correspoing entry, then

$$\begin{cases} c_1 + c_2 + c_3 = 2 \\ 2c_1 + c_3 = 1 \\ c_1 + 2c_2 = 5 \end{cases} \implies c_1 = 1, c_2 = 2 \text{ and } c_3 = -1$$

Thus

$$\underline{v} = \underline{v}_1 + 2\underline{v}_2 - \underline{v}_3$$

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Lecture 17 – 2016/2017

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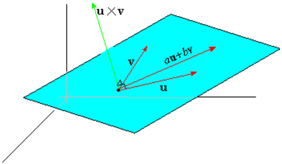
Span

If $S = \{\underline{v}_1, \underline{v}_2 \ldots, \underline{v}_k$ is a set of vectors in a vector space V, then the set of all vectors in V that are linear combinations of the vectors in S is denoted by

$$\text{span } S \quad \text{or} \quad \text{span}\{\underline{v}_1, \underline{v}_2 \ldots, \underline{v}_k\}$$

- The figure shows a portion of span $\{\underline{u}, \underline{v}\}$, where $\underline{u}$ and $\underline{v}$ are noncollinear vectors in R^3
- span $\{\underline{u}, \underline{v}\}$ is a plane that contains the vectors $\underline{u}$ and $\underline{v}$



Example

Consider the set of 2×3 matrices given by

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

The span S is the set in M_{23} consisting of all vectors of the form

$$a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}, \quad \text{where } a, b, c, d \in \mathbb{R}$$

Theorem

Let $S = \{\underline{v}_1, \underline{v}_2 \ldots, \underline{v}_k\}$ be a set of vectors in a vector space V. Then span S is a subspace of V.

Example

In P_2 let $\underline{v}_1 = 2t^2 + t + 2$, $\underline{v}_2 = t^2 - 2t$, $\underline{v}_3 = 5t^2 - 5t + 2$, $\underline{v}_4 = -t^2 - 3t - 2$. Determine if the vector $\underline{u} = t^2 + t + 2$ belongs to span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$

Solution

If we can find scalars c_1, c_2, c_3 and c_4 so that

$$c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + c_4\underline{v}_4 = \underline{u}$$

then $\underline{u}$ belongs to span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$
We have

$$c_1(2t^2 + t + 2) + c_2(t^2 - 2t) + c_3(5t^2 - 5t + 2) + c_4(-t^2 - 3t - 2) \\ = t^2 + t + 2$$

Example cont.

The two polynomials agree for all values of t only if the coefficients of respective powers of t agree, thus

$$\begin{aligned} 2c_1 + c_2 + 5c_3 - c_4 &= 1 \\ c_1 - 2c_2 - 5c_3 - 3c_4 &= 1 \\ 2c_1 + 2c_3 - 2c_4 &= 2 \end{aligned}$$

Form the augmented matrix and transform into rref, we obtain

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which tells us that the system is **inconsistent**, means no solution and hence $\underline{u}$ does not belong to span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$.

Definition

The vectors in set $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$ in a vector space V are said to **span** V if every vector in V is a linear combination of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$. Then we also say that the set S **spans** V, or that $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$ **spans** V, or that V is **spanned by** S, or span $S = V$

Example

Let V be the vector space R^3 and let $\underline{v}_1 = (1, 2, 1)$, $\underline{v}_2 = (1, 0, 2)$, and $\underline{v}_3 = (1, 1, 0)$, Do $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$ span V?

Solution

Let $\underline{v} = (a, b, c)$ be any vector in R^3, where a, b and c are arbitrary real numbers. We need to find out whether there are constants c_1, c_2 and c_3 such that

$$c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = \underline{v}$$

Solution cont.

This takes us to

$$\begin{aligned} c_1 + c_2 + c_3 &= a \\ 2c_1 + c_3 &= b \\ c_1 + 2c_2 &= c \end{aligned}$$

Whose solution is

$$c_1 = \frac{-2a + 2b + c}{3}, \ c_2 = \frac{a - b + c}{3}, \ c_3 = \frac{4a - b - 2c}{3}$$

For every choice of a, b and c we can find the scalars c_i's, therefore $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_3$ spans R^3 or $\text{span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\} = R^3$

Example

Let V be the vector space P_2. Let $S = \{p_1(t), p_2(t)\}$, where $p_1(t) = t^2 + 2t + 1$ and $p_2(t) = t^2 + 2$. Does S span P_2?

Solution

Let $p(t) = at^2 + bt + c$ be any polynomial in P_2, where a, b and c are any real numbers. We must find out whether there are numbers c_1 and c_2 such that

$$p(t) = c_1 p_1(t) + c_2 p_2(t), \text{ then}$$

$$at^2 + bt + c = c_1(t^2 + 2t + 1) + c_2(t^2 + 2)$$

$$\Longrightarrow (c_1 + c_2)t^2 + (2c_1)t + (c_1 + 2c_2) = at^2 + bt + c$$

Using equality of polynomials

$$\begin{cases} c_1 + c_2 = a \\ 2c_1 = b \\ c_1 + 2c_2 = c \end{cases}$$

Example

Write the augmented matrix and transform into rref, we obtain

$$\left[\begin{array}{ccc} 1 & 0 & 2a-c \\ 0 & 1 & c-a \\ 0 & 0 & b-4a+2c \end{array}\right]$$

If $b-4a+2c \neq 0$, then the system is inconsistent and there is no solution. Hence $S = \{p_1(t), p_2(t)\}$ does not span P_2.

Example

The vectors $\underline{i} = (1, 0)$ and $\underline{j} = (0, 1)$ span R^2, then if $\underline{u} = (u_1, u_2) \in R^2$, then it is possible to express

$$\underline{u} = u_1\underline{i} + u_2\underline{j}$$

Similarly for R^3, if $\underline{i} = (1, 0, 0)$, $\underline{j} = (0, 1, 0)$ and $\underline{k} = (0, 0, 1)$ span R^3, then a vector $\underline{u} = (u_1, u_2, u_3) \in R^3$ can be written as

$$\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$$

Generally, if $\underline{e}_1 = (1, 0, \dots, 0)$, $\underline{e}_2 = (0, 1, \dots, 0), \dots$ $\underline{e}_n = (0, 0, \dots, 1)$ span R^n, any vector $\underline{u} = (u_1, u_2, \dots, u_n)$ can be written as

$$\underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + \cdots + u_n\underline{e}_n$$

Example

The set $S = \{t^n, t^{n-1}, \ldots, t, 1\}$ span P_n, since any polynomial in P_n is of the form

$$a_0 t_n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_{n-1} t + a_n$$

Example

Consider the homogeneous linear system $A\underline{x} = \underline{0}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{bmatrix}$$

The set of all solutions to $A\underline{x} = \underline{0}$ forms a subspace of R^4. To determine a spanning set for the solution space, we find that the rref of the augmented matrix is

Example cont.

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is therefore given by

$$\begin{cases} x_1 = -r - 2s \\ x_2 = r \\ x_3 = s \\ x_4 = s \end{cases} \quad \text{or} \quad \underline{x} = r \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence the vectors $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ span the **solution space**.

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Lecture 18 – 2016/2017

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Linear indenpendence

Definition

The vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ in a vector space V are said to be **lineary dependent** if there exit constants $c_1, c_2, \ldots . c_k$, not all zeros, such that

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_k\underline{v}_k = \underline{0}$$

Otherwise, $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ are said to be **linearly independent**, i.e $c_1 = c_2 = \cdots = c_k = 0$.

Example

Determine whether the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ are linearly dependent or lineary independent.

Linear indenpendence

Solution

Forming the equation

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We obtain the homogeneous system

$$\begin{cases} -c_1 - 2c_2 & = 0 \\ c_1 + 0c_2 & = 0 \\ 0c_1 + c_2 & = 0 \\ 0c_1 + c_2 & = 0 \end{cases} \Longrightarrow c_1 = c_2 = 0$$

Hence the vectors are linearly independent.

Example

Consider the vectors $\underline{v}_1 = (1, 2, -1)$, $\underline{v}_2 = (1, -2, 1)$, $\underline{v}_3 = (-3, 2, -1)$ and $\underline{v}_4 = (2, 0, 0)$ in $\mathbb{R}^4$. Is $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ linearly dependent or linearly independent?

Solution

Setting up the test equation,

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Giving

$$\begin{cases} c_1 + c_2 - 3c_3 + 2c_4 & = 0 \\ 2c_1 - 2c_2 + 2c_3 & = 0 \\ -c_1 + c_2 - c_3 & = 0 \end{cases}$$

Solution cont.

homogeneous equation with 3 equations and 4 unknowns, it has non-trivial solution (confirm!). e.g $c_1 = 1, c_2 = 2, c_3 = 1, c_4 = 0$.

Example

If $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ are k vectors in any vector space and $\underline{v}_i$ is the zero vector, then linearly independent test equation holds by letting $c_i = 1$ and $c_j = 0$ for $j \neq i$. Thus $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is linearly dependent.
Note: Every set of vectors containing the zero vector is linearly dependent.

Example

Consider the set of vectors $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ in $\mathbb{R}^4$, where

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Basis and dimension

Example cont.

and let $W = \text{span } S$. Since $\underline{v}_4 = \underline{v}_1 + \underline{v}_2$, we conclude that $W = \text{span } S_1$ where $S_1 = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$.

Basis

The vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ in a vector space V form basis for V if

1. $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ span V, and
2. $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ are linearly independent

Example

1. The vectors $\underline{e}_1 = (1, 0)$ and $\underline{e}_2 = (0, 1)$ form a basis for $\mathbb{R}^2$
2. The vectors $\underline{e}_1 = (1, 0, 0)$, $\underline{e}_2 = (0, 1, 0)$ and $\underline{e}_3 = (0, 0, 1)$ form a basis for $\mathbb{R}^3$
3. Generally, the vectors $\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n$ form a basis for $\mathbb{R}^n$

Each set of vectors is called the **natural basis** or **standard basis**.

Basis and dimension

Example

Show that the set $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$, where $\underline{v}_1 = (1, 0, 1, 0)$, $\underline{v}_2 = (0, 1, -1, 2)$, $\underline{v}_3 = (0, 2, 2, 1)$ and $\underline{v}_4 = (1, 0, 0, 1)$ is a basis for $\mathbb{R}^4$.

Solution

We need to show two things, first the set $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ are linearly independent and secondly the $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ span $\mathbb{R}^4$
To show that S is linearly independent, we form the equation

$$c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + c_4\underline{v}_4 = \underline{0}$$

substituting into the equation

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution cont.

Then we need to solve the linear system

$$\begin{cases} c_1 + c_4 & = 0 \\ c_2 + 2c_3 & = 0 \\ c_1 - c_2 + 2c_3 & = 0 \\ 2c_2 + c_3 + c_4 & = 0 \end{cases} \implies c_1 = c_2 = c_3 = c_4 = 0 \quad \textbf{(verify!)}$$

This tells us that S is linearly independent.
Does S span $\mathbb{R}^4$?, we need to form an equation

$$\underline{v} = k_1\underline{v}_1 + k_2\underline{v}_2 + k_3\underline{v}_3 + k_4\underline{v}_4, \quad \underline{v} = (a, b, c, d) \in \mathbb{R}^4$$

substituting

Solution cont.

$$k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} + k_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

which gives us

$$\begin{cases} k_1 + k_4 & = a \\ k_2 + 2k_3 & = b \\ k_1 - k_2 + 2k_3 & = c \\ 2k_2 + k_3 + k_4 & = d \end{cases}$$

(Home work), verify that the system has solution for k_1, k_2, k_3 and k_4 for any a, b, c and d. Hence S span $\mathbb{R}^4$ and is a basis for $\mathbb{R}^4$.

Example

Show that the set $S = \{t^2 + 1, t - 1, 2t + 2\}$ is a basis for the vector space P_2.

Solution

Is S independent?, we form an equation

$$c_1(t^2 + 1) + c_2(t - 1) + c_3(2t + 2) = 0$$

$$c_1 t^2 + (c_2 + 2c_3)t + (c_1 - c_2 + 2c_3) = 0 \Longrightarrow \begin{cases} c_1 & = 0 \\ c_2 + 2c_3 & = 0 \\ c_1 - c_2 + 2c_3 & = 0 \end{cases}$$

The only solution to the system is $c_1 = c_2 = c_3 = 0$ **(verify!)**. Therefore S is linearly independent.

Solution cont.

Does S span P_2? Let $p(t) = at^2 + bt + c$ be any polynomial in P_2, we must find constants k_1, k_2 and k_3 s.t

$$at^2 + bt + c = k_1(t^2 + 1) + k_2(t - 1) + k_3(2t + 2)$$

Gives

$$\begin{cases} k_1 & = a \\ k_2 + 2k_3 & = b \\ k_1 - k_2 + 2k_3 & = c \end{cases}$$

Which has solution

$$k_1 = a, \quad k_2 = \frac{a + b - c}{2}, \quad k_3 = \frac{c + b - a}{4}$$

Hence S span P_2

Note

The set of vectors $\{t^n, t^{n-1}, \dots, t, 1\}$ forms a basis for the vector space P_n called the **natural basis** or **standard basis** for P_n

Definition

A vector space V is called **finite-dimensional** if there is a finite subset of V that is a basis for V. If there no such finite subset of V, then V is called **infinite-dimensional**.

Theorem

If $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of the vectors in S.

Proof.

Because S span V, it follows that every vector $\underline{v}$ in V can be written as

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n \quad \text{and}$$
$$\underline{v} = d_1\underline{v}_1 + d_2\underline{v}_2 + \cdots + d_n\underline{v}_n$$

Proof cont.

subtracting the two

$$\underline{0} = (c_1 - d_1)\underline{v}_1 + (c_2 - d_2)\underline{v}_2 + \cdots + (c_n - d_n)\underline{v}_n$$

Since S is linearly independent, it follows that $c_i - d_i = 0, 1 \leq i \leq n$, so $c_i = d_i$, $1 \leq i \leq n$. Hence there is only one way to express $\underline{v}$ as a linear combination of the vectors in S.

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Basis and dimension

Basis from a set of vectors

Let $V = R^m$ and let $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ be set of nonzero vectors in V. The procedure for finding a subset of S that is a basis for $W = \text{span } S$ is as follows

1. Form the equation

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n = \underline{0}$$

2. Construct the augmented matrix associated with homogeneous system of equation above, and transform it to rref
3. The vectors corresponding to the columns containing the leading ones form a basis for $W = \text{span } S$

Example

Let $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5\}$ be a set of vectors in R^4, where
$\underline{v}_1 = (1, 2, -2, 1)$, $\underline{v}_2 = (-3, 0, -4, 3)$, $\underline{v}_3 = (2, 1, 1, -1)$,
$\underline{v}_4 = (-3, 3, -9, 6)$, , $\underline{v}_5 = (9, 3, 7, -6)$

Solution

Form the equation

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} -3 \\ 3 \\ -9 \\ 6 \end{bmatrix} + c_5 \begin{bmatrix} 9 \\ 3 \\ 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution cont

equating corresponding components

$$\begin{cases} c_1 - 3c_2 + 2c_3 - 3c_4 + 9c_5 & = 0 \\ 2c_1 + 0c_2 + c_3 + 3c_4 + 3c_5 & = 0 \\ -2c_1 - 4c_2 + c_3 - 9c_4 + 7c_5 & = 0 \\ c_1 + 3c_2 - c_3 + 6c_4 - 6c_5 & = 0 \end{cases}$$

Form the augmented matrix and transform into rref

$$\begin{bmatrix} 1 & -3 & 2 & -3 & 9 & 0 \\ 2 & 0 & 1 & 3 & 3 & 0 \\ -2 & -4 & 1 & -9 & 7 & 0 \\ 1 & 3 & -1 & 6 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} & -\frac{}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading ones appear in column 1 and 2, so $\{\underline{v}_1, \underline{v}_2\}$ is a basis for $W = \text{span } S$

Basis

Theorem

If $S = \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ is a basis for a vector space V and $T = \{\underline{w}_1, \underline{w}_2, \cdots, \underline{w}_r\}$ is linearly independent set of vectors in V, then $r \leq n$.

Theorem

If $S = \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ and $T = \{\underline{w}_1, \underline{w}_2, \cdots, \underline{w}_m\}$ are bases for a vector space, then $n = m$.

Dimension

The dimension of a nonzero vector space V, denoted by dim V is the number of vectors in a basis for V.

Example

The dimension of R^2 is 2, the dimension of R^3 is 3; and in general of R^n is n

Basis

Example

The dimension of P_2 is 3, the dimension of P_3 is 4.

Theorem

If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there is a basis T for V, which contains S.

Example

Find basis for R^4 that contains the vectors $\underline{v}_1 = (1, 0, 1, 0)$ and $\underline{v}_2 = (-1, 1, -1, 0)$

Solution

Let $\{\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4\}$ be the natural basis for R^4, where
$\underline{e}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T, \underline{e}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T, \underline{e}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$ and
$\underline{e}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$

Example cont

Form the set $S = \{\underline{v}_1, \underline{v}_2, \underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4\}$, since $\{\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4\}$ spans R^4, so does S. To find a subset of S that is a basis for R^4, we form equation

$$c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{e}_1 + c_4\underline{e}_2 + c_5\underline{e}_3 + c_6\underline{e}_4 = \underline{0}$$

which leads to the homogeneous equation

$$\begin{cases} c_1 - c_2 + c_3 & = 0 \\ -c_2 + c_4 & = 0 \\ c_1 - c_2 + c_5 & = 0 \\ c_6 & = 0 \end{cases}$$

Example cont

Write into augmented matrix and transform into rref, gives

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The leading ones appear in the columns 1,2,3 and 6, therefore $S = \{\underline{v}_1, \underline{v}_2, \underline{e}_1, \underline{e}_4\}$ is a basis for R^4 containing $\underline{v}_1$ and $\underline{v}_2$

Row space

The set of all linear combinations of the row vectors of an $m \times n$ matrix is called the **row space** of A and is denoted by Row$A$
Each row has n entries, so Row$A$ is a subspace of R^n.

Basis

Example

Let $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

The row space of A is the subspace of R^5 spanned by $\{\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_3\}$, where $\underline{r}_1 = (-2, -5, 8, 0, -17)$, $\underline{r}_2 = (1, 3, -5, 1, 5)$, $\underline{r}_3 = (3, 11, -19, 7, 1)$, $\underline{r}_4 = (1, 7, -13, 5, -3)$

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as B.

Basis

Example

Find bases for the row space, column space, and the null space of the matrix Let $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

Solution

To find bases for the row space and the column space, we reduce A into echelon form $A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Thus, the first three rows of B form a basis for the row space of B as well as A. Then,

basis for Row $A = \{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$

Example continue

For the column space, observe from B, that the pivots are in columns 1,2 and 4, hence columns 1,2 and 4 of A (not B) form basis for Col A:

$$\text{Basis for Col } A = \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

For Nul A, we need the rref of A, further operations on B, yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example continue

The equation $A\underline{x} = \underline{0}$ is equivalent to $C\underline{x} = \underline{0}$, that is

$$\begin{cases} x_1 + x_3 + x_5 = 0 \\ x_2 - 2x_3 + 3x_5 = 0 \\ x_4 - 5x_5 = 0 \end{cases}$$

So $x_3 = t$, $x_5 = r$ are the free variables. Then

$$\begin{cases} x_4 = 5r \\ x_2 = 2t - 3r \\ x_1 = -t - r \end{cases} \quad t, r \in R \Longrightarrow \underline{x} = t \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

Example continue

$$\text{Basis for Nul}A = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Note

1. Unlike the basis for Col A, the bases for Row A and Nul A have no simple connections with the entries in A itself.
2. dim (Basis for Row$A$) $= 3$, dim (Basis for Col$A$) $= 3$, dim (Basis for Nul$A$) $= 2$

Rank and Nullity

The **rank** of A is the dimension of the column space of A.
The dimension of the null space is also called the **nullity** of A.

Rank theorem

The dimension of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also satisfies the equation

$$\text{rank } A + \text{nullity } A = n$$

Take away

1. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?
2. Could a 6×9 matrix have a two-dimensional null space?

MT127 – Linear Algebra I
Lecture 20 – 2016/2017

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Linear Transformation

Let V and W are vector spaces. **A linear transformation** L of V into W is a function assigning a unique vector $L(\underline{u})$ in W to each $\underline{u}$ in V such that

1. $L(\underline{u} + \underline{v}) = L(\underline{u}) + L(\underline{v})$, for every $\underline{u}$ in V
2. $L(k\underline{u}) = kL(\underline{u})$

NonLinear Transformation

A function T of V into W is said to be nonlinear if it is not linear transformation

Image and range

The vector $L(\underline{u})$ in W is called the **image** of $\underline{u}$
The set of all images in W of the vectors in V is called the **range** of L

Linear Transformation

Image and range

We write that L maps V into W, even if it is not linear as

$$L : V \to W$$

If $V = W$, a linear transformation $L : V \to V$ is called **linear operator** on R^n

Matrix transformation

Let A be an $m \times n$ matrix, we define a matrix transformation as a function $L : R^m \to R^m$ as $L(\underline{u}) = A(\underline{u})$.
If $\underline{u}$ and $\underline{v}$ are vectors in R^n and c is a scalar then

1. $L(\underline{u} + \underline{v}) = A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = L(\underline{u}) + L(\underline{v})$
2. $L(c\underline{u}) = A(c\underline{u}) = cA\underline{u} = cL(\underline{u})$

Hence, every matrix transformation is a linear transformation

Linear Transformation

Common transformations

Reflection with respect to the x-axis: $L : R^2 \to R^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

Projection into the x-y plane: $L : R^3 \to R^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Dilation: $L : R^3 \to R^3$ is defined by $L(\underline{u}) = r\underline{u}$ for $r > 1$
Contraction: $L : R^3 \to R^3$ is defined by $L(\underline{u}) = r\underline{u}$ for $0 < r < 1$

Linear Transformation

Common transformations cont.

Rotation counterclockwise through an angle ϕ: $L : R^2 \to R^2$ is defined by

$$L(\underline{u}) = \begin{bmatrix} \cos\phi & -sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Example

Let $L : R^3 \to R^2$ be defined by $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 1 \\ u_2 - u_3 \end{bmatrix}$. Determine whether L is linear transformation.

Solution

Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then

Common transformations cont.

$$L(\underline{u}+\underline{v}) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = L\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \right) = \begin{bmatrix} (u_1 + v_1) + 1 \\ (u_2 + v_2) - (u_3 + v_3) \end{bmatrix}$$

On the other hand

$$L(\underline{u}) + L(\underline{v}) = \begin{bmatrix} u_1 + 1 \\ u_2 - u_3 \end{bmatrix} + \begin{bmatrix} v_1 + 1 \\ v_2 - v_3 \end{bmatrix} = \begin{bmatrix} (u_1 + v_1) + 2 \\ (u_2 - u_3) + (v_2 - v_3) \end{bmatrix}$$

Since the first coordinates are different, $L(\underline{u} + \underline{v}) \neq \underline{u} + \underline{v}$ and therefore L is not linear transformation

Linear Transformation

Theorem

If $L : V \to W$ is a linear transformation, then

$$L(c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_k\underline{u}_k) = c_1L(\underline{u}_1) + c_2L(\underline{u}_2) + \cdots + c_kL(\underline{u}_k)$$

for any vectors $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k$ in V and any scalars $c_1, c_2, \ldots, c_k$

Theorem

Let $L : V \to W$ be a linear transformation, then

1. $L(\underline{0}_V) = \underline{0}_W$
2. $L(\underline{u} - \underline{v}) = L(\underline{u}) - L(\underline{v})$

Example

The previous example could be solved more easily by using the theorem that if L is linear then $L(\underline{0}_{R^3}) = \underline{0}_{R^2}$ Since

$$L \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

Let $L : R^3 \to R^2$ be linear transformation for which we know that $L(1,0,0) = (2,-1)$, $L(0,1,0) = (3,1)$ and $L(0,0,1) = (-1,2)$. Find $L(-3,4,2)$.

Linear Transformation

Solution

First write $(-3, 4, 2)$ as a linear combination of the given transformed vectors, for this example we have natural basis, otherwise must be computed, then

$$\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then

$$L\left(\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} \right) = -3L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + 4L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + 2L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$
$$= -3 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

Theorem

Let $L : V \to W$ be a linear transformation of an $n-$dimensional vector space V into a vector space W. Also, let $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ be basis for V. If $\underline{u}$ is any vector in V, then $L(\underline{u})$ is completely determined by $S = \{L(\underline{v}_1), L(\underline{v}_2), \ldots, L(\underline{v}_n)\}$.

Proof

Since $\underline{u}$ is in V, we can write

$$\underline{u} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n$$

where $c_1, c_2, \ldots, c_n$ are uniquely determined scalars. Then using properties of linear transformation, we have

$$L(\underline{u}) = L(c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n) = c_1L(\underline{v}_1) + c_2L(\underline{v}_2) + \cdots + c_nL(\underline{v}_n)$$

Example

Let $L : P_1 \to p_2$ be a linear transformation for which we know that

$$L(t+1) = t^2 - 1 \quad \text{and} \quad L(t-1) = t^2 + t$$

What is $L(7t+3)$ and $L(at+b)$

Solution

First note that $\{t+1, t-1\}$ form basis for P_1 (**verify!**)
Then we find that (**verify!**)

$$7t + 3 = 5(t+1) + 2(t-1)$$

Then

$$\begin{aligned} L(7t+3) &= L\big(5(t+1) + 2(t-1)\big) \\ &= 7L(t+1) + 2L(t-1) \\ &= 5(t^2-1) + 2(t^2+1) = 7t^2 + 2t - 5 \end{aligned}$$

Linear Transformation

One to one

A linear transformation $L : V \to W$ is said to be **one-to-one** if for all $\underline{v}_1, \underline{v}_2$ in V, $\underline{v}_1 \neq \underline{v}_2$ implies that $L(\underline{v}_1) \neq L(\underline{v}_2)$. Equivalently
L is one-to-one if for all $\underline{v}_1, \underline{v}_2$ in V, $L(\underline{v}_1) = L(\underline{v}_2)$ implies that $\underline{v}_1 = \underline{v}_2$

Example

Determine whether the operation $L : R^2 \to R^2$ be defined by

$$L(x, y) = (x + y, x - y)$$

is one to one.

Solution

Let $\underline{v}_1 = (a_1, a_2), \underline{v}_2 = (b_1, b_2)$, then if $L(\underline{v}_1) = L(\underline{v}_2)$, we have

$$a_1 + a_2 = b_1 + b_2$$
$$a_1 - a_2 = b_1 - b_2$$

Solution cont.

Adding the two equations

$$2a_1 = 2a_2 \implies a_1 = a_2$$

Also

$$b_1 = b_2$$

Hence, L is one-to-one.

Example

Let $L : R^3 \to R^2$ be linear transformation defined by $L(x, y, z) = (x, y)$. Is L one-to-one?

Solution

Since $L(1, 3, 2) = L(1, 3, -10) = (1, 3)$, we conclude that L is not one-to-one.

MT127 – Linear Algebra I
Lecture 21 – 2016/2017

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Kernel and range of a Linear Transformation

Kernel

Let $L : V \to W$ be a linear transformation. The **kernel** of L, ker L, is the subset of V consisting of all vectors $\underline{v}$ such that $L(\underline{v}) = \underline{0}_W$

Kernel is not empty

Ker L is never empty! (why?). Since $\underline{0}_V$ is in ker L.

Example 1

Use the previous example; $L : R^3 \to R^2$ be linear transformation defined by $L(x, y, z) = (x, y)$. Find ker L.

Solution

To find ker L, we must determine all $\underline{x}$ in R^3 such that $L(\underline{x}) = \underline{0}$.
Let $\underline{x} = (x_1, x_2, x_3)$, then

$$L(\underline{x}) = L(x_1, x_2, x_3) = \underline{0} = (0, 0)$$

Example cont.

but $L(\underline{x}) = (x_1, x_2)$, thus $(x_1, x_2) = (0, 0) \implies x_1 = 0, x_2 = 0$, x_3 can take any real number. Hence ker L consists of all vectors in R^3 of the form $(0, 0, r)$, where $r \in R$

Example 2

Assume the transformation is given by $L : R^2 \to R^2$ be defined by $L(x, y) = (x + y, x - y)$. Find ker L.

Solution

ker L consists of all vectors in R^2 such that $L(\underline{x}) = \underline{0}$. We must solve the linear system

$$\begin{cases} x + y = 0 \\ x - y = 0 \end{cases} \implies x = 0, y = 0$$

so ker $L = \{\underline{0}\}$

Example 3

Find the ker L if $L : R^4 \to R^2$ is defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+y \\ z+w \end{bmatrix}$$

Solution

ker L is the set of all vectors $\underline{u}$ in R^4 such that $L(\underline{u}) = \underline{0}$.
This leads to the system

$$\begin{cases} x+y=0 \\ z+w=0 \end{cases} \implies \underline{u} = r\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Example cont.

where r and s are any real numbers. So the ker L consists of all linear combinations of $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, a subspace of R^4.

Theorem

If $L : V \to W$ is a linear transformation, then ker L is a subspace of V.

Proof

First observe that ker L is not an empty set, since $\underline{0}_V \in$ ker L.
Let $\underline{u}$ and $\underline{v}$ be in ker L, then since L is a linear transformation

$$L(\underline{u} + \underline{v}) = L(\underline{u}) + L(\underline{v}) = \underline{0}_W + \underline{0}_W = \underline{0}_W$$

so $\underline{u} + \underline{v}$ is in ker L.

Proof cont.

If c is a scalar, then since L is a linear transformation

$$L(c\underline{u}) = cL(\underline{u}) = c\underline{0}_W = \underline{0}_W$$

Hence $c\underline{u}$ is in $\ker L$. Hence $\ker L$ is a subspace of V.

Example 4

If L is given in Example 2, then $\ker L$ is the subspace $\{\underline{0}\}$, **its dimension is zero**.

Example 5

If L is given in Example 1, then basis for $\ker L$ is $(0, 0, 1)$ and $\dim(\ker L) = 1$

Example 6

If L is given in Example 3, then basis for ker L consists of the vectors
$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, then $\dim(\ker L) = 2$.

Application of ker L

An examination of the elements in ker L allows us to decide whether L is one-to-one or not.

Theorem

A linear transformation $L : V \to W$ is one-to-one if and only if $\ker L = \{\underline{0}_W\}$.

Example 7

The linear transformation in Example 2 is one-to-one, while that in Example 1 is not.

Range

If $L : V \to W$ is a linear transformation, then the **range** of L, denoted by range L, is the set of all vectors in W that are images, under L, of vectors in V.
Thus a vector $\underline{w}$ is in range L if there exists some vector $\underline{v}$ in V such that $L(\underline{v}) = \underline{w}$.

Onto

If range $L = W$, we say that L is **onto**. That is, L is onto if and only if, given any $\underline{w}$ in W, there is a $\underline{v}$ in V such that $L(\underline{v}) = \underline{w}$.

Theorem

If $L : V \to W$ is a linear transformation, then range L is subspace of W.

Proof

First, observe that range L is not an empty set, since $\underline{0}_W = L(\underline{0}_V)$, so $\underline{0}_W$ is in range L.

Proof cont.

Let $\underline{w}_1$ and $\underline{w}_2$ be in range L, then $\underline{w}_1 = L(\underline{v}_1)$ and $\underline{w}_2 = L(\underline{v}_2)$ for some $\underline{v}_1$ and $\underline{v}_2$ in V, now

$$\underline{w}_1 + \underline{w}_2 = L(\underline{v}_1) + L(\underline{v}_2) = L(\underline{v}_1 + \underline{v}_1)$$

$\implies \underline{w}_1 + \underline{w}_2$ is in range L
If c is a scalar, then

$$c\underline{w}_1 = cL(\underline{v}_1) = L(c\underline{v}_1)$$

$\implies c\underline{w}_1$ is in range L. Hence range L is a subspace of W.

Example

Let $L : R^3 \to R^3$ be defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Example cont.

1. Is L onto?
2. Find basis for range L
3. Find ker L
4. Is L one-to-one?

Solution

1. Given any vector $\underline{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in R^3, where a, b, c are any real numbers. Can we find vector $\underline{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ in R^3 so that $L(\underline{v}) = \underline{w}$?

Solution cont.

We seek a solution to the linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

we find rref of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a \end{bmatrix}$$

Thus a solution exists only when $c - b - a = 0$. Hence L is not onto. That means there exist values a, b, c for which there is no vector $\underline{v}$ in R^3 such that $L(\underline{v}) = [a\ b\ c]^T$.

Solution cont.

2. To find basis for range L, we note that

$$L\left(\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\right)=\begin{bmatrix}1&0&1\\1&1&2\\2&1&3\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}a_1+a_3\\a_1+a_2+2a_3\\2a_1+a_2+3a_3\end{bmatrix}$$
$$=a_1\begin{bmatrix}1\\1\\2\end{bmatrix}+a_2\begin{bmatrix}0\\1\\1\end{bmatrix}+a_3\begin{bmatrix}1\\2\\3\end{bmatrix}$$

That means $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ spans range L. That is, range L is the subspace of R^3 spanned by the columns of the matrix defining L.

basis for range $L=\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ **WHY?** and $\dim(\text{range } L)=2$

Solution cont.

3. To find $\ker L$, we wish to find all $\underline{v}$ in R^3 so that $L(\underline{v}) = \underline{0}_{R^3}$. Solving the resulting homogeneous system,

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we find that **(verify!)**

$$\underline{v} = \begin{bmatrix} -r \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad r \in R$$

and that, $\dim(\ker L) = 1$

Solution cont.

4. Since $\ker L \neq \{0\}$, it follows that L is not one-to-one.

Theorem

If $L : V \to W$ is a linear transformation of an $n-$dimensional vector space V into a vector space W, then

$$\dim(\ker L) + \dim(\text{range } L) = \dim V$$

Theorem

The dimension of $\ker L$ is also called the nullity of L, and the dimension of range L is called the rank of L.

Example

Let $L : P_2 \to P_2$ be the linear transformation defined by

$$L(at^2 + bt + c) = (a + 2b)t + (b + c)$$

1. Is $-4t^2 + 2t - 2$ in ker L?
2. Is $t^2 + 2t + 1$ in range L?
3. Find basis for ker L
4. Is L one-to-one?
5. Find basis for range L
6. Is L onto?
7. Verify the theorem that

$$\dim(\ker L) + \dim(\text{range } L) = \dim V$$

Solution

1. Since $L(-4t^2 + 2t - 2) = (-4 + 2 \times 2)t + (-2 + 2) = 0$
$\Longrightarrow -4t^2 + 2t - 2 \in \ker L$

2. The vector $t^2 + 2t + 1$ is in range L if we can find a vector $at^2 + bt + c$ in P_2 st

$$L(at^2 + bt + c) = t^2 + 2t + 1 = (a + 2b)t + (b + c)$$

Equating corresponding coefficients, we have

$$\begin{cases} 0 = 1 \\ a + 2b = 2 \\ b + c = 1 \end{cases}$$

The system has no solution, and therefore the given vector is not in range L

Solution cont

3.) The vector $L(at^2 + bt + c)$ is in ker L if

$$L(at^2 + bt + c) = \underline{0}$$

that is, if

$$(a + 2b)t + (b + c) = 0$$

$$\implies \begin{cases} a + 2b = 0 \\ b + c = 0 \end{cases}$$

Transform the augmented matrix into rref, we find (verify)

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \implies \underline{v} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \ t \in R$$

so a basis for ker L is $\{2t^2 - t + 1\}$

Solution cont

4.) Since ker L does not contain only of the zero vector, L is not one-to-one.

5.) Every vector in range L has the form

$$(a + 2b)t + (b + c)$$

so the vectors $\{t, 1\}$ span range L. Since these vectors are linearly independent, they form basis for range L.

6.) The dimension of P_2 is 3, while range L is a subspace of P_2 of dimension 2, so range $L \neq P_2$. Hence L is not onto.

7.) From part **3.)** and from part **5.)** we have

$$\dim(\ker\ L) = 1, \qquad \dim(\text{range}\ L) = 2$$

so

$$3 = \dim\ P_2 = \dim(\ker\ L) + \dim(\text{range}\ L)$$

MT127 – Linear Algebra I
Lecture 22 – 2016/2017

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Inner-Product Spaces

Inner-Product Spaces

An inner product on a real vector space V is a function that assigns a real number $\langle x, y \rangle$, to each pair of vectors $\underline{u}$ and $\underline{v}$ in V, and that satisfies these properties

1. $\langle \underline{u}, \underline{u} \rangle \geq 0$ and $\langle \underline{u}, \underline{u} \rangle = 0$ if and only if $\underline{u} = \underline{0}$
2. $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$
3. $\langle a\underline{u}, \underline{v} \rangle = a\langle \underline{v}, \underline{u} \rangle$
4. $\langle \underline{u}, \underline{v} + \underline{w} \rangle = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$

Dot product

The dot product in R^n is an inner product in the sense of the above definition, where $\langle \underline{x}, \underline{y} \rangle = \underline{x}^T \underline{y}$

Inner-Product Spaces

Example

Let V be the vector space $\mathbb{R}^2$, and let A be 2×2 matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$.

Verify that the function $\langle \underline{u}, \underline{v} \rangle = \underline{u}^T A \underline{v}$ is an inner product for $\mathbb{R}^2$.

Solution

We need to verify the four properties of the inner product. Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$, then

$$\begin{aligned} \langle \underline{u}, \underline{u} \rangle = \underline{u}^T A \underline{u} &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= 3u_1^2 + 4u_1 u_2 + 4u_2^2 \\ &= 2u_1^2 + (u_1 + 2u_2)^2 \geq 0 \end{aligned}$$

Solution cont.

and $\langle \underline{u}, \underline{v} \rangle = 0$ if and only if $u_1 = u_2 = 0$, i.e $\underline{u} = \underline{0}$ (**Prop. 1 OK**)
Note that A is symmetric, that is $A = A^T$, also observe that if $\underline{u}$ and $\underline{v}$ are in $\mathbb{R}^2$, then $\underline{u}^T A \underline{v}$ is a 1×1 matrix, so

$$(\underline{u}^T A \underline{v})^T = \underline{u}^T A \underline{v}, \quad \text{then}$$

$$\langle \underline{u}, \underline{v} \rangle = \underline{u}^T A \underline{v} = (\underline{u}^T A \underline{v})^T = \underline{v}^T A^T (\underline{u}^T)^T = \underline{v}^T A^T \underline{u} = \underline{v}^T A \underline{u} = \langle \underline{v}, \underline{u} \rangle$$

(Prop 2 OK)
$\langle a\underline{u}, \underline{v} \rangle = (a\underline{u})^T A \underline{v} = a(\underline{u}^T A \underline{v}) = a\langle \underline{u}, \underline{v} \rangle$ **(Prop 3 OK)**
$\langle \underline{u}, \underline{v} + \underline{w} \rangle = \underline{u}^T A(\underline{v} + \underline{w}) = \underline{u}^T A \underline{v} + \underline{u}^T A \underline{w} = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$ **(Prop 4 OK)**
Therefore, $\langle \underline{u}, \underline{v} \rangle$ is an inner product for $\mathbb{R}^2$.

Example

For $p(t)$ and $q(t)$ in P_2, verify that

$$\langle p, q\rangle = \int_0^1 p(t)q(t)dt$$

is an inner product.

Solution

Start with property 1, Let $p(t), q(t), r(t)$ are in P_2

$$\langle p, p\rangle = \int_0^1 p(t)^2 dt \geq 0$$

and that $\langle p, p\rangle = 0$ if and only if $p(t) = 0$, $0 \leq t \leq 1$ **(Prop. 1 OK)**
$\langle p, q\rangle = \int_0^1 p(t)q(t)dt = \int_0^1 q(t)p(t)dt = \langle q, p\rangle$ **(Prop. 2 OK)**
$\langle ap, q\rangle = \int_0^1 (ap(t))q(t)dt = a\int_0^1 p(t)q(t)dt = a\langle p, q\rangle$ **(Prop. 3 OK)**

Solution cont.

$$
\begin{aligned}
\langle p, q + r\rangle = \int_0^1 p(t)[q(t) + r(t)]dt &= \int_0^1 \Big(p(t)q(t) + p(t)r(t)\Big)\,dt \\
&= \int_0^1 p(t)q(t)dt + \int_0^1 p(t)r(t)dt \\
&= \langle p, q\rangle + \langle p, r\rangle
\end{aligned}
$$

(**Prop. 4 OK**)
Therefore $\langle p, q\rangle$ is an inner product.

Inner-Product Spaces

Inner product space

A vector space with an inner product is called an **inner-product space**.
In $\mathbb{R}^n$, we can use the inner-product as a measure of size.
If V is an inner-product space, then for each $\underline{v}$ in V we define $||\underline{v}||$ (**the norm** of $\underline{v}$) as

$$||\underline{v}|| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$$

Note that $\langle \underline{v}, \underline{v} \rangle \geq 0$ for all $\underline{v} \in V$, so the norm function is always defined.

Example

Use the inner product for P_2 defined in previous example to determine $||t^2||$.

Solution

$$||t^2|| = \sqrt{\langle t^2, t^2 \rangle} = \sqrt{\int_0^1 t^2 t^2 dt} = \sqrt{\int_0^1 t^4 dt} = \frac{1}{\sqrt{5}}$$

Orthogonal bases

If $\underline{u}$ and $\underline{v}$ are vectors in an inner-product space V, we say that $\underline{u}$ and $\underline{v}$ are **orthogonal** if $\langle \underline{u}, \underline{v} \rangle = 0$. Similarly, $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is an **orthogonal set** in V if $\langle \underline{v}_i, \underline{v}_j \rangle = 0$ when $i \neq j$. In addition, if an orthogonal set of vectors S is a basis for V, we call S an **orthogonal basis**.

Orthonormal basis

Let W be a subspace of V. A basis $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ for W is **orthonormal** if

1. $\langle \underline{v}_i, \underline{v}_j \rangle = 0$ for $i \neq j$, (Mutually perpendicular)
2. $\langle \underline{v}_i, \underline{v}_i \rangle = 1$ (Length 1)

Inner-Product Spaces

Theorem

Let $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ be an orthogonal basis for an inner product space V. IF $\underline{u}$ is any vector in V, then

$$\underline{u} = \frac{\langle \underline{v}_1, \underline{u} \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle}\underline{v}_1 + \frac{\langle \underline{v}_2, \underline{u} \rangle}{\langle \underline{v}_2, \underline{v}_2 \rangle}\underline{v}_2 + \cdots + \frac{\langle \underline{v}_n, \underline{u} \rangle}{\langle \underline{v}_n, \underline{v}_n \rangle}\underline{v}_n$$

Gram-Schmidt Orthogonalization

Let V be an inner-product space, and let $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n\}$ be basis for V. Let $\underline{v}_1 = \underline{u}_1$, and for $2 \leq k \leq n$ define $\underline{v}_k$ by

$$\underline{v}_k = \underline{u}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{u}_k, \underline{v}_j \rangle}{\langle \underline{v}_j, \underline{v}_j \rangle}\underline{v}_j$$

Then $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is an orthogonal basis for V.

Example

Let the inner product on P_2 be the one given in previous example. Starting with the natural basis $\{1, x, x^2\}$, use Gram-Schmidt orthogonalization to obtain an orthogonal basis for P_2.

Solution

If we let $\{p_0, p_1, p_2\}$ denote the orthogonal basis, we have $p_0(x) = 1$ and find $p_1(x)$ from

$$p_1(x) = u_1(x) - \frac{\langle \underline{p}_0, \underline{u}_1 \rangle}{\langle \underline{p}_0, \underline{p}_0 \rangle} \underline{p}_0(x)$$

we have $\underline{p}_0 = 1, \underline{u}_1 = x$, $\langle \underline{p}_0, \underline{u}_1 \rangle = \langle 1, x \rangle = \int_0^1 x dx = 1/2$, and $\langle \underline{p}_0, \underline{p}_0 \rangle = \langle 1, 1 \rangle = \int_0^1 dx = 1$, so

$$p_1(x) = x - \frac{1}{2}/1 = x - \frac{1}{2}$$

Solution cont.

Then $p_2(x)$ is obtained by

$$p_2(x) = u_2(x) - \frac{\langle \underline{p}_1, \underline{u}_2 \rangle}{\langle \underline{p}_1, \underline{p}_1 \rangle} \underline{p}_1(x) - \frac{\langle \underline{p}_0, \underline{u}_2 \rangle}{\langle \underline{p}_0, \underline{p}_0 \rangle} \underline{p}_0(x)$$

we have the following:
$u_2(x) = x^2$, and

$$\langle \underline{p}_1, \underline{u}_2 \rangle = \langle x - \frac{1}{2}, x^2 \rangle = \int_0^1 (x^3 - x^2/2)dx = 1/12$$

$$\langle \underline{p}_1, \underline{p}_1 \rangle = \langle x - \frac{1}{2}, x - \frac{1}{2} \rangle = \int_0^1 (x^2 - x + 1/4)dx = 1/12$$

$$\langle \underline{p}_0, \underline{u}_2 \rangle = \langle 1, x^2 \rangle = = \int_0^1 x^2 dx = 1/3$$

Solution cont.

Therefore,

$$p_2(x) = x^2 - p_1(x) - \frac{p_0(x)}{3} = x^2 - \left(x + \frac{1}{2}\right) - \frac{1}{3}$$
$$= x^2 - x + \frac{1}{6}$$

and $\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$ is an orthogonal basis for P_2 with respect to the inner product.

END OF LECTURES

MT127 – Linear Algebra I
Lecture 23 – 2017/2018

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February 4, 2018

Kernel and range of a Linear Transformation

Kernel

Let $L : V \to W$ be a linear transformation. The **kernel** of L, ker L, is the subset of V consisting of all vectors $\underline{v}$ such that $L(\underline{v}) = \underline{0}_W$

Kernel is not empty

Ker L is never empty! (why?). Since $\underline{0}_V$ is in ker L.

Example 1

Use the previous example; $L : \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformation defined by $L(x, y, z) = (x, y)$. Find ker L.

Solution

To find ker L, we must determine all $\underline{x}$ in $\mathbb{R}^3$ such that $L(\underline{x}) = \underline{0}$.
Let $\underline{x} = (x_1, x_2, x_3)$, then

$$L(\underline{x}) = L(x_1, x_2, x_3) = \underline{0} = (0, 0)$$

Example cont.

but $L(\underline{x}) = (x_1, x_2)$, thus $(x_1, x_2) = (0, 0) \Longrightarrow x_1 = 0, x_2 = 0$, x_3 can take any real number. Hence ker L consists of all vectors in $\mathbb{R}^3$ of the form $(0, 0, r)$, where $r \in \mathbb{R}$

Example 2

Assume the transformation is given by $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L(x, y) = (x + y, x - y)$. Find ker L.

Solution

ker L consists of all vectors in $\mathbb{R}^2$ such that $L(\underline{x}) = \underline{0}$. We must solve the linear system

$$\begin{cases} x + y = 0 \\ x - y = 0 \end{cases} \Longrightarrow x = 0, y = 0$$

so ker $L = \{\underline{0}\}$

Example 3

Find the ker L if $L : \mathbb{R}^4 \to \mathbb{R}^2$ is defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+y \\ z+w \end{bmatrix}$$

Solution

ker L is the set of all vectors $\underline{u}$ in $\mathbb{R}^4$ such that $L(\underline{u}) = \underline{0}$.
This leads to the system

$$\begin{cases} x+y=0 \\ z+w=0 \end{cases} \implies \underline{u} = r\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Example cont.

where r and s are any real numbers. So the ker L consists of all linear combinations of $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, a subspace of $\mathbb{R}^4$.

Theorem

If $L : V \to W$ is a linear transformation, then ker L is a subspace of V.

Proof

First observe that ker L is not an empty set, since $\underline{0}_V \in$ ker L.
Let $\underline{u}$ and $\underline{v}$ be in ker L, then since L is a linear transformation

$$L(\underline{u} + \underline{v}) = L(\underline{u}) + L(\underline{v}) = \underline{0}_W + \underline{0}_W = \underline{0}_W$$

so $\underline{u} + \underline{v}$ is in ker L.

Proof cont.

If c is a scalar, then since L is a linear transformation

$$L(c\underline{u}) = cL(\underline{u}) = c\underline{0}_W = \underline{0}_W$$

Hence $c\underline{u}$ is in ker L. Hence ker L is a subspace of V.

Example 4

If L is given in Example 2, then ker L is the subspace $\{\underline{0}\}$, **its dimension is zero**.

Example 5

If L is given in Example 1, then basis for ker L is $(0, 0, 1)$ and $\dim(\ker L) = 1$

Example 6

If L is given in Example 3, then basis for ker L consists of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, then $\dim(\ker L) = 2$.

Application of ker L

An examination of the elements in ker L allows us to decide whether L is one-to-one or not.

Theorem

A linear transformation $L : V \to W$ is one-to-one if and only if $\ker L = \{\underline{0}_W\}$.

Example 7

The linear transformation in Example 2 is one-to-one, while that in Example 1 is not.

Range

If $L : V \to W$ is a linear transformation, then the **range** of L, denoted by range L, is the set of all vectors in W that are images, under L, of vectors in V.
Thus a vector $\underline{w}$ is in range L if there exists some vector $\underline{v}$ in V such that $L(\underline{v}) = \underline{w}$.

Onto

If range $L = W$, we say that L is **onto**. That is, L is onto if and only if, given any $\underline{w}$ in W, there is a $\underline{v}$ in V such that $L(\underline{v}) = \underline{w}$.

Theorem

If $L : V \to W$ is a linear transformation, then range L is subspace of W.

Proof

First, observe that range L is not an empty set, since $\underline{0}_W = L(\underline{0}_V)$, so $\underline{0}_W$ is in range L.

Proof cont.

Let $\underline{w}_1$ and $\underline{w}_2$ be in range L, then $\underline{w}_1 = L(\underline{v}_1)$ and $\underline{w}_2 = L(\underline{v}_2)$ for some $\underline{v}_1$ and $\underline{v}_2$ in V, now

$$\underline{w}_1 + \underline{w}_2 = L(\underline{v}_1) + L(\underline{v}_2) = L(\underline{v}_1 + \underline{v}_1)$$

$\Longrightarrow \underline{w}_1 + \underline{w}_2$ is in range L
If c is a scalar, then

$$c\underline{w}_1 = cL(\underline{v}_1) = L(c\underline{v}_1)$$

$\Longrightarrow c\underline{w}_1$ is in range L. Hence range L is a subspace of W.

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Example cont.

1. Is L onto?
2. Find basis for range L
3. Find ker L
4. Is L one-to-one?

Solution

1. Given any vector $\underline{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$, where a, b, c are any real numbers. Can we find vector $\underline{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ in $\mathbb{R}^3$ so that $L(\underline{v}) = \underline{w}$?

Solution cont.

We seek a solution to the linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

we find rref of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a \end{bmatrix}$$

Thus a solution exists only when $c - b - a = 0$. Hence L is not onto. That means there exist values a, b, c for which there is no vector $\underline{v}$ in $\mathbb{R}^3$ such that $L(\underline{v}) = [a\ b\ c]^T$.

Solution cont.

2. To find basis for range L, we note that

$$L\left(\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\right)=\begin{bmatrix}1&0&1\\1&1&2\\2&1&3\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}a_1+a_3\\a_1+a_2+2a_3\\2a_1+a_2+3a_3\end{bmatrix}$$

$$=a_1\begin{bmatrix}1\\1\\2\end{bmatrix}+a_2\begin{bmatrix}0\\1\\1\end{bmatrix}+a_3\begin{bmatrix}1\\2\\3\end{bmatrix}$$

That means $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ spans range L. That is, range L is the subspace of $\mathbb{R}^3$ spanned by the columns of the matrix defining L.

basis for range $L=\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ **WHY?** and $\dim(\text{range } L)=2$

Solution cont.

3. To find ker L, we wish to find all $\underline{v}$ in $\mathbb{R}^3$ so that $L(\underline{v}) = \underline{0}_{\mathbb{R}^3}$. Solving the resulting homogeneous system,

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we find that **(verify!)**

$$\underline{v} = \begin{bmatrix} -r \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}$$

and that, $\dim(\ker L) = 1$

Solution cont.

4. Since $\ker L \neq \{0\}$, it follows that L is not one-to-one.

Theorem

If $L : V \to W$ is a linear transformation of an $n-$dimensional vector space V into a vector space W, then

$$\dim(\ker L) + \dim(\text{range } L) = \dim V$$

Theorem

The dimension of $\ker L$ is also called the nullity of L, and the dimension of range L is called the rank of L.

Example

Let $L : P_2 \to P_2$ be the linear transformation defined by

$$L(at^2 + bt + c) = (a + 2b)t + (b + c)$$

1. Is $-4t^2 + 2t - 2$ in ker L?
2. Is $t^2 + 2t + 1$ in range L?
3. Find basis for ker L
4. Is L one-to-one?
5. Find basis for range L
6. Is L onto?
7. Verify the theorem that

$$\dim(\ker L) + \dim(\text{range } L) = \dim V$$

Solution

1. Since $L(-4t^2 + 2t - 2) = (-4 + 2 \times 2)t + (-2 + 2) = 0$ $\Longrightarrow -4t^2 + 2t - 2 \in \ker L$
2. The vector $t^2 + 2t + 1$ is in range L if we can find a vector $at^2 + bt + c$ in P_2 st

$$L(at^2 + bt + c) = t^2 + 2t + 1 = (a + 2b)t + (b + c)$$

Equating corresponding coefficients, we have

$$\begin{cases} 0 = 1 \\ a + 2b = 2 \\ b + c = 1 \end{cases}$$

The system has no solution, and therefore the given vector is not in range L

Solution cont

3.) The vector $L(at^2 + bt + c)$ is in ker L if

$$L(at^2 + bt + c) = \underline{0}$$

that is, if

$$(a + 2b)t + (b + c) = 0$$

$$\Longrightarrow \begin{cases} a + 2b = 0 \\ b + c = 0 \end{cases}$$

Transform the augmented matrix into rref, we find (verify)

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Longrightarrow \underline{v} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

so a basis for ker L is $\{2t^2 - t + 1\}$

Solution cont

4.) Since ker L does not contain only of the zero vector, L is not one-to-one.

5.) Every vector in range L has the form

$$(a+2b)t+(b+c)$$

so the vectors $\{t, 1\}$ span range L. Since these vectors are linearly independent, they form basis for range L.

6.) The dimension of P_2 is 3, while range L is a subspace of P_2 of dimension 2, so range $L \neq P_2$. Hence L is not onto.

7.) From part **3.)** and from part **5.)** we have

$$\dim(\ker\ L) = 1, \qquad \dim(\text{range}\ L) = 2$$

so

$$3 = \dim\ P_2 = \dim(\ker\ L) + \dim(\text{range}\ L)$$

MT127 – Linear Algebra I
Lecture 24 – 2017/2018

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Matrix of a linear transformation

Theorem

Let $L : V \to W$ be a linear transformation of an n-dim vector space V into an m-dim vector space W $(n \neq 0, m \neq 0)$ and let $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ and $T = \{\underline{w}_1, \underline{w}_2, \dots, \underline{w}_m\}$ be basis for V and W, respectively. Then the $m \times n$ matrix A, whose jth column is the coordinate vector $[L(\underline{v}_j)]_T$ of $L(\underline{v}_j)$ with respect to T, is associated with L and has the following property: if $\underline{x} \in V$, then

$$[L(\underline{x})]_T = A[\underline{x}]_S$$

where $[\underline{x}]_S$ and $[L(\underline{x})]_T$ are the coordinate vectors of $\underline{x}$ and $L(\underline{x})$ with respect to the respective bases S and T. Moreover, A is the only matix with this property.

Definition

The matrix A is called the matrix respresting L with respect to the bases S and T, or the matrix of L with respect to S and T.

Matrix of a linear transformation

Summary

The procedure for computing the matrix A of $L : V \to W$ with respect to bases $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ and $T = \{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_n\}$ for V and W:

1. Compute $L(\underline{v}_j)$ for $j = 1, 2, \ldots, n$
2. Find coordinate vector $[L(\underline{v}_j)]_T$ of $L(\underline{v}_j)$ with respect to the basis T
3. The matrix A of L with respect to S and T is formed by choosing $[L(\underline{v}_j)]_T$ as the jth column of A.

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \end{bmatrix}$$

Let $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ and $T = \{\underline{w}_1, \underline{w}_2\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, where

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \underline{w}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

1. Find the matrix of L with respect to S and T.
2. Verify $[L(\underline{x})]_T = A[\underline{x}]_S$, using $\underline{x} = (1, 6, 3)$

Solution

We have

$$L(\underline{v}_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, L(\underline{v}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, L(\underline{v}_3) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

To find the coordinate vectors $[L(\underline{v}_1)]_T$, $[L(\underline{v}_2)]_T$ and $[L(\underline{v}_3)]_T$, we write

$$L(\underline{v}_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = a_1\underline{w}_1 + a_2\underline{w}_2 = a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$L(\underline{v}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = b_1\underline{w}_1 + b_2\underline{w}_2 = b_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$L(\underline{v}_3) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1\underline{w}_1 + c_2\underline{w}_2 = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The 3 linear systems have same coefficient matrix, we can solve at once

Matrix of a linear transformation

Solution

The augmented matrix with its rref are

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -1 & -\frac{2}{3} & -\frac{4}{3} \end{bmatrix}$$

Hence the matrix A of L with respect to S and T is

$$A = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & -\frac{2}{3} & -\frac{4}{3} \end{bmatrix}$$

Now, we have

$$[L(\underline{x})]_T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & -\frac{2}{3} & -\frac{4}{3} \end{bmatrix} [\underline{x}]_S$$

Given $\underline{x} = (1, 6, 3)$, then $L(\underline{x}) = \begin{bmatrix} 1+6 \\ 6-3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

Solution

To Verify, we compute $[\underline{x}]_S = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ **(verify!)**. Then,

$$[L(\underline{x})]_T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & -\frac{2}{3} & -\frac{4}{3} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ -\frac{11}{3} \end{bmatrix}$$

then

$$L(\underline{x}) = \frac{10}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{11}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

which agrees with the previous value for $L(\underline{x})$

Matrix of a linear transformation

Example 2

Let $L : P_1 \to P_2$ be defined by $L[p(t)] = tp(t)$.

1. Find the matrix of L with respect to the bases $S = \{t, 1\}$ and $T = \{t^2, t-1, t+1\}$ for P_1 and P_2, respectively
2. If $p(t) = 3t - 2$, compute $L[p(t)]$ using the matrix obtained in 1.

Solution

We have **(verify!)**

$$L(t) = t^2 = 1(t^2) + 0(t-1) + 0(t+1) \Longrightarrow [L(t)]_T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$L(1) = t = 0(t^2) + \frac{1}{2}(t-1) + \frac{1}{2}(t+1) \Longrightarrow [L(1)]_T = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Solution cont.

Then, the matrix of L with respect to S and T is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Now, we compute $[L(p(t))]_T$, using

$$[L(p(t))]_T = A[p(t)]_S = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

Hence

$$L(p(t)) = 3t^2 + (-1)(t-1) + (-1)(t+1) = 3t^2 - 2t$$

Theorem

Let $L : V \to V$ be linear operator, where V is an n-dimensional vector space. Let $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ and $T = \{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_n\}$ be bases for V and let P be the transition matrix from T to S. If A is the matrix representing L with respect to S, then $P^{-1}AP$ is the matrix representing L with respect to the basis T.

Proof

If P is the transition matrix from T to S, and $\underline{x} \in V$, then

$$[\underline{x}]_S = P[\underline{x}]_T \tag{1}$$

where jth column of P is the coordinate vector $[\underline{w}_j]_S$ of $\underline{w}_j$ to S-basis. We know that P^{-1} is the transition matrix from S to T, whose jth column is the coordinate vector $[\underline{v}_j]_T$ of $\underline{v}_j$ to T-basis, then for $\underline{y} \in V$,

$$[\underline{y}]_T = P^{-1}[\underline{y}]_S \tag{2}$$

Proof cont.

If A is the matrix representing L with respect to S, then

$$[L(\underline{x})]_S = A[\underline{x}]_S, \quad \text{for } \underline{x} \in V \tag{3}$$

Substituting $\underline{y} = L(\underline{x})$ in 2, we have

$$[L(\underline{x})]_T = P^{-1}[L(\underline{x})]_S$$

Using first 3 and then 1 to this last equation, we obtain

$$[L(\underline{x})]_T = P^{-1}[L(\underline{x})]_S = P^{-1}A[\underline{x}]_S = P^{-1}AP[\underline{x}]_T$$

The equation

$$[L(\underline{x})]_T = P^{-1}AP[\underline{x}]_T$$

means that $B = P^{-1}AP$ is the matrix representing L with respect to T.

Example 3

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 \\ a_1 - 2a_2 \end{bmatrix}$$

Let

$$S = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \text{and } T = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

be bases for $\mathbb{R}^2$.

1. Find a matrix L representing L with respect to S
2. Use the matrix above to find the matrix of L with respect to T
3. Verify the above result using direct method

Solution

Since the S basis is natural, we can easily find (verify), that

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

as the matrix representing L with respect to S
The transition matrix P from T to S is given by **(verify!)**

$$P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

The transition matrix from S to T is P^{-1},

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

Change of basis $\Longrightarrow$ new matrix of L

Solution cont.

Then the matrix B representing L with respect to T is

$$B = P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

Using direct method, we compute the matrix of L with respect to T as follows:

$$L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix},\ L\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

we form the augmented matrix whose rref is given too

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & 1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$\therefore B = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ **which agrees with the previous!**

MT127 – Linear Algebra I
Lecture 25 – 2017/2018

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Inner-Product Spaces

An inner product on a real vector space V is a function that assigns a real number $\langle x, y\rangle$, to each pair of vectors $\underline{u}$ and $\underline{v}$ in V, and that satisfies these properties

1. $\langle \underline{u}, \underline{u}\rangle \geq 0$ and $\langle \underline{u}, \underline{u}\rangle = 0$ if and only if $\underline{u} = \underline{0}$
2. $\langle \underline{u}, \underline{v}\rangle = \langle \underline{v}, \underline{u}\rangle$
3. $\langle a\underline{u}, \underline{v}\rangle = a\langle \underline{u}, \underline{v}\rangle$
4. $\langle \underline{u}, \underline{v} + \underline{w}\rangle = \langle \underline{u}, \underline{v}\rangle + \langle \underline{u}, \underline{w}\rangle$

Dot product

The dot product in $\mathbb{R}^n$ is an inner product in the sense of the above definition, where $\langle \underline{x}, \underline{y}\rangle = \underline{x}^T\underline{y}$

Inner-Product Spaces

Example

Let V be the vector space $\mathbb{R}^2$, and let A be 2×2 matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$.

Verify that the function $\langle \underline{u}, \underline{v} \rangle = \underline{u}^T A \underline{v}$ is an inner product for $\mathbb{R}^2$.

Solution

We need to verify the four properties of the inner product. Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$, then

$$\begin{aligned} \langle \underline{u}, \underline{u} \rangle = \underline{u}^T A \underline{u} &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= 3u_1^2 + 4u_1 u_2 + 4u_2^2 \\ &= 2u_1^2 + (u_1 + 2u_2)^2 \geq 0 \end{aligned}$$

Solution cont.

and $\langle \underline{u}, \underline{u} \rangle = 0$ if and only if $u_1 = u_2 = 0$, i.e $\underline{u} = \underline{0}$ **(Prop. 1 OK)**
Note that A is symmetric, that is $A = A^T$, also observe that if $\underline{u}$ and $\underline{v}$ are in $\mathbb{R}^2$, then $\underline{u}^T A \underline{v}$ is a 1×1 matrix, so

$$(\underline{u}^T A \underline{v})^T = \underline{u}^T A \underline{v}, \quad \text{then}$$

$$\langle \underline{u}, \underline{v} \rangle = \underline{u}^T A \underline{v} = (\underline{u}^T A \underline{v})^T = \underline{v}^T A^T (\underline{u}^T)^T = \underline{v}^T A^T \underline{u} = \underline{v}^T A \underline{u} = \langle \underline{v}, \underline{u} \rangle$$

(Prop 2 OK)
$\langle a\underline{u}, \underline{v} \rangle = (a\underline{u})^T A \underline{v} = a(\underline{u}^T A \underline{v}) = a\langle \underline{u}, \underline{v} \rangle$ **(Prop 3 OK)**
$\langle \underline{u}, \underline{v} + \underline{w} \rangle = \underline{u}^T A(\underline{v} + \underline{w}) = \underline{u}^T A \underline{v} + \underline{u}^T A \underline{w} = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$ **(Prop 4 OK)**
Therefore, $\langle \underline{u}, \underline{v} \rangle$ is an inner product for $\mathbb{R}^2$.

Example

For $p(t)$ and $q(t)$ in P_2, verify that

$$\langle p, q\rangle = \int_0^1 p(t)q(t)dt$$

is an inner product.

Solution

Start with property 1, Let $p(t), q(t), r(t)$ are in P_2

$$\langle p, p\rangle = \int_0^1 p(t)^2 dt \geq 0$$

and that $\langle p, p\rangle = 0$ if and only if $p(t) = 0$, $0 \leq t \leq 1$ **(Prop. 1 OK)**
$\langle p, q\rangle = \int_0^1 p(t)q(t)dt = \int_0^1 q(t)p(t)dt = \langle q, p\rangle$ **(Prop. 2 OK)**
$\langle ap, q\rangle = \int_0^1 (ap(t))q(t)dt = a\int_0^1 p(t)q(t)dt = a\langle p, q\rangle$ **(Prop. 3 OK)**

Solution cont.

$$\begin{aligned}\langle p, q+r\rangle = \int_0^1 p(t)[q(t)+r(t)]dt &= \int_0^1 \Big(p(t)q(t)+p(t)r(t)\Big)dt \\ &= \int_0^1 p(t)q(t)dt + \int_0^1 p(t)r(t)dt \\ &= \langle p, q\rangle + \langle p, r\rangle\end{aligned}$$

(**Prop. 4 OK**)
Therefore $\langle p, q\rangle$ is an inner product.

Inner product space

A vector space with an inner product is called an **inner-product space**.
In $\mathbb{R}^n$, we can use the inner-product as a measure of size.
If V is an inner-product space, then for each $\underline{v}$ in V we define $||\underline{v}||$ (**the norm** of $\underline{v}$) as

$$||\underline{v}|| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$$

Note that $\langle \underline{v}, \underline{v} \rangle \geq 0$ for all $\underline{v} \in V$, so the norm function is always defined.

Example

Use the inner product for P_2 defined in previous example to determine $||t^2||$.

Solution

$$||t^2|| = \sqrt{\langle t^2, t^2 \rangle} = \sqrt{\int_0^1 t^2 t^2 dt} = \sqrt{\int_0^1 t^4 dt} = \frac{1}{\sqrt{5}}$$

Orthogonal bases

If $\underline{u}$ and $\underline{v}$ are vectors in an inner-product space V, we say that $\underline{u}$ and $\underline{v}$ are **orthogonal** if $\langle \underline{u}, \underline{v} \rangle = 0$. Similarly, $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is an **orthogonal set** in V if $\langle \underline{v}_i, \underline{v}_j \rangle = 0$ when $i \neq j$. In addition, if an orthogonal set of vectors S is a basis for V, we call S an **orthogonal basis**.

Orthonormal basis

Let W be a subspace of V. A basis $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ for W is **orthonormal** if

1. $\langle \underline{v}_i, \underline{v}_j \rangle = 0$ for $i \neq j$, (Mutually perpendicular)
2. $\langle \underline{v}_i, \underline{v}_i \rangle = 1$ (Length 1)

Inner-Product Spaces

Theorem

Let $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ be an orthogonal basis for an inner product space V. IF $\underline{u}$ is any vector in V, then

$$\underline{u} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n$$

where

$$c_j = \frac{\langle \underline{u}, \underline{v}_j \rangle}{\langle \underline{v}_j, \underline{v}_j \rangle}, \; j = 1, 2, \ldots, n$$

Example

Let $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ be orthonormal basis for $\mathbb{R}^3$, where

$$\underline{v}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \; \underline{v}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \; \underline{v}_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

Write the vector $\underline{u} = (3, 4, 5)$ as a linear combination of the vectors in S.

Solution

We have

$$\underline{u} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3$$

where the values of c_1, c_2 and c_3 can be obtained without solving linear independence problem!

$$c_j = \frac{\langle \underline{u}, \underline{v}_j \rangle}{\langle \underline{v}_j, \underline{v}_j \rangle},\ j = 1, 2, 3$$

Since S is orthonormal basis, $\langle \underline{v}_j, \underline{v}_j \rangle = 1$, so
$c_1 = \langle \underline{u}, \underline{v}_1 \rangle = 1$, $c_2 = \langle \underline{u}, \underline{v}_2 \rangle = 0$ and $c_3 = \langle \underline{u}, \underline{v}_3 \rangle = 7$, therefore

$$\underline{u} = \underline{v}_1 + 7\underline{v}_3$$

Gram-Schmidt Orthogonalization

Let V be an inner-product space, and let $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ be basis for V. Let $\underline{u}_1 = \underline{v}_1$, and for $2 \leq k \leq n$ define $\underline{u}_k$ by

$$\underline{u}_k = \underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j$$

Then $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n\}$ is an orthogonal basis for V.
Moreover, if we let $\underline{w}_i = \frac{1}{||\underline{u}_i||} \underline{u}_i$, then $T = \{\underline{w}_1, \underline{w}_1, \ldots, \underline{w}_n\}$ is an othornomal basis for V

Example

Let the inner product on P_2 be the one given in previous example. Starting with the natural basis $\{1, x, x^2\}$, use Gram-Schmidt orthogonalization to obtain an orthogonal basis for P_2.

Solution

If we let $\{p_1, p_2, p_3\}$ denote the orthogonal basis, we have from the set of basis that $v_1(x) = 1, v_2(x) = x, v_3(x) = x^2$, then $p_1(x) = v_1(x) = 1$ and find $p_2(x)$ from

$$p_2(x) = v_2(x) - \frac{\langle \underline{p}_1, \underline{v}_2 \rangle}{\langle \underline{p}_1, \underline{p}_1 \rangle} \underline{p}_1(x)$$

we have $\underline{p}_1 = 1, \underline{v}_2 = x$, $\langle \underline{p}_1, \underline{v}_2 \rangle = \langle 1, x \rangle = \int_0^1 x dx = 1/2$, and $\langle \underline{p}_1, \underline{p}_1 \rangle = \langle 1, 1 \rangle = \int_0^1 dx = 1$, so

$$p_2(x) = x - \frac{1}{2}/1 = x - \frac{1}{2}$$

Solution cont.

Then $p_3(x)$ is obtained by

$$p_3(x) = v_3(x) - \frac{\langle \underline{p}_1, \underline{v}_3 \rangle}{\langle \underline{p}_1, \underline{p}_1 \rangle} \underline{p}_1(x) - \frac{\langle \underline{p}_2, \underline{v}_3 \rangle}{\langle \underline{p}_2, \underline{p}_2 \rangle} \underline{p}_2(x)$$

we have the following:
$v_3(x) = x^2$, and

$$\langle \underline{p}_2, \underline{v}_3 \rangle = \langle x - \frac{1}{2}, x^2 \rangle = \int_0^1 (x^3 - x^2/2)dx = 1/12$$

$$\langle \underline{p}_2, \underline{p}_2 \rangle = \langle x - \frac{1}{2}, x - \frac{1}{2} \rangle = \int_0^1 (x^2 - x + 1/4)dx = 1/12$$

$$\langle \underline{p}_1, \underline{v}_3 \rangle = \langle 1, x^2 \rangle = \int_0^1 x^2 dx = 1/3$$

Solution cont.

Therefore,

$$p_3(x) = v_3(x) - \frac{p_1(x)}{3} - p_2(x) = x^2 - \left(x - \frac{1}{2}\right) - \frac{1}{3}$$
$$= x^2 - x + \frac{1}{6}$$

and $\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$ is an **orthogonal** basis for P_2 with respect to the inner product.

Take away: Find an orthonormal basis for P_2 with respect to the inner product defined above.

MT127 – Linear Algebra I
Lecture 26 – 2017/2018

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Orthogonal Complements

Definition

Let W be a subspace of $\mathbb{R}^n$. A vector $\underline{u} \in V$ is said to be orthogonal to W if it is orthogonal to every vector in W. The set of all vectors in $\mathbb{R}^n$ that are orthogonal to to all vectors in W is called the orthogonal complement of W in $\mathbb{R}^n$ and is denoted by $W^\perp$ (read "W perp")

Example

Let W be the subspace of $\mathbb{R}^3$ consisting of all multiples of the vector $\underline{w} = (2, -3, 4)$. Find $W^\perp$.

Solution

$W = \text{span } \underline{w}$, so W is 1-dim subspace of $\mathbb{R}^3$.
A vector $\underline{u}$ belongs to $W^\perp$ if and only if $\underline{u}$ is orthogonal to $c\underline{w}$, i.e
$\underline{u}.(\underline{cw}) = 0$.
Let $\underline{u} = (x, y, z)$, then the $W^\perp$ is the plane with normal $\underline{w}$
$2x - 3y + 4z = 0$

Orthogonal Complements

Theorem

Let W be a subspace of $\mathbb{R}^n$, then

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $W \cap W^\perp = \{\underline{0}\}$

Proof

Let $\underline{u}_1$ and $\underline{u}_2$ be in $W^\perp$ and let $\underline{w} \in W$, this means $\underline{u}_1.\underline{w} = 0$ and $\underline{u}_2.\underline{w} = 0$, then

$$(\underline{u}_1 + \underline{u}_2).\underline{w} = (\underline{u}_1.\underline{w} + \underline{u}_2.\underline{w} = 0 + 0 = 0$$

so $\underline{u}_1 + \underline{u}_2$ is in $W^\perp$

$$(c\underline{u}).\underline{w} = c(\underline{u}.\underline{w}) = 0$$

so $c\underline{u}$ is in $W^\perp$ and therefore $W^\perp$ is a subspace

Orthogonal Complements

Example

Let W be the subspace of $\mathbb{R}^4$ with basis $\{\underline{w}_1, \underline{w}_2\}$, where $\underline{w}_1 = (1,1,0,1)$ and $\underline{w}_2 = (0,-1,1,1)$. Find basis for $W^\perp$.

Example

Let $\underline{u} = (a,b,c,d) \in W^\perp$, then $\underline{u}.\underline{w}_1 = \underline{u}.\underline{w}_2 = 0$

$$\underline{u}.\underline{w}_1 = a+b+d = 0 \text{ and } \underline{u}.\underline{w}_2 = -b+c+d = 0$$

Solving the homogeneous system, we have **(verify!)**

$$\underline{u} = \begin{bmatrix} -r-2s \\ r+s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix};\ \ r,s \in \mathbb{R}$$

Hence the vectors $(-1,1,1,0)$ and $(-2,1,0,1)$ form basis for $W^\perp$.

Orthogonal Complements

Theorem

Let W be a subspace of $\mathbb{R}^n$, then

$$\mathbb{R}^n = W \oplus W^\perp$$

Proof

Let $\dim W = m$, then W has basis of m vectors. With Gram-Schmidt process we can transform this basis to orthonormal basis $S = \{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_m\}$. If $\underline{v} \in \mathbb{R}^n$, let

$$\underline{w} = (\underline{v}.\underline{w}_1)\underline{w}_1 + (\underline{v}.\underline{w}_2)\underline{w}_2 + \cdots + (\underline{v}.\underline{w}_m)\underline{w}_m \tag{1}$$

$$\underline{u} = \underline{v} - \underline{w} \tag{2}$$

Since $\underline{w}$ is a linear combination of vectors in S, then $\underline{w}$ belongs to W. We need to show that $\underline{u}$ lies in $W^\perp$ by showing that $\underline{u}$ is orthogonal to every vector in S.

Orthogonal Complements

Proof

For each vector $\underline{w}_i$ in S, we have

$$\begin{aligned}\underline{u}.\underline{w}_i =&(\underline{v}-\underline{w}).\underline{w}_i = \underline{v}.\underline{w}_i - \underline{w}.\underline{w}_i \\ =&\underline{v}.\underline{w}_i - \left[(\underline{v}.\underline{w}_1)\underline{w}_1 + (\underline{v}.\underline{w}_2)\underline{w}_2 + \cdots + (\underline{v}.\underline{w}_m)\right]\underline{w}_i \\ =&\underline{v}.\underline{w}_i - (\underline{v}.\underline{w}_i)(\underline{w}_i.\underline{w}_i) = \underline{v}.\underline{w}_i - \underline{v}.\underline{w}_i = 0\end{aligned}$$

Since $\underline{w}_i.\underline{w}_j = 0, \forall i \neq j$ and $\underline{w}_i.\underline{w}_i = 1$, $1 \leq i \leq m$. Thus $\underline{u}$ is orthogonal to every vector in W and so lies in $W^\perp$. Hence

$$\underline{v} = \underline{w} + \underline{u}$$

Which means $\mathbb{R}^n = W + W^\perp$. Since $W \cap W^\perp = \{\underline{0}\}$, it follows that

$$\mathbb{R}^n = W \oplus W^\perp$$

Example

From previous example, subspace W had the basis $\{\underline{w}_1, \underline{w}_2\} = \{(1,1,0,1),(0,-1,1,1)\}$ and we determined that $W^{\perp}$ had basis $\{\underline{w}_3, \underline{w}_4\} = \{(-1,1,1,0),(-2,1,0,1)\}$. If $\underline{v} = (-1,1,4,3)$, find a vector $\underline{w}$ in W and a vector $\underline{u}$ in $W^{\perp}$ so that $\underline{v} = \underline{w} + \underline{u}$.

Solution

Use Gram-Schmidt process to determine an orthonormal basis for W, as

$$\underline{u}_1 = \frac{1}{\sqrt{3}}(1,1,0,1) \quad \text{and} \quad \underline{u}_2 = \frac{1}{\sqrt{3}}(0,-1,1,1). \text{ Let}$$

$$\begin{aligned}\underline{w} =& (\underline{v}.\underline{u}_1)\underline{u}_1 + (\underline{v}.\underline{u}_2)\underline{u}_2 \\ =& \sqrt{3}\underline{u}_1 + 2\sqrt{3}\underline{u}_2 = (1,1,0,1) + 2(0,-1,1,1) = (1,-1,2,3)\end{aligned}$$

Solution cont.

Then we compute

$$\underline{u} = \underline{v} - \underline{w} = (-1, 1, 4, 3) - (1, -1, 2, 3) = (-2, 2, 2, 0)$$

It follows that $\underline{v} = \underline{w} + \underline{u}$, for $\underline{w} \in W$ and $\underline{u} \in W^{\perp}$

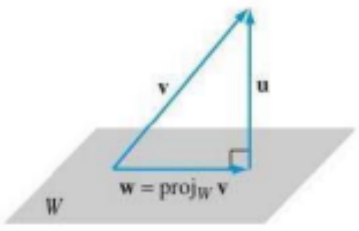
Projections and applications

Projections

Let W is a subspace of $\mathbb{R}^n$ with orthonormal basis $\{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_m\}$ and $\underline{v}$ is any vector in $\mathbb{R}^n$, then there exist unique vectors $\underline{w} \in W$ and $\underline{u} \in W^{\perp}$ such that $\underline{v} = \underline{w} + \underline{u}$
Moreover, we saw from eqn 1 that

$$\underline{w} = (\underline{v}.\underline{w}_1)\underline{w}_1 + (\underline{v}.\underline{w}_2)\underline{w}_2 + \cdots + (\underline{v}.\underline{w}_m)\underline{w}_m$$

which is called the orthogonal projection of $\underline{v}$ on W and is denoted by $\text{proj}_W \underline{v}$



Projections and applications

Projections cont.

When $\{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_m\}$ are orthogonal basis for W, then

$$\text{proj}_W \underline{v} = \frac{\underline{v}.\underline{w}_1}{\underline{w}_1.\underline{w}_1}\underline{w}_1 + \frac{\underline{v}.\underline{w}_2}{\underline{w}_2.\underline{w}_2}\underline{w}_2 + \cdots + \frac{\underline{v}.\underline{w}_m}{\underline{w}_m.\underline{w}_m}\underline{w}_m$$

Exercise

Rephrase the Gram-Schmidt theorem using the projection notations.

Example

Let W be the two-dim subspace of $\mathbb{R}^3$ with orthonormal basis $\{\underline{w}_1, \underline{w}_2\}$, where

$$\underline{w}_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right) \text{ and } \underline{w}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Using standard inner product on $\mathbb{R}^3$, find the orthogonal projection of $\underline{v} = (2, 1, 3)$ on W and the vector $\underline{u}$ that is orthogonal to every vector in W.

Projections and applications

Solution

We have

$$\begin{aligned}\underline{w} = \mathrm{proj}_W \underline{v} =& (\underline{v}.\underline{w}_1)\underline{w}_1 + (\underline{v}.\underline{w}_2) \\ =& -1\underline{w}_1 + \frac{5}{\sqrt{2}}\underline{w}_2 = \left(\frac{11}{6}, \frac{1}{3}, \frac{19}{6}\right)\end{aligned}$$

and

$$\underline{u} = \underline{v} - \underline{w} = \left(\frac{1}{6}, \frac{2}{3}, -\frac{1}{6}\right)$$

Distance from a point to a plane

It is clear from the figure that, the distance from $\underline{v}$ to the plane W is given by the length of the vector $\underline{u} = \underline{v} - \underline{w}$, thus, that is

$$||\underline{v} - \mathrm{proj}_W \underline{v}||$$

Example

Let W be the subspace of $\mathbb{R}^3$ defined in previous example, and let $\underline{v} = (1,1,0)$. Find the distance from $\underline{v}$ to W.

Solution

We first compute $\text{proj}_W\underline{v}$ and then $\underline{v} - \text{proj}_W\underline{v}$:

$$\text{proj}_W\underline{v} = (\underline{v}.\underline{w}_1)\underline{w}_1 + (\underline{v}.\underline{w}_2)\underline{w}_2 = \frac{1}{3}\underline{w}_1 + \frac{1}{\sqrt{2}}\underline{w}_2 = \left(\frac{13}{18}, -\frac{1}{9}, \frac{5}{18}\right)$$

$$\underline{v} - \text{proj}_W\underline{v} = (1,1,0) - \left(\frac{13}{18}, -\frac{1}{9}, \frac{5}{18}\right) = \left(\frac{5}{18}, \frac{10}{9}, -\frac{5}{18}\right) \text{ and}$$

$$||\underline{v} - \text{proj}_W\underline{v}|| = \sqrt{\frac{25}{324} + \frac{100}{81} + \frac{25}{324}} = \frac{5\sqrt{2}}{6}$$

So the distance from $\underline{v}$ to W is $\frac{5\sqrt{2}}{6}$

Note: This is the **closest distance** from $\underline{v}$ to W

MT127 – Linear Algebra I
Lecture 27 – 2017/2018

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LEAST SQUARES METHOD

Consistency vs Approximate solution

- $m \times n$ linear system $A\underline{x} = \underline{b}$ is inconsistent if it has no solution
- $A\underline{x} = \underline{b}$ is consistent if and only if $\underline{b} \in \text{col } A$.
- inconsistent systems are common and we must find a way to deal with it.
- One approach is to change the problem, $A\underline{x} = \underline{b}$ not to be strictly satisfied.
- Instead we seek a vector $\hat{x} \in \mathbb{R}^n$ such that $A\underline{\hat{x}}$ is as close to $\underline{b}$ as possible.
- If $W =$ col A, then the vector in W that is closest to $\underline{b}$ is $\text{proj}_W \underline{b}$
- That is $||\underline{b} - \underline{w}||$ for some $\underline{w} \in W$ is minimized when $\underline{w} = \text{proj}_W \underline{b}$
- If we find $\underline{\hat{x}}$ such that $A\underline{\hat{x}} = \text{proj}_W \underline{b}$, then we are assured that $||\underline{b} - A\underline{\hat{x}}||$ will be minimum.
- As shown from last lecture, $\underline{b}$ is $\text{proj}_W \underline{b} = \underline{b} - A\underline{\hat{x}}$ is orthogonal to every vector in W.

Consistency vs Approximate solution

- If follows that $\underline{b} - A\hat{\underline{x}}$ is orthogonal to every vector in W. It follows that $\underline{b} - A\hat{\underline{x}}$ is orthogonal to each column of A,

$$A^T(A\hat{\underline{x}} - \underline{b}) = \underline{0}$$

  Equivalently

$$A^T A\hat{\underline{x}} = A^T\underline{b}$$

  Tha is $\hat{\underline{x}}$ is a solution to $A^T A\underline{x} = A^T\underline{b}$

- The above equation is called the normal system of equations. whose solution is called a least squares solution to the linear system.

Theorem

If A is an $m \times n$ matrix with rank $A = n$, then $A^T A$ is nonsingular and the linear system $A\underline{x} = \underline{b}$ has a unique solution least square solution given by

$$\underline{\hat{x}} = (A^T A)^{-1} A^T \underline{b}$$

Example

Determine a least squares solution to $A\underline{x} = \underline{b}$, where

LEAST SQUARES LINE FIT

Theorem

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Example

Determine a least squares solution to $A\underline{x} = \underline{b}$, where