1. Defining a POVM

Consider an n dimensional quantum system whose state can be one of a set of $m \leq n$ linearly independent states $|\psi_j\rangle$. Construct a POVM $(E_1, E_2, \ldots, E_m, E_{m+1})$ such that the outcome $j \leq m$ occurs only for the state $|\psi_j\rangle$.

2. Subadditivity

(a) Prove the subadditivity property of the Shannon entropy: $H(XY) \leq H(X) + H(Y)$.

(b) Prove the subadditivity property of the Von Neumann entropy: $S(AB) \leq S(A) + S(B)$. Include the proof of Klein's inequality. (*Hint: it's all in Nielsen and Chuang*)

3. More entropies

(a) Show that $S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$ for any density matrices ρ and σ and for systems of any dimension.

(b) Show that $S(|\psi\rangle\langle\psi|) = 0$ for any pure state for a system of any dimension.