PDE 第二章第二次作业

yjyxbhsx

2024 年 11 月 18 日

【题目 1】(课本第 9 题) 求解并证明初值问题

$$\begin{cases} u_{tt} - a^2 u_{xx} + cu = f(x,t) & (x,t) \in \mathbb{R}^3 \times (0,+\infty) \\ u|_{t=0} = \phi(x) & x \in \mathbb{R} \\ u_t|_{t=0} = \psi(x) & x \in \mathbb{R} \end{cases}$$

解的唯一性.

证明. 令设 $u(x,t)$ 为上述方程的解，令

$$v(x,y,t) = u(x,t) \cdot \exp\left(i\frac{\sqrt{c}}{a}y\right).$$

直接计算，v 满足

$$v_{tt} - a^2 v_{xx} - a^2 v_{yy} = \left(u_{tt} - a^2 u_{xx} + cu\right) \cdot \exp\left(i\frac{\sqrt{c}}{a}y\right) = f(x,t)e^{i\frac{c}{a}y}.$$

$$v|_{t=0} = \phi(x)e^{i\frac{c}{a}y}, \qquad v_t|_{t=0} = \psi(x)e^{i\frac{c}{a}y}.$$

由 Poisson 公式可解出 v，从而得到 u。

要证明改解是唯一的，只需证明 $f = \psi = \phi = 0$ 时只有零解。任意给定 $(x_0, t_0) \in \mathbb{R} \times (0,+\infty)$，定义能量

$$E(t) = \frac{1}{2}\int_{x_0-a(t-t_0)}^{x_0+a(t-t_0)} u_t^2 + a^2 u_x^2 \,\mathrm{d}x.$$

求导得

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} u_t u_{tt} + a^2 u_x u_{xt} \,\mathrm{d}x - \frac{a}{2}\left(u_t^2 + a^2 u_x^2\right)\Big|_{x_0+a(t_0-t)} - \frac{a}{2}\left(u_t^2 + a^2 u_x^2\right)\Big|_{x_0-a(t_0-t)} \\
&= \int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} u_t(u_{tt} - a^2 u_{xx}) \,\mathrm{d}x - \frac{a}{2}\left(u_t^2 + a^2 u_x^2\right)\Big|_{x_0+a(t_0-t)} - \frac{a}{2}\left(u_t^2 + a^2 u_x^2\right)\Big|_{x_0-a(t_0-t)} \\
&\quad + a^2 u_x u_t\Big|_{x_0+a(t_0-t)} - a^2 u_x u_t\Big|_{x_0-a(t_0-t)} \\
&= -c\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} u u_t \,\mathrm{d}x \\
&\quad - \frac{a}{2}\left(u_t^2 + a^2 u_x^2 - 2a u_x u_t\right)\Big|_{x_0+a(t_0-t)} - \frac{a}{2}\left(u_t^2 + a^2 u_x^2 + 2u_x u_t\right)\Big|_{x_0-a(t_0-t)}
\end{aligned}$$

由

$$|2a u_x u_t| \le u_t^2 + a^2 u_x^2$$

知

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le -c\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} u u_t \,\mathrm{d}x \le \frac{c^2}{2}\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} u^2 \,\mathrm{d}x + E(t).$$

由 Gronwell 不等式有

$$E(t) \leq \frac{c^2 e^{t_0}}{2} \int_0^t \int_{x_0-a(t_0-\tau)}^{x_0+a(t_0-\tau)} u^2 \,\mathrm{d}x\,\mathrm{d}\tau.$$

另一方面

$$\begin{aligned}\frac{1}{2}\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} u^2 \,\mathrm{d}x &= \frac{1}{2}\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} u^2 - u^2|_{t=0} \,\mathrm{d}x = \frac{1}{2}\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} \int_0^t \frac{\partial}{\partial t} u^2(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x \\ &\leq \frac{1}{2}\int_0^t \int_{x_0-a(t_0-\tau)}^{x_0+a(t_0-\tau)} u^2 \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_0^t \int_{x_0-a(t_0-\tau)}^{x_0+a(t_0-\tau)} u_t^2 \,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

从而

$$\frac{1}{2}\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} \left(u^2+u_t^2+a^2u_x^2\right)\mathrm{d}x \leq \frac{c^2e^{t_0}+1}{2}\int_0^t \int_{x_0-a(t_0-\tau)}^{x_0+a(t_0-\tau)} u^2 \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_0^t \int_{x_0-a(t_0-\tau)}^{x_0+a(t_0-\tau)} u_t^2 \,\mathrm{d}x\,\mathrm{d}\tau.$$

再次用 Gronwell 不等式有

$$\begin{aligned}\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} \left(u^2+u_t^2+a^2u_x^2\right)\mathrm{d}x &\leq c^2e^{t_0}\left(e^{t_0}+1\right)\int_0^t \int_{x_0-a(t_0-\tau)}^{x_0+a(t_0-\tau)} u^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &\leq c^2e^{t_0}\left(e^{t_0}+1\right) t_0 \sup_{0\leq\tau\leq t_0}\int_{x_0-a(t_0-\tau)}^{x_0+a(t_0-\tau)} u^2 \,\mathrm{d}x\end{aligned}$$

取 t_0 满足

$$c^2\left(e^{t_0}-1\right)e^{t_0} < 1$$

则可证明在

$$\{(x,t) : x\in\mathbb{R}, 0\leq t\leq t_0\}$$

上 $u=0$。以 $kt_0(k\in\mathbb{Z}_{>0})$ 为初始点重复这个过程即可。 □

【题目 2】(课本第 19 题) 求解三维波动方程的 Cauchy 问题

$$\begin{cases} u_{tt} = a^2\left(u_{xx}+u_{yy}+u_{zz}\right) & \text{in } \mathbb{R}^3\times(0,+\infty) \\ u=0 \quad u_t = x^3+y^2z & \text{on } \mathbb{R}^3\times\{t=0\} \end{cases}.$$

解. 记 $x=(x_1,x_2,x_3)^{\mathrm{T}}$, $y=(u,v,w)^{\mathrm{T}}$, $h(y)=u^3+v^2w$. 由 Kirchhoff 公式知

$$u(x,t) = t \fint_{\partial B(x,at)} u^3+v^2w\,\mathrm{d}S(y) = \frac{1}{4\pi a^2 t}\int_{\partial B(x,at)} u^3+v^2w\,\mathrm{d}S(y).$$

作变量替换

$$\begin{cases} u = x_1 + at\sin\phi\cos\theta \\ v = x_2 + at\sin\phi\sin\theta \\ w = x_3 + at\cos\phi \end{cases}$$

则

$$
\begin{aligned}
u(x,t) =& \frac{1}{4\pi a^2 t}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \left((x_1 + at\sin\phi\cos\theta)^3 + (x_2 + at\sin\phi\sin\theta)^2 (x_3 + at\cos\phi) \right) a^2t^2 \sin\phi \,\mathrm{d}\phi \\
=& \frac{(x_1^3 + x_2^2 x_3)t}{4\pi} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\phi \,\mathrm{d}\phi + \frac{t}{4\pi}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} x_2 at \sin\phi \,\mathrm{d}\sin\phi \\
& + \frac{t}{4\pi}\int_0^{2\pi} \cos\theta \,\mathrm{d}\theta \int_0^{\pi} 2x_1^2 at\sin^2\phi \,\mathrm{d}\phi + \frac{t}{4\pi}\int_0^{2\pi} \sin\theta \,\mathrm{d}\theta \int_0^{\pi} 2atx_2x_3\sin^2\phi \,\mathrm{d}\phi \\
& + \frac{t}{4\pi}\int_0^{2\pi} \sin^2\theta \,\mathrm{d}\theta \int_0^{\pi} x_3 a^2t^2\sin^3\phi \,\mathrm{d}\phi + \frac{t}{4\pi}\int_0^{2\pi} \cos^2\theta \,\mathrm{d}\theta \int_0^{\pi} 3x_1a^2t^2\sin^3\phi \,\mathrm{d}\phi \\
& + \frac{t}{4\pi}\int_0^{2\pi} \sin\theta \,\mathrm{d}\theta \int_0^{\pi} 2x_2a^2t^2\sin^2\phi \,\mathrm{d}\sin\phi + \frac{t}{4\pi}\int_0^{2\pi} \sin^2\theta \,\mathrm{d}\theta \int_0^{\pi} a^3t^3\sin^3\phi \,\mathrm{d}\sin\phi \\
& + \frac{t}{4\pi}\int_0^{2\pi} \cos^3\theta \,\mathrm{d}\theta \int_0^{\pi} a^3t^3\sin^4\phi \,\mathrm{d}\phi \\
=& \left(x_1^3 + x_2^2 x_3\right) t + \frac{a^2t^3}{3}(3x_1 + x_3).
\end{aligned}
$$

□

【题目 3】(课本第 20 题) 用降维法导出一维波动方程 Cauchy 问题的求解公式。

证明. 只需对 Cauchy 问题

$$
\begin{cases}
u_{tt} - a^2 u_{xx} = 0 & \text{in } \mathbb{R}^2 \times (0, +\infty) \\
u = 0,\ u_t = h & \text{on } \mathbb{R}^2 \times t = 0
\end{cases}
$$

证明即可。令

$$
\bar{u}(x,y,t) = u(x,t), \qquad \bar{h}(x,y) = h(x).
$$

则 $\bar{u}$ 满足 Cauchy 问题

$$
\begin{cases}
u_{tt} - a^2(u_{xx} + u_{yy}) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\
u = 0,\ u_t = h & \text{on } \mathbb{R} \times t = 0
\end{cases}
$$

由 Poissin 公式知

$$
\begin{aligned}
u(x,t) = \bar{u}(x,0,t) &= \frac{1}{2\pi a}\int_{B((x,y),at)} \frac{\bar{h}(u,v)}{\sqrt{a^2t^2 - (x-u)^2 - v^2}} \,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2\pi a}\int_{x-at}^{x+at} h(u)\,\mathrm{d}u \int_{-\sqrt{a^2t^2-(u-x)^2}}^{\sqrt{a^2t^2-(u-x)^2}} \frac{1}{\sqrt{a^2t^2 - (x-u)^2 - v^2}} \,\mathrm{d}v
\end{aligned}
$$

作变量替换

$$
v = \sqrt{a^2t^2 - (x-u)^2 - v^2} \cdot \sin\theta
$$

则

$$
u(x,t) = \frac{1}{2\pi a}\int_{x-at}^{x+at} h(u)\,\mathrm{d}u \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\cos\theta}\,\mathrm{d}\theta = \frac{1}{2a}\int_{x-at}^{x+at} h(u)\,\mathrm{d}u.
$$

□

【题目 4】(课本第 21 题) 求解二维波动方程的 Cauchy 问题

$$
\begin{cases}
u_{tt} - a^2\left(u_{xx} + u_{yy}\right) = 0 & \text{in } \mathbb{R}^2 \times (0, +\infty) \\
u = x^2(x+y),\ u_t = 0 & \text{on } \mathbb{R}^2 \times \{t = 0\}
\end{cases}.
$$

解. 由 Poissin 公式知

$$u(x,y,t)=\frac{\partial}{\partial t}\left(\frac{1}{2\pi a}\int_{B((x,y),at)}\frac{u^2(u+v)}{\sqrt{a^2t^2-(u-x)^2+(v-y)^2}}\,\mathrm{d}u\,\mathrm{d}v\right)$$

其中作变量替换

$$\begin{cases}u=x+r\cos\theta\\ v=y+r\sin\theta\end{cases}$$

那么

$$\begin{aligned}&\int_{B((x,y),at)}\frac{u^2(u+v)}{\sqrt{a^2t^2-(u-x)^2+(v-y)^2}}\,\mathrm{d}u\,\mathrm{d}v\\ =&\int_0^{2\pi}\mathrm{d}\theta\int_0^{at}\frac{(x+r\cos\theta)^2(x+y+r(\sin\theta+\cos\theta))}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}r\\ =&\int_0^{2\pi}\mathrm{d}\theta\int_0^{at}\frac{x^2(x+y)+(3x+y)r^2\cos^2\theta}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}r\end{aligned}$$

令 $r=at\sin\phi$，那么上式化为

$$\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\left(x^2(x+y)+(3x+y)a^2t^2\sin^2\phi\cos^2\theta\right)at\cdot\sin\phi\,\mathrm{d}\phi$$

故

$$\begin{aligned}u(x,y,t)&=\frac{\partial}{\partial t}\left(\frac{1}{2\pi a}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\left(x^2(x+y)+(3x+y)a^2t^2\sin^2\phi\cos^2\theta\right)at\cdot\sin\phi\,\mathrm{d}\phi\right)\\ &=\frac{1}{2\pi a}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\left(ax^2(x+y)+3(3x+y)a^3t^2\sin^2\phi\cos^2\theta\right)\sin\phi\,\mathrm{d}\phi\\ &=x^2(x+y)+(3x+y)a^2t^2.\end{aligned}$$

□

【题目 5】(课本第 22 题) 求以下特征问题的特征函数

$$\begin{cases}X''(x)+\lambda X(x)=0, & x\in(0,\ell)\\ X(0)=X'(\ell)+hX(\ell)=0 & h>0\end{cases}$$

解. 该问题为 Sturm - Liouville 问题，因此 $\lambda>0$。从而

$$X(x)=C_1\cos(\mu x)+C_2\sin(\mu x),\qquad x\in(0,\ell).$$

其中 $\mu=\sqrt{\lambda}$. 将初始条件带入上式得

$$\begin{pmatrix}1 & 0\\ h\cos\mu\ell-\mu\sin\mu\ell & h\sin\mu\ell+\mu\cos\mu\ell\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

从而 $C_1=0$。又该方程有非零解，故系数行列式为 0，从而

$$h\sin\mu\ell+\mu\cos\mu\ell=0\qquad\Longrightarrow\qquad\tan\ell\mu=-\frac{1}{h}\cdot\mu.$$

从而 $\forall\, n\in\mathbb{Z}_{>0}$，存在 $\mu_n\in((n-1)\pi,n\pi)$ 使得

$$\tan\ell\mu_n=-\frac{1}{h}\cdot\mu_n,$$

即 $\{\lambda_n=\mu_n^2\}_{n=1}^{\infty}$ 为所有特征值。取 $C_2=1$，则特征函数为

$$X_n(x)=\sin(\mu_n x),$$

其中 μ_n 为方程 $\tan(\ell x) = -x/h$ 的所有正解。 □

以下题目中均设

$$L = \frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}, \qquad Q = (0,\ell)\times(0,+\infty).$$

【题目 6】(课本第 23 题) 用分离变量法求解：

$$\begin{cases} Lu = 0 & (x,t)\in Q \\ u|_{x=0} = u_x|_{x=\ell} = 0, & t\geq 0 \\ u|_{t=0} = x(x-2\ell), u_t|_{t=0}, & 0\leq x\leq \ell \end{cases}$$

解. 令 $u(x,t) = X(x)T(t)$，则原问题化为

$$\begin{cases} X'' + \lambda X = 0 & x\in(0,\ell) \\ X(0) = X'(\ell) = 0 \end{cases}$$

$$T'' + a^2\lambda T = 0, \qquad t > 0.$$

该问题为 Sturm - Liouville 问题，因此 $\lambda > 0$。从而

$$X(x) = C_1\cos\mu x + C_2\sin\mu x.$$

其中 $\mu = \sqrt{\lambda}$。将初始条件带入上式得

$$\begin{pmatrix} 1 & 0 \\ -\mu\sin\mu\ell & \mu\cos\mu\ell \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

从而 $C_1 = 0$。又该方程有非零解，从而系数行列式为 0，即

$$\mu\cos\mu\ell = 0, \qquad \Longrightarrow \qquad \mu_n = \frac{(n-\frac{1}{2})\pi}{\ell}, \ \forall\, n\in\mathbb{Z}_{\geq 0}.$$

从而特征函数为

$$X_n(x) = \sin\left(\frac{(2n-1)\pi}{2\ell}x\right), \qquad \Longrightarrow \qquad T_n(t) = A_n\cos\left(\frac{(2n-1)a\pi}{2\ell}t\right) + B_n\sin\left(\frac{(2n-1)a\pi}{2\ell}t\right)$$

故

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{(2n-1)a\pi}{2\ell}t\right) + B_n\sin\left(\frac{(2n-1)a\pi}{2\ell}t\right)\right)\sin\left(\frac{(2n-1)\pi}{2\ell}x\right)$$

又

$$x(x-2\ell) = u(x,0) = \sum_{n=1}^{\infty}A_n\sin\left(\frac{(2n-1)\pi}{2\ell}x\right)$$

$$0 = u_t(x,0) = \sum_{n=1}^{\infty}\frac{(2n-1)a\pi}{2\ell}B_n\sin\left(\frac{(2n-1)\pi}{2\ell}x\right)$$

故

$$A_n = \frac{2}{\ell}\int_0^{\ell}x(x-2\ell)\sin\left(\frac{(2n-1)\pi}{2\ell}x\right)\,\mathrm{d}x = -\frac{32\ell^3}{(2n-1)^3\pi^3}, \qquad B_n = 0 \qquad \forall\, n\in\mathbb{Z}_{>0}.$$

从而

$$u(x,t) = -\sum_{n=1}^{\infty}\frac{32\ell^3}{(2n-1)^3\pi^3}\cos\left(\frac{(2n-1)a\pi}{2\ell}t\right)\sin\left(\frac{(2n-1)\pi}{2\ell}x\right)$$

□

【题目 7】(课本第 25 题) 设 $u(x,t)$ 适合定解问题

$$\begin{cases} Lu = f(x,t), & (x,t) \in Q \\ -u_x + \alpha u|_{x=0} = \mu_1(t) & t \geq 0 \\ u_x + \beta u|_{x=\ell} = \mu(t) & t \geq 0 \\ u|_{t=0} = \phi(x),\ u_t|_{t=0} = 0, & x \in [0,\ell] \end{cases}.$$

试引进辅助函数，把边界条件齐次化，设

1. $\alpha, \beta > 0$;
2. $\alpha = \beta = 0$.

解.

1. $\alpha, \beta > 0$

   设

$$v(x,t) = u(x,t) + h(x,t)$$

   将原问题的边界齐次化。即

$$(-u_x + \alpha u - h_x + \alpha h)|_{x=0} = 0 \qquad (u_x + \beta u + h_x + \beta h)|_{x=\ell} = 0$$

   由于 u 为原问题的解，故

$$(-h_x + \alpha h)|_{x=0} + \mu_1(t) = 0, \qquad (h_x + \beta h)|_{x=\ell} + \mu(t) = 0.$$

   现令

$$h(x,t) = (mx+n)\mu(t) + (px+q)\mu_1(t).$$

   带入上面两式得

$$\begin{pmatrix} -m + \alpha n & -p + \alpha q + 1 \\ (\beta\ell + 1)m + \beta n + 1 & (1 + \beta\ell)p + q \end{pmatrix} \begin{pmatrix} \mu(t) \\ \mu_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   从而

$$\begin{pmatrix} -1 & \alpha & & \\ & & -1 & \alpha \\ \beta\ell + 1 & \beta & & \\ & & 1 + \beta\ell & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

   解得

$$m = \frac{\alpha}{\alpha + \beta + \alpha\beta\ell}, \qquad n = \frac{1}{\alpha + \beta + \alpha\beta\ell}, \qquad p = \frac{-\beta}{\alpha + \beta + \alpha\beta\ell}, \qquad q = \frac{1 + \beta\ell}{\alpha + \beta + \alpha\beta\ell}$$

   从而

$$v(x,t) = u(x,t) + \frac{(\alpha x + 1)\mu(t) + (-\beta x + 1 + \beta\ell)\mu_1(t)}{\alpha + \beta + \alpha\beta\ell}$$

2. $\alpha = \beta = 0$

   此时边界条件为

$$u_x(0,t) + \mu_1(t) = 0, \qquad u_x(\ell,t) - \mu(t) = 0.$$

   设

$$v(x,t) = u(x,t) + h(x,t)$$

将原问题的边界齐次化。即

$$h_x(0,t) - \mu_1(t) = 0, \qquad h_x(\ell,t) + \mu(t) = 0.$$

不妨设

$$h_x(x,t) = \frac{\ell - x}{\ell}\mu_1(t) - \frac{x}{\ell}\mu(t).$$

两侧对 x 积分，得到一个解为

$$h(x,t) = \left(x - \frac{x^2}{2\ell}\right)\mu_1(t) - \frac{x^2}{2\ell}\mu(t).$$

从而

$$v(x,t) = u(x,t) + \left(x - \frac{x^2}{2\ell}\right)\mu_1(t) - \frac{x^2}{2\ell}\mu(t)$$

□

【题目 8】(课本第 26 题) 用分离变量法求解

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 & (x,t) \in Q \\ u_x|_{x=0} = A\sin\omega t,\ u|_{x=\ell} = 0, & t \geq 0 \\ u|_{t=0} = 1,\ u_t|_{t=0} = 0, & x \in [x,\ell] \end{cases}.$$

<u>**解.**</u> 令

$$v(x,t) = u(x,t) - A(x-\ell)\sin\omega t,$$

那么 v 满足方程

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial^2 v}{\partial x^2} = A\omega^2(x-\ell)\sin\omega t & (x,t) \in Q \\ v_x|_{x=0} = 0,\ v|_{x=\ell} = 0, & t \geq 0 \\ v|_{t=0} = 1,\ v_t|_{t=0} = -A\omega(x-\ell), & x \in [x,\ell] \end{cases}.$$

记 $v(x,t) = X(x)T(t)$. 其特征方程为

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0,\ X(\ell) = 0 \end{cases}$$

解得特征方程为

$$X_n(x) = \cos\frac{\left(n - \frac{1}{2}\right)\pi x}{\ell}, \qquad \forall\, n \in \mathbb{Z}_{>0}. \tag{1}$$

从而

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\cos\frac{\left(n-\frac{1}{2}\right)\pi x}{\ell}$$

带入边界条件有

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty}\left(T_n''(t) + \frac{\left(n-\frac{1}{2}\right)^2\pi^2}{\ell^2}T_n(t)\right)\cos\frac{\left(n-\frac{1}{2}\right)\pi x}{\ell} = A\omega^2(x-\ell)\sin\omega t, \\ \displaystyle\sum_{n=1}^{\infty} T_n(0)\cos\frac{\left(n-\frac{1}{2}\right)\pi x}{\ell} = 1, \\ \displaystyle\sum_{n=1}^{\infty} T_n'(0)\cos\frac{\left(n-\frac{1}{2}\right)\pi x}{\ell} = -A\omega(x-\ell) \end{cases}$$

即 $\forall\, n \in \mathbb{Z}_{>0}$，$T_n$ 满足微分方程

$$\begin{cases} T_n''(t) + \dfrac{\left(n-\frac{1}{2}\right)^2 \pi^2}{\ell^2} T_n(t) = f_n(t) \\ T_n(0) = \alpha_n, \qquad T_n'(0) = \beta_n. \end{cases}$$

其中

$$f_n(t) = \frac{2}{\ell}\int_0^\ell A\omega^2(x-\ell)\sin\omega t\cos\frac{\left(n-\frac{1}{2}\right)\pi x}{\ell}\,\mathrm{d}x = -\frac{8A\omega^2\ell}{(2n-1)^2\pi^2}\sin\omega t = -\omega\beta_n\sin\omega t,$$

$$\alpha_n = \frac{2}{\ell}\int_0^\ell \cos\frac{\left(n-\frac{1}{2}\right)\pi x}{\ell}\,\mathrm{d}x = (-1)^{n+1}\cdot\frac{4}{(2n+1)\pi}, \tag{2}$$

$$\beta_n = \frac{2}{\ell}\int_0^\ell -A\omega(x-\ell)\cos\frac{\left(n-\frac{1}{2}\right)\pi x}{\ell}\,\mathrm{d}x = \frac{8A\omega\ell}{(2n-1)^2\pi^2}. \tag{3}$$

记

$$\omega_n = \frac{2n-1}{2\ell}\pi. \tag{4}$$

解上面这个初值问题，得

$$\begin{aligned} T_n(t) =& \alpha_n\cos\omega_n t + \frac{\beta_n}{\omega_n}\sin\omega_n t + \frac{1}{\omega_n}\cdot\int_0^t f_n(\tau)\sin\omega_n(t-\tau)\,\mathrm{d}\tau \\ =& \alpha_n\cos\omega_n t + \frac{\beta_n}{\omega_n}\sin\omega_n t - \frac{\omega\beta_n}{\omega_n}\cdot\int_0^t \sin\omega\tau\sin\omega_n(t-\tau)\,\mathrm{d}\tau \end{aligned}$$

其中

$$\int_0^t \sin\omega\tau\sin\omega_n(t-\tau)\,\mathrm{d}\tau = \frac{1}{2}\int_0^t \cos((\omega+\omega_n)\tau-\omega_n t) - \cos((\omega-\omega_n)\tau+\omega_n t)\,\mathrm{d}\tau$$

首先设若 $\omega \neq \omega_n$，那么

$$\int_0^t \sin\omega\tau\sin\omega_n(t-\tau)\,\mathrm{d}\tau = \frac{\omega\sin\omega_n t - \omega_n\sin\omega t}{\omega^2-\omega_n^2}$$

从而

$$T_n(t) = \alpha_n\cos\omega_n t + \frac{\beta_n}{\omega_n}\sin\omega_n t - \frac{\omega\beta_n}{\omega_n\left(\omega^2-\omega_n^2\right)}\cdot(\omega\sin\omega_n t - \omega_n\sin\omega t) \tag{5}$$

其中 $\alpha_n, \beta_n, \omega_n$ 由式2, 3, 4 给出

1. 若不产生共振现象，即 $\omega \neq \omega_n, \forall\, n \in \mathbb{Z}_{>0}$，那么

   $$u(x,t) = A(x-\ell)\sin\omega t + \sum_{n=1}^\infty X_n(x)T_n(t).$$

   其中 T_n, X_n 由式5, 1 给出

2. 若产生共振，设 $\omega = \omega_k$，则

   $$\widetilde{T}_k(t) = \lim_{\omega\to\omega_k} T_k(t) = \alpha_k\cos\omega_k t + \frac{\beta_k}{\omega_k}\sin\omega_k t - \beta_k\left(\frac{\sin\omega_k t}{2\omega_k} - \frac{t\cos\omega_k t}{2}\right)$$

   那么

   $$u(x,t) = A(x-\ell)\sin\omega t + \sum_{\substack{n\geq 1\\ n\neq k}} X_n(x)T_n(t) + X_k(x)\widetilde{T}_k(t).$$

□

【题目 9】(课本第 27 题) 考虑定解问题

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & (x,t) \in Q \\ u(0,t) = u(\ell,t) = 0 & t \geq 0; \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & x \in [0,\ell] \end{cases}$$

要使解为古典解，那么 ϕ, ψ, f 要加什么条件?

证明.

$\phi \in C^2[0,\ell], \psi \in C^3[0,\ell], f \in C^2\left([0,\ell] \times (0,+\infty)\right)$ 令 $u(x,t) = X(x)T(t)$ 并且满足

$$\phi(0) = \phi(\ell) = \phi''(0) = \phi''(\ell) = \psi(0) = \psi(\ell) = f(0,t) = f(\ell,t) = 0.$$

下面给出证明。

原问题的特征方程为

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\ell) = 0 \end{cases}$$

解得

$$X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right), \qquad \forall\, n \in \mathbb{Z}_{>0}.$$

记

$$f_n(t) = \frac{2}{\ell}\int_0^\ell f(x,t)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x, \qquad \phi_n = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x, \qquad \psi_n = \frac{2}{\ell}\int_0^\ell \psi(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x,$$

那么 T_n 满足微分方程

$$\begin{cases} T_n'' + \left(\frac{n\pi}{\ell}\right)^2 T_n = f_n(t) \\ T_n(0) = \phi_n,\ T_n'(0) = \psi_n \end{cases}$$

解得

$$T_n(t) = \phi_n \cos\frac{n\pi}{\ell}t + \frac{\ell}{n\pi}\psi_n \sin\frac{n\pi}{\ell}t + \frac{\ell}{n\pi}\int_0^t f_n(\tau)\sin\frac{n\pi}{\ell}(t-\tau)\,\mathrm{d}\tau.$$

记

$$A_n = \phi_n = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{n\pi x}{\ell}\right)\,\mathrm{d}x = \frac{2}{\ell}\cdot\left(\frac{\ell}{n\pi}\right)^3\int_0^\ell \phi'''(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x$$

$$B_n = \frac{\ell}{n\pi}\psi_n = \frac{2}{\ell}\cdot\frac{\ell}{n\pi}\int_0^\ell \psi(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x = -\frac{2}{\ell}\cdot\left(\frac{\ell}{n\pi}\right)^3\int_0^\ell \psi''(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x.$$

若记

$$v_n(x,t) = \left(\phi_n \cos\frac{n\pi}{\ell}t + \frac{\ell}{n\pi}\psi_n \sin\frac{n\pi}{\ell}t\right)\sin\frac{n\pi}{\ell}x.$$

那么

$$|v_n| \leq |A_n| + |B_n| = \mathcal{O}\left(\frac{1}{n^3}\right)$$

$$|Dv_n| = \mathcal{O}\left(\frac{1}{n^2}\right)$$

$$|D^2 v_n| \leq \left(\int_0^\ell \phi'''(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x\right)^2 + \left(\int_0^\ell \psi''(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x\right)^2 + \mathcal{O}\left(\frac{1}{n^2}\right)$$

但由 Bessel 不等式知

$$\sum_{n=1}^\infty \left(\int_0^\ell \phi'''(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x\right)^2 \leq \frac{2}{\ell}\int_0^\ell |\phi'''(x)|^2\,\mathrm{d}x < +\infty$$

$$\sum_{n=1}^{\infty}\left(\int_0^{\ell}\psi''(x)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x\right)^2 \le \frac{2}{\ell}\int_0^{\ell}|\psi''(x)|^2\,\mathrm{d}x < +\infty$$

从而 v_n, Dv_n, D^2v_n 求和一致收敛。下面证明对 $w_n = u_n - v_n$ 也有同样的结论。

记

$$g_n(t) = \frac{\ell}{n\pi}\int_0^t f_n(\tau)\sin\frac{n\pi}{\ell}(t-\tau)\,\mathrm{d}\tau.$$

则

$$f_n(\tau) = -\frac{2}{\ell}\cdot\left(\frac{\ell}{n\pi}\right)^2\int_0^{\ell} f_{xx}(x,\tau)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x = \mathcal{O}\left(\frac{\gamma_n}{n^2}\right).$$

其中

$$\gamma_n = \gamma_n(t) = \int_0^{\ell} f_{xx}(x,\tau)\sin\left(\frac{n\pi}{\ell}x\right)\,\mathrm{d}x.$$

由 Bessel 不等式有

$$\sum_{n=1}^{\infty}\gamma_n^2 \le \frac{2}{\ell}\int_0^{\ell}|f_{xx}|^2\,\mathrm{d}x \le 2\|f_{xx}\|_{\infty}.$$

故

$$D^{\alpha}\left(g_n(t)\sin\frac{n\pi}{\ell}x\right) = \mathcal{O}\left(\frac{\gamma_n}{n}\right)$$

其中 $\alpha = 0,1,2$。但

$$2\cdot\frac{\gamma_n}{n^2} \le \sum_{n=1}^{\infty}\left(\gamma_n^2 + 1/n^2\right) < +\infty.$$

从而一致收敛。

□

【题目 10】(课本第 28 题) 用能量不等式证明一维波动方程带有第三边值条件的初边值问题解的唯一性。

证明. 设一维波动方程带有第三边值条件的初边值问题为

$$\begin{cases} u_{tt} - a^2u_{xx} = f(x,t) & (x,t)\in(0,\ell)\times(0,+\infty) \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & x\in[0,\ell] \\ -u_x + \alpha u\Big|_{x=0} = g_1,\ u_x + \beta u\Big|_{x=\ell} = g_2 & \end{cases}$$

只需证明当 $\phi = \psi = g_1 = g_2 = 0$ 即可。

定义能量为

$$E(t) = \frac{1}{2}\int_0^{\ell} u_t^2 + a^2u_x^2\,\mathrm{d}x.$$

则

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_0^{\ell} u_tu_{tt} + a^2u_xu_{xt}\,\mathrm{d}t = \int_0^{\ell} u_t\left(u_{tt} - a^2u_{xx}\right)\,\mathrm{d}x + a^2u_xu_t\Big|_0^{\ell} \\ &= a^2(u_x(\ell,t)u_t(\ell,t) - u_x(0,t)u_t(0,t)) = a^2(-\beta u(\ell,t)u_t(\ell,t) - \alpha u(0,t)u_t(0,t)) \\ &= -\frac{1}{2}a^2\left(\alpha(u^2)_t\Big|_{x=0} + \beta(u^2)_t\Big|_{x=\ell}\right)\end{aligned}$$

从而

$$E(t) - E(0) = -\frac{1}{2}\left(\alpha u^2(0,t) + \beta u^2(\ell,t)\right) \le 0.$$

即

$$E(t) \le E(0) = 0.$$

但恒有 $E(t)\ge 0$，故 $E(t) = 0$，即

$$u_t = u_x = 0.$$

这说明 $u = 0$。 □