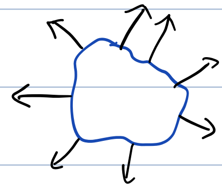
三.(3).

证明 $\bigstar \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{n}} = f & \text{on } \partial\Omega =: \Gamma \end{cases}$



除常数外解唯一.

其中 $u \in C^1(\overline{\Omega})$, $\Gamma = \partial\Omega$. f 连续.

Proof: 设 u, v 均为 $(\bigstar)$ 的解, 则只需证

$$u - v = C \iff \nabla(u-v) = 0.$$

$$\begin{cases} \Delta(u-v) = 0 & \text{in } \Omega \\ \dfrac{\partial(u-v)}{\partial \vec{n}} = \dfrac{\partial u}{\partial \vec{n}} - \dfrac{\partial v}{\partial \vec{n}} = f - f = 0 & \text{on } \partial\Omega \end{cases}$$

只需证: $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega \end{cases}$

的解为常数 $\iff \nabla u = 0$ $\bigstar$.

对 $\Delta u = 0$ 两侧乘 u 再积分:

$$0 = \int_\Omega u \Delta u \, dx = -\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} u \, dS(x)$$

$$= -\int_\Omega |\nabla u|^2 dx.$$

$$\Longrightarrow |\nabla u|^2 = 0 \Longrightarrow \nabla u = 0.$$

广义函数(分布).

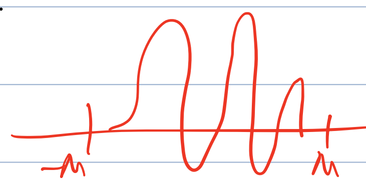
$$\mathcal{D} = (C_0^\infty(\mathbb{R}), \tau)$$

有界闭集

$C_0^\infty(\mathbb{R}) = \{ f \in C^\infty : \text{supp}(f) \text{ 为 compact} \}$.

$\text{supp}(f) = \overline{\{x : f(x) \neq 0\}} \rightarrow$ 闭

$\exists M > 0$, $|x| > M \Longrightarrow f(x) = 0$.



定义 $\varphi, \varphi_n \in C_0^\infty(\mathbb{R})$ $n \geq 1$

① $\exists M > 0$, $|x| \geq M \Rightarrow \varphi(x) = \varphi_n(x) = 0$

② $\lim\limits_{n\to\infty} \max\limits_{[-M,M]} |\varphi_n^{(k)}(x) - \varphi^{(k)}(x)| = 0$. $\forall k \geq 0$.

$k=0$ 时 $\lim\limits_{n\to\infty} \max\limits_{[-M,M]} |\varphi_n(x) - \varphi(x)| = 0$.

称 $\{\varphi_n\}$ 收敛于 φ. ~D了. 规定了上述收敛性的线性空间

$C_0^\infty(\mathbb{R})$ 称为基本空间 $\mathcal{D}(\mathbb{R})$, $\varphi \in \mathcal{D}(\mathbb{R})$ 试验函数.

目的是：使 $\mathcal{D}(\mathbb{R})$ 为完备空间

↓

Cauchy 列都收敛

定义(广义函数). $\mathcal{D}(\mathbb{R})$ 上的连续线性泛函称为广义函数.

$\mathcal{D}'(\mathbb{R})$. i.e. $\forall f \in \mathcal{D}'(\mathbb{R})$

① $f: \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$.

② $f(\alpha\varphi + \beta\psi) = \alpha f(\varphi) + \beta f(\psi)$ $\alpha, \beta \in \mathbb{R}$, $\varphi, \psi \in \mathcal{D}(\mathbb{R})$.

③ $\varphi_n \to \varphi$ (in $\mathcal{D}$) 有 $f(\varphi_n) \to f(\varphi)$. (数列收敛)

$\langle f, \varphi \rangle := f(\varphi)$ (对偶积)

$(\alpha f + \beta g)(\varphi) = \alpha f(\varphi) + \beta g(\varphi)$

$\langle \alpha f + \beta g, \varphi \rangle = \alpha \langle f, \varphi \rangle + \beta \langle g, \varphi \rangle$

例: $\delta(x) \in \mathcal{D}'$

① $\langle \delta, \varphi \rangle = \varphi(0)$ $\varphi \in \mathcal{D}(\mathbb{R})$

② $\langle \delta, \alpha\varphi + \beta\psi \rangle = (\alpha\varphi + \beta\psi)|_0 = \alpha\varphi(0) + \beta\psi(0)$

$= \alpha\langle \delta, \varphi \rangle + \beta \langle \delta, \psi \rangle$

③ $\varphi_n \to \varphi$ $\max\limits_{[-M,M]} |\varphi_n(x) - \varphi(x)| \to 0$ $(n\to\infty)$

$|\langle \delta, \varphi_n - \varphi \rangle| = |\varphi_n(0) - \varphi(0)| \leq \max\limits_{[-M,M]} |\varphi_n(x) - \varphi(x)| \to 0$ $(n\to\infty)$

综上：$\delta \in \mathcal{D}'$

例：$f \in L^1_{loc}(\mathbb{R})$，$\forall M > 0$.

$$\int_{-M}^{M} |f(x)|\,dx < \infty.$$

$$\langle f, \varphi \rangle =: \int_{\mathbb{R}} f(x)\varphi(x)\,dx \qquad \forall \varphi \in \mathcal{D}(\mathbb{R})$$

$\mathrm{supp}(\varphi)$ 紧 $\Rightarrow$ $\exists M>0$ $\mathrm{supp}(\varphi) \subset [-M, M]$.

$$\Rightarrow \quad |\langle f, \varphi \rangle| \le \int_{-M}^{M} |f(x)|\,dx \cdot \|\varphi\|_\infty$$

$$< +\infty.$$

$\Rightarrow$ 定义合理.

一般的，可将 $\langle f, \varphi \rangle$ 均理解成

$$\int_{\mathbb{R}} f(x)\varphi(x)\,dx$$

$\delta \rightsquigarrow$. $\displaystyle\int_{\mathbb{R}} \delta(x)\varphi(x)\,dx = \varphi(0)$.

Remark δ 不是局部可积函数 P124

不存在 $f \in L^1_{loc}(\mathbb{R})$ s.t. $\int f(x)\varphi(x)\,dx = \varphi(0)$.

广义函数收敛.

$f \in \mathcal{D}'(\mathbb{R})$, $f_n \in \mathcal{D}'(\mathbb{R})$ 若

$$\lim_{n\to\infty} \langle f_n, \varphi \rangle = \langle f, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\mathbb{R})$$

称 f_n 收敛于 f. x.

(数分) $\{f_n\}$ 逐点收敛于 f 指：

$\forall x \in \mathbb{R}$ $\displaystyle\lim_{n\to\infty} f_n(x) = f(x)$

$f, f_\lambda \in \mathcal{D}'(\mathbb{R})$ $\lambda \to \lambda_0$

$$\lim_{\lambda\to\lambda_0} \langle f_\lambda, \varphi \rangle = \langle f, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

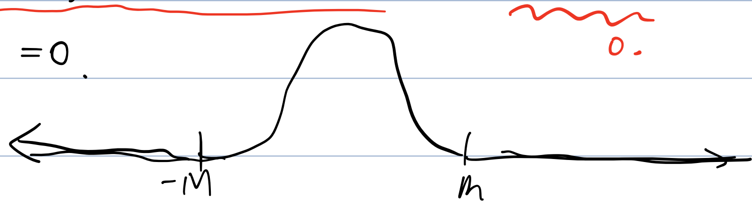
称 f_λ 收敛于 f.

广义函数运算

$\forall f \in C^k(\mathbb{R})$, $\varphi \in \mathcal{D}(\mathbb{R})$.

$$\int_{\mathbb{R}} f'\varphi \, dx = -\int f\varphi' \, dx + \underbrace{f\varphi \Big|_{-\infty}^{+\infty}}_{0.}$$

$\lim\limits_{x\to\infty} \varphi(x) = 0$.



$$\Longrightarrow \quad \int_{\mathbb{R}} f'\varphi \, dx = -\int f\varphi' \, dx.$$

$$\int_{\mathbb{R}} f^{(k)}\varphi \, dx = (-1)^k \int f \varphi^{(k)} \, dx.$$

$f \in L_{loc}(\mathbb{R})$，若存在 $g \in L_{loc}(\mathbb{R})$ s.t.

$$(-1)^k \int_{\mathbb{R}} f(x)\, \varphi^{(k)}(x) \, dx = \int_{\mathbb{R}} g(x)\, \varphi(x) \, dx. \qquad \forall \varphi \in \mathcal{D}$$

称 g 为 f 的弱(k)阶导.

$f^{(k)}$

广义函数 k 阶导

$\in \mathcal{D}.$

$f \in \mathcal{D}'(\mathbb{R})$，定义 $f^{(k)}$ 为满足下述等式

$$\langle f^{(k)}, \varphi \rangle = (-1)^k \langle f, \varphi^{(k)} \rangle \qquad \forall \varphi \in \mathcal{D}.$$

验证 $f^{(k)}$ 确实是广义函数，只需验证连续性.

令 $\varphi_n \xrightarrow{\mathcal{D}} \varphi$，有

$$\lim_{n\to\infty} \max_{[-M,M]} |\varphi_n^{(k+\ell)}(x) - \varphi_n^{(k+\ell)}(x)| \to 0 \qquad (\ell \geq 0)$$

记 $\varphi^{(k)} = \psi$，则由上述收敛

$$\varphi_n^{(k)} =: \psi_n \quad \lim_{n\to\infty} \max_{[-M,M]} |\psi_n^{(\ell)}(x) - \psi_n^{(\ell)}(x)| \to 0 \qquad (\ell \geq 0).$$

即 $\varphi_n \xrightarrow{\mathcal{D}} \varphi$

$$|\langle f^{(k)}, \varphi_n - \varphi\rangle| = |\langle f, \varphi_n^{(k)} - \varphi^{(k)}\rangle|$$

$$= |\langle f, \psi_n - \psi\rangle| \to 0$$

☆

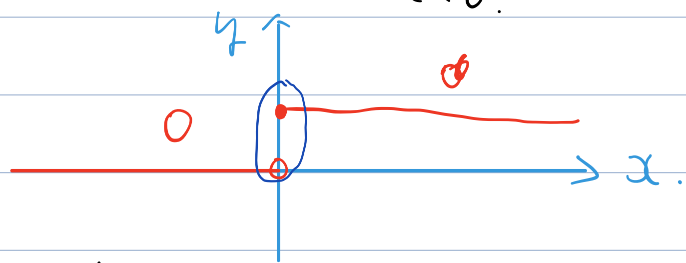
例：δ 广义微商

$$\langle \delta^{(k)}, \varphi\rangle = (-1)^k \langle \delta, \varphi^{(k)}\rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

$$= (-1)^k \varphi^{(k)}(0).$$

例：

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$



$$\langle H', \varphi\rangle = (-1)\langle H, \varphi'\rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R})$$

$$= -\int_{\mathbb{R}} H(x)\varphi'(x)\,dx$$

$$= -\int_0^{+\infty} \varphi'(x)\,dx$$

$$= -\varphi(x)\Big|_0^{+\infty}$$

$$= \varphi(0)$$

$$= \langle \delta, \varphi\rangle$$

δ 是卷积幺元.

$$\langle \delta, \varphi\rangle = \int \delta(y)\varphi(y)\,dy = \varphi(0)$$

$$\delta * \varphi(x) = \int \delta(y)\varphi(x-y)\,dx = \varphi(x)$$

$$\delta * \varphi = \varphi.$$

热方程的解：

$$u(x,t) = g_{(t)}*\varphi(x) + \int_0^t f_\tau * g_{t-\tau}(x)\,d\tau.$$

$$g_{(t)} = \begin{cases} \dfrac{1}{2a\sqrt{\pi t}}\, e^{\frac{-x^2}{4a^2t}} & t>0. \\ 0 & t\le 0. \end{cases}$$

若 $f(x,t)=0$ 即 $\begin{cases} u_t - u_{xx} = 0 \\ u|_{t=0} = \varphi. \end{cases}$

解为：$u(x,t) = g_{(t)} * \varphi(x).$

$u(x,0) = \lim\limits_{t\to 0} u(x,t) = \lim\limits_{t\to 0} g_{(t)} * \varphi(x) = \varphi.$ ☆

$\delta * \varphi = \varphi.$

$\lim\limits_{t\to 0} g_t = \delta$

$\boxed{f(x) \longrightarrow f_t(x) = \frac{1}{t} f\left(\frac{x}{t}\right)}$

$g_{(1)}(x) = \dfrac{1}{2a\sqrt{\pi}} e^{-\frac{x^2}{4a}} =: K(x)$

$g_{(t)}(x) = \dfrac{1}{2a\sqrt{\pi}\cdot\sqrt{t}} e^{-\frac{x^2}{4at}} = \frac{1}{\sqrt{t}} K\left(\frac{x}{\sqrt{t}}\right) = K_{\sqrt{t}}(x).$

$\lim\limits_{t\to 0} K_{\sqrt{t}} \to \delta.$ ☆

$u = g_{(t)} * \varphi(x) = \int_{\mathbb{R}} \boxed{g_{(t)}(x-y)}\, \varphi(y)\, dy.$

$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) g_{(t)} = 0 \qquad (t>0).$

$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u = \int_{\mathbb{R}} \left(\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) g_{(t)}(x-y)\right) \varphi(y)\, dy = 0.$ ☆

$K_{\sqrt{t}} * \varphi \to \varphi$

恒等逼近

$\phi \in L^1(\mathbb{R}^n)$, $\int \phi\, dx = 1$, $t>0$ 定义

$\phi_t(x) = \frac{1}{t^n} \phi\left(\frac{x}{t}\right).$

称 $\{\phi_t\}$ 为一个恒等逼近，则

$$\lim_{t\to 0} \|\phi_t * f - f\|_u = 0 \qquad \forall f \in C_0(\mathbb{R}^n)$$

$$\|f\|_u = \sup_{x\in\mathbb{R}^n} |f(x)|$$

$$\Longrightarrow \lim_{t\to 0} \phi_t * f = f \qquad \lim_{t\to 0}\phi_t = \delta.$$

Proof.

$$\int_{\mathbb{R}^n} \phi_t(x)\,dx = \int_{\mathbb{R}^n} \frac{1}{t^n}\phi\Big(\frac{x}{t}\Big)dx \overset{\frac{x}{t}=y}{=} \int_{\mathbb{R}^n}\phi(y)\,dy = 1.$$

$$\phi_t * f(x) - f(x) = -f(x) + \int_{\mathbb{R}^n} \frac{1}{t^n}\phi\Big(\frac{y}{t}\Big) f(x-y)\,dy. \qquad \frac{y}{t}\mapsto y.$$

$$= \int_{\mathbb{R}^n} \phi(y) f(x-ty)\,dy - f(x) \qquad \int\phi = 1.$$

$$= \int_{\mathbb{R}^n} \phi(y)\big(f(x-ty) - f(x)\big)dy$$

$$f \in C_0(\mathbb{R}^n) \Longrightarrow \forall \varepsilon>0,\ \exists\delta.$$

$$|h|<\delta \Longrightarrow \|f(x-h) - f(x)\|_u < \varepsilon \quad (\text{一致连续})$$

$$\Longrightarrow \sup|\phi_t * f(x) - f(x)| \qquad |ty|<\delta \qquad |ty|\geq\delta.$$

$$\leq \int_{\mathbb{R}^n} |\phi(y)|\,|f(x-ty) - f(x)|\,dy$$

$$= \int_{|y|<\delta/t} |\phi(y)|\,|f(x-ty) - f(x)|\,dy$$

$$+ \int_{|y|\geq\delta/t} |\phi(y)|\,|f(x-ty) - f(x)|\,dy$$

$$\leq \varepsilon\cdot\int_{\mathbb{R}^n}|\phi(y)|\,dy \qquad \frac{\delta}{t}\to+\infty$$

$$+ 2\|f\|_u \cdot \int_{|y|\geq\delta/t}|\phi(y)|\,dy$$

$$\leq \|\phi\|_1\cdot\varepsilon + 2\|f\|_u \int_{|y|\geq\delta/t}|\phi(y)|\,dy$$

$$\sup_x |\phi_t * f(x) - f(x)| \leq \varepsilon\cdot\|\phi\|_1 + 2\|f\|_u \int_{|y|\geq\delta/t}|\phi(y)|\,dy$$

$$\to \varepsilon\cdot\|\phi\|_1 \qquad (t\to 0)$$

$$\lim_{t\to 0} \underline{\sup |\varphi_t * f(x) - f(x)|} \leq \varepsilon \cdot \|\varphi\|_1$$

由 ε 任意性： $\lim_{t\to 0} \|\varphi_t * f - f\|_{L^1} = 0.$

$\|h_{\sqrt{t}} * f - f\|_{L^1} \Rightarrow 0.$ $h_{\sqrt{t}} \to \delta$

$$☆\ K(x;t) = \begin{cases} \dfrac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} & t>\tau \\ 0. & t\leq\tau \end{cases}$$

$h_{\sqrt{t}}(x).$

热方程基本解 (131, 133)

$$\begin{cases} u_t - a^2 u_{xx} = 0. \\ u|_{t=0} = \delta(x-\xi). \end{cases}$$ 参数. ☆

若 $f(x,t;\xi)$ 满足 (☆)，则称 f 为热方程基本解

可取为 $K(x-\xi;t)$

$$\lim_{t\to 0} K(x-\xi;t) = \delta(x-\xi).$$

$$\begin{cases} u_t - a^2 u_{xx} = \delta(x-\xi, t-\tau) \\ u|_{t=0} = 0. \end{cases} \Rightarrow \text{可取 } K(x-\xi, t-\tau)$$

Laplace 方程基本解：

$$-\Delta u = \delta(x-\xi).$$

- 满足 $-\Delta u = 0$ 的函数. 是径向的，即 $u(x) = u(Tx)$
  $u(|x|) = v(x)$ 　　即 $-\Delta$ 作用两侧相等.
- Δ 写成 新形式 　　$r = \sqrt{x_1^2 + \cdots + x_n^2}$

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x_i} = \frac{x_i}{|x|}\frac{\partial}{\partial r}.$$

$$\frac{\partial^2}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|}\frac{\partial}{\partial r}\right) = \frac{|x|^2-x_i^2}{|x|^3}\frac{\partial}{\partial r} + \frac{x_i^2}{|x|^2}\frac{\partial^2}{\partial r^2}.$$

$$\Longrightarrow \Delta = \sum \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial^2 r} + \frac{n-1}{r}\frac{\partial}{\partial r}$$

- 变为 ODE： $u(x) = v(|x|)$

$$-\Delta u = 0 \Longleftrightarrow v''(r) + \frac{n-1}{r}v'(r) = 0.$$

$$\Longrightarrow \frac{dv'}{dr} = -\frac{n-1}{r}v'(r)$$

$\log =: \ln$

$$\Longleftrightarrow \frac{dv'}{v'} = -\frac{n-1}{r}dr$$

$$\Longleftrightarrow d\log(v') = d\log\frac{1}{r^{n-1}}$$

$$\Longleftrightarrow v' = \frac{C_1}{r^{n-1}}$$

$$\Longleftrightarrow v = \begin{cases} \frac{\tilde{C}_1}{r^{n-2}} + C_2, & n \geq 3 \\ \tilde{C}_1 \log r + C_2 & n = 2 \end{cases}$$

$$\Phi(x) = \begin{cases} \underline{\frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}}} & n \geq 3 \\ -\frac{1}{2\pi}\log|x| & n = 2. \end{cases}$$

$\alpha(n)$: n 维球体积 例：3-d: $\frac{4}{3}\pi$

半径为1的

结论： $-\Delta u = \delta(x-\xi)$ 基本解为 $\Phi(x-\xi) =: \Gamma(x;\xi)$.

2-d: $-\frac{1}{2\pi}\log|x-\xi|$

3-d: $\frac{1}{3\times1\times\frac{4}{3}\pi}\frac{1}{|x-\xi|} = \frac{1}{4\pi|x-\xi|}$

Green 公式：

☆ $$\int_\Omega u\underline{\Delta v} - v\Delta u \, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} dS(x).$$

$$\begin{cases} -\Delta u = f & \text{in } \Omega. \end{cases}$$

$u = \varphi \quad \text{on } \partial\Omega.$

$$\boxed{\begin{aligned} &-\Delta\Gamma(x;\xi) = \delta(x-\xi). \\ &\int -\Delta\Gamma(x;\xi)\, u(x)\,dx = \int \delta(x-\xi)\, u(x)\,dx = u(\xi). \end{aligned}}$$

$$\int_\Omega u\,\Delta\Gamma(x;\xi) - \Gamma(x;\xi)\,\Delta u(x)\,dx = \int_{\partial\Omega} u(x)\frac{\partial\Gamma}{\partial n}(x;\xi) - \Gamma(x;\xi)\frac{\partial u(x)}{\partial n}\,ds(x)$$

$$\Rightarrow \quad -u(\xi) + \int_\Omega \Gamma(x;\xi) f(x)\,dx = \int_{\partial\Omega} u(x)\frac{\partial\Gamma}{\partial n}(x;\xi) - \Gamma(x;\xi)\frac{\partial u}{\partial n}\,ds$$

$$\Rightarrow \quad u(\xi) = -\int_\Omega \Gamma(x;\xi) f(x)\,dx + \int_{\partial\Omega} \Gamma(x;\xi)\frac{\partial u}{\partial n}(x) - \varphi(x)\frac{\partial\Gamma}{\partial n}(x;\xi)\,ds \quad ①$$

· Γ 已知. f 已知 $u|_{\partial\Omega}=\varphi$ 已知

· $\frac{\partial u}{\partial n}$ 未知. ⟶ 想法:去掉这一项

若 $\exists\ g(x;\xi)$ 使得

$$-\Delta g(x;\xi) = 0 \quad \text{in } \Omega.$$

则.

$$0 = -\int_\Omega g(x;\xi) f(x)\,dx + \int_{\partial\Omega} g(x;\xi)\frac{\partial u}{\partial n}(x) - \varphi(x)\frac{\partial g}{\partial n}(x;\xi)\,ds. \quad ②$$

① + ② $\Rightarrow$ $\Gamma + g = G$

$$u(\xi) = -\int_\Omega G(x;\xi) f(x)\,dx + \int_{\partial\Omega} G(x;\xi)\frac{\partial u}{\partial n} - \varphi(x)\frac{\partial G}{\partial n}(x;\xi)\,ds$$

可令 $G(x;\xi)|_{\partial\Omega} = 0. \Leftrightarrow g(x;\xi)|_{\partial\Omega} = -\Gamma(x;\xi)$

$$\begin{cases} -\Delta g(x;\xi) = 0. & \Omega. \\ g(x;\xi)|_{\partial\Omega} = -\Gamma(x;\xi)|_{\partial\Omega} \end{cases}$$

★ $$\begin{cases} -\Delta G(x;\xi) = \delta(x;\xi) & \Omega. \\ G(x,\xi)|_{\partial\Omega} = 0. \end{cases}$$

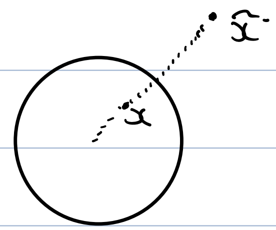
$G(x;\xi)$ ⟶ Green function

Green function. (镜象法).

① 球面上 Green function.

$B(0,R)$. $x \in \mathbb{R}^n$, $x \neq 0$. ，定义其关于 $\partial B(0,R)$ 对称点为 $\tilde{x}$，满足

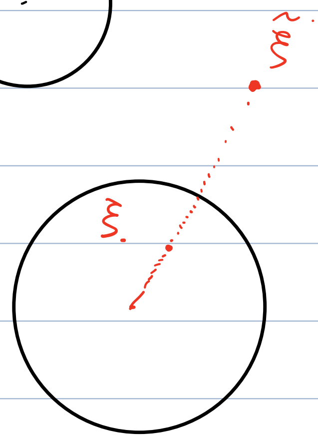
$$x \cdot \tilde{x} = R^2.$$



$$\Longrightarrow \quad \tilde{x} = kx$$

$$\tilde{x} \cdot x = k|x|^2 = R^2.$$

$$\Longrightarrow \quad \tilde{x} = \frac{R^2}{|x|^2} x$$



$$G(x;\xi) = \Gamma(x;\xi) + g(x;\xi).$$

$$-\Delta\Gamma = \delta(x-\xi) \qquad -\Delta g = 0.$$

$\Phi(k(x-\tilde{\xi}))$ 在 $B(0,R)$ 内调和.

$$-\Delta\Phi = 0 \quad \text{in } B(0,R).$$

$$\begin{cases} -\Delta g = 0 \\ g|_{\partial B} = -\Gamma|_{\partial B} \end{cases} \quad ☆$$

$$\Phi(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}$$

$|x| = R$ $\quad |k(x-\tilde{\xi})|^2 = |x-\xi|^2$ 时 $\Phi(k(x-\tilde{\xi})) = \Phi(x-\xi)$

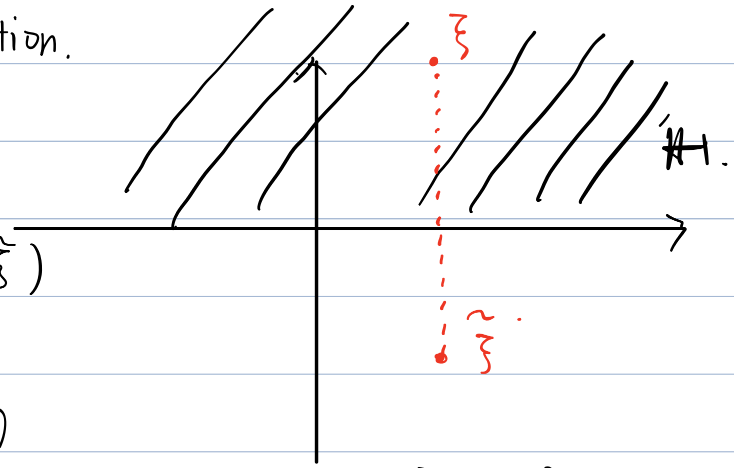
$$\tilde{\xi} = \frac{R^2}{|\xi|^2}\xi \qquad \partial B = \{x : |x| = R\}$$

$$\begin{aligned} |x-\tilde{\xi}|^2 &= |x|^2 - 2x\cdot\tilde{\xi} + |\tilde{\xi}|^2 \\ &= R^2 - 2x\cdot\xi\frac{R^2}{|\xi|^2} + \frac{R^2|x|^2}{|\xi|^2} \\ &= \frac{R^2}{|\xi|^2}\left(|\xi|^2 - 2x\cdot\xi + |x|^2\right) \\ &= \frac{R^2}{|\xi|^2}|x-\xi|^2 \end{aligned}$$

$$k = \frac{|\xi|}{R}.$$

$$G(x;\xi) = \Gamma(x;\xi) - \Phi\left(\frac{|\xi|}{R}(x-\tilde{\xi})\right).$$

② 半平面 Green function.



$$G(x;\xi) = \Gamma(x;\xi) - \Phi(x-\tilde{\xi})$$

当 $x \in \partial\mathbb{H}$ 时

$$\Gamma(x;\xi) = \Phi(x-\xi)$$

$$|x-\xi|^2 = (x_1-\xi_1)^2 + \cdots + (x_{n-1}-\xi_{n-1})^2 + \xi_n^2.$$

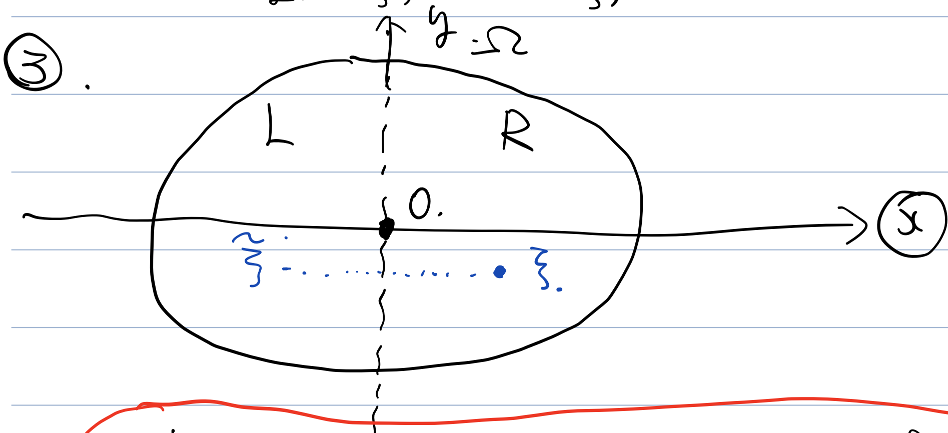
$$\parallel$$

$$|x-\tilde{\xi}|^2 = (x_1-\xi_1)^2 + \cdots + (x_{n-1}-\xi_{n-1})^2 + (-\xi_n)^2$$

即: $\Gamma(x;\xi) - \Phi(x-\tilde{\xi})$

$$= \Phi(x-\xi) - \Phi(x-\tilde{\xi}) = 0,$$

③.



已知 Ω 上的 Green，求 R 上 Green function.

$$\begin{cases} -\Delta G(x;\xi) = \delta(x-\xi) \\ G|_{\partial R} = 0. \end{cases}$$

延拓之后 $\tilde{G}(x;\xi)$，则

$$\begin{cases} -\Delta \tilde{G} = \delta(x-\xi) - \delta(x-\tilde{\xi}) \\ \tilde{G}|_{\partial\Omega} = 0. \end{cases}$$

$$\begin{cases} -\Delta \widetilde{G}_1 = \delta(x-\xi) \\ \widetilde{G}_1|_{\partial\Omega} = 0 \end{cases} \qquad \begin{cases} -\Delta \widetilde{G}_2 = \delta(x-\tilde{\xi}) \\ \widetilde{G}_2|_{\partial\Omega} = 0 \end{cases}$$

则 $\widetilde{G} = \widetilde{G}_1 + \widetilde{G}_2$.

$G = \widetilde{G}|_R$.