((Calderón - Zygmund)

Let $f \in L^1$, $\alpha > 0$ with.

$$\boxed{\alpha > \frac{1}{\mu(\mathbb{R}^n)} \int_{\mathbb{R}^n} f(x)\, d\mu(x)}$$

若 $\mu(\mathbb{R}^n) = \infty$

Then $\exists$ g, $\{b_k\}$, $\{B_k^*\}_{k=1}^{\infty}$ "balls" such that.

g $\qquad$ b_k $\qquad$ cube.

① $f = g + b \qquad b = \sum_{k=1}^{\infty} b_k$

② $\exists C > 0$, $\quad |g(x)| \leq C\alpha \quad$ a.e. $x \in \mathbb{R}^n$.

③ $\operatorname{supp}(b_k) \subset B_k^*$

$$\int_{\mathbb{R}^n} b_k(x)\, d\mu(x) = 0 \qquad \int_{\mathbb{R}^n} |b_k(x)|\, d\mu(x) \leq C\alpha\, \mu(B_k^*)$$

$\mathbb{R}^n = B_k^*$

④ $\sum_k \mu(B_k^*) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)|\, d\mu(x)$.

<u>Proof</u>: Let $\quad E_\alpha = \{x \in \mathbb{R}^n : \tilde{M}f(x) > \alpha\}$.

Where: $\quad \tilde{M}f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y)$

• E_α is open. $\quad \{B(x,r)\}$.

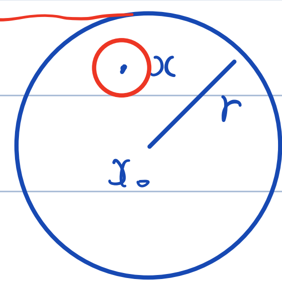
$\forall x \in E_\alpha = \{x \in \mathbb{R}^n : \tilde{M}f(x) > \alpha\}$.

then. $\exists$ $B = B(x_0, r_0)$ s.t.

· $x \in B(x_0, r_0) \qquad$ · $\frac{1}{\mu(B(x_0,r_0))} \int_{B(x_0,r_0)} |f(y)|\, d\mu(y) > \alpha$.

$\Rightarrow \exists\, \delta > 0$ s.t.

$$B(x, \delta) \subset B(x_0, r_0)$$



$\forall y \in B(x,\delta)$ 有 $\quad y \in B(x_0, r_0)$

$\Rightarrow \quad \tilde{M}f(y) = \sup_{B \ni y} \frac{1}{\mu(B)} \int_B |f(z)|\, dz$

$$\geq \frac{1}{\mu(B(y_0,r_0))}\int_{B(y_0,r_0)}|f(z)|dz$$

$$> \alpha$$

$\Rightarrow \quad y \in E_\alpha$

$\Rightarrow \quad B(x,\delta) \subset E_\alpha \Rightarrow \quad E_\alpha$ is open.

- Assume. $E_\alpha \neq \mathbb{R}^n$.

Then apply lemma of § 3.2.

F closed. $F \neq \phi$, $\exists \{B_k\}_{k=1}^\infty \quad \{B_k^*\}_{k=1}^\infty \quad \{B_k^{**}\}_{k=1}^\infty$

① $i \neq j \Rightarrow B_i \neq B_j$

②. $\bigcup B_k^* = \mathcal{O} = F^c$

③ $\boxed{B_k^{**} \cap F \neq \phi}$ ✓ $\forall k \geq 1$.

Remark. $B_k^* = B(x_k, \frac{\delta(x_k)}{2}) \quad B_k^{**} = B(x_k, 2\delta(x_k))$.

We can take $\widetilde{B_k^{**}} = B(x_k, \beta\delta(x_k))$. where $\delta > 2$. satisfying ③.

Take $\mathcal{O} = E_\alpha$, $F = E_\alpha^c \neq \phi$,

$\exists \{B_k\} \quad \{B_k^*\} \quad \{B_k^{**}\}. \quad \{Q_k\}$ "cube"

s.t. $\quad B_k \subset Q_k \subset B_k^*$

$$Q_k : \begin{cases} i \neq j \Rightarrow Q_i \neq Q_j & \checkmark \quad i \neq j \Rightarrow B_i \neq B_j \\ \bigcup Q_i = \mathcal{O} = E_\alpha. & \checkmark \quad \bigcup B_k^* = \mathcal{O} = E_\alpha \end{cases}$$

$$\sum \mu(B_k) \leq \sum \mu(Q_k) = \mu(\bigcup Q_k) = \mu(E_\alpha)$$

Define: $\quad g(x) = \begin{cases} f(x) & x \in E_\alpha^c \\ \frac{1}{\mu(Q_j)}\int_{Q_j} f(y)dy & x \in Q_j \end{cases}$

$$b_k(x) = \left(f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y)\,dy \right) \chi_{Q_k}(x).$$

$$f = g + b = g + \sum b_k.$$

To prove ②: $|g| \leq C\alpha$ a.e. $x \in \mathbb{R}^n$.

By Lebesgue Differential theorem: for a.e. $x \in \boxed{E_\alpha^c = [\tilde{M}f \leq \alpha]}$.

$$|f(x)| = \left| \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\,dy \right|$$

$$\leq \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\,dy.$$

$$\leq \tilde{M}f(x) \leq \alpha.$$

i.e. $|g(x)| = |f(x)| \leq \alpha$ a.e. $x \in E_\alpha^c$.

$\forall\, k \geq 1$ $\quad B_k^{**} \cap E_\alpha^c \neq \phi \Rightarrow \exists\, y_k \in B_k^{**} \cap E_\alpha^c$.

$$\Rightarrow \begin{cases} \cdot\ y_k \in B_k^{**} \\ \cdot\ \alpha \geq \tilde{M}f(y_k) = \sup_{B \ni y_k} \frac{1}{\mu(B)} \int_B |f(z)|\,dz. \end{cases}$$

$$\Rightarrow \alpha \geq \tilde{M}f(y_k) \geq \frac{1}{\mu(B_k^{**})} \int_{B_k^{**}} |f(z)|\,dz.$$

$$\Rightarrow \int_{Q_k} |f(z)|\,dz \leq \int_{B_k^{**}} |f(z)|\,dz.$$

$$\leq \mu(B_k^{**}) \cdot \alpha \qquad B_k \subset Q_k$$

$$\leq C \cdot \alpha\, \mu(B_k).$$

$$\leq C \cdot \alpha\, \mu(Q_k).$$

$$\Rightarrow |g(x)| \leq \frac{1}{\mu(Q_k)} \int_{Q_k} |f(z)|\,dz \leq C\alpha \qquad \forall\, x \in Q_k.$$

$$\Rightarrow |g(x)| \leq C\alpha \qquad \text{a.e. } x \in \mathbb{R}^n.$$

To prove ③. $\begin{cases} \operatorname{Supp}(b_k) \subset B_k^{*} \\ \int b_k\,d\mu = 0 \\ \int |b_k|\,d\mu \leq C \cdot \alpha\, \mu(B_k^{*}) \end{cases}$

$$\int_{Q_k} |b_k(x)|\, d\mu(x) = \int_{Q_k} \left| f(x) - \frac{1}{|Q_k|}\int_{Q_k} f(y)\,d\mu(y) \right| d\mu(x)$$

$$\leq \int_{Q_k} |f(x)|\, d\mu(x) + \int_{Q_k} |f(y)|\, d\mu(y)$$

$$= 2\int_{Q_k} |f(x)|\, d\mu(x).$$

$$\leq 2C\cdot\alpha\, \mu(Q_k)$$

$$\leq C\cdot\alpha\, \mu(B_k^*).$$

To prove ④: $\sum_k \mu(B_k^*) \leq \frac{C}{\alpha}\int_{\mathbb{R}^n} |f(x)|\, d\mu(x)$

$$\sum_k \mu(B_k^*) \leq C\sum_k \mu(B_k)$$

$[\tilde{M}f(x) > \alpha]$.

$\shortparallel$

$$\leq C\sum_k \mu(Q_k) = C\mu(E_\alpha)$$

$$\leq C\frac{1}{\alpha}\|f\|_1$$

$$= C\frac{1}{\alpha}\int_{\mathbb{R}^n} |f(x)|\, d\mu(x).$$

- Consider $\{x\in\mathbb{R}^n : \tilde{M}f(x) > \alpha\} = \mathbb{R}^n$ then by weak $(1,1)$ of Maximal function:

$$\mu(\mathbb{R}^n) = \mu(E_\alpha) \leq \frac{C}{\alpha}\int_{\mathbb{R}^n} |f(x)|\, d\mu(x).$$

Let ② $g(x) = \frac{1}{\mu(\mathbb{R}^n)}\int_{\mathbb{R}^n} f(y)\, d\mu(y)$ $\qquad B_1 = \mathbb{R}^n$.

$b(x) = f(x) - g(x).$

then. $|g(x)| \leq \frac{1}{\mu(\mathbb{R}^n)}\int_{\mathbb{R}^n} |f(y)|\, d\mu(y) \leq \alpha$

③ $$\int_{\mathbb{R}^n} |b(x)|\, d\mu(x) \leq \int_{\mathbb{R}^n} \left| f(x) - \frac{1}{\mu(\mathbb{R}^n)}\int_{\mathbb{R}^n} f(y)\,d\mu(y) \right| d\mu(x).$$

$$\leq 2\int_{\mathbb{R}^n} |f(x)|\, dy.$$

$$\leq 2\alpha\, \mu(\mathbb{R}^n)$$

④ $\mu(\mathbb{R}^n) = \mu(B_1) \leq \frac{C}{\alpha}\int_{\mathbb{R}^n} |f(x)|\, d\mu(x).$ $\square$

$$Q_k = B_k^* \cap \left(\bigcup_{j<k} Q_j\right)^c \cap \left(\bigcup_{j>k} B_j\right)^c$$

$$Q_1 = \underbrace{B_1^*}_{\text{open}} \cap \underline{\left(\bigcup_{j>1} B_j\right)}^{c}.$$

Singular Integral.

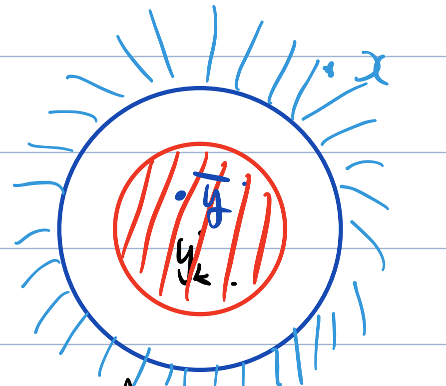
Theorem. Let T be a bounded operator on $L^q(\mathbb{R}^n)$ $(1<q\le\infty)$.

Let K be a function on $\mathbb{R}^n\times\mathbb{R}^n\setminus\Delta$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,d\mu(y) \quad \text{for } x\notin \operatorname{supp}(f)$$

where $f\in L^q(\mathbb{R}^n)$, has compact support.

$$\Delta = \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x=y\}.$$



if $x \in \operatorname{Supp}(f)$ $\boxed{K(x,x)}$

Further, suppose K also satisfies that:

$\exists\, C, A>0$

$$\int_{\mathbb{R}^n\setminus B(y_k, c\delta)} |K(x,y_k) - K(x,\bar y)|\,d\mu(x) \le A$$

for all $y\in\mathbb{R}^n$, $\delta>0$, $\bar y \in B(y_k,\delta)$.

$$\left(\int_{|x-y|>C|y-\bar y|} |K(x,y)-K(x,\bar y)|\,dx \le A\right).$$

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\,d\mu(y)$$

$K(x,y)$

$$\int_{|x-y|>C|y-\bar y|} |K(x-y) - K(x-\bar y)|\,dx \le A.$$

$$\int_{|z|>C|w|} |K(z) - K(z+w)|\,dx \le A.$$

Then T is weak $(1,1)$ and strong (p,p) $(1<p<q)$

Proof: T strong $(q,q) \rightsquigarrow$ weak (q,q) $\Big\}\xrightarrow{\text{Interpolation}}$ strong (p,p).

weak $(1,1)$

It suffices to check that T is weak $(1,1)$

i.e. $\boxed{\mu(\{x : |Tf(x)| > C'\alpha\}) \leq \frac{C}{\alpha}\int |f|\, d\mu.}$ ☆

Let $f = g + b$ be the C-Z decomposition at height $\boxed{\alpha}$

$|Tf| \leq |Tg| + |Tb|$

$\mu([|Tf| > C'\alpha]) \leq \boxed{\mu([|Tg| > \frac{C'\alpha}{2}])} + \boxed{\mu([|Tb| > \frac{C'\alpha}{2}])}.$

- $q < \infty$

$$\mu([|Tg| > \frac{C'\alpha}{2}]) \leq \left(\frac{2}{C'\alpha}\right)^q \int |Tg|^q\, d\mu.$$

$$\leq \frac{C}{\alpha^q}\int |g|^q\, d\mu. \to g \in L^q.\ g = \begin{cases} & E_\alpha^c \\ & E_\alpha \end{cases}$$

$$\leq \frac{C}{\alpha^q}\left(\int_{E_\alpha^c} |g|^q d\mu + \sum \int_{Q_k} |g|^q\, d\mu.\right).$$

$|g| \leq C\alpha$ a.e. $\qquad |g|^{q-1} \leq (C\alpha)^{q-1}$.

$$\leq \frac{C}{\alpha}\left(\int_{E_\alpha^c} |f|\, d\mu + \sum \int_{Q_k} |g|\, d\mu\right)$$

$$\leq \frac{C}{\alpha}\left(\int_{E_\alpha^c} |f|\, d\mu + \sum \int_{Q_k} \left|\frac{1}{\mu(Q_k)}\int_{Q_k} f\, d\mu\right| d\mu.\right.$$

$$\leq \frac{C}{\alpha}\left(\int_{E_\alpha^c} |f|\, d\mu + \sum \int_{Q_k} |f|\, d\mu\right).$$

$$= \frac{C}{\alpha}\int_{\mathbb{R}^n} |f|\, d\mu.$$

- $q = \infty$

$T \in \mathcal{L}(L^\infty(\mathbb{R}^n)) \Longrightarrow \exists\, A > 0$ s.t. $\qquad |g| \leq C\alpha$ a.e. x.

$$\|Tg\|_\infty \leq A\|g\|_\infty \leq AC\alpha$$

Take $\boxed{\frac{C'\alpha}{2} > AC\alpha}$ $\qquad C' > 2AC$

then $[|Tg| > \frac{C'\alpha}{2}] \subset [|Tg| > AC\alpha]$.

$\|Tg\|_\infty \leq AC\alpha$

$$\Rightarrow \quad \mu\left([|Tg| > A(\alpha)]\right) = 0$$

$$\Rightarrow \quad \mu\left([|Tg| > \tfrac{c'\alpha}{2}]\right) \le \mu\left([|Tg| > A(\alpha)]\right) = 0.$$

$$\Rightarrow \mu\left([|Tg| > \tfrac{c'\alpha}{2}]\right) \le \frac{C}{\alpha}\|f\|_{L^1(\mu)}$$

$$\boxed{\mu\left([|Tb| > \tfrac{c'\alpha}{2}]\right)} = \mu\left(\{x \in \textstyle\bigcup B_k^{**} \,;\, |Tb| > \tfrac{c'\alpha}{2}\}\right) + \mu\left(\{x \notin \textstyle\bigcup B_k^{**} : |Tb| > \tfrac{c'\alpha}{2}\}\right).$$

$$\le \mu\left(\textstyle\bigcup B_k^{**}\right) + \mu\left(\{x \notin \textstyle\bigcup B_k^{**} : |Tb| > \tfrac{c'\alpha}{2}\}\right).$$

$$\le \sum \mu(B_k^{**}) + \frac{C}{\alpha}\int_{\mathbb{R}^n \setminus \bigcup B_k^{**}} |Tb(x)|\, d\mu(x).$$

$$\le C\sum \mu(B_k) + \frac{C}{\alpha}\sum \int_{\mathbb{R}^n \setminus \bigcup B_k^{**}} |Tb_k(x)|\, d\mu(x).$$

$$\le C\sum \mu(Q_k) + \boxed{\frac{C}{\alpha}\sum \int_{\mathbb{R}^n \setminus B_k^{**}} |Tb_k(x)|\, d\mu(x).}$$

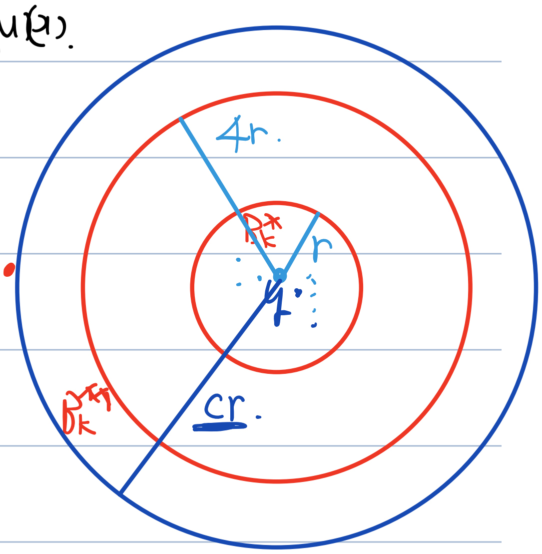
$$\sum \mu(Q_k) = \mu(E_\alpha) = \mu\left([\tilde{M}f(x) > \alpha]\right)$$

$$\le \boxed{\frac{C}{\alpha}\|f\|_{L^1(\mu)}}.$$

$$\int_{\mathbb{R}^n \setminus B_k^{**}} \left| \int_{B_k^*} K(x,y)\, b_k(y)\, d\mu(y) \right| d\mu(x).$$

$B_k^{**} = B(y_k, r_k)$ $y \in B_k^*$. $\quad x \notin B_k^{**}$

x



$x \in \operatorname{supp}(b_k)$

$$\int_{B_k^*} b_k(y)\, d\mu(y) = 0$$

$$\Rightarrow \int_{B_k^*} b_k(y)\, K(x, y_k)\, d\mu(y) = 0.$$

$$\Rightarrow \int_{\mathbb{R}^n \setminus B_k^{**}} \left| \int_{B_k^*} K(x,y)\, b_k(y)\, d\mu(y) \right| d\mu(x)$$

$$= \int_{\mathbb{R}^n \setminus B_k^{**}} \left| \int_{B_k^*} b_k(y)\left(K(x,y) - K(x,y_k)\right) d\mu(y) \right| d\mu(x).$$

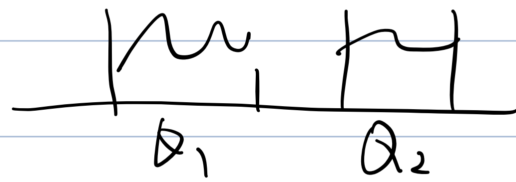
$$\leq \int_{B_k^*} |b_k(y)| \left(\int_{\mathbb{R}^n \setminus B_k^{**}} |K(x,y) - K(x,y_k)| \, d\mu(x) \right) d\mu(y).$$

$$\leq A \cdot \int_{B_k^*} |b_k(y)| \, d\mu(y)$$

$$\frac{C}{\alpha} \sum \left(\qquad \qquad \right)$$

$$\leq \frac{A \cdot C}{\alpha} \sum_k \int_{B_k^*} |b_k(y)| \, d\mu(y)$$

$$= \frac{AC}{\alpha} \sum_k \int_{Q_k} |b_k(y)| \, d\mu(y)$$



$$= \frac{AC}{\alpha} \sum_k \int_{\mathbb{R}^n} |b_k(y) \chi_{Q_k}(y)| \, d\mu(y)$$

$$= \frac{AC}{\alpha} \int_{\mathbb{R}^n} \left| \sum_k b_k(y) \chi_{Q_k}(y) \right| d\mu(y).$$

$$b_k = (\qquad) \chi_{Q_k}(y)$$

$$= \frac{AC}{\alpha} \int_{\mathbb{R}^n} |b(y)| \, d\mu(y).$$

$$\leq \frac{C}{\alpha} \|f\|_{L^1(\mu)}.$$

$$\Rightarrow \quad \mu([\,|Tf| > C'\alpha\,]) \leq \frac{C}{\alpha} \|f\|_{L^1(\mu)}. \qquad \square.$$