Weak L^p space.

- $\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\})$ $\quad (X, \mathcal{M}, \mu)$

  distribution function of f.

- $[f]_p = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{\frac{1}{p}}$ $\quad 1 \le p < \infty$

  $f \in$ weak L^p if
  $$[f]_p < \infty.$$

  $L^p \subset$ weak L^p

$$\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\})$$
$$\leq \int \left|\frac{f(x)}{\alpha}\right|^p dx = \frac{1}{\alpha^p} \|f\|_p^p$$
$$\Rightarrow \underbrace{\left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{\frac{1}{p}}}_{= [f]_p} \leq \|f\|_p$$

$T : (X, \mathcal{M}, \mu) \longrightarrow (Y, \mathcal{N}, \nu)$ sublinear

$$|T(f+g)| \leq |Tf| + |Tg| \quad f, g$$
$$|T(cf)| = c|Tf| \quad c > 0$$

- T strong (p, q) $\quad L^p \to L^q$

$$\|Tf\|_q \leq C\|f\|_p \quad f \in L^p$$

- T weak (p, q) $\quad L^p \longrightarrow$ weak L^q. $\quad$ weak (p, ∞) $=$ strong (p, ∞)

$$[Tf]_q \leq C\|f\|_p.$$

∞.

Weak $(1,1)$

$$[Tf]_1 \leq C \|f\|_1$$

$$\|$$

$$\sup_{\alpha>0} \alpha\, \mu(\{x : |Tf(x)| > \alpha\}) \leq C \|f\|_1$$

$$\Updownarrow$$

$$\boxed{\mu(\{x : |Tf(x)| > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1}$$

Theorem (Marcinkiewicz Interpolation). Let $(X, \mu), (Y, \nu)$ be measure space. $1 \leq p_0 < p_1 \leq \infty$. Let T be a sublinear operator from $L^{p_0}(X,\mu) + L^{p_1}(X,\mu)$ to the measure function on Y that is weak (p_0, p_0) and weak (p_1, p_1). Then

T is strong (p,p) for $p_0 < p < p_1$

Proof:

Weak (p_0, p_0)

$$\nu([|Tf| > \lambda]) \leq \left(\frac{A_0 \|f\|_{p_0}}{\lambda}\right)^{p_0}.$$

Weak (p_1, p_1)

$$\nu([|Tf| > \lambda]) \leq \left(\frac{A_1 \|f\|_{p_1}}{\lambda}\right)^{p_1} \qquad p_1 < \infty.$$

$$\|Tf\|_\infty \leq A_2 \|f\|_\infty. \qquad \boxed{p_1 = \infty}$$

$$\boxed{\text{Strong } (p,p) \qquad \|Tf\|_p \leq M \cdot \|f\|_p.}$$

Let $f \in L^p$. $\forall \lambda > 0$,

$$f_0 = f \chi_{[|f| > \alpha]} \rightsquigarrow L^{p_0}.$$

$f_1 = f\chi_{[|f|\leq c\lambda]}$. $\longrightarrow L^{p_1}$ $\quad p_0 < p < p_1$

then: $\quad f = f_0 + f_1$

$$\|f_0\|_{p_0}^{p_0} = \int_{[|f|>c\lambda]} |f|^{p_0} d\mu \leq \int_{[|f|>c\lambda]} |f|^{p_0} \left(\frac{|f|}{c\lambda}\right)^{p-p_0} d\mu.$$

$$= \left(\frac{1}{c\lambda}\right)^{p-p_0} \int |f|^p d\mu < \infty.$$

$$\|f_1\|_{p_1}^{p_1} < \infty.$$

$$\begin{cases} f_0 \in L^{p_0} \\ f_1 \in L^{p_1} \end{cases}$$

$Tf_0 \quad Tf_1$

$f = f_0 + f_1$

① $p_1 < \infty$.

$$\|Tf\|_p^p = \int |Tf|^p d\nu = p\int_0^\infty \lambda^{p-1} \boxed{\nu(\{|Tf(x)| > \lambda\})} d\lambda.$$

$$\leq p\int_0^\infty \lambda^{p-1} \nu(\{|\underbrace{Tf_1}_{L_{p_1}}(x)| > \lambda/2\}) d\lambda$$

$$+ p\int_0^\infty \lambda^{p-1} \nu(\{|\underbrace{Tf_0}_{L^{p_0}}(x)| > \lambda/2\}) d\lambda.$$

$$\leq p\int_0^\infty \lambda^{p-1} \left(\frac{2A_1}{\lambda}\right)^{p_1} \|f_1\|_{p_1}^{p_1} d\lambda$$

$$+ p\int_0^\infty \lambda^{p-1} \left(\frac{2A_0}{\lambda}\right)^{p_0} \|f_0\|_{p_0}^{p_0} d\lambda.$$

$$p\int_0^\infty \lambda^{p-1} \left(\frac{2A_0}{\lambda}\right)^{p_0} \left(\int |f_0|^{p_0} d\mu\right) d\lambda.$$

$\lambda < \frac{|f|}{c}$.

$$= p\int_0^\infty \lambda^{p-1} \left(\frac{2A_0}{\lambda}\right)^{p_0} \left(\int_{[|f|>c\lambda]} |f|^{p_0} d\mu\right) d\lambda$$

$$= p\int |f|^{p_0} \left(\int_0^{\frac{|f|}{c}} \lambda^{p-1-p_0} d\lambda\right) d\mu. \quad (2A_0)^{p_0}.$$

$$= \frac{p}{p-p_0} \int |f|^{p_0} \left(\frac{|f|}{c}\right)^{p-p_0} d\mu \quad (2A_0)^{p_0}.$$

$$= \frac{p}{p-p_0} \frac{1}{c^{p-p_0}} \cdot (2A_0)^{p_0} \underbrace{\int |f|^p \, d\mu}.$$

$$= \underline{\frac{p}{p-p_0} \frac{1}{c^{p-p_0}} (2A_0)^{p_0} \|f\|_p^p}. \quad \forall$$

$$p\int_0^\infty \lambda^{p-1} \left(\frac{2A_1}{\lambda}\right)^{p_1} \left(\int |f_1|^{p_1} d\mu\right) d\lambda$$

$$\le \underline{\frac{p}{p_1-p} \frac{1}{c^{p-p_1}} (2A_1)^{p_1} \|f\|_p^p}.$$

$$\Rightarrow \|Tf\|_p^p \le \left(\frac{p(2A_0)^{p_0}}{p-p_0} \frac{1}{c^{p-p_0}} + \frac{p(2A_1)^{p_1}}{p_1-p} \frac{1}{c^{p-p_1}} \right) \|f\|_p^p.$$

$p_1 = \infty$:

$$\|Tf\|_p^p = p\int_0^\infty \lambda^{p-1} \nu(\{|Tf(x)| > \lambda\}) \, d\lambda$$

$$\le \underbrace{p\int_0^\infty \lambda^{p-1} \nu(\{|Tf_0(x)| > \lambda/2\}) \, d\lambda} \le M \cdot \|f\|_p^p$$

$$+ p\int_0^\infty \lambda^{p-1} \nu\left(\boxed{\{|Tf_1(x)| > \lambda/2\}}\right) d\lambda.$$

T: weak (∞, ∞) $\qquad f_1 = f\chi_{[|f| \le c\lambda]}$

$$\|Tf_1\|_\infty \le A_1\|f_1\|_\infty \le A_1 c\lambda.$$

Let $\quad A_1 c\lambda < \lambda/2 \iff c < \frac{1}{2A_1}$

then $\quad \{|Tf_1(x)| > \lambda/2\} = \phi.$

since $\quad \|Tf_1\|_\infty \le \frac{\lambda}{2},$

$$\Rightarrow \|Tf\|_p^p \le p\int_0^\infty \lambda^{p-1} \nu(\{|Tf_0(x)| > \lambda/2\}) \, d\lambda.$$

$$\leq C \|f\|_p^p.$$

Theorem (Calderón - Zygmund decomposition).

Let $f \in L^1$ and $\alpha > 0$ with

$$\alpha > \frac{1}{\mu(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f| \, d\mu$$

Then $\exists$ g, $\{b_k\}_{k=1}^{\infty}$, balls $\{B_k^*\}_{k=1}^{\infty}$ such that:

① $f = \underbrace{g}_{\text{good}} + \underbrace{b}_{\text{bad}}$ $\qquad b = \sum b_k$, $\operatorname{supp}(b_k) \subset B_k^*$.

② $|g(x)| \leq C\alpha$ $\quad$ a.e $x \in \mathbb{R}^n$.

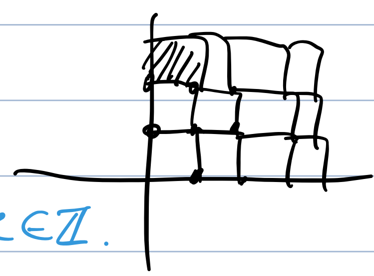
③.

$$\int |b_k(x)| \, d\mu(x) \leq C \mu(B_k^*)$$

$$\int b_k(x) \, d\mu(x) = 0.$$

④. $\sum_k \mu(B_k^*) \leq \frac{C}{\alpha} \int |f(x)| \, d\mu(x).$

Dyadic maximal function.



$$\mathcal{Q}_0 = \{x + [0,1)^n : x_i \in \mathbb{Z}\}.$$

$$\mathcal{Q}_k = \{2^{-k}(x + [0,1)^n) : x_i \in \mathbb{Z}\}. \leftarrow k \in \mathbb{Z}.$$

$$\mathcal{Q} = \bigcup_{k \in \mathbb{Z}} \mathcal{Q}_k. \longrightarrow \text{dyadic cube}.$$

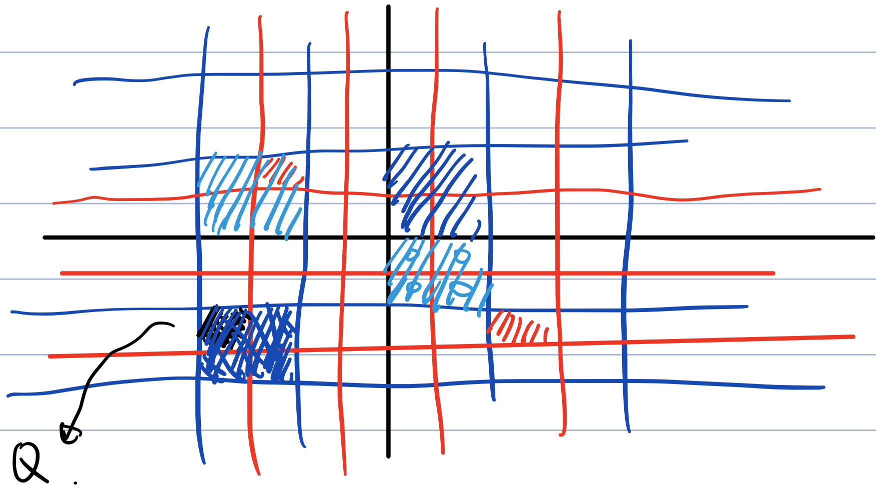
Properties:

① $\forall x \in \mathbb{R}^n$, $\forall k \in \mathbb{Z}$, $\exists Q \in \mathcal{Q}_k$ s.t.

$x \in Q.$

② . $\forall P, Q \in \mathcal{Q}.$

either $P \cap Q \neq \phi$ or $P \subset Q \,/\, Q \subset P.$



Q .

③ . for $j < k$, given $Q \in \mathcal{Q}_k$. $\exists ! P \in \mathcal{Q}_j$

s.t. $Q \subset P.$

and Q contain 2^n dyadic cube of $\mathcal{Q}_{k+1}$.

Give $f \in L^1_{loc}(\mathbb{R}^n)$, define.

$$E_k f(x) = \sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x). \rightarrow \text{simple function.}$$

then : if Ω is the union of cubes of $\mathcal{Q}_k$. then

$$\int_\Omega E_k f(x) = \int_\Omega f$$

$$\Omega = \bigsqcup_{j=1}^{\infty} Q_j \qquad \mathcal{Q}_k$$

$$\int_\Omega E_k f(x) = \sum \int_{Q_j} E_k f(x). = \sum \int_{Q_j} \left(\frac{1}{|Q_j|} \int_{Q_j} f \right) \chi_{Q_j}$$

$$= \sum \int_{Q_j} f = \int_\Omega f.$$

dyadic maximal function :

$$M_d f(x) =: \sup_k |E_k f(x)|$$

given x $\quad M_d f(x) = \sup \left\{ \left| \frac{1}{|Q|}\int_Q f \right| ; \underline{x \in Q,\ Q \in \mathcal{Q}_k} \right\}$

Theorem.

① M_d is weak $(1,1)$.

②. If $f \in L^1_{loc}(\mathbb{R}^n)$, then.

$$\lim_{k\to\infty} E_k f(x) = f(x) \quad a.e. \quad x \in \mathbb{R}^n.$$

Proof. $\quad \left|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}\right| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|\,dx$

We can assume that f is nonnegative. Claim:

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \bigcup_k \Omega_k.$$

Where :

$$\boxed{\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \lambda ;\ j < k \Rightarrow E_j f(x) \leq \lambda\}}.$$

Since $f \in L^1$, $\quad E_k f(x) = \frac{1}{|Q|}\int_Q f(y)\,dy \qquad (x \in Q,\ Q \in \mathcal{Q}_k)$

$$\leq \frac{1}{\underbrace{|Q|}_{\substack{|Q| \\ (2^{-k})^n}}}\int_{\mathbb{R}^n} f(y)\,dy$$

$$= \underline{2^{nk}\int_{\mathbb{R}^n} f(y)\,dy} . \longrightarrow 0 \quad (\text{as } k \to -\infty).$$

Such k exists.

$\forall x \in \bigcup_k \Omega_k, \quad \exists k_0,$ s.t. $x \in \Omega_{k_0}$

$\Longrightarrow \quad E_{k_0} f(x) > \lambda.$

$M_d f(x) = \sup_k |E_k f(x)| \geqslant E_{k_0} f(x) > \lambda$

$\Longrightarrow \quad x \in \{ M_d f > \lambda \}.$

$\forall x \in \{ M_d f > \lambda \}, \quad \sup E_k f(x) > \lambda \Rightarrow \exists k_0 \in \mathbb{Z}$ s.t.

$E_{k_0} f(x) > \lambda.$

$\Longrightarrow \quad \{ x : M_d f(x) > \lambda \} = \bigcup_k \Omega_k$

- $\{\Omega_j\}$ are disjoint

$\Omega_k = \{ x : E_k f(x) > \lambda ; \ j < k \Rightarrow E_j f(x) \leq \lambda \}$

$\forall x \in \Omega_k \Longrightarrow E_k f(x) > \lambda \quad \exists Q \in \mathcal{Q}_k, \ x \in Q$

$\Longrightarrow \forall y \in Q, \quad E_k f(y) = E_k f(x) > \lambda.$

i.e. $\quad \Omega_k = \bigsqcup_{j=1}^{\infty} Q_j^{(k)} \qquad Q_j^{(k)} \in \mathcal{Q}_k.$

Hence:

$|\{ x \in \mathbb{R}^n : M_d f(x) > \lambda \}| = \sum_k |\Omega_k|.$

$\leqslant \sum_k \frac{1}{\lambda} \int_{\Omega_k} E_k f.$

$= \frac{1}{\lambda} \sum_k \int_{\Omega_k} f.$

$= \frac{1}{\lambda} \int_{\bigcup_k \Omega_k} f. \quad \leqslant \frac{1}{\lambda} \int_{\mathbb{R}^n} f.$

(Calderón - Zygmund) Given $f \in L^1(\mathbb{R}^n)$ and $f \geq 0$, $\lambda > 0$, $\exists \{Q_j\}$ disjoint dyadic cube such that

① $f(x) \leq \lambda$ a.e. $x \notin \bigcup Q_j$ ☆

② $|\bigcup_j Q_j| \leq \frac{1}{\lambda} \|f\|_1$

③.

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} f \leq 2^n \lambda \quad ☆$$

Remark:
$$g(x) = \begin{cases} f(x) & x \notin \bigcup Q_j \\ \frac{1}{|Q_j|} \int_{Q_j} f. & x \in Q_j \end{cases}$$

$$\Rightarrow \quad g(x) \leq 2^n \lambda.$$

$$b(x) = f(x) - g(x) = \sum_j \left(f(x) - \frac{1}{|Q_j|} \int_{Q_j} f \right) \chi_{Q_j}(x)$$

$$b_k(x) = \left(f(x) - \frac{1}{|Q_k|} \int_{Q_k} f \right) \chi_{Q_k}(x).$$

Proof

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \bigcup \Omega_k = \bigcup Q_j$$

$$\Omega_k = \{x : E_k f(x) > \lambda \ ; \ j < k \Rightarrow E_j f(x) \leq \lambda\}.$$

①. $x \in (\bigcup Q_j)^c = \{x : M_d f(x) \leq \lambda\}.$

$$f(x) = \lim_{k \to \infty} E_k f(x) \qquad \text{a.e. } x \in (\bigcup Q_j)^c.$$
$$\leq \sup_k |E_k f(x)| = M_d f(x) \leq \lambda$$

②. $|\bigcup Q_j| = |\{x : M_d f(x) > \lambda\}| \leq \frac{1}{\lambda} \|f\|_1$

③. $\forall j \in \mathbb{Z}_{>0}$ $\exists k \in \mathbb{Z}$ s.t.

$$Q_j \subset \Omega_k = \{x : E_k f(x) > \lambda ; \ j < k \Rightarrow E_j f(x) \leq \lambda\}.$$

$$\Rightarrow \quad |Q_j| < \frac{1}{\lambda} \int_{Q_j} E_k f = \frac{1}{\lambda} \int_{Q_j} f.$$

i.e. $\lambda < \frac{1}{|Q_j|} \int_{Q_j} f.$

$Q_j \subset \Omega_k \Longrightarrow Q_j \in \mathcal{Q}_k$ $\quad \frac{1}{2^k}$ $\quad \exists!\ \tilde{Q} \in \mathcal{Q}_{k-1}$ s.t.: $Q_j \subset \tilde{Q}$. $\quad \frac{2}{2^k}$. $\quad 4$

$Q_j \subset \tilde{Q}$

$$\int_{Q_j} f \le \int_{\tilde{Q}} f = |\tilde{Q}| \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} f$$

$$= 2^n |Q_j| \cdot \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} f. \qquad \tilde{Q} \in \mathcal{Q}_{k-1}$$

$$= 2^n |Q_j|\, E_{k-1} f(x)\, \chi_{\tilde{Q}}(x) \qquad \forall x \in Q_j$$

$\forall x \in Q_j$ 有: $E_{k-1} f(x) \le \lambda$

$$\le 2^n |Q_j| \lambda$$

$$\Longrightarrow \quad \frac{1}{|Q_j|} \int_{Q_j} f \le 2^n \lambda$$