Workshop Lecture

for

Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators

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March 14, 2024

Contents

1 Inner Product and Hilbert spaces

1.1 Inner-Product Space

Definition 1 (inner product) A function $(\cdot,\cdot)$ on a vertor space $\mathbb{V}$ is called an inner product if

1. (positive) $(v,v) \geq 0$
2. (definite) $(v,v) = 0 \Longleftrightarrow v = 0$
3. (symmetry) $(v_1, v_2) = (v_2, v_1)$
4. (bilinear) $(a_1 v_1 + a_2 v_2, v) = a_1(v_1, v) + a_2(v_2, v)$

for all $v, v_1, v_2 \in \mathbb{V}$, $a_2, a_2 \in \mathbb{R}$.

A vector space with an asssociated inner product is called an inner-product space.

Theorem 2 (Cauchy-Schwarz inequality) Let $\mathbb{V}$ be a inner space. Then for all $u, v \in \mathbb{V}$, we have

$$|(u,v)| \leq (u,u)^{1/2}(v,v)^{1/2}.$$

The equality holds if and only if $a_1 u + a_2 v = 0$ for some $a_1, a_2 \in \mathbb{R}$.

Proof. If $v = 0$, then it is obvious that the inequality holds.

If $v \neq 0$, let $w = u - ((u,v)/(v,v)^2)v$, then $(w,u) = 0$. Denote $((u,v)/(v,v)^2)v = v'$, then

$$\begin{aligned}(u,u) &= (w+v', w+v') = (w,w) + (v'v') \\ &\geq (v',v') = (u,v)^2/(v,v)\end{aligned}$$

The equality holds if and onlf if $(w,w) = 0$. □

Definition 3 (norm induced by inner product) It follows that

$$|u| = (u,u)^{1/2}$$

is a norm. Indeed, we have

$$|u+v|^2 = (u+v, u+v) = |u|^2 + 2(u,v) + |v|^2 \leq |u|^2 + 2|u||v| + |v|^2.$$

Proposition 4 (parallelogram rule)

$$\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2 = \frac{1}{2}\left(|a|^2 + |b|^2\right), \qquad \forall\, a, b \in \mathbb{V}.$$

Proof. The proof form that

$$\begin{aligned}\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2 &= \frac{1}{4}\left((a+b,a+b)+(a-b,a-b)\right) \\ &= \frac{1}{4}(2(a,a)+2(b,b)).\end{aligned}$$

□

Remark 5 Although any inner product naturally define a metric, not every metric space exhibits the structure that is necessary to also be an inner-produce space. Shown as following.

Example 6 The norm

$$\|f\| = \sup_{x\in[0,1]} |f(x)|$$

on $\mathcal{C}[0,1]$ is not induced by an inner product.

Proof. Let

$$f(x)=1, \qquad g(x)=x, \qquad \forall\, x\in[0,1].$$

Then $f,g\in\mathcal{C}[0,1]$. But

$$\|f\|^2+\|g\|^2 = 2$$

$$\|f+g\|^2+\|f-g\|^2 = 4+1=5$$

the parallelogram rule fails. □

Theorem 7 (continuity of inner product) Let $\mathbb{V}$ be a inner product space with inner product $(\cdot,\cdot)$, $\{v_n\}$, $\{u_n\}$ be sequences and $v,u\in\mathbb{V}$. Then that $|v-v_n|\to 0$, $|u-u_n|\to 0$ implies

$$(u_n,v_n)\to(u,v).$$

Proof. Since

$$\begin{aligned}|(u_n,v_n)-(u,v)| &\le |(u_n,v_n)-(u_n,v)|+|(u_n,v)-(u,v)| \\ &\le |u_n||v_n-v|+|v||u_n-u|\end{aligned}$$

and $\{u_n\}$ is bounded since

$$|u_n|\le|u_n-u|+|u|.$$

□

1.2 Hilbert Space

Definition 8 (Hilbert space) Hilbert space is a complete inner-product space.

Example 9 Any finnite-dimensional inner-product space is Hilbert space.

Example 10 The ℓ^2 space is a Hilbert space. The inner product is

$$(u,v) = \sum_{i=1}^{\infty} u_i v_i,$$

for any

$$u = (u_1, \cdots),\ v = (v_1, \cdots) \in \ell^2.$$

Definition 11 Element u, v of an inner-product space $\mathbb{X}$ are said to be orthogonal if $(u,v) = 0$.

A countalbe collection of element $\{e_j\}_{j=1}^{\infty}$ is said to be an orthonormal sequence if

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \forall\, i, j \in \mathbb{Z}_{>0}$$

Theorem 12 (Pythagoras) If $(f,g) = 0$, then

$$|f+g|^2 = |f|^2 + |g|^2.$$

Morever, if $\{e_j\}$ is an orthonormal sequence

$$\left|\sum_{i=1}^{n} a_i e_i\right| = \sum_{i=1}^{n} a_i^2.$$

Theorem 13 (Bessel inequality) Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal sequence in an inner-product space $\mathbb{X}$. For any $x \in \mathbb{X}$,

$$\sum_{i=1}^{\infty} (x, e_i)^2 \leq |x|^2,$$

and $\sum_{i=1}^{\infty} (x, e_i) e_i$ converges if $\mathbb{X}$ is a Hilbert space.

Proof. Bessel inequality holds since

$$0 \leq \left| x - \sum_{i=1}^{n} (x, e_i) e_i \right|^2 = |x|^2 - \sum_{i=1}^{n} (x, e_i)^2.$$

This implies that $\sum_{i=1}^{\infty} (x, e_i)^2 < \infty$. It follows that $\{S_n = \sum_{i=1}^{n} (x, e_i) e_i\}$ is Cauchy since

$$|S_n| = \sum_{i=1}^{n} (x, e_i)^2.$$

$\square$

Theorem 14 Let $\{x_n\}_{n=1}^{\infty}$ be a countable collection of a Hilbert space $\mathbb{H}$ such that every finite subcollection of $\{x_n\}$ is linearly independent. Define $e_1 = x_1/|x_1|, e_n = v_n/|v_n|$ for

$$v_n = x_n - \sum_{j=1}^{n-1} (x_j, e_j) e_j.$$

Then, $\{e_n\}$ is an orthonormal sequence and $\overline{\text{span}\{x_n\}} = \overline{\text{span}\{e_n\}}$.

1.2.1 Orthogonality

Definition 15 An orthonormal sequence $\{e_n\}$ in a Hilbert space $\mathbb{H}$ is called an orthonormal basis or a complete orthogonal system (CONS) if $\overline{\text{span}\{e_n\}} = \mathbb{H}$.

Theorem 16 The following properties of an orthonormal sequence $\{e_n\}$ in Hilbert space $\mathbb{H}$ are equivalent.

1. (sparable) $\{e_n\}$ is a CONS
2. (complete)
$$(f, e_n) = 0 \text{ for all } n \qquad \Longrightarrow \qquad f = 0.$$
3. (Fourier expandansion) If $f \in \mathbb{H}$, then
$$\sum_{j=1}^{N} (f, e_j) e_j \to f \qquad \text{in norm.}$$
4. (Parseval's identity) If $a_k = (f, e_k)$, then
$$|f| = \sum_{k=1}^{\infty} |a_k|^2.$$

Proof.

$1 \Longrightarrow 2$. Given $f \in \mathbb{H}$ with $(f, e_n) = 0$ for all n. By assumption, there exists a sequence $\{g_n\}$ of elements in $\mathbb{H}$ that are linear combinations of elements in $\{e_n\}$, and such that $|f - g_n| \to 0$ as $n \to \infty$. Thus

$$|f|^2 = (f, f) = (f, f - g_n) \leq |f||f - g_n| \to 0, \qquad \text{as } n \to \infty$$

Thus $f = 0$.

$2 \Longrightarrow 3$. By theorem 13, $\exists\, g \in \mathbb{H}$ such that

$$S_N = \sum_{j=1}^{N} (f, e_j) e_j \to g, \qquad \text{as } N \to \infty.$$

It suffices to prove that $g = f$.

Given n, let $N > n$, then

$$(f - S_N, e_n) = 0.$$

Letting $N \to \infty$, we have

$$(f - g, e_n) = 0, \qquad \forall\, n$$

which implies that $f = g$.

$3 \Longrightarrow 1$ is obvious.

$3 \Longleftrightarrow 4$. Since

$$\left| f - \sum_{k=1}^{N} a_k e_k \right| = |f| - \sum_{k=1}^{N} a_k^2$$

□

1.2.2 Isometrically Isomorphi

Definition 17 Two metric spaces $(\mathbb{M}_1, d_1)$ and $(\mathbb{M}_2, d_2)$ are said to be isometrically isomorphi or congruent if there exists a bijective function $T : \mathbb{M}_2 \longrightarrow \mathbb{M}_1$ such that

$$d_2(x_1, x_2) = d_1(Tx_1, Tx_2), \qquad \forall\, x_1, x_2 \in \mathbb{M}_2.$$

Theorem 18 Let $\mathbb{H}_1, \mathbb{H}_2$ be Hilbert spaces with inner products $(\cdot,\cdot)_i$, $i = 1, 2$. Suppose for some index set E there are collections of vectors $\mathbb{U}_i = \{u_t^{(i)}\}_{t\in E}$ such that $\overline{\text{span}(\mathbb{U}_i)} = \mathbb{H}_i$. If for every $i, j \in E$

$$(u_i^{(1)}, u_j^{(1)})_1 = (u_i^{(2)}, u_j^{(2)})_2,$$

$\mathbb{H}_1, \mathbb{H}_2$ congruent.

Proof. Let $\mathbb{U}_1 = \{u_\alpha\}_{\alpha\in E}$ $\mathbb{U}_2 = \{v_\alpha\}_{\alpha\in E}$. Given $f \in \mathbb{H}_1$, there exists a sequence $\{g_n\}$ of elements in $\mathbb{H}$ that are linear combinations of elements in $\{u_n\}$, such that $|f - g_n| \to 0$.

For all $g \in \text{span}(\mathbb{U}_1)$,

$$g = \sum_{i=1}^{n} a_i u_i,$$

define $Tg = \sum_{i=1}^{n} a_i v_i \in \text{span}(\mathbb{U}_2)$. Since g_n converges to f, thus Cauchy, which implies $\{Tg_n\}$ is also Cauchy since $(g_n, g_n)_1 = (Tg_n, Tg_n)_2$. Thus there exists an element of $\mathbb{H}_2$, to which $\{Tg_n\}$ converges, denoted by Tf. Thus $(f, f)_1 = (Tf, Tf)_2$. And it is obvious that T is bijective. □

Theorem 19 Any infinite-dimensional separable Hilbert space is congruent to $\ell^2(\mathbb{Z}_{>0})$.

Proof. Any infinite-dimensional separable Hilbert space $\mathbb{H}$ has a CONS. Let $\mathbb{U} = \{e_j\}_{j=1}^{n}$ be the CONS of $\mathbb{H}$. Let $\mathbb{V} = \{\varepsilon_j\}_{j=1}^{\infty}$, where ε_j is a sequence of all zeros except for a 1 as its jth entry. Then it is obvious that

$$(e_i, e_j) = (\varepsilon_i, \varepsilon_j).$$

□

1.2.3 Example: $\mathbb{L}^2$ Space

Example 20 $\mathbb{L}^2(E, \mathscr{B}, \mu)$ is a Hilbert space with the inner product

$$(f, g) = \int_E fg \, \mathrm{d}\mu, \qquad \forall f, g \in \mathbb{L}^2(E, \mathscr{B}, \mu).$$

Here, we focuse on $\mathbb{L}^2[0,1]$, $E = [0,1]$, $\mathscr{B}$ the Borel σ-field of $[0,1]$ and μ Lebesgue measure.

Theorem 21 The following sets of functions

$$B_1 = \{f_0(x) = 1, f_n(x) = \sqrt{2}\cos(n\pi x), n \geq 1\},$$

$$B_2 = \{g_n(x) = \sqrt{2}\sin(n\pi x), n \geq 1\}$$

and

$$B_3 = \{h_0(x) = 1, h_{2n-1} = \sqrt{2}\sin(2n\pi x), h_{2n}(x) = \sqrt{2}\cos(2n\pi x), n \geq 1\}$$

are all orthonormal bases for $\mathbb{L}^2[0,1]$.

Proof. It is clear that B_1, B_2, B_3 are orthonormal. Hence, we need only show that they are bases.

B_1: $\forall f \in \mathbb{L}^2[0,1]$ and $\varepsilon > 0$, there exists $g \in \mathcal{C}[0,1]$, such that

$$\|f - g\| < \varepsilon/2.$$

Set

$$h(s) = g\left(\frac{\arccos s}{\pi}\right) \qquad s \in [-1,1].$$

By Weierstrass Theorem, there exists a polynomial p such that

$$|h(s) - p(s)| < \varepsilon/2 \qquad \forall s \in [-1,1].$$

Let $k(x) = p(\cos \pi x)$, then

$$\|g - k\| = \left(\int_{[0,1]} |g - k|^2\right)^{1/2} \leq \varepsilon/2.$$

Thus $\|f - p(\cos \pi x)\| < \varepsilon$.

B_2: $\forall\ f \in \mathbb{L}^2[0,1]$ and $\varepsilon > 0$, let $f_\delta = f\chi_{[\delta,1-\delta]}$. There is $\delta > 0$ such that

$$\|f - f_\delta\| < \varepsilon/2.$$

Now $h(x) = f_\delta(x)/\sin(\pi x) \in \mathbb{L}^2[0,1]$ and so there is a function $k(x) = \sum_{i=0}^m a_i \cos(i\pi x)$ such that $\|k - h\| < \varepsilon/2$. But

$$\begin{aligned}\|h-k\|^2 &= \int_0^\delta k^2(x)\,\mathrm{d}x + \int_\delta^{1-\delta} |k(x) - f_\delta(x)/\sin(\pi x)|^2\,\mathrm{d}x + \int_{1-\delta}^1 k^2(x)\,\mathrm{d}x \\ &\geq \int_0^\delta k^2(x)\sin^2(\pi x)dx + \int_\delta^{1-\delta} |k\sin(\pi x) - f_\delta|^2(x)\,\mathrm{d}x + \int_{1-\delta}^1 k^2(x)\sin^2(\pi x)\,\mathrm{d}x \\ &= \int_0^1 |k\sin(\pi x) - f_\delta(x)|^2\,\mathrm{d}x = \|k(x)\sin(\pi x) - f_\delta(x)\|^2.\end{aligned}$$

B_3: Suppose, by contradiction, that B_3 is not a basis. Thus there exists a nonzero function $f \in \mathbb{L}^2[0,1]$ such that

$$(f, h_0) = (f, h_n) = 0 \qquad \forall\ n.$$

i.e.

$$0 = \int_0^1 f(x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 f\left(\frac{x+1}{2}\right) + f\left(\frac{-x+1}{2}\right)\,\mathrm{d}x$$

$$0 = \frac{1}{2}\cdot(-1)^n \int_0^1 \left(f\left(\frac{x+1}{2}\right) + f\left(\frac{-x+1}{2}\right)\right)\cos(n\pi x)\,\mathrm{d}x.$$

$$0 = \frac{1}{2}\cdot(-1)^n \int_0^1 \left(f\left(\frac{x+1}{2}\right) - f\left(\frac{-x+1}{2}\right)\right)\sin(n\pi x)\,\mathrm{d}x$$

Since B_1 , B_2 are basis in $\mathbb{L}^2[0,1]$, we conclude that

$$f\left(\frac{x+1}{2}\right) - f\left(\frac{-x+1}{2}\right) = f\left(\frac{x+1}{2}\right) + f\left(\frac{-x+1}{2}\right) = 0.$$

Since $x \in [0,1]$ is arbitrary, $f = 0$ in $[0,1]$. $\square$

2 The projection theorem and orthogonal decomposition

2.1 Projection theorem

Theorem 22 Let $\mathbb{M} \subset \mathbb{H}$ be a noempty closed convex set. Then for every $f \in \mathbb{H}$, there exists a unique element $u \in \mathbb{M}$ such that

$$|f - u| = \min_{v\in\mathbb{M}} |f - v| = \mathrm{dist}(f, \mathbb{M}). \tag{1}$$

Morever, u is characterized by the property

$$u \in \mathbb{M}, \qquad \text{and} \qquad (f-u, v-u) \le 0, \qquad \forall\, v \in \mathbb{M}. \tag{2}$$

Remark 23 The element u is called the projection of f in $\mathbb{M}$, denoted by

$$\mathcal{P}_{\mathbb{M}} f = u.$$

<u>**Proof.**</u>

1. *Existence.* Let $(v_n) \in \mathbb{M}$ such that

   $$d_n = |f - v_n| \to d = \inf_{v \in \mathbb{M}} |f - v|.$$

   By parallelogram rule, we have

   $$\left|f - \frac{v_n + v_m}{2}\right| + \left|\frac{v_n - v_m}{2}\right| = \frac{1}{2}\left(d_n^2 + d_m^2\right), \qquad \forall\, m, n \in \mathbb{Z}_{>0}$$

   But $(v_n + v_m)/2 \in \mathbb{M}$ since $\mathbb{M}$ is convex and thus $|f - \frac{v_n+v_m}{2}| \ge d$. It follows that

   $$\left|\frac{v_n - v_m}{2}\right| \le \frac{1}{2}(d_m^2 + d_n^2) - d^2.$$

   i.e. $\{v_n\}$ Cauchy.

2. *Equivalence of 1 and 2.*Assume that $u \in \mathbb{M}$ satisfies 2 and let $v \in \mathbb{M}$. We have

   $$w = (1-t)u + tv \in \mathbb{M}, \qquad \forall\, t \in [0,1].$$

   Then

   $$|f - u| \le |f - w| = |(f-u) - t(u-v)|$$

   Therefore

   $$|f-u|^2 \le |f-u|^2 - 2t(f-u, u-v) + t^2|u-v|^2$$

   which implies $2(f-u, u-v) \le t^2|u-v|^2$. Letting $t \to 0$, we obtain 2.

   Conversely, assume that u satisfies 2, then

   $$\begin{aligned} |f-u|^2 - |f-v|^2 &= |f-u|^2 - |(f-u) - (v-u)|^2 \\ &= 2(f-u, v-u) - |v-u|^2 \le 0. \end{aligned}$$

3. *Uniqueness.* If u_1, u_2 satisfy 2. Then

$$(f - u_1, v - u_1) \leq 0 \qquad \forall\, v \in \mathbb{M}$$
$$(f - u_2, v - u_2) \leq 0 \qquad \forall\, v \in \mathbb{M}$$

Choose $v = u_2, v = u_1$ represently.

□

Corollary 24 Let $\mathbb{M} \subset \mathbb{H}$ be a noempty closed linear subspace. Then for every $f \in \mathbb{H}$, there exists a unique element $u \in \mathbb{M}$ such that

$$|f - u| = \min_{v \in \mathbb{M}} |f - v| = \operatorname{dist}(f, \mathbb{M}). \tag{3}$$

Morever, u is characterized by the property

$$u \in \mathbb{M}, \qquad \text{and} \qquad (f - u, v - u) = 0, \qquad \forall\, v \in \mathbb{M}. \tag{4}$$

Proof. Let $v = u \pm u$ in 2,

$$(f - u, u) = 0$$

It follows that

$$(f - u, v) \leq 0 \qquad \forall\, v \in \mathbb{M}.$$

Replace v by $-v$. □

2.2 Orthogonal decomposition

Definition 25 Let $\mathbb{X}$ be a inner-product space with $\mathbb{M} \subset \mathbb{X}$. The orthogonal complement of $\mathbb{M}$ is

$$\mathbb{M}^{\perp} = \{x \in \mathbb{X} : (x, y) = 0 \text{ for all } x \in \mathbb{M}\}.$$

Remark 26 If $\mathbb{M}$ is closed subspace, every $x \in \mathbb{H}$ can be uniquely expressed as

$$x = \mathcal{P}_{\mathbb{M}} x + (x - \mathcal{P}_{\mathbb{M}} x) \in \mathbb{M} + \mathbb{M}^{\perp}.$$

And it is clear that

$$\mathbb{M} \cap \mathbb{M}^{\perp} = \{0\}.$$

Definition 27 Let $\mathbb{M}_1, \mathbb{M}_2$ be orthogonal subspace of $\mathbb{X}$; i.e.

$$x_1 \perp x_2, \qquad \forall\, x_1 \in \mathbb{M}_1, x_2 \in \mathbb{M}_2.$$

Then the collection

$$\{x_1 + x_2 : x_i \in \mathbb{M}_i, i = 1, 2\}$$

is denoted by $\mathbb{M}_1 \oplus \mathbb{M}_2$ and is referred to as the orthogonal direct sum of $\mathbb{M}_1$ and $\mathbb{M}_2$.

Theorem 28 Let $\mathbb{M}$ be the closed subspace of Hilbert space $\mathbb{H}$. Then

$$\mathbb{M} \oplus \mathbb{M}^{\perp} = \mathbb{H}.$$

Theorem 29 Let $\mathbb{H}$ be a Hilbert space with $\mathbb{M}$ a subset of $\mathbb{H}$. Then

1. $\mathbb{M}^{\perp}$ is a closed subspace;
2. $\mathbb{M} \subset \left(\mathbb{M}^{\perp}\right)^{\perp}$;
3. $\left(\mathbb{M}^{\perp}\right)^{\perp} = \overline{\mathbb{M}}$ if $\mathbb{M}$ is a subspace.

Proof.

1. It suffices to prove that the limit points of $\mathbb{M}^{\perp}$ are in $\mathbb{M}^{\perp}$. Let $(x_n) \subset \mathbb{M}^{\perp}$ and $x_n \to x$. Then
$$(x_n, y) = 0, \qquad \forall\, y \in \mathbb{H}.$$
Thus by continuity of $(\cdot, \cdot)$,
$$(x, y) = (\lim x_n, y) = \lim(x_n, y) = 0, \qquad \forall\, y \in \mathbb{M}.$$
which implies that $x \in \mathbb{M}^{\perp}$.

2. It is clear that $\forall\, x \in \mathbb{M}$
$$x \in \left(\mathbb{M}^{\perp}\right)^{\perp} = \{x \in \mathbb{H} : (x, y) = 0\ \forall\, y \in \mathbb{M}^{\perp}\}.$$

3. Since $\left(\mathbb{M}^{\perp}\right)^{\perp}$ is closed, we have $\overline{M} \subseteq \left(\mathbb{M}^{\perp}\right)^{\perp}$ by the definition of closure. By theorem 28, we have
$$\left(\mathbb{M}^{\perp}\right)^{\perp} = \overline{\mathbb{M}} \oplus \left(\mathbb{M}^{\perp}\right)^{\perp} \cap \overline{\mathbb{M}}^{\perp}.$$
But
$$\left(\mathbb{M}^{\perp}\right)^{\perp} \cap \overline{\mathbb{M}}^{\perp} \subset \left(\mathbb{M}^{\perp}\right)^{\perp} \cap \mathbb{M}^{\perp} = \{0\}.$$

$\square$

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