PDE 第三章作业

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课本习题

【题目 1】(课本第 1 题) 求 Fourier 变换

1. $f(x) = \begin{cases} 0 & |x| > a \\ 1 - \dfrac{|x|}{a} & |x| \le a \end{cases}$.
2. $f(x) = \exp(-a|x|)$.

解.

1. $$\widehat{f}(\lambda) = \int_{-a}^{a} \left(1 - \frac{|x|}{a}\right) e^{-i\lambda x}\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{a\lambda^2}(1 - \cos\lambda a).$$
2. $$\widehat{f}(\lambda) = \int_{\mathbb{R}} \exp(-a|x| - i\lambda x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + \lambda^2}.$$

□

【题目 2】(课本第 2 题) 利用 Fourier 变换的性质求下列函数的 Fourier 变换

1. $f(x) = \begin{cases} e^{\mu x} & |x| < a \\ 0 & |x| \ge a \end{cases}$
2. $f(x) = \sin(\lambda_0 x)\, e^{-a|x|}$.
3. $f(x) = \begin{cases} e^{i\lambda_0 x} & |x| < L \\ 0 & |x| \ge L \end{cases}$

解.

1. 由于
$$\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{(\mu - i\lambda)x}\,\mathrm{d}x.$$
故
$$\widehat{f}(\lambda - i\mu) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\lambda x}\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\lambda a}{\lambda}.$$
从而
$$\widehat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(\lambda + i\mu)a}{\lambda + i\mu}.$$

2. 由于

$$\widehat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\sin\lambda_0 x e^{-a|x|}e^{-i\lambda x}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{i\lambda_0 x}-e^{-i\lambda_0 x}}{2i}e^{-a|x|}e^{-i\lambda x}\,\mathrm{d}x$$
$$=\frac{1}{2i}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-a|x|}\left(e^{-i(\lambda-\lambda_0)x}-e^{-i(\lambda+\lambda_0)x}\right)\mathrm{d}x\right)$$

令 $g(x)=e^{-a|x|}$，那么

$$\widehat{f}(\lambda)=\frac{1}{2i}\left(\widehat{g}(\lambda-\lambda_0)-\widehat{g}(\lambda+\lambda_0)\right)=\frac{i}{\sqrt{2\pi}}\left(\frac{a}{a^2+(\lambda-\lambda_0)^2}-\frac{a}{a^2+(\lambda+\lambda_0)^2}\right).$$

3. 由于

$$\widehat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-L}^{L}e^{i\lambda_0 x}e^{-i\lambda x}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\frac{1}{i(\lambda_0-\lambda)}e^{i(\lambda_0-\lambda)x}\Big|_{-L}^{L}=-\sqrt{\frac{2}{\pi}}\frac{\sin(L(\lambda-\lambda_0))}{\lambda-\lambda_0}$$

□

【题目 3】(课本第 3 题) 求 Fourier 逆变换

1. $f(\lambda)=e^{-a^2\lambda^2 t}$
2. $f(\lambda)=e^{(-a^2\lambda^2+ib\lambda+c)t}$
3. $f(\lambda)=e^{-|\lambda|y}$

解.

1. 记

$$g(x)=\exp(-x^2)$$

那么

$$\widehat{g}(\lambda)=\frac{1}{\sqrt{2}}\exp\left(-\frac{x^2}{4}\right).$$

并且

$$((g(Ax)))^{\wedge}=\frac{1}{A}\widehat{g}\left(\frac{\lambda}{A}\right)=\frac{1}{\sqrt{2}A}\exp\left(-\frac{\lambda^2}{4A^2}\right)$$

令

$$\frac{1}{4A^2}=a^2 t$$

那么

$$A=\frac{1}{2a\sqrt{t}}$$

从而

$$\left(g\left(\frac{x}{2a\sqrt{t}}\right)\right)^{\wedge}=a\sqrt{2t}\exp\left(-a^2\lambda^2 t\right).$$

即

$$\left(\frac{1}{a\sqrt{2t}}g\left(\frac{x}{2a\sqrt{t}}\right)\right)^{\wedge}=f(\lambda)$$

故

$$f^{\vee}(x)=\frac{1}{a\sqrt{2t}}g\left(\frac{x}{2a\sqrt{t}}\right)=\frac{1}{a\sqrt{2t}}\exp\left(-\frac{x^2}{4a^2 t}\right).$$

2. 注意到

$$f(x)=\exp\left(-a^2\left(\lambda-\frac{ib}{2a^2}\right)^2 t\right)\cdot\exp\left(-\frac{tb^2}{4a^2}+ct\right)$$

从而

$$
\begin{aligned}
f^\vee(x) &= \exp\left(-\frac{tb^2}{4a^2}+ct\right)\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left(-a^2\left(\lambda-\frac{ib}{2a^2}\right)^2 t\right)e^{i\lambda x}\,\mathrm{d}\lambda \\
&= \exp\left(-\frac{tb^2}{4a^2}+ct\right)\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left(-a^2\lambda^2 t\right)\cdot\exp\left(i\left(\lambda+\frac{ib}{2a^2}\right)x\right)\mathrm{d}\lambda \\
&= \exp\left(-\frac{tb^2}{4a^2}+ct-\frac{b}{2a^2}x\right)\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left(-a^2\lambda^2 t\right)\cdot\exp\left(i\lambda x\right)\mathrm{d}\lambda \\
&= \exp\left(-\frac{tb^2}{4a^2}+ct-\frac{b}{2a^2}x\right)\cdot\left(\exp\left(-a^2\lambda^2 t\right)\right)^\vee \\
&= \frac{1}{\sqrt{2t}a}\exp\left(ct-\frac{1}{t}\left(\frac{bt+x}{2a}\right)^2\right)
\end{aligned}
$$

3.

$$
f^\vee(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-|\lambda|y}e^{-i\lambda x}\,\mathrm{d}\lambda = \sqrt{\frac{2}{\pi}}\cdot\frac{y}{y^2+x^2}.
$$

□

【题目 4】(课本第 4 题) 用 Fourier 变换求解

$$
\begin{cases}
\dfrac{\partial u}{\partial t}-a^2\dfrac{\partial^2 u}{\partial x^2}-b\dfrac{\partial u}{\partial x}-cu=f(x,t) & (x,t)\in\mathbb{R}\times(0,+\infty)\\
u|_{t=0}=\phi(x) & x\in\mathbb{R}
\end{cases}
$$

解. 方程两遍对 x 进行 Fourier 变换有

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t}\widehat{u}+a^2\lambda^2\widehat{u}-ib\lambda\widehat{u}-c\widehat{u}=\widehat{f}\\
\widehat{u}|_{t=0}=\widehat{\phi}
\end{cases}
$$

记

$$
g_t(\lambda)=\exp\left(-(a^2\lambda^2-ib\lambda+c)t\right)
$$

那么原方程的解为

$$
\widehat{u}=\widehat{\phi}\widehat{g}_t+\int_0^t\widehat{f}\widehat{g}_{t-\tau}\,\mathrm{d}\tau=\frac{1}{\sqrt{2\pi}}\widehat{\phi*g_t}+\frac{1}{\sqrt{2\pi}}\int_0^t\widehat{f*g_{t-\tau}}\,\mathrm{d}\tau
$$

从而

$$
u=\frac{1}{\sqrt{2\pi}}\phi*g_t+\frac{1}{\sqrt{2\pi}}\int_0^t f*g_{t-\tau}\,\mathrm{d}\tau
$$

□

【题目 5】(课本第 5 题) 证明在 $\mathscr{D}'(\mathbb{R})$ 的意义下

1. $\phi(x)\delta'(x)=-\phi'(0)\delta(x)+\phi(0)\delta'(x)$;
2. $x^m\delta^{(m)}(x)=(-1)^m m!\delta(x)$.

证明.

1. 这是因为 $\forall\,\psi\in\mathscr{D}(\mathbb{R})$ 有

$$
\begin{aligned}
<\phi(x)\delta'(x),\psi(x)> &=<\delta'(x),\phi(x)\psi(x)>=-<\delta(x),\left(\phi(x)\psi(x)\right)'> \\
&=-<\delta(x),\phi(x)\psi'(x)+\psi(x)\phi'(x)> \\
&=-\phi(0)\psi'(0)-\psi(0)\phi'(0) \\
&=-\phi(0)<\delta(x),\psi'(x)>-\phi'(0)<\delta(x),\psi(x)> \\
&=<\phi(0)\delta'(x)-\phi'(0)\delta(x),\psi(x)>.
\end{aligned}
$$

2. 这是因为 $\forall\, \psi \in \mathscr{D}(\mathbb{R})$

$$< x^m\delta^{(m)}(x), \psi(x) > = < \delta^{(m)}(x), x^m\psi(x) > = (-1)^m \left\langle \delta(x), \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m (x^m\psi(x)) \right\rangle$$
$$= (-1)^m m!\psi(0) = < (-1)^m m!\delta(x), \psi(x) > .$$

□

【题目 6】(课本第 6 题) 计算

1. $(|x|)^{(m)}$，其中 $m \in \mathbb{Z}_{>0}$;
2. $\left(H(x)e^{ax}\right)''$

解.

1. $\forall\, \psi \in \mathscr{D}(\mathbb{R})$，由于

$$\int_{\mathbb{R}} |x|\psi(x)\,\mathrm{d}x = -\int_0^{\infty} \psi(x)\,\mathrm{d}x + \int_{-\infty}^{0} \psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}} \psi(x)(H(x) - H(-x))\,\mathrm{d}x,$$

故

$$|x|' = H(x) - H(-x).$$

又由于 $H'(x) = \delta(x)$ 故

$$(|x|)^{(m)} = \begin{cases} H(x) - H(-x) & m = 1 \\ 2\delta^{(m-2)}(x) & m > 1 \end{cases}$$

2.

$$\left(H(x)e^{ax}\right)'' = \delta'(x)e^{ax} + 2a\delta(x)e^{ax} + a^2H(x)e^{ax}.$$

□

【题目 7】(课本第 7 题) 求广义导数，其中

1. $f(x) = \begin{cases} \sin x & x \geq 0 \\ 0 & x < 0 \end{cases}$
2. $f(x) = \begin{cases} x^2 & |x| \leq 1 \\ 0 & |x| > 1. \end{cases}$

解.

1. 由于 $f(x) = \sin x H(x)$，因此

$$f'(x) = \cos x H(x) + \sin x\delta x = \cos x H(x) + \sin 0\delta x = \cos x H(x).$$

2. 由于 $f(x) = x^2(H(x-1) - H(x+1))$，故

$$f'(x) = 2x(H(x-1) - H(x+1)) + x^2(\delta(x-1) - \delta(x+1))$$
$$= 2x(H(x-1) - H(x+1)) + \delta(x-1) - \delta(x+1).$$

□

【题目 8】(课本第 9 题) 用分离变量法求解

1.
$$\begin{cases} u_t = a^2 u_{xx} & 0 < x < \pi, t > 0 \\ u|_{t=0} = \sin x & 0 \le x \le \pi \\ u_x|_{x=0} = u_x|_{x=\pi} = 0 & t > 0. \end{cases}$$

2.
$$\begin{cases} u_t = a^2 u_{xx} & 0 < x < \ell, t > 0 \\ u|_{t=0} = 0 & 0 \le x \le \ell \\ u|_{x=0} = 0,\ u|_{x=\ell} = At & t > 0 \end{cases}$$

3.
$$\begin{cases} u_t - a^2 u_{xx} = 0 & 0 < x < \ell, t > 0 \\ u|_{t=0} = 0 & 0 \le x \le \ell \\ u_x|_{x=0} = 0,\ u_x|_{x=\ell} = q & t > 0 \end{cases}$$

证明.

1. 令 $u(x,t) = X(x)T(t)$，则特征问题为

$$X'' + \lambda X = 0, \qquad X'(0) = X'(\pi) = 0$$

解得

$$X_n(x) = \cos n\pi, \qquad \lambda_n = n^2.$$

那么 T_n 满足

$$Tn' + n^2 a^2 T_n = 0$$

解得

$$T_n(t) = C_n \exp\left(-n^2 a^2 t\right).$$

进而

$$u(x,t) = \sum_{n=0}^{\infty} C_n \exp\left(-n^2 a^2 t\right) \cos nx$$

带入边值条件有

$$C_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos n\pi \, \mathrm{d}x = \begin{cases} \dfrac{4}{\pi} \cdot \dfrac{1}{1-n^2} & 2 \mid n \\ 0 & 2 \nmid n \end{cases}$$

从而

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4}{\pi} \cdot \frac{1}{1-4n^2} \exp\left(-4n^2 a^2 t\right) \cos(2nx)$$

2. 令

$$v(x,t) = u(x,t) - \frac{x}{\ell} At,$$

则 v 满足方程

$$\begin{cases} v_t - a^2 v_{xx} = -\dfrac{x}{\ell} A & 0 < x < \ell, t > 0 \\ v|_{t=0} = 0 & 0 \le x \le \ell \\ v|_{x=0} = v|_{x=\ell} = 0 & t > 0 \end{cases}$$

令 $v(x,t) = X(x)T(t)$，从而特征方程

$$X'' + \lambda X = 0, \qquad X(0) = X(\ell) = 0$$

解得

$$X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right), \qquad \forall\, n \in \mathbb{Z}_{>0}.$$

从而

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\left(\frac{n\pi}{\ell}x\right).$$

又由于

$$\frac{2}{\ell}\int_0^{\ell} -\frac{x}{\ell}At\sin\left(\frac{n\pi}{\ell}x\right)\mathrm{d}x = (-1)^n\frac{2\ell}{n\pi},$$

因此得到 T_n 满足的微分方程为

$$T_n' + \left(\frac{an\pi}{\ell}\right)^2 T_n = (-1)\frac{2\ell}{n\pi}, \qquad T_n(0) = 0.$$

解得

$$T_n(t) = (-1)^n\frac{2A}{n\pi}(-1)^n\left(\frac{\ell}{an\pi}\right)^2\left(1-\exp\left(-\left(\frac{an\pi}{\ell}\right)^2 t\right)\right)$$

即

$$u(x,t) = \frac{x}{\ell}At + \sum_{n=1}^{\infty}(-1)^n\frac{2A}{n\pi}(-1)^n\left(\frac{\ell}{an\pi}\right)^2\left(1-\exp\left(-\left(\frac{an\pi}{\ell}\right)^2 t\right)\right)\sin\left(\frac{n\pi}{\ell}x\right).$$

3. 令

$$v(x,t) = u(x,t) - \frac{x^2}{2\ell}q$$

那么 v 满足

$$\begin{cases} v_t - a^2 v_{xx} = \dfrac{qa^2}{\ell} & 0 < x < \ell, t > 0 \\ v|_{t=0} = -\dfrac{x^2}{2\ell}q & 0 \le x \le \ell \\ v_x|_{x=0} = v_x|_{x=\ell} = 0, & t > 0 \end{cases}$$

特征方程为

$$X'' + \lambda X = 0 \qquad X'(0) = X'(\ell) = 0$$

从而

$$X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right)$$

故

$$\sum_{n=0}^{\infty}\left(T_n' + \left(\frac{n\pi a}{\ell}\right)^2 T_n\right)\cos\left(\frac{n\pi x}{\ell}\right) = \frac{qa^2}{\ell}$$

当 $n \neq 0$ 时

$$T_n' + \left(\frac{n\pi a}{\ell}\right)^2 T_n = 0$$

$$T_n(0) = \frac{2}{\ell}\int_0^{\ell} -\frac{x^2}{2\ell}q\cos\left(\frac{n\pi x}{\ell}\right)\mathrm{d}x = -\frac{2q\ell(-1)^n}{n^2\pi^2}$$

解得

$$T_n(t) = -\frac{2q(-1)^n}{n^2\pi^2}\exp\left(-\left(\frac{n\pi a}{\ell}\right)^2 t\right)$$

当 $n = 0$ 时

$$T_n' = \frac{qa^2}{\ell}, \qquad T_0(t) = -\frac{q}{2\ell}\cdot\frac{2}{\ell}\int_0^{\ell} x^2\,\mathrm{d}x = -\frac{q\ell}{3}$$

故

$$t_0(t) = \frac{qa^2}{\ell}t - \frac{q\ell}{3}.$$

从而

$$u(x,t) = \frac{qa^2}{\ell}t - \frac{q\ell}{3} + \frac{q}{2\ell}x - \frac{2q\ell}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\exp\left(-\left(\frac{n\pi a}{\ell}\right)^2 t\right)\cos\left(\frac{n\pi x}{\ell}\right)$$

□

【题目 9】(课本第 13 题) 设 $u \in \mathscr{C}^{2,1}(\overline{Q}), u_t \in \mathscr{C}^{2,1}(Q)$ 且满足定解问题

$$\begin{cases} u_t - u_x x = f(x,t) & (x,t) \in Q \\ u|_{t=0} = \phi(x) & 0 \le x \le \ell \\ u|_{x=0} = u|_{x=\ell} = 0 & 0 \le t \le T \end{cases}$$

则有以下估计

$$\max_{\overline{Q}} |u_t| \le C\left(\|f\|_{\mathscr{C}^1(\overline{Q})} + \|\phi''\|_{\mathscr{C}[0,\ell]}\right).$$

证明. 令 $v = u_t$，那么 v 满足方程

$$\begin{cases} u_t - u_x x = f_t(x,t) & (x,t) \in Q \\ u|_{t=0} = f(x,0) + \phi''(x) & 0 \le x \le \ell \\ u|_{x=0} = u|_{x=\ell} = 0 & 0 \le t \le T \end{cases}$$

从而

$$\max_{\overline{Q}} |u_t| = \max_{\overline{Q}} |v| \le T \cdot \|f_t\|_\infty + \|f(\cdot,0) + \phi''\|_\infty \le T \cdot \|f_t\|_\infty + \|f\|_\infty + \|\phi''\|_\infty$$

由于

$$\|f\|_{\mathscr{C}^1} = \|f\|_\infty + \|f_t\|_\infty + \|f_x\|_\infty$$

因此若令 $C = \max(T,1)$，那么

$$\max_{\overline{Q}} \|u_t\| \le C\left(\|f\|_{\mathscr{C}^1(Q)} + \|\phi''\|_{\mathscr{C}[0,\ell]}\right)$$

□

【题目 10】(课本第 15 题) 设 $u, u_x \in \mathscr{C}(\overline{Q}) \cap \mathscr{C}^{2,1}(Q)$，$u$ 满足第三边值问题

$$\begin{cases} Lu = u_t - u_{xx} = f(x,t), & (x,t) \in Q \\ u|_{t=0} = \phi & 0 \le x \le \ell \\ -\dfrac{\partial u}{\partial x} + \alpha u\Big|_{x=0} = g_1 & 0 \le t \le T \\ \dfrac{\partial u}{\partial x} + \beta u\Big|_{x=\ell} = g_2 & 0 \le t \le t. \end{cases}$$

其中 $\alpha, \beta \ge 0$ 给出 $\max_{\overline{Q}} |u_x|$ 的估计。

证明. 令 $v = u_x$，那么 v 满足方程

$$\begin{cases} Lu = f_x & (x,t) \in Q \\ v|_{t=0} = \phi' & 0 \le x \le \ell \\ v|_{x=0} = -g_1(t) + \alpha u(0,t) & 0 \le t \le T \\ v|_{x=\ell} = g_2 - \beta u(\ell,t) & 0 \le t \le T \end{cases}$$

从而 $v = u_x$ 有估计

$$\max_{\overline{Q}} |u_x| = \max_{\overline{Q}} |v| \le T \cdot \|f_x\|_\infty + \max\left(\|\phi'\|_\infty, \|g_1\|_\infty + \alpha\|u\|_\infty, \|g_2\|_\infty + \beta\|u\|_\infty\right)$$

由于 u 满足的条件知

$$\|u\|_\infty = \max_{\overline{Q}} |u| \le C\left(\|f\|_\infty + \max\left(\|\phi\|_\infty, \|g_1\|\infty, \|g_2\|_\infty\right)\right)$$

其中 C 只与 T,ℓ 有关。综上

$$\max_{\overline{Q}}|u_x| \le C_1\left(\|g_1\|_\infty + \|g_2\|_\infty + \|\phi\|_{\mathscr{C}^1} + \|f\|_{\mathscr{C}^1}\right),$$

其中

$$C_1 = \max\left(T, \alpha C + \beta C + 1\right).$$

□

【题目 11】(课本第 18 题) 设 $u \in \mathscr{C}(\overline{Q}) \cap \mathscr{C}^{2,1}(Q)$ 且满足

$$Lu = u_t - a^2 u_{xx} + c(x,t)u \le 0, \qquad (x,t) \in Q,$$

其中 $c(x,t)$ 非负有界。证明：若 u 在 $\overline{Q}$ 上取得非负最大值，则 u 必然在抛物边界 Γ 上达到它在 $\overline{Q}$ 上的非负最大值。

证明. 任意给定 $\varepsilon > 0$，令 $v = u + \varepsilon g$，其中 $g = g(t) \ge 0$。由于 u 在 $\overline{Q}$ 上取得非负最大值，那么 v 也在 $\overline{Q}$ 上取得非负最大值。

并且

$$Lv = Lu + \varepsilon\left(g' + c(x,t)g\right).$$

我们希望 $g' + c(x,t)g < 0$，因此可取

$$g(t) = \exp(-Mt).$$

其中 $M > 0$ 为 c 的一个上界。从而

$$Lv < 0.$$

这将导致 v 的非负最大值必然在抛物边界取到，从而

$$\max_{\overline{Q}} u = \max_{\overline{Q}}(v - \varepsilon g) \le \max_{\overline{Q}} v \le \max_{\Gamma} v \le \max_{\Gamma} u + \varepsilon.$$

令 $\varepsilon \to 0^+$ 即得。 □

【题目 12】(课本第 21 题) 证明半无界问题

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) & 0 < x < \infty, t > 0 \\ u|_{t=0} = \phi & 0 \le x < \infty \\ u|_{x=0} = \mu & t \ge 0 \end{cases}$$

的有界解是唯一的。

证明. 只需证明当 $f = \phi = \mu = 0$ 时只有零解即可。记

$$K = \sup|u|.$$

$\forall L > 0$ 考虑区域

$$Q_L = \{(x,t) : 0 < x < L, 0 < t \le t\}$$

以及辅助函数

$$v(x,t) = \pm u(x,t) + \frac{K}{L^2}\left(2a^2 t + x^2\right)$$

则

$$Lv = 0 \qquad v|_{t=0} = \frac{K}{L^2}x^2 \ge 0$$

$$v|_{x=0} = \frac{K}{L^2} \cdot 2a^2 t \ge 0 \qquad v|_{x=L} = \frac{K}{L^2}\left(2a^2 t + L^2\right) \pm u|_{x=L} \ge 0$$

由弱极值原理知

$$\min_{Q_L} v \geq \min_{\partial Q_L} v \geq 0$$

从而

$$|u| \leq \frac{K}{L}\left(2a^2 t + x^2\right), \qquad \forall\,(x,t) \in Q_L$$

从而对任意给定的 $(x_0, t_0) \in (0, +\infty) \times (0, T]$

$$|u| \leq \frac{K}{L}\left(2a^2 t_0 + x_0^2\right), \qquad \forall\, L > x_0.$$

令 $L \to \infty$ 则

$$|u(x_0, t_0)| = 0.$$

由 (x_0, t_0) 的任意性知 $u = 0$。从而解唯一。 □

【题目 13】(课本第 22 题) 设 $u \in \mathscr{C}^{2,1}(\overline{Q})$ 是问题

$$\begin{cases} u_t - u_{xx} = f & (x,t) \in Q \\ u(x,0) = \phi(x), & 0 \leq x \leq \ell \\ u(0,t) = u(\ell,t) = 0, & 0 \leq t \leq T. \end{cases}$$

的解，则 u 满足以下估计

$$\sup_{0\leq t\leq T} \int_0^\ell u_x^2 \,\mathrm{d}x + \int_0^T \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t \leq M\left(\int_0^\ell (\phi')^2 \,\mathrm{d}x + \int_0^T \int_0^\ell f^2 \,\mathrm{d}x\,\mathrm{d}t\right).$$

证明. 在

$$u_t - u_{xx} = f$$

两边乘 u_t 并积分得

$$\begin{aligned} \int_0^\tau \int_0^\ell f u_t \,\mathrm{d}x\,\mathrm{d}t &= \int_0^\tau \int_0^\ell u_t^2 - u_{xx} u_t \,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \left(u_x u_t|_0^\ell + \int_0^\ell u_x u_{xt} \,\mathrm{d}x \right) \mathrm{d}t \\ &= \int_0^\tau \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_0^\ell u_x u_{xt} \,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\ell \left(u_x^2|_0^\tau \right) \mathrm{d}x \\ &= \int_0^\tau \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\ell u_x^2(x,\tau) \,\mathrm{d}x - \frac{1}{2}\int_0^\ell (\phi')^2 \,\mathrm{d}x. \end{aligned}$$

从而由平均值不等式知

$$\int_0^\tau \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\ell u_x^2(x,\tau) \,\mathrm{d}x \leq \frac{1}{2}\int_0^\ell (\phi')^2 \,\mathrm{d}x + \frac{1}{2}\int_0^\tau \int_0^\ell f^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\tau \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t.$$

进而

$$\int_0^\tau \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^\ell u_x^2(x,\tau) \,\mathrm{d}x \leq \int_0^\ell (\phi')^2 \,\mathrm{d}x + \int_0^\tau \int_0^\ell f^2 \,\mathrm{d}x\,\mathrm{d}t$$

不等式两侧对 $\tau \in (0, T]$ 取上确界则有

$$\sup_{0\leq t\leq T} \int_0^\ell u_x^2 \,\mathrm{d}x + \int_0^T \int_0^\ell u_t^2 \,\mathrm{d}x\,\mathrm{d}t \leq \int_0^\ell (\phi')^2 \,\mathrm{d}x + \int_0^T \int_0^\ell f^2 \,\mathrm{d}x\,\mathrm{d}t.$$

□

【题目 14】(课本第 23 题) 设 $u \in \mathscr{C}^{1,0}(\overline{Q}) \cap \mathscr{C}^{2,1}(Q)$ 且满足以下定解问题

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) & (x,t) \in Q \\ u(x,0) = \phi(x) & 0 \le x \le \ell \\ -\dfrac{\partial u}{\partial x} + \alpha u\Big|_{x=0} = \dfrac{\partial u}{\partial x} + \beta u\Big|_{x=\ell} = 0 & 0 \le t \le T \end{cases}$$

其中 $\alpha, \beta \ge 0$，证明

$$\sup_{0\le t\le T} \int_0^\ell u^2 \,\mathrm{d}x + \int_0^T \int_0^\ell u_x^2 \,\mathrm{d}x\,\mathrm{d}t \le M\left(\int_0^\ell \phi^2 \,\mathrm{d}x + \int_0^T \int_0^\ell f^2 \,\mathrm{d}x\,\mathrm{d}t\right).$$

其中 M 只依赖于 T, α。

证明. 在

$$u_t - a^2 u_{xx} = f$$

两边乘 u 并积分得

$$\begin{aligned} \int_0^\tau \int_0^\ell uf \,\mathrm{d}x\,\mathrm{d}t &= \int_0^\tau \int_0^\ell u_t u - a^2 u_{xx} u \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_0^\ell \left(u^2\big|_0^\tau\right)\mathrm{d}x - a^2 \int_0^\tau \left(u_x u|_0^\ell - \int_0^\ell u_x^2 \,\mathrm{d}x\right)\mathrm{d}t \\ &= \frac{1}{2}\int_0^\ell u^2(x,\tau)\,\mathrm{d}x - \frac{1}{2}\int_0^\ell \phi^2 \,\mathrm{d}x + \int_0^\tau \int_0^\ell u_x^2 \,\mathrm{d}x\,\mathrm{d}t + a^2 \int_0^\tau \alpha u^2(0,t) + \beta u^2(\ell,t)\,\mathrm{d}t. \end{aligned}$$

进而由平均值不等式得

$$\int_0^\ell u^2(x,\tau)\,\mathrm{d}x + 2a^2 \int_0^\tau \int_0^\ell u_x^2 \,\mathrm{d}x\,\mathrm{d}t \le \int_0^\tau \int_0^\ell u^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_0^\ell f^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^\ell \phi^2 \,\mathrm{d}x.$$

记

$$\Omega(\tau) = \int_0^\tau \int_0^\ell u^2 \,\mathrm{d}x\,\mathrm{d}t, \qquad F(\tau) = \int_0^\ell \phi^2 \,\mathrm{d}x + \int_0^\tau \int_0^\ell f^2 \,\mathrm{d}x$$

那么 $\Omega(0) = 0$。由上述不等式可以得到

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} \le \Omega(\tau) + F(\tau).$$

故由 Gronwell 不等式知

$$\Omega(\tau) \le e^\tau F(\tau)$$

从而

$$\int_0^\ell u^2(x,\tau)\,\mathrm{d}x + 2a^2 \int_0^\tau \int_0^\ell u_x^2 \,\mathrm{d}x\,\mathrm{d}t \le \left(1 + e^\tau\right)\left(\int_0^\tau \int_0^\ell f^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^\ell \phi^2 \,\mathrm{d}x\right)$$

从而对 τ 取上确界有

$$\sup_{0\le\tau\le T} \int_0^\ell u^2(x,\tau)\,\mathrm{d}x + \int_0^T \int_0^\ell u_x^2 \,\mathrm{d}x\,\mathrm{d}t \le M\left(\int_0^T \int_0^\ell f^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^\ell \phi^2 \,\mathrm{d}x\right)$$

其中

$$M = \frac{1 + e^T}{\min\left(1, 2a^2\right)}$$

□

补充练习

【题目 15】 利用 Fourier 变换求解一维波动方程的初值问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t), & \text{in } \mathbb{R} \times (0, +\infty) \\ u = \phi,\ u_t = \psi, & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

解. 在方程两遍对 x 进行 Fourier 变换有

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{u} + a^2\lambda^2\widehat{u} = \widehat{f} \\ \widehat{u}|_{t=0} = \widehat{\phi},\ \dfrac{\mathrm{d}}{\mathrm{d}t}\widehat{u}\Big|_{t=0} = \widehat{\psi} \end{cases}$$

解得

$$\widehat{u}(\lambda,t) = \widehat{\phi}(\lambda)\cos(a\lambda t) + \frac{1}{a\lambda}\widehat{\psi}(\lambda)\sin(a\lambda t) + \frac{1}{a\lambda}\int_0^t \widehat{f}(\lambda,t)\sin(a\lambda(t-\tau))\,\mathrm{d}\tau.$$

由于

$$(\cos(a\lambda t))^{\vee} = \sqrt{\frac{\pi}{2}}\left(\delta(x-at) + \delta(x+at)\right), \qquad \left(\frac{\sin(a\lambda t)}{a\lambda}\right)^{\vee} = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{a}\chi_{(-at,at]}(x)$$

故

$$\begin{aligned} u(x,t) &= \frac{1}{2}\phi * (\delta(\cdot - at) + \delta(\cdot + at)) + \frac{1}{2}\psi * \chi_{(-at,at]} + \frac{1}{2}\int_0^t f * \chi_{(-a(t-\tau),a(t-\tau)]}\,\mathrm{d}\tau \\ &= \frac{1}{2}\big(\phi(x+at) + \phi(x-at)\big) + \frac{1}{2}\int_{x-at}^{x+at}\phi(y)\,\mathrm{d}y + \frac{1}{2}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(y)\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned}$$

□