

## PDE 第二章第二次作业

应数 2101 杨嘉昱 2216113458

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【题目 1】 求解三维波动方程的 Cauchy 问题

$$\begin{cases} u_{tt} = a^2 (u_{xx} + u_{yy} + u_{zz}) & \text{in } \mathbb{R}^3 \times (0, +\infty) \\ u = 0 \quad u_t = x^3 + y^2 z & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}.$$

解. 记  $x = (x_1, x_2, x_3)^T$ ,  $y = (u, v, w)^T$ ,  $h(y) = u^3 + v^2 w$ . 由 Kirchhoff 公式知

$$u(x, t) = t \oint_{\partial B(x, at)} u^3 + v^2 w \, dS(y) = \frac{1}{4\pi a^2 t} \int_{\partial B(x, at)} u^3 + v^2 w \, dS(y).$$

作变量替换

$$\begin{cases} u = x_1 + at \sin \phi \cos \theta \\ v = x_2 + at \sin \phi \sin \theta \\ w = x_3 + at \cos \phi \end{cases}$$

则

$$\begin{aligned} u(x, t) &= \frac{1}{4\pi a^2 t} \int_0^{2\pi} d\theta \int_0^\pi \left( (x_1 + at \sin \phi \cos \theta)^3 + (x_2 + at \sin \phi \sin \theta)^2 (x_3 + at \cos \phi) \right) a^2 t^2 \sin \phi \, d\phi \\ &= \frac{(x_1^3 + x_2^2 x_3)t}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \sin \phi \, d\phi + \frac{t}{4\pi} \int_0^{2\pi} d\theta \int_0^\pi x_2 at \sin \phi \, d\sin \phi \\ &\quad + \frac{t}{4\pi} \int_0^{2\pi} \cos \theta \, d\theta \int_0^\pi 2x_1^2 at \sin^2 \phi \, d\phi + \frac{t}{4\pi} \int_0^{2\pi} \sin \theta \, d\theta \int_0^\pi 2atx_2 x_3 \sin^2 \phi \, d\phi \\ &\quad + \frac{t}{4\pi} \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^\pi x_3 a^2 t^2 \sin^3 \phi \, d\phi + \frac{t}{4\pi} \int_0^{2\pi} \cos^2 \theta \, d\theta \int_0^\pi 3x_1 a^2 t^2 \sin^3 \phi \, d\phi \\ &\quad + \frac{t}{4\pi} \int_0^{2\pi} \sin \theta \, d\theta \int_0^\pi 2x_2 a^2 t^2 \sin^2 \phi \, d\sin \phi + \frac{t}{4\pi} \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^\pi a^3 t^3 \sin^3 \phi \, d\sin \phi \\ &\quad + \frac{t}{4\pi} \int_0^{2\pi} \cos^3 \theta \, d\theta \int_0^\pi a^3 t^3 \sin^4 \phi \, d\phi \\ &= (x_1^3 + x_2^2 x_3)t + \frac{a^2 t^3}{3} (3x_1 + x_3). \end{aligned}$$

□

【题目 2】 用降纬法导出一维波动方程 Cauchy 问题的求解公式。

证明. 只需对 Cauchy 问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & \text{in } \mathbb{R}^2 \times (0, +\infty) \\ u = 0, \quad u_t = h & \text{on } \mathbb{R}^2 \times t = 0 \end{cases}$$

证明即可。令

$$\bar{u}(x, y, t) = u(x, t), \quad \bar{h}(x, y) = h(x).$$

则  $\bar{u}$  满足 Cauchy 问题

$$\begin{cases} u_{tt} - a^2(u_{xx} + u_{yy}) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ u = 0, u_t = h & \text{on } \mathbb{R} \times t = 0 \end{cases}$$

由 Poisson 公式知

$$\begin{aligned} u(x, t) = \bar{u}(x, 0, t) &= \frac{1}{2\pi a} \int_{B((x, y), at)} \frac{\bar{h}(u, v)}{\sqrt{a^2 t^2 - (x - u)^2 - v^2}} du dv \\ &= \frac{1}{2\pi a} \int_{x-at}^{x+at} h(u) du \int_{-\sqrt{a^2 t^2 - (u-x)^2}}^{\sqrt{a^2 t^2 - (u-x)^2}} \frac{1}{\sqrt{a^2 t^2 - (x - u)^2 - v^2}} dv \end{aligned}$$

作变量替换

$$v = \sqrt{a^2 t^2 - (x - u)^2} \cdot \sin \theta$$

则

$$u(x, t) = \frac{1}{2\pi a} \int_{x-at}^{x+at} h(u) du \int_{-\pi/2}^{\pi/2} \frac{\cos \theta}{\cos \theta} d\theta = \frac{1}{2a} \int_{x-at}^{x+at} h(u) du.$$

□

**【题目 3】** 求解二维波动方程的 Cauchy 问题

$$\begin{cases} u_{tt} - a^2(u_{xx} + u_{yy}) = 0 & \text{in } \mathbb{R}^2 \times (0, +\infty) \\ u = x^2(x + y), u_t = 0 & \text{on } \mathbb{R}^2 \times \{t = 0\} \end{cases}.$$

解. 由 Poisson 公式知

$$u(x, y, t) = \frac{\partial}{\partial t} \left( \frac{1}{2\pi a} \int_{B((x, y), at)} \frac{u^2(u + v)}{\sqrt{a^2 t^2 - (u - x)^2 + (v - y)^2}} du dv \right)$$

其中作变量替换

$$\begin{cases} u = x + r \cos \theta \\ v = y + r \sin \theta \end{cases}$$

那么

$$\begin{aligned} & \int_{B((x, y), at)} \frac{u^2(u + v)}{\sqrt{a^2 t^2 - (u - x)^2 + (v - y)^2}} du dv \\ &= \int_0^{2\pi} d\theta \int_0^{at} \frac{(x + r \cos \theta)^2 (x + y + r(\sin \theta + \cos \theta))}{\sqrt{a^2 t^2 - r^2}} r dr \\ &= \int_0^{2\pi} d\theta \int_0^{at} \frac{x^2(x + y) + (3x + y)r^2 \cos^2 \theta}{\sqrt{a^2 t^2 - r^2}} r dr \end{aligned}$$

令  $r = at \sin \phi$ , 那么上式化为

$$\int_0^{2\pi} d\theta \int_0^{\pi/2} (x^2(x + y) + (3x + y)a^2 t^2 \sin^2 \phi \cos^2 \theta) at \cdot \sin \phi d\phi$$

故

$$\begin{aligned} u(x, y, t) &= \frac{\partial}{\partial t} \left( \frac{1}{2\pi a} \int_0^{2\pi} d\theta \int_0^{\pi/2} (x^2(x + y) + (3x + y)a^2 t^2 \sin^2 \phi \cos^2 \theta) at \cdot \sin \phi d\phi \right) \\ &= \frac{1}{2\pi a} \int_0^{2\pi} d\theta \int_0^{\pi/2} (ax^2(x + y) + 3(3x + y)a^3 t^2 \sin^2 \phi \cos^2 \theta) \sin \phi d\phi \\ &= x^2(x + y) + (3x + y)a^2 t^2. \end{aligned}$$

□

【题目 4】求以下特征问题的特征函数

$$\begin{cases} X''(x) + \lambda X(x) = 0, & x \in (0, \ell) \\ X(0) = X'(\ell) + hX(\ell) = 0 & h > 0 \end{cases}$$

解. 该问题为 Sturm - Liouville 问题, 因此  $\lambda > 0$ 。从而

$$X(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x), \quad x \in (0, \ell).$$

其中  $\mu = \sqrt{\lambda}$ . 将初始条件带入上式得

$$\begin{pmatrix} 1 & 0 \\ h \cos \mu \ell - \mu \sin \mu \ell & h \sin \mu \ell + \mu \cos \mu \ell \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

从而  $C_1 = 0$ 。又该方程有非零解, 故系数行列式为 0, 从而

$$h \sin \mu \ell + \mu \cos \mu \ell = 0 \implies \tan \ell \mu = -\frac{1}{h} \cdot \mu.$$

从而  $\forall n \in \mathbb{Z}_{>0}$ , 存在  $\mu_n \in ((n-1)\pi, n\pi)$  使得

$$\tan \ell \mu_n = -\frac{1}{h} \cdot \mu_n,$$

即  $\{\lambda_n = \mu_n^2\}_{n=1}^{\infty}$  为所有特征值。取  $C_2 = 1$ , 则特征函数为

$$X_n(x) = \sin(\mu_n x),$$

其中  $\mu_n$  为方程  $\tan(\ell x) = -x/h$  的所有正解。□

以下题目中均设

$$L = \frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}, \quad Q = (0, \ell) \times (0, +\infty).$$

【题目 5】用分离变量法求解:

$$\begin{cases} Lu = 0 & (x, t) \in Q \\ u|_{x=0} = u_x|_{x=\ell} = 0, & t \geq 0 \\ u|_{t=0} = x(x-2\ell), u_t|_{t=0} = 0, & 0 \leq x \leq \ell \end{cases}$$

解. 令  $u(x, t) = X(x)T(t)$ , 则原问题化为

$$\begin{cases} X'' + \lambda X = 0 & x \in (0, \ell) \\ X(0) = X'(\ell) = 0 \\ T'' + a^2 \lambda T = 0, & t > 0. \end{cases}$$

该问题为 Sturm - Liouville 问题, 因此  $\lambda > 0$ 。从而

$$X(x) = C_1 \cos \mu x + C_2 \sin \mu x.$$

其中  $\mu = \sqrt{\lambda}$ 。将初始条件带入上式得

$$\begin{pmatrix} 1 & 0 \\ -\mu \sin \mu \ell & \mu \cos \mu \ell \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

从而  $C_1 = 0$ 。又该方程有非零解，从而系数行列式为 0，即

$$\mu \cos \mu \ell = 0, \quad \implies \quad \mu_n = \frac{(n - \frac{1}{2})\pi}{\ell}, \quad \forall n \in \mathbb{Z}_{\geq 0}.$$

从而特征函数为

$$X_n(x) = \sin\left(\frac{(2n-1)\pi}{2\ell}x\right), \quad \implies \quad T_n(t) = A_n \cos\left(\frac{(2n-1)a\pi}{2\ell}t\right) + B_n \sin\left(\frac{(2n-1)a\pi}{2\ell}t\right)$$

故

$$u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos\left(\frac{(2n-1)a\pi}{2\ell}t\right) + B_n \sin\left(\frac{(2n-1)a\pi}{2\ell}t\right) \right) \sin\left(\frac{(2n-1)\pi}{2\ell}x\right)$$

又

$$\begin{aligned} x(x-2\ell) &= u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi}{2\ell}x\right) \\ 0 &= u_t(x, 0) = \sum_{n=1}^{\infty} \frac{(2n-1)a\pi}{2\ell} B_n \sin\left(\frac{(2n-1)\pi}{2\ell}x\right) \end{aligned}$$

故

$$A_n = \frac{2}{\ell} \int_0^{\ell} x(x-2\ell) \sin\left(\frac{(2n-1)\pi}{2\ell}x\right) dx = -\frac{32\ell^3}{(2n-1)^3\pi^3}, \quad B_n = 0 \quad \forall n \in \mathbb{Z}_{>0}.$$

从而

$$u(x, t) = -\sum_{n=1}^{\infty} \frac{32\ell^3}{(2n-1)^3\pi^3} \cos\left(\frac{(2n-1)a\pi}{2\ell}t\right) \sin\left(\frac{(2n-1)\pi}{2\ell}x\right)$$

□

**【题目 6】** 设  $u(x, t)$  适合定解问题

$$\begin{cases} Lu = f(x, t), & (x, t) \in Q \\ -u_x + \alpha u|_{x=0} = \mu_1(t) & t \geq 0 \\ u_x + \beta u|_{x=\ell} = \mu(t) & t \geq 0 \\ u|_{t=0} = \phi(x), \quad u_t|_{t=0} = 0, & x \in [0, \ell] \end{cases}.$$

试引进辅助函数，把边界条件齐次化，设

1.  $\alpha, \beta > 0$ ;
2.  $\alpha = \beta = 0$ .

解.

1.  $\alpha, \beta > 0$

设

$$v(x, t) = u(x, t) + h(x, t)$$

将原问题的边界齐次化。即

$$(-u_x + \alpha u - h_x + \alpha h)|_{x=0} = 0 \quad (u_x + \beta u + h_x + \beta h)|_{x=\ell} = 0$$

由于  $u$  为原问题的解，故

$$(-h_x + \alpha h)|_{x=0} + \mu_1(t) = 0, \quad (h_x + \beta h)|_{x=\ell} + \mu(t) = 0.$$

现令

$$h(x, t) = (mx + n)\mu(t) + (px + q)\mu_1(t).$$

带入上面两式得

$$\begin{pmatrix} -m + \alpha n & -p + \alpha q + 1 \\ (\beta\ell + 1)m + \beta n + 1 & (1 + \beta\ell)p + q \end{pmatrix} \begin{pmatrix} \mu(t) \\ \mu_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

从而

$$\begin{pmatrix} -1 & \alpha & & \\ & & -1 & \alpha \\ \beta\ell + 1 & \beta & & \\ & & 1 + \beta\ell & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

解得

$$m = \frac{\alpha}{\alpha + \beta + \alpha\beta\ell}, \quad n = \frac{1}{\alpha + \beta + \alpha\beta\ell}, \quad p = \frac{-\beta}{\alpha + \beta + \alpha\beta\ell}, \quad q = \frac{1 + \beta\ell}{\alpha + \beta + \alpha\beta\ell}$$

从而

$$v(x, t) = u(x, t) + \frac{(\alpha x + 1)\mu(t) + (-\beta x + 1 + \beta\ell)\mu_1(t)}{\alpha + \beta + \alpha\beta\ell}$$

## 2. $\alpha = \beta = 0$

此时边界条件为

$$u_x(0, t) + \mu_1(t) = 0, \quad u_x(\ell, t) - \mu(t) = 0.$$

设

$$v(x, t) = u(x, t) + h(x, t)$$

将原问题的边界齐次化。即

$$h_x(0, t) - \mu_1(t) = 0, \quad h_x(\ell, t) + \mu(t) = 0.$$

不妨设

$$h_x(x, t) = \frac{\ell - x}{\ell} \mu_1(t) - \frac{x}{\ell} \mu(t).$$

两侧对  $x$  积分, 得到一个解为

$$h(x, t) = \left(x - \frac{x^2}{2\ell}\right) \mu_1(t) - \frac{x^2}{2\ell} \mu(t).$$

从而

$$v(x, t) = u(x, t) + \left(x - \frac{x^2}{2\ell}\right) \mu_1(t) - \frac{x^2}{2\ell} \mu(t)$$

□

## 【题目 7】用分离变量法求解

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & (x, t) \in Q \\ u_x|_{x=0} = A \sin \omega t, \quad u|_{x=\ell} = 0, & t \geq 0 \\ u|_{t=0} = 1, \quad u_t|_{t=0} = 0, & x \in [x, \ell] \end{cases}.$$

解. 令

$$v(x, t) = u(x, t) - A(x - \ell) \sin \omega t,$$

那么  $v$  满足方程

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = A\omega^2(x-\ell) \sin \omega t & (x, t) \in Q \\ v_x|_{x=0} = 0, v|_{x=\ell} = 0, & t \geq 0 \\ v|_{t=0} = 1, v_t|_{t=0} = -A\omega(x-\ell), & x \in [x, \ell] \end{cases}.$$

记  $v(x, t) = X(x)T(t)$ . 其特征方程为

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, X(\ell) = 0 \end{cases}$$

解得特征方程为

$$X_n(x) = \cos \frac{\left(n - \frac{1}{2}\right) \pi x}{\ell}, \quad \forall n \in \mathbb{Z}_{>0}. \quad (1)$$

从而

$$v(x, t) = \sum_{n=1}^{\infty} T_n(t) \cos \frac{\left(n - \frac{1}{2}\right) \pi x}{\ell}$$

带入边界条件有

$$\begin{cases} \sum_{n=1}^{\infty} \left( T_n''(t) + \frac{\left(n - \frac{1}{2}\right)^2 \pi^2}{\ell^2} T_n(t) \right) \cos \frac{\left(n - \frac{1}{2}\right) \pi x}{\ell} = A\omega^2(x-\ell) \sin \omega t, \\ \sum_{n=1}^{\infty} T_n(0) \cos \frac{\left(n - \frac{1}{2}\right) \pi x}{\ell} = 1, \\ \sum_{n=1}^{\infty} T_n'(0) \cos \frac{\left(n - \frac{1}{2}\right) \pi x}{\ell} = -A\omega(x-\ell) \end{cases}$$

即  $\forall n \in \mathbb{Z}_{>0}$ ,  $T_n$  满足微分方程

$$\begin{cases} T_n''(t) + \frac{\left(n - \frac{1}{2}\right)^2 \pi^2}{\ell^2} T_n(t) = f_n(t) \\ T_n(0) = \alpha_n, \quad T_n'(0) = \beta_n. \end{cases}$$

其中

$$f_n(t) = \frac{2}{\ell} \int_0^\ell A\omega^2(x-\ell) \sin \omega t \cos \frac{\left(n - \frac{1}{2}\right) \pi x}{\ell} dx = -\frac{8A\omega^2\ell}{(2n-1)^2\pi^2} \sin \omega t = -\omega\beta_n \sin \omega t,$$

$$\alpha_n = \frac{2}{\ell} \int_0^\ell \cos \frac{\left(n - \frac{1}{2}\right) \pi x}{\ell} dx = (-1)^{n+1} \cdot \frac{4}{(2n+1)\pi}, \quad (2)$$

$$\beta_n = \frac{2}{\ell} \int_0^\ell -A\omega(x-\ell) \cos \frac{\left(n - \frac{1}{2}\right) \pi x}{\ell} dx = \frac{8A\omega\ell}{(2n-1)^2\pi^2}. \quad (3)$$

记

$$\omega_n = \frac{2n-1}{2\ell} \pi. \quad (4)$$

解上面这个初值问题, 得

$$\begin{aligned} T_n(t) &= \alpha_n \cos \omega_n t + \frac{\beta_n}{\omega_n} \sin \omega_n t + \frac{1}{\omega_n} \cdot \int_0^t f_n(\tau) \sin \omega_n(t-\tau) d\tau \\ &= \alpha_n \cos \omega_n t + \frac{\beta_n}{\omega_n} \sin \omega_n t - \frac{\omega\beta_n}{\omega_n} \cdot \int_0^t \sin \omega\tau \sin \omega_n(t-\tau) d\tau \end{aligned}$$

其中

$$\int_0^t \sin \omega \tau \sin \omega_n(t-\tau) d\tau = \frac{1}{2} \int_0^t \cos((\omega + \omega_n)\tau - \omega_n t) - \cos((\omega - \omega_n)\tau + \omega_n t) d\tau$$

首先设若  $\omega \neq \omega_n$ , 那么

$$\int_0^t \sin \omega \tau \sin \omega_n(t-\tau) d\tau = \frac{\omega \sin \omega_n t - \omega_n \sin \omega t}{\omega^2 - \omega_n^2}$$

从而

$$T_n(t) = \alpha_n \cos \omega_n t + \frac{\beta_n}{\omega_n} \sin \omega_n t - \frac{\omega \beta_n}{\omega_n (\omega^2 - \omega_n^2)} \cdot (\omega \sin \omega_n t - \omega_n \sin \omega t) \quad (5)$$

其中  $\alpha_n, \beta_n, \omega_n$  由式2, 3, 4 给出

1. 若不产生共振现象, 即  $\omega \neq \omega_n, \forall n \in \mathbb{Z}_{>0}$ , 那么

$$u(x, t) = A(x - \ell) \sin \omega t + \sum_{n=1}^{\infty} X_n(x) T_n(t).$$

其中  $T_n, X_n$  由式5, 1 给出

2. 若产生共振, 设  $\omega = \omega_k$ , 则

$$\tilde{T}_k(t) = \lim_{\omega \rightarrow \omega_k} T_k(t) = \alpha_k \cos \omega_k t + \frac{\beta_k}{\omega_k} \sin \omega_k t - \beta_k \left( \frac{\sin \omega_k t}{2\omega_k} - \frac{t \cos \omega_k t}{2} \right)$$

那么

$$u(x, t) = A(x - \ell) \sin \omega t + \sum_{\substack{n \geq 1 \\ n \neq k}} X_n(x) T_n(t) + X_k(x) \tilde{T}_k(t).$$

□

**【题目 8】** 考虑定解问题

$$\begin{cases} u_{tt} - u_{xx} = f(x, t) & (x, t) \in Q \\ u(0, t) = u(\ell, t) = 0 & t \geq 0; \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x) & x \in [0, \ell] \end{cases}$$

要使解为古典解, 那么  $\phi, \psi, f$  要加什么条件?

**证明.**

$\phi \in C^2[0, \ell], \psi \in C^3[0, \ell], f \in C^2([0, \ell] \times (0, +\infty))$  令  $u(x, t) = X(x)T(t)$  并且满足

$$\phi(0) = \phi(\ell) = \phi''(0) = \phi''(\ell) = \psi(0) = \psi(\ell) = f(0, t) = f(\ell, t) = 0.$$

下面给出证明。

原问题的特征方程为

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\ell) = 0 \end{cases}$$

解得

$$X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right), \quad \forall n \in \mathbb{Z}_{>0}.$$

记

$$f_n(t) = \frac{2}{\ell} \int_0^\ell f(x, t) \sin\left(\frac{n\pi}{\ell}x\right) dx, \quad \phi_n = \frac{2}{\ell} \int_0^\ell \phi(x) \sin\left(\frac{n\pi}{\ell}x\right) dx, \quad \psi_n = \frac{2}{\ell} \int_0^\ell \psi(x) \sin\left(\frac{n\pi}{\ell}x\right) dx,$$

那么  $T_n$  满足微分方程

$$\begin{cases} T_n'' + \left(\frac{n\pi}{\ell}\right)^2 T_n = f_n(t) \\ T_n(0) = \phi_n, T_n'(0) = \psi_n \end{cases}$$

解得

$$T_n(t) = \phi_n \cos \frac{n\pi}{\ell} t + \frac{\ell}{n\pi} \psi_n \sin \frac{n\pi}{\ell} t + \frac{\ell}{n\pi} \int_0^t f_n(\tau) \sin \frac{n\pi}{\ell} (t - \tau) d\tau.$$

记

$$\begin{aligned} A_n = \phi_n &= \frac{2}{\ell} \int_0^\ell \phi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx = \frac{2}{\ell} \cdot \left(\frac{\ell}{n\pi}\right)^3 \int_0^\ell \phi'''(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \\ B_n = \frac{\ell}{n\pi} \psi_n &= \frac{2}{\ell} \cdot \frac{\ell}{n\pi} \int_0^\ell \psi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx = -\frac{2}{\ell} \cdot \left(\frac{\ell}{n\pi}\right)^3 \int_0^\ell \psi''(x) \sin\left(\frac{n\pi x}{\ell}\right) dx. \end{aligned}$$

若记

$$v_n(x, t) = \left( \phi_n \cos \frac{n\pi}{\ell} t + \frac{\ell}{n\pi} \psi_n \sin \frac{n\pi}{\ell} t \right) \sin \frac{n\pi}{\ell} x.$$

那么

$$\begin{aligned} |v_n| &\leq |A_n| + |B_n| = \mathcal{O}\left(\frac{1}{n^3}\right) \\ |Dv_n| &= \mathcal{O}\left(\frac{1}{n^2}\right) \\ |D^2v_n| &\leq \left( \int_0^\ell \phi'''(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \right)^2 + \left( \int_0^\ell \psi''(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \right)^2 + \mathcal{O}\left(\frac{1}{n^2}\right) \end{aligned}$$

但由 Bessel 不等式知

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \int_0^\ell \phi'''(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \right)^2 &\leq \frac{2}{\ell} \int_0^\ell |\phi'''(x)|^2 dx < +\infty \\ \sum_{n=1}^{\infty} \left( \int_0^\ell \psi''(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \right)^2 &\leq \frac{2}{\ell} \int_0^\ell |\psi''(x)|^2 dx < +\infty \end{aligned}$$

从而  $v_n, Dv_n, D^2v_n$  求和一致收敛。下面证明对  $w_n = u_n - v_n$  也有同样的结论。

记

$$g_n(t) = \frac{\ell}{n\pi} \int_0^t f_n(\tau) \sin \frac{n\pi}{\ell} (t - \tau) d\tau.$$

则

$$f_n(\tau) = -\frac{2}{\ell} \cdot \left(\frac{\ell}{n\pi}\right)^2 \int_0^\ell f_{xx}(x, \tau) \sin\left(\frac{n\pi x}{\ell}\right) dx = \mathcal{O}\left(\frac{\gamma_n}{n^2}\right).$$

其中

$$\gamma_n = \gamma_n(t) = \int_0^\ell f_{xx}(x, \tau) \sin\left(\frac{n\pi x}{\ell}\right) dx.$$

由 Bessel 不等式有

$$\sum_{n=1}^{\infty} \gamma_n^2 \leq \frac{2}{\ell} \int_0^\ell |f_{xx}|^2 dx \leq 2 \|f_{xx}\|_\infty.$$

故

$$D^\alpha \left( g_n(t) \sin \frac{n\pi}{\ell} x \right) = \mathcal{O}\left(\frac{\gamma_n}{n}\right)$$

其中  $\alpha = 0, 1, 2$ 。但

$$2 \cdot \frac{\gamma_n}{n^2} \leq \sum_{n=1}^{\infty} (\gamma_n^2 + 1/n^2) < +\infty.$$

从而一致收敛。

□



**【题目 9】** 用能量不等式证明一维波动方程带有第三边值条件的初边值问题解的唯一性。

**证明.** 设一维波动方程带有第三边值条件的初边值问题为

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x, t) & (x, t) \in (0, \ell) \times (0, +\infty) \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x) & x \in [0, \ell] \\ -u_x + \alpha u \Big|_{x=0} = g_1, u_x + \beta u \Big|_{x=\ell} = g_2 \end{cases}$$

只需证明当  $\phi = \psi = g_1 = g_2 = 0$  即可。

定义能量为

$$E(t) = \frac{1}{2} \int_0^\ell u_t^2 + a^2 u_x^2 \, dx.$$

则

$$\begin{aligned} \frac{dE}{dt} &= \int_0^\ell u_t u_{tt} + a^2 u_x u_{xt} \, dx = \int_0^\ell u_t (u_{tt} - a^2 u_{xx}) \, dx + a^2 u_x u_t \Big|_0^\ell \\ &= a^2 (u_x(\ell, t) u_t(\ell, t) - u_x(0, t) u_t(0, t)) = a^2 (-\beta u(\ell, t) u_t(\ell, t) - \alpha u(0, t) u_t(0, t)) \\ &= -\frac{1}{2} a^2 \left( \alpha (u^2)_t \Big|_{x=0} + \beta (u^2)_t \Big|_{x=\ell} \right) \end{aligned}$$

从而

$$E(t) - E(0) = -\frac{1}{2} (\alpha u^2(0, t) + \beta u^2(\ell, t)) \leq 0.$$

即

$$E(t) \leq E(0) = 0.$$

但恒有  $E(t) \geq 0$ , 故  $E(t) = 0$ , 即

$$u_t = u_x = 0.$$

这说明  $u = 0$ 。

□