Second gravity

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A theory of a new gravitational interaction is described. This theory follows naturally from a new Lagrangian formulation of Maxwell's theory for photons and electrons (and positrons) whose associated Euler Lagrange equations imply the conventional Maxwell equations, but which possesses new *bosonic* spinor degrees of freedom that may be associated with a new type of fundamental gravitational interaction. The precise character of this gravitational interaction with a photon vector potential is explicitly defined in terms of a local U(1)-invariant Lagrangian in Eq. (86). However, in Sec. VI B 1, in order to parallel the well known Friedmann model in cosmology, a phenomenological description of the new gravitational interaction coupled to Newton–Einstein gravity that is sourced by an ideal fluid is discussed. To lay the foundation for a description of the new gravitational interaction, our new formulation of Maxwell's theory must first be described. It is cast on the real, eight-dimensional pseudo-Euclidean vector space defined by the split octonion algebra, regarded as a vector space over $\mathbb{R}$ and denoted as $\mathbb{R}^{4,4} \cong M_{3,1} \oplus {}^*M_{3,1}$. (Here $M_{3,1}$ denotes real four-dimensional Minkowski space-time and ${}^*M_{3,1}$ denotes its dual; $\mathbb{R}^{4,4}$ resembles the phase space of a single relativistic particle.) The new gravitational interaction is carried by a field that defines an algebraically distinguished element of the split octonion algebra, namely, the multiplicative unit element. We call this interaction the "unit" interaction and more descriptively refer to it as "second gravity."
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I. INTRODUCTION

This paper first describes a new Lagrangian formulation of Maxwell's theory for photons and electrons (and positrons) whose associated Euler Lagrange equations imply the conventional Maxwell equations. Central to this new formalism is an algebraic construction that possesses new *bosonic* degrees of freedom that comprise of an eight-component "unit" spinor field u^a, $a=1,\ldots,8$. In the latter part of this paper, we conjecture that this unit spinor field $u^a(x^\mu)$ may be associated with a new fundamental gravitational interaction. For brevity, we say that the unit spinor field carries a massless "second gravity" interaction. This idea is based on the observation that, as will be argued below, we are free to regard the u field as dimensionless, and so are not compelled to introduce a nonzero characteristic mass for u field quanta. Moreover, the invariant norm of u, denoted by $\sqrt{\tilde{u}\sigma u}$, of the unit spinor field should never vanish. This suggests that the Lagrangian density, whose associated Euler–Lagrange equations govern the dynamical evolution of the unit field, should be constructed from a term that contains a factor such as $(\tilde{u}\sigma u)^{-1}$. This is somewhat analogous to the Einstein–Hilbert Lagrangian density, which is constructed from the metric and its inverse; this often has the practical effect of enforcing the nonvanishing of the determinant of the metric when the field equations for the metric are satisfied. Even if the u field is not dimensionless, the factors of u arising from the norm of u in the denominator of the unit

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Lagrangian may be employed to nondimensionalize the free-field unit Lagrangian density, except for the dimensional derivatives that must appear. In this case, a multiplicative dimensional coupling constant is required in the free-field unit Lagrangian density to give the action the correct dimension. A free-field unit Lagrangian density may be chosen so that this multiplicative dimensional coupling constant is proportional to the coupling constant employed in the Einstein–Hilbert Lagrangian density. Other choices are, of course, possible.

The precise character of the new formulation of the electromagnetic interaction is explicitly defined in terms of a local U(1)-invariant Lagrangian in Eq. (86). In Sec. VI B 1 we go on to begin a discussion of a phenomenological description of the "second" gravity interaction with Newton–Einstein gravity sourced by an ideal fluid that is characterized by its density and pressure, and which satisfies a phenomenological equation of state.

In Sec. VI we list several possible "free-field" Lagrangians for the unit field and propose that the coupling constant of the unit field to the rest of the universe is through a function of the Planck mass. As such, this theory may ultimately provide a new model of the {"inflaton," "dark" (energy, matter)} interactions with ordinary matter, unified through the unit field. This conjecture requires further study. However, we note that, in the simplest model, this unification is accomplished without introducing a new characteristic mass into the theory[1,21,14] or a "Chameleon" field whose varying mass depends on the local matter density.[35,27] Also, the new model does not invoke supersymmetry.

In addition, path integration with respect to certain of these new fields yields a new formulation of nonlinear electrodynamics. There is a growing interest in the cosmological effects of nonlinear electrodynamics because some theories of nonlinear electrodynamics give rise to cosmological inflation,[39,16] spawn a period of cosmic acceleration,[39,46,17,40] may avoid the problem of initial singularity,[39,18] and may explain the generation of astrophysically observed primeval magnetic fields during inflation era.[10] Large-scale magnetic fields with intensities of the order of microgauss have been observed by Faraday rotation measurements in galaxies (at both high and low redshifts) as well as in clusters of galaxies.[4,28,42] One of the most challenging problems in modern astrophysics and cosmology is the origin of galactic and extragalactic magnetic fields.[44]

We study a new formulation of Maxwell's theory that is cast on the split octonion algebra, regarded as a vector space over $\mathbb{R}$. Our convention is that this vector space is isomorphic to the real, eight-dimensional pseudo-Euclidean vector space $\mathbb{E}_{4,4} \cong \mathbb{R}^{4,4} \cong M_{3,1} \oplus {}^{*}M_{3,1}$, where $M_{3,1}$ denotes real four-dimensional Minkowski space-time and ${}^{*}M_{3,1}$ denotes its dual. We assume that $M_{3,1}$ is endowed with the pseudo-Euclidean metric $\eta_{3,1} = \mathrm{diag}(1,1,1,-1)$ and that ${}^{*}M_{3,1}$ is endowed with the pseudo-Euclidean metric $-\eta_{3,1} = \mathrm{diag}(-1,-1,-1,1)$. Thus $\mathbb{R}^{4,4}$ resembles the phase space of a single relativistic particle (appropriate restrictions on the automorphism group are implied). It is well known[2] that $\mathbb{R}^{4,4}$ carries representations of two types of basic spinor and one type of vector. In Sec. V we employ the two types of $\mathbb{R}^{4,4}$ spinor to represent the fundamental electromagnetic field degrees of freedom. We assume that the multiplicative unit element of the split octonion algebra is defined in terms of a constant type-1 $\mathbb{R}^{4,4}$ spinor with components u^a, $a = 1, \ldots, 8$. Given these assumptions, the Lagrangian for the Maxwell theory is constructed from a type-2 $\mathbb{R}^{4,4}$ spinor that represents a generalization of the electromagnetic four-vector potential and a type-1 $\mathbb{R}^{4,4}$ spinor that is a Lagrangian multiplier whose physical interpretation represents a generalization of the electromagnetic field tensor. In Sec. VI we associate the multiplicative unit element of the split octonion algebra with a type-1 $\mathbb{R}^{4,4}$ bosonic spinor field $u^a = u^a(x)$, $a = 1, \ldots, 8$, $x \in \mathbb{R}^{4,4}$, that is postulated to carry the unit second gravity interaction. First, we restrict attention to flat Minkowski space-time. A Lagrangian is studied that describes the unit interaction with the photon. It is shown that the electromagnetic interaction with the unit field is uniquely defined independent of any coupling constants. Lastly we turn to defining a simple cosmological model.

A word concerning the history of the (split) octonions is employed in this model. In 1925 Cartan[12,13] described "triality" as a symmetry between three types of geometrical objects that exist in real, eight-dimensional Euclidean space $\mathbb{E}_8 \cong \mathbb{R}^8$, namely, between vectors and two types of spinor (semispinors of the first type and semispinors of the second type, in the terminology of

Cartan). Vectors with components V^A, $A=1,\ldots,8$, may be thought of as defining directional derivatives $V^A(\partial/\partial x^A)$ with respect to Cartesian coordinates x^A, $A=1,\ldots,8$, of C^∞ functions on $\mathbb{E}_8$. It is natural to identify $\mathbb{E}_8$ with the nonassociative octonion algebra $\mathbb{O}$ invented by John T. Graves in late 1843, regarded as a vector space over $\mathbb{R}$. Analogous to the normed vector space $\mathbb{E}_8$, the vector space $\mathbb{E}_{4,4}$ (with indefinite norm) carries representations of vectors and two types of spinor; the latter is employed in this paper to carry the fundamental electromagnetic field degrees of freedom. Also, as mentioned above, in Sec. VI the unit interaction is associated with a type-1 $\mathbb{R}^{4,4}$ spinor field u^a.

In this paper, no attempt is made to resolve the photon field into contributions from the gauge field tensors describing weak hypercharge and the third component of the weak isospin. This action is premature. Indeed, the very successful standard model (SM) is also rather empirical, requiring some 19 (or more) independent (running) numerical parameters to define the theory (for example, the Weinberg angle is a parameter that has a value that varies as a function of the momentum transfer at which it is measured). Integration of the new formulation of Maxwell's theory with the final (extended) form of the SM may be postponed.

In order to make this paper self-contained and to establish our notation and conventions, we summarize some of the preliminary works established and discussed in detail in Ref. 36.

A. Maxwell's equations

As is very well known, Maxwell's equations in Gaussian units[26,3] for the electromagnetic field tensor $\mathbb{F}_{\alpha\beta}$ and its dual ${}^*\mathbb{F}^{\mu\nu}=-\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\mathbb{F}_{\alpha\beta}$ may be written as $\partial_\beta\mathbb{F}^{\alpha\beta}=(4\pi/c)\mathbb{J}^\alpha$ and $\partial_\beta\,{}^*\mathbb{F}^{\alpha\beta}=0$, the latter being "solved" by $\mathbb{F}_{\alpha\beta}=\partial_\alpha\mathbb{A}_\beta-\partial_\beta\mathbb{A}_\alpha$. Here, α, β, μ, and ν run from 1 to 4; we also assume that lower case Latin indices run from 1 to 3. The electric $\{\vec{\mathbb{E}}\}_j=\mathbb{E}_j=\mathbb{F}_{j4}=\mathbb{F}^j_{\ 4}=-\mathbb{F}^{j4}$ and magnetic $\{\vec{\mathbb{B}}\}^j=\mathbb{B}^j=\mathbb{B}_j=\frac{1}{2}\epsilon^{jhk}\mathbb{F}_{hk}=\frac{1}{2}\epsilon^{hkj4}\mathbb{F}_{hk}$ space-time fields do not, of course, transform as vectors, but as components of the antisymmetric rank 2 tensor $\mathbb{F}_{\alpha\beta}$. Projection onto the local time axis of $\mathbb{F}_{\alpha\beta}$ and its dual ${}^*\mathbb{F}^{\mu\nu}$ provides the conventional (noncovariant) procedures for defining, respectively, the electric and magnetic fields. For later use, we record

$$\mathbb{F}^{\alpha\beta}=\eta^{\alpha\mu}\eta^{\beta\nu}\mathbb{F}_{\mu\nu}=\begin{pmatrix} 0 & \mathbb{B}_3 & -\mathbb{B}_2 & -\mathbb{E}_1 \\ -\mathbb{B}_3 & 0 & \mathbb{B}_1 & -\mathbb{E}_2 \\ \mathbb{B}_2 & -\mathbb{B}_1 & 0 & -\mathbb{E}_3 \\ \mathbb{E}_1 & \mathbb{E}_2 & \mathbb{E}_3 & 0 \end{pmatrix} \tag{1}$$

and

$${}^*\mathbb{F}^{\alpha\beta}=\begin{pmatrix} 0 & -\mathbb{E}_3 & \mathbb{E}_2 & -\mathbb{B}_1 \\ \mathbb{E}_3 & 0 & -\mathbb{E}_1 & -\mathbb{B}_2 \\ -\mathbb{E}_2 & \mathbb{E}_1 & 0 & -\mathbb{B}_3 \\ \mathbb{B}_1 & \mathbb{B}_2 & \mathbb{B}_3 & 0 \end{pmatrix}. \tag{2}$$

Here, $\eta=\eta_{3,1}=\mathrm{diag}(1,1,1,-1)$ is the pseudo-Euclidean metric on flat four-dimensional Minkowski space-time $M_{3,1}$. When constructing the conventional Lagrangian L for the electromagnetic field, one usually defines $\mathbb{F}_{\alpha\beta}=\partial_\alpha\mathbb{A}_\beta-\partial_\beta\mathbb{A}_\alpha$, which ensures that $\partial_\beta\,{}^*\mathbb{F}^{\alpha\beta}=0$, and then defines $L=-(1/16\pi)\mathbb{F}^{\alpha\beta}\mathbb{F}_{\alpha\beta}-(q/c)\mathbb{J}^\alpha\mathbb{A}_\alpha$ from $\mathbb{A}_\alpha$, its derivatives, and a conserved electric current $\mathbb{J}^\alpha$.

There is a very long history of the work that seeks to generalize the realization of the photon field. One noteworthy approach is the massless Duffin–Kemmer–Petaiu (DKP) equations[24] for a ten-component wave function. The DKP equations imply both sets of Maxwell equations, $\partial_\beta\,{}^*\mathbb{F}^{\alpha\beta}=0$ and $\partial_\beta\mathbb{F}^{\alpha\beta}=(4\pi/c)\mathbb{J}^\alpha$. The ten-component DKP employs 10×10 β^μ matrices that satisfy the DKP relations

$$\beta^\mu\beta^\alpha\beta^\nu + \beta^\nu\beta^\alpha\beta^\mu = \eta^{\alpha\mu}\beta^\nu + \eta^{\alpha\nu}\beta^\mu, \tag{3}$$

where $\eta_{\mu\nu}=\mathrm{diag}(1,1,1,-1)$. Massless DKP also employs a projection matrix $\gamma=\gamma^2$ such that $\gamma\beta^\alpha+\beta^\alpha\gamma=\beta^\alpha$ and a conserved source $J(x)\in\mathbb{R}^{10}$ that verifies $\gamma J=0$. Our new formulation is completely different from massless DKP.

Almost 80 years ago, Mignani *et al.*[32] investigated an analogy between the Maxwell equations (Gaussian units) formulated in terms of a complex Riemann–Silberstein vector $\sqrt{2}\vec{\mathfrak{E}}=\vec{\mathbb{E}}+i\vec{\mathbb{B}}$ and the Dirac equation. Shortly thereafter Maxwell's equations were cast into a spinor form[29] that employed two symmetric second rank spinors associated with massless helicity-one fields,

$$\Phi_{\alpha\beta}=\begin{pmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{pmatrix}$$

and

$$\Phi^{\dot\alpha\dot\beta}=\begin{pmatrix}\phi^{\dot1\dot1} & \phi^{\dot1\dot2}\\ \phi^{\dot2\dot1} & \phi^{\dot2\dot2}\end{pmatrix}.$$

The components of these spinors were defined as particular linear combinations of the components of $\vec{\mathfrak{E}}$. Almost simultaneously, four-component constructions χ were introduced in literature in order to realize a Dirac-spinor-like representation of the photon field and a Dirac equation form for Maxwell's equations. These objects were constructed according to an object equivalent to $\chi=(\phi_{11},\phi_{12},\phi_{21},\phi_{22})^T$ and possess a Lorentz transformation law more complex than that of a symmetric second rank spinor. This formalism was studied by Oppenheimer in 1931.[41] This was later developed by Moses[33,34] and others.[5,20] See Ref. 5 for a more complete review of the four-component Dirac-spinor-like representation (and other representations) of the photon wave function.

There are (split) octonion realizations of electromagnetism that seek to generalize the representation of the photon field and its interactions with matter. Among many contributions, Refs. 6, 19, 15, 22, 23, 11, and 45 and references therein are particularly noteworthy. While this body of work contains many significant results, none follow the approach discussed here, which emphasizes the role of the multiplicative unit of the split octonions, and which is outlined in the first two paragraphs of this section.

In summary, the focus of this paper is to develop an alternative to the approaches of Mignani *et al.* and others that gives the accepted Maxwell equations and Lorentz force for the source, and which also possesses new degrees of freedom that may be associated with a model description of the gravitational unit interaction. These new bosonic degrees of freedom are arranged into a type-1 $\mathbb{R}^{4,4}$ spinor field with components u^a, $a=1,\ldots,8$, which we refer to as the unit (spinor) field. This unit spinor field is associated with an algebraically distinguished split octonion, the multiplicative unit element in the algebra. An advantage of our approach is that our reformulation of Maxwell's equations naturally defines the gravitational interaction of the unit field with the electromagnetic field. However, the Lagrangian for the free unit field is completely undetermined and must be defined using other arguments.

II. REPRESENTATIONS OF $\overline{\mathrm{SO}(4,4;\mathbb{R})}$

Since this theory depends on an unconventional representation of bosonic fields, it may be worthwhile to first recall several well known results concerning the representations of $\mathrm{SO}(8,\mathbb{C})$.

A. Representations of $\mathrm{SO}(8,\mathbb{C})$

Summarizing and applying a few facts that may be found in, for example, Boerner, *Representations of Groups*,[7] the Clifford algebra C_8 may be defined as the algebra generated by a set of eight elements e_j, $j=1,\ldots,8$, that anticommute with each other and have unit square $e_je_k+e_ke_j=2\delta_{jk}\mathbb{I}_{16\times16}$, $\mathbb{I}$=unit matrix. The scaled commutators $\frac{1}{4}(e_je_k-e_ke_j)$ computed from the 16-

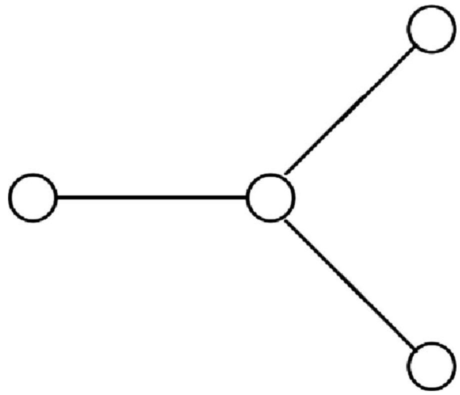

FIG. 1. Dynkin diagram for D_4.

dimensional irreducible representation of the e_j are the infinitesimal generators of a reducible 16-dimensional representation of spin $(8,\mathbb{C})$, which is the universal double covering of the special orthogonal group $\mathrm{SO}(8,\mathbb{C})$. This 16-dimensional representation is fully reducible to the direct sum of two inequivalent irreducible 8×8 spin representations of the infinitesimal generators of spin $(8,\mathbb{C})$, which leads to the identification of type 1 and type 2 spinors. The fundamental irreducible vector representation of $\mathrm{SO}(8,\mathbb{C})$ is also 8×8. The Dynkin diagram for $D_4 \cong \mathrm{SO}(8)$ is symmetrical and pictured in Fig. 1. The three outer nodes correspond to the vector representation (leftmost node), type 1 spinor and type 2 spinor representations of spin(8), and the central node corresponds to the adjoint representation.

B. Spinor representations of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$

Let $\mathrm{SO}(4,4;\mathbb{R})$ denote the group of all matrices with unit determinant that preserves the quadratic form

$$(x^8)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 - [(x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2],$$

where the x^A, $A,B,\ldots=1,2,\ldots,8$ comprise of a standard Cartesian coordinate system for $\mathbb{R}^{4,4}$. This quadratic form defines the standard pseudo-Euclidean norm on $\mathbb{R}^{4,4}$ and associated indefinite inner product $\langle\cdot,\cdot\rangle$ such that $\mathbb{R}^{4,4}\times\mathbb{R}^{4,4} \ni (u,v)\mapsto\langle u,v\rangle\in\mathbb{R}$. Equivalently, $\mathbb{R}^{4,4}$ may be endowed with the $\mathrm{SO}(4,4;\mathbb{R})$-invariant pseudo-Euclidean metric $\mathbb{G}$ whose components $\mathbb{G}_{AB}$ are numerically defined according to

$$\mathbb{G}_{AB} = \mathbb{G}^{AB} = \begin{pmatrix} \eta_{3,1} & 0 \\ 0 & -\eta_{3,1} \end{pmatrix}, \tag{4}$$

where $\eta_{3,1}=\mathrm{diag}(1,1,1,-1)$ is the pseudo-Euclidean metric on flat four-dimensional Minkowski space-time $M_{3,1}$. Here, $A,B,\ldots=1,\ldots,8$ may be regarded as $\mathbb{R}^{4,4}$ vector indices; $\mathbb{G}\leftrightarrow\mathbb{G}_{AB}$ $(\mathbb{G}^{-1}\leftrightarrow\mathbb{G}^{AB})$ will be used to lower (raise) upper case Latin indices.

$\mathrm{SO}(4,4;\mathbb{R})$ is a pseudo-orthogonal Lie group that possess two connected components.[7,25] The 2-1 covering group spin $(4,4,\mathbb{R})$, alternatively denoted as $\overline{\mathrm{SO}(4,4;\mathbb{R})}$, is distinguished by a bar placed over $\mathrm{SO}(4,4;\mathbb{R})$. $\mathbb{R}^{4,4}$ may be endowed with a $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ invariant metric σ (Ref. 37) that we represent as

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5}$$

where 0 denotes the 4×4 zero matrix and 1 denotes the 4×4 unit matrix. The matrix elements of σ are $\sigma_{ab}=\sigma_{ba}$, where $a,b,\ldots=1,\ldots,8$ are $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ spinor indices [elaborated in Eqs. (18) and

(23) below]. Note that σ^2 is equal to the unit matrix, so that the eigenvalues of σ of are ± 1. Since the trace of σ is zero, these eigenvalues occur with equal multiplicity. Given a spinor $\psi \in \mathbb{R}^{4,4}$ that is expanded as a linear combination of the eigenvectors of σ, computation of the $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ invariant norm-squared $\psi^a \sigma_{ab} \psi^b$ yields a $\overline{\mathrm{SO}(4,4;\mathbb{R})}$-invariant quadratic form.

A spinor element $\psi \in \mathbb{R}^{4,4}$ is defined to have components $\psi^a \in \mathbb{R}$. The "left-handed" and "right-handed" spin(8) spinors have $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ counterparts that are denoted as ψ_1 and ψ_2 in this paper, and transform, respectively, under two inequivalent real 8×8 irreducible spinor representations of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ that we call $D_{(1)}$ (type 1) and $D_{(2)}$ (type 2).

For simplicity, we do not distinguish the spinor index on ψ_2 from that on ψ_1 using a convention such as ψ_{1a} and $\psi_2{}^{\dot{a}}$ for spinor components. As long as no confusion arises, this simplified notation may be justified.

As is the case with $\mathrm{SO}(2n;\mathbb{C})$, the basic spinor representation of the pseudo-orthogonal group $\mathrm{SO}(4,4;\mathbb{R})$ may be constructed from the irreducible generators t^A, $A=1,\ldots,8$, of the pseudo-Clifford algebra $C_{4,4}$.[7,8,31] Lord[31] indicated a general procedure for constructing the spinor representations of $\mathrm{SO}(2n;\mathbb{C})$ of the first and second kind (handedness) from the generators of C_{2n-2}. Following Lord, we shall call such irreducible C_{2n-2} generators "reduced Brauer–Weyl generators."[8]

Using Lord's procedure, we construct the representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ by first defining the real 8×8 matrix reduced Brauer–Weyl generators τ^A, $\overline{\tau}^A$, $A,B,\ldots=1,\ldots,8$, of the pseudo-Clifford algebra $C_{4,4}$ that anticommute and have square ± 1. We realize this by requiring that the tau matrices satisfy (the tilde denotes transpose)

$$\sigma \overline{\tau}^A = \widetilde{\sigma \tau^A} = \widetilde{\tau}^A \sigma \tag{6}$$

and

$$\tau^A \overline{\tau}^B + \tau^B \overline{\tau}^A = 2\mathbb{I}_{8\times 8}\mathbb{G}^{AB} = \overline{\tau}^A \tau^B + \overline{\tau}^B \tau^A, \tag{7}$$

where $\mathbb{I}_{8\times 8}$ denotes the 8×8 unit matrix. Denoting the matrix elements of τ^A by $\tau^{Aa}{}_b$, we may write Eq. (6) as

$$\overline{\tau}^A{}_{ab} = \tau^A{}_{ba} \tag{8}$$

where we have used σ to lower the spinor indices. In general, σ (σ^{-1}) will be employed to lower (raise) lower case Latin indices [i.e., a $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ spinor index of either type).

We adopt a real irreducible 8×8 matrix representation of the tau matrices (see the Appendix) in which $\overline{\tau}^8 = \mathbb{I}_{8\times 8} = \tau^8$. Then by Eq. (7) $\overline{\tau}^A = -\tau^A$ for $A=1,\ldots,7$. Hence, again by Eq. (7), $(\tau^A)^2$ is equal to $-\mathbb{I}_{8\times 8}$ for $A=1,2,3$ and is equal to $+\mathbb{I}_{8\times 8}$ for $A=4,5,6,7,8$.

C. Transformation under action of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$

The special Lorentz transformation properties of the theory may be determined by constructing a real reducible 16×16 matrix representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$, utilizing the irreducible generators t^A, $A=1,\ldots,8$ of the (pseudo) Clifford algebra $C_{4,4}$. Continuing to employ Lord's general procedure,[31] we define the irreducible generators t^A as

$$t^A = \begin{pmatrix} 0 & \overline{\tau}^A \\ \tau^A & 0 \end{pmatrix}. \tag{9}$$

Let $g \in \overline{\mathrm{SO}(4,4;\mathbb{R})}$. The 16×16 basic spinor representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ is reducible into the two real 8×8 inequivalent irreducible spinor representations $D_{(1)}(g)$ and $D_{(2)}(g)$ of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$. The reduced generators of the two real 8×8 spinor representations $D_{(1)}(g)$ and $D_{(2)}(g)$ of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ follow from the calculation of the infinitesimal generators

$$t^A t^B - t^B t^A = \begin{pmatrix} \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A & 0 \\ 0 & \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A \end{pmatrix} = 4 \begin{pmatrix} D_{(1)}{}^{AB} & 0 \\ 0 & D_{(2)}{}^{AB} \end{pmatrix} \tag{10}$$

of the 16-component spinor representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$. We see that, as is, in fact, well known from the general theory, the 16-component spinor representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ is the direct sum of two (inequivalent) real 8×8 irreducible spinor representations $D_{(1)}=D_{(1)}(g)$ and $D_{(2)}=D_{(2)}(g)$ of $\overline{\mathrm{SO}(4,4;\mathbb{R})} \ni g$ that are generated by $D_{(1)}{}^{AB}$ and $D_{(2)}{}^{AB}$, respectively. This we record as

$$4D_{(1)}{}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A \tag{11}$$

and

$$4D_{(2)}{}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A. \tag{12}$$

For completeness, we remark that the generators of the two spinor types are images of the projection operators

$$\chi_\pm = \tfrac{1}{2}(1 \pm t^9),$$

$$\chi_+ = \begin{pmatrix} \mathbb{I}_{8\times 8} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\chi_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{8\times 8} \end{pmatrix}, \tag{13}$$

where

$$t^9 = t^1 t^2 t^3 t^4 t^5 t^6 t^7 t^8 = \begin{pmatrix} \bar{\tau}^0 & 0 \\ 0 & \tau^0 \end{pmatrix}. \tag{14}$$

Here

$$\bar{\tau}^0 = \bar{\tau}^1 \tau^2 \bar{\tau}^3 \tau^4 \bar{\tau}^5 \tau^6 \bar{\tau}^7 = \tau^1 \tau^2 \tau^3 \tau^4 \tau^5 \tau^6 \tau^7 \tag{15}$$

and

$$\tau^0 = \tau^1 \bar{\tau}^2 \tau^3 \bar{\tau}^4 \tau^5 \bar{\tau}^6 \tau^7 = -\tau^1 \tau^2 \tau^3 \tau^4 \tau^5 \tau^6 \tau^7 = -\bar{\tau}^0 \tag{16}$$

The representation of the tau matrices is irreducible. $\bar{\tau}^0$ has square equal to $+\mathbb{I}_{8\times 8}$ and commutes with each of the τ^A matrices (and therefore with all of their products). Therefore, we conclude that $\bar{\tau}^0 = \pm \mathbb{I}_{8\times 8}$ in any irreducible representation. $\bar{\tau}^0 = \mathbb{I}_{8\times 8}$ in the irreducible representation given in the Appendix.

Let $\omega_{AB} = -\omega_{BA} \in \mathbb{R}$, $A,B=1,\ldots,8$, enumerate a set of 28 real parameters that coordinatize $g=g(\omega) \in \overline{\mathrm{SO}(4,4;\mathbb{R})}$. Also, let $L=L(g) \in \mathrm{SO}(4,4;\mathbb{R})$ have matrix elements L^A_B, $\omega^\#$ denotes the real 8×8 matrix with matrix elements $\omega^A{}_B = \mathbb{G}^{AC}\omega_{CB}$, $\omega_1 = \frac{1}{2}\omega_{AB} D_{(1)}{}^{AB}$, and $\omega_2 = \frac{1}{2}\omega_{AB} D_{(2)}{}^{AB}$. We find that

$$D_{(1)} = D_{(1)}(g) = \exp\left(\tfrac{1}{2}\omega_1\right),$$

$$D_{(2)} = D_{(2)}(g) = \exp\left(\tfrac{1}{2}\omega_2\right),$$

$$L^A{}_B = L^A{}_B(g) = \{\exp(\omega^\#)\}^A{}_B, \tag{17}$$

where, under the action of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$,

$$\widetilde{D_{(1)}{}^{AB}}\sigma = -\sigma D_{(1)}{}^{AB} \Rightarrow \widetilde{D}_{(1)}\sigma = \sigma D_{(1)}{}^{-1}, \tag{18}$$

$$\widetilde{D_{(2)}{}^{AB}}\sigma = -\sigma D_{(2)}{}^{AB} \Rightarrow \widetilde{D}_{(2)}\sigma = \sigma D_{(2)}{}^{-1}, \tag{19}$$

$$L^{A}{}_{C}\mathbb{G}_{AB}L^{B}{}_{D} = \mathbb{G}_{CD} = \{\widetilde{L}\mathbb{G}L\}_{CD}, \tag{20}$$

$$L^{A}{}_{B}\overline{\tau}^{B} = D_{(1)}{}^{-1}\overline{\tau}^{A}D_{(2)}, \tag{21}$$

$$L^{A}{}_{B}\tau^{B} = D_{(2)}{}^{-1}\tau^{A}D_{(1)}, \tag{22}$$

The canonical 2-1 homomorphism $\overline{\mathrm{SO}(4,4;\mathbb{R})} \to \mathrm{SO}(4,4;\mathbb{R}) : g \mapsto L(g)$ is given by

$$8L^{A}{}_{B} = \mathrm{tr}(D_{(1)}^{-1}\overline{\tau}^{A}D_{(2)}\tau^{C})\mathbb{G}_{CB}, \tag{23}$$

where tr denotes the trace. Note that $D_{(1)}(g(\omega)) = D_{(2)}(g(\omega))$ when $\omega_{A8}=0$, i.e., when one restricts $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to

$$\overline{\mathrm{SO}(3,4;\mathbb{R})} = \{g \in \overline{\mathrm{SO}(4,4;\mathbb{R})} \mid ,$$

$$\left. g = \begin{pmatrix} \exp\left(\frac{1}{4}\omega_{AB}D_{(1)}{}^{AB}\right) & 0 \\ 0 & \exp\left(\frac{1}{4}\omega_{AB}D_{(2)}{}^{AB}\right) \end{pmatrix} \quad \text{and} \quad \omega_{A8} = 0 \right\} \tag{24}$$

[one of the real forms of spin $(7,\mathbb{C})$].

III. $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ COVARIANT MULTIPLICATIONS

Let V_1, V_2, and V_3 be vector spaces over $\mathbb{R}$. A duality is a nondegenerate bilinear map $V_1 \times V_2 \to \mathbb{R}$. A triality is a nondegenerate trilinear map $V_1 \times V_2 \times V_3 \to \mathbb{R}$. A triality may be associated with a bilinear map that some authors call a "multiplication"[2] by dualizing, $V_1 \times V_2 \to {}^{*}V_3 \cong V_3$.

Recall that $\mathbb{R}^{4,4}$ is a special real eight-dimensional pseudo-Euclidean space-time that may be endowed with the indefinite metric σ and a concomitant indefinite inner product $\langle \cdot , \cdot \rangle$ and which carries real irreducible representations $D_{(1)}$ and $D_{(2)}$ of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$. It may also be regarded as a real Minkowski space-time that is endowed with the indefinite metric $\mathbb{G}$ and carries a real irreducible vector representation of $\mathrm{SO}(4,4;\mathbb{R})$.

Let $u, \psi_1, \psi_2 \in \mathbb{R}^{4,4}$. In this section we assume that u is fixed and satisfies $\langle u,u \rangle = \widetilde{u}\sigma u = 1$ (the tilde denotes transpose). A natural choice for u is one of the normalized eigenvectors of σ whose eigenvalue is +1. (The algebraic significance of the spinor u is described in Sec. IV.) Under the action of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ we assume that $u \mapsto \overline{u} = D_{(1)}u$, $\psi_1 \mapsto \overline{\psi}_1 = D_{(1)}\psi_1$, and $\psi_2 \mapsto \overline{\psi}_2 = D_{(2)}\psi_2$.

We study two multiplications that possess covariant transformation laws under the action of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$. The first multiplication $m_1{}^{A} : \mathbb{R}^{4,4} \times \mathbb{R}^{4,4} \to \mathbb{R}^{4,4}$ is defined by

$$Q^{A} = \widetilde{u}\sigma\overline{\tau}^{A}\psi_2. \tag{25}$$

For fixed u^{a}, $Q^{A} \in \mathbb{R}^{4,4}$ depends on eight real parameters arranged into the type-2 spinor ψ_2.

The second multiplication $m_2{}^{AB} : \mathbb{R}^{4,4} \times \mathbb{R}^{4,4} \to V_3$ has an image in $V_3 \cong \mathbb{R}^{4,4} \times \mathbb{R}^{4,4}$ and depends on eight real parameters (for fixed u^{a}) arranged into the type-1 spinor ψ_1,

$$Q^{AB} = \widetilde{u}\sigma\overline{\tau}^{A}\tau^{B}\psi_1. \tag{26}$$

We emphasize that, for fixed u, Q^{AB} possesses only eight degrees of freedom corresponding to the eight independent degrees of freedom of ψ_1, so we also refer to this map as a multiplication.

Conversely, Eq. (26) may be solved for the components ${\psi_1}^a = {\psi_1}^a(Q^{AB})$. If we employ the representation of the tau matrices given in the Appendix and then solve for ${\psi_1}^a$, we obtain

$$\psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -Q^{23}+Q^{48} \\ -Q^{31}+Q^{43} \\ Q^{12}+Q^{42} \\ Q^{41}+Q^{88} \\ -Q^{23}-Q^{48} \\ -Q^{31}-Q^{43} \\ Q^{12}-Q^{42} \\ -Q^{41}+Q^{88} \end{pmatrix}. \tag{27}$$

Suppose we break the $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ symmetry down to $\overline{\mathrm{SO}(3,1;\mathbb{R})}$ according to Sec. III B. We note that $\mathbb{G}$ is invariant under $\overline{\mathrm{SO}(3,1;\mathbb{R})}$ since it is invariant under $\overline{\mathrm{SO}(4,4;\mathbb{R})}$. It is convenient to define a $\mathrm{SO}(3,1;\mathbb{R})$-invariant symplectic structure Ω on $\mathbb{R}^{4,4}$ (and a complex structure on the split octonion algebra) by

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{28}$$

where 0 denotes the 4×4 zero matrix and 1 denotes the 4×4 unit matrix.

Q^{AB} may be expressed in terms of a linear combination of "simple" objects that transform as $M_{3,1}$ tensors of appropriate rank. Consider first two arbitrary $M_{3,1}$ vectors with components $\mathbb{A}_\mu$ and $\mathbb{B}_\mu$. Q^{AB} may be written as a linear combination of $\mathbb{A}^\alpha\mathbb{B}^\beta \pm \mathbb{A}^\beta\mathbb{B}^\alpha$, $\epsilon^{\alpha\beta\mu\nu}\mathbb{A}_\mu\mathbb{B}_\nu$, and $\eta^{\alpha\beta}\eta^{\mu\nu}\mathbb{A}_\mu\mathbb{B}_\nu$, but this is does not yield a linear relationship between Q^{AB} and $\{\mathbb{A}_\mu,\mathbb{B}_\mu\}$. Also, since Q^{AB} has exactly eight independent degrees of freedom, it cannot be expressed in terms of a traceless symmetric $M_{3,1}$ rank 2 tensor, which has nine independent components. However, we find that the Q^{AB} may be represented in terms of an arbitrary antisymmetric $M_{3,1}$ rank 2 tensor $\mathbb{F}^{\beta\alpha} = -\mathbb{F}^{\alpha\beta}$ and two $\mathrm{SO}(3,1;\mathbb{R})$ scalars q_4 and q_8 according to

$$Q^{AB} = \begin{pmatrix} \mathbb{F}^{\alpha\beta} & {}^*\mathbb{F}^{\alpha}{}_{\beta} \\ {}^*\mathbb{F}_{\alpha}{}^{\beta} & \mathbb{F}_{\alpha\beta} \end{pmatrix} + q_4\Omega^{AB} + q_8\mathbb{G}^{AB}, \tag{29}$$

where ${}^*\mathbb{F}^{\alpha\beta}$ is dual to $\mathbb{F}^{\alpha\beta}$ and defined by ${}^*\mathbb{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\mathbb{F}_{\alpha\beta}$. Note that $Q^{[AB]} = \frac{1}{2}(Q^{AB} - Q^{BA})$ is independent of q_8, that q_8 may be expressed as Q^{88}, and that q_4 may be expressed as Q^{48}.

If we identify this arbitrary antisymmetric $M_{3,1}$ rank 2 tensor $\mathbb{F}^{\alpha\beta}$ with the $\mathbb{F}^{\alpha\beta}$ of Eq. (1), as the notation already suggests, then a short calculation yields

$$-\frac{\partial}{\partial x^\beta}Q^{[\beta A]} = -\frac{\partial}{\partial x^B}Q^{[BA]} = \begin{pmatrix} \vec{\nabla}\times\vec{\mathbb{B}} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\mathbb{E}} \\ \vec{\nabla}\cdot\vec{\mathbb{E}} \\ -\left(\vec{\nabla}\times\vec{\mathbb{E}} + \frac{1}{c}\frac{\partial}{\partial t}\vec{\mathbb{B}} + \vec{\nabla}\mathbb{B}^4\right) \\ -\left(\vec{\nabla}\cdot\vec{\mathbb{B}} + \frac{1}{c}\frac{\partial}{\partial t}\mathbb{B}^4\right) \end{pmatrix}. \tag{30}$$

Here, $\mathbb{B}^4 = Q^{48}$ and $\mathbb{E}^4 = Q^{88}$. Therefore, this representation may be employed in a realization of Maxwell's equations when $\mathbb{B}^4 = 0$ [$\mathbb{B}^4$ is shown below to transform as a scalar under proper Lorentz transformations $\mathrm{SO}(3,1;\mathbb{R})$]. In general, $\mathbb{E}^4$ and $\mathbb{B}^4$ are both scalars under proper Lorentz transformations, as discussed below.

A word on the notation is employed in this paper. Let $x,\ \mathbb{T}\in\mathbb{R}^{4,4}$. All field functions $f(x)\equiv f(x^A)$, $x\in\mathbb{R}^{4,4}$ are actually only functions of x^μ, $\mu=1,2,3,4$. Therefore, for expressions involving contractions, such as $\mathbb{T}^B f_{,B}$, $\mathbb{T}^B(\partial/\partial x^B)f(x^A)=\mathbb{T}^\beta(\partial/\partial x^\beta)f(x^A)$.

A. Covariance of maps under $\overline{\mathrm{SO}(4,4;\mathbb{R})}$

Consider the transformation law for the $\bar{Q}^A$,

$$\bar{Q}^A=\tilde{\bar{u}}\sigma\overline{\tau}^A\bar{\psi}_2=\widetilde{D_{(1)}u}\sigma\overline{\tau}^A D_{(2)}\psi=\tilde{u}\sigma D_{(1)}{}^{-1}\overline{\tau}^A D_{(2)}\psi=L^A{}_B\tilde{u}\sigma\overline{\tau}^B\psi=L^A{}_BQ^B,$$

which follows from Eq. (21). Next we compute the transformation law for the $\bar{Q}^{AB}$,

$$\begin{aligned}\bar{Q}^{AB}&=\tilde{\bar{u}}\sigma\overline{\tau}^A\tau^B\bar{\psi}_1=\widetilde{D_{(1)}u}\sigma\overline{\tau}^A\tau^B D_{(1)}\psi_1\\&=\tilde{u}\widetilde{D_{(1)}}\sigma\overline{\tau}^A D_{(2)}D_{(2)}{}^{-1}\tau^B D_{(1)}\psi_1\\&=\tilde{u}\sigma(D_{(1)}{}^{-1}\overline{\tau}^A D_{(2)})(D_{(2)}{}^{-1}\tau^B D_{(1)})\psi_1\\&=L^A{}_CL^B{}_D\tilde{u}\sigma\overline{\tau}^C\tau^D\psi_1=L^A{}_CL^B{}_DQ^{CD},\end{aligned}$$

which follows from Eqs. (21) and (22). In summary, under the action of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$,

$$u\mapsto\bar{u}=D_{(1)}u,$$

$$\psi_1\mapsto\bar{\psi}_1=D_{(1)}\psi_1,$$

$$\psi_2\mapsto\bar{\psi}_2=D_{(2)}\psi_2,$$

$$Q^A\mapsto\bar{Q}^A=L^A{}_BQ^B,$$

$$Q^{AB}\mapsto\bar{Q}^{AB}=L^A{}_CL^B{}_DQ^{CD}=\{LQ\tilde{L}\}^{AB}. \tag{31}$$

B. Restriction of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to $\overline{\mathrm{SO}(3,1;\mathbb{R})}$

Clearly, in order for Eq. (29) to possess physical significance, the action of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ must be restricted to $\overline{\mathrm{SO}(3,1;\mathbb{R})}$ in a manner that links transformations of x^5,x^6,x^7,x^8 to x^1,x^2,x^3,x^4. Therefore, we are led to identify $\mathbb{R}^{4,4}$ with the phase space $M_{3,1}\oplus{}^*M_{3,1}$, $\mathbb{R}^{4,4}\cong M_{3,1}\oplus{}^*M_{3,1}$, and to restrict $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to the subgroup

$$\mathbb{K}=\left\{g\in\overline{\mathrm{SO}(4,4;\mathbb{R})}\middle|g=\begin{pmatrix}h&0\\0&\tilde{h}^{-1}\end{pmatrix},\ h\in\overline{\mathrm{SO}(3,1;\mathbb{R})}\right\}. \tag{32}$$

The metric $\mathbb{G}$ on $M_{3,1}\oplus{}^*M_{3,1}$ is

$$\mathbb{G}=\begin{pmatrix}\eta&0\\0&-\tilde{\eta}^{-1}\end{pmatrix}, \tag{33}$$

where we recall that $\eta=\eta_{3,1}=\mathrm{diag}(1,1,1,-1)$ is the pseudo-Euclidean metric on flat four-dimensional Minkowski space-time $M_{3,1}$. Under the action of $L\in\mathrm{SO}(3,1;\mathbb{R})$, $x^\alpha\mapsto\bar{x}^\alpha=L^\alpha_\beta x^\beta$ and $x^{4+\alpha}\equiv p_\alpha\mapsto\bar{p}_\alpha=p_\beta L^\beta_\alpha$; $\bar{p}_\alpha\bar{x}^\alpha=p_\alpha x^\alpha$ is an invariant since for $g\in\mathbb{K}$

$$M_{3,1} \oplus {}^*M_{3,1} \ni \begin{pmatrix} x \\ \tilde{p} \end{pmatrix} \mapsto g\begin{pmatrix} x \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} hx \\ \tilde{h}^{-1}\tilde{p} \end{pmatrix} = \begin{pmatrix} hx \\ \widetilde{ph^{-1}} \end{pmatrix}. \tag{34}$$

Moreover, a simple calculation proves that Ω is invariant under the action of $\mathbb{K}$. Therefore, $\mathbb{E}^4$ and $\mathbb{B}^4$ are both scalars under proper Lorentz transformations $\mathrm{SO}(3,1;\mathbb{R})$. Thus, by setting the scalars $\mathbb{E}^4$ and $\mathbb{B}^4$ in Eq. (27) equal to zero the multiplication $m_2{}^{AB}(u,\psi_1)$ of Eq. (26) yields a natural Lorentz covariant representation of both the electromagnetic field tensor $\mathbb{F}_{\alpha\beta}$ and it dual ${}^*\mathbb{F}^{\alpha\beta}$.

IV. ALGEBRAIC SIGNIFICANCE OF THE SPINOR *u*

Using the tau matrices and a constant type-1 spinor $u \in \mathbb{R}^{4,4}$, with $\langle u,u\rangle = \tilde{u}\sigma u = 1$, we may define a nonassociative product on a spinor basis ε_a of $\mathbb{R}^{4,4}$ that endows the real vector space $\mathbb{R}^{4,4}$ with the structure of the split octonion algebra over the reals. Moreover, we demonstrate that a realization of the multiplicative unit element of the split octonion algebra is the linear function of u^a given by $\varepsilon_a u^a$. Hence, the spinor u plays a distinguished role in this formalism.

A. Fundamental identity

The tau matrices verify an important identity[37] which we record as follows.

Lemma 1: Let M be any 8×8 matrix satisfying

$$\widetilde{\sigma M} = \sigma M \tag{35}$$

and moreover transforming under $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ according to

$$M \mapsto D_{(1)} M D_{(1)}{}^{-1}. \tag{36}$$

Then

$$\tau_A M \overline{\tau}^A = \mathbb{I}_{8\times 8}\,\mathrm{tr}(M) \tag{37}$$

(recall that $\mathbb{I}_{8\times 8}$ denotes the 8×8 unit matrix).

Let ε_a be an oriented spinor basis of $\mathbb{R}^{4,4}$ normalized according to

$$\langle \varepsilon_a, \varepsilon_b\rangle = \sigma_{ab}. \tag{38}$$

We write a type-1 spinor element $\Psi \in \mathbb{R}^{4,4}$ as

$$\Psi = \varepsilon_a \psi^a. \tag{39}$$

Let us choose a constant spinor $\varepsilon_a u^a = U \in \mathbb{R}^{4,4}$ with constant components u^a normalized to

$$1 = \langle U,U\rangle = u^a \sigma_{ab} u^b = \tilde{u}\sigma u = \mathrm{tr}(u \otimes \tilde{u}\sigma) = \mathrm{tr}(u\tilde{u}\sigma) \tag{40}$$

everywhere on $\mathbb{R}^{4,4}$, but being otherwise arbitrary. We may define a $\mathbb{R}^{4,4}$ frame in terms of U and the tau matrices as follows. Let M be the real 8×8 matrix defined by

$$M = u \otimes \tilde{u}\sigma = u\tilde{u}\sigma,$$

$$M^a{}_b = u^a u^c \sigma_{cb}. \tag{41}$$

Then M obeys Eq. (35) and transforms under $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ according to Eq. (36). We may therefore apply the Lemma 1 and thereby obtain

$$\mathbb{I}_{8\times 8} = \tau_A (u\tilde{u}\sigma)\overline{\tau}^A = (\tau_A u)(\tilde{u}\sigma\overline{\tau}^A). \tag{42}$$

This is a resolution of the identity on $\mathbb{R}^{4,4}$. Alternatively, this relation may be interpreted as a completeness condition verified by the (components of the) orthogonal frame $\mathfrak{F}$,

$$\mathfrak{F}^a_A = \tau_A{}^a{}_b u^b \tag{43}$$

and its inverse

$$\mathfrak{F}^A_a = u^c \sigma_{cb} \overline{\tau}^{Ab}{}_a \tag{44}$$

Accordingly, Eq. (42) may be expressed in index notation as

$$\{\mathbb{I}_{8\times 8}\}^a_b = \delta^a_b = \mathfrak{F}^a_A \mathfrak{F}^A_b. \tag{45}$$

Since a matrix commutes with its inverse, we also have

$$\delta^A_B = \mathfrak{F}^A_a \mathfrak{F}^a_B. \tag{46}$$

Given an oriented spinor basis ε_a of $\mathbb{R}^{4,4}$, define the oriented vector basis $\varepsilon_A = \varepsilon_a \mathfrak{F}^a_A$. Since this definition creates a notational ambiguity in a term such as ε_3, we define the oriented vector basis as

$$\epsilon_A = \varepsilon_a \mathfrak{F}^a_A. \tag{47}$$

These bases are normalized according to

$$\langle \varepsilon_a, \varepsilon_b \rangle = \sigma_{ab} \Leftrightarrow \langle \epsilon_A, \epsilon_B \rangle = \mathbb{G}_{AB}. \tag{48}$$

An element $\Psi \in \mathbb{R}^{4,4}$ is realized as $\Psi = \varepsilon_a \psi^a = \epsilon_A \psi^A$. Since this convention creates an ambiguity in a term such as ψ^3, we change this relation to

$$\Psi = \varepsilon_a \psi^a = \epsilon_A \hat{\psi}^A,$$

$$\hat{\psi}^A = \mathfrak{F}^A_a \psi^a \Leftrightarrow \psi^a = \mathfrak{F}^a_A \hat{\psi}^A. \tag{49}$$

We note that the fiducial $\varepsilon_a u^a = U \in \mathbb{R}^{4,4}$ with components u^a normalized to $1 = \langle U, U \rangle = u^a \sigma_{ab} u^b$ may be expressed as $U = \varepsilon_a u^a = \varepsilon_a \tau_8{}^a{}_b u^b = \varepsilon_a \mathfrak{F}^a_8 = \epsilon_8$, which we record as

$$id = \varepsilon_a u^a = \epsilon_8, \tag{50}$$

where id will be shown to be a realization of the multiplicative unit element of the split octonion algebra when the multiplication constants of the algebra are constructed from $\varepsilon_a u^a = U$ and the tau matrices [see Eq. (54)]. Accordingly, the conjugate of a split octonion $\Psi = \epsilon_A \hat{\psi}^A$ may conventionally be defined with respect to a vector basis as

$$\Psi^* = \epsilon_8 \hat{\psi}^8 - \sum_{A=1}^{7} \epsilon_A \hat{\psi}^A, \tag{51}$$

just as for the octonions. Looking ahead, the square of the norm of the element $\Psi \in \mathbb{R}^{4,4}$ may be identified using the split octonion product (for clarity, sometimes explicitly expressed using $\star$), defined in Sec. IV B, as

$$\langle \Psi, \Psi \rangle id = \Psi^* \Psi = \Psi^* \star \Psi$$

since

$$
\begin{aligned}
\langle\Psi,\Psi\rangle id &= \Psi^{*}\Psi = \Psi^{*}\star\Psi \\
&= \left\{\epsilon_8\hat{\psi}^8 - \sum_{A=1}^{7}\epsilon_A\hat{\psi}^A\right\}\star\left\{\epsilon_8\hat{\psi}^8 + \sum_{B=1}^{7}\epsilon_B\hat{\psi}^B\right\} \\
&= \{\epsilon_8\}^2\{\hat{\psi}^8\}^2 - \sum_{A=1}^{7}\{\epsilon_A\}^2\{\hat{\psi}^A\}^2 \\
&= \epsilon_8[\{\hat{\psi}^8\}^2 + \{\hat{\psi}^1\}^2 + \{\hat{\psi}^2\}^2 + \{\hat{\psi}^3\}^2 - (\{\hat{\psi}^4\}^2 + \{\hat{\psi}^5\}^2 + \{\hat{\psi}^6\}^2 + \{\hat{\psi}^7\}^2)]
\end{aligned}
\tag{52}
$$

(here $\{\epsilon_A\}^2=\epsilon_A\star\epsilon_A$). Note that a somewhat unfortunate terminology may arise here: if $\Psi=\epsilon_A\hat{\psi}^A \in \mathbb{R}^{4,4}$, then $\hat{\psi}^A\in\mathbb{R}$; if $\hat{\psi}^8=0$ then Ψ is imaginary (i.e., Ψ is an imaginary split octonion). More importantly, we also note that the parameter $\mathbb{E}_4$ is the real part of $\varepsilon_a\psi_1^a\in\mathbb{R}^{4,4}$ in Eqs. (27) and (29), so setting $\mathbb{E}_4=0$ implies that $\varepsilon_a\psi_1^a$ is an imaginary split octonion.

B. Split octonion algebra

We may define a nonassociative alternative multiplication of the spinor basis ε_a (vector basis ϵ_A) that endows the real vector space $\mathbb{R}^{4,4}$ with the structure of a normed algebra with multiplicative unit.[38] This is accomplished by specifying the multiplication constants m_{ab}^{c} (m_{AB}^{C}) of the algebra, which verify

$$
\begin{aligned}
\varepsilon_a\varepsilon_b &= \varepsilon_c m_{ab}^{c}, \\
\epsilon_A\epsilon_B &= \epsilon_C m_{AB}^{C}.
\end{aligned}
\tag{53}
$$

We adopt a set of m_{ab}^{c} (m_{AB}^{C}) defined by

$$
\begin{aligned}
m_{ab}^{c} &= \mathfrak{F}_a^A\tau_A{}^c{}_b, \\
m_{AB}^{C} &= \mathfrak{F}_c^C\tau_A{}^c{}_b\mathfrak{F}_B^b,
\end{aligned}
\tag{54}
$$

as a realization of the multiplication constants of the algebra. We have proven that the nonassociative product defined by Eqs. (53) and (54) of the spinor basis ε_a (of the vector basis ϵ_A) endows the real vector space $\mathbb{R}^{4,4}$ with the structure of the split octonion algebra over the reals. A realization of the multiplicative unit element of the split octonion algebra

$$
id = \varepsilon_a u^a = \epsilon_8, \tag{55}
$$

is explicit in the following multiplication table, which employs the representation of u^a and the tau matrices given in the Appendix.

Multiplication Table.

$\epsilon_A \times \epsilon_B=$

$\epsilon_A=\backslash\epsilon_B=$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7	ϵ_8
ϵ_1	$-\epsilon_8$	ϵ_3	$-\epsilon_2$	$-\epsilon_5$	ϵ_4	$-\epsilon_7$	ϵ_6	ϵ_1
ϵ_2	$-\epsilon_3$	$-\epsilon_8$	ϵ_1	$-\epsilon_6$	ϵ_7	ϵ_4	$-\epsilon_5$	ϵ_2
ϵ_3	ϵ_2	$-\epsilon_1$	$-\epsilon_8$	$-\epsilon_7$	$-\epsilon_6$	ϵ_5	ϵ_4	ϵ_3
ϵ_4	ϵ_5	ϵ_6	ϵ_7	ϵ_8	ϵ_1	ϵ_2	ϵ_3	ϵ_4
ϵ_5	$-\epsilon_4$	$-\epsilon_7$	ϵ_6	$-\epsilon_1$	ϵ_8	ϵ_3	$-\epsilon_2$	ϵ_5
ϵ_6	ϵ_7	$-\epsilon_4$	$-\epsilon_5$	$-\epsilon_2$	$-\epsilon_3$	ϵ_8	ϵ_1	ϵ_6
ϵ_7	$-\epsilon_6$	ϵ_5	$-\epsilon_4$	$-\epsilon_3$	ϵ_2	$-\epsilon_1$	ϵ_8	ϵ_7
ϵ_8	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7	ϵ_8

V. MAXWELL'S EQUATIONS

Let $\mathbb{J}^\alpha(x^\mu)=(\vec{\mathbb{J}},c\rho)^T$ denote the (conserved) current density of an electrically charged particle of charge q. We define $\hat{\mathbb{J}}^A(x^\mu)=(\mathbb{J}^\alpha(x^\mu),0,0,0,0)$; this decomposition is preserved under $\overline{\mathrm{SO}(3,1;\mathbb{R})}$. The spinor form is $\mathbb{J}^a(x^\mu)=\mathfrak{F}^a_A\hat{\mathbb{J}}^A(x^\mu)$.

The classical form of $\mathbb{J}^\alpha(x^\mu)$ for a charged (point) particle that is located at $y^\mu=y^\mu(\tau)$ and possesses four-velocity $\dot{w}^\alpha(\tau)=(d/d\tau)y^\alpha(\tau)$ is $\mathbb{J}^\alpha(x^\mu)=qc\int_{-\infty}^{\infty}\dot{w}^\alpha(\tau)\delta^{(4)}(x^\mu-y^\mu(\tau))d\tau$. Here $c=1$ is the speed of light in vacuum and τ is the proper time. The quantum form of $\hat{\mathbb{J}}^\alpha(x^\mu)$ may be computed using Schwinger's real, eight-component formalism of QED.[43] Thus the current may be either classical or quantum.

We recall that Eq. (30),

$$-\frac{\partial}{\partial x^B}Q^{[BA]}=\frac{\partial}{\partial x^B}\frac{1}{2}(Q^{BA}-Q^{AB})=\begin{pmatrix}\vec{\nabla}\times\vec{\mathbb{B}}-\dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{\mathbb{E}}\\ \vec{\nabla}\cdot\vec{\mathbb{E}}\\ -\left(\vec{\nabla}\times\vec{\mathbb{E}}+\dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{\mathbb{B}}+\vec{\nabla}\mathbb{B}^4\right)\\ -\left(\vec{\nabla}\cdot\vec{\mathbb{B}}+\dfrac{1}{c}\dfrac{\partial}{\partial t}\mathbb{B}^4\right)\end{pmatrix}, \tag{56}$$

is independent of $\mathbb{E}^4$, and that B^4 is a Lorentz scalar. Setting $B^4\equiv 0$, we find that the Maxwell equations (Gaussian units) are realized by

$$-\frac{\partial}{\partial x^B}Q^{[BA]}=\frac{\partial}{\partial x^B}Q^{[AB]}=\frac{4\pi}{c}\begin{pmatrix}\vec{\mathbb{J}}\\ c\rho\\ \vec{0}\\ 0\end{pmatrix}\equiv\frac{4\pi}{c}\hat{\mathbb{J}}^A, \tag{57}$$

with

$$\hat{\mathbb{J}}^A=\begin{pmatrix}\vec{\mathbb{J}}\\ c\rho\\ \vec{0}\\ 0\end{pmatrix}. \tag{58}$$

To map these equations into their spinor form, we calculate $\tau_A u Q^{BA}=\tau_A u\tilde{u}\sigma\tilde{\tau}^B\tau^A\psi_1=\tau_A u\tilde{u}\sigma($ $-\tilde{\tau}^A\tau^B+2\mathbb{G}^{BA}\mathbb{I}_{8\times 8})\psi_1=\tau^B(-\mathbb{I}_{8\times 8}+2u\tilde{u}\sigma)\psi_1$ and $\tau_A uQ^{AB}=\tau_A u\tilde{u}\sigma\tilde{\tau}^A\tau^B\psi_1=\tau^B\psi_1$, where we have used Eq. (42). Therefore,

$$\tau_A uQ^{[AB]}=\tau^B(\mathbb{I}_{8\times 8}-u\tilde{u}\sigma)\psi_1=\tau^B\mathbb{P}_\perp\psi_1, \tag{59}$$

where $\mathbb{P}_\perp$ is defined by

$$\mathbb{P}_\perp=\left[\mathbb{I}_{8\times 8}-\frac{u\tilde{u}\sigma}{\tilde{u}\sigma u}\right]. \tag{60}$$

In passing, we note that given an arbitrary element $\Psi=\varepsilon_a\psi^a\in\mathbb{R}^{4,4}$ [restricting $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to the subgroup $\overline{\mathrm{SO}(3,4;\mathbb{R})}$ defined in Eq. (24)], the so-called imaginary part of the split octonion Ψ is given by $\varepsilon_a\{[\mathbb{I}_{8\times 8}-u\tilde{u}\sigma]\psi\}^a$, where $\{[\mathbb{I}_{8\times 8}-u\tilde{u}\sigma]\psi\}^a\in\mathbb{R}$.

The spinor form of Maxwell's equations is obtained by mapping Eq. (57),

$$\frac{\partial}{\partial x^B}\tau^B(\mathbb{I}_{8\times 8}-u\tilde{u}\sigma)\psi_1=\frac{4\pi}{c}[(\tau_A u)\hat{\mathbb{J}}^A]\equiv\frac{4\pi}{c}\mathbb{J}, \tag{61}$$

where $\mathbb{J}^a=\mathfrak{F}^a_A\hat{\mathbb{J}}^A$.

When the multiplicative unit element of the split octonion algebra is redefined in Sec. VI in terms of a type-1 $\mathbb{R}^{4,4}$ spinor field $u^a=u^a(x)$, $a=1,\dots,8$, $x\in\mathbb{R}^{4,4}$, then the normalization constraint $\langle U,U\rangle=u^a\sigma_{ab}u^b=\tilde{u}\sigma u=1$ must be replaced by the weaker constraint $\tilde{u}\sigma u>0$, and Eq. (61) must be replaced by

$$\frac{\partial}{\partial x^B}\tau^B\left(\mathbb{I}_{8\times 8}-\frac{u\tilde{u}\sigma}{\tilde{u}\sigma u}\right)\psi_1=\frac{\partial}{\partial x^B}\tau^B(\mathbb{P}_\perp\psi_1)=\frac{4\pi}{c}\left[\left(\tau_A\frac{u}{\sqrt{\tilde{u}\sigma u}}\right)\hat{\mathbb{J}}^A\right]\equiv\frac{4\pi}{c}\mathbb{J}. \tag{62}$$

Also, the second definition of the frame and its inverse

$$\mathfrak{F}^a_A=\frac{1}{\sqrt{\tilde{u}\sigma u}}\tau_A{}^a{}_b u^b \tag{63}$$

and

$$\mathfrak{F}^A_a=\frac{1}{\sqrt{\tilde{u}\sigma u}}u^c\sigma_{cb}\overline{\tau}^{Ab}{}_a \tag{64}$$

should be used in constructing a Lagrangian.

A. Charge conservation

The local expression of charge conservation through the continuity equation

$$\frac{4\pi}{c}\hat{\mathbb{J}}^A_{,A}=\frac{4\pi}{c}\mathbb{J}^\alpha_{,\alpha}=\frac{4\pi}{c}\left\{\vec{\nabla}\cdot\vec{\mathbb{J}}+\frac{\partial}{\partial ct}c\rho\right\}=\frac{4\pi}{c}\left\{\vec{\nabla}\cdot\vec{\mathbb{J}}+\frac{\partial\rho}{\partial t}\right\}=\frac{\partial}{\partial x^A}\frac{\partial}{\partial x^B}Q^{[BA]}\equiv 0 \tag{65}$$

is satisfied identically since $Q^{[AB]}$ is skew symmetric in $\{A,B\}$ and $\partial^2/\partial x^A\partial x^B$ is symmetric in $\{A,B\}$.

B. Lagrangian

In this section we assume that we are *a priori* given a constant (global) type-1 spinor u that satisfies $\tilde{u}\sigma u=1$ and also given a type-2 spinor field $\mathbb{J}$ that maps to a conserved electric charge current density $\hat{\mathbb{J}}^A$. We further assume that the spinor fields (ψ_1,ψ_2) carry the fundamental dynamical field degrees of freedom of the electromagnetic field. The type-2 spinor fields are related to conventional vector quantities through the mapping by the frame determined by u defined in Eq. (25). The conserved electric charge current density is

$$\hat{\mathbb{J}}^A=\mathfrak{F}^A_a\mathbb{J}^a\Leftrightarrow\mathbb{J}^a=\mathfrak{F}^a_A\hat{\mathbb{J}}^A. \tag{66}$$

It should be emphasized that the current $\hat{\mathbb{J}}^A$ may be either classical or quantum.[43] In addition,

$$\mathbb{A}^B=\mathfrak{F}^B_a\psi^a_2=(\vec{\mathbb{A}},\mathbb{A}_4,\vec{\mathbb{C}},\mathbb{C}_4)_D\mathbb{G}^{DB}=\mathfrak{F}^B_a\psi^a_2=\frac{1}{\sqrt{\tilde{u}\sigma u}}\tilde{u}\sigma\overline{\tau}^B\psi_2\Leftrightarrow\psi^a_2=\mathfrak{F}^a_B\mathbb{A}^B \tag{67}$$

represents a generalization of the electromagnetic four-vector potential.

Both $\mathbb{J}^B$ and $\mathbb{A}^B$ are true $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ vector fields since $\mathbb{J}$ and ψ_2 are both type-2 spinors. In addition, we assume that under a local U(1) gauge transformation generated by $\chi=\chi(x^A)$,

$$\mathbb{A}_B \mapsto \mathbb{A}'_B = \mathbb{A}_B + \frac{\partial}{\partial x^B}\chi,$$

$$\psi_2 \mapsto \psi'_2 = \psi_2 + \tau^B \frac{u}{\sqrt{\tilde{u}\sigma u}}\frac{\partial}{\partial x^B}\chi,$$

$$u \mapsto u' = u,$$

$$\psi_1 \mapsto \psi'_1 = \psi_1. \tag{68}$$

A Lagrangian for this problem is

$$\begin{aligned} L_M &= \frac{1}{2}\widetilde{\psi_1}\sigma\psi_1 + \widetilde{\psi_{2,A}}\sigma\tau^A\left[\mathbb{I}_{8\times 8} - \frac{u\tilde{u}\sigma}{\tilde{u}\sigma u}\right]\psi_1 + \frac{4\pi}{c}\widetilde{\psi_2}\sigma\mathbb{J} \\ &= \frac{1}{2}\widetilde{\psi_1}\sigma\psi_1 + \widetilde{\psi_1}\sigma\left[\mathbb{I}_{8\times 8} - \frac{u\tilde{u}\sigma}{\tilde{u}\sigma u}\right]\overline{\tau}^A\psi_{2,A} + \frac{4\pi}{c}\widetilde{\mathbb{J}}\sigma\psi_2 \\ &= \frac{1}{2}\widetilde{\psi_1}\sigma\psi_1 + \widetilde{\psi_1}\sigma\mathbb{P}_\perp\overline{\tau}^A\psi_{2,A} + \frac{4\pi}{c}\widetilde{\mathbb{J}}\sigma\psi_2, \end{aligned} \tag{69}$$

Here, ψ_1 may be thought of as a set of eight Lagrange multipliers whose physical interpretation follows from the evaluation of the Euler–Lagrange equations. However, if we substitute for ψ_1 using the parametrization of Eq. (27) into in the scalar product

$$\langle \Psi_1, \Psi_1 \rangle = \widetilde{\psi}_1\sigma\psi_1 = \vec{\mathbb{B}}\cdot\vec{\mathbb{B}} - \{\mathbb{B}^4\}^2 - \vec{\mathbb{E}}\cdot\vec{\mathbb{E}} + \{\mathbb{E}^4\}^2, \tag{70}$$

then we recognize the familiar expression $\vec{\mathbb{B}}\cdot\vec{\mathbb{B}}-\vec{\mathbb{E}}\cdot\vec{\mathbb{E}}$.

Under a local U(1) gauge transformation generated by $\chi=\chi(x^A)$, $L_M \mapsto L'_M = L_M + \Delta L_M$, where

$$\begin{aligned} \Delta L_M &= \widetilde{\psi_1}\sigma\left[\mathbb{I}_{8\times 8} - \frac{u\tilde{u}\sigma}{\tilde{u}\sigma u}\right]\left[\overline{\tau}^\mu\tau^\beta\left\{\chi\frac{u}{\sqrt{\tilde{u}\sigma u}}\right\}_{,\mu\beta}\right] + \frac{4\pi}{c}\widetilde{\mathbb{J}}\sigma\frac{\partial}{\partial x^B}\left\{\chi\tau^B\frac{u}{\sqrt{\tilde{u}\sigma u}}\right\} \\ &= \widetilde{\psi_1}\sigma\mathbb{P}_\perp\frac{1}{2}[\overline{\tau}^\mu\tau^\beta + \overline{\tau}^\beta\tau^\mu]\frac{u}{\sqrt{\tilde{u}\sigma u}}\chi_{,\mu\beta} + \frac{4\pi}{c}\left\{\widetilde{\mathbb{J}}\sigma\tau^B\frac{u}{\sqrt{\tilde{u}\sigma u}}\right\}\chi_{,B} \\ &= \widetilde{\psi_1}\sigma\mathbb{P}_\perp\frac{1}{2}[2\mathbb{I}_{8\times 8}\eta^{\mu\beta}]\frac{u}{\sqrt{\tilde{u}\sigma u}}\chi_{,\mu\beta} + \frac{4\pi}{c}\hat{\mathbb{J}}^B\{\chi_{,B}\} \\ &= \widetilde{\psi_1}\sigma[\mathbb{P}_\perp u]\frac{1}{\sqrt{\tilde{u}\sigma u}}\chi_{,\mu\beta}\eta^{\mu\beta} + \frac{4\pi}{c}\{\hat{\mathbb{J}}^B\chi\}_{,B} - \frac{4\pi}{c}\chi\hat{\mathbb{J}}^B_{,B} \\ &= \left\{\frac{4\pi}{c}\hat{\mathbb{J}}^B\chi\right\}_{,B}. \end{aligned} \tag{71}$$

Here, we have used $\hat{\mathbb{J}}^B_{,B}=0$ [Eqs. (7) and (60)]. Since this is a (total) divergence, it is annihilated by the Euler–Lagrange operator and does not change the field equations.

The Euler–Lagrange equations for ψ_2 give exactly Eq. (62). The Euler–Lagrange equations for ψ_1 imply that

$$\psi_1 = -\left[\mathbb{I}_{8\times 8} - \frac{u\tilde{u}\sigma}{\tilde{u}\sigma u}\right]\overline{\tau}^A\psi_{2,A} = -\mathbb{P}_\perp\overline{\tau}^A\psi_{2,A}. \tag{72}$$

A simple interpretation of this equation is obtained when u^a is constant by substituting for ψ_1 using the parametrization of Eq. (27) and setting $\psi_2^a = \mathfrak{F}^a_B \mathbb{A}^B$. Substituting into Eq. (72) and solving for $\{\mathbb{B}^\mu, \mathbb{E}^\mu\}$ yields

$$\begin{pmatrix} \vec{\mathbb{B}} \\ \mathbb{B}^4 \end{pmatrix} = \begin{pmatrix} \vec{\nabla}\times\vec{\mathbb{A}} + \dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{\mathbb{C}} + \vec{\nabla}\mathbb{C}_4 \\ -\vec{\nabla}\cdot\vec{\mathbb{C}} - \dfrac{1}{c}\dfrac{\partial}{\partial t}\mathbb{C}_4 \end{pmatrix} \tag{73}$$

and

$$\begin{pmatrix} \vec{\mathbb{E}} \\ \mathbb{E}^4 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{\mathbb{A}} + \vec{\nabla}\mathbb{A}_4 + \vec{\nabla}\times\vec{\mathbb{C}} \\ 0 \end{pmatrix} \tag{74}$$

Since the field $\Psi = \Psi_1 \in \mathbb{R}^{4,4}$ is realized as $\Psi = \varepsilon_a \psi_1{}^a = \epsilon_A \hat{\psi}_1{}^A$ with

$$\mathbb{E}^4 = \tilde{u}\sigma\psi_1 = \hat{\psi}_1{}^8 = 0, \tag{75}$$

we see that the so-called real part of Ψ is zero, as required. Thus, the electromagnetic field in this formalism is represented by a pure imaginary split octonion. We emphasize that "real" and "imaginary" only have meaning if we restrict the automorphism group of the vector space $\mathbb{R}^{4,4}$ to those norm-preserving transformations that preserve both the one-dimensional real and seven-dimensional imaginary subalgebras of the eight-dimensional split octonion algebra. This corresponds to the restriction of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to the subgroup $\overline{\mathrm{SO}(3,4;\mathbb{R})}$ defined in Eq. (24).

In passing, we remark that the dual nature of the realizations of $\vec{\mathbb{E}}$ and $\vec{\mathbb{B}}$ is noteworthy. If the requirement of manifest charge conservation is dropped, then the projection operator $\mathbb{I}_{8\times 8} - u\tilde{u}\sigma/\tilde{u}\sigma u$ in Eqs. (62) and (69) should be replaced by the identity operator $\mathbb{I}_{8\times 8}$; then the Euler Lagrange equations yield $\mathbb{E}^4 = -\vec{\nabla}\cdot\vec{\mathbb{A}} + (\partial/\partial x^4)\mathbb{A}_4 = -\mathbb{A}^\mu_{,\mu}$, which has the same general form as the realization of $\mathbb{B}^4$.

Let $\Delta = (\partial/\partial x^j)(\partial/\partial x^j)$ denote the Laplacian operator in Cartesian coordinates, and $\Box = \eta^{\mu\nu}(\partial/\partial x^\mu)(\partial/\partial x^\nu)$ denote the D'Alembertian operator (or wave) operator. Substituting Eq. (72) into Eq. (62) yields

$$= \begin{pmatrix} -\Box\vec{\mathbb{A}} + \vec{\nabla}\left\{\vec{\nabla}\cdot\vec{\mathbb{A}} - \dfrac{\partial}{\partial x^4}\mathbb{A}_4\right\} \\ \Delta\mathbb{A}_4 - \dfrac{\partial}{\partial x^4}\vec{\nabla}\cdot\vec{\mathbb{A}} \\ \Box\vec{\mathbb{C}} \\ -\Box\mathbb{C}_4 \end{pmatrix} = \frac{4\pi}{c}\begin{pmatrix} \vec{\mathbb{J}} \\ c\rho \\ \vec{0} \\ 0 \end{pmatrix}, \tag{76}$$

Since $\vec{\mathbb{C}} \equiv \vec{0}$ and $\mathbb{C}^4 \equiv 0$ solve the last half of these equations, this formulation of the Maxwell theory, with constant u^a, provides a solution that is completely consistent with, and equivalent to, the conventional classical theory. The first half of these equations are precisely the conventional Maxwell's equations formulated in terms of the conventional scalar and three-vector potentials. In particular, $\vec{\mathbb{A}}$ and $\mathbb{A}_4$ are sourced by $\vec{\mathbb{J}}$ and ρ, which, in turn, couple only to $\vec{\mathbb{A}}$ and $\mathbb{A}_4$ and not to $\vec{\mathbb{C}}$ and $\mathbb{C}_4$. For example, when the Lorenz condition $\vec{\nabla}\cdot\vec{\mathbb{A}} - (\partial/\partial x^4)\mathbb{A}_4 = \mathbb{A}^\mu_{,\mu} = 0$ is satisfied, all components of the conventional vector potential $\mathbb{A}_\mu$ satisfy the usual (sourced) wave equation [see, for example, Eq. (11.130) in Ref. 26 with $\mathbb{A}_4 = -\Phi$].

VI. THE UNIT FIELD

Let $u^a = u^a(x^\mu)$ denote the interacting local second gravity field. We initially restrict our work to flat Minkowski space-time. We drop the assumption of Sec. V B that u is a given constant (global) spinor and instead assume that u is a type-1 spinor field $u^a = u^a(x^\mu) \in \mathbb{R}^{4,4}$ that satisfies

$$\tilde{u}\sigma u = \sigma_{ab}u^a(x)u^b(x) > 0 \; \forall \; x \in M_{3,1}. \tag{77}$$

We further assume that $\psi_1 = \psi_1(x^\mu)$ is a local real eight-component Lagrange multiplier that transforms as a type-1 spinor field (and whose physical significance should be derived *a posteriori* and is not given *a priori*) and that $\psi_2 = \psi_2(x^\mu)$ is a real eight-component type-2 spinor field that represents the photon wave function. We assume that the bosonic spinor fields (u, ψ_1, ψ_2) carry the fundamental dynamical degrees of freedom for the (semi)classical theory.

The photon field is described using spinor fields. An immediate implication of this assumption is that the vector photon field

$$\mathbb{A}^B = \mathfrak{F}_a^B {\psi_2}^a = \frac{1}{\sqrt{\tilde{u}\sigma u}}\tilde{u}\sigma\overline{\tau}^B\psi_2 \tag{78}$$

now represents a composite particle. If a free photon Hamiltonian $\mathbb{H}_\gamma$ is constructed solely from $\mathbb{A}^B$, its partial derivatives and $\eta^{\mu\nu}$, $\mathbb{H}_\gamma = \mathbb{H}_\gamma(\eta; \mathbb{A}_\alpha, \mathbb{A}_{\alpha,\mu}, \ldots)$, then photon thermodynamics is modified when the composite system is excited out of its ground state. While it may not be clear how these internal degrees of freedom are to be excited, this may provide an experimental check for evaluating this unconventional theory.

A. Interaction with electromagnetic field

When u^a is constant, then the Maxwell Lagrangian L_M is given by Eq. (69) and possesses local U(1) invariance. When u^a is not constant, but instead comprise of the components of a position-dependent type-1 spinor field, then Eq. (69) loses local U(1) invariance. This invariance may be restored by replacing the gradient operator ∂_A in L_M with

$$D_A = \mathbb{I}_{8\times 8}\partial_A - \tau^C\left(\frac{u}{\sqrt{\tilde{u}\sigma u}}\right)_{,A}\frac{\tilde{u}}{\sqrt{\tilde{u}\sigma u}}\sigma\overline{\tau}_C. \tag{79}$$

If we define

$$v = \frac{1}{\sqrt{\tilde{u}\sigma u}}u, \tag{80}$$

then we find that

$$\tilde{v}\sigma\overline{\tau}_C\tau^B v = \delta_C^B,$$

$$\tau^A v\tilde{v}\sigma\overline{\tau}_A = \mathbb{I}_{8\times 8},$$

$$\tau^A(dv)\tilde{v}\sigma\overline{\tau}_A + \tau^A v(\widetilde{dv})\sigma\overline{\tau}_A = 0_{8\times 8},$$

$$D_A(\psi_2) = \psi_{2,A} - \tau^C v_{,A}\tilde{v}\sigma\overline{\tau}_C\psi_2 = \psi_{2,A} + \tau^C v\tilde{v}_{,A}\sigma\overline{\tau}_C\psi_2$$

$$\widetilde{D_A(\psi_2)}\sigma = (\widetilde{\psi_2}\sigma)_{,A} + (\widetilde{\psi_2}\sigma)\tau^C v_{,A}\tilde{v}\sigma\overline{\tau}_C$$

$$\mathfrak{F}_a^B[D_A(\psi_2)]^a = \partial_A(\mathfrak{F}_a^B\psi_2^a) \Leftrightarrow \mathfrak{F}_a^B D_{Ab}^a = \partial_A\mathfrak{F}_b^B \quad \text{on type 2 spinors},$$

$$D_A(v)=0 \quad (v \text{ is a special type 1 spinor}). \tag{81}$$

Under a local U(1) gauge transformation generated by $\chi=\chi(x^A)$,

$$\psi_2 \mapsto \psi_2' = \psi_2 + \tau^B v \frac{\partial \chi}{\partial x^B},$$

$$\begin{aligned} D_A(\psi_2) \mapsto D_A(\psi_2') &= D_A(\psi_2) + D_A(\tau^B v \chi_{,B}) \\ &= \psi_{2,A} - \tau^C v_{,A}\tilde{v}\sigma\bar{\tau}_C\psi_2 + \tau^B v_{,A}\chi_{,B} + \tau^B v \chi_{,BA} - \tau^C v_{,A}\tilde{v}\sigma\bar{\tau}_C(\tau^B v\chi_{,B}) \\ &= \psi_{2,A} - \tau^C v_{,A}\tilde{v}\sigma\bar{\tau}_C\psi_2 + \tau^B v \chi_{,BA}, \end{aligned} \tag{82}$$

since $\tau^B v_{,A}\chi_{,B} - \tau^C v_{,A}\tilde{v}\sigma\bar{\tau}_C(\tau^B v\chi_{,B}) = \tau^B v_{,A}\chi_{,B} - \tau^C v_{,A}(\tilde{v}\sigma\bar{\tau}_C\tau^B v)\chi_{,B} = \tau^B v_{,A}\chi_{,B} - \tau^C v_{,A}(\delta_C^B)\chi_{,B}=0$.

The important fact to glean from all of this is simply that the operator $\mathbb{P}_\perp \bar{\tau}^A D_A$ possesses local U(1) gauge invariance,

$$\mathbb{P}_\perp \bar{\tau}^A D_A(\psi_2') = \mathbb{P}_\perp \bar{\tau}^A D_A(\psi_2) \tag{83}$$

[note that $\mathbb{P}_\perp \bar{\tau}^A(\tau^B v \chi_{,BA}) = (\mathbb{P}_\perp v)\Box\chi = 0$ since $\mathbb{P}_\perp v = 0$].

A local U(1)-invariant (modulo the total divergence exhibited in the last section) Lagrangian for this problem is therefore

$$\begin{aligned} L_M^{(0)} &= \frac{1}{2}\widetilde{\psi_1}\sigma\psi_1 + \widetilde{\psi_1}\sigma\mathbb{P}_\perp\bar{\tau}^A D_A(\psi_2) + \frac{4\pi}{c}\tilde{\mathbb{J}}\sigma\psi_2 \\ &= \frac{1}{2}\widetilde{\psi_1}\sigma\psi_1 + \widetilde{\psi_1}\sigma\mathbb{P}_\perp\bar{\tau}^A\psi_{2,A} - \widetilde{\psi_1}\sigma\mathbb{P}_\perp\bar{\tau}^A\left(\tau^C\left(\frac{u}{\sqrt{\tilde{u}\sigma u}}\right)_{,A}\frac{\tilde{u}}{\sqrt{\tilde{u}\sigma u}}\sigma\bar{\tau}_C\right)\psi_2 + \frac{4\pi}{c}\tilde{\mathbb{J}}\sigma\psi_2 \\ &= \frac{1}{2}\widetilde{\psi_1}\sigma\psi_1 + \widetilde{\psi_1}\sigma\mathbb{P}_\perp\bar{\tau}^A\psi_{2,A} + \frac{1}{\sqrt{\tilde{u}\sigma u}}\widetilde{\psi_1}\sigma\mathbb{P}_\perp\bar{\tau}^A\tau^C u\left[\left(\frac{\tilde{u}}{\sqrt{\tilde{u}\sigma u}}\right)_{,A}\sigma\bar{\tau}_C\psi_2\right] + \frac{4\pi}{c}\tilde{\mathbb{J}}\sigma\psi_2 \\ &= \frac{1}{2}\widetilde{\psi_1}\sigma\psi_1 + \widetilde{\psi_1}\sigma\bar{\tau}^A\psi_{2,A} - \frac{1}{\tilde{u}\sigma u}\widetilde{\psi_1}\sigma u(\tilde{u}\sigma\bar{\tau}^A\psi_{2,A}) + \frac{1}{\sqrt{\tilde{u}\sigma u}}\widetilde{\psi_1}\sigma\bar{\tau}^A\tau^C u\left[\left(\frac{\tilde{u}}{\sqrt{\tilde{u}\sigma u}}\right)_{,A}\sigma\bar{\tau}_C\psi_2\right] \\ &\quad - \frac{1}{\sqrt{\tilde{u}\sigma u}}\widetilde{\psi_1}\sigma u\left[\left(\frac{\tilde{u}}{\sqrt{\tilde{u}\sigma u}}\right)_{,A}\sigma\bar{\tau}^A\psi_2\right] + \frac{4\pi}{c}\tilde{\mathbb{J}}\sigma\psi_2. \end{aligned} \tag{84}$$

A linear (in ψ_2) physical theory of electrodynamics in terms of the spinor fields (u,ψ_2) alone may be defined in terms of an effective Lagrangian $L_M = L_M(u,\psi_2)$, which may be obtained by functional integration with respect to the Lagrange multiplier spinor field ψ_1,

$$e^{(i/\hbar)\int L_M d^4x} = N' \int e^{(i/\hbar)\int L_M^{(0)} d^4x} D[\psi_1(x)], \tag{85}$$

where N' is a normalization constant. The path integral may be performed by mapping from spinor coordinates ψ_1^a to vector component $\hat{\psi}_1^A$, followed by a generalized Wick rotation from the split octonions to the octonions. The projection operator in L, $\mathbb{P}_\perp = [\mathbb{I}_{8\times 8} - u\tilde{u}\sigma/\tilde{u}\sigma u]$, annihilates the real part of ψ_1 so that the transformed functional integrand is explicitly a simple Gaussian with respect to $\hat{\psi}_1^8$ and a translated Gaussian with respect to $\hat{\psi}_1^A$, $A=1,\ldots,7$. The mapping from spinor coordinates ${\psi_1}^a(x)$ to vector components $\hat{\psi}_1^A(x)$, for each $x\in\mathbb{R}^{4,4}$ introduces a unit Jacobian determinant into the measure. We obtain

$$L_M = -\frac{1}{2}\{\overline{\tau}^B D_B(\psi_2)\}^T \sigma \mathbb{P}_\perp \overline{\tau}^A D_A(\psi_2) + \frac{4\pi}{c}\tilde{\mathbb{J}}\sigma\psi_2 \tag{86}$$

as a local U(1)-invariant Lagrangian for the electromagnetic sector.

B. Unit field Lagrangians on pseudo-Riemannian space-time

In this section Greek indices run from 1 to 4. When enclosed in parentheses, they are flat Minkowski space-time SO(3,1) indices; otherwise, they are $\mathbb{X}_{3,1}$ vector indices. We assume that our physical Universe may be approximately modeled in terms of fields defined on a pseudo-Riemannian manifold $\mathbb{X}_{3,1}$ whose metric tensor $g_{\alpha\beta}$ carries the Newton–Einstein gravitational degrees of freedom. The signature of the metric $g_{\alpha\beta}$ is (−+++). Let $x^\alpha=(x,y,z,t)$ be local comoving coordinates for a neighborhood of $p\in\mathbb{X}_{3,1}$. Let G denote the Newtonian gravitational constant, c the speed of light in vacuum, and $\sqrt{\hbar G/c^3}$ is the Planck length ℓ. We often employ units in which $8\pi G=1=\hbar=c$.

The unit spinor $u^a=u^a(x^\mu)$ field carries the massless second gravity interaction. We do not introduce a characteristic mass for the u field quanta, and hence we regard the u field to be dimensionless. The reasoning is that if the u field is massless then the only mass parameter available to give u a nontrivial dimension is the Planck mass. This is a multiplicative factor and may be factored into an overall coupling constant. Alternatively, one may argue that if the u field does not satisfy $\tilde{u}\sigma u=1$ then

$$v^a = \frac{u^a}{\sqrt{\tilde{u}\sigma u}} \tag{87}$$

is employed to construct the frame $\mathfrak{F}_A^a$. The v field is clearly dimensionless. A "minimalist" postulate is asserted to assign u the same dimension as v, namely, zero.

Following Refs. 9 and 30, we introduce an orthonormal tetrad $\hat{E}_{(\mu)}=E^\alpha_{(\mu)}(\partial/\partial x^\alpha)$ that defines a set of four linearly independent vector fields such that $\mathbb{E}\equiv\det(E^\alpha_{(\mu)})=1/\sqrt{-g}\neq 0$. The components are assumed to be orthonormal: $g_{\alpha\beta}E^\alpha_{(\mu)}E^\beta_{(\nu)}=\eta_{(\mu)(\nu)}$ with respect to the $\mathbb{X}_{3,1}$ metric $g_{\alpha\beta}$. Let $\tau^\alpha=E^\alpha_{(\mu)}\tau^{(\mu)}$. Our Lorentz connection $\omega_\mu^{(\alpha)(\beta)}$ and spin connection $\Gamma_\mu=\frac{1}{8}\omega_{\mu(\alpha)(\beta)}[\overline{\tau}^{(\alpha)}\tau^{(\beta)}-\overline{\tau}^{(\beta)}\tau^{(\alpha)}]$ are defined[30] using

$$2\omega_\mu^{(\alpha)(\beta)} = E^{(\alpha)\kappa}(\partial_\mu E^{(\beta)}_\kappa - \partial_\kappa E^{(\beta)}_\mu) - E^{(\beta)\kappa}(\partial_\mu E^{(\alpha)}_\kappa - \partial_\kappa E^{(\alpha)}_\mu) + E^{(\alpha)\kappa}E^{(\beta)\sigma}(\partial_\sigma E^{(\lambda)}_\kappa - \partial_\kappa E^{(\lambda)}_\sigma)E_{(\lambda)\mu}. \tag{88}$$

Note that the covariant derivative $\nabla_\beta\tau^\alpha=\tau^\alpha_{;\beta}=\partial_\beta\tau^\alpha+\{{}^{\alpha}_{\nu\beta}\}\tau^\nu+\tau^\alpha\Gamma_\beta-\Gamma_\beta\tau^\alpha\equiv 0$. Here $\{{}^{\alpha}_{\nu\beta}\}$ is the Christoffel symbol of the second kind. For brevity, we write $\tau^\alpha\nabla_\alpha u=\tau^\alpha u_{|\alpha}$.

We write the total Lagrangian density for a simple model cosmological problem as

$$L=\sqrt{-g}\left[\frac{1}{2}\frac{1}{8\pi G}(L_{\text{Einstein-Hilbert}}+\lambda^3 L_u)+L_\rho\right], \tag{89}$$

where L_u is a Lagrangian for the u^a second gravity field and λ is a dimensionless parameter; here, L_ρ describes a perfect fluid.

We bear in mind the fact that a Lagrangian should be defined that supports the nonvanishing of $\tilde{u}\sigma u$. If we demand that L_u be a $\mathrm{SO}(4,4;\mathbb{R})$ scalar, then three obvious choices for L_u are

$$(\mathrm{I})L_u=\tfrac{1}{2}\tilde{u}_{|A}\sigma\overline{\tau}^A\tau^B u_{|B}=\tfrac{1}{2}\sigma_{ab}\mathbb{G}^{\alpha\beta}u^a_{|\alpha}u^b_{|\beta}+\mu\tilde{u}\sigma u, \tag{90}$$

where μ is a nondimensional parameter,

$$(\mathrm{II})\ \ L_u=\frac{1}{2\tilde{u}\sigma u}\tilde{u}_{|A}\sigma\overline{\tau}^A\tau^B u_{|B}=\frac{1}{2\tilde{u}\sigma u}\sigma_{ab}\mathbb{G}^{\alpha\beta}u^a{}_{|\alpha}u^b{}_{|\beta}, \tag{91}$$

$$\text{(III)} \;\; L_u = \frac{1}{2}\left[\frac{\tilde{u}}{\sqrt{\tilde{u}\sigma u}}\right]_{|A} \sigma\overline{\tau}^A\tau^B\left[\frac{u}{\sqrt{\tilde{u}\sigma u}}\right]_{|B} = \frac{1}{2}\sigma_{ab}\mathbb{G}^{\alpha\beta}v^a{}_{|\alpha}v^b{}_{|\beta}. \tag{92}$$

If we demand that L_u be a $\overline{\mathrm{SO}(3,1;\mathbb{R})}$ scalar, then other obvious choices for L_u are

$$\text{(IV)} \;\; L_u = \frac{1}{2\ell}(\tilde{u}\sigma\tau^A u_{|A} + \mu\tilde{u}\sigma u), \tag{93}$$

$$\text{(V)} \;\; L_u = \frac{1}{2\ell}\left[\frac{\tilde{u}}{\sqrt{\tilde{u}\sigma u}}\right]\sigma\tau^A\nabla_A\left[\frac{u}{\sqrt{\tilde{u}\sigma u}}\right] = \frac{1}{2\ell\tilde{u}\sigma u}\tilde{u}\sigma\tau^\alpha u_{|\alpha}. \tag{94}$$

These choices all imply a dimensional coupling constant that is inversely proportional to a positive power of the gravitational constant. Hence, this is a theory of a gravitational interaction. Since this constant is not dimensionless, a full quantum field theory for the unit field u^a may have renormalization issues.

Most importantly, a nonlinear effective theory of electrodynamics in terms of the spinor field ψ_2 alone may be defined in terms of an effective Lagrangian $L_{\text{eff}}=L_{\text{eff}}(\psi_2,\ldots)$, which may be obtained by functional integration with respect to the Lagrange multiplier spinor field ψ_1 and an additional functional integration with respect to the unit spinor field u,

$$e^{(i/\hbar)\int L_{\text{eff}}d^4x} = N\int e^{(i/\hbar)\int L d^4x}D[\psi_1(x)]D[u(x)], \tag{95}$$

where N is another normalization constant.

1. Simple cosmological model incorporating the $u^a(x^\mu)$

We consider a problem that may be modeled using a simple flat Robertson–Walker line element

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2). \tag{96}$$

We write the total Lagrangian density for this problem as

$$L = \sqrt{-g}\left[\frac{1}{2}\frac{1}{8\pi G}(L_{\text{Einstein-Hilbert}}[g_{\alpha\beta}(E^\alpha_{(\mu)})] + \lambda^3 L_u) + L_\rho\right], \tag{97}$$

where L_u is the Lagrangian of Eq. (92) for the u^a and λ is a dimensionless parameter. We shall find that the scale factor $a(t)$ is continuous and has continuous first and second derivatives except when $a=0$ and $\lambda a=1$. $L_{\text{Einstein-Hilbert}}$ and L_u each have dimensional coupling constants that are proportional to $1/G$ because both the $E^\alpha_{(\mu)}$ and the u^a fields are dimensionless.

Here, L_ρ is the Lagrangian for a perfect fluid that has a timelike four-velocity streamline field $V^\mu(x^\beta)$. The associated perfect fluid stress-energy tensor $T^{\mu\nu}\equiv(-2/\sqrt{-g})[\delta(L_\rho\sqrt{-g})/\delta g_{\mu\nu}]$ is $T^{\mu\nu}=(\rho+p)V^\mu(x^\mu)V^\mu(x^\nu)+pg^{\mu\nu}$, where ρ and p are the proper energy density and pressure in the fluid rest frame. We assume that the fluid pressure p is related to the density ρ by an equation of state $p=\alpha\rho$. In this example, we consider only radiation and relativistic matter that satisfy $p=\frac{1}{3}\rho$. This type of source is often used in Friedmann type cosmological models. We should emphasize that in this example, for simplicity, we have assumed that the direct interaction of the unit field with the (p,ρ,V^μ) terms may be neglected to zeroth order.

We ask that L_u be a $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ scalar whose associated Euler–Lagrange equations preserve $\tilde{u}\sigma u$ and, as mentioned above, choose the Lagrangian of Eq. (92) for investigation,

$$L_u = \frac{1}{2}\left[\frac{\tilde{u}}{\sqrt{\tilde{u}\sigma u}}\right]_{|\alpha} \sigma\overline{\tau}^\alpha\tau^\beta\left[\frac{u}{\sqrt{\tilde{u}\sigma u}}\right]_{|\beta} = \lambda\frac{1}{2}\sigma_{ab}\mathbb{G}^{\alpha\beta}v^a{}_{|\alpha}v^b{}_{|\beta}. \tag{98}$$

We seek a solution to the coupled field equations of the special form $\rho(t)=\rho[a(t)]$. We denote $\rho(0)=\rho[a(0)]$ and sometimes use $a(0)=a_0$ and $\rho(0)=\rho_0$. Assuming a solution of the form $\tilde{u}=(0,0,0,e^{-(1/2)F(t)}/\sqrt{2},0,0,0,e^{(1/2)F(t)}/\sqrt{2})$ the Euler–Lagrange equations for u are equivalent to $3a'(t)F'(t)+a(t)F''(t)=0$. This has a solution

$$F(t) = F(0) + \sqrt{\frac{a(0)^3\rho(0)}{\lambda^3}}\int_0^t \frac{1}{a(t')^3}dt'.$$

Substituting this solution into the Euler–Lagrange equations for a yields

$$\rho(0)a(0)^3 - a(t)^3\rho[a(t)] - 3a(t)(-1+\lambda^3 a(t)^3)\dot{a}(t)^2 = 0 \tag{99}$$

and

$$-\rho(0)a(0)^3 + a(t)(\dot{a}(t)^2 + a(t)(2\ddot{a}(t) + a(t)(\alpha\rho[a(t)] - \lambda^3 a(t)(\dot{a}(t)^2 + 2a(t)\ddot{a}(t))))) = 0. \tag{100}$$

Solving Eqs. (99) and (100) for $(\dot{a},\ddot{a})$, and then differentiating the expression for $\dot{a}$ and setting it equal to the expression for $\ddot{a}$ yields the consistency condition

$$a^7\lambda^3\dot{\rho}(a) + a^6\lambda^3\rho(a) - a^4\dot{\rho}'(a) - 4a^3\rho(a) + 3a0^3\rho(0) = 0. \tag{101}$$

This may be integrated and yields $\rho=\rho(a,a_0,\rho_0,\lambda)=\rho(\lambda a,\lambda a_0,\rho_0,1)$, where

$$\rho(a,a_0,\rho_0,1) = \rho_0\frac{a_0^3}{a^3}\left\{1 + \frac{1-a^3}{3a}\left[2\sqrt{3}\left(a\tan\left[\frac{1+2a}{\sqrt{3}}\right] - a\tan\left[\frac{1+2a_0}{\sqrt{3}}\right]\right) + \log\left(\frac{(1+a(1+a))(-1+a_0)^2}{(1+a_0(1+a_0))(-1+a)^2}\right)\right]\right\}. \tag{102}$$

This is substituted into the abovementioned expression for $\ddot{a}$ to yield

$$\ddot{a} = \frac{1}{\lambda}\mathrm{acc}[a\lambda, a_0\lambda, \rho_0],$$

where

$$\mathrm{acc}[x,x_0,\rho_0] = -\frac{x_0^3\rho_0}{3x^3}\left(\frac{x}{x^3-1} + \frac{1}{3}\left(\log\left[\frac{(x(x+1)+1)(x_0-1)^2}{(x_0(x_0+1)+1)(x-1)^2}\right] + 2\sqrt{3}\left(\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{2x_0+1}{\sqrt{3}}\right)\right)\right)\right) \tag{103}$$

and into the abovementioned expression for $\dot{a}$ to yield

$$\dot{a}^2 = \frac{1}{\lambda^2}\mathrm{vsq}[a\lambda, a_0\lambda, \rho_0],$$

where

$$\mathrm{vsq}[x,x_0,\rho_0] = \frac{x_0^3\rho_0\left(2\sqrt{3}\left(a\tan\left[\frac{1+2x}{\sqrt{3}}\right] - a\tan\left[\frac{1+2x_0}{\sqrt{3}}\right]\right) + \log\left[\frac{(1+x+x^2)(-1+x_0)^2}{(1+x_0+x_0^2)(-1+x)^2}\right]\right)}{9x^2}. \tag{104}$$

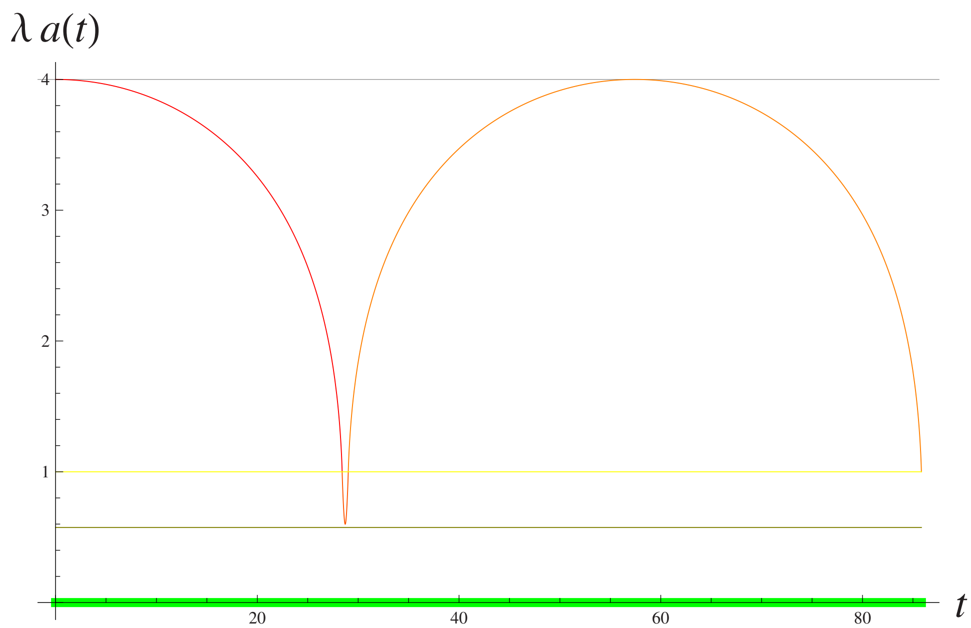

FIG. 2. (Color online) Time evolution of $\lambda a(t)$; $\lambda a(0)=4$, $\dot{a}(0)=0$.

Equation (103) may be numerically integrated. Solution graphs for the case $\lambda a(0)=4$, $\dot{a}(0)=0$ are given in Figs. 1–3. A solution graph for the case $\lambda a(0)=\frac{1}{100}$, $\dot{a}(0)=0$ is given in Fig. 4 (see also Fig. 5). It is beyond the scope of this paper to discuss the implications and extensions of this model.

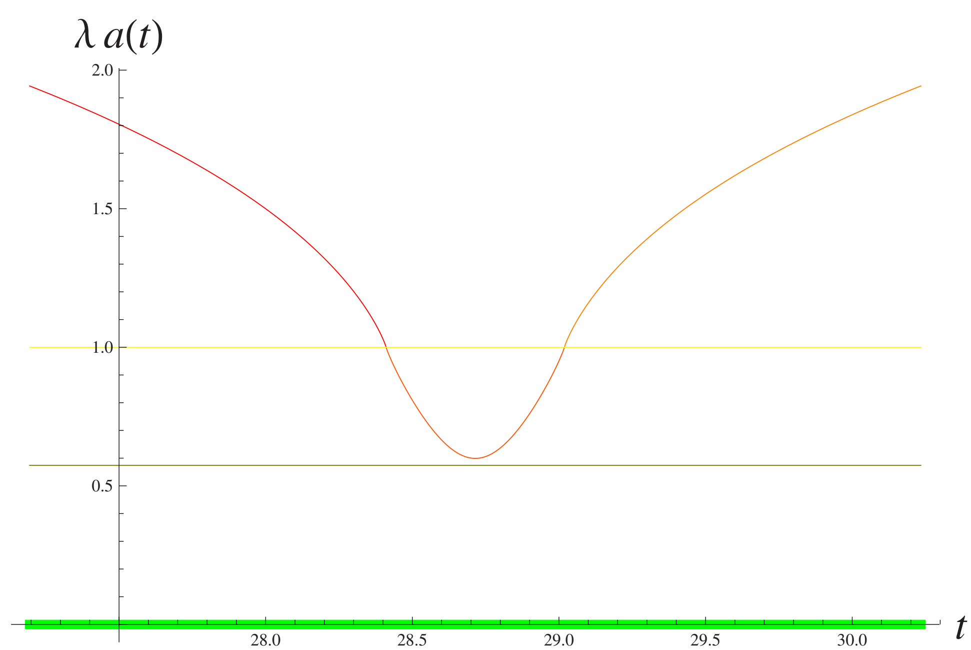

FIG. 3. (Color online) Detail of time evolution of $\lambda a(t)$; $\lambda a(0)=4$, $\dot{a}(0)=0$ at a turn-around point.

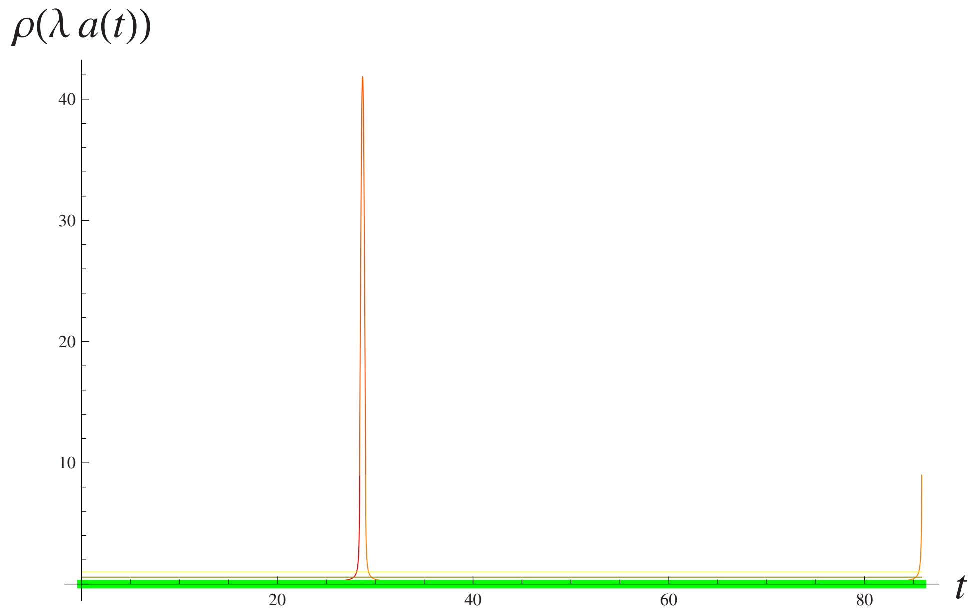

FIG. 4. (Color online) Time evolution of the proper energy density ρ; $\lambda a(0)=4$, $\dot{a}(0)=0$.

VII. CONCLUSION AND FURTHER WORK

Our model cosmological problem has demonstrated that the new unit spinor field coupled to Newton–Einstein gravity that is sourced by an ideal fluid yields a qualitatively compelling model that requires further study.

In this paper, we have first described a Lorentz covariant reformulation of the Maxwell–Lorentz theory on $\mathbb{M}_{3,1}$ that is completely consistent with the accepted theory, but that possesses new unit bosonic degrees of freedom that may be employed to model a second, non-Newton–Einstein, gravitational interaction. This realization of Maxwell's equations explicitly defines,

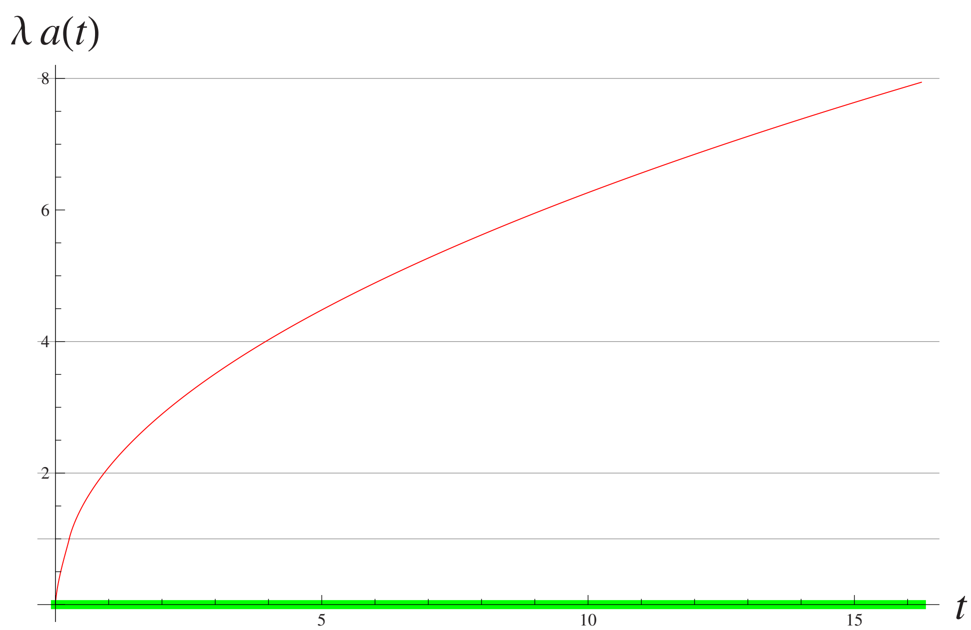

FIG. 5. (Color online) Time evolution of $\lambda a(t)$; $\lambda a(0)=0.01$, $\dot{a}(0)=0$.

uniquely and unambiguously, the second gravitational interaction of the new unit field degrees of freedom with the photon field potential ψ_2. Given the conserved electronic current vector $\mathbb{J}^A(x^\mu)$, the spinor fields (u,ψ_1,ψ_2) carry the fundamental *bosonic* dynamical degrees of freedom for a flat space-time model of (semiclassical) electrodynamics. The precise character of the second gravitational unit interaction with a photon vector potential has been explicitly defined in terms of a local U(1)-invariant Lagrangian in Eq. (86). Interesting problems are to extend the field equations so that the unused dimensions (x^5,x^6,x^7,x^8) also play a role and to answer the question as to whether ψ_2 may be generalized to include both the photon and some other gauge field.

Although there are other (split) octonion realizations of electromagnetism (see Refs. 6, 19, 15, 22, 23, 11, and 45 and references therein), this formalism is manifestly distinct. The distinguished role of the split octonion unit element, realized by the real eight-component spinor u^a and the space-time frame that is constructed from it, is unique to this theory.

Lastly we mention that path integration with respect to (ψ_1,u) may yield a new formulation of nonlinear electrodynamics. There is a growing interest in the cosmological effects of nonlinear electrodynamics because some theories of nonlinear electrodynamics give rise to cosmological inflation,[39,16] spawn a period of cosmic acceleration,[39,46,17,40] may avoid the problem of initial singularity,[39,18] and may explain the generation of astrophysically observed primeval magnetic fields during the inflation era.[10] Large-scale magnetic fields with intensities of the order of microgauss have been observed by Faraday rotation measurements in galaxies (at both high and low redshifts) as well as in clusters of galaxies.[4,28,42] One of the most challenging problems in modern astrophysics and cosmology is the origin of galactic and extragalactic magnetic fields.[44] As discussed in Ref. 10, there is a growing interest in the cosmological effects of nonlinear electrodynamics because some theories of nonlinear electrodynamics give rise to cosmological inflation,[39,16] spawn a period of cosmic acceleration,[39,46,17,40] may avoid the problem of initial singularity,[39,19] and may explain the generation of astrophysically observed primeval magnetic fields during the inflation era.[9] Large-scale magnetic fields with intensities of the order of microgauss have been observed by Faraday rotation measurements in galaxies (at both high and low redshifts) as well as in clusters of galaxies.[4,28,42] One of the most challenging problems in modern astrophysics and cosmology is the origin of galactic and extragalactic magnetic fields.[44] At present, there is no generally accepted form for the Lagrangian governing nonlinear electrodynamics (see, for example, Refs. 39 and 10). L_{eff} in Eq. (95) will yield a unique choice for the Lagrangian of nonlinear electrodynamics once L_u is uniquely identified (an association of $\mathbb{R}^{4,4}$ with a representative fiber of a bundle may be implied).

APPENDIX

The type-1 spinor with components u^a is represented by

$$\tilde{u} = \frac{1}{\sqrt{2}}(0,0,0,1,0,0,0,1). \tag{A1}$$

The particular irreducible representation of the tau matrices employed in the examples is

$$\tau^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tau^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tau^5 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tau^7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau^8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \text{(A2)}$$

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