Identities satisfied by the generators of the Dirac algebra

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The geometry of real four-dimensional spinor space and its symmetry groups are reviewed from the perspective of $\overline{\mathrm{SO}(3,3)}$. Two identities that concern the matrix generators of $\overline{\mathrm{SO}(3,3)}$, and which were first proved by Dirac, are generalized.

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1. INTRODUCTION

This paper contains several new results relating to $\overline{\mathrm{SO}(3,1)}$ spinor algebra that may be of general use. The main result, the lemma of Sec. 3, is a straightforward generalization of an identity discovered by Dirac, which is satisfied by the 4×4 matrix generators of the Dirac algebra. Section 2 is expository, $\overline{\mathrm{SO}(3,1)}$ spinor algebra is discussed in detail from the perspective of $\overline{\mathrm{SO}(3,3)}$. Section 4, provided in the interest of completeness, records the transformation properties of various geometric objects under $\overline{\mathrm{SO}(3,1)}$.

Notations and conventions used in this paper are as follows: upper case Latin indices run from 1 to 6, while both Greek and early lower case Latin indices run from 1 to 4. If M is a matrix, then $\widetilde{M}$ denotes the transpose of M. We work in a coordinate system such that the metric tensor $g_{\alpha\beta}$ on M_4 has components $g_{\alpha\beta} = \mathrm{diag}(1,1,1,-1)$.

2. THE γ^{AB} MATRICES AND $\overline{\mathrm{SO}(3,3)}$

Let $\gamma^{AB} = -\gamma^{BA}A,B,... = 1,...,6$ denote 15 elements, which are defined by[1–3]

$$\gamma^{AB}\gamma^{CD} = \gamma_0(g^{AD}g^{BC} - g^{AC}g^{BD}) - g^{AC}\gamma^{BD} + g^{AD}\gamma^{BC} + g^{BC}\gamma^{AD} - g^{BD}\gamma^{AC} - \tfrac{1}{2}\epsilon^{ABCDEF}g_{EG}g_{FH}\gamma^{GH}, \tag{1}$$

where

$$g_{AB} = g^{AB} = \mathrm{diag}(1,1,1,-1,-1,-1), \tag{2}$$

γ_0 is the identity element, and ϵ^{ABCDEF} is the totally antisymmetric Levi–Cività tensor-density of weight $+1$ in six dimensions, $\epsilon^{123456} = +1$. In virtue of Eq. (1), the set of elements $\{\pm\gamma_0, \pm\gamma^{AB}\}$ forms a finite group of order 32. We shall consider only real irreducible representations of this group in which the $\{\gamma_0,\gamma^{AB}\}$ are linearly independent. By Burnside's theorem,[4] a representation of a finite group of degree f is irreducible if and only if there occur f^2 linearly independent matrices in it; hence, the degree of this representation is four. Thus, each of the γ^{AB} is a real 4×4 matrix, and γ_0 is the 4×4 identity matrix. We shall denote the real four-dimensional vector space that carries this irreducible representation as D_4, and refer to D_4 as (real four-dimensional) Dirac space. The vectors of D_4 will be called (real) contravariant spinors, for reasons that will become apparent below. The elements of D_4^*, the vector space dual to D_4, will be called (real) covariant spinors.

On account of the defining relations of Eq. (1), one finds that

$$\gamma^{AB}\gamma^{CD} - \gamma^{CD}\gamma^{AB} = [\gamma^{AB},\gamma^{CD}] = -2(g^{AC}\gamma^{BD} - g^{AD}\gamma^{BC} - g^{BC}\gamma^{AD} + g^{BD}\gamma^{AC}), \tag{3}$$

so that the $-\tfrac{1}{2}\gamma^{AB}$ comprise a real 4×4 irreducible representation of a linearly independent basis of the $\overline{\mathrm{SO}(3,3)}$ Lie algebra, so(3,3). Moreover, Eq. (1) implies that each of the γ^{AB} matrices has square equal to $\pm\gamma_0$, and either commutes or anticommutes with any other γ^{RS} matrix. Given a particular γ^{AB}, there exists another γ matrix, say τ, which anticommutes with it. Thus trace $(\gamma^{AB}) = \mathrm{tr}(\tau^{-1}\tau\gamma^{AB}) = \mathrm{tr}(\tau\gamma^{AB}\tau^{-1}) = -\mathrm{tr}(\gamma^{AB}) = 0$. Since the γ^{AB} are trace-free and linearly independent, one deduces the well-known real Lie algebra isomorphism $\mathrm{so}(3,3)\cong\mathrm{sl}(4,\mathbb{R})$. Hence D_4 carries an irreducible representation of $\overline{\mathrm{SL}(4,\mathbb{R})}\cong\overline{\mathrm{SO}(3,3)}$; the vectors of D_4 are reduced $\overline{\mathrm{SO}(3,3)}$ spinors.

Under the involutive automorphism $\gamma^{AB}\to-\widetilde{\gamma}^{AB}$ of so(3,3), the Lie algebra decomposes into the eigenvalue (-1) and eigenvalue $(+1)$ subspaces corresponding to, respectively, the nine linearly independent real traceless symmetric 4×4 matrices, and the six linearly independent real skew-symmetric 4×4 matrices. The eigenvalue $(+1)$ subspace is the subalgebra so(4), which is the Lie algebra of $\overline{\mathrm{SO}(4)}$, the maximal compact subgroup of $\overline{\mathrm{SL}(4,\mathbb{R})}$. The subalgebra $\mathrm{so}(4)\cong\mathrm{su}(2)+\mathrm{su}(2)$ may be further decomposed into the even (eigenvalue $+1$) and odd (eigenvalue -1) subspaces of the linear transformation of so(4) whereby $\tau\in\mathrm{so}(4)$ is mapped into its dual, ${}^*\tau$. The even subspace under $*$ of so(4) corresponds to self-dual tensors, and, say, the first su(2) in the direct sum; the odd subspace corresponds to anti-self-dual tensors, and the second su(2) in the direct sum. A basis for so(4) may be chosen as follows. Each of the six skew-symmetric γ matrices has the property that the square of the matrix is equal to $-\gamma_0$. By Eq. (1), these six matrices are given by $(h = 1,2,3)$,

$$2s^h = (\gamma^{23},\gamma^{31},\gamma^{21}), \tag{4}$$

and

$$2t^h = (\gamma^{45},\gamma^{64},\gamma^{65}). \tag{5}$$

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From Eq. (3), these matrices verify ($h,k,m = 1,2,3$)

$$[s^h,t^k] = 0, \quad (6)$$

$$[s^h,s^k] = \epsilon^{hkm}s^m, \quad (7)$$

$$[t^h,t^k] = \epsilon^{hkm}t^m. \quad (8)$$

The s^h (resp. t^h) are anti-Hermitian generators of a real reducible unitary representation of SU(2). We shall assume that the s^h are self-dual, and the t^h are anti-self-dual. The six matrices s^h, t^h, comprise a linearly independent basis for the six-dimensional subalgebra so(4) of so(3,3).

The nine symmetric trace-free γ matrices may be denoted as $\gamma^{h'k}$, $h' = 1',2',3'$, where $1' = 6$, $2' = 5$, $3' = 4$. The $\gamma^{h'k}$ comprise a linearly independent basis for the nine-dimensional symmetric subspace of so(3,3). They may be expressed in terms of s^h, t^h as follows: contracting Eq. (1) with ϵ_{RSABCD} yields:

$$\gamma_{RS} = -(1/4!)\epsilon_{RSABCD}\gamma^{AB}\gamma^{CD}, \quad (9)$$

where $\gamma_{RS} = g_{RA}g_{SB}\gamma^{AB}$. Evaluating the left-hand side of Eq. (9) for $\gamma^{h'k}$ [after repeated use of Eq. (9)] gives

$$\gamma^{h'k} = -4g^{hn}t^n s^m g^{mk}, \quad (10)$$

where

$$g^{hk} = \mathrm{diag}(1,1,-1), \quad (11)$$

and, as we have heretofore implicitly assumed, the summation convention is operative for repeated indices; here m and n assume the values 1, 2, and 3.

Let $\gamma^{ABa}{}_b$ denote the ath row and bth column of γ^{AB}, where $a,b = 1,2,3,4$. A concrete representation of the γ^{AB} is ($h,k,m,n = 1,2,3$),

$$2(s^h)^a{}_b = -\epsilon_{hab4} - \delta_{ah}\delta_{b4} + \delta_{a4}\delta_{bh} \quad \text{(self-dual)}, \quad (12)$$

$$2(t^h)^a{}_b = -\epsilon_{hab4} + \delta_{ah}\delta_{b4} - \delta_{a4}\delta_{bh} \quad \text{(anti-self-dual)}, \quad (13)$$

and

$$(\gamma^{h'k})^a{}_b = g^{hm}g^{kn}(\delta_{ab}\delta_{mn} - \delta_{am}\delta_{bn} - \delta_{an}\delta_{bm} - 2\delta_{a4}\delta_{b4}\delta_{mn} + \delta_{a4}\epsilon_{mnb4} + \delta_{b4}\epsilon_{mna4}), \quad (14)$$

where ϵ_{abcd} is the totally antisymmetric Levi-Cività tensor density of weight (-1) on D_4; $\epsilon_{1234} = +1$, g^{hm} as defined in Eq. (11); and we have substituted Eqs. (12) and (13) into Eq. (10) to obtain Eq. (14). Denoting the right-hand side of Eq. (12) by s^h_{ab}, by self-dual we mean that $s^h_{ab} = \frac{1}{2}\epsilon_{abcd}s^h_{cd}$.

There does not exist a $\overline{\mathrm{SO}(3,3)}$ invariant bilinear form (inner product) on D_4. The $\overline{\mathrm{SO}(3,3)}$ symmetry must be broken down to, say, $\overline{\mathrm{SO}(4)}$, or $\overline{\mathrm{SO}(3,2)}$ or $\overline{\mathrm{SO}(3,1)}$ in order to define an invariant bilinear form on D_4. To see this, suppose that $\tilde{\lambda}\epsilon\lambda'$ is a $\overline{\mathrm{SO}(3,3)}$ invariant bilinear form, where $\lambda,\lambda' \in D_4$, $\tilde{\lambda}$ denotes the transpose of λ, and ϵ is the "metric" spinor of covariant-rank two. Under $S = \exp(-\frac{1}{4}\omega_{AB}\gamma^{AB}) \in \overline{\mathrm{SO}(3,3)}$ (the $\omega_{AB} = -\omega_{BA}$ are 15 real parameters), $\lambda' \to S\lambda$ and $\tilde{\lambda} \to \tilde{\lambda}\tilde{S}$; in order for $\tilde{\lambda}\epsilon\lambda'$ to be an invariant under $\overline{\mathrm{SO}(3,3)}$ ϵ must be invariant under automorphism by S: $\epsilon \to \tilde{S}\epsilon S = \epsilon$. This is equivalent to $\tilde{\gamma}^{AB}\epsilon = -\epsilon\gamma^{AB}$ ($\tilde{\gamma}^{AB}$ denotes the transpose of γ^{AB}).

Let $\gamma^{AB} = s^h$ or t^h; then ϵ must commute with each of these matrices, since each is skew-symmetric. Hence ϵ commutes also with the products, as defined in Eq. (10), and thus ϵ commutes with every matrix in the irreducible representation. Therefore, by the second part of Schur's lemma,[4] ϵ is a numerical multiple of the unit matrix. However, each $\gamma^{h'k}$ is symmetric, and must therefore anticommute with ϵ: $\tilde{\gamma}^{h'k}\epsilon = \gamma^{h'k}\epsilon = -\epsilon\gamma^{h'k}$. Hence ϵ must be zero; there is no $\overline{\mathrm{SO}(3,3)}$ invariant bilinear form on the real vector space D_4. Another way to show this is to note that γ^{12}, γ^{34}, and γ^{56} commute and satisfy $\tilde{\gamma}^{12} = -\gamma^{12}$, $\tilde{\gamma}^{34} = \gamma^{34}$, $\tilde{\gamma}^{56} = -\gamma^{56}$, and $\gamma^{12}\gamma^{56} = \gamma^{34}$. However, $(-\tilde{\gamma}^{12})(-\tilde{\gamma}^{56}) = \gamma^{34} \neq -\tilde{\gamma}^{34}$, so that $-\tilde{\gamma}^{AB}$ is not equivalent to γ^{AB}: $-\tilde{\gamma}^{AB}\epsilon = \epsilon\gamma^{AB} \Rightarrow \epsilon = 0$.

There are a number of bilinear forms on D_4 that are defined by a nonsingular covariant rank-two spinor ϵ, which are invariant under a subgroup of $\overline{\mathrm{SO}(3,3)}$. If ϵ is symmetric, $\tilde{\epsilon} = \epsilon$, then $\frac{1}{2}\epsilon\gamma^{AB}\omega_{AB}$ is skew-symmetric:

$$\tfrac{1}{2}\omega_{AB}\tilde{\gamma}^{AB}\epsilon = \tfrac{1}{2}\omega_{AB}\tilde{\gamma}^{AB}\tilde{\epsilon} = \widetilde{\tfrac{1}{2}\epsilon\gamma^{AB}\omega_{AB}} = -\tfrac{1}{2}\epsilon\gamma^{AB}\omega_{AB}.$$

Since there are six linearly-independent skew-symmetric real 4×4 matrices, the maximal subgroup of $\overline{\mathrm{SO}(3,3)}$ that leaves ϵ invariant corresponds to the six-parameter subgroup of $\overline{\mathrm{SO}(3,3)}$ generated by $\{s^h,t^h\}$, namely, a $\overline{\mathrm{SO}(4)}$ subgroup of $\overline{\mathrm{SO}(3,3)}$. A $\overline{\mathrm{SO}(4)}$ invariant inner product may be defined on D_4 utilizing a symmetric ϵ.

If ϵ is skew-symmetric, $\tilde{\epsilon} = -\epsilon$; then $\frac{1}{2}\epsilon\gamma^{AB}\omega_{AB}$ is symmetric

$$\tfrac{1}{2}\omega_{AB}\tilde{\gamma}^{AB}\epsilon = -\tfrac{1}{2}\tilde{\gamma}^{AB}\tilde{\epsilon} = -\widetilde{\tfrac{1}{2}\epsilon\gamma^{AB}\omega_{AB}} = -\tfrac{1}{2}\epsilon\gamma^{AB}\omega_{AB}.$$

Since there are ten linearly-independent real symmetric 4×4 matrices, ϵ defines a nonsingular skew-symmetric bilinear form on D_4 whose maximal invariance group is one of the six possible ten-parameter subgroups $\overline{\mathrm{SO}(3,2)}$ and $\overline{\mathrm{SO}(2,3)}$ of $\overline{\mathrm{SO}(3,3)}$ that are generated by ten of the fifteen $-\frac{1}{2}\gamma^{AB}$. (Which particular subgroup, of course, depends upon the choice of ϵ.) Since ϵ defines a symplectic form on D_4, one deduces the real Lie algebra isomorphisms $\mathrm{so}(3,2) \cong \mathrm{sp}(2,\mathbf{R}) \cong \mathrm{so}(2,3)$, where $\mathrm{sp}(n,\mathbf{R})$ is the real symplectic Lie algebra whose defining representation is of degree $2n$.

$\overline{\mathrm{SO}(3,1)}$ is a subgroup of $\overline{\mathrm{SO}(3,2)}$, but not of $\overline{\mathrm{SO}(2,3)}$, so that most interest lies with $\overline{\mathrm{SO}(3,2)}$ invariant-symplectic forms ϵ. There are essentially three distinct choices for ϵ, namely, γ^{45}, γ^{56}, or γ^{64}. From

$$\tfrac{1}{2}\omega_{AB}\tilde{\gamma}^{AB}\epsilon = -\tfrac{1}{2}\epsilon\gamma^{AB}\omega_{AB} \quad (15)$$

and Eq. (1), one concludes the following:

(i) If $\epsilon = \gamma^{45}$, then one must set $\omega_{A6} = 0$ in order to satisfy Eq. (15); the generators of this $\overline{\mathrm{SO}(3,2)}$ are therefore $\{-\frac{1}{2}\gamma^{\alpha\beta}, -\frac{1}{2}\gamma^{\alpha5}\}$, where $\alpha,\beta = 1,2,3,4$.

(ii) If $\epsilon = \gamma^{56}$, then one must set $\omega_{A4} = 0$; the generators are $\{-\frac{1}{2}\gamma^{hk}, -\frac{1}{2}\gamma^{h5}, -\frac{1}{2}\gamma^{h6}, -\frac{1}{2}\gamma^{56}; h,k = 1,2,3\}$.

(iii) If $\epsilon = \gamma^{64}$, then one must set $\omega_{A5} = 0$; the generators are $\{-\frac{1}{2}\gamma^{\alpha\beta}, -\frac{1}{2}\gamma^{\alpha6}\}$.

$\overline{\mathrm{SO}(3,3)}$ transformations on D_4 may be associated with SO(3,3) transformations on a flat six-dimensional (three space, three time) Minkowski space-time M_6, whose metric tensor is given by Eq. (2). By restriction to an appropriate four-dimensional affine subspace of M_6, we can realize M_4. For the sake of simplicity, we shall assume that the x^4 axis of M_4 coincides with the x^4 axis of M_6 in every coordinate system. It is customary to exclude choice (ii), $\epsilon = \gamma^{56}$, as an interesting symplectic form on D_4. γ^{56} is invariant under those automorphisms of D_4 that correspond with the automorphisms of M_6 that leave the x^4 axis of M_6 invariant.

Which of the candidates, γ^{45} or γ^{64}, that is adopted for ϵ depends upon the association defined between $\gamma^{\alpha 5}$ and $\gamma^{\alpha 6}$, and Dirac's γ^α matrices, and is also based on the fact that one must restrict the $\overline{\mathrm{SO}(3,2)}$ symmetry to a $\overline{\mathrm{SO}(3,1)}$ subgroup in order to be in accordance with relativity. As things stand, case (i) $\epsilon = \gamma^{45}$, implies that $\epsilon\gamma^{\alpha 6}$ is antisymmetric; $\gamma^{\alpha 5}$ mixes with $\gamma^{\alpha\beta}$ under SO(3,2), while $\{\gamma^{\alpha 6}, \gamma^{56}\}$ is a SO(3,2) vector ($\omega_{A6} = 0$) (the transformation properties of the γ matrices are discussed in Sec. 4). Case (iii), $\epsilon = \gamma^{64}$, implies that $\epsilon\gamma^{\alpha 6}$ is symmetric, while $\epsilon\gamma^{\alpha 5}$ is skew-symmetric; $\gamma^{\alpha 6}$ mixes with $\gamma^{\alpha\beta}$ under SO(3,2), while $\{\gamma^{\alpha 5}, \gamma^{65}\}$ is a SO(3,2) vector ($\omega_{A5} = 0$).

Equivalent formalisms are: case (i), $\epsilon = \gamma^{45}$; define $\gamma^\alpha = \gamma^{\alpha 5}$, and append to the constraint $\omega_{A6} = 0$, the restriction $\omega_{A5} = 0$, so that $\gamma^{\alpha 5}$ transforms as a vector under SO(3,1); case (iii), $\epsilon = \gamma^{64}$; define $\gamma^\alpha = \gamma^{\alpha 6}$, and append to the constraint $\omega_{A5} = 0$ the restriction $\omega_{A6} = 0$, so that $\gamma^{\alpha 6}$ transforms as a vector under SO(3,1). In both cases, the $\overline{\mathrm{SO}(3,2)}$ symmetry is reduced to $\overline{\mathrm{SO}(3,1)}$.

Without loss of generality, we shall utilize $\epsilon = \gamma^{64}$ as the symplectic form. In order to make contact with the usual conventions found in the literature, it is convenient to make the following definitions.

Let γ^α (Greek indices run from 1 to 4) denote four real 4×4 matrices (Dirac's γ matrices) that generate an irreducible representation of the pseudo-Clifford algebra C_4 (also known as the Dirac algebra). The γ^α are defined by

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\gamma_0 g^{\alpha\beta}, \tag{16}$$

where

$$g^{\alpha\beta} = g_{\alpha\beta} = \mathrm{diag}(1,1,1,-1) \tag{17}$$

is the metric tensor on M_4, in a Cartesian coordinate system. Let

$$\begin{aligned}\gamma^5 &= -(1/4!)\epsilon_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu \\ &= -\gamma^1\gamma^2\gamma^3\gamma^4,\end{aligned} \tag{18}$$

where $\epsilon_{\alpha\beta\mu\nu}$ is the totally antisymmetric Levi-Cività tensor density of weight (-1) in four dimensions, $\epsilon_{1234} = +1$. A representation of a linearly independent basis for C_4 is

$$\gamma^\alpha = \gamma^{\alpha 6}, \tag{19}$$

$$\gamma^5 = \gamma^{56}, \tag{20}$$

$$\gamma^\alpha\gamma^5 = \gamma^{\alpha 5}, \tag{21}$$

and defining

$$S^{\alpha\beta} = -\tfrac{1}{4}[\gamma^\alpha, \gamma^\beta], \tag{22}$$

$$S^{\alpha\beta} = -\tfrac{1}{2}\gamma^{\alpha\beta}. \tag{23}$$

The symplectic form ϵ on D_4 is defined to be

$$\epsilon = \gamma^{64}. \tag{24}$$

As a consequence of Eq. (1), and the definitions of Eqs. (16)–(24), are the identities

$$\tilde{\gamma}^\alpha\epsilon = -\epsilon\gamma^\alpha, \tag{25}$$

$$\tilde{S}^{\alpha\beta}\epsilon = -\epsilon S^{\alpha\beta}, \tag{26}$$

$$[S^{\alpha\beta}, \gamma_\mu] = \delta^\alpha_\mu\gamma^\beta - \delta^\beta_\mu\gamma^\alpha, \tag{27}$$

$$[S^{\alpha\beta}, S^{\mu\nu}] = g^{\alpha\mu}S^{\beta\nu} - g^{\alpha\nu}S^{\beta\mu} - g^{\beta\mu}S^{\alpha\nu} + g^{\beta\nu}S^{\alpha\mu}, \tag{28}$$

and

$$\gamma^5 S^{\alpha\beta} = \tfrac{1}{2}g^{\alpha\mu}g^{\beta\nu}\epsilon_{\mu\nu\lambda\sigma}S^{\lambda\sigma}. \tag{29}$$

We introduce a $\overline{\mathrm{SO}(3,1)}$ index notation to compliment the matrix notation which we have been using. Associate $\overline{\mathrm{SO}(3,1)}$ indices as follows: $D_4 \ni \lambda \leftrightarrow \lambda^a$; $D_4^* \ni \xi \leftrightarrow \xi_a$ (note that in matrix notation, $\xi\lambda$ denotes $\xi_a\lambda^a$, while $\lambda\xi$ denotes the 4×4 matrix with elements $\lambda^a\xi_b$; one has $\mathrm{tr}\,\lambda\xi = \xi\lambda$); $\epsilon \leftrightarrow \epsilon_{ab} = -\epsilon_{ba}$; $\tilde{\lambda}\epsilon \leftrightarrow \lambda_b = \lambda^a\epsilon_{ab}$, where the tilde denotes the transpose of a matrix (mnemonic $b\leftrightarrow$ below); raise $\overline{\mathrm{SO}(3,1)}$ indices with ϵ^{ab} according as $\xi^a = \epsilon^{ab}\xi_b$ (mnemonic: $a\leftrightarrow$ above). According to this convention $\epsilon^{ab} = \epsilon^{ac}\epsilon^{bd}\epsilon_{cd} = \epsilon^{ac}(\epsilon^{bd}\epsilon_{cd}) = \epsilon^{ac}\delta^b_c$; therefore,

$$(\epsilon^{-1})^{ab} = \epsilon^{ba} = -\epsilon^{ab}, \tag{30}$$

$$\epsilon^{ac}\epsilon_{cb} = -\delta^a_b, \tag{31}$$

and we find the correspondence $\xi^a \leftrightarrow -\epsilon^{-1}\tilde{\xi}$. $\gamma^{AB} \leftrightarrow \gamma^{ABa}{}_b$. In index notation, Eq. (25) is $\gamma^{\alpha c}{}_a\epsilon_{cb} = -\epsilon_{ac}\gamma^{\alpha c}{}_b = \epsilon_{ca}\gamma^{\alpha c}{}_b$, i.e.,

$$\gamma^\alpha{}_{ba} = \gamma^\alpha{}_{ab}. \tag{32}$$

Equation (26) is $S^{\alpha\beta c}{}_a\epsilon_{cb} = -\epsilon_{ac}S^{\alpha\beta c}{}_b$,

$$S^{\alpha\beta}_{ba} = S^{\alpha\beta}_{ab}. \tag{33}$$

$\tilde{S}\epsilon S = \epsilon, S \in \overline{\mathrm{SO}(3,1)}$, reads

$$S^c{}_a\epsilon_{cd}S^d{}_b = \epsilon_{ab}. \tag{34}$$

Since $\tilde{\gamma}^{\alpha 5}\epsilon = \epsilon\gamma^{\alpha 5}$,

$$\gamma^{\alpha 5}_{ba} = -\gamma^{\alpha 5}_{ab}; \tag{35}$$

similarly, $\tilde{\gamma}^5\epsilon = \epsilon\gamma^5$, so that

$$\gamma^5_{ba} = -\gamma^5_{ab}. \tag{36}$$

The determinant of ϵ is given by $\det\epsilon = \epsilon^{abcd}\epsilon_{a1}\epsilon_{b2}\epsilon_{c3}\epsilon_{d4}$, or equivalently, $\epsilon_{a'b'c'd'}\det\epsilon = \epsilon^{abcd}\epsilon_{aa'}\epsilon_{bb'}\epsilon_{cc'}\epsilon_{dd'}$; since $\epsilon^2 = -\gamma_0$, ϵ has eigenvalues $\pm i$; since $\mathrm{tr}\,\epsilon = 0$, the eigenvalues occur with equal multiplicity. Hence $\det\epsilon = 1$, and thus

$$\epsilon_{a'b'c'd'} = \epsilon_{aa'}\epsilon_{bb'}\epsilon_{cc'}\epsilon_{dd'}\epsilon^{abcd}. \tag{37}$$

The fact that the s^h are self-dual, and the t^h are anti-self-dual, may be expressed covariantly in both matrix and index notation. Since $\epsilon^{-1} = -\epsilon$, ${}^*\epsilon = -\epsilon$ may be written as

$$\epsilon = {}^*\epsilon^{-1}; \tag{38}$$

in index notation, Eq. (38) is $\epsilon_{ab} = \frac{1}{2}\epsilon_{abcd}(\epsilon^{-1})^{cd} = \frac{1}{2}\epsilon_{abcd}\epsilon^{dc}$, or

$$\epsilon_{ab} = -\tfrac{1}{2}\epsilon_{abcd}\epsilon^{cd}. \tag{39}$$

Using Eq. (1), one finds that $-\epsilon\gamma^\alpha\gamma^5 = \tilde{\gamma}^\alpha\gamma^5\epsilon^{-1} = 2\delta^\alpha_h g^{hk} s^k - \delta^\alpha_4\gamma^5$. Therefore,

$$-\epsilon\gamma^\alpha\gamma^5 = {}^*\gamma^\alpha\gamma^5\epsilon^{-1} \tag{40}$$

expresses the fact that the s^h are self-dual for $\alpha = 1,2,3$, and that γ^5 is anti-self-dual when $\alpha = 4$. Noting that

$$(-\epsilon\gamma^{AB})_{ab} = \gamma^{AB}_{ab}(= -\epsilon_{ac}\gamma^{ABc}{}_b = \epsilon_{ca}\gamma^{ABc}{}_b = \gamma^{AB}_{ab}),$$

and

$$(\gamma^{AB}\epsilon^{-1})^{ab} = \gamma^{ABab}(= \gamma^{ABa}{}_c\epsilon^{-1cb} = \gamma^{ABa}{}_c\epsilon^{bc} = \gamma^{ABab}),$$

Eq. (40) may be expressed as $(\gamma^\alpha\gamma^5)_{ab} = \frac{1}{2}\epsilon_{abcd}(\gamma^\alpha\gamma^5)^{cd}$, which, using Eq. (21) yields

$$\gamma^{\alpha 5}_{ab} = \tfrac{1}{2}\epsilon_{abcd}\gamma^{\alpha 5cd}. \tag{41}$$

Lastly, from Eq. (1), $-\epsilon\gamma^5 = \gamma^{45} = -\gamma^5\epsilon^{-1}$, which combined with ${}^*\gamma^{45} = -\gamma^{45}$ gives

$$-\epsilon\gamma^5 = {}^*\gamma^5\epsilon^{-1}, \tag{42}$$

i.e.,

$$\gamma^5_{ab} = \tfrac{1}{2}\epsilon_{abcd}\gamma^{5cd}. \tag{43}$$

As an application of Eq. (39), we evaluate

$$\begin{aligned}
\epsilon^{da}\epsilon^{bc} + \epsilon^{db}\epsilon^{ca} + \epsilon^{dc}\epsilon^{ab} &= \tfrac{1}{2}\epsilon^{da'}\epsilon^{b'c'}\delta^{abc}_{a'b'c'} \\
&= \tfrac{1}{2}\epsilon^{da'}\epsilon^{b'c'}\delta^{abce}_{a'b'c'e} \\
&= \tfrac{1}{2}\epsilon^{abce}\epsilon_{a'b'c'e}\epsilon^{da'}\epsilon^{b'c'} \\
&= -\epsilon^{abce}\epsilon^{da'}\epsilon_{a'e} \text{ [using Eq. (39)]} \\
&= \epsilon^{abcd} \text{ [using Eq. (31)]}.
\end{aligned}$$

Thus

$$\epsilon^{abcd} = \epsilon^{da}\epsilon^{bc} + \epsilon^{db}\epsilon^{ca} + \epsilon^{dc}\epsilon^{ab}. \tag{44}$$

(See Ref. 5 for a clear exposition of the properties of the generalized Kronecker delta, ϵ^{abcd}, and ϵ_{abcd}.)

3. A BASIC LEMMA

Lemma: Let X be an arbitrary 4×4 matrix; then

$$\gamma^{56}X\gamma^{56} + \gamma^{64}\tilde{X}\gamma^{64} + \gamma^{45}\tilde{X}\gamma^{45} = X - \gamma_0 \operatorname{tr} X + \gamma^{56}\operatorname{tr}\gamma^{56}X, \tag{45}$$

where $\tilde{X}$ denotes the transpose of X, and tr X is the trace of X. This identity is valid for any cyclic permutation of $(\gamma^{56},\gamma^{64},\gamma^{45})$, and under the replacement $\gamma^{56}\to\gamma^{12}$, $\gamma^{64}\to\gamma^{31'}$, and $\gamma^{45}\to\gamma^{23}$.

Proof: Eq. (45) is linear in X; we verify that this equation is true for $X = \gamma_0$, γ^{56}, $\gamma^{\alpha 5}$, $\gamma^{\alpha 6}$, and $\gamma^{\alpha\beta}$. Note that only for $X = \gamma_0$ (resp γ^{56}) is tr X (resp tr $\gamma^{56}X$) nonvanishing.

(i) $X = \gamma_0$; since $(\gamma^{56})^2 = -\gamma_0 = (\gamma^{64})^2 = (\gamma^{45})^2$, Eq. (45) yields

$$\begin{aligned}
&(\gamma^{56})^2 + (\gamma^{64})^2 + (\gamma^{45})^2 = -3\gamma_0 \\
&\quad = \gamma_0 - \gamma_0\operatorname{tr}\gamma_0 + \gamma^{56}\operatorname{tr}\gamma^{56} = \gamma_0 - 4\gamma_0;
\end{aligned}$$

(ii) $X = \gamma^{56}$; since $\tilde{\gamma}^{56} = -\gamma^{56}$, and γ^{56} anticommutes with both γ^{64} and γ^{45}; Eq. (45) gives

$$\begin{aligned}
&-\gamma^{56} - \gamma^{64}\gamma^{56}\gamma^{64} - \gamma^{45}\gamma^{56}\gamma^{45} = -3\gamma^{56} \\
&\quad = \gamma^{56} - \gamma_0\operatorname{tr}\gamma^{56} + \gamma^{56}\operatorname{tr}(-\gamma_0) = \gamma^{56} - 4\gamma^{56};
\end{aligned}$$

(iii) $X = \gamma^{\alpha 5}$; from Eq. (1),

$\gamma^{\alpha 5}\gamma^{56} = -\gamma^{56}\gamma^{\alpha 5}$, $\tilde{\gamma}^{\alpha 5}\gamma^{64} = \gamma^{64}\gamma^{\alpha 5}$, and

$\tilde{\gamma}^{\alpha 5}\gamma^{45} = -\gamma^{45}\gamma^{\alpha 5}$; by Eq. (45),

$$\begin{aligned}
&\gamma^{56}\gamma^{\alpha 5}\gamma^{56} + \gamma^{64}\tilde{\gamma}^{\alpha 5}\gamma^{64} + \gamma^{45}\tilde{\gamma}^{\alpha 5}\gamma^{45} \\
&\quad = (\gamma^{56})^2(-\gamma^{\alpha 5}) + (\gamma^{64})^2\gamma^{\alpha 5} + (\gamma^{45})^2(-\gamma^{\alpha 5}) = \gamma^{\alpha 5};
\end{aligned}$$

(iv) $X = \gamma^{\alpha 6}$; one deduces from Eq. (1) that

$\gamma^{\alpha 6}\gamma^{56} = -\gamma^{56}\gamma^{\alpha 6}$, $\tilde{\gamma}^{\alpha 6}\gamma^{64} = -\gamma^{64}\gamma^{\alpha 6}$, and

$\tilde{\gamma}^{\alpha 6}\gamma^{45} = \gamma^{45}\gamma^{\alpha 6}$; Eq. (45) becomes

$$\begin{aligned}
&\gamma^{56}\gamma^{\alpha 6}\gamma^{56} + \gamma^{64}\tilde{\gamma}^{\alpha 6}\gamma^{64} + \gamma^{45}\tilde{\gamma}^{\alpha 6}\gamma^{45} \\
&\quad = (\gamma^{56})^2(-\gamma^{\alpha 6}) + (\gamma^{64})^2(-\gamma^{\alpha 6}) + (\gamma^{45})^2\gamma^{\alpha 6} = \gamma^{\alpha 6};
\end{aligned}$$

(v) $X = \gamma^{\alpha\beta}$; from Eq. (1),

$\gamma^{\alpha\beta}\gamma^{56} = \gamma^{56}\gamma^{\alpha\beta}$, $\tilde{\gamma}^{\alpha\beta}\gamma^{64} = -\gamma^{64}\gamma^{\alpha\beta}$, and

$\tilde{\gamma}^{\alpha\beta}\gamma^{45} = -\gamma^{45}\gamma^{\alpha\beta}$; Eq. (45) gives

$$\begin{aligned}
&\gamma^{56}\gamma^{\alpha\beta}\gamma^{56} + \gamma^{64}\tilde{\gamma}^{\alpha\beta}\gamma^{64} + \gamma^{45}\tilde{\gamma}^{\alpha\beta}\gamma^{45} \\
&\quad = (\gamma^{56})^2\gamma^{\alpha\beta} + (\gamma^{64})^2(-\gamma^{\alpha\beta}) + (\gamma^{45})^2(-\gamma^{\alpha\beta}) = \gamma^{\alpha\beta}.
\end{aligned}$$

■

Since $\gamma^{56} = \gamma^5$, $\gamma^{64} = \epsilon = -\epsilon^{-1}$, and $\gamma^{45} = \gamma^{46}\gamma^{56} = -\epsilon\gamma^5 = -\gamma^5\epsilon^{-1}$, Eq. (45) may be written covariantly as

$$\gamma^5X\gamma^5 - \epsilon^{-1}\tilde{X}\epsilon + \gamma^5\epsilon^{-1}\tilde{X}\epsilon\gamma^5 = X - \gamma_0\operatorname{tr}X + \gamma^5\operatorname{tr}\gamma^5X.$$

Bringing X to the left-hand side of this equation, and then multiplying by $-\gamma^5$ gives

$$[X + \epsilon^{-1}\tilde{X}\epsilon, \gamma^5]_+ = \gamma^5\operatorname{tr}X + \gamma_0\operatorname{tr}\gamma^5X, \tag{46}$$

where $[A,B]_+ = AB + BA$ denotes the anticommutator of A and B. In index notation, $X + \epsilon^{-1}\tilde{X}\epsilon$ is $X^a{}_b + \epsilon^{-1ac}X^d{}_c\epsilon_{db} = X^a{}_b + \epsilon^{ca}X^d{}_c\epsilon_{db} = X^a{}_b - \epsilon_{db}X^d{}_c\epsilon^{ac} = X^a{}_b - X_b{}^a$; Eq. (46) can be written as

$$\begin{aligned}
&(X^a{}_c - X_c{}^a)\gamma^{5c}{}_b + \gamma^{5a}{}_c(X^c{}_b - X_b{}^c) \\
&\quad = \gamma^{5a}{}_bX^c{}_c + \delta^a_b\gamma^{5c}{}_dX^d{}_c.
\end{aligned} \tag{47}$$

Eq. (45) is a simple but useful identity, and is a generalization of an identity first proved by Dirac[2] in 1963. The assertion that Eq. (45) is valid under permutation of γ^{56}, γ^{64}, γ^{45} is true because, as far as so(3,3) is concerned, no su(2) generator is to be preferred over the remaining two. Eq. (45) remains valid under the replacement $\gamma^{h'k'}\to\gamma^{hk}$ because of the symmetric roles played by the two su(2) subalgebras in the direct sum of so(4) [self-dual and anti-self-dual, required in order that the six skew-symmetric matrices be linearly independent, plays no role in Eq. (45)].

As an application of this lemma, we prove that

$$\gamma_\alpha\lambda\xi\gamma^\alpha = \gamma_0\xi\lambda + \gamma^5\xi\gamma^5\lambda + \epsilon^{-1}\tilde{\xi}\tilde{\lambda}\epsilon + \gamma^5\epsilon^{-1}\tilde{\xi}\tilde{\lambda}\epsilon\gamma^5. \tag{48}$$

The starting point of this evaluation is to replace γ^{56}, γ^{64}, and γ^{45} in Eq. (45) with, respectively, γ^{12}, γ^{23}, γ^{31}. This yields

$$\gamma^{12}X\gamma^{12} + \gamma^{23}\tilde{X}\gamma^{23} + \gamma^{31}\tilde{X}\gamma^{31} = X - \gamma_0\operatorname{tr}X + \gamma^{12}\operatorname{tr}\gamma^{12}X. \tag{49}$$

Let X be an arbitrary symmetric matrix, $\tilde{X} = X$; then $\operatorname{tr}\gamma^{12}X = 0$, because X may be expanded in terms of γ_0 and $\gamma^{h'k}$; each of these matrices, when multiplied by γ^{12}, has vanishing trace [see Eq. (10)]. Consider

$$\gamma^{34}(\gamma^{12}X\gamma^{12}+\gamma^{23}X\gamma^{23}+\gamma^{31}X\gamma^{31})\gamma^{34}$$
$$=\gamma^5X\gamma^5-\gamma^{24}X\gamma^{24}-\gamma^{14}X\gamma^{14}$$
[using Eqs. (1) and (9)]
$$=\gamma^{34}(X-\gamma_0 \operatorname{tr} X)\gamma^{34}=\gamma^{34}X\gamma^{34}-\gamma_0 \operatorname{tr} X.$$

Hence $\gamma^{h4}X\gamma^{h4}=\gamma_0 \operatorname{tr} X+\gamma^5X\gamma^5$; since X is an arbitrary symmetric matrix, this implies that (for convenience we write $\gamma^{ABa}{}_b$ as $\gamma^{AB}{}_{ab}$ in this paragraph)

$$\gamma^{h4}_{ab}\gamma^{h4}_{cd}+\gamma^{h4}_{ac}\gamma^{h4}_{bd}=2\delta_{ad}\delta_{bc}+\gamma^5_{ab}\gamma^5_{cd}+\gamma^5_{ac}\gamma^5_{bd}.$$

Holding d fixed, one may obtain two similar equations by cyclically permuting (a,b,c). Upon adding two of these equations and subtracting the third, one finds that

$$\gamma^{h4}_{ab}\gamma^{h4}_{cd}=-\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}+\gamma^5_{ac}\gamma^5_{bd}+\gamma^5_{ad}\gamma^5_{bc}.$$

Contracting this result with $\lambda_b\xi_c$ yields
$$\gamma^{h4}\lambda\xi\gamma^{h4}=-\lambda\xi+\tilde{\xi}\tilde{\lambda}+\gamma_0\xi\lambda+\gamma^5\tilde{\xi}\tilde{\lambda}\gamma^5-\gamma^5\xi\gamma^5\lambda.$$
Therefore,

$$\gamma_4(\gamma^{h4}\lambda\xi\gamma^{h4}+\lambda\xi)\gamma^4=\gamma^{h4}\gamma^{46}\lambda\xi\gamma^{h4}\gamma^{46}+\gamma_4\lambda\xi\gamma^4,$$
$$\gamma_\alpha\lambda\xi\gamma^\alpha=\epsilon^{-1}(\tilde{\xi}\tilde{\lambda}+\gamma_0\xi\lambda+\gamma^5\tilde{\xi}\tilde{\lambda}\gamma^5-\gamma^5\xi\gamma^5\lambda)\epsilon$$
$$=\gamma_0\xi\lambda+\gamma^5\xi\gamma^5\lambda+\epsilon^{-1}\tilde{\xi}\tilde{\lambda}\epsilon$$
$$+\gamma^5\epsilon^{-1}\tilde{\xi}\tilde{\lambda}\epsilon\gamma^5.$$

An orthogonality relationship satisfied by the γ^{AB} is

$$-\tfrac{1}{2}\gamma^{ABa}{}_b\gamma_{AB}{}^c{}_d=4\delta^a_d\delta^c_b-\delta^a_b\delta^c_d. \tag{50}$$

To prove this, construct a matrix $Y(X)=-\frac{1}{2}\gamma^{AB}X\gamma_{AB}+X$, where X is an arbitrary 4×4 matrix. Since $(\gamma^{AB})^{-1}=-\gamma_{AB}$, this may be written as $Y=X+\Sigma_{i=1}^{15}\gamma^iX(\gamma^i)^{-1}$, where $\gamma^{AB}\leftrightarrow\gamma^i, i=1,\ldots,15$. If τ is any γ^{AB} matrix, then $\tau Y=\tau X\tau^{-1}\tau+\Sigma_{i=1}^{15}$ $\tau\gamma^iX(\tau\gamma^i)^{-1}\tau=Y\tau$; hence $\gamma^{AB}Y=Y\gamma^{AB}$, and since the γ^{AB} comprise an irreducible set, Y is a multiple $T^c{}_bX^b{}_c$ of γ_0: $\delta^a_dT^c_bX^b_c=X^b_c(-\frac{1}{2}\gamma^{ABa}{}_b\gamma_{AB}{}^c{}_d+\delta^a_b\delta^c_d)$. Thus $\delta^a_dT^c{}_b=-\frac{1}{2}\gamma^{ABa}{}_b\gamma_{AB}{}^c{}_d+\delta^a_b\delta^c_d$, which implies $T^a{}_b=-\frac{1}{2}\gamma^{ABc}{}_b\gamma_{AB}{}^a{}_c+\delta^a_b=4\delta^a_b$, which in turn implies Eq. (50).

4. TRANSFORMATION PROPERTIES OF γ^{AB}

The $\gamma^{ABa}{}_b$ are numerically invariant under combined SO(3,3) transformations of M_6 indices $\{A,B\}$, and $\overline{\mathrm{SO}(3,3)}$ transformations of spinor indices $\{a,b\}$. To see this, suppose $L^A{}_B\in\mathrm{SO}(3,3)$; then the metric on M_6, g_{AB} of Eq. (2), is invariant under automorphism by L:

$$g_{AB}\to L^C{}_Ag_{CD}L^D{}_B=g_{AB}. \tag{51}$$

The matrices $L^A{}_CL^B{}_D\gamma^{CD}$ satisfy Eq. (1) on account of Eq. (51), and the fact that $(L^A{}_B$ $\det(L^A{}_B)=1$: $\epsilon^{A'B'C'D'E'F'}L^A{}_{A'}L^B{}_{B'}L^C{}_{C'}$ $L^D{}_{D'}L^E{}_{E'}L^F{}_{F'}=\det L\cdot\epsilon^{ABCDEF}$. Hence the $L^A{}_CL^B{}_D\gamma^{CD}$ provide a real 4×4 irreducible representation of the group. The sum of the squares of the degrees of the irreducible representations of the group equals the order of the group, $32=1^2+4^2+\cdots$ (*the degree one irrep is the trivial representation*), so that there can be only one irreducible representation of degree four. Therefore, $L^A{}_CL^B{}_D\gamma^{CD}$ is equivalent to γ^{AB}, there exists a real nonsingular 4×4 matrix $S=S(L)$ such that

$$\gamma^{AB}=L^A{}_CL^B{}_DS\gamma^{CD}S^{-1}. \tag{52}$$

S may be assumed to have determinant equal to $+1$, and is determined up to a factor of ±1. The set of all such matrices S provides an irreducible representation $\overline{\mathrm{SL}(4,\mathbb{R})}\cong\overline{\mathrm{SO}(3,3)}$.

A special Lorentz transformation $x\to x'=Lx$ on M_6 is accompanied by a $\overline{\mathrm{SO}(3,3)}$ transformation on D_4: $\lambda\to\lambda'=S\lambda$. By Eq. (3), S is generated by $-\frac{1}{2}\gamma^{AB}$; for if $L^A{}_B=\delta^A{}_B-\omega^A{}_B+\cdots=(e^{-\omega})^A{}_B$, where $\omega_{AB}=-\omega_{BA}$ are 15 real parameters, then

$$S=\gamma_0-\tfrac{1}{4}\omega_{AB}\gamma^{AB}+\cdots=\exp\{-\tfrac{1}{4}\omega_{AB}\gamma^{AB}\} \tag{53}$$

satisfies Eq. (52).

One can construct a 2–1 representation of $\overline{\mathrm{SO}(3,3)}$ onto SO(3,3) as follows. Let Γ^A denote six matrices defined by

$$\Gamma^h=2g^{hk}\begin{pmatrix}0 & s^k\\ -s^k & 0\end{pmatrix} \tag{54}$$

and

$$\Gamma^{h'}=2g^{hk}\begin{pmatrix}0 & t^k\\ t^k & 0\end{pmatrix}, \tag{55}$$

where $h,k=1,2,3$; $h'=1',2',3'$ and $1'=6$, $2'=5$, $3'=4$; and $g^{hk}=\mathrm{diag}(1,1,-1)$. One may easily verify that the Γ^A satisfy

$$\Gamma^A\Gamma^B+\Gamma^B\Gamma^A=2g^{AB}I, \tag{56}$$

where I denotes the 8×8 unit matrix

$$-\tfrac{1}{4}[\Gamma^A,\Gamma^B]=-\tfrac{1}{2}\begin{pmatrix}\gamma^{AB} & 0\\ 0 & -\tilde{\gamma}^{AB}\end{pmatrix}; \tag{57}$$

and

$$\Gamma^A=L^A{}_BM\Gamma^BM^{-1}, \tag{58}$$

where

$$M=\begin{pmatrix}S & 0\\ 0 & \tilde{S}^{-1}\end{pmatrix}, \tag{59}$$

S being defined in Eq. (53). Therefore, given $S\in\overline{\mathrm{SO}(3,3)}$, the map $\overline{\mathrm{SO}(3,3)}\to\mathrm{SO}(3,3)$ defined by

$$L^A{}_B=\tfrac{1}{8}\operatorname{tr}(M^{-1}\Gamma^AM\Gamma_B) \tag{60}$$

is a 2–1 representation of $\overline{\mathrm{SO}(3,3)}$ onto SO(3,3).

Concomitant with the identification of γ^{64} as a $\overline{\mathrm{SO}(3,1)}$ invariant symplectic form on D_4 is the reduction of $\overline{\mathrm{SO}(3,3)}$ symmetry to $\overline{\mathrm{SO}(3,1)}$ defined by setting $\omega_{A5}=0=\omega_{A6}$. According to this restriction, we have $L^A{}_6=\delta^A_6$, $L^A{}_5=\delta^A_5$, and $L^\alpha{}_\beta=\delta^\alpha_\beta-\omega^A{}_\beta+\cdots=(e^{-\omega})^\alpha{}_\beta$, where $\omega_{\alpha\beta}=-\omega_{\beta\alpha}$ are six real parameters. $S\in\overline{\mathrm{SO}(3,1)}$ is given by

$$S(\omega)=\exp\{\tfrac{1}{2}\omega_{\alpha\beta}S^{\alpha\beta}\}, \tag{61}$$

where $S^{\alpha\beta}$ is defined in Eqs. (22) and (23). Under the restriction to a $\overline{\mathrm{SO}(3,1)}$ subgroup of $\overline{\mathrm{SO}(3,3)}$, the γ^{AB} decompose into sets transforming as tensors under $\overline{\mathrm{SO}(3,1)}$:

$$\epsilon\to\tilde{S}\epsilon S=\epsilon, \tag{62}$$
$$\gamma^\alpha\to L^\alpha{}_\beta S\gamma^\beta S^{-1}=\gamma^\alpha, \tag{63}$$

$$\gamma^5 \to S\gamma^5 S^{-1} = \gamma^5, \tag{64}$$

and

$$S^{\alpha\beta} \to L^{\alpha}{}_{\mu} L^{\beta}{}_{\nu} S S^{\mu\nu} S^{-1} = S^{\alpha\beta}. \tag{65}$$

Let $\lambda \in D_4$ and $\xi \in D_4^*$; under $\overline{\mathrm{SO}(3,1)}$,

$$\lambda \to S\lambda \tag{66}$$

and

$$\xi \to \xi S^{-1}. \tag{67}$$

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