Dirac Identity; generalized by Nash; plus other scratch [of course]

(* ===================== Two references: ============================== *)
(*JOURNAL OF NATHENATICAL PHYSICS, VOLUME 4, NUMBER 7, JULY 1963*)
(*"A Remarkable Representation of the 3 + 2 de Sitter Group"*)
(*P. A. M. DIrac*)
(* === *)
(*J. Math. Phys. 25 (2), February 1984*)
(*"Identities satisfied by the generators of the Dirac algebra"*)
(*Patrick L. Nash*)
(* === *)

3. A BASIC LEMMA

Lemma: Let X be an arbitrary 4×4 matrix; then

$$\gamma^{56}X\gamma^{56}+\gamma^{64}\widetilde{X}\gamma^{64}+\gamma^{45}\widetilde{X}\gamma^{45} = X-\gamma_0 \operatorname{tr} X+\gamma^{56}\operatorname{tr}\gamma^{56}X, \tag{45}$$

where $\widetilde{X}$ denotes the transpose of X, and tr X is the trace of X. This identity is valid for any cyclic permutation of $(\gamma^{56},\gamma^{64},\gamma^{45})$, and under the replacement $\gamma^{56}\rightarrow\gamma^{12}$, $\gamma^{64}\rightarrow\gamma^{31'}$, and $\gamma^{45}\rightarrow\gamma^{23}$.

Proof: Eq. (45) is linear in X; we verify that this equation is true for $X=\gamma_0$, γ^{56}, γ^{a5}, γ^{a6}, and $\gamma^{\alpha\beta}$. Note that only for $X=\gamma_0$ (resp γ^{56}) is tr X (resp tr $\gamma^{56}X$) nonvanishing.

(i) $X=\gamma_0$; since $(\gamma^{56})^2=-\gamma_0=(\gamma^{64})^2=(\gamma^{45})^2$, Eq. (45) yields

$$(\gamma^{56})^2+(\gamma^{64})^2+(\gamma^{45})^2=-3\gamma_0 = \gamma_0-\gamma_0\operatorname{tr}\gamma_0+\gamma^{56}\operatorname{tr}\gamma^{56}=\gamma_0-4\gamma_0;$$

$$S^{\alpha\beta} = -\tfrac{1}{4}[\gamma^\alpha,\gamma^\beta], \tag{22}$$

$$S^{\alpha\beta} = -\tfrac{1}{2}\gamma^{\alpha\beta}. \tag{23}$$

The symplectic form ϵ on D_4 is defined to be

$$\epsilon = \gamma^{64}. \tag{24}$$

As a consequence of Eq. (1), and the definitions of Eqs. (16)–(24), are the identities

$$\tilde{\gamma}^\alpha\epsilon = -\epsilon\gamma^\alpha, \tag{25}$$

$$\tilde{S}^{\alpha\beta}\epsilon = -\epsilon S^{\alpha\beta}, \tag{26}$$

$$[S^{\alpha\beta},\gamma_\mu] = \delta^\alpha_\mu\gamma^\beta - \delta^\beta_\mu\gamma^\alpha, \tag{27}$$

$$[S^{\alpha\beta},S^{\mu\nu}] = g^{\alpha\mu}S^{\beta\nu} - g^{\alpha\nu}S^{\beta\mu} - g^{\beta\mu}S^{\alpha\nu} + g^{\beta\nu}S^{\alpha\mu}, \tag{28}$$

and

$$\gamma^5 S^{\alpha\beta} = \tfrac{1}{2}g^{\alpha\mu}g^{\beta\nu}\epsilon_{\mu\nu\lambda\sigma}S^{\lambda\sigma}. \tag{29}$$

$$g^{\alpha\beta} = g_{\alpha\beta} = \mathrm{diag}(1,1,1,-1) \tag{5}$$

Under the involutive automorphism $\gamma^{AB} \rightarrow -\tilde{\gamma}^{AB}$ of so(3,3), the Lie algebra decomposes into the eigenvalue (-1) and eigenvalue $(+1)$ subspaces corresponding to, respectively, the nine linearly independent real traceless symmetric 4×4 matrices, and the six linearly independent real skew-symmetric 4×4 matrices. The eigenvalue $(+1)$ subspace is the subalgebra so(4), which is the Lie algebra of $\overline{\mathrm{SO}(4)}$, the maximal compact subgroup of $\overline{\mathrm{SL}(4,\mathbf{R})}$. The subalgebra $\mathrm{so}(4) \cong \mathrm{su}(2) \dot{+} \mathrm{su}(2)$ may be further decomposed into the even (eigenvalue $+1$) and odd (eigenvalue -1) subspaces of the linear transformation of so(4) whereby $\tau \in \mathrm{so}(4)$ is mapped into its dual, $^*\tau$. The even subspace under * of so(4) corresponds to self-dual tensors, and, say, the first su(2) in the direct sum; the odd subspace corresponds to anti-self-dual tensors, and the second su(2) in the direct sum. A basis for so(4) may be chosen as follows. Each of the six skew-symmetric γ matrices has the property that the square of the matrix is equal to $-\gamma_0$. By Eq. (1), these six matrices are given by $(h=1,2,3)$,

$$2s^h = (\gamma^{23},\gamma^{31},\gamma^{21}), \tag{4}$$

and

$$2t^h = (\gamma^{45},\gamma^{64},\gamma^{65}). \tag{5}$$

In[1]:=
```
Needs["Notation`"]
```

In[2]:=
```
Symbolize[ σ22̅ ]
```

$$g^{\alpha\beta} = g_{\alpha\beta} = \mathrm{diag}(1,1,1,-1) \tag{5}$$

In[3]:=
```
Symbolize[ gαβ ]
```

In[4]:=
```
X = {x1, x2, x3, x4, x5, x6};
Protect[X];
Protect[x1, x2, x3, x4, x5, x6];
```

In[7]:=
```
Ψ4 = f4[#][x1, x2, x3, x4, x5, x6] & /@ Range[4]
```

Out[7]=
```
{f4[1][x1, x2, x3, x4, x5, x6], f4[2][x1, x2, x3, x4, x5, x6],
 f4[3][x1, x2, x3, x4, x5, x6], f4[4][x1, x2, x3, x4, x5, x6]}
```

In[8]:=
```
(*psi1=f8[#][x1,x2,x3,x4,x5,x6]&/@Range[4]*)
```

In[9]:=
```
(*psi2=h8[#][x1,x2,x3,x4,x5,x6]&/@Range[4]*)
```

In[10]:=
```
psi1 = f8[#][x1, x2, x3, x4] + ⅈ f8c[#][x1, x2, x3, x4] & /@ Range[4]
```

Out[10]=
```
{f8[1][x1, x2, x3, x4] + ⅈ f8c[1][x1, x2, x3, x4],
 f8[2][x1, x2, x3, x4] + ⅈ f8c[2][x1, x2, x3, x4],
 f8[3][x1, x2, x3, x4] + ⅈ f8c[3][x1, x2, x3, x4],
 f8[4][x1, x2, x3, x4] + ⅈ f8c[4][x1, x2, x3, x4]}
```

In[11]:=
```
psi2 = h8[#][x1, x2, x3, x4] + ⅈ h8c[#][x1, x2, x3, x4] & /@ Range[4]
```

Out[11]=
```
{h8[1][x1, x2, x3, x4] + ⅈ h8c[1][x1, x2, x3, x4],
 h8[2][x1, x2, x3, x4] + ⅈ h8c[2][x1, x2, x3, x4],
 h8[3][x1, x2, x3, x4] + ⅈ h8c[3][x1, x2, x3, x4],
 h8[4][x1, x2, x3, x4] + ⅈ h8c[4][x1, x2, x3, x4]}
```

In[12]:=
```
(*processRawSets[rawSets_]:=
 Module[{s1,s2,J,ret,L=Range[Length[rawSets]],r={}},
  While[Length[L]>1,
   s1=Union[rawSets⟦L⟦1⟧⟧];
   L=Drop[L,1];
   J=L⟦1⟧;
   While[J≤16,
    If[MemberQ[L,J],
     s2=Union[rawSets⟦J⟧];
     If[Intersection[s1,s2]≠{},s1=Union[Flatten[{s1,s2}]];
      L=Complement[L,{J}];,##&[]];,
     ##&[]];
    J++;];

   AppendTo[r,s1];
  ];
  ret=Sort[r,#1⟦1⟧<#2⟦1⟧&];
  Return[ret];
 ]*)
```

In[13]:=
```
rawSetsΨ4 = Cases[
  #,

    h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], "f4"] :→ n,
  Infinity,
  Heads → True
] & /@ Ψ4
```

Out[13]=
```
{{1}, {2}, {3}, {4}}
```

In[14]:=
```
(*processRawSets[rawSetsΨ16]*)
```

---

some function definitions :

In[15]:=
```
expr_* := expr /. Complex[a_, b_] :→ Complex[a, -b]
```

In[16]:= ra[z_] := $\frac{1}{2}$ (z + (z /. {Complex[a_, b_] :→ Complex[a, -b]}))

In[17]:= ia[z_] := $\frac{1}{2\,i}$ (z - (z /. {Complex[a_, b_] :→ Complex[a, -b]}))

In[18]:=
```
Clear[rawSets];
rawSets[l_, f_, o_] := Module[{t, h},
   t = Table[Cases[
  l[[j]],
h_Symbol[n_Integer] /;
        StringEndsQ[SymbolName[h], f]  :→ n + o,
  Infinity,
  Heads → True
], {j, 1, Length[l]}];
   Return[t];
  ]
```

In[20]:=
```
(*rawSetsel16=rawSets[el16,"f16",0]*)
```

In[21]:=
```
(*couplings=showCoupledEquations[rawSetsel16]
 %//Flatten//Sort
 %===Range[0,15]*)
```

In[22]:=
```
Clear[dualRawSets];
dualRawSets[l_, f_, o_, h_, oh_] := Module[{r, r1, r2},
  r1 = rawSets[l, f, o];
  r2 = rawSets[l, h, oh];
  r = Union[Flatten[{r1, r2}, 1]] // Drop[#, 1] &;
  Return[r];
 ]
```

In[24]:=
```
(* Helper to perform one pass of merging *)
MergeSetsStep[currentSets_List] :=
  Module[{n = Length[currentSets], i, j, merged = currentSets},
 (* We look for the first pair (i, j) that intersects *)
 Catch[
  For[i = 1, i ≤ Length[merged], i++,
   For[j = i + 1, j ≤ Length[merged], j++,
    If[Intersection[merged⟦i⟧, merged⟦j⟧] =!= {},
      (* Found intersection:
             Merge and Throw to restart/finish this step *)
      merged = Delete[merged, {{i}, {j}}];
      AppendTo[merged,
           Sort[Union[currentSets⟦i⟧, currentSets⟦j⟧]]];
      Throw[Sort[merged]]; (* Sort for canonical form *)
    ];
   ];
  ];
  (* If no intersection found, return original sorted *)
  Sort[merged]
 ]
];
```

In[25]:=
```
showCoupledEquations[items_List] :=
  FixedPoint[MergeSetsStep, items];
```

constants

← ERROR : 08similarityTransformation

has 1 st index that transforms as $\frac{\partial}{\partial \Psi}$, not as Ψs

In[26]:=
```
(* O8similarityTransformation has 1st
 index that transforms as ∂/∂Ψ , not as Ψ *)
```

O8similarityTransformation has 1 st index that transforms as $\frac{\partial}{\partial \Psi}$,

not as Ψ :

In[27]:=
```
(*ArrayFlatten[{{IdentityMatrix[4],0},{0,-IdentityMatrix[4]}}]*)
```

In[28]:=
```
ID4 = IdentityMatrix[4]; Protect[ID4];
ID8 = IdentityMatrix[8];
Protect[ID8];
```

In[30]:=
```
η4488 =
 ArrayFlatten[{{IdentityMatrix[4], 0}, {0, -IdentityMatrix[4]}}];
Protect[η4488];
```

In[31]:=
```
η4488 // MatrixForm
```

Out[31]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[32]:=
```
η4488[[#, #]] & /@ Range[8]
```

Out[32]=
```
{1, 1, 1, 1, -1, -1, -1, -1}
```

In[33]:=
```
Unprotect[ϵ3]; Unprotect[ϵ4]
```

Out[33]=
```
{}
```

In[34]:=
```
ϵ3 = Array[Signature[{##}] &, {3, 3, 3}]
```

Out[34]=
```
{{{0, 0, 0}, {0, 0, 1}, {0, -1, 0}},
 {{0, 0, -1}, {0, 0, 0}, {1, 0, 0}}, {{0, 1, 0}, {-1, 0, 0}, {0, 0, 0}}}
```

In[35]:=
```
ϵ4 = Array[Signature[{##}] &, {4, 4, 4, 4}]
```
Out[35]=
```
{{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 0, 0}, {0, 1, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 1, 0}, {0, -1, 0, 0}, {0, 0, 0, 0}}},
  {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 1}, {0, 0, 0, 0}, {0, 0, 0, 0}, {-1, 0, 0, 0}},
   {{0, 0, -1, 0}, {0, 0, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 0}}},
  {{{0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}, {0, -1, 0, 0}},
   {{0, 0, 0, -1}, {0, 0, 0, 0}, {0, 0, 0, 0}, {1, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}},
  {{{0, 0, 0, 0}, {0, 0, -1, 0}, {0, 1, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 1, 0}, {0, 0, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, -1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}}
```

In[36]:=
```
Protect[ϵ3, ϵ4]
```
Out[36]=
```
{ϵ3, ϵ4}
```

In[37]:=
```
ϵ3〚1, 2, 3〛
```
Out[37]=
```
1
```

In[38]:=
```
Unprotect[σ8, σ88]
```
Out[38]=
```
{}
```

In[39]:=
```
σ88 = ArrayFlatten[{{ConstantArray[0, {4, 4}], IdentityMatrix[4]},
    {IdentityMatrix[4], ConstantArray[0, {4, 4}]}}];
Protect[σ88];
```

In[40]:=
```
σ88.σ88 === ID8
Tr[σ88] === 0
```
Out[40]=
```
True
```
Out[41]=
```
True
```
In[42]:=
```
(*mid=ExpandAll[-1*ID4];*)
(*MId=ExpandAll[-1*ID8];
simp={Zero4→0,ID4→1,mid→-1};
Simp={Zero8→ZERO,ID8→ONE,MId→MONE};*)
```
In[43]:=
```
Zero4 = ConstantArray[0, {4, 4}];
Protect[Zero4];
Zero8 = ConstantArray[0, {8, 8}];
Protect[Zero8];
```
In[44]:=
```
(*Zero16=ConstantArray[0,{16,16}];
G16=ArrayFlatten[{{η8,0},{0,-η8}}];
Id16=IdentityMatrix[16];MId16=ExpandAll[-1*Id16];
Simp16={Zero16→ZERO16,Id16→ONE16,MId16→MONE16};*)
```
In[45]:=
```
(*ZERO16=ConstantArray[0,{16,16}];
G16=ArrayFlatten[{{η4488,0},{0,-η4488}}];
ID16=IdentityMatrix[16];MId16=ExpandAll[-1*ID16];
Simp16={ZERO16→ZERO16,ID16→ONE16,MId16→MONE16};*)
```
In[46]:=
```
(*ωs=Flatten[Table[ToExpression["ω"<>ToString[A1]<>ToString[B1]],
    {A1,1,7},{B1,A1+1,8}]]
 Length[ωs]*)
```

In[47]:=
```
η4488
Dimensions[%]
gαβ = η4488⟦2 ;; 5, 2 ;; 5⟧
```
Out[47]=
```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0},
 {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}}
```
Out[48]=
```
{8, 8}
```
Out[49]=
```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}}
```
In[50]:=
```
Protect[gαβ]
```
Out[50]=
```
{g␣Subscript␣αβ}
```
In[51]:=
```
Qa[h_, p_, q_] := Signature[{h, p, q, 4}];
Qb[h_, p_, q_] := ID4⟦p, 4⟧ * ID4⟦q, h⟧ - ID4⟦p, h⟧ * ID4⟦q, 4⟧;
SelfDualAntiSymmetric[h_, p_, q_] := Qa[h, p, q] - Qb[h, p, q] ;
AntiSelfDualAntiSymmetric[h_, p_, q_] := (Qa[h, p, q] + Qb[h, p, q] );
```
In[55]:=
```
Qa[1, 2, 3]
```
Out[55]=
```
1
```
In[56]:=
```
ϵ4⟦1, 2, 3, 4⟧
```
Out[56]=
```
1
```
In[57]:=
```
Unprotect[s4by4, t4by4]
```
Out[57]=
```
{}
```
In[58]:=
```
Do[s4by4[h] = Table[Table[SelfDualAntiSymmetric[h, p, q], {q, 4}],
   {p, 4}], {h, 1, 3} ]
```
In[59]:=
```
Do[
 t4by4[h] = Table[Table[AntiSelfDualAntiSymmetric[h, p, q], {q, 4}],
   {p, 4}], {h, 1, 3} ]
```
In[60]:=
```
(*Protect[
 {s4by4[1],s4by4[2],s4by4[3],t4by4[1],t4by4[2],t4by4[3]}]*)
```

In[61]:=
```
Protect[s4by4, t4by4]
```
Out[61]=
```
{s4by4, t4by4}
```

Recall Self Dual Anti-Symmetric

In[62]:=

$$\text{Table}\left[\left(\frac{1}{2}\sum_{j1=1}^{4}\left(\sum_{j2=1}^{4}\epsilon4[[p, q, j1, j2]]\times \text{s4by4}[h][[j1, j2]]\right)\right) - \text{s4by4}[h][[p, q]], \{h, 1, 3\}, \{q, 4\}, \{p, 4\}\right]$$

Out[62]=
```
{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}
```

Recall Anti SelfDual Anti-Symmetric

In[63]:=

$$\text{Table}\left[\left(\frac{1}{2}\sum_{j1=1}^{4}\left(\sum_{j2=1}^{4}\epsilon4[[p, q, j1, j2]]\times \text{t4by4}[h][[j1, j2]]\right)\right) + \text{t4by4}[h][[p, q]], \{h, 1, 3\}, \{q, 4\}, \{p, 4\}\right]$$

Out[63]=
```
{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}
```

In[64]:= $, h' = 1', 2', 3'$, where $1' = 6, 2' = 5, 3' = 4.$

Out[64]=

$, h' = 1', 2', 3'$, where $1' = 6, 2' = 5, 3' = 4.$

In[65]:=
```
hprime = {6, 5, 4}; Protect[hprime]
```
Out[65]=
```
{hprime}
```

In[66]:=
```
g3 = DiagonalMatrix[{1, 1, -1}] ; Protect[g3];
```

In[67]:=

Under the involutive automorphism $\gamma^{AB} \rightarrow -\tilde{\gamma}^{AB}$ of so(3,3), the Lie algebra decomposes into the eigenvalue (-1) and eigenvalue $(+1)$ subspaces corresponding to, respectively, the nine linearly independent real traceless symmetric 4×4 matrices, and the six linearly independent real skew-symmetric 4×4 matrices. The eigenvalue $(+1)$ subspace is the subalgebra so(4), which is the Lie algebra of $\overline{\mathrm{SO}(4)}$, the maximal compact subgroup of $\overline{\mathrm{SL}(4,\mathbf{R})}$. The subalgebra $\mathrm{so}(4) \cong \mathrm{su}(2) \dot{+} \mathrm{su}(2)$ may be further decomposed into the even (eigenvalue $+1$) and odd (eigenvalue -1) subspaces of the linear transformation of so(4) whereby $\tau \in \mathrm{so}(4)$ is mapped into its dual, ${}^*\tau$. The even subspace under * of so(4) corresponds to self-dual tensors, and, say, the first su(2) in the direct sum; the odd subspace corresponds to anti-self-dual tensors, and the second su(2) in the direct sum. A basis for so(4) may be chosen as follows. Each of the six skew-symmetric γ matrices has the property that the square of the matrix is equal to $-\gamma_0$. By Eq. (1), these six matrices are given by $(h = 1,2,3)$,

$$2s^h = (\gamma^{23}, \gamma^{31}, \gamma^{21}), \tag{4}$$

and

$$2t^h = (\gamma^{45}, \gamma^{64}, \gamma^{65}). \tag{5}$$

Out[67]=

Under the involutive automorphism $\gamma^{AB} \to -\tilde{\gamma}^{AB}$ of so(3,3), the Lie algebra decomposes into the eigenvalue (-1) and eigenvalue $(+1)$ subspaces corresponding to, respectively, the nine linearly independent real traceless symmetric 4×4 matrices, and the six linearly independent real skew-symmetric 4×4 matrices. The eigenvalue $(+1)$ subspace is the subalgebra so(4), which is the Lie algebra of $\overline{\mathrm{SO}(4)}$, the maximal compact subgroup of $\overline{\mathrm{SL}(4,\mathbf{R})}$. The subalgebra $\mathrm{so}(4) \cong \mathrm{su}(2) \dot{+} \mathrm{su}(2)$ may be further decomposed into the even (eigenvalue $+1$) and odd (eigenvalue -1) subspaces of the linear transformation of so(4) whereby $\tau \in \mathrm{so}(4)$ is mapped into its dual, ${}^*\tau$. The even subspace under $*$ of so(4) corresponds to self-dual tensors, and, say, the first su(2) in the direct sum; the odd subspace corresponds to anti-self-dual tensors, and the second su(2) in the direct sum. A basis for so(4) may be chosen as follows. Each of the six skew-symmetric γ matrices has the property that the square of the matrix is equal to $-\gamma_0$. By Eq. (1), these six matrices are given by $(h = 1,2,3)$,

$$2s^h = (\gamma^{23}, \gamma^{31}, \gamma^{21}), \tag{4}$$

and

$$2t^h = (\gamma^{45}, \gamma^{64}, \gamma^{65}). \tag{5}$$

ϵ3 = Array[Signature[{##}] &, {3, 3, 3}] Protect[]

In[68]:=
```
g3 // MatrixForm
```

Out[68]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In[69]:=
```
Transpose[{{2, 3}, {3, 1}, {1, 2}}].g3
```

Out[69]=

{{2, 3, -1}, {3, 1, -2}}

In[70]:=
```
g3.{{2, 3}, {3, 1}, {1, 2}}
```

Out[70]=

{{2, 3}, {3, 1}, {-1, -2}}

```
In[71]:= sToAB = (g3.{{2, 3}, {3, 1}, {1, 2}} /. {{-1, -2} → -{1, 2}}) // Hold
Out[71]=
Hold[g3.{{2, 3}, {3, 1}, {1, 2}} /. {{-1, -2} → -{1, 2}}]

In[72]:= sToAB = {{2, 3}, {3, 1}, ({1, 2} // Times[-1, #] &) // Inactivate}
Out[72]=
{{2, 3}, {3, 1}, ((-1) × #1 &)[{1, 2}]}

In[73]:=
```

The nine symmetric trace-free γ matrices may be denoted as $\gamma^{h'k}$, $h' = 1', 2', 3'$, where $1' = 6, 2' = 5, 3' = 4$. The $\gamma^{h'k}$ comprise a linearly independent basis for the nine-dimensional symmetric subspace of so(3,3). They may be ex-

```
Out[73]=
```

The nine symmetric trace-free γ matrices may be denoted as $\gamma^{h'k}$, $h' = 1', 2', 3'$, where $1' = 6, 2' = 5, 3' = 4$. The $\gamma^{h'k}$ comprise a linearly independent basis for the nine-dimensional symmetric subspace of so(3,3). They may be ex-

```
In[74]:= Do[Do[constructSymmetric[h, k] =
            t4by4[h].s4by4[k] * (-g3[[h, h]] * g3[[k, k]]), {k, 1, 3}], {h, 1, 3}]

In[75]:= Table[
          {{{j, k}, {hprime[[j]], k}}, constructSymmetric[j, k] // MatrixForm},
          {j, 3}, {k, 3}] // MatrixForm
Out[75]//MatrixForm=
```

{{1, 1}, {6, 1}} $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	{{1, 2}, {6, 2}} $\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	{{1, 3}, {6, 3}} $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
{{2, 1}, {5, 1}} $\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	{{2, 2}, {5, 2}} $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	{{2, 3}, {5, 3}} $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$
{{3, 1}, {4, 1}} $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	{{3, 2}, {4, 2}} $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	{{3, 3}, {4, 3}} $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

γ

In[76]:=
```
Unprotect[γ];
```

In[77]:=
```
Clear[γ];
γ = {constructSymmetric[1, 1],
   constructSymmetric[1, 2], constructSymmetric[1, 3], t4by4[2]}
```

Out[78]=
```
{{{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}},
 {{0, -1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}}}
```

WAIT until γ^5 to Protect[γ]

In[79]:=
```
gαβ // MatrixForm
```

Out[79]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In[80]:=
```
Table[FullSimplify[γ[[h]].γ[[h]] === gαβ[[h, h]] * ID4], {h, 1, 4}]
```

Out[80]=
```
{True, True, True, True}
```

In[81]:=
```
Table[FullSimplify[γ[[h]].γ[[k]] + γ[[k]].γ[[h]] === 2 gαβ[[h, k]] * ID4],
  {h, 1, 4}, {k, 1, 4}]
```

Out[81]=
```
{{True, True, True, True}, {True, True, True, True},
 {True, True, True, True}, {True, True, True, True}}
```

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\gamma_0 g^{\alpha\beta}, \tag{4}$$

where

$$g^{\alpha\beta} = g_{\alpha\beta} = \mathrm{diag}(1,1,1,-1) \tag{5}$$

and γ_0 denotes the 4×4 unit matrix. The $\Sigma^{\alpha\beta}$ of Eq. (1)

In[82]:=
```
Table[{{h, k}, FullSimplify[(γ[[h]].γ[[k]] + γ[[k]].γ[[h]])/2] === g_αβ[[h, k]] ID4},
  {h, 1, 4}, {k, 1, 4}] // MatrixForm
```

Out[82]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

In[83]:=
```
Table[{{h, k}, FullSimplify[(γ[[h]].γ[[k]] + γ[[k]].γ[[h]])/2] ===
   η4488[[1 + h, 1 + k]] ID4}, {h, 1, 4}, {k, 1, 4}] // MatrixForm
```

Out[83]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

$$\gamma^5 = -\gamma^1 \gamma^2 \gamma^3 \gamma^4$$

In[84]:=
```
-γ[[1]].γ[[2]].γ[[3]].γ[[4]]
```

Out[84]=
```
{{0, -1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}}
```

In[85]:=
```
AppendTo[γ, (-γ[[1]].γ[[2]].γ[[3]].γ[[4]])]
```

Out[85]=
```
{{{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}},
 {{0, -1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, -1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}}}
```

In[86]:=
```
Protect[γ]
```
Out[86]=
```
{γ}
```

γ5, ...

In[87]:=
```
γ⟦5⟧
```
Out[87]=
```
{{0, -1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}}
```
In[88]:=
```
γ⟦5⟧ === -t4by4[3]
```
Out[88]=
```
True
```
In[89]:=
```
γ⟦4⟧ === t4by4[2]
```
Out[89]=
```
True
```
In[90]:=
```
γ⟦4⟧ // MatrixForm
```
Out[90]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[91]:=
```
sϵ = {ϵ → t4by4[2]}
```
Out[91]=
```
{ϵ → {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}}}
```
In[92]:=
```
Transpose[ϵ /. sϵ] === (-ϵ /. sϵ)
```
Out[92]=
```
True
```
In[93]:=
```
(*Parallel*)Table[{{A1}, FullSimplify[
   (ϵ /. sϵ).γ⟦A1⟧ === Transpose[(ϵ /. sϵ).γ⟦A1⟧]]}, {A1, 1, 5}]
```
Out[93]=
```
{{{1}, True}, {{2}, True}, {{3}, True}, {{4}, True}, {{5}, False}}
```
In[94]:=
```
γ⟦4⟧ === t4by4[2]
```
Out[94]=
```
True
```

In[95]:=
```
t4by4[1]
```
Out[95]=
```
{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}}
```

In[96]:=
```
(*γ⟦5⟧===γs⟦5,6⟧*)
```

In[97]:=
```
Table[{{h, k}, FullSimplify[γ⟦h⟧.γ⟦k⟧ + γ⟦k⟧.γ⟦h⟧] ===
    2 * η4488⟦1 + h, 1 + k⟧ ID4}, {h, 1, 5}, {k, 1, 5}] // MatrixForm
```
Out[97]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,5\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,5\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,5\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,5\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{5,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,5\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

In[98]:=
```
(γ56 = -γ⟦5⟧) // MatrixForm
(γ64 = -γ⟦4⟧) // MatrixForm
(*(γ45=t4by4[2])//MatrixForm*)
(γ45 = γ⟦1⟧.γ⟦2⟧.γ⟦3⟧) // MatrixForm
```
Out[98]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[99]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Out[100]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

In[101]:=

```
γ456 = {γ45, γ56, γ64}
```

Out[101]=

```
{{{0, 0, 0, 1}, {0, 0, -1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 0}, {0, -1, 0, 0}}}
```

In[102]:=

$$\text{Table}\left[\left\{\{j, k\}, \frac{\gamma456[\![j]\!].\gamma456[\![k]\!] + \gamma456[\![k]\!].\gamma456[\![j]\!]}{2} === \text{If}[j == k, -\text{ID4}, \text{Zero4}]\right\}, \{j, 1, 3\}, \{k, 1, 3\}\right] \text{ // MatrixForm}$$

Out[102]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1, 1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1, 2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1, 3\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2, 1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2, 2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2, 3\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3, 1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3, 2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3, 3\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

In[103]:=

```
σ22 = Flatten[{{IdentityMatrix[2]},
    If[# == 2, i, 1] * PauliMatrix[#] & /@ Range[3]}, 1]
```

Out[103]=

```
{{{1, 0}, {0, 1}}, {{0, 1}, {1, 0}},
 {{0, 1}, {-1, 0}}, {{1, 0}, {0, -1}}}
```

In[104]:=

$\overline{\sigma 22}$ = Flatten[{{-IdentityMatrix[2]},
    If[# == 2, i, 1] * PauliMatrix[#] & /@ Range[3]}, 1]

Out[104]=

```
{{{-1, 0}, {0, -1}}, {{0, 1}, {1, 0}},
 {{0, 1}, {-1, 0}}, {{1, 0}, {0, -1}}}
```

In[105]:=

```
Table[{{A, B},
   FullSimplify[1/2 Tr[1/2 (σ22⟦A⟧.OverBar[σ22]⟦B⟧ + σ22⟦B⟧.OverBar[σ22]⟦A⟧)]]},
  {A, 1, 4}, {B, 1, 4}] // MatrixForm
```

Out[105]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ -1 \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ 1 \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ -1 \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ 1 \end{pmatrix} \end{pmatrix}$$

In[106]:=

```
η2244 = DiagonalMatrix[{-1, 1, -1, 1}]
```

Out[106]=

```
{{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}}
```

SO(4);   evalues, evecs of σ

In[107]:=

```
σ8 = σ = γ64
```

Out[107]=

```
{{0, 0, 1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 0}, {0, -1, 0, 0}}
```

In[108]:=

```
{evalues, evecs} = Eigensystem[σ]
```

Out[108]=

```
{{i, i, -i, -i},
 {{0, -i, 0, 1}, {-i, 0, 1, 0}, {0, i, 0, 1}, {i, 0, 1, 0}}}
```

In[109]:=

```
(u = ExpandAll[1/√2 evecs]) // MatrixForm
```

Out[109]//MatrixForm=

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

In[110]:=

```
Table[u[[h]]*.σ.u[[h]], {h, 1, Length[u]}]
```

Out[110]=

$$\{\mathbb{i}, \mathbb{i}, -\mathbb{i}, -\mathbb{i}\}$$

In[111]:=

```
Table[(Transpose[u][[h]]).σ.u[[h]] // FullSimplify, {h, 1, Length[u]}]
```

Out[111]=

$$\left\{-\frac{1}{2}-\frac{\mathbb{i}}{2}, -\frac{1}{2}-\frac{\mathbb{i}}{2}, \frac{1}{2}-\frac{\mathbb{i}}{2}, \frac{1}{2}-\frac{\mathbb{i}}{2}\right\}$$

In[112]:=

```
(rePr = KroneckerProduct[u[[3]], u[[3]]*.σ]) // MatrixForm
```

Out[112]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathbb{i}}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{\mathbb{i}}{2} \end{pmatrix}$$

In[113]:=

```
psi2im = psi2 - rePr.psi2 // FullSimplify
```

Out[113]=

$$\Big\{h8[1][x1, x2, x3, x4] + \mathbb{i}\, h8c[1][x1, x2, x3, x4],$$
$$\frac{1}{2}\big((2+\mathbb{i})\, h8[2][x1, x2, x3, x4] - h8[4][x1, x2, x3, x4] -$$
$$(1-2\,\mathbb{i})\, h8c[2][x1, x2, x3, x4] - \mathbb{i}\, h8c[4][x1, x2, x3, x4]\big),$$
$$h8[3][x1, x2, x3, x4] + \mathbb{i}\, h8c[3][x1, x2, x3, x4],$$
$$\frac{1}{2}\big(h8[2][x1, x2, x3, x4] + (2+\mathbb{i})\, h8[4][x1, x2, x3, x4] +$$
$$\mathbb{i}\,(h8c[2][x1, x2, x3, x4] + (2+\mathbb{i})\, h8c[4][x1, x2, x3, x4])\big)\Big\}$$

In[114]:=

```
psi2cc = 2 rePr.psi2 - psi2 // FullSimplify
```

Out[114]=

$$\{-h8[1][x1, x2, x3, x4] - \mathbb{i}\, h8c[1][x1, x2, x3, x4],$$
$$(-1-\mathbb{i})\, h8[2][x1, x2, x3, x4] + h8[4][x1, x2, x3, x4] +$$
$$(1-\mathbb{i})\, h8c[2][x1, x2, x3, x4] + \mathbb{i}\, h8c[4][x1, x2, x3, x4],$$
$$-h8[3][x1, x2, x3, x4] - \mathbb{i}\, h8c[3][x1, x2, x3, x4],$$
$$-h8[2][x1, x2, x3, x4] - (1+\mathbb{i})\, h8[4][x1, x2, x3, x4] -$$
$$\mathbb{i}\,(h8c[2][x1, x2, x3, x4] + (1+\mathbb{i})\, h8c[4][x1, x2, x3, x4])\}$$

Dirac Identity; generalized by Nash

BELOW:

anti4by4[[#]][[2]] & /@ {1, 5, 6}==={{{5}, {4}↔γ64}, {{12}, {1, 2, 3}}, {{16}, {1, 2, 3, 4}↔γ56}}

VERY CAREFULL :

$$\{\gamma 64, \quad \gamma 45, \quad \gamma 56\}:$$

$$\left\{\underbrace{\gamma[\![4]\!]}, \underbrace{\gamma[\![1]\!].\gamma[\![2]\!].\gamma[\![3]\!]} \; \underbrace{\underbrace{\gamma[\![1]\!].\gamma[\![2]\!].\gamma[\![3]\!].\gamma[\![4]\!]}}\right\}$$

$$\left\{\overbrace{\{4\}}, \quad \overbrace{\{1, 2, 3\}}, \quad \overbrace{x5, x6, x7}\right\}:$$

{{{5}, {4}}, {{12}, {1, 2, 3}}, {{16}, {1, 2, 3, 4}}}

In[ ]:= anti4by4[[#]][[2]] & /@ {2, 3, 4}

Out[ ]=

{{{6}, {1, 2}}, {{7}, {1, 3}}, {{9}, {2, 3}}}

In[ ]:=

```
(γ56 = -γ[[5]]) // MatrixForm
(γ64 = -γ[[4]]) // MatrixForm
(*(γ45=t4by4[2])//MatrixForm*)
(γ45 = γ[[1]].γ[[2]].γ[[3]]) // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

In[∘]:=

SOME PROPERTIES OF THE OPERATORS χ

Lemma. For any symmetrical 4×4 matrix S,

$$\gamma_{12} S \gamma_{12} + \gamma_{23} S \gamma_{23} + \gamma_{31} S \gamma_{31} = S - \langle S \rangle \gamma_0, \qquad (22)$$

and similarly,

$$\gamma_{45} S \gamma_{45} + \gamma_{56} S \gamma_{56} + \gamma_{64} S \gamma_{64} = S - \langle S \rangle \gamma_0. \qquad (23)$$

To prove (22), we note that any symmetric γ_{AB} has one of its suffixes equal to 1, 2, or 3 and the other equal to 4, 5, or 6 and therefore it commutes with one of the three quantities γ_{12}, γ_{23}, γ_{31} and anticommutes with the other two. Thus if such a γ_{AB} is substituted for S in (22), two of the terms on the left become S and the third one $-S$, and the $\langle S \rangle$ on the right is zero. Also (22) holds with $S = \gamma_0$, so it holds for any symmetric S. The proof of (23) is similar.

Out[∘]=

SOME PROPERTIES OF THE OPERATORS χ

Lemma. For any symmetrical 4×4 matrix S,

$$\gamma_{12} S \gamma_{12} + \gamma_{23} S \gamma_{23} + \gamma_{31} S \gamma_{31} = S - \langle S \rangle \gamma_0, \qquad (22)$$

and similarly,

$$\gamma_{45} S \gamma_{45} + \gamma_{56} S \gamma_{56} + \gamma_{64} S \gamma_{64} = S - \langle S \rangle \gamma_0. \qquad (23)$$

To prove (22), we note that any symmetric γ_{AB} has one of its suffixes equal to 1, 2, or 3 and the other equal to 4, 5, or 6 and therefore it commutes with one of the three quantities γ_{12}, γ_{23}, γ_{31} and anticommutes with the other two. Thus if such a γ_{AB} is substituted for S in (22), two of the terms on the left become S and the third one $-S$, and the $\langle S \rangle$ on the right is zero. Also (22) holds with $S = \gamma_0$, so it holds for any symmetric S. The proof of (23) is similar.

```
Clear[DiracNash];
DiracNash[M_] :=
 Zero4 === FullSimplify[γ56.M.γ56 + γ64.Transpose[M].γ64 +
    γ45.Transpose[M].γ45 - M + ID4 * Tr[M] - γ56 * Tr[γ56.M]]
```

```
DiracNash[#] & /@ {γ56, γ64, γ45}
```

```
{True, True, True}
```

```
Table[DiracNash[γ[[j]]], {j, 1, 5}]
```

```
{True, True, True, True, True}
```

SOME PROPERTIES OF THE OPERATORS χ

Lemma. For any symmetrical 4×4 matrix S,

$$\gamma_{12}S\gamma_{12} + \gamma_{23}S\gamma_{23} + \gamma_{31}S\gamma_{31} = S - \langle S\rangle\gamma_0, \tag{22}$$

and similarly,

$$\gamma_{45}S\gamma_{45} + \gamma_{56}S\gamma_{56} + \gamma_{64}S\gamma_{64} = S - \langle S\rangle\gamma_0. \tag{23}$$

To prove (22), we note that any symmetric γ_{AB} has one of its suffixes equal to 1, 2, or 3 and the other equal to 4, 5, or 6 and therefore it commutes with one of the three quantities γ_{12}, γ_{23}, γ_{31} and anticommutes with the other two. Thus if such a γ_{AB} is substituted for S in (22), two of the terms on the left become S and the third one $-S$, and the $\langle S\rangle$ on the right is zero. Also (22) holds with $S = \gamma_0$, so it holds for any symmetric S. The proof of (23) is similar.

In[•]:=

3. A BASIC LEMMA

Lemma: Let X be an arbitrary 4×4 matrix; then

$$\gamma^{56}X\gamma^{56}+\gamma^{64}\widetilde{X}\gamma^{64}+\gamma^{45}\widetilde{X}\gamma^{45}$$
$$=X-\gamma_0\,\mathrm{tr}\,X+\gamma^{56}\,\mathrm{tr}\,\gamma^{56}X, \qquad (45)$$

where $\widetilde{X}$ denotes the transpose of X, and tr X is the trace of X. This identity is valid for any cyclic permutation of $(\gamma^{56},\gamma^{64},\gamma^{45})$, and under the replacement $\gamma^{56}\to\gamma^{12}$, $\gamma^{64}\to\gamma^{31}$, and $\gamma^{45}\to\gamma^{23}$.

Proof: Eq. (45) is linear in X; we verify that this equation is true for $X=\gamma_0, \gamma^{56}, \gamma^{\alpha5}, \gamma^{\alpha6}$, and $\gamma^{\alpha\beta}$. Note that only for $X=\gamma_0$ (resp γ^{56}) is tr X (resp tr $\gamma^{56}X$) nonvanishing.

(i) $X=\gamma_0$; since $(\gamma^{56})^2=-\gamma_0=(\gamma^{64})^2=(\gamma^{45})^2$, Eq. (45) yields

$$(\gamma^{56})^2+(\gamma^{64})^2+(\gamma^{45})^2=-3\gamma_0$$
$$=\gamma_0-\gamma_0\,\mathrm{tr}\,\gamma_0+\gamma^{56}\,\mathrm{tr}\,\gamma^{56}=\gamma_0-4\gamma_0;$$

207 J. Math. Phys., Vol. 25, No. 2, February 1984

In[•]:=
```
(*DiracNash[M_]:=γ56.M.γ56+γ64.Transpose[M].γ64+
  γ45.Transpose[M].γ45-M+ID4*Tr[M]-γ56*Tr[γ56.M]*)
```

In[•]:=
```
Ξ = Array[ξ, 4]
```
Out[•]=
```
{ξ[1], ξ[2], ξ[3], ξ[4]}
```

In[•]:=
```
MX = Array[m, {4, 4}]
```
Out[•]=
```
{{m[1, 1], m[1, 2], m[1, 3], m[1, 4]},
 {m[2, 1], m[2, 2], m[2, 3], m[2, 4]},
 {m[3, 1], m[3, 2], m[3, 3], m[3, 4]},
 {m[4, 1], m[4, 2], m[4, 3], m[4, 4]}}
```

γ

In[•]:=
```
Do[beta[j] = γ⟦j⟧, {j, 4}]
```

In[ ]:=
```
txA0 = {{IdentityMatrix[4], {0}}};
Length[txA0]
txA = {}; Do[AppendTo[txA, {beta[j], {j}}], {j, 1, 4}]
Length[txA]
txAB = {};
Do[AppendTo[txAB, {beta[j].beta[k], {j, k}}],
  {j, 1, 3}, {k, j + 1, 4}]
Length[txAB]
txABC = {};
Do[AppendTo[txABC, {beta[j].beta[k].beta[h], {j, k, h}}],
  {j, 1, 2}, {k, j + 1, 3}, {h, k + 1, 4}];
Length[txABC]
txABCD = {{beta[1].beta[2].beta[3].beta[4], {1, 2, 3, 4}}};
Length[txABCD]
```

Out[ ]=
1

Out[ ]=
4

Out[ ]=
6

Out[ ]=
4

Out[ ]=
1

In[ ]:=
```
betabase4by4 = Flatten[{txA0, txA, txAB, txABC, txABCD}, 1];
Length[betabase4by4]
```

Out[ ]=
16

In[ ]:=
```
Select[betabase4by4, #[[1]] === -γ[[5]] & → "Index"]
```

Out[ ]=
{16}

In[ ]:=
```
Select[betabase4by4, #[[1]] === γ45 & → "Index"]
```

Out[ ]=
{12}

In[ ]:=
```
Select[betabase4by4, #[[2]] === {1, 2} & → "Index"]
```
Out[ ]=
```
{6}
```

In[ ]:=
```
Select[betabase4by4, #[[2]] === {1, 3} & → "Index"]
```
Out[ ]=
```
{7}
```

In[ ]:=
```
Select[betabase4by4, #[[2]] === {2, 3} & → "Index"]
```
Out[ ]=
```
{9}
```

In[ ]:=
```
Select[betabase4by4, #[[1]] === γ[[1]] & → "Index"]
```
Out[ ]=
```
{2}
```

In[ ]:=
```
Select[betabase4by4, #[[1]] === γ[[2]] & → "Index"]
```
Out[ ]=
```
{3}
```

In[ ]:=
```
Select[betabase4by4, #[[1]] === γ[[3]] & → "Index"]
```
Out[ ]=
```
{4}
```

In[ ]:=
```
Select[betabase4by4, #[[1]] === γ[[4]] & → "Index"]
```
Out[ ]=
```
{5}
```

In[ ]:=
```
Select[betabase4by4, #[[1]] === -γ[[5]] & → "Index"]
```
Out[ ]=
```
{16}
```

In[ ]:=
```
Select[betabase4by4, #[[1]] === -γ[[5]] & → "Index"]
```
Out[ ]=
```
{16}
```

In[ ]:=
```
Select[betabase4by4, #[[2]] === {2, 4} & → "Index"]
```
Out[ ]=
```
{10}
```

In[ ]:=
```
Select[betabase4by4, #[[2]] === {3, 4} & → "Index"]
```
Out[ ]=
```
{11}
```

In[ ]:=
```
DiracNash[#⟦1⟧] & /@ betabase4by4
```
Out[ ]=
```
{True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True}
```

In[ ]:=
```
betabase4by4⟦16⟧⟦1⟧
```
Out[ ]=
```
{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}}
```

In[ ]:=
```
anti4by4 = {};
Do[If[betabase4by4⟦k, 1⟧ == -Transpose[betabase4by4⟦k, 1⟧],
  AppendTo[anti4by4, {betabase4by4⟦k, 1⟧,
    {{k}, betabase4by4⟦k, 2⟧}}] ], {k, 1, Length[betabase4by4]}];
Length[anti4by4]
anti4by4⟦%⟧
```
Out[ ]=
```
6
```
Out[ ]=
```
{{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
 {{16}, {1, 2, 3, 4}}}
```

In[∘]:=
```
Eigensystem[#] & /@ anti4by4[[All, 1]]
MatrixForm[#[[2]]] & /@ %
```

Out[∘]=
```
{{{i, i, -i, -i}, {{0, i, 0, 1}, {i, 0, 1, 0},
    {0, -i, 0, 1}, {-i, 0, 1, 0}}}, {{i, i, -i, -i},
   {{0, 0, -i, 1}, {-i, 1, 0, 0}, {0, 0, i, 1}, {i, 1, 0, 0}}},
  {{i, i, -i, -i}, {{0, -i, 0, 1}, {i, 0, 1, 0},
    {0, i, 0, 1}, {-i, 0, 1, 0}}}, {{i, i, -i, -i},
   {{i, 0, 0, 1}, {0, i, 1, 0}, {-i, 0, 0, 1}, {0, -i, 1, 0}}},
  {{i, i, -i, -i}, {{-i, 0, 0, 1}, {0, i, 1, 0},
    {i, 0, 0, 1}, {0, -i, 1, 0}}}, {{i, i, -i, -i},
   {{0, 0, i, 1}, {-i, 1, 0, 0}, {0, 0, -i, 1}, {i, 1, 0, 0}}}}
```

Out[∘]=

$$\left\{\begin{pmatrix} 0 & i & 0 & 1 \\ i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -i & 1 \\ -i & 1 & 0 & 0 \\ 0 & 0 & i & 1 \\ i & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 & 1 \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \\ -i & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & 1 \\ -i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \\ i & 1 & 0 & 0 \end{pmatrix}\right\}$$

In[∘]:=
```
Zero4
```

Out[∘]=
```
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

In[∘]:=
```
ID4
```

Out[∘]=
```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

In[∘]:=
```
UnsameQ
```

Out[∘]=
```
UnsameQ
```

two distinct su (2) ' s :

In[∘]:=
```
Table[Block[{r}, r =
    (anti4by4[[j]][[1]].anti4by4[[k]][[1]] + anti4by4[[k]][[1]].anti4by4[[j]][[1]])/2;
   {{j, k}, r}], {j, 1, Length[anti4by4]},
  {k, 1, Length[anti4by4]}] // MatrixForm
```

Out[∘]//MatrixForm=
```
({1, 1}  {{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}})   ({{0, 0
({2, 1}  {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}})    ({{-1, 0
({3, 1}  {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}})    ({{0,
({4, 1}  {{0, -1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}})  ({{0,
({5, 1}  {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}})      ({{0, 0,
({6, 1}  {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}})      ({{-1,
```

In[∘]:=
```
(*Table[Block[{r},r=(anti4by4[[j]][[1]].anti4by4[[k]][[1]]+anti4by4[[k]][[1]].anti4by4[[j]][[1]])/2;
  {{j,k},r===If[j=!=k,Zero4,-ID4]}],
  {j,1,Length[anti4by4]},{k,1,Length[anti4by4]}]//MatrixForm*)
```

In[∘]:=
```
(*Table[
  Block[{r},{{j,k},r=(anti4by4[[j]][[1]].anti4by4[[k]][[1]]+anti4by4[[k]][[1]].anti4by4[[j]][[1]])/2===
      If[j=!=k,Zero4,r]}],{j,1,Length[anti4by4]},
  {k,1,Length[anti4by4]}]//MatrixForm*)
```

In[∘]:=
```
(*Table[
  {{j,k},Zero4===(anti4by4[[j]][[1]].anti4by4[[k]][[1]]+anti4by4[[k]][[1]].anti4by4[[j]][[1]])/2},
  {j,1,Length[anti4by4]-1},{k,j+1,Length[anti4by4]}]//MatrixForm*)
```

In[∘]:=
```
anti4by4[[#]][[2]] & /@ {1, 5, 6}
```

Out[∘]=
```
{{{5}, {4}}, {{12}, {1, 2, 3}}, {{16}, {1, 2, 3, 4}}}
```

In[∘]:=
```
anti4by4[[#]][[2]] & /@ {2, 3, 4}
```

Out[∘]=
```
{{{6}, {1, 2}}, {{7}, {1, 3}}, {{9}, {2, 3}}}
```

In[ ]:=
```
symm4by4 = {};
Do[If[betabase4by4[[k, 1]] == Transpose[betabase4by4[[k, 1]]],
  AppendTo[symm4by4, {betabase4by4[[k, 1]],
    {{k}, betabase4by4[[k, 2]]}}] ], {k, 1, Length[betabase4by4]}];
Length[symm4by4]
symm4by4[[%]]
```

Out[ ]=
```
10
```

Out[ ]=
```
{{{0, -1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}},
 {{15}, {2, 3, 4}}}
```

In[ ]:=
```
Eigensystem[#] & /@ symm4by4[[All, 1]]
MatrixForm[#[[2]]] & /@ %
```

Out[ ]=
```
{{{1, 1, 1, 1}, {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}},
 {{-1, -1, 1, 1},
  {{0, 0, 0, 1}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}}},
 {{-1, -1, 1, 1}, {{0, 0, -1, 1}, {1, 1, 0, 0},
   {0, 0, 1, 1}, {-1, 1, 0, 0}}}, {{-1, -1, 1, 1},
  {{0, -1, 0, 1}, {-1, 0, 1, 0}, {0, 1, 0, 1}, {1, 0, 1, 0}}},
 {{-1, -1, 1, 1}, {{0, 1, 0, 1}, {-1, 0, 1, 0},
   {0, -1, 0, 1}, {1, 0, 1, 0}}}, {{-1, -1, 1, 1},
  {{-1, 0, 0, 1}, {0, -1, 1, 0}, {1, 0, 0, 1}, {0, 1, 1, 0}}},
 {{-1, -1, 1, 1}, {{0, 0, 0, 1}, {0, 0, 1, 0},
   {0, 1, 0, 0}, {1, 0, 0, 0}}}, {{-1, -1, 1, 1},
  {{1, 0, 0, 1}, {0, -1, 1, 0}, {-1, 0, 0, 1}, {0, 1, 1, 0}}},
 {{-1, -1, 1, 1}, {{0, 0, 1, 0}, {1, 0, 0, 0},
   {0, 0, 0, 1}, {0, 1, 0, 0}}}, {{-1, -1, 1, 1},
  {{0, 0, 1, 1}, {1, 1, 0, 0}, {0, 0, -1, 1}, {-1, 1, 0, 0}}}}
```

Out[ ]=

$$\left\{\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&-1&1\\1&1&0&0\\0&0&1&1\\-1&1&0&0\end{pmatrix},\begin{pmatrix}0&-1&0&1\\-1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\-1&0&1&0\\0&-1&0&1\\1&0&1&0\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}-1&0&0&1\\0&-1&1&0\\1&0&0&1\\0&1&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&1\\0&-1&1&0\\-1&0&0&1\\0&1&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&1&1\\1&1&0&0\\0&0&-1&1\\-1&1&0&0\end{pmatrix}\right\}$$

In[ ]:=
```
η16 = DiagonalMatrix[
  Table[FullSimplify[1/4 Tr[symm4by4⟦A, 1⟧.symm4by4⟦A, 1⟧]],
   {A, Length[symm4by4]}]]
Tr[η16]
```

Out[ ]=
```
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}}
```

Out[ ]=
```
10
```

In[ ]:=
```
γ⟦5⟧
```

Out[ ]=
```
{{0, -1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}}
```

In[ ]:=
```
γ⟦5⟧ === -t4by4[3]
```

Out[ ]=
```
True
```

In[ ]:=
```
(*γ⟦5⟧===γs⟦5,6⟧*)
```

In[ ]:=
```
Select[anti4by4, #⟦1⟧ === -γ⟦5⟧ & → "Index"]
```

Out[ ]=
```
{6}
```

In[ ]:=
```
anti4by4⟦6⟧
-γ⟦5⟧
```

Out[ ]=
```
{{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
 {{16}, {1, 2, 3, 4}}}
```

Out[ ]=
```
{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}}
```

In[ ]:=

3. A BASIC LEMMA

Lemma: Let X be an arbitrary 4×4 matrix; then

$$\gamma^{56}X\gamma^{56}+\gamma^{64}\widetilde{X}\gamma^{64}+\gamma^{45}\widetilde{X}\gamma^{45} = X-\gamma_0 \operatorname{tr} X+\gamma^{56}\operatorname{tr}\gamma^{56}X, \quad (45)$$

where $\widetilde{X}$ denotes the transpose of X, and tr X is the trace of X. This identity is valid for any cyclic permutation of $(\gamma^{56},\gamma^{64},\gamma^{45})$, and under the replacement $\gamma^{56}\to\gamma^{12}$, $\gamma^{64}\to\gamma^{31}$, and $\gamma^{45}\to\gamma^{23}$.

Proof: Eq. (45) is linear in X; we verify that this equation is true for $X=\gamma_0$, γ^{56}, γ^{a5}, γ^{a6}, and $\gamma^{\alpha\beta}$. Note that only for $X=\gamma_0$ (resp γ^{56}) is tr X (resp tr $\gamma^{56}X$) nonvanishing.

(i) $X=\gamma_0$; since $(\gamma^{56})^2=-\gamma_0=(\gamma^{64})^2=(\gamma^{45})^2$, Eq. (45) yields

$$(\gamma^{56})^2+(\gamma^{64})^2+(\gamma^{45})^2=-3\gamma_0 = \gamma_0-\gamma_0\operatorname{tr}\gamma_0+\gamma^{56}\operatorname{tr}\gamma^{56}=\gamma_0-4\gamma_0;$$

In[ ]:= X

Out[ ]=

```
{x1, x2, x3, x4, x5, x6}
```

gamma

In[∘]:=
```
Table[{{A, B},
   FullSimplify[ExpandAll[1/2 (σ22⟦A⟧.OverBar[σ22]⟦B⟧ + σ22⟦B⟧.OverBar[σ22]⟦A⟧) ==
      η2244⟦A, B⟧ * IdentityMatrix[2]]]},
  {A, 1, 4}, {B, 1, 4}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

In[∘]:=
```
Do[gamma[j] = ArrayFlatten[{{0, OverBar[σ22]⟦j⟧}, {σ22⟦j⟧, 0}}], {j, 4}]
```

In[ ]:=
```
ggtxA0 = {{IdentityMatrix[4], {0}}};
Length[ggtxA0]
ggtxA = {};
Do[AppendTo[ggtxA, {gamma[j], {j}}], {j, 1, 4}]
Length[gggtxA]
ggtxAB = {};
Do[AppendTo[ggtxAB, {gamma[j].gamma[k], {j, k}}],
 {j, 1, 3}, {k, j + 1, 4}]
Length[ggtxAB]
ggtxABC = {};
Do[AppendTo[ggtxABC, {gamma[j].gamma[k].gamma[h], {j, k, h}}],
 {j, 1, 2}, {k, j + 1, 3}, {h, k + 1, 4}];
Length[ggtxABC]
ggtxABCD = {{gamma[1].gamma[2].gamma[3].gamma[4], {1, 2, 3, 4}}};
Length[ggtxABCD]
```

Out[ ]=
```
1
```

Out[ ]=
```
0
```

Out[ ]=
```
6
```

Out[ ]=
```
4
```

Out[ ]=
```
1
```

In[ ]:=
```
ggbase4by4 = Flatten[{ggtxA0, ggtxA, ggtxAB, ggtxABC, ggtxABCD}, 1];
Length[ggbase4by4]
```

Out[ ]=
```
16
```

In[ ]:=
```
DiracNash[#[[1]]] & /@ ggbase4by4
```

Out[ ]=
```
{True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True}
```

In[ ]:=
```
Table[{{j, k}, Block[{r},
    r = FullSimplify[1/2 gamma[j].gamma[k] + 1/2 gamma[k].gamma[j]];
    (*{r,If[r==0,Style[r,Red],Style[r,Darker[Green]]]}*)
    MatrixForm[r]]}, {j, 1, 4}, {k, 1, 4}] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix}
\begin{pmatrix}\{1,1\}\\ \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{1,2\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{1,3\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{1,4\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} \\
\begin{pmatrix}\{2,1\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{2,2\}\\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{2,3\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{2,4\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} \\
\begin{pmatrix}\{3,1\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{3,2\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{3,3\}\\ \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{3,4\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} \\
\begin{pmatrix}\{4,1\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{4,2\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{4,3\}\\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{pmatrix} &
\begin{pmatrix}\{4,4\}\\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\end{pmatrix}
\end{pmatrix}$$

In[∘]:=

```
Table[{{A, B}, Block[{r},
    r = FullSimplify[1/4 Tr[1/2 (ggbase4by4⟦A, 1⟧.ggbase4by4⟦B, 1⟧ +
            ggbase4by4⟦B, 1⟧.ggbase4by4⟦A, 1⟧)]];
    If[r == 0, Style[r, Red], Style[r, Darker[Green]]]]}, {A, 1,
   Length[ggbase4by4]}, {B, 1, Length[ggbase4by4]}] // MatrixForm
```

Out[∘]//MatrixForm=

{1, 1} 1	{1, 2} 0	{1, 3} 0	{1, 4} 0	{1, 5} 0	{1, 6} 0	{
{2, 1} 0	{2, 2} -1	{2, 3} 0	{2, 4} 0	{2, 5} 0	{2, 6} 0	{
{3, 1} 0	{3, 2} 0	{3, 3} 1	{3, 4} 0	{3, 5} 0	{3, 6} 0	{
{4, 1} 0	{4, 2} 0	{4, 3} 0	{4, 4} -1	{4, 5} 0	{4, 6} 0	{
{5, 1} 0	{5, 2} 0	{5, 3} 0	{5, 4} 0	{5, 5} 1	{5, 6} 0	{
{6, 1} 0	{6, 2} 0	{6, 3} 0	{6, 4} 0	{6, 5} 0	{6, 6} 1	{
{7, 1} 0	{7, 2} 0	{7, 3} 0	{7, 4} 0	{7, 5} 0	{7, 6} 0	{
{8, 1} 0	{8, 2} 0	{8, 3} 0	{8, 4} 0	{8, 5} 0	{8, 6} 0	{
{9, 1} 0	{9, 2} 0	{9, 3} 0	{9, 4} 0	{9, 5} 0	{9, 6} 0	{
{10, 1} 0	{10, 2} 0	{10, 3} 0	{10, 4} 0	{10, 5} 0	{10, 6} 0	{1
{11, 1} 0	{11, 2} 0	{11, 3} 0	{11, 4} 0	{11, 5} 0	{11, 6} 0	{1
{12, 1} 0	{12, 2} 0	{12, 3} 0	{12, 4} 0	{12, 5} 0	{12, 6} 0	{1
{13, 1} 0	{13, 2} 0	{13, 3} 0	{13, 4} 0	{13, 5} 0	{13, 6} 0	{1
{14, 1} 0	{14, 2} 0	{14, 3} 0	{14, 4} 0	{14, 5} 0	{14, 6} 0	{1
{15, 1} 0	{15, 2} 0	{15, 3} 0	{15, 4} 0	{15, 5} 0	{15, 6} 0	{1
{16, 1} 0	{16, 2} 0	{16, 3} 0	{16, 4} 0	{16, 5} 0	{16, 6} 0	{1

In[ ]:=
```
ggbase4by4[[16]][[1]]
```
Out[ ]=
```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}
```

In[ ]:=
```
gganti4by4 = {};
Do[If[ggbase4by4[[k, 1]] == -Transpose[ggbase4by4[[k, 1]]], AppendTo[
    gganti4by4, {ggbase4by4[[k, 1]], {{k}, ggbase4by4[[k, 2]]}}] ],
  {k, 1, Length[ggbase4by4]}];
Length[gganti4by4]
gganti4by4[[%]]
```
Out[ ]=
```
6
```
Out[ ]=
```
{{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}},
 {{14}, {1, 3, 4}}}
```

In[ ]:=
```
ggsymm4by4 = {};
Do[If[ggbase4by4[[k, 1]] == Transpose[ggbase4by4[[k, 1]]], AppendTo[
    ggsymm4by4, {ggbase4by4[[k, 1]], {{k}, ggbase4by4[[k, 2]]}}] ],
  {k, 1, Length[ggbase4by4]}];
Length[ggsymm4by4]
ggsymm4by4[[%]]
```
Out[ ]=
```
10
```
Out[ ]=
```
{{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}},
 {{16}, {1, 2, 3, 4}}}
```

In[ ]:=
```
DiagonalMatrix[
 Table[FullSimplify[1/4 Tr[ggsymm4by4[[A, 1]].ggsymm4by4[[A, 1]]]],
  {A, Length[ggsymm4by4]}]]
Tr[η16]
```

Out[ ]=
```
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}}
```

Out[ ]=
```
10
```

S44αβ commutation relations; misc :

$$S^{\alpha\beta} = -\tfrac{1}{4}[\gamma^\alpha,\gamma^\beta], \qquad (22)$$

$$S^{\alpha\beta} = -\tfrac{1}{2}\gamma^{\alpha\beta}. \qquad (23)$$

The symplectic form ϵ on D_4 is defined to be

$$\epsilon = \gamma^{64}. \qquad (24)$$

As a consequence of Eq. (1), and the definitions of Eqs. (16)–(24), are the identities

$$\tilde{\gamma}^\alpha\epsilon = -\epsilon\gamma^\alpha, \qquad (25)$$

$$\tilde{S}^{\alpha\beta}\epsilon = -\epsilon S^{\alpha\beta}, \qquad (26)$$

$$[S^{\alpha\beta},\gamma_\mu] = \delta^\alpha_\mu\gamma^\beta - \delta^\beta_\mu\gamma^\alpha, \qquad (27)$$

$$[S^{\alpha\beta},S^{\mu\nu}] = g^{\alpha\mu}S^{\beta\nu} - g^{\alpha\nu}S^{\beta\mu} - g^{\beta\mu}S^{\alpha\nu} + g^{\beta\nu}S^{\alpha\mu}, \qquad (28)$$

and

$$\gamma^5 S^{\alpha\beta} = \tfrac{1}{2}g^{\alpha\mu}g^{\beta\nu}\epsilon_{\mu\nu\lambda\sigma}S^{\lambda\sigma}. \qquad (29)$$

```
S44αβ =
  -1/4 Table[FullSimplify[γ[[h]].γ[[k]] - γ[[k]].γ[[h]]], {h, 1, 4}, {k, 1, 4}];
```

In[ ]:=
```
(*SAB=Table[1/4 (T16^A[A1].T16^A[B1]-T16^A[B1].T16^A[A1]),
   {A1,0, 7},{B1,0, 7}];*)
```

In[ ]:=
```
(*ParallelTable[
   FullSimplify[σ16.SAB⟦A1,B1⟧===-Transpose[σ16.SAB⟦A1,B1⟧]],
   {A1,1, 8},{B1,1,8}]//Flatten//Union*)
```

In[ ]:=
```
(*ParallelTable[
 {{A1,B1},FullSimplify[SAB⟦A1,B1⟧===-Transpose[SAB⟦A1,B1⟧]]},
 {A1,1, 8},{B1,1,8}]*)
```

In[ ]:=
```
S44αβ =
  -1/4 Table[FullSimplify[γ⟦h⟧.γ⟦k⟧ - γ⟦k⟧.γ⟦h⟧], {h, 1, 4}, {k, 1, 4}];
```

In[ ]:=
```
η44 = g_αβ
```

Out[ ]=
```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}}
```

In[ ]:=
```
Table[FullSimplify[
    S44αβ⟦A1, B1⟧.S44αβ⟦A2, B2⟧ - S44αβ⟦A2, B2⟧.S44αβ⟦A1, B1⟧ ==
     (η44⟦A1, A2⟧ × S44αβ⟦B1, B2⟧ - η44⟦A1, B2⟧ × S44αβ⟦B1, A2⟧ -
       η44⟦B1, A2⟧ × S44αβ⟦A1, B2⟧ + η44⟦B1, B2⟧ × S44αβ⟦A1, A2⟧)],
   {A1, 1, 3}, {B1, A1 + 1, 4}, {A2, 1, 3},
   {B2, A2 + 1, 4}] // Flatten // Union
```

Out[ ]=
```
{True}
```

In[ ]:=
```
(*ParallelTable[{{A1,B1,A2,B2},FullSimplify[
    S44αβ⟦A1,B1⟧.S44αβ⟦A2,B2⟧ -S44αβ⟦A2,B2⟧.S44αβ⟦A1,B1⟧==
     (η44⟦A1,A2⟧S44αβ⟦B1,B2⟧-η44⟦A1,B2⟧S44αβ⟦B1,A2⟧-
       η44⟦B1,A2⟧S44αβ⟦A1,B2⟧+η44⟦B1,B2⟧S44αβ⟦A1,A2⟧)]},
  {A1,1, 4},{B1,1,4},{A2,1, 4},{B2,1,4}]//MatrixForm*)
```

In[ ]:=
```
(*Parallel*)
Table[FullSimplify[S44αβ⟦A1, B1⟧.γ⟦B2⟧ - γ⟦B2⟧.S44αβ⟦A1, B1⟧ ==
     (η44⟦B2, A1⟧ × γ⟦B1⟧ - η44⟦B2, B1⟧ × γ⟦A1⟧)],
   {A1, 1, 4}, {B1, 1, 4}, {B2, 1, 4}] // Flatten // Union
```

Out[ ]=
```
{True}
```

In[ ]:=
```
(*ParallelTable[{{A1,B1,B2},
   FullSimplify[S44αβ⟦A1,B1⟧.γ⟦B2⟧ -γ⟦B2⟧.S44αβ⟦A1,B1⟧==
     (η44⟦B2,A1⟧γ⟦B1⟧-η44⟦B2,B1⟧γ⟦A1⟧)]},
  {A1,1, 4},{B1,1,4},{B2,1,4}]//MatrixForm*)
```

In[ ]:=
```
sϵ
(ϵ /. sϵ)
% === γ⟦4⟧
```

Out[ ]=
```
{ϵ → {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}}}
```

Out[ ]=
```
{{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}}
```

Out[ ]=
```
True
```

In[ ]:=
```
Table[
 {{j}, FullSimplify[(ϵ /. sϵ).γ⟦j⟧ === Transpose[(ϵ /. sϵ).γ⟦j⟧]]},
 {j, 1, 5}]
```

Out[ ]=
```
{{{1}, True}, {{2}, True}, {{3}, True}, {{4}, True}, {{5}, False}}
```

```
(*Parallel*)Table[{{A1, B1}, FullSimplify[
    (ϵ /. sϵ).S44αβ⟦A1, B1⟧ === Transpose[(ϵ /. sϵ).S44αβ⟦A1, B1⟧]]},
 {A1, 1, 3}, {B1, A1 + 1, 4}]
```

Out[ ]=
```
{{{{1, 2}, True}, {{1, 3}, True}, {{1, 4}, True}},
 {{{2, 3}, True}, {{2, 4}, True}}, {{{3, 4}, True}}}
```

In[∘]:=
```
(*Parallel*)Table[{{A1, B1},
  FullSimplify[FullSimplify[(ϵ /. sϵ).(ⅈ * S44αβ⟦A1, B1⟧)] -
    FullSimplify[-Transpose[(ϵ /. sϵ).(ⅈ * S44αβ⟦A1, B1⟧)]*]]},
 {A1, 1, 3}, {B1, A1 + 1, 4}]
```

Out[∘]=
```
{{{{1, 2}, {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}},
  {{1, 3}, {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}},
  {{1, 4}, {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}},
 {{{2, 3}, {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}},
  {{2, 4}, {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}},
 {{{3, 4}, {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}}}
```

In[∘]:=
```
(*Parallel*)Table[{{A1, B1},
  FullSimplify[FullSimplify[(ϵ /. sϵ).(ⅈ * S44αβ⟦A1, B1⟧)] ===
    FullSimplify[-Transpose[(ϵ /. sϵ).(ⅈ * S44αβ⟦A1, B1⟧)]*]]},
 {A1, 1, 3}, {B1, A1 + 1, 4}]
```

Out[∘]=
```
{{{{1, 2}, True}, {{1, 3}, True}, {{1, 4}, True}},
 {{{2, 3}, True}, {{2, 4}, True}}, {{{3, 4}, True}}}
```

In[∘]:=
```
(ϵ /. sϵ).(ⅈ * S44αβ⟦1, 2⟧) // FullSimplify // MatrixForm
Transpose[(ϵ /. sϵ).(ⅈ * S44αβ⟦1, 2⟧)]* // FullSimplify // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & \frac{i}{2} \\ 0 & 0 & -\frac{i}{2} & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & \frac{i}{2} & 0 \\ 0 & \frac{i}{2} & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 \end{pmatrix}$$

The commutation relations for the generators J_{ab} of the $\mathfrak{so}(3,3)$ Lie algebra (using the physics convention where generators are Hermitian) are given by the formula:

$$[J_{ab}, J_{cd}] = i(\eta_{ac} J_{bd} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac})$$

where:

- The indices a, b, c, d range from 1 to 6.
- η_{ab} is the **metric tensor** with signature $(+,+,+,-,-,-)$ (or the reverse, depending on the convention). A common choice is a diagonal matrix $\eta = \mathbf{diag}(1,1,1,-1,-1,-1)$.
- J_{ab} are the **generators** of the algebra, which are antisymmetric, $J_{ab} = -J_{ba}$.
- The terms on the right-hand side involve Kronecker deltas weighted by the metric, ensuring the resulting generator is also an element of the algebra and satisfies the required antisymmetry and Jacobi identity properties.

The final answer is $\mathbf{[J_{ab}, J_{cd}] = i(\eta_{ac} J_{bd} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac})}$.

Step 1: Define the generators and metric

The **so(4,4)** Lie algebra has 28 generators, which can be denoted as $J_{\mu\nu}$ where the indices μ, ν range from 0 to 7, and $J_{\mu\nu} = -J_{\nu\mu}$. The Lie algebra is defined in an 8-dimensional real vector space equipped with a metric $\eta_{\mu\nu}$ with signature (4, 4).

- **Generators:** $J_{\mu\nu}$ with $\mu, \nu \in \{0, 1, \ldots, 7\}$
- **Metric tensor:** $\eta_{\mu\nu} = \mathrm{diag}(-1,-1,-1,-1,+1,+1,+1,+1)$ (or the opposite sign convention)

Step 2: State the general commutation relation

The commutation relations for the generators of the **so(p,q)** Lie algebra are a generalization of the **so(n)** algebra relations, incorporating the metric tensor $\eta_{\mu\nu}$.

Answer:

The commutation relations for the generators $J_{\mu\nu}$ of the **so(4,4)** Lie algebra are:

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho}$$

where $\eta_{\mu\nu}$ is the metric tensor with a (4,4) signature (four plus signs and four minus signs, or vice versa).

(Note: In some physics conventions, an extra factor of i or $i\hbar$ is included depending on whether the generators are defined as Hermitian or not).

what are the commutation relation for the generators of so(4,4)?

The commutation relations for the generators $J_{\mu\nu}$ of the $so(4,4)$ Lie algebra, where $\mu, \nu \in \{0, 1, \ldots, 7\}$ and $J_{\mu\nu} = -J_{\nu\mu}$, are the standard relations for orthogonal Lie algebras, given by a generalization of the angular momentum commutation relations:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$

where $\eta_{\mu\nu}$ is the flat metric of 8-dimensional Minkowski space with signature (4,4) (or some other convention, the structure constants are the same as for $so(8)$ in an appropriate basis), which is a diagonal matrix with four positive and four negative entries (e.g., $\eta = \text{diag}(+1, +1, +1, +1, -1, -1, -1, -1)$).

↪ **Why is the metric signature significant for $so(4,4)$ commutation relations?**

Why is the metric signature significant for $so(4,4)$ commutation relations?

The metric signature is significant for the $so(p, q)$ commutation relations because the metric tensor $\eta_{\mu\nu}$ acts as the **structure constants** of the Lie algebra, defining its specific algebraic structure and physical properties (if applicable).

Here is a detailed explanation of its significance:

- **Definition of the Algebra:** The commutation relation for the generators $J_{\mu\nu}$ of any orthogonal Lie algebra $so(p, q)$ is given by:
  $[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$
  The values of the structure constants depend directly on the components of the metric $\eta_{\mu\nu}$. This means that the specific algebraic properties of $so(4, 4)$ are distinct from $so(8)$ (which has a (8,0) signature, i.e., a Euclidean metric), even though they operate in the same 8-dimensional vector space.

- **Physical Interpretation:** In physics, the metric signature determines the nature of spacetime. A mixed signature (like (4,4) or the more common (1,3) Minkowski signature) implies a causal structure, distinguishing between timelike and spacelike dimensions.
  - Generators associated with rotations in spacelike dimensions (positive $\eta_{\mu\nu}$ entries) typically lead to compact symmetry groups (like $SO(4)$ or $SO(8)$).

In[ ]:=
```
η4488
```

Out[ ]=
```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0},
 {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}}
```

orthogonal Lie algebra $so(p,q)$ is given by:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$

$$t^A = \begin{pmatrix} 0 & \overline{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

In[ ]:=
```
(*Table[
 SAB⟦A1,B1⟧ == ArrayFlatten[{{SAB1⟦A1,B1⟧,0},{0,SAB2⟦A1,B1⟧}}],
 {A1,0, 7},{B1,0,7}]*)
```

$$4D_{(1)}{}^{AB} = \overline{\tau}^A\tau^B - \overline{\tau}^B\tau^A$$

$$4D_{(2)}{}^{AB} = \tau^A\overline{\tau}^B - \tau^B\overline{\tau}^A.$$

Let $g \in \overline{SO(4,4)}$ and $L \in SO(4,4)$. The canonical 2-1 homomorphism $\overline{SO(4,4)} \to SO(4,4)$ is given by

(2.8) $$8L^{A'}{}_{B'}(g) = \mathrm{tr}\,\widetilde{D}^{(2)}(g)\,\tau^{A'}\,D^{(1)}(g)\,\overline{\tau}_{B'}\,.$$

As usual tr denotes the trace. Under the action of $\overline{SO(4,4)}$

(2.9) $$\widetilde{D}^{(1)}\sigma = \sigma D^{(1)-1},$$

(2.10) $$\sigma^{-1}\widetilde{D}^{(2)} = D^{(2)-1}\sigma^{-1},$$

(2.11) $$L^{A'}{}_{C'}\,G_{A'B'}\,L^{B'}{}_{D'} = G_{C'D'},$$

(2.12) $$L^{A'}{}_{B'}\,\tau^{B'} = \widetilde{D}^{(2)}\,\tau^{A'}\,D^{(1)},$$

(2.13) $$L^{A'}{}_{B'}\,\bar{\tau}^{B'} = D^{(1)-1}\,\bar{\tau}^{A'}\,\widetilde{D}^{(2)-1}\,.$$

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A Remarkable Representation of the 3 + 2 de Sitter Group

P. A. M. Dirac
Cambridge University, Cambridge, England and
*Institute for Advanced Study, Princeton, New Jersey**
(Received 20 February 1963)

Identities satisfied by the generators of the Dirac algebra

Patrick L. Nash
M. S. 474, NASA Langley Research Center, Hampton, Virginia 23665

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In[∘]:=
```
Unprotect[γABs];
γABs = Flatten[{s4by4[#] & /@ Range[3], t4by4[#] & /@ Range[3],
   Flatten[Table[st[J, K], {J, 1, 3}, {K, 1, 3}], 1], {ID4}}, 1]
```
Out[∘]=
```
{{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
 {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}}, st[1, 1],
 st[1, 2], st[1, 3], st[2, 1], st[2, 2], st[2, 3], st[3, 1], st[3, 2],
 st[3, 3], {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}
```

In[∘]:=
```
Length[γABs]
```
Out[∘]=
```
16
```

In[∘]:=
```
Protect[γABs]
```
Out[∘]=
```
{γABs}
```

In[∘]:=
```
γABs[[1]]
```
Out[∘]=
```
{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}}
```

In[ ]:= γABs[[-1]]

Out[ ]=

$$\{\{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\}$$

Lagrangian

In[ ]:= ψim = psi2im

Out[ ]=

$$\Big\{\text{h8}[1][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + i\,\text{h8c}[1][\text{x1}, \text{x2}, \text{x3}, \text{x4}],\ \frac{1}{2}\big((2+i)\,\text{h8}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - \text{h8}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - (1-2\,i)\,\text{h8c}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - i\,\text{h8c}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}]\big),\ \text{h8}[3][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + i\,\text{h8c}[3][\text{x1}, \text{x2}, \text{x3}, \text{x4}],\ \frac{1}{2}\big(\text{h8}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + (2+i)\,\text{h8}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + i\,(\text{h8c}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + (2+i)\,\text{h8c}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}])\big)\Big\}$$

In[ ]:= ψcc = psi2cc

Out[ ]=

$$\{-\text{h8}[1][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - i\,\text{h8c}[1][\text{x1}, \text{x2}, \text{x3}, \text{x4}],\ (-1-i)\,\text{h8}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + \text{h8}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + (1-i)\,\text{h8c}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + i\,\text{h8c}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}],\ -\text{h8}[3][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - i\,\text{h8c}[3][\text{x1}, \text{x2}, \text{x3}, \text{x4}],\ -\text{h8}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - (1+i)\,\text{h8}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - i\,(\text{h8c}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + (1+i)\,\text{h8c}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}])\}$$

In[ ]:= ψccc = psi2cc*

Out[ ]=

$$\{-\text{h8}[1][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + i\,\text{h8c}[1][\text{x1}, \text{x2}, \text{x3}, \text{x4}],\ (-1+i)\,\text{h8}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + \text{h8}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + (1+i)\,\text{h8c}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - i\,\text{h8c}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}],\ -\text{h8}[3][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + i\,\text{h8c}[3][\text{x1}, \text{x2}, \text{x3}, \text{x4}],\ -\text{h8}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] - (1-i)\,\text{h8}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + i\,(\text{h8c}[2][\text{x1}, \text{x2}, \text{x3}, \text{x4}] + (1-i)\,\text{h8c}[4][\text{x1}, \text{x2}, \text{x3}, \text{x4}])\}$$

```
In[ ]:= (* Kinetic
         term: psi1Imag^T . σ8 . Sum[tauBar[A].D[psi2,X⟦A+1⟧],{A,0,7}] *)
        L8kinetic = Transpose[psi2*].σ8.Sum[γ⟦A⟧.D[psi2, X⟦A⟧], {A, 1, 4}];
        L8kinetic12 =
          Transpose[psi1].σ8.Sum[γ⟦A⟧.D[psi2, X⟦A⟧], {A, 1, 4}];
        L8kineticim = Transpose[ψim].σ8.Sum[γ⟦A⟧.D[psi2, X⟦A⟧], {A, 1, 4}];
        L8kineticcc = Transpose[ψcc].σ8.Sum[γ⟦A⟧.D[psi2, X⟦A⟧], {A, 1, 4}];
        L8kineticccc =
          Transpose[ψcc*].σ8.Sum[γ⟦A⟧.D[psi2, X⟦A⟧], {A, 1, 4}];

In[ ]:= L8mass = M*Transpose[psi2*].σ8.psi2;
        L8mass12 = M*Transpose[psi1].σ8.psi2;
        L8massim = M*Transpose[ψim].σ8.psi2;
        L8masscc = M* Transpose[ψcc].σ8.psi2;
        L8massccc = M*Transpose[ψcc*].σ8.psi2;
```

```
In[]:= L812 = L8mass12 + L8kinetic12
Out[]=
M ((-f8[3][x1, x2, x3, x4] - ⅈ f8c[3][x1, x2, x3, x4])
      (h8[1][x1, x2, x3, x4] + ⅈ h8c[1][x1, x2, x3, x4]) +
    (-f8[4][x1, x2, x3, x4] - ⅈ f8c[4][x1, x2, x3, x4])
      (h8[2][x1, x2, x3, x4] + ⅈ h8c[2][x1, x2, x3, x4]) +
    (f8[1][x1, x2, x3, x4] + ⅈ f8c[1][x1, x2, x3, x4])
      (h8[3][x1, x2, x3, x4] + ⅈ h8c[3][x1, x2, x3, x4]) +
    (f8[2][x1, x2, x3, x4] + ⅈ f8c[2][x1, x2, x3, x4])
      (h8[4][x1, x2, x3, x4] + ⅈ h8c[4][x1, x2, x3, x4])) +
 (-f8[3][x1, x2, x3, x4] - ⅈ f8c[3][x1, x2, x3, x4])
  (-h8[3]^(0,0,0,1)[x1, x2, x3, x4] - ⅈ h8c[3]^(0,0,0,1)[x1, x2, x3, x4] +
    h8[3]^(0,0,1,0)[x1, x2, x3, x4] + ⅈ h8c[3]^(0,0,1,0)[x1, x2, x3, x4] -
    h8[2]^(0,1,0,0)[x1, x2, x3, x4] - ⅈ h8c[2]^(0,1,0,0)[x1, x2, x3, x4] -
    h8[1]^(1,0,0,0)[x1, x2, x3, x4] - ⅈ h8c[1]^(1,0,0,0)[x1, x2, x3, x4]) +
 (-f8[4][x1, x2, x3, x4] - ⅈ f8c[4][x1, x2, x3, x4])
  (-h8[4]^(0,0,0,1)[x1, x2, x3, x4] - ⅈ h8c[4]^(0,0,0,1)[x1, x2, x3, x4] +
    h8[4]^(0,0,1,0)[x1, x2, x3, x4] + ⅈ h8c[4]^(0,0,1,0)[x1, x2, x3, x4] -
    h8[1]^(0,1,0,0)[x1, x2, x3, x4] - ⅈ h8c[1]^(0,1,0,0)[x1, x2, x3, x4] +
    h8[2]^(1,0,0,0)[x1, x2, x3, x4] + ⅈ h8c[2]^(1,0,0,0)[x1, x2, x3, x4]) +
 (f8[1][x1, x2, x3, x4] + ⅈ f8c[1][x1, x2, x3, x4])
  (h8[1]^(0,0,0,1)[x1, x2, x3, x4] + ⅈ h8c[1]^(0,0,0,1)[x1, x2, x3, x4] +
    h8[1]^(0,0,1,0)[x1, x2, x3, x4] + ⅈ h8c[1]^(0,0,1,0)[x1, x2, x3, x4] +
    h8[4]^(0,1,0,0)[x1, x2, x3, x4] + ⅈ h8c[4]^(0,1,0,0)[x1, x2, x3, x4] +
    h8[3]^(1,0,0,0)[x1, x2, x3, x4] + ⅈ h8c[3]^(1,0,0,0)[x1, x2, x3, x4]) +
 (f8[2][x1, x2, x3, x4] + ⅈ f8c[2][x1, x2, x3, x4])
  (h8[2]^(0,0,0,1)[x1, x2, x3, x4] + ⅈ h8c[2]^(0,0,0,1)[x1, x2, x3, x4] +
    h8[2]^(0,0,1,0)[x1, x2, x3, x4] + ⅈ h8c[2]^(0,0,1,0)[x1, x2, x3, x4] +
    h8[3]^(0,1,0,0)[x1, x2, x3, x4] + ⅈ h8c[3]^(0,1,0,0)[x1, x2, x3, x4] -
    h8[4]^(1,0,0,0)[x1, x2, x3, x4] - ⅈ h8c[4]^(1,0,0,0)[x1, x2, x3, x4])
```

In[ ]:=
```
(* Full L8 Lagrangian *)
L8 = L8mass + L8kinetic
L8im = L8massim + L8kineticim
L8cc = 1 / 2 * L8masscc + L8kineticcc
(*Print["\n=== Full L8 Lagrangian ==="]
 Print[Simplify[L8]]*)
```

Out[ ]=

$$
\begin{aligned}
&M\,\big((h8[1][x1, x2, x3, x4] + i\,h8c[1][x1, x2, x3, x4])\\
&\quad(-h8[3][x1, x2, x3, x4] + i\,h8c[3][x1, x2, x3, x4]) +\\
&\quad(h8[1][x1, x2, x3, x4] - i\,h8c[1][x1, x2, x3, x4])\\
&\quad(h8[3][x1, x2, x3, x4] + i\,h8c[3][x1, x2, x3, x4]) +\\
&\quad(h8[2][x1, x2, x3, x4] + i\,h8c[2][x1, x2, x3, x4])\\
&\quad(-h8[4][x1, x2, x3, x4] + i\,h8c[4][x1, x2, x3, x4]) +\\
&\quad(h8[2][x1, x2, x3, x4] - i\,h8c[2][x1, x2, x3, x4])\\
&\quad(h8[4][x1, x2, x3, x4] + i\,h8c[4][x1, x2, x3, x4])\big) +\\
&(-h8[3][x1, x2, x3, x4] + i\,h8c[3][x1, x2, x3, x4])\\
&\Big(-h8[3]^{(0,0,0,1)}[x1, x2, x3, x4] - i\,h8c[3]^{(0,0,0,1)}[x1, x2, x3, x4] +\\
&\quad h8[3]^{(0,0,1,0)}[x1, x2, x3, x4] + i\,h8c[3]^{(0,0,1,0)}[x1, x2, x3, x4] -\\
&\quad h8[2]^{(0,1,0,0)}[x1, x2, x3, x4] - i\,h8c[2]^{(0,1,0,0)}[x1, x2, x3, x4] -\\
&\quad h8[1]^{(1,0,0,0)}[x1, x2, x3, x4] - i\,h8c[1]^{(1,0,0,0)}[x1, x2, x3, x4]\Big) +\\
&(-h8[4][x1, x2, x3, x4] + i\,h8c[4][x1, x2, x3, x4])\\
&\Big(-h8[4]^{(0,0,0,1)}[x1, x2, x3, x4] - i\,h8c[4]^{(0,0,0,1)}[x1, x2, x3, x4] +\\
&\quad h8[4]^{(0,0,1,0)}[x1, x2, x3, x4] + i\,h8c[4]^{(0,0,1,0)}[x1, x2, x3, x4] -\\
&\quad h8[1]^{(0,1,0,0)}[x1, x2, x3, x4] - i\,h8c[1]^{(0,1,0,0)}[x1, x2, x3, x4] +\\
&\quad h8[2]^{(1,0,0,0)}[x1, x2, x3, x4] + i\,h8c[2]^{(1,0,0,0)}[x1, x2, x3, x4]\Big) +\\
&(h8[1][x1, x2, x3, x4] - i\,h8c[1][x1, x2, x3, x4])\\
&\Big(h8[1]^{(0,0,0,1)}[x1, x2, x3, x4] + i\,h8c[1]^{(0,0,0,1)}[x1, x2, x3, x4] +\\
&\quad h8[1]^{(0,0,1,0)}[x1, x2, x3, x4] + i\,h8c[1]^{(0,0,1,0)}[x1, x2, x3, x4] +\\
&\quad h8[4]^{(0,1,0,0)}[x1, x2, x3, x4] + i\,h8c[4]^{(0,1,0,0)}[x1, x2, x3, x4] +\\
&\quad h8[3]^{(1,0,0,0)}[x1, x2, x3, x4] + i\,h8c[3]^{(1,0,0,0)}[x1, x2, x3, x4]\Big) +\\
&(h8[2][x1, x2, x3, x4] - i\,h8c[2][x1, x2, x3, x4])\\
&\Big(h8[2]^{(0,0,0,1)}[x1, x2, x3, x4] + i\,h8c[2]^{(0,0,0,1)}[x1, x2, x3, x4] +\\
&\quad h8[2]^{(0,0,1,0)}[x1, x2, x3, x4] + i\,h8c[2]^{(0,0,1,0)}[x1, x2, x3, x4] +\\
&\quad h8[3]^{(0,1,0,0)}[x1, x2, x3, x4] + i\,h8c[3]^{(0,1,0,0)}[x1, x2, x3, x4] -\\
&\quad h8[4]^{(1,0,0,0)}[x1, x2, x3, x4] - i\,h8c[4]^{(1,0,0,0)}[x1, x2, x3, x4]\Big)
\end{aligned}
$$

Out[ ]=

$$
\begin{aligned}
&M\,\Big((h8[1][x1, x2, x3, x4] + i\,h8c[1][x1, x2, x3, x4])\\
&\quad(-h8[3][x1, x2, x3, x4] - i\,h8c[3][x1, x2, x3, x4]) +\\
&\quad(h8[1][x1, x2, x3, x4] + i\,h8c[1][x1, x2, x3, x4])\\
&\quad(h8[3][x1, x2, x3, x4] + i\,h8c[3][x1, x2, x3, x4]) +
\end{aligned}
$$

$$\frac{1}{2}\left((2+i)\,h8[2][x1,x2,x3,x4]-h8[4][x1,x2,x3,x4]-(1-2i)\,h8c[2][x1,x2,x3,x4]-i\,h8c[4][x1,x2,x3,x4]\right)$$
$$(h8[4][x1,x2,x3,x4]+i\,h8c[4][x1,x2,x3,x4])+$$
$$\frac{1}{2}\left(h8[2][x1,x2,x3,x4]+i\,h8c[2][x1,x2,x3,x4]\right)$$
$$\left.\left(-h8[2][x1,x2,x3,x4]-(2+i)\,h8[4][x1,x2,x3,x4]-i\,(h8c[2][x1,x2,x3,x4]+(2+i)\,h8c[4][x1,x2,x3,x4])\right)\right)+$$
$$(-h8[3][x1,x2,x3,x4]-i\,h8c[3][x1,x2,x3,x4])$$
$$\left(-h8[3]^{(0,0,0,1)}[x1,x2,x3,x4]-i\,h8c[3]^{(0,0,0,1)}[x1,x2,x3,x4]+h8[3]^{(0,0,1,0)}[x1,x2,x3,x4]+i\,h8c[3]^{(0,0,1,0)}[x1,x2,x3,x4]-h8[2]^{(0,1,0,0)}[x1,x2,x3,x4]-i\,h8c[2]^{(0,1,0,0)}[x1,x2,x3,x4]-h8[1]^{(1,0,0,0)}[x1,x2,x3,x4]-i\,h8c[1]^{(1,0,0,0)}[x1,x2,x3,x4]\right)+$$
$$\frac{1}{2}\left(-h8[2][x1,x2,x3,x4]-(2+i)\,h8[4][x1,x2,x3,x4]-i\,(h8c[2][x1,x2,x3,x4]+(2+i)\,h8c[4][x1,x2,x3,x4])\right)$$
$$\left(-h8[4]^{(0,0,0,1)}[x1,x2,x3,x4]-i\,h8c[4]^{(0,0,0,1)}[x1,x2,x3,x4]+h8[4]^{(0,0,1,0)}[x1,x2,x3,x4]+i\,h8c[4]^{(0,0,1,0)}[x1,x2,x3,x4]-h8[1]^{(0,1,0,0)}[x1,x2,x3,x4]-i\,h8c[1]^{(0,1,0,0)}[x1,x2,x3,x4]+h8[2]^{(1,0,0,0)}[x1,x2,x3,x4]+i\,h8c[2]^{(1,0,0,0)}[x1,x2,x3,x4]\right)+$$
$$(h8[1][x1,x2,x3,x4]+i\,h8c[1][x1,x2,x3,x4])$$
$$\left(h8[1]^{(0,0,0,1)}[x1,x2,x3,x4]+i\,h8c[1]^{(0,0,0,1)}[x1,x2,x3,x4]+h8[1]^{(0,0,1,0)}[x1,x2,x3,x4]+i\,h8c[1]^{(0,0,1,0)}[x1,x2,x3,x4]+h8[4]^{(0,1,0,0)}[x1,x2,x3,x4]+i\,h8c[4]^{(0,1,0,0)}[x1,x2,x3,x4]+h8[3]^{(1,0,0,0)}[x1,x2,x3,x4]+i\,h8c[3]^{(1,0,0,0)}[x1,x2,x3,x4]\right)+$$
$$\frac{1}{2}\left((2+i)\,h8[2][x1,x2,x3,x4]-h8[4][x1,x2,x3,x4]-(1-2i)\,h8c[2][x1,x2,x3,x4]-i\,h8c[4][x1,x2,x3,x4]\right)$$
$$\left(h8[2]^{(0,0,0,1)}[x1,x2,x3,x4]+i\,h8c[2]^{(0,0,0,1)}[x1,x2,x3,x4]+h8[2]^{(0,0,1,0)}[x1,x2,x3,x4]+i\,h8c[2]^{(0,0,1,0)}[x1,x2,x3,x4]+h8[3]^{(0,1,0,0)}[x1,x2,x3,x4]+i\,h8c[3]^{(0,1,0,0)}[x1,x2,x3,x4]-h8[4]^{(1,0,0,0)}[x1,x2,x3,x4]-i\,h8c[4]^{(1,0,0,0)}[x1,x2,x3,x4]\right)$$

Out[ ]=

$$
\begin{aligned}
&\frac{1}{2} M \big((-h8[1][x1, x2, x3, x4] - \mathbb{i}\, h8c[1][x1, x2, x3, x4]) \\
&\quad (h8[3][x1, x2, x3, x4] + \mathbb{i}\, h8c[3][x1, x2, x3, x4]) + \\
&\quad (h8[1][x1, x2, x3, x4] + \mathbb{i}\, h8c[1][x1, x2, x3, x4]) \\
&\quad (h8[3][x1, x2, x3, x4] + \mathbb{i}\, h8c[3][x1, x2, x3, x4]) + \\
&\quad (h8[4][x1, x2, x3, x4] + \mathbb{i}\, h8c[4][x1, x2, x3, x4]) \\
&\quad ((-1 - \mathbb{i})\, h8[2][x1, x2, x3, x4] + h8[4][x1, x2, x3, x4] + \\
&\qquad (1 - \mathbb{i})\, h8c[2][x1, x2, x3, x4] + \mathbb{i}\, h8c[4][x1, x2, x3, x4]) + \\
&\quad (h8[2][x1, x2, x3, x4] + \mathbb{i}\, h8c[2][x1, x2, x3, x4]) \\
&\quad (h8[2][x1, x2, x3, x4] + (1 + \mathbb{i})\, h8[4][x1, x2, x3, x4] + \\
&\qquad \mathbb{i}\, (h8c[2][x1, x2, x3, x4] + (1 + \mathbb{i})\, h8c[4][x1, x2, x3, x4]))) + \\
&(h8[3][x1, x2, x3, x4] + \mathbb{i}\, h8c[3][x1, x2, x3, x4]) \\
&\big(-h8[3]^{(0,0,0,1)}[x1, x2, x3, x4] - \mathbb{i}\, h8c[3]^{(0,0,0,1)}[x1, x2, x3, x4] + \\
&\quad h8[3]^{(0,0,1,0)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[3]^{(0,0,1,0)}[x1, x2, x3, x4] - \\
&\quad h8[2]^{(0,1,0,0)}[x1, x2, x3, x4] - \mathbb{i}\, h8c[2]^{(0,1,0,0)}[x1, x2, x3, x4] - \\
&\quad h8[1]^{(1,0,0,0)}[x1, x2, x3, x4] - \mathbb{i}\, h8c[1]^{(1,0,0,0)}[x1, x2, x3, x4]\big) + \\
&(h8[2][x1, x2, x3, x4] + (1 + \mathbb{i})\, h8[4][x1, x2, x3, x4] + \\
&\quad \mathbb{i}\, (h8c[2][x1, x2, x3, x4] + (1 + \mathbb{i})\, h8c[4][x1, x2, x3, x4])) \\
&\big(-h8[4]^{(0,0,0,1)}[x1, x2, x3, x4] - \mathbb{i}\, h8c[4]^{(0,0,0,1)}[x1, x2, x3, x4] + \\
&\quad h8[4]^{(0,0,1,0)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[4]^{(0,0,1,0)}[x1, x2, x3, x4] - \\
&\quad h8[1]^{(0,1,0,0)}[x1, x2, x3, x4] - \mathbb{i}\, h8c[1]^{(0,1,0,0)}[x1, x2, x3, x4] + \\
&\quad h8[2]^{(1,0,0,0)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[2]^{(1,0,0,0)}[x1, x2, x3, x4]\big) + \\
&(-h8[1][x1, x2, x3, x4] - \mathbb{i}\, h8c[1][x1, x2, x3, x4]) \\
&\big(h8[1]^{(0,0,0,1)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[1]^{(0,0,0,1)}[x1, x2, x3, x4] + \\
&\quad h8[1]^{(0,0,1,0)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[1]^{(0,0,1,0)}[x1, x2, x3, x4] + \\
&\quad h8[4]^{(0,1,0,0)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[4]^{(0,1,0,0)}[x1, x2, x3, x4] + \\
&\quad h8[3]^{(1,0,0,0)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[3]^{(1,0,0,0)}[x1, x2, x3, x4]\big) + \\
&((-1 - \mathbb{i})\, h8[2][x1, x2, x3, x4] + h8[4][x1, x2, x3, x4] + \\
&\quad (1 - \mathbb{i})\, h8c[2][x1, x2, x3, x4] + \mathbb{i}\, h8c[4][x1, x2, x3, x4]) \\
&\big(h8[2]^{(0,0,0,1)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[2]^{(0,0,0,1)}[x1, x2, x3, x4] + \\
&\quad h8[2]^{(0,0,1,0)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[2]^{(0,0,1,0)}[x1, x2, x3, x4] + \\
&\quad h8[3]^{(0,1,0,0)}[x1, x2, x3, x4] + \mathbb{i}\, h8c[3]^{(0,1,0,0)}[x1, x2, x3, x4] - \\
&\quad h8[4]^{(1,0,0,0)}[x1, x2, x3, x4] - \mathbb{i}\, h8c[4]^{(1,0,0,0)}[x1, x2, x3, x4]\big)
\end{aligned}
$$

In[∘]:=
```
Clear[eL8];
eL8[l_] := Module[{L, t, t2, useDSQRT = 1},
  L = l; (*Lagr[j];*)
  t = (*Parallel*)Table[(*Full*)
    Simplify[1/useDSQRT (D[L, f8[k][x1, x2, x3, x4, x5, x6]] -
        Sum[D[D[L, D[f8[k][x1, x2, x3, x4, x5, x6], X[[j]]]], X[[j]]],
         {j, 1, 4}])], {k, 4}];

  t2 = (*Parallel*)Table[(*Full*)
    Simplify[1/useDSQRT (D[L, h8[k][x1, x2, x3, x4, x5, x6]] -
        Sum[D[D[L, D[h8[k][x1, x2, x3, x4, x5, x6], X[[j]]]], X[[j]]],
         {j, 1, 4}])], {k, 4}];

  Return[Flatten[{t, t2}]];
 ]
```

In[∘]:=
```
eL8[L812]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0}
```

In[∘]:=
```
eL8[L8]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0}
```

In[∘]:=
```
eL8[L8im]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0}
```

In[∘]:=
```
eL8[L8cc]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0}
```