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On the Structure of the Split Octonion Algebra.

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Summary. — The known equivalence of special spinors and vector-scalar sets is discussed within the context of the algebra of the split octonions. One implication of this equivalence is that the usual Dirac spinor field can be recast as a vector-scalar field, and this construction is outlined. A process of structure constant factorization is illustrated by the realization of the split octonion multiplication constants (with respect to a spinor basis) as products of certain matrix generators and an arbitrary normalized spinor.

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1. – Introduction.

It is well known that the basic spinor representation of the complex orthogonal group $O(2n, \boldsymbol{C})$ may be constructed from the irreducible generators t_i, $i = 1, \ldots, 2n$ of the Clifford algebra C_{2n}. These generators anticommute and have square equal to $+1$ [1, 2]. The antisymmetrized products of pairs of these generators (times a constant factor of 1/4), when computed from an irreducible representation, generate the basic spinor representation of the full group $O(2n, \boldsymbol{C})$. This representation is of degree 2^n and reducible with respect to $SO(2n, \boldsymbol{C})$. Two irreducible $SO(2n, \boldsymbol{C})$ representations of degree $(1/2)\,2^n$ comprise this faithful representation of $O(2n, \boldsymbol{C})$, namely the spinor representations of the first and second kind determined, respectively, by the weights $m_+ = (1/2, \ldots, 1/2)$ [n factors] and $m_- = (1/2, \ldots, 1/2, -1/2)$ [n factors]. Lord [3] has indicated a general procedure for constructing the spinor representations of $SO(2n)$ of the first and second kind from the generators of C_{2n-2}. We shall call such irreducible C_{2n-2} generators «reduced Brauer-Weyl generators».

In this paper we shall discuss a fundamental relationship between the reduced Brauer-Weyl generators of a certain pseudo-orthogonal subgroup of $SL(8, \boldsymbol{R})$ and the multiplication constants of the split octonion algebra over the reals. It will be seen that every choice of a suitably normalized $\boldsymbol{R}^8$ spinor leads directly to a set of multiplication constants for the split octonions whose algebraic properties are governed by these reduced Brauer-Weyl generators.

The reduced generators will also be seen to provide an elegant formulation of Cartan's principle of triality [4] (*i.e.* the equivalence of $\boldsymbol{R}^8$ «vectors», «semi-spinors of the first type» and «semi-spinors of the second type» in the terminology of Cartan) in terms of an orthonormal frame determined by these same generators and the fiducial $\boldsymbol{R}^8$ spinor.

Many researchers have incorporated the nonassociative alternative octonion algebra $\boldsymbol{O}$ (the algebra of the Cayley numbers) into the structure of a quantum-mechanical theory. This paper is not intended as a history of this work, but several notable efforts should be pointed out. In 1935 Jordan, von Neumann and Wigner [5], following earlier work (1933) of Jordan [6], studied the algebra of 3×3 Hermitian matrices over the Cayley numbers as a possible algebraic generalization of the formalism of quantum mechanics. Later, after the advent of the color quark model, Gamba [7, 8], Horwitz and Biederharn [9], Gursey [10] and Gunaydin and Gursey [11], among others, studied the possibility of relating quark structure to the octonions. This program was undertaken because of the intimate relation between $SU(3)$ and the octonions: the octonion algebra may be endowed with a Spin (7) structure; $G_2 \subset \mathrm{Spin}(7)$ is the compact subgroup that leaves the unit element of $\boldsymbol{O}$ fixed. As is well known, there exists a map $G_2 \rightarrow S^6$ that turns G_2 into a principal $SU(3)$-bundle over S^6. Under the action of $SU(3)$ the generators of G_2 decompose into sets that transform as a $SU(3)$ triplet, an anti-triplet, and an octet. The well-known «quark structure» of G_2 is readily apparent.

The octonions also occur in some spontaneous compactifications of Kaluza-Klein supergravity [12-17]. The differential-geometric properties of the split octonions discussed in this paper may be relevant to the solution and interpretation of supergravity field equations that involve octonions.

2. – Basic spinor representations of $SO(4, 4)$.

Let $SO(4, 4)$ denote the group of all matrices in $SL(8, \boldsymbol{R})$ that preserve the quadratic form $(x^8)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 - \{(x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2\}$, where the $x^{A'}$, $A', B', \ldots = 1, 2, \ldots, 8$ comprise a standard Cartesian coordinate system for $\boldsymbol{R}^8$. We denote by $SO(3, 4)$ the subgroup of $SO(4, 4)$ that leaves x^8 invariant. Both $SO(3, 4)$ and $SO(4, 4)$ are pseudo-orthogonal Lie groups that possess two connected components [18]. Their respective 2-1 covering groups are distinguished by a bar placed over them.

We begin by reviewing the construction of the Brauer-Weyl generators of the two inequivalent irreducible real 8×8 basic spinor representations of $\overline{SO(4,4)} \supset \overline{SO(3,4)}$ [1, 3, 18]. As mentioned in the introduction, the reduced generators of these irreducible representations play a fundamental role in providing us with a concrete realization of the multiplication constants of the split octonion algebra, as well as in the formulation of Cartan's exceptional equivalence of $\boldsymbol{R}^8$ vectors and spinors (these vectors and spinors will ultimately be viewed as elements of the split octonion algebra over the reals).

Let us consider $\boldsymbol{R}^8$ endowed with the $SO(4,4)$-invariant pseudo-Riemannian metric

$$G_{A'B'} = G^{A'B'} = \begin{pmatrix} g_{3,1} & 0 \\ 0 & -g_{3,1} \end{pmatrix}, \tag{2.1}$$

where $g_{3,1} = \mathrm{diag}(1,1,1,-1)$ is the pseudo-Riemannian metric on flat Minkowski space-time $M_4 = M_{3,1}$. Here A', B', $\ldots = 1, \ldots, 8$ may be regarded as $\boldsymbol{R}^8$ vector indices; G (respectively, G^{-1}) will be used to lower (respectively, raise) primed upper case Latin indices.

$\boldsymbol{R}^8$ may also be endowed with a $\overline{SO(4,4)}$ invariant metric σ

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

where 0 denotes the 4×4 zero matrix and 1 denotes the 4×4 unit matrix. The matrix elements of σ are $\sigma_{ab} = \sigma_{ba}$, where $a, b, \ldots = 1, \ldots, 8$ are $\overline{SO(4,4)}$ spinor indices. We define the reduced Brauer-Weyl generators $\{\bar{\tau}^{A'}, \tau^{A'}\}$ of the generators of the two real 8×8 inequivalent irreducible representations of $\overline{SO(4,4)}$ (see eqs. (2.6) and (2.7)) by demanding that the tau matrices satisfy (the tilde denotes transpose)

$$\tilde{\bar{\tau}}^{A'}\sigma = \sigma\bar{\tau}^{A'} \tag{2.3}$$

and

$$\tau^{A'}\bar{\tau}^{B'} + \tau^{B'}\bar{\tau}^{A'} = 2IG^{A'B'} = \bar{\tau}^{A'}\tau^{B'} + \bar{\tau}^{B'}\tau^{A'}, \tag{2.4}$$

where I denotes the 8×8 unit matrix. Denoting the matrix elements of $\tau^{A'}$ by $\tau^{A'a}{}_b$, we may write eq. (3) as

$$\bar{\tau}^{A'}{}_{ab} = \tau^{A'}{}_{ba}, \tag{2.5}$$

where we have used σ to lower the spinor index. In general, σ (respectively, σ^{-1}) will be employed to lower (respectively, raise) lower case Latin indices (*i.e.* $\overline{SO(4,4)}$ spinor indices).

Let $\boldsymbol{O}_S$ denote the split octonion algebra over the reals. As a vector space $\boldsymbol{O}_S$ may be decomposed into the direct sum $\boldsymbol{O}_S = \mathrm{Re}\,\{\boldsymbol{O}_S\} \oplus \mathrm{Im}\,\{\boldsymbol{O}_S\} \simeq \boldsymbol{R} \oplus \mathrm{Im}\,\{\boldsymbol{O}_S\}$. It is convenient to define an «admissible» representation of the tau matrices that permits one to define an orientation on $\boldsymbol{R}^8$ in terms of the tau matrices and (in sect. 4) to identify $\mathrm{Re}\,\{\boldsymbol{O}_S\}$ with the x^8-axis of $\boldsymbol{R}^8$. We proceed as follows.

First we adopt a real irreducible 8×8 matrix representation of the tau matrices in which $\tau^8 = I = \bar{\tau}^8$. Then by eq. (2.3) $\bar{\tau}^A = -\tau^A$ for $A = 1, \ldots, 7$. Hence by eq. (2.4) $(\tau^A)^2$ is equal to $-I$ for $A = 1, 2, 3$ and is equal to $+I$ for $A = 4, 5, 6, 7$. To complete the definition of an admissible representation we note that the product $H = \tau^1\tau^2\tau^3\tau^4\tau^5\tau^6\tau^7$ commutes with each of the tau matrices (and therefore with all of their products) and has square equal to $+I$. Since the representation of the tau matrices is irreducible we find that $H = \pm I$ in any irreducible representation. We say that a representation is admissible if $\tau^8 = I = \bar{\tau}^8$ and $H = +I$. The latter criterion selects one of the two inequivalent irreducible representations of the tau matrices.

A concrete admissible representation of the tau matrices has been given in ref. [19]. (We are using a slight change in notation from ref. [19]: «tau bar» is presently denoted «tau» and conversely, and the overall sign of τ^7 has been changed.) For the sake of completeness we list several other results from that paper.

The reduced generators of the two real 8×8 basic spinor representations $D^{(1)}$ and $D^{(2)}$ of $\overline{SO(4,4)}$ may be defined in terms of the tau matrices according to

$$-4D^{(1)A'B'} = \bar{\tau}^{A'}\tau^{B'} - \bar{\tau}^{B'}\tau^{A'} \tag{2.6}$$

and

$$4\widetilde{D}^{(2)A'B'} = \tau^{A'}\bar{\tau}^{B'} - \tau^{B'}\bar{\tau}^{A'}. \tag{2.7}$$

Let $g \in \overline{SO(4,4)}$ and $L \in SO(4,4)$. The canonical 2-1 homomorphism $\overline{SO(4,4)} \to SO(4,4)$ is given by

$$8L^{A'}{}_{B'}(g) = \mathrm{tr}\,\widetilde{D}^{(2)}(g)\,\tau^{A'}D^{(1)}(g)\,\bar{\tau}_{B'}. \tag{2.8}$$

As usual tr denotes the trace. Under the action of $\overline{SO(4,4)}$

$$\widetilde{D}^{(1)}\sigma = \sigma D^{(1)-1}, \tag{2.9}$$

$$\sigma^{-1}\widetilde{D}^{(2)} = D^{(2)-1}\sigma^{-1}, \tag{2.10}$$

$$L^{A'}{}_{C'}G_{A'B'}L^{B'}{}_{D'} = G_{C'D'}, \tag{2.11}$$

$$L^{A'}{}_{B'}\tau^{B'} = \widetilde{D}^{(2)}\tau^{A'}D^{(1)}, \tag{2.12}$$

and

$$L^{A'}{}_{B'}\bar{\tau}^{B'} = D^{(1)-1}\bar{\tau}^{A'}\widetilde{D}^{(2)-1}. \tag{2.13}$$

The tau matrices verify two identities that will enable us to concisely formulate Cartan's equivalence of $\boldsymbol{R}^8$ (endowed with a pseudo-Riemannian metric) vectors and spinors. These we record in a lemma.

Lemma [19]. Let T and W be any 8×8 matrices satisfying

$$(\sigma T)^{\mathrm{T}} = \sigma T, \tag{2.14}$$

where $(\ )^{\mathrm{T}}$ denotes the transpose and

$$(W\sigma^{-1})^{\mathrm{T}} = W\sigma^{-1}, \tag{2.15}$$

and moreover transforming under $\overline{SO(4,4)}$ according to

$$T \to D^{(1)}TD^{(1)-1} \tag{2.16}$$

and

$$W \to D^{(2)-1}WD^{(2)}. \tag{2.17}$$

Then

$$\tau_{A'}T\bar{\tau}^{A'} = I\,\mathrm{tr}(T) \tag{2.18}$$

and

$$\bar{\tau}^{A'}\widetilde{W}\tau_{A'} = I\,\mathrm{tr}(W). \tag{2.19}$$

Equation (2.18) will be employed in the next section to define a one-to-one invertible map from a $\overline{SO(3,4)}$ spinor to a $SO(3,4)$ vector-scalar pair. In this process orthogonal frames on $\boldsymbol{R}^8$ arise that carry one $\overline{SO(3,4)}$ spinor index «a» and one $SO(3,4)$ vector-scalar index «A'». In sect. 4 these frames and the tau matrices will be combined to generate the multiplication constants for the split octonion algebra.

3. – Cartan's exceptional equivalence.

Let $V_{4,4}$ denote the real pseudo-Riemannian space $\boldsymbol{R}^8$ endowed with the indefinite metric σ of the last section. We shall assume that $V_{4,4}$ carries the real irreducible representation $D^{(1)}$ of $\overline{SO(4,4)}$. Let e_a be an oriented (spinor) basis of $V_{4,4}$ normalized according to

$$\langle e_a, e_b \rangle = \sigma_{ab} . \tag{3.1}$$

We write an element X of $V_{4,4}$ as

$$X = e_a x^a . \tag{3.2}$$

Under the action of $\overline{SO(4,4)}$ $x^a \to x'^a = D^{(1)a}{}_b x^b$. We call x a real (eight component) reduced $\overline{SO(4,4)}$ spinor of the first kind (Cartan's semi-spinor of the first type), or more briefly, a $V_{4,4}$ spinor.

Let us choose $U \in V_{4,4}$ with components u^a normalized to

$$\langle U, U \rangle = \tilde{u}\sigma u = 1 \tag{3.3}$$

everywhere on $V_{4,4}$, but being otherwise arbitrary (matrix multiplication in terms of components is implied in the middle equality). We can define a $V_{4,4}$ frame in terms of u and the tau matrices as follows. Let T be the real 8×8 matrix defined by

$$T = u\tilde{u}\sigma . \tag{3.4}$$

Then T obeys eq. (2.14), $(\sigma T)^{\mathrm{T}} = \sigma T$, and transforms under $\overline{SO(4,4)}$ according to eq. (2.16). We may therefore apply the *Lemma* and thereby obtain

$$I = \tau_{A'} u\tilde{u}\sigma\bar{\tau}^{A'} . \tag{3.5}$$

This is a resolution of the identity on $V_{4,4}$. Alternatively this relation may be interpreted as a completeness condition verified by the (components of the) orthogonal frame

$$E^a_{A'} = \tau^a_{A'b} u^b \tag{3.6}$$

and its inverse

$$E^{A'}_a = u^b \sigma_{bc} \bar{\tau}^{A'c}{}_a . \tag{3.7}$$

Accordingly eq. (3.5) may be expressed as

$$E^a_{A'} E^{A'}_b = \delta^a_b . \tag{3.8}$$

Since a matrix commutes with its inverse we also have

$$E^{A'}_a E^a_{B'} = \delta^{A'}_{B'} . \tag{3.9}$$

Evidently the components of any $V_{4,4}$ spinor with respect to this frame comprise a $\boldsymbol{R}^8$ vector, and conversely. This is essentially a statement of Cartan's principle of triality [4], also called the exceptional equivalence of $\boldsymbol{R}^8$ spinors and vectors [19]. However, we must be more careful here in spelling out the transformation properties of the various geometric objects. Let $\overline{SO(3,4)}$ be the subgroup of $\overline{SO(4,4)}$ that preserves $\delta_8^{A'}$. Then $\widetilde{D}^{(2)}(g) = D^{(1)}(g^{-1})$ for $g \in \overline{SO(3,4)}$. For $X \in V_{4,4}$, consider (in matrix notation)

$$x = Ix = (\tau_{A'} u\widetilde{u}\sigma\bar{\tau}^{A'})x = \tau_{A'} u(\widetilde{u}\sigma\bar{\tau}^{A'} x) = \tau_{A'} u x^{A'}, \tag{3.10}$$

where we have defined

$$x^{A'} = \widetilde{u}\sigma\bar{\tau}^{A'} x = E_a^{A'} x^a . \tag{3.11}$$

On the face of it eqs. (3.10) and (3.11) define an equivalence of spinors and vectors. However, in virtue of eqs. (2.9) and (2.13) we see that the $x^{A'}$ do not quite transform as the components of a $SO(4,4)$ vector. Closer examination shows that under the restriction to the $SO(3,4)$ subgroup of $SO(4,4)$ defined above, the x^A, $A = 1, \dots, 7$ are seen to transform as a $SO(3,4)$ vector, while x^8 ($A' = 8$, not $a = 8$; our simplified notation carries some unfortunate overhead) is a $SO(3,4)$ scalar. Since the representation $D^{(1)}$ of $\overline{SO(4,4)}$ is irreducible when restricted to $\overline{SO(3,4)}$ we obtain from eqs. (3.10) and (3.11) the equivalence of a real eight component $\overline{SO(3,4)}$ spinor and a real $SO(3,4)$ vector-scalar pair.

We observe that the vector-scalar basis

$$e_{A'} = e_a E_{A'}^a = e_a \tau^a_{A'b} u^b \tag{3.12}$$

comprise an oriented orthonormal frame of $V_{4,4}$ (oriented because $\det(E_{A'}^a) = 1$ for every admissible representation of the tau matrices). Using eqs. (3.1), (2.3) and (2.4), we find that

$$\langle e_{A'}, e_{B'} \rangle = G_{A'B'} . \tag{3.13}$$

4. – Split octonion algebra.

We shall define a nonassociative alternative multiplication of the spinor basis e_a of the previous section that turns the real vector space $V_{4,4}$ into a normed algebra with multiplicative unit. This may be accomplished by specifying the multiplication constants m_{ab}^c of the algebra, which verify

$$e_a e_b = e_c m_{ab}^c . \tag{4.1}$$

We shall adopt a set of m^c_{ab} defined by

$$m^c_{ab} = \tau^c_{A'b} E^{A'}_a \tag{4.2}$$

as the multiplication constants of the algebra.

Theorem. The product defined by eqs. (4.1) and (4.2) of the spinor basis e_a endows the real vector space $V_{4,4}$ with the structure of the split octonion algebra over the reals.

Remark. The field of real numbers may be generalized to any field of characteristic $\neq 2$.

Proof. We must show that eqs. (4.1) and (4.2) define a multiplication with unit element that satisfies $\langle XY, XY\rangle = \langle X, X\rangle\langle Y, Y\rangle$ for every X, Y in the algebra. (An algebra that satisfies these two properties is said to be a composition algebra. As is well known, Hurwitz [20] has shown that there exist only four nonisomorphic composition algebras that are also division algebras over a field of characteristic $\neq 2$, namely, $\boldsymbol{R}$, $\boldsymbol{C}$, the quaternions and the octonions.)

Let us begin the proof by identifying the multiplicative unit element, which we shall show is given by

$$1 = e_a u^a = U. \tag{4.3}$$

To this end, consider $e_a(e_b u^b) = e_c m^c_{ab} u^b = e_c \delta^c_a$ [by eq. (3.5)] $= e_a$. Also $(e_a u^a) e_b = e_c m^c_{ab} u^a = e_c \tau^c_{A'b}(E^{A'}_a u^a) = e_c \delta^c_b$ (since $\tau^8 = I = \bar{\tau}^8$ and $u^a = E^a_8) = e_b$. Hence $1 = e_a u^a = U$.

Next, the multiplication of eqs. (4.1) and (4.2) defines a composition algebra: for $X, Y \in V_{4,4}$ we put $X = e_a x^a$ and $Y = e_a y^a$ and consider $\langle XY, XY\rangle$; we seek to show that this equals $\langle X,X\rangle\langle Y,Y\rangle$, where $\langle Y, Y\rangle = \langle e_a y^a, e_b y^b\rangle = y^a y^b \langle e_a, e_b\rangle = y^a y^b \sigma_{ab} = \tilde{y}\sigma y$. Similarly $\langle X, X\rangle = \tilde{x}\sigma x = x^{A'} G_{A'B'} x^{B'}$, where $x^{A'} = E^{A'}_a x^a$. We observe that $XY = (e_a x^a)(e_b y^b) = e_c m^c_{ab} x^a y^b = (e\tau_{A'} y) x^{A'}$. Therefore $\langle XY, XY\rangle = x^{A'} x^{B'} \langle e\tau_{A'} y, e\tau_{B'} y\rangle = x^{A'} x^{B'} (\tau_{A'} y)^{\mathrm{T}} \sigma(\tau_{B'} y) = x^{A'} x^{B'} \tilde{y}\sigma\bar{\tau}_{A'}\tau_{B'} y$ by eq. (2.3). But $\tilde{y}\sigma\bar{\tau}_{A'}\tau_{B'} y = (\tilde{y}\sigma\bar{\tau}_{A'}\tau_{B'} y)^{\mathrm{T}} = \tilde{y}\sigma\bar{\tau}_{B'}\tau_{A'} y$ [using eq. (2.3) again] $= (1/2)\tilde{y}\sigma\{\bar{\tau}_{A'}\tau_{B'} + \bar{\tau}_{B'}\tau_{A'}\} y = \tilde{y}\sigma y G_{A'B'} = G_{A'B'}\langle Y, Y\rangle$ by eq. (2.4). Hence $\langle XY, XY\rangle = x^{A'} x^{B'} G_{A'B'}\langle Y, Y\rangle = \langle X, X\rangle\langle Y, Y\rangle$.

This eight-dimensional composition algebra with unit possesses divisors of zero since the metric is indefinite. It is known that there is only one distinct algebra possessing these properties, namely the split octonion algebra [21]. *QED.*

We have seen that the choice of a $U \in V_{4,4}$ with components u^a normalized to $\langle U, U\rangle = 1$ gives rise to an oriented orthonormal $V_{4,4}$ frame that explicitly realizes Cartan's exceptional equivalence. Moreover we have seen that the spinor components of the tau matrices (the reduced Brauer-Weyl generators of the Lie algebra $so(4,4)$), as expressed in eq. (4.2), give rise to a set of multiplication constants that turns $V_{4,4}$ into a copy of the split octonion algebra over the reals. The $V_{4,4}$ spinor u corresponds to the multiplicative unit $1 = e_a u^a$ of the algebra. Suppressing the spinor indices, we may also write $1 = eu = eIu = e\tau_8 u = e_{A'}$ for $A' = 8$, so that $\boldsymbol{R} \simeq \mathrm{Re}\{\boldsymbol{O}_S\}$ is the linear span of $1 = U$. If $X \in V_{4,4}$ then $X = e_{A'} x^{A'}$ and $\mathrm{Re}\{X\} = x^8$ ($A' = 8$, not $a = 8$).

Lastly we remark that

$$\bar{e}_{A'} = e_a \bar{\tau}^a_{A'b} u^b \tag{4.4}$$

is the octonion conjugate of $e_{A'}$, whence the octonion conjugate of e_a is

$$\bar{e}_a = \bar{e}_{A'} E^{A'}_a . \tag{4.5}$$

5. – Potential physical applications.

It is possible to recast the Dirac equation for the relativistic electron into its vector-scalar form using the equivalence of eqs. (3.10) and (3.11). This yields a set of equations not for a spinor wave function, but for a $SO(3,4)$ (restricted to $O(3,1)$) vector-scalar wave function. The electron wave function in this case is a real $O(3,1)$ vector plus four real $O(3,1)$ scalars. This may be of practical importance in particular physical problems where the interactions between electron and external fields are such that a scalar component of the electron wave function or a projection of the electron vector wave function onto some axis is noninteracting and hence obeys the free-field equations.

The vector-scalar Lagrangian for the (non–second-quantized) electron has been given elsewhere [22]. However, the vector-scalar version of the Dirac Lagrangian for the second-quantized electron field is perhaps more natural, so let us briefly consider this case. We write the electron field as

$$\Psi = e_a \psi^a . \tag{5.1}$$

The ψ^a comprise a Hermitian eight-component spinor electron-positron field (the charge operator is not diagonal in this representation). Schwinger [23] has formulated quantum electrodynamics in terms of this type of field operator. Using eq. (3.11) we may define anticommuting vector-scalar electron-positron fields according to

$$\psi^{A'} = E^{A'}_a \psi^a . \tag{5.2}$$

Substituting the inverse of this identity into the fermion Lagrangian

$$L = (i/2)\langle \Psi, \tau^6 \tau^7[\tau^\alpha(\partial_\alpha - ie\tau^5\tau^6 A_\alpha) + m]\,\Psi\rangle \tag{5.3}$$

yields a spin-1/2 fermion Lagrangian constructed from vector and scalar anticommuting operators. We mention in passing that the multiplication constants for the $e_{A'}$ of eq. (3.12) explicitly arise in this vector-scalar Lagrangian, and the expression for the Lagrangian can be simplified considerably when one takes advantage of their many remarkable properties. However, a discussion of these properties lies outside the scope of this paper, in which we have emphasized the nature of the multiplication constants for the spinor basis e_a.

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● RIASSUNTO (*)

Si discute la nota equivalenza di set vettoriali-scalari e spinoriali speciali nel contesto dell'algebra degli ottonioni separati. Un'implicazione di questa equivalenza è che il campo spinoriale di Dirac consueto può essere rimodellato come campo vettoriale-scalare e si sottolinea questa costruzione. Si illustra un processo di fattorizzazione costante di struttura con la realizzazione delle costanti di moltiplicazione degli ottonioni separati (rispetto ad una base spinoriale) come prodotti di certi generatori di matrici e uno spinore normalizzato arbitrario.

(*) *Traduzione a cura della Redazione.*

О структуре алгебры расщепленных октонионов.

Резюме (*). — Обсуждается известная эквивалентность специальных спиноров и системы векторов-скаляров в контексте алгебры расщепленных октонионов. Одно из применений этой эквивалентности заключается в том, что обычно дираковское спинорное поле может быть преобразовано, как векторно-скалярное поле. Предлагается указанная конструкция. Иллюстрируется процесс факторизации структурной постоянной с помощью представления мультипликационных постоянных расщепленных октонионов (по отношению с спинорному базису), как произведений некоторых матричных генераторов и произвольного нормированного спинора.

(*) *Переведено редакцией.*