

Scratch work of Author; CopyRight (C) 2022, Patrick L. Nash, under the General Public License ;Please cite this work, and this web page, if you use it

WARNING : syncope, presyncope AHEAD
(yes, i can use Google, too)

Begin

```
In[1]:= Print["CopyRight (C) 2022, Patrick L. Nash, under the General Public License."]
CopyRight (C) 2022, Patrick L. Nash, under the General Public License.

In[2]:= Print["Please cite this work, and this web page, if you use it"]
Please cite this work, and this web page, if you use it

ConvertMapleToMathematicaV2

In[3]:= FileNameJoin[Drop[FileNameSplit[NotebookFileName[]], -1]]
mapleToMathematicaV2PATH = FileNameJoin[{%, "ConvertMapleToMathematicaV2.wl"}]
Out[3]= C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX

Out[4]= C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX\
          ConvertMapleToMathematicaV2.wl

In[5]:= Get[mapleToMathematicaV2PATH]      (*Loads new maple to mathematica parser*)
ConvertMapleToMathematicaV2 loaded successfully!
BUT, WARNING: DO NOT USE IF YOU WANT A CORRECT RESULT!

Load the Maple output strings.

In[864]:= 
FileNameJoin[Drop[FileNameSplit[NotebookFileName[]], -1]]
etoExpPATH = FileNameJoin[{%, "EtoExp.wl"}]
Out[864]= C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX

Out[865]= C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX\EtoExp.wl

In[866]:= 
Get[etoExpPATH]
EtoExp.wl loaded. BUT, WARNING: DO NOT USE IF YOU WANT A CORRECT RESULT!

In[6]:= NotebookFileName[]
Out[6]= C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX\Pre-
          gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes.nb

In[7]:= Unprotect[dir];
```

```
In[8]:= dir = FileNameJoin[Drop[FileNameSplit[NotebookFileName[]], -1]];
Protect[dir];
If[$Failed == SetDirectory[dir], {CreateDirectory[dir], SetDirectory[dir]}];

In[11]:= (*FileNameJoin[Drop[FileNameSplit[NotebookFileName[]], -1]]*)

In[12]:= names = FileNameSplit[NotebookFileName[]]

Out[12]= {C:, Users, nsh, Documents, 8-dim, 2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX,
Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes.nb}

In[13]:= name = StringReplace[names[[-1]], "nb" → "mx"]

Out[13]= Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes.mx

In[14]:= header = StringReplace[names[[-1]], ".nb" → "-"]

Out[14]= Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes-

In[15]:= SetOptions[Simplify, TimeConstraint → 1]

Out[15]= {Assumptions → $Assumptions, ComplexityFunction → Automatic, ExcludedForms → {},
TimeConstraint → 1, TransformationFunctions → Automatic, Trig → True}

In[16]:= SetOptions[FullSimplify, TimeConstraint → 3]

Out[16]= {Assumptions → $Assumptions, ComplexityFunction → Automatic, ExcludedForms → {},
TimeConstraint → 3, TransformationFunctions → Automatic, Trig → True}

In[17]:= Options[Simplify]

Out[17]= {Assumptions → $Assumptions, ComplexityFunction → Automatic, ExcludedForms → {},
TimeConstraint → 1, TransformationFunctions → Automatic, Trig → True}

In[18]:= Options[FullSimplify]

Out[18]= {Assumptions → $Assumptions, ComplexityFunction → Automatic, ExcludedForms → {},
TimeConstraint → 3, TransformationFunctions → Automatic, Trig → True}

In[19]:= {Off[General::spell], Off[General::spell1]};

In[20]:= DIM8 = 8;

In[21]:= Protect[DIM8, M, K, H]

Out[21]= {DIM8, M, K, H}

Coordinates : from old ideas in git clone https://
```

github.com / 43 d168f3e / Eternal – DEFLATION – Inflation.git
 $\{ \underbrace{\text{hidden space}}, \underbrace{\text{3-space}}, \underbrace{\text{time}}, \underbrace{\text{superluminal deflating time}} \}$
 $\{ x_0, \overbrace{x_1, x_2, x_3}, x_4, \overbrace{x_5, x_6, x_7} \} :$

```
In[22]:= X = {x0, x1, x2, x3, x4, x5, x6, x7};  

Protect[X];  

Protect[x0, x1, x2, x3, x4, x5, x6, x7];
```

```
In[25]:= sx0 = And @@ Thread[X > 0]  

Out[25]= x0 > 0 && x1 > 0 && x2 > 0 && x3 > 0 && x4 > 0 && x5 > 0 && x6 > 0 && x7 > 0
```

```
In[26]:= ssX = H > 0 && sx0 && 6 H x0 > 0 && 2 la[x4] > 0 && Cot[6 H x0] > 0 && Sin[6 H x0] > 0  

Out[26]= H > 0 && x0 > 0 && x1 > 0 && x2 > 0 && x3 > 0 && x4 > 0 && x5 > 0 && x6 > 0 &&  

x7 > 0 && 6 H x0[] > 0 && 2 la[x4[]] > 0 && Cot[6 H x0[]] > 0 && Sin[6 H x0[]] > 0
```

For Maple_2025 :

```
In[27]:= sreplaceZ = Z[#] → ToExpression["Z" <> ToString[#]] & /@ Range[0, 15]  

Out[27]= {Z[0] → Z0, Z[1] → Z1, Z[2] → Z2, Z[3] → Z3, Z[4] → Z4,  

Z[5] → Z5, Z[6] → Z6, Z[7] → Z7, Z[8] → Z8, Z[9] → Z9, Z[10] → Z10,  

Z[11] → Z11, Z[12] → Z12, Z[13] → Z13, Z[14] → Z14, Z[15] → Z15}  

In[28]:= sreplacenZ = nZ[#] → ToExpression["nZ" <> ToString[#]] & /@ Range[0, 15]  

Out[28]= {nZ[0] → nZ0, nZ[1] → nZ1, nZ[2] → nZ2, nZ[3] → nZ3, nZ[4] → nZ4,  

nZ[5] → nZ5, nZ[6] → nZ6, nZ[7] → nZ7, nZ[8] → nZ8, nZ[9] → nZ9, nZ[10] → nZ10,  

nZ[11] → nZ11, nZ[12] → nZ12, nZ[13] → nZ13, nZ[14] → nZ14, nZ[15] → nZ15}
```

lie; speed up Simplify? :

```
In[29]:= constraintVars = x0 > 0 && x4 > 0 && z > 0 && t > 0
```

```
Out[29]= x0 > 0 && x4 > 0 && z > 0 && t > 0
```

```
In[30]:= subsDefects = {}  

Out[30]= {}
```

future coordinate transformation :

```
In[31]:= 6 * H * x0 == z && H x4 == t
szt = Solve[%, {z, t}][[1]]
sx0x4 = Solve[%%, {x0, x4}][[1]]
Protect[sx0x4, szt];
```

Out[31]=
6 H x0 == z && H x4 == t

Out[32]=
{z → 6 H x0, t → H x4}

Out[33]=
 $\left\{x0 \rightarrow \frac{z}{6H}, x4 \rightarrow \frac{t}{H}\right\}$

SO (4, 4) Minkowski Lorentz metric η 4488 : 4 + 4 spacetime; 8 × 8 dimensional :

```
In[35]:= (η4488 = ArrayFlatten[{{IdentityMatrix[4], 0}, {0, -IdentityMatrix[4]}}]) // MatrixForm
```

Out[35]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

SO (4, 4) Spinor Lorentz metric σ ;

τ^A are analogous to the Dirac gamma matrices :

all true :

```
Table[{A, B, FullSimplify[1/2 (τ[A].τ[B] + τ[B].τ[A]) == η4488[[A + 1, B + 1]] * ID8]}, {A, 0, 7}, {B, 0, 7}]
```

Type - 1 and type - 2 employ σ (the usual notational abuse),

since $\sigma = \sigma^{-1}$: 4 + 4 spacetime, 8 × 8 dimensional and 16 × 16 dimensional :

indices. We define the reduced Brauer-Weyl generators $\{\bar{\tau}^{A'}, \tau^{A'}\}$ of the generators of the two real 8×8 inequivalent irreducible representations of $\overline{SO(4, 4)}$ (see eqs. (2.6) and (2.7)) by demanding that the tau matrices satisfy (the tilde denotes transpose)

$$(2.3) \quad \tilde{\tau}^{A'} \sigma = \sigma \bar{\tau}^{A'}$$

and

$$(2.4) \quad \tau^{A'} \bar{\tau}^{B'} + \tau^{B'} \bar{\tau}^{A'} = 2IG^{A'B'} = \bar{\tau}^{A'} \tau^{B'} + \bar{\tau}^{B'} \tau^{A'},$$

where I denotes the 8×8 unit matrix. Denoting the matrix elements of $\tau^{A'}$ by $\tau^{A'a}_{b}$, we may write eq. (3) as

$$(2.5) \quad \bar{\tau}^{A'}_{ab} = \tau^{A'}_{ba},$$

$$t^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

$$t^A t^B - t^B t^A = \begin{pmatrix} \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A & 0 \\ 0 & \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A \end{pmatrix} = 4 \begin{pmatrix} D_{(1)}^{AB} & 0 \\ 0 & D_{(2)}^{AB} \end{pmatrix}$$

$$4D_{(1)}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A$$

$$4D_{(2)}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A.$$

Let $g \in \overline{SO(4, 4)}$ and $L \in SO(4, 4)$. The canonical 2-1 homomorphism $\overline{SO(4, 4)} \rightarrow SO(4, 4)$ is given by

$$(2.8) \quad 8L^{A'}{}_{B'}(g) = \text{tr} \tilde{D}^{(2)}(g) \tau^{A'} D^{(1)}(g) \bar{\tau}_{B'}.$$

As usual tr denotes the trace. Under the action of $\overline{SO(4, 4)}$

$$(2.9) \quad \tilde{D}^{(1)} \sigma = \sigma D^{(1)-1},$$

$$(2.10) \quad \sigma^{-1} \tilde{D}^{(2)} = D^{(2)-1} \sigma^{-1},$$

$$(2.11) \quad L^{A'}{}_{C'} G_{A'B'} L^{B'}{}_{D'} = G_{C'D'},$$

$$(2.12) \quad L^{A'}{}_{B'} \tau^{B'} = \tilde{D}^{(2)} \tau^{A'} D^{(1)},$$

$$(2.13) \quad L^{A'}{}_{B'} \bar{\tau}^{B'} = D^{(1)-1} \bar{\tau}^{A'} \tilde{D}^{(2)-1}.$$

```
In[36]:= (σ = ArrayFlatten[{{ConstantArray[0, {4, 4}], IdentityMatrix[4]}, {IdentityMatrix[4], ConstantArray[0, {4, 4}]}}]) // MatrixForm
```

Out[36]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

defs; some Symbols; metric

?todo: NEED DefScalarFunction[#]; & /@ (f16[#]&/@Range[0,15]), and allow xact to compute?

Introduce the wave function, Ψ16, for a Universe:

```
In[37]:= Ψ16 = f16[#] [x0, x4] & /@ Range[0, 15]
```

Out[37]=

$$\{f16[0][x0, x4], f16[1][x0, x4], f16[2][x0, x4], f16[3][x0, x4], f16[4][x0, x4], f16[5][x0, x4], f16[6][x0, x4], f16[7][x0, x4], f16[8][x0, x4], f16[9][x0, x4], f16[10][x0, x4], f16[11][x0, x4], f16[12][x0, x4], f16[13][x0, x4], f16[14][x0, x4], f16[15][x0, x4]\}$$

```
In[38]:= (*processRawSets[rawSets_]:=Module[{s1,s2,J,ret,L=Range[Length[rawSets]],r={}},  
  While[Length[L]>1,  
    s1=Union[rawSets[[L[[1]]]];  
    L=Drop[L,1];  
    J=L[[1]];  
    While[J≤16,  
      If[MemberQ[L,J],  
        s2=Union[rawSets[[J]]];  
        If[Intersection[s1,s2]≠{},s1=Union[Flatten[{s1,s2}]];  
        L=Complement[L,{J}];,##&[]];  
        ##&[]];  
      J++];  
  
      AppendTo[r,s1];  
    ];  
    ret=Sort[r,#1[[1]]<#2[[1]]&];  
    Return[ret];  
  ]*)
```

```
In[39]:= (*rawSets@16=Cases[  
  #,  
  h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], "f16"] :> n,  
  Infinity,  
  Heads → True  
 ]&/@f16*)
```

```
In[40]:= (*processRawSets[rawSets@16]*)
```

```
In[41]:= (*newf16=f16[#][x0,x4]&/@Range[0,15]*)
```

```
In[42]:= Clear[sf16Aa];  
sf16Aa = f16[#] → ToExpression["((Z[" <> ToString[#] <> "] [6*H#1,H#2])&)"] &/@  
Range[0, 15]
```

```
Out[42]= {f16[0] → (Z[0] [6 H#1, H#2] &), f16[1] → (Z[1] [6 H#1, H#2] &),  
f16[2] → (Z[2] [6 H#1, H#2] &), f16[3] → (Z[3] [6 H#1, H#2] &),  
f16[4] → (Z[4] [6 H#1, H#2] &), f16[5] → (Z[5] [6 H#1, H#2] &),  
f16[6] → (Z[6] [6 H#1, H#2] &), f16[7] → (Z[7] [6 H#1, H#2] &),  
f16[8] → (Z[8] [6 H#1, H#2] &), f16[9] → (Z[9] [6 H#1, H#2] &),  
f16[10] → (Z[10] [6 H#1, H#2] &), f16[11] → (Z[11] [6 H#1, H#2] &),  
f16[12] → (Z[12] [6 H#1, H#2] &), f16[13] → (Z[13] [6 H#1, H#2] &),  
f16[14] → (Z[14] [6 H#1, H#2] &), f16[15] → (Z[15] [6 H#1, H#2] &)}
```

```
In[43]:= (*Clear[sf16Aa];  
sf16Aa=f16[#]→ToExpression[  
  "((Z["<>ToString[#]<>"] [6*H#1,H#2] 1 Sin[6*H#1]1/2 )&)"]&/@Range[0,15]*)
```

```
In[44]:= Clear[snewf16Aa];
snewf16Aa = f16[#] & /@ Range[0, 15]
```

$$\begin{aligned} \text{Out}[44]= & \left\{ \begin{aligned} f16[0] &\rightarrow \left(\frac{nZ[0][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[1] &\rightarrow \left(\frac{nZ[1][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[2] &\rightarrow \left(\frac{nZ[2][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[3] &\rightarrow \left(\frac{nZ[3][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[4] &\rightarrow \left(\frac{nZ[4][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[5] &\rightarrow \left(\frac{nZ[5][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[6] &\rightarrow \left(\frac{nZ[6][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[7] &\rightarrow \left(\frac{nZ[7][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[8] &\rightarrow \left(\frac{nZ[8][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[9] &\rightarrow \left(\frac{nZ[9][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[10] &\rightarrow \left(\frac{nZ[10][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[11] &\rightarrow \left(\frac{nZ[11][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[12] &\rightarrow \left(\frac{nZ[12][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[13] &\rightarrow \left(\frac{nZ[13][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[14] &\rightarrow \left(\frac{nZ[14][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right), \\ f16[15] &\rightarrow \left(\frac{nZ[15][6H\#1, H\#2]}{\sqrt{\sin[6H\#1]}} \& \right) \end{aligned} \right\} \end{aligned}$$

0 (4, 4) related :

```
In[45]:= Needs["Notation`"]
```

```
In[46]:= Symbolize[\bar{\tau}]
```

```
In[47]:= Symbolize[T16^A]
```

```
In[48]:= Symbolize[T16^\alpha]
```

```
In[49]:= Symbolize[m^c_{ab}]
```

```
In[50]:= Symbolize[m^c_{AB}]
```

```
In[51]:= Symbolize[\sigma g^a_{1A}]
```

```
In[52]:= Symbolize[\sigma g^a_{2A}]
```

```
In[53]:= Symbolize[u^a]
```

```
In[54]:= Symbolize[Jy]
```

```
In[55]:= Symbolize[Fa]
```

```
In[56]:= Symbolize[Fa]
```

```
In[57]:= Symbolize[overline{o22}]
```

$$g^{\alpha\beta} = g_{\alpha\beta} = \text{diag}(1, 1, 1, -1) \quad (5)$$

```
In[58]:= Symbolize[gab]
```

note below: η_{AB} is really $g_{(A)(B)}$:: Minkowski – Lorentz metric

```
In[59]:= Symbolize[etaAB]
```

note below: G_{AB} is really $G_{(A)(B)}$:: Minkowski – Lorentz

```
In[60]:= (*Symbolize[GAB]*)
```

```
In[61]:= Symbolize[S44ab]
```

$$\Sigma^{\alpha\beta} = \xi S^{\alpha\beta} u,$$

```
In[62]:= Symbolize[Sigmaab]
```

```
In[63]:= (*Symbolize[Sumab]*)
```

$$\frac{1}{2} \sum^{jk} \Sigma_{jk} = \frac{1}{4} \partial_a u^a u^b \partial_b - \frac{1}{4} \partial_a u^b u^b \partial_a,$$

```
In[64]:= Symbolize[Sigma]
```

NEED DefScalarFunction[#]; & /@ Flatten[Universe]

```
In[65]:= Universe = ToExpression["U" <> ToString[#]] & /@ Range[0, 7]
```

```
In[65]:= Symbolize[Fa]
```

```
In[66]:= Symbolize[Fa]
```

```
In[67]:= (*Clear[sf16Aa];
sf16Aa=f16[#]ToExpression[
"((Z["<>ToString[#]<>"] [6*H#1,3*H#2]\frac{1}{\sin[6*H#1]^{1/2}})&)" ]&/@Range[0,15]*)
```

```
In[68]:= (*Clear[sf16Aa];
sfψ16Aa=f16[#]→ToExpression["((Z["<>ToString[#]<>"] [6*H*#1,3*H*#2])&)"]&/@Range[0,15]*)
```

```
In[69]:= (*Clear[sf16Aa];
sfψ16Aa=f16[#]→ToExpression[
"((Z["<>ToString[#]<>"] [6*H*#1,H*#2] 1/Sin[6*H*#1]^(1/2))&)"]&/@Range[0,15]*)
```

```
In[70]:= (*Inactivate[((q)&)]Evaluate[Activate[Evaluate[q]]]*)
```

```
In[71]:= Symbolize[Eα(A)]
```

```
In[72]:= Symbolize[eα(A)]
```

```
In[73]:= eα(A) = Array[Eα(A) [#1 - 1, #2 - 1] &, {8, 8}]
```

```
Out[73]= {{Eα(A) [0, 0], Eα(A) [0, 1], Eα(A) [0, 2], Eα(A) [0, 3], Eα(A) [0, 4], Eα(A) [0, 5], Eα(A) [0, 6], Eα(A) [0, 7]}, {Eα(A) [1, 0], Eα(A) [1, 1], Eα(A) [1, 2], Eα(A) [1, 3], Eα(A) [1, 4], Eα(A) [1, 5], Eα(A) [1, 6], Eα(A) [1, 7]}, {Eα(A) [2, 0], Eα(A) [2, 1], Eα(A) [2, 2], Eα(A) [2, 3], Eα(A) [2, 4], Eα(A) [2, 5], Eα(A) [2, 6], Eα(A) [2, 7]}, {Eα(A) [3, 0], Eα(A) [3, 1], Eα(A) [3, 2], Eα(A) [3, 3], Eα(A) [3, 4], Eα(A) [3, 5], Eα(A) [3, 6], Eα(A) [3, 7]}, {Eα(A) [4, 0], Eα(A) [4, 1], Eα(A) [4, 2], Eα(A) [4, 3], Eα(A) [4, 4], Eα(A) [4, 5], Eα(A) [4, 6], Eα(A) [4, 7]}, {Eα(A) [5, 0], Eα(A) [5, 1], Eα(A) [5, 2], Eα(A) [5, 3], Eα(A) [5, 4], Eα(A) [5, 5], Eα(A) [5, 6], Eα(A) [5, 7]}, {Eα(A) [6, 0], Eα(A) [6, 1], Eα(A) [6, 2], Eα(A) [6, 3], Eα(A) [6, 4], Eα(A) [6, 5], Eα(A) [6, 6], Eα(A) [6, 7]}, {Eα(A) [7, 0], Eα(A) [7, 1], Eα(A) [7, 2], Eα(A) [7, 3], Eα(A) [7, 4], Eα(A) [7, 5], Eα(A) [7, 6], Eα(A) [7, 7]}}
```

```
In[74]:= (*Symbolize[seα(A)]*)
```

```
In[75]:= Symbolize[sgeα(A)]
```

```
In[76]:= Symbolize[sgtryeα(A)]
```

```
In[77]:= Symbolize[E(A)α]
```

```
In[78]:= Symbolize[sge(A)α]
```

```
In[79]:= Symbolize[sgtrye(A)α]
```

```
In[80]:= Symbolize[e(A)α]
```

```

In[81]:= eα(A) = Array[Eα(A) [##1 - 1, ##2 - 1] &, {8, 8}]

Out[81]= { {Eα(A) [0, 0], Eα(A) [0, 1], Eα(A) [0, 2], Eα(A) [0, 3], Eα(A) [0, 4], Eα(A) [0, 5], Eα(A) [0, 6], Eα(A) [0, 7]}, {Eα(A) [1, 0], Eα(A) [1, 1], Eα(A) [1, 2], Eα(A) [1, 3], Eα(A) [1, 4], Eα(A) [1, 5], Eα(A) [1, 6], Eα(A) [1, 7]}, {Eα(A) [2, 0], Eα(A) [2, 1], Eα(A) [2, 2], Eα(A) [2, 3], Eα(A) [2, 4], Eα(A) [2, 5], Eα(A) [2, 6], Eα(A) [2, 7]}, {Eα(A) [3, 0], Eα(A) [3, 1], Eα(A) [3, 2], Eα(A) [3, 3], Eα(A) [3, 4], Eα(A) [3, 5], Eα(A) [3, 6], Eα(A) [3, 7]}, {Eα(A) [4, 0], Eα(A) [4, 1], Eα(A) [4, 2], Eα(A) [4, 3], Eα(A) [4, 4], Eα(A) [4, 5], Eα(A) [4, 6], Eα(A) [4, 7]}, {Eα(A) [5, 0], Eα(A) [5, 1], Eα(A) [5, 2], Eα(A) [5, 3], Eα(A) [5, 4], Eα(A) [5, 5], Eα(A) [5, 6], Eα(A) [5, 7]}, {Eα(A) [6, 0], Eα(A) [6, 1], Eα(A) [6, 2], Eα(A) [6, 3], Eα(A) [6, 4], Eα(A) [6, 5], Eα(A) [6, 6], Eα(A) [6, 7]}, {Eα(A) [7, 0], Eα(A) [7, 1], Eα(A) [7, 2], Eα(A) [7, 3], Eα(A) [7, 4], Eα(A) [7, 5], Eα(A) [7, 6], Eα(A) [7, 7]}}

In[82]:= (*gtryeα(A)=(eα(A)/.sgtryeα(A))*)

In[83]:= (*gtryeα(A)=(eα(A)/.sgtryeα(A))*)

In[84]:= (*Symbolize[Γκα β]*)

In[85]:= (*preSpinConnection=Array[Γκβ[##]&,{8,8,8}];*)

In[86]:= (*Symbolize[ EAa]*)

In[87]:= (*Symbolize[ EaA]*)

In[88]:= (* FAa FaA *)

In[89]:= (*Do[ FAa=EAa[h]=Table[(u[[h]].σ.(τ[B])),{B,1,8}],{h,1,Length[u]}];
Do[ EaA[h]=FullSimplify[Inverse[EAa[h]]],{h,1,Length[u]}];
Table[ FaA=EaA[h]==(u[[h]].σ.u[[h]])Transpose[Table[
FullSimplify[ExpandAll[ η8[[B,B]]τ[B].u[[h]] ]],{B,1,8}]],{h,1,Length[u]}]*)

In[90]:= (*Table[
FullSimplify[ExpandAll[Transpose[EAa[h]].η8.EAa[h]]]- (u[[h]].σ.u[[h]])σ==Zero,
{h,1,Length[u}])*)

In[91]:= (*inη88=FullSimplify[Inverse[η8]];*)

In[92]:= (*Clear[ω(a)(b)];*) Clear[ω]; Symbolize[ω(a)(b)]

In[93]:= ω = Array[ω(a)(b) [##]&, {8, 8, 8}];
Protect[ω]

Out[94]= {ω}

```

constants

```

← ERROR : 08similarityTransformation has 1 st index that transforms as  $\frac{\partial}{\partial \Psi}$  , not as  $\Psi$ 
In[95]:= (* 08similarityTransformation has 1st index that transforms as  $\frac{\partial}{\partial \Psi}$  , not as  $\Psi$  *)
          08similarityTransformation has 1 st index that transforms as  $\frac{\partial}{\partial \Psi}$  , not as  $\Psi$  :

In[96]:= (*ArrayFlatten[{{IdentityMatrix[4],0},{0,-IdentityMatrix[4]}}]*)

In[97]:= ID4 = IdentityMatrix[4];
          ID8 = IdentityMatrix[8];

In[99]:= η4488 // MatrixForm
Out[99]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$


In[100]:= η4488[[#, #]] & /@ Range[8]
Out[100]= {1, 1, 1, 1, -1, -1, -1, -1}

In[101]:= 

Zero4 = ConstantArray[0, {4, 4}]; Zero8 = ConstantArray[0, {8, 8}];
mid = ExpandAll[-1 * ID4];
ε3 = Array[Signature[{##}] &, {3, 3, 3}];
ε4 = Array[Signature[{##}] &, {4, 4, 4, 4}];
MId = ExpandAll[-1 * ID8];
simp = {Zero4 → 0, ID4 → 1, mid → -1};
Simp = {Zero8 → ZERO, ID8 → ONE, MId → MONE};

Out[103]= {{0, 0, 0}, {0, 0, 1}, {0, -1, 0}},
{{0, 0, -1}, {0, 0, 0}, {1, 0, 0}}, {{0, 1, 0}, {-1, 0, 0}, {0, 0, 0}}
```

In[108]:=

```

Zero16 = ConstantArray[0, {16, 16}];
G16 = ArrayFlatten[{{η8, 0}, {0, -η8}}];
Id16 = IdentityMatrix[16]; MId16 = ExpandAll[-1 * Id16];
Simp16 = {Zero16 → ZERO16, Id16 → ONE16, MId16 → MONE16};

In[112]:= 
```

```

ZERO16 = ConstantArray[0, {16, 16}];
G16 = ArrayFlatten[{{η4488, 0}, {0, -η4488}}];
ID16 = IdentityMatrix[16]; MId16 = ExpandAll[-1 * ID16];
Simp16 = {ZERO16 → ZERO16, ID16 → ONE16, MId16 → MONE16};
```

```
In[116]:= ws = Flatten[Table[
  ToExpression["w" <> ToString[A1] <> ToString[B1]], {A1, 1, 7}, {B1, A1 + 1, 8}]]
Length[ws]

Out[116]= {w12, w13, w14, w15, w16, w17, w18, w23, w24, w25, w26, w27, w28,
w34, w35, w36, w37, w38, w45, w46, w47, w48, w56, w57, w58, w67, w68, w78}

Out[117]= 28

In[118]:= σ.σ - ID8 == Zero8
Tr[σ] == 0

Out[118]= True

Out[119]= True
```

some function definitions:

```
In[120]:= Clear[rawSets];
rawSets[l_, f_, o_: 0] := Module[{t, h},
  t = Table[Cases[
    l[[j]],
    h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], f] & n + o,
    Infinity,
    Heads → True
  ], {j, 1, Length[l]}];
  Return[t];
]

In[121]:= (*rawSetsel16=rawSets[e116,"f16",0]*)
```

```
In[123]:= (*couplings=showCoupledEquations[rawSetsel16]
%//Flatten//Sort
%==Range[0,15]*)
```

```
In[124]:= Clear[dualRawSets];
dualRawSets[l_, f_, o_, h_, oh_] := Module[{r, r1, r2},
  r1 = rawSets[l, f, o];
  r2 = rawSets[l, h, oh];
  r = Union[Flatten[{r1, r2}, 1]] // Drop[#, 1] &;
  Return[r];
]
```

'FixedPoint', below, uses 'MergeSetsStep', below.

According to Wolfram docs, one uses 'Throw' to exit from 'FixedPoint' before it is finished.

▼ Details and Options

* FixedPoint always returns the last result it gets

FixedPoint really uses Catch and Throw, so what could go wrong

▼ Details and Options

- FixedPoint always returns the last result it gets.
- You can use Throw to exit from FixedPoint before it is finished.
- FixedPoint[f, expr] applies SameQ to successive pairs of results to determine whether a fixed point has been reached.
- FixedPoint[f, expr, ..., SameTest -> s] applies s to successive pairs of results.

In[126]:=

```
(* Helper to perform one pass of merging *)
MergeSetsStep[currentSets_List] :=
  Module[{n = Length[currentSets], i, j, merged = currentSets},
  (* We look for the first pair (i, j) that intersects *)
  Catch[
    For[i = 1, i <= Length[merged], i++,
      For[j = i + 1, j <= Length[merged], j++,
        If[Intersection[merged[[i]], merged[[j]]] != {},
          (* Found intersection: Merge and Throw to restart/finish this step *)
          merged = Delete[merged, {{i}, {j}}];
          AppendTo[merged, Sort[Union[currentSets[[i]], currentSets[[j]]]]];
          Throw[Sort[merged]]; (* Sort for canonical form *)
        ];
      ];
    ];
    (* If no intersection found, return original sorted *)
    Sort[merged]
  ];
];
```

In[127]:=

```
showCoupledEquations[items_List] := FixedPoint[MergeSetsStep, items];
```

In[128]:=

```
Clear[safeInverse];
safeInverse[m_?MatrixQ] := Module[{a, b},
  {a, b} = Dimensions[m];
  Catch[If[a != b, Throw[Return[{a, b}]], ## &[]]];
  Catch[If[Det[m] == 0, Throw[Return["Error: Singular Matrix"]], ## &[]]];
  Return[Inverse[m]];];
```

```
In[130]:= blockPartition[mat_] := Module[{(*blocks,block11,block12,block21,block22,*)ret},
  blocks = Partition[mat, {8, 8}];
  (*block11=blocks[[1,1]] ;*) (*Top-left block*)
  (*block12=blocks[[1,2]] ;*) (*Top-right block*)
  (*block21=blocks[[2,1]] ;*) (*Bottom-left block*)
  (*block22=blocks[[2,2]] ;*) (*Bottom-right block*)
  ret = ArrayFlatten[
    {{blocks[[1, 1]], blocks[[1, 2]]}, {blocks[[2, 1]], blocks[[2, 2]]}}] === mat;
  If[ret,
    Return[{{blocks[[1, 1]], blocks[[1, 2]]}, {blocks[[2, 1]], blocks[[2, 2]]}}], ##&[]];
  Return[Null];]
```

In[131]:=

```
eextract[a_] := Extract[a, {#}] & /@ Range[0, Length[a]]
```

Sequence[Subscript[expr, 1], Subscript[expr, 2], ...]

represents a sequence of arguments to be spliced automatically into any function .

In[132]:=

```
skelx[x_, matQ_ : True] := Module[{t = {666}},
  If[matQ == True, t = Block[{r},
    r = x;
    Table[Block[{q}, q = SameQ[0, r[[j, i]]];
      If[True == q, Style[0, Blue], Style[Length[r[[j, i]]], Red]]],
      {j, 1, Length[r]}, {i, 1, Length[r[[1]]]}]];
    If[matQ == False, t = Block[{r},
      r = x;
      Table[Block[{q}, q = SameQ[0, r[[j]]];
        If[True == q, Style[0, {RGBColor -> {0, 1/3, 0}, Bold}],
          Style[Length[r[[j]]], Red]]], {j, 1, Length[r]}]];
      MatrixForm[t, TableAlignments -> Left]]]
```

In[133]:=

```
makeSym[size_, fn_] := Module[{rtmp}, rtmp = Table[fn[i, j], {i, 1, size}, {j, 1, i}];
  MapThread[Join, {rtmp, Rest /@ Flatten[rtmp, {{2}, {1}}]}]]
```

In[134]:=

```
makeAntiSym[size_, fn_] :=
Module[{rtmp}, rtmp = Table[If[i == j, 0, fn[i, j]], {i, 1, size}, {j, 1, i}];
  MapThread[Join, {rtmp, Rest /@ Flatten[-rtmp, {{2}, {1}}]}]]
```

```
In[135]:= 
Block[{MX}, MX = makeAntiSym[8, Subscript[\[omega], ##] &]]
```

```
Out[135]=
{ {0, -\omega_{2,1}, -\omega_{3,1}, -\omega_{4,1}, -\omega_{5,1}, -\omega_{6,1}, -\omega_{7,1}, -\omega_{8,1}},
{ \omega_{2,1}, 0, -\omega_{3,2}, -\omega_{4,2}, -\omega_{5,2}, -\omega_{6,2}, -\omega_{7,2}, -\omega_{8,2} },
{ \omega_{3,1}, \omega_{3,2}, 0, -\omega_{4,3}, -\omega_{5,3}, -\omega_{6,3}, -\omega_{7,3}, -\omega_{8,3} },
{ \omega_{4,1}, \omega_{4,2}, \omega_{4,3}, 0, -\omega_{5,4}, -\omega_{6,4}, -\omega_{7,4}, -\omega_{8,4} },
{ \omega_{5,1}, \omega_{5,2}, \omega_{5,3}, \omega_{5,4}, 0, -\omega_{6,5}, -\omega_{7,5}, -\omega_{8,5} },
{ \omega_{6,1}, \omega_{6,2}, \omega_{6,3}, \omega_{6,4}, \omega_{6,5}, 0, -\omega_{7,6}, -\omega_{8,6} },
{ \omega_{7,1}, \omega_{7,2}, \omega_{7,3}, \omega_{7,4}, \omega_{7,5}, \omega_{7,6}, 0, -\omega_{8,7} },
{ \omega_{8,1}, \omega_{8,2}, \omega_{8,3}, \omega_{8,4}, \omega_{8,5}, \omega_{8,6}, \omega_{8,7}, 0 } }
```

If ‘paste’ Maple MathML to ToExpression[MathMLForm[]] fails, then try to edit this :

O(4,4); evals, evecs of σ

```
In[136]:= 
{evals, evecs} = Eigensystem[\[sigma]]
```

```
Out[136]=
{ {-1, -1, -1, -1, 1, 1, 1, 1}, {{0, 0, 0, -1, 0, 0, 0, 1}, {0, 0, -1, 0, 0, 0, 1, 0},
{0, -1, 0, 0, 1, 0, 0}, {-1, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 1},
{0, 0, 1, 0, 0, 0, 1, 0}, {0, 1, 0, 0, 0, 1, 0, 0}, {1, 0, 0, 0, 1, 0, 0, 0}}}
```

```
In[137]:= 
u = ExpandAll[1/Sqrt[2] evecs] // MatrixForm
```

```
Out[137]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

```

```
In[138]:= 
Table[u[[h]].\[sigma].u[[h]], {h, 1, Length[u]}]
```

```
Out[138]=
{-1, -1, -1, -1, 1, 1, 1, 1}
```

```
In[139]:= 
Table[(Transpose[u][[h]]).\[sigma].u[[h]], {h, 1, Length[u]}]
```

```
Out[139]=
{-1, -1, -1, -1, 1, 1, 1, 1}
```

```
In[140]:= Table[(Transpose[u][[A]]) == -σ.u[[A]], {A, 1, 4}]
Table[(Transpose[u][[A]]) == σ.u[[A]], {A, 5, Length[u]}]

Out[140]= {True, True, True, True}

Out[141]= {True, True, True, True}
```

$\text{SO}(4), \gamma; M8, \{0, 1, 2, 3, 4, 5, 6, 7\}, \{+, +, +, -, -, -, -\}, \{x0, x1, x2, x3, x4, x5, x6, x7\}$

In[142]:=

Under the involutive automorphism $\gamma^{AB} \rightarrow -\tilde{\gamma}^{AB}$ of $\text{so}(3,3)$, the Lie algebra decomposes into the eigenvalue (-1) and eigenvalue $(+1)$ subspaces corresponding to, respectively, the nine linearly independent real traceless symmetric 4×4 matrices, and the six linearly independent real skew-symmetric 4×4 matrices. The eigenvalue $(+1)$ subspace is the subalgebra $\text{so}(4)$, which is the Lie algebra of $\overline{\text{SO}(4)}$, the maximal compact subgroup of $\overline{\text{SL}(4, \mathbb{R})}$. The subalgebra $\text{so}(4) \cong \text{su}(2) + \text{su}(2)$ may be further decomposed into the even (eigenvalue $+1$) and odd (eigenvalue -1) subspaces of the linear transformation of $\text{so}(4)$ whereby $\tau \in \text{so}(4)$ is mapped into its dual, ${}^*\tau$. The even subspace under * of $\text{so}(4)$ corresponds to self-dual tensors, and, say, the first $\text{su}(2)$ in the direct sum; the odd subspace corresponds to anti-self-dual tensors, and the second $\text{su}(2)$ in the direct sum. A basis for $\text{so}(4)$ may be chosen as follows. Each of the six skew-symmetric γ matrices has the property that the square of the matrix is equal to $-\gamma_0$. By Eq. (1), these six matrices are given by ($h = 1, 2, 3$),

$$2s^h = (\gamma^{23}, \gamma^{31}, \gamma^{21}), \quad (4)$$

and

$$2t^h = (\gamma^{45}, \gamma^{64}, \gamma^{65}). \quad (5)$$

Out[142]=

Under the involutive automorphism $\gamma^{AB} \rightarrow -\tilde{\gamma}^{AB}$ of $so(3,3)$, the Lie algebra decomposes into the eigenvalue (-1) and eigenvalue $(+1)$ subspaces corresponding to, respectively, the nine linearly independent real traceless symmetric 4×4 matrices, and the six linearly independent real skew-symmetric 4×4 matrices. The eigenvalue $(+1)$ subspace is the subalgebra $so(4)$, which is the Lie algebra of $\overline{SO(4)}$, the maximal compact subgroup of $\overline{SL(4,R)}$. The subalgebra $so(4) \cong su(2) + su(2)$ may be further decomposed into the even (eigenvalue $+1$) and odd (eigenvalue -1) subspaces of the linear transformation of $so(4)$ whereby $\tau \in so(4)$ is mapped into its dual, ${}^*\tau$. The even subspace under * of $so(4)$ corresponds to self-dual tensors, and, say, the first $su(2)$ in the direct sum; the odd subspace corresponds to anti-self-dual tensors, and the second $su(2)$ in the direct sum. A basis for $so(4)$ may be chosen as follows. Each of the six skew-symmetric γ matrices has the property that the square of the matrix is equal to $-\gamma_0$. By Eq. (1), these six matrices are given by ($h = 1, 2, 3$),

$$2s^h = (\gamma^{23}, \gamma^{31}, \gamma^{21}), \quad (4)$$

and

$$2t^h = (\gamma^{45}, \gamma^{64}, \gamma^{65}). \quad (5)$$

In[143]:=

```
Qa[h_, p_, q_] := Signature[{h, p, q, 4}];
Qb[h_, p_, q_] := ID4[p, 4] * ID4[q, h] - ID4[p, h] * ID4[q, 4];
SelfDualAntiSymmetric[h_, p_, q_] := Qa[h, p, q] - Qb[h, p, q];
AntiSelfDualAntiSymmetric[h_, p_, q_] := (Qa[h, p, q] + Qb[h, p, q]);
```

In[147]:=

```
Qa[1, 2, 3]
```

Out[147]=

```
1
```

In[148]:=

```
e4[[1, 2, 3, 4]]
```

Out[148]=

```
1
```

In[149]:=

```
Do[
```

```
s4by4[h] = Table[Table[SelfDualAntiSymmetric[h, p, q], {q, 4}], {p, 4}], {h, 1, 3}]
```

```
In[150]:= Do[t4by4[h] = Table[Table[AntiSelfDualAntiSymmetric[h, p, q], {q, 4}], {p, 4}], {h, 1, 3}]

In[151]:= (*Protect[{s4by4[1],s4by4[2],s4by4[3],t4by4[1],t4by4[2],t4by4[3]}]*)

In[152]:= Protect[s4by4, t4by4]

Out[152]= {s4by4, t4by4}

In[153]:= Clear[sixAntiSymmetric8by8];
sixAntiSymmetric8by8 = Flatten[{Table[ArrayFlatten[{{0, s4by4[h]}, {s4by4[h], 0}}], {h, 1, 3}], Table[ArrayFlatten[{{0, t4by4[h]}, {-t4by4[h], 0}}], {h, 3, 1, -1}]}, 1];

In[154]:= sixAntiSymmetric8by8

Out[154]= {{ {0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0},
{0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
{0, 0, 1, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 1, 0, 0, 0},
{0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0},
{0, 0, 0, 1, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1},
{0, 0, 0, 0, 0, 0, -1, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1},
{0, 0, 0, 0, 0, 0, 1, 0}, {0, -1, 0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}, {0, 0, 0, 0, 1, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
{-1, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, -1}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, -1, 0, 0},
{0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0},
{0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}}}

In[155]:= {s4by4[#] // MatrixForm, MatrixForm[t4by4[#]]} & /@ Range[3]

Out[155]= {{ {0, 0, 0, 1}, {0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0} },
{ {0, 0, -1, 0}, {0, 0, 0, 1}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0} },
{ {0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}, {0, 0, 0, -1} },
{ {0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 0, -1}, {0, 0, 1, 0} } }
```

```
In[156]:= Table[{{J, K}, MatrixForm[s4by4[J].t4by4[K]]}, {J, 1, 3}, {K, 1, 3}]
Out[156]=
{{{{1, 1}, {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}}}, {{1, 2}, {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}}, {{1, 3}, {{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}}}}, {{{2, 1}, {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}}}, {{2, 2}, {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}}}, {{2, 3}, {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}}}, {{{3, 1}, {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}}}, {{3, 2}, {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}}}, {{3, 3}, {{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}}}]
```

```
In[157]:= Do[st[J, K] = s4by4[J].t4by4[K], {J, 1, 3}, {K, 1, 3}]
```

```
In[158]:= Table[{{J, K}, MatrixForm[st[J, K]]}, {J, 1, 3}, {K, 1, 3}]
```

```
Out[158]=
{{{{1, 1}, {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}}}, {{1, 2}, {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}}, {{1, 3}, {{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}}}}, {{{2, 1}, {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}}}, {{2, 2}, {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}}}, {{2, 3}, {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}}}, {{{3, 1}, {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}}}, {{3, 2}, {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}}}, {{3, 3}, {{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}}}]
```

```
In[159]:= Protect[st]
```

```
Out[159]= {st}
```

```
In[160]:= Table[{{J, K}, MatrixForm[st[J, K] - Transpose[st[J, K]]]}, {J, 1, 3}, {K, 1, 3}]
Out[160]=
{{{{1, 1}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}, {{1, 2}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}, {{1, 3}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}}, {{{{2, 1}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}, {{2, 2}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}, {{2, 3}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}}, {{{{3, 1}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}, {{3, 2}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}, {{3, 3}, {0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0}}}}
```

In[161]:= Permutations[Range[3], 2]

```
Out[161]= {{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 1}, {2, 3}, {3, 1}, {3, 2}}
```

In[162]:= KroneckerProduct[Range[3], Range[3]]

```
Out[162]= {{1, 2, 3}, {2, 4, 6}, {3, 6, 9}}
```

In[163]:= Table[{J, K}, {J, 1, 3}, {K, 1, 3}]
st[##]& /@%
st[##]& @@ %

```
Out[163]= {{{1, 1}, {1, 2}, {1, 3}}, {{2, 1}, {2, 2}, {2, 3}}, {{3, 1}, {3, 2}, {3, 3}}}

Out[164]= {st[{{1, 1}, {1, 2}, {1, 3}}], st[{{2, 1}, {2, 2}, {2, 3}}], st[{{3, 1}, {3, 2}, {3, 3}}]}

Out[165]= st[{{1, 1}, {1, 2}, {1, 3}}, {{2, 1}, {2, 2}, {2, 3}}, {{3, 1}, {3, 2}, {3, 3}}]}
```

```
In[166]:= Flatten[{s4by4[#] & /@ Range[3], t4by4[#] & /@ Range[3],
  Flatten[Table[st[J, K], {J, 1, 3}, {K, 1, 3}], 1], {ID4}], 1]
Length[%]

Out[166]= {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}},
{{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}},
{{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
{{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
{{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
{{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
{{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}},
{{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}},
{{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}},
{{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}

Out[167]= 16

In[168]:= Unprotect[yx];
yx = Flatten[{s4by4[#] & /@ Range[3], t4by4[#] & /@ Range[3],
  Flatten[Table[st[J, K], {J, 1, 3}, {K, 1, 3}], 1], {ID4}], 1]

Out[169]= {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}},
{{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}},
{{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
{{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
{{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
{{0, 0, 1, 0}, {0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
{{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
{{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}},
{{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}},
{{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}},
{{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}
```

```
In[170]:= Length[ $\gamma x$ ]
Out[170]= 16

In[171]:= Protect[ $\gamma x$ ]
Out[171]= { $\gamma x$ }

In[172]:=  $\gamma x[[1]]$ 
Out[172]= {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}}
```

```
In[173]:=  $\gamma x[[-1]]$ 
Out[173]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

Self Dual Anti-Symmetric

```
In[174]:= Table

```

Anti SelfDual Anti-Symmetric

```
In[175]:= Table

```

$\text{SO}(4,4), \text{Spin}(4,4) = \overline{\text{SO}(4,4)}$, τ ; if using xact: M8, {0, 1, 2, 3, 4, 5, 6, 7}, {+, +, +, -, -, -}, {x0, x1, x2, x3, x4, x5, x6, x7}

In[176]:=

$$\sigma \bar{\tau}^A = \widetilde{\sigma \tau^A} = \widetilde{\tau}^A \sigma$$

$$\tau^A \bar{\tau}^B + \tau^B \bar{\tau}^A = 2\mathbb{I}_{8 \times 8} G^{AB} = \bar{\tau}^A \tau^B + \bar{\tau}^B \tau^A,$$

8 unit matrix. Denoting the matrix elemen

$$\bar{\tau}^A_{ab} = \tau^A_{ba}$$

042501-7 Second gravity

J. Math. Phys. 51, 042501 (2010)

$$t^A t^B - t^B t^A = \begin{pmatrix} \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A & 0 \\ 0 & \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A \end{pmatrix} = 4 \begin{pmatrix} D_{(1)}^{AB} & 0 \\ 0 & D_{(2)}^{AB} \end{pmatrix} \quad (10)$$

of the 16-component spinor representation of $\overline{\text{SO}(4,4;\mathbb{R})}$. We see that, as is, in fact, well known from the general theory, the 16-component spinor representation of $\overline{\text{SO}(4,4;\mathbb{R})}$ is the direct sum of two (inequivalent) real 8×8 irreducible spinor representations $D_{(1)} = D_{(1)}(g)$ and $D_{(2)} = D_{(2)}(g)$ of $\overline{\text{SO}(4,4;\mathbb{R})} \ni g$ that are generated by $D_{(1)}^{AB}$ and $D_{(2)}^{AB}$, respectively. This we record as

$$4D_{(1)}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A \quad (11)$$

and

$$4D_{(2)}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A. \quad (12)$$

Out[176]=

$$\sigma \bar{\tau}^A = \widetilde{\sigma \tau^A} = \widetilde{\tau}^A \sigma$$

$$\tau^A \bar{\tau}^B + \tau^B \bar{\tau}^A = 2\mathbb{I}_{8 \times 8} G^{AB} = \bar{\tau}^A \tau^B + \bar{\tau}^B \tau^A,$$

8 unit matrix. Denoting the matrix elements

$$\bar{\tau}^A_{ab} = \tau^A_{ba}$$

042501-7 Second gravity

J. Math. Phys. 51, 042501 (2010)

$$t^A t^B - t^B t^A = \begin{pmatrix} \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A & 0 \\ 0 & \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A \end{pmatrix} = 4 \begin{pmatrix} D_{(1)}^{AB} & 0 \\ 0 & D_{(2)}^{AB} \end{pmatrix} \quad (10)$$

of the 16-component spinor representation of $\overline{\text{SO}(4,4;\mathbb{R})}$. We see that, as is, in fact, well known from the general theory, the 16-component spinor representation of $\overline{\text{SO}(4,4;\mathbb{R})}$ is the direct sum of two (inequivalent) real 8×8 irreducible spinor representations $D_{(1)} = D_{(1)}(g)$ and $D_{(2)} = D_{(2)}(g)$ of $\overline{\text{SO}(4,4;\mathbb{R})} \ni g$ that are generated by $D_{(1)}^{AB}$ and $D_{(2)}^{AB}$, respectively. This we record as

$$4D_{(1)}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A \quad (11)$$

and

$$4D_{(2)}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A. \quad (12)$$

In[177]:=

042501-8 Patrick L. Nash

J. Math. Phys. 51, 042501 (2010)

$$\widetilde{D_{(1)}}^{AB}\sigma = -\sigma D_{(1)}^{AB} \Rightarrow \tilde{D}_{(1)}\sigma = \sigma D_{(1)}^{-1}, \quad (18)$$

$$\widetilde{D_{(2)}}^{AB}\sigma = -\sigma D_{(2)}^{AB} \Rightarrow \tilde{D}_{(2)}\sigma = \sigma D_{(2)}^{-1}, \quad (19)$$

$$L^A{}_C G_{AB} L^B{}_D = G_{CD} = \{\tilde{L}GL\}_{CD}, \quad (20)$$

$$L^A{}_B \bar{\tau}^B = D_{(1)}^{-1} \bar{\tau}^A D_{(2)}, \quad (21)$$

$$L^A{}_B \tau^B = D_{(2)}^{-1} \tau^A D_{(1)}, \quad (22)$$

The canonical 2-1 homomorphism $\overline{\text{SO}(4,4;\mathbb{R})} \rightarrow \text{SO}(4,4;\mathbb{R}): g \mapsto L(g)$ is given by

$$8L^A{}_B = \text{tr}(D_{(1)}^{-1} \bar{\tau}^A D_{(2)} \tau^B) G_{CB}, \quad (23)$$

where tr denotes the trace. Note that $D_{(1)}(g(\omega)) = D_{(2)}(g(\omega))$ when $\omega_{A8} = 0$, i.e., when one restricts $\overline{\text{SO}(4,4;\mathbb{R})}$ to

$$\overline{\text{SO}(3,4;\mathbb{R})} = \{g \in \overline{\text{SO}(4,4;\mathbb{R})} \mid ,$$

$$g = \begin{pmatrix} \exp\left(\frac{1}{4}\omega_{AB} D_{(1)}^{AB}\right) & 0 \\ 0 & \exp\left(\frac{1}{4}\omega_{AB} D_{(2)}^{AB}\right) \end{pmatrix} \quad \text{and} \quad \omega_{A8} = 0 \} \quad (24)$$

[one of the real forms of spin (7, C)].

Out[177]:=

042501-8 Patrick L. Nash

J. Math. Phys. 51, 042501 (2010)

$$\widetilde{D}_{(1)}^{AB}\sigma = -\sigma D_{(1)}^{AB} \Rightarrow \widetilde{D}_{(1)}\sigma = \sigma D_{(1)}^{-1}, \quad (18)$$

$$\widetilde{D}_{(2)}^{AB}\sigma = -\sigma D_{(2)}^{AB} \Rightarrow \widetilde{D}_{(2)}\sigma = \sigma D_{(2)}^{-1}, \quad (19)$$

$$L^A{}_C G_{AB} L^B{}_D = G_{CD} = \{\tilde{L}GL\}_{CD}, \quad (20)$$

$$L^A{}_B \bar{\tau}^B = D_{(1)}^{-1} \bar{\tau}^A D_{(2)}, \quad (21)$$

$$L^A{}_B \tau^B = D_{(2)}^{-1} \tau^A D_{(1)}, \quad (22)$$

The canonical 2-1 homomorphism $\overline{\mathrm{SO}(4,4;\mathbb{R})} \rightarrow \mathrm{SO}(4,4;\mathbb{R}): g \mapsto L(g)$ is given by

$$8L^A{}_B = \mathrm{tr}(D_{(1)}^{-1} \bar{\tau}^A D_{(2)} \tau^B) G_{CB}, \quad (23)$$

where tr denotes the trace. Note that $D_{(1)}(g(\omega)) = D_{(2)}(g(\omega))$ when $\omega_{A8}=0$, i.e., when one restricts $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to

$$\overline{\mathrm{SO}(3,4;\mathbb{R})} = \{g \in \overline{\mathrm{SO}(4,4;\mathbb{R})} \mid$$

$$g = \begin{pmatrix} \exp\left(\frac{1}{4}\omega_{AB} D_{(1)}^{AB}\right) & 0 \\ 0 & \exp\left(\frac{1}{4}\omega_{AB} D_{(2)}^{AB}\right) \end{pmatrix} \quad \text{and} \quad \omega_{A8} = 0 \} \quad (24)$$

[one of the real forms of spin (7, C)].

In[178]:=

 $\eta4488 // \mathrm{MatrixForm}$

Out[178]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[179]:=

 $\eta4488 == \mathrm{DiagonalMatrix}[\mathrm{RotateRight}[\mathrm{values}, 4]]$

Out[179]=

True

In[180]:=

```
DiagonalMatrix[RotateRight[evalues, 3]] // MatrixForm
```

Out[180]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[181]:=

```

Clear[\tau];
\tau[0] = ID8;
Do[\tau[h] = sixAntiSymmetric8by8[h], {h, 1, 6}];
(\tau[7] = \tau[1].\tau[2].\tau[3].\tau[4].\tau[5].\tau[6]) // MatrixForm
(*\bar{\tau}[#] = \sigma. Transpose[\sigma.\tau[#]]&@Range[0,7]*)
Do[\bar{\tau}[h] = \sigma. Transpose[\sigma.\tau[h]], {h, 0, 7}]
Table[\{\{A1, B1}, \frac{1}{2} (\tau[A1].\bar{\tau}[B1] + \tau[B1].\bar{\tau}[A1]) === \eta^4
    \{A1, 0, 7\}, \{B1, 0, 7\}\} // MatrixForm

```

Out[184]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[186]//MatrixForm=

```
In[187]:= σ == τ[1].τ[2].τ[3]
Out[187]= True

In[188]:= Table[{A, -η4488[A+1, A+1]*τ[A] == Transpose[τ[A]]}, {A, 1, 7}]
Out[188]= {{1, True}, {2, True}, {3, True}, {4, True}, {5, True}, {6, True}, {7, True}}
```

In[189]:= σ.τ[1].τ[2].τ[3] // MatrixForm

```
Out[189]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```

In[190]:= τ[7] // MatrixForm

```
Out[190]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```

In[191]:= (Ω = σ.τ[7]) // MatrixForm

```
Out[191]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

```

In[192]:= Ω == τ[4].τ[5].τ[6]

```
Out[192]= True
```

In[193]:=

$$\tau[5].\tau[6].\tau[7] // \text{MatrixForm}$$

Out[193]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

In[194]:=

$$(\sigma\Omega = \sigma.\Omega) // \text{MatrixForm}$$

Out[194]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[195]:=

$$\text{Inverse}[\sigma\Omega] // \text{MatrixForm}$$

Out[195]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[196]:=

$$\tau[1].\tau[2].\tau[3].\tau[4].\tau[5].\tau[6].\tau[7] == \tau[0] == \text{ID8}$$

Out[196]=

True

In[197]:=

$$(\sigma == \tau[1].\tau[2].\tau[3])$$

Out[197]=

True

In[198]:=

$$(\sigma == \tau[4].\tau[5].\tau[6].\tau[7])$$

Out[198]=

True

In[199]:=

$$\tau[1].\tau[2].\tau[3].\tau[\#] == -\text{Transpose}[\tau[1].\tau[2].\tau[3].\tau[\#]] \& /@ \text{Range}[0, 7]$$

Out[199]=

{False, True, True, True, True, True, True}

In[200]:=

$$\sigma.\bar{\tau}[\#] == \text{Transpose}[\sigma.\tau[\#]] \& /@ \text{Range}[0, 7]$$

Out[200]=

{True, True, True, True, True, True, True}

In[201]:=

$$\tau[5].\tau[6].\tau[7].\bar{\tau}[\#] == \text{Transpose}[\tau[5].\tau[6].\tau[7].\tau[\#]] \& /@ \text{Range}[0, 7]$$

Out[201]=

{False, True, True, True, False, True, True}

In[202]:=

$$\tau[4].\tau[\#] == \text{Transpose}[\tau[4].\tau[\#]] \& /@ \text{Range}[0, 7]$$

Out[202]=

{True, True, True, True, True, False, False}

In[203]:=

$$\tau[4].\bar{\tau}[\#] == \text{Transpose}[\tau[4].\tau[\#]] \& /@ \text{Range}[0, 7]$$

Out[203]=

{True, False, False, False, True, True, True}

In[204]:=

$$\tau[4].\tau[\#] == \text{Transpose}[\tau[4].\tau[\#]] \& /@ \text{Range}[0, 7]$$

Out[204]=

{True, True, True, True, True, False, False}

$$t^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}. \quad ; \quad 0(4, 4) : \text{SAB} ;$$

$$\text{covariantDiffMatrix} = T16^A[5].T16^A[6].T16^A[7]$$

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix} \cdot \begin{pmatrix} \tilde{a} & \tilde{c} \\ \tilde{b} & \tilde{d} \end{pmatrix} = \begin{pmatrix} b.\tau^A & a.\bar{\tau}^A \\ d.\tau^A & c.\bar{\tau}^A \end{pmatrix} \cdot \begin{pmatrix} \tilde{a} & \tilde{c} \\ \tilde{b} & \tilde{d} \end{pmatrix}$$

$$= \begin{pmatrix} b.\tau^A.\tilde{a} + a.\bar{\tau}^A.\tilde{b} & b.\tau^A.\tilde{c} + a.\bar{\tau}^A.\tilde{d} \\ d.\tau^A.\tilde{a} + c.\bar{\tau}^A.\tilde{b} & d.\tau^A.\tilde{c} + c.\bar{\tau}^A.\tilde{d} \end{pmatrix}$$

Need these in order to define the Lagrangian for the "universe," later :

In[205]:=

```
(σ16 = T16A[0].T16A[1].T16A[2].T16A[3]) // MatrixForm
```

Out[205]//MatrixForm=

`T16A[0].T16A[1].T16A[2].T16A[3]`

In[206]:=

```
Table[T16A[A1] = ArrayFlatten[{{{0, τ[A1]}, {τ[A1], 0}}}], {A1, 0, 7}];
```

In[207]:=

```
(T16A[8] = T16A[0].T16A[1].T16A[2].T16A[3].T16A[4].T16A[5].T16A[6].T16A[7]) //  
MatrixForm
```

Out[207]//MatrixForm=

In[208]:=

$\sigma_{16} // \text{MatrixForm}$

Out[208]//MatrixForm=

In[209]:=

$$\begin{aligned} \mathbf{T16^A}[0] \cdot \mathbf{T16^A}[1] \cdot \mathbf{T16^A}[2] \cdot \mathbf{T16^A}[3] \cdot \mathbf{T16^A}[\#] &= \\ -\text{Transpose}[\mathbf{T16^A}[0] \cdot \mathbf{T16^A}[1] \cdot \mathbf{T16^A}[2] \cdot \mathbf{T16^A}[3] \cdot \mathbf{T16^A}[\#]] &\& /@ \text{Range}[0, 8] \end{aligned}$$

Out[209]=

{True, True, True, True, True, True, True, False}

In[210]:=

$$\begin{aligned} \mathbf{T16^A}[4] \cdot \mathbf{T16^A}[5] \cdot \mathbf{T16^A}[6] \cdot \mathbf{T16^A}[7] \cdot \mathbf{T16^A}[\#] &= \\ \text{Transpose}[\mathbf{T16^A}[4] \cdot \mathbf{T16^A}[5] \cdot \mathbf{T16^A}[6] \cdot \mathbf{T16^A}[7] \cdot \mathbf{T16^A}[\#]] &\& /@ \text{Range}[0, 8] \end{aligned}$$

Out[210]=

{True, True, True, True, True, True, True, True}

In[211]:=

$$\mathbf{T16^A}[8] = \sigma_{16} \cdot \mathbf{T16^A}[4] \cdot \mathbf{T16^A}[5] \cdot \mathbf{T16^A}[6] \cdot \mathbf{T16^A}[7]$$

Out[211]=

True

In[212]:=

$$\mathbf{T16^A}[\#] = \text{Transpose}[\mathbf{T16^A}[\#]] \& /@ \text{Range}[0, 8]$$

Out[212]=

{True, True, True, True, False, False, False, True}

In[213]:=

$$\mathbf{T16^A}[\#] = -\text{Transpose}[\mathbf{T16^A}[\#]] \& /@ \text{Range}[0, 8]$$

Out[213]=

{False, False, False, False, True, True, True, False}

In[214]:=

$$\sigma_{16} \cdot \mathbf{T16^A}[\#] = -\text{Transpose}[\sigma_{16} \cdot \mathbf{T16^A}[\#]] \& /@ \text{Range}[0, 7]$$

Out[214]=

{True, True, True, True, True, True, True, True}

In[215]:=

```
(covariantDiffMatrix = T16A[5].T16A[6].T16A[7]) // MatrixForm
```

Out[215]//MatrixForm=

In[216]:=

```
σ16.covariantDiffMatrix = Transpose[σ16.covariantDiffMatrix]
```

Out[216]=

True

In[217]:=

ID16 // MatrixForm

Out[217]//MatrixForm=

```
MatrixForm[  
 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0  
 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0  
 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0  
 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0  
 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0  
 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0  
 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0  
 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0  
 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0  
 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0  
 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0  
 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0  
 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0  
 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0  
 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0  
 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1  
 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
```

In[218]:=

$$(\sigma16 (*=\text{T16}^A[0].\text{T16}^A[1].\text{T16}^A[2].\text{T16}^A[3]*)) // \text{MatrixForm}$$

Out[218]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[219]:=

$$\sigma16 = \text{ArrayFlatten}[\{\{-\sigma, 0\}, \{0, \sigma\}\}]$$

Out[219]=

True

later, after defining $e_{(A)}^\alpha$:

In[220]:=

$$(*\text{Table}[\text{T16}^\alpha[\alpha1-1]=\text{Sum}[(e_{(A)}^\alpha[[\alpha1,A1]] \) \ \text{T16}^A[A1-1],\{A1,1,8\}],\{\alpha1,1,8\}];*)$$

In[221]:=

$$(* (\text{T16}^\alpha[8]=\text{T16}^\alpha[0].\text{T16}^\alpha[1].\text{T16}^\alpha[2].\text{T16}^\alpha[3].\text{T16}^\alpha[4].\text{T16}^\alpha[5].\text{T16}^\alpha[6].\text{T16}^\alpha[7]) // \text{MatrixForm}*)$$

In[222]:=

$$(*\text{eLpairs}*)$$

In[223]:=

$$(*\Omega16a=\text{Array}[0\&, \{16,16\}]; \\ \{1+\#[[1]],1+\#[[2]]\} \&/@eLpairs \\ (\Omega16a[[\#[[1]],\#[[2]]]]=1) \&/@%*)$$

In[224]:=

$$(*\Omega16a//\text{MatrixForm}*)$$

In[225]:=

$$(* (\Omega16=\sigma16. (\Omega16a+\text{Transpose}[\Omega16a])) //\text{MatrixForm}*)$$

In[226]:=

```
T16A[0].T16A[4] // MatrixForm
T16A[0].T16A[4].σ16
(Ω16 = %) // MatrixForm
```

Out[226]//MatrixForm=

0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	-1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-1	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	-1	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	-1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-1	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0

Out[227]=

Out[228]//MatrixForm=

In[229]:=

Det [Ω16]

Out[229]=

1

In[230]:=

σ16.Ω16 // MatrixForm

Out[230]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[231]:=

σ16.T16^A[#] == -Transpose[σ16.T16^A[#]] & /@ Range[0, 7]

Out[231]=

{True, True, True, True, True, True, True}

In[232]:=

σ16.T16^A[#] == -Transpose[σ16.T16^A[#]] & /@ Range[0, 7]

Out[232]=

{True, True, True, True, True, True, True}

In[233]:=

σ16.T16^A[#] == -Transpose[σ16.T16^A[#]] & /@ Range[0, 7]

Out[233]=

{True, True, True, True, True, True, True}

In[234]:=

σ16.newT16^A[#] == -Transpose[σ16.newT16^A[#]] & /@ Range[0, 7]

Out[234]=

$$\left\{ \{\{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}\} \right\}$$

In[235]:=

```
Out[235]=
{newo16.newT16A[0] == -(newo16.newT16A[0])T, newo16.newT16A[1] == -(newo16.newT16A[1])T,
 newo16.newT16A[2] == -(newo16.newT16A[2])T, newo16.newT16A[3] == -(newo16.newT16A[3])T,
 newo16.newT16A[4] == -(newo16.newT16A[4])T, newo16.newT16A[5] == -(newo16.newT16A[5])T,
 newo16.newT16A[6] == -(newo16.newT16A[6])T, newo16.newT16A[7] == -(newo16.newT16A[7])T}
```

In[236]:=

```
Table[  
  { {A1, B1}, FullSimplify[ExpandAll[ $\frac{1}{2} (T16^A[A1].T16^A[B1] + T16^A[B1].T16^A[A1]) ==$   
     $\eta 4488 [A1 + 1, B1 + 1] * ID16$  ] ] }, {A1, 0, 7}, {B1, 0, 7} ] // MatrixForm
```

Out[236]//MatrixForm=

In[237]:=

```
Table[{ {A1, B1}, FullSimplify[
    ExpandAll[  $\frac{1}{2} (\text{newT16}^A[\text{A1}] . \text{newT16}^A[\text{B1}] + \text{newT16}^A[\text{B1}] . \text{newT16}^A[\text{A1}]) ==$ 
     $\eta4488[[\text{A1} + 1, \text{B1} + 1]] * \text{ID16}] }], {A1, 0, 7}, {B1, 0, 7}] // MatrixForm$ 
```

Out[237]//MatrixForm=

$$\left(\begin{array}{l} \text{newT16}^A[0]^2 = \{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\ \frac{1}{2} (\text{newT16}^A[0] . \text{newT16}^A[1] + \text{newT16}^A[1] . \text{newT16}^A[0]) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\ \frac{1}{2} (\text{newT16}^A[0] . \text{newT16}^A[2] + \text{newT16}^A[2] . \text{newT16}^A[0]) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\ \frac{1}{2} (\text{newT16}^A[0] . \text{newT16}^A[3] + \text{newT16}^A[3] . \text{newT16}^A[0]) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\ \frac{1}{2} (\text{newT16}^A[0] . \text{newT16}^A[4] + \text{newT16}^A[4] . \text{newT16}^A[0]) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\ \frac{1}{2} (\text{newT16}^A[0] . \text{newT16}^A[5] + \text{newT16}^A[5] . \text{newT16}^A[0]) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\ \frac{1}{2} (\text{newT16}^A[0] . \text{newT16}^A[6] + \text{newT16}^A[6] . \text{newT16}^A[0]) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\ \frac{1}{2} (\text{newT16}^A[0] . \text{newT16}^A[7] + \text{newT16}^A[7] . \text{newT16}^A[0]) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \end{array} \right)$$

In[238]:=

```
 $\Psi16upper = \text{Take}[\Psi16, 8]$ 
 $\Psi16lower = \text{Take}[\Psi16, -8]$ 
 $\Psi16 == \text{Flatten}[\{\Psi16upper, \Psi16lower\}]$ 
```

Out[238]=

```
{f16[0][x0, x4], f16[1][x0, x4], f16[2][x0, x4], f16[3][x0, x4],
 f16[4][x0, x4], f16[5][x0, x4], f16[6][x0, x4], f16[7][x0, x4]}
```

Out[239]=

```
{f16[8][x0, x4], f16[9][x0, x4], f16[10][x0, x4], f16[11][x0, x4],
 f16[12][x0, x4], f16[13][x0, x4], f16[14][x0, x4], f16[15][x0, x4]}
```

Out[240]=

True

$$t^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

```
Table[T16^A[A1] = ArrayFlatten[{{0, \bar{\tau}[A1]}, {\tau[A1], 0}}], {A1, 0, 7}]
o16 = ArrayFlatten[{{-\sigma, 0}, {0, \sigma}}]
 $\begin{pmatrix} 0 & \bar{\tau}[A1] \\ \tau[A1] & 0 \end{pmatrix} \begin{pmatrix} \Psi16upper \\ \Psi16lower \end{pmatrix} = \begin{pmatrix} \bar{\tau}[A1].\Psi16upper \\ \tau[A1].\Psi16upper \end{pmatrix}$ 
```

$$\circ16. \left(\begin{array}{cc} 0 & \tau[A1] \\ \tau[A1] & 0 \end{array} \right) \left(\begin{array}{c} \circ16upper \\ \circ16lower \end{array} \right) = \left(\begin{array}{c} -\sigma\tau[A1].\circ16lower \\ \sigma\tau[A1].\circ16upper \end{array} \right)$$

$$\circ16.\circ16. \left(\begin{array}{cc} 0 & \tau[A1] \\ \tau[A1] & 0 \end{array} \right). \circ16 = \left(\begin{array}{c} -\circ16upper.\sigma\tau[A1].\circ16lower \\ \circ16lower.\sigma\tau[A1].\circ16upper \end{array} \right)$$

In[241]:= (* $\sigma16.T16^A[\#]$ &/@Range[0,7]*)

In[242]:=

```
(*  
(T16^A[8]=FullSimplify[T16^A[0].T16^A[1].T16^A[2].  
T16^A[3].T16^A[4].T16^A[5].T16^A[6].T16^A[7]])//MatrixForm*)
```

In[243]:=

η4488

Out[243]=

```
{ {1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},  
{0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0},  
{0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1} }
```

In[244]:=

η99 = DiagonalMatrix[{1, 1, 1, 1, -1, -1, -1, -1, 1}]

Out[244]=

```
{ {1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1} }
```

In[245]:=

```
(*Table[{A1,B1,FullSimplify[  
ExpandAll[1/2 ( T16^A[A1].T16^A[B1]+T16^A[B1].T16^A[A1])= η99[[A1,B1]]*ID16]]}],  
{A1,0, 8},{B1,0, 8}]//MatrixForm*)
```

In[246]:=

```
(*Table[1/4 (T16^A[A1].T16^A[B1]+T16^A[B1].T16^A[A1]), {A1, 8}, {B1,8}]*)
```

```
In[247]:= 
$$\left( P_L = \frac{1}{2} (ID16 - T16^A[8]) \right) // MatrixForm$$

```

```
In[248]:= 
$$\left( P_R = \frac{1}{2} (ID16 + T16^A[8]) \right) // \text{MatrixForm}$$

```

```
In[249]:= {ID16 == PL + PR, PL.PL == PL, PL.PR == PR.PL == ZERO16}
```

Out[249]=

```
SAB = Table[ $\frac{1}{4} (T16^A[A1].T16^A[B1] - T16^A[B1].T16^A[A1])$ , {A1, 0, 7}, {B1, 0, 7}];
```

In[250]:=

```
SAB = Table[ $\frac{1}{4} (T16^A[A1].T16^A[B1] - T16^A[B1].T16^A[A1])$ , {A1, 0, 7}, {B1, 0, 7}];
```

In[251]:=

```
Table[FullSimplify[σ16.SAB[[A1, B1]] == -Transpose[σ16.SAB[[A1, B1]]]], {A1, 1, 8}, {B1, 1, 8}] // Flatten // Union
```

Out[251]=

```
{True}
```

In[252]:=

```
Table[FullSimplify[SAB[[A1, B1]].SAB[[A2, B2]] - SAB[[A2, B2]].SAB[[A1, B1]] == - (η4488[[A1, A2]] × SAB[[B1, B2]] - η4488[[A1, B2]] × SAB[[B1, A2]] - η4488[[B1, A2]] × SAB[[A1, B2]] + η4488[[B1, B2]] × SAB[[A1, A2]])], {A1, 1, 7}, {B1, A1 + 1, 8}, {A2, 1, 7}, {B2, A2 + 1, 8}] // Flatten // Union
```

Out[252]=

```
{True}
```

In[253]:=

```
Table[FullSimplify[SAB[[A1, B1]].T16^A[B2 - 1] - T16^A[B2 - 1].SAB[[A1, B1]] == (-η4488[[B2, A1]] T16^A[B1 - 1] + η4488[[B2, B1]] T16^A[A1 - 1])], {A1, 1, 8}, {B1, 1, 8}, {B2, 1, 8}] // Flatten // Union
```

Out[253]=

```
{True}
```

In[254]:=

```
(*Table[{{A1,B1}},FullSimplify[σ16.SAB[[A1,B1]]== -Transpose[σ16.SAB[[A1,B1]]]]], {A1,1, 8},{B1,1,8}]*)
```

In[255]:=

```
(*Table[ {{A1,B1,A2,B2}},FullSimplify[SAB[[A1,B1]].SAB[[A2,B2]] - SAB[[A2,B2]].SAB[[A1,B1]] == - (η4488[[A1,A2]]SAB[[B1,B2]] - η4488[[A1,B2]]SAB[[B1,A2]] - η4488[[B1,A2]]SAB[[A1,B2]] + η4488[[B1,B2]]SAB[[A1,A2]]) ]], {A1,1, 7},{B1,A1+1,8},{A2,1, 7},{B2,A2+1,8}]*)
```

In[256]:=

```
(*Table[ {{A1,B1,B2}},FullSimplify[SAB[[A1,B1]].T16^A[B2-1] - T16^A[B2-1].SAB[[A1,B1]] == (-η4488[[B2,A1]] T16^A[B1-1] + η4488[[B2,B1]] T16^A[A1-1]) ]], {A1,1, 8},{B1,1,8},{B2,1,8}]*)
```

In[257]:=

```
Do[sAB[A1][B1] = 1/4 (T16^A[A1].T16^A[B1] - T16^A[B1].T16^A[A1]),  
{A1, 0, 7}, {B1, 0, 7}];
```

In[258]:=

Dimensions [ArrayFlatten[{{0, ID8}, {ID8, 0}}]]

Out[258]=

{16, 16}

In[259]:=

sAB [-1 + 1] [-1 + 2] == **SAB** [1, 2]

Out[259]=

True

In[260]:=

$\sigma16.SAB[1, 2] // \text{MatrixForm}$

Out[260]/MatrixForm=

In[261]:=

s16.sAB[1] [2] // MatrixForm

Out[261]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The commutation relations for the generators J_{ab} of the $\mathfrak{so}(3, 3)$ Lie algebra (using the physics convention where generators are Hermitian) are given by the formula: $\textcircled{2}$

$$[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad} + \eta_{bd}J_{ac})$$

where:

- The indices a, b, c, d range from 1 to 6.
- η_{ab} is the **metric tensor** with signature $(+, +, +, -, -, -)$ (or the reverse, depending on the convention). A common choice is a diagonal matrix $\eta = \text{diag}(1, 1, 1, -1, -1, -1)$.
- J_{ab} are the **generators** of the algebra, which are antisymmetric, $J_{ab} = -J_{ba}$.
- The terms on the right-hand side involve Kronecker deltas weighted by the metric, ensuring the resulting generator is also an element of the algebra and satisfies the required antisymmetry and Jacobi identity properties. $\textcircled{2}$

The final answer is $[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad} + \eta_{bd}J_{ac})$.

Step 1: Define the generators and metric

The **so(4,4)** Lie algebra has 28 generators, which can be denoted as $J_{\mu\nu}$ where the indices μ, ν range from 0 to 7, and $J_{\mu\nu} = -J_{\nu\mu}$. The Lie algebra is defined in an 8-dimensional real vector space equipped with a metric $\eta_{\mu\nu}$ with signature (4, 4). \circlearrowleft

- **Generators:** $J_{\mu\nu}$ with $\mu, \nu \in \{0, 1, \dots, 7\}$
- **Metric tensor:** $\eta_{\mu\nu} = \text{diag}(-1, -1, -1, -1, +1, +1, +1, +1)$ (or the opposite sign convention) \circlearrowleft

Step 2: State the general commutation relation

The commutation relations for the generators of the **so(p,q)** Lie algebra are a generalization of the **so(n)** algebra relations, incorporating the metric tensor $\eta_{\mu\nu}$. \circlearrowleft

Answer:

The commutation relations for the generators $J_{\mu\nu}$ of the **so(4,4)** Lie algebra are:

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho}$$

where $\eta_{\mu\nu}$ is the metric tensor with a (4,4) signature (four plus signs and four minus signs, or vice versa). \circlearrowleft

(Note: In some physics conventions, an extra factor of i or $i\hbar$ is included depending on whether the generators are defined as Hermitian or not). \circlearrowleft

what are the commutation relation for the generators of **so(4,4)**?

The commutation relations for the generators $J_{\mu\nu}$ of the **so(4,4)** Lie algebra, where $\mu, \nu \in \{0, 1, \dots, 7\}$ and $J_{\mu\nu} = -J_{\nu\mu}$, are the **standard relations for orthogonal Lie algebras**, given by a generalization of the angular momentum commutation relations: \circlearrowleft

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$

where $\eta_{\mu\nu}$ is the flat metric of 8-dimensional Minkowski space with signature (4,4) (or some other convention, the structure constants are the same as for **so(8)** in an appropriate basis), which is a diagonal matrix with four positive and four negative entries (e.g., $\eta = \text{diag}(+1, +1, +1, +1, -1, -1, -1, -1)$).



\hookrightarrow Why is the metric signature significant for $\$so(4,4)\$$ commutation relations?

Why is the metric signature significant for \$so(4,4)\$ commutation relations?

The metric signature is significant for the $so(p, q)$ commutation relations because the metric tensor $\eta_{\mu\nu}$ acts as the **structure constants** of the Lie algebra, defining its specific algebraic structure and physical properties (if applicable). ²

Here is a detailed explanation of its significance:

- **Definition of the Algebra:** The commutation relation for the generators $J_{\mu\nu}$ of any orthogonal Lie algebra $so(p, q)$ is given by:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$
The values of the structure constants depend directly on the components of the metric $\eta_{\mu\nu}$. This means that the specific algebraic properties of $so(4, 4)$ are distinct from $so(8)$ (which has a (8,0) signature, i.e., a Euclidean metric), even though they operate in the same 8-dimensional vector space.
- **Physical Interpretation:** In physics, the metric signature determines the nature of spacetime. A mixed signature (like (4,4) or the more common (1,3) Minkowski signature) implies a causal structure, distinguishing between timelike and spacelike dimensions.
 - Generators associated with rotations in spacelike dimensions (positive $\eta_{\mu\nu}$ entries) typically lead to compact symmetry groups (like $SO(4)$ or $SO(8)$).

In[262]:=

η4488

Out[262]=

```
{ {1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
{0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0},
{0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1} }
```

orthogonal Lie algebra $so(p, q)$ is given by:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$

In[263]:=

$$\text{SAB}[6, 2] \cdot \text{SAB}[6, 7] - \text{SAB}[6, 7] \cdot \text{SAB}[6, 2] = \eta4488[6, 6] \times \text{SAB}[7, 2]$$

Out[263]=

True

In[264]:=

$$\{ \{6, 2, 6, 7\}, \text{False} \}$$

Out[264]=

$$\{ \{6, 2, 6, 7\}, \text{False} \}$$

In[265]:=

$$\text{SAB}[1, 2] \cdot \text{SAB}[1, 3] - \text{SAB}[1, 3] \cdot \text{SAB}[1, 2] = \eta4488[1, 1] \times \text{SAB}[2, 3]$$

Out[265]=

False

In[266]:=

sAB[0][0]

Out[266]=

$$\{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}\}$$

In[267]:=

sAB[0][1]

Out[267]=

$$\left\{\left\{0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \right. \\ \left\{0, 0, 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \\ \left\{0, 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \\ \left\{0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \left\{0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \\ \left\{0, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \\ \left\{-\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \\ \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}\right\}, \\ \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0\right\}, \\ \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0\right\}, \\ \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0\right\}, \\ \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0\right\}, \\ \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0\right\}\}$$

```
In[268]:= Table[Transpose[T16^A[8].sAB[A1][B1]] == T16^A[8].sAB[A1][B1], {A1, 0, 7}, {B1, 0, 7}]
Out[268]= {{True, False, False, False, True, True, True}, {False, True, False, False, True, True, True}, {False, False, True, False, True, True, True}, {False, False, False, True, True, True, True}, {True, True, True, True, False, False, False}, {True, True, True, True, False, True, False}, {True, True, True, True, False, False, True}, {True, True, True, True, False, False, False}}
In[269]:= Table[Transpose[T16^A[7].sAB[A1][B1]] == T16^A[7].sAB[A1][B1], {A1, 0, 7}, {B1, 0, 7}]
Out[269]= {{True, True, True, True, False, False, True}, {True, True, True, True, False, False, False}, {True, True, True, True, False, False, True}, {True, True, True, True, False, False, False}, {False, False, False, False, True, True, False}, {False, False, False, False, True, True, True}, {False, False, False, False, True, True, False}, {True, True, True, True, False, False, True}}
In[270]:= Table[Transpose[\sigma16.sAB[A1][B1]] == \sigma16.sAB[A1][B1], {A1, 0, 7}, {B1, 0, 7}]
Out[270]= {{True, False, False, False, False, False, False}, {False, True, False, False, False, False, False}, {False, False, True, False, False, False, False}, {False, False, False, True, False, False, False}, {False, False, False, False, True, False, False}, {False, False, False, False, False, True, False}, {False, False, False, False, False, False, True}, {False, False, False, False, False, False, False}}
In[271]:= Table[Transpose[\sigma16.SAB[A1, B1]] == \sigma16.SAB[A1, B1], {A1, 1, 8}, {B1, 1, 8}]
Out[271]= {{True, False, False, False, False, False, False}, {False, True, False, False, False, False, False}, {False, False, True, False, False, False, False}, {False, False, False, True, False, False, False}, {False, False, False, False, True, False, False}, {False, False, False, False, False, True, False}, {False, False, False, False, False, False, True}, {False, False, False, False, False, False, False}}
```

In[272]:=

SAB[[1, 2][1]]

Out[272]=

$$\left\{0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}$$

In[273]:=

SAB[[1, 2][2]]

Out[273]=

$$\left\{0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}$$

$$t^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

In[274]:=

SAB1 = Table[Take[SAB[[A1, B1]][C1], 8], {A1, 8}, {B1, 8}, {C1, 1, 8}];

In[275]:=

SAB2 = Table[Take[SAB[[A1, B1]][C1], -8], {A1, 8}, {B1, 8}, {C1, 9, 16}];

In[276]:=

SAB[[1, 2]] == ArrayFlatten[{SAB1[[1, 2]], 0}, {0, SAB2[[1, 2]]}]

Out[276]=

True

In[277]:=

```
(*Table[SAB[[A1,B1]]== ArrayFlatten[{SAB1[[1,2]],0},{0,SAB2[[1,2]]}]),{A1,0,7},{B1,0,7}]*)
```

$$4D_{(1)}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A$$

$$4D_{(2)}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A.$$

Let $g \in \overline{SO(4, 4)}$ and $L \in SO(4, 4)$. The canonical 2-1 homomorphism $\overline{SO(4, 4)} \rightarrow SO(4, 4)$ is given by

$$(2.8) \quad 8L^{A'}{}_{B'}(g) = \text{tr} \tilde{D}^{(2)}(g) \tau^{A'} D^{(1)}(g) \bar{\tau}_{B'}.$$

As usual tr denotes the trace. Under the action of $\overline{SO(4, 4)}$

$$(2.9) \quad \tilde{D}^{(1)} \sigma = \sigma D^{(1)-1},$$

$$(2.10) \quad \sigma^{-1} \tilde{D}^{(2)} = D^{(2)-1} \sigma^{-1},$$

$$(2.11) \quad L^{A'}{}_{C'} G_{A'B'} L^{B'}{}_{D'} = G_{C'D'},$$

$$(2.12) \quad L^{A'}{}_{B'} \tau^{B'} = \tilde{D}^{(2)} \tau^{A'} D^{(1)},$$

$$(2.13) \quad L^{A'}{}_{B'} \bar{\tau}^{B'} = D^{(1)-1} \bar{\tau}^{A'} \tilde{D}^{(2)-1}.$$

In[278]:=

SAB2[[1, 2]]

Out[278]=

$$\left\{ \left\{ 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2} \right\}, \left\{ 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2} \right\}, \right. \\ \left\{ 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0 \right\}, \left\{ 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0 \right\}, \left\{ 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0 \right\}, \\ \left. \left\{ 0, 0, -\frac{1}{2}, 0, 0, 0, 0, 0 \right\}, \left\{ 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0 \right\}, \left\{ \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0 \right\} \right\}$$

In[279]:=

 $\tau[\theta]$

Out[279]=

$$\{ \{ 1, 0, 0, 0, 0, 0, 0, 0 \}, \{ 0, 1, 0, 0, 0, 0, 0, 0 \}, \\ \{ 0, 0, 1, 0, 0, 0, 0, 0 \}, \{ 0, 0, 0, 1, 0, 0, 0, 0 \}, \{ 0, 0, 0, 0, 1, 0, 0, 0 \}, \\ \{ 0, 0, 0, 0, 0, 1, 0, 0 \}, \{ 0, 0, 0, 0, 0, 0, 1, 0 \}, \{ 0, 0, 0, 0, 0, 0, 0, 1 \} \}$$

In[280]:=

```
(*Block[{b,A,F_a^(μ),F_a^a,ψ,Ψ},
b=Array[A,{8}];
ψ[1]=Array[Ψ[1],{8}];
ψ[2]=Array[Ψ[2],{8}];
F_a^(μ)=Table[(1/(Sqrt[ψ[1].σ.ψ[1]])ψ[1].σ.(τ[μL])),{μL,1,8}];
F_a^a=Transpose[Table[(1/(Sqrt[ψ[1].σ.ψ[1]])η4488[[μL,μL]τ[μL].ψ[1]]),{μL,1,8}]];
b==FullSimplify[F_a^(μ).ψ[2]/.Thread[ψ[2]→F_a^a.b]]]*)
```

In[281]:=

```
(*Block[{b,A,B,q,Fa(μ),ψ,φ,Π},
b=Array[A,{8}];
ψ[1]=Array[Π[1],{8}];
φ[1]=Array[Π[2],{8}];
q=Table[FullSimplify[({1/(ψ[1].σ.ψ[1]) ψ[1].σ.(1/2 (τ[A1].τ[B1]-τ[B1].τ[A1]).φ[1])}),
{A1,1,8},{B1,1,8}]];
B=FullSimplify[
(1/(ψ[1].σ.ψ[1]) ParallelSum[η4488[[μL,μL]]τ[μL]b[[μL]],{μL,1,8}].ψ[1]);
FullSimplify[q/.Thread[φ[1]→B]]]*)
```

In[282]:=

```
(*Block[{b,A,B,q,Fa(μ),ψ,φ,Π},
b=Array[A,{8}];
ψ[1]=Array[Π[1],{8}];
φ[1]=Array[Π[2],{8}];
q=Table[FullSimplify[(ψ[1].σ.(τ[A1].τ[B1]).φ[1])],{A1,1,8},{B1,1,8}];
B=FullSimplify[( ParallelSum[η4488[[μL,μL]]τ[μL]b[[μL]],{μL,1,8}].ψ[1])];
FullSimplify[({1/(ψ[1].σ.ψ[1]) q})/.Thread[φ[1]→B]]]*)
```

CHECK Table[T16^α[α1-1] = Sum[(e^α_(A)[[α1, A1]]) T16^A[A1-1], {A1, 1, 8}], {α1, 1, 8}];
and ● ● ● :

In[283]:=

```
(*Table[T16α[α1-1]= Sum[(eα(A)[[α1,A1]]) T16A[A1-1],{A1,1,8}],{α1,1,8}];*)
```

In[284]:=

```
(* (T16α[8]=T16α[0].T16α[1].T16α[2].T16α[3].T16α[4].T16α[5].T16α[6].T16α[7]) //
MatrixForm*)
```

In[285]:=

```
Table[
  { {A1, B1}, FullSimplify[ExpandAll[ $\frac{1}{2} (T16^A[A1].T16^A[B1] + T16^A[B1].T16^A[A1]) ==$ 
     $\eta 4488 [A1 + 1, B1 + 1] * ID16]$  ] } }, {A1, 0, 7}, {B1, 0, 7}] // MatrixForm
```

Out[285]//MatrixForm=

$\begin{pmatrix} \{0, 0\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{0, 1\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{0, 2\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{0, 3\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{0, 4\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{0, 5\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{0, 6\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{0, 7\} \\ \text{True} \end{pmatrix}$
$\begin{pmatrix} \{1, 0\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{1, 1\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{1, 2\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{1, 3\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{1, 4\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{1, 5\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{1, 6\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{1, 7\} \\ \text{True} \end{pmatrix}$
$\begin{pmatrix} \{2, 0\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{2, 1\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{2, 2\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{2, 3\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{2, 4\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{2, 5\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{2, 6\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{2, 7\} \\ \text{True} \end{pmatrix}$
$\begin{pmatrix} \{3, 0\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{3, 1\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{3, 2\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{3, 3\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{3, 4\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{3, 5\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{3, 6\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{3, 7\} \\ \text{True} \end{pmatrix}$
$\begin{pmatrix} \{4, 0\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{4, 1\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{4, 2\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{4, 3\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{4, 4\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{4, 5\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{4, 6\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{4, 7\} \\ \text{True} \end{pmatrix}$
$\begin{pmatrix} \{5, 0\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{5, 1\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{5, 2\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{5, 3\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{5, 4\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{5, 5\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{5, 6\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{5, 7\} \\ \text{True} \end{pmatrix}$
$\begin{pmatrix} \{6, 0\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{6, 1\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{6, 2\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{6, 3\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{6, 4\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{6, 5\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{6, 6\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{6, 7\} \\ \text{True} \end{pmatrix}$
$\begin{pmatrix} \{7, 0\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{7, 1\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{7, 2\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{7, 3\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{7, 4\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{7, 5\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{7, 6\} \\ \text{True} \end{pmatrix}$	$\begin{pmatrix} \{7, 7\} \\ \text{True} \end{pmatrix}$

BASIS of 16×16 matrices:

In[286]:=

16 * 16

Out[286]=

256

In[287]:=

 2^8

Out[287]=

256

In[288]:=

Sum[Binomial[8, p], {p, 0, 8}]

Out[288]=

256

In[289]:=

ID16 == ID16 == Dot @@ (T16^A[#] & /@ Range[0, 8])

Out[289]=

True

In[290]:=

```
Clear[t160]; t160 = {{Id16, {Range[0, 8]}}}; Length[t160]
Clear[t16A];
t16A = {};
```

```

Do[AppendTo[t16A, {T16A[j], {j}}], {j, 0, 7}];
Length[t16A]
Clear[t16AB];
t16AB = {};
Do[AppendTo[t16AB, {T16A[j].T16A[k], {j, k}}], {j, 0, 6}, {k, j + 1, 7}];
Length[t16AB]
Clear[t16ABC];
t16ABC = {};
Do[AppendTo[t16ABC, {T16A[j].T16A[k].T16A[h], {j, k, h}}], {j, 0, 5}, {k, j + 1, 6}, {h, k + 1, 7}];
Length[t16ABC]
Clear[t16ABCD];
t16ABCD = {};
Do[AppendTo[t16ABCD, {T16A[j].T16A[k].T16A[h].T16A[i], {j, k, h, i}}], {j, 0, 4}, {k, j + 1, 5}, {h, k + 1, 6}, {i, h + 1, 7}];
Length[t16ABCD]
Clear[t16ABCDE];
t16ABCDE = {};
Do[AppendTo[t16ABCDE, {T16A[j].T16A[k].T16A[h].T16A[i].T16A[l], {j, k, h, i, l}}], {j, 0, 3}, {k, j + 1, 4}, {h, k + 1, 5}, {i, h + 1, 6}, {l, i + 1, 7}];
Length[t16ABCDE]
Clear[t16ABCDEF];
t16ABCDEF = {};
Do[AppendTo[t16ABCDEF,
{T16A[j].T16A[k].T16A[h].T16A[i].T16A[l].T16A[q], {j, k, h, i, l, q}}], {j, 0, 2}, {k, j + 1, 3}, {h, k + 1, 4}, {i, h + 1, 5}, {l, i + 1, 6}, {q, l + 1, 7}];
Length[t16ABCDEF]
Clear[t16ABCDEFG];
t16ABCDEFG = {};
Do[AppendTo[t16ABCDEFG, {T16A[j].T16A[k].T16A[h].T16A[i].T16A[l].T16A[q].T16A[r],
{j, k, h, i, l, q, r}]], {j, 0, 1}, {k, j + 1, 2},
{h, k + 1, 3}, {i, h + 1, 4}, {l, i + 1, 5}, {q, l + 1, 6}, {r, q + 1, 7}];
Length[t16ABCDEFG]
Clear[t16ABCDEFGH];
t16ABCDEFGH = {};
Do[AppendTo[t16ABCDEFGH, {T16A[j].T16A[k].T16A[h].T16A[i].T16A[l].T16A[q].T16A[r].
T16A[s], {j, k, h, i, l, q, r, s}}], {j, 0, 1}, {k, j + 1, 1}, {h, k + 1, 2},
{i, h + 1, 3}, {l, i + 1, 4}, {q, l + 1, 5}, {r, q + 1, 6}, {s, r + 1, 7}];
Length[t16ABCDEFGH]
% + %% + %% + %%% + %%%% + %%%%% + %%%%%% + %%%%%%% + %%%%%%%%

```

```

Out[290]=
1

Out[291]=
8

Out[292]=
28

Out[293]=
56

Out[294]=
70

Out[295]=
56

Out[296]=
28

Out[297]=
8

Out[298]=
1

Out[299]=
256

In[300]:= (*base16=Flatten[
  {t160, t16A, t16AB, t16ABC, t16ABCD, t16ABCDE, t16ABCDEF, t16ABCDEFG, t16ABCDEFGH}, 1];
Length[base16]*)

In[301]:= base16 = Flatten[{t16A, t16AB, t16ABC, t16ABCD,
  t16ABCDE, t16ABCDEF, t16ABCDEFG, t16ABCDEFGH, t160}, 1];
Length[base16]

Out[301]=
256

In[302]:= Length[Union[base16[[All, 1]]]]
Out[302]=
256

In[303]:= Tr[#.#/16] & /@ base16[[All, 1]] // Flatten // Union
Out[303]=
{-1, 1}

```

```
In[304]:= positiveTrMM = Select[base16, Tr[#[[1]] . #[[1]]] > 0 & → "Index"]  
Out[304]= {1, 2, 3, 4, 12, 13, 14, 15, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51, 59, 60, 61, 62, 63, 64, 65, 66, 73, 74, 75, 76, 89, 90, 91, 92, 93, 102, 103, 104, 105, 106, 107, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 148, 149, 150, 151, 152, 153, 162, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 183, 184, 185, 186, 187, 188, 197, 198, 199, 200, 201, 202, 203, 212, 217, 218, 225, 226, 227, 228, 229, 230, 231, 232, 234, 235, 236, 237, 240, 241, 242, 243, 247, 248, 249, 250, 255, 256}
```

```
In[305]:= negativeTrMM = Select[base16, Tr[#[[1]] . #[[1]]] < 0 & → "Index"]

Out[305]= {5, 6, 7, 8, 9, 10, 11, 16, 17, 22, 31, 32, 33, 34, 35, 36, 37, 38, 43, 52, 53, 54, 55, 56, 57,
58, 67, 68, 69, 70, 71, 72, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 94, 95, 96, 97,
98, 99, 100, 101, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 130, 131, 144, 145,
146, 147, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 179, 180, 181, 182,
189, 190, 191, 192, 193, 194, 195, 196, 204, 205, 206, 207, 208, 209, 210, 211, 213, 214,
215, 216, 219, 220, 221, 222, 223, 224, 233, 238, 239, 244, 245, 246, 251, 252, 253, 254}
```

```
In[306]:= {Length[positiveTrMM], Length[negativeTrMM]}

Out[306]= {136, 120}
```

```
In[307]:= dups = Block[{s}, s = base16[[All, 1]]; Select[s, MemberQ[s, -#] &]]; Length[dups]

Out[307]= 0
```

```
In[308]:= dups = Select[base16, MemberQ[base16[[All, 1]], -#[[1]]] &]; Length[dups]  
Out[308]= 0
```

```
In[309]:= BASE16 = use16 = Complement[base16, dups]; Length[use16]
```

Out[309]=

```
In[310]:= (* (T16^A[8]=T16^A[0].T16^A[1].T16^A[2].T16^A[3].T16^A[4].T16^A[5].T16^A[6].T16^A[7]) // MatrixForm*)
```

```
In[311]:= MatrixForm[T16^A[[#]]]& /@ Range[0, 7]
```



```

In[312]:= (*BASE16[-1]*)

In[313]:= (*BASE16[-2]
%[[1]]==BASE16[-1][[1]]*)

In[314]:= 16 * (16 + 1) / 2
16 * (16 - 1) / 2
% + %%

Out[314]= 136

Out[315]= 120

Out[316]= 256

In[317]:= Clear[antisymm16];
antisymm16 = {};
Do[If[base16[[k, 1]] === -Transpose[base16[[k, 1]]], AppendTo[antisymm16,
{base16[[k, 1]], {{k}, base16[[k, 2]]}}], {k, 1, Length[base16]}];
Length[antisymm16]
antisymm16[%]

Out[318]= 120

Out[319]= {{ {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}, {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}}, {{254}, {1, 2, 3, 4, 5, 6, 7}}}

```

In[320]:=

```

Clear[symmm16];
symmm16 = {};
Do[If[base16[[k, 1]] === Transpose[base16[[k, 1]]], AppendTo[symmm16,
    {base16[[k, 1]], {{k}, base16[[k, 2]]}}] ], {k, 1, Length[base16]}];
Length[symmm16]
symmm16[[%]]

```

Out[321]=

136

Out[322]=

```

{{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {{256}}, {{0, 1, 2, 3, 4, 5, 6, 7, 8}}}}

```

In[323]:=

```

dupssymmm16 = Select[symmm16, MemberQ[symmm16[[All, 1]], -#[[1]]] &]; Length[dupssymmm16]

```

Out[323]=

0

In[324]:=

#[[2]] & /@ symm16

Out[324]=

```

{{{{1}, {0}}, {{2}, {1}}, {{3}, {2}}, {{4}, {3}}, {{12}, {0, 4}}, {{13}, {0, 5}}, {{14}, {0, 6}}, {{15}, {0, 7}}, {{18}, {1, 4}}, {{19}, {1, 5}}, {{20}, {1, 6}}, {{21}, {1, 7}}, {{23}, {2, 4}}, {{24}, {2, 5}}, {{25}, {2, 6}}, {{26}, {2, 7}}, {{27}, {3, 4}}, {{28}, {3, 5}}, {{29}, {3, 6}}, {{30}, {3, 7}}, {{39}, {0, 1, 4}}, {{40}, {0, 1, 5}}, {{41}, {0, 1, 6}}, {{42}, {0, 1, 7}}, {{44}, {0, 2, 4}}, {{45}, {0, 2, 5}}, {{46}, {0, 2, 6}}, {{47}, {0, 2, 7}}, {{48}, {0, 3, 4}}, {{49}, {0, 3, 5}}, {{50}, {0, 3, 6}}, {{51}, {0, 3, 7}}, {{59}, {1, 2, 4}}, {{60}, {1, 2, 5}}, {{61}, {1, 2, 6}}, {{62}, {1, 2, 7}}, {{63}, {1, 3, 4}}, {{64}, {1, 3, 5}}, {{65}, {1, 3, 6}}, {{66}, {1, 3, 7}}, {{73}, {2, 3, 4}}, {{74}, {2, 3, 5}}, {{75}, {2, 3, 6}}, {{76}, {2, 3, 7}}, {{89}, {4, 5, 6}}, {{90}, {4, 5, 7}}, {{91}, {4, 6, 7}}, {{92}, {5, 6, 7}}, {{93}, {0, 1, 2, 3}}, {{102}, {0, 1, 4, 5}}, {{103}, {0, 1, 4, 6}}, {{104}, {0, 1, 4, 7}}, {{105}, {0, 1, 5, 6}}, {{106}, {0, 1, 5, 7}}, {{107}, {0, 1, 6, 7}}, {{112}, {0, 2, 4, 5}}, {{113}, {0, 2, 4, 6}}, {{114}, {0, 2, 4, 7}}, {{115}, {0, 2, 5, 6}}, {{116}, {0, 2, 5, 7}}, {{117}, {0, 2, 6, 7}}, {{118}, {0, 3, 4, 5}}, {{119}, {0, 3, 4, 6}}, {{120}, {0, 3, 4, 7}}, {{121}, {0, 3, 5, 6}}, {{122}, {0, 3, 5, 7}}, {{123}, {0, 3, 6, 7}}, {{132}, {1, 2, 4, 5}}, {{133}, {1, 2, 4, 6}}, {{134}, {1, 2, 4, 7}}, {{135}, {1, 2, 5, 6}}, {{136}, {1, 2, 5, 7}}, {{137}, {1, 2, 6, 7}}, {{138}, {1, 3, 4, 5}}, {{139}, {1, 3, 4, 6}}, {{140}, {1, 3, 4, 7}}, {{141}, {1, 3, 5, 6}}, {{142}, {1, 3, 5, 7}}, {{143}, {1, 3, 6, 7}}, {{148}, {2, 3, 4, 5}}, {{149}, {2, 3, 4, 6}}, {{150}, {2, 3, 4, 7}}, {{151}, {2, 3, 5, 6}}, {{152}, {2, 3, 5, 7}}, {{153}, {2, 3, 6, 7}}, {{162}, {4, 5, 6, 7}}, {{167}, {0, 1, 2, 4, 5}}, {{168}, {0, 1, 2, 4, 6}}, {{169}, {0, 1, 2, 4, 7}}, {{170}, {0, 1, 2, 5, 6}}, {{171}, {0, 1, 2, 5, 7}}, {{172}, {0, 1, 2, 6, 7}}, {{173}, {0, 1, 3, 4, 5}}, {{174}, {0, 1, 3, 4, 6}}, {{175}, {0, 1, 3, 4, 7}}, {{176}, {0, 1, 3, 5, 6}}, {{177}, {0, 1, 3, 5, 7}}, {{178}, {0, 1, 3, 6, 7}}, {{183}, {0, 2, 3, 4, 5}}, {{184}, {0, 2, 3, 4, 6}}, {{185}, {0, 2, 3, 4, 7}}, {{186}, {0, 2, 3, 5, 6}}, {{187}, {0, 2, 3, 5, 7}}, {{188}, {0, 2, 3, 6, 7}}, {{197}, {0, 4, 5, 6, 7}}, {{198}, {1, 2, 3, 4, 5}}, {{199}, {1, 2, 3, 4, 6}}, {{200}, {1, 2, 3, 4, 7}}, {{201}, {1, 2, 3, 5, 6}}, {{202}, {1, 2, 3, 5, 7}}, {{203}, {1, 2, 3, 6, 7}}, {{212}, {1, 4, 5, 6, 7}}, {{217}, {2, 4, 5, 6, 7}}, {{218}, {3, 4, 5, 6, 7}}, {{225}, {0, 1, 2, 4, 5, 6}}, {{226}, {0, 1, 2, 4, 5, 7}}, {{227}, {0, 1, 2, 4, 6, 7}}, {{228}, {0, 1, 2, 5, 6, 7}}, {{229}, {0, 1, 3, 4, 5, 6}}, {{230}, {0, 1, 3, 4, 5, 7}}, {{231}, {0, 1, 3, 4, 6, 7}}, {{232}, {0, 1, 3, 5, 6, 7}}, {{234}, {0, 2, 3, 4, 5, 6}}, {{235}, {0, 2, 3, 4, 5, 7}}, {{236}, {0, 2, 3, 4, 6, 7}}, {{237}, {0, 2, 3, 5, 6, 7}}, {{240}, {1, 2, 3, 4, 5, 6}}, {{241}, {1, 2, 3, 4, 5, 7}}, {{242}, {1, 2, 3, 4, 6, 7}}, {{243}, {1, 2, 3, 5, 6, 7}}, {{247}, {0, 1, 2, 3, 4, 5, 6}}, {{248}, {0, 1, 2, 3, 4, 5, 7}}, {{249}, {0, 1, 2, 3, 4, 6, 7}}, {{250}, {0, 1, 2, 3, 5, 6, 7}}, {{255}, {0, 1, 2, 3, 4, 5, 6, 7}}, {{256}, {0, 1, 2, 3, 4, 5, 6, 7, 8}}}}

```

In[325]:=

 $\sigma16 = T16^A[0].T16^A[1].T16^A[2].T16^A[3]$

Out[325]=

True

```
In[326]:= σ16 == base16[[93]][1]
Out[326]= True

In[327]:= Select[symmm16, #[2][1][1] == 93 & → "Index"]
Out[327]= {49}

In[328]:= σ16 == symmm16[[49]][1]
Out[328]= True

In[329]:= symmm16[[134]]
Out[329]= {{ {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}, {{250}, {0, 1, 2, 3, 5, 6, 7}}}}
```

```
In[330]:= Select[symmm16, #[2][1][1] == 250 & → "Index"]
Out[330]= {134}
```

```
In[331]:= T16A
Out[331]= T16A
```

```
In[332]:= σ16.T16A[5].T16A[6].T16A[7] == symmm16[[134]][1]
Out[332]= True
```

```
In[333]:= T16A[8] == base16[[255]][1]
Out[333]= True
```

```
In[334]:= Select[symmm16, #[2][1][1] == 255 & → "Index"]
Out[334]= {135}
```

similarly:

```
In[335]:= coupled1 = {4, 5, 0, 9, 9, 12, 4, 5, 1, 8, 8, 13} // Union
Out[335]= {0, 1, 4, 5, 8, 9, 12, 13}

In[336]:= coupled2 = Complement[Range[0, 15], coupled1]
Out[336]= {2, 3, 6, 7, 10, 11, 14, 15}

In[337]:= coupled = {coupled1, coupled2}
Out[337]= {{0, 1, 4, 5, 8, 9, 12, 13}, {2, 3, 6, 7, 10, 11, 14, 15} }

In[338]:= Block[{m, vars, ebs, ebs1, ebs2, s, ¶16},
  ¶16 = f[#] & /@ Range[16];
  m = Array[qb[#1, #2] &, {16, 16}];
  vars = Flatten[m];
  ebs1 =
    And @@ Thread[m.¶16 == Flatten[{¶16[# + 1] & /@ coupled1, ¶16[# + 1] & /@ coupled2}]];
  ebs2 = And @@ Thread[m.Transpose[m] == Flatten[ID16]];
  ebs = ebs1 && ebs2;
  s = Solve[ebs, vars] // FullSimplify;
  s
]
Out[338]= {}
```



```
In[341]:= linTrans.Transpose[linTrans] // FullSimplify // MatrixForm
Out[341]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$


In[342]:= linTrans.¶16 // Column
Union[%[[1]]]
Out[342]=
f16[0][x0, x4]
f16[1][x0, x4]
f16[8][x0, x4]
f16[9][x0, x4]
f16[2][x0, x4]
f16[3][x0, x4]
f16[10][x0, x4]
f16[11][x0, x4]
f16[4][x0, x4]
f16[5][x0, x4]
f16[12][x0, x4]
f16[13][x0, x4]
f16[6][x0, x4]
f16[7][x0, x4]
f16[14][x0, x4]
f16[15][x0, x4]

Out[343]=
{f16[0][x0, x4], f16[1][x0, x4], f16[2][x0, x4], f16[3][x0, x4],
 f16[4][x0, x4], f16[5][x0, x4], f16[6][x0, x4], f16[7][x0, x4],
 f16[8][x0, x4], f16[9][x0, x4], f16[10][x0, x4], f16[11][x0, x4],
 f16[12][x0, x4], f16[13][x0, x4], f16[14][x0, x4], f16[15][x0, x4]}

In[344]:= positiveTrMM = Select[base16, Tr[#[[1]] . #[[1]]] > 0 & -> "Index"]
Out[344]=
{1, 2, 3, 4, 12, 13, 14, 15, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51, 59,
 60, 61, 62, 63, 64, 65, 66, 73, 74, 75, 76, 89, 90, 91, 92, 93, 102, 103, 104, 105, 106, 107, 112, 113, 114, 115, 116,
 117, 118, 119, 120, 121, 122, 123, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 148, 149, 150, 151,
 152, 153, 162, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 183, 184, 185, 186, 187, 188, 197, 198}
```

```

199, 200, 201, 202, 203, 212, 217, 218, 225, 226, 227, 228, 229, 230, 231, 232, 234, 235, 236, 237, 240, 241, 242,
243, 247, 248, 249, 250, 255, 256}

In[343]:= negativeTrMM = Select[base16, Tr[#[[1]] . #[[1]]] < 0 & -> "Index"]

Out[343]= {5, 6, 7, 8, 9, 10, 11, 16, 17, 22, 31, 32, 33, 34, 35, 36, 37, 38, 43, 52, 53, 54, 55, 56, 57, 58, 67, 68, 69, 70, 71, 72, 77,
78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 94, 95, 96, 97, 98, 99, 100, 101, 108, 109, 110, 111, 124, 125, 126, 127, 128,
129, 130, 131, 144, 145, 146, 147, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 179, 180, 181, 182,
189, 190, 191, 192, 193, 194, 195, 196, 204, 205, 206, 207, 208, 209, 210, 211, 213, 214, 215, 216, 219, 220, 221,
222, 223, 224, 233, 238, 239, 244, 245, 246, 251, 252, 253, 254}

In[344]:= {Length[positiveTrMM], Length[negativeTrMM]}

Out[344]= {136, 120}

In[344]:= positiveTrMM = Select[base16, Tr[#[[1]].#[[1]]] > 0 & -> "Index"]

Out[344]= {1, 2, 3, 4, 12, 13, 14, 15, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 39, 40, 41, 42, 44,
45, 46, 47, 48, 49, 50, 51, 59, 60, 61, 62, 63, 64, 65, 66, 73, 74, 75, 76, 89, 90, 91,
92, 93, 102, 103, 104, 105, 106, 107, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121,
122, 123, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 148, 149, 150, 151,
152, 153, 162, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 183, 184, 185,
186, 187, 188, 197, 198, 199, 200, 201, 202, 203, 212, 217, 218, 225, 226, 227, 228, 229,
230, 231, 232, 234, 235, 236, 237, 240, 241, 242, 243, 247, 248, 249, 250, 255, 256}

In[345]:= negativeTrMM = Select[base16, Tr[#[[1]].#[[1]]] < 0 & -> "Index"]

Out[345]= {5, 6, 7, 8, 9, 10, 11, 16, 17, 22, 31, 32, 33, 34, 35, 36, 37, 38, 43, 52, 53, 54, 55, 56, 57,
58, 67, 68, 69, 70, 71, 72, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 94, 95, 96, 97,
98, 99, 100, 101, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 130, 131, 144, 145,
146, 147, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 179, 180, 181, 182,
189, 190, 191, 192, 193, 194, 195, 196, 204, 205, 206, 207, 208, 209, 210, 211, 213, 214,
215, 216, 219, 220, 221, 222, 223, 224, 233, 238, 239, 244, 245, 246, 251, 252, 253, 254}

In[346]:= {Length[positiveTrMM], Length[negativeTrMM]}

Out[346]= {136, 120}

```

```
In[347]:= base16[[#]] & /@ positiveTrMM;
{#[[2]], Tr[Transpose[#[[1]].linTrans]]} & /@ %
Select[%, #[[2]] ≠ 0 &]
linTransSymmetric = Table[Select[base16, #[[2]] == %[[j]][[1]] &], {j, 1, Length[%]}];

Out[348]= {{{0}, 0}, {{1}, 0}, {{2}, 0}, {{3}, 0}, {{0, 4}, 0}, {{0, 5}, 0}, {{0, 6}, 0}, {{0, 7}, -4},
{{1, 4}, 0}, {{1, 5}, 0}, {{1, 6}, 0}, {{1, 7}, 0}, {{2, 4}, 0}, {{2, 5}, 0}, {{2, 6}, 0},
{{2, 7}, 0}, {{3, 4}, -4}, {{3, 5}, 0}, {{3, 6}, 0}, {{3, 7}, 0}, {{0, 1, 4}, 0},
{{0, 1, 5}, 0}, {{0, 1, 6}, 0}, {{0, 1, 7}, 0}, {{0, 2, 4}, 0}, {{0, 2, 5}, 0}, {{0, 2, 6}, 0},
{{0, 2, 7}, 0}, {{0, 3, 4}, 0}, {{0, 3, 5}, 0}, {{0, 3, 6}, 0}, {{0, 3, 7}, 0},
{{1, 2, 4}, 0}, {{1, 2, 5}, 0}, {{1, 2, 6}, 0}, {{1, 2, 7}, 0}, {{1, 3, 4}, 0},
{{1, 3, 5}, 0}, {{1, 3, 6}, 0}, {{1, 3, 7}, 0}, {{2, 3, 4}, 0}, {{2, 3, 5}, 0},
{{2, 3, 6}, 0}, {{2, 3, 7}, 0}, {{4, 5, 6}, -4}, {{4, 5, 7}, 0}, {{4, 6, 7}, 0},
{{5, 6, 7}, 0}, {{0, 1, 2, 3}, 0}, {{0, 1, 4, 5}, 0}, {{0, 1, 4, 6}, 0}, {{0, 1, 4, 7}, 0},
{{0, 1, 5, 6}, 0}, {{0, 1, 5, 7}, 0}, {{0, 1, 6, 7}, 0}, {{0, 2, 4, 5}, 0}, {{0, 2, 4, 6}, 0},
{{0, 2, 4, 7}, 0}, {{0, 2, 5, 6}, 0}, {{0, 2, 5, 7}, 0}, {{0, 2, 6, 7}, 0}, {{0, 3, 4, 5}, 0},
{{0, 3, 4, 6}, 0}, {{0, 3, 4, 7}, 4}, {{0, 3, 5, 6}, 0}, {{0, 3, 5, 7}, 0}, {{0, 3, 6, 7}, 0},
{{1, 2, 4, 5}, 0}, {{1, 2, 4, 6}, 0}, {{1, 2, 4, 7}, 0}, {{1, 2, 5, 6}, 0}, {{1, 2, 5, 7}, 0},
{{1, 2, 6, 7}, -4}, {{1, 3, 4, 5}, 0}, {{1, 3, 4, 6}, 0}, {{1, 3, 4, 7}, 0}, {{1, 3, 5, 6}, 0},
{{1, 3, 5, 7}, 0}, {{1, 3, 6, 7}, 0}, {{2, 3, 4, 5}, 0}, {{2, 3, 4, 6}, 0}, {{2, 3, 4, 7}, 0},
{{2, 3, 5, 6}, 0}, {{2, 3, 5, 7}, 0}, {{2, 3, 6, 7}, 0}, {{4, 5, 6, 7}, 0}, {{0, 1, 2, 4, 5}, 4},
{{0, 1, 2, 4, 6}, 0}, {{0, 1, 2, 4, 7}, 0}, {{0, 1, 2, 5, 6}, 0}, {{0, 1, 2, 5, 7}, 0},
{{0, 1, 2, 6, 7}, 0}, {{0, 1, 3, 4, 5}, 0}, {{0, 1, 3, 4, 6}, 0}, {{0, 1, 3, 4, 7}, 0},
{{0, 1, 3, 5, 6}, 0}, {{0, 1, 3, 5, 7}, 0}, {{0, 1, 3, 6, 7}, 0}, {{0, 2, 3, 4, 5}, 0},
{{0, 2, 3, 4, 6}, 0}, {{0, 2, 3, 4, 7}, 0}, {{0, 2, 3, 5, 6}, 0}, {{0, 2, 3, 5, 7}, 0},
{{0, 2, 3, 6, 7}, 0}, {{0, 4, 5, 6, 7}, 4}, {{1, 2, 3, 4, 5}, 0}, {{1, 2, 3, 4, 6}, 0},
{{1, 2, 3, 4, 7}, 0}, {{1, 2, 3, 5, 6}, 0}, {{1, 2, 3, 5, 7}, -4}, {{1, 2, 3, 6, 7}, 0},
{{1, 4, 5, 6, 7}, 0}, {{2, 4, 5, 6, 7}, 0}, {{3, 4, 5, 6, 7}, 0}, {{0, 1, 2, 4, 5, 6}, 0},
{{0, 1, 2, 4, 5, 7}, 0}, {{0, 1, 2, 4, 6, 7}, 0}, {{0, 1, 2, 5, 6, 7}, 0},
{{0, 1, 3, 4, 5, 6}, 0}, {{0, 1, 3, 4, 5, 7}, 0}, {{0, 1, 3, 4, 6, 7}, 0},
{{0, 1, 3, 5, 6, 7}, 0}, {{0, 2, 3, 4, 5, 6}, 0}, {{0, 2, 3, 4, 5, 7}, 0},
{{0, 2, 3, 4, 6, 7}, 0}, {{0, 2, 3, 5, 6, 7}, 0}, {{1, 2, 3, 4, 5, 6}, 0},
{{1, 2, 3, 4, 5, 7}, 0}, {{1, 2, 3, 4, 6, 7}, -4}, {{1, 2, 3, 5, 6, 7}, 0},
{{0, 1, 2, 3, 4, 5, 6}, 0}, {{0, 1, 2, 3, 4, 5, 7}, 0}, {{0, 1, 2, 3, 4, 6, 7}, 0},
{{0, 1, 2, 3, 5, 6, 7}, 0}, {{0, 1, 2, 3, 4, 5, 6, 7}, 0}, {{{0, 1, 2, 3, 4, 5, 6, 7, 8}}, 4}

Out[349]= {{0, 7}, -4}, {3, 4}, -4}, {4, 5, 6}, -4}, {0, 3, 4, 7}, 4},
{1, 2, 6, 7}, -4}, {0, 1, 2, 4, 5}, 4}, {0, 4, 5, 6, 7}, 4},
{1, 2, 3, 5, 7}, -4}, {1, 2, 3, 4, 6, 7}, -4}, {{{0, 1, 2, 3, 4, 5, 6, 7, 8}}, 4}}
```

In[351]:=

linTransSymmetric[[1]]

Out[351]=

```

{ {{ {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}}, {0, 7} } }

```

```
In[352]:= base16[[#]] & /@ negativeTrMM;
{#[[2]], Tr[Transpose[#[[1]].linTrans]]} & /@ %
Select[%, #[[2]] ≠ 0 &]
linTransAntiSymmetric = Table[Select[base16, #[[2]] == %[[j]][[1]] &], {j, 1, Length[%]}];
Out[353]= {{{4}, 0}, {{5}, 0}, {{6}, 0}, {{7}, 0}, {{0, 1}, 0}, {{0, 2}, 0}, {{0, 3}, 0},
{{1, 2}, 0}, {{1, 3}, 0}, {{2, 3}, 0}, {{4, 5}, 0}, {{4, 6}, 0}, {{4, 7}, 0},
{{5, 6}, 0}, {{5, 7}, 0}, {{6, 7}, 0}, {{0, 1, 2}, 0}, {{0, 1, 3}, 0}, {{0, 2, 3}, 0},
{{0, 4, 5}, 0}, {{0, 4, 6}, 0}, {{0, 4, 7}, 0}, {{0, 5, 6}, 0}, {{0, 5, 7}, 0},
{{0, 6, 7}, 0}, {{1, 2, 3}, 0}, {{1, 4, 5}, 0}, {{1, 4, 6}, 0}, {{1, 4, 7}, 0},
{{1, 5, 6}, 0}, {{1, 5, 7}, 0}, {{1, 6, 7}, 0}, {{2, 4, 5}, 0}, {{2, 4, 6}, 0},
{{2, 4, 7}, 0}, {{2, 5, 6}, 0}, {{2, 5, 7}, 0}, {{2, 6, 7}, 0}, {{3, 4, 5}, 0},
{{3, 4, 6}, 0}, {{3, 4, 7}, 0}, {{3, 5, 6}, -4}, {{3, 5, 7}, 0}, {{3, 6, 7}, 0},
{{0, 1, 2, 4}, 0}, {{0, 1, 2, 5}, 0}, {{0, 1, 2, 6}, -4}, {{0, 1, 2, 7}, 0},
{{0, 1, 3, 4}, 0}, {{0, 1, 3, 5}, 0}, {{0, 1, 3, 6}, 0}, {{0, 1, 3, 7}, 0}, {{0, 2, 3, 4}, 0},
{{0, 2, 3, 5}, 0}, {{0, 2, 3, 6}, 0}, {{0, 2, 3, 7}, 0}, {{0, 4, 5, 6}, 0}, {{0, 4, 5, 7}, 0},
{{0, 4, 6, 7}, 0}, {{0, 5, 6, 7}, 0}, {{1, 2, 3, 4}, 0}, {{1, 2, 3, 5}, 0}, {{1, 2, 3, 6}, 0},
{{1, 2, 3, 7}, 0}, {{1, 4, 5, 6}, 0}, {{1, 4, 5, 7}, 0}, {{1, 4, 6, 7}, 0}, {{1, 5, 6, 7}, 0},
{{2, 4, 5, 6}, 0}, {{2, 4, 5, 7}, 0}, {{2, 4, 6, 7}, 0}, {{2, 5, 6, 7}, 0}, {{3, 4, 5, 6}, 0},
{{3, 4, 5, 7}, 0}, {{3, 4, 6, 7}, 0}, {{3, 5, 6, 7}, 0}, {{0, 1, 2, 3, 4}, 0},
{{0, 1, 2, 3, 5}, -4}, {{0, 1, 2, 3, 6}, 0}, {{0, 1, 2, 3, 7}, 0}, {{0, 1, 4, 5, 6}, 0},
{{0, 1, 4, 5, 7}, 0}, {{0, 1, 4, 6, 7}, 0}, {{0, 1, 5, 6, 7}, 0}, {{0, 2, 4, 5, 6}, 0},
{{0, 2, 4, 5, 7}, 0}, {{0, 2, 4, 6, 7}, 0}, {{0, 2, 5, 6, 7}, 0}, {{0, 3, 4, 5, 6}, 0},
{{0, 3, 4, 5, 7}, 0}, {{0, 3, 4, 6, 7}, 0}, {{0, 3, 5, 6, 7}, 4}, {{1, 2, 4, 5, 6}, 0},
{{1, 2, 4, 5, 7}, 4}, {{1, 2, 4, 6, 7}, 0}, {{1, 2, 5, 6, 7}, 0}, {{1, 3, 4, 5, 6}, 0},
{{1, 3, 4, 5, 7}, 0}, {{1, 3, 4, 6, 7}, 0}, {{1, 3, 5, 6, 7}, 0}, {{2, 3, 4, 5, 6}, 0},
{{2, 3, 4, 5, 7}, 0}, {{2, 3, 4, 6, 7}, 0}, {{2, 3, 5, 6, 7}, 0}, {{0, 1, 2, 3, 4, 5}, 0},
{{0, 1, 2, 3, 4, 6}, -4}, {{0, 1, 2, 3, 4, 7}, 0}, {{0, 1, 2, 3, 5, 6}, 0},
{{0, 1, 2, 3, 5, 7}, 0}, {{0, 1, 2, 3, 6, 7}, 0}, {{0, 1, 4, 5, 6, 7}, 0},
{{0, 2, 4, 5, 6, 7}, 0}, {{0, 3, 4, 5, 6, 7}, 0}, {{1, 2, 4, 5, 6, 7}, 0},
{{1, 3, 4, 5, 6, 7}, 0}, {{2, 3, 4, 5, 6, 7}, 0}, {{0, 1, 2, 4, 5, 6, 7}, 0},
{{0, 1, 3, 4, 5, 6, 7}, 0}, {{0, 2, 3, 4, 5, 6, 7}, 0}, {{1, 2, 3, 4, 5, 6, 7}, 0}}
Out[354]= {{3, 5, 6}, -4}, {{0, 1, 2, 6}, -4}, {{0, 1, 2, 3, 5}, -4},
{{0, 3, 5, 6, 7}, 4}, {{1, 2, 4, 5, 7}, 4}, {{0, 1, 2, 3, 4, 6}, -4}}
In[356]:= {Length[linTransSymmetric], Length[linTransAntiSymmetric]}
Out[356]= {10, 6}
```

BASIS of 8×8 matrices :

In[357]:=

```
tA = {};
Do[AppendTo[tA, {τ[j], {j}}], {j, 1, 7}]
Length[tA]
tAB = {};
Do[AppendTo[tAB, {τ[j].τ[k], {j, k}}], {j, 1, 6}, {k, j + 1, 7}]
Length[tAB]
tABC = {};
Do[AppendTo[tABC, {τ[j].τ[k].τ[h], {j, k, h}}],
{j, 1, 5}, {k, j + 1, 6}, {h, k + 1, 7}];
Length[tABC]
```

Out[358]=

7

Out[360]=

21

Out[362]=

35

In[363]:=

tA[[1]]

Out[363]=

```
{ {{0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0},
{0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
{0, 0, 1, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}}, {1} }
```

In[364]:=

tA[[#]] [[1]] == -Transpose[tA[[#]] [[1]]] & /@ Range[Length[tA]]

Out[364]=

{True, True, True, False, False, False}

In[365]:=

bas64 = Flatten[{tA, tAB, tABC, {{Id, {64}}}}, 1]; Length[bas64]

Out[365]=

64

In[366]:=

```
determineIF8AntiSymmetric[m_] := Module[{t, tt, sum, res, r},
t = m.τ[#] & /@ Range[0, 7];
tt = Transpose[#] & /@ t;
sum = FullSimplify[t[[#]] + tt[[#]]] & /@ Range[Length[t]];
res = Union[Flatten[#]] & /@ sum;
If[Length[#] > 1, Return[False]] & /@ res;
r = Union[Flatten[res]];
Return[r == {0}];
]
```


In[376]:=

```

η64 =
DiagonalMatrix[Table[FullSimplify[ $\frac{1}{8} \text{Tr}[\text{bas64}[A, 1].\text{bas64}[A, 1]]$ ], {A, 64}]];
Tr[η64]

```

Out[376]=

$$7 + \frac{1}{8} \text{Tr}[\text{Id}.\text{Id}]$$

In[377]:=

```

(* (* (* (* (* (* (countTrace=Table[{A,B,FullSimplify[ $\frac{1}{8} \text{Tr}[\text{bas64}[A,1].\text{bas64}[B,1]$ ]] },
{A,1, 64},{B,A, 64}]) //MatrixForm*)*)*)*)*)*)

```

In[378]:=

$$(*■*)$$

In[379]:=

$$32 \times 65$$

Out[379]=

$$2080$$

In[380]:=

$$(*\text{Length}[\text{Flatten}[\text{countTrace}, 1]]*)$$

In[381]:=

$$(*\{\text{Length}[\text{countTrace}], \text{Length}[\text{countTrace}\llbracket 1 \rrbracket], \text{Length}[\text{countTrace}\llbracket 1, 1 \rrbracket]\}*)$$

In[382]:=

$$(*\{\text{Length}[\text{countTrace}], \text{Length}[\text{countTrace}\llbracket 2 \rrbracket], \text{Length}[\text{countTrace}\llbracket 2, 1 \rrbracket]\}*)$$

In[383]:=

$$(*\{\text{Length}[\text{countTrace}], \text{Length}[\text{countTrace}\llbracket 64 \rrbracket], \text{Length}[\text{countTrace}\llbracket 64, 1 \rrbracket]\}*)$$

In[384]:=

$$(*\{\text{Length}[\text{Flatten}[\text{countTrace}, 1]\llbracket \text{All}, 3 \rrbracket], \text{Count}[\text{Flatten}[\text{countTrace}, 1]\llbracket \text{All}, 3 \rrbracket, 0], \\ \text{Count}[\text{Flatten}[\text{countTrace}, 1]\llbracket \text{All}, 3 \rrbracket, 1], \text{Count}[\text{Flatten}[\text{countTrace}, 1]\llbracket \text{All}, 3 \rrbracket, -1]\}*)$$

In[385]:=

$$32 \times 63$$

Out[385]=

$$2016$$

In[386]:=

```

anti = {};
Do[If[bas64[[k, 1]] == -Transpose[bas64[[k, 1]]],
    AppendTo[anti, {bas64[[k, 1]], {{k}, bas64[[k, 2]]}}]], {k, 1, 64}];
Length[anti]
anti[[28]]

```

Out[387]=

28

Out[388]=

```

{{{0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0},
{0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, -1, 0}}, {{63}, {5, 6, 7}}}

```

In[389]:=

anti[[28, 1]]

Out[389]=

```

{{{0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, -1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0},
{0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, -1, 0}}

```

In[390]:=

anti[[28, 2]]

Out[390]=

```

{{63}, {5, 6, 7}}

```

In[391]:=

```

symm = {};
Do[If[bas64[[k, 1]] == Transpose[bas64[[k, 1]]],
    AppendTo[symm, {bas64[[k, 1]], {{k}, bas64[[k, 2]]}}]], {k, 1, 64}];
Length[symm]
symm[[28]]

```

Out[392]=

35

Out[393]=

```

{{{ -1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0},
{0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0},
{0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 1}}, {{52}, {2, 5, 7}}}

```

4×4 Dirac γ matrices:

$$S^{\alpha\beta} = -\frac{1}{4}[\gamma^\alpha, \gamma^\beta], \quad (22)$$

$$S^{\alpha\beta} = -\frac{1}{2}\gamma^{\alpha\beta}. \quad (23)$$

The symplectic form ϵ on D_4 is defined to be

$$\epsilon = \gamma^{64}. \quad (24)$$

As a consequence of Eq. (1), and the definitions of Eqs. (16)–(24), are the identities

$$\tilde{\gamma}^\alpha \epsilon = -\epsilon \gamma^\alpha, \quad (25)$$

$$\tilde{S}^{\alpha\beta} \epsilon = -\epsilon S^{\alpha\beta}, \quad (26)$$

$$[S^{\alpha\beta}, \gamma_\mu] = \delta_\mu^\alpha \gamma^\beta - \delta_\mu^\beta \gamma^\alpha, \quad (27)$$

$$[S^{\alpha\beta}, S^{\mu\nu}] = g^{\alpha\mu} S^{\beta\nu} - g^{\alpha\nu} S^{\beta\mu} - g^{\beta\mu} S^{\alpha\nu} + g^{\beta\nu} S^{\alpha\mu}, \quad (28)$$

and

$$\gamma^5 S^{\alpha\beta} = \frac{1}{2} g^{\alpha\mu} g^{\beta\nu} \epsilon_{\mu\nu\lambda\sigma} S^{\lambda\sigma}. \quad (29)$$

$$g^{\alpha\beta} = g_{\alpha\beta} = \text{diag}(1, 1, 1, -1) \quad (5)$$

In[394]:=

(*Symbolize[g_{αβ}]*)

In[395]:=

$\eta 4488$

Dimensions [%]

$g_{\alpha\beta} = \eta 4488 [2 ; 5, 2 ; 5]$

Out[395]=

$\{ \{1, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0\},$
 $\{0, 0, 1, 0, 0, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 0, 0, -1, 0, 0, 0\},$
 $\{0, 0, 0, 0, 0, -1, 0, 0\}, \{0, 0, 0, 0, 0, 0, -1, 0\}, \{0, 0, 0, 0, 0, 0, 0, -1\} \}$

Out[396]=

{8, 8}

Out[397]=

$\{ \{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, -1\} \}$

In[398]:=

Protect[g_{αβ}]

Out[398]=

{g_{Subscript}αβ}

In[399]:=

g3 = DiagonalMatrix[{1, 1, -1}]

Out[399]=

$\{ \{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, -1\} \}$

In[400]:=

Do[Do[Gr[h, k] = t4by4[h].s4by4[k]*(-g3[[h, h]]*g3[[k, k]]), {k, 1, 3}], {h, 1, 3}]

In[401]:= **Gr[1, 1] // MatrixForm**

Out[401]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In[402]:=

$$\gamma = \{\text{Gr}[1, 1], \text{Gr}[1, 2], \text{Gr}[1, 3], \text{t4by4}[2]\};$$

In[403]:=

$$\text{Table}[\text{FullSimplify}[\gamma[h].\gamma[h] == g_{\alpha\beta}[h, h] * \text{ID4}], \{h, 1, 4\}]$$

Out[403]=

$$\{\text{True}, \text{True}, \text{True}, \text{True}\}$$

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\gamma_0 g^{\alpha\beta}, \quad (4)$$

where

$$g^{\alpha\beta} = g_{\alpha\beta} = \text{diag}(1, 1, 1, -1) \quad (5)$$

and γ_0 denotes the 4×4 unit matrix. The $\Sigma^{\alpha\beta}$ of Eq. (1)

In[404]:=

$$\text{Table}\left[\left\{\{h, k\}, \text{FullSimplify}\left[\frac{\gamma[h].\gamma[k] + \gamma[k].\gamma[h]}{2}\right] == g_{\alpha\beta}[h, k] \text{ID4}\right\}, \{h, 1, 4\}, \{k, 1, 4\}\right] // \text{MatrixForm}$$

Out[404]//MatrixForm=

$$\begin{pmatrix} (\{1, 1\}) & (\{1, 2\}) & (\{1, 3\}) & (\{1, 4\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{2, 1\}) & (\{2, 2\}) & (\{2, 3\}) & (\{2, 4\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{3, 1\}) & (\{3, 2\}) & (\{3, 3\}) & (\{3, 4\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{4, 1\}) & (\{4, 2\}) & (\{4, 3\}) & (\{4, 4\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \end{pmatrix}$$

In[405]:=

$$\text{Table}\left[\left\{\{h, k\}, \text{FullSimplify}\left[\frac{\gamma[h] \cdot \gamma[k] + \gamma[k] \cdot \gamma[h]}{2}\right] == \eta 4488 [1+h, 1+k] \text{ ID4}\right\}, \{h, 1, 4\}, \{k, 1, 4\}\right] // \text{MatrixForm}$$

Out[405]//MatrixForm=

$$\begin{pmatrix} (\{1, 1\}) & (\{1, 2\}) & (\{1, 3\}) & (\{1, 4\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{2, 1\}) & (\{2, 2\}) & (\{2, 3\}) & (\{2, 4\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{3, 1\}) & (\{3, 2\}) & (\{3, 3\}) & (\{3, 4\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{4, 1\}) & (\{4, 2\}) & (\{4, 3\}) & (\{4, 4\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \end{pmatrix}$$

$$\gamma^5 = -\gamma^1 \gamma^2 \gamma^3 \gamma^4$$

In[406]:=

$$-\gamma[1] \cdot \gamma[2] \cdot \gamma[3] \cdot \gamma[4]$$

Out[406]=

$$\{\{0, -1, 0, 0\}, \{1, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, -1, 0\}\}$$

In[407]:=

$$\text{AppendTo}[\gamma, (-\gamma[1] \cdot \gamma[2] \cdot \gamma[3] \cdot \gamma[4])]$$

Out[407]=

$$\{\{\{-1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, -1\}\}, \{\{0, -1, 0, 0\}, \{-1, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 1, 0\}\}, \{\{0, 0, 1, 0\}, \{0, 0, 0, 1\}, \{1, 0, 0, 0\}, \{0, 1, 0, 0\}\}, \{\{0, 0, -1, 0\}, \{0, 0, 0, -1\}, \{1, 0, 0, 0\}, \{0, 1, 0, 0\}\}, \{\{0, -1, 0, 0\}, \{1, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, -1, 0\}\}\}$$

In[408]:=

$$\text{Protect}[\gamma]$$

Out[408]=

$$\{\gamma\}$$

In[409]:=

$$\gamma[5]$$

Out[409]=

$$\{\{0, -1, 0, 0\}, \{1, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, -1, 0\}\}$$

In[410]:=

$$\gamma[5] == -t4by4[3]$$

Out[410]=

$$\text{True}$$

In[411]:=

```
Table[{{h, k}, FullSimplify[\gamma[h].\gamma[k] + \gamma[k].\gamma[h]] === 2 * \eta4488[[1 + h, 1 + k]] ID4}, {h, 1, 5}, {k, 1, 5}] // MatrixForm
```

Out[411]//MatrixForm=

$$\begin{pmatrix} (\{1, 1\}) & (\{1, 2\}) & (\{1, 3\}) & (\{1, 4\}) & (\{1, 5\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{2, 1\}) & (\{2, 2\}) & (\{2, 3\}) & (\{2, 4\}) & (\{2, 5\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{3, 1\}) & (\{3, 2\}) & (\{3, 3\}) & (\{3, 4\}) & (\{3, 5\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{4, 1\}) & (\{4, 2\}) & (\{4, 3\}) & (\{4, 4\}) & (\{4, 5\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \\ (\{5, 1\}) & (\{5, 2\}) & (\{5, 3\}) & (\{5, 4\}) & (\{5, 5\}) \\ (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) & (\text{True}) \end{pmatrix}$$

$$\gamma^\alpha = \gamma^{\alpha 6}, \quad (19)$$

$$\gamma^5 = \gamma^{56}, \quad (20)$$

$$\gamma^\alpha \gamma^5 = \gamma^{\alpha 5}, \quad (21)$$

$$\epsilon = \gamma^{64}.$$

In[412]:=

```
-\gamma[4] // MatrixForm
```

Out[412]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

In[413]:=

```
s\epsilon = {\epsilon \rightarrow t4by4[2]}
```

Out[413]=

```
{\epsilon \rightarrow {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}}}
```

In[414]:=

```
Transpose[\epsilon /. s\epsilon] === (-\epsilon /. s\epsilon)
```

Out[414]=

```
True
```

In[415]:=

```
Table[
 {{A1}, FullSimplify[(\epsilon /. s\epsilon).\gamma[A1] === Transpose[(\epsilon /. s\epsilon).\gamma[A1]]]}, {A1, 1, 5}]
```

Out[415]=

```
{{{1}, True}, {{2}, True}, {{3}, True}, {{4}, True}, {{5}, False}}
```

```
In[416]:=  $\gamma[[4]] == \text{t4by4}[2]$ 
Out[416]= True

In[417]:=  $\text{t4by4}[1]$ 
Out[417]=  $\{\{0, 0, 0, -1\}, \{0, 0, 1, 0\}, \{0, -1, 0, 0\}, \{1, 0, 0, 0\}\}$ 
```

$$S^{\alpha\beta} = -\frac{1}{4}[\gamma^\alpha, \gamma^\beta]. \quad (3)$$

```
In[418]:=  $S44\alpha\beta = -\frac{1}{4} \text{Table}[\text{FullSimplify}[\gamma[h].\gamma[k] - \gamma[k].\gamma[h]], \{h, 1, 4\}, \{k, 1, 4\}];$ 
```

```
In[419]:= Protect[S44 $\alpha\beta$ ]
```

```
Out[419]= {S44 $\alpha\beta$ }
```

```
In[420]:=  $\eta44 = \text{DiagonalMatrix}[\{1, 1, 1, -1\}]$ 
```

```
Out[420]=  $\{\{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, -1\}\}$ 
```

```
In[421]:= Protect[\mathbf{\eta44}]
```

```
Out[421]= {\mathbf{\eta44}}
```

```
 $S44\alpha\beta$  commutation relations; misc :
```

$$S^{\alpha\beta} = -\frac{1}{4}[\gamma^\alpha, \gamma^\beta], \quad (22)$$

$$S^{\alpha\beta} = -\frac{1}{2}\gamma^{\alpha\beta}. \quad (23)$$

The symplectic form ϵ on D_4 is defined to be

$$\epsilon = \gamma^{64}. \quad (24)$$

As a consequence of Eq. (1), and the definitions of Eqs. (16)–(24), are the identities

$$\tilde{\gamma}^\alpha \epsilon = -\epsilon \gamma^\alpha, \quad (25)$$

$$\tilde{S}^{\alpha\beta} \epsilon = -\epsilon S^{\alpha\beta}, \quad (26)$$

$$[S^{\alpha\beta}, \gamma_\mu] = \delta_\mu^\alpha \gamma^\beta - \delta_\mu^\beta \gamma^\alpha, \quad (27)$$

$$[S^{\alpha\beta}, S^{\mu\nu}] = g^{\alpha\mu} S^{\beta\nu} - g^{\alpha\nu} S^{\beta\mu} - g^{\beta\mu} S^{\alpha\nu} + g^{\beta\nu} S^{\alpha\mu}, \quad (28)$$

and

$$\gamma^5 S^{\alpha\beta} = \frac{1}{4} g^{\alpha\mu} g^{\beta\nu} \epsilon_{\mu\nu\lambda\sigma} S^{\lambda\sigma}. \quad (29)$$

$$S44\alpha\beta = -\frac{1}{4} \text{Table}[\text{FullSimplify}[\gamma[h].\gamma[k] - \gamma[k].\gamma[h]], \{h, 1, 4\}, \{k, 1, 4\}];$$

In[422]:=

$$(*SAB=\text{Table}[\frac{1}{4} (\text{T16}^A[A1].\text{T16}^A[B1]-\text{T16}^A[B1].\text{T16}^A[A1]), \{A1,0, 7\}, \{B1,0, 7\}];*)$$

In[423]:=

$$(*\text{Table}[\text{FullSimplify}[\sigma16.SAB[A1,B1]===\text{Transpose}[\sigma16.SAB[A1,B1]]], \{A1,1, 8\}, \{B1,1,8\}]\//\text{Flatten}\//\text{Union}*)$$

In[424]:=

$$(*\text{Table}[\{\{A1,B1\}, \text{FullSimplify}[SAB[A1,B1]===\text{Transpose}[SAB[A1,B1]]]\}, \{A1,1, 8\}, \{B1,1,8\}]*)$$

In[425]:=

$$\begin{aligned} \text{Table}[\text{FullSimplify}[S44\alpha\beta[A1, B1].S44\alpha\beta[A2, B2] - S44\alpha\beta[A2, B2].S44\alpha\beta[A1, B1]] == \\ (\eta44[A1, A2] \times S44\alpha\beta[B1, B2] - \eta44[A1, B2] \times S44\alpha\beta[B1, A2] - \\ \eta44[B1, A2] \times S44\alpha\beta[A1, B2] + \eta44[B1, B2] \times S44\alpha\beta[A1, A2])], \\ \{A1, 1, 3\}, \{B1, A1 + 1, 4\}, \{A2, 1, 3\}, \{B2, A2 + 1, 4\}] // \text{Flatten} // \text{Union} \end{aligned}$$

Out[425]=

$$\{\text{True}\}$$

In[426]:=

```
Table[{{A1, B1, A2, B2},  
      FullSimplify[S44 $\alpha\beta$ [[A1, B1]].S44 $\alpha\beta$ [[A2, B2]] - S44 $\alpha\beta$ [[A2, B2]].S44 $\alpha\beta$ [[A1, B1]] ==  
      ( $\eta_{44}$ [[A1, A2]]  $\times$  S44 $\alpha\beta$ [[B1, B2]] -  $\eta_{44}$ [[A1, B2]]  $\times$  S44 $\alpha\beta$ [[B1, A2]] -  
        $\eta_{44}$ [[B1, A2]]  $\times$  S44 $\alpha\beta$ [[A1, B2]] +  $\eta_{44}$ [[B1, B2]]  $\times$  S44 $\alpha\beta$ [[A1, A2]]) ]},  
      {A1, 1, 4}, {B1, 1, 4}, {A2, 1, 4}, {B2, 1, 4}] // MatrixForm
```

Out[426]//MatrixForm=

In[427]:=

```
Table[FullSimplify[S44 $\alpha\beta$ [[A1, B1]]. $\gamma$ [[B2]] -  $\gamma$ [[B2]].S44 $\alpha\beta$ [[A1, B1]] ==
  ( $\eta$ 44[[B2, A1]]  $\times$   $\gamma$ [[B1]] -  $\eta$ 44[[B2, B1]]  $\times$   $\gamma$ [[A1]])], {A1, 1, 4}, {B1, 1, 4}, {B2, 1, 4}] // Flatten // Union
```

Out[427]=

{ True }

In[428]:=

```
Table[{{A1, B1, B2}, FullSimplify[S44 $\alpha\beta$ [A1, B1]. $\gamma$ [B2] -  $\gamma$ [B2].S44 $\alpha\beta$ [A1, B1] ==
(η44[B2, A1]  $\times$   $\gamma$ [B1] - η44[B2, B1]  $\times$   $\gamma$ [A1])]}, {A1, 1, 4}, {B1, 1, 4}, {B2, 1, 4}] // MatrixForm
```

Out[428]//MatrixForm=

$$\begin{array}{cccc} \left(\begin{array}{cc} \{1, 1, 1\} & \text{True} \\ \{1, 1, 2\} & \text{True} \\ \{1, 1, 3\} & \text{True} \\ \{1, 1, 4\} & \text{True} \\ \{2, 1, 1\} & \text{True} \\ \{2, 1, 2\} & \text{True} \\ \{2, 1, 3\} & \text{True} \\ \{2, 1, 4\} & \text{True} \\ \{3, 1, 1\} & \text{True} \\ \{3, 1, 2\} & \text{True} \\ \{3, 1, 3\} & \text{True} \\ \{3, 1, 4\} & \text{True} \\ \{4, 1, 1\} & \text{True} \\ \{4, 1, 2\} & \text{True} \\ \{4, 1, 3\} & \text{True} \\ \{4, 1, 4\} & \text{True} \end{array} \right) & \left(\begin{array}{cc} \{1, 2, 1\} & \text{True} \\ \{1, 2, 2\} & \text{True} \\ \{1, 2, 3\} & \text{True} \\ \{1, 2, 4\} & \text{True} \\ \{2, 2, 1\} & \text{True} \\ \{2, 2, 2\} & \text{True} \\ \{2, 2, 3\} & \text{True} \\ \{2, 2, 4\} & \text{True} \\ \{3, 2, 1\} & \text{True} \\ \{3, 2, 2\} & \text{True} \\ \{3, 2, 3\} & \text{True} \\ \{3, 2, 4\} & \text{True} \\ \{4, 2, 1\} & \text{True} \\ \{4, 2, 2\} & \text{True} \\ \{4, 2, 3\} & \text{True} \\ \{4, 2, 4\} & \text{True} \end{array} \right) & \left(\begin{array}{cc} \{1, 3, 1\} & \text{True} \\ \{1, 3, 2\} & \text{True} \\ \{1, 3, 3\} & \text{True} \\ \{1, 3, 4\} & \text{True} \\ \{2, 3, 1\} & \text{True} \\ \{2, 3, 2\} & \text{True} \\ \{2, 3, 3\} & \text{True} \\ \{2, 3, 4\} & \text{True} \\ \{3, 3, 1\} & \text{True} \\ \{3, 3, 2\} & \text{True} \\ \{3, 3, 3\} & \text{True} \\ \{3, 3, 4\} & \text{True} \\ \{4, 3, 1\} & \text{True} \\ \{4, 3, 2\} & \text{True} \\ \{4, 3, 3\} & \text{True} \\ \{4, 3, 4\} & \text{True} \end{array} \right) & \left(\begin{array}{cc} \{1, 4, 1\} & \text{True} \\ \{1, 4, 2\} & \text{True} \\ \{1, 4, 3\} & \text{True} \\ \{1, 4, 4\} & \text{True} \\ \{2, 4, 1\} & \text{True} \\ \{2, 4, 2\} & \text{True} \\ \{2, 4, 3\} & \text{True} \\ \{2, 4, 4\} & \text{True} \\ \{3, 4, 1\} & \text{True} \\ \{3, 4, 2\} & \text{True} \\ \{3, 4, 3\} & \text{True} \\ \{3, 4, 4\} & \text{True} \\ \{4, 4, 1\} & \text{True} \\ \{4, 4, 2\} & \text{True} \\ \{4, 4, 3\} & \text{True} \\ \{4, 4, 4\} & \text{True} \end{array} \right) \end{array}$$

In[429]:=

```
Table[{{A1, B1},
FullSimplify[(ε /. se).S44 $\alpha\beta$ [A1, B1] === Transpose[(ε /. se).S44 $\alpha\beta$ [A1, B1]]]}, {A1, 1, 3}, {B1, A1 + 1, 4}]
```

Out[429]=

$$\{\{\{1, 2\}, \text{True}\}, \{\{1, 3\}, \text{True}\}, \{\{1, 4\}, \text{True}\}\}, \{\{\{2, 3\}, \text{True}\}, \{\{2, 4\}, \text{True}\}\}, \{\{\{3, 4\}, \text{True}\}\}$$

The commutation relations for the generators J_{ab} of the $\mathfrak{so}(3,3)$ Lie algebra (using the physics convention where generators are Hermitian) are given by the formula: \otimes

$$[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad} + \eta_{bd}J_{ac})$$

where:

- The indices a, b, c, d range from 1 to 6.
- η_{ab} is the **metric tensor** with signature $(+, +, +, -, -, -)$ (or the reverse, depending on the convention). A common choice is a diagonal matrix $\eta = \text{diag}(1, 1, 1, -1, -1, -1)$.
- J_{ab} are the **generators** of the algebra, which are antisymmetric, $J_{ab} = -J_{ba}$.
- The terms on the right-hand side involve Kronecker deltas weighted by the metric, ensuring the resulting generator is also an element of the algebra and satisfies the required antisymmetry and Jacobi identity properties. \otimes

The final answer is $[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad} + \eta_{bd}J_{ac})$.

Step 1: Define the generators and metric

The **so(4,4)** Lie algebra has 28 generators, which can be denoted as $J_{\mu\nu}$ where the indices μ, ν range from 0 to 7, and $J_{\mu\nu} = -J_{\nu\mu}$. The Lie algebra is defined in an 8-dimensional real vector space equipped with a metric $\eta_{\mu\nu}$ with signature (4, 4). \circlearrowleft

- **Generators:** $J_{\mu\nu}$ with $\mu, \nu \in \{0, 1, \dots, 7\}$
- **Metric tensor:** $\eta_{\mu\nu} = \text{diag}(-1, -1, -1, -1, +1, +1, +1, +1)$ (or the opposite sign convention) \circlearrowleft

Step 2: State the general commutation relation

The commutation relations for the generators of the **so(p,q)** Lie algebra are a generalization of the **so(n)** algebra relations, incorporating the metric tensor $\eta_{\mu\nu}$. \circlearrowleft

Answer:

The commutation relations for the generators $J_{\mu\nu}$ of the **so(4,4)** Lie algebra are:

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho}$$

where $\eta_{\mu\nu}$ is the metric tensor with a (4,4) signature (four plus signs and four minus signs, or vice versa). \circlearrowleft

(Note: In some physics conventions, an extra factor of i or $i\hbar$ is included depending on whether the generators are defined as Hermitian or not). \circlearrowleft

what are the commutation relation for the generators of $\text{so}(4,4)$?

The commutation relations for the generators $J_{\mu\nu}$ of the $so(4,4)$ Lie algebra, where $\mu, \nu \in \{0, 1, \dots, 7\}$ and $J_{\mu\nu} = -J_{\nu\mu}$, are the **standard relations for orthogonal Lie algebras**, given by a generalization of the angular momentum commutation relations: \circlearrowleft

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$

where $\eta_{\mu\nu}$ is the flat metric of 8-dimensional Minkowski space with signature (4,4) (or some other convention, the structure constants are the same as for $so(8)$ in an appropriate basis), which is a diagonal matrix with four positive and four negative entries (e.g., $\eta = \text{diag}(+1, +1, +1, +1, -1, -1, -1, -1)$).



\hookrightarrow Why is the metric signature significant for $\$so(4,4)\$$ commutation relations?

Why is the metric signature significant for \$so(4,4)\$ commutation relations?

The metric signature is significant for the $so(p, q)$ commutation relations because the metric tensor $\eta_{\mu\nu}$ acts as the **structure constants** of the Lie algebra, defining its specific algebraic structure and physical properties (if applicable). ^②

Here is a detailed explanation of its significance:

- **Definition of the Algebra:** The commutation relation for the generators $J_{\mu\nu}$ of any orthogonal Lie algebra $so(p, q)$ is given by:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$
The values of the structure constants depend directly on the components of the metric $\eta_{\mu\nu}$. This means that the specific algebraic properties of $so(4, 4)$ are distinct from $so(8)$ (which has a (8,0) signature, i.e., a Euclidean metric), even though they operate in the same 8-dimensional vector space.
- **Physical Interpretation:** In physics, the metric signature determines the nature of spacetime. A mixed signature (like (4,4) or the more common (1,3) Minkowski signature) implies a causal structure, distinguishing between timelike and spacelike dimensions.
 - Generators associated with rotations in spacelike dimensions (positive $\eta_{\mu\nu}$ entries) typically lead to compact symmetry groups (like $SO(4)$ or $SO(8)$).

In[430]:=

η4488

Out[430]=

```
{ {1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
{0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0},
{0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1} }
```

orthogonal Lie algebra $so(p, q)$ is given by:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$

$$t^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

In[431]:=

```
(*Table[SAB[[A1,B1]]== ArrayFlatten[{{SAB1[[A1,B1]],0},{0,SAB2[[A1,B1]]}}],{A1,{0, 7},{B1,{0,7}}]*)
```

$$4D_{(1)}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A$$

$$4D_{(2)}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A.$$

Let $g \in \overline{SO(4, 4)}$ and $L \in SO(4, 4)$. The canonical 2-1 homomorphism $\overline{SO(4, 4)} \rightarrow SO(4, 4)$ is given by

$$(2.8) \quad 8L^{A'}{}_{B'}(g) = \text{tr } \bar{D}^{(2)}(g) \tau^{A'} D^{(1)}(g) \bar{\tau}_{B'}.$$

As usual tr denotes the trace. Under the action of $\overline{SO(4, 4)}$

$$(2.9) \quad \bar{D}^{(1)} \sigma = \sigma D^{(1)-1},$$

$$(2.10) \quad \sigma^{-1} \bar{D}^{(2)} = D^{(2)-1} \sigma^{-1},$$

$$(2.11) \quad L^{A'}{}_{C'} G_{A'B'} L^{B'}{}_{D'} = G_{C'D'},$$

$$(2.12) \quad L^{A'}{}_{B'} \tau^{B'} = \bar{D}^{(2)} \tau^{A'} D^{(1)},$$

$$(2.13) \quad L^{A'}{}_{B'} \bar{\tau}^{B'} = D^{(1)-1} \bar{\tau}^{A'} \bar{D}^{(2)-1}.$$

JOURNAL OF MATHEMATICAL PHYSICS

VOLUME 4, NUMBER 7

JULY 1963

A Remarkable Representation of the 3 + 2 de Sitter Group

P. A. M. DIRAC

*Cambridge University, Cambridge, England and
Institute for Advanced Study, Princeton, New Jersey**
(Received 20 February 1963)

Identities satisfied by the generators of the Dirac algebra

Patrick L. Nash

M. S. 474, NASA Langley Research Center, Hampton, Virginia 23665

(Received 26 July 1983; accepted for publication 23 September 1983)

```
In[432]:= Unprotect[ $\gamma$ ABs];
 $\gamma$ ABs = Flatten[{s4by4[#] & /@ Range[3], t4by4[#] & /@ Range[3],
Flatten[Table[st[J, K], {J, 1, 3}, {K, 1, 3}], 1], {ID4}], 1]

Out[433]= {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}},
{{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}},
{{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
{{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
{{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
{{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
{{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
{{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}},
{{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}},
{{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}},
{{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}

In[434]:= Length[ $\gamma$ ABs]
Out[434]= 16

In[435]:= Protect[ $\gamma$ ABs]
Out[435]= { $\gamma$ ABs}

In[436]:=  $\gamma$ ABs[[1]]
Out[436]= {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}}

In[437]:=  $\gamma$ ABs[[ -1 ]]
Out[437]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}]

Recall Self Dual Anti-Symmetric
```

In[438]:=

```
Table[ $\left(\frac{1}{2} \sum_{j_1=1}^4 \left( \sum_{j_2=1}^4 \epsilon 4[p, q, j_1, j_2] \times s4by4[h][j_1, j_2] \right) - s4by4[h][p, q],\right.$   

 $\left. \{h, 1, 3\}, \{q, 4\}, \{p, 4\}\right]$ 
```

Out[438]=

```
{ {{ {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0} }},  

  {{ {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0} }},  

  {{ {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0} }}}
```

Recall Anti SelfDual Anti-Symmetric

In[439]:=

```
Table[ $\left(\frac{1}{2} \sum_{j_1=1}^4 \left( \sum_{j_2=1}^4 \epsilon 4[p, q, j_1, j_2] \times t4by4[h][j_1, j_2] \right) + t4by4[h][p, q],\right.$   

 $\left. \{h, 1, 3\}, \{q, 4\}, \{p, 4\}\right)$ 
```

Out[439]=

```
{ {{ {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0} }},  

  {{ {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0} }},  

  {{ {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0} }}}
```

BASIS of 4×4 matrices :

Proc. Camb. Phil. Soc. (1968), 64, 765

765

PCPS 64–96

Printed in Great Britain

The Dirac spinor in six dimensions

BY E. A. LORD

Department of Mathematics, King's College, University of London

(Received 17 November 1966)

3. Decomposition of the basic spinor representations. We have already stated that the basic spinor representations of $SO(n, c)$, for n even, are reducible. We shall consider here the reduction of the basic 8-component spinor representation of $SO(6, c)$ into constituent 4-component representations. The procedure is easily generalized to apply to any even-dimensional space. An irreducible representation of the Clifford algebra C_6 can be constructed from a given irreducible representation of C_5 . The $\gamma_\mu (\mu = 1, \dots, 5)$ of an irreducible representation of C_5 satisfy (2.5), so that the quantities $e_\alpha (\alpha = 1, \dots, 6)$ defined by

$$\left. \begin{aligned} e_\mu &= (-i) \begin{pmatrix} & \gamma_\mu B \\ -B^{-1}\gamma_\mu & \end{pmatrix} \quad (\mu = 1, \dots, 5), \\ e_6 &= \begin{pmatrix} & B \\ B^{-1} & \end{pmatrix}, \end{aligned} \right\} \quad (3.1)$$

satisfy

$$\frac{1}{2}(e_\alpha e_\beta + e_\beta e_\alpha) = \delta_{\alpha\beta} \quad (\alpha, \beta = 1, \dots, 6).$$

For the moment the matrix B is completely arbitrary. The e_α therefore give an irreducible representation of C_6 . They can be written more concisely in terms of the sets of matrices a_α and \bar{a}_α , defined as follows:

$$\left. \begin{aligned} a_\mu &= \gamma_\mu B, \quad \bar{a}_\mu = -B^{-1}\gamma_\mu \quad (\mu = 1, \dots, 5), \\ a_6 &= iB, \quad \bar{a}_6 = iB^{-1}. \end{aligned} \right\} \quad (3.2)$$

The generator e_α of C_6 can then be written

$$e_\alpha = (-i) \begin{pmatrix} & a_\alpha \\ \bar{a}_\alpha & \end{pmatrix} \quad (\alpha = 1, \dots, 6). \quad (3.3)$$

With $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ we then find

$$e_7 = ie_1e_2e_3e_4e_5e_6 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

and $e_{\alpha\beta} = \frac{1}{2}(e_\alpha e_\beta - e_\beta e_\alpha) = -\frac{1}{2} \begin{pmatrix} a_\alpha \bar{a}_\beta - a_\beta \bar{a}_\alpha & \\ \bar{a}_\alpha a_\beta - \bar{a}_\beta a_\alpha & \end{pmatrix}.$

This latter expression shows that the infinitesimal generators of the 8-component spinor representation of $SO(6, c)$ is the direct sum of two 4-component representations. The basic spinor can be written

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

where ϕ and χ are four component spinors which transform according to representations with infinitesimal generators

and $G_{\alpha\beta}^{(1)} = -\frac{1}{4}(a_\alpha \bar{a}_\beta - a_\beta \bar{a}_\alpha) \quad \left. \begin{aligned} G_{\alpha\beta}^{(2)} &= -\frac{1}{4}(\bar{a}_\alpha a_\beta - \bar{a}_\beta a_\alpha), \end{aligned} \right\} \quad (3.4)$

$$\underbrace{\text{sy}}_1, \underbrace{\text{sy}}_4, \underbrace{\text{asy}}_6, \underbrace{\text{sy}}_4, \underbrace{\text{sy}}_1$$

```
In[440]:= Plus @@ {1, 4, 6, 4, 1}
Out[440]= 16
{sy, sy, asy, sy, sy}
  1   4   6   4   1

In[441]:= Plus @@ {1, 4, 6, 4, 1}
Out[441]= 16

In[442]:= MatrixForm[If[#, i, 1] * PauliMatrix[#]] & /@ Range[3]
Out[442]= {{{0, 1}, {1, 0}}, {{0, 1}, {-1, 0}}, {{1, 0}, {0, -1}}}

In[443]:= σ22 = Flatten[{ {IdentityMatrix[2]}, If[#, i, 1] * PauliMatrix[#] & /@ Range[3]}, 1]
Out[443]= {{{1, 0}, {0, 1}}, {{0, 1}, {1, 0}}, {{0, 1}, {-1, 0}}, {{1, 0}, {0, -1}}}

In[444]:= σ22 =
Flatten[{ {-IdentityMatrix[2]}, If[#, i, 1] * PauliMatrix[#] & /@ Range[3]}, 1]
Out[444]= {{{-1, 0}, {0, -1}}, {{0, 1}, {1, 0}}, {{0, 1}, {-1, 0}}, {{1, 0}, {0, -1}}}

In[445]:= -IdentityMatrix[2] === Dot @@ (σ22[[#]] & /@ Range[Length[σ22]])
Out[445]= True

In[446]:= Table[{{A, B}, FullSimplify[1/2 Tr[1/2 (σ22[[A]].σ22[[B]] + σ22[[B]].σ22[[A]])]]}], {{A, 1, 4}, {B, 1, 4}}] // MatrixForm
Out[446]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1, 1\} \\ -1 \end{pmatrix} & \begin{pmatrix} \{1, 2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{1, 3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{1, 4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{2, 1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{2, 2\} \\ 1 \end{pmatrix} & \begin{pmatrix} \{2, 3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{2, 4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{3, 1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{3, 2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{3, 3\} \\ -1 \end{pmatrix} & \begin{pmatrix} \{3, 4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{4, 1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4, 2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4, 3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4, 4\} \\ 1 \end{pmatrix} \end{pmatrix}$$


In[447]:= η2244 = DiagonalMatrix[{-1, 1, -1, 1}]
Out[447]= {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}}
```

In[448]:=

```
Table[{{A, B}, FullSimplify[ExpandAll[ $\frac{1}{2} (\sigma_{22}[A].\overline{\sigma_{22}}[B] + \sigma_{22}[B].\overline{\sigma_{22}}[A]) ==$   

 $\eta_{2244}[A, B]*IdentityMatrix[2]]]}}, {A, 1, 4}, {B, 1, 4}] // MatrixForm$ 
```

Out[448]//MatrixForm=

$$\begin{pmatrix} \{1, 1\} & \{1, 2\} & \{1, 3\} & \{1, 4\} \\ \text{True} & \text{True} & \text{True} & \text{True} \\ \{2, 1\} & \{2, 2\} & \{2, 3\} & \{2, 4\} \\ \text{True} & \text{True} & \text{True} & \text{True} \\ \{3, 1\} & \{3, 2\} & \{3, 3\} & \{3, 4\} \\ \text{True} & \text{True} & \text{True} & \text{True} \\ \{4, 1\} & \{4, 2\} & \{4, 3\} & \{4, 4\} \\ \text{True} & \text{True} & \text{True} & \text{True} \end{pmatrix}$$

Proc. Camb. Phil. Soc. (1968), 64, 765

765

PCPS 64–96

Printed in Great Britain

The Dirac spinor in six dimensions

BY E. A. LORD

Department of Mathematics, King's College, University of London

(Received 17 November 1966)

$$\left. \begin{aligned} e_\mu &= (-i) \begin{pmatrix} \gamma_\mu B \\ -B^{-1}\gamma_\mu \end{pmatrix} \quad (\mu = 1, \dots, 5), \\ e_6 &= \begin{pmatrix} B \\ B^{-1} \end{pmatrix}, \end{aligned} \right\}$$

$$\frac{1}{2}(e_\alpha e_\beta + e_\beta e_\alpha) = \delta_{\alpha\beta} \quad (\alpha, \beta = 1, \dots, 6).$$

$$\left. \begin{aligned} \alpha_\mu &= \gamma_\mu B, \quad \bar{a}_\mu = -B^{-1}\gamma_\mu \quad (\mu = 1, \dots, 5), \\ a_6 &= iB, \quad \bar{a}_6 = iB^{-1}. \end{aligned} \right\}$$

generator e_α of C_6 can then be written

$$e_\alpha = (-i) \begin{pmatrix} a_\alpha \\ \bar{a}_\alpha \end{pmatrix} \quad (\alpha = 1, \dots, 6).$$

 $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ we then find

$$e_7 = ie_1e_2e_3e_4e_5e_6 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$e_{\alpha\beta} = \frac{1}{2}(e_\alpha e_\beta - e_\beta e_\alpha) = -\frac{1}{2} \begin{pmatrix} a_\alpha \bar{a}_\beta - a_\beta \bar{a}_\alpha & \\ \bar{a}_\alpha a_\beta - \bar{a}_\beta a_\alpha & \end{pmatrix}.$$

```
In[449]:= Do[yyy[j] = ArrayFlatten[{{0, σ22[[j]]}, {-σ22[[j]], 0}}], {j, 4}];
yyy[5] = ArrayFlatten[{{0, IdentityMatrix[2]}, {IdentityMatrix[2], 0}}]
Out[449]= {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}}
In[450]:= yyy[5] = {{0, 0, -1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 0}, {0, 1, 0, 0}}
Out[450]= {{0, 0, -1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 0}, {0, 1, 0, 0}}
In[451]:= (yyy[6] = Dot @@ (yyy[#[ ] & /@ Range[5])) // MatrixForm
Out[451]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

In[452]:= MatrixForm[yyy[#[ ] ] & /@ Range[6]
Out[452]= {{0 0 1 0}, {0 0 0 1}, {-1 0 0 0}, {0 -1 0 0}, {0 0 0 1}, {0 0 1 0}, {0 0 -1 0}, {0 0 0 -1}, {-1 0 0 0}, {0 1 0 0}, {0 0 0 1}, {0 0 0 -1}, {1 0 0 0}, {0 -1 0 0}}
In[453]:= Do[xxx[j] = ArrayFlatten[{{0, σ22[[j]]}, {σ22[[j]], 0}}], {j, 4}]
In[454]:= MatrixForm[xxx[#[ ] ] & /@ Range[4]
Out[454]= {{0 0 -1 0}, {0 0 0 -1}, {1 0 0 0}, {0 1 0 0}, {0 0 0 1}, {0 0 1 0}, {0 0 -1 0}, {0 1 0 0}, {-1 0 0 0}, {0 -1 0 0}}
In[455]:= xxx[5] = Dot @@ (xxx[#[ ] & /@ Range[4])
Out[455]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}
In[456]:= IdentityMatrix[4]
Out[456]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

In[457]:=

```

txA0 = {{IdentityMatrix[4], {0}}};
Length[txA0]
txA = {} ; Do[AppendTo[txA, {xxx[j], {j}}], {j, 1, 4}]
Length[txA]
txAB = {};
Do[AppendTo[txAB, {xxx[j].xxx[k], {j, k}}], {j, 1, 3}, {k, j + 1, 4}]
Length[txAB]
txABC = {};
Do[AppendTo[txABC, {xxx[j].xxx[k].xxx[h], {j, k, h}}],
{j, 1, 2}, {k, j + 1, 3}, {h, k + 1, 4}];
Length[txABC]
txABCD = {{xxx[1].xxx[2].xxx[3].xxx[4], {1, 2, 3, 4}}};
Length[txABCD]

```

Out[458]=

1

Out[460]=

4

Out[462]=

6

Out[464]=

4

Out[466]=

1

In[467]:=

```
base4by4 = Flatten[{txA0, txA, txAB, txBC, txABCD}, 1]; Length[base4by4]
```

Out[467]=

16

In[468]:=

```
Table[{{A, B}, Block[{r}, r = FullSimplify[
  1/4 Tr[1/2 (base4by4[[A, 1]].base4by4[[B, 1]] + base4by4[[B, 1]].base4by4[[A, 1]])]];
  If[r == 0, Style[r, Red], Style[r, Darker[Green]]]], {A, 1, Length[base4by4]}, {B, 1, Length[base4by4]}] // MatrixForm
```

Out[468]//MatrixForm=

In[469]:=

base4by4 [16] [1]

Out[469]=

$$\{\{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, -1, 0\}, \{0, 0, 0, -1\}\}$$

In[470]:=

```

anti4by4 = {};
Do[If[base4by4[[k, 1]] == -Transpose[base4by4[[k, 1]]], AppendTo[anti4by4,
    {base4by4[[k, 1]], {{k}, base4by4[[k, 2]]}}]], {k, 1, Length[base4by4]}];
Length[anti4by4]
anti4by4[[%]]

```

Out[471]=

6

Out[472]=

 $\{\{ \{0, 0, 0, 1\}, \{0, 0, 1, 0\}, \{0, -1, 0, 0\}, \{-1, 0, 0, 0\} \}, \{\{14\}, \{1, 3, 4\}\}\}$

In[473]:=

```

symm4by4 = {};
Do[If[base4by4[[k, 1]] == Transpose[base4by4[[k, 1]]], AppendTo[symm4by4,
    {base4by4[[k, 1]], {{k}, base4by4[[k, 2]]}}]], {k, 1, Length[base4by4]}];
Length[symm4by4]
symm4by4[[%]]

```

Out[474]=

10

Out[475]=

 $\{\{ \{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, -1, 0\}, \{0, 0, 0, -1\} \}, \{\{16\}, \{1, 2, 3, 4\}\}\}$

In[476]:=

```

η16 = DiagonalMatrix[Table[
  FullSimplify[ $\frac{1}{4} \text{Tr}[\text{symm4by4}[[A, 1]] . \text{symm4by4}[[A, 1]]]$ ], {A, Length[symm4by4]}]];
Tr[η16]

```

Out[476]=

10

O(4,4); values, evecs of σ

In[477]:=

```
{values, evecs} = Eigensystem[σ]

```

Out[477]=

 $\{\{-1, -1, -1, -1, 1, 1, 1, 1\}, \{0, 0, 0, -1, 0, 0, 0, 1\}, \{0, 0, -1, 0, 0, 0, 1, 0\},$
 $\{0, -1, 0, 0, 1, 0, 0, 0\}, \{-1, 0, 0, 1, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0, 0, 1\},$
 $\{0, 0, 1, 0, 0, 1, 0\}, \{0, 1, 0, 0, 0, 1, 0, 0\}, \{1, 0, 0, 0, 1, 0, 0, 0\}\}\}$

In[478]:= $\text{ui} = \text{ExpandAll}\left[\frac{1}{\sqrt{2}} \text{evecs}\right] // \text{MatrixForm}$

Out[478]/MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

In[479]:= $\text{Table}[\text{ui}[[h]].\sigma.\text{ui}[[h]], \{h, 1, \text{Length}[\text{ui}]\}]$

Out[479]= $\{-1, -1, -1, -1, 1, 1, 1, 1\}$

In[480]:= $\text{Table}[(\text{Transpose}[\text{ui}] [[h]]).\sigma.\text{ui}[[h]], \{h, 1, \text{Length}[\text{ui}]\}]$

Out[480]= $\{-1, -1, -1, -1, 1, 1, 1, 1\}$

In[481]:= $\text{Table}[(\text{Transpose}[\text{ui}] [[A]]) == -\sigma.\text{ui}[[A]], \{A, 1, 4\}]$
 $\text{Table}[(\text{Transpose}[\text{ui}] [[A]]) == \sigma.\text{ui}[[A]], \{A, 5, \text{Length}[\text{ui}]\}]$

Out[481]= $\{\text{True}, \text{True}, \text{True}, \text{True}\}$

Out[482]= $\{\text{True}, \text{True}, \text{True}, \text{True}\}$

In[483]:=

hUSE = 8;

In[484]:= $\text{Protect}[\text{hUSE}]$

Out[484]= $\{\text{hUSE}\}$

In[485]:=

unit = ui[[hUSE]]

Out[485]= $\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}$

```
In[486]:= Protect[unit]
Out[486]= {unit}

In[487]:= (Transpose[unit].σ.unit)
Out[487]= 1

In[488]:= (Transpose[unit].σ.(τ[θ]))
Out[488]= {1/√2, 0, 0, 0, 1/√2, 0, 0, 0}


---



$$\mathfrak{F}_a^A = \frac{1}{\sqrt{\tilde{u} \sigma u}} u^c \sigma_{cb} \bar{\tau}^{Ab}{}_a$$


```

$$\mathfrak{F}_A^a = \frac{1}{\sqrt{\tilde{u} \sigma u}} \tau_A{}^a{}_b u^b$$

```
In[489]:= Clear[resolution];
resolution[v_List] := Module[{EAa, EaA, FaA, id, u},
  If[Length[v] == 8, ## &[], Print["ERROR Length[v] ", Length[v]];
   Return[Length[v]]];
  v
  u = v/Abs[Transpose[v].σ.v] // FullSimplify;
  EAa = Table[u.σ.τ[A1], {A1, 0, 7}];  $\hat{\sigma}_a^b = \frac{1}{\sqrt{u \sigma u}} \sigma_{ab} \tau^{cd} u_d^b$ ;
  EaA = FullSimplify[Inverse[EAa]];
  (*FaA=FullSimplify[Transpose[η4488[[#+1,#+1]*({τ[#].unit})]&/@Range[0,7]];*)
  FaA = FullSimplify[
    Transpose[Table[η4488[[A1 + 1, A1 + 1]] * ({τ[A1].unit}), {A1, 0, 7}]]];
  If[FaA === EaA, Print["u^A = ", EAa.u], Print["ERROR"]];
  If[F_A^a.F_a^A === ID8, ## &[], Print["ERROR F_A^a.F_a^A != ID8"]];
  Return[Union[Flatten[FullSimplify[FaA - EaA]]]];
  Return[{FaA, EaA, FaA - EaA}];
]
```

```
In[491]:= resolution[unit]
```

```

u^A = {1, 0, 0, 0, 0, 0, 0, 0}
ERROR F_A^a.F_a^A != ID8
Out[491]=
{0}

```

In[492]:=

```

Clear[resolution2];
resolution2[u_List] := Module[{EAa, EaA, FaA, id},
  If[Length[u] == 8, ##&[], Print["ERROR Length[u] ", Length[u]];
   Return[Length[u]]];
  (*u=----- //FullSimplify;*)
  EAa = Table[u.σ.τ[A1], {A1, 0, 7}];
  
$$\tilde{\delta}_\mu^A = \frac{1}{\sqrt{\sigma \tau_\mu}} \mu^\nu \sigma_{\nu B} \tau_A^{B\mu};$$

  Print["u^A = ", EAa.u];
  Return[EAa];
  EaA = FullSimplify[Inverse[EAa]]; 
$$\tilde{\delta}_A^B = \frac{1}{\sqrt{\sigma \tau_\mu}} \tau_A^{B\mu} \mu^\nu;$$

  (*FaA=FullSimplify[Transpose[η4488[[#+1,#+1]*(\tau[[#].unit)]&/@Range[0,7]]];*)
  FaA = FullSimplify[
    Transpose[Table[η4488[[A1 + 1, A1 + 1]] * (\tau[A1].unit), {A1, 0, 7}]]];
  If[FaA === EaA, Print["u^A = ", EAa.u], Print["ERROR"]];
  If[F_A^a.F_a^A === ID8, ##&[], Print["ERROR F_A^a.F_a^A != ID8"]];
  Return[Union[Flatten[FullSimplify[FaA - EaA]]];
  Return[{FaA, EaA, FaA - EaA}];
]

```

In[494]:=

```

W[#] & /@ Range[0, 7];
resolution2[%]
u^A = {2 W[0] × W[4] + 2 W[1] × W[5] + 2 W[2] × W[6] + 2 W[3] × W[7], 0, 0, 0, 0, 0, 0}

```

Out[495]=

```

{W[4], W[5], W[6], W[7], W[0], W[1], W[2], W[3]},
{W[3], W[2], -W[1], -W[0], W[7], W[6], -W[5], -W[4]},
{-W[2], W[3], W[0], -W[1], -W[6], W[7], W[4], -W[5]},
{W[1], -W[0], W[3], -W[2], W[5], -W[4], W[7], -W[6]},
{-W[1], W[0], W[3], -W[2], W[5], -W[4], -W[7], W[6]},
{W[2], W[3], -W[0], -W[1], -W[6], -W[7], W[4], W[5]},
{W[3], -W[2], W[1], -W[0], -W[7], W[6], -W[5], W[4]},
{W[4], W[5], W[6], W[7], -W[0], -W[1], -W[2], -W[3]}}

```

In[496]:=

```

myid[MX_] := Sum[τ[A].MX.τ[A] η4488[[A, A]], {A, 1, 8}]
Myid[MX_] := Sum[τ[A].MX.τ[A] × η448[[A, A]], {A, 1, 8}]

```

```
Union[
Table[Sum[\tau[A].bas64[[k, 1]].\sigma.\bar{\tau}[A] \times \eta8[[A, A], {A, 1, 8}] - IdTr[bas64[[k, 1]].\sigma] +
4 eg[[k]] \frac{bas64[[k, 1]] - Transpose[bas64[[k, 1]]]}{2} .\sigma, {k, 1, 64}], ]]
```

```
{{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}}
```

In[498]:=

 $\bar{\tau}[0]$

Out[498]=

```
{ {1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 1} }
```

In[499]:=

 $(Transpose[unit].\sigma.(\bar{\tau}[0]))$

Out[499]=

```
{ \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0 }
```

FaA = FullSimplify[Transpose[Table[\eta4488[[A1 + 1, A1 + 1]]*(\tau[A1].unit), {A1, 0, 7}]]]

In[500]:=

```
FAa = Transpose[\eta4488[[# + 1, # + 1]] * (\tau[#].unit) & /@ Range[0, 7]]
```

Out[500]=

```
{ { \frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{2}} }, { 0, 0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0 }, { 0, 0, \frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0 }, { 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0 }, { \frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, -\frac{1}{\sqrt{2}} }, { 0, 0, 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, 0 }, { 0, 0, \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0, 0 }, { 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}, 0 } }
```

In[501]:=

```
FAa = Transpose[unit].\sigma.(\bar{\tau}[\#]) & /@ Range[0, 7]
```

Out[501]=

```
{ { \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0 }, { 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}} }, { 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0 }, { 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0 }, { 0, \frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0 }, { 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0 }, { 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}} }, { \frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0 } }
```

```
In[502]:=  $F_A^a \cdot F_a^A == ID8$ 
```

```
Out[502]= True
```

pretend that the X are Minkowski Coordinates :

```
In[503]:= Block[{U, u = U[#] & /@ Range[8], fx = F_A^a.X, eq, sx, su, ret},
  eq = And @@ Thread[u == fx];
  sx = Solve[eq, X][[-1]];
  su = Solve[eq, u][[-1]];
  ret = {sx, su};
  ret]
```

```
Out[503]=  $\left\{ \begin{array}{l} x0 \rightarrow \frac{U[1]}{\sqrt{2}} + \frac{U[5]}{\sqrt{2}}, x1 \rightarrow -\frac{U[4]}{\sqrt{2}} - \frac{U[8]}{\sqrt{2}}, x2 \rightarrow \frac{U[3]}{\sqrt{2}} + \frac{U[7]}{\sqrt{2}}, \\ x3 \rightarrow -\frac{U[2]}{\sqrt{2}} - \frac{U[6]}{\sqrt{2}}, x4 \rightarrow \frac{1}{2} (\sqrt{2} U[2] - \sqrt{2} U[6]), x5 \rightarrow \frac{1}{2} (-\sqrt{2} U[3] + \sqrt{2} U[7]), \\ x6 \rightarrow \frac{1}{2} (-\sqrt{2} U[4] + \sqrt{2} U[8]), x7 \rightarrow \frac{1}{2} (\sqrt{2} U[1] - \sqrt{2} U[5]) \end{array} \right\},$ 
 $\left\{ \begin{array}{l} U[1] \rightarrow \frac{1}{2} (\sqrt{2} x0 + \sqrt{2} x7), U[2] \rightarrow \frac{1}{2} (-\sqrt{2} x3 + \sqrt{2} x4), U[3] \rightarrow \frac{1}{2} (\sqrt{2} x2 - \sqrt{2} x5), \\ U[4] \rightarrow \frac{1}{2} (-\sqrt{2} x1 - \sqrt{2} x6), U[5] \rightarrow \frac{1}{2} (\sqrt{2} x0 - \sqrt{2} x7), U[6] \rightarrow \frac{1}{2} (-\sqrt{2} x3 - \sqrt{2} x4), \\ U[7] \rightarrow \frac{1}{2} (\sqrt{2} x2 + \sqrt{2} x5), U[8] \rightarrow \frac{1}{2} (-\sqrt{2} x1 + \sqrt{2} x6) \end{array} \right\}$ 
```

```
In[504]:= (sgtrye(A) = Thread[Flatten[e(A)] → Flatten[eAa04]]) (*//Column*)
```

```
In[504]:= subsDefects
```

```
Out[504]= {}
```

```
In[505]:= Thread[X → 0]
```

```
Out[505]= {x0 → 0, x1 → 0, x2 → 0, x3 → 0, x4 → 0, x5 → 0, x6 → 0, x7 → 0}
```

split octonions; evals, evecs of σ

```
# Type - 1 spinor basis (constructed from Clifford algebra considerations)
# These create a maximal isotropic subspace with respect to the split form
self.type1_spinor_matrix = sqrt2_inv*np.array([[1, 1, 0, 0, 0, 0, 0, 0], # s1_0[1, -1, 0, 0, 0, 0, 0, 0],
                                                # s1_1[0, 0, 1, 1, 0, 0, 0, 0], # s1_2[0, 0, 1, -1, 0, 0, 0, 0], # s1_3[0, 0, 0, 1, 1, 0, 0], # s1_4[0, 0, 0, 1, -1, 0, 0], # s1_5[0, 0, 0, 0, 0, 1, 1], # s1_6[0, 0, 0, 0, 0, 1, -1], # s1_7])
```

```
# Type - 2 spinor basis (the complementary spinor representation)
# Related to Type - 1 by another triality automorphism
self.type2_spinor_matrix = sqrt2_inv*np.array([[1, 0, 1, 0, 0, 0, 0, 0], # s2_0[1, 0, -1, 0, 0, 0, 0, 0], # s2_
1[0, 1, 0, 1, 0, 0, 0, 0], # s2_2[0, 1, 0, -1, 0, 0, 0, 0], # s2_3[0, 0, 0, 1, 0, 1, 0], # s2_4[0, 0, 0, 1, 0, -1, 0], # s2_
5[0, 0, 0, 0, 1, 0, 1], # s2_6[0, 0, 0, 0, 1, 0, -1], # s2_7])


$$M_{S_1 \rightarrow S_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

```

4 Transformation Matrices

We provide explicit 8×8 matrices that transform coordinates between the three bases.

4.1 Vector to Type-1 Spinor

The transformation matrix from vector basis to type-1 spinor basis is:

$$M_{V \rightarrow S_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad (11)$$

4.2 Vector to Type-2 Spinor

The transformation matrix from vector basis to type-2 spinor basis is:

$$M_{V \rightarrow S_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \quad (12)$$

In[506]:=

```
(type1SpinorBasisMatrix = {{1, 1, 0, 0, 0, 0, 0, 0}, {1, -1, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 1, 0, 0, 0, 0}, {0, 0, 1, -1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 1, 0, 0}, {0, 0, 0, 0, 1, -1, 0, 0}, {0, 0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 0, 0, 1, -1}}) // MatrixForm
```

Out[506]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

In[507]:=

```
(type2SpinorBasisMatrix = {{1, 0, 1, 0, 0, 0, 0, 0}, {1, 0, -1, 0, 0, 0, 0, 0}, {0, 1, 0, 1, 0, 0, 0, 0}, {0, 1, 0, -1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 1, 0}, {0, 0, 0, 0, 1, 0, -1, 0}, {0, 0, 0, 0, 0, 1, 0, 1}, {0, 0, 0, 0, 0, 1, 0, -1}}) // MatrixForm
```

Out[507]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

$$M_{S_1 \rightarrow S_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} M_{S_1 \rightarrow S_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

In[508]:=

```
MS1S2 = type1SpinorBasisMatrix.Transpose[type2SpinorBasisMatrix] // MatrixForm
```

Out[508]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

In[509]:=

$\eta_{AB} = \eta_{4488}$

Out[509]=

```
{ {1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1} }
```

In[510]:=

```
{evalsAB, evecsAB} = Eigensystem[\etaAB]
```

Out[510]=

```
{ {-1, -1, -1, -1, 1, 1, 1, 1}, {{0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0}}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0}} }
```

In[511]:=

```
{evals, evecs} = Eigensystem[σ]
```

Out[511]=

```
{ {-1, -1, -1, -1, 1, 1, 1, 1}, {{0, 0, 0, -1, 0, 0, 0, 1}, {0, 0, -1, 0, 0, 0, 1, 0}, {0, -1, 0, 0, 0, 1, 0, 0}, {-1, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 1}, {0, 0, 1, 0, 0, 0, 1, 0}, {0, 1, 0, 0, 0, 1, 0, 0}, {1, 0, 0, 0, 1, 0, 0, 0}}}
```

In[512]:=

$$\left(\text{u} = \text{ExpandAll} \left[\frac{1}{\sqrt{2}} \text{evecs} \right] \right) // \text{MatrixForm}$$

Out[512]/MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

In[513]:=

```
Table[u[[h]].σ.u[[h]], {h, 1, Length[u]}]
```

Out[513]=

```
{-1, -1, -1, -1, 1, 1, 1, 1}
```

In[514]:=

```
Table[(Transpose[u][[h]]).σ.u[[h]], {h, 1, Length[u]}]
```

Out[514]=

```
{-1, -1, -1, -1, 1, 1, 1, 1}
```

In[515]:=

```
Table[(Transpose[u][[A]]) == -σ.u[[A]], {A, 1, 4}]
```

```
Table[(Transpose[u][[A]]) == σ.u[[A]], {A, 5, Length[u]}]
```

Out[515]=

```
{True, True, True, True}
```

Out[516]=

```
{True, True, True, True}
```

In[517]:=

```
hUSE = 8;
```

Set: Symbol hUSE is Protected. *i*

In[518]:=

```
Protect[hUSE]
```

Out[518]=

```
{}
```

```
In[519]:= unit = u[[hUSE]]
*** Set: Symbol unit is Protected. i
Out[519]=  $\left\{ \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0 \right\}$ 
```

```
In[520]:= Protect[unit]
Out[520]= {}
```

```
In[521]:= (Transpose[unit].σ.unit)
Out[521]= 1
```

```
In[522]:= (Transpose[unit].σ.(τ[θ]))
Out[522]=  $\left\{ \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0 \right\}$ 
```

```
In[523]:= Do[EAa[h] = Table[(u[[h]].σ.(τ[B])), {B, 0, 7}], {h, 1, Length[u]}];
 $\tilde{\delta}_a^b = \frac{1}{\sqrt{u} \sigma u} u^c \sigma_{cb} \tau^b_a$ 
Do[EaA[h] = FullSimplify[Inverse[EAa[h]]], {h, 1, Length[u]}];
 $\tilde{\delta}_A^a = \frac{1}{\sqrt{u} \sigma u} \tau_A^a b^b$ 
Table[EaA[h] === (u[[h]].σ.u[[h]])
    Transpose[Table[FullSimplify[ExpandAll[n4488[B, B] × τ[B - 1].u[[h]]]],
        {B, 1, 8}]], {h, 1, Length[u]}]
```

```
Out[523]=  $\tilde{\delta}_a^b = \frac{1}{\sqrt{u} \sigma u} u^c \sigma_{cb} \tau^b_a$ 
```

```
Out[524]=  $\tilde{\delta}_A^a = \frac{1}{\sqrt{u} \sigma u} \tau_A^a b^b$ 
```

```
Out[525]= {True, True, True, True, True, True, True, True}
```

```
In[526]:= (*Do[EAa[h] = Table[(u[[h]].σ.(τ[B])), {B, 0, 7}], {h, 1, Length[u]}];
Do[EaA[h] = FullSimplify[Inverse[EAa[h]]], {h, 1, Length[u]}];
Table[EaA[h] === (u[[h]].σ.u[[h]])
    Transpose[Table[FullSimplify[ExpandAll[n4488[B, B] τ[B - 1].u[[h]]]], {B, 1, 8}]],
        {h, 1, Length[u]}]*)
```

In[527]:=

```
myid[MX_] := Sum[ $\bar{\tau}[A].MX.\tau[A]$   $\eta4488[[A, A], \{A, 1, 8\}]$ ]
Myid[MX_] := Sum[ $\tau[A].MX.\bar{\tau}[A] \times \eta448[[A, A], \{A, 1, 8\}]$ ]
```

```
Union[
  Table[ $\text{Sum}[\tau[A].\text{bas64}[[k, 1]].\sigma.\bar{\tau}[A] \times \eta8[[A, A], \{A, 1, 8\}] - \text{IdTr}[\text{bas64}[[k, 1]].\sigma]] +$ 
     $4 \text{ eg}[[k]] \frac{\text{bas64}[[k, 1]] - \text{Transpose}[\text{bas64}[[k, 1]]]}{2} .\sigma, \{k, 1, 64\}]$ ]
```

```
{\{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}\}}
```

In[529]:=

 $\bar{\tau}[0]$

Out[529]=

```
\{\{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 1}\}
```

In[530]:=

 $(\text{Transpose}[\text{unit}].\sigma.(\bar{\tau}[0]))$

Out[530]=

```
\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\}
```

In[531]:=

```
FaA = Transpose[η4488[[#, # + 1]] * (τ[#[].unit) & /@ Range[0, 7]]
```

Out[531]=

```
\{\{\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\}, \{0, 0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\}, \{0, 0, \frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\}, {0, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0\}, \{\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}\}, {0, 0, 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, 0\}, \{0, 0, \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0, 0\}, \{0, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}, 0\}\}
```

In[532]:=

```
Faa = Transpose[unit].σ.(bar τ[#]) & /@ Range[0, 7]
```

Out[532]=

```
\{\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\}, \{0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}\}, {0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0\}, \{0, -\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\}, \{0, \frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\}, {0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0\}, \{0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}\}, \{\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0\}\}
```

In[533]:= $\mathbf{F}_A^a \cdot \mathbf{F}_a^A == \mathbf{ID8}$

Out[533]= True

In[534]:= $E_A^a [hUSE] == F_a^A$
 $E_A^a [hUSE] == F_A^a$

Out[534]= True

Out[535]= True

$$\text{gtrye}_{\alpha}^{(A)} = (e_{\alpha}^{(A)} /. \text{sgtrye}_{\alpha}^{(A)}) /. \text{subsDefects}$$

$$\text{gtrye}_{(A)}^{\alpha} = (e_{(A)}^{\alpha} /. \text{sgtrye}_{(A)}^{\alpha}) /. \text{subsDefects}$$

$$(\text{sgtrye}_{(A)}^{\alpha} = \text{Thread}[\text{Flatten}[e_{(A)}^{\alpha}] \rightarrow \text{Flatten}[\text{Inverse}[e_{\alpha}^{(A)} /. \text{sgtrye}_{\alpha}^{(A)}]]])$$

$$(* // \text{Column} *)$$

$$\text{gtrye}_{(A)}^{\alpha} = (e_{(A)}^{\alpha} /. \text{sgtrye}_{(A)}^{\alpha}) /. \text{subsDefects}$$

$$\left\{ \{ \text{Tan}[6 H x 0], 0, 0, 0, 0, 0, 0, 0, 0 \}, \left\{ 0, \frac{e^{-a4[H x 4]}}{\text{Sin}[6 H x 0]^{1/6}}, 0, 0, 0, 0, 0, 0, 0 \right\}, \right.$$

$$\left. \left\{ 0, 0, \frac{e^{-a4[H x 4]}}{\text{Sin}[6 H x 0]^{1/6}}, 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, \frac{e^{-a4[H x 4]}}{\text{Sin}[6 H x 0]^{1/6}}, 0, 0, 0, 0, 0 \right\}, \right.$$

$$\left. \left\{ 0, 0, 0, 0, 1, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0, \frac{e^{a4[H x 4]}}{\text{Sin}[6 H x 0]^{1/6}}, 0, 0 \right\}, \right.$$

$$\left. \left\{ 0, 0, 0, 0, 0, 0, \frac{e^{a4[H x 4]}}{\text{Sin}[6 H x 0]^{1/6}}, 0 \right\}, \left\{ 0, 0, 0, 0, 0, 0, 0, \frac{e^{a4[H x 4]}}{\text{Sin}[6 H x 0]^{1/6}} \right\} \right\}$$

pretend that the X are Minkowski Coordinates :

In[536]:=

```
Block[{U, u = U[#] & /@ Range[8], fx = Fa.X, eq, sx, su, ret},
  eq = And @@ Thread[u == fx];
  sx = Solve[eq, X][[-1]];
  su = Solve[eq, u][[-1]];
  ret = {sx, su};
  ret]
```

Out[536]=

$$\begin{aligned} \left\{ \begin{aligned} x0 &\rightarrow \frac{U[1]}{\sqrt{2}} + \frac{U[5]}{\sqrt{2}}, & x1 &\rightarrow -\frac{U[4]}{\sqrt{2}} - \frac{U[8]}{\sqrt{2}}, & x2 &\rightarrow \frac{U[3]}{\sqrt{2}} + \frac{U[7]}{\sqrt{2}}, \\ x3 &\rightarrow -\frac{U[2]}{\sqrt{2}} - \frac{U[6]}{\sqrt{2}}, & x4 &\rightarrow \frac{1}{2} (\sqrt{2} U[2] - \sqrt{2} U[6]), & x5 &\rightarrow \frac{1}{2} (-\sqrt{2} U[3] + \sqrt{2} U[7]), \\ x6 &\rightarrow \frac{1}{2} (-\sqrt{2} U[4] + \sqrt{2} U[8]), & x7 &\rightarrow \frac{1}{2} (\sqrt{2} U[1] - \sqrt{2} U[5]) \end{aligned} \right\}, \\ \left\{ \begin{aligned} U[1] &\rightarrow \frac{1}{2} (\sqrt{2} x0 + \sqrt{2} x7), & U[2] &\rightarrow \frac{1}{2} (-\sqrt{2} x3 + \sqrt{2} x4), & U[3] &\rightarrow \frac{1}{2} (\sqrt{2} x2 - \sqrt{2} x5), \\ U[4] &\rightarrow \frac{1}{2} (-\sqrt{2} x1 - \sqrt{2} x6), & U[5] &\rightarrow \frac{1}{2} (\sqrt{2} x0 - \sqrt{2} x7), & U[6] &\rightarrow \frac{1}{2} (-\sqrt{2} x3 - \sqrt{2} x4), \\ U[7] &\rightarrow \frac{1}{2} (\sqrt{2} x2 + \sqrt{2} x5), & U[8] &\rightarrow \frac{1}{2} (-\sqrt{2} x1 + \sqrt{2} x6) \end{aligned} \right\} \end{aligned}$$

In[536]:=

```
(sgtrye(A) = Thread[Flatten[e(A)] → Flatten[eAa04]]) (*//Column*)
```

In[537]:=

subsDefects

Out[537]=

{ }

In[538]:=

Thread[X → 0]

Out[538]=

{x0 → 0, x1 → 0, x2 → 0, x3 → 0, x4 → 0, x5 → 0, x6 → 0, x7 → 0}

Stop pretending:

```
(sgtrye(A) = Thread[Flatten[e(A)] → Flatten[Inverse[e(A) /. sgtrye(A)]]]) (*//Column*)
```

gtrye^(A) = (e^(A) /. sgtrye^(A)) /. subsDefects

$$\begin{aligned} &\left\{ \{ \text{Tan}[6 H x0], 0, 0, 0, 0, 0, 0, 0, 0 \}, \left\{ 0, \frac{e^{-a4[H x4]}}{\text{Sin}[6 H x0]^{1/6}}, 0, 0, 0, 0, 0, 0 \right\}, \right. \\ &\left. \left\{ 0, 0, \frac{e^{-a4[H x4]}}{\text{Sin}[6 H x0]^{1/6}}, 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, \frac{e^{-a4[H x4]}}{\text{Sin}[6 H x0]^{1/6}}, 0, 0, 0, 0 \right\}, \right. \\ &\left. \left\{ 0, 0, 0, 0, 1, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0, \frac{e^{a4[H x4]}}{\text{Sin}[6 H x0]^{1/6}}, 0, 0 \right\}, \right. \end{aligned}$$

$$\left\{ 0, 0, 0, 0, 0, 0, \frac{e^{a4[H \times 4]}}{\sin[6H \times 0]^{1/6}}, 0 \right\}, \left\{ 0, 0, 0, 0, 0, 0, 0, 0, \frac{e^{a4[H \times 4]}}{\sin[6H \times 0]^{1/6}} \right\}$$

In[539]:=

$$(* e_{\alpha}^{(A)} = eA\alpha = gtrye_{\alpha}^{(A)} *)$$

In[540]:=

$$(* \{gtrye_{\alpha}^{(A)}, gtrye_{\alpha}^{\alpha} \} // MatrixForm *)$$

In[541]:=

$$(* F_A^a . e_{\alpha}^{(A)} . X / . sgtrye_{\alpha}^{(A)} // FullSimplify[\#, constraintVars] &; \\ frameOctadX = \# / . \left\{ \sqrt{e^{2 a4[H \times 4]}} \rightarrow e^{a4[H \times 4]}, \sqrt{e^{-2 a4[H \times 4]}} \rightarrow e^{-a4[H \times 4]} \right\} & / @ \% \\ \% / . \{x1 \rightarrow 0, x2 \rightarrow 0, x3 \rightarrow 0, x5 \rightarrow 0, x6 \rightarrow 0, x7 \rightarrow 0\} *)$$

In[542]:=

$$(* F_A^a . eA\alpha . X - frameOctadX // FullSimplify[\#, constraintVars] & *)$$

In[543]:=

$$(* eA\alpha *)$$

In[544]:=

$$(* \{ \{ F_A^a \}, \{ eA\alpha \}, \{ X \} \} // Column *)$$

split octonion multiplication constants :

$$m_{ab}^c = \mathbb{F}_a^A \tau_A^c b$$

$$m_{AB}^C = \mathbb{F}_c^C \tau_A^c \mathbb{F}_B^b$$

In[545]:=

$$EA = \text{Array}[eA, 8]$$

Out[545]=

$$\{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]\}$$

In[546]:=

$$\{E^A_a[hUSE] == F_a^A, E^a_A[hUSE] == F_A^a\}$$

Out[546]=

$$\{\text{True}, \text{True}\}$$

$$m_{ab}^c = \mathbb{F}_a^A \tau_A^c b$$

$$m_{AB}^C = \mathbb{F}_c^C \tau_A^c \mathbb{F}_B^b$$

$$\varepsilon_a \varepsilon_b = \varepsilon_c m_{ab}^c,$$

$$\epsilon_A \epsilon_B = \epsilon_C m_{AB}^C. \quad (53)$$

ed by

$$m_{ab}^c = \mathfrak{F}_a^A \tau_A{}^c{}_b,$$

$$m_{AB}^C = \mathfrak{F}_c^C \tau_A{}^c{}_b \mathfrak{F}_B^b, \quad (54)$$

In[547]:=

$$(*\text{Symbolize}\left[m_{ab}^c\right] *)$$

In[548]:=

$$(*\text{Symbolize}\left[m_{AB}^C\right] *)$$

In[549]:=

$$\mathbf{mabc} = \text{Array}[U, \{8, 8, 8\}];$$

In[550]:=

$$\begin{aligned} \tau A &= \eta_{AB}[\#, \#] * \tau [\# - 1] \& /@ \text{Range}[8]; \\ \text{Do}[\text{Do}[\mathbf{mabc}[[a1, b1, c1]] = \text{Sum}[E_a^A[hUSE][A1, a1] * \tau A[[A1]][[c1, b1]], \{A1, 1, 8\}], \{c1, 1, 8\}], \\ &\{a1, 1, 8\}, \{b1, 1, 8\}] \end{aligned}$$

In[552]:=

$$m_{ab}^c = \mathbf{mabc};$$

$$\text{NonCommutativeMultiply} \quad **$$

$$a \text{**} b$$

$$a \otimes b \otimes c$$

is a general associative, but non-commutative, form of multiplication.

Expand and Simplify do not operate on expressions with NonCommutativeMultiply :

In[553]:=

$$\begin{aligned} \text{unit.EA} // \text{Simplify} \\ \% \otimes \% // \text{Expand} \end{aligned}$$

Out[553]=

$$\frac{eA[1] + eA[5]}{\sqrt{2}}$$

Out[554]=

$$\frac{eA[1] + eA[5]}{\sqrt{2}} \otimes \frac{eA[1] + eA[5]}{\sqrt{2}}$$

```

In[555]:= 
$$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \text{Plus @@ } \{\text{mabc}\llbracket 1, 1, \text{All} \rrbracket, \text{mabc}\llbracket 1, 5, \text{All} \rrbracket, \text{mabc}\llbracket 5, 1, \text{All} \rrbracket, \text{mabc}\llbracket 5, 5, \text{All} \rrbracket\}$$

%.EA // Simplify
unit.EA // Simplify
% == %%

Out[555]= 
$$\left\{ \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0 \right\}$$


Out[556]= 
$$\frac{\text{eA}[1] + \text{eA}[5]}{\sqrt{2}}$$


Out[557]= 
$$\frac{\text{eA}[1] + \text{eA}[5]}{\sqrt{2}}$$


Out[558]= True

In[559]:= (*EAa[hUSE];  

ηAB[[#, #]*τ[#-1]&/@Range[8];  

mabc=Transpose[%%].Transpose[%]  

Dimensions[%]  

mABC=( EAa[hUSE].(EaA[hUSE].mabc)).EaA[hUSE]*)



---



$$m_{ab}^c = \mathbb{F}_a^A \tau_A^c b$$


$$m_{AB}^C = \mathbb{F}_c^C \tau_A^c \mathbb{F}_B^b$$


In[560]:= mABC = Array[R, {8, 8, 8}];

In[561]:= τA = ηAB[[#, #]*τ[#-1]&/@Range[8];
Do[Do[mABC[[A1, B1, C1]] = Sum[EAa[hUSE][[C1, c1]]*τA[[A1]][[c1, b1]]*EaA[hUSE][[b1, B1]],
{b1, 1, 8}, {c1, 1, 8}], {C1, 1, 8}], {A1, 1, 8}, {B1, 1, 8}]

```

In[563]:=

```
gridSplitOctonion =
Grid[Prepend[Drop[Reap[For[A1 = 1, A1 < 9, A1++, Sow[Flatten[{EA[[A1]], Table[
Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}}]}]]], 1] [[1]] [[1]], Flatten[{"A/B", EA}]]], Frame -> All]
```

Out[563]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

1	i	j	k	l	li	lj	lk
i	-1	k	-j	-li	l	-lk	lj
j	-k	-1	i	-lj	lk	l	-li
k	j	-i	-1	-lk	-lj	li	l
l	li	lj	lk	1	i	j	k
li	-l	-lk	lj	-i	1	k	-j
lj	lk	-l	-li	-j	-k	1	i
lk	-lj	li	-l	-k	j	-i	1

In[564]:=

$$m_{AB}^C = mABC;$$

In[565]:=

$$\begin{aligned} m_{ab}^c &= \mathbb{F}_a^A \tau_A^c b \\ m_{AB}^C &= \mathbb{F}_c^C \tau_A^c b \mathbb{F}_B^b \end{aligned}$$

Out[565]=

$$\begin{aligned} m_{ab}^c &= \mathbb{F}_a^A \tau_A^c b \\ m_{AB}^C &= \mathbb{F}_c^C \tau_A^c b \mathbb{F}_B^b \end{aligned}$$

```
In[566]:= E^A_a[hUSE].Transpose[(E^a_A[hUSE].mabc).E^a_A[hUSE]] === mABC
Out[566]= False

In[567]:= Transpose[(E^a_A[hUSE].mabc).E^a_A[hUSE]].E^A_a[hUSE] === mABC
Out[567]= False

In[568]:= Transpose[(E^A_a[hUSE].(E^a_A[hUSE].mabc)).E^a_A[hUSE]] === mABC
Out[568]= False

In[569]:= (*τA=ηAB[[#, #]*τ[#-1]&/@Range[8];
Table[Table[
  mABC[[A1,B1,C1]]=Sum[E^A_a[hUSE][[C1,c1]]*τA[[A1]][[c1,b1]]*E^a_A[hUSE][[b1,B1]],{b1,1,8},{c1,1,8}],
  {C1,1,8}],{A1,1,8},{B1,1,8}]*)

```

1	i	j	k	ℓ	ℓi	ℓj	ℓk
i	-1	k	$-j$	$-\ell i$	ℓ	$-\ell k$	ℓj
j	$-k$	-1	i	$-\ell j$	ℓk	ℓ	$-\ell i$
k	j	$-i$	-1	$-\ell k$	$-\ell j$	ℓi	ℓ
ℓ	ℓi	ℓj	ℓk	1	i	j	k
ℓi	$-\ell$	$-\ell k$	ℓj	$-i$	1	k	$-j$
ℓj	ℓk	$-\ell$	$-\ell i$	$-j$	$-k$	1	i
ℓk	$-\ell j$	ℓi	$-\ell$	$-k$	j	$-i$	1

```
In[570]:= Table[Table[{{A1, B1, C1}, mABC[[A1, B1, C1]], {C1, 1, 8}}, {A1, 1, 8}, {B1, 1, 8}]
Out[570]= {{{{{1, 1, 1}, 1}, {{1, 1, 2}, 0}, {{1, 1, 3}, 0}, {{1, 1, 4}, 0},
{{{1, 1, 5}, 0}, {{1, 1, 6}, 0}, {{1, 1, 7}, 0}, {{1, 1, 8}, 0}}}, {{{1, 2, 1}, 0}, {{1, 2, 2}, 1}, {{1, 2, 3}, 0}, {{1, 2, 4}, 0},
{{{1, 2, 5}, 0}, {{1, 2, 6}, 0}, {{1, 2, 7}, 0}, {{1, 2, 8}, 0}}}, {{{1, 3, 1}, 0}, {{1, 3, 2}, 0}, {{1, 3, 3}, 1}, {{1, 3, 4}, 0},
{{{1, 3, 5}, 0}, {{1, 3, 6}, 0}, {{1, 3, 7}, 0}, {{1, 3, 8}, 0}}}, {{{1, 4, 1}, 0}, {{1, 4, 2}, 0}, {{1, 4, 3}, 0}, {{1, 4, 4}, 1},
{{{1, 4, 5}, 0}, {{1, 4, 6}, 0}, {{1, 4, 7}, 0}, {{1, 4, 8}, 0}}}, {{{1, 5, 1}, 0}, {{1, 5, 2}, 0}, {{1, 5, 3}, 0}, {{1, 5, 4}, 0}}}
```

```

{{1, 5, 5}, 1}, {{1, 5, 6}, 0}, {{1, 5, 7}, 0}, {{1, 5, 8}, 0}},  

{{{1, 6, 1}, 0}, {{1, 6, 2}, 0}, {{1, 6, 3}, 0}, {{1, 6, 4}, 0},  

 {{1, 6, 5}, 0}, {{1, 6, 6}, 1}, {{1, 6, 7}, 0}, {{1, 6, 8}, 0}},  

{{{1, 7, 1}, 0}, {{1, 7, 2}, 0}, {{1, 7, 3}, 0}, {{1, 7, 4}, 0},  

 {{1, 7, 5}, 0}, {{1, 7, 6}, 0}, {{1, 7, 7}, 1}, {{1, 7, 8}, 0}},  

{{{1, 8, 1}, 0}, {{1, 8, 2}, 0}, {{1, 8, 3}, 0}, {{1, 8, 4}, 0},  

 {{1, 8, 5}, 0}, {{1, 8, 6}, 0}, {{1, 8, 7}, 0}, {{1, 8, 8}, 1}}},  

{{{2, 1, 1}, 0}, {{2, 1, 2}, 1}, {{2, 1, 3}, 0}, {{2, 1, 4}, 0},  

 {{2, 1, 5}, 0}, {{2, 1, 6}, 0}, {{2, 1, 7}, 0}, {{2, 1, 8}, 0}},  

{{{2, 2, 1}, -1}, {{2, 2, 2}, 0}, {{2, 2, 3}, 0}, {{2, 2, 4}, 0},  

 {{2, 2, 5}, 0}, {{2, 2, 6}, 0}, {{2, 2, 7}, 0}, {{2, 2, 8}, 0}},  

{{{2, 3, 1}, 0}, {{2, 3, 2}, 0}, {{2, 3, 3}, 0}, {{2, 3, 4}, -1},  

 {{2, 3, 5}, 0}, {{2, 3, 6}, 0}, {{2, 3, 7}, 0}, {{2, 3, 8}, 0}},  

{{{2, 4, 1}, 0}, {{2, 4, 2}, 0}, {{2, 4, 3}, 1}, {{2, 4, 4}, 0},  

 {{2, 4, 5}, 0}, {{2, 4, 6}, 0}, {{2, 4, 7}, 0}, {{2, 4, 8}, 0}},  

{{{2, 5, 1}, 0}, {{2, 5, 2}, 0}, {{2, 5, 3}, 0}, {{2, 5, 4}, 0},  

 {{2, 5, 5}, 0}, {{2, 5, 6}, -1}, {{2, 5, 7}, 0}, {{2, 5, 8}, 0}},  

{{{2, 6, 1}, 0}, {{2, 6, 2}, 0}, {{2, 6, 3}, 0}, {{2, 6, 4}, 0},  

 {{2, 6, 5}, 1}, {{2, 6, 6}, 0}, {{2, 6, 7}, 0}, {{2, 6, 8}, 0}},  

{{{2, 7, 1}, 0}, {{2, 7, 2}, 0}, {{2, 7, 3}, 0}, {{2, 7, 4}, 0},  

 {{2, 7, 5}, 0}, {{2, 7, 6}, 0}, {{2, 7, 7}, 0}, {{2, 7, 8}, 1}},  

{{{2, 8, 1}, 0}, {{2, 8, 2}, 0}, {{2, 8, 3}, 0}, {{2, 8, 4}, 0},  

 {{2, 8, 5}, 0}, {{2, 8, 6}, 0}, {{2, 8, 7}, -1}, {{2, 8, 8}, 0}}},  

{{{3, 1, 1}, 0}, {{3, 1, 2}, 0}, {{3, 1, 3}, 1}, {{3, 1, 4}, 0},  

 {{3, 1, 5}, 0}, {{3, 1, 6}, 0}, {{3, 1, 7}, 0}, {{3, 1, 8}, 0}},  

{{{3, 2, 1}, 0}, {{3, 2, 2}, 0}, {{3, 2, 3}, 0}, {{3, 2, 4}, 1},  

 {{3, 2, 5}, 0}, {{3, 2, 6}, 0}, {{3, 2, 7}, 0}, {{3, 2, 8}, 0}},  

{{{3, 3, 1}, -1}, {{3, 3, 2}, 0}, {{3, 3, 3}, 0}, {{3, 3, 4}, 0},  

 {{3, 3, 5}, 0}, {{3, 3, 6}, 0}, {{3, 3, 7}, 0}, {{3, 3, 8}, 0}},  

{{{3, 4, 1}, 0}, {{3, 4, 2}, -1}, {{3, 4, 3}, 0}, {{3, 4, 4}, 0},  

 {{3, 4, 5}, 0}, {{3, 4, 6}, 0}, {{3, 4, 7}, 0}, {{3, 4, 8}, 0}},  

{{{3, 5, 1}, 0}, {{3, 5, 2}, 0}, {{3, 5, 3}, 0}, {{3, 5, 4}, 0},  

 {{3, 5, 5}, 0}, {{3, 5, 6}, 0}, {{3, 5, 7}, -1}, {{3, 5, 8}, 0}},  

{{{3, 6, 1}, 0}, {{3, 6, 2}, 0}, {{3, 6, 3}, 0}, {{3, 6, 4}, 0},  

 {{3, 6, 5}, 0}, {{3, 6, 6}, 0}, {{3, 6, 7}, 0}, {{3, 6, 8}, -1}},  

{{{3, 7, 1}, 0}, {{3, 7, 2}, 0}, {{3, 7, 3}, 0}, {{3, 7, 4}, 0},  

 {{3, 7, 5}, 1}, {{3, 7, 6}, 0}, {{3, 7, 7}, 0}, {{3, 7, 8}, 0}},  

{{{3, 8, 1}, 0}, {{3, 8, 2}, 0}, {{3, 8, 3}, 0}, {{3, 8, 4}, 0},  

 {{3, 8, 5}, 0}, {{3, 8, 6}, 1}, {{3, 8, 7}, 0}, {{3, 8, 8}, 0}}},  

{{{4, 1, 1}, 0}, {{4, 1, 2}, 0}, {{4, 1, 3}, 0}, {{4, 1, 4}, 1},  

 {{4, 1, 5}, 0}, {{4, 1, 6}, 0}, {{4, 1, 7}, 0}, {{4, 1, 8}, 0}},  

{{{4, 2, 1}, 0}, {{4, 2, 2}, 0}, {{4, 2, 3}, -1}, {{4, 2, 4}, 0},  

 {{4, 2, 5}, 0}, {{4, 2, 6}, 0}, {{4, 2, 7}, 0}, {{4, 2, 8}, 0}},  

{{{4, 3, 1}, 0}, {{4, 3, 2}, 1}, {{4, 3, 3}, 0}, {{4, 3, 4}, 0},  

 {{4, 3, 5}, 0}, {{4, 3, 6}, 0}, {{4, 3, 7}, 0}, {{4, 3, 8}, 0}},  

{{{4, 4, 1}, -1}, {{4, 4, 2}, 0}, {{4, 4, 3}, 0}, {{4, 4, 4}, 0},  

 {{4, 4, 5}, 0}, {{4, 4, 6}, 0}, {{4, 4, 7}, 0}, {{4, 4, 8}, 0}}}

```

```

{{{{4, 5, 1}, 0}, {{4, 5, 2}, 0}, {{4, 5, 3}, 0}, {{4, 5, 4}, 0},
 {{{4, 5, 5}, 0}, {{4, 5, 6}, 0}, {{4, 5, 7}, 0}, {{4, 5, 8}, -1}},
 {{{4, 6, 1}, 0}, {{4, 6, 2}, 0}, {{4, 6, 3}, 0}, {{4, 6, 4}, 0},
 {{{4, 6, 5}, 0}, {{4, 6, 6}, 0}, {{4, 6, 7}, 1}, {{4, 6, 8}, 0}},
 {{{4, 7, 1}, 0}, {{4, 7, 2}, 0}, {{4, 7, 3}, 0}, {{4, 7, 4}, 0},
 {{{4, 7, 5}, 0}, {{4, 7, 6}, -1}, {{4, 7, 7}, 0}, {{4, 7, 8}, 0}},
 {{{4, 8, 1}, 0}, {{4, 8, 2}, 0}, {{4, 8, 3}, 0}, {{4, 8, 4}, 0},
 {{{4, 8, 5}, 1}, {{4, 8, 6}, 0}, {{4, 8, 7}, 0}, {{4, 8, 8}, 0}}}},
 {{{5, 1, 1}, 0}, {{5, 1, 2}, 0}, {{5, 1, 3}, 0}, {{5, 1, 4}, 0},
 {{{5, 1, 5}, 1}, {{5, 1, 6}, 0}, {{5, 1, 7}, 0}, {{5, 1, 8}, 0}},
 {{{5, 2, 1}, 0}, {{5, 2, 2}, 0}, {{5, 2, 3}, 0}, {{5, 2, 4}, 0},
 {{{5, 2, 5}, 0}, {{5, 2, 6}, 1}, {{5, 2, 7}, 0}, {{5, 2, 8}, 0}},
 {{{5, 3, 1}, 0}, {{5, 3, 2}, 0}, {{5, 3, 3}, 0}, {{5, 3, 4}, 0},
 {{{5, 3, 5}, 0}, {{5, 3, 6}, 0}, {{5, 3, 7}, 1}, {{5, 3, 8}, 0}},
 {{{5, 4, 1}, 0}, {{5, 4, 2}, 0}, {{5, 4, 3}, 0}, {{5, 4, 4}, 0},
 {{{5, 4, 5}, 0}, {{5, 4, 6}, 0}, {{5, 4, 7}, 0}, {{5, 4, 8}, 1}},
 {{{5, 5, 1}, 1}, {{5, 5, 2}, 0}, {{5, 5, 3}, 0}, {{5, 5, 4}, 0},
 {{{5, 5, 5}, 0}, {{5, 5, 6}, 0}, {{5, 5, 7}, 0}, {{5, 5, 8}, 0}},
 {{{5, 6, 1}, 0}, {{5, 6, 2}, 1}, {{5, 6, 3}, 0}, {{5, 6, 4}, 0},
 {{{5, 6, 5}, 0}, {{5, 6, 6}, 0}, {{5, 6, 7}, 0}, {{5, 6, 8}, 0}},
 {{{5, 7, 1}, 0}, {{5, 7, 2}, 0}, {{5, 7, 3}, 1}, {{5, 7, 4}, 0},
 {{{5, 7, 5}, 0}, {{5, 7, 6}, 0}, {{5, 7, 7}, 0}, {{5, 7, 8}, 0}},
 {{{5, 8, 1}, 0}, {{5, 8, 2}, 0}, {{5, 8, 3}, 0}, {{5, 8, 4}, 1},
 {{{5, 8, 5}, 0}, {{5, 8, 6}, 0}, {{5, 8, 7}, 0}, {{5, 8, 8}, 0}}}},
 {{{6, 1, 1}, 0}, {{6, 1, 2}, 0}, {{6, 1, 3}, 0}, {{6, 1, 4}, 0},
 {{{6, 1, 5}, 0}, {{6, 1, 6}, 1}, {{6, 1, 7}, 0}, {{6, 1, 8}, 0}},
 {{{6, 2, 1}, 0}, {{6, 2, 2}, 0}, {{6, 2, 3}, 0}, {{6, 2, 4}, 0},
 {{{6, 2, 5}, -1}, {{6, 2, 6}, 0}, {{6, 2, 7}, 0}, {{6, 2, 8}, 0}},
 {{{6, 3, 1}, 0}, {{6, 3, 2}, 0}, {{6, 3, 3}, 0}, {{6, 3, 4}, 0},
 {{{6, 3, 5}, 0}, {{6, 3, 6}, 0}, {{6, 3, 7}, 0}, {{6, 3, 8}, 1}},
 {{{6, 4, 1}, 0}, {{6, 4, 2}, 0}, {{6, 4, 3}, 0}, {{6, 4, 4}, 0},
 {{{6, 4, 5}, 0}, {{6, 4, 6}, 0}, {{6, 4, 7}, -1}, {{6, 4, 8}, 0}},
 {{{6, 5, 1}, 0}, {{6, 5, 2}, -1}, {{6, 5, 3}, 0}, {{6, 5, 4}, 0},
 {{{6, 5, 5}, 0}, {{6, 5, 6}, 0}, {{6, 5, 7}, 0}, {{6, 5, 8}, 0}},
 {{{6, 6, 1}, 1}, {{6, 6, 2}, 0}, {{6, 6, 3}, 0}, {{6, 6, 4}, 0},
 {{{6, 6, 5}, 0}, {{6, 6, 6}, 0}, {{6, 6, 7}, 0}, {{6, 6, 8}, 0}},
 {{{6, 7, 1}, 0}, {{6, 7, 2}, 0}, {{6, 7, 3}, 0}, {{6, 7, 4}, -1},
 {{{6, 7, 5}, 0}, {{6, 7, 6}, 0}, {{6, 7, 7}, 0}, {{6, 7, 8}, 0}},
 {{{6, 8, 1}, 0}, {{6, 8, 2}, 0}, {{6, 8, 3}, 1}, {{6, 8, 4}, 0},
 {{{6, 8, 5}, 0}, {{6, 8, 6}, 0}, {{6, 8, 7}, 0}, {{6, 8, 8}, 0}}}},
 {{{7, 1, 1}, 0}, {{7, 1, 2}, 0}, {{7, 1, 3}, 0}, {{7, 1, 4}, 0},
 {{{7, 1, 5}, 0}, {{7, 1, 6}, 0}, {{7, 1, 7}, 1}, {{7, 1, 8}, 0}},
 {{{7, 2, 1}, 0}, {{7, 2, 2}, 0}, {{7, 2, 3}, 0}, {{7, 2, 4}, 0},
 {{{7, 2, 5}, 0}, {{7, 2, 6}, 0}, {{7, 2, 7}, 0}, {{7, 2, 8}, -1}},
 {{{7, 3, 1}, 0}, {{7, 3, 2}, 0}, {{7, 3, 3}, 0}, {{7, 3, 4}, 0},
 {{{7, 3, 5}, -1}, {{7, 3, 6}, 0}, {{7, 3, 7}, 0}, {{7, 3, 8}, 0}},
 {{{7, 4, 1}, 0}, {{7, 4, 2}, 0}, {{7, 4, 3}, 0}, {{7, 4, 4}, 0}},

```

```

{{{7, 4, 5}, 0}, {{7, 4, 6}, 1}, {{7, 4, 7}, 0}, {{7, 4, 8}, 0}},  

{{{7, 5, 1}, 0}, {{7, 5, 2}, 0}, {{7, 5, 3}, -1}, {{7, 5, 4}, 0}},  

{{{7, 5, 5}, 0}, {{7, 5, 6}, 0}, {{7, 5, 7}, 0}, {{7, 5, 8}, 0}},  

{{{7, 6, 1}, 0}, {{7, 6, 2}, 0}, {{7, 6, 3}, 0}, {{7, 6, 4}, 1}},  

{{{7, 6, 5}, 0}, {{7, 6, 6}, 0}, {{7, 6, 7}, 0}, {{7, 6, 8}, 0}},  

{{{7, 7, 1}, 1}, {{7, 7, 2}, 0}, {{7, 7, 3}, 0}, {{7, 7, 4}, 0}},  

{{{7, 7, 5}, 0}, {{7, 7, 6}, 0}, {{7, 7, 7}, 0}, {{7, 7, 8}, 0}},  

{{{7, 8, 1}, 0}, {{7, 8, 2}, -1}, {{7, 8, 3}, 0}, {{7, 8, 4}, 0}},  

{{{7, 8, 5}, 0}, {{7, 8, 6}, 0}, {{7, 8, 7}, 0}, {{7, 8, 8}, 0}}},  

{{{8, 1, 1}, 0}, {{8, 1, 2}, 0}, {{8, 1, 3}, 0}, {{8, 1, 4}, 0}},  

{{{8, 1, 5}, 0}, {{8, 1, 6}, 0}, {{8, 1, 7}, 0}, {{8, 1, 8}, 1}},  

{{{8, 2, 1}, 0}, {{8, 2, 2}, 0}, {{8, 2, 3}, 0}, {{8, 2, 4}, 0}},  

{{{8, 2, 5}, 0}, {{8, 2, 6}, 0}, {{8, 2, 7}, 1}, {{8, 2, 8}, 0}},  

{{{8, 3, 1}, 0}, {{8, 3, 2}, 0}, {{8, 3, 3}, 0}, {{8, 3, 4}, 0}},  

{{{8, 3, 5}, 0}, {{8, 3, 6}, -1}, {{8, 3, 7}, 0}, {{8, 3, 8}, 0}},  

{{{8, 4, 1}, 0}, {{8, 4, 2}, 0}, {{8, 4, 3}, 0}, {{8, 4, 4}, 0}},  

{{{8, 4, 5}, -1}, {{8, 4, 6}, 0}, {{8, 4, 7}, 0}, {{8, 4, 8}, 0}},  

{{{8, 5, 1}, 0}, {{8, 5, 2}, 0}, {{8, 5, 3}, 0}, {{8, 5, 4}, -1}},  

{{{8, 5, 5}, 0}, {{8, 5, 6}, 0}, {{8, 5, 7}, 0}, {{8, 5, 8}, 0}},  

{{{8, 6, 1}, 0}, {{8, 6, 2}, 0}, {{8, 6, 3}, -1}, {{8, 6, 4}, 0}},  

{{{8, 6, 5}, 0}, {{8, 6, 6}, 0}, {{8, 6, 7}, 0}, {{8, 6, 8}, 0}},  

{{{8, 7, 1}, 0}, {{8, 7, 2}, 1}, {{8, 7, 3}, 0}, {{8, 7, 4}, 0}},  

{{{8, 7, 5}, 0}, {{8, 7, 6}, 0}, {{8, 7, 7}, 0}, {{8, 7, 8}, 0}},  

{{{8, 8, 1}, 1}, {{8, 8, 2}, 0}, {{8, 8, 3}, 0}, {{8, 8, 4}, 0}},  

{{{8, 8, 5}, 0}, {{8, 8, 6}, 0}, {{8, 8, 7}, 0}, {{8, 8, 8}, 0}}}}

```

```
In[571]:= (*FAtA=Sum[ E^A_a[hUSE][[A1,A11]]*\eta_{AB}[[A1,A1]]*\tau[A1-1],{A1,1,8}]//FullSimplify  
Dimensions[FAtA]*)
```

```
In[572]:= (*E^A_a[hUSE]  
 \eta_{AB}[[#,#]]*\tau[#+-1]&/@Range[8]  
 Transpose[%%].%  
 Dimensions[%]*)
```

```
In[573]:= (*FAtA[[1]]  
 Sum[ E^A_a[hUSE][[A1,1]]*\eta_{AB}[[A1,A1]]*\tau[A1-1],{A1,1,8}]//FullSimplify*)
```

```
In[574]:= (*mcab=Table[Sum[ E^A_a[hUSE][[A1,a1]]*\eta_{AB}[[A1,A1]]*\tau[A1-1][[c,b]],{A1,1,8}],  
 {a1,1,8},{b,1,8},{c,1,8}]*)
```

```
In[575]:= (*m^C_{ab}=  
 Table[Sum[ E^A_a[hUSE][[A1,a1]]*\eta_{AB}[[A1,A1]]*\tau[A1-1][[c,b]],{A1,1,8}],{a1,1,8},{b,1,8},{c,1,8}]  
 mCAB=( E^A_a[hUSE].(E^a_A[hUSE].%)).E^a_A[hUSE]*)
```

```
In[576]:= (*mCAB[[1,2,3]]
mCAB[[3,1,2]]
mCAB[[2,1,3]]*)

In[577]:= (xw)w = xw^2
```

is described as follows: If we let $\text{Im } \mathbf{O} \subseteq \mathbf{O}$ be the hyperplane (through 0) orthogonal to $1 \in \mathbf{O}$, and we let $S^6 \subseteq \text{Im } \mathbf{O}$ be the space of unit vectors, then right multiplication by $u \in S^6$ induces a linear transformation $R_u: \mathbf{O} \rightarrow \mathbf{O}$ which is orthogonal and satisfies $(R_u)^2 = -1$. Thus, associated to each $u \in S^6$ is a complex structure on \mathbf{O} (induced by $J = R_u$) which is compatible with the natural inner product on \mathbf{O} . We denote by \mathbf{O}_u the Hermitian vector space whose underlying real vector space (with inner product) is \mathbf{O} and whose complex structure is given by R_u .

We let $S^6 = \{u \in \text{Im } \mathbf{O} \mid \langle u, u \rangle = 1\}$. The elements of S^6 are called the *imaginary units* of \mathbf{O} . For any $u \in S^6$, we have $u = -\bar{u}$, so $u^2 = -u\bar{u} = -\langle u, u \rangle = -1$. We may use u to define a map $J_u: \mathbf{O} \rightarrow \mathbf{O}$ given by

$$(1.15) \quad J_u(x) = xu.$$

The identity (1.9) shows that $J_u^2(x) = (xu)u = xu^2 = -x$, so J_u defines a complex structure on \mathbf{O} . We write \mathbf{O}_u to denote \mathbf{O} endowed with the complex structure J_u . If $u \neq v$, then clearly $J_u \neq J_v$, so we actually have a six-sphere of *distinct* complex structures on \mathbf{O} . However, because S^6 is connected, we see that the orientation of \mathbf{O} induced by the natural orientation of \mathbf{O}_u as a complex vector space does not depend on u . We refer to this orientation as the natural orientation of \mathbf{O} .

Using (1.3), we see that if $u \in S$, then

$$\langle J_u(x), J_u(y) \rangle = \langle xu, yu \rangle = \langle x, y \rangle \langle u, u \rangle = \langle x, y \rangle$$

so J_u is an isometry for each $u \in S$. Moreover, it follows that \mathbf{O}_u is endowed with a natural Hermitian structure with respect to the inner product \langle , \rangle . We denote the group of complex linear transformations of \mathbf{O}_u by $GL(\mathbf{O}_u)$ and the special unitary transformations of \mathbf{O}_u with its Hermitian metric by $SU(\mathbf{O}_u)$.

J_u

In[577]:=

```
Table[FullSimplify[
Sum[\psi[A1] - J[B1] \eta_{AB} [A1, A1] E^A_a[hUSE][C][c] \tau[A1 - 1][c][b] E^a_A[hUSE][b][B1],
{c, 1, 8}, {b, 1, 8}, {A1, 1, 8}, {B1, 1, 8}], {C, 1, 8}]
```

Out[577]=

$$\begin{aligned}
& \{J[1] - \psi[1] - J[2] - \psi[2] - J[3] - \psi[3] - J[4] - \psi[4] + \\
& J[5] - \psi[5] + J[6] - \psi[6] + J[7] - \psi[7] + J[8] - \psi[8], J[2] - \psi[1] + J[1] - \psi[2] - \\
& J[4] - \psi[3] + J[3] - \psi[4] + J[6] - \psi[5] - J[5] - \psi[6] - J[8] - \psi[7] + J[7] - \psi[8], \\
& J[3] - \psi[1] + J[4] - \psi[2] + J[1] - \psi[3] - J[2] - \psi[4] + J[7] - \psi[5] + J[8] - \psi[6] - \\
& J[5] - \psi[7] - J[6] - \psi[8], J[4] = \psi[1] - J[3] = \psi[2] + J[2] = \psi[3] + \\
& J[1] - \psi[4] + J[8] - \psi[5] - J[7] - \psi[6] + J[6] - \psi[7] - J[5] - \psi[8], \\
& J[5] - \psi[1] + J[6] - \psi[2] + J[7] - \psi[3] + J[8] - \psi[4] + J[1] - \psi[5] - J[2] - \psi[6] - \\
& J[3] - \psi[7] - J[4] - \psi[8], J[6] - \psi[1] - J[5] - \psi[2] + J[8] - \psi[3] - \\
& J[7] - \psi[4] + J[2] - \psi[5] + J[1] - \psi[6] + J[4] - \psi[7] - J[3] - \psi[8], \\
& J[7] - \psi[1] - J[8] - \psi[2] - J[5] - \psi[3] + J[6] - \psi[4] + J[3] - \psi[5] - J[4] - \psi[6] + \\
& J[1] - \psi[7] + J[2] - \psi[8], J[8] - \psi[1] + J[7] - \psi[2] - J[6] - \psi[3] - \\
& J[5] - \psi[4] + J[4] - \psi[5] + J[3] - \psi[6] - J[2] - \psi[7] + J[1] - \psi[8]\}
\end{aligned}$$

In[578]:=

B = 1;

```
Grid[{EA, FullSimplify[ExpandAll[\eta_{AB}[B, B] EA.E^A_a[h].\tau[B - 1].E^a_A[h]]]}, Frame \rightarrow All]
```

Out[578]=

```
Grid[\{\{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]\},
{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}.e^A_a[h].
{\{1, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0, 0\},
\{0, 0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0\}, \{0, 0, 0, 0, 0, 1, 0, 0\},
\{0, 0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 0, 0, 1\}}.e^a_A[h]\}, Frame \rightarrow All]
```

```
In[579]:= AB = Array[AtB, {8, 8}];  
productsAB =  
Table[Table[{{A1, B1}, Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 8}]}, {A1, 1, 8}], {B1, 1, 8}]  
Partition[Flatten[%], 8] === %  
(*AB[[#][1], #][2]] = #][3]&/@Partition[% , 8]*)  
  
Out[580]= {{{{1, 1}, eA[1]}, {{2, 1}, eA[2]}, {{3, 1}, eA[3]}, {{4, 1}, eA[4]},  
{{{5, 1}, eA[5]}, {{6, 1}, eA[6]}, {{7, 1}, eA[7]}, {{8, 1}, eA[8]}},  
{{{1, 2}, eA[2]}, {{2, 2}, -eA[1]}, {{3, 2}, eA[4]}, {{4, 2}, -eA[3]},  
{{{5, 2}, eA[6]}, {{6, 2}, -eA[5]}, {{7, 2}, -eA[8]}, {{8, 2}, eA[7]}},  
{{{1, 3}, eA[3]}, {{2, 3}, -eA[4]}, {{3, 3}, -eA[1]}, {{4, 3}, eA[2]},  
{{{5, 3}, eA[7]}, {{6, 3}, eA[8]}, {{7, 3}, -eA[5]}, {{8, 3}, -eA[6]}},  
{{{1, 4}, eA[4]}, {{2, 4}, eA[3]}, {{3, 4}, -eA[2]}, {{4, 4}, -eA[1]},  
{{{5, 4}, eA[8]}, {{6, 4}, -eA[7]}, {{7, 4}, eA[6]}, {{8, 4}, -eA[5]}},  
{{{1, 5}, eA[5]}, {{2, 5}, -eA[6]}, {{3, 5}, -eA[7]}, {{4, 5}, -eA[8]},  
{{{5, 5}, eA[1]}, {{6, 5}, -eA[2]}, {{7, 5}, -eA[3]}, {{8, 5}, -eA[4]}},  
{{{1, 6}, eA[6]}, {{2, 6}, eA[5]}, {{3, 6}, -eA[8]}, {{4, 6}, eA[7]},  
{{{5, 6}, eA[2]}, {{6, 6}, eA[1]}, {{7, 6}, eA[4]}, {{8, 6}, -eA[3]}},  
{{{1, 7}, eA[7]}, {{2, 7}, eA[8]}, {{3, 7}, eA[5]}, {{4, 7}, -eA[6]},  
{{{5, 7}, eA[3]}, {{6, 7}, -eA[4]}, {{7, 7}, eA[1]}, {{8, 7}, eA[2]}},  
{{{1, 8}, eA[8]}, {{2, 8}, -eA[7]}, {{3, 8}, eA[6]}, {{4, 8}, eA[5]},  
{{{5, 8}, eA[4]}, {{6, 8}, eA[3]}, {{7, 8}, -eA[2]}, {{8, 8}, eA[1]}}}  
  
Out[581]= True  
  
In[582]:= Table[AB[[productsAB[[A1, B1, 1]]][1], productsAB[[A1, B1, 1]][2]] = productsAB[[A1, B1, 2]],  
{A1, 1, 8}, {B1, 1, 8}]  
  
Out[582]= {{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},  
{eA[2], -eA[1], eA[4], -eA[3], eA[6], -eA[5], -eA[8], eA[7]},  
{eA[3], -eA[4], -eA[1], eA[2], eA[7], eA[8], -eA[5], -eA[6]},  
{eA[4], eA[3], -eA[2], -eA[1], eA[8], -eA[7], eA[6], -eA[5]},  
{eA[5], -eA[6], -eA[7], -eA[8], eA[1], -eA[2], -eA[3], -eA[4]},  
{eA[6], eA[5], -eA[8], eA[7], eA[2], eA[1], eA[4], -eA[3]},  
{eA[7], eA[8], eA[5], -eA[6], eA[3], -eA[4], eA[1], eA[2]},  
{eA[8], -eA[7], eA[6], eA[5], eA[4], eA[3], -eA[2], eA[1]}}
```

In[583]:=

```

gridx = Grid[{Flatten[{"A/B", EA}],
  Block[{A1 = 1}, Flatten[
    {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 2}, Flatten[
    {EA[[2]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 3}, Flatten[
    {EA[[3]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 4}, Flatten[
    {EA[[4]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 5}, Flatten[
    {EA[[5]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 6}, Flatten[
    {EA[[6]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 7}, Flatten[
    {EA[[7]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 8}, Flatten[
    {EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]],
  Frame -> All]
}
```

Out[583]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

1	i	j	k	ℓ	ℓi	ℓj	ℓk
i	-1	k	$-j$	$-\ell i$	ℓ	$-\ell k$	ℓj
j	$-k$	-1	i	$-\ell j$	ℓk	ℓ	$-\ell i$
k	j	$-i$	-1	$-\ell k$	$-\ell j$	ℓi	ℓ
ℓ	ℓi	ℓj	ℓk	1	i	j	k
ℓi	$-\ell$	$-\ell k$	ℓj	$-i$	1	k	$-j$
ℓj	ℓk	$-\ell$	$-\ell i$	$-j$	$-k$	1	i
ℓk	$-\ell j$	ℓi	$-\ell$	$-k$	j	$-i$	1

In[584]:=

```

{Flatten[{"A/B", EA}],
 Block[{A1 = 1}, Flatten[
 {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 2}, Flatten[
 {EA[[2]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 3}, Flatten[
 {EA[[3]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 4}, Flatten[
 {EA[[4]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 5}, Flatten[
 {EA[[5]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 6}, Flatten[
 {EA[[6]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 7}, Flatten[
 {EA[[7]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 8},
 Flatten[{EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]]
};

gridx = Grid[%, Frame -> All]

```

Out[585]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

In[586]:=

```

{Flatten[{"A/B", EA}],
 Block[{A1 = 1}, Flatten[
 {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 2}, Flatten[
 {EA[[2]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 3}, Flatten[
 {EA[[3]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 4}, Flatten[
 {EA[[4]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 5}, Flatten[
 {EA[[5]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 6}, Flatten[
 {EA[[6]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 7}, Flatten[
 {EA[[7]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 8},
 Flatten[{EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]
];
gridx = Grid[%, Frame -> All]

```

Out[587]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

```
In[588]:= 
{Flatten[{ "A/B", EA}], 
 Block[{A1 = 1}, 
  Flatten[{EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 2}, 
  Flatten[{EA[[2]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 3}, 
  Flatten[{EA[[3]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 4}, 
  Flatten[{EA[[4]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 5}, 
  Flatten[{EA[[5]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 6}, 
  Flatten[{EA[[6]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 7}, 
  Flatten[{EA[[7]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 8}, 
  Flatten[{EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}}]]} 
}
Dimensions[%]
gridx = Grid[%%, Frame -> All]
```

```
Out[588]=
{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, 
 {eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, 
 {eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, 
 {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, 
 {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, 
 {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, 
 {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, 
 {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, 
 {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, 
 {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}
```

```
Out[589]=
{9, 9}
```

```
Out[590]=
```

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

```
In[591]:= 
{Flatten[{ "A/B", EA}], 
 Block[{A1 = 1}, 
  Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 2}, 
  Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 3}, 
  Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 4}, 
  Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 5}, 
  Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 6}, 
  Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 7}, 
  Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}}], 
 Block[{A1 = 8}, 
  Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]})]}]
}
Dimensions[%]
gridx = Grid[%%, Frame -> All]
```

```
Out[591]=
{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, 
 {eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, 
 {eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, 
 {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, 
 {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, 
 {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, 
 {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, 
 {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, 
 {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, 
 {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}
```

```
Out[592]=
{9, 9}
```

```
Out[593]=
```

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

```
In[594]:= Reap[For[A1 = 1, A1 < 9, A1++,
Sow[Flatten[{EA[A1], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}]]]
```

Out[594]= {Null, {{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}}

```
In[595]:= Drop[Reap[For[A1 = 1, A1 < 9, A1++, Sow[Flatten[{EA[A1],
Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}]]], 1][[1]][[1]]]
```

Dimensions[%]

```
Out[595]= {{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}
```

```
Out[596]= {8, 9}
```

In[597]:=

```

Prepend[
Drop[Reap[For[A1 = 1, A1 < 9, A1++, Sow[Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] *
EA[[C1]], {C1, 1, 8}], {B1, 1, 8}}]}]]], 1][[1]][[1]], Flatten[{"A/B", EA}]]
Dimensions[%]
gridx = Grid[%%, Frame -> All]

```

Out[597]=

```

{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
{eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},
{eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},
{eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},
{eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},
{eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},
{eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]},
{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}

```

Out[598]=

```
{9, 9}
```

Out[599]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

In[600]:=

```

{Flatten[{"A/B", EA}],
 Reap[Do[Sow[Flatten[
   {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]], ,
 {A1, 1, 8}]];
};

gridx = Grid[%, Frame -> All]

```

Out[601]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
	{ {{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, {eA[1], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, {eA[1], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, {eA[1], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, {eA[1], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, {eA[1], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, {eA[1], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[1], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]} } }							

In[602]:=

```
{Flatten[{"A/B", EA}],
 Reap[Do[Sow[Flatten[
   {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}], {
   A1, 1, 8}]] // Drop[#, 1] &
];
 gridx = Grid[% , Frame -> All]
```

Out[603]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
$\{\{eA[1], eA[1], eA[2], eA[3],$ $eA[4], eA[5], eA[6], eA[7], eA[8]\},$ $\{eA[1], eA[2], -eA[1], -eA[4], eA[3],$ $-eA[6], eA[5], eA[8], -eA[7]\},$ $\{eA[1], eA[3], eA[4], -eA[1], -eA[2],$ $-eA[7], -eA[8], eA[5], eA[6]\},$ $\{eA[1], eA[4], -eA[3], eA[2], -eA[1],$ $-eA[8], eA[7], -eA[6], eA[5]\},$ $\{eA[1], eA[5], eA[6], eA[7], eA[8],$ $eA[1], eA[2], eA[3], eA[4]\},$ $\{eA[1], eA[6], -eA[5], eA[8], -eA[7],$ $-eA[2], eA[1], -eA[4], eA[3]\},$ $\{eA[1], eA[7], -eA[8], -eA[5], eA[6],$ $-eA[3], eA[4], eA[1], -eA[2]\},$ $\{eA[1], eA[8], eA[7], -eA[6], -eA[5],$ $-eA[4], -eA[3], eA[2], eA[1]\}\}$								

In[604]:=

```
{Flatten[{"A/B", EA}],
 Table[
  Flatten[{EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}], {
  A1, 1, 8}]
];
 gridx = Grid[% , Frame -> All]
```

Out[605]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
$\{eA[1],$								
$eA[1],$	$eA[2],$	$eA[3],$	$eA[4],$	$eA[5],$	$eA[6],$	$eA[7],$	$eA[8],$	
$eA[2],$	$-eA[1],$	$eA[4],$	$-eA[3],$	$eA[6],$	$-eA[5],$	$-eA[8],$	$eA[7],$	
$eA[3],$	$-eA[4],$	$-eA[1],$	$eA[2],$	$eA[7],$	$eA[8],$	$-eA[5],$	$-eA[6],$	
$eA[4],$	$eA[3],$	$-eA[2],$	$-eA[1],$	$eA[8],$	$-eA[7],$	$eA[6],$	$-eA[5],$	
$eA[5],$	$-eA[6],$	$-eA[7],$	$-eA[8],$	$eA[1],$	$-eA[2],$	$-eA[3],$	$-eA[4],$	
$eA[6],$	$eA[5],$	$-eA[8],$	$eA[7],$	$eA[2],$	$eA[1],$	$eA[4],$	$-eA[3],$	
$eA[7],$	$eA[8],$	$eA[5],$	$-eA[6],$	$eA[3],$	$-eA[4],$	$eA[1],$	$eA[2],$	
$eA[8]\}$	$-eA[7]\}$	$eA[6]\}$	$eA[5]\}$	$eA[4]\}$	$eA[3]\}$	$-eA[2]\}$	$eA[1]\}$	

In[606]:=

```

{Flatten[{"A/B", EA}],
 Flatten[Reap[Do[Sow[
   Flatten[{EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]]},
   {A1, 1, 8}]] // Drop[#, 1] &, 1]
];
 gridx = Grid[%, Frame -> All]

```

Out[607]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{ {eA[8], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, {eA[8], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, {eA[8], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, {eA[8], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, {eA[8], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, {eA[8], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, {eA[8], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}								

In[608]:=

```

{Flatten[{"A/B", EA}],
 Flatten[{Reap[Do[Sow[
   Block[{A1 = 8}, Flatten[
     {EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]
   ], {A1, 1, 8}]]}, 1]
};

gridx = Grid[% , Frame -> All]

```

Out[609]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
	{ {{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}							

In[610]:=

```
{Flatten[{"A/B", EA}],
 Reap[Do[Sow[
  Block[{A1 = 8}, Flatten[
   {EA[8], Table[Sum[mABC[A1, B1, C1] * EA[C1], {C1, 1, 8}], {B1, 1, 8}]}]
 ], {A1, 1, 8}]];
 ];
 gridx = Grid[%, Frame -> All]
```

Out[611]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
	{ {{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]} } }							

In[612]:=

```
// Drop[#, 1] &
```

In[612]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],  
  Flatten[Reap[Do[Sow[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}],  
    {B1, 1, 8}]}], {A1, 1, 8}]] // Drop[#, 1] &] // Partition[#, 1 + 8] &  
, Frame -> All]
```

Out[612]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}	{eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}	{eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}	{eA[4], eA[4], -eA[3], eA[2], -eA[1], eA[8], -eA[8], eA[7], eA[5]}	{eA[5], eA[5], eA[6], eA[7], eA[1], -eA[7], -eA[2], eA[2], eA[4]}	{eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], -eA[3], eA[1], eA[3]}	{eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], -eA[4], eA[1], -eA[2]}	{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}	

In[613]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],  
  Reap[Do[Sow[Flatten[{EA[[A1]], Table[  
    Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]], {A1, 1, 8}]]  
, Frame -> All]
```

Out[613]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
	{ { {eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]} } }							

In[614]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],  
  Reap[For[A1 = 1, A1 < 9, A1++,  
    Sow[Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}],  
      {B1, 1, 8}]}]]] // Drop[#, 1] &,  
  }, Frame -> All]
```

Out[614]=

```
Grid[{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},  
  {{{{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},  
    {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},  
    {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},  
    {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},  
    {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},  
    {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},  
    {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[8], eA[8],  
      eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}}, Null}, Frame -> All]
```

Remove duplicates from a list, keeping original order :

In[615]:=

```
unsortedUnion[x_] := Reap[Sow[1, x], _, #1 &][[2]]
```

In[616]:=

```
unsortedUnion[x_] := Reap[Sow[1, x], _, #1 &]
```

In[617]:=

```
unsortedUnion[{b, b, c, a, c, a, b, d}]
```

Out[617]=

```
{1, {b, c, a, d}}
```

In[618]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],  
  Reap[For[A1 = 1, A1 < 9, A1++, Sow[1, Flatten[{EA[[A1]], Table[  
    Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]], _, #1 &][[2]],  
  }, Frame -> All]
```

Out[618]=

```
Grid[{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},  
  {{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8], -eA[1], -eA[4],  
    -eA[6], -eA[7], -eA[2], -eA[8], -eA[3], -eA[5]}}, Null}, Frame -> All]
```

In[619]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],  
  For[A1 = 1, A1 < 9, A1++, Catch[Throw[Flatten[  
    {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]]],  
  }, Frame -> All]
```

Out[619]=

```
Grid[{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, Null, Null}, Frame -> All]
```

In[620]:=

```
gridxy = Grid[List[Flatten[{"A/B", EA}],  
Table[Flatten[  
{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]  
, {A1, 1, 8}]], Frame -> All]
```

Out[620]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}	{eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}	{eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}	{eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}	{eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}	{eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], eA[4], eA[3]}	{eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[1], eA[2], -eA[2]}	{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}	

In[621]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],  
Flatten[{Table[Flatten[  
{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]  
, {A1, 1, 8}]], 1}], Frame -> All]
```

Out[621]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}	{eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}	{eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[7], eA[5]}	{eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[8], eA[3], -eA[6]}	{eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], -eA[2], -eA[7], -eA[5]}	{eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[6], eA[4], -eA[3]}	{eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[1], eA[2], -eA[2]}	{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[1]}	

In[622]:=

```
(*Block[{A1=1,B1=2,C1=3},  
Sum[ EA_a[hUSE][[C1,c]]*(((ηAB[[A1,A1]]*τ[A1-1][[c,b]])*E^a_A[hUSE][[b,B1]]),{c,1,8},{b,1,8}]]*)
```

In[623]:=

```
(* E^A_a[hUSE].E^a_A[hUSE]==E^a_A[hUSE].E^A_a[hUSE]==ID8*)
```

In[624]:=

```
(*Symbolize[m^c_AB]*)
```

In[625]:=

```
(*m^c_AB=Table[Sum[ EA_a[hUSE][[C1,c]]*(((ηAB[[A1,A1]]*τ[A1-1][[c,b]])*E^a_A[hUSE][[b,B1]])),  
{c,1,8},{b,1,8}],{A1,1,8},{B1,1,8},{C1,1,8}]*)
```

```
In[626]:= EA
Out[626]= {eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}

In[627]:= (*Table[E^A_a[hUSE].(η_AB[[A1,A1]]*τ[A1-1]).E^a_A[hUSE],{A1,1,8}]
%[[1]]
%[[8]]
%%%[[2]]*)
```

Spinor Lagrangian

```
In[628]:= Protect[H, K]
Out[628]= {}

In[629]:= (*Clear[Lg];
Lg[]:=Sqrt[detgg]*(
Transpose[Ω16].σ16.Sum[FullSimplify[(T16^α[α1-1])/sg],constraintVars].
(D[Ω16,X[[α1]]]+(Ω1/2)*Sum[ωmat[[α1,a,b]]*SAB[[a,b]].Ω16,{a,1,8},{b,1,8}]),
{α1,1,Length[X]}]+(H*M)*Transpose[Ω16].σ16.
Ω16)//Simplify[#,constraintVars]&*)
```

```
In[630]:= sx0x4
Out[630]= {x0 → z/(6 H), x4 → t/H}
```

```
In[631]:= szt
Out[631]= {z → 6 H x0, t → H x4}
```

```
In[632]:= Clear[La];
La[] := 
  (1/H Transpose[Ω16].σ16.Sum[T16^A[α1-1].(D[Ω16,X[[α1]]]),{α1,1,Length[X]}] +
  (2 M/H)*Transpose[Ω16].σ16.Ω16)//Simplify[#,constraintVars]&
```

In[634]:=

```

La[];
- Sum[(D[Transpose[\!f16], X[[\alpha1]]]).\!\sigma16.\!\text{T16}^A[\alpha1 - 1], {\alpha1, 1, Length[X]}].\!\f16 +
(2 * H * M) * Transpose[\!f16].\!\sigma16.\!\f16 // FullSimplify[#, constraintVars] &;
ExpandAll[% - %%] // FullSimplify[#, constraintVars] &

```

Out[636]=

$$\begin{aligned}
& \frac{1}{H} (-1 + H) (4 M f16[2][x0, x4] \times f16[6][x0, x4] + 4 H M f16[2][x0, x4] \times f16[6][x0, x4] + \\
& 4 M f16[3][x0, x4] \times f16[7][x0, x4] + 4 H M f16[3][x0, x4] \times f16[7][x0, x4] - \\
& 4 M f16[8][x0, x4] \times f16[12][x0, x4] - 4 H M f16[8][x0, x4] \times f16[12][x0, x4] - \\
& 4 M f16[9][x0, x4] \times f16[13][x0, x4] - 4 H M f16[9][x0, x4] \times f16[13][x0, x4] - \\
& 4 M f16[10][x0, x4] \times f16[14][x0, x4] - 4 H M f16[10][x0, x4] \times f16[14][x0, x4] - \\
& 4 M f16[11][x0, x4] \times f16[15][x0, x4] - 4 H M f16[11][x0, x4] \times f16[15][x0, x4] - \\
& f16[9][x0, x4] f16[0]^{(0,1)}[x0, x4] + f16[8][x0, x4] f16[1]^{(0,1)}[x0, x4] + \\
& f16[11][x0, x4] f16[2]^{(0,1)}[x0, x4] - f16[10][x0, x4] f16[3]^{(0,1)}[x0, x4] + \\
& f16[13][x0, x4] f16[4]^{(0,1)}[x0, x4] - f16[12][x0, x4] f16[5]^{(0,1)}[x0, x4] - \\
& f16[15][x0, x4] f16[6]^{(0,1)}[x0, x4] + f16[14][x0, x4] f16[7]^{(0,1)}[x0, x4] + \\
& f16[3][x0, x4] f16[10]^{(0,1)}[x0, x4] - f16[2][x0, x4] f16[11]^{(0,1)}[x0, x4] - \\
& f16[7][x0, x4] f16[14]^{(0,1)}[x0, x4] + f16[6][x0, x4] f16[15]^{(0,1)}[x0, x4] - \\
& f16[12][x0, x4] f16[0]^{(1,0)}[x0, x4] - f16[13][x0, x4] f16[1]^{(1,0)}[x0, x4] - \\
& f16[14][x0, x4] f16[2]^{(1,0)}[x0, x4] - f16[15][x0, x4] f16[3]^{(1,0)}[x0, x4] - \\
& f16[8][x0, x4] f16[4]^{(1,0)}[x0, x4] - f16[9][x0, x4] f16[5]^{(1,0)}[x0, x4] - \\
& f16[10][x0, x4] f16[6]^{(1,0)}[x0, x4] - f16[11][x0, x4] f16[7]^{(1,0)}[x0, x4] + \\
& f16[4][x0, x4] (-f16[13]^{(0,1)}[x0, x4] + f16[8]^{(1,0)}[x0, x4]) + \\
& f16[5][x0, x4] (f16[12]^{(0,1)}[x0, x4] + f16[9]^{(1,0)}[x0, x4]) + \\
& f16[6][x0, x4] f16[10]^{(1,0)}[x0, x4] + f16[7][x0, x4] f16[11]^{(1,0)}[x0, x4] + \\
& f16[0][x0, x4] (4 (1 + H) M f16[4][x0, x4] + f16[9]^{(0,1)}[x0, x4] + f16[12]^{(1,0)}[x0, x4]) + \\
& f16[1][x0, x4] (4 (1 + H) M f16[5][x0, x4] - f16[8]^{(0,1)}[x0, x4] + f16[13]^{(1,0)}[x0, x4]) + \\
& f16[2][x0, x4] f16[14]^{(1,0)}[x0, x4] + f16[3][x0, x4] f16[15]^{(1,0)}[x0, x4]
\end{aligned}$$

In[637]:=

La[]

Out[637]=

$$\frac{1}{H} \left(-4 M (f16[0][x0, x4] \times f16[4][x0, x4] + f16[1][x0, x4] \times f16[5][x0, x4] + f16[2][x0, x4] \times f16[6][x0, x4] + f16[3][x0, x4] \times f16[7][x0, x4] - f16[8][x0, x4] \times f16[12][x0, x4] - f16[9][x0, x4] \times f16[13][x0, x4] - f16[10][x0, x4] \times f16[14][x0, x4] - f16[11][x0, x4] \times f16[15][x0, x4]) + f16[12][x0, x4] (f16[5]^{(0,1)}[x0, x4] + f16[0]^{(1,0)}[x0, x4]) + f16[13][x0, x4] (-f16[4]^{(0,1)}[x0, x4] + f16[1]^{(1,0)}[x0, x4]) + f16[14][x0, x4] (-f16[7]^{(0,1)}[x0, x4] + f16[2]^{(1,0)}[x0, x4]) + f16[15][x0, x4] (f16[6]^{(0,1)}[x0, x4] + f16[3]^{(1,0)}[x0, x4]) + f16[8][x0, x4] (-f16[1]^{(0,1)}[x0, x4] + f16[4]^{(1,0)}[x0, x4]) + f16[9][x0, x4] (f16[0]^{(0,1)}[x0, x4] + f16[5]^{(1,0)}[x0, x4]) + f16[10][x0, x4] (f16[3]^{(0,1)}[x0, x4] + f16[6]^{(1,0)}[x0, x4]) + f16[11][x0, x4] (-f16[2]^{(0,1)}[x0, x4] + f16[7]^{(1,0)}[x0, x4]) + f16[4][x0, x4] (f16[13]^{(0,1)}[x0, x4] - f16[8]^{(1,0)}[x0, x4]) - f16[5][x0, x4] (f16[12]^{(0,1)}[x0, x4] + f16[9]^{(1,0)}[x0, x4]) - f16[6][x0, x4] (f16[15]^{(0,1)}[x0, x4] + f16[10]^{(1,0)}[x0, x4]) + f16[7][x0, x4] (f16[14]^{(0,1)}[x0, x4] - f16[11]^{(1,0)}[x0, x4]) - f16[0][x0, x4] (f16[9]^{(0,1)}[x0, x4] + f16[12]^{(1,0)}[x0, x4]) + f16[1][x0, x4] (f16[8]^{(0,1)}[x0, x4] - f16[13]^{(1,0)}[x0, x4]) + f16[2][x0, x4] (f16[11]^{(0,1)}[x0, x4] - f16[14]^{(1,0)}[x0, x4]) - f16[3][x0, x4] (f16[10]^{(0,1)}[x0, x4] + f16[15]^{(1,0)}[x0, x4]) \right)$$

In[638]:=

```
(*Clear[La];
La[]:=useDSQRT *
( Transpose[\!`16].\!\>16.Sum[useT16[\!\>\alpha1].(D[\!\>\alpha1,X[\!\>\alpha1]]+(\!\>\frac{\!\>\alpha1}{2})*Sum[\!\>\omega\!\>\mu IJ[\!\>\alpha1][A1,B1]]*SAB[A1,B1],{A1,1,8},{B1,1,8}].\!`16),{\!\>\alpha1,1,Length[X]}]+
(H*M)*Transpose[\!`16].\!\>16) // Simplify[#,constraintVars]&*)
```

In[639]:=

```
(Transpose[\!`16].\!\>16.Sum[T16^A[\!\>\alpha1-1].(D[\!\>\alpha1,X[\!\>\alpha1]]),{\!\>\alpha1,1,Length[X]}]+
(H*M)*Transpose[\!`16].\!\>16.\!`16) // FullSimplify[#,constraintVars]&;
-Sum[(D[Transpose[\!`16],X[\!\>\alpha1]]).\!\>16.T16^A[\!\>\alpha1-1],{\!\>\alpha1,1,Length[X]}].\!`16+
(H*M)*Transpose[\!`16].\!\>16.\!`16 // FullSimplify[#,constraintVars]&;
ExpandAll[%-%] // FullSimplify[#,constraintVars]&
```

Out[641]=

0

In[642]:=

```
(*Clear[Lj];
Lj[j_]:=useDSQRT *((((
Transpose[\!16].\!\!16.useT16[1].D[\ \!16,X[1]]+(
Transpose[\!16].\!\!16.useT16[5].D[\ \!16,X[5]]+(
\!\!2)*Sum[(\!usew[1,A1,B1]*)\!\!\muIJ[1][A1,B1] Transpose[\!16].\!\!16.
useT16[1].SAB[A1,B1].(base16[j,1]).\!16,\{A1,1,8\},\{B1,1,8\}]+
Transpose[\!16].\!\!16.useT16[5].D[\ \!16,X[5]]+(\!\!2)*
Sum[(\!usew[5,A1,B1]*)\!\!\muIJ[5][A1,B1]Transpose[\!16].\!\!16.
useT16[5].SAB[A1,B1].(base16[j,1]).\!16,\{A1,1,8\},\{B1,1,8\}]+
(H*M)*Transpose[\!16].\!\!16.\!16)//Simplify[#,constraintVars]&*)
```

In[643]:=

(*Lj[92]*)

In[644]:=

detgg

Out[644]=

detgg

In[645]:=

```
Clear[eL];
eL[Lagrangian_Symbol, detsqrt_]:=Module[{L, t},
L=Lagrangian[];
t=Table[
FullSimplify[\frac{1}{detsqrt}\left(D[L, f16[k][x0, x4]]-D[D[L, f16[k]^{(1,0)}[x0, x4]], x0]-D[D[L, f16[k]^{(0,1)}[x0, x4]], x4]\right), constraintVars], {k, 0, 15}];
Return[t/.subsDefects];
]
```

eLa

In[647]:=

```
(*Get[
"2025-12-14_Q2=0-WaveFunctionUniverse-4+4-Einstein-Lovelock-Nash-L9i-eLa.mx"]*)
```

In[648]:=

eLa=eL[La, 1];

In[649]:=

eLa

Out[649]=

$$\left\{ -\frac{2 (2 M f16[4] [x0, x4] + f16[9]^{(0,1)} [x0, x4] + f16[12]^{(1,0)} [x0, x4])}{H}, \right.$$

$$-\frac{2 (2 M f16[5] [x0, x4] - f16[8]^{(0,1)} [x0, x4] + f16[13]^{(1,0)} [x0, x4])}{H},$$

$$-\frac{2 (2 M f16[6] [x0, x4] - f16[11]^{(0,1)} [x0, x4] + f16[14]^{(1,0)} [x0, x4])}{H},$$

$$-\frac{2 (2 M f16[7] [x0, x4] + f16[10]^{(0,1)} [x0, x4] + f16[15]^{(1,0)} [x0, x4])}{H},$$

$$-\frac{2 (2 M f16[0] [x0, x4] - f16[13]^{(0,1)} [x0, x4] + f16[8]^{(1,0)} [x0, x4])}{H},$$

$$-\frac{2 (2 M f16[1] [x0, x4] + f16[12]^{(0,1)} [x0, x4] + f16[9]^{(1,0)} [x0, x4])}{H},$$

$$-\frac{2 (2 M f16[2] [x0, x4] + f16[15]^{(0,1)} [x0, x4] + f16[10]^{(1,0)} [x0, x4])}{H},$$

$$-\frac{2 (2 M f16[3] [x0, x4] - f16[14]^{(0,1)} [x0, x4] + f16[11]^{(1,0)} [x0, x4])}{H},$$

$$\frac{2 (2 M f16[12] [x0, x4] - f16[1]^{(0,1)} [x0, x4] + f16[4]^{(1,0)} [x0, x4])}{H},$$

$$\frac{2 (2 M f16[13] [x0, x4] + f16[0]^{(0,1)} [x0, x4] + f16[5]^{(1,0)} [x0, x4])}{H},$$

$$\frac{2 (2 M f16[14] [x0, x4] + f16[3]^{(0,1)} [x0, x4] + f16[6]^{(1,0)} [x0, x4])}{H},$$

$$\frac{2 (2 M f16[15] [x0, x4] - f16[2]^{(0,1)} [x0, x4] + f16[7]^{(1,0)} [x0, x4])}{H},$$

$$\frac{2 (2 M f16[8] [x0, x4] + f16[5]^{(0,1)} [x0, x4] + f16[0]^{(1,0)} [x0, x4])}{H},$$

$$\frac{2 (2 M f16[9] [x0, x4] - f16[4]^{(0,1)} [x0, x4] + f16[1]^{(1,0)} [x0, x4])}{H},$$

$$\frac{2 (2 M f16[10] [x0, x4] - f16[7]^{(0,1)} [x0, x4] + f16[2]^{(1,0)} [x0, x4])}{H},$$

$$\left. \frac{2 (2 M f16[11] [x0, x4] + f16[6]^{(0,1)} [x0, x4] + f16[3]^{(1,0)} [x0, x4])}{H} \right\}$$

In[650]:=

Length[eLa]

Out[650]=

16

```
In[651]:= DumpSave[ToString[header <> "eLa.mx"], eLa];
In[652]:= Print[ToString[header <> "eLa.mx"]];
Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes-eLa.mx
```

```
In[653]:= (*Cases[
  eLa,
  h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], "f16"] :> n,
  Infinity,
  Heads → True
] *)
```

WRITE OUT THE VARIABLES THAT ARE COUPLED TO ANOTHER VARIABLE :

```
In[654]:= eLaRawSets = Cases[
  #,
  h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], "f16"] :> n,
  Infinity,
  Heads → True
] & /@ eLa
Out[654]= {{4, 9, 12}, {5, 8, 13}, {6, 11, 14}, {7, 10, 15},
{0, 13, 8}, {1, 12, 9}, {2, 15, 10}, {3, 14, 11}, {12, 1, 4}, {13, 0, 5},
{14, 3, 6}, {15, 2, 7}, {8, 5, 0}, {9, 4, 1}, {10, 7, 2}, {11, 6, 3}}
```

```
In[655]:= Dimensions[eLaRawSets]
Out[655]= {16, 3}
```

```
In[656]:= rawSetsel16 = rawSets[eLa, "f16", 0]
% === eLaRawSets
Out[656]= {{4, 9, 12}, {5, 8, 13}, {6, 11, 14}, {7, 10, 15},
{0, 13, 8}, {1, 12, 9}, {2, 15, 10}, {3, 14, 11}, {12, 1, 4}, {13, 0, 5},
{14, 3, 6}, {15, 2, 7}, {8, 5, 0}, {9, 4, 1}, {10, 7, 2}, {11, 6, 3}}
```

```
Out[657]= True
```

Kähler. We show that such M^6 are foliated by 4-planes in \mathbf{O} in a unique way.

In[658]:=

```
eLaCouplings = showCoupledEquations[eLaRawSets]
% // Flatten // Sort
% === Range[0, 15]
```

Out[658]=

 $\{\{0, 5, 8, 13\}, \{1, 4, 9, 12\}, \{2, 7, 10, 15\}, \{3, 6, 11, 14\}\}$

Out[659]=

 $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

Out[660]=

 True

In[661]:=

```
eLaLastCouplings = eLaCouplings
```

Out[661]=

 $\{\{0, 5, 8, 13\}, \{1, 4, 9, 12\}, \{2, 7, 10, 15\}, \{3, 6, 11, 14\}\}$

J. DIFFERENTIAL GEOMETRY
17 (1982) 185–232

SUBMANIFOLDS AND SPECIAL STRUCTURES ON THE OCTONIANS

ROBERT L. BRYANT

An inner product algebra over \mathbf{R} is a vector space \mathbf{A} over \mathbf{R} which possesses a nondegenerate inner product $\langle \cdot, \cdot \rangle: \mathbf{A} \times \mathbf{A} \rightarrow \mathbf{R}$ and a multiplication $\mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$ with unit $1 \in \mathbf{A}$ so that for all $x, y \in \mathbf{A}$

$$(1.1) \quad \langle xy, xy \rangle = \langle x, x \rangle \langle y, y \rangle.$$

For convenience's sake we will identify \mathbf{R} with the 1-dimensional subalgebra of \mathbf{A} generated by $1 \in \mathbf{A}$. By (1.1), we have $\langle 1, 1 \rangle = \langle 1, 1 \rangle^2$. If $\langle 1, 1 \rangle = 0$, then

$$(1.28) \quad x \times y = \frac{1}{2}(\bar{y}x - \bar{x}y).$$

$$(1.28) \quad x \times y = \frac{1}{2}(\bar{y}x - \bar{x}y).$$

$\langle x, x \rangle = 0$ for all $x \in A$, contradicting the nondegeneracy assumption. Hence $\langle 1, 1 \rangle = 1$. We define the orthogonal complement of 1 to be $\text{Im } A \subseteq A$. It is a proper subspace and we have $A = \mathbb{R} \oplus \text{Im } A$. Give $x \in A$, we define $\bar{x} \in A$,

$$(1.2) \quad \bar{x} = 2\langle x, 1 \rangle - x.$$

We denote $\langle x, 1 \rangle$ by $\text{Re } x$ and $(x - \text{Re } x)$ by $\text{Im } x$. Clearly $x \in \text{Im } A$ if and only if $x = -\bar{x}$ or $x = \text{Im } x$ or $\text{Re } x = 0$.

If we polarize (1.1) in the x -variable, we get the identity

$$(1.3) \quad \langle xy, zy \rangle = \langle x, z \rangle \langle y, y \rangle.$$

If we expand $\langle x(1+w), y(1+w) \rangle$ in two ways and compare terms, we find

$$\langle xw, y \rangle = \langle x, y(2\langle w, 1 \rangle - w) \rangle$$

or

$$(1.4) \quad \langle xw, y \rangle = \langle x, y\bar{w} \rangle$$

for all $x, y, w \in A$. In the same way, we get

$$(1.4') \quad \langle wx, y \rangle = \langle x, \bar{w}y \rangle.$$

Using (1.4) and (1.4') repeatedly, we get

$$\begin{aligned} \langle w, \bar{y}\bar{x} \rangle &= \langle yw, \bar{x} \rangle = \langle y, \bar{x}\bar{w} \rangle = \langle xy, \bar{w} \rangle \\ &= \langle w(xy), 1 \rangle = \langle w, \overline{(xy)} \rangle \end{aligned}$$

for all $x, y, w \in A$. It follows that

$$(1.5) \quad \overline{(xy)} = \bar{y}\bar{x}.$$

From (1.5), we conclude that $x\bar{x}$ is real for all $x \in A$, but then $\langle x, x \rangle = \langle x\bar{x}, 1 \rangle = x\bar{x}$.

$$(1.6) \quad x\bar{x} = \langle x, x \rangle = \bar{x}x.$$

Polarizing (1.6) we get

$$(1.7) \quad \langle x, y \rangle = \frac{1}{2}(x\bar{y} + y\bar{x}).$$

We also compute

$$\langle (xw)\bar{w}, y \rangle = \langle xw, yw \rangle = \langle x, y \rangle \langle w, w \rangle = \langle x \langle w, w \rangle, y \rangle$$

so

$$(1.8) \quad (xw)\bar{w} = x(w\bar{w}) \quad (= x \langle w, w \rangle).$$

by subtracting $2(xw)\langle w, 1 \rangle$ from both sides of (1.8), we get

$$(1.9) \quad (xw)w = xw^2$$

in spite of the fact that we have not assumed that A is associative. In a similar manner, we get

$$(1.8') \quad \bar{w}(wx) = (\bar{w}w)x,$$

$$(1.9') \quad w(wx) = w^2x.$$

in spite of the fact that we have not assumed that \mathbf{A} is associative. In a similar manner, we get

$$(1.8') \quad \bar{w}(wx) = (\bar{w}w)x,$$

$$(1.9') \quad w(wx) = w^2x.$$

By polarizing (1.8) and (1.8'), we get the identities

$$(1.10) \quad (xu)\bar{v} + (xv)\bar{u} = 2x\langle u, v \rangle,$$

$$(1.11) \quad u(\bar{v}x) + v(\bar{u}x) = 2x\langle u, v \rangle.$$

In particular, if $\langle u, v \rangle = 0$, then $(xu)\bar{v} = -(xv)\bar{u}$ and $u(\bar{v}x) = -v(\bar{u}x)$. We may use these facts to prove the following lemma (see [12]).

Lemma 1.1. *If $\mathbf{B} \subseteq \mathbf{A}$ is an inner product subalgebra and $u \in \mathbf{A}$ is orthogonal to \mathbf{B} , then $Bu \perp \mathbf{B}$ and $\mathbf{B} \oplus Bu$ is a subalgebra of \mathbf{A} which satisfies*

$$(1.12) \quad (a + bu)(c + du) = (ac - \langle u, u \rangle \bar{d}b) + (da + b\bar{c})u.$$

This lemma allows us to start with $\mathbf{B} = \mathbf{R}$ and “build up” to \mathbf{A} by successively adding on orthogonal subspaces. Using this technique, one can show that if we assume that \langle , \rangle is positive definite, then there are only four inner product algebras over \mathbf{R} , namely \mathbf{R} , \mathbf{C} , \mathbf{H} (the quaternions) and \mathbf{O} (the octonians).

Explicitly, we may regard \mathbf{O} as the vector space $\mathbf{H} \oplus \mathbf{H}$. If we write 1 for $(1,0) \in \mathbf{O}$ and ϵ for $(0,1) \in \mathbf{O}$, the above lemma shows that the multiplication in \mathbf{O} must be given by

$$(1.13) \quad (a + b\epsilon), (c + d\epsilon) = (ac - \bar{d}b) + (da + b\bar{c})\epsilon$$

where the inner product satisfies

$$(1.4) \quad \langle (a + b\epsilon), (a + b\epsilon) \rangle = a\bar{a} + b\bar{b}$$

whenever $a, b, c, d \in \mathbf{H}$.

We let $S^6 = \{u \in \text{Im } \mathbf{O} \mid \langle u, u \rangle = 1\}$. The elements of S^6 are called the *imaginary units* of \mathbf{O} . For any $u \in S^6$, we have $u = -\bar{u}$, so $u^2 = -u\bar{u} = -\langle u, u \rangle = -1$. We may use u to define a map $J_u: \mathbf{O} \rightarrow \mathbf{O}$ given by

$$(1.15) \quad J_u(x) = xu.$$

The identity (1.9) shows that $J_u^2(x) = (xu)u = xu^2 = -x$, so J_u defines a complex structure on \mathbf{O} . We write \mathbf{O}_u to denote \mathbf{O} endowed with the complex structure J_u . If $u \neq v$, then clearly $J_u \neq J_v$, so we actually have a six-sphere of *distinct* complex structures on \mathbf{O} . However, because S^6 is connected, we see that the orientation of \mathbf{O} induced by the natural orientation of \mathbf{O}_u as a complex vector space does not depend on u . We refer to this orientation as the natural orientation of \mathbf{O} .

Using (1.3), we see that if $u \in S$, then

$$\langle J_u(x), J_u(y) \rangle = \langle xu, yu \rangle = \langle x, y \rangle \langle u, u \rangle = \langle x, y \rangle$$

so J_u is an isometry for each $u \in S$. Moreover, it follows that \mathbf{O}_u is endowed with a natural Hermitian structure with respect to the inner product \langle , \rangle . We denote the group of complex linear transformations of \mathbf{O}_u by $GL(\mathbf{O}_u)$ and the special unitary transformations of \mathbf{O}_u with its Hermitian metric by $SU(\mathbf{O}_u)$.

We let $\text{Spin}(7) \subseteq SO(8)$ denote the subgroup generated by the set $\{J_u \mid u \in S^6\} \subseteq SO(8)$. It is known (see [12]) that $\text{Spin}(7)$ is a connected, simply connected, compact Lie group of real dimension 21. Its center is $\{\pm I_8\} \cong \mathbf{Z}/2$ and $\text{Spin}(7)/\{\pm I_8\}$ is isomorphic to $SO(7)$, a simple group. We want to make explicit the structure equations of $\text{Spin}(7)$ as a subgroup of $SO(8)$ in such a way that its relationship with the complex structures J_u is made clear.

Let $u \in S^6$ be an imaginary unit which is orthogonal to $\varepsilon \in \mathbf{O}$. For each $\lambda \in \mathbf{R}$, $(\cos \lambda \varepsilon + \sin \lambda u)$ is an imaginary unit. Hence $J_\varepsilon \circ J_{(\cos \lambda \varepsilon + \sin \lambda u)} = -\cos \lambda I + \sin \lambda J_\varepsilon \circ J_u$ is an element of $\text{Spin}(7)$. We easily compute that $J_\varepsilon \circ J_u + J_u \circ J_\varepsilon = 0$ by using (1.10). Thus $(J_\varepsilon \circ J_u)^2 = J_\varepsilon \circ J_u \circ J_\varepsilon \circ J_u = -J_\varepsilon^2 \circ J_u^2 = -I$. It follows that

$$(1.16) \quad \exp(\lambda J_\varepsilon \circ J_u) = \cos \lambda I + \sin \lambda J_\varepsilon \circ J_u.$$

Thus, if $\text{spin}(7) \subseteq so(8)$ is the Lie algebra of $\text{Spin}(7)$, we see that $J_\varepsilon \circ J_u \in \text{spin}(7)$ for all $u \in S^6$ with $\langle u, \varepsilon \rangle = 0$. Since $\text{spin}(7)$ is a vector space, we see that $L \subseteq \text{spin}(7)$ where

$$(1.17) \quad L = \{J_\varepsilon \circ J_w \mid w \in \text{Im } \mathbf{O}, \langle \varepsilon, w \rangle = 0\}.$$

Note that $\dim_{\mathbf{R}} L = 6$.

To go further, we will choose a basis and exhibit L as a vector space of matrices. In order to do this, let j and k be orthogonal imaginary units in \mathbf{H} .¹ We define the standard basis of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$, $(N, E, \bar{N}, \bar{E}) = (N, E_1, E_2, E_3, \bar{N}, \bar{E}_1, \bar{E}_2, \bar{E}_3)$ as follows: We set $N = \frac{1}{2}(1 - i\varepsilon)$, $\bar{N} = \frac{1}{2}(1 + i\varepsilon)$ and

$$(1.18) \quad \begin{aligned} E_1 &= jN, & \bar{E}_1 &= j\bar{N}, \\ E_2 &= kN, & \bar{E}_2 &= k\bar{N}, \\ E_3 &= (kj)N, & \bar{E}_3 &= (kj)\bar{N}. \end{aligned}$$

(Note that conjugation in $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$ occurs *only* in the \mathbf{C} -factor.) By using the formulae in (1.13) and some elementary calculation, we see that if we set

¹In all that follows, we *never* use i to denote a quaternion or an octonian. For us, $i \in \mathbf{C}$ and $\mathbf{C} \not\subseteq \mathbf{H}$.

$w = 2\text{Re}(a^1 E_1 + a^2 E_2 + a^3 E_3)$ where $a_i \in \mathbf{C}$ and $\text{Re}: \mathbf{C} \otimes_{\mathbf{R}} \mathbf{O} \rightarrow \mathbf{O}$ is the real projection, then $w \in \text{Im } \mathbf{O}$, $\langle \varepsilon, w \rangle = 0$ and

$$J_\varepsilon \circ J_w(N, E_1, E_2, E_3)$$

$$= \begin{pmatrix} 1 & 0 & -1 & -i & -i^2 & -i^3 \end{pmatrix}$$

$w = 2\operatorname{Re}(a^1 E_1 + a^2 E_2 + a^3 E_3)$ where $a_i \in \mathbf{C}$ and $\operatorname{Re}: \mathbf{C} \otimes_{\mathbf{R}} \mathbf{O} \rightarrow \mathbf{O}$ is the real projection, then $w \in \operatorname{Im} \mathbf{O}$, $\langle e, w \rangle = 0$ and

$$(1.19) \quad J_e \circ J_w(N, E_1, E_2, E_3) = (\bar{N}, \bar{E}_1, \bar{E}_2, \bar{E}_3) \begin{pmatrix} 0 & i\bar{a}^1 & i\bar{a}^2 & i\bar{a}^3 \\ -i\bar{a}^1 & 0 & i\bar{a}^3 & -i\bar{a}^2 \\ -i\bar{a}^2 & -i\bar{a}^3 & 0 & i\bar{a}^1 \\ -i\bar{a}^3 & i\bar{a}^2 & -i\bar{a}^1 & 0 \end{pmatrix}.$$

To simplify matters, if $a = (a^i)$ is any column vector of height 3 (with complex entries), we define $[a]$ to be the 3×3 skew symmetric matrix

$$(1.20) \quad [a] = \begin{pmatrix} 0 & a^3 & -a^2 \\ -a^3 & 0 & a^1 \\ a^2 & -a^1 & 0 \end{pmatrix}.$$

Note that $[a]$ is the matrix of the linear transformation from \mathbf{C}^3 to \mathbf{C}^3 determined by cross product with $a \in \mathbf{C}^3$. We will eventually need the following identities for $a, b \in \mathbf{C}^3$ and $A \in M_{3 \times 3}(\mathbf{C})$.

$$(1.21) \quad \begin{aligned} [a]b + [b]a &= 0, \\ [Aa] &= (\operatorname{tr} A)[a] - 'A[a] - [a]A, \\ [a][b] &= b'a - 'abI_3 \end{aligned}$$

(I_3 is the 3×3 identity matrix).

We may now rewrite (1.19) in the more compact form

$$(1.19') \quad J_e \circ J_w(N, E) = (\bar{N}, \bar{E}) \begin{pmatrix} 0 & i\bar{a} \\ -i\bar{a} & [ia] \end{pmatrix}$$

where $w = 2\operatorname{Re}(aE)$ (row by column multiplication is understood). It follows that, expressed in the full basis (N, E, \bar{N}, \bar{E}) we have

$$(1.19'') \quad J_e \circ J_w(N, E, \bar{N}, \bar{E}) = (N, E, \bar{N}, \bar{E}) \begin{pmatrix} 0 & 0 & 0 & -i'a \\ 0 & 0 & ia & [-i\bar{a}] \\ 0 & i'\bar{a} & 0 & 0 \\ -i\bar{a} & [ia] & 0 & 0 \end{pmatrix}.$$

Thus, imbedding $\operatorname{End}(\mathbf{O}) \hookrightarrow M_{8 \times 8}(\mathbf{C})$, the space of 8×8 complex matrices, via the standard basis, we get

$$(1.22) \quad L = \left\{ \begin{pmatrix} 0 & 0 & 0 & -i\bar{a} \\ 0 & 0 & a & [\bar{a}] \\ 0 & -i\bar{a} & 0 & 0 \\ \bar{a} & [a] & 0 & 0 \end{pmatrix} \middle| a \in \mathbf{C}^3 = M_{3 \times 1}(\mathbf{C}) \right\}.$$

An easy computation, using (1.21), then shows

$$(1.23) \quad [L, L] = \left\{ \begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix} \middle| \kappa + ' \bar{\kappa} = 0 \text{ and } \operatorname{tr} \kappa = 0 \right\} \text{ and } \kappa \in M_{4 \times 4}(\mathbf{C}).$$

Since $L \subseteq \operatorname{spin}(7)$, $[L, L] \subseteq \operatorname{spin}(7)$, and $L \cap [L, L] = 0$; and since

$$\dim_{\mathbf{R}} L = 6,$$

$$(1.24) \quad \dim_{\mathbf{R}} [L, L] = 15,$$

$$\dim_{\mathbf{R}} \operatorname{spin}(7) = 21,$$

$$(1.22) \quad L = \left\{ \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & a & [\bar{a}] \\ 0 & -\bar{a} & 0 & 0 \\ \bar{a} & [a] & 0 & 0 \end{pmatrix} \middle| a \in \mathbf{C}^3 = M_{3 \times 1}(\mathbf{C}) \right\}. \quad \text{Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes.nb | 147}$$

An easy computation, using (1.21), then shows

$$(1.23) \quad [L, L] = \left\{ \begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix} \middle| \kappa + \bar{\kappa} = 0 \text{ and } \text{tr } \kappa = 0 \right\}.$$

Since $L \subseteq \text{spin}(7)$, $[L, L] \subseteq \text{spin}(7)$, and $L \cap [L, L] = 0$; and since

$$(1.24) \quad \begin{aligned} \dim_{\mathbf{R}} L &= 6, \\ \dim_{\mathbf{R}} [L, L] &= 15, \\ \dim_{\mathbf{R}} \text{spin}(7) &= 21, \end{aligned}$$

we conclude that

$$(1.25) \quad \text{spin}(7) = L \oplus [L, L].$$

Finally, note that $[L, L] = su(\mathbf{O}_e)$, the Lie algebra of $SU(\mathbf{O}_e)$. If we note that

$$(1.26) \quad gl(\mathbf{O}_e) \cap \text{spin}(7) = su(\mathbf{O}_e)$$

and that $\text{Spin}(7)$ is connected, we deduce that

$$(1.27) \quad GL(\mathbf{O}_e) \cap \text{Spin}(7) = SU(\mathbf{O}_e).$$

We record our main result so far:

Proposition 1.1. *Extend the elements of $\text{Spin}(7) \subseteq \text{End}(\mathbf{O})$ complex linearly so that $\text{Spin}(7) \subseteq \text{End}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O})$. If we use the standard basis (N, E, \bar{N}, \bar{E}) of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$ to represent $\text{End}_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O})$ as the 8×8 complex valued matrices, then*

$$\text{spin}(7) = \left\{ \begin{pmatrix} ic & -\bar{b} & 0 & -\bar{a} \\ b & d & a & [\bar{a}] \\ 0 & -\bar{a} & -ic & -\bar{b} \\ \bar{a} & [a] & \bar{b} & \bar{d} \end{pmatrix} \middle| \begin{array}{l} a, b \in M_{3 \times 1}(\mathbf{C}), \\ c \in \mathbf{R}, d \in M_{3 \times 3}(\mathbf{C}), \\ d + \bar{d} = 0, \\ \text{tr } d + ic = 0. \end{array} \right\}.$$

As we will see below, $\text{Spin}(7)$ actually satisfies $GL(\mathbf{O}_u) \cap \text{Spin}(7) = SU(\mathbf{O}_u)$ for all $u \in S^6$.

For $x, y \in \mathbf{O}$, we define $x \times y$ by the formula

$$(1.28) \quad x \times y = \frac{1}{2}(\bar{y}x - \bar{x}y).$$

$x \times y$ is called the cross product of x and y . Clearly $x \times y \in \text{Im } \mathbf{O}$. We have the useful identities

$$(1.29) \quad \langle x, y \rangle = 0 \Rightarrow x \times y = \bar{y}x = -\bar{x}y,$$

$$(1.30) \quad E_i \times \bar{N} = N \times \bar{E}_i = 0.$$

For each $u \in S^6$, we let $r_u: \text{Im } \mathbf{O} \rightarrow \text{Im } \mathbf{O}$ be defined by $r_u(x) = \bar{u}(xu) = (\bar{u}x)u$ (this last association formula follows easily from (1.8) and (1.8')). Using the Moufang identities (see Appendix B of [12]), one can verify that there

We will also have occasion to study the geometry of $\text{Im } \mathbf{O}$ under a slightly smaller group than $\text{Spin}(7)$. We get $G_2 \subseteq \text{Spin}(7)$ be the subgroup which leaves $1 \in \mathbf{O}$ fixed. Thus G_2 is a compact subgroup of $\text{Spin}(7)$. If we define $p: \text{Spin}(7) \rightarrow S^7 \subseteq \mathbf{O}$ by setting $p(g) = n + \bar{n}$ where $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$, then clearly $p^{-1}(1) = G_2$. Computing the differential of p , we get

$$(1.31) \quad p' = i(-\bar{f}) + f(\bar{x} + \bar{a}) + \bar{f}(\bar{x} + \bar{a}).$$

We will also have occasion to study the geometry of $\text{Im } \mathbf{O}$ under a slightly smaller group than $\text{Spin}(7)$. We get $G_2 \subseteq \text{Spin}(7)$ be the subgroup which leaves $1 \in \mathbf{O}$ fixed. Thus G_2 is a compact subgroup of $\text{Spin}(7)$. If we define $p: \text{Spin}(7) \rightarrow S^7 \subseteq \mathbf{O}$ by setting $p(g) = n + \bar{n}$ where $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$, then clearly $p^{-1}(1) = G_2$. Computing the differential of p , we get

$$(2.14) \quad dp = i(n - \bar{n})\rho + f(\bar{h} + \theta) + \bar{f}(\bar{\bar{h}} + \bar{\theta}).$$

It follows that p has rank 7 and gives $\text{Spin}(7)$ the structure of a G_2 -bundle over S^7 . The connectedness and simple-connectedness of $\text{Spin}(7)$ and S^7 shows that G_2 must be connected and that the Lie algebra of G_2 is obtained from that of $\text{Spin}(7)$ by setting $\rho = \bar{h} + \theta = 0$.

For $g \in G_2$, we say an admissible basis of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$, $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$ is G_2 -admissible. Since $n + \bar{n} \equiv 1$ for such bases, we remove this information and set $u = i(n - \bar{n})$. We then have the following proposition whose proof is an easy computation and is omitted.

Proposition 2.3 (*The Structure Equations of G_2*). *The map $u: G_2 \rightarrow S^6$ makes G_2 into a principal right $SU(3)$ -bundle over S^6 . In fact, we have the structure equations*

$$(2.15) \quad du = f(-2i\theta) + \bar{f}(2i\bar{\theta}),$$

$$(2.16) \quad df = u(-i'\bar{\theta}) + f\kappa + \bar{f}[\theta],$$

$$(2.17) \quad d\theta = -\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta},$$

$$(2.18) \quad d\kappa = -\kappa \wedge \kappa + 3\theta \wedge i'\bar{\theta} - i'\theta \wedge \bar{\theta}I_3.$$

It follows that S^6 possesses a unique nonintegrable almost complex structure so that a complex-valued 1-form $\alpha \in \Omega_{\mathbf{C}}^1(S^6)$ is of type $(1,0)$ if and only if $u^*(\alpha)$ is a linear combination of $\{\theta^1, \theta^2, \theta^3\}$.

prove that the almost complex structure on S^6 determines the metric structure of S^6 so that any invariant of the *local* almost complex structure is also a metric invariant (for a more precise statement, see Proposition 4.1 and its proof). (This is the compact-form analogue of Cartan's characterization of the split form of G_2 as the pseudo-group of a certain differential system on a five manifold.) This justifies our use of the metric structure on S^6 to study the almost complex structure of S^6 .

multiplication. Since $G_2(\mathbf{C})$ acts as algebra automorphisms of $\mathbf{O}_{\mathbf{C}}$, this gives us another proof that $G_2(\mathbf{C})$ leaves \mathcal{L} invariant. (Cartan in [6], proves a striking

multiplication. Since $G_2(\mathbb{C})$ acts as algebra automorphisms of $\mathbf{O}_{\mathbb{C}}$, this gives us another proof that $G_2(\mathbb{C})$ leaves \mathcal{L} invariant. (Cartan in [6], proves a striking

SPECIAL STRUCTURES ON THE OCTONIANS

229

converse: The pseudo-group of bi-holomorphic transformations of $\tilde{G}(2, \text{Im } \mathbf{O})$ which preserve \mathcal{L} is exactly the pseudo-group generated by the action of $G_2(\mathbb{C})$. This is the complex analogue of Proposition 4.1, but it is much harder to prove. We refer the interested reader to [6].)

- [3] E. Cartan, *La théorie des groupes finis et continus et la géométrie différentielle*, Gauthier-Villars, Paris, 1937.
- [4] ———, *Les systèmes différentielles extérieurs*, Hermann, Paris, 1945.
- [5] ———, *Sur les systèmes en involution d'équations aux dérivées partielles du second ordre à une fonction inconnue de trois variables indépendantes*, Oeuvres Complètes, Vol. 2, Pt. 2, pp. 1035–1125.
- [6] ———, *Über die Einfachen Transformationsgruppen*, Oeuvres Complètes, Vol. 1, Pt. 1, pp. 107–132.

0. Introduction

Geometries associated to the exceptional groups and “exceptional” representations of classical groups often display interesting features closely related to (but distinctly different from) the more familiar features of the classical groups. This paper centers on the geometries in E^7 and E^8 whose groups of symmetries are $G_2 \subseteq SO(7)$ and $\text{Spin}(7) \subseteq SO(8)$. Both of these groups are related to the octonians (sometimes called Cayley numbers) and may be defined in terms of octonionic multiplication. In particular, G_2 , the compact exceptional group of (real) dimension 14, is the subgroup of algebra automorphisms of \mathbf{O} (the octonians) and $\text{Spin}(7) \subseteq SO(8)$ may be defined as the subgroup of $GL_{\mathbb{R}}(\mathbf{O})$ generated by right multiplication by unit octonians which are purely imaginary.

The geometry of the algebra \mathbf{O} is closely related to the complex numbers. In §1, we develop some of the properties of \mathbf{O} that we need for later sections. (Our presentation is essentially borrowed from Appendix A of [12], but any mistakes are, of course, due to the author.) A particularly interesting property

0. Introduction

Geometries associated to the exceptional groups and “exceptional” representations of classical groups often display interesting features closely related to (but distinctly different from) the more familiar features of the classical groups. This paper centers on the geometries in E^7 and E^8 whose groups of symmetries are $G_2 \subseteq SO(7)$ and $\text{Spin}(7) \subseteq SO(8)$. Both of these groups are related to the octonians (sometimes called Cayley numbers) and may be defined in terms of octonionic multiplication. In particular, G_2 , the compact exceptional group of (real) dimension 14, is the subgroup of algebra automorphisms of \mathbf{O} (the octonians) and $\text{Spin}(7) \subseteq SO(8)$ may be defined as the subgroup of $GL_{\mathbb{R}}(\mathbf{O})$ generated by right multiplication by unit octonians which are purely imaginary.

The geometry of the algebra \mathbf{O} is closely related to the complex numbers. In §1, we develop some of the properties of \mathbf{O} that we need for later sections. (Our presentation is essentially borrowed from Appendix A of [12], but any mistakes are, of course, due to the author.) A particularly interesting property is described as follows: If we let $\text{Im } \mathbf{O} \subseteq \mathbf{O}$ be the hyperplane (through 0) orthogonal to $1 \in \mathbf{O}$, and we let $S^6 \subseteq \text{Im } \mathbf{O}$ be the space of unit vectors, then right multiplication by $u \in S^6$ induces a linear transformation $R_u: \mathbf{O} \rightarrow \mathbf{O}$ which is orthogonal and satisfies $(R_u)^2 = -1$. Thus, associated to each $u \in S^6$ is a complex structure on \mathbf{O} (induced by $J = R_u$) which is compatible with the natural inner product on \mathbf{O} . We denote by \mathbf{O}_u the Hermitian vector space whose underlying real vector space (with inner product) is \mathbf{O} and whose complex structure is given by R_u .

Classically, this observation was used to define an almost complex structure on S^6 as follows: If $u \in S^6$, then R_u preserves the 2-plane spanned by 1 and u and therefore preserves its orthogonal 6-plane, which may be identified with $T_u S^6 \subseteq \text{Im } \mathbf{O}$ after translation to the origin. Thus R_u induces a complex

In the present paper, after some preliminary work establishing the structure equations of $\text{Spin}(7) \subseteq SO(8)$, we study oriented manifolds $M^6 \subseteq \mathbf{O}$. As is pointed out in [12], every oriented 6-plane in $\mathbf{O} \simeq \mathbb{R}^8$ is a complex three-plane in \mathbf{O}_u for a unique $u \in S^6$. Thus, every oriented six-manifold in \mathbf{O} inherits a natural $U(3)$ -structure generalizing the case where $M^6 \subseteq \text{Im } \mathbf{O}$. In this case, we decompose the second fundamental form II of M into *three* pieces and prove the analogues of Calabi’s theorems concerning when the $U(3)$ -structure is complex integrable and when $d\Omega = 0$. In particular, we show that the induced $U(3)$ -structure on $M^6 \subseteq \mathbf{O}$ is Kähler if and only if M^6 is a complex hypersurface in \mathbf{O}_u for some fixed $u \in S^6$. We then go further in the study of those $M^6 \subseteq \mathbf{O}$ for which the $U(3)$ -structure is complex integrable but which are not Kähler. We show that such M^6 are foliated by 4-planes in \mathbf{O} in a unique way. We refer to this foliation as the *asymptotic ruling* of M^6 . Using the moving frame, we prove that if the asymptotic ruling is parallel then M^6 is the product of a fixed 4-plane in \mathbf{O} with a minimal surface in the orthogonal 4-plane. In

In the present paper, after some preliminary work establishing the structure equations of $\text{Spin}(7) \subseteq SO(8)$, we study oriented manifolds $M^6 \subseteq \mathbf{O}$. As is pointed out in [12], every oriented 6-plane in $\mathbf{O} \simeq \mathbb{R}^8$ is a complex three-plane in \mathbf{O}_u for a unique $u \in S^6$. Thus, every oriented six-manifold in \mathbf{O} inherits a natural $U(3)$ -structure generalizing the case where $M^6 \subseteq \text{Im } \mathbf{O}$. In this case, we decompose the second fundamental form Π of M into *three* pieces and prove the analogues of Calabi's theorems concerning when the $U(3)$ -structure is complex integrable and when $d\Omega = 0$. In particular, we show that the induced $U(3)$ -structure on $M^6 \subseteq \mathbf{O}$ is Kähler if and only if M^6 is a complex hypersurface in \mathbf{O}_u for some fixed $u \in S^6$. We then go further in the study of those $M^6 \subseteq \mathbf{O}$ for which the $U(3)$ -structure is complex integrable but which are not Kähler. We show that such M^6 are foliated by 4-planes in \mathbf{O} in a unique way. We refer to this foliation as the *asymptotic ruling* of M^6 . Using the moving frame, we prove that if the asymptotic ruling is parallel then M^6 is the product of a fixed 4-plane in \mathbf{O} with a minimal surface in the orthogonal 4-plane. In

particular, we show that Calabi's examples are exactly the M^6 with parallel asymptotic ruling which lie in the hyperplane $\text{Im } \mathbf{O} \subseteq \mathbf{O}$. We then use Cartan's theory of differential systems in involution to show that the analytic non-Kähler but complex $M^6 \subseteq \mathbf{O}$ "depend" on 12 analytic functions of 1 (real) variable. (For a more precise statement, see §3).

We observe, as did Gray, that the canonical 2-form on $M^6 \subseteq \mathbf{O}$ is *always* co-closed. Finally, we show that any $M^6 \subseteq \mathbf{O}$ for which the canonical 2-form Ω is closed is necessarily Kähler (and therefore must be a complex hypersurface in \mathbf{O}_u for some fixed $u \in S^6$). In particular, such $M^6 \subseteq \text{Im } \mathbf{O}$ must be hyperplanes. This recovers a result of Gray (see [10]).

In the final section of the paper, we study the "complex curves" in S^6 , i.e., those maps $\phi: M^2 \rightarrow S^6$ where M^2 is a Riemann surface and $d\phi$ is complex linear with respect to the almost complex structure on S^6 induced by the inclusion $S^6 \subseteq \text{Im } \mathbf{O}$. This study is motivated by the fact that the cone on such a complex curve gives a 3-fold in $\text{Im } \mathbf{O}$ which is *associative* in the sense of [12]. Such cones are absolutely mass minimizing and their singular structure reflects the singular structure of general associative varieties in $\mathbb{R}^7 \subseteq \text{Im } \mathbf{O}$. We first prove that the almost complex structure on S^6 determines the metric structure of S^6 so that any invariant of the *local* almost complex structure is also a metric invariant (for a more precise statement, see Proposition 4.1 and its proof). (This is the compact-form analogue of Cartan's characterization of the split form of G_2 as the pseudo-group of a certain differential system on a five manifold.) This justifies our use of the metric structure on S^6 to study the almost complex structure of S^6 .

Kähler. We show that such M^6 are foliated by 4-planes in \mathbf{O} in a unique way. We refer to this foliation as the *asymptotic ruling* of M^6 . Using the moving frame, we prove that if the asymptotic ruling is parallel then M^6 is the product of a fixed 4-plane in \mathbf{O} with a minimal surface in the orthogonal 4-plane. In

particular, we show that Calabi's examples are exactly the M^6 with parallel asymptotic ruling which lie in the hyperplane $\text{Im } \mathbf{O} \subseteq \mathbf{O}$. We then use Cartan's theory of differential systems in involution to show that the analytic non-Kähler but complex $M^6 \subseteq \mathbf{O}$ "depend" on 12 analytic functions of 1 (real) variable. (For a more precise statement, see §3).

We observe, as did Gray, that the canonical 2-form on $M^6 \subseteq \mathbf{O}$ is *always* co-closed. Finally, we show that any $M^6 \subseteq \mathbf{O}$ for which the canonical 2-form Ω is closed is necessarily Kähler (and therefore must be a complex hypersurface in \mathbf{O}_u for some fixed $u \in S^6$). In particular, such $M^6 \subseteq \text{Im } \mathbf{O}$ must be hyperplanes. This recovers a result of Gray (see [10]).

In the final section of the paper, we study the "complex curves" in S^6 , i.e., those maps $\phi: M^2 \rightarrow S^6$ where M^2 is a Riemann surface and $d\phi$ is complex linear with respect to the almost complex structure on S^6 induced by the inclusion $S^6 \subseteq \text{Im } \mathbf{O}$. This study is motivated by the fact that the cone on such a complex curve gives a 3-fold in $\text{Im } \mathbf{O}$ which is *associative* in the sense of [12]. Such cones are absolutely mass minimizing and their singular structure reflects the singular structure of general associative varieties in $\mathbf{R}^7 \subseteq \text{Im } \mathbf{O}$. We first prove that the almost complex structure on S^6 determines the metric structure of S^6 so that any invariant of the *local* almost complex structure is also a metric invariant (for a more precise statement, see Proposition 4.1 and its proof). (This is the compact-form analogue of Cartan's characterization of the split form of G_2 as the pseudo-group of a certain differential system on a five manifold.) This justifies our use of the metric structure on S^6 to study the almost complex structure of S^6 .

3. Oriented 6-manifolds in \mathbf{O}

Let M^6 be an abstract oriented 6-manifold with a smooth differentiable structure. Let $X: M^6 \rightarrow \mathbf{O}$ be a smooth immersion of M^6 into \mathbf{O} . We say that an admissible frame $(y; n, f, \bar{n}, \bar{f}) \in \mathcal{F}$ is *adapted* at $p \in M$ if $X(p) = y$ and if (f_1, f_2, f_3) is a $(1,0)$ basis of $X_*(T_p M^6)$ with its induced orientation from M^6 and complex structure induced from right multiplication by $2in \times \bar{n}$. We let $\mathcal{F}_X(M)$ denote the space of pairs $(p, (y; n, f, \bar{n}, \bar{f}))$, $p \in M^6$, $(y; n, f, \bar{n}, \bar{f}) \in \mathcal{F}$ where $(y; n, f, \bar{n}, \bar{f})$ is adapted at p . We call $\mathcal{F}_X(M)$ the *adapted frame bundle of the immersion $X: M^6 \rightarrow \mathbf{O}$* . We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\tilde{x}} & \mathcal{F} \\ \downarrow p & & \downarrow x \\ M & \xrightarrow{x} & \mathbf{O} \end{array}$$

We see that $p: \mathcal{F}_X(M) \rightarrow M$ is a right $U(3)$ -bundle over M which may be regarded as a subbundle of the $GL(6, \mathbf{R})$ bundle of the tangential frames of M . We simply refer to this G -structure as the $U(3)$ -structure on M induced by the immersion $X: M \rightarrow \mathbf{O}$. The reader should be aware that other authors have called such structures "almost hermitian".

3. Oriented 6-manifolds in \mathbf{O}

Let M^6 be an abstract oriented 6-manifold with a smooth differentiable structure. Let $X: M^6 \rightarrow \mathbf{O}$ be a smooth immersion of M^6 into \mathbf{O} . We say that an admissible frame $(y; n, f, \bar{n}, \bar{f}) \in \mathcal{F}$ is *adapted* at $p \in M$ if $X(p) = y$ and if (f_1, f_2, f_3) is a $(1,0)$ basis of $X_*(T_p M^6)$ with its induced orientation from M^6 and complex structure induced from right multiplication by $2in \times \bar{n}$. We let $\mathcal{F}_X(M)$ denote the space of pairs $(p, (y; n, f, \bar{n}, \bar{f}))$, $p \in M^6$, $(y; n, f, \bar{n}, \bar{f}) \in \mathcal{F}$ where $(y; n, f, \bar{n}, \bar{f})$ is adapted at p . We call $\mathcal{F}_X(M)$ the *adapted frame bundle of the immersion $X: M^6 \rightarrow \mathbf{O}$* . We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\tilde{X}} & \mathcal{F} \\ \downarrow p & & \downarrow x \\ M & \xrightarrow{X} & \mathbf{O} \end{array}$$

We see that $p: \mathcal{F}_X(M) \rightarrow M$ is a right $U(3)$ -bundle over M which may be regarded as a subbundle of the $GL(6, \mathbb{R})$ bundle of the tangential frames of M . We simply refer to this G -structure as the $U(3)$ -structure on M induced by the immersion $X: M \rightarrow \mathbf{O}$. The reader should be aware that other authors have called such structures "almost hermitian".

The forms on \mathcal{F} pullback under \tilde{X}^* to give forms on $\mathcal{F}_X(M)$ which we continue to denote by the same letters. The following basic theorem follows immediately from the theory of moving frames and the structure equations of \mathbf{O} (see §1, (1.35), (1.36), (1.40), (1.41)).

Theorem 3.1. *Let $X: M^6 \rightarrow \mathbf{O}$ be an oriented immersion and let $p: \mathcal{F}_X(M) \rightarrow M$ be the adapted frame bundle. Then M inherits a $U(3)$ -structure where $\mathcal{F}_X(M)$ is the bundle of unitary frames and whose features are described as follows:*

- (i) $\nu = \bar{\nu} = 0$ on $\mathcal{F}_X(M)$.
- (ii) A form $\alpha \in \Omega_C^1(M)$ is of type $(1,0)$ if and only if $p^*(\alpha) \equiv 0 \pmod{(\omega^1, \omega^2, \omega^3)}$.
- (iii) A canonical 2-form, Ω , of type $(1,1)$ is associated to the $U(3)$ -structure and is characterized by the condition $p^*(\Omega) = (i/2)\omega \wedge \bar{\omega}$.
- (iv) The metric g on M induced by X from \mathbf{O} satisfies $p^*(g) = \omega \circ \bar{\omega}$.
- (v) The structure equations hold:

$$(3.1) \quad dx = f\omega + \bar{f}\bar{\omega},$$

$$(3.2) \quad dn = n\rho + f\bar{\eta} + \bar{f}\bar{\theta},$$

$$(3.3) \quad df = -n^i\bar{\eta} + fk - \bar{n}^i\bar{\theta} + \bar{f}[\theta],$$

(and the equations gotten from these by conjugation).

Of course, a $U(3)$ -structure has many invariants and those $U(3)$ -structures which satisfy extra conditions are of particular interest. Among these, the most important for us will be the following: A $U(3)$ -structure on M will be said to be

(i) *complex* if the underlying almost complex structure is integrable to a complex structure (by the Newlander-Nirenberg theorem, this is equivalent to the condition $d\alpha \equiv 0 \pmod{\Omega^{1,0}(M)}$ for all $\alpha \in \Omega^{1,0}(M)$; see [7]);

(ii) *symplectic* if the canonical two-form Ω is closed;

(iii) *co-symplectic* if Ω is co-closed, i.e., $\delta\Omega = 0$ (this is equivalent to either of

$$(3.2) \quad dn = n\omega + f\bar{h} + \bar{f}\theta,$$

$$(3.3) \quad df = -n'\bar{h} + f\kappa - \bar{n}'\bar{\theta} + \bar{f}[\theta],$$

154 | Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes.nb
(and the equations gotten from these by conjugation).

Of course, a $U(3)$ -structure has many invariants and those $U(3)$ -structures which satisfy extra conditions are of particular interest. Among these, the most important for us will be the following: A $U(3)$ -structure on M will be said to be

- (i) *complex* if the underlying almost complex structure is integrable to a complex structure (by the Newlander-Nirenberg theorem, this is equivalent to the condition $d\alpha \equiv 0 \pmod{\Omega^{1,0}(M)}$ for all $\alpha \in \Omega^{1,0}(M)$; see [7]);
- (ii) *symplectic* if the canonical two-form Ω is closed;
- (iii) *co-symplectic* if Ω is co-closed, i.e., $\delta\Omega = 0$ (this is equivalent to either of the conditions $d\Omega^2 = 0$ or $d * \Omega = 0$);
- (iv) *Kähler* if it is both complex and symplectic;
- (v) *co-Kähler* if it is both complex and co-symplectic.

Note that symplectic implies co-symplectic, but not conversely (see below). Complex $U(3)$ -structures are often called “Hermitian”.³

Our analysis of $U(3)$ -structures induced by oriented immersions $X: M^6 \rightarrow \mathbf{O}$ begins with the second fundamental form. If we differentiate the equation $\nu = 0$ on $\mathcal{F}_X(M)$, the structure equations (1.41) give

$$(3.4) \quad {}' \bar{h} \wedge \omega + {}' \theta \wedge \bar{\omega} = 0.$$

Applying Cartan’s Lemma, we conclude that there exist 3×3 matrices of functions, A, B, C on $\mathcal{F}_X(M)$ (with complex values) satisfying

$$(3.5) \quad A = {}^t A, \quad C = {}^t C,$$

$$(3.6) \quad \begin{pmatrix} \bar{h} \\ \theta \end{pmatrix} = \begin{pmatrix} \bar{B} & \bar{A} \\ {}' B & \bar{C} \end{pmatrix} \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}.$$

Using these formulae, we easily compute the second fundamental form of $X: M^6 \rightarrow \mathbf{O}$ as an Euclidean immersion as

$$(3.7) \quad \text{II} = -2 \operatorname{Re}\{({}' \bar{h} \circ \omega + {}' \theta \circ \bar{\omega})n\}.$$

Classically, one views II as a linear map $\text{II}: S^2(TM) \rightarrow NM$ where TM is the tangent bundle of the immersion X . Using the almost complex structure on M and the orientation of the 2-plane bundle $N_x M$, we have canonical splittings

$$\mathbf{C} \otimes_{\mathbf{R}} S^2(TM) = S_{\mathbf{C}}^{2,0}(M) \oplus S_{\mathbf{C}}^{1,1}(M) \oplus S_{\mathbf{C}}^{0,2}(M),$$

$$\mathbf{C} \otimes_{\mathbf{R}} NM = N^{1,0}M \oplus N^{0,1}M,$$

where the bundles on the right are complex vector bundles over M . For example, $S_{\mathbf{C},q}^{2,0}(M)$ for $q \in M$ is spanned by products of the form $e_1 \circ e_2$ where

e_1 and e_2 are $(1,0)$ vectors in $T_{\mathbf{C},q} M$. If we extend II complex linearly to a map $\mathbf{C} \otimes_{\mathbf{R}} S^2(TM) \rightarrow \mathbf{C} \otimes_{\mathbf{R}} NM$, and split it into components via the above splittings, we see that II has three independent pieces, the rest being determined by symmetry and reality of II. These components are $\text{II}^{2,0}: S_{\mathbf{C}}^{2,0}(M) \rightarrow (M) \rightarrow N^{1,0}M$ given on $\mathcal{F}_X(M)$ by

$$(3.7a) \quad \text{II}^{2,0} = (-{}' \omega \circ A\omega)n,$$

$\text{II}^{1,1}: S_{\mathbf{C}}^{1,1}(M) \rightarrow N^{1,0}M$ given by

$$(3.7b) \quad \text{II}^{1,1} = (-{}' \bar{\omega} \circ {}' B\omega - {}' \omega \circ B\bar{\omega})n,$$

and $\text{II}^{0,2}: S_{\mathbf{C}}^{0,2}(M) \rightarrow N^{1,0}M$ given by

$$(3.7c) \quad \text{II}^{0,2} = -({}' \bar{\omega} \circ \bar{C}\bar{\omega})n.$$

From this, one easily computes the trace of II with respect to the first fundamental form $I = {}' \omega \circ \bar{\omega}$ as

$$\mathbf{C} \otimes_{\mathbf{R}} S^2(TM) = S_{\mathbf{C}}^{2,0}(M) \oplus S_{\mathbf{C}}^{1,1}(M) \oplus S_{\mathbf{C}}^{0,2}(M),$$

$$\mathbf{C} \otimes_{\mathbf{R}} NM = N^{1,0}M \oplus N^{0,1}M,$$

where the bundles on the right are complex vector bundles over M . For example, $S_{\mathbf{C},q}^{2,0}(M)$ for $q \in M$ is spanned by products of the form $e_1 \circ e_2$ where

e_1 and e_2 are $(1,0)$ vectors in $T_{\mathbf{C},q}M$. If we extend II complex linearly to a map $\mathbf{C} \otimes_{\mathbf{R}} S^2(TM) \rightarrow \mathbf{C} \otimes_{\mathbf{R}} NM$, and split it into components via the above splittings, we see that II has three independent pieces, the rest being determined by symmetry and reality of II . These components are $\text{II}^{2,0}: S_{\mathbf{C}}^{2,0}(M) \rightarrow (M) \rightarrow N^{1,0}M$ given on $\mathcal{F}_X(M)$ by

$$(3.7a) \quad \text{II}^{2,0} = (-{}^t\omega \circ A\omega)n,$$

$\text{II}^{1,1}: S_{\mathbf{C}}^{1,1}(M) \rightarrow N^{1,0}M$ given by

$$(3.7b) \quad \text{II}^{1,1} = (-{}^t\bar{\omega} \circ {}^tB\omega - {}^t\omega \circ B\bar{\omega})n,$$

and $\text{II}^{0,2}: S_{\mathbf{C}}^{0,2}(M) \rightarrow N^{1,0}M$ given by

$$(3.7c) \quad \text{II}^{0,2} = -({}^t\bar{\omega} \circ \bar{C}\bar{\omega})n.$$

From this, one easily computes the trace of II with respect to the first fundamental form $\text{I} = {}^t\omega \circ \bar{\omega}$ as

$$(3.8) \quad H = \frac{1}{6}\text{tr}_{\text{I}}\text{II} = -\frac{1}{3}(\text{tr } Bn + \text{tr } \bar{B}\bar{n}).$$

H is often called the mean curvature vector of the immersion X . The above discussion gives us a geometric interpretation of the components of II with respect to the $U(3)$ -structure. We will now relate these components to the special conditions discussed above for $U(3)$ -structures.

Theorem 3.2. *Let $X: M^6 \rightarrow \mathbf{O}$ be an immersion of the oriented manifold M^6 . The induced $U(3)$ -structure is complex if and only if $B = 0$.*

eLazt

In[662]:=

```
(*Get[
"2025-12-14_Q2=0-WaveFunctionUniverse-4+4-Einstein-Lovelock-Nash-L9i-eLazt.mx"];
*)
```

In[663]:=

```
time1 = Now
AbsoluteTiming[
eLazt = 
$$\frac{1}{2 * H} * eLa /. sfy16Aa /. sx0x4 // FullSimplify[\#, constraintVars] &]
Now - time1$$

```

Out[663]=

Fri 20 Feb 2026 07:05:08 GMT-8

```

Out[664]=
{0.126555, { - $\frac{2 M Z[4][z,t] + H Z[9]^{(0,1)}[z,t] + 6 H Z[12]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{-2 M Z[5][z,t] + H (Z[8]^{(0,1)}[z,t] - 6 Z[13]^{(1,0)}[z,t])}{H^2}$ ,
 $\frac{-2 M Z[6][z,t] + H (Z[11]^{(0,1)}[z,t] - 6 Z[14]^{(1,0)}[z,t])}{H^2}$ ,
 $\frac{2 M Z[7][z,t] + H Z[10]^{(0,1)}[z,t] + 6 H Z[15]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{-2 M Z[0][z,t] + H (Z[13]^{(0,1)}[z,t] - 6 Z[8]^{(1,0)}[z,t])}{H^2}$ ,
 $\frac{2 M Z[1][z,t] + H Z[12]^{(0,1)}[z,t] + 6 H Z[9]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{-2 M Z[2][z,t] + H Z[15]^{(0,1)}[z,t] + 6 H Z[10]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{-2 M Z[3][z,t] + H (Z[14]^{(0,1)}[z,t] - 6 Z[11]^{(1,0)}[z,t])}{H^2}$ ,
 $\frac{2 M Z[12][z,t] - H Z[1]^{(0,1)}[z,t] + 6 H Z[4]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{2 M Z[13][z,t] + H Z[0]^{(0,1)}[z,t] + 6 H Z[5]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{2 M Z[14][z,t] + H Z[3]^{(0,1)}[z,t] + 6 H Z[6]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{2 M Z[15][z,t] - H Z[2]^{(0,1)}[z,t] + 6 H Z[7]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{2 M Z[8][z,t] + H Z[5]^{(0,1)}[z,t] + 6 H Z[0]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{2 M Z[9][z,t] - H Z[4]^{(0,1)}[z,t] + 6 H Z[1]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{2 M Z[10][z,t] - H Z[7]^{(0,1)}[z,t] + 6 H Z[2]^{(1,0)}[z,t]}{H^2}$ ,
 $\frac{2 M Z[11][z,t] + H Z[6]^{(0,1)}[z,t] + 6 H Z[3]^{(1,0)}[z,t]}{H^2} \}}$ 
```

```

Out[665]=
0.1575708 s

```

```

In[666]:= DumpSave[ToString[header <> "eLazt.mx"], eLazt];
In[667]:= Print[ToString[header <> "eLazt.mx"]]
Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes-eLazt.mx

```

In[668]:=

eLast // Column

Out[668]=

$$\begin{aligned}
 & -\frac{2 M Z[4][z,t] + H Z[9]^{(0,1)}[z,t] + 6 H Z[12]^{(1,0)}[z,t]}{H^2} \\
 & -2 M Z[5][z,t] + H (Z[8]^{(0,1)}[z,t] - 6 Z[13]^{(1,0)}[z,t]) \\
 & -2 M Z[6][z,t] + H (Z[11]^{(0,1)}[z,t] - 6 Z[14]^{(1,0)}[z,t]) \\
 & -\frac{2 M Z[7][z,t] + H Z[10]^{(0,1)}[z,t] + 6 H Z[15]^{(1,0)}[z,t]}{H^2} \\
 & -2 M Z[0][z,t] + H (Z[13]^{(0,1)}[z,t] - 6 Z[8]^{(1,0)}[z,t]) \\
 & -\frac{2 M Z[1][z,t] + H Z[12]^{(0,1)}[z,t] + 6 H Z[9]^{(1,0)}[z,t]}{H^2} \\
 & -\frac{2 M Z[2][z,t] + H Z[15]^{(0,1)}[z,t] + 6 H Z[10]^{(1,0)}[z,t]}{H^2} \\
 & -2 M Z[3][z,t] + H (Z[14]^{(0,1)}[z,t] - 6 Z[11]^{(1,0)}[z,t]) \\
 & -\frac{2 M Z[12][z,t] - H Z[1]^{(0,1)}[z,t] + 6 H Z[4]^{(1,0)}[z,t]}{H^2} \\
 & 2 M Z[13][z,t] + H Z[0]^{(0,1)}[z,t] + 6 H Z[5]^{(1,0)}[z,t] \\
 & 2 M Z[14][z,t] + H Z[3]^{(0,1)}[z,t] + 6 H Z[6]^{(1,0)}[z,t] \\
 & 2 M Z[15][z,t] - H Z[2]^{(0,1)}[z,t] + 6 H Z[7]^{(1,0)}[z,t] \\
 & 2 M Z[8][z,t] + H Z[5]^{(0,1)}[z,t] + 6 H Z[0]^{(1,0)}[z,t] \\
 & 2 M Z[9][z,t] - H Z[4]^{(0,1)}[z,t] + 6 H Z[1]^{(1,0)}[z,t] \\
 & 2 M Z[10][z,t] - H Z[7]^{(0,1)}[z,t] + 6 H Z[2]^{(1,0)}[z,t] \\
 & 2 M Z[11][z,t] + H Z[6]^{(0,1)}[z,t] + 6 H Z[3]^{(1,0)}[z,t]
 \end{aligned}$$

In[669]:=

varZ = Z[#] & /@ Range[0, 15]

Out[669]=

$$\{Z[0], Z[1], Z[2], Z[3], Z[4], Z[5], Z[6], \\ Z[7], Z[8], Z[9], Z[10], Z[11], Z[12], Z[13], Z[14], Z[15]\}$$

In[670]:=

varZzt = #[z, t] & /@ varZ

Out[670]=

$$\{Z[0][z,t], Z[1][z,t], Z[2][z,t], Z[3][z,t], Z[4][z,t], \\ Z[5][z,t], Z[6][z,t], Z[7][z,t], Z[8][z,t], Z[9][z,t], Z[10][z,t], \\ Z[11][z,t], Z[12][z,t], Z[13][z,t], Z[14][z,t], Z[15][z,t]\}$$

In[671]:=

DzvarZ = D[varZzt, z]

Out[671]=

$$\{Z[0]^{(1,0)}[z,t], Z[1]^{(1,0)}[z,t], Z[2]^{(1,0)}[z,t], Z[3]^{(1,0)}[z,t], \\ Z[4]^{(1,0)}[z,t], Z[5]^{(1,0)}[z,t], Z[6]^{(1,0)}[z,t], Z[7]^{(1,0)}[z,t], \\ Z[8]^{(1,0)}[z,t], Z[9]^{(1,0)}[z,t], Z[10]^{(1,0)}[z,t], Z[11]^{(1,0)}[z,t], \\ Z[12]^{(1,0)}[z,t], Z[13]^{(1,0)}[z,t], Z[14]^{(1,0)}[z,t], Z[15]^{(1,0)}[z,t]\}$$

```
In[672]:= DtvarZ = D[varZzt, t]

Out[672]= {Z[0]^(0,1)[z, t], Z[1]^(0,1)[z, t], Z[2]^(0,1)[z, t], Z[3]^(0,1)[z, t],
Z[4]^(0,1)[z, t], Z[5]^(0,1)[z, t], Z[6]^(0,1)[z, t], Z[7]^(0,1)[z, t],
Z[8]^(0,1)[z, t], Z[9]^(0,1)[z, t], Z[10]^(0,1)[z, t], Z[11]^(0,1)[z, t],
Z[12]^(0,1)[z, t], Z[13]^(0,1)[z, t], Z[14]^(0,1)[z, t], Z[15]^(0,1)[z, t]}

In[673]:= (DtvarZsubs = Solve[And @@ Thread[θ == eLazt], DtvarZ][[1]] // 
FullSimplify[#, constraintVars] &) // Column
DtvarZEQS = %[[1]] /. {Rule → Equal}
DtvarZrelations = DtvarZsubs /. {Rule → Subtract}

Out[673]=
Z[0]^(0,1)[z, t] → - $\frac{2M Z[13][z,t]}{H} - 6 Z[5]^{(1,0)}[z, t]$ 
Z[1]^(0,1)[z, t] →  $\frac{2M Z[12][z,t]}{H} + 6 Z[4]^{(1,0)}[z, t]$ 
Z[2]^(0,1)[z, t] →  $\frac{2M Z[15][z,t]}{H} + 6 Z[7]^{(1,0)}[z, t]$ 
Z[3]^(0,1)[z, t] → - $\frac{2M Z[14][z,t]}{H} - 6 Z[6]^{(1,0)}[z, t]$ 
Z[4]^(0,1)[z, t] →  $\frac{2M Z[9][z,t]}{H} + 6 Z[1]^{(1,0)}[z, t]$ 
Z[5]^(0,1)[z, t] → - $\frac{2M Z[8][z,t]}{H} - 6 Z[0]^{(1,0)}[z, t]$ 
Z[6]^(0,1)[z, t] → - $\frac{2M Z[11][z,t]}{H} - 6 Z[3]^{(1,0)}[z, t]$ 
Z[7]^(0,1)[z, t] →  $\frac{2M Z[10][z,t]}{H} + 6 Z[2]^{(1,0)}[z, t]$ 
Z[8]^(0,1)[z, t] →  $\frac{2M Z[5][z,t]}{H} + 6 Z[13]^{(1,0)}[z, t]$ 
Z[9]^(0,1)[z, t] → - $\frac{2M Z[4][z,t]}{H} - 6 Z[12]^{(1,0)}[z, t]$ 
Z[10]^(0,1)[z, t] → - $\frac{2M Z[7][z,t]}{H} - 6 Z[15]^{(1,0)}[z, t]$ 
Z[11]^(0,1)[z, t] →  $\frac{2M Z[6][z,t]}{H} + 6 Z[14]^{(1,0)}[z, t]$ 
Z[12]^(0,1)[z, t] → - $\frac{2M Z[1][z,t]}{H} - 6 Z[9]^{(1,0)}[z, t]$ 
Z[13]^(0,1)[z, t] →  $\frac{2M Z[0][z,t]}{H} + 6 Z[8]^{(1,0)}[z, t]$ 
Z[14]^(0,1)[z, t] →  $\frac{2M Z[3][z,t]}{H} + 6 Z[11]^{(1,0)}[z, t]$ 
Z[15]^(0,1)[z, t] → - $\frac{2M Z[2][z,t]}{H} - 6 Z[10]^{(1,0)}[z, t]$ 
```

Out[674]=

$$\begin{aligned} \left\{ Z[0]^{(0,1)}[z, t] = & -\frac{2 M Z[13][z, t]}{H} - 6 Z[5]^{(1,0)}[z, t], \right. \\ Z[1]^{(0,1)}[z, t] = & \frac{2 M Z[12][z, t]}{H} + 6 Z[4]^{(1,0)}[z, t], \\ Z[2]^{(0,1)}[z, t] = & \frac{2 M Z[15][z, t]}{H} + 6 Z[7]^{(1,0)}[z, t], \\ Z[3]^{(0,1)}[z, t] = & -\frac{2 M Z[14][z, t]}{H} - 6 Z[6]^{(1,0)}[z, t], \\ Z[4]^{(0,1)}[z, t] = & \frac{2 M Z[9][z, t]}{H} + 6 Z[1]^{(1,0)}[z, t], \\ Z[5]^{(0,1)}[z, t] = & -\frac{2 M Z[8][z, t]}{H} - 6 Z[0]^{(1,0)}[z, t], \\ Z[6]^{(0,1)}[z, t] = & -\frac{2 M Z[11][z, t]}{H} - 6 Z[3]^{(1,0)}[z, t], \\ Z[7]^{(0,1)}[z, t] = & \frac{2 M Z[10][z, t]}{H} + 6 Z[2]^{(1,0)}[z, t], \\ Z[8]^{(0,1)}[z, t] = & \frac{2 M Z[5][z, t]}{H} + 6 Z[13]^{(1,0)}[z, t], \\ Z[9]^{(0,1)}[z, t] = & -\frac{2 M Z[4][z, t]}{H} - 6 Z[12]^{(1,0)}[z, t], \\ Z[10]^{(0,1)}[z, t] = & -\frac{2 M Z[7][z, t]}{H} - 6 Z[15]^{(1,0)}[z, t], \\ Z[11]^{(0,1)}[z, t] = & \frac{2 M Z[6][z, t]}{H} + 6 Z[14]^{(1,0)}[z, t], \\ Z[12]^{(0,1)}[z, t] = & -\frac{2 M Z[1][z, t]}{H} - 6 Z[9]^{(1,0)}[z, t], \\ Z[13]^{(0,1)}[z, t] = & \frac{2 M Z[0][z, t]}{H} + 6 Z[8]^{(1,0)}[z, t], \\ Z[14]^{(0,1)}[z, t] = & \frac{2 M Z[3][z, t]}{H} + 6 Z[11]^{(1,0)}[z, t], \\ \left. Z[15]^{(0,1)}[z, t] = & -\frac{2 M Z[2][z, t]}{H} - 6 Z[10]^{(1,0)}[z, t] \right\} \end{aligned}$$

```
Out[675]=
{
$$\frac{2 M Z[13][z, t]}{H} + Z[0]^{(0,1)}[z, t] + 6 Z[5]^{(1,0)}[z, t],$$


$$-\frac{2 M Z[12][z, t]}{H} + Z[1]^{(0,1)}[z, t] - 6 Z[4]^{(1,0)}[z, t],$$


$$-\frac{2 M Z[15][z, t]}{H} + Z[2]^{(0,1)}[z, t] - 6 Z[7]^{(1,0)}[z, t],$$


$$\frac{2 M Z[14][z, t]}{H} + Z[3]^{(0,1)}[z, t] + 6 Z[6]^{(1,0)}[z, t],$$


$$-\frac{2 M Z[9][z, t]}{H} + Z[4]^{(0,1)}[z, t] - 6 Z[1]^{(1,0)}[z, t],$$


$$\frac{2 M Z[8][z, t]}{H} + Z[5]^{(0,1)}[z, t] + 6 Z[0]^{(1,0)}[z, t],$$


$$\frac{2 M Z[11][z, t]}{H} + Z[6]^{(0,1)}[z, t] + 6 Z[3]^{(1,0)}[z, t],$$


$$-\frac{2 M Z[10][z, t]}{H} + Z[7]^{(0,1)}[z, t] - 6 Z[2]^{(1,0)}[z, t],$$


$$-\frac{2 M Z[5][z, t]}{H} + Z[8]^{(0,1)}[z, t] - 6 Z[13]^{(1,0)}[z, t],$$


$$\frac{2 M Z[4][z, t]}{H} + Z[9]^{(0,1)}[z, t] + 6 Z[12]^{(1,0)}[z, t],$$


$$\frac{2 M Z[7][z, t]}{H} + Z[10]^{(0,1)}[z, t] + 6 Z[15]^{(1,0)}[z, t],$$


$$-\frac{2 M Z[6][z, t]}{H} + Z[11]^{(0,1)}[z, t] - 6 Z[14]^{(1,0)}[z, t],$$


$$\frac{2 M Z[1][z, t]}{H} + Z[12]^{(0,1)}[z, t] + 6 Z[9]^{(1,0)}[z, t],$$


$$-\frac{2 M Z[0][z, t]}{H} + Z[13]^{(0,1)}[z, t] - 6 Z[8]^{(1,0)}[z, t],$$


$$-\frac{2 M Z[3][z, t]}{H} + Z[14]^{(0,1)}[z, t] - 6 Z[11]^{(1,0)}[z, t],$$


$$\frac{2 M Z[2][z, t]}{H} + Z[15]^{(0,1)}[z, t] + 6 Z[10]^{(1,0)}[z, t]\}$$

```

In[676]:=

```
(*ToString[FullForm[#]]&/@DtvarZsubs;
StringReplace[#, "Rule"→ "Subtract"]&/@%;
(DtvarZrelations=ToExpression[#]&/@%) //Column*)
```

WRITE OUT THE VARIABLES THAT ARE COUPLED TO ANOTHER VARIABLE:

In[677]:=

```
eLaztCouplings = eLaCouplings
```

Out[677]=

```
{ {0, 5, 8, 13}, {1, 4, 9, 12}, {2, 7, 10, 15}, {3, 6, 11, 14} }
```

SEPARATE VARIABLES THAT ARE COUPLED TO EACH OTHER (reorder):

```

In[678]:= yZdef = yZ[#] & /@ Range[0, 15]
Out[678]= {yZ[0], yZ[1], yZ[2], yZ[3], yZ[4], yZ[5], yZ[6], yZ[7],
           yZ[8], yZ[9], yZ[10], yZ[11], yZ[12], yZ[13], yZ[14], yZ[15]}

In[679]:= Zdef = Z[#] & /@ Range[0, 15]
Out[679]= {Z[0], Z[1], Z[2], Z[3], Z[4], Z[5], Z[6],
           Z[7], Z[8], Z[9], Z[10], Z[11], Z[12], Z[13], Z[14], Z[15]}

In[680]:= sZtOyZ = Z[#\[If[1]]] \[Rule] yZ[#\[If[2]]] & /@ Transpose[{Flatten[eLastCouplings], Range[0, 15]}]
syZtoZ = Solve[And @@ (sZtOyZ /. {Rule \[Rule] Equal}), (yZ[#] & /@ Range[0, 15])] \[Leftarrow] -1]
zeroZyZforCaExpression = -sZtOyZ /. {Rule \[Rule] Subtract}
zeroZyZEQS = sZtOyZ /. {Rule \[Rule] Equal}

Out[680]= {Z[0] \[Rule] yZ[0], Z[5] \[Rule] yZ[1], Z[8] \[Rule] yZ[2], Z[13] \[Rule] yZ[3], Z[1] \[Rule] yZ[4], Z[4] \[Rule] yZ[5],
           Z[9] \[Rule] yZ[6], Z[12] \[Rule] yZ[7], Z[2] \[Rule] yZ[8], Z[7] \[Rule] yZ[9], Z[10] \[Rule] yZ[10],
           Z[15] \[Rule] yZ[11], Z[3] \[Rule] yZ[12], Z[6] \[Rule] yZ[13], Z[11] \[Rule] yZ[14], Z[14] \[Rule] yZ[15]}

Out[681]= {yZ[0] \[Rule] Z[0], yZ[1] \[Rule] Z[5], yZ[2] \[Rule] Z[8], yZ[3] \[Rule] Z[13], yZ[4] \[Rule] Z[1], yZ[5] \[Rule] Z[4],
           yZ[6] \[Rule] Z[9], yZ[7] \[Rule] Z[12], yZ[8] \[Rule] Z[2], yZ[9] \[Rule] Z[7], yZ[10] \[Rule] Z[10],
           yZ[11] \[Rule] Z[15], yZ[12] \[Rule] Z[3], yZ[13] \[Rule] Z[6], yZ[14] \[Rule] Z[11], yZ[15] \[Rule] Z[14]}

Out[682]= {yZ[0] - Z[0], yZ[1] - Z[5], yZ[2] - Z[8], yZ[3] - Z[13], yZ[4] - Z[1],
           yZ[5] - Z[4], yZ[6] - Z[9], yZ[7] - Z[12], yZ[8] - Z[2], yZ[9] - Z[7], yZ[10] - Z[10],
           yZ[11] - Z[15], yZ[12] - Z[3], yZ[13] - Z[6], yZ[14] - Z[11], yZ[15] - Z[14]}

Out[683]= {Z[0] == yZ[0], Z[5] == yZ[1], Z[8] == yZ[2], Z[13] == yZ[3], Z[1] == yZ[4], Z[4] == yZ[5],
           Z[9] == yZ[6], Z[12] == yZ[7], Z[2] == yZ[8], Z[7] == yZ[9], Z[10] == yZ[10],
           Z[15] == yZ[11], Z[3] == yZ[12], Z[6] == yZ[13], Z[11] == yZ[14], Z[14] == yZ[15]}

```

In[684]:=

(eLyZ = eLazt /. sZtOyZ // FullSimplify) // Column

Out[684]=

$$\begin{aligned}
 & -\frac{2 M y Z[5][z,t] + H y Z[6]^{(0,1)}[z,t] + 6 H y Z[7]^{(1,0)}[z,t]}{H^2} \\
 & -\frac{-2 M y Z[1][z,t] + H(y Z[2]^{(0,1)}[z,t] - 6 y Z[3]^{(1,0)}[z,t])}{H^2} \\
 & -\frac{-2 M y Z[13][z,t] + H(y Z[14]^{(0,1)}[z,t] - 6 y Z[15]^{(1,0)}[z,t])}{H^2} \\
 & -\frac{2 M y Z[9][z,t] + H y Z[10]^{(0,1)}[z,t] + 6 H y Z[11]^{(1,0)}[z,t]}{H^2} \\
 & -\frac{-2 M y Z[0][z,t] + H(y Z[3]^{(0,1)}[z,t] - 6 y Z[2]^{(1,0)}[z,t])}{H^2} \\
 & -\frac{2 M y Z[4][z,t] + H y Z[7]^{(0,1)}[z,t] + 6 H y Z[6]^{(1,0)}[z,t]}{H^2} \\
 & -\frac{-2 M y Z[8][z,t] + H y Z[11]^{(0,1)}[z,t] + 6 H y Z[10]^{(1,0)}[z,t]}{H^2} \\
 & -\frac{-2 M y Z[12][z,t] + H(y Z[15]^{(0,1)}[z,t] - 6 y Z[14]^{(1,0)}[z,t])}{H^2} \\
 & \frac{2 M y Z[7][z,t] - H y Z[4]^{(0,1)}[z,t] + 6 H y Z[5]^{(1,0)}[z,t]}{H^2} \\
 & \frac{2 M y Z[3][z,t] + H y Z[0]^{(0,1)}[z,t] + 6 H y Z[1]^{(1,0)}[z,t]}{H^2} \\
 & \frac{2 M y Z[15][z,t] + H y Z[12]^{(0,1)}[z,t] + 6 H y Z[13]^{(1,0)}[z,t]}{H^2} \\
 & \frac{2 M y Z[11][z,t] - H y Z[8]^{(0,1)}[z,t] + 6 H y Z[9]^{(1,0)}[z,t]}{H^2} \\
 & \frac{2 M y Z[2][z,t] + H y Z[1]^{(0,1)}[z,t] + 6 H y Z[0]^{(1,0)}[z,t]}{H^2} \\
 & \frac{2 M y Z[6][z,t] - H y Z[5]^{(0,1)}[z,t] + 6 H y Z[4]^{(1,0)}[z,t]}{H^2} \\
 & \frac{2 M y Z[10][z,t] - H y Z[9]^{(0,1)}[z,t] + 6 H y Z[8]^{(1,0)}[z,t]}{H^2} \\
 & \frac{2 M y Z[14][z,t] + H y Z[13]^{(0,1)}[z,t] + 6 H y Z[12]^{(1,0)}[z,t]}{H^2}
 \end{aligned}$$

In[685]:=

```
(DtyZsubsA =
  Solve[And @@ Thread[θ == eLyZ], (D[(yZ[#][z, t] & /@ Range[θ, 15]), t])[[1]] // FullSimplify[#, constraintVars] &] // Column
```

Out[685]=

$$\begin{aligned} yZ[0]^{(0,1)}[z, t] &\rightarrow -\frac{2MyZ[3][z, t]}{H} - 6yZ[1]^{(1,0)}[z, t] \\ yZ[1]^{(0,1)}[z, t] &\rightarrow -\frac{2MyZ[2][z, t]}{H} - 6yZ[0]^{(1,0)}[z, t] \\ yZ[2]^{(0,1)}[z, t] &\rightarrow \frac{2MyZ[1][z, t]}{H} + 6yZ[3]^{(1,0)}[z, t] \\ yZ[3]^{(0,1)}[z, t] &\rightarrow \frac{2MyZ[0][z, t]}{H} + 6yZ[2]^{(1,0)}[z, t] \\ yZ[4]^{(0,1)}[z, t] &\rightarrow \frac{2MyZ[7][z, t]}{H} + 6yZ[5]^{(1,0)}[z, t] \\ yZ[5]^{(0,1)}[z, t] &\rightarrow \frac{2MyZ[6][z, t]}{H} + 6yZ[4]^{(1,0)}[z, t] \\ yZ[6]^{(0,1)}[z, t] &\rightarrow -\frac{2MyZ[5][z, t]}{H} - 6yZ[7]^{(1,0)}[z, t] \\ yZ[7]^{(0,1)}[z, t] &\rightarrow -\frac{2MyZ[4][z, t]}{H} - 6yZ[6]^{(1,0)}[z, t] \\ yZ[8]^{(0,1)}[z, t] &\rightarrow \frac{2MyZ[11][z, t]}{H} + 6yZ[9]^{(1,0)}[z, t] \\ yZ[9]^{(0,1)}[z, t] &\rightarrow \frac{2MyZ[10][z, t]}{H} + 6yZ[8]^{(1,0)}[z, t] \\ yZ[10]^{(0,1)}[z, t] &\rightarrow -\frac{2MyZ[9][z, t]}{H} - 6yZ[11]^{(1,0)}[z, t] \\ yZ[11]^{(0,1)}[z, t] &\rightarrow -\frac{2MyZ[8][z, t]}{H} - 6yZ[10]^{(1,0)}[z, t] \\ yZ[12]^{(0,1)}[z, t] &\rightarrow -\frac{2MyZ[15][z, t]}{H} - 6yZ[13]^{(1,0)}[z, t] \\ yZ[13]^{(0,1)}[z, t] &\rightarrow -\frac{2MyZ[14][z, t]}{H} - 6yZ[12]^{(1,0)}[z, t] \\ yZ[14]^{(0,1)}[z, t] &\rightarrow \frac{2MyZ[13][z, t]}{H} + 6yZ[15]^{(1,0)}[z, t] \\ yZ[15]^{(0,1)}[z, t] &\rightarrow \frac{2MyZ[12][z, t]}{H} + 6yZ[14]^{(1,0)}[z, t] \end{aligned}$$

In[686]:=

```
DtvaryZEQS = DtyZsubsA /. {Rule → Equal};
zeroDtyZeqs = DtyZsubsA /. {Rule → Subtract};
coupledyeZeqs = Partition[DtvaryZEQS, 4]
```

$$\text{Out}[687]=$$

$$\left\{ \begin{aligned} & \frac{2 M y Z[3][z, t]}{H} + y Z[0]^{(0,1)}[z, t] + 6 y Z[1]^{(1,0)}[z, t], \\ & \frac{2 M y Z[2][z, t]}{H} + y Z[1]^{(0,1)}[z, t] + 6 y Z[0]^{(1,0)}[z, t], \\ & -\frac{2 M y Z[1][z, t]}{H} + y Z[2]^{(0,1)}[z, t] - 6 y Z[3]^{(1,0)}[z, t], \\ & -\frac{2 M y Z[0][z, t]}{H} + y Z[3]^{(0,1)}[z, t] - 6 y Z[2]^{(1,0)}[z, t], \\ & -\frac{2 M y Z[7][z, t]}{H} + y Z[4]^{(0,1)}[z, t] - 6 y Z[5]^{(1,0)}[z, t], \\ & -\frac{2 M y Z[6][z, t]}{H} + y Z[5]^{(0,1)}[z, t] - 6 y Z[4]^{(1,0)}[z, t], \\ & \frac{2 M y Z[5][z, t]}{H} + y Z[6]^{(0,1)}[z, t] + 6 y Z[7]^{(1,0)}[z, t], \\ & \frac{2 M y Z[4][z, t]}{H} + y Z[7]^{(0,1)}[z, t] + 6 y Z[6]^{(1,0)}[z, t], \\ & -\frac{2 M y Z[11][z, t]}{H} + y Z[8]^{(0,1)}[z, t] - 6 y Z[9]^{(1,0)}[z, t], \\ & -\frac{2 M y Z[10][z, t]}{H} + y Z[9]^{(0,1)}[z, t] - 6 y Z[8]^{(1,0)}[z, t], \\ & \frac{2 M y Z[9][z, t]}{H} + y Z[10]^{(0,1)}[z, t] + 6 y Z[11]^{(1,0)}[z, t], \\ & \frac{2 M y Z[8][z, t]}{H} + y Z[11]^{(0,1)}[z, t] + 6 y Z[10]^{(1,0)}[z, t], \\ & \frac{2 M y Z[15][z, t]}{H} + y Z[12]^{(0,1)}[z, t] + 6 y Z[13]^{(1,0)}[z, t], \\ & \frac{2 M y Z[14][z, t]}{H} + y Z[13]^{(0,1)}[z, t] + 6 y Z[12]^{(1,0)}[z, t], \\ & -\frac{2 M y Z[13][z, t]}{H} + y Z[14]^{(0,1)}[z, t] - 6 y Z[15]^{(1,0)}[z, t], \\ & -\frac{2 M y Z[12][z, t]}{H} + y Z[15]^{(0,1)}[z, t] - 6 y Z[14]^{(1,0)}[z, t] \end{aligned} \right\}$$

Out[688]:=

$$\begin{aligned} \left\{ \begin{aligned} yZ[0]^{(0,1)}[z, t] &= -\frac{2 M yZ[3][z, t]}{H} - 6 yZ[1]^{(1,0)}[z, t], \\ yZ[1]^{(0,1)}[z, t] &= -\frac{2 M yZ[2][z, t]}{H} - 6 yZ[0]^{(1,0)}[z, t], \\ yZ[2]^{(0,1)}[z, t] &= \frac{2 M yZ[1][z, t]}{H} + 6 yZ[3]^{(1,0)}[z, t], \\ yZ[3]^{(0,1)}[z, t] &= \frac{2 M yZ[0][z, t]}{H} + 6 yZ[2]^{(1,0)}[z, t] \end{aligned} \right\}, \\ \left\{ \begin{aligned} yZ[4]^{(0,1)}[z, t] &= \frac{2 M yZ[7][z, t]}{H} + 6 yZ[5]^{(1,0)}[z, t], \\ yZ[5]^{(0,1)}[z, t] &= \frac{2 M yZ[6][z, t]}{H} + 6 yZ[4]^{(1,0)}[z, t], \\ yZ[6]^{(0,1)}[z, t] &= -\frac{2 M yZ[5][z, t]}{H} - 6 yZ[7]^{(1,0)}[z, t], \\ yZ[7]^{(0,1)}[z, t] &= -\frac{2 M yZ[4][z, t]}{H} - 6 yZ[6]^{(1,0)}[z, t] \end{aligned} \right\}, \\ \left\{ \begin{aligned} yZ[8]^{(0,1)}[z, t] &= \frac{2 M yZ[11][z, t]}{H} + 6 yZ[9]^{(1,0)}[z, t], \\ yZ[9]^{(0,1)}[z, t] &= \frac{2 M yZ[10][z, t]}{H} + 6 yZ[8]^{(1,0)}[z, t], \\ yZ[10]^{(0,1)}[z, t] &= -\frac{2 M yZ[9][z, t]}{H} - 6 yZ[11]^{(1,0)}[z, t], \\ yZ[11]^{(0,1)}[z, t] &= -\frac{2 M yZ[8][z, t]}{H} - 6 yZ[10]^{(1,0)}[z, t] \end{aligned} \right\}, \\ \left\{ \begin{aligned} yZ[12]^{(0,1)}[z, t] &= -\frac{2 M yZ[15][z, t]}{H} - 6 yZ[13]^{(1,0)}[z, t], \\ yZ[13]^{(0,1)}[z, t] &= -\frac{2 M yZ[14][z, t]}{H} - 6 yZ[12]^{(1,0)}[z, t], \\ yZ[14]^{(0,1)}[z, t] &= \frac{2 M yZ[13][z, t]}{H} + 6 yZ[15]^{(1,0)}[z, t], \\ yZ[15]^{(0,1)}[z, t] &= \frac{2 M yZ[12][z, t]}{H} + 6 yZ[14]^{(1,0)}[z, t] \end{aligned} \right\} \end{aligned}$$

In[689]:=

```
(*DtvaryZEQS=DtyZsubs/.{Rule→Equal};
zeroDtyZeqs=DtyZsubs/.{Rule→Subtract}
coupledyZeqs=Partition[DtvaryZEQS,4]*)
```

DSolve

```
In[690]:= Clear[yZ, yZvar];
yZvar = (yZ[#[z, t] & /@ Range[0, 15])

Out[691]= {yZ[0][z, t], yZ[1][z, t], yZ[2][z, t], yZ[3][z, t], yZ[4][z, t],
yZ[5][z, t], yZ[6][z, t], yZ[7][z, t], yZ[8][z, t], yZ[9][z, t], yZ[10][z, t],
yZ[11][z, t], yZ[12][z, t], yZ[13][z, t], yZ[14][z, t], yZ[15][z, t]}

In[692]:= DSolve[coupledYzeqs[[1]], yZvar[[1 ;; 4]], {z, t}]

Out[692]= DSolve[{yZ[0]^(0,1)[z, t] == - $\frac{2 M yZ[3][z, t]}{H}$  - 6 yZ[1]^(1,0)[z, t],
yZ[1]^(0,1)[z, t] == - $\frac{2 M yZ[2][z, t]}{H}$  - 6 yZ[0]^(1,0)[z, t],
yZ[2]^(0,1)[z, t] ==  $\frac{2 M yZ[1][z, t]}{H}$  + 6 yZ[3]^(1,0)[z, t],
yZ[3]^(0,1)[z, t] ==  $\frac{2 M yZ[0][z, t]}{H}$  + 6 yZ[2]^(1,0)[z, t]}, {yZ[0][z, t], yZ[1][z, t], yZ[2][z, t], yZ[3][z, t]}, {z, t}]

In[693]:= DSolve[coupledYzeqs[[2]], yZvar[[5 ;; 8]], {z, t}]

Out[693]= DSolve[{yZ[4]^(0,1)[z, t] ==  $\frac{2 M yZ[7][z, t]}{H}$  + 6 yZ[5]^(1,0)[z, t],
yZ[5]^(0,1)[z, t] ==  $\frac{2 M yZ[6][z, t]}{H}$  + 6 yZ[4]^(1,0)[z, t],
yZ[6]^(0,1)[z, t] == - $\frac{2 M yZ[5][z, t]}{H}$  - 6 yZ[7]^(1,0)[z, t],
yZ[7]^(0,1)[z, t] == - $\frac{2 M yZ[4][z, t]}{H}$  - 6 yZ[6]^(1,0)[z, t]}, {yZ[4][z, t], yZ[5][z, t], yZ[6][z, t], yZ[7][z, t]}, {z, t}]
```

In[694]:=

```
DSolve[coupledyZeqs[[3]], yZvar[[9 ;; 12]], {z, t}]
```

Out[694]=

$$\begin{aligned} \text{DSolve}\left[\left\{\begin{aligned} yZ[8]^{(0,1)}[z, t] &= \frac{2 M yZ[11][z, t]}{H} + 6 yZ[9]^{(1,0)}[z, t], \\ yZ[9]^{(0,1)}[z, t] &= \frac{2 M yZ[10][z, t]}{H} + 6 yZ[8]^{(1,0)}[z, t], \\ yZ[10]^{(0,1)}[z, t] &= -\frac{2 M yZ[9][z, t]}{H} - 6 yZ[11]^{(1,0)}[z, t], \\ yZ[11]^{(0,1)}[z, t] &= -\frac{2 M yZ[8][z, t]}{H} - 6 yZ[10]^{(1,0)}[z, t] \end{aligned}\right\}, \{z, t\}\right] \end{aligned}$$

In[695]:=

```
DSolve[coupledyZeqs[[4]], yZvar[[13 ;; 16]], {z, t}]
```

Out[695]=

$$\begin{aligned} \text{DSolve}\left[\left\{\begin{aligned} yZ[12]^{(0,1)}[z, t] &= -\frac{2 M yZ[15][z, t]}{H} - 6 yZ[13]^{(1,0)}[z, t], \\ yZ[13]^{(0,1)}[z, t] &= -\frac{2 M yZ[14][z, t]}{H} - 6 yZ[12]^{(1,0)}[z, t], \\ yZ[14]^{(0,1)}[z, t] &= \frac{2 M yZ[13][z, t]}{H} + 6 yZ[15]^{(1,0)}[z, t], \\ yZ[15]^{(0,1)}[z, t] &= \frac{2 M yZ[12][z, t]}{H} + 6 yZ[14]^{(1,0)}[z, t] \end{aligned}\right\}, \{z, t\}\right] \end{aligned}$$

TRY MAPLE: MUST use ` List[Equal[Derivative[0,1][yZ8][z,t],Pl...`

```
In[696]:= syZzt = yZ[#] → ToExpression["((yZ" <> ToString[#] <> "[#1,#2])&)" ] & /@ Range[0, 15]
maplexyZeqs = FullForm[coupledelyZeqs /. % // FullSimplify]

Out[696]= {yZ[0] → (yZ0[#1, #2] &), yZ[1] → (yZ1[#1, #2] &),
yZ[2] → (yZ2[#1, #2] &), yZ[3] → (yZ3[#1, #2] &), yZ[4] → (yZ4[#1, #2] &),
yZ[5] → (yZ5[#1, #2] &), yZ[6] → (yZ6[#1, #2] &), yZ[7] → (yZ7[#1, #2] &),
yZ[8] → (yZ8[#1, #2] &), yZ[9] → (yZ9[#1, #2] &), yZ[10] → (yZ10[#1, #2] &),
yZ[11] → (yZ11[#1, #2] &), yZ[12] → (yZ12[#1, #2] &),
yZ[13] → (yZ13[#1, #2] &), yZ[14] → (yZ14[#1, #2] &), yZ[15] → (yZ15[#1, #2] &)}

Out[697]//FullForm=
List[List[Equal[Plus[Times[2, Power[H, -1], M, yZ3[z, t]], Derivative[0, 1][yZ0][z, t], Times[6, Derivative[1, 0][yZ1][z, t]]], 0], Equal[Plus[Times[2, Power[H, -1], M, yZ2[z, t]], Derivative[0, 1][yZ1][z, t], Times[6, Derivative[1, 0][yZ0][z, t]]], 0], Equal[Derivative[0, 1][yZ2][z, t], Plus[Times[2, Power[H, -1], M, yZ1[z, t]], Times[6, Derivative[1, 0][yZ3][z, t]]]], Equal[Derivative[0, 1][yZ3][z, t], Plus[Times[2, Power[H, -1], M, yZ0[z, t]], Times[6, Derivative[1, 0][yZ2][z, t]]]], List[Equal[Derivative[0, 1][yZ4][z, t], Plus[Times[2, Power[H, -1], M, yZ7[z, t]], Times[6, Derivative[1, 0][yZ5][z, t]]], Equal[Derivative[0, 1][yZ5][z, t], Plus[Times[2, Power[H, -1], M, yZ6[z, t]], Times[6, Derivative[1, 0][yZ4][z, t]]]], Equal[Plus[Times[2, Power[H, -1], M, yZ5[z, t]], Derivative[0, 1][yZ6][z, t], Times[6, Derivative[1, 0][yZ7][z, t]]], 0], Equal[Plus[Times[2, Power[H, -1], M, yZ4[z, t]], Derivative[0, 1][yZ7][z, t], Times[6, Derivative[1, 0][yZ6][z, t]]], 0]], List[Equal[Derivative[0, 1][yZ8][z, t], Plus[Times[2, Power[H, -1], M, yZ11[z, t]], Times[6, Derivative[1, 0][yZ9][z, t]]], Equal[Derivative[0, 1][yZ9][z, t], Plus[Times[2, Power[H, -1], M, yZ10[z, t]], Times[6, Derivative[1, 0][yZ8][z, t]]]], Equal[Plus[Times[2, Power[H, -1], M, yZ9[z, t]], Derivative[0, 1][yZ10][z, t], Times[6, Derivative[1, 0][yZ11][z, t]]], 0], Equal[Plus[Times[2, Power[H, -1], M, yZ8[z, t]], Derivative[0, 1][yZ11][z, t], Times[6, Derivative[1, 0][yZ10][z, t]]], 0]], List[Equal[Plus[Times[2, Power[H, -1], M, yZ15[z, t]], Derivative[0, 1][yZ12][z, t], Times[6, Derivative[1, 0][yZ13][z, t]]], 0], Equal[Plus[Times[2, Power[H, -1], M, yZ14[z, t]], Derivative[0, 1][yZ13][z, t], Times[6, Derivative[1, 0][yZ12][z, t]]], 0], Equal[Derivative[0, 1][yZ15][z, t], Plus[Times[2, Power[H, -1], M, yZ12[z, t]], Times[6, Derivative[1, 0][yZ14][z, t]]]]]
```

```
In[698]:= mapleyZvars =
  FullForm[yZ[#[z, t] & /@ Range[4 * (# - 1), 4 * (# - 1) + 3] /. syZzt] & /@ Range[4]
Out[698]= {List[yZ0[z, t], yZ1[z, t], yZ2[z, t], yZ3[z, t]],  

          List[yZ4[z, t], yZ5[z, t], yZ6[z, t], yZ7[z, t]],  

          List[yZ8[z, t], yZ9[z, t], yZ10[z, t], yZ11[z, t]],  

          List[yZ12[z, t], yZ13[z, t], yZ14[z, t], yZ15[z, t]]}
```

FOR MAPLE : A set is an unordered sequence of distinct expressions enclosed in braces {}, representing a set in the mathematical sense.

A list is an ordered sequence of expressions enclosed in square brackets []. The ordering of the expressions is the ordering of es.

```
“maple-output.py”:
import textwrap
filename = "maple-output.txt"
maple_long_string = "paste maple string here"
# Wraps the text and joins with newlines
with open (filename, "w") as f:
    f . write ("\n" . join (textwrap . wrap (maple_long_string, width = 80)))
```

FOR MAPLE : A set is an unordered sequence of distinct expressions enclosed in braces {}, representing a set in the mathematical sense.

A list is an ordered sequence of expressions enclosed in square brackets []. The ordering of the expressions is the ordering of es.

"Solution 15feb" by Maple

```
In[699]:= syZtoZ
Out[699]= {yZ[0] → Z[0], yZ[1] → Z[5], yZ[2] → Z[8], yZ[3] → Z[13], yZ[4] → Z[1], yZ[5] → Z[4],  

          yZ[6] → Z[9], yZ[7] → Z[12], yZ[8] → Z[2], yZ[9] → Z[7], yZ[10] → Z[10],  

          yZ[11] → Z[15], yZ[12] → Z[3], yZ[13] → Z[6], yZ[14] → Z[11], yZ[15] → Z[14]}
```

```
In[700]:= (*Z[#]&/@Range[0,15]*)
Table[Z[j] → ToExpression["((Z[" <> ToString[j] <> "] [#1,#2])&)"], {j, 0, 15}];
ssyZtoZ = syZtoZ /. %

Out[701]= {yZ[0] → (Z[0] [#1, #2] &), yZ[1] → (Z[5] [#1, #2] &),
yZ[2] → (Z[8] [#1, #2] &), yZ[3] → (Z[13] [#1, #2] &), yZ[4] → (Z[1] [#1, #2] &),
yZ[5] → (Z[4] [#1, #2] &), yZ[6] → (Z[9] [#1, #2] &), yZ[7] → (Z[12] [#1, #2] &),
yZ[8] → (Z[2] [#1, #2] &), yZ[9] → (Z[7] [#1, #2] &), yZ[10] → (Z[10] [#1, #2] &),
yZ[11] → (Z[15] [#1, #2] &), yZ[12] → (Z[3] [#1, #2] &), yZ[13] → (Z[6] [#1, #2] &),
yZ[14] → (Z[11] [#1, #2] &), yZ[15] → (Z[14] [#1, #2] &)}

In[702]:= §16;
% /. sfψ16Aa /. sx0x4;
% /. sZt0yZ;
Thread[% == %%];
Solve[And @@ %, yZ[#][z, t] & /@ Range[0, 15]] [[1]]
% /. szt /. {Rule → Equal};
Solve[And @@ %, f16[#][x0, x4] & /@ Range[0, 15]] [[1]];
% /. {Rule → Equal}

Out[706]= {yZ[0][z, t] → f16[0][x0, x4], yZ[1][z, t] → f16[5][x0, x4],
yZ[2][z, t] → f16[8][x0, x4], yZ[3][z, t] → f16[13][x0, x4],
yZ[4][z, t] → f16[1][x0, x4], yZ[5][z, t] → f16[4][x0, x4], yZ[6][z, t] → f16[9][x0, x4],
yZ[7][z, t] → f16[12][x0, x4], yZ[8][z, t] → f16[2][x0, x4], yZ[9][z, t] → f16[7][x0, x4],
yZ[10][z, t] → f16[10][x0, x4], yZ[11][z, t] → f16[15][x0, x4],
yZ[12][z, t] → f16[3][x0, x4], yZ[13][z, t] → f16[6][x0, x4],
yZ[14][z, t] → f16[11][x0, x4], yZ[15][z, t] → f16[14][x0, x4]}

Out[709]= {f16[0][x0, x4] == yZ[0][6 H x0, H x4], f16[1][x0, x4] == yZ[4][6 H x0, H x4],
f16[2][x0, x4] == yZ[8][6 H x0, H x4], f16[3][x0, x4] == yZ[12][6 H x0, H x4],
f16[4][x0, x4] == yZ[5][6 H x0, H x4], f16[5][x0, x4] == yZ[1][6 H x0, H x4],
f16[6][x0, x4] == yZ[13][6 H x0, H x4], f16[7][x0, x4] == yZ[9][6 H x0, H x4],
f16[8][x0, x4] == yZ[2][6 H x0, H x4], f16[9][x0, x4] == yZ[6][6 H x0, H x4],
f16[10][x0, x4] == yZ[10][6 H x0, H x4], f16[11][x0, x4] == yZ[14][6 H x0, H x4],
f16[12][x0, x4] == yZ[7][6 H x0, H x4], f16[13][x0, x4] == yZ[3][6 H x0, H x4],
f16[14][x0, x4] == yZ[15][6 H x0, H x4], f16[15][x0, x4] == yZ[11][6 H x0, H x4]}

Load the Maple output strings. Get is a lexer and parer and ... , so use ReadString
```

```
In[710]:= SetDirectory[NotebookDirectory[]];
```

```
In[711]:= maplestringEQ1 = "{ {yZ0(z, t) = exp((6*(z/6 + t)*H^2*C1^2 - 6*(-z/6 + t)*M^2)/(6*C1*H^2))*(H^2*(c11*c12 - c13*c14)*C1^2 - M^2*(c11*c12 + c13*c14))/(2*H*C1*M), yZ1(z, t) = -((H^2*(c11*c12 - c13*c14)*C1^2 + M^2*(c11*c12 + c13*c14))*exp((6*(z/6 + t)*H^2*C1^2 - 6*(-z/6 + t)*M^2)/(6*C1*H^2)))/(2*H*C1*M), yZ2(z, t) = c13*exp((6*(z/6 + t)*H^2*C1^2 - 6*(-z/6 + t)*M^2)/(6*C1*H^2))*c14, yZ3(z, t) = c11*exp((6*(z/6 + t)*H^2*C1^2 - 6*(-z/6 + t)*M^2)/(6*C1*H^2))*c12} }";
```

```
In[712]:= maplestringEQ2 = "{ {yZ4(z, t) = -exp((6*(z/6 + t)*H^2*C2^2 - 6*(-z/6 + t)*M^2)/(6*C2*H^2))*(H^2*(c21*c22 + c23*c24)*C2^2 - M^2*(c21*c22 - c23*c24))/(2*H*C2*M), yZ5(z, t) = -((H^2*(c21*c22 + c23*c24)*C2^2 + M^2*(c21*c22 - c23*c24))*exp((6*(z/6 + t)*H^2*C2^2 - 6*(-z/6 + t)*M^2)/(6*C2*H^2)))/(2*H*C2*M), yZ6(z, t) = c23*exp((6*(z/6 + t)*H^2*C2^2 - 6*(-z/6 + t)*M^2)/(6*C2*H^2))*c24, yZ7(z, t) = c21*exp((6*(z/6 + t)*H^2*C2^2 - 6*(-z/6 + t)*M^2)/(6*C2*H^2))*c22} }";
```

```
In[713]:= maplestringEQ3 = "
{{yZ10(z, t) = c33*exp((6*H^2*(z/6 + t)*C3^2 - 6*(-z/6 + t)*M^2)/(6*C3*H^2))*c34,
yZ11(z, t) = c31*exp((6*H^2*(z/6 + t)*C3^2 - 6*(-z/6 + t)*M^2)/(6*C3*H^2))*c32, yZ8(z, t) = -exp((6*H^2*(z/6 + t)*C3^2 - 6*(-z/6 + t)*M^2)/(6*C3*H^2))*(H^2*(c31*c32 + c33*c34)*C3^2 - M^2*(c31*c32 - c33*c34))/(2*H*C3*M), yZ9(z, t) = -((H^2*(c31*c32 + c33*c34)*C3^2 + M^2*(c31*c32 - c33*c34))*exp((6*H^2*(z/6 + t)*C3^2 - 6*(-z/6 + t)*M^2)/(6*C3*H^2)))/(2*H*C3*M)} }";
```

```
In[714]:= maplestringEQ4 = "{ {yZ12(z, t) = exp((6*H^2*(z/6 + t)*C4^2 - 6*(-z/6 + t)*M^2)/(6*C4*H^2))*(H^2*(c41*c42 - c43*c44)*C4^2 - M^2*(c41*c42 + c43*c44))/(2*H*C4*M), yZ13(z, t) = -((H^2*(c41*c42 - c43*c44)*C4^2 + M^2*(c41*c42 + c43*c44))*exp((6*H^2*(z/6 + t)*C4^2 - 6*(-z/6 + t)*M^2)/(6*C4*H^2)))/(2*H*C4*M), yZ14(z, t) = c43*exp((6*H^2*(z/6 + t)*C4^2 - 6*(-z/6 + t)*M^2)/(6*C4*H^2))*c44, yZ15(z, t) = c41*exp((6*H^2*(z/6 + t)*C4^2 - 6*(-z/6 + t)*M^2)/(6*C4*H^2))*c42} }";
```

```
In[715]:= solved13FebEQ1 = ConvertMapleToMathematicaV2[maplestringEQ1]
```

```
Out[715]=

$$\left\{ \begin{aligned} YZ0[z, t] &= \frac{e^{\frac{-6M^2\left(t-\frac{z}{6}\right)+6C1^2H^2\left(t+\frac{z}{6}\right)}{6C1H^2}} \left(C1^2(c11c12 - c13c14)H^2 - (c11c12 + c13c14)M^2\right)}{2C1HM}, \\ YZ1[z, t] &= -\frac{e^{\frac{-6M^2\left(t-\frac{z}{6}\right)+6C1^2H^2\left(t+\frac{z}{6}\right)}{6C1H^2}} \left(C1^2(c11c12 - c13c14)H^2 + (c11c12 + c13c14)M^2\right)}{2C1HM}, \\ YZ2[z, t] &= c13c14 e^{\frac{-6M^2\left(t-\frac{z}{6}\right)+6C1^2H^2\left(t+\frac{z}{6}\right)}{6C1H^2}}, \quad YZ3[z, t] = c11c12 e^{\frac{-6M^2\left(t-\frac{z}{6}\right)+6C1^2H^2\left(t+\frac{z}{6}\right)}{6C1H^2}} \end{aligned} \right\}$$

```

In[716]:=

solved13FebEQ2 = ConvertMapleToMathematicaV2[maplestringEQ2]

Out[716]=

$$\left\{ \begin{array}{l} YZ4[z, t] = -\frac{e^{\frac{-6M^2(t-\frac{z}{6})+6C2^2H^2(t+\frac{z}{6})}{6C2H^2}} (C2^2(c21c22+c23c24)H^2 - (c21c22-c23c24)M^2)}{2C2HM}, \\ YZ5[z, t] = -\frac{e^{\frac{-6M^2(t-\frac{z}{6})+6C2^2H^2(t+\frac{z}{6})}{6C2H^2}} (C2^2(c21c22+c23c24)H^2 + (c21c22-c23c24)M^2)}{2C2HM}, \\ YZ6[z, t] = c23c24 e^{\frac{-6M^2(t-\frac{z}{6})+6C2^2H^2(t+\frac{z}{6})}{6C2H^2}}, YZ7[z, t] = c21c22 e^{\frac{-6M^2(t-\frac{z}{6})+6C2^2H^2(t+\frac{z}{6})}{6C2H^2}} \end{array} \right\}$$

In[717]:=

solved13FebEQ3 = ConvertMapleToMathematicaV2[maplestringEQ3]

Out[717]=

$$\left\{ \begin{array}{l} YZ10[z, t] = c33c34 e^{\frac{-6M^2(t-\frac{z}{6})+6C3^2H^2(t+\frac{z}{6})}{6C3H^2}}, YZ11[z, t] = c31c32 e^{\frac{-6M^2(t-\frac{z}{6})+6C3^2H^2(t+\frac{z}{6})}{6C3H^2}}, \\ YZ8[z, t] = -\frac{e^{\frac{-6M^2(t-\frac{z}{6})+6C3^2H^2(t+\frac{z}{6})}{6C3H^2}} (C3^2(c31c32+c33c34)H^2 - (c31c32-c33c34)M^2)}{2C3HM}, \\ YZ9[z, t] = -\frac{e^{\frac{-6M^2(t-\frac{z}{6})+6C3^2H^2(t+\frac{z}{6})}{6C3H^2}} (C3^2(c31c32+c33c34)H^2 + (c31c32-c33c34)M^2)}{2C3HM} \end{array} \right\}$$

In[718]:=

solved13FebEQ4 = ConvertMapleToMathematicaV2[maplestringEQ4]

Out[718]=

$$\left\{ \begin{array}{l} YZ12[z, t] = \frac{e^{\frac{-6M^2(t-\frac{z}{6})+6C4^2H^2(t+\frac{z}{6})}{6C4H^2}} (C4^2(c41c42-c43c44)H^2 - (c41c42+c43c44)M^2)}{2C4HM}, \\ YZ13[z, t] = -\frac{e^{\frac{-6M^2(t-\frac{z}{6})+6C4^2H^2(t+\frac{z}{6})}{6C4H^2}} (C4^2(c41c42-c43c44)H^2 + (c41c42+c43c44)M^2)}{2C4HM}, \\ YZ14[z, t] = c43c44 e^{\frac{-6M^2(t-\frac{z}{6})+6C4^2H^2(t+\frac{z}{6})}{6C4H^2}}, YZ15[z, t] = c41c42 e^{\frac{-6M^2(t-\frac{z}{6})+6C4^2H^2(t+\frac{z}{6})}{6C4H^2}} \end{array} \right\}$$

In[719]:=

YZvar = ToExpression["YZ" <> ToString[#] <> "[z,t]"] & /@ Range[0, 15]

Out[719]=

{YZ0[z, t], YZ1[z, t], YZ2[z, t], YZ3[z, t], YZ4[z, t], YZ5[z, t], YZ6[z, t], YZ7[z, t], YZ8[z, t], YZ9[z, t], YZ10[z, t], YZ11[z, t], YZ12[z, t], YZ13[z, t], YZ14[z, t], YZ15[z, t]}

```
In[720]:= sYZvar = Solve[And @@
Flatten[{solved13FebEQ1, solved13FebEQ2, solved13FebEQ3, solved13FebEQ4}], YZvar] [[1]]
Out[720]=

$$\begin{aligned} \text{YZ0}[z, t] &\rightarrow \frac{e^{-\frac{M^2(t-\frac{z}{6})}{c1 H^2} + c1(t+\frac{z}{6})} (C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2)}{2 C1 H M}, \\ \text{YZ1}[z, t] &\rightarrow -\frac{e^{-\frac{M^2(t-\frac{z}{6})}{c1 H^2} + c1(t+\frac{z}{6})} (C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 + c11 c12 M^2 + c13 c14 M^2)}{2 C1 H M}, \\ \text{YZ2}[z, t] &\rightarrow c13 c14 e^{-\frac{M^2(t-\frac{z}{6})}{c1 H^2} + c1(t+\frac{z}{6})}, \text{YZ3}[z, t] \rightarrow c11 c12 e^{-\frac{M^2(t-\frac{z}{6})}{c1 H^2} + c1(t+\frac{z}{6})}, \\ \text{YZ4}[z, t] &\rightarrow -\frac{e^{-\frac{M^2(t-\frac{z}{6})}{c2 H^2} + c2(t+\frac{z}{6})} (C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 - c21 c22 M^2 + c23 c24 M^2)}{2 C2 H M}, \\ \text{YZ5}[z, t] &\rightarrow -\frac{e^{-\frac{M^2(t-\frac{z}{6})}{c2 H^2} + c2(t+\frac{z}{6})} (C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 + c21 c22 M^2 - c23 c24 M^2)}{2 C2 H M}, \\ \text{YZ6}[z, t] &\rightarrow c23 c24 e^{-\frac{M^2(t-\frac{z}{6})}{c2 H^2} + c2(t+\frac{z}{6})}, \text{YZ7}[z, t] \rightarrow c21 c22 e^{-\frac{M^2(t-\frac{z}{6})}{c2 H^2} + c2(t+\frac{z}{6})}, \\ \text{YZ8}[z, t] &\rightarrow -\frac{e^{-\frac{M^2(t-\frac{z}{6})}{c3 H^2} + c3(t+\frac{z}{6})} (C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 - c31 c32 M^2 + c33 c34 M^2)}{2 C3 H M}, \\ \text{YZ9}[z, t] &\rightarrow -\frac{e^{-\frac{M^2(t-\frac{z}{6})}{c3 H^2} + c3(t+\frac{z}{6})} (C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 + c31 c32 M^2 - c33 c34 M^2)}{2 C3 H M}, \\ \text{YZ10}[z, t] &\rightarrow c33 c34 e^{-\frac{M^2(t-\frac{z}{6})}{c3 H^2} + c3(t+\frac{z}{6})}, \text{YZ11}[z, t] \rightarrow c31 c32 e^{-\frac{M^2(t-\frac{z}{6})}{c3 H^2} + c3(t+\frac{z}{6})}, \\ \text{YZ12}[z, t] &\rightarrow -\frac{e^{-\frac{M^2(t-\frac{z}{6})}{c4 H^2} + c4(t+\frac{z}{6})} (C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 - c41 c42 M^2 - c43 c44 M^2)}{2 C4 H M}, \\ \text{YZ13}[z, t] &\rightarrow -\frac{e^{-\frac{M^2(t-\frac{z}{6})}{c4 H^2} + c4(t+\frac{z}{6})} (C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 + c41 c42 M^2 + c43 c44 M^2)}{2 C4 H M}, \\ \text{YZ14}[z, t] &\rightarrow c43 c44 e^{-\frac{M^2(t-\frac{z}{6})}{c4 H^2} + c4(t+\frac{z}{6})}, \text{YZ15}[z, t] \rightarrow c41 c42 e^{-\frac{M^2(t-\frac{z}{6})}{c4 H^2} + c4(t+\frac{z}{6})} \} \end{aligned}$$

```

```
In[721]:= yZ[#] & /@ Range[0, 15]
sYZvar[[All, 2]] /. {z → #1, t → #2};
Table[ToExpression["((" <> ToString[FullForm[%[[j]]]] <> ")& )"], {j, 1, Length[%]}];
ssyZ = Thread[%%% → %]
Out[721]=
{yZ[0], yZ[1], yZ[2], yZ[3], yZ[4], yZ[5], yZ[6], yZ[7],
 yZ[8], yZ[9], yZ[10], yZ[11], yZ[12], yZ[13], yZ[14], yZ[15]}
```

```
Out[724]=

$$\left\{ \text{yz}[0] \rightarrow \left( \frac{1}{C1 H M} \text{Rational}[1, 2] e^{-\frac{M^2(\text{Rational}[-1, 6] \#1 \#2)}{c1 H^2} + c1(\text{Rational}[1, 6] \#1 \#2)} \right) \right.$$

```

$$\begin{aligned}
& \left(C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2 \right) \& \Big), \\
yZ[1] & \rightarrow \left(\frac{1}{C1 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \#1+\#2)} \right. \\
& \quad \left. (C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 + c11 c12 M^2 + c13 c14 M^2) \& \Big), \\
yZ[2] & \rightarrow \left(c13 c14 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \#1+\#2)} \& \Big), \\
yZ[3] & \rightarrow \left(c11 c12 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \#1+\#2)} \& \Big), \\
yZ[4] & \rightarrow \left(\frac{1}{C2 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \#1+\#2)} \right. \\
& \quad \left. (C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 - c21 c22 M^2 + c23 c24 M^2) \& \Big), \\
yZ[5] & \rightarrow \left(\frac{1}{C2 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \#1+\#2)} \right. \\
& \quad \left. (C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 + c21 c22 M^2 - c23 c24 M^2) \& \Big), \\
yZ[6] & \rightarrow \left(c23 c24 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \#1+\#2)} \& \Big), \\
yZ[7] & \rightarrow \left(c21 c22 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \#1+\#2)} \& \Big), \\
yZ[8] & \rightarrow \left(\frac{1}{C3 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \#1+\#2)} \right. \\
& \quad \left. (C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 - c31 c32 M^2 + c33 c34 M^2) \& \Big), \\
yZ[9] & \rightarrow \left(\frac{1}{C3 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \#1+\#2)} \right. \\
& \quad \left. (C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 + c31 c32 M^2 - c33 c34 M^2) \& \Big), \\
yZ[10] & \rightarrow \left(c33 c34 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \#1+\#2)} \& \Big), \\
yZ[11] & \rightarrow \left(c31 c32 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \#1+\#2)} \& \Big), \\
yZ[12] & \rightarrow \left(\frac{1}{C4 H M} \text{Rational}[1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C4 H^2} + C4 (\text{Rational}[1, 6] \#1+\#2)} \right. \\
& \quad \left. (C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 - c41 c42 M^2 - c43 c44 M^2) \& \Big), \\
yZ[13] & \rightarrow \left(\frac{1}{C4 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C4 H^2} + C4 (\text{Rational}[1, 6] \#1+\#2)} \right. \\
& \quad \left. (C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 + c41 c42 M^2 + c43 c44 M^2) \& \Big), \\
yZ[14] & \rightarrow \left(c43 c44 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C4 H^2} + C4 (\text{Rational}[1, 6] \#1+\#2)} \& \Big), \\
yZ[15] & \rightarrow \left(c41 c42 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{C4 H^2} + C4 (\text{Rational}[1, 6] \#1+\#2)} \& \Big) \Big\}
\end{aligned}$$

This page is intentionally left blank.

CHECK MAPLE SOLUTIONS:

In[725]:=

```
FullSimplify[coupledYZeqs[[#]] /. ssyZ] & /@ Range[4]
```

Out[725]=

```
{ {True, True, True, True}, {True, True, True, True},
{True, True, True, True}, {True, True, True, True} }
```

In[726]:=

```
ssyZ
sZt0yz
sZ = % /. %% // Sort
```

Out[726]=

```
yZ[0] →  $\left( \frac{1}{C1 H M} \text{Rational}[1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \right.$   

 $\left. (C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2) \& \right),$   

yZ[1] →  $\left( \frac{1}{C1 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \right.$   

 $\left. (C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 + c11 c12 M^2 + c13 c14 M^2) \& \right),$   

yZ[2] →  $\left( c13 c14 e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \& \right),$   

yZ[3] →  $\left( c11 c12 e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \& \right),$   

yZ[4] →  $\left( \frac{1}{C2 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \right.$   

 $\left. (C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 - c21 c22 M^2 + c23 c24 M^2) \& \right),$   

yZ[5] →  $\left( \frac{1}{C2 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \right.$   

 $\left. (C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 + c21 c22 M^2 - c23 c24 M^2) \& \right),$   

yZ[6] →  $\left( c23 c24 e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \& \right),$   

yZ[7] →  $\left( c21 c22 e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \& \right),$   

yZ[8] →  $\left( \frac{1}{C3 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \right.$   

 $\left. (C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 - c31 c32 M^2 + c33 c34 M^2) \& \right),$   

yZ[9] →  $\left( \frac{1}{C3 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \right.$   

 $\left. (C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 + c31 c32 M^2 - c33 c34 M^2) \& \right),$   

yZ[10] →  $\left( c33 c34 e^{-\frac{M^2 (\text{Rational}[-1, 6] \oplus 1 \oplus 2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \oplus 1 \oplus 2)} \& \right),$ 
```

$$\begin{aligned}
yZ[11] &\rightarrow \left(c31 c32 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c3 H^2}} + C3 (\text{Rational}[1, 6] \#1+\#2) \& \right), \\
yZ[12] &\rightarrow \left(\frac{1}{C4 H M} \text{Rational}[1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c4 H^2}} + C4 (\text{Rational}[1, 6] \#1+\#2) \right. \\
&\quad \left. (C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 - c41 c42 M^2 - c43 c44 M^2) \& \right), \\
yZ[13] &\rightarrow \left(\frac{1}{C4 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c4 H^2}} + C4 (\text{Rational}[1, 6] \#1+\#2) \right. \\
&\quad \left. (C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 + c41 c42 M^2 + c43 c44 M^2) \& \right), \\
yZ[14] &\rightarrow \left(c43 c44 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c4 H^2}} + C4 (\text{Rational}[1, 6] \#1+\#2) \& \right), \\
yZ[15] &\rightarrow \left(c41 c42 e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c4 H^2}} + C4 (\text{Rational}[1, 6] \#1+\#2) \& \right) \}
\end{aligned}$$

Out[727]=
{Z[0] → yZ[0], Z[5] → yZ[1], Z[8] → yZ[2], Z[13] → yZ[3], Z[1] → yZ[4], Z[4] → yZ[5],
Z[9] → yZ[6], Z[12] → yZ[7], Z[2] → yZ[8], Z[7] → yZ[9], Z[10] → yZ[10],
Z[15] → yZ[11], Z[3] → yZ[12], Z[6] → yZ[13], Z[11] → yZ[14], Z[14] → yZ[15]}

Out[728]=
{Z[0] → $\left(\frac{1}{C1 H M} \text{Rational}[1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c1 H^2}} + C1 (\text{Rational}[1, 6] \#1+\#2) \right. \\ \left. (C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2) \& \right)$,
Z[1] → $\left(\frac{1}{C2 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c2 H^2}} + C2 (\text{Rational}[1, 6] \#1+\#2) \right. \\ \left. (C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 - c21 c22 M^2 + c23 c24 M^2) \& \right)$,
Z[2] → $\left(\frac{1}{C3 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c3 H^2}} + C3 (\text{Rational}[1, 6] \#1+\#2) \right. \\ \left. (C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 - c31 c32 M^2 + c33 c34 M^2) \& \right)$,
Z[3] → $\left(\frac{1}{C4 H M} \text{Rational}[1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c4 H^2}} + C4 (\text{Rational}[1, 6] \#1+\#2) \right. \\ \left. (C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 - c41 c42 M^2 - c43 c44 M^2) \& \right)$,
Z[4] → $\left(\frac{1}{C2 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c2 H^2}} + C2 (\text{Rational}[1, 6] \#1+\#2) \right. \\ \left. (C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 + c21 c22 M^2 - c23 c24 M^2) \& \right)$,
Z[5] → $\left(\frac{1}{C1 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c1 H^2}} + C1 (\text{Rational}[1, 6] \#1+\#2) \right. \\ \left. (C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 + c11 c12 M^2 + c13 c14 M^2) \& \right)$,
Z[6] → $\left(\frac{1}{C4 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \#1+\#2)}{c4 H^2}} + C4 (\text{Rational}[1, 6] \#1+\#2) \right. \\ \left. (C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 + c41 c42 M^2 + c43 c44 M^2) \& \right)$,

$$\begin{aligned}
Z[7] &\rightarrow \left(\frac{1}{C3 H M} \text{Rational}[-1, 2] e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \# 1 \# 2)} \right. \\
&\quad \left. (C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 + c31 c32 M^2 - c33 c34 M^2) \& \right), \\
Z[8] &\rightarrow \left(c13 c14 e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \# 1 \# 2)} \& \right), \\
Z[9] &\rightarrow \left(c23 c24 e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \# 1 \# 2)} \& \right), \\
Z[10] &\rightarrow \left(c33 c34 e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \# 1 \# 2)} \& \right), \\
Z[11] &\rightarrow \left(c43 c44 e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C4 H^2} + C4 (\text{Rational}[1, 6] \# 1 \# 2)} \& \right), \\
Z[12] &\rightarrow \left(c21 c22 e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C2 H^2} + C2 (\text{Rational}[1, 6] \# 1 \# 2)} \& \right), \\
Z[13] &\rightarrow \left(c11 c12 e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C1 H^2} + C1 (\text{Rational}[1, 6] \# 1 \# 2)} \& \right), \\
Z[14] &\rightarrow \left(c41 c42 e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C4 H^2} + C4 (\text{Rational}[1, 6] \# 1 \# 2)} \& \right), \\
Z[15] &\rightarrow \left(c31 c32 e^{-\frac{M^2 (\text{Rational}[-1, 6] \# 1 \# 2)}{C3 H^2} + C3 (\text{Rational}[1, 6] \# 1 \# 2)} \& \right) \}
\end{aligned}$$

In[729]:=

```
(¶16a = ¶16 /. sf¶16Aa /. sx0x4 /. sz /. szt // FullSimplify) // Column
```

```

Out[729]:= 

$$\begin{aligned}
& \frac{\frac{H^2(x_0-x_4)}{C1 H} + C1 H (x_0+x_4)}{2 C1 H M} (C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2) \\
& - \frac{\frac{H^2(x_0-x_4)}{C2 H} + C2 H (x_0+x_4)}{2 C2 H M} (C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2) \\
& - \frac{\frac{H^2(x_0-x_4)}{C3 H} + C3 H (x_0+x_4)}{2 C3 H M} (C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2) \\
& - \frac{\frac{H^2(x_0-x_4)}{C4 H} + C4 H (x_0+x_4)}{2 C4 H M} (C4^2 (c41 c42 - c43 c44) H^2 - (c41 c42 + c43 c44) M^2) \\
& - \frac{\frac{H^2(x_0-x_4)}{C2 H} + C2 H (x_0+x_4)}{2 C2 H M} (C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2) \\
& - \frac{\frac{H^2(x_0-x_4)}{C1 H} + C1 H (x_0+x_4)}{2 C1 H M} (C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2) \\
& - \frac{\frac{H^2(x_0-x_4)}{C4 H} + C4 H (x_0+x_4)}{2 C4 H M} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \\
& - \frac{\frac{H^2(x_0-x_4)}{C3 H} + C3 H (x_0+x_4)}{2 C3 H M} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2) \\
c13 c14 & \in \frac{H^2(x_0-x_4)}{C1 H} + C1 H (x_0+x_4) \\
c23 c24 & \in \frac{H^2(x_0-x_4)}{C2 H} + C2 H (x_0+x_4) \\
c33 c34 & \in \frac{H^2(x_0-x_4)}{C3 H} + C3 H (x_0+x_4) \\
c43 c44 & \in \frac{H^2(x_0-x_4)}{C4 H} + C4 H (x_0+x_4) \\
c21 c22 & \in \frac{H^2(x_0-x_4)}{C2 H} + C2 H (x_0+x_4) \\
c11 c12 & \in \frac{H^2(x_0-x_4)}{C1 H} + C1 H (x_0+x_4) \\
c41 c42 & \in \frac{H^2(x_0-x_4)}{C4 H} + C4 H (x_0+x_4) \\
c31 c32 & \in \frac{H^2(x_0-x_4)}{C3 H} + C3 H (x_0+x_4)
\end{aligned}$$


```

```

In[851]:= 
ξadv == x0 - x4 && ξret == x0 + x4
sξ = Solve[%, {ξadv, ξret}] [[1]]
ssx0x4 = Solve[%%, {x0, x4}] [[1]]

```

```

Out[851]= 
ξadv == x0 - x4 && ξret == x0 + x4

Out[852]= 
{ξadv → x0 - x4, ξret → x0 + x4}

Out[853]= 
{x0 →  $\frac{\xi_{\text{adv}} + \xi_{\text{ret}}}{2}$ , x4 →  $\frac{1}{2}(-\xi_{\text{adv}} + \xi_{\text{ret}})$ }

```

```

In[854]:= 
(*Clear[ssx0x4]; ssx0x4={}*)

```

$\psi1 = \Psi16a[[1;;8]] // FullSimplify; \psi2 = \Psi16a[[8+1;;8+8]] // FullSimplify$

Out[734]=

$$\left\{ c13 c14 \in \frac{\frac{H^2 (x0-x4)}{C1 H} + C1 H (x0+x4)}{C1 H}, c23 c24 \in \frac{\frac{H^2 (x0-x4)}{C2 H} + C2 H (x0+x4)}{C2 H}, \right.$$

$$c33 c34 \in \frac{\frac{H^2 (x0-x4)}{C3 H} + C3 H (x0+x4)}{C3 H}, c43 c44 \in \frac{\frac{H^2 (x0-x4)}{C4 H} + C4 H (x0+x4)}{C4 H}, c21 c22 \in \frac{\frac{H^2 (x0-x4)}{C2 H} + C2 H (x0+x4)}{C2 H},$$

$$\left. c11 c12 \in \frac{\frac{H^2 (x0-x4)}{C1 H} + C1 H (x0+x4)}{C1 H}, c41 c42 \in \frac{\frac{H^2 (x0-x4)}{C4 H} + C4 H (x0+x4)}{C4 H}, c31 c32 \in \frac{\frac{H^2 (x0-x4)}{C3 H} + C3 H (x0+x4)}{C3 H} \right\}$$

In[735]:=

type1DIVBYtype2 = $\Psi16a[[1;;8]] / \Psi16a[[8+1;;8+8]] // FullSimplify$

Out[735]=

$$\left\{ \begin{aligned} & \frac{C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2}{2 C1 c13 c14 H M}, \\ & - \frac{C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2}{2 C2 c23 c24 H M}, \\ & - \frac{C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2}{2 C3 c33 c34 H M}, \\ & \frac{C4^2 (c41 c42 - c43 c44) H^2 - (c41 c42 + c43 c44) M^2}{2 C4 c43 c44 H M}, \\ & - \frac{C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2}{2 C2 c21 c22 H M}, \\ & - \frac{C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2}{2 C1 c11 c12 H M}, \\ & - \frac{C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2}{2 C4 c41 c42 H M}, \\ & - \frac{C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2}{2 C3 c31 c32 H M} \end{aligned} \right\}$$

In[736]:=

realPartOfType2 = unit. $\psi2 // FullSimplify$

Out[736]=

$$\frac{c13 c14 \in \frac{\frac{M^2 (x0-x4)}{C1 H} + C1 H (x0+x4)}{C1 H} + c21 c22 \in \frac{\frac{M^2 (x0-x4)}{C2 H} + C2 H (x0+x4)}{C2 H}}{\sqrt{2}}$$

In[737]:=

realPartOfType1 = unit. $\psi1 // FullSimplify$

Out[737]=

$$\frac{\frac{H^2 (x0-x4)}{C1 H} + C1 H (x0+x4) (C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2)}{C1} - \frac{\frac{H^2 (x0-x4)}{C2 H} + C2 H (x0+x4) (C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2)}{C2}$$

$$2 \sqrt{2} H M$$

$$\mathfrak{F}_A^a = \frac{1}{\sqrt{\tilde{u}\sigma u}} \tau_A^a{}_b u^b \quad (63)$$

$$\mathfrak{F}_a^A = \frac{1}{\sqrt{\tilde{u}\sigma u}} u^c \sigma_{cb} \bar{\tau}^{Ab}{}_a \quad (64)$$

In[738]:=

$$F_a^A \cdot F_A^a == F_A^a \cdot F_a^A == \text{ID8}$$

Out[738]=

True

In[739]:=

$$(\text{vectorFormOfType2} = F_a^A \cdot \psi^2 // \text{FullSimplify}) // \text{Column}$$

Out[739]=

$$\begin{aligned} & c13 c14 \in \frac{\frac{H^2 (x0-x4)}{c1 H} + c1 H (x0+x4)}{\sqrt{2}} + c21 c22 \in \frac{\frac{H^2 (x0-x4)}{c2 H} + c2 H (x0+x4)}{\sqrt{2}} \\ & - c31 c32 \in \frac{\frac{H^2 (x0-x4)}{c3 H} + c3 H (x0+x4)}{\sqrt{2}} + c43 c44 \in \frac{\frac{H^2 (x0-x4)}{c4 H} + c4 H (x0+x4)}{\sqrt{2}} \\ & c33 c34 \in \frac{\frac{H^2 (x0-x4)}{c3 H} + c3 H (x0+x4)}{\sqrt{2}} + c41 c42 \in \frac{\frac{H^2 (x0-x4)}{c4 H} + c4 H (x0+x4)}{\sqrt{2}} \\ & - c11 c12 \in \frac{\frac{H^2 (x0-x4)}{c1 H} + c1 H (x0+x4)}{\sqrt{2}} + c23 c24 \in \frac{\frac{H^2 (x0-x4)}{c2 H} + c2 H (x0+x4)}{\sqrt{2}} \\ & - c11 c12 \in \frac{\frac{H^2 (x0-x4)}{c1 H} + c1 H (x0+x4)}{\sqrt{2}} + c23 c24 \in \frac{\frac{H^2 (x0-x4)}{c2 H} + c2 H (x0+x4)}{\sqrt{2}} \\ & - c33 c34 \in \frac{\frac{H^2 (x0-x4)}{c3 H} + c3 H (x0+x4)}{\sqrt{2}} + c41 c42 \in \frac{\frac{H^2 (x0-x4)}{c4 H} + c4 H (x0+x4)}{\sqrt{2}} \\ & c31 c32 \in \frac{\frac{H^2 (x0-x4)}{c3 H} + c3 H (x0+x4)}{\sqrt{2}} - c43 c44 \in \frac{\frac{H^2 (x0-x4)}{c4 H} + c4 H (x0+x4)}{\sqrt{2}} \\ & c13 c14 \in \frac{\frac{H^2 (x0-x4)}{c1 H} + c1 H (x0+x4)}{\sqrt{2}} - c21 c22 \in \frac{\frac{H^2 (x0-x4)}{c2 H} + c2 H (x0+x4)}{\sqrt{2}} \end{aligned}$$

In[740]:=

$$\eta 4488 // \text{MatrixForm}$$

Out[740]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[741]:=

$$\text{vectorFormOfType1} = (\eta4488. \text{Transpose}[\psi1].\sigma.F_A^a) // \text{FullSimplify} // \text{Column}$$

Out[741]=

$$\begin{aligned} & \frac{\frac{H^2(x0-x4)}{C1 H(x0+x4)} (C1^2(c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2)}{C1} - \frac{\frac{H^2(x0-x4)}{C2 H(x0+x4)} (C2^2(c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2)}{C2} \\ & \quad 2 \sqrt{2} H M \\ & \frac{\frac{H^2(x0-x4)}{C3 H(x0+x4)} (C3^2(c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2)}{C3} + \frac{\frac{H^2(x0-x4)}{C4 H(x0+x4)} (C4^2(-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2)}{C4} \\ & \quad 2 \sqrt{2} H M \\ & - \frac{\frac{H^2(x0-x4)}{C3 H(x0+x4)} (C3^2(c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2)}{C3} - \frac{\frac{H^2(x0-x4)}{C4 H(x0+x4)} (C4^2(c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2)}{C4} \\ & \quad 2 \sqrt{2} H M \\ & \frac{\frac{H^2(x0-x4)}{C1 H(x0+x4)} (C1^2(c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2)}{C1} + \frac{\frac{H^2(x0-x4)}{C2 H(x0+x4)} (C2^2(c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2)}{C2} \\ & \quad 2 \sqrt{2} H M \\ & \frac{\frac{H^2(x0-x4)}{C1 H(x0+x4)} (C1^2(c11 c12 - c13 c14) H^2 + (c11 c12 - c13 c14) M^2)}{C1} - \frac{\frac{H^2(x0-x4)}{C2 H(x0+x4)} (C2^2(c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2)}{C2} \\ & \quad 2 \sqrt{2} H M \\ & \frac{\frac{H^2(x0-x4)}{C3 H(x0+x4)} (C3^2(c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2)}{C3} - \frac{\frac{H^2(x0-x4)}{C4 H(x0+x4)} (C4^2(c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2)}{C4} \\ & \quad 2 \sqrt{2} H M \\ & - \frac{\frac{H^2(x0-x4)}{C3 H(x0+x4)} (C3^2(c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2)}{C3} + \frac{\frac{H^2(x0-x4)}{C4 H(x0+x4)} (C4^2(-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2)}{C4} \\ & \quad 2 \sqrt{2} H M \end{aligned}$$

7 - plane example :

In[744]:=

$$\text{Solve}[0 == \text{vectorFormOfType1.EA}, \text{EA}] // \text{FullSimplify}$$

Solve: Equations may not give solutions for all "solve" variables. [?](#)

Out[744]=

$$\begin{aligned} & \left\{ \left\{ \text{EA}[8] \rightarrow \left((c11 c12 + c13 c14) C2 C3 C4 e^{\frac{H^2(x0-x4)}{C1 H(x0+x4)}} M^2 (\text{EA}[1] - \text{EA}[4] - \text{EA}[5]) - \right. \right. \right. \\ & \quad C1^2 (c11 c12 - c13 c14) C2 C3 C4 e^{\frac{H^2(x0-x4)}{C1 H(x0+x4)}} H^2 (\text{EA}[1] + \text{EA}[4] + \text{EA}[5]) + \\ & \quad C1 \left((c21 c22 - c23 c24) C3 C4 e^{\frac{H^2(x0-x4)}{C2 H(x0+x4)}} M^2 (\text{EA}[1] + \text{EA}[4] - \text{EA}[5]) + \right. \\ & \quad C2^2 (c21 c22 + c23 c24) C3 C4 e^{\frac{H^2(x0-x4)}{C2 H(x0+x4)}} H^2 (\text{EA}[1] - \text{EA}[4] + \text{EA}[5]) + \\ & \quad C2 \left(- \left((c31 c32 - c33 c34) C4 e^{\frac{H^2(x0-x4)}{C3 H(x0+x4)}} M^2 (\text{EA}[2] + \text{EA}[3] - \text{EA}[6] - \text{EA}[7]) \right) - \right. \\ & \quad C3^2 (c31 c32 + c33 c34) C4 e^{\frac{H^2(x0-x4)}{C3 H(x0+x4)}} H^2 (\text{EA}[2] - \text{EA}[3] + \text{EA}[6] - \text{EA}[7]) + \\ & \quad C3 e^{\frac{H^2(x0-x4)}{C4 H(x0+x4)}} ((c41 c42 + c43 c44) M^2 (-\text{EA}[2] + \text{EA}[3] + \text{EA}[6] - \text{EA}[7]) + C4^2 \\ & \quad \left. \left. \left. (c41 c42 - c43 c44) H^2 (\text{EA}[2] + \text{EA}[3] + \text{EA}[6] + \text{EA}[7]) \right) \right) \right) \Bigg) \Bigg) \\ & \quad \left(C3 C4 \left(C2 e^{\frac{H^2(x0-x4)}{C1 H(x0+x4)}} (C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2) + \right. \right. \\ & \quad \left. \left. C1 e^{\frac{H^2(x0-x4)}{C2 H(x0+x4)}} (C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2) \right) \right) \Bigg) \Bigg\} \end{aligned}$$

6 - plane = 7 - plane \cap 7 - plane example :

```
In[991]:= EAa [hUSE].unit
%.EA
% === FAa.unit.EA
Out[991]= {1, 0, 0, 0, 0, 0, 0, 0}
Out[992]= eA[1]
Out[993]= True

In[842]:= Binomial[8, 2]
Out[842]= 28

vectorFormOfType1 is dual to 7-plane 71 ;
vectorFormOfType2 is dual to 7-plane 72 ;
Assuming that 71 and 72 are not parallel,
M6 = 71 ∩ 72
M6 has 28 parameterizations; R = linear span of eA[1]; skip R and start with {eA[6],
eA[7]} = ...

In[855]:= {sξ, ssx0x4}
Out[855]= {ξadv → x0 - x4, ξret → x0 + x4}, {x0 → (ξadv + ξret)/2, x4 → 1/2 (-ξadv + ξret) }

In[856]:= time1 = Now
timeA = AbsoluteTiming[M6 = Solve[
And @@ Thread[θ == Flatten[{vectorFormOfType1.EA, vectorFormOfType2.EA}]],
{eA[6], eA[7]}] /. ssx0x4 // (*Full*)Simplify];
Print[timeA, "; ", Now - time1];

Out[856]= Fri 20 Feb 2026 08:46:25 GMT-8
{17.6559, {eA[6] → -((c31 c32 e(M2 ξadv + C3 H ξret)/C3 H) - c43 c44 e(M2 ξadv + C4 H ξret)/C4 H), eA[1] + 
C3 C4 (C2 e(M2 ξadv + C1 H ξret)/C1 H) (C12 (c11 c12 - c13 c14) H2 - (c11 c12 + c13 c14) M2) - 
C1 e(M2 ξadv + C2 H ξret)/C2 H (C22 (c21 c22 + c23 c24) H2 + (c21 c22 - c23 c24) M2)}, eA[2] - 
C1 C2 (C4 e(M2 ξadv + C3 H ξret)/C3 H) (C32 (c31 c32 + c33 c34) H2 + (c31 c32 - c33 c34) M2) + 
C3 e(M2 ξadv + C4 H ξret)/C4 H (C42 (-c41 c42 + c43 c44) H2 + (c41 c42 + c43 c44) M2)}}
```

$$\begin{aligned}
& C1 \cdot C2 \left(C4 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} \left(C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2 \right) + \right. \\
& \quad \left. C3 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \left(C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2 \right) \right) eA[3] + \\
& C3 \cdot C4 \left(C2 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \left(C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2 \right) + \right. \\
& \quad \left. C1 \cdot e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \left(C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2 \right) \right) eA[4] + \\
& C3 \cdot C4 \left(C2 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \left(C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2 \right) - \right. \\
& \quad \left. C1 \cdot e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \left(C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2 \right) \right) eA[5] + \\
& C3 \cdot C4 \left(C2 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \left(C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2 \right) + \right. \\
& \quad \left. C1 \cdot e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \left(C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2 \right) \right) eA[8] + \\
& C1 \cdot C2 \left(C4 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} \left(C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2 \right) - \right. \\
& \quad \left. C3 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \left(C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2 \right) \right) \\
& \quad \left(-c31 c32 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] - c43 c44 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] + c33 c34 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[3] + \right. \\
& \quad c41 c42 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[3] - c11 c12 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[4] - \\
& \quad c23 c24 \cdot e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[4] - c11 c12 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[5] + c23 c24 \cdot e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] + \\
& \quad c21 c22 \cdot e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (eA[1] - eA[8]) + c13 c14 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (eA[1] + eA[8]) \Big) \Big) / \\
& \quad \left(C1 \cdot C2 \left(\left(c31 c32 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \left(C4 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} \right. \right. \right. \\
& \quad \left. \left. \left. \left(C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2 \right) - C3 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right. \right. \right. \\
& \quad \left. \left. \left. \left(C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2 \right) \right) - \left(c33 c34 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c41 c42 \right. \right. \\
& \quad \left. \left. \left. \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \left(C4 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} \left(C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2 \right) - C3 \right. \right. \\
& \quad \left. \left. \left. \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \left(C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2 \right) \right) \right) \right), \\
& eA[7] \rightarrow \left((c11 c12 + c13 c14) \cdot C2 \cdot C3 \cdot C4 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \left(c33 c34 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c41 c42 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \right. \\
& \quad M^2 (eA[1] - eA[4] - eA[5] + eA[8]) - \\
& C1^2 (c11 c12 - c13 c14) \cdot C2 \cdot C3 \cdot C4 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \\
& \quad \left(c33 c34 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c41 c42 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \\
& \quad H^2 (eA[1] + eA[4] + eA[5] + eA[8]) + \\
& C1 \left(C2 \left(- \left((c31 c32 - c33 c34) \cdot C4 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} M^2 \left(c31 c32 \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] + c33 c34 \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \cdot e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] - c41 c42 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] + c43 c44 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] - \right. \right. \right. \right. \\
& \quad 2 c41 c42 \cdot e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[3] + c11 c12 \cdot e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[4] + c23 c24 \cdot e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] \right) \Big) +
\end{aligned}$$

$$\begin{aligned}
& C3^2 (c31 c32 + c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} H^2 \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] - \right. \\
& c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] + c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] + c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] - \\
& 2 c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[3] + c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[4] + c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[4] + \\
& \left. c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[5] - c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] \right) - C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \\
& \left(C4^2 (c41 c42 - c43 c44) H^2 \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] + c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] + \right. \right. \\
& c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] - c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (eA[2] + 2 eA[3]) + c11 c12 \\
& e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[4] + c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[4] + c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[5] - \\
& \left. \left. c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] \right) - (c41 c42 + c43 c44) M^2 \left(-c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] + \right. \right. \\
& c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] - c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} \\
& (eA[2] - 2 eA[3]) - c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[4] - c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[4] - \\
& \left. \left. c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[5] + c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] \right) \right) - c21 c22 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \\
& \left(C3^2 (c31 c32 + c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} H^2 - (c31 c32 - c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} M^2 - \right. \\
& C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \left. \right) (eA[1] - eA[8]) + \\
& (c21 c22 - c23 c24) C3 C4 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \left(c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \\
& M^2 (eA[1] + eA[4] - eA[5] - eA[8]) + \\
& C2^2 (c21 c22 + c23 c24) C3 C4 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \left(c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \\
& H^2 (eA[1] - eA[4] + eA[5] - eA[8]) - c13 c14 C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \\
& \left(C3^2 (c31 c32 + c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} H^2 - (c31 c32 - c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} M^2 - \right. \\
& \left. \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) (eA[1] + eA[8]) \right) / \\
& \left(C1 C2 \left(C3^2 (c31 c32 + c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} + \right. \right. \right. \right. \\
& \left. \left. \left. \left. (c41 c42 - c43 c44) e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) H^2 - (c31 c32 - c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} \right. \right. \\
& \left. \left. \left. \left. (c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} + c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - (c41 c42 + c43 c44) e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}}) M^2 + \right. \right. \right. \\
& \left. \left. \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \left((c41^2 c42^2 - c43^2 c44^2) e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 H^2 - M^2) - \right. \right. \right. \\
& \left. \left. \left. c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) + c33 c34 \right. \right. \right. \\
& \left. \left. \left. e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) \right) \right) \right) \right) \right); 17.6600852 s
\end{aligned}$$

R = linear span of eA[1]; try solving for eA[1] and eA[7]

In[859]:=

```

time1 = Now
timeA = AbsoluteTiming[M617 = Solve[
    And @@ Thread[θ == Flatten[{vectorFormOfType1.EA, vectorFormOfType2.EA}]],
    {eA[1], eA[7]}] /. ssx0x4 // (*Full*)Simplify];
Print[timeA, "; ", Now - time1];

```

Out[859]=

Fri 20 Feb 2026 08:46:47 GMT-8

$$\begin{aligned}
& \left\{ 15.8984, \left\{ \left\{ eA[1] \rightarrow \right. \right. \right. \\
& \left. \left. \left. - \left(\left(-C1 C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2) - C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right. \right. \right. \right. \right. \\
& \left. \left. \left. \left. \left. \left. (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) \right) \right. \right. \right. \\
& \left. \left. \left. \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] + c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] - c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[3] - \right. \right. \right. \\
& \left. \left. \left. c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[3] + c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[4] + c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[4] + \right. \right. \right. \\
& \left. \left. \left. c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[5] - c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] + c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[6] - \right. \right. \right. \\
& \left. \left. \left. c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[6] - c13 c14 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[8] + c21 c22 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[8] \right) + \right. \right. \right. \\
& \left. \left. \left. \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \right. \right. \right. \\
& \left. \left. \left. \left(C1 C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2) + \right. \right. \right. \right. \\
& \left. \left. \left. \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) eA[2] - \right. \right. \right. \\
& \left. \left. \left. C1 C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2) + \right. \right. \right. \\
& \left. \left. \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) eA[3] + \right. \right. \right. \\
& \left. \left. \left. C3 C4 \left(C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2) + \right. \right. \right. \\
& \left. \left. \left. C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2) \right) eA[4] + \right. \right. \right. \\
& \left. \left. \left. C3 C4 \left(C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2) - \right. \right. \right. \\
& \left. \left. \left. C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2) \right) eA[5] + \right. \right. \right. \\
& \left. \left. \left. C1 C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2) - \right. \right. \right. \\
& \left. \left. \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) eA[6] + \right. \right. \right. \\
& \left. \left. \left. C3 C4 \left(C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2) + \right. \right. \right. \\
& \left. \left. \left. C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2) \right) eA[8] \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
 & \left(C1 \, C2 \, C3 \, C4 \left(\left(c31 \, c32 \, e^{\frac{M^2 \xi_{adv}}{C3H} + C3H \xi_{ret}} - c43 \, c44 \, e^{\frac{M^2 \xi_{adv}}{C4H} + C4H \xi_{ret}} \right) \right. \right. \\
 & \left. \left. - \frac{e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} (C1^2 (c11 \, c12 - c13 \, c14) H^2 - (c11 \, c12 + c13 \, c14) M^2)}{C1} - \right. \right. \\
 & \left. \left. \frac{e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} (C2^2 (c21 \, c22 + c23 \, c24) H^2 + (c21 \, c22 - c23 \, c24) M^2)}{C2} \right) - \right. \\
 & \left. \left(c13 \, c14 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} + c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} \right) \right. \\
 & \left. \left. - \frac{e^{\frac{M^2 \xi_{adv}}{C3H} + C3H \xi_{ret}} (C3^2 (c31 \, c32 + c33 \, c34) H^2 + (c31 \, c32 - c33 \, c34) M^2)}{C3} + \right. \right. \\
 & \left. \left. \frac{e^{\frac{M^2 \xi_{adv}}{C4H} + C4H \xi_{ret}} (C4^2 (-c41 \, c42 + c43 \, c44) H^2 + (c41 \, c42 + c43 \, c44) M^2)}{C4} \right) \right) \right), \\
 eA[7] \rightarrow & \left(- \left((c11 \, c12 + c13 \, c14) \, C2 \, C3 \, C4 \, M^2 \left(c31 \, c32 \, e^{\frac{(C1+C3) (M^2 \xi_{adv} + C1C3 H^2 \xi_{ret})}{C1C3H}} eA[2] + \right. \right. \right. \right. \\
 & \left. \left. \left. \left. c43 \, c44 \, e^{\frac{(C1+C4) (M^2 \xi_{adv} - C1C4 H^2 \xi_{ret})}{C1C4H}} eA[2] - e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} (-c11 \, c12 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[4] + \right. \right. \right. \\
 & \left. \left. \left. \left. c13 \, c14 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[4] + c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[4] - c23 \, c24 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[4] - \right. \right. \right. \\
 & \left. \left. \left. \left. c11 \, c12 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[5] + c13 \, c14 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[5] + c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[5] + \right. \right. \right. \\
 & \left. \left. \left. \left. c23 \, c24 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[5] + c33 \, c34 \, e^{\frac{M^2 \xi_{adv}}{C3H} + C3H \xi_{ret}} (eA[3] - eA[6]) + \right. \right. \right. \\
 & \left. \left. \left. \left. c41 \, c42 \, e^{\frac{M^2 \xi_{adv}}{C4H} + C4H \xi_{ret}} (eA[3] + eA[6]) - 2 \, c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[8] \right) \right) \right) + \right. \\
 & \left. C1^2 (c11 \, c12 - c13 \, c14) \, C2 \, C3 \, C4 \, H^2 \left(c31 \, c32 \, e^{\frac{(C1+C3) (M^2 \xi_{adv} + C1C3 H^2 \xi_{ret})}{C1C3H}} eA[2] + \right. \right. \\
 & \left. \left. c43 \, c44 \, e^{\frac{(C1+C4) (M^2 \xi_{adv} - C1C4 H^2 \xi_{ret})}{C1C4H}} eA[2] + e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} \right. \right. \\
 & \left. \left. \left(c11 \, c12 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[4] + c13 \, c14 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[4] + c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[4] + \right. \right. \right. \\
 & \left. \left. \left. c23 \, c24 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[4] + c11 \, c12 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[5] + c13 \, c14 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[5] + \right. \right. \right. \\
 & \left. \left. \left. c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[5] - c23 \, c24 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[5] - c33 \, c34 \, e^{\frac{M^2 \xi_{adv}}{C3H} + C3H \xi_{ret}} \right. \right. \right. \\
 & \left. \left. \left. (eA[3] - eA[6]) - c41 \, c42 \, e^{\frac{M^2 \xi_{adv}}{C4H} + C4H \xi_{ret}} (eA[3] + eA[6]) + 2 \, c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[8] \right) \right) + \right. \\
 & \left. C1 \left(- \left((c21 \, c22 - c23 \, c24) \, C3 \, C4 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} M^2 \left(c31 \, c32 \, e^{\frac{M^2 \xi_{adv}}{C3H} + C3H \xi_{ret}} eA[2] + \right. \right. \right. \right. \\
 & \left. \left. \left. c43 \, c44 \, e^{\frac{M^2 \xi_{adv}}{C4H} + C4H \xi_{ret}} eA[2] - c33 \, c34 \, e^{\frac{M^2 \xi_{adv}}{C3H} + C3H \xi_{ret}} eA[3] - c41 \, c42 \, e^{\frac{M^2 \xi_{adv}}{C4H} + C4H \xi_{ret}} eA[3] + \right. \right. \right. \\
 & \left. \left. \left. c11 \, c12 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[4] + c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[4] + c23 \, c24 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[4] + \right. \right. \right. \\
 & \left. \left. \left. c11 \, c12 \, e^{\frac{M^2 \xi_{adv}}{C1H} + C1H \xi_{ret}} eA[5] - c21 \, c22 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[5] - c23 \, c24 \, e^{\frac{M^2 \xi_{adv}}{C2H} + C2H \xi_{ret}} eA[5] + \right. \right. \right. \\
 & \left. \left. \left. c33 \, c34 \, e^{\frac{M^2 \xi_{adv}}{C3H} + C3H \xi_{ret}} eA[6] - c41 \, c42 \, e^{\frac{M^2 \xi_{adv}}{C4H} + C4H \xi_{ret}} eA[6] \right) \right) - \right.
 \end{aligned}$$

$$\begin{aligned}
& C2^2 (c21 c22 + c23 c24) C3 C4 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} H^2 \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] + \right. \\
& \quad c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] - c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[3] - c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[3] + \\
& \quad c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[4] - c21 c22 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[4] + c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[4] + \\
& \quad c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[5] + c21 c22 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] - c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] + \\
& \quad \left. c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[6] - c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[6] \right) + \\
& C2 c21 c22 \left((c31 c32 - c33 c34) C4 e^{\frac{(C2 \cdot C3) (M^2 \xi_{adv} \cdot C2 C3 H^2 \xi_{ret})}{C2 C3 H}} M^2 (eA[2] + eA[3] - eA[6]) + \right. \\
& \quad C3^2 (c31 c32 + c33 c34) C4 e^{\frac{(C2 \cdot C3) (M^2 \xi_{adv} \cdot C2 C3 H^2 \xi_{ret})}{C2 C3 H}} H^2 (eA[2] - eA[3] + eA[6]) - \\
& \quad C3 e^{\frac{(C2 \cdot C4) (M^2 \xi_{adv} \cdot C2 C4 H^2 \xi_{ret})}{C2 C4 H}} \left(- ((c41 c42 + c43 c44) M^2 (eA[2] - eA[3] - eA[6])) + \right. \\
& \quad \left. \left. C4^2 (c41 c42 - c43 c44) H^2 (eA[2] + eA[3] + eA[6]) \right) + \right. \\
& \quad c13 c14 \left(C2 \left((c31 c32 - c33 c34) C4 e^{\frac{(C1 \cdot C3) (M^2 \xi_{adv} \cdot C1 C3 H^2 \xi_{ret})}{C1 C3 H}} M^2 (eA[2] + eA[3] - eA[6]) + \right. \right. \\
& \quad C3^2 (c31 c32 + c33 c34) C4 e^{\frac{(C1 \cdot C3) (M^2 \xi_{adv} \cdot C1 C3 H^2 \xi_{ret})}{C1 C3 H}} H^2 (eA[2] - eA[3] + eA[6]) - \\
& \quad C3 e^{\frac{(C1 \cdot C4) (M^2 \xi_{adv} \cdot C1 C4 H^2 \xi_{ret})}{C1 C4 H}} \left(- ((c41 c42 + c43 c44) M^2 (eA[2] - eA[3] - eA[6])) + \right. \\
& \quad \left. \left. C4^2 (c41 c42 - c43 c44) H^2 (eA[2] + eA[3] + eA[6]) \right) - \right. \\
& \quad (c21 c22 - c23 c24) C3 C4 e^{\frac{(C1 \cdot C2) (M^2 \xi_{adv} \cdot C1 C2 H^2 \xi_{ret})}{C1 C2 H}} M^2 (eA[4] - eA[5] - 2 eA[8]) + \\
& \quad C2^2 (c21 c22 + c23 c24) C3 C4 e^{\frac{(C1 \cdot C2) (M^2 \xi_{adv} \cdot C1 C2 H^2 \xi_{ret})}{C1 C2 H}} H^2 (eA[4] - eA[5] + 2 eA[8]) \left) \right) \Bigg) / \\
& \quad \left(C1^2 (c11 c12 - c13 c14) C2 C3 C4 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) H^2 - \right. \\
& \quad (c11 c12 + c13 c14) C2 C3 C4 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \\
& \quad \left. \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \right. \\
& \quad M^2 + C1 c13 c14 C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} \\
& \quad \left. \left(C3^2 (c31 c32 + c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} H^2 + (c31 c32 - c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} M^2 + \right. \right. \\
& \quad C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \left(C4^2 (c41 c42 - c43 c44) H^2 - (c41 c42 + c43 c44) M^2 \right) + \\
& \quad C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \left(-C2^2 (c21 c22 + c23 c24) C3 C4 \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) H^2 - \right. \\
& \quad (c21 c22 - c23 c24) C3 C4 \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) M^2 + \\
& \quad C2 c21 c22 \left(C3^2 (c31 c32 + c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} H^2 + (c31 c32 - c33 c34) C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} M^2 + \right. \\
& \quad \left. \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \left(C4^2 (c41 c42 - c43 c44) H^2 - (c41 c42 + c43 c44) M^2 \right) \right) \right) \Bigg) \Bigg) \Bigg); \quad 15.9064204 s
\end{aligned}$$

checks :

```
In[751]:= EAa[hUSE].EaA[hUSE] === EaA[hUSE].EAa[hUSE] === ID8
```

```
Out[751]= True
```

In[752]:= $F_A^a \cdot F_a^A == F_a^A \cdot F_A^a == \text{ID8}$

Out[752]=

True

$$M^6 \subseteq \text{Im } \mathbf{O}, \text{ if } 0 ==$$

In[764]:=

(eA /. M6[[1]]).unit

Out[764]=

$$\frac{eA[1]}{\sqrt{2}} + \frac{eA[5]}{\sqrt{2}}$$

$$M617[[1]] \subseteq \text{Im } \mathbf{O}, \text{ if } 0 == eA[1] /. M617[[1]]$$

In[997]:=

eA[1] /. M617[[1]] // (*Full*) Simplify

Out[997]=

$$\begin{aligned} & - \left(-C1 C2 \left(C4 e^{\frac{M^2 \xi_{\text{adv}}}{C3 H} + C3 H \xi_{\text{ret}}} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2) - \right. \right. \\ & \quad \left. \left. C3 e^{\frac{M^2 \xi_{\text{adv}}}{C4 H} + C4 H \xi_{\text{ret}}} (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) \right. \\ & \quad \left(c31 c32 e^{\frac{M^2 \xi_{\text{adv}}}{C3 H} + C3 H \xi_{\text{ret}}} eA[2] + c43 c44 e^{\frac{M^2 \xi_{\text{adv}}}{C4 H} + C4 H \xi_{\text{ret}}} eA[2] - c33 c34 e^{\frac{M^2 \xi_{\text{adv}}}{C3 H} + C3 H \xi_{\text{ret}}} eA[3] - \right. \\ & \quad \left. c41 c42 e^{\frac{M^2 \xi_{\text{adv}}}{C4 H} + C4 H \xi_{\text{ret}}} eA[3] + c11 c12 e^{\frac{M^2 \xi_{\text{adv}}}{C1 H} + C1 H \xi_{\text{ret}}} eA[4] + c23 c24 e^{\frac{M^2 \xi_{\text{adv}}}{C2 H} + C2 H \xi_{\text{ret}}} eA[4] + \right. \\ & \quad \left. c11 c12 e^{\frac{M^2 \xi_{\text{adv}}}{C1 H} + C1 H \xi_{\text{ret}}} eA[5] - c23 c24 e^{\frac{M^2 \xi_{\text{adv}}}{C2 H} + C2 H \xi_{\text{ret}}} eA[5] + c33 c34 e^{\frac{M^2 \xi_{\text{adv}}}{C3 H} + C3 H \xi_{\text{ret}}} eA[6] - \right. \\ & \quad \left. c41 c42 e^{\frac{M^2 \xi_{\text{adv}}}{C4 H} + C4 H \xi_{\text{ret}}} eA[6] - c13 c14 e^{\frac{M^2 \xi_{\text{adv}}}{C1 H} + C1 H \xi_{\text{ret}}} eA[8] + c21 c22 e^{\frac{M^2 \xi_{\text{adv}}}{C2 H} + C2 H \xi_{\text{ret}}} eA[8] \right) + \\ & \quad \left(c31 c32 e^{\frac{M^2 \xi_{\text{adv}}}{C3 H} + C3 H \xi_{\text{ret}}} - c43 c44 e^{\frac{M^2 \xi_{\text{adv}}}{C4 H} + C4 H \xi_{\text{ret}}} \right) \\ & \quad \left(C1 C2 \left(C4 e^{\frac{M^2 \xi_{\text{adv}}}{C3 H} + C3 H \xi_{\text{ret}}} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2) + \right. \right. \\ & \quad \left. \left. C3 e^{\frac{M^2 \xi_{\text{adv}}}{C4 H} + C4 H \xi_{\text{ret}}} (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) eA[2] - \right. \\ & \quad \left. C1 C2 \left(C4 e^{\frac{M^2 \xi_{\text{adv}}}{C3 H} + C3 H \xi_{\text{ret}}} (C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2) + \right. \right. \\ & \quad \left. \left. C3 e^{\frac{M^2 \xi_{\text{adv}}}{C4 H} + C4 H \xi_{\text{ret}}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) eA[3] + \right. \\ & \quad \left. C3 C4 \left(C2 e^{\frac{M^2 \xi_{\text{adv}}}{C1 H} + C1 H \xi_{\text{ret}}} (C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2) + \right. \right. \\ & \quad \left. \left. C1 e^{\frac{M^2 \xi_{\text{adv}}}{C2 H} + C2 H \xi_{\text{ret}}} (C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2) \right) eA[4] + \right. \end{aligned}$$

$$\begin{aligned}
& C3 \ C4 \left(C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2) - \right. \\
& \quad \left. C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2) \right) eA[5] + \\
& C1 \ C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2) - \right. \\
& \quad \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) eA[6] + \\
& C3 \ C4 \left(C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2) + \right. \\
& \quad \left. C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2) \right) eA[8] \Bigg) \Bigg) / \\
& \left(C1 \ C2 \ C3 \ C4 \left(\left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \right. \right. \\
& \quad \left. \left. \frac{e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2)}{C1} - \right. \right. \\
& \quad \left. \left. \frac{e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2)}{C2} \right) - \right. \\
& \quad \left(c13 c14 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} + c21 c22 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \right) \\
& \quad \left. \left. \left. - \frac{e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2)}{C3} + \right. \right. \right. \\
& \quad \left. \left. \left. \frac{e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2)}{C4} \right) \right) \right) \Bigg)
\end{aligned}$$

M617[1] $\subseteq \text{Im } \mathbf{O}$, if $\theta ==$

In[996]:=

(EA /. M617[1]).unit // (*Full*)Simplify

Out[996]=

$$\begin{aligned}
& \frac{1}{\sqrt{2}} \left(eA[5] - \right. \\
& \quad \left. - C1 \ C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2) - C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right. \right. \\
& \quad \left. \left. (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) \right)
\end{aligned}$$

$$\begin{aligned}
& \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[2] + c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[2] - c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[3] - \right. \\
& \quad c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[3] + c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[4] + c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[4] + \\
& \quad c11 c12 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[5] - c23 c24 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[5] + c33 c34 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} eA[6] - \\
& \quad \left. c41 c42 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} eA[6] - c13 c14 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} eA[8] + c21 c22 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} eA[8] \right) + \\
& \left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \\
& \left(C1 C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2) + \right. \right. \\
& \quad C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \Big) eA[2] - \\
& \quad C1 C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2) + \right. \\
& \quad \left. \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) eA[3] + \right. \\
& \quad C3 C4 \left(C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2) + \right. \\
& \quad \left. \left. C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2) \right) eA[4] + \right. \\
& \quad C3 C4 \left(C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 + (c11 c12 + c13 c14) M^2) - \right. \\
& \quad \left. \left. C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (-c21 c22 + c23 c24) M^2) \right) eA[5] + \right. \\
& \quad C1 C2 \left(C4 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} (C3^2 (c31 c32 + c33 c34) H^2 + (-c31 c32 + c33 c34) M^2) - \right. \\
& \quad \left. \left. C3 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} (C4^2 (c41 c42 - c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) eA[6] + \right. \\
& \quad C3 C4 \left(C2 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2) + \right. \\
& \quad \left. \left. C1 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2) \right) eA[8] \right) \Big) \\
& \left(C1 C2 C3 C4 \left(\left(c31 c32 e^{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}} - c43 c44 e^{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}} \right) \right. \right. \\
& \quad \left. \left. \left(e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} (C1^2 (c11 c12 - c13 c14) H^2 - (c11 c12 + c13 c14) M^2) \right) - \right. \right. \\
& \quad \left. \left. \left. C1 \right. \right. \right. \\
& \quad \left. \left. \left. \left(e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} (C2^2 (c21 c22 + c23 c24) H^2 + (c21 c22 - c23 c24) M^2) \right) \right. \right. \right. \\
& \quad \left. \left. \left. \left. C2 \right. \right. \right. \right. \\
& \quad \left(c13 c14 e^{\frac{M^2 \xi_{adv}}{C1 H} + C1 H \xi_{ret}} + c21 c22 e^{\frac{M^2 \xi_{adv}}{C2 H} + C2 H \xi_{ret}} \right)
\end{aligned}$$

$$\left(-\frac{\frac{M^2 \xi_{adv}}{C3 H} + C3 H \xi_{ret}}{C3} (C3^2 (c31 c32 + c33 c34) H^2 + (c31 c32 - c33 c34) M^2) + \right. \\ \left. \left. \left. \frac{\frac{M^2 \xi_{adv}}{C4 H} + C4 H \xi_{ret}}{C4} (C4^2 (-c41 c42 + c43 c44) H^2 + (c41 c42 + c43 c44) M^2) \right) \right) \right)$$

try

In[814]:=

$$\text{EtoExp}[YZ0[z, t] \rightarrow \frac{e^{-\frac{M^2 (t-\frac{z}{6})}{C1 H^2} + C1 (t+\frac{z}{6})} (C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2)}{2 C1 H M}]$$

FullForm[%]

Out[814]=

$$YZ0[z, t] \rightarrow \frac{(C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C1 H^2} + C1 (t + \frac{z}{6})\right]}{2 C1 H M}$$

Out[815]/.FullForm=

```
Rule[YZ0[z, t], Times[Rational[1, 2], Power[C1, -1], Power[H, -1], Power[M, -1],
Plus[Times[Power[C1, 2], c11, c12, Power[H, 2]], Times[-1, Power[C1, 2], c13, c14,
Power[H, 2]], Times[-1, c11, c12, Power[M, 2]], Times[-1, c13, c14, Power[M, 2]]],
Inactive[Exp][Plus[Times[-1, Power[C1, -1], Power[H, -2], Power[M, 2],
Plus[t, Times[Rational[-1, 6], z]]], Times[C1, Plus[t, Times[Rational[1, 6], z]]]]]]]
```

In[816]:=

```
EtoExp[##] & /@ sYZvar
## /. {Exp -> Sin} & //.%
```

Out[816]=

$$\begin{aligned}
YZ0[z, t] &\rightarrow \frac{(C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C1 H^2} + C1 \left(t + \frac{z}{6}\right)\right]}{2 C1 H M}, \\
YZ1[z, t] &\rightarrow -\frac{(C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 + c11 c12 M^2 + c13 c14 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C1 H^2} + C1 \left(t + \frac{z}{6}\right)\right]}{2 C1 H M}, \\
YZ2[z, t] &\rightarrow c13 c14 \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C1 H^2} + C1 \left(t + \frac{z}{6}\right)\right], \\
YZ3[z, t] &\rightarrow c11 c12 \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C1 H^2} + C1 \left(t + \frac{z}{6}\right)\right], \\
YZ4[z, t] &\rightarrow -\frac{(C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 - c21 c22 M^2 + c23 c24 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C2 H^2} + C2 \left(t + \frac{z}{6}\right)\right]}{2 C2 H M}, \\
YZ5[z, t] &\rightarrow -\frac{(C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 + c21 c22 M^2 - c23 c24 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C2 H^2} + C2 \left(t + \frac{z}{6}\right)\right]}{2 C2 H M}, \\
YZ6[z, t] &\rightarrow c23 c24 \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C2 H^2} + C2 \left(t + \frac{z}{6}\right)\right], \\
YZ7[z, t] &\rightarrow c21 c22 \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C2 H^2} + C2 \left(t + \frac{z}{6}\right)\right], \\
YZ8[z, t] &\rightarrow -\frac{(C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 - c31 c32 M^2 + c33 c34 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C3 H^2} + C3 \left(t + \frac{z}{6}\right)\right]}{2 C3 H M}, \\
YZ9[z, t] &\rightarrow -\frac{(C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 + c31 c32 M^2 - c33 c34 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C3 H^2} + C3 \left(t + \frac{z}{6}\right)\right]}{2 C3 H M}, \\
YZ10[z, t] &\rightarrow c33 c34 \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C3 H^2} + C3 \left(t + \frac{z}{6}\right)\right], \\
YZ11[z, t] &\rightarrow c31 c32 \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C3 H^2} + C3 \left(t + \frac{z}{6}\right)\right], \\
YZ12[z, t] &\rightarrow -\frac{(C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 - c41 c42 M^2 - c43 c44 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C4 H^2} + C4 \left(t + \frac{z}{6}\right)\right]}{2 C4 H M}, \\
YZ13[z, t] &\rightarrow -\frac{(C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 + c41 c42 M^2 + c43 c44 M^2) \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C4 H^2} + C4 \left(t + \frac{z}{6}\right)\right]}{2 C4 H M}, \\
YZ14[z, t] &\rightarrow c43 c44 \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C4 H^2} + C4 \left(t + \frac{z}{6}\right)\right], \\
YZ15[z, t] &\rightarrow c41 c42 \text{Exp}\left[-\frac{M^2 (t-\frac{z}{6})}{C4 H^2} + C4 \left(t + \frac{z}{6}\right)\right]\}
\end{aligned}$$

Out[817]=

$$\begin{aligned}
YZ0[z, t] &\rightarrow \frac{(C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2) \sin\left[-\frac{M^2(t-\frac{z}{6})}{C1 H^2} + C1 \left(t + \frac{z}{6}\right)\right]}{2 C1 H M}, \\
YZ1[z, t] &\rightarrow -\frac{(C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 + c11 c12 M^2 + c13 c14 M^2) \sin\left[-\frac{M^2(t-\frac{z}{6})}{C1 H^2} + C1 \left(t + \frac{z}{6}\right)\right]}{2 C1 H M}, \\
YZ2[z, t] &\rightarrow c13 c14 \sin\left[-\frac{M^2(t-\frac{z}{6})}{C1 H^2} + C1 \left(t + \frac{z}{6}\right)\right], \\
YZ3[z, t] &\rightarrow c11 c12 \sin\left[-\frac{M^2(t-\frac{z}{6})}{C1 H^2} + C1 \left(t + \frac{z}{6}\right)\right], \\
YZ4[z, t] &\rightarrow -\frac{(C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 - c21 c22 M^2 + c23 c24 M^2) \sin\left[-\frac{M^2(t-\frac{z}{6})}{C2 H^2} + C2 \left(t + \frac{z}{6}\right)\right]}{2 C2 H M}, \\
YZ5[z, t] &\rightarrow -\frac{(C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 + c21 c22 M^2 - c23 c24 M^2) \sin\left[-\frac{M^2(t-\frac{z}{6})}{C2 H^2} + C2 \left(t + \frac{z}{6}\right)\right]}{2 C2 H M}, \\
YZ6[z, t] &\rightarrow c23 c24 \sin\left[-\frac{M^2(t-\frac{z}{6})}{C2 H^2} + C2 \left(t + \frac{z}{6}\right)\right], \\
YZ7[z, t] &\rightarrow c21 c22 \sin\left[-\frac{M^2(t-\frac{z}{6})}{C2 H^2} + C2 \left(t + \frac{z}{6}\right)\right], \\
YZ8[z, t] &\rightarrow -\frac{(C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 - c31 c32 M^2 + c33 c34 M^2) \sin\left[-\frac{M^2(t-\frac{z}{6})}{C3 H^2} + C3 \left(t + \frac{z}{6}\right)\right]}{2 C3 H M}, \\
YZ9[z, t] &\rightarrow -\frac{(C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 + c31 c32 M^2 - c33 c34 M^2) \sin\left[-\frac{M^2(t-\frac{z}{6})}{C3 H^2} + C3 \left(t + \frac{z}{6}\right)\right]}{2 C3 H M}, \\
YZ10[z, t] &\rightarrow c33 c34 \sin\left[-\frac{M^2(t-\frac{z}{6})}{C3 H^2} + C3 \left(t + \frac{z}{6}\right)\right], \\
YZ11[z, t] &\rightarrow c31 c32 \sin\left[-\frac{M^2(t-\frac{z}{6})}{C3 H^2} + C3 \left(t + \frac{z}{6}\right)\right], \\
YZ12[z, t] &\rightarrow -\frac{(C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 - c41 c42 M^2 - c43 c44 M^2) \sin\left[-\frac{M^2(t-\frac{z}{6})}{C4 H^2} + C4 \left(t + \frac{z}{6}\right)\right]}{2 C4 H M}, \\
YZ13[z, t] &\rightarrow -\frac{(C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 + c41 c42 M^2 + c43 c44 M^2) \sin\left[-\frac{M^2(t-\frac{z}{6})}{C4 H^2} + C4 \left(t + \frac{z}{6}\right)\right]}{2 C4 H M}, \\
YZ14[z, t] &\rightarrow c43 c44 \sin\left[-\frac{M^2(t-\frac{z}{6})}{C4 H^2} + C4 \left(t + \frac{z}{6}\right)\right], \\
YZ15[z, t] &\rightarrow c41 c42 \sin\left[-\frac{M^2(t-\frac{z}{6})}{C4 H^2} + C4 \left(t + \frac{z}{6}\right)\right]\}
\end{aligned}$$

```
In[822]:= EtoExp[ToString[#, InputForm] & /@ sYZvar
Out[822]= {YZ0[z, t] -> ((C1^2*c11*c12*H^2 - C1^2*c13*c14*H^2 - c11*c12*M^2 - c13*c14*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C1*H^2)) + C1*(t + z/6)])/(2*C1*H*M),
YZ1[z, t] -> -1/2*((C1^2*c11*c12*H^2 - C1^2*c13*c14*H^2 + c11*c12*M^2 + c13*c14*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C1*H^2)) + C1*(t + z/6)])/(C1*H*M),
YZ2[z, t] -> c13*c14*Inactive[Exp][-((M^2*(t - z/6))/(C1*H^2)) + C1*(t + z/6)],
YZ3[z, t] -> c11*c12*Inactive[Exp][-((M^2*(t - z/6))/(C1*H^2)) + C1*(t + z/6)],
YZ4[z, t] -> -1/2*((C2^2*c21*c22*H^2 + C2^2*c23*c24*H^2 - c21*c22*M^2 + c23*c24*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C2*H^2)) + C2*(t + z/6)])/(C2*H*M),
YZ5[z, t] -> -1/2*((C2^2*c21*c22*H^2 + C2^2*c23*c24*H^2 + c21*c22*M^2 - c23*c24*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C2*H^2)) + C2*(t + z/6)])/(C2*H*M),
YZ6[z, t] -> c23*c24*Inactive[Exp][-((M^2*(t - z/6))/(C2*H^2)) + C2*(t + z/6)],
YZ7[z, t] -> c21*c22*Inactive[Exp][-((M^2*(t - z/6))/(C2*H^2)) + C2*(t + z/6)],
YZ8[z, t] -> -1/2*((C3^2*c31*c32*H^2 + C3^2*c33*c34*H^2 - c31*c32*M^2 + c33*c34*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C3*H^2)) + C3*(t + z/6)])/(C3*H*M),
YZ9[z, t] -> -1/2*((C3^2*c31*c32*H^2 + C3^2*c33*c34*H^2 + c31*c32*M^2 - c33*c34*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C3*H^2)) + C3*(t + z/6)])/(C3*H*M),
YZ10[z, t] -> c33*c34*Inactive[Exp][-((M^2*(t - z/6))/(C3*H^2)) + C3*(t + z/6)],
YZ11[z, t] -> c31*c32*Inactive[Exp][-((M^2*(t - z/6))/(C3*H^2)) + C3*(t + z/6)],
YZ12[z, t] -> ((C4^2*c41*c42*H^2 - C4^2*c43*c44*H^2 - c41*c42*M^2 - c43*c44*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C4*H^2)) + C4*(t + z/6)])/(2*C4*H*M),
YZ13[z, t] -> -1/2*((C4^2*c41*c42*H^2 - C4^2*c43*c44*H^2 + c41*c42*M^2 + c43*c44*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C4*H^2)) + C4*(t + z/6)])/(C4*H*M),
YZ14[z, t] -> c43*c44*Inactive[Exp][-((M^2*(t - z/6))/(C4*H^2)) + C4*(t + z/6)],
YZ15[z, t] -> c41*c42*Inactive[Exp][-((M^2*(t - z/6))/(C4*H^2)) + C4*(t + z/6)]}

In[823]:= yZ[#] & /@ Range[0, 15]
sYZvar[[All, 2]] /. {z -> #1, t -> #2};
Table[ToExpression["((" <> ToString[FullForm[%[[j]]]] <> ")")"], {j, 1, Length[%]}];
ssyZ = Thread[%%% -> %]
```

E

is the exponential constant E (base of natural logarithms), with numerical value ≈ 2.71828 .

```
In[982]:= yZ[#][z, t] & /@ Range[0, 15];
% /. ssyZ;
EtoExp[#] & /@ %;
% /. {Exp -> QF} // Activate;
% /. {z -> #1, t -> #2};
syZ = Table[
  ssyZ[[j, 1]] -> ToExpression["((" <> ToString[%[[j]], InputForm] <> ")")"], {j, 1, Length[%]}]
(*Thread[ssyZ[[All, 1]] ->%] /. {z -> #1, t -> #2}*)
```

Out[987]=
$$\left\{ yZ[0] \rightarrow \frac{\left(C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 - c11 c12 M^2 - c13 c14 M^2 \right) QF \left[-\frac{M^2 \left(\frac{\#1}{6} + \#2 \right)}{C1 H^2} + C1 \left(\frac{\#1}{6} + \#2 \right) \right]}{2 C1 H M} \&, \right.$$

$yZ[1] \rightarrow$

$$\left(-\frac{(C1^2 c11 c12 H^2 - C1^2 c13 c14 H^2 + c11 c12 M^2 + c13 c14 M^2) QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C1 H^2} + C1 \left(\frac{\#1}{6} + \#2 \right) \right]}{2 (C1 H M)} \& \right),$$

$$yZ[2] \rightarrow \left(c13 c14 QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C1 H^2} + C1 \left(\frac{\#1}{6} + \#2 \right) \right] \& \right),$$

$$yZ[3] \rightarrow \left(c11 c12 QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C1 H^2} + C1 \left(\frac{\#1}{6} + \#2 \right) \right] \& \right), yZ[4] \rightarrow$$

$$\left(-\frac{(C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 - c21 c22 M^2 + c23 c24 M^2) QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C2 H^2} + C2 \left(\frac{\#1}{6} + \#2 \right) \right]}{2 (C2 H M)} \& \right), yZ[$$

$$5] \rightarrow \left(-\frac{(C2^2 c21 c22 H^2 + C2^2 c23 c24 H^2 + c21 c22 M^2 - c23 c24 M^2) QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C2 H^2} + C2 \left(\frac{\#1}{6} + \#2 \right) \right]}{2 (C2 H M)} \& \right),$$

$$yZ[6] \rightarrow \left(c23 c24 QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C2 H^2} + C2 \left(\frac{\#1}{6} + \#2 \right) \right] \& \right),$$

$$yZ[7] \rightarrow \left(c21 c22 QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C2 H^2} + C2 \left(\frac{\#1}{6} + \#2 \right) \right] \& \right), yZ[8] \rightarrow$$

$$\left(-\frac{(C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 - c31 c32 M^2 + c33 c34 M^2) QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C3 H^2} + C3 \left(\frac{\#1}{6} + \#2 \right) \right]}{2 (C3 H M)} \& \right), yZ[$$

$$9] \rightarrow \left(-\frac{(C3^2 c31 c32 H^2 + C3^2 c33 c34 H^2 + c31 c32 M^2 - c33 c34 M^2) QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C3 H^2} + C3 \left(\frac{\#1}{6} + \#2 \right) \right]}{2 (C3 H M)} \& \right),$$

$$yZ[10] \rightarrow \left(c33 c34 QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C3 H^2} + C3 \left(\frac{\#1}{6} + \#2 \right) \right] \& \right),$$

$$yZ[11] \rightarrow \left(c31 c32 QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C3 H^2} + C3 \left(\frac{\#1}{6} + \#2 \right) \right] \& \right), yZ[12] \rightarrow$$

$$\left(-\frac{(C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 - c41 c42 M^2 - c43 c44 M^2) QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C4 H^2} + C4 \left(\frac{\#1}{6} + \#2 \right) \right]}{2 C4 H M} \& \right),$$

$yZ[13] \rightarrow$

$$\left(-\frac{\left(C4^2 c41 c42 H^2 - C4^2 c43 c44 H^2 + c41 c42 M^2 + c43 c44 M^2 \right) QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C4 H^2} + C4 \left(\frac{\#1}{6} + \#2 \right) \right] \& }{2 (C4 H M)} , \right.$$

$$yZ[14] \rightarrow \left(c43 c44 QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C4 H^2} + C4 \left(\frac{\#1}{6} + \#2 \right) \right] \& , \right)$$

$$yZ[15] \rightarrow \left(c41 c42 QF \left[-\frac{M^2 \left(-\frac{\#1}{6} + \#2 \right)}{C4 H^2} + C4 \left(\frac{\#1}{6} + \#2 \right) \right] \& \right)$$

Z to yZ mapping

In[771]:=

```
cayZ = CoefficientArrays[zeroZyzforCaExpression, yZdef]
```

Out[771]=

```
{SparseArray[ Specified elements:  Dimensions: ], SparseArray[ Specified elements:  Dimensions: ]}}
```

In[772]:=

```
caZ = CoefficientArrays[zeroZyzforCaExpression, Zdef]
```

Out[772]=

```
{SparseArray[ Specified elements:  Dimensions: ], SparseArray[ Specified elements:  Dimensions: ]}}
```

In[773]:=

```
zeroZyzforCaExpression === cayZ[[1]] + cayZ[[2]].yZdef // Expand
```

Out[773]=

```
True
```

In[774]:=

```
zeroZyzforCaExpression === caZ[[1]] + caZ[[2]].Zdef // Expand
```

Out[774]=

```
True
```

In[775]:=

```
(cayZ2 = cayZ[[2]] // Normal) === ID16
```

Out[775]=

```
True
```

In[776]:=

(cayZ1 = cayZ[[1]] // Normal)

Out[776]=

{-Z[0], -Z[5], -Z[8], -Z[13], -Z[1], -Z[4], -Z[9],
-Z[12], -Z[2], -Z[7], -Z[10], -Z[15], -Z[3], -Z[6], -Z[11], -Z[14]}

In[777]:=

(caZ2 = caZ[[2]] // Normal) // MatrixForm

Out[777]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[778]:=

Transpose[caZ2].caZ2 === Id16

Out[778]=

True

In[779]:=

sZt0yZ

Out[779]=

{Z[0] → yZ[0], Z[5] → yZ[1], Z[8] → yZ[2], Z[13] → yZ[3], Z[1] → yZ[4], Z[4] → yZ[5],
Z[9] → yZ[6], Z[12] → yZ[7], Z[2] → yZ[8], Z[7] → yZ[9], Z[10] → yZ[10],
Z[15] → yZ[11], Z[3] → yZ[12], Z[6] → yZ[13], Z[11] → yZ[14], Z[14] → yZ[15]}

In[780]:=

```
Transpose[caZ2].caZ[[1]] + Transpose[caZ2].caZ[[2]].Zdef // Expand
sZyZ = Solve[And @@ Thread[\theta == %], Zdef][[1]]
sZyZa = Thread[Zdef -> -Transpose[caZ2].caZ[[1]]]
% == %%
```

Out[780]=

$$\{-yZ[0] + Z[0], -yZ[4] + Z[1], -yZ[8] + Z[2], -yZ[12] + Z[3], -yZ[5] + Z[4], -yZ[1] + Z[5], \\ -yZ[13] + Z[6], -yZ[9] + Z[7], -yZ[2] + Z[8], -yZ[6] + Z[9], -yZ[10] + Z[10], \\ -yZ[14] + Z[11], -yZ[7] + Z[12], -yZ[3] + Z[13], -yZ[15] + Z[14], -yZ[11] + Z[15]\}$$

Out[781]=

$$\{Z[0] \rightarrow yZ[0], Z[1] \rightarrow yZ[4], Z[2] \rightarrow yZ[8], Z[3] \rightarrow yZ[12], Z[4] \rightarrow yZ[5], Z[5] \rightarrow yZ[1], \\ Z[6] \rightarrow yZ[13], Z[7] \rightarrow yZ[9], Z[8] \rightarrow yZ[2], Z[9] \rightarrow yZ[6], Z[10] \rightarrow yZ[10], \\ Z[11] \rightarrow yZ[14], Z[12] \rightarrow yZ[7], Z[13] \rightarrow yZ[3], Z[14] \rightarrow yZ[15], Z[15] \rightarrow yZ[11]\}$$

Out[782]=

$$\{Z[0] \rightarrow yZ[0], Z[1] \rightarrow yZ[4], Z[2] \rightarrow yZ[8], Z[3] \rightarrow yZ[12], Z[4] \rightarrow yZ[5], Z[5] \rightarrow yZ[1], \\ Z[6] \rightarrow yZ[13], Z[7] \rightarrow yZ[9], Z[8] \rightarrow yZ[2], Z[9] \rightarrow yZ[6], Z[10] \rightarrow yZ[10], \\ Z[11] \rightarrow yZ[14], Z[12] \rightarrow yZ[7], Z[13] \rightarrow yZ[3], Z[14] \rightarrow yZ[15], Z[15] \rightarrow yZ[11]\}$$

Out[783]=

True

In[784]:=

almightyS = -Transpose[caZ2]

Out[784]=

$$\{\{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}\}$$

O(8) not O(4, 4) :

In[785]:=

Transpose[almightyS].almightyS === ID16

Out[785]=

True

In[786]:=

Transpose[almightyS].σ16.almightyS === σ16

Out[786]=

False

In[787]:=

 $\sigma_{16} // \text{MatrixForm}$

Out[787]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[788]:=

```
Transpose[almightyS].ArrayFlatten[{{0, σ}, {σ, 0}}].almightyS ===
ArrayFlatten[{{0, σ}, {σ, 0}}]
```

Out[788]=

False

In[789]:=

```
Clear[La];
La[] := 
(Transpose[Ψ16].σ16.Sum[T16^A[α1 - 1].(D[Ψ16, X[α1]]), {α1, 1, Length[X]}] +
(H * M) * Transpose[Ψ16].σ16.Ψ16) // Simplify[#, constraintVars] &
```

In[789]:=

 $\sigma_{16}.T16^A[\#] == -\text{Transpose}[\sigma_{16}.T16^A[\#]] \& /@ \text{Range}[0, 7]$

Out[789]=

{True, True, True, True, True, True, True}

In[790]:=

```

La[];
- Sum[(D[Transpose[\!f16], X[\!\alpha1]]) . \!f16.T16^{\!\alpha1 - 1}, {\alpha1, 1, Length[X]}] . \!f16 +
(H * M) * Transpose[\!f16].\!f16 // FullSimplify[#, constraintVars] &;
ExpandAll[% - %%] // FullSimplify[#, constraintVars] &

```

Out[792]=

$$\begin{aligned}
& \frac{1}{H} \left(4 M f16[2][x0, x4] \times f16[6][x0, x4] - \right. \\
& 2 H^2 M f16[2][x0, x4] \times f16[6][x0, x4] + 4 M f16[3][x0, x4] \times f16[7][x0, x4] - \\
& 2 H^2 M f16[3][x0, x4] \times f16[7][x0, x4] - 4 M f16[8][x0, x4] \times f16[12][x0, x4] + \\
& 2 H^2 M f16[8][x0, x4] \times f16[12][x0, x4] - 4 M f16[9][x0, x4] \times f16[13][x0, x4] + \\
& 2 H^2 M f16[9][x0, x4] \times f16[13][x0, x4] - 4 M f16[10][x0, x4] \times f16[14][x0, x4] + \\
& 2 H^2 M f16[10][x0, x4] \times f16[14][x0, x4] - 4 M f16[11][x0, x4] \times f16[15][x0, x4] + \\
& 2 H^2 M f16[11][x0, x4] \times f16[15][x0, x4] - f16[9][x0, x4] f16[0]^{(0,1)}[x0, x4] + \\
& H f16[9][x0, x4] f16[0]^{(0,1)}[x0, x4] + f16[8][x0, x4] f16[1]^{(0,1)}[x0, x4] - \\
& H f16[8][x0, x4] f16[1]^{(0,1)}[x0, x4] + f16[11][x0, x4] f16[2]^{(0,1)}[x0, x4] - \\
& H f16[11][x0, x4] f16[2]^{(0,1)}[x0, x4] - f16[10][x0, x4] f16[3]^{(0,1)}[x0, x4] + \\
& H f16[10][x0, x4] f16[3]^{(0,1)}[x0, x4] + f16[13][x0, x4] f16[4]^{(0,1)}[x0, x4] - \\
& H f16[13][x0, x4] f16[4]^{(0,1)}[x0, x4] - f16[12][x0, x4] f16[5]^{(0,1)}[x0, x4] + \\
& H f16[12][x0, x4] f16[5]^{(0,1)}[x0, x4] - f16[15][x0, x4] f16[6]^{(0,1)}[x0, x4] + \\
& H f16[15][x0, x4] f16[6]^{(0,1)}[x0, x4] + f16[14][x0, x4] f16[7]^{(0,1)}[x0, x4] - \\
& H f16[14][x0, x4] f16[7]^{(0,1)}[x0, x4] + f16[3][x0, x4] f16[10]^{(0,1)}[x0, x4] - \\
& H f16[3][x0, x4] f16[10]^{(0,1)}[x0, x4] - f16[2][x0, x4] f16[11]^{(0,1)}[x0, x4] + \\
& H f16[2][x0, x4] f16[11]^{(0,1)}[x0, x4] + f16[5][x0, x4] f16[12]^{(0,1)}[x0, x4] - \\
& H f16[5][x0, x4] f16[12]^{(0,1)}[x0, x4] - f16[4][x0, x4] f16[13]^{(0,1)}[x0, x4] + \\
& H f16[4][x0, x4] f16[13]^{(0,1)}[x0, x4] - f16[7][x0, x4] f16[14]^{(0,1)}[x0, x4] + \\
& H f16[7][x0, x4] f16[14]^{(0,1)}[x0, x4] + f16[6][x0, x4] f16[15]^{(0,1)}[x0, x4] - \\
& H f16[6][x0, x4] f16[15]^{(0,1)}[x0, x4] - f16[12][x0, x4] f16[0]^{(1,0)}[x0, x4] + \\
& H f16[12][x0, x4] f16[0]^{(1,0)}[x0, x4] - f16[13][x0, x4] f16[1]^{(1,0)}[x0, x4] + \\
& H f16[13][x0, x4] f16[1]^{(1,0)}[x0, x4] - f16[14][x0, x4] f16[2]^{(1,0)}[x0, x4] + \\
& H f16[14][x0, x4] f16[2]^{(1,0)}[x0, x4] - f16[15][x0, x4] f16[3]^{(1,0)}[x0, x4] + \\
& H f16[15][x0, x4] f16[3]^{(1,0)}[x0, x4] - f16[8][x0, x4] f16[4]^{(1,0)}[x0, x4] + \\
& H f16[8][x0, x4] f16[4]^{(1,0)}[x0, x4] - f16[9][x0, x4] f16[5]^{(1,0)}[x0, x4] + \\
& H f16[9][x0, x4] f16[5]^{(1,0)}[x0, x4] - f16[10][x0, x4] f16[6]^{(1,0)}[x0, x4] + \\
& H f16[10][x0, x4] f16[6]^{(1,0)}[x0, x4] - f16[11][x0, x4] f16[7]^{(1,0)}[x0, x4] + \\
& H f16[11][x0, x4] f16[7]^{(1,0)}[x0, x4] + f16[4][x0, x4] f16[8]^{(1,0)}[x0, x4] - \\
& H f16[4][x0, x4] f16[8]^{(1,0)}[x0, x4] + f16[5][x0, x4] f16[9]^{(1,0)}[x0, x4] - \\
& H f16[5][x0, x4] f16[9]^{(1,0)}[x0, x4] + f16[6][x0, x4] f16[10]^{(1,0)}[x0, x4] - \\
& H f16[6][x0, x4] f16[10]^{(1,0)}[x0, x4] + f16[7][x0, x4] f16[11]^{(1,0)}[x0, x4] - \\
& H f16[7][x0, x4] f16[11]^{(1,0)}[x0, x4] + f16[0][x0, x4] (-2 (-2 + H^2) M f16[4][x0, x4] - \\
& (-1 + H) (f16[9]^{(0,1)}[x0, x4] + f16[12]^{(1,0)}[x0, x4])) + f16[1][x0, x4] \\
& (-2 (-2 + H^2) M f16[5][x0, x4] + (-1 + H) (f16[8]^{(0,1)}[x0, x4] - f16[13]^{(1,0)}[x0, x4])) + \\
& f16[2][x0, x4] f16[14]^{(1,0)}[x0, x4] - H f16[2][x0, x4] f16[14]^{(1,0)}[x0, x4] - \\
& (-1 + H) f16[3][x0, x4] f16[15]^{(1,0)}[x0, x4])
\end{aligned}$$

In[793]:=

```
( Transpose[\!`16].\!\>16.Sum[T16^{\alpha1 - 1}.(D[\!`16, X[\alpha1]], {\alpha1, 1, Length[X]}] +
 (H*M)*Transpose[\!`16].\!\>16.\!`16) /. sf\!\>16Aa /.
 sx0x4 // FullSimplify[#, constraintVars] &;
 - Sum[(D[Transpose[\!`16], X[\alpha1]]).\!\>16.T16^{\alpha1 - 1}, {\alpha1, 1, Length[X]}].\!`16 +
 (H*M)*Transpose[\!`16].\!\>16.\!`16 /. sf\!\>16Aa /.
 sx0x4 // FullSimplify[#, constraintVars] &;
 ExpandAll[% - %] // FullSimplify[#, constraintVars] &
```

Out[795]=

0

In[796]:=

Thread[Zdef \(\rightarrow\) -Transpose[caZ2].caZ[1]]

Out[796]=

$$\{Z[0] \rightarrow yZ[0], Z[1] \rightarrow yZ[4], Z[2] \rightarrow yZ[8], Z[3] \rightarrow yZ[12], Z[4] \rightarrow yZ[5], Z[5] \rightarrow yZ[1],$$

$$Z[6] \rightarrow yZ[13], Z[7] \rightarrow yZ[9], Z[8] \rightarrow yZ[2], Z[9] \rightarrow yZ[6], Z[10] \rightarrow yZ[10],$$

$$Z[11] \rightarrow yZ[14], Z[12] \rightarrow yZ[7], Z[13] \rightarrow yZ[3], Z[14] \rightarrow yZ[15], Z[15] \rightarrow yZ[11]\}$$

In[797]:=

zeroZyZforCaExpression === caZ[1] + caZ[2].Zdef // Expand

Out[797]=

True

In[798]:=

(caZ1 = caZ[1] // Normal) === yZdef

Out[798]=

True

In[799]:=

caZ[1] + caZ[2].Zdef // Expand
% /. sZt0yZ

Out[799]=

$$\{yZ[0] - Z[0], yZ[1] - Z[5], yZ[2] - Z[8], yZ[3] - Z[13], yZ[4] - Z[1],$$

$$yZ[5] - Z[4], yZ[6] - Z[9], yZ[7] - Z[12], yZ[8] - Z[2], yZ[9] - Z[7], yZ[10] - Z[10],$$

$$yZ[11] - Z[15], yZ[12] - Z[3], yZ[13] - Z[6], yZ[14] - Z[11], yZ[15] - Z[14]\}$$

Out[800]=

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
In[801]:= almightyS.Zdef
% /. sZt0yZ
yZdef === (% // Sort)
Out[801]= {Z[0], Z[4], Z[8], Z[12], Z[5], Z[1], Z[13],
Z[9], Z[2], Z[6], Z[10], Z[14], Z[7], Z[3], Z[15], Z[11]}

Out[802]= {yZ[0], yZ[5], yZ[2], yZ[7], yZ[1], yZ[4], yZ[3], yZ[6],
yZ[8], yZ[13], yZ[10], yZ[15], yZ[9], yZ[12], yZ[11], yZ[14]}

Out[803]= True
```

```
In[804]:= cayZ2 === ID16
```

```
Out[804]= True
```

```
In[805]:= caZ2 // MatrixForm
```

```
Out[805]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

```

ORTHOGONAL O(16) SIMILARITY TRANSFORMATION :

```
In[806]:= caZ2.Transpose[caZ2] === ID16
```

```
Out[806]= True
```

BUT NOT ORTHOGONAL O(8,8) SIMILARITY TRANSFORMATION :

```
In[807]:= caZ2.o16.Transpose[caZ2] === o16
```

```
Out[807]= False
```

Also, Not a Direct Sum :

```
In[808]:= caZ2 // MatrixForm
Out[808]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$


In[809]:= caZ2 === ArrayFlatten[{{caZ2[[1;;8, 1;;8]], caZ2[[1;;8, 9;;16]], {caZ2[[9;;16, 1;;8]], caZ2[[9;;16, 9;;16]]}}]
Out[809]= True

In[810]:= caZ2 === ArrayFlatten[{{caZ2[[1;;8, 1;;8], 0], {0, caZ2[[9;;16, 9;;16]]}}}]
Out[810]= False
```

Notes

Scratch work of Author;

CopyRight (C) 2022, Patrick L . Nash, under the General Public License ;Please cite this work, and this web page, if you use it

Scratch work only!!

Patrick L . Nash, Ph . D .

(c) 2022, under GPL

Professor, UTSA Physics and Astronomy, Retired (UTSA)

Patrick299Nash at gmail ...

Do not read if you are irascible; I apologize for typos and duplications and everthing else .

git clone https://github.com/43d168f3e/Pre-Universe.git

We hope to employ the formalism developed in this notebook to study ideas put forward in "The Particle Problem in the General Theory of Relativity," published in Physical Review 48, 73–77:

JULY 1, 1935

PHYSICAL REVIEW

The Particle Problem in the General Theory of Relativity

A. EINSTEIN AND N. ROSEN, *Institute for Advanced Study, Princeton*

(Received May 8, 1935)

If one tries to interpret the regular solution (5a) in the space of r, θ, ϕ, t , one arrives at the following conclusion. The four-dimensional space is described mathematically by two congruent parts or "sheets," corresponding to $u > 0$ and $u < 0$, which are joined by a hyperplane $r = 2m$ or $u = 0$ in which g vanishes.² We call such a connection between the two sheets a "bridge."

We see now in the given solution, free from singularities, the mathematical representation of an elementary particle (neutron or neutrino). Characteristic of the theory we are presenting is the description of space by means of two sheets. A bridge, spatially finite, which connects these sheets characterizes the presence of an electrically neutral elementary particle. With this conception one not only obtains the representation of an elementary particle by using only the field equations, that is, without introducing new field quantities to describe the density of matter; one

The origin of this calculation has its roots in "A Remarkable Representation of the 3+2 de Sitter Group" by P.A.M. Dirac, J.Math.Phys.4,901–909 (1963).

The term "reduced Brauer–Weyl generators", refers to a specific set of irreducible generators for certain complex Clifford algebras, particularly in the context of mathematical physics and the study of Dirac spinors. These generators were introduced by **E.A. Lord** (E. A. Lord. "The Dirac spinor in six dimensions". Mathematical Proceedings of the Cambridge Philosophical Society, vol. 64, pp. 765–778, 1968) and are described as "reduced" because they form an irreducible representation of the Clifford algebra C_{2n} from $C_{2(n-1)}$.

Here, we employ Lord's idea of reduced Brauer–Weyl generators to extend Dirac's 4×4 gamma matrices into this notebook's real tau8 8×8 and tau16 16×16 matrices, which are employed to write down the 'Dirac Equation for the Pre-Universe'

[remember, this is before the Big Bang, and before the particles of the standard model exist]
 formulated in terms of a 16-component $\overline{O(4,4)}$ spinor Ψ_{16}
 (Ψ_{16} AKA 'WAVE FUNCTION OF the UN-UNIVERSE,' which could possibly be named something more
 pompous).

This equation is used to investigate EINSTEIN AND ROSEN's idea

If one tries to interpret the regular solution (5a) in the space of r, θ, ϕ, t , one arrives at the following conclusion. The four-dimensional space is described mathematically by two congruent parts or "sheets," corresponding to $u>0$ and $u<0$, which are joined by a hyperplane $r=2m$ or $u=0$ in which g vanishes.² We call such a connection between the two sheets a "bridge."

We see now in the given solution, free from singularities, the mathematical representation of an elementary particle (neutron or neutrino). Characteristic of the theory we are presenting is the description of space by means of two sheets. A bridge, spatially finite, which connects these sheets characterizes the presence of an electrically neutral elementary particle. With this conception one not only obtains the representation of an elementary particle by using only the field equations, that is, without introducing new field quantities to describe the density of matter; one

The four-dimensional space is described mathematically by two congruent parts or "sheets," corresponding to $u>0$ and $u<0$, which are joined by a hyperplane $r=2m$ or $u=0$ in which g vanishes.² We call such a connection between the two sheets a "bridge", which marketing departments now call a "wormhole".

THIS WORK IS PARTIALLY BASED ON :

JOURNAL OF MATHEMATICAL PHYSICS

VOLUME 4, NUMBER 7

A Remarkable Representation of the 3 + 2 de Sitter Group

P. A. M. DIRAC

*Cambridge University, Cambridge, England and
Institute for Advanced Study, Princeton, New Jersey**
(Received 20 February 1963)

Proc. Camb. Phil. Soc. (1968), 64, 765

765

PCPS 64-96

Printed in Great Britain

The Dirac spinor in six dimensions

By E. A. LORD

Department of Mathematics, King's College, University of London

(Received 17 November 1966)

JOURNAL OF MATHEMATICAL PHYSICS 51, 042501 (2010)

Second gravityPatrick L. Nash^{a)}*Department of Physics and Astronomy, The University of Texas at San Antonio, San Antonio, Texas 78249-0697, USA*

(Received 29 July 2009; accepted 8 February 2010; published online 5 April 2010)

IL NUOVO CIMENTO

VOL. 105 B, N. 1

Gennaio 1990

On the Structure of the Split Octonion Algebra.

P. L. NASH

*Division of Earth and Physical Sciences
University of Texas at San Antonio, TX 78285-0663*

(ricevuto il 24 Luglio 1989)

A new spin- $\frac{1}{2}$ wave equation

Patrick L. Nash

Division of Earth and Physical Sciences, University of Texas at San Antonio, San Antonio, Texas 78285

(ricevuto il 24 Luglio 1989)

Identities satisfied by the generators of the Dirac algebra

Patrick L. Nash
 M. S. 474, NASA Langley Research Center, Hampton, Virginia 23665

(Received 26 July 1983; accepted for publication 23 September 1983)

A new spin- $\frac{1}{2}$ wave equation

Patrick L. Nash
 Division of Earth and Physical Sciences, University of Texas at San Antonio, San Antonio, Texas 78285
 (Received 24 January 1984; accepted for publication 16 November 1984)

The split Octonion algebra carries basic 8-component representations of $\overline{\text{SO}(4,4;\mathbb{R})}$ (left-spinor, right-spinor and vector), which are equivalent [**Cartan's triality**].

In[836]:=

A. Representations of $\text{SO}(8,\mathbb{C})$

Summarizing and applying a few facts that may be found in, for example, Boerner, *Representations of Groups*,⁷ the Clifford algebra C_8 may be defined as the algebra generated by a set of eight elements e_j , $j=1, \dots, 8$, that anticommute with each other and have unit square $e_j e_k + e_k e_j = 2\delta_{jk} \mathbb{I}_{16 \times 16}$, \mathbb{I} =unit matrix. The scaled commutators $\frac{1}{4}(e_j e_k - e_k e_j)$ computed from the 16-

042501-5 Second gravity

J. Math. Phys. 51. 042501 (2010)

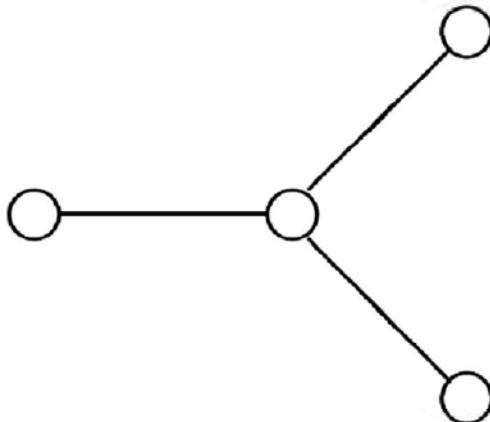


FIG. 1. Dynkin diagram for D_4 .

dimensional irreducible representation of the e_j are the infinitesimal generators of a reducible 16-dimensional representation of $\text{spin}(8,\mathbb{C})$, which is the universal double covering of the special orthogonal group $\text{SO}(8,\mathbb{C})$. This 16-dimensional representation is fully reducible to the direct sum of two inequivalent irreducible 8×8 spin representations of the infinitesimal generators of $\text{spin}(8,\mathbb{C})$, which leads to the identification of type 1 and type 2 spinors. The fundamental irreducible vector representation of $\text{SO}(8,\mathbb{C})$ is also 8×8 . The Dynkin diagram for $D_4 \cong \text{SO}(8)$ is symmetrical and pictured in Fig. 1. The three outer nodes correspond to the vector representation (leftmost node), type 1 spinor and type 2 spinor representations of $\text{spin}(8)$, and the central node corresponds

The canonical 2-1 homomorphism $\overline{\text{SO}(4,4;\mathbb{R})} \rightarrow \text{SO}(4,4;\mathbb{R}) : g \mapsto L(g)$ is given by

$$8L_A^B = \text{tr}(D_{(1)}^{-1} \bar{\tau}^A D_{(2)} \tau^C) G_{CB}, \quad (23)$$

to the adjoint representation.

The canonical 2-1 homomorphism $\overline{\text{SO}(4,4;\mathbb{R})} \rightarrow \text{SO}(4,4;\mathbb{R}) : g \mapsto L(g)$ is given by

$$8L^A_B = \text{tr}(D_{(1)}^{-1} \bar{\tau}^A D_{(2)} \tau^C) G_{CB}, \quad (23)$$

where tr denotes the trace. Note that $D_{(1)}(g(\omega)) = D_{(2)}(g(\omega))$ when $\omega_{A8}=0$, i.e., when one restricts $\overline{\text{SO}(4,4;\mathbb{R})}$ to

$$\begin{aligned} \overline{\text{SO}(3,4;\mathbb{R})} &= \{g \in \overline{\text{SO}(4,4;\mathbb{R})} \mid \\ g &= \begin{pmatrix} \exp\left(\frac{1}{4}\omega_{AB}D_{(1)}^{AB}\right) & 0 \\ 0 & \exp\left(\frac{1}{4}\omega_{AB}D_{(2)}^{AB}\right) \end{pmatrix} \quad \text{and} \quad \omega_{A8}=0 \end{aligned} \quad (24)$$

[one of the real forms of spin (7, C)].

$M_{4,4}$ = Minkowski spacetime, signature = {4, 4}

$O_{4,4}$ = split octonion algebra

$M_{4,4}$ = Minkowski spacetime, signature = {4, 4}

$O_{4,4}$ = split octonion algebra

with vector and spinor bases :

$o \in O_{4,4}; o = eA_j o^j = ea\psi_1_j o\psi_1^j = ea\psi_2_j o\psi_2^j$ [Cartan's triality]

$= \psi_1_j ea^j = ea_j \psi_2^j$ [restrict to Spin (3, 4)]

$\langle ea_a, ea_b \rangle = \sigma_{ab}$

$\langle eA_A, eA_B \rangle = \eta_{AB}$

$\{X, EA\} \in M_{4,4} \otimes O_{4,4}$

$\{X, EA\}$

$\{x0, x1, x2, x3, x4, x5, x6, x7\},$

$\{ea[1], ea[2], ea[3], ea[4], ea[5], ea[6], ea[7], ea[8]\}$

In[839]:=

```
gridSplitOctonion =
Grid[Prepend[Drop[Reap[For[A1 = 1, A1 < 9, A1++, Sow[Flatten[{EA[[A1]], Table[
Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}}]}], 1] [[1]] [[1]], Flatten[{"A/B", EA}]]], Frame -> All]
```

Out[839]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

citations for the 1935 Einstein and Rosen article,

"The Particle Problem in the General Theory of Relativity," published in Physical Review 48, 73–77:

APA/Chicago : Einstein, A., & Rosen,
N.(1935).The Particle Problem in the General Theory of Relativity.Physical Review,
48 (1), 73–77. <https://doi.org/10.1103/PhysRev.48.x.73>

BibTeX:

```
@article {EinsteinRosen1935, author = {Einstein, A. and Rosen, N.},
title = {The Particle Problem in the General Theory of Relativity},
journal = {Phys. Rev.}, volume = {48}, issue = {1}, pages = {73--\times 77}, year = {1935},
publisher = {American Physical Society}, doi = {10.1103/PhysRev.x.48.x.73}}
```

BibTeX :

```
@article {EinsteinRosen1935, author = {Einstein, A. and Rosen, N.},
title = {The Particle Problem in the General Theory of Relativity},
journal = {Phys. Rev.}, volume = {48}, issue = {1}, pages = {73--\times 77},
year = {1935}, publisher = {American Physical Society},
doi = {10.1103 / PhysRev.x.48.x.73}}
```

BibTeX:

```
@article{EinsteinRosen1935,
  author = {Einstein, A. and Rosen, N.},
  title = {The Particle Problem in the General Theory of Relativity},
  journal = {Phys. Rev.},
  volume = {48},
  issue = {1},
  pages = {73--77},
  year = {1935},
  publisher = {American Physical Society},
  doi = {10.1103/PhysRev.48.73}
}
```

74

A. EINSTEIN AND N. ROSEN

In the following we shall show that it is possible to do this in a natural way, that the question we are raising can be answered in the affirmative.

§1. A SPECIAL KIND OF SINGULARITY AND ITS REMOVAL

The first step to the general theory of relativity was to be found in the so-called "Principle of Equivalence": If in a space free from gravitation a reference system is uniformly accelerated, the reference system can be treated as being "at rest," provided one interprets the condition of the space with respect to it as a homogeneous gravitational field. As is well known the latter is exactly described by the metric field¹

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + \alpha^2 x_1^2 dx_4^2. \quad (1)$$

The $g_{\mu\nu}$ of this field satisfy in general the equations

$$R^l_{klm} = 0, \quad (2)$$

and hence the equations

$$R_{kl} = R^m_{klm} = 0. \quad (3)$$

The $g_{\mu\nu}$ corresponding to (1) are regular for all finite points of space-time. Nevertheless one cannot assert that Eqs. (3) are satisfied by (1) for all finite values of x_1, \dots, x_4 . This is due to the fact that the determinant g of the $g_{\mu\nu}$ vanishes for $x_1=0$. The contravariant $g^{\mu\nu}$ therefore become infinite and the tensors R^l_{klm} and R_{kl} take on the form 0/0. From the standpoint of Eqs. (3) the hyperplane $x_1=0$ then represents a singularity of the field.

We now ask whether the field law of gravitation (and later on the field law of gravitation and electricity) could not be modified in a natural way without essential change so that the solution (1) would satisfy the field equations for all finite points, i.e., also for $x_1=0$. W. Mayer has called our attention to the fact that one can make R^l_{klm} and R_{kl} into rational functions of the $g_{\mu\nu}$ and their first two derivatives by multiplying them by suitable powers of g . It is easy to show

¹ It is worth pointing out that this metric field does not represent the whole Minkowski space but only part of it. Thus, the transformation that converts

$ds^2 = -d\xi_1^2 - d\xi_2^2 - d\xi_3^2 + d\xi_4^2$

into (1) is $\xi_1 = x_1 \cosh \sigma x_4$, $\xi_2 = x_2$,

that in $g^2 R_{kl}$ there is no longer any denominator. If then we replace (3) by

$$R_{kl} = g^2 R_{kl} = 0, \quad (3a)$$

this system of equations is satisfied by (1) at all finite points. This amounts to introducing in place of the $g^{\mu\nu}$ the cofactors $[g_{\mu\nu}]$ of the $g_{\mu\nu}$ in g in order to avoid the occurrence of denominators. One is therefore operating with tensor densities of a suitable weight instead of with tensors. In this way one succeeds in avoiding singularities of that special kind which is characterized by the vanishing of g .

The solution (1) naturally has no deeper physical significance insofar as it extends into spatial infinity. It allows one to see however to what extent the regularization of the hypersurfaces $g=0$ leads to a theoretical representation of matter, regarded from the standpoint of the original theory. Thus, in the framework of the original theory one has the gravitational equations

$$R_{ik} - \frac{1}{2} g_{ik} R = -T_{ik}, \quad (4)$$

where T_{ik} is the tensor of mass or energy density. To interpret (1) in the framework of this theory we must approximate the line element by a slightly different one which avoids the singularity $g=0$. Accordingly we introduce a small constant σ and let

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + (\alpha^2 x_1^2 + \sigma) dx_4^2; \quad (1a)$$

the smaller $\sigma (> 0)$ is chosen, the nearer does this gravitational field come to that of (1). If one calculates from this the (fictitious) energy tensor T_{ik} one obtains as nonvanishing components

$$T_{22} = T_{23} = \alpha^2 / \sigma / (1 + \alpha^2 x_1^2 / \sigma)^2.$$

We see then that the smaller one takes σ the more is the tensor concentrated in the neighborhood of the hypersurface $x_1=0$. From the standpoint of the original theory the solution (1) contains a singularity which corresponds to an energy or mass concentrated in the surface $x_1=0$; from the standpoint of the modified theory, however, (1) is a solution of (3a), free from singulari-

In the following we shall show that it is possible to do this in a natural way, that the question we are raising can be answered in the affirmative.

§1. A SPECIAL KIND OF SINGULARITY AND ITS REMOVAL

The first step to the general theory of relativity was to be found in the so-called "Principle of Equivalence": If in a space free from gravitation a reference system is uniformly accelerated, the reference system can be treated as being "at rest," provided one interprets the condition of the space with respect to it as a homogeneous gravitational field. As is well known the latter is exactly described by the metric field¹

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + \alpha^2 x_1^2 dx_4^2. \quad (1)$$

The $g_{\mu\nu}$ of this field satisfy in general the equations

$$R^i{}_{klm} = 0, \quad (2)$$

and hence the equations

$$R_{kl} = R^m{}_{klm} = 0. \quad (3)$$

The $g_{\mu\nu}$ corresponding to (1) are regular for all finite points of space-time. Nevertheless one cannot assert that Eqs. (3) are satisfied by (1) for all finite values of x_1, \dots, x_4 . This is due to the fact that the determinant g of the $g_{\mu\nu}$ vanishes for $x_1 = 0$. The contravariant $g^{\mu\nu}$ therefore become infinite and the tensors $R^i{}_{klm}$ and R_{kl} take on the form 0/0. From the standpoint of Eqs. (3) the hyperplane $x_1 = 0$ then represents a singularity of the field.

We now ask whether the field law of gravitation (and later on the field law of gravitation and electricity) could not be modified in a natural way without essential change so that the solution (1) would satisfy the field equations for all finite points, i.e., also for $x_1 = 0$. W. Mayer has called our attention to the fact that one can make $R^i{}_{klm}$ and R_{kl} into rational functions of the $g_{\mu\nu}$ and their first two derivatives by multiplying them by suitable powers of g . It is easy to show

¹ It is worth pointing out that this metric field does not represent the whole Minkowski space but only part of it. Thus, the transformation that converts

into (1) is

$$\begin{aligned} ds^2 &= -d\xi_1^2 - d\xi_2^2 - d\xi_3^2 + d\xi_4^2 \\ \xi_1 &= x_1 \cosh \alpha x_4, & \xi_3 &= x_3, \\ \xi_2 &= x_2, & \xi_4 &= x_1 \sinh \alpha x_4. \end{aligned}$$

It follows that only those points for which $\xi_4^2 \geq \xi_1^2$ correspond to points for which (1) is the metric.

that in $g^2 R_{kl}$ there is no longer any denominator. If then we replace (3) by

$$R_{kl}^* = g^2 R_{kl} = 0, \quad (3a)$$

this system of equations is satisfied by (1) at all finite points. This amounts to introducing in place of the $g^{\mu\nu}$ the cofactors $[g_{\mu\nu}]$ of the $g_{\mu\nu}$ in g in order to avoid the occurrence of denominators. One is therefore operating with tensor densities of a suitable weight instead of with tensors. In this way one succeeds in avoiding singularities of that special kind which is characterized by the vanishing of g .

The solution (1) naturally has no deeper physical significance insofar as it extends into spatial infinity. It allows one to see however to what extent the regularization of the hypersurfaces $g=0$ leads to a theoretical representation of matter, regarded from the standpoint of the original theory. Thus, in the framework of the original theory one has the gravitational equations

$$R_{ik} - \frac{1}{2} g_{ik} R = -T_{ik}, \quad (4)$$

where T_{ik} is the tensor of mass or energy density. To interpret (1) in the framework of this theory we must approximate the line element by a slightly different one which avoids the singularity $g=0$. Accordingly we introduce a small constant σ and let

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + (\alpha^2 x_1^2 + \sigma) dx_4^2; \quad (1a)$$

the smaller $\sigma (> 0)$ is chosen, the nearer does this gravitational field come to that of (1). If one calculates from this the (fictitious) energy tensor T_{ik} one obtains as nonvanishing components

$$T_{22} = T_{33} = \alpha^2 / \sigma / (1 + \alpha^2 x_1^2 / \sigma)^2.$$

We see then that the smaller one takes σ the more is the tensor concentrated in the neighborhood of the hypersurface $x_1 = 0$. From the standpoint of the original theory the solution (1) contains a singularity which corresponds to an energy or mass concentrated in the surface $x_1 = 0$; from the standpoint of the modified theory, however, (1) is a solution of (3a), free from singularities, which describes the "field-producing mass," without requiring for this the introduction of any new field quantities.

§2. THE SCHWARZSCHILD SOLUTION

As is well known, Schwarzschild found the spherically symmetric static solution of the gravitational equations

$$\begin{aligned} ds^2 &= -\frac{1}{1-2m/r} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad + (1-2m/r)dt^2, \quad (5) \end{aligned}$$

($r > 2m$, θ from 0 to π , ϕ from 0 to 2π); the variables x_1, x_2, x_3, x_4 are here r, θ, ϕ, t . The vanishing

represent the whole Minkowski space but only part of it. Thus, the transformation that converts

$$ds^2 = -d\xi_1^2 - d\xi_2^2 - d\xi_3^2 + d\xi_4^2$$

into (1) is

$$\xi_1 = x_1 \cosh \alpha x_4, \quad \xi_2 = x_2, \quad \xi_3 = x_3,$$

212 | Pre-gravity_Pre-Big_Bang_M6=3 Generations_of_Einstein_Rosen_2-planes_and_3-Planes

It follows that only those points for which $\xi_1^2 \geq \xi_4^2$ correspond to points for which (1) is the metric.

energy or mass concentrated in the surface $x_1=0$; from the standpoint of the modified theory, however, (1) is a solution of (3a), free from singularities, which describes the "field-producing mass," without requiring for this the introduction of any new field quantities.

§2. THE SCHWARZSCHILD SOLUTION

As is well known, Schwarzschild found the spherically symmetric static solution of the gravitational equations

$$ds^2 = -\frac{1}{1-2m/r} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1-2m/r)dt^2, \quad (5)$$

($r > 2m$, θ from 0 to π , ϕ from 0 to 2π); the variables x_1, x_2, x_3, x_4 are here r, θ, ϕ, t . The vanishing of the determinant of the $g_{\mu\nu}$ for $\theta=0$ is unimportant, since the corresponding (spatial) direction is not preferred. On the other hand g_{11} for $r=2m$ becomes infinite and hence we have there a singularity.

If one introduces in place of r a new variable according to the equation

$$u^2 = r - 2m,$$

one obtains for ds^2 the expression

$$ds^2 = -4(u^2 + 2m)du^2 - (u^2 + 2m)^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{u^2}{u^2 + 2m}dt^2. \quad (5a)$$

These new $g_{\mu\nu}$ are regular functions for all values of the variables. For $u=0$, however, g_{44} vanishes, hence also the determinant g . This does not prevent the field equations (3a), which have no denominators, from being satisfied for all values of the independent variables. We are therefore dealing with a solution of the (new) field equations, which is free from singularities for all finite

points. The hypersurface $u=0$ (or in the original variables, $r=2m$) plays here the same role as the hypersurface $x_1=0$ in the previous example.

As u varies from $-\infty$ to $+\infty$, r varies from $+\infty$ to $2m$ and then again from $2m$ to $+\infty$. If one tries to interpret the regular solution (5a) in the space of r, θ, ϕ, t , one arrives at the following conclusion. The four-dimensional space is described mathematically by two congruent parts or "sheets," corresponding to $u>0$ and $u<0$, which are joined by a hyperplane $r=2m$ or $u=0$ in which g vanishes.² We call such a connection between the two sheets a "bridge."

We see now in the given solution, free from singularities, the mathematical representation of an elementary particle (neutron or neutrino). Characteristic of the theory we are presenting is the description of space by means of two sheets. A bridge, spatially finite, which connects these sheets characterizes the presence of an electrically neutral elementary particle. With this conception one not only obtains the representation of an elementary particle by using only the field equations, that is, without introducing new field quantities to describe the density of matter; one is also able to understand the atomistic character of matter as well as the fact that there can be no particles of negative mass. The latter is made clear by the following considerations. If we had started from a Schwarzschild solution with negative m , we should not have been able to make the solution regular by introducing a new variable u instead of r ; that is to say, no "bridge" is possible that corresponds to a particle of negative mass.

If we consider once more the solution (1) from the standpoint of the information we have acquired from the Schwarzschild solution, we see that there also the two congruent halves of the space for $x_1>0$ and $x_1<0$ can be interpreted as two sheets each corresponding to the same physical space. In this sense the example represents a gravitational field, independent of x_2 and x_3 ,

SUBMANIFOLDS AND SPECIAL STRUCTURES
ON THE OCTONIANS

ROBERT L. BRYANT

SUBMANIFOLDS AND SPECIAL STRUCTURES
ON THE OCTONIANS

ROBERT L. BRYANT

0. Introduction

Geometries associated to the exceptional groups and “exceptional” representations of classical groups often display interesting features closely related to (but distinctly different from) the more familiar features of the classical groups. This paper centers on the geometries in E^7 and E^8 whose groups of symmetries are $G_2 \subseteq SO(7)$ and $\text{Spin}(7) \subseteq SO(8)$. Both of these groups are related to the octonians (sometimes called Cayley numbers) and may be defined in terms of octonionic multiplication. In particular, G_2 , the compact exceptional group of (real) dimension 14, is the subgroup of algebra automorphisms of \mathbf{O} (the octonians) and $\text{Spin}(7) \subseteq SO(8)$ may be defined as the subgroup of $GL_R(\mathbf{O})$ generated by right multiplication by unit octonians which are purely imaginary.

The geometry of the algebra \mathbf{O} is closely related to the complex numbers. In §1, we develop some of the properties of \mathbf{O} that we need for later sections. (Our presentation is essentially borrowed from Appendix A of [12], but any mistakes are, of course, due to the author.) A particularly interesting property is described as follows: If we let $\text{Im } \mathbf{O} \subseteq \mathbf{O}$ be the hyperplane (through 0) orthogonal to $1 \in \mathbf{O}$, and we let $S^6 \subseteq \text{Im } \mathbf{O}$ be the space of unit vectors, then right multiplication by $u \in S^6$ induces a linear transformation $R_u: \mathbf{O} \rightarrow \mathbf{O}$ which is orthogonal and satisfies $(R_u)^2 = -1$. Thus, associated to each $u \in S^6$ is a complex structure on \mathbf{O} (induced by $J = R_u$) which is compatible with the natural inner product on \mathbf{O} . We denote by \mathbf{O}_u the Hermitian vector space whose underlying real vector space (with inner product) is \mathbf{O} and whose complex structure is given by R_u .

Classically, this observation was used to define an almost complex structure on S^6 as follows: If $u \in S^6$, then R_u preserves the 2-plane spanned by 1 and u and therefore preserves its orthogonal 6-plane, which may be identified with $T_u S^6 \subseteq \text{Im } \mathbf{O}$ after translation to the origin. Thus R_u induces a complex

structure on $T_u S^6$ for each $u \in S^6$. This almost complex structure is not integrable (even locally) to a complex structure (see below). In [Ca], Calabi noticed that for any oriented $M^6 \subseteq \text{Im } \mathbf{O}$, $R_{N(x)}$ induces a complex structure on $T_x M^6$ (where $N(x)$ is the oriented unit normal). Thus every oriented $M^6 \subseteq \text{Im } \mathbf{O}$ inherits an almost complex structure. Moreover, M^6 inherits a metric from $\text{Im } \mathbf{O}$, so we actually have a $U(3)$ -structure on M^6 . (Calabi calls these structures "almost Hermitian." He also proves that such M^6 possess a canonical $SU(3)$ -substructure but we will not need this.) Calabi shows that the second fundamental form Π of M decomposes with respect to the $U(3)$ -structure into a piece $\Pi^{1,1}$ of type $(1,1)$ and a piece $\Pi^{0,2}$ of type $(0,2)$. He then shows that the almost complex structure of M is integrable if and only if $\Pi^{1,1} = 0$ and that the canonical 2-form of the $U(3)$ -structure, say Ω , is closed if and only if $\Pi^{0,2} = 0$ and $\text{tr}_1 \Pi^{1,1} = 0$. From this it follows that the $U(3)$ -structure on M^6 is Kähler if and only if $\Pi \equiv 0$, so that M^6 is a hyperplane (or a union of pieces of hyperplanes). Calabi then constructs nontrivial examples of $M^6 \subseteq \text{Im } \mathbf{O}$ for which the almost complex structure is integrable. His examples are of the form $S \times \mathbf{R}^4 \subseteq \text{Im } \mathbf{O}$, where $S \subseteq \mathbf{R}^3$ is a minimal surface, $\mathbf{R}^3 \subseteq \text{Im } \mathbf{O}$ is an associative 3-plane, and $\mathbf{R}^4 = (\mathbf{R}^3)^\perp$. Calabi leaves open the problem of determining whether or not there are nontrivial $M^6 \subseteq \text{Im } \mathbf{O}$ for which the canonical 2-form is closed.

In [10], Gray generalized Calabi's construction somewhat by considering hypersurfaces in N^7 where $T_x N^7$ has a vector cross product modeled on $\text{Im } \mathbf{O} \simeq \mathbf{R}^7$. In the case $N = \text{Im } \mathbf{O}$, Gray observes that the canonical 2-form Ω on $M^6 \subseteq \text{Im } \mathbf{O}$ is always co-closed, i.e., $\delta\Omega = 0$ (or equivalently $d\Omega^2 = 0$).

In the present paper, after some preliminary work establishing the structure equations of $\text{Spin}(7) \subseteq SO(8)$, we study oriented manifolds $M^6 \subseteq \mathbf{O}$. As is pointed out in [12], every oriented 6-plane in $\mathbf{O} \simeq \mathbf{R}^8$ is a complex three-plane in \mathbf{O}_u for a unique $u \in S^6$. Thus, every oriented six-manifold in \mathbf{O} inherits a natural $U(3)$ -structure generalizing the case where $M^6 \subseteq \text{Im } \mathbf{O}$. In this case, we decompose the second fundamental form Π of M into three pieces and prove the analogues of Calabi's theorems concerning when the $U(3)$ -structure is complex integrable and when $d\Omega = 0$. In particular, we show that the induced $U(3)$ -structure on $M^6 \subseteq \mathbf{O}$ is Kähler if and only if M^6 is a complex hypersurface in \mathbf{O}_u for some fixed $u \in S^6$. We then go further in the study of those $M^6 \subseteq \mathbf{O}$ for which the $U(3)$ -structure is complex integrable but which are not Kähler. We show that such M^6 are foliated by 4-planes in \mathbf{O} in a unique way. We refer to this foliation as the *asymptotic ruling* of M^6 . Using the moving frame, we prove that if the asymptotic ruling is parallel then M^6 is the product of a fixed 4-plane in \mathbf{O} with a minimal surface in the orthogonal 4-plane. In

particular, we show that Calabi's examples are exactly the M^6 with parallel asymptotic ruling which lie in the hyperplane $\text{Im } \mathbf{O} \subseteq \mathbf{O}$. We then use Cartan's theory of differential systems in involution to show that the analytic non-Kähler

but complex $M^6 \subseteq \mathbf{O}$ “depend” on 12 analytic functions of 1 (real) variable. (For a more precise statement, see §3).

We observe, as did Gray, that the canonical 2-form on $M^6 \subseteq \mathbf{O}$ is *always* co-closed. Finally, we show that any $M^6 \subseteq \mathbf{O}$ for which the canonical 2-form Ω is closed is necessarily Kähler (and therefore must be a complex hypersurface in \mathbf{O}_u for some fixed $u \in S^6$). In particular, such $M^6 \subseteq \text{Im } \mathbf{O}$ must be hyperplanes. This recovers a result of Gray (see [10]).

In the final section of the paper, we study the “complex curves” in S^6 , i.e., those maps $\phi: M^2 \rightarrow S^6$ where M^2 is a Riemann surface and $d\phi$ is complex linear with respect to the almost complex structure on S^6 induced by the inclusion $S^6 \subseteq \text{Im } \mathbf{O}$. This study is motivated by the fact that the cone on such a complex curve gives a 3-fold in $\text{Im } \mathbf{O}$ which is *associative* in the sense of [12]. Such cones are absolutely mass minimizing and their singular structure reflects the singular structure of general associative varieties in $\mathbf{R}^7 \subseteq \text{Im } \mathbf{O}$. We first prove that the almost complex structure on S^6 determines the metric structure of S^6 so that any invariant of the *local* almost complex structure is also a metric invariant (for a more precise statement, see Proposition 4.1 and its proof). (This is the compact-form analogue of Cartan’s characterization of the split form of G_2 as the pseudo-group of a certain differential system on a five manifold.) This justifies our use of the metric structure on S^6 to study the almost complex structure of S^6 .

Since the generalized Cauchy-Riemann equations for local mappings of Riemann surfaces into an *almost* complex manifold form a determined elliptic system (which is first order, quasi-linear) we expect the local theory of complex curves in S^6 to be analogous to the local theory of complex curves in \mathbf{C}^3 . (In the analytic category, this is certainly the case.) Along these lines, we develop a Frenet formalism for complex curves in S^6 analogous to that developed for complex curves in \mathbf{CP}^3 . We define the first, second and the third fundamental forms of $\phi: M^2 \rightarrow S^6$ as holomorphic sections of line bundles over M^2 . In particular, the third fundamental form III, analogous to the torsion of a space curve, plays a crucial role. The assumption that $\text{III} \neq 0$ places severe restrictions on the divisors of the three fundamental forms (see [11] for terminology concerning Riemann surfaces). We are able to prove, for example, that if $M^2 = \mathbf{P}^1$, then $\text{III} \equiv 0$ is impossible. It seems likely that for fixed genus g , the space of complex curves $\phi: M^2 \rightarrow S^6$ (where M^2 has genus g) with $\text{III} \equiv 0$ is finite dimensional, but we have not proven this.

1. The octonians and Spin(7)

In this section, we give a brief description of the octonian algebra \mathbf{O} and derive a few of its properties. We then go on to define the group $\text{Spin}(7) \subseteq SO(8)$ by octonian multiplication and to derive its Lie algebra and structure equations in a form suitable for our differential geometric investigations in the following sections. For more details on \mathbf{O} and $\text{Spin}(7)$, the reader is encouraged to consult Appendix A in [12] and the classical references listed in its bibliography.

An inner product algebra over \mathbf{R} is a vector space \mathbf{A} over \mathbf{R} which possesses a nondegenerate inner product $\langle , \rangle: \mathbf{A} \times \mathbf{A} \rightarrow \mathbf{R}$ and a multiplication $\mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$ with unit $1 \in \mathbf{A}$ so that for all $x, y \in \mathbf{A}$

$$(1.1) \quad \langle xy, xy \rangle = \langle x, x \rangle \langle y, y \rangle.$$

For convenience's sake we will identify \mathbf{R} with the 1-dimensional subalgebra of \mathbf{A} generated by $1 \in \mathbf{A}$. By (1.1), we have $\langle 1, 1 \rangle = \langle 1, 1 \rangle^2$. If $\langle 1, 1 \rangle = 0$, then

show that if we assume that \langle , \rangle is positive definite, then there are only four inner product algebras over \mathbf{R} , namely \mathbf{R} , \mathbf{C} , \mathbf{H} (the quaternions) and \mathbf{O} (the octonians).

Explicitly, we may regard \mathbf{O} as the vector space $\mathbf{H} \oplus \mathbf{H}$. If we write 1 for $(1,0) \in \mathbf{O}$ and ε for $(0,1) \in \mathbf{O}$, the above lemma shows that the multiplication in \mathbf{O} must be given by

$$(1.13) \quad (a + b\varepsilon), (c + d\varepsilon) = (ac - \bar{d}\bar{b}) + (da + b\bar{c})\varepsilon$$

where the inner product satisfies

$$(1.4) \quad \langle (a + b\varepsilon), (a + b\varepsilon) \rangle = a\bar{a} + b\bar{b}$$

whenever $a, b, c, d \in \mathbf{H}$.

We let $S^6 = \{u \in \text{Im } \mathbf{O} \mid \langle u, u \rangle = 1\}$. The elements of S^6 are called the *imaginary units* of \mathbf{O} . For any $u \in S^6$, we have $u = -\bar{u}$, so $u^2 = -u\bar{u} = -\langle u, u \rangle = -1$. We may use u to define a map $J_u: \mathbf{O} \rightarrow \mathbf{O}$ given by

$$(1.15) \quad J_u(x) = xu.$$

The identity (1.9) shows that $J_u^2(x) = (xu)u = xu^2 = -x$, so J_u defines a complex structure on \mathbf{O} . We write \mathbf{O}_u to denote \mathbf{O} endowed with the complex structure J_u . If $u \neq v$, then clearly $J_u \neq J_v$, so we actually have a six-sphere of *distinct* complex structures on \mathbf{O} . However, because S^6 is connected, we see that the orientation of \mathbf{O} induced by the natural orientation of \mathbf{O}_u as a complex vector space does not depend on u . We refer to this orientation as the natural orientation of \mathbf{O} .