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WARNING : syncope, presyncope AHEAD
(yes, i can use Google, too)

Begin

In[1]:=
```
Print["CopyRight (C) 2022, Patrick L. Nash, under the General Public License."]
```
```
CopyRight (C) 2022, Patrick L. Nash, under the General Public License.
```

In[2]:=
```
Print["Please  cite  this  work,  and this web page, if you use it"]
```
```
Please  cite  this  work,  and this web page, if you use it
```

ConvertMapleToMathematicaV2

In[3]:=
```
FileNameJoin[Drop[FileNameSplit[NotebookFileName[]], -1]]
mapleToMathematicaV2PATH = FileNameJoin[{%, "ConvertMapleToMathematicaV2.wl"}]
```

Out[3]=
```
C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX
```

Out[4]=
```
C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX\
  ConvertMapleToMathematicaV2.wl
```

In[5]:=
```
Get[mapleToMathematicaV2PATH]    (*Loads new maple to mathematica parser*)
```
```
ConvertMapleToMathematicaV2 loaded successfully!
  BUT, WARNING:  DO NOT USE IF YOU WANT A CORRECT RESULT!
```

Load the Maple output strings.

In[864]:=
```
FileNameJoin[Drop[FileNameSplit[NotebookFileName[]], -1]]
etoExpPATH = FileNameJoin[{%, "EtoExp.wl"}]
```

Out[864]=
```
C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX
```

Out[865]=
```
C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX\EtoExp.wl
```

In[866]:=
```
Get[etoExpPATH]
```
```
EtoExp.wl loaded.  BUT, WARNING:  DO NOT USE IF YOU WANT A CORRECT RESULT!
```

In[6]:=
```
NotebookFileName[]
```

Out[6]=
```
C:\Users\nsh\Documents\8-dim\2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX\Pre-
  gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes.nb
```

In[7]:=
```
Unprotect[dir];
```

In[8]:=
```
dir = FileNameJoin[Drop[FileNameSplit[NotebookFileName[]], -1]];
Protect[dir];
If[$Failed == SetDirectory[dir], {CreateDirectory[dir], SetDirectory[dir]}];
```

In[11]:=
```
(*FileNameJoin[Drop[FileNameSplit[NotebookFileName[]],-1]]*)
```

In[12]:=
```
names = FileNameSplit[NotebookFileName[]]
```

Out[12]=
```
{C:, Users, nsh, Documents, 8-dim, 2026-02-09-Eternal-DEFLATION-Inflation-M4-MAX,
 Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes.nb}
```

In[13]:=
```
name = StringReplace[names[[-1]], "nb" → "mx"]
```

Out[13]=
```
Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes.mx
```

In[14]:=
```
header = StringReplace[names[[-1]], ".nb" → "-"]
```

Out[14]=
```
Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes-
```

In[15]:=
```
SetOptions[Simplify, TimeConstraint → 1]
```

Out[15]=
```
{Assumptions :→ $Assumptions, ComplexityFunction → Automatic, ExcludedForms → {},
 TimeConstraint → 1, TransformationFunctions → Automatic, Trig → True}
```

In[16]:=
```
SetOptions[FullSimplify, TimeConstraint → 3]
```

Out[16]=
```
{Assumptions :→ $Assumptions, ComplexityFunction → Automatic, ExcludedForms → {},
 TimeConstraint → 3, TransformationFunctions → Automatic, Trig → True}
```

In[17]:=
```
Options[Simplify]
```

Out[17]=
```
{Assumptions :→ $Assumptions, ComplexityFunction → Automatic, ExcludedForms → {},
 TimeConstraint → 1, TransformationFunctions → Automatic, Trig → True}
```

In[18]:=
```
Options[FullSimplify]
```

Out[18]=
```
{Assumptions :→ $Assumptions, ComplexityFunction → Automatic, ExcludedForms → {},
 TimeConstraint → 3, TransformationFunctions → Automatic, Trig → True}
```

In[19]:=
```
{Off[General::spell], Off[General::spell1]};
```

In[20]:=
```
DIM8 = 8;
```

In[21]:=
```
Protect[DIM8, M, K, H]
```

Out[21]=
```
{DIM8, M, K, H}
```

Coordinates : from old ideas in git clone https : //

github.com / 43 d168f3e / Eternal – DEFLATION – Inflation.git

{hidden space, 3 – space, time, superluminal deflating time}

{x0, x1, x2, x3, x4, x5, x6, x7} :

In[22]:=
```
X = {x0, x1, x2, x3, x4, x5, x6, x7};
Protect[X];
Protect[x0, x1, x2, x3, x4, x5, x6, x7];
```

In[25]:=
```
sX0 = And @@ Thread[X > 0]
```

Out[25]=
```
x0 > 0 && x1 > 0 && x2 > 0 && x3 > 0 && x4 > 0 && x5 > 0 && x6 > 0 && x7 > 0
```

In[26]:=
```
ssX = H > 0 && sX0 && 6 H x0 > 0 && 2 la[x4] > 0 && Cot[6 H x0] > 0 && Sin[6 H x0] > 0
```

Out[26]=
```
H > 0 && x0 > 0 && x1 > 0 && x2 > 0 && x3 > 0 && x4 > 0 && x5 > 0 && x6 > 0 &&
 x7 > 0 && 6 H x0[] > 0 && 2 la[x4[]] > 0 && Cot[6 H x0[]] > 0 && Sin[6 H x0[]] > 0
```

For Maple_2025 :

In[27]:=
```
sreplaceZ = Z[#] → ToExpression["Z" <> ToString[#]] & /@ Range[0, 15]
```

Out[27]=
```
{Z[0] → Z0, Z[1] → Z1, Z[2] → Z2, Z[3] → Z3, Z[4] → Z4,
 Z[5] → Z5, Z[6] → Z6, Z[7] → Z7, Z[8] → Z8, Z[9] → Z9, Z[10] → Z10,
 Z[11] → Z11, Z[12] → Z12, Z[13] → Z13, Z[14] → Z14, Z[15] → Z15}
```

In[28]:=
```
sreplacenZ = nZ[#] → ToExpression["nZ" <> ToString[#]] & /@ Range[0, 15]
```

Out[28]=
```
{nZ[0] → nZ0, nZ[1] → nZ1, nZ[2] → nZ2, nZ[3] → nZ3, nZ[4] → nZ4,
 nZ[5] → nZ5, nZ[6] → nZ6, nZ[7] → nZ7, nZ[8] → nZ8, nZ[9] → nZ9, nZ[10] → nZ10,
 nZ[11] → nZ11, nZ[12] → nZ12, nZ[13] → nZ13, nZ[14] → nZ14, nZ[15] → nZ15}
```

lie; speed up Simplify? :

In[29]:=
```
constraintVars = x0 > 0 && x4 > 0 && z > 0 && t > 0
```

Out[29]=
```
x0 > 0 && x4 > 0 && z > 0 && t > 0
```

In[30]:=
```
subsDefects = {}
```

Out[30]=
```
{}
```

future coordinate transformation :

In[31]:=
```
6 * H * x0 == z && H x4 == t
szt = Solve[%, {z, t}][[1]]
sx0x4 = Solve[%%, {x0, x4}][[1]]
Protect[sx0x4, szt];
```

Out[31]=

$6\,H\,x0 == z \,\&\&\, H\,x4 == t$

Out[32]=

$\{z \to 6\,H\,x0,\ t \to H\,x4\}$

Out[33]=

$\left\{x0 \to \frac{z}{6\,H},\ x4 \to \frac{t}{H}\right\}$

SO (4, 4) Minkowski Lorentz metric $\eta 4488$: 4 + 4 spacetime; 8 × 8 dimensional :

In[35]:=
```
(η4488 = ArrayFlatten[{{IdentityMatrix[4], 0}, {0, -IdentityMatrix[4]}}]) //
 MatrixForm
```

Out[35]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

SO (4, 4) Spinor Lorentz metric σ;

τ^A are analogous to the Dirac gamma matrices :

all true :

```
Table[{A, B, FullSimplify[1/2 (τ[A].τ̄[B] + τ[B].τ̄[A]) == η4488[[A + 1, B + 1]] * ID8]},
 {A, 0, 7}, {B, 0, 7}]
```

Type – 1 and type – 2 employ σ (the ususal notational abuse),

since $\sigma == \sigma^{-1}$: 4 + 4 spacetime, 8 × 8 dimensional and 16 × 16 dimensional :

indices. We define the reduced Brauer-Weyl generators $\{\bar{\tau}^{A'}, \tau^{A'}\}$ of the generators of the two real 8×8 inequivalent irreducible representations of $\overline{SO(4,4)}$ (see eqs. (2.6) and (2.7)) by demanding that the tau matrices satisfy (the tilde denotes transpose)

$$\tilde{\bar{\tau}}^{A'} \sigma = \sigma \bar{\tau}^{A'} \tag{2.3}$$

and

$$\tau^{A'} \bar{\tau}^{B'} + \tau^{B'} \bar{\tau}^{A'} = 2IG^{A'B'} = \bar{\tau}^{A'} \tau^{B'} + \bar{\tau}^{B'} \tau^{A'}, \tag{2.4}$$

where I denotes the 8×8 unit matrix. Denoting the matrix elements of $\tau^{A'}$ by $\tau^{A'a}{}_b$, we may write eq. (3) as

$$\bar{\tau}^{A'}{}_{ab} = \tau^{A'}{}_{ba}, \tag{2.5}$$

$$t^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

$$t^A t^B - t^B t^A = \begin{pmatrix} \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A & 0 \\ 0 & \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A \end{pmatrix} = 4 \begin{pmatrix} D_{(1)}{}^{AB} & 0 \\ 0 & D_{(2)}{}^{AB} \end{pmatrix}$$

$$4D_{(1)}{}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A$$

$$4D_{(2)}{}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A.$$

Let $g \in \overline{SO(4,4)}$ and $L \in SO(4,4)$. The canonical 2-1 homomorphism $\overline{SO(4,4)} \to SO(4,4)$ is given by

$$8L^{A'}{}_{B'}(g) = \operatorname{tr} \widetilde{D}^{(2)}(g)\, \tau^{A'} D^{(1)}(g)\, \bar{\tau}_{B'} . \tag{2.8}$$

As usual tr denotes the trace. Under the action of $\overline{SO(4,4)}$

$$\widetilde{D}^{(1)} \sigma = \sigma D^{(1)-1} , \tag{2.9}$$

$$\sigma^{-1} \widetilde{D}^{(2)} = D^{(2)-1} \sigma^{-1} , \tag{2.10}$$

$$L^{A'}{}_{C'} G_{A'B'} L^{B'}{}_{D'} = G_{C'D'} , \tag{2.11}$$

$$L^{A'}{}_{B'} \tau^{B'} = \widetilde{D}^{(2)} \tau^{A'} D^{(1)} , \tag{2.12}$$

$$L^{A'}{}_{B'} \bar{\tau}^{B'} = D^{(1)-1} \bar{\tau}^{A'} \widetilde{D}^{(2)-1} . \tag{2.13}$$

In[36]:=
```
(σ = ArrayFlatten[{{ConstantArray[0, {4, 4}], IdentityMatrix[4]},
    {IdentityMatrix[4], ConstantArray[0, {4, 4}]}}]) // MatrixForm
```

Out[36]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

---

defs; some Symbols; metric

?todo: NEED DefScalarFunction[#]; & /@ (f16[#]&/@Range[0,15]), and allow xact to compute?

Introduce the wave function, Ψ16, for a Universe:

In[37]:=
```
Ψ16 = f16[#][x0, x4] & /@ Range[0, 15]
```

Out[37]=
```
{f16[0][x0, x4], f16[1][x0, x4], f16[2][x0, x4], f16[3][x0, x4],
 f16[4][x0, x4], f16[5][x0, x4], f16[6][x0, x4], f16[7][x0, x4],
 f16[8][x0, x4], f16[9][x0, x4], f16[10][x0, x4], f16[11][x0, x4],
 f16[12][x0, x4], f16[13][x0, x4], f16[14][x0, x4], f16[15][x0, x4]}
```

In[38]:=
```
(*processRawSets[rawSets_]:=Module[{s1,s2,J,ret,L=Range[Length[rawSets]],r={}},
  While[Length[L]>1,
   s1=Union[rawSets⟦L⟦1⟧⟧];
   L=Drop[L,1];
   J=L⟦1⟧;
   While[J≤16,
    If[MemberQ[L,J],
     s2=Union[rawSets⟦J⟧];
     If[Intersection[s1,s2]≠{},s1=Union[Flatten[{s1,s2}]];
      L=Complement[L,{J}];,##&[]];,
     ##&[]];
    J++;];

   AppendTo[r,s1];
  ];
  ret=Sort[r,#1⟦1⟧<#2⟦1⟧&];
  Return[ret];
 ]*)
```

In[39]:=
```
(*rawSetsΨ16=Cases[
    #,
    h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], "f16"] :→ n,
    Infinity,
    Heads → True
]&/@Ψ16*)
```

In[40]:=
```
(*processRawSets[rawSetsΨ16]*)
```

In[41]:=
```
(*Ψnew16=F16[#][x0,x4]&/@Range[0,15]*)
```

In[42]:=
```
Clear[sf16Aa];
sfψ16Aa = f16[#] → ToExpression["((Z[" <> ToString[#] <> "][6*H*#1,H*#2])&)"] & /@
  Range[0, 15]
```

Out[42]=
```
{f16[0] → (Z[0][6 H #1, H #2] &), f16[1] → (Z[1][6 H #1, H #2] &),
 f16[2] → (Z[2][6 H #1, H #2] &), f16[3] → (Z[3][6 H #1, H #2] &),
 f16[4] → (Z[4][6 H #1, H #2] &), f16[5] → (Z[5][6 H #1, H #2] &),
 f16[6] → (Z[6][6 H #1, H #2] &), f16[7] → (Z[7][6 H #1, H #2] &),
 f16[8] → (Z[8][6 H #1, H #2] &), f16[9] → (Z[9][6 H #1, H #2] &),
 f16[10] → (Z[10][6 H #1, H #2] &), f16[11] → (Z[11][6 H #1, H #2] &),
 f16[12] → (Z[12][6 H #1, H #2] &), f16[13] → (Z[13][6 H #1, H #2] &),
 f16[14] → (Z[14][6 H #1, H #2] &), f16[15] → (Z[15][6 H #1, H #2] &)}
```

In[43]:=
```
(*Clear[sf16Aa];
sfψ16Aa=f16[#]→ToExpression[
      "((Z["<>ToString[#]<>"][6*H*#1,H*#2] 1/(Sin[6*H*#1]^(1/2)))&)"]&/@Range[0,15]*)
```

In[44]:=
```
Clear[snewfψ16Aa];
snewfψ16Aa = f16[#] → ToExpression["((nZ[" <> ToString[#] <>
    "][6*H*#1,H*#2] 1/(Sin[6*H*#1]^(1/2))&)"] & /@ Range[0, 15]
```

Out[44]=

$$\left\{f16[0] \to \left(\frac{nZ[0][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\ f16[1] \to \left(\frac{nZ[1][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\right.$$
$$f16[2] \to \left(\frac{nZ[2][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\ f16[3] \to \left(\frac{nZ[3][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),$$
$$f16[4] \to \left(\frac{nZ[4][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\ f16[5] \to \left(\frac{nZ[5][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),$$
$$f16[6] \to \left(\frac{nZ[6][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\ f16[7] \to \left(\frac{nZ[7][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),$$
$$f16[8] \to \left(\frac{nZ[8][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\ f16[9] \to \left(\frac{nZ[9][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),$$
$$f16[10] \to \left(\frac{nZ[10][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\ f16[11] \to \left(\frac{nZ[11][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),$$
$$f16[12] \to \left(\frac{nZ[12][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\ f16[13] \to \left(\frac{nZ[13][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),$$
$$\left.f16[14] \to \left(\frac{nZ[14][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right),\ f16[15] \to \left(\frac{nZ[15][6\,H\,\#1,\ H\,\#2]}{\sqrt{\mathrm{Sin}[6\,H\,\#1]}}\ \&\right)\right\}$$

O (4, 4) related :

In[45]:= `Needs["Notation`"]`

In[46]:= Symbolize[$\bar{\tau}$]

In[47]:= Symbolize[$T16^{A}$]

In[48]:= Symbolize[$T16^{\alpha}$]

In[49]:= Symbolize[m^{C}_{ab}]

In[50]:= Symbolize[m^{C}_{AB}]

In[51]:= Symbolize[$\sigma g^{a}_{1\,A}$]

In[52]:= Symbolize[$\sigma g^{a}_{2\,A}$]

In[53]:= Symbolize[u^{a}]

In[54]:= Symbolize[J_y]

In[55]:= Symbolize[F_a^A]

In[56]:= Symbolize[F_A^a]

In[57]:= Symbolize[$\overline{\sigma 22}$]

$$g^{\alpha\beta} = g_{\alpha\beta} = \mathrm{diag}(1,1,1,-1) \tag{5}$$

In[58]:= Symbolize[$g_{\alpha\beta}$]

note below : η_{AB} is really $g_{(A)\ (B)}$ :: Minkowski – Lorentz metric

In[59]:= Symbolize[η_{AB}]

note below : G_{AB} is really $G_{(A)\ (B)}$ :: Minkowski – Lorentz

In[60]:= (*Symbolize[G_{AB}]*)

In[61]:= Symbolize[S44$\alpha\beta$]

$$\Sigma^{\alpha\beta} = \xi S^{\alpha\beta} u,$$

In[62]:= Symbolize[$\Sigma^{\alpha\beta}$]

In[63]:= (*Symbolize[$\Sigma^{\alpha\beta}$]*)

$$\tfrac{1}{2}\Sigma^{jk}\Sigma_{jk} = \tfrac{1}{4}\,\partial_a\, u^a u^b\, \partial_b - \tfrac{1}{4}\partial_a\, u^b u^b\, \partial_a,$$

In[64]:= Symbolize[Σ^2]

NEED DefScalarFunction[#]; & /@ Flatten[Universe]

In[ ]:= Universe = ToExpression["U" <> ToString[#]] & /@ Range[0, 7]

In[65]:= Symbolize[$\mathcal{F}_a^A$]

In[66]:= Symbolize[$\mathcal{F}_A^a$]

In[67]:=
```
(*Clear[sf16Aa];
sfψ16Aa=f16[#]→ToExpression[
    "((Z["<>ToString[#]<>"][6*H*#1,3*H*#2] 1/(Sin[6*H*#1]^(1/2)))&)"]&/@Range[0,15]*)
```

In[68]:=
```
(*Clear[sf16Aa];
sfψ16Aa=f16[#]→ToExpression["((Z["<>ToString[#]<>"][6*H*#1,3*H*#2])&)"]&/@
  Range[0,15]*)
```

In[69]:=
```
(*Clear[sf16Aa];
sfψ16Aa=f16[#]→ToExpression[
    "((Z["<>ToString[#]<>"][6*H*#1,H*#2] 1/Sin[6*H*#1]^(1/2))&)"]&/@Range[0,15]*)
```

In[70]:=
```
(*Inactivate[((q)&)]Evaluate[Activate[Evaluate[q]]]*)
```

In[71]:= Symbolize[$E_{\alpha}^{(A)}$]

In[72]:= Symbolize[$e_{\alpha}^{(A)}$]

In[73]:= $e_{\alpha}^{(A)}$ = Array[$E_{\alpha}^{(A)}$[#1 - 1, #2 - 1] &, {8, 8}]

Out[73]=

$$\{\{E_{\alpha}^{(A)}[0,0], E_{\alpha}^{(A)}[0,1], E_{\alpha}^{(A)}[0,2], E_{\alpha}^{(A)}[0,3], E_{\alpha}^{(A)}[0,4], E_{\alpha}^{(A)}[0,5], E_{\alpha}^{(A)}[0,6], E_{\alpha}^{(A)}[0,7]\},$$
$$\{E_{\alpha}^{(A)}[1,0], E_{\alpha}^{(A)}[1,1], E_{\alpha}^{(A)}[1,2], E_{\alpha}^{(A)}[1,3], E_{\alpha}^{(A)}[1,4], E_{\alpha}^{(A)}[1,5], E_{\alpha}^{(A)}[1,6], E_{\alpha}^{(A)}[1,7]\},$$
$$\{E_{\alpha}^{(A)}[2,0], E_{\alpha}^{(A)}[2,1], E_{\alpha}^{(A)}[2,2], E_{\alpha}^{(A)}[2,3], E_{\alpha}^{(A)}[2,4], E_{\alpha}^{(A)}[2,5], E_{\alpha}^{(A)}[2,6], E_{\alpha}^{(A)}[2,7]\},$$
$$\{E_{\alpha}^{(A)}[3,0], E_{\alpha}^{(A)}[3,1], E_{\alpha}^{(A)}[3,2], E_{\alpha}^{(A)}[3,3], E_{\alpha}^{(A)}[3,4], E_{\alpha}^{(A)}[3,5], E_{\alpha}^{(A)}[3,6], E_{\alpha}^{(A)}[3,7]\},$$
$$\{E_{\alpha}^{(A)}[4,0], E_{\alpha}^{(A)}[4,1], E_{\alpha}^{(A)}[4,2], E_{\alpha}^{(A)}[4,3], E_{\alpha}^{(A)}[4,4], E_{\alpha}^{(A)}[4,5], E_{\alpha}^{(A)}[4,6], E_{\alpha}^{(A)}[4,7]\},$$
$$\{E_{\alpha}^{(A)}[5,0], E_{\alpha}^{(A)}[5,1], E_{\alpha}^{(A)}[5,2], E_{\alpha}^{(A)}[5,3], E_{\alpha}^{(A)}[5,4], E_{\alpha}^{(A)}[5,5], E_{\alpha}^{(A)}[5,6], E_{\alpha}^{(A)}[5,7]\},$$
$$\{E_{\alpha}^{(A)}[6,0], E_{\alpha}^{(A)}[6,1], E_{\alpha}^{(A)}[6,2], E_{\alpha}^{(A)}[6,3], E_{\alpha}^{(A)}[6,4], E_{\alpha}^{(A)}[6,5], E_{\alpha}^{(A)}[6,6], E_{\alpha}^{(A)}[6,7]\},$$
$$\{E_{\alpha}^{(A)}[7,0], E_{\alpha}^{(A)}[7,1], E_{\alpha}^{(A)}[7,2], E_{\alpha}^{(A)}[7,3], E_{\alpha}^{(A)}[7,4], E_{\alpha}^{(A)}[7,5], E_{\alpha}^{(A)}[7,6], E_{\alpha}^{(A)}[7,7]\}\}$$

In[74]:= (*Symbolize[$se_{\alpha}^{(A)}$]*)

In[75]:= Symbolize[$sge_{\alpha}^{(A)}$]

In[76]:= Symbolize[$sgtrye_{\alpha}^{(A)}$]

In[77]:= Symbolize[$E_{(A)}^{\alpha}$]

In[78]:= Symbolize[$sge_{(A)}^{\alpha}$]

In[79]:= Symbolize[$sgtrye_{(A)}^{\alpha}$]

In[80]:= Symbolize[$e_{(A)}^{\alpha}$]

In[81]:=
```
e^α_(A) = Array[E^α_(A)[#1 - 1, #2 - 1] &, {8, 8}]
```

Out[81]=

$$\{\{E^{\alpha}_{(A)}[0,0], E^{\alpha}_{(A)}[0,1], E^{\alpha}_{(A)}[0,2], E^{\alpha}_{(A)}[0,3], E^{\alpha}_{(A)}[0,4], E^{\alpha}_{(A)}[0,5], E^{\alpha}_{(A)}[0,6], E^{\alpha}_{(A)}[0,7]\},$$
$$\{E^{\alpha}_{(A)}[1,0], E^{\alpha}_{(A)}[1,1], E^{\alpha}_{(A)}[1,2], E^{\alpha}_{(A)}[1,3], E^{\alpha}_{(A)}[1,4], E^{\alpha}_{(A)}[1,5], E^{\alpha}_{(A)}[1,6], E^{\alpha}_{(A)}[1,7]\},$$
$$\{E^{\alpha}_{(A)}[2,0], E^{\alpha}_{(A)}[2,1], E^{\alpha}_{(A)}[2,2], E^{\alpha}_{(A)}[2,3], E^{\alpha}_{(A)}[2,4], E^{\alpha}_{(A)}[2,5], E^{\alpha}_{(A)}[2,6], E^{\alpha}_{(A)}[2,7]\},$$
$$\{E^{\alpha}_{(A)}[3,0], E^{\alpha}_{(A)}[3,1], E^{\alpha}_{(A)}[3,2], E^{\alpha}_{(A)}[3,3], E^{\alpha}_{(A)}[3,4], E^{\alpha}_{(A)}[3,5], E^{\alpha}_{(A)}[3,6], E^{\alpha}_{(A)}[3,7]\},$$
$$\{E^{\alpha}_{(A)}[4,0], E^{\alpha}_{(A)}[4,1], E^{\alpha}_{(A)}[4,2], E^{\alpha}_{(A)}[4,3], E^{\alpha}_{(A)}[4,4], E^{\alpha}_{(A)}[4,5], E^{\alpha}_{(A)}[4,6], E^{\alpha}_{(A)}[4,7]\},$$
$$\{E^{\alpha}_{(A)}[5,0], E^{\alpha}_{(A)}[5,1], E^{\alpha}_{(A)}[5,2], E^{\alpha}_{(A)}[5,3], E^{\alpha}_{(A)}[5,4], E^{\alpha}_{(A)}[5,5], E^{\alpha}_{(A)}[5,6], E^{\alpha}_{(A)}[5,7]\},$$
$$\{E^{\alpha}_{(A)}[6,0], E^{\alpha}_{(A)}[6,1], E^{\alpha}_{(A)}[6,2], E^{\alpha}_{(A)}[6,3], E^{\alpha}_{(A)}[6,4], E^{\alpha}_{(A)}[6,5], E^{\alpha}_{(A)}[6,6], E^{\alpha}_{(A)}[6,7]\},$$
$$\{E^{\alpha}_{(A)}[7,0], E^{\alpha}_{(A)}[7,1], E^{\alpha}_{(A)}[7,2], E^{\alpha}_{(A)}[7,3], E^{\alpha}_{(A)}[7,4], E^{\alpha}_{(A)}[7,5], E^{\alpha}_{(A)}[7,6], E^{\alpha}_{(A)}[7,7]\}\}$$

In[82]:=
```
(*gtrye^(A)_α = (e^(A)_α /.sgtrye^(A)_α)*)
```

In[83]:=
```
(*gtrye^α_(A) = (e^α_(A) /.sgtrye^α_(A))*)
```

In[84]:=
```
(*Symbolize[Γ^κ_α β]*)
```

In[85]:=
```
(*preSpinConnection=Array[Γ^κ_α β[##]&,{8,8,8}];*)
```

In[86]:=
```
(*Symbolize[ E^A_a ]*)
```

In[87]:=
```
(*Symbolize[ E^a_A ]*)
```

In[88]:=
```
(* F̄^A_a  F̄^a_A *)
```

In[89]:=
```
(*Do[ F̄^A_a =E^A_a[h]=Table[(u[[h]].σ.(τ̄[B])),{B,1,8}],{h,1,Length[u]}];
Do[E^a_A[h]=FullSimplify[Inverse[E^A_a[h]]],{h,1,Length[u]}];
Table[ F̄^a_A =E^a_A[h]== (u[[h]].σ.u[[h]])Transpose[Table[
      FullSimplify[ExpandAll[ η8[[B,B]]τ[B].u[[h]] ]],{B,1,8}]],{h,1,Length[u]}]*)
```

In[90]:=
```
(*Table[
 FullSimplify[ExpandAll[Transpose[E^A_a[h]].η8.E^A_a[h]]]- (u[[h]].σ.u[[h]])σ== Zero,
 {h,1,Length[u]}]*)
```

In[91]:=
```
(*iη88=FullSimplify[Inverse[η8]];*)
```

In[92]:=
```
(*Clear[ω^(a)_μ (b)];*)Clear[ω]; Symbolize[ω^(a)_μ (b)]
```

In[93]:=
```
ω = Array[ω^(a)_μ (b)[##] &, {8, 8, 8}];
Protect[ω]
```

Out[94]=

$$\{\omega\}$$

constants

← ERROR : O8similarityTransformation has 1 st index that transforms as $\frac{\partial}{\partial \Psi}$, not as Ψs

In[95]:=
```
(* O8similarityTransformation has 1st index that transforms as ∂/∂Ψ , not as Ψ *)
```

O8similarityTransformation has 1 st index that transforms as $\frac{\partial}{\partial \Psi}$, not as Ψ :

In[96]:=
```
(*ArrayFlatten[{{IdentityMatrix[4],0},{0,-IdentityMatrix[4]}}]*)
```

In[97]:=
```
ID4 = IdentityMatrix[4];
ID8 = IdentityMatrix[8];
```

In[99]:=
```
η4488 // MatrixForm
```

Out[99]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[100]:=
```
η4488[[#, #]] & /@ Range[8]
```

Out[100]=
```
{1, 1, 1, 1, -1, -1, -1, -1}
```

In[101]:=
```
Zero4 = ConstantArray[0, {4, 4}]; Zero8 = ConstantArray[0, {8, 8}];
mid = ExpandAll[-1 * ID4];
ϵ3 = Array[Signature[{##}] &, {3, 3, 3}]
ϵ4 = Array[Signature[{##}] &, {4, 4, 4, 4}];
MId = ExpandAll[-1 * ID8];
simp = {Zero4 → 0, ID4 → 1, mid → -1};
Simp = {Zero8 → ZERO, ID8 → ONE, MId → MONE};
```

Out[103]=
```
{{{0, 0, 0}, {0, 0, 1}, {0, -1, 0}},
 {{0, 0, -1}, {0, 0, 0}, {1, 0, 0}}, {{0, 1, 0}, {-1, 0, 0}, {0, 0, 0}}}
```

In[108]:=
```
Zero16 = ConstantArray[0, {16, 16}];
G16 = ArrayFlatten[{{η8, 0}, {0, -η8}}];
Id16 = IdentityMatrix[16]; MId16 = ExpandAll[-1 * Id16];
Simp16 = {Zero16 → ZERO16, Id16 → ONE16, MId16 → MONE16};
```

In[112]:=
```
ZERO16 = ConstantArray[0, {16, 16}];
G16 = ArrayFlatten[{{η4488, 0}, {0, -η4488}}];
ID16 = IdentityMatrix[16]; MId16 = ExpandAll[-1 * ID16];
Simp16 = {ZERO16 → ZERO16, ID16 → ONE16, MId16 → MONE16};
```

In[116]:=

```
ωs = Flatten[Table[
   ToExpression["ω" <> ToString[A1] <> ToString[B1]], {A1, 1, 7}, {B1, A1 + 1, 8}]]
Length[ωs]
```

Out[116]=

```
{ω12, ω13, ω14, ω15, ω16, ω17, ω18, ω23, ω24, ω25, ω26, ω27, ω28,
 ω34, ω35, ω36, ω37, ω38, ω45, ω46, ω47, ω48, ω56, ω57, ω58, ω67, ω68, ω78}
```

Out[117]=

```
28
```

In[118]:=

```
σ.σ - ID8 == Zero8
Tr[σ] == 0
```

Out[118]=

```
True
```

Out[119]=

```
True
```

---

some function definitions :

In[120]:=

```
Clear[rawSets];
rawSets[l_, f_, o_ : 0] := Module[{t, h},
  t = Table[Cases[
  l[[j]],
h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], f] :> n + o,
  Infinity,
  Heads → True
], {j, 1, Length[l]}];
  Return[t];
 ]
```

In[122]:=

```
(*rawSetsel16=rawSets[el16,"f16",0]*)
```

In[123]:=

```
(*couplings=showCoupledEquations[rawSetsel16]
 %//Flatten//Sort
 %===Range[0,15]*)
```

In[124]:=

```
Clear[dualRawSets];
dualRawSets[l_, f_, o_, h_, oh_] := Module[{r, r1, r2},
  r1 = rawSets[l, f, o];
  r2 = rawSets[l, h, oh];
  r = Union[Flatten[{r1, r2}, 1]] // Drop[#, 1] &;
  Return[r];
 ]
```

'FixedPoint', below, uses 'MergeSetsStep', below.
According to Wolfram docs, one uses 'Throw' to exit from 'FixedPoint' before it is finished.

Details and Options

- FixedPoint always returns the last result it gets

FixedPoint really uses Catch and Throw, so what could go wrong

Details and Options

- FixedPoint always returns the last result it gets.
- You can use Throw to exit from FixedPoint before it is finished.
- FixedPoint[*f*, *expr*] applies SameQ to successive pairs of results to determine whether a fixed point has been reached.
- FixedPoint[*f*, *expr*, ..., SameTest -> *s*] applies *s* to successive pairs of results.

In[126]:=

```
(* Helper to perform one pass of merging *)
MergeSetsStep[currentSets_List] :=
  Module[{n = Length[currentSets], i, j, merged = currentSets},
 (* We look for the first pair (i, j) that intersects *)
 Catch[
  For[i = 1, i ≤ Length[merged], i++,
   For[j = i + 1, j ≤ Length[merged], j++,
    If[Intersection[merged[[i]], merged[[j]]] =!= {},
     (* Found intersection: Merge and Throw to restart/finish this step *)
     merged = Delete[merged, {{i}, {j}}];
     AppendTo[merged, Sort[Union[currentSets[[i]], currentSets[[j]]]]];
     Throw[Sort[merged]]; (* Sort for canonical form *)
    ];
   ];
  ];
  (* If no intersection found, return original sorted *)
  Sort[merged]
 ]
];
```

In[127]:=

```
showCoupledEquations[items_List] := FixedPoint[MergeSetsStep, items];
```

In[128]:=

```
Clear[safeInverse];
safeInverse[m_?MatrixQ] := Module[{a, b},
 {a, b} = Dimensions[m];
  Catch[If[a ≠ b, Throw[Return[{a, b}]], ## &[]]];
 Catch[If[Det[m] == 0, Throw[Return["Error: Singular Matrix"]];, ## &[]]];
 Return[ Inverse[m]];];
```

In[130]:=

```
blockPartition[mat_] := Module[{(*blocks,block11,block12,block21,block22,*)ret},
  blocks = Partition[mat, {8, 8}];
  (*block11=blocks⟦1,1⟧ ;*)(*Top-left block*)
  (*block12=blocks⟦1,2⟧ ;*)(*Top-right block*)
  (*block21=blocks⟦2,1⟧ ;*)(*Bottom-left block*)
  (*block22=blocks⟦2,2⟧ ;*)(*Bottom-right block*)
  ret = ArrayFlatten[
     {{blocks⟦1, 1⟧, blocks⟦1, 2⟧}, {blocks⟦2, 1⟧, blocks⟦2, 2⟧}}] === mat;
  If[ret,
   Return[{{blocks⟦1, 1⟧, blocks⟦1, 2⟧}, {blocks⟦2, 1⟧, blocks⟦2, 2⟧}}], ## &[]];
  Return[Null];]
```

In[131]:=

```
eextract[a_] := Extract[a, {#}] & /@ Range[0, Length[a]]
```

Sequence[Subscript[expr, 1], Subscript[expr, 2], …]
represents a sequence of arguments to be spliced automatically into any function .

In[132]:=

```
skelx[x_, matQ_ : True] := Module[{t = {666}},
  If[matQ == True, t = Block[{r},
     r = x;
     Table[Block[{q}, q = SameQ[0, r⟦j, i⟧];
       If[True == q, Style[0, Blue], Style[Length[r⟦j, i⟧], Red]]],
      {j, 1, Length[r]}, {i, 1, Length[r⟦-1⟧]}]]];
  If[matQ == False, t = Block[{r},
     r = x;
     Table[Block[{q}, q = SameQ[0, r⟦j⟧];
       If[True == q, Style[0, {RGBColor → {0, 1 / 3, 0}, Bold}],
        Style[Length[r⟦j⟧], Red]]], {j, 1, Length[r]}]]];
  MatrixForm[t, TableAlignments → Left]]
```

In[133]:=

```
makeSym[size_, fn_] := Module[{rtmp}, rtmp = Table[fn[i, j], {i, 1, size}, {j, 1, i}];
  MapThread[Join, {rtmp, Rest /@ Flatten[rtmp, {{2}, {1}}]}]]
```

In[134]:=

```
makeAntiSym[size_, fn_] :=
 Module[{rtmp}, rtmp = Table[If[i == j, 0, fn[i, j]], {i, 1, size}, {j, 1, i}];
  MapThread[Join, {rtmp, Rest /@ Flatten[-rtmp, {{2}, {1}}]}]]
```

In[135]:=

```
Block[{MX}, MX = makeAntiSym[8, Subscript[ωω, ##] &]]
```

Out[135]=

$$\{\{0, -\omega\omega_{2,1}, -\omega\omega_{3,1}, -\omega\omega_{4,1}, -\omega\omega_{5,1}, -\omega\omega_{6,1}, -\omega\omega_{7,1}, -\omega\omega_{8,1}\},$$
$$\{\omega\omega_{2,1}, 0, -\omega\omega_{3,2}, -\omega\omega_{4,2}, -\omega\omega_{5,2}, -\omega\omega_{6,2}, -\omega\omega_{7,2}, -\omega\omega_{8,2}\},$$
$$\{\omega\omega_{3,1}, \omega\omega_{3,2}, 0, -\omega\omega_{4,3}, -\omega\omega_{5,3}, -\omega\omega_{6,3}, -\omega\omega_{7,3}, -\omega\omega_{8,3}\},$$
$$\{\omega\omega_{4,1}, \omega\omega_{4,2}, \omega\omega_{4,3}, 0, -\omega\omega_{5,4}, -\omega\omega_{6,4}, -\omega\omega_{7,4}, -\omega\omega_{8,4}\},$$
$$\{\omega\omega_{5,1}, \omega\omega_{5,2}, \omega\omega_{5,3}, \omega\omega_{5,4}, 0, -\omega\omega_{6,5}, -\omega\omega_{7,5}, -\omega\omega_{8,5}\},$$
$$\{\omega\omega_{6,1}, \omega\omega_{6,2}, \omega\omega_{6,3}, \omega\omega_{6,4}, \omega\omega_{6,5}, 0, -\omega\omega_{7,6}, -\omega\omega_{8,6}\},$$
$$\{\omega\omega_{7,1}, \omega\omega_{7,2}, \omega\omega_{7,3}, \omega\omega_{7,4}, \omega\omega_{7,5}, \omega\omega_{7,6}, 0, -\omega\omega_{8,7}\},$$
$$\{\omega\omega_{8,1}, \omega\omega_{8,2}, \omega\omega_{8,3}, \omega\omega_{8,4}, \omega\omega_{8,5}, \omega\omega_{8,6}, \omega\omega_{8,7}, 0\}\}$$

If ‘paste’ Maple MathML to ToExpression[MathMLForm[]] fails, then try to edit this :

O(4,4); evalues, evecs of σ

In[136]:=

```
{evalues, evecs} = Eigensystem[σ]
```

Out[136]=

```
{{-1, -1, -1, -1, 1, 1, 1, 1}, {{0, 0, 0, -1, 0, 0, 0, 1}, {0, 0, -1, 0, 0, 0, 1, 0},
  {0, -1, 0, 0, 0, 1, 0, 0}, {-1, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 1},
  {0, 0, 1, 0, 0, 0, 1, 0}, {0, 1, 0, 0, 0, 1, 0, 0}, {1, 0, 0, 0, 1, 0, 0, 0}}}
```

In[137]:=

$$\left(\mathbf{u} = \mathbf{ExpandAll}\left[\frac{1}{\sqrt{2}}\ \mathbf{evecs}\right]\right)\ //\ \mathbf{MatrixForm}$$

Out[137]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

In[138]:=

```
Table[u[[h]].σ.u[[h]], {h, 1, Length[u]}]
```

Out[138]=

```
{-1, -1, -1, -1, 1, 1, 1, 1}
```

In[139]:=

```
Table[(Transpose[u][[h]]).σ.u[[h]], {h, 1, Length[u]}]
```

Out[139]=

```
{-1, -1, -1, -1, 1, 1, 1, 1}
```

In[140]:=

```
Table[(Transpose[u][[A]]) == -σ.u[[A]], {A, 1, 4}]
Table[(Transpose[u][[A]]) == σ.u[[A]], {A, 5, Length[u]}]
```

Out[140]=

```
{True, True, True, True}
```

Out[141]=

```
{True, True, True, True}
```

SO(4), γ; M8, {0, 1, 2, 3, 4, 5, 6, 7}, {+,+,+,+,-,-,-,-}, {x0, x1, x2, x3, x4, x5, x6, x7}

In[142]:=

Under the involutive automorphism $\gamma^{AB} \to -\tilde{\gamma}^{AB}$ of so(3,3), the Lie algebra decomposes into the eigenvalue (-1) and eigenvalue $(+1)$ subspaces corresponding to, respectively, the nine linearly independent real traceless symmetric 4×4 matrices, and the six linearly independent real skew-symmetric 4×4 matrices. The eigenvalue $(+1)$ subspace is the subalgebra so(4), which is the Lie algebra of $\overline{\mathrm{SO}(4)}$, the maximal compact subgroup of $\overline{\mathrm{SL}(4,\mathrm{R})}$. The subalgebra $\mathrm{so}(4) \cong \mathrm{su}(2) \dot{+} \mathrm{su}(2)$ may be further decomposed into the even (eigenvalue $+1$) and odd (eigenvalue -1) subspaces of the linear transformation of so(4) whereby $\tau \in \mathrm{so}(4)$ is mapped into its dual, $^*\tau$. The even subspace under * of so(4) corresponds to self-dual tensors, and, say, the first su(2) in the direct sum; the odd subspace corresponds to anti-self-dual tensors, and the second su(2) in the direct sum. A basis for so(4) may be chosen as follows. Each of the six skew-symmetric γ matrices has the property that the square of the matrix is equal to $-\gamma_0$. By Eq. (1), these six matrices are given by $(h = 1,2,3)$,

$$2s^h = (\gamma^{23}, \gamma^{31}, \gamma^{21}), \tag{4}$$

and

$$2t^h = (\gamma^{45}, \gamma^{64}, \gamma^{65}). \tag{5}$$

Out[142]=

Under the involutive automorphism $\gamma^{AB} \to -\tilde{\gamma}^{AB}$ of so(3,3), the Lie algebra decomposes into the eigenvalue (-1) and eigenvalue $(+1)$ subspaces corresponding to, respectively, the nine linearly independent real traceless symmetric 4×4 matrices, and the six linearly independent real skew-symmetric 4×4 matrices. The eigenvalue $(+1)$ subspace is the subalgebra so(4), which is the Lie algebra of $\overline{SO(4)}$, the maximal compact subgroup of $\overline{SL(4,R)}$. The subalgebra $so(4) \cong su(2) \dot{+} su(2)$ may be further decomposed into the even (eigenvalue $+1$) and odd (eigenvalue -1) subspaces of the linear transformation of so(4) whereby $\tau \in so(4)$ is mapped into its dual, $^*\tau$. The even subspace under * of so(4) corresponds to self-dual tensors, and, say, the first su(2) in the direct sum; the odd subspace corresponds to anti-self-dual tensors, and the second su(2) in the direct sum. A basis for so(4) may be chosen as follows. Each of the six skew-symmetric γ matrices has the property that the square of the matrix is equal to $-\gamma_0$. By Eq. (1), these six matrices are given by $(h = 1,2,3)$,

$$2s^h = (\gamma^{23}, \gamma^{31}, \gamma^{21}), \tag{4}$$

and

$$2t^h = (\gamma^{45}, \gamma^{64}, \gamma^{65}). \tag{5}$$

In[143]:=

```
Qa[h_, p_, q_] := Signature[{h, p, q, 4}];
Qb[h_, p_, q_] := ID4[[p, 4]] * ID4[[q, h]] - ID4[[p, h]] * ID4[[q, 4]];
SelfDualAntiSymmetric[h_, p_, q_] := Qa[h, p, q] - Qb[h, p, q] ;
AntiSelfDualAntiSymmetric[h_, p_, q_] := (Qa[h, p, q] + Qb[h, p, q] );
```

In[147]:=

```
Qa[1, 2, 3]
```

Out[147]=

```
1
```

In[148]:=

```
ϵ4[[1, 2, 3, 4]]
```

Out[148]=

```
1
```

In[149]:=

```
Do[
 s4by4[h] = Table[Table[SelfDualAntiSymmetric[h, p, q], {q, 4}], {p, 4}], {h, 1, 3} ]
```

In[150]:=

```
Do[t4by4[h] = Table[Table[AntiSelfDualAntiSymmetric[h, p, q], {q, 4}], {p, 4}],
 {h, 1, 3} ]
```

In[151]:=

```
(*Protect[{s4by4[1],s4by4[2],s4by4[3],t4by4[1],t4by4[2],t4by4[3]}]*)
```

In[152]:=

```
Protect[s4by4, t4by4]
```

Out[152]=

```
{s4by4, t4by4}
```

In[153]:=

```
Clear[sixAntiSymmetric8by8];
sixAntiSymmetric8by8 = Flatten[{Table[ ArrayFlatten[{{0, s4by4[h]}, {s4by4[h], 0}}], {h,
      1, 3} ], Table[ ArrayFlatten[{{0, t4by4[h]}, {-t4by4[h], 0}}], {h, 3, 1, -1} ]}, 1];
```

In[154]:=

```
sixAntiSymmetric8by8
```

Out[154]=

```
{{{0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0},
  {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0},
  {0, 0, 0, 1, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1},
  {0, 0, 0, 0, 0, 0, -1, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1},
  {0, 0, 0, 0, 0, 0, 1, 0}, {0, -1, 0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}, {0, 0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
  {-1, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0, 0, 0, -1}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, -1, 0, 0},
  {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0},
  {0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}}}
```

In[155]:=

```
{s4by4[#] // MatrixForm, MatrixForm[t4by4[#]]} & /@ Range[3]
```

Out[155]=

$$\left\{\left\{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right\},\right.$$

$$\left.\left\{\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right\}\right\}$$

In[156]:=

```
Table[{{J, K}, MatrixForm[s4by4[J].t4by4[K]]}, {J, 1, 3}, {K, 1, 3}]
```

Out[156]=

$$\left\{\left\{\left\{\{1,1\},\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\right\},\left\{\{1,2\},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right\},\left\{\{1,3\},\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\right\}\right\},\right.$$

$$\left\{\left\{\{2,1\},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}\right\},\left\{\{2,2\},\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\right\},\left\{\{2,3\},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\right\}\right\},$$

$$\left.\left\{\left\{\{3,1\},\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\right\},\left\{\{3,2\},\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\right\},\left\{\{3,3\},\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}\right\}\right\}$$

In[157]:=

```
Do[st[J, K] = s4by4[J].t4by4[K], {J, 1, 3}, {K, 1, 3}]
```

In[158]:=

```
Table[{{J, K}, MatrixForm[st[J, K]]}, {J, 1, 3}, {K, 1, 3}]
```

Out[158]=

$$\left\{\left\{\left\{\{1,1\},\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\right\},\left\{\{1,2\},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right\},\left\{\{1,3\},\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\right\}\right\},\right.$$

$$\left\{\left\{\{2,1\},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}\right\},\left\{\{2,2\},\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\right\},\left\{\{2,3\},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\right\}\right\},$$

$$\left.\left\{\left\{\{3,1\},\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\right\},\left\{\{3,2\},\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\right\},\left\{\{3,3\},\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}\right\}\right\}$$

In[159]:=

```
Protect[st]
```

Out[159]=

```
{st}
```

In[160]:=

```
Table[{{J, K}, MatrixForm[st[J, K] - Transpose[st[J, K]]]}, {J, 1, 3}, {K, 1, 3}]
```

Out[160]=

$$\left\{\left\{\left\{\{1,1\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\},\left\{\{1,2\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\},\left\{\{1,3\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}\right\},\right.$$

$$\left\{\left\{\{2,1\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\},\left\{\{2,2\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\},\left\{\{2,3\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}\right\},$$

$$\left.\left\{\left\{\{3,1\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\},\left\{\{3,2\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\},\left\{\{3,3\},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}\right\}\right\}$$

In[161]:=

```
Permutations[Range[3], 2]
```

Out[161]=

```
{{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 1}, {2, 3}, {3, 1}, {3, 2}}
```

In[162]:=

```
KroneckerProduct[Range[3], Range[3]]
```

Out[162]=

```
{{1, 2, 3}, {2, 4, 6}, {3, 6, 9}}
```

In[163]:=

```
Table[{J, K}, {J, 1, 3}, {K, 1, 3}]
st[##] & /@ %
st[##] & @@ %%
```

Out[163]=

```
{{{1, 1}, {1, 2}, {1, 3}}, {{2, 1}, {2, 2}, {2, 3}}, {{3, 1}, {3, 2}, {3, 3}}}
```

Out[164]=

```
{st[{{1, 1}, {1, 2}, {1, 3}}], st[{{2, 1}, {2, 2}, {2, 3}}], st[{{3, 1}, {3, 2}, {3, 3}}]}
```

Out[165]=

```
st[{{1, 1}, {1, 2}, {1, 3}}, {{2, 1}, {2, 2}, {2, 3}}, {{3, 1}, {3, 2}, {3, 3}}]
```

In[166]:=

```
Flatten[{s4by4[#] & /@ Range[3], t4by4[#] & /@ Range[3],
  Flatten[Table[st[J, K], {J, 1, 3}, {K, 1, 3}], 1], {ID4}}, 1]
Length[%]
```

Out[166]=

```
{{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
 {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
 {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
 {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
 {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}},
 {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
 {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}},
 {{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
 {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}
```

Out[167]=

```
16
```

In[168]:=

```
Unprotect[γx];
γx = Flatten[{s4by4[#] & /@ Range[3], t4by4[#] & /@ Range[3],
   Flatten[Table[st[J, K], {J, 1, 3}, {K, 1, 3}], 1], {ID4}}, 1]
```

Out[169]=

```
{{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
 {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
 {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
 {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
 {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}},
 {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
 {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}},
 {{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
 {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}
```

In[170]:=

```
Length[γx]
```

Out[170]=

```
16
```

In[171]:=

```
Protect[γx]
```

Out[171]=

```
{γx}
```

In[172]:=

```
γx[[1]]
```

Out[172]=

```
{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}}
```

In[173]:=

```
γx[[-1]]
```

Out[173]=

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

Self Dual Anti-Symmetric

In[174]:=

$$\text{Table}\left[\left(\frac{1}{2}\sum_{j1=1}^{4}\left(\sum_{j2=1}^{4}\epsilon 4[[p, q, j1, j2]]\times \text{s4by4}[h][[j1, j2]]\right)\right) - \text{s4by4}[h][[p, q]],\ \{h, 1, 3\}, \{q, 4\}, \{p, 4\}\right]$$

Out[174]=

```
{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}
```

Anti SelfDual Anti-Symmetric

In[175]:=

$$\text{Table}\left[\left(\frac{1}{2}\sum_{j1=1}^{4}\left(\sum_{j2=1}^{4}\epsilon 4[[p, q, j1, j2]]\times \text{t4by4}[h][[j1, j2]]\right)\right) + \text{t4by4}[h][[p, q]],\ \{h, 1, 3\}, \{q, 4\}, \{p, 4\}\right]$$

Out[175]=

```
{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}
```

SO(4,4), Spin(4,4) = $\overline{\mathrm{SO}(4,4)}$, τ ; if using xact: M8, {0, 1, 2, 3, 4, 5, 6, 7}, {+,+,+,+,-,-,-,-}, {x0, x1, x2, x3, x4, x5, x6, x7}

In[176]:=

$$\sigma\bar{\tau}^A = \widetilde{\sigma\tau^A} = \tilde{\tau}^A\sigma$$

$$\tau^A\bar{\tau}^B + \tau^B\bar{\tau}^A = 2\mathrm{I}_{8\times 8}\mathrm{G}^{AB} = \bar{\tau}^A\tau^B + \bar{\tau}^B\tau^A,$$

8 unit matrix. Denoting the matrix elemei

$$\bar{\tau}^A{}_{ab} = \tau^A{}_{ba}$$

$$t^A t^B - t^B t^A = \begin{pmatrix} \bar{\tau}^A\tau^B - \bar{\tau}^B\tau^A & 0 \\ 0 & \tau^A\bar{\tau}^B - \tau^B\bar{\tau}^A \end{pmatrix} = 4\begin{pmatrix} D_{(1)}{}^{AB} & 0 \\ 0 & D_{(2)}{}^{AB} \end{pmatrix} \tag{10}$$

of the 16-component spinor representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$. We see that, as is, in fact, well known from the general theory, the 16-component spinor representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ is the direct sum of two (inequivalent) real 8×8 irreducible spinor representations $D_{(1)}=D_{(1)}(g)$ and $D_{(2)}=D_{(2)}(g)$ of $\overline{\mathrm{SO}(4,4;\mathbb{R})} \ni g$ that are generated by $D_{(1)}{}^{AB}$ and $D_{(2)}{}^{AB}$, respectively. This we record as

$$4D_{(1)}{}^{AB} = \bar{\tau}^A\tau^B - \bar{\tau}^B\tau^A \tag{11}$$

and

$$4D_{(2)}{}^{AB} = \tau^A\bar{\tau}^B - \tau^B\bar{\tau}^A. \tag{12}$$

Out[176]=

$$\sigma\bar{\tau}^A = \widetilde{\sigma\tau^A} = \bar{\tau}^A\sigma$$

$$\tau^A\bar{\tau}^B + \tau^B\bar{\tau}^A = 2\mathrm{I}_{8\times 8}\mathrm{G}^{AB} = \bar{\tau}^A\tau^B + \bar{\tau}^B\tau^A,$$

8 unit matrix. Denoting the matrix elemei

$$\bar{\tau}^A{}_{ab} = \tau^A{}_{ba}$$

$$t^A t^B - t^B t^A = \begin{pmatrix} \bar{\tau}^A\tau^B - \bar{\tau}^B\tau^A & 0 \\ 0 & \tau^A\bar{\tau}^B - \tau^B\bar{\tau}^A \end{pmatrix} = 4\begin{pmatrix} D_{(1)}{}^{AB} & 0 \\ 0 & D_{(2)}{}^{AB} \end{pmatrix} \tag{10}$$

of the 16-component spinor representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$. We see that, as is, in fact, well known from the general theory, the 16-component spinor representation of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ is the direct sum of two (inequivalent) real 8×8 irreducible spinor representations $D_{(1)}=D_{(1)}(g)$ and $D_{(2)}=D_{(2)}(g)$ of $\overline{\mathrm{SO}(4,4;\mathbb{R})} \ni g$ that are generated by $D_{(1)}{}^{AB}$ and $D_{(2)}{}^{AB}$, respectively. This we record as

$$4D_{(1)}{}^{AB} = \bar{\tau}^A\tau^B - \bar{\tau}^B\tau^A \tag{11}$$

and

$$4D_{(2)}{}^{AB} = \tau^A\bar{\tau}^B - \tau^B\bar{\tau}^A. \tag{12}$$

In[177]:=

$$\widetilde{D_{(1)}{}^{AB}}\sigma = -\sigma D_{(1)}{}^{AB} \Rightarrow \widetilde{D}_{(1)}\sigma = \sigma D_{(1)}{}^{-1}, \tag{18}$$

$$\widetilde{D_{(2)}{}^{AB}}\sigma = -\sigma D_{(2)}{}^{AB} \Rightarrow \widetilde{D}_{(2)}\sigma = \sigma D_{(2)}{}^{-1}, \tag{19}$$

$$L^A{}_C \mathrm{G}_{AB} L^B{}_D = \mathrm{G}_{CD} = \{\widetilde{L}\mathrm{G}L\}_{CD}, \tag{20}$$

$$L^A{}_B \overline{\tau}^B = D_{(1)}{}^{-1}\overline{\tau}^A D_{(2)}, \tag{21}$$

$$L^A{}_B \tau^B = D_{(2)}{}^{-1}\tau^A D_{(1)}, \tag{22}$$

The canonical 2-1 homomorphism $\overline{\mathrm{SO}(4,4;\mathbb{R})} \to \mathrm{SO}(4,4;\mathbb{R}): g \mapsto L(g)$ is given by

$$8L^A{}_B = \mathrm{tr}(D_{(1)}^{-1}\overline{\tau}^A D_{(2)}\tau^C)\mathrm{G}_{CB}, \tag{23}$$

where tr denotes the trace. Note that $D_{(1)}(g(\omega)) = D_{(2)}(g(\omega))$ when $\omega_{A8}=0$, i.e., when one restricts $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to

$$\overline{\mathrm{SO}(3,4;\mathbb{R})} = \left\{ g \in \overline{\mathrm{SO}(4,4;\mathbb{R})} \,\middle|\, g = \begin{pmatrix} \exp\left(\frac{1}{4}\omega_{AB}D_{(1)}{}^{AB}\right) & 0 \\ 0 & \exp\left(\frac{1}{4}\omega_{AB}D_{(2)}{}^{AB}\right) \end{pmatrix} \text{ and } \omega_{A8} = 0 \right\} \tag{24}$$

[one of the real forms of spin $(7,\mathbb{C})$].

Out[177]=

$$\widetilde{D_{(1)}{}^{AB}}\sigma = -\sigma D_{(1)}{}^{AB} \Rightarrow \tilde{D}_{(1)}\sigma = \sigma D_{(1)}{}^{-1}, \tag{18}$$

$$\widetilde{D_{(2)}{}^{AB}}\sigma = -\sigma D_{(2)}{}^{AB} \Rightarrow \tilde{D}_{(2)}\sigma = \sigma D_{(2)}{}^{-1}, \tag{19}$$

$$L^A{}_C \mathbb{G}_{AB} L^B{}_D = \mathbb{G}_{CD} = \{\tilde{L}\mathbb{G}L\}_{CD}, \tag{20}$$

$$L^A{}_B \bar{\tau}^B = D_{(1)}{}^{-1}\bar{\tau}^A D_{(2)}, \tag{21}$$

$$L^A{}_B \tau^B = D_{(2)}{}^{-1}\tau^A D_{(1)}, \tag{22}$$

The canonical 2-1 homomorphism $\overline{\mathrm{SO}(4,4;\mathbb{R})} \to \mathrm{SO}(4,4;\mathbb{R}): g \mapsto L(g)$ is given by

$$8L^A{}_B = \mathrm{tr}(D_{(1)}^{-1}\bar{\tau}^A D_{(2)}\tau^C)\mathbb{G}_{CB}, \tag{23}$$

where tr denotes the trace. Note that $D_{(1)}(g(\omega)) = D_{(2)}(g(\omega))$ when $\omega_{A8}=0$, i.e., when one restricts $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to

$$\overline{\mathrm{SO}(3,4;\mathbb{R})} = \left\{ g \in \overline{\mathrm{SO}(4,4;\mathbb{R})} \,\middle|\, g = \begin{pmatrix} \exp\left(\frac{1}{4}\omega_{AB}D_{(1)}{}^{AB}\right) & 0 \\ 0 & \exp\left(\frac{1}{4}\omega_{AB}D_{(2)}{}^{AB}\right) \end{pmatrix} \text{ and } \omega_{A8} = 0 \right\}, \tag{24}$$

[one of the real forms of spin $(7,\mathbb{C})$].

In[178]:=

```
η4488 // MatrixForm
```

Out[178]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[179]:=

```
η4488 == DiagonalMatrix[RotateRight[evalues, 4]]
```

Out[179]=

```
True
```

In[180]:=

```
DiagonalMatrix[RotateRight[evalues, 3]] // MatrixForm
```

Out[180]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[181]:=

```
Clear[τ];
τ[0] = ID8;
Do[τ[h] = sixAntiSymmetric8by8[[h]], {h, 1, 6} ];
(τ[7] = τ[1].τ[2].τ[3].τ[4].τ[5].τ[6]) // MatrixForm
(*τ̄[#] =σ. Transpose[σ.τ[#]]&/@Range[0,7]*)
Do[τ̄[h] = σ. Transpose[σ.τ[h]], {h, 0, 7}]
Table[{{A1, B1}, 1/2 (τ[A1].τ̄[B1] + τ[B1].τ̄[A1]) === η4488[[A1 + 1, B1 + 1]] * ID8},
  {A1, 0, 7}, {B1, 0, 7}] // MatrixForm
```

Out[184]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[186]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{0,0\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{0,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{0,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{0,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{0,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{0,5\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{0,6\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{0,7\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{1,0\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,5\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,6\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,7\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,0\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,5\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,6\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,7\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,0\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,5\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,6\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,7\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,0\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,5\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,6\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,7\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{5,0\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,5\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,6\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,7\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{6,0\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{6,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{6,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{6,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{6,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{6,5\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{6,6\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{6,7\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{7,0\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{7,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{7,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{7,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{7,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{7,5\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{7,6\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{7,7\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

In[187]:=

```
σ == τ[1].τ[2].τ[3]
```

Out[187]=

```
True
```

In[188]:=

```
Table[{A, -η4488[[A + 1, A + 1]] * τ[A] == Transpose[τ[A]]} , {A, 1, 7}]
```

Out[188]=

```
{{1, True}, {2, True}, {3, True}, {4, True}, {5, True}, {6, True}, {7, True}}
```

In[189]:=

```
σ.τ[1].τ[2].τ[3] // MatrixForm
```

Out[189]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[190]:=

```
τ[7] // MatrixForm
```

Out[190]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[191]:=

```
(Ω = σ.τ[7]) // MatrixForm
```

Out[191]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[192]:=

```
Ω == τ[4].τ[5].τ[6]
```

Out[192]=

```
True
```

In[193]:=

```
τ[5].τ[6].τ[7] // MatrixForm
```

Out[193]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

In[194]:=

```
(σΩ = σ.Ω) // MatrixForm
```

Out[194]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[195]:=

```
Inverse[σΩ] // MatrixForm
```

Out[195]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[196]:=

```
τ[1].τ[2].τ[3].τ[4].τ[5].τ[6].τ[7] == τ[0] == ID8
```

Out[196]=

True

In[197]:=

```
(σ == τ[1].τ[2].τ[3])
```

Out[197]=

True

In[198]:=

```
(σ == τ[4].τ[5].τ[6].τ[7])
```

Out[198]=

True

In[199]:=

```
τ[1].τ[2].τ[3].τ[#] == -Transpose[τ[1].τ[2].τ[3].τ[#]] & /@ Range[0, 7]
```

Out[199]=

```
{False, True, True, True, True, True, True, True}
```

In[200]:=

```
σ.τ̄[#] == Transpose[σ.τ[#]] & /@ Range[0, 7]
```

Out[200]=

```
{True, True, True, True, True, True, True, True}
```

In[201]:=

```
τ[5].τ[6].τ[7].τ̄[#] == Transpose[τ[5].τ[6].τ[7].τ[#]] & /@ Range[0, 7]
```

Out[201]=

```
{False, True, True, True, False, True, True, True}
```

In[202]:=

```
τ[4].τ[#] == Transpose[τ[4].τ[#]] & /@ Range[0, 7]
```

Out[202]=

```
{True, True, True, True, True, False, False, False}
```

In[203]:=

```
τ[4].τ̄[#] == Transpose[τ[4].τ[#]] & /@ Range[0, 7]
```

Out[203]=

```
{True, False, False, False, False, True, True, True}
```

In[204]:=

```
τ[4].τ[#] == Transpose[τ[4].τ[#]] & /@ Range[0, 7]
```

Out[204]=

```
{True, True, True, True, True, False, False, False}
```

$$t^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$; O (4, 4) : SAB ;

covariantDiffMatrix = $T16^A[5].T16^A[6].T16^A[7]$

$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.\begin{pmatrix} \tilde{a} & \tilde{c} \\ \tilde{b} & \tilde{d} \end{pmatrix} = \begin{pmatrix} b.\tau^A & a.\bar{\tau}^A \\ d.\tau^A & c.\bar{\tau}^A \end{pmatrix}.\begin{pmatrix} \tilde{a} & \tilde{c} \\ \tilde{b} & \tilde{d} \end{pmatrix}$$

$$= \begin{pmatrix} b.\tau^A.\tilde{a} + a.\bar{\tau}^A.\tilde{b} & b.\tau^A.\tilde{c} + a.\bar{\tau}^A.\tilde{d} \\ d.\tau^A.\tilde{a} + c.\bar{\tau}^A.\tilde{b} & d.\tau^A.\tilde{c} + c.\bar{\tau}^A.\tilde{d} \end{pmatrix}$$

Need these in order to define the Lagrangian for the "universe," later :

In[205]:=

```
(σ16 = T16^A[0].T16^A[1].T16^A[2].T16^A[3]) // MatrixForm
```

Out[205]//MatrixForm=

$$T16^A[0].T16^A[1].T16^A[2].T16^A[3]$$

In[206]:=

```
Table[T16^A[A1] = ArrayFlatten[{{0, τ̄[A1]}, {τ[A1], 0}}], {A1, 0, 7}];
```

In[207]:=

```
(T16^A[8] = T16^A[0].T16^A[1].T16^A[2].T16^A[3].T16^A[4].T16^A[5].T16^A[6].T16^A[7]) //
 MatrixForm
```

Out[207]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[208]:=

```
σ16 // MatrixForm
```

Out[208]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[209]:=

```
T16^A[0].T16^A[1].T16^A[2].T16^A[3].T16^A[#] ==
   -Transpose[T16^A[0].T16^A[1].T16^A[2].T16^A[3].T16^A[#]] & /@ Range[0, 8]
```

Out[209]=

```
{True, True, True, True, True, True, True, True, False}
```

In[210]:=

```
T16^A[4].T16^A[5].T16^A[6].T16^A[7].T16^A[#] ==
   Transpose[T16^A[4].T16^A[5].T16^A[6].T16^A[7].T16^A[#]] & /@ Range[0, 8]
```

Out[210]=

```
{True, True, True, True, True, True, True, True, True}
```

In[211]:=

```
T16^A[8] == σ16.T16^A[4].T16^A[5].T16^A[6].T16^A[7]
```

Out[211]=

```
True
```

In[212]:=

```
T16^A[#] == Transpose[T16^A[#]] & /@ Range[0, 8]
```

Out[212]=

```
{True, True, True, True, False, False, False, False, True}
```

In[213]:=

```
T16^A[#] == -Transpose[T16^A[#]] & /@ Range[0, 8]
```

Out[213]=

```
{False, False, False, False, True, True, True, True, False}
```

In[214]:=

```
σ16.T16^A[#] == -Transpose[σ16.T16^A[#]] & /@ Range[0, 7]
```

Out[214]=

```
{True, True, True, True, True, True, True, True}
```

In[215]:=

```
(covariantDiffMatrix = T16^A[5].T16^A[6].T16^A[7]) // MatrixForm
```

Out[215]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[216]:=

```
σ16.covariantDiffMatrix == Transpose[σ16.covariantDiffMatrix]
```

Out[216]=

True

In[217]:=

```
ID16 // MatrixForm
```

Out[217]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[218]:=

```
(σ16(*=T16^A[0].T16^A[1].T16^A[2].T16^A[3]*)) // MatrixForm
```

Out[218]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[219]:=

```
σ16 == ArrayFlatten[{{-σ, 0}, {0, σ}}]
```

Out[219]=

True

later, after defining $e^{\alpha}_{(A)}$:

In[220]:=

```
(*Table[T16^α[α1-1]=  Sum[(e^α_(A)[[α1,A1]]) T16^A[A1-1],{A1,1,8}],{α1,1,8}];*)
```

In[221]:=

```
(*(T16^α[8]=T16^α[0].T16^α[1].T16^α[2].T16^α[3].T16^α[4].T16^α[5].T16^α[6].T16^α[7])//
 MatrixForm*)
```

In[222]:=

```
(*eLpairs*)
```

In[223]:=

```
(*Ω16a=Array[0&,{16,16}];
 {1+#[[1]],1+#[[2]]}&/@eLpairs
 (Ω16a[[#[[1]],#[[2]]]]=1)&/@%*)
```

In[224]:=

```
(*Ω16a//MatrixForm*)
```

In[225]:=

```
(*(Ω16=σ16.(Ω16a+Transpose[Ω16a]))//MatrixForm*)
```

In[226]:=

```
T16^A[0].T16^A[4] // MatrixForm
T16^A[0].T16^A[4].σ16
(Ω16 = %) // MatrixForm
```

Out[226]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[227]=

```
{{0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}}
```

Out[228]//MatrixForm=

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In[229]:=

```
Det[Ω16]
```

Out[229]=

1

In[230]:=

```
σ16.Ω16 // MatrixForm
```

Out[230]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[231]:=

```
σ16.T16^A[#] == -Transpose[σ16.T16^A[#]] & /@ Range[0, 7]
```

Out[231]=

{True, True, True, True, True, True, True, True}

In[232]:=

```
σ16.T16^A[#] == -Transpose[σ16.T16^A[#]] & /@ Range[0, 7]
```

Out[232]=

{True, True, True, True, True, True, True, True}

In[233]:=

```
σ16.T16^A[#] == -Transpose[σ16.T16^A[#]] & /@ Range[0, 7]
```

Out[233]=

{True, True, True, True, True, True, True, True}

In[234]:=

```
σ16.newT16^A[#] == -Transpose[σ16.newT16^A[#]] & /@ Range[0, 7]
```

Out[234]=

```
{{{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
     0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
    {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
```

```
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
     0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[0] ==

 -({{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},                                ,
      {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
       0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
      {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
      {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
       0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[0])^T

{{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
   {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
    1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[1] ==

 -({{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},                                ,
      {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
       0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
      {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
      {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
       0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[1])^T

{{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
```

```
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
  {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
   1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[2] ==

 -({{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},                                ,
     {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
      0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
     {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
      0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[2])^T

{{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
  {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
   1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[3] ==

 -({{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},                                ,
     {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
      0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
     {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
     {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
     {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
      0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[3])^T

{{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0,
```

```
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
   {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
    1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[4] ==

 -({{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},                          ,
      {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
       0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
      {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
      {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
       0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[4])^T

{{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
   {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
    1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[5] ==

 -({{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},                          ,
      {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
       0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
      {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
      {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
       0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[5])^T

{{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
```

```
{0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[6] ==

-({{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},                          ,
    {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
     0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
    {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
     0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[6])^T

{{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
   {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
    1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[7] ==

-({{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},                          }
    {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0,
     0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
    {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
     0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}.newT16^A[7])^T
```

In[235]:=

```
newσ16.newT16^A[#] == -Transpose[newσ16.newT16^A[#]] & /@ Range[0, 7]
```

Out[235]=

```
{newσ16.newT16^A[0] == -(newσ16.newT16^A[0])^T, newσ16.newT16^A[1] == -(newσ16.newT16^A[1])^T,
 newσ16.newT16^A[2] == -(newσ16.newT16^A[2])^T, newσ16.newT16^A[3] == -(newσ16.newT16^A[3])^T,
 newσ16.newT16^A[4] == -(newσ16.newT16^A[4])^T, newσ16.newT16^A[5] == -(newσ16.newT16^A[5])^T,
 newσ16.newT16^A[6] == -(newσ16.newT16^A[6])^T, newσ16.newT16^A[7] == -(newσ16.newT16^A[7])^T}
```

In[236]:=

```
Table[
  {{A1, B1}, FullSimplify[ExpandAll[1/2 (T16^A[A1].T16^A[B1] + T16^A[B1].T16^A[A1]) ==
      η4488[[A1 + 1, B1 + 1]] * ID16]]}, {A1, 0, 7}, {B1, 0, 7}] // MatrixForm
```

Out[236]//MatrixForm=

({0, 0}, True)	({0, 1}, True)	({0, 2}, True)	({0, 3}, True)	({0, 4}, True)	({0, 5}, True)	({0, 6}, True)	({0, 7}, True)
({1, 0}, True)	({1, 1}, True)	({1, 2}, True)	({1, 3}, True)	({1, 4}, True)	({1, 5}, True)	({1, 6}, True)	({1, 7}, True)
({2, 0}, True)	({2, 1}, True)	({2, 2}, True)	({2, 3}, True)	({2, 4}, True)	({2, 5}, True)	({2, 6}, True)	({2, 7}, True)
({3, 0}, True)	({3, 1}, True)	({3, 2}, True)	({3, 3}, True)	({3, 4}, True)	({3, 5}, True)	({3, 6}, True)	({3, 7}, True)
({4, 0}, True)	({4, 1}, True)	({4, 2}, True)	({4, 3}, True)	({4, 4}, True)	({4, 5}, True)	({4, 6}, True)	({4, 7}, True)
({5, 0}, True)	({5, 1}, True)	({5, 2}, True)	({5, 3}, True)	({5, 4}, True)	({5, 5}, True)	({5, 6}, True)	({5, 7}, True)
({6, 0}, True)	({6, 1}, True)	({6, 2}, True)	({6, 3}, True)	({6, 4}, True)	({6, 5}, True)	({6, 6}, True)	({6, 7}, True)
({7, 0}, True)	({7, 1}, True)	({7, 2}, True)	({7, 3}, True)	({7, 4}, True)	({7, 5}, True)	({7, 6}, True)	({7, 7}, True)

In[237]:=

```
Table[{{A1, B1}, FullSimplify[
    ExpandAll[1/2 (newT16^A[A1].newT16^A[B1] + newT16^A[B1].newT16^A[A1]) ==
      η4488[[A1 + 1, B1 + 1]] * ID16]]}, {A1, 0, 7}, {B1, 0, 7}] // MatrixForm
```

Out[237]//MatrixForm=

```
(newT16^A[0]^2 == {{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0,
(1/2 (newT16^A[0].newT16^A[1] + newT16^A[1].newT16^A[0]) == {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
(1/2 (newT16^A[0].newT16^A[2] + newT16^A[2].newT16^A[0]) == {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
(1/2 (newT16^A[0].newT16^A[3] + newT16^A[3].newT16^A[0]) == {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
(1/2 (newT16^A[0].newT16^A[4] + newT16^A[4].newT16^A[0]) == {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
(1/2 (newT16^A[0].newT16^A[5] + newT16^A[5].newT16^A[0]) == {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
(1/2 (newT16^A[0].newT16^A[6] + newT16^A[6].newT16^A[0]) == {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
(1/2 (newT16^A[0].newT16^A[7] + newT16^A[7].newT16^A[0]) == {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
```

In[238]:=

```
Ψ16upper = Take[Ψ16, 8]
Ψ16lower = Take[Ψ16, -8]
Ψ16 == Flatten[{Ψ16upper, Ψ16lower}]
```

Out[238]=

```
{f16[0][x0, x4], f16[1][x0, x4], f16[2][x0, x4], f16[3][x0, x4],
 f16[4][x0, x4], f16[5][x0, x4], f16[6][x0, x4], f16[7][x0, x4]}
```

Out[239]=

```
{f16[8][x0, x4], f16[9][x0, x4], f16[10][x0, x4], f16[11][x0, x4],
 f16[12][x0, x4], f16[13][x0, x4], f16[14][x0, x4], f16[15][x0, x4]}
```

Out[240]=

```
True
```

$$\Gamma^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

```
Table[T16^A[A1] == ArrayFlatten[{{0, τ̄[A1]}, {τ[A1], 0}}], {A1, 0, 7}]
σ16 == ArrayFlatten[{{-σ, 0}, {0, σ}}]
(0  τ̄[A1]) (Ψ16upper) == (τ̄[A1].Ψ16lower)
(τ[A1]  0) (Ψ16lower)    (τ[A1].Ψ16upper)
```

$$\sigma 16.\begin{pmatrix} 0 & \tau[A1] \\ \tau[A1] & 0 \end{pmatrix}\begin{pmatrix} \psi 16upper \\ \psi 16lower \end{pmatrix} = \begin{pmatrix} -\sigma\,\tau[A1].\psi 16lower \\ \sigma\,\tau[A1].\psi 16upper \end{pmatrix}$$

$$\psi 16.\sigma 16.\begin{pmatrix} 0 & \tau[A1] \\ \tau[A1] & 0 \end{pmatrix}.\psi 16 = \begin{pmatrix} -\psi 16upper.\sigma\,\tau[A1].\psi 16lower \\ \psi 16lower.\sigma\,\tau[A1].\psi 16upper \end{pmatrix}$$

In[241]:=

```
(*σ16.T16^A[#]&/@Range[0,7]*)
```

In[242]:=

```
(*
(T16^A[8]=FullSimplify[T16^A[0].T16^A[1].T16^A[2].
    T16^A[3].T16^A[4].T16^A[5].T16^A[6].T16^A[7]])//MatrixForm*)
```

In[243]:=

```
η4488
```

Out[243]=

```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0},
 {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}}
```

In[244]:=

```
η99 = DiagonalMatrix[{1, 1, 1, 1, -1, -1, -1, -1, 1}]
```

Out[244]=

```
{{1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1}}
```

In[245]:=

```
(*Table[{A1,B1,FullSimplify[
    ExpandAll[1/2 (T16^A[A1].T16^A[B1]+T16^A[B1].T16^A[A1]) == η99[[A1,B1]]*ID16]]},
  {A1,0, 8},{B1,0, 8}]//MatrixForm*)
```

In[246]:=

```
(*Table[1/4 (T16^A[A1].T16^A[B1]+T16^A[B1].T16^A[A1]), {A1, 8},{B1,8}]*)
```

In[247]:=

$$\left(P_L = \frac{1}{2}\left(ID16 - T16^A[8]\right)\right) \text{ // MatrixForm}$$

Out[247]//MatrixForm=

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

In[248]:=

$$\left(P_R = \frac{1}{2}\left(ID16 + T16^A[8]\right)\right) \text{ // MatrixForm}$$

Out[248]//MatrixForm=

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

In[249]:=

$$\{ID16 == P_L + P_R,\ P_L.P_L == P_L,\ P_L.P_R == P_R.P_L == ZERO16\}$$

Out[249]=

{True, True, True}

```
SAB = Table[1/4 (T16^A[A1].T16^A[B1] - T16^A[B1].T16^A[A1]), {A1, 0, 7}, {B1, 0, 7}];
```

In[250]:=

```
SAB = Table[1/4 (T16^A[A1].T16^A[B1] - T16^A[B1].T16^A[A1]), {A1, 0, 7}, {B1, 0, 7}];
```

In[251]:=

```
Table[FullSimplify[σ16.SAB⟦A1, B1⟧ === -Transpose[σ16.SAB⟦A1, B1⟧]],
   {A1, 1, 8}, {B1, 1, 8}] // Flatten // Union
```

Out[251]=

```
{True}
```

In[252]:=

```
Table[FullSimplify[SAB⟦A1, B1⟧.SAB⟦A2, B2⟧ - SAB⟦A2, B2⟧.SAB⟦A1, B1⟧ ==
      -(η4488⟦A1, A2⟧ × SAB⟦B1, B2⟧ - η4488⟦A1, B2⟧ × SAB⟦B1, A2⟧ -
         η4488⟦B1, A2⟧ × SAB⟦A1, B2⟧ + η4488⟦B1, B2⟧ × SAB⟦A1, A2⟧)],
   {A1, 1, 7}, {B1, A1 + 1, 8}, {A2, 1, 7}, {B2, A2 + 1, 8}] // Flatten // Union
```

Out[252]=

```
{True}
```

In[253]:=

```
Table[FullSimplify[SAB⟦A1, B1⟧.T16^A[B2 - 1] - T16^A[B2 - 1].SAB⟦A1, B1⟧ ==
      (-η4488⟦B2, A1⟧ T16^A[B1 - 1] + η4488⟦B2, B1⟧ T16^A[A1 - 1])],
   {A1, 1, 8}, {B1, 1, 8}, {B2, 1, 8}] // Flatten // Union
```

Out[253]=

```
{True}
```

In[254]:=

```
(*Table[{{A1,B1},FullSimplify[σ16.SAB⟦A1,B1⟧==-Transpose[σ16.SAB⟦A1,B1⟧]]},
 {A1,1, 8},{B1,1,8}]*)
```

In[255]:=

```
(*Table[
 {{A1,B1,A2,B2},FullSimplify[SAB⟦A1,B1⟧.SAB⟦A2,B2⟧ -SAB⟦A2,B2⟧.SAB⟦A1,B1⟧==
    -(η4488⟦A1,A2⟧SAB⟦B1,B2⟧-η4488⟦A1,B2⟧SAB⟦B1,A2⟧-
       η4488⟦B1,A2⟧SAB⟦A1,B2⟧+η4488⟦B1,B2⟧SAB⟦A1,A2⟧)]},
 {A1,1, 7},{B1,A1+1,8},{A2,1, 7},{B2,A2+1,8}]*)
```

In[256]:=

```
(*Table[{{A1,B1,B2},FullSimplify[SAB⟦A1,B1⟧.T16^A[B2-1] -T16^A[B2-1].SAB⟦A1,B1⟧==
     (-η4488⟦B2,A1⟧T16^A[B1-1]+η4488⟦B2,B1⟧T16^A[A1-1])]},
 {A1,1, 8},{B1,1,8},{B2,1,8}]*)
```

In[257]:=

```
Do[sAB[A1][B1] = 1/4 (T16^A[A1].T16^A[B1] - T16^A[B1].T16^A[A1]),
  {A1, 0, 7}, {B1, 0, 7}];
```

In[258]:=

```
Dimensions[ArrayFlatten[{{0, ID8}, {ID8, 0}}]]
```

Out[258]=

{16, 16}

In[259]:=

```
sAB[-1 + 1][-1 + 2] == SAB[[1, 2]]
```

Out[259]=

True

In[260]:=

```
σ16.SAB[[1, 2]] // MatrixForm
```

Out[260]//MatrixForm=

$$\begin{pmatrix}
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0
\end{pmatrix}$$

In[261]:=

```
σ16.sAB[1][2] // MatrixForm
```

Out[261]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The commutation relations for the generators J_{ab} of the $\mathfrak{so}(3,3)$ Lie algebra (using the physics convention where generators are Hermitian) are given by the formula:

$$[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad} + \eta_{bd}J_{ac})$$

where:

- The indices a, b, c, d range from 1 to 6.
- η_{ab} is the **metric tensor** with signature $(+,+,+,-,-,-)$ (or the reverse, depending on the convention). A common choice is a diagonal matrix $\eta = \mathbf{diag}(1,1,1,-1,-1,-1)$.
- J_{ab} are the **generators** of the algebra, which are antisymmetric, $J_{ab} = -J_{ba}$.
- The terms on the right-hand side involve Kronecker deltas weighted by the metric, ensuring the resulting generator is also an element of the algebra and satisfies the required antisymmetry and Jacobi identity properties.

The final answer is $[\mathbf{J_{ab}}, \mathbf{J_{cd}}] = \mathbf{i}(\boldsymbol{\eta}_{\mathbf{ac}}\mathbf{J_{bd}} - \boldsymbol{\eta}_{\mathbf{ad}}\mathbf{J_{bc}} - \boldsymbol{\eta}_{\mathbf{bc}}\mathbf{J_{ad}} + \boldsymbol{\eta}_{\mathbf{bd}}\mathbf{J_{ac}})$.

Step 1: Define the generators and metric

The **so(4,4)** Lie algebra has 28 generators, which can be denoted as $J_{\mu\nu}$ where the indices μ, ν range from 0 to 7, and $J_{\mu\nu} = -J_{\nu\mu}$. The Lie algebra is defined in an 8-dimensional real vector space equipped with a metric $\eta_{\mu\nu}$ with signature (4, 4).

- **Generators:** $J_{\mu\nu}$ with $\mu, \nu \in \{0, 1, \ldots, 7\}$
- **Metric tensor:** $\eta_{\mu\nu} = \mathrm{diag}(-1, -1, -1, -1, +1, +1, +1, +1)$ (or the opposite sign convention)

Step 2: State the general commutation relation

The commutation relations for the generators of the **so(p,q)** Lie algebra are a generalization of the **so(n)** algebra relations, incorporating the metric tensor $\eta_{\mu\nu}$.

Answer:

The commutation relations for the generators $J_{\mu\nu}$ of the **so(4,4)** Lie algebra are:

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho}$$

where $\eta_{\mu\nu}$ is the metric tensor with a (4,4) signature (four plus signs and four minus signs, or vice versa).

(Note: In some physics conventions, an extra factor of i or $i\hbar$ is included depending on whether the generators are defined as Hermitian or not).

> what are the commutation relation for the generators of so(4,4)?

The commutation relations for the generators $J_{\mu\nu}$ of the $so(4,4)$ Lie algebra, where $\mu, \nu \in \{0, 1, \ldots, 7\}$ and $J_{\mu\nu} = -J_{\nu\mu}$, are the standard relations for orthogonal Lie algebras, given by a generalization of the angular momentum commutation relations:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho})$$

where $\eta_{\mu\nu}$ is the flat metric of 8-dimensional Minkowski space with signature (4,4) (or some other convention, the structure constants are the same as for $so(8)$ in an appropriate basis), which is a diagonal matrix with four positive and four negative entries (e.g., $\eta = \mathrm{diag}(+1, +1, +1, +1, -1, -1, -1, -1)$).

↪ **Why is the metric signature significant for $so(4,4)$ commutation relations?**

Why is the metric signature significant for $so(4,4)$ commutation relations?

The metric signature is significant for the $so(p,q)$ commutation relations because the metric tensor $\eta_{\mu\nu}$ acts as the **structure constants** of the Lie algebra, defining its specific algebraic structure and physical properties (if applicable).

Here is a detailed explanation of its significance:

- **Definition of the Algebra:** The commutation relation for the generators $J_{\mu\nu}$ of any orthogonal Lie algebra $so(p,q)$ is given by:
  $[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$
  The values of the structure constants depend directly on the components of the metric $\eta_{\mu\nu}$. This means that the specific algebraic properties of $so(4,4)$ are distinct from $so(8)$ (which has a (8,0) signature, i.e., a Euclidean metric), even though they operate in the same 8-dimensional vector space.
- **Physical Interpretation:** In physics, the metric signature determines the nature of spacetime. A mixed signature (like (4,4) or the more common (1,3) Minkowski signature) implies a causal structure, distinguishing between timelike and spacelike dimensions.
  - Generators associated with rotations in spacelike dimensions (positive $\eta_{\mu\nu}$ entries) typically lead to compact symmetry groups (like $SO(4)$ or $SO(8)$).

In[262]:=

```
η4488
```

Out[262]=

```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0},
 {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}}
```

orthogonal Lie algebra $so(p,q)$ is given by:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$

In[263]:=

```
SAB[[6, 2]].SAB[[6, 7]] - SAB[[6, 7]].SAB[[6, 2]] == η4488[[6, 6]] × SAB[[7, 2]]
```

Out[263]=

```
True
```

In[264]:=

```
{{6, 2, 6, 7}, False}
```

Out[264]=

```
{{6, 2, 6, 7}, False}
```

In[265]:=

```
SAB[[1, 2]].SAB[[1, 3]] - SAB[[1, 3]].SAB[[1, 2]] == η4488[[1, 1]] × SAB[[2, 3]]
```

Out[265]=

```
False
```

In[266]:=

```
sAB[0][0]
```

Out[266]=

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

In[267]:=

```
sAB[0][1]
```

Out[267]=

$$\left\{\left\{0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\},\right.$$
$$\left\{0, 0, 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{0, 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \left\{0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{0, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{-\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}\right\},$$
$$\left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0\right\},$$
$$\left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0\right\},$$
$$\left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0\right\},$$
$$\left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0, 0\right\},$$
$$\left.\left\{0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0\right\}\right\}$$

In[268]:=

```
Table[Transpose[T16^A[8].sAB[A1][B1]] == T16^A[8].sAB[A1][B1], {A1, 0, 7}, {B1, 0, 7}]
```

Out[268]=

```
{{True, False, False, False, True, True, True, True},
 {False, True, False, False, True, True, True, True},
 {False, False, True, False, True, True, True, True},
 {False, False, False, True, True, True, True, True},
 {True, True, True, True, True, False, False, False},
 {True, True, True, True, False, True, False, False},
 {True, True, True, True, False, False, True, False},
 {True, True, True, True, False, False, False, True}}
```

In[269]:=

```
Table[Transpose[T16^A[7].sAB[A1][B1]] == T16^A[7].sAB[A1][B1], {A1, 0, 7}, {B1, 0, 7}]
```

Out[269]=

```
{{True, True, True, True, False, False, False, True},
 {True, True, True, True, False, False, False, True},
 {True, True, True, True, False, False, False, True},
 {True, True, True, True, False, False, False, True},
 {False, False, False, False, True, True, True, False},
 {False, False, False, False, True, True, True, False},
 {False, False, False, False, True, True, True, False},
 {True, True, True, True, False, False, False, True}}
```

In[270]:=

```
Table[Transpose[σ16.sAB[A1][B1]] == σ16.sAB[A1][B1], {A1, 0, 7}, {B1, 0, 7}]
```

Out[270]=

```
{{True, False, False, False, False, False, False, False},
 {False, True, False, False, False, False, False, False},
 {False, False, True, False, False, False, False, False},
 {False, False, False, True, False, False, False, False},
 {False, False, False, False, True, False, False, False},
 {False, False, False, False, False, True, False, False},
 {False, False, False, False, False, False, True, False},
 {False, False, False, False, False, False, False, True}}
```

In[271]:=

```
Table[Transpose[σ16.SAB[[A1, B1]]] == σ16.SAB[[A1, B1]], {A1, 1, 8}, {B1, 1, 8}]
```

Out[271]=

```
{{True, False, False, False, False, False, False, False},
 {False, True, False, False, False, False, False, False},
 {False, False, True, False, False, False, False, False},
 {False, False, False, True, False, False, False, False},
 {False, False, False, False, True, False, False, False},
 {False, False, False, False, False, True, False, False},
 {False, False, False, False, False, False, True, False},
 {False, False, False, False, False, False, False, True}}
```

In[272]:=

```
SAB[[1, 2]][[1]]
```

Out[272]=

$$\left\{0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0\right\}$$

In[273]:=

```
SAB[[1, 2]][[2]]
```

Out[273]=

$$\left\{0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}$$

$$\tau^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

In[274]:=

```
SAB1 = Table[Take[SAB[[A1, B1]][[C1]], 8],  {A1, 8}, {B1, 8}, {C1, 1, 8}];
```

In[275]:=

```
SAB2 = Table[Take[SAB[[A1, B1]][[C1]], -8],  {A1, 8}, {B1, 8}, {C1, 9, 16}];
```

In[276]:=

```
SAB[[1, 2]] == ArrayFlatten[{{SAB1[[1, 2]], 0}, {0, SAB2[[1, 2]]}}]
```

Out[276]=

True

In[277]:=

```
(*Table[SAB[[A1,B1]]== ArrayFlatten[{{SAB1[[A1,B1]],0},{0,SAB2[[A1,B1]]}}],
 {A1,0, 7},{B1,0,7}]*)
```

$$4D_{(1)}{}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A$$

$$4D_{(2)}{}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A.$$

Let $g \in \overline{SO(4,4)}$ and $L \in SO(4,4)$. The canonical 2-1 homomorphism $\overline{SO(4,4)} \to SO(4,4)$ is given by

$$8L^{A'}{}_{B'}(g) = \operatorname{tr} \widetilde{D}^{(2)}(g)\, \tau^{A'} D^{(1)}(g)\, \overline{\tau}_{B'} . \tag{2.8}$$

As usual tr denotes the trace. Under the action of $\overline{SO(4,4)}$

$$\widetilde{D}^{(1)} \sigma = \sigma D^{(1)-1}, \tag{2.9}$$

$$\sigma^{-1} \widetilde{D}^{(2)} = D^{(2)-1} \sigma^{-1}, \tag{2.10}$$

$$L^{A'}{}_{C'} G_{A'B'} L^{B'}{}_{D'} = G_{C'D'}, \tag{2.11}$$

$$L^{A'}{}_{B'} \tau^{B'} = \widetilde{D}^{(2)} \tau^{A'} D^{(1)}, \tag{2.12}$$

$$L^{A'}{}_{B'} \overline{\tau}^{B'} = D^{(1)-1} \overline{\tau}^{A'} \widetilde{D}^{(2)-1} . \tag{2.13}$$

In[278]:=

```
SAB2[[1, 2]]
```

Out[278]=

$$\left\{\left\{0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}\right\}, \left\{0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0\right\}, \left\{0, 0, 0, 0, 0, \frac{1}{2}, 0, 0\right\}, \left\{0, 0, 0, 0, \frac{1}{2}, 0, 0, 0\right\}, \left\{0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0\right\}, \left\{0, 0, -\frac{1}{2}, 0, 0, 0, 0, 0\right\}, \left\{0, \frac{1}{2}, 0, 0, 0, 0, 0, 0\right\}, \left\{\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0\right\}\right\}$$

In[279]:=

```
τ[0]
```

Out[279]=

```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0},
 {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 1}}
```

In[280]:=

```
(*Block[{b,A,F_a^(μ),F^a_(μ),ψ,Ψ},
 b=Array[A,{8}];
 ψ[1]=Array[Ψ[1],{8}];
 ψ[2]=Array[Ψ[2],{8}];
 F_a^(μ)=Table[(1/Sqrt[ψ[1].σ.ψ[1]] ψ[1].σ.(τ[μL])),{μL,1,8}];
 F^a_(μ)= Transpose[Table[(1/Sqrt[ψ[1].σ.ψ[1]] η4488[[μL,μL]] τ[μL].ψ[1]),{μL,1,8}]];
 b===FullSimplify[F_a^(μ).ψ[2]/.Thread[ψ[2]→F^a_(μ).b]]]*)
```

In[281]:=

```
(*Block[{b,A,B,q,F^a_(μ),ψ,φ,Ψ},
 b=Array[A,{8}];
 ψ[1]=Array[Ψ[1],{8}];
 φ[1]=Array[Ψ[2],{8}];
 q=Table[FullSimplify[(1/Sqrt[ψ[1].σ.ψ[1]] ψ[1].σ.1/2 (τ̄[A1].τ[B1]-τ̄[B1].τ[A1]).φ[1])],
   {A1,1,8},{B1,1,8}];
 B=FullSimplify[
   (1/Sqrt[ψ[1].σ.ψ[1]] ParallelSum[η4488[[μL,μL]]τ[μL]b[[μL]],{μL,1,8}].ψ[1])];
 FullSimplify[q/.Thread[φ[1]→B]]]*)
```

In[282]:=

```
(*Block[{b,A,B,q,F^a_(μ),ψ,φ,Ψ},
 b=Array[A,{8}];
 ψ[1]=Array[Ψ[1],{8}];
 φ[1]=Array[Ψ[2],{8}];
 q=Table[FullSimplify[(ψ[1].σ.(τ̄[A1].τ[B1]).φ[1])],{A1,1,8},{B1,1,8}];
 B=FullSimplify[( ParallelSum[η4488[[μL,μL]]τ[μL]b[[μL]],{μL,1,8}].ψ[1])];
 FullSimplify[1/Sqrt[ψ[1].σ.ψ[1]] q/.Thread[φ[1]→B]]]*)
```

CHECK Table[T16^α[α1 - 1] = Sum[(e^α_(A) [[α1, A1]]) T16^A[A1 - 1], {A1, 1, 8}], {α1, 1, 8}];
and ● ● ● :

In[283]:=

```
(*Table[T16^α[α1-1]=  Sum[(e^α_(A) [[α1,A1]] ) T16^A[A1-1],{A1,1,8}],{α1,1,8}];*)
```

In[284]:=

```
(*(T16^α[8]=T16^α[0].T16^α[1].T16^α[2].T16^α[3].T16^α[4].T16^α[5].T16^α[6].T16^α[7])//
 MatrixForm*)
```

In[285]:=

```
Table[
 {{A1, B1}, FullSimplify[ExpandAll[1/2 (T16^A[A1].T16^A[B1] + T16^A[B1].T16^A[A1]) ==
     η4488[[A1 + 1, B1 + 1]] * ID16]]}, {A1, 0, 7}, {B1, 0, 7}] // MatrixForm
```

Out[285]//MatrixForm=

{0, 0} True	{0, 1} True	{0, 2} True	{0, 3} True	{0, 4} True	{0, 5} True	{0, 6} True	{0, 7} True
{1, 0} True	{1, 1} True	{1, 2} True	{1, 3} True	{1, 4} True	{1, 5} True	{1, 6} True	{1, 7} True
{2, 0} True	{2, 1} True	{2, 2} True	{2, 3} True	{2, 4} True	{2, 5} True	{2, 6} True	{2, 7} True
{3, 0} True	{3, 1} True	{3, 2} True	{3, 3} True	{3, 4} True	{3, 5} True	{3, 6} True	{3, 7} True
{4, 0} True	{4, 1} True	{4, 2} True	{4, 3} True	{4, 4} True	{4, 5} True	{4, 6} True	{4, 7} True
{5, 0} True	{5, 1} True	{5, 2} True	{5, 3} True	{5, 4} True	{5, 5} True	{5, 6} True	{5, 7} True
{6, 0} True	{6, 1} True	{6, 2} True	{6, 3} True	{6, 4} True	{6, 5} True	{6, 6} True	{6, 7} True
{7, 0} True	{7, 1} True	{7, 2} True	{7, 3} True	{7, 4} True	{7, 5} True	{7, 6} True	{7, 7} True

BASIS of 16 × 16 matrices :

In[286]:=

```
16 * 16
```

Out[286]=

256

In[287]:=

2^8

Out[287]=

256

In[288]:=

```
Sum[Binomial[8, p], {p, 0, 8}]
```

Out[288]=

256

In[289]:=

```
Id16 == ID16 == Dot @@ (T16^A[#] & /@ Range[0, 8])
```

Out[289]=

True

In[290]:=

```
Clear[t160]; t160 = {{Id16, {Range[0, 8]}}}; Length[t160]
Clear[t16A];
t16A = {};
```

```
Do[AppendTo[t16A, {T16^A[j], {j}}], {j, 0, 7}];
Length[t16A]
Clear[t16AB];
t16AB = {};
Do[AppendTo[t16AB, {T16^A[j].T16^A[k], {j, k}}], {j, 0, 6}, {k, j + 1, 7}];
Length[t16AB]
Clear[t16ABC];
t16ABC = {};
Do[AppendTo[t16ABC, {T16^A[j].T16^A[k].T16^A[h], {j, k, h}}],
 {j, 0, 5}, {k, j + 1, 6}, {h, k + 1, 7}];
Length[t16ABC]
Clear[t16ABCD];
t16ABCD = {};
Do[AppendTo[t16ABCD, {T16^A[j].T16^A[k].T16^A[h].T16^A[i], {j, k, h, i}}],
 {j, 0, 4}, {k, j + 1, 5}, {h, k + 1, 6}, {i, h + 1, 7}];
Length[t16ABCD]
Clear[t16ABCDE];
t16ABCDE = {};
Do[AppendTo[t16ABCDE, {T16^A[j].T16^A[k].T16^A[h].T16^A[i].T16^A[l], {j, k, h, i, l}}],
 {j, 0, 3}, {k, j + 1, 4}, {h, k + 1, 5}, {i, h + 1, 6}, {l, i + 1, 7}];
Length[t16ABCDE]
Clear[t16ABCDEF];
t16ABCDEF = {};
Do[AppendTo[t16ABCDEF,
  {T16^A[j].T16^A[k].T16^A[h].T16^A[i].T16^A[l].T16^A[q], {j, k, h, i, l, q}}],
 {j, 0, 2}, {k, j + 1, 3}, {h, k + 1, 4}, {i, h + 1, 5}, {l, i + 1, 6}, {q, l + 1, 7}];
Length[t16ABCDEF]
Clear[t16ABCDEFG];
t16ABCDEFG = {};
Do[AppendTo[t16ABCDEFG, {T16^A[j].T16^A[k].T16^A[h].T16^A[i].T16^A[l].T16^A[q].T16^A[r],
   {j, k, h, i, l, q, r}}], {j, 0, 1}, {k, j + 1, 2},
 {h, k + 1, 3}, {i, h + 1, 4}, {l, i + 1, 5}, {q, l + 1, 6}, {r, q + 1, 7}];
Length[t16ABCDEFG]
Clear[t16ABCDEFGH];
t16ABCDEFGH = {};
Do[AppendTo[t16ABCDEFGH, {T16^A[j].T16^A[k].T16^A[h].T16^A[i].T16^A[l].T16^A[q].T16^A[r].
    T16^A[s], {j, k, h, i, l, q, r, s}}], {j, 0, 1}, {k, j + 1, 1}, {h, k + 1, 2},
 {i, h + 1, 3}, {l, i + 1, 4}, {q, l + 1, 5}, {r, q + 1, 6}, {s, r + 1, 7}];
Length[t16ABCDEFGH]
% + %% + %%% + %%%% + %%%%% + %%%%%% + %%%%%%% + %%%%%%%% + %%%%%%%%%
```

Out[290]=

```
1
```

Out[291]=

```
8
```

Out[292]=

```
28
```

Out[293]=

```
56
```

Out[294]=

```
70
```

Out[295]=

```
56
```

Out[296]=

```
28
```

Out[297]=

```
8
```

Out[298]=

```
1
```

Out[299]=

```
256
```

In[300]:=

```
(*base16=Flatten[
  {t160,t16A,t16AB,t16ABC,t16ABCD,t16ABCDE,t16ABCDEF,t16ABCDEFG,t16ABCDEFGH},1];
Length[base16]*)
```

In[301]:=

```
base16 = Flatten[{t16A, t16AB, t16ABC, t16ABCD,
   t16ABCDE, t16ABCDEF, t16ABCDEFG, t16ABCDEFGH, t160}, 1];
Length[base16]
```

Out[301]=

```
256
```

In[302]:=

```
Length[Union[base16[[All, 1]]]]
```

Out[302]=

```
256
```

In[303]:=

```
Tr[#.# / 16] & /@ base16[[All, 1]] // Flatten // Union
```

Out[303]=

```
{-1, 1}
```

In[304]:=

```
positiveTrMM = Select[base16, Tr[#[[1]].#[[1]]] > 0 & → "Index"]
```

Out[304]=

```
{1, 2, 3, 4, 12, 13, 14, 15, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 39, 40, 41, 42, 44,
 45, 46, 47, 48, 49, 50, 51, 59, 60, 61, 62, 63, 64, 65, 66, 73, 74, 75, 76, 89, 90, 91,
 92, 93, 102, 103, 104, 105, 106, 107, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121,
 122, 123, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 148, 149, 150, 151,
 152, 153, 162, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 183, 184, 185,
 186, 187, 188, 197, 198, 199, 200, 201, 202, 203, 212, 217, 218, 225, 226, 227, 228, 229,
 230, 231, 232, 234, 235, 236, 237, 240, 241, 242, 243, 247, 248, 249, 250, 255, 256}
```

In[305]:=

```
negativeTrMM = Select[base16, Tr[#[[1]].#[[1]]] < 0 & → "Index"]
```

Out[305]=

```
{5, 6, 7, 8, 9, 10, 11, 16, 17, 22, 31, 32, 33, 34, 35, 36, 37, 38, 43, 52, 53, 54, 55, 56, 57,
 58, 67, 68, 69, 70, 71, 72, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 94, 95, 96, 97,
 98, 99, 100, 101, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 130, 131, 144, 145,
 146, 147, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 179, 180, 181, 182,
 189, 190, 191, 192, 193, 194, 195, 196, 204, 205, 206, 207, 208, 209, 210, 211, 213, 214,
 215, 216, 219, 220, 221, 222, 223, 224, 233, 238, 239, 244, 245, 246, 251, 252, 253, 254}
```

In[306]:=

```
{Length[positiveTrMM], Length[negativeTrMM]}
```

Out[306]=

```
{136, 120}
```

In[307]:=

```
dups = Block[{s}, s = base16[[All, 1]]; Select[s, MemberQ[s, -#] &]]; Length[dups]
```

Out[307]=

```
0
```

In[308]:=

```
dups = Select[base16, MemberQ[base16[[All, 1]], -#[[1]]] &]; Length[dups]
```

Out[308]=

```
0
```

In[309]:=

```
BASE16 = use16 = Complement[base16, dups]; Length[use16]
```

Out[309]=

```
256
```

In[310]:=

```
(*(T16^A[8]=T16^A[0].T16^A[1].T16^A[2].T16^A[3].T16^A[4].T16^A[5].T16^A[6].T16^A[7])//
 MatrixForm*)
```

In[311]:=

```
MatrixForm[T16^A[#]] & /@ Range[0, 7]
```

Out[311]=

$$\left\{\begin{pmatrix} 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\left.\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}\right\}$$

In[312]:=

```
(*BASE16[[-1]]*)
```

In[313]:=

```
(*BASE16[[-2]]
 %[[1]]==BASE16[[-1]][[1]]*)
```

In[314]:=

```
16 * (16 + 1) / 2
16 * (16 - 1) / 2
% + %%
```

Out[314]=

```
136
```

Out[315]=

```
120
```

Out[316]=

```
256
```

In[317]:=

```
Clear[antisymm16];
antisymm16 = {};
Do[If[base16[[k, 1]] === - Transpose[base16[[k, 1]]], AppendTo[antisymm16,
   {base16[[k, 1]], {{k}, base16[[k, 2]]}}] ], {k, 1, Length[base16]}];
Length[antisymm16]
antisymm16[[%]]
```

Out[318]=

```
120
```

Out[319]=

```
{{{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}, {-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0}}, {{254}, {1, 2, 3, 4, 5, 6, 7}}}
```

In[320]:=

```
Clear[symm16];
symm16 = {};
Do[If[base16[[k, 1]] === Transpose[base16[[k, 1]]], AppendTo[symm16,
   {base16[[k, 1]], {{k}, base16[[k, 2]]}}] ], {k, 1, Length[base16]}];
Length[symm16]
symm16[[%]]
```

Out[321]=

```
136
```

Out[322]=

```
{{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}},
 {{256}, {{0, 1, 2, 3, 4, 5, 6, 7, 8}}}}
```

In[323]:=

```
dupssymm16 = Select[symm16, MemberQ[symm16[[All, 1]], -#[[1]] &]; Length[dupssymm16]
```

Out[323]=

```
0
```

In[324]:=

```
#[[2]] & /@ symm16
```

Out[324]=

```
{{{1}, {0}}, {{2}, {1}}, {{3}, {2}}, {{4}, {3}}, {{12}, {0, 4}}, {{13}, {0, 5}},
 {{14}, {0, 6}}, {{15}, {0, 7}}, {{18}, {1, 4}}, {{19}, {1, 5}}, {{20}, {1, 6}},
 {{21}, {1, 7}}, {{23}, {2, 4}}, {{24}, {2, 5}}, {{25}, {2, 6}}, {{26}, {2, 7}},
 {{27}, {3, 4}}, {{28}, {3, 5}}, {{29}, {3, 6}}, {{30}, {3, 7}}, {{39}, {0, 1, 4}},
 {{40}, {0, 1, 5}}, {{41}, {0, 1, 6}}, {{42}, {0, 1, 7}}, {{44}, {0, 2, 4}}, {{45}, {0, 2, 5}},
 {{46}, {0, 2, 6}}, {{47}, {0, 2, 7}}, {{48}, {0, 3, 4}}, {{49}, {0, 3, 5}},
 {{50}, {0, 3, 6}}, {{51}, {0, 3, 7}}, {{59}, {1, 2, 4}}, {{60}, {1, 2, 5}},
 {{61}, {1, 2, 6}}, {{62}, {1, 2, 7}}, {{63}, {1, 3, 4}}, {{64}, {1, 3, 5}},
 {{65}, {1, 3, 6}}, {{66}, {1, 3, 7}}, {{73}, {2, 3, 4}}, {{74}, {2, 3, 5}},
 {{75}, {2, 3, 6}}, {{76}, {2, 3, 7}}, {{89}, {4, 5, 6}}, {{90}, {4, 5, 7}},
 {{91}, {4, 6, 7}}, {{92}, {5, 6, 7}}, {{93}, {0, 1, 2, 3}}, {{102}, {0, 1, 4, 5}},
 {{103}, {0, 1, 4, 6}}, {{104}, {0, 1, 4, 7}}, {{105}, {0, 1, 5, 6}}, {{106}, {0, 1, 5, 7}},
 {{107}, {0, 1, 6, 7}}, {{112}, {0, 2, 4, 5}}, {{113}, {0, 2, 4, 6}}, {{114}, {0, 2, 4, 7}},
 {{115}, {0, 2, 5, 6}}, {{116}, {0, 2, 5, 7}}, {{117}, {0, 2, 6, 7}}, {{118}, {0, 3, 4, 5}},
 {{119}, {0, 3, 4, 6}}, {{120}, {0, 3, 4, 7}}, {{121}, {0, 3, 5, 6}}, {{122}, {0, 3, 5, 7}},
 {{123}, {0, 3, 6, 7}}, {{132}, {1, 2, 4, 5}}, {{133}, {1, 2, 4, 6}}, {{134}, {1, 2, 4, 7}},
 {{135}, {1, 2, 5, 6}}, {{136}, {1, 2, 5, 7}}, {{137}, {1, 2, 6, 7}}, {{138}, {1, 3, 4, 5}},
 {{139}, {1, 3, 4, 6}}, {{140}, {1, 3, 4, 7}}, {{141}, {1, 3, 5, 6}}, {{142}, {1, 3, 5, 7}},
 {{143}, {1, 3, 6, 7}}, {{148}, {2, 3, 4, 5}}, {{149}, {2, 3, 4, 6}}, {{150}, {2, 3, 4, 7}},
 {{151}, {2, 3, 5, 6}}, {{152}, {2, 3, 5, 7}}, {{153}, {2, 3, 6, 7}}, {{162}, {4, 5, 6, 7}},
 {{167}, {0, 1, 2, 4, 5}}, {{168}, {0, 1, 2, 4, 6}}, {{169}, {0, 1, 2, 4, 7}},
 {{170}, {0, 1, 2, 5, 6}}, {{171}, {0, 1, 2, 5, 7}}, {{172}, {0, 1, 2, 6, 7}},
 {{173}, {0, 1, 3, 4, 5}}, {{174}, {0, 1, 3, 4, 6}}, {{175}, {0, 1, 3, 4, 7}},
 {{176}, {0, 1, 3, 5, 6}}, {{177}, {0, 1, 3, 5, 7}}, {{178}, {0, 1, 3, 6, 7}},
 {{183}, {0, 2, 3, 4, 5}}, {{184}, {0, 2, 3, 4, 6}}, {{185}, {0, 2, 3, 4, 7}},
 {{186}, {0, 2, 3, 5, 6}}, {{187}, {0, 2, 3, 5, 7}}, {{188}, {0, 2, 3, 6, 7}},
 {{197}, {0, 4, 5, 6, 7}}, {{198}, {1, 2, 3, 4, 5}}, {{199}, {1, 2, 3, 4, 6}},
 {{200}, {1, 2, 3, 4, 7}}, {{201}, {1, 2, 3, 5, 6}}, {{202}, {1, 2, 3, 5, 7}},
 {{203}, {1, 2, 3, 6, 7}}, {{212}, {1, 4, 5, 6, 7}}, {{217}, {2, 4, 5, 6, 7}},
 {{218}, {3, 4, 5, 6, 7}}, {{225}, {0, 1, 2, 4, 5, 6}}, {{226}, {0, 1, 2, 4, 5, 7}},
 {{227}, {0, 1, 2, 4, 6, 7}}, {{228}, {0, 1, 2, 5, 6, 7}}, {{229}, {0, 1, 3, 4, 5, 6}},
 {{230}, {0, 1, 3, 4, 5, 7}}, {{231}, {0, 1, 3, 4, 6, 7}}, {{232}, {0, 1, 3, 5, 6, 7}},
 {{234}, {0, 2, 3, 4, 5, 6}}, {{235}, {0, 2, 3, 4, 5, 7}}, {{236}, {0, 2, 3, 4, 6, 7}},
 {{237}, {0, 2, 3, 5, 6, 7}}, {{240}, {1, 2, 3, 4, 5, 6}}, {{241}, {1, 2, 3, 4, 5, 7}},
 {{242}, {1, 2, 3, 4, 6, 7}}, {{243}, {1, 2, 3, 5, 6, 7}}, {{247}, {0, 1, 2, 3, 4, 5, 6}},
 {{248}, {0, 1, 2, 3, 4, 5, 7}}, {{249}, {0, 1, 2, 3, 4, 6, 7}}, {{250}, {0, 1, 2, 3, 5, 6, 7}},
 {{255}, {0, 1, 2, 3, 4, 5, 6, 7}}, {{256}, {{0, 1, 2, 3, 4, 5, 6, 7, 8}}}}
```

In[325]:=

```
σ16 == T16^A[0].T16^A[1].T16^A[2].T16^A[3]
```

Out[325]=

```
True
```

In[326]:=

```
σ16 == base16[[93]][[1]]
```

Out[326]=

```
True
```

In[327]:=

```
Select[symm16, #[[2]][[1]][[1]] == 93 & → "Index"]
```

Out[327]=

```
{49}
```

In[328]:=

```
σ16 == symm16[[49]][[1]]
```

Out[328]=

```
True
```

In[329]:=

```
symm16[[134]]
```

Out[329]=

```
{{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}, {{250}, {0, 1, 2, 3, 5, 6, 7}}}
```

In[330]:=

```
Select[symm16, #[[2]][[1]][[1]] == 250 & → "Index"]
```

Out[330]=

```
{134}
```

In[331]:=

```
T16__Superscript__A
```

Out[331]=

$T16^A$

In[332]:=

$\sigma16.T16^A[5].T16^A[6].T16^A[7] == symm16[[134]][[1]]$

Out[332]=

```
True
```

In[333]:=

$T16^A[8] == base16[[255]][[1]]$

Out[333]=

```
True
```

In[334]:=

```
Select[symm16, #[[2]][[1]][[1]] == 255 & → "Index"]
```

Out[334]=

```
{135}
```

similarly :

In[335]:=

```
coupled1 = {4, 5, 0, 9, 9, 12, 4, 5, 1, 8, 8, 13} // Union
```

Out[335]=

```
{0, 1, 4, 5, 8, 9, 12, 13}
```

In[336]:=

```
coupled2 = Complement[Range[0, 15], coupled1]
```

Out[336]=

```
{2, 3, 6, 7, 10, 11, 14, 15}
```

In[337]:=

```
coupled = {coupled1, coupled2}
```

Out[337]=

```
{{0, 1, 4, 5, 8, 9, 12, 13}, {2, 3, 6, 7, 10, 11, 14, 15}}
```

In[338]:=

```
Block[{m, vars, eqs, eqs1, eqs2, s, Φ16},
 Φ16 = f[#] & /@ Range[16];
 m = Array[qb[#1, #2] &, {16, 16}];
 vars = Flatten[m];
 eqs1 =
  And @@ Thread[m.Φ16 == Flatten[{Φ16[[# + 1]] & /@ coupled1, Φ16[[# + 1]] & /@ coupled2}]];
 eqs2 = And @@ Thread[m.Transpose[m] == Flatten[ID16]];
 eqs = eqs1 && eqs2;
 s = Solve[eqs, vars] // FullSimplify;
 s
]
```

Out[338]=

```
{}
```

In[339]:=

```
linTrans = Block[{m, b, vars, eqs, eqs1, eqs2, s, Φ16, t},
  Φ16 = f[#] & /@ Range[16];
  m = Array[qb[#1, #2] &, {16, 16}];
  b = Flatten[{Φ16⟦# + 1⟧ & /@ coupled1, Φ16⟦# + 1⟧ & /@ coupled2}];
  t = Table[Block[{sv, eq, d}, d = D[ m⟦j⟧.b, Φ16⟦j⟧];
      sv = Flatten[{d → 1, Thread[Complement[m⟦j⟧, {d}] → 0]}];
      (*Print[{m⟦j⟧.b},{Φ16⟦j⟧},{sv}];*)
      (*eq=Thread[m⟦j⟧.b==Φ16⟦j⟧];
      sv=Solve[eq,Flatten[m⟦j⟧]]//FullSimplify;
      sv*)(*{v.b,Φ16⟦j⟧}*)sv], {j, 1, 16}];
  (*
  vars=Flatten[m];
  eqs1=And@@Thread[m.Φ16==Flatten[{Φ16⟦#+1⟧&/@coupled1,Φ16⟦#+1⟧&/@coupled2}]];
  eqs2=And@@Thread[m.Transpose[m]==Flatten[ID16]];
  eqs=eqs1&&eqs2;
  s=Solve[eqs,vars]//FullSimplify;*)
  m /. Flatten[t]
 ]
```

Out[339]=

```
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
 {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}}
```

In[340]:=

```
linTrans // MatrixForm
```

Out[340]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[341]:=

```
linTrans.Transpose[linTrans] // FullSimplify // MatrixForm
```

Out[341]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[342]:=

```
linTrans.Ψ16 // Column
Union[%[[1]]]
```

Out[342]=

```
f16[0][x0, x4]
f16[1][x0, x4]
f16[8][x0, x4]
f16[9][x0, x4]
f16[2][x0, x4]
f16[3][x0, x4]
f16[10][x0, x4]
f16[11][x0, x4]
f16[4][x0, x4]
f16[5][x0, x4]
f16[12][x0, x4]
f16[13][x0, x4]
f16[6][x0, x4]
f16[7][x0, x4]
f16[14][x0, x4]
f16[15][x0, x4]
```

Out[343]=

```
{f16[0][x0, x4], f16[1][x0, x4], f16[2][x0, x4], f16[3][x0, x4],
 f16[4][x0, x4], f16[5][x0, x4], f16[6][x0, x4], f16[7][x0, x4],
 f16[8][x0, x4], f16[9][x0, x4], f16[10][x0, x4], f16[11][x0, x4],
 f16[12][x0, x4], f16[13][x0, x4], f16[14][x0, x4], f16[15][x0, x4]}
```

In[ ]:=

```
positiveTrMM = Select[base16, Tr[#[[1]] . #[[1]]] > 0 & -> "Index"]
```

Out[ ]=

{1, 2, 3, 4, 12, 13, 14, 15, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51, 59, 60, 61, 62, 63, 64, 65, 66, 73, 74, 75, 76, 89, 90, 91, 92, 93, 102, 103, 104, 105, 106, 107, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 148, 149, 150, 151, 152, 153, 162, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 183, 184, 185, 186, 187, 188, 197, 198,

199, 200, 201, 202, 203, 212, 217, 218, 225, 226, 227, 228, 229, 230, 231, 232, 234, 235, 236, 237, 240, 241, 242, 243, 247, 248, 249, 250, 255, 256}

In[ ]:=
```
negativeTrMM = Select[base16, Tr[#[[1]] . #[[1]]] < 0 & -> "Index"]
```

Out[ ]=

{5, 6, 7, 8, 9, 10, 11, 16, 17, 22, 31, 32, 33, 34, 35, 36, 37, 38, 43, 52, 53, 54, 55, 56, 57, 58, 67, 68, 69, 70, 71, 72, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 94, 95, 96, 97, 98, 99, 100, 101, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 130, 131, 144, 145, 146, 147, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 179, 180, 181, 182, 189, 190, 191, 192, 193, 194, 195, 196, 204, 205, 206, 207, 208, 209, 210, 211, 213, 214, 215, 216, 219, 220, 221, 222, 223, 224, 233, 238, 239, 244, 245, 246, 251, 252, 253, 254}

In[ ]:=
```
{Length[positiveTrMM], Length[negativeTrMM]}
```

Out[ ]=

{136, 120}

In[344]:=
```
positiveTrMM = Select[base16, Tr[#[[1]].#[[1]]] > 0 & → "Index"]
```

Out[344]=

{1, 2, 3, 4, 12, 13, 14, 15, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51, 59, 60, 61, 62, 63, 64, 65, 66, 73, 74, 75, 76, 89, 90, 91, 92, 93, 102, 103, 104, 105, 106, 107, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 148, 149, 150, 151, 152, 153, 162, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 183, 184, 185, 186, 187, 188, 197, 198, 199, 200, 201, 202, 203, 212, 217, 218, 225, 226, 227, 228, 229, 230, 231, 232, 234, 235, 236, 237, 240, 241, 242, 243, 247, 248, 249, 250, 255, 256}

In[345]:=
```
negativeTrMM = Select[base16, Tr[#[[1]].#[[1]]] < 0 & → "Index"]
```

Out[345]=

{5, 6, 7, 8, 9, 10, 11, 16, 17, 22, 31, 32, 33, 34, 35, 36, 37, 38, 43, 52, 53, 54, 55, 56, 57, 58, 67, 68, 69, 70, 71, 72, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 94, 95, 96, 97, 98, 99, 100, 101, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 130, 131, 144, 145, 146, 147, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 179, 180, 181, 182, 189, 190, 191, 192, 193, 194, 195, 196, 204, 205, 206, 207, 208, 209, 210, 211, 213, 214, 215, 216, 219, 220, 221, 222, 223, 224, 233, 238, 239, 244, 245, 246, 251, 252, 253, 254}

In[346]:=
```
{Length[positiveTrMM], Length[negativeTrMM]}
```

Out[346]=

{136, 120}

In[347]:=

```
base16⟦#⟧ & /@ positiveTrMM;
{#⟦2⟧, Tr[Transpose[#⟦1⟧].linTrans]} & /@ %
Select[%, #⟦2⟧ ≠ 0 &]
linTransSymmetric = Table[Select[base16, #⟦2⟧ == %⟦j⟧⟦1⟧ &], {j, 1, Length[%]}];
```

Out[348]=

```
{{{0}, 0}, {{1}, 0}, {{2}, 0}, {{3}, 0}, {{0, 4}, 0}, {{0, 5}, 0}, {{0, 6}, 0}, {{0, 7}, -4},
 {{1, 4}, 0}, {{1, 5}, 0}, {{1, 6}, 0}, {{1, 7}, 0}, {{2, 4}, 0}, {{2, 5}, 0}, {{2, 6}, 0},
 {{2, 7}, 0}, {{3, 4}, -4}, {{3, 5}, 0}, {{3, 6}, 0}, {{3, 7}, 0}, {{0, 1, 4}, 0},
 {{0, 1, 5}, 0}, {{0, 1, 6}, 0}, {{0, 1, 7}, 0}, {{0, 2, 4}, 0}, {{0, 2, 5}, 0}, {{0, 2, 6}, 0},
 {{0, 2, 7}, 0}, {{0, 3, 4}, 0}, {{0, 3, 5}, 0}, {{0, 3, 6}, 0}, {{0, 3, 7}, 0},
 {{1, 2, 4}, 0}, {{1, 2, 5}, 0}, {{1, 2, 6}, 0}, {{1, 2, 7}, 0}, {{1, 3, 4}, 0},
 {{1, 3, 5}, 0}, {{1, 3, 6}, 0}, {{1, 3, 7}, 0}, {{2, 3, 4}, 0}, {{2, 3, 5}, 0},
 {{2, 3, 6}, 0}, {{2, 3, 7}, 0}, {{4, 5, 6}, -4}, {{4, 5, 7}, 0}, {{4, 6, 7}, 0},
 {{5, 6, 7}, 0}, {{0, 1, 2, 3}, 0}, {{0, 1, 4, 5}, 0}, {{0, 1, 4, 6}, 0}, {{0, 1, 4, 7}, 0},
 {{0, 1, 5, 6}, 0}, {{0, 1, 5, 7}, 0}, {{0, 1, 6, 7}, 0}, {{0, 2, 4, 5}, 0}, {{0, 2, 4, 6}, 0},
 {{0, 2, 4, 7}, 0}, {{0, 2, 5, 6}, 0}, {{0, 2, 5, 7}, 0}, {{0, 2, 6, 7}, 0}, {{0, 3, 4, 5}, 0},
 {{0, 3, 4, 6}, 0}, {{0, 3, 4, 7}, 4}, {{0, 3, 5, 6}, 0}, {{0, 3, 5, 7}, 0}, {{0, 3, 6, 7}, 0},
 {{1, 2, 4, 5}, 0}, {{1, 2, 4, 6}, 0}, {{1, 2, 4, 7}, 0}, {{1, 2, 5, 6}, 0}, {{1, 2, 5, 7}, 0},
 {{1, 2, 6, 7}, -4}, {{1, 3, 4, 5}, 0}, {{1, 3, 4, 6}, 0}, {{1, 3, 4, 7}, 0}, {{1, 3, 5, 6}, 0},
 {{1, 3, 5, 7}, 0}, {{1, 3, 6, 7}, 0}, {{2, 3, 4, 5}, 0}, {{2, 3, 4, 6}, 0}, {{2, 3, 4, 7}, 0},
 {{2, 3, 5, 6}, 0}, {{2, 3, 5, 7}, 0}, {{2, 3, 6, 7}, 0}, {{4, 5, 6, 7}, 0}, {{0, 1, 2, 4, 5}, 4},
 {{0, 1, 2, 4, 6}, 0}, {{0, 1, 2, 4, 7}, 0}, {{0, 1, 2, 5, 6}, 0}, {{0, 1, 2, 5, 7}, 0},
 {{0, 1, 2, 6, 7}, 0}, {{0, 1, 3, 4, 5}, 0}, {{0, 1, 3, 4, 6}, 0}, {{0, 1, 3, 4, 7}, 0},
 {{0, 1, 3, 5, 6}, 0}, {{0, 1, 3, 5, 7}, 0}, {{0, 1, 3, 6, 7}, 0}, {{0, 2, 3, 4, 5}, 0},
 {{0, 2, 3, 4, 6}, 0}, {{0, 2, 3, 4, 7}, 0}, {{0, 2, 3, 5, 6}, 0}, {{0, 2, 3, 5, 7}, 0},
 {{0, 2, 3, 6, 7}, 0}, {{0, 4, 5, 6, 7}, 4}, {{1, 2, 3, 4, 5}, 0}, {{1, 2, 3, 4, 6}, 0},
 {{1, 2, 3, 4, 7}, 0}, {{1, 2, 3, 5, 6}, 0}, {{1, 2, 3, 5, 7}, -4}, {{1, 2, 3, 6, 7}, 0},
 {{1, 4, 5, 6, 7}, 0}, {{2, 4, 5, 6, 7}, 0}, {{3, 4, 5, 6, 7}, 0}, {{0, 1, 2, 4, 5, 6}, 0},
 {{0, 1, 2, 4, 5, 7}, 0}, {{0, 1, 2, 4, 6, 7}, 0}, {{0, 1, 2, 5, 6, 7}, 0},
 {{0, 1, 3, 4, 5, 6}, 0}, {{0, 1, 3, 4, 5, 7}, 0}, {{0, 1, 3, 4, 6, 7}, 0},
 {{0, 1, 3, 5, 6, 7}, 0}, {{0, 2, 3, 4, 5, 6}, 0}, {{0, 2, 3, 4, 5, 7}, 0},
 {{0, 2, 3, 4, 6, 7}, 0}, {{0, 2, 3, 5, 6, 7}, 0}, {{1, 2, 3, 4, 5, 6}, 0},
 {{1, 2, 3, 4, 5, 7}, 0}, {{1, 2, 3, 4, 6, 7}, -4}, {{1, 2, 3, 5, 6, 7}, 0},
 {{0, 1, 2, 3, 4, 5, 6}, 0}, {{0, 1, 2, 3, 4, 5, 7}, 0}, {{0, 1, 2, 3, 4, 6, 7}, 0},
 {{0, 1, 2, 3, 5, 6, 7}, 0}, {{0, 1, 2, 3, 4, 5, 6, 7}, 0}, {{{0, 1, 2, 3, 4, 5, 6, 7, 8}}, 4}}
```

Out[349]=

```
{{{0, 7}, -4}, {{3, 4}, -4}, {{4, 5, 6}, -4}, {{0, 3, 4, 7}, 4},
 {{1, 2, 6, 7}, -4}, {{0, 1, 2, 4, 5}, 4}, {{0, 4, 5, 6, 7}, 4},
 {{1, 2, 3, 5, 7}, -4}, {{1, 2, 3, 4, 6, 7}, -4}, {{{0, 1, 2, 3, 4, 5, 6, 7, 8}}, 4}}
```

In[351]:=

```
linTransSymmetric[[1]]
```

Out[351]=

```
{{{{-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1}}, {0, 7}}}
```

In[352]:=

```
base16[[#]] & /@ negativeTrMM;
{#[[2]], Tr[Transpose[#[[1]]].linTrans]} & /@ %
Select[%, #[[2]] ≠ 0 &]
linTransAntiSymmetric = Table[Select[base16, #[[2]] == %[[j]][[1]] &], {j, 1, Length[%]}];
```

Out[353]=

```
{{{4}, 0}, {{5}, 0}, {{6}, 0}, {{7}, 0}, {{0, 1}, 0}, {{0, 2}, 0}, {{0, 3}, 0},
 {{1, 2}, 0}, {{1, 3}, 0}, {{2, 3}, 0}, {{4, 5}, 0}, {{4, 6}, 0}, {{4, 7}, 0},
 {{5, 6}, 0}, {{5, 7}, 0}, {{6, 7}, 0}, {{0, 1, 2}, 0}, {{0, 1, 3}, 0}, {{0, 2, 3}, 0},
 {{0, 4, 5}, 0}, {{0, 4, 6}, 0}, {{0, 4, 7}, 0}, {{0, 5, 6}, 0}, {{0, 5, 7}, 0},
 {{0, 6, 7}, 0}, {{1, 2, 3}, 0}, {{1, 4, 5}, 0}, {{1, 4, 6}, 0}, {{1, 4, 7}, 0},
 {{1, 5, 6}, 0}, {{1, 5, 7}, 0}, {{1, 6, 7}, 0}, {{2, 4, 5}, 0}, {{2, 4, 6}, 0},
 {{2, 4, 7}, 0}, {{2, 5, 6}, 0}, {{2, 5, 7}, 0}, {{2, 6, 7}, 0}, {{3, 4, 5}, 0},
 {{3, 4, 6}, 0}, {{3, 4, 7}, 0}, {{3, 5, 6}, -4}, {{3, 5, 7}, 0}, {{3, 6, 7}, 0},
 {{0, 1, 2, 4}, 0}, {{0, 1, 2, 5}, 0}, {{0, 1, 2, 6}, -4}, {{0, 1, 2, 7}, 0},
 {{0, 1, 3, 4}, 0}, {{0, 1, 3, 5}, 0}, {{0, 1, 3, 6}, 0}, {{0, 1, 3, 7}, 0}, {{0, 2, 3, 4}, 0},
 {{0, 2, 3, 5}, 0}, {{0, 2, 3, 6}, 0}, {{0, 2, 3, 7}, 0}, {{0, 4, 5, 6}, 0}, {{0, 4, 5, 7}, 0},
 {{0, 4, 6, 7}, 0}, {{0, 5, 6, 7}, 0}, {{1, 2, 3, 4}, 0}, {{1, 2, 3, 5}, 0}, {{1, 2, 3, 6}, 0},
 {{1, 2, 3, 7}, 0}, {{1, 4, 5, 6}, 0}, {{1, 4, 5, 7}, 0}, {{1, 4, 6, 7}, 0}, {{1, 5, 6, 7}, 0},
 {{2, 4, 5, 6}, 0}, {{2, 4, 5, 7}, 0}, {{2, 4, 6, 7}, 0}, {{2, 5, 6, 7}, 0}, {{3, 4, 5, 6}, 0},
 {{3, 4, 5, 7}, 0}, {{3, 4, 6, 7}, 0}, {{3, 5, 6, 7}, 0}, {{0, 1, 2, 3, 4}, 0},
 {{0, 1, 2, 3, 5}, -4}, {{0, 1, 2, 3, 6}, 0}, {{0, 1, 2, 3, 7}, 0}, {{0, 1, 4, 5, 6}, 0},
 {{0, 1, 4, 5, 7}, 0}, {{0, 1, 4, 6, 7}, 0}, {{0, 1, 5, 6, 7}, 0}, {{0, 2, 4, 5, 6}, 0},
 {{0, 2, 4, 5, 7}, 0}, {{0, 2, 4, 6, 7}, 0}, {{0, 2, 5, 6, 7}, 0}, {{0, 3, 4, 5, 6}, 0},
 {{0, 3, 4, 5, 7}, 0}, {{0, 3, 4, 6, 7}, 0}, {{0, 3, 5, 6, 7}, 4}, {{1, 2, 4, 5, 6}, 0},
 {{1, 2, 4, 5, 7}, 4}, {{1, 2, 4, 6, 7}, 0}, {{1, 2, 5, 6, 7}, 0}, {{1, 3, 4, 5, 6}, 0},
 {{1, 3, 4, 5, 7}, 0}, {{1, 3, 4, 6, 7}, 0}, {{1, 3, 5, 6, 7}, 0}, {{2, 3, 4, 5, 6}, 0},
 {{2, 3, 4, 5, 7}, 0}, {{2, 3, 4, 6, 7}, 0}, {{2, 3, 5, 6, 7}, 0}, {{0, 1, 2, 3, 4, 5}, 0},
 {{0, 1, 2, 3, 4, 6}, -4}, {{0, 1, 2, 3, 4, 7}, 0}, {{0, 1, 2, 3, 5, 6}, 0},
 {{0, 1, 2, 3, 5, 7}, 0}, {{0, 1, 2, 3, 6, 7}, 0}, {{0, 1, 4, 5, 6, 7}, 0},
 {{0, 2, 4, 5, 6, 7}, 0}, {{0, 3, 4, 5, 6, 7}, 0}, {{1, 2, 4, 5, 6, 7}, 0},
 {{1, 3, 4, 5, 6, 7}, 0}, {{2, 3, 4, 5, 6, 7}, 0}, {{0, 1, 2, 4, 5, 6, 7}, 0},
 {{0, 1, 3, 4, 5, 6, 7}, 0}, {{0, 2, 3, 4, 5, 6, 7}, 0}, {{1, 2, 3, 4, 5, 6, 7}, 0}}
```

Out[354]=

```
{{{3, 5, 6}, -4}, {{0, 1, 2, 6}, -4}, {{0, 1, 2, 3, 5}, -4},
 {{0, 3, 5, 6, 7}, 4}, {{1, 2, 4, 5, 7}, 4}, {{0, 1, 2, 3, 4, 6}, -4}}
```

In[356]:=

```
{Length[linTransSymmetric], Length[linTransAntiSymmetric]}
```

Out[356]=

```
{10, 6}
```

BASIS of 8 × 8 matrices :

In[357]:=

```
tA = {}; Do[AppendTo[tA, {τ[j], {j}}], {j, 1, 7}]
Length[tA]
tAB = {};
Do[AppendTo[tAB, {τ[j].τ[k], {j, k}}], {j, 1, 6}, {k, j + 1, 7}]
Length[tAB]
tABC = {};
Do[AppendTo[tABC, {τ[j].τ[k].τ[h], {j, k, h}}],
 {j, 1, 5}, {k, j + 1, 6}, {h, k + 1, 7}];
Length[tABC]
```

Out[358]=

```
7
```

Out[360]=

```
21
```

Out[362]=

```
35
```

In[363]:=

```
tA[[1]]
```

Out[363]=

```
{{{0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0},
  {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}}, {1}}
```

In[364]:=

```
tA[[#]][[1]] == -Transpose[tA[[#]][[1]]] & /@ Range[Length[tA]]
```

Out[364]=

```
{True, True, True, False, False, False, False}
```

In[365]:=

```
bas64 = Flatten[{tA, tAB, tABC, {{Id, {64}}}}, 1]; Length[bas64]
```

Out[365]=

```
64
```

In[366]:=

```
determineIF8AntiSymmetric[m_] := Module[{t, tt, sum, res, r},
  t = m.τ[#] & /@ Range[0, 7];
  tt = Transpose[#] & /@ t;
  sum = FullSimplify[t[[#]] + tt[[#]]] & /@ Range[Length[t]];
  res = Union[Flatten[#]] & /@ sum;
  If[Length[#] > 1, Return[False]] & /@ res;
  r = Union[Flatten[res]];
  Return[r == {0}];
 ]
```

In[367]:=

```
determineIF8AntiSymmetric[Zero8]
```

Out[367]=

```
True
```

In[368]:=

```
determineIF8AntiSymmetric[ID8]
```

Out[368]=

```
False
```

In[369]:=

```
find8AntiSymmetric = If[determineIF8AntiSymmetric[#[[1]]], #[[2]], ## &] & /@ bas64
```

Out[369]=

```
{###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &,
 ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &,
 ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &,
 ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &,
 ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &, ###1 &}
```

In[370]:=

```
anti = {};
Do[If[bas64[[k, 1]] == -Transpose[bas64[[k, 1]]],
  AppendTo[anti, {bas64[[k, 1]], {{k}, bas64[[k, 2]]}}] ], {k, 1, 64}];
Length[anti]
anti[[28]]
```

Out[371]=

```
28
```

Out[372]=

```
{{{0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, -1, 0}}, {{63}, {5, 6, 7}}}
```

In[373]:=

```
symm = {};
Do[If[bas64[[k, 1]] == Transpose[bas64[[k, 1]]],
  AppendTo[symm, {bas64[[k, 1]], {{k}, bas64[[k, 2]]}}] ], {k, 1, 64}];
Length[symm]
symm[[%]]
```

Out[374]=

```
35
```

Out[375]=

```
{{{0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0},
  {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 1},
  {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0}}, {{59}, {3, 6, 7}}}
```

In[376]:=

```
η64 =
 DiagonalMatrix[Table[FullSimplify[1/8 Tr[bas64[[A, 1]].bas64[[A, 1]]]], {A, 64}]];
Tr[η64]
```

Out[376]=

$$7 + \frac{1}{8}\,\mathrm{Tr}[\mathrm{Id}.\mathrm{Id}]$$

In[377]:=

```
(*(*(*(*(*(*(countTrace=Table[{A,B,FullSimplify[1/8 Tr[bas64[[A,1]].bas64[[B,1]]]] },
    {A,1, 64},{B,A, 64}])//MatrixForm*)*)*)*)*)*)
```

In[378]:=

```
(*■*)
```

In[379]:=

```
32 × 65
```

Out[379]=

2080

In[380]:=

```
(*Length[Flatten[countTrace,1]]*)
```

In[381]:=

```
(*{Length[countTrace],Length[countTrace[[1]]],Length[countTrace[[1,1]]]}*)
```

In[382]:=

```
(*{Length[countTrace],Length[countTrace[[2]]],Length[countTrace[[2,1]]]}*)
```

In[383]:=

```
(*{Length[countTrace],Length[countTrace[[64]]],Length[countTrace[[64,1]]]}*)
```

In[384]:=

```
(*{Length[Flatten[countTrace,1][[All,3]]],Count[Flatten[countTrace,1][[All,3]],0],
 Count[Flatten[countTrace,1][[All,3]],1],Count[Flatten[countTrace,1][[All,3]],-1]}*)
```

In[385]:=

```
32 × 63
```

Out[385]=

2016

In[386]:=

```
anti = {};
Do[If[bas64[[k, 1]] == -Transpose[bas64[[k, 1]]],
  AppendTo[anti, {bas64[[k, 1]], {{k}, bas64[[k, 2]]}}] ], {k, 1, 64}];
Length[anti]
anti[[28]]
```

Out[387]=

```
28
```

Out[388]=

```
{{{0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, -1, 0}}, {{63}, {5, 6, 7}}}
```

In[389]:=

```
anti[[28, 1]]
```

Out[389]=

```
{{0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, -1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0},
 {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, -1, 0}}
```

In[390]:=

```
anti[[28, 2]]
```

Out[390]=

```
{{63}, {5, 6, 7}}
```

In[391]:=

```
symm = {};
Do[If[bas64[[k, 1]] == Transpose[bas64[[k, 1]]],
  AppendTo[symm, {bas64[[k, 1]], {{k}, bas64[[k, 2]]}}] ], {k, 1, 64}];
Length[symm]
symm[[28]]
```

Out[392]=

```
35
```

Out[393]=

```
{{{-1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0},
  {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0},
  {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, 1}}, {{52}, {2, 5, 7}}}
```

4×4 Dirac γ matrices:

$$S^{\alpha\beta} = -\tfrac{1}{4}[\gamma^{\alpha},\gamma^{\beta}], \tag{22}$$

$$S^{\alpha\beta} = -\tfrac{1}{2}\gamma^{\alpha\beta}. \tag{23}$$

The symplectic form ϵ on D_4 is defined to be

$$\epsilon = \gamma^{64}. \tag{24}$$

As a consequence of Eq. (1), and the definitions of Eqs. (16)–(24), are the identities

$$\tilde{\gamma}^{\alpha}\epsilon = -\epsilon\gamma^{\alpha}, \tag{25}$$

$$\tilde{S}^{\alpha\beta}\epsilon = -\epsilon S^{\alpha\beta}, \tag{26}$$

$$[S^{\alpha\beta},\gamma_{\mu}] = \delta^{\alpha}_{\mu}\gamma^{\beta} - \delta^{\beta}_{\mu}\gamma^{\alpha}, \tag{27}$$

$$[S^{\alpha\beta},S^{\mu\nu}] = g^{\alpha\mu}S^{\beta\nu} - g^{\alpha\nu}S^{\beta\mu} - g^{\beta\mu}S^{\alpha\nu} + g^{\beta\nu}S^{\alpha\mu}, \tag{28}$$

and

$$\gamma^{5}S^{\alpha\beta} = \tfrac{1}{2}g^{\alpha\mu}g^{\beta\nu}\epsilon_{\mu\nu\lambda\sigma}S^{\lambda\sigma}. \tag{29}$$

$$g^{\alpha\beta} = g_{\alpha\beta} = \mathrm{diag}(1,1,1,-1) \tag{5}$$

In[394]:=

```
(*Symbolize[ gαβ ]*)
```

In[395]:=

```
η4488
Dimensions[%]
gαβ = η4488[[2 ;; 5, 2 ;; 5]]
```

Out[395]=

```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0},
 {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}}
```

Out[396]=

```
{8, 8}
```

Out[397]=

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}}
```

In[398]:=

```
Protect[gαβ]
```

Out[398]=

```
{g__Subscript__αβ}
```

In[399]:=

```
g3 = DiagonalMatrix[{1, 1, -1}]
```

Out[399]=

```
{{1, 0, 0}, {0, 1, 0}, {0, 0, -1}}
```

In[400]:=

```
Do[Do[Gr[h, k] = t4by4[h].s4by4[k] * (-g3[[h, h]] * g3[[k, k]]), {k, 1, 3}], {h, 1, 3}]
```

In[401]:=

```
Gr[1, 1] // MatrixForm
```

Out[401]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In[402]:=

```
γ = {Gr[1, 1], Gr[1, 2], Gr[1, 3], t4by4[2]};
```

In[403]:=

```
Table[FullSimplify[γ[[h]].γ[[h]] === g_αβ[[h, h]] * ID4], {h, 1, 4}]
```

Out[403]=

{True, True, True, True}

$$\gamma^{\alpha}\gamma^{\beta} + \gamma^{\beta}\gamma^{\alpha} = 2\gamma_0 g^{\alpha\beta}, \tag{4}$$

where

$$g^{\alpha\beta} = g_{\alpha\beta} = \mathrm{diag}(1,1,1,-1) \tag{5}$$

and γ_0 denotes the 4×4 unit matrix. The $\Sigma^{\alpha\beta}$ of Eq. (1)

In[404]:=

```
Table[{{h, k}, FullSimplify[(γ[[h]].γ[[k]] + γ[[k]].γ[[h]])/2] === g_αβ[[h, k]] ID4},
  {h, 1, 4}, {k, 1, 4}] // MatrixForm
```

Out[404]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

In[405]:=

```
Table[{{h, k}, FullSimplify[(γ[[h]].γ[[k]] + γ[[k]].γ[[h]])/2] === η4488[[1 + h, 1 + k]] ID4},
  {h, 1, 4}, {k, 1, 4}] // MatrixForm
```

Out[405]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

$$\gamma^5 = -\gamma^1 \gamma^2 \gamma^3 \gamma^4$$

In[406]:=

```
-γ[[1]].γ[[2]].γ[[3]].γ[[4]]
```

Out[406]=

```
{{0, -1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}}
```

In[407]:=

```
AppendTo[γ, (-γ[[1]].γ[[2]].γ[[3]].γ[[4]])]
```

Out[407]=

```
{{{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}},
 {{0, -1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, -1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}}}
```

In[408]:=

```
Protect[γ]
```

Out[408]=

```
{γ}
```

In[409]:=

```
γ[[5]]
```

Out[409]=

```
{{0, -1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}}
```

In[410]:=

```
γ[[5]] === -t4by4[3]
```

Out[410]=

```
True
```

In[411]:=

```
Table[{{h, k}, FullSimplify[γ[[h]].γ[[k]] + γ[[k]].γ[[h]]] === 2 * η4488[[1 + h, 1 + k]] ID4},
  {h, 1, 5}, {k, 1, 5}] // MatrixForm
```

Out[411]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,5\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,5\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,5\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,5\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{5,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,4\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{5,5\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

$$\gamma^{\alpha} = \gamma^{\alpha 6}, \tag{19}$$

$$\gamma^{5} = \gamma^{56}, \tag{20}$$

$$\gamma^{\alpha}\gamma^{5} = \gamma^{\alpha 5}, \tag{21}$$

$$\epsilon = \gamma^{64}.$$

In[412]:=

```
-γ[[4]] // MatrixForm
```

Out[412]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

In[413]:=

```
sϵ = {ϵ → t4by4[2]}
```

Out[413]=

```
{ϵ → {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}}}
```

In[414]:=

```
Transpose[ϵ /. sϵ] === (-ϵ /. sϵ)
```

Out[414]=

```
True
```

In[415]:=

```
Table[
 {{A1}, FullSimplify[(ϵ /. sϵ).γ[[A1]] === Transpose[(ϵ /. sϵ).γ[[A1]]]]}, {A1, 1, 5}]
```

Out[415]=

```
{{{1}, True}, {{2}, True}, {{3}, True}, {{4}, True}, {{5}, False}}
```

In[416]:=

```
γ[[4]] === t4by4[2]
```

Out[416]=

```
True
```

In[417]:=

```
t4by4[1]
```

Out[417]=

```
{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}}
```

$$S^{\alpha\beta} = -\tfrac{1}{4}[\gamma^{\alpha}, \gamma^{\beta}]. \qquad (3)$$

In[418]:=

```
S44αβ = -1/4 Table[FullSimplify[γ[[h]].γ[[k]] - γ[[k]].γ[[h]]], {h, 1, 4}, {k, 1, 4}];
```

In[419]:=

```
Protect[S44αβ]
```

Out[419]=

```
{S44αβ}
```

In[420]:=

```
η44 = DiagonalMatrix[{1, 1, 1, -1}]
```

Out[420]=

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}}
```

In[421]:=

```
Protect[η44]
```

Out[421]=

```
{η44}
```

S44αβ commutation relations; misc :

$$S^{\alpha\beta} = -\tfrac{1}{4}[\gamma^{\alpha},\gamma^{\beta}], \tag{22}$$

$$S^{\alpha\beta} = -\tfrac{1}{2}\gamma^{\alpha\beta}. \tag{23}$$

The symplectic form ϵ on D_4 is defined to be

$$\epsilon = \gamma^{64}. \tag{24}$$

As a consequence of Eq. (1), and the definitions of Eqs. (16)–(24), are the identities

$$\tilde{\gamma}^{\alpha}\epsilon = -\epsilon\gamma^{\alpha}, \tag{25}$$

$$\tilde{S}^{\alpha\beta}\epsilon = -\epsilon S^{\alpha\beta}, \tag{26}$$

$$[S^{\alpha\beta},\gamma_{\mu}] = \delta^{\alpha}_{\mu}\gamma^{\beta} - \delta^{\beta}_{\mu}\gamma^{\alpha}, \tag{27}$$

$$[S^{\alpha\beta},S^{\mu\nu}] = g^{\alpha\mu}S^{\beta\nu} - g^{\alpha\nu}S^{\beta\mu} - g^{\beta\mu}S^{\alpha\nu} + g^{\beta\nu}S^{\alpha\mu}, \tag{28}$$

and

$$\gamma^{5}S^{\alpha\beta} = \tfrac{1}{2}g^{\alpha\mu}g^{\beta\nu}\epsilon_{\mu\nu\lambda\sigma}S^{\lambda\sigma}. \tag{29}$$

```
S44αβ = -1/4 Table[FullSimplify[γ[[h]].γ[[k]] - γ[[k]].γ[[h]]], {h, 1, 4}, {k, 1, 4}];
```

In[422]:=

```
(*SAB=Table[1/4 (T16^A[A1].T16^A[B1]-T16^A[B1].T16^A[A1]), {A1,0, 7},{B1,0, 7}];*)
```

In[423]:=

```
(*Table[FullSimplify[σ16.SAB[[A1,B1]]===-Transpose[σ16.SAB[[A1,B1]]]],
   {A1,1, 8},{B1,1,8}]//Flatten//Union*)
```

In[424]:=

```
(*Table[{{A1,B1},FullSimplify[SAB[[A1,B1]]===-Transpose[SAB[[A1,B1]]]]},
 {A1,1, 8},{B1,1,8}]*)
```

In[425]:=

```
Table[FullSimplify[S44αβ[[A1, B1]].S44αβ[[A2, B2]] - S44αβ[[A2, B2]].S44αβ[[A1, B1]] ==
      (η44[[A1, A2]] × S44αβ[[B1, B2]] - η44[[A1, B2]] × S44αβ[[B1, A2]] -
        η44[[B1, A2]] × S44αβ[[A1, B2]] + η44[[B1, B2]] × S44αβ[[A1, A2]])],
   {A1, 1, 3}, {B1, A1 + 1, 4}, {A2, 1, 3}, {B2, A2 + 1, 4}] // Flatten // Union
```

Out[425]=

```
{True}
```

In[426]:=

```
Table[{{A1, B1, A2, B2},
   FullSimplify[S44αβ⟦A1, B1⟧.S44αβ⟦A2, B2⟧ - S44αβ⟦A2, B2⟧.S44αβ⟦A1, B1⟧ ==
     (η44⟦A1, A2⟧ × S44αβ⟦B1, B2⟧ - η44⟦A1, B2⟧ × S44αβ⟦B1, A2⟧ -
       η44⟦B1, A2⟧ × S44αβ⟦A1, B2⟧ + η44⟦B1, B2⟧ × S44αβ⟦A1, A2⟧)]},
  {A1, 1, 4}, {B1, 1, 4}, {A2, 1, 4}, {B2, 1, 4}] // MatrixForm
```

Out[426]//MatrixForm=

{1, 1, 1, 1} True	{1, 1, 1, 2} True	{1, 1, 1, 3} True	{1, 1, 1, 4} True	{1, 2, 1, 1} True	{1, 2, 1 True
{1, 1, 2, 1} True	{1, 1, 2, 2} True	{1, 1, 2, 3} True	{1, 1, 2, 4} True	{1, 2, 2, 1} True	{1, 2, 2 True
{1, 1, 3, 1} True	{1, 1, 3, 2} True	{1, 1, 3, 3} True	{1, 1, 3, 4} True	{1, 2, 3, 1} True	{1, 2, 3 True
{1, 1, 4, 1} True	{1, 1, 4, 2} True	{1, 1, 4, 3} True	{1, 1, 4, 4} True	{1, 2, 4, 1} True	{1, 2, 4 True
{2, 1, 1, 1} True	{2, 1, 1, 2} True	{2, 1, 1, 3} True	{2, 1, 1, 4} True	{2, 2, 1, 1} True	{2, 2, 1 True
{2, 1, 2, 1} True	{2, 1, 2, 2} True	{2, 1, 2, 3} True	{2, 1, 2, 4} True	{2, 2, 2, 1} True	{2, 2, 2 True
{2, 1, 3, 1} True	{2, 1, 3, 2} True	{2, 1, 3, 3} True	{2, 1, 3, 4} True	{2, 2, 3, 1} True	{2, 2, 3 True
{2, 1, 4, 1} True	{2, 1, 4, 2} True	{2, 1, 4, 3} True	{2, 1, 4, 4} True	{2, 2, 4, 1} True	{2, 2, 4 True
{3, 1, 1, 1} True	{3, 1, 1, 2} True	{3, 1, 1, 3} True	{3, 1, 1, 4} True	{3, 2, 1, 1} True	{3, 2, 1 True
{3, 1, 2, 1} True	{3, 1, 2, 2} True	{3, 1, 2, 3} True	{3, 1, 2, 4} True	{3, 2, 2, 1} True	{3, 2, 2 True
{3, 1, 3, 1} True	{3, 1, 3, 2} True	{3, 1, 3, 3} True	{3, 1, 3, 4} True	{3, 2, 3, 1} True	{3, 2, 3 True
{3, 1, 4, 1} True	{3, 1, 4, 2} True	{3, 1, 4, 3} True	{3, 1, 4, 4} True	{3, 2, 4, 1} True	{3, 2, 4 True
{4, 1, 1, 1} True	{4, 1, 1, 2} True	{4, 1, 1, 3} True	{4, 1, 1, 4} True	{4, 2, 1, 1} True	{4, 2, 1 True
{4, 1, 2, 1} True	{4, 1, 2, 2} True	{4, 1, 2, 3} True	{4, 1, 2, 4} True	{4, 2, 2, 1} True	{4, 2, 2 True
{4, 1, 3, 1} True	{4, 1, 3, 2} True	{4, 1, 3, 3} True	{4, 1, 3, 4} True	{4, 2, 3, 1} True	{4, 2, 3 True
{4, 1, 4, 1} True	{4, 1, 4, 2} True	{4, 1, 4, 3} True	{4, 1, 4, 4} True	{4, 2, 4, 1} True	{4, 2, 4 True

In[427]:=

```
Table[FullSimplify[S44αβ⟦A1, B1⟧.γ⟦B2⟧ - γ⟦B2⟧.S44αβ⟦A1, B1⟧ ==
     (η44⟦B2, A1⟧ × γ⟦B1⟧ - η44⟦B2, B1⟧ × γ⟦A1⟧)],
   {A1, 1, 4}, {B1, 1, 4}, {B2, 1, 4}] // Flatten // Union
```

Out[427]=

{True}

In[428]:=

```
Table[{{A1, B1, B2}, FullSimplify[S44αβ[[A1, B1]].γ[[B2]] - γ[[B2]].S44αβ[[A1, B1]] ==
      (η44[[B2, A1]] × γ[[B1]] - η44[[B2, B1]] × γ[[A1]])]},
  {A1, 1, 4}, {B1, 1, 4}, {B2, 1, 4}] // MatrixForm
```

Out[428]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1,1\} & \text{True} \\ \{1,1,2\} & \text{True} \\ \{1,1,3\} & \text{True} \\ \{1,1,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2,1\} & \text{True} \\ \{1,2,2\} & \text{True} \\ \{1,2,3\} & \text{True} \\ \{1,2,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3,1\} & \text{True} \\ \{1,3,2\} & \text{True} \\ \{1,3,3\} & \text{True} \\ \{1,3,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4,1\} & \text{True} \\ \{1,4,2\} & \text{True} \\ \{1,4,3\} & \text{True} \\ \{1,4,4\} & \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1,1\} & \text{True} \\ \{2,1,2\} & \text{True} \\ \{2,1,3\} & \text{True} \\ \{2,1,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2,1\} & \text{True} \\ \{2,2,2\} & \text{True} \\ \{2,2,3\} & \text{True} \\ \{2,2,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3,1\} & \text{True} \\ \{2,3,2\} & \text{True} \\ \{2,3,3\} & \text{True} \\ \{2,3,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4,1\} & \text{True} \\ \{2,4,2\} & \text{True} \\ \{2,4,3\} & \text{True} \\ \{2,4,4\} & \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1,1\} & \text{True} \\ \{3,1,2\} & \text{True} \\ \{3,1,3\} & \text{True} \\ \{3,1,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2,1\} & \text{True} \\ \{3,2,2\} & \text{True} \\ \{3,2,3\} & \text{True} \\ \{3,2,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3,1\} & \text{True} \\ \{3,3,2\} & \text{True} \\ \{3,3,3\} & \text{True} \\ \{3,3,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4,1\} & \text{True} \\ \{3,4,2\} & \text{True} \\ \{3,4,3\} & \text{True} \\ \{3,4,4\} & \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1,1\} & \text{True} \\ \{4,1,2\} & \text{True} \\ \{4,1,3\} & \text{True} \\ \{4,1,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2,1\} & \text{True} \\ \{4,2,2\} & \text{True} \\ \{4,2,3\} & \text{True} \\ \{4,2,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3,1\} & \text{True} \\ \{4,3,2\} & \text{True} \\ \{4,3,3\} & \text{True} \\ \{4,3,4\} & \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4,1\} & \text{True} \\ \{4,4,2\} & \text{True} \\ \{4,4,3\} & \text{True} \\ \{4,4,4\} & \text{True} \end{pmatrix} \end{pmatrix}$$

In[429]:=

```
Table[{{A1, B1},
  FullSimplify[(ϵ /. sϵ).S44αβ[[A1, B1]] === Transpose[(ϵ /. sϵ).S44αβ[[A1, B1]]]]},
 {A1, 1, 3}, {B1, A1 + 1, 4}]
```

Out[429]=

```
{{{{1, 2}, True}, {{1, 3}, True}, {{1, 4}, True}},
 {{{2, 3}, True}, {{2, 4}, True}}, {{{3, 4}, True}}}
```

The commutation relations for the generators J_{ab} of the $\mathfrak{so}(3,3)$ Lie algebra (using the physics convention where generators are Hermitian) are given by the formula:

$$[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad} + \eta_{bd}J_{ac})$$

where:

- The indices a, b, c, d range from 1 to 6.
- η_{ab} is the **metric tensor** with signature $(+,+,+,-,-,-)$ (or the reverse, depending on the convention). A common choice is a diagonal matrix $\eta = \mathbf{diag}(1,1,1,-1,-1,-1)$.
- J_{ab} are the **generators** of the algebra, which are antisymmetric, $J_{ab} = -J_{ba}$.
- The terms on the right-hand side involve Kronecker deltas weighted by the metric, ensuring the resulting generator is also an element of the algebra and satisfies the required antisymmetry and Jacobi identity properties.

The final answer is $[\mathbf{J_{ab}}, \mathbf{J_{cd}}] = \mathbf{i}(\boldsymbol{\eta}_{\mathbf{ac}}\mathbf{J_{bd}} - \boldsymbol{\eta}_{\mathbf{ad}}\mathbf{J_{bc}} - \boldsymbol{\eta}_{\mathbf{bc}}\mathbf{J_{ad}} + \boldsymbol{\eta}_{\mathbf{bd}}\mathbf{J_{ac}})$.

Step 1: Define the generators and metric

The **so(4,4)** Lie algebra has 28 generators, which can be denoted as $J_{\mu\nu}$ where the indices μ, ν range from 0 to 7, and $J_{\mu\nu} = -J_{\nu\mu}$. The Lie algebra is defined in an 8-dimensional real vector space equipped with a metric $\eta_{\mu\nu}$ with signature (4, 4).

- **Generators:** $J_{\mu\nu}$ with $\mu, \nu \in \{0, 1, \ldots, 7\}$
- **Metric tensor:** $\eta_{\mu\nu} = \mathrm{diag}(-1, -1, -1, -1, +1, +1, +1, +1)$ (or the opposite sign convention)

Step 2: State the general commutation relation

The commutation relations for the generators of the **so(p,q)** Lie algebra are a generalization of the **so(n)** algebra relations, incorporating the metric tensor $\eta_{\mu\nu}$.

Answer:

The commutation relations for the generators $J_{\mu\nu}$ of the **so(4,4)** Lie algebra are:

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho}$$

where $\eta_{\mu\nu}$ is the metric tensor with a (4,4) signature (four plus signs and four minus signs, or vice versa).

(Note: In some physics conventions, an extra factor of i or $i\hbar$ is included depending on whether the generators are defined as Hermitian or not).

> what are the commutation relation for the generators of so(4,4)?

The commutation relations for the generators $J_{\mu\nu}$ of the $so(4,4)$ Lie algebra, where $\mu, \nu \in \{0, 1, \ldots, 7\}$ and $J_{\mu\nu} = -J_{\nu\mu}$, are the standard relations for orthogonal Lie algebras, given by a generalization of the angular momentum commutation relations:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho})$$

where $\eta_{\mu\nu}$ is the flat metric of 8-dimensional Minkowski space with signature (4,4) (or some other convention, the structure constants are the same as for $so(8)$ in an appropriate basis), which is a diagonal matrix with four positive and four negative entries (e.g., $\eta = \mathrm{diag}(+1, +1, +1, +1, -1, -1, -1, -1)$).

↪ **Why is the metric signature significant for $so(4,4)$ commutation relations?**

Why is the metric signature significant for $so(4,4)$ commutation relations?

The metric signature is significant for the $so(p,q)$ commutation relations because the metric tensor $\eta_{\mu\nu}$ acts as the **structure constants** of the Lie algebra, defining its specific algebraic structure and physical properties (if applicable).

Here is a detailed explanation of its significance:

- **Definition of the Algebra:** The commutation relation for the generators $J_{\mu\nu}$ of any orthogonal Lie algebra $so(p,q)$ is given by:
  $[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$
  The values of the structure constants depend directly on the components of the metric $\eta_{\mu\nu}$. This means that the specific algebraic properties of $so(4,4)$ are distinct from $so(8)$ (which has a (8,0) signature, i.e., a Euclidean metric), even though they operate in the same 8-dimensional vector space.

- **Physical Interpretation:** In physics, the metric signature determines the nature of spacetime. A mixed signature (like (4,4) or the more common (1,3) Minkowski signature) implies a causal structure, distinguishing between timelike and spacelike dimensions.
  - Generators associated with rotations in spacelike dimensions (positive $\eta_{\mu\nu}$ entries) typically lead to compact symmetry groups (like $SO(4)$ or $SO(8)$).

In[430]:=

η4488

Out[430]=

```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0},
 {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}}
```

orthogonal Lie algebra $so(p,q)$ is given by:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})$$

$$t^A = \begin{pmatrix} 0 & \bar{\tau}^A \\ \tau^A & 0 \end{pmatrix}.$$

In[431]:=

```
(*Table[SAB⟦A1,B1⟧== ArrayFlatten[{{SAB1⟦A1,B1⟧,0},{0,SAB2⟦A1,B1⟧}}],
 {A1,0, 7},{B1,0,7}]*)
```

$$4D_{(1)}{}^{AB} = \bar{\tau}^A \tau^B - \bar{\tau}^B \tau^A$$

$$4D_{(2)}{}^{AB} = \tau^A \bar{\tau}^B - \tau^B \bar{\tau}^A .$$

Let $g \in \overline{SO(4,4)}$ and $L \in SO(4,4)$. The canonical 2-1 homomorphism $\overline{SO(4,4)} \to SO(4,4)$ is given by

$$8L^{A'}{}_{B'}(g) = \operatorname{tr} \widetilde{D}^{(2)}(g)\, \tau^{A'} D^{(1)}(g)\, \bar{\tau}_{B'} . \tag{2.8}$$

As usual tr denotes the trace. Under the action of $\overline{SO(4,4)}$

$$\widetilde{D}^{(1)} \sigma = \sigma D^{(1)-1} , \tag{2.9}$$

$$\sigma^{-1} \widetilde{D}^{(2)} = D^{(2)-1} \sigma^{-1} , \tag{2.10}$$

$$L^{A'}{}_{C'} G_{A'B'} L^{B'}{}_{D'} = G_{C'D'} , \tag{2.11}$$

$$L^{A'}{}_{B'} \tau^{B'} = \widetilde{D}^{(2)} \tau^{A'} D^{(1)} , \tag{2.12}$$

$$L^{A'}{}_{B'} \bar{\tau}^{B'} = D^{(1)-1} \bar{\tau}^{A'} \widetilde{D}^{(2)-1} . \tag{2.13}$$

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A Remarkable Representation of the 3 + 2 de Sitter Group

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(Received 20 February 1963)

Identities satisfied by the generators of the Dirac algebra

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(Received 26 July 1983; accepted for publication 23 September 1983)

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In[432]:=

```
Unprotect[γABs];
γABs = Flatten[{s4by4[#] & /@ Range[3], t4by4[#] & /@ Range[3],
   Flatten[Table[st[J, K], {J, 1, 3}, {K, 1, 3}], 1], {ID4}}, 1]
```

Out[433]=

```
{{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
 {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {1, 0, 0, 0}},
 {{0, 0, -1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 1, 0, 0}, {-1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}},
 {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
 {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, -1}, {1, 0, 0, 0}, {0, -1, 0, 0}},
 {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, -1, 0}},
 {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}},
 {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}},
 {{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
 {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}}
```

In[434]:=

```
Length[γABs]
```

Out[434]=

16

In[435]:=

```
Protect[γABs]
```

Out[435]=

{γABs}

In[436]:=

```
γABs[[1]]
```

Out[436]=

```
{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}}
```

In[437]:=

```
γABs[[-1]]
```

Out[437]=

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

Recall Self Dual Anti-Symmetric

In[438]:=

```
Table[(1/2 Sum[(Sum[ϵ4[[p, q, j1, j2]] × s4by4[h][[j1, j2]], {j2, 1, 4}]), {j1, 1, 4}]) - s4by4[h][[p, q]],
 {h, 1, 3}, {q, 4}, {p, 4}]
```

Out[438]=

```
{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}
```

Recall Anti SelfDual Anti-Symmetric

In[439]:=

```
Table[(1/2 Sum[(Sum[ϵ4[[p, q, j1, j2]] × t4by4[h][[j1, j2]], {j2, 1, 4}]), {j1, 1, 4}]) + t4by4[h][[p, q]],
 {h, 1, 3}, {q, 4}, {p, 4}]
```

Out[439]=

```
{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}
```

BASIS of 4 × 4 matrices :

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The Dirac spinor in six dimensions

By E. A. LORD
Department of Mathematics, King's College, University of London

(*Received* 17 *November* 1966)

3. *Decomposition of the basic spinor representations.* We have already stated that the basic spinor representations of $SO(n,c)$, for n even, are reducible. We shall consider here the reduction of the basic 8-component spinor representation of $SO(6,c)$ into constituent 4-component representations. The procedure is easily generalized to apply to any even-dimensional space. An irreducible representation of the Clifford algebra C_6 can be constructed from a given irreducible representation of C_5. The $\gamma_\mu (\mu = 1, \dots 5)$ of an irreducible representation of C_5 satisfy (2·5), so that the quantities e_α $(\alpha = 1, \dots 6)$ defined by

$$\left.\begin{aligned} e_\mu &= (-i)\begin{pmatrix} & \gamma_\mu B \\ -B^{-1}\gamma_\mu & \end{pmatrix} \quad (\mu = 1, \dots 5), \\ e_6 &= \begin{pmatrix} & B \\ B^{-1} & \end{pmatrix}, \end{aligned}\right\} \tag{3·1}$$

satisfy
$$\tfrac{1}{2}(e_\alpha e_\beta + e_\beta e_\alpha) = \delta_{\alpha\beta} \quad (\alpha, \beta = 1, \dots 6).$$

For the moment the matrix B is completely arbitrary. The e_α therefore give an irreducible representation of C_6. They can be written more concisely in terms of the sets of matrices a_α and $\bar{a}_\alpha$, defined as follows:

$$\left.\begin{aligned} \alpha_\mu &= \gamma_\mu B, \quad \bar{a}_\mu = -B^{-1}\gamma_\mu \quad (\mu = 1, \dots 5), \\ a_6 &= iB, \quad \bar{a}_6 = iB^{-1}. \end{aligned}\right\} \tag{3·2}$$

The generator e_α of C_6 can then be written

$$e_\alpha = (-i)\begin{pmatrix} & a_\alpha \\ \bar{a}_\alpha & \end{pmatrix} \quad (\alpha = 1, \dots 6). \tag{3·3}$$

With $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ we then find

$$e_7 = ie_1e_2e_3e_4e_5e_6 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

and
$$e_{\alpha\beta} = \tfrac{1}{2}(e_\alpha e_\beta - e_\beta e_\alpha) = -\frac{1}{2}\begin{pmatrix} a_\alpha\bar{a}_\beta - a_\beta\bar{a}_\alpha & \\ & \bar{a}_\alpha a_\beta - \bar{a}_\beta a_\alpha \end{pmatrix}.$$

This latter expression shows that the infinitesimal generators of the 8-component spinor representation of $SO(6,c)$ is the direct sum of two 4-component representations. The basic spinor can be written

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

where ϕ and χ are four component spinors which transform according to representations with infinitesimal generators

$$\left.\begin{aligned} G^{(1)}_{\alpha\beta} &= -\tfrac{1}{4}(a_\alpha\bar{a} - a_\beta\bar{a}_\alpha) \\ G^{(2)}_{\alpha\beta} &= -\tfrac{1}{4}(\bar{a}_\alpha a_\beta - \bar{a}_\beta a_\alpha), \end{aligned}\right\} \tag{3·4}$$

and

$\{\underbrace{\text{sy}}_{1}, \underbrace{\text{sy}}_{4}, \underbrace{\text{asy}}_{6}, \underbrace{\text{sy}}_{4}, \underbrace{\text{sy}}_{1}\}$

In[440]:=

```
Plus @@ {1, 4, 6, 4, 1}
```

Out[440]=

16

$$\{\underbrace{sy}_{1}, \underbrace{sy}_{4}, \underbrace{asy}_{6}, \underbrace{sy}_{4}, \underbrace{sy}_{1}\}$$

In[441]:=

```
Plus @@ {1, 4, 6, 4, 1}
```

Out[441]=

16

In[442]:=

```
MatrixForm[If[# == 2, ⅈ, 1] * PauliMatrix[#]] & /@ Range[3]
```

Out[442]=

$$\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$$

In[443]:=

```
σ22 = Flatten[{{IdentityMatrix[2]}, If[# == 2, ⅈ, 1] * PauliMatrix[#] & /@ Range[3]}, 1]
```

Out[443]=

{{{1, 0}, {0, 1}}, {{0, 1}, {1, 0}}, {{0, 1}, {-1, 0}}, {{1, 0}, {0, -1}}}

In[444]:=

```
σ22̄ =
 Flatten[{{-IdentityMatrix[2]}, If[# == 2, ⅈ, 1] * PauliMatrix[#] & /@ Range[3]}, 1]
```

Out[444]=

{{{-1, 0}, {0, -1}}, {{0, 1}, {1, 0}}, {{0, 1}, {-1, 0}}, {{1, 0}, {0, -1}}}

In[445]:=

```
-IdentityMatrix[2] === Dot @@ (σ22[[#]] & /@ Range[Length[σ22]])
```

Out[445]=

True

In[446]:=

```
Table[{{A, B}, FullSimplify[1/2 Tr[1/2 (σ22[[A]].σ22̄[[B]] + σ22[[B]].σ22̄[[A]])]]},
  {A, 1, 4}, {B, 1, 4}] // MatrixForm
```

Out[446]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1, 1\} \\ -1 \end{pmatrix} & \begin{pmatrix} \{1, 2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{1, 3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{1, 4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{2, 1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{2, 2\} \\ 1 \end{pmatrix} & \begin{pmatrix} \{2, 3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{2, 4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{3, 1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{3, 2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{3, 3\} \\ -1 \end{pmatrix} & \begin{pmatrix} \{3, 4\} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \{4, 1\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4, 2\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4, 3\} \\ 0 \end{pmatrix} & \begin{pmatrix} \{4, 4\} \\ 1 \end{pmatrix} \end{pmatrix}$$

In[447]:=

```
η2244 = DiagonalMatrix[{-1, 1, -1, 1}]
```

Out[447]=

{{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}}

In[448]:=

```
Table[{{A, B}, FullSimplify[ExpandAll[1/2 (σ22[[A]].σ22̄[[B]] + σ22[[B]].σ22̄[[A]]) ==
      η2244[[A, B]] * IdentityMatrix[2]]]}, {A, 1, 4}, {B, 1, 4}] // MatrixForm
```

Out[448]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} \{1,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{1,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{2,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{2,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{3,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{3,4\} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \{4,1\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,2\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,3\} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \{4,4\} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

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The Dirac spinor in six dimensions

By E. A. LORD
Department of Mathematics, King's College, University of London

(*Received* 17 *November* 1966)

$$\left.\begin{aligned} e_\mu &= (-i)\begin{pmatrix} & \gamma_\mu B \\ -B^{-1}\gamma_\mu & \end{pmatrix} \quad (\mu = 1, \ldots 5), \\ e_6 &= \begin{pmatrix} & B \\ B^{-1} & \end{pmatrix}, \end{aligned}\right\}$$

$$\tfrac{1}{2}(e_\alpha e_\beta + e_\beta e_\alpha) = \delta_{\alpha\beta} \quad (\alpha, \beta = 1, \ldots 6).$$

$$\left.\begin{aligned} \alpha_\mu &= \gamma_\mu B, \quad \bar{a}_\mu = -B^{-1}\gamma_\mu \quad (\mu = 1, \ldots 5), \\ a_6 &= iB, \quad \bar{a}_6 = iB^{-1}. \end{aligned}\right\}$$

;enerator e_α of C_6 can then be written

$$e_\alpha = (-i)\begin{pmatrix} & a_\alpha \\ \bar{a}_\alpha & \end{pmatrix} \quad (\alpha = 1, \ldots 6).$$

$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ we then find

$$e_7 = ie_1e_2e_3e_4e_5e_6 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$e_{\alpha\beta} = \tfrac{1}{2}(e_\alpha e_\beta - e_\beta e_\alpha) = -\frac{1}{2}\begin{pmatrix} a_\alpha\bar{a}_\beta - a_\beta\bar{a}_\alpha & \\ & \bar{a}_\alpha a_\beta - \bar{a}_\beta a_\alpha \end{pmatrix}.$$

In[449]:=

```
Do[yyy[j] = ArrayFlatten[{{0, σ22[[j]]}, {-σ22[[j]], 0}}], {j, 4}];
yyy[5] = ArrayFlatten[{{0, IdentityMatrix[2]}, {IdentityMatrix[2], 0}}]
```

Out[449]=

```
{{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}}
```

In[450]:=

```
yyy[5] = {{0, 0, -1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 0}, {0, 1, 0, 0}}
```

Out[450]=

```
{{0, 0, -1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 0}, {0, 1, 0, 0}}
```

In[451]:=

```
(yyy[6] = Dot @@ (yyy[#] & /@ Range[5])) // MatrixForm
```

Out[451]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

In[452]:=

```
MatrixForm[yyy[#]] & /@ Range[6]
```

Out[452]=

$$\left\{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right\}$$

In[453]:=

```
Do[xxx[j] = ArrayFlatten[{{0, σ22̅[[j]]}, {σ22[[j]], 0}}], {j, 4}]
```

In[454]:=

```
MatrixForm[xxx[#]] & /@ Range[4]
```

Out[454]=

$$\left\{\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right\}$$

In[455]:=

```
xxx[5] = Dot @@ (xxx[#] & /@ Range[4])
```

Out[455]=

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}
```

In[456]:=

```
IdentityMatrix[4]
```

Out[456]=

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

In[457]:=

```
txA0 = {{IdentityMatrix[4], {0}}};
Length[txA0]
txA = {}; Do[AppendTo[txA, {xxx[j], {j}}], {j, 1, 4}]
Length[txA]
txAB = {};
Do[AppendTo[txAB, {xxx[j].xxx[k], {j, k}}], {j, 1, 3}, {k, j + 1, 4}]
Length[txAB]
txABC = {};
Do[AppendTo[txABC, {xxx[j].xxx[k].xxx[h], {j, k, h}}],
 {j, 1, 2}, {k, j + 1, 3}, {h, k + 1, 4}];
Length[txABC]
txABCD = {{xxx[1].xxx[2].xxx[3].xxx[4], {1, 2, 3, 4}}};
Length[txABCD]
```

Out[458]=

1

Out[460]=

4

Out[462]=

6

Out[464]=

4

Out[466]=

1

In[467]:=

```
base4by4 = Flatten[{txA0, txA, txAB, txABC, txABCD}, 1]; Length[base4by4]
```

Out[467]=

16

In[468]:=

```
Table[{{A, B}, Block[{r}, r = FullSimplify[
      1/4 Tr[1/2 (base4by4[[A, 1]].base4by4[[B, 1]] + base4by4[[B, 1]].base4by4[[A, 1]])]];
    If[r == 0, Style[r, Red], Style[r, Darker[Green]]]]},
  {A, 1, Length[base4by4]}, {B, 1, Length[base4by4]}] // MatrixForm
```

Out[468]//MatrixForm=

({1, 1}, 1)	({1, 2}, 0)	({1, 3}, 0)	({1, 4}, 0)	({1, 5}, 0)	({1, 6}, 0)	({1, 7}, 0)	({1, 8}, 0)	(
({2, 1}, 0)	({2, 2}, -1)	({2, 3}, 0)	({2, 4}, 0)	({2, 5}, 0)	({2, 6}, 0)	({2, 7}, 0)	({2, 8}, 0)	(
({3, 1}, 0)	({3, 2}, 0)	({3, 3}, 1)	({3, 4}, 0)	({3, 5}, 0)	({3, 6}, 0)	({3, 7}, 0)	({3, 8}, 0)	(
({4, 1}, 0)	({4, 2}, 0)	({4, 3}, 0)	({4, 4}, -1)	({4, 5}, 0)	({4, 6}, 0)	({4, 7}, 0)	({4, 8}, 0)	(
({5, 1}, 0)	({5, 2}, 0)	({5, 3}, 0)	({5, 4}, 0)	({5, 5}, 1)	({5, 6}, 0)	({5, 7}, 0)	({5, 8}, 0)	(
({6, 1}, 0)	({6, 2}, 0)	({6, 3}, 0)	({6, 4}, 0)	({6, 5}, 0)	({6, 6}, 1)	({6, 7}, 0)	({6, 8}, 0)	(
({7, 1}, 0)	({7, 2}, 0)	({7, 3}, 0)	({7, 4}, 0)	({7, 5}, 0)	({7, 6}, 0)	({7, 7}, -1)	({7, 8}, 0)	(
({8, 1}, 0)	({8, 2}, 0)	({8, 3}, 0)	({8, 4}, 0)	({8, 5}, 0)	({8, 6}, 0)	({8, 7}, 0)	({8, 8}, 1)	(
({9, 1}, 0)	({9, 2}, 0)	({9, 3}, 0)	({9, 4}, 0)	({9, 5}, 0)	({9, 6}, 0)	({9, 7}, 0)	({9, 8}, 0)	(
({10, 1}, 0)	({10, 2}, 0)	({10, 3}, 0)	({10, 4}, 0)	({10, 5}, 0)	({10, 6}, 0)	({10, 7}, 0)	({10, 8}, 0)	(
({11, 1}, 0)	({11, 2}, 0)	({11, 3}, 0)	({11, 4}, 0)	({11, 5}, 0)	({11, 6}, 0)	({11, 7}, 0)	({11, 8}, 0)	(
({12, 1}, 0)	({12, 2}, 0)	({12, 3}, 0)	({12, 4}, 0)	({12, 5}, 0)	({12, 6}, 0)	({12, 7}, 0)	({12, 8}, 0)	(
({13, 1}, 0)	({13, 2}, 0)	({13, 3}, 0)	({13, 4}, 0)	({13, 5}, 0)	({13, 6}, 0)	({13, 7}, 0)	({13, 8}, 0)	(
({14, 1}, 0)	({14, 2}, 0)	({14, 3}, 0)	({14, 4}, 0)	({14, 5}, 0)	({14, 6}, 0)	({14, 7}, 0)	({14, 8}, 0)	(
({15, 1}, 0)	({15, 2}, 0)	({15, 3}, 0)	({15, 4}, 0)	({15, 5}, 0)	({15, 6}, 0)	({15, 7}, 0)	({15, 8}, 0)	(
({16, 1}, 0)	({16, 2}, 0)	({16, 3}, 0)	({16, 4}, 0)	({16, 5}, 0)	({16, 6}, 0)	({16, 7}, 0)	({16, 8}, 0)	(

In[469]:=

```
base4by4[[16]][[1]]
```

Out[469]=

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}
```

In[470]:=

```
anti4by4 = {};
Do[If[base4by4[[k, 1]] == -Transpose[base4by4[[k, 1]]], AppendTo[anti4by4,
   {base4by4[[k, 1]], {{k}, base4by4[[k, 2]]}}] ], {k, 1, Length[base4by4]}];
Length[anti4by4]
anti4by4[[%]]
```

Out[471]=

6

Out[472]=

{{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}}, {{14}, {1, 3, 4}}}

In[473]:=

```
symm4by4 = {};
Do[If[base4by4[[k, 1]] == Transpose[base4by4[[k, 1]]], AppendTo[symm4by4,
   {base4by4[[k, 1]], {{k}, base4by4[[k, 2]]}}] ], {k, 1, Length[base4by4]}];
Length[symm4by4]
symm4by4[[%]]
```

Out[474]=

10

Out[475]=

{{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}, {{16}, {1, 2, 3, 4}}}

In[476]:=

```
η16 = DiagonalMatrix[Table[
   FullSimplify[1/4 Tr[symm4by4[[A, 1]].symm4by4[[A, 1]]]], {A, Length[symm4by4]}]];
Tr[η16]
```

Out[476]=

10

O(4,4); evalues, evecs of σ

In[477]:=

```
{evalues, evecs} = Eigensystem[σ]
```

Out[477]=

{{-1, -1, -1, -1, 1, 1, 1, 1}, {{0, 0, 0, -1, 0, 0, 0, 1}, {0, 0, -1, 0, 0, 0, 1, 0},
  {0, -1, 0, 0, 0, 1, 0, 0}, {-1, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 1},
  {0, 0, 1, 0, 0, 0, 1, 0}, {0, 1, 0, 0, 0, 1, 0, 0}, {1, 0, 0, 0, 1, 0, 0, 0}}}

In[478]:=

```
(𝕦 = ExpandAll[1/√2 evecs]) // MatrixForm
```

Out[478]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

In[479]:=

```
Table[𝕦[[h]].σ.𝕦[[h]], {h, 1, Length[𝕦]}]
```

Out[479]=

```
{-1, -1, -1, -1, 1, 1, 1, 1}
```

In[480]:=

```
Table[(Transpose[𝕦][[h]]).σ.𝕦[[h]], {h, 1, Length[𝕦]}]
```

Out[480]=

```
{-1, -1, -1, -1, 1, 1, 1, 1}
```

In[481]:=

```
Table[(Transpose[𝕦][[A]]) == -σ.𝕦[[A]], {A, 1, 4}]
Table[(Transpose[𝕦][[A]]) == σ.𝕦[[A]], {A, 5, Length[𝕦]}]
```

Out[481]=

```
{True, True, True, True}
```

Out[482]=

```
{True, True, True, True}
```

In[483]:=

```
hUSE = 8;
```

In[484]:=

```
Protect[hUSE]
```

Out[484]=

```
{hUSE}
```

In[485]:=

```
unit = 𝕦[[hUSE]]
```

Out[485]=

$$\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}$$

In[486]:=

```
Protect[unit]
```

Out[486]=

```
{unit}
```

In[487]:=

```
(Transpose[unit].σ.unit)
```

Out[487]=

1

In[488]:=

```
(Transpose[unit].σ.(τ̄[0]))
```

Out[488]=

$$\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}$$

$$\mathfrak{F}^{A}_{a} = \frac{1}{\sqrt{\tilde{u}\sigma u}} u^{c} \sigma_{cb} \bar{\tau}^{Ab}{}_{a}$$

$$\mathfrak{F}^{a}_{A} = \frac{1}{\sqrt{\tilde{u}\sigma u}} \tau_{A}{}^{a}{}_{b} u^{b}$$

In[489]:=

```
Clear[resolution];
resolution[v_List] := Module[{EAa, EaA, FaA, id, u},
  If[Length[v] == 8, ##&[], Print["ERROR Length[v] ", Length[v]];
   Return[Length[v]]];
  u = v / Sqrt[Abs[Transpose[v].σ.v]] // FullSimplify;
  EAa = Table[u.σ.τ̄[A1], {A1, 0, 7}]; 𝔉^A_a = 1/Sqrt[ũσu] u^c σ_cb τ̄^Ab_a;
  EaA = FullSimplify[Inverse[EAa]]; 𝔉^a_A = 1/Sqrt[ũσu] τ_A^a_b u^b;
  (*FaA=FullSimplify[Transpose[η4488[[#+1,#+1]]*(τ[#].unit)]&/@Range[0,7]];*)
  FaA = FullSimplify[
    Transpose[Table[η4488[[A1 + 1, A1 + 1]] * (τ[A1].unit), {A1, 0, 7}]]];
  If[FaA === EaA, Print["u^A = ", EAa.u], Print["ERROR"]];
  If[F^a_A.F^A_a === ID8, ##&[], Print["ERROR F^a_A.F^A_a != ID8"]];
  Return[Union[Flatten[FullSimplify[FaA - EaA]]]];
  Return[{FaA, EaA, FaA - EaA}];
 ]
```

In[491]:=

```
resolution[unit]
```

```
u^A = {1, 0, 0, 0, 0, 0, 0, 0}
ERROR F^a_A.F^A_a != ID8
```

Out[491]=

```
{0}
```

In[492]:=

```
Clear[resolution2];
resolution2[u_List] := Module[{EAa, EaA, FaA, id},
  If[Length[u] == 8, ## &[], Print["ERROR Length[u] ", Length[u]];
   Return[Length[u]]];
  (*u=v/Sqrt[Abs[Transpose[v].σ.v]] //FullSimplify;*)
  EAa = Table[u.σ.τ̄[A1], {A1, 0, 7}];
  δ̄^A_a = 1/Sqrt[u^μ σ_μν u^ν] u^ν σ_νa δ^AB τ̄^a_B;
  Print["u^A = ", EAa.u];
  Return[EAa];
  EaA = FullSimplify[Inverse[EAa]]; δ̄^a_A = 1/Sqrt[u^μ σ_μν u^ν] τ^ν_A u^μ;
  (*FaA=FullSimplify[Transpose[η4488[[#+1,#+1]]*(τ[#].unit)]&/@Range[0,7]];*)
  FaA = FullSimplify[
    Transpose[Table[η4488[[A1 + 1, A1 + 1]] * (τ[A1].unit), {A1, 0, 7}]]];
  If[FaA === EaA, Print["u^A = ", EAa.u], Print["ERROR"]];
  If[F^a_A.F^A_a === ID8, ## &[], Print["ERROR F^a_A.F^A_a != ID8"]];
  Return[Union[Flatten[FullSimplify[FaA - EaA]]]];
  Return[{FaA, EaA, FaA - EaA}];
 ]
```

In[494]:=

```
W[#] & /@ Range[0, 7];
resolution2[%]
u^A = {2 W[0] × W[4] + 2 W[1] × W[5] + 2 W[2] × W[6] + 2 W[3] × W[7], 0, 0, 0, 0, 0, 0, 0}
```

Out[495]=

```
{{W[4], W[5], W[6], W[7], W[0], W[1], W[2], W[3]},
 {W[3], W[2], -W[1], -W[0], W[7], W[6], -W[5], -W[4]},
 {-W[2], W[3], W[0], -W[1], -W[6], W[7], W[4], -W[5]},
 {W[1], -W[0], W[3], -W[2], W[5], -W[4], W[7], -W[6]},
 {-W[1], W[0], W[3], -W[2], W[5], -W[4], -W[7], W[6]},
 {W[2], W[3], -W[0], -W[1], -W[6], -W[7], W[4], W[5]},
 {W[3], -W[2], W[1], -W[0], -W[7], W[6], -W[5], W[4]},
 {W[4], W[5], W[6], W[7], -W[0], -W[1], -W[2], -W[3]}}
```

In[496]:=

```
myid[MX_] := Sum[τ̄[A].MX.τ[A] η4488[[A, A]], {A, 1, 8}]
Myid[MX_] := Sum[τ[A].MX.τ̄[A] × η448[[A, A]], {A, 1, 8}]
```

```
Union[
 Table[Sum[τ[A].bas64[[k, 1]].σ.τ̄[A] × η8[[A, A]], {A, 1, 8}] - Id Tr[bas64[[k, 1]].σ] +
   4 eg[[k]] (bas64[[k, 1]] - Transpose[bas64[[k, 1]]])/2 .σ, {k, 1, 64}]]
```

```
{{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}}
```

In[498]:=

```
τ̄[0]
```

Out[498]=

```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0},
 {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 1}}
```

In[499]:=

```
(Transpose[unit].σ.(τ̄[0]))
```

Out[499]=

$$\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}$$

```
FaA = FullSimplify[Transpose[Table[η4488[[A1 + 1, A1 + 1]]*(τ[A1] . unit), {A1, 0, 7}]]]
```

In[500]:=

$$F^a_A = \text{Transpose}[\eta 4488[[\# + 1, \# + 1]] * (\tau[\#].\text{unit}) \;\&\; /@\; \text{Range}[0, 7]]$$

Out[500]=

$$\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\right\}, \left\{0, 0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}, \left\{0, 0, \frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\right\},\right.$$
$$\left\{0, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}\right\},$$
$$\left.\left\{0, 0, 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, 0\right\}, \left\{0, 0, \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0, 0\right\}, \left\{0, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}, 0\right\}\right\}$$

In[501]:=

$$F^A_a = \text{Transpose}[\text{unit}].\sigma.(\bar{\tau}[\#]) \;\&\; /@\; \text{Range}[0, 7]$$

Out[501]=

$$\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}, \left\{0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}\right\},\right.$$
$$\left\{0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0\right\}, \left\{0, -\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\right\}, \left\{0, \frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\right\},$$
$$\left.\left\{0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0\right\}, \left\{0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0\right\}\right\}$$

In[502]:=

```
F^a_A . F^A_a === ID8
```

Out[502]=

```
True
```

pretend that the X are Minkowski Coordinates :

In[503]:=

```
Block[{U, u = U[#] & /@ Range[8], fx = F^a_A.X, eq, sx, su, ret},
 eq = And @@ Thread[u == fx];
 sx = Solve[eq, X][[-1]];
 su = Solve[eq, u][[-1]];
 ret = {sx, su};
 ret]
```

Out[503]=

$$\left\{\left\{x0 \to \frac{U[1]}{\sqrt{2}} + \frac{U[5]}{\sqrt{2}},\ x1 \to -\frac{U[4]}{\sqrt{2}} - \frac{U[8]}{\sqrt{2}},\ x2 \to \frac{U[3]}{\sqrt{2}} + \frac{U[7]}{\sqrt{2}},\ x3 \to -\frac{U[2]}{\sqrt{2}} - \frac{U[6]}{\sqrt{2}},\ x4 \to \frac{1}{2}\left(\sqrt{2}\,U[2] - \sqrt{2}\,U[6]\right),\ x5 \to \frac{1}{2}\left(-\sqrt{2}\,U[3] + \sqrt{2}\,U[7]\right),\ x6 \to \frac{1}{2}\left(-\sqrt{2}\,U[4] + \sqrt{2}\,U[8]\right),\ x7 \to \frac{1}{2}\left(\sqrt{2}\,U[1] - \sqrt{2}\,U[5]\right)\right\},\right.$$
$$\left.\left\{U[1] \to \frac{1}{2}\left(\sqrt{2}\,x0 + \sqrt{2}\,x7\right),\ U[2] \to \frac{1}{2}\left(-\sqrt{2}\,x3 + \sqrt{2}\,x4\right),\ U[3] \to \frac{1}{2}\left(\sqrt{2}\,x2 - \sqrt{2}\,x5\right),\ U[4] \to \frac{1}{2}\left(-\sqrt{2}\,x1 - \sqrt{2}\,x6\right),\ U[5] \to \frac{1}{2}\left(\sqrt{2}\,x0 - \sqrt{2}\,x7\right),\ U[6] \to \frac{1}{2}\left(-\sqrt{2}\,x3 - \sqrt{2}\,x4\right),\ U[7] \to \frac{1}{2}\left(\sqrt{2}\,x2 + \sqrt{2}\,x5\right),\ U[8] \to \frac{1}{2}\left(-\sqrt{2}\,x1 + \sqrt{2}\,x6\right)\right\}\right\}$$

In[ ]:=

```
(sgtrye^(A)_α = Thread[Flatten[e^(A)_α] → Flatten[eAa04]]) (*//Column*)
```

In[504]:=

```
subsDefects
```

Out[504]=

```
{}
```

In[505]:=

```
Thread[X → 0]
```

Out[505]=

```
{x0 → 0, x1 → 0, x2 → 0, x3 → 0, x4 → 0, x5 → 0, x6 → 0, x7 → 0}
```

---

split octonions; evalues, evecs of σ

Type - 1 spinor basis (constructed from Clifford algebra considerations)
These create a maximal isotropic subspace with respect to the split form
self . type1_spinor _basis _matrix = sqrt2_inv*np . array ([[1, 1, 0, 0, 0, 0, 0, 0], # s1_ 0[1, -1, 0, 0, 0, 0, 0, 0], # s1_ 1[0, 0, 1, 1, 0, 0, 0, 0], # s1_ 2[0, 0, 1, -1, 0, 0, 0, 0], # s1_ 3[0, 0, 0, 0, 1, 1, 0, 0], # s1_ 4[0, 0, 0, 0, 1, -1, 0, 0], # s1_ 5[0, 0, 0, 0, 0, 0, 1, 1], # s1_ 6[0, 0, 0, 0, 0, 0, 1, -1], # s1_ 7])

Type - 2 spinor basis (the complementary spinor representation)
Related to Type - 1 by another triality automorphism
self . type2_spinor _basis _matrix = sqrt2_inv*np . array ([[1, 0, 1, 0, 0, 0, 0, 0], # s2_ 0[1, 0, -1, 0, 0, 0, 0, 0], # s2_ 1[0, 1, 0, 1, 0, 0, 0, 0], # s2_ 2[0, 1, 0, -1, 0, 0, 0, 0], # s2_ 3[0, 0, 0, 0, 1, 0, 1, 0], # s2_ 4[0, 0, 0, 0, 1, 0, -1, 0], # s2_ 5[0, 0, 0, 0, 0, 1, 0, 1], # s2_ 6[0, 0, 0, 0, 0, 1, 0, -1], # s2_ 7])

$$M_{S_1 \to S_2} = \frac{1}{2}\begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{matrix}$$

4 Transformation Matrices

We provide explicit 8×8 matrices that transform coordinates between the three bases.

4.1 Vector to Type-1 Spinor

The transformation matrix from vector basis to type-1 spinor basis is:

$$M_{V \to S_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{11}$$

4.2 Vector to Type-2 Spinor

The transformation matrix from vector basis to type-2 spinor basis is:

$$M_{V \to S_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \tag{12}$$

In[506]:=

```
(type1SpinorBasisMatrix = {{1, 1, 0, 0, 0, 0, 0, 0}, {1, -1, 0, 0, 0, 0, 0, 0},
    {0, 0, 1, 1, 0, 0, 0, 0}, {0, 0, 1, -1, 0, 0, 0, 0},
    {0, 0, 0, 0, 1, 1, 0, 0}, {0, 0, 0, 0, 1, -1, 0, 0},
    {0, 0, 0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 0, 0, 1, -1}}) // MatrixForm
```

Out[506]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

In[507]:=

```
(type2SpinorBasisMatrix = {{1, 0, 1, 0, 0, 0, 0, 0}, {1, 0, -1, 0, 0, 0, 0, 0},
    {0, 1, 0, 1, 0, 0, 0, 0}, {0, 1, 0, -1, 0, 0, 0, 0},
    {0, 0, 0, 0, 1, 0, 1, 0}, {0, 0, 0, 0, 1, 0, -1, 0},
    {0, 0, 0, 0, 0, 1, 0, 1}, {0, 0, 0, 0, 0, 1, 0, -1}}) // MatrixForm
```

Out[507]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

In[ ]:=

$$M_{s_1 \to s_2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \qquad M_{s_1 \to s_2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

In[508]:=

```
MS1S2 = type1SpinorBasisMatrix.Transpose[type2SpinorBasisMatrix] // MatrixForm
```

Out[508]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

In[509]:=

```
ηAB = η4488
```

Out[509]=

```
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0},
 {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, -1}}
```

In[510]:=

```
{evaluesAB, evecsAB} = Eigensystem[ηAB]
```

Out[510]=

```
{{-1, -1, -1, -1, 1, 1, 1, 1}, {{0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0},
  {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0}}}
```

In[511]:=

```
{evalues, evecs} = Eigensystem[σ]
```

Out[511]=

```
{{-1, -1, -1, -1, 1, 1, 1, 1}, {{0, 0, 0, -1, 0, 0, 0, 1}, {0, 0, -1, 0, 0, 0, 1, 0},
  {0, -1, 0, 0, 0, 1, 0, 0}, {-1, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 1},
  {0, 0, 1, 0, 0, 0, 1, 0}, {0, 1, 0, 0, 0, 1, 0, 0}, {1, 0, 0, 0, 1, 0, 0, 0}}}
```

In[512]:=

```
(u = ExpandAll[1/Sqrt[2] evecs]) // MatrixForm
```

Out[512]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

In[513]:=

```
Table[u[[h]].σ.u[[h]], {h, 1, Length[u]}]
```

Out[513]=

```
{-1, -1, -1, -1, 1, 1, 1, 1}
```

In[514]:=

```
Table[(Transpose[u][[h]]).σ.u[[h]], {h, 1, Length[u]}]
```

Out[514]=

```
{-1, -1, -1, -1, 1, 1, 1, 1}
```

In[515]:=

```
Table[(Transpose[u][[A]]) === -σ.u[[A]], {A, 1, 4}]
Table[(Transpose[u][[A]]) === σ.u[[A]], {A, 5, Length[u]}]
```

Out[515]=

```
{True, True, True, True}
```

Out[516]=

```
{True, True, True, True}
```

In[517]:=

```
hUSE = 8;
```

Set: Symbol hUSE is Protected.

In[518]:=

```
Protect[hUSE]
```

Out[518]=

```
{}
```

In[519]:=

```
unit = 𝕦[[hUSE]]
```

Set: Symbol unit is Protected.

Out[519]=

$$\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}$$

In[520]:=

```
Protect[unit]
```

Out[520]=

{}

In[521]:=

```
(Transpose[unit].σ.unit)
```

Out[521]=

1

In[522]:=

```
(Transpose[unit].σ.(τ̄[0]))
```

Out[522]=

$$\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}$$

In[523]:=

```
Do[E^A_a[h] = Table[(𝕦[[h]].σ.(τ̄[B])), {B, 0, 7}], {h, 1, Length[𝕦]}];
```

$\mathfrak{E}^A_a = \frac{1}{\sqrt{\tilde{u}\sigma u}} u^c \sigma_{cb} \bar{\tau}^{Ab}{}_a$

```
Do[E^a_A[h] = FullSimplify[Inverse[E^A_a[h]]], {h, 1, Length[𝕦]}];
```

$\mathfrak{E}^a_A = \frac{1}{\sqrt{\tilde{u}\sigma u}} \tau_A{}^a{}_b u^b$

```
Table[E^a_A[h] === (𝕦[[h]].σ.𝕦[[h]])
   Transpose[Table[FullSimplify[ExpandAll[ η4488[[B, B]] × τ[B - 1].𝕦[[h]] ]],
     {B, 1, 8}]], {h, 1, Length[𝕦]}]
```

Out[523]=

$$\mathfrak{E}^A_a = \frac{1}{\sqrt{\tilde{u}\sigma u}} u^c \sigma_{cb} \bar{\tau}^{Ab}{}_a$$

Out[524]=

$$\mathfrak{E}^a_A = \frac{1}{\sqrt{\tilde{u}\sigma u}} \tau_A{}^a{}_b u^b$$

Out[525]=

{True, True, True, True, True, True, True, True}

In[526]:=

```
(*Do[E^A_a[h]=Table[(u[[h]].σ.(τ̄[B])),{B,0,7}],{h,1,Length[u]}];
Do[E^a_A[h]=FullSimplify[Inverse[E^A_a[h]]],{h,1,Length[u]}];
 Table[E^a_A[h]=== (u[[h]].σ.u[[h]])
    Transpose[Table[FullSimplify[ExpandAll[ η4488[[B,B]]τ[B-1].u[[h]] ]],{B,1,8}]],
  {h,1,Length[u]}]*)
```

In[527]:=

```
myid[MX_] := Sum[τ̄[A].MX.τ[A] η4488⟦A, A⟧, {A, 1, 8}]
Myid[MX_] := Sum[τ[A].MX.τ̄[A] × η448⟦A, A⟧, {A, 1, 8}]
```

```
Union[
 Table[Sum[τ[A].bas64⟦k, 1⟧.σ.τ̄[A] × η8⟦A, A⟧, {A, 1, 8}] - Id Tr[bas64⟦k, 1⟧.σ] +
   4 eg⟦k⟧ (bas64⟦k, 1⟧ - Transpose[bas64⟦k, 1⟧])/2 .σ, {k, 1, 64}]]
```

{{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}}

In[529]:=

```
τ̄[0]
```

Out[529]=

{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0},
 {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 1}}

In[530]:=

```
(Transpose[unit].σ.(τ̄[0]))
```

Out[530]=

$$\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}$$

In[531]:=

$$F_A^a = \text{Transpose}[\eta 4488[[\# + 1, \# + 1]] * (\tau[\#].\text{unit}) \,\&\, /@\, \text{Range}[0, 7]]$$

Out[531]=

$$\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\right\}, \left\{0, 0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}, \left\{0, 0, \frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\right\},\right.$$
$$\left\{0, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}\right\},$$
$$\left.\left\{0, 0, 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, 0\right\}, \left\{0, 0, \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0, 0\right\}, \left\{0, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}, 0\right\}\right\}$$

In[532]:=

$$F_a^A = \text{Transpose}[\text{unit}].\sigma.(\bar{\tau}[\#]) \,\&\, /@\, \text{Range}[0, 7]$$

Out[532]=

$$\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}, \left\{0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}\right\},\right.$$
$$\left\{0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0\right\}, \left\{0, -\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\right\}, \left\{0, \frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0\right\},$$
$$\left.\left\{0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0\right\}, \left\{0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0\right\}\right\}$$

In[533]:=

```
F^a_A . F^A_a === ID8
```

Out[533]=

True

In[534]:=

```
E^A_a[hUSE] === F^A_a
E^a_A[hUSE] === F^a_A
```

Out[534]=

True

Out[535]=

True

```
gtrye^(A)_α = (e^(A)_α /. sgtrye^(A)_α) /. subsDefects
gtrye^α_(A) = (e^α_(A) /. sgtrye^α_(A)) /. subsDefects
```

```
(sgtrye^α_(A) = Thread[Flatten[e^α_(A)] → Flatten[Inverse[e^(A)_α /. sgtrye^(A)_α]]])
(*//Column*)
```

```
gtrye^α_(A) = (e^α_(A) /. sgtrye^α_(A)) /. subsDefects
```

Out[ ]=

$$\left\{\{\mathrm{Tan}[6\,H\,x0],0,0,0,0,0,0,0\},\left\{0,\frac{e^{-a4[H\,x4]}}{\mathrm{Sin}[6\,H\,x0]^{1/6}},0,0,0,0,0,0\right\},\right.$$
$$\left\{0,0,\frac{e^{-a4[H\,x4]}}{\mathrm{Sin}[6\,H\,x0]^{1/6}},0,0,0,0,0\right\},\left\{0,0,0,\frac{e^{-a4[H\,x4]}}{\mathrm{Sin}[6\,H\,x0]^{1/6}},0,0,0,0\right\},$$
$$\{0,0,0,0,1,0,0,0\},\left\{0,0,0,0,0,\frac{e^{a4[H\,x4]}}{\mathrm{Sin}[6\,H\,x0]^{1/6}},0,0\right\},$$
$$\left.\left\{0,0,0,0,0,0,\frac{e^{a4[H\,x4]}}{\mathrm{Sin}[6\,H\,x0]^{1/6}},0\right\},\left\{0,0,0,0,0,0,0,\frac{e^{a4[H\,x4]}}{\mathrm{Sin}[6\,H\,x0]^{1/6}}\right\}\right\}$$

pretend that the X are Minkowski Coordinates :

In[536]:=

```
Block[{U, u = U[#] & /@ Range[8], fx = F^a_A.X, eq, sx, su, ret},
 eq = And @@ Thread[u == fx];
 sx = Solve[eq, X][[-1]];
 su = Solve[eq, u][[-1]];
 ret = {sx, su};
 ret]
```

Out[536]=

$$\left\{\left\{x0 \to \frac{U[1]}{\sqrt{2}} + \frac{U[5]}{\sqrt{2}},\ x1 \to -\frac{U[4]}{\sqrt{2}} - \frac{U[8]}{\sqrt{2}},\ x2 \to \frac{U[3]}{\sqrt{2}} + \frac{U[7]}{\sqrt{2}},\right.\right.$$
$$x3 \to -\frac{U[2]}{\sqrt{2}} - \frac{U[6]}{\sqrt{2}},\ x4 \to \frac{1}{2}\left(\sqrt{2}\,U[2] - \sqrt{2}\,U[6]\right),\ x5 \to \frac{1}{2}\left(-\sqrt{2}\,U[3] + \sqrt{2}\,U[7]\right),$$
$$\left.x6 \to \frac{1}{2}\left(-\sqrt{2}\,U[4] + \sqrt{2}\,U[8]\right),\ x7 \to \frac{1}{2}\left(\sqrt{2}\,U[1] - \sqrt{2}\,U[5]\right)\right\},$$
$$\left\{U[1] \to \frac{1}{2}\left(\sqrt{2}\,x0 + \sqrt{2}\,x7\right),\ U[2] \to \frac{1}{2}\left(-\sqrt{2}\,x3 + \sqrt{2}\,x4\right),\ U[3] \to \frac{1}{2}\left(\sqrt{2}\,x2 - \sqrt{2}\,x5\right),\right.$$
$$U[4] \to \frac{1}{2}\left(-\sqrt{2}\,x1 - \sqrt{2}\,x6\right),\ U[5] \to \frac{1}{2}\left(\sqrt{2}\,x0 - \sqrt{2}\,x7\right),\ U[6] \to \frac{1}{2}\left(-\sqrt{2}\,x3 - \sqrt{2}\,x4\right),$$
$$\left.\left.U[7] \to \frac{1}{2}\left(\sqrt{2}\,x2 + \sqrt{2}\,x5\right),\ U[8] \to \frac{1}{2}\left(-\sqrt{2}\,x1 + \sqrt{2}\,x6\right)\right\}\right\}$$

In[ ]:=

$$\left(\text{sgtrye}^{(A)}_{\alpha} = \text{Thread}\left[\text{Flatten}\left[e^{(A)}_{\alpha}\right] \to \text{Flatten}[\text{eAa04}]\right]\right)\ (*//\text{Column}*)$$

In[537]:=

```
subsDefects
```

Out[537]=

{}

In[538]:=

```
Thread[X → 0]
```

Out[538]=

{x0 → 0, x1 → 0, x2 → 0, x3 → 0, x4 → 0, x5 → 0, x6 → 0, x7 → 0}

Stop pretending:

$$\left(\text{sgtrye}^{\alpha}_{(A)} = \text{Thread}\left[\text{Flatten}\left[e^{\alpha}_{(A)}\right] \to \text{Flatten}\left[\text{Inverse}\left[e^{(A)}_{\alpha}\ /.\ \text{sgtrye}^{(A)}_{\alpha}\right]\right]\right)$$
$$(*//\text{Column}*)$$

$$\text{gtrye}^{\alpha}_{(A)} = \left(e^{\alpha}_{(A)}\ /.\ \text{sgtrye}^{\alpha}_{(A)}\right)\ /.\ \text{subsDefects}$$

$$\left\{\{\text{Tan}[6\,H\,x0], 0, 0, 0, 0, 0, 0, 0\},\ \left\{0, \frac{e^{-a4[H\,x4]}}{\text{Sin}[6\,H\,x0]^{1/6}}, 0, 0, 0, 0, 0, 0\right\},\right.$$
$$\left\{0, 0, \frac{e^{-a4[H\,x4]}}{\text{Sin}[6\,H\,x0]^{1/6}}, 0, 0, 0, 0, 0\right\},\ \left\{0, 0, 0, \frac{e^{-a4[H\,x4]}}{\text{Sin}[6\,H\,x0]^{1/6}}, 0, 0, 0, 0\right\},$$
$$\{0, 0, 0, 0, 1, 0, 0, 0\},\ \left\{0, 0, 0, 0, 0, \frac{e^{a4[H\,x4]}}{\text{Sin}[6\,H\,x0]^{1/6}}, 0, 0\right\},$$

$$\left\{0, 0, 0, 0, 0, 0, \frac{e^{a4[H\,x4]}}{\text{Sin}[6\,H\,x0]^{1/6}}, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, \frac{e^{a4[H\,x4]}}{\text{Sin}[6\,H\,x0]^{1/6}}\right\}\}$$

In[539]:=

```
(*e_α^(A) =eAα=gtrye_α^(A) *)
```

In[540]:=

```
(*{gtrye_α^(A) ,gtrye^α_(A) }//MatrixForm*)
```

In[541]:=

```
(*F_A^a.e_α^(A) .X/.sgtrye_α^(A) //FullSimplify[#,constraintVars]&;
 frameOctadX=#/.{Sqrt[e^(2 a4[H x4])]→e^(a4[H x4]), Sqrt[e^(-2 a4[H x4])]→e^(-a4[H x4])}&/@%
 %/.{x1→0,x2→0,x3→0,x5→0,x6→0,x7→0}*)
```

In[542]:=

```
(*F_A^a.eAα.X-frameOctadX//FullSimplify[#,constraintVars]&*)
```

In[543]:=

```
(*eAα*)
```

In[544]:=

```
(*{{F_A^a},{eAα},{X}}//Column*)
```

split octonion multiplication constants :

$$m^{c}_{ab} = \mathbb{F}^{A}_{a}\,\tau_{A}{}^{c}{}_{b}$$
$$m^{C}_{AB} = \mathbb{F}^{C}_{c}\,\tau_{A}{}^{c}{}_{b}\,\mathbb{F}^{b}_{B}$$

In[545]:=

```
EA = Array[eA, 8]
```

Out[545]=

```
{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}
```

In[546]:=

```
{E^A_a[hUSE] === F^A_a, E^a_A[hUSE] === F^a_A}
```

Out[546]=

```
{True, True}
```

$$m^{c}_{ab} = \mathbb{F}^{A}_{a}\,\tau_{A}{}^{c}{}_{b}$$
$$m^{C}_{AB} = \mathbb{F}^{C}_{c}\,\tau_{A}{}^{c}{}_{b}\,\mathbb{F}^{b}_{B}$$

$$\varepsilon_a \varepsilon_b = \varepsilon_c m^c_{ab},$$

$$\epsilon_A \epsilon_B = \epsilon_C m^C_{AB}. \tag{53}$$

ed by

$$m^c_{ab} = \mathfrak{F}^A_a \tau_A{}^c{}_b,$$

$$m^C_{AB} = \mathfrak{F}^C_c \tau_A{}^c{}_b \mathfrak{F}^b_B, \tag{54}$$

In[547]:=

```
(*Symbolize[ m^c_ab ]*)
```

In[548]:=

```
(*Symbolize[ m^C_AB ]*)
```

In[549]:=

```
mabc = Array[U, {8, 8, 8}];
```

In[550]:=

```
τA = η_AB[[#, #]] * τ[# - 1] & /@ Range[8];
Do[Do[mabc[[a1, b1, c1]] = Sum[E^A_a[hUSE][[A1, a1]] * τA[[A1]][[c1, b1]], {A1, 1, 8}], {c1, 1, 8}],
  {a1, 1, 8}, {b1, 1, 8}]
```

In[552]:=

```
m^c_ab = mabc;
```

NonCommutativeMultiply **

a ** b

$a \boxtimes b \boxtimes c$

is a general associative, but non-commutative, form of multiplication.

Expand and Simplify do not operate on expressions with NonCommutativeMultiply :

In[553]:=

```
unit.EA // Simplify
% ⊠ % // Expand
```

Out[553]=

$$\frac{eA[1] + eA[5]}{\sqrt{2}}$$

Out[554]=

$$\frac{eA[1] + eA[5]}{\sqrt{2}} \boxtimes \frac{eA[1] + eA[5]}{\sqrt{2}}$$

In[555]:=

```
1/√2 1/√2 Plus @@ {mabc⟦1, 1, All⟧, mabc⟦1, 5, All⟧, mabc⟦5, 1, All⟧, mabc⟦5, 5, All⟧}
%.EA // Simplify
unit.EA // Simplify
% === %%
```

Out[555]=

$$\left\{\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right\}$$

Out[556]=

$$\frac{eA[1] + eA[5]}{\sqrt{2}}$$

Out[557]=

$$\frac{eA[1] + eA[5]}{\sqrt{2}}$$

Out[558]=

True

In[559]:=

```
(*E^A_a[hUSE];
η_AB⟦#,#⟧*τ[#-1]&/@Range[8];
mabc=Transpose[%%].Transpose[%]
  Dimensions[%]
  mABC=( E^A_a[hUSE].(E^a_A[hUSE].mabc)).E^a_A[hUSE]*)
```

$$m^{c}_{ab} = \mathbb{F}^{A}_{a}\,\tau_{A}{}^{c}{}_{b}$$
$$m^{C}_{AB} = \mathbb{F}^{C}_{c}\,\tau_{A}{}^{c}{}_{b}\,\mathbb{F}^{b}_{B}$$

In[560]:=

```
mABC = Array[R, {8, 8, 8}];
```

In[561]:=

```
τA = η_AB⟦#, #⟧ * τ[# - 1] & /@ Range[8];
Do[Do[mABC⟦A1, B1, C1⟧ = Sum[E^A_a[hUSE]⟦C1, c1⟧ * τA⟦A1⟧⟦c1, b1⟧ * E^a_A[hUSE]⟦b1, B1⟧,
    {b1, 1, 8}, {c1, 1, 8}], {C1, 1, 8}], {A1, 1, 8}, {B1, 1, 8}]
```

In[563]:=

```
gridSplitOctonion =
 Grid[Prepend[Drop[Reap[For[A1 = 1, A1 < 9, A1++, Sow[Flatten[{EA[[A1]], Table[
            Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]]],
      1][[1]][[1]], Flatten[{"A/B", EA}]], Frame → All]
```

Out[563]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

1	i	j	k	ℓ	ℓi	ℓj	ℓk
i	-1	k	$-j$	$-\ell i$	ℓ	$-\ell k$	ℓj
j	$-k$	-1	i	$-\ell j$	ℓk	ℓ	$-\ell i$
k	j	$-i$	-1	$-\ell k$	$-\ell j$	ℓi	ℓ
ℓ	ℓi	ℓj	ℓk	1	i	j	k
ℓi	$-\ell$	$-\ell k$	ℓj	$-i$	1	k	$-j$
ℓj	ℓk	$-\ell$	$-\ell i$	$-j$	$-k$	1	i
ℓk	$-\ell j$	ℓi	$-\ell$	$-k$	j	$-i$	1

In[564]:=

```
m^C_AB = mABC;
```

In[565]:=

$$m^c_{ab} = \mathbb{F}^A_a \tau_A{}^c{}_b$$
$$m^C_{AB} = \mathbb{F}^C_c \tau_A{}^c{}_b \mathbb{F}^b_B$$

Out[565]=

$$m^c_{ab} = \mathbb{F}^A_a \tau_A{}^c{}_b$$
$$m^C_{AB} = \mathbb{F}^C_c \tau_A{}^c{}_b \mathbb{F}^b_B$$

In[566]:=

```
E^A_a[hUSE].Transpose[(E^a_A[hUSE].mabc).E^a_A[hUSE]] === mABC
```

Out[566]=

```
False
```

In[567]:=

```
Transpose[(E^a_A[hUSE].mabc).E^a_A[hUSE]].E^A_a[hUSE] === mABC
```

Out[567]=

```
False
```

In[568]:=

```
Transpose[( E^A_a[hUSE].(E^a_A[hUSE].mabc))].E^a_A[hUSE] === mABC
```

Out[568]=

```
False
```

In[569]:=

```
(*τA=η_AB[[#,#]]*τ[[#-1]]&/@Range[8];
Table[Table[
  mABC[[A1,B1,C1]]=Sum[E^A_a[hUSE][[C1,c1]]*τA[[A1]][[c1,b1]]*E^a_A[hUSE][[b1,B1]],{b1,1,8},{c1,1,8}],
  {C1,1,8}],{A1,1,8},{B1,1,8}]*)
```

1	i	j	k	ℓ	ℓi	ℓj	ℓk
i	-1	k	$-j$	$-\ell i$	ℓ	$-\ell k$	ℓj
j	$-k$	-1	i	$-\ell j$	ℓk	ℓ	$-\ell i$
k	j	$-i$	-1	$-\ell k$	$-\ell j$	ℓi	ℓ
ℓ	ℓi	ℓj	ℓk	1	i	j	k
ℓi	$-\ell$	$-\ell k$	ℓj	$-i$	1	k	$-j$
ℓj	ℓk	$-\ell$	$-\ell i$	$-j$	$-k$	1	i
ℓk	$-\ell j$	ℓi	$-\ell$	$-k$	j	$-i$	1

In[570]:=

```
Table[Table[{{A1, B1, C1}, mABC[[A1, B1, C1]]}, {C1, 1, 8}], {A1, 1, 8}, {B1, 1, 8}]
```

Out[570]=

```
{{{{{1, 1, 1}, 1}, {{1, 1, 2}, 0}, {{1, 1, 3}, 0}, {{1, 1, 4}, 0},
    {{1, 1, 5}, 0}, {{1, 1, 6}, 0}, {{1, 1, 7}, 0}, {{1, 1, 8}, 0}},
  {{{1, 2, 1}, 0}, {{1, 2, 2}, 1}, {{1, 2, 3}, 0}, {{1, 2, 4}, 0},
    {{1, 2, 5}, 0}, {{1, 2, 6}, 0}, {{1, 2, 7}, 0}, {{1, 2, 8}, 0}},
  {{{1, 3, 1}, 0}, {{1, 3, 2}, 0}, {{1, 3, 3}, 1}, {{1, 3, 4}, 0},
    {{1, 3, 5}, 0}, {{1, 3, 6}, 0}, {{1, 3, 7}, 0}, {{1, 3, 8}, 0}},
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    {{1, 4, 5}, 0}, {{1, 4, 6}, 0}, {{1, 4, 7}, 0}, {{1, 4, 8}, 0}},
  {{{1, 5, 1}, 0}, {{1, 5, 2}, 0}, {{1, 5, 3}, 0}, {{1, 5, 4}, 0},
```

```
    {{1, 5, 5}, 1}, {{1, 5, 6}, 0}, {{1, 5, 7}, 0}, {{1, 5, 8}, 0}},
   {{{1, 6, 1}, 0}, {{1, 6, 2}, 0}, {{1, 6, 3}, 0}, {{1, 6, 4}, 0},
    {{1, 6, 5}, 0}, {{1, 6, 6}, 1}, {{1, 6, 7}, 0}, {{1, 6, 8}, 0}},
   {{{1, 7, 1}, 0}, {{1, 7, 2}, 0}, {{1, 7, 3}, 0}, {{1, 7, 4}, 0},
    {{1, 7, 5}, 0}, {{1, 7, 6}, 0}, {{1, 7, 7}, 1}, {{1, 7, 8}, 0}},
   {{{1, 8, 1}, 0}, {{1, 8, 2}, 0}, {{1, 8, 3}, 0}, {{1, 8, 4}, 0},
    {{1, 8, 5}, 0}, {{1, 8, 6}, 0}, {{1, 8, 7}, 0}, {{1, 8, 8}, 1}}},
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    {{2, 1, 5}, 0}, {{2, 1, 6}, 0}, {{2, 1, 7}, 0}, {{2, 1, 8}, 0}},
   {{{2, 2, 1}, -1}, {{2, 2, 2}, 0}, {{2, 2, 3}, 0}, {{2, 2, 4}, 0},
    {{2, 2, 5}, 0}, {{2, 2, 6}, 0}, {{2, 2, 7}, 0}, {{2, 2, 8}, 0}},
   {{{2, 3, 1}, 0}, {{2, 3, 2}, 0}, {{2, 3, 3}, 0}, {{2, 3, 4}, -1},
    {{2, 3, 5}, 0}, {{2, 3, 6}, 0}, {{2, 3, 7}, 0}, {{2, 3, 8}, 0}},
   {{{2, 4, 1}, 0}, {{2, 4, 2}, 0}, {{2, 4, 3}, 1}, {{2, 4, 4}, 0},
    {{2, 4, 5}, 0}, {{2, 4, 6}, 0}, {{2, 4, 7}, 0}, {{2, 4, 8}, 0}},
   {{{2, 5, 1}, 0}, {{2, 5, 2}, 0}, {{2, 5, 3}, 0}, {{2, 5, 4}, 0},
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   {{{2, 6, 1}, 0}, {{2, 6, 2}, 0}, {{2, 6, 3}, 0}, {{2, 6, 4}, 0},
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    {{2, 7, 5}, 0}, {{2, 7, 6}, 0}, {{2, 7, 7}, 0}, {{2, 7, 8}, 1}},
   {{{2, 8, 1}, 0}, {{2, 8, 2}, 0}, {{2, 8, 3}, 0}, {{2, 8, 4}, 0},
    {{2, 8, 5}, 0}, {{2, 8, 6}, 0}, {{2, 8, 7}, -1}, {{2, 8, 8}, 0}}},
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```

```
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  {{6, 8, 5}, 0}, {{6, 8, 6}, 0}, {{6, 8, 7}, 0}, {{6, 8, 8}, 0}}},
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  {{7, 1, 5}, 0}, {{7, 1, 6}, 0}, {{7, 1, 7}, 1}, {{7, 1, 8}, 0}},
 {{{7, 2, 1}, 0}, {{7, 2, 2}, 0}, {{7, 2, 3}, 0}, {{7, 2, 4}, 0},
  {{7, 2, 5}, 0}, {{7, 2, 6}, 0}, {{7, 2, 7}, 0}, {{7, 2, 8}, -1}},
 {{{7, 3, 1}, 0}, {{7, 3, 2}, 0}, {{7, 3, 3}, 0}, {{7, 3, 4}, 0},
  {{7, 3, 5}, -1}, {{7, 3, 6}, 0}, {{7, 3, 7}, 0}, {{7, 3, 8}, 0}},
 {{{7, 4, 1}, 0}, {{7, 4, 2}, 0}, {{7, 4, 3}, 0}, {{7, 4, 4}, 0},
```

```
    {{7, 4, 5}, 0}, {{7, 4, 6}, 1}, {{7, 4, 7}, 0}, {{7, 4, 8}, 0}},
   {{{7, 5, 1}, 0}, {{7, 5, 2}, 0}, {{7, 5, 3}, -1}, {{7, 5, 4}, 0},
    {{7, 5, 5}, 0}, {{7, 5, 6}, 0}, {{7, 5, 7}, 0}, {{7, 5, 8}, 0}},
   {{{7, 6, 1}, 0}, {{7, 6, 2}, 0}, {{7, 6, 3}, 0}, {{7, 6, 4}, 1},
    {{7, 6, 5}, 0}, {{7, 6, 6}, 0}, {{7, 6, 7}, 0}, {{7, 6, 8}, 0}},
   {{{7, 7, 1}, 1}, {{7, 7, 2}, 0}, {{7, 7, 3}, 0}, {{7, 7, 4}, 0},
    {{7, 7, 5}, 0}, {{7, 7, 6}, 0}, {{7, 7, 7}, 0}, {{7, 7, 8}, 0}},
   {{{7, 8, 1}, 0}, {{7, 8, 2}, -1}, {{7, 8, 3}, 0}, {{7, 8, 4}, 0},
    {{7, 8, 5}, 0}, {{7, 8, 6}, 0}, {{7, 8, 7}, 0}, {{7, 8, 8}, 0}}},
  {{{{8, 1, 1}, 0}, {{8, 1, 2}, 0}, {{8, 1, 3}, 0}, {{8, 1, 4}, 0},
    {{8, 1, 5}, 0}, {{8, 1, 6}, 0}, {{8, 1, 7}, 0}, {{8, 1, 8}, 1}},
   {{{8, 2, 1}, 0}, {{8, 2, 2}, 0}, {{8, 2, 3}, 0}, {{8, 2, 4}, 0},
    {{8, 2, 5}, 0}, {{8, 2, 6}, 0}, {{8, 2, 7}, 1}, {{8, 2, 8}, 0}},
   {{{8, 3, 1}, 0}, {{8, 3, 2}, 0}, {{8, 3, 3}, 0}, {{8, 3, 4}, 0},
    {{8, 3, 5}, 0}, {{8, 3, 6}, -1}, {{8, 3, 7}, 0}, {{8, 3, 8}, 0}},
   {{{8, 4, 1}, 0}, {{8, 4, 2}, 0}, {{8, 4, 3}, 0}, {{8, 4, 4}, 0},
    {{8, 4, 5}, -1}, {{8, 4, 6}, 0}, {{8, 4, 7}, 0}, {{8, 4, 8}, 0}},
   {{{8, 5, 1}, 0}, {{8, 5, 2}, 0}, {{8, 5, 3}, 0}, {{8, 5, 4}, -1},
    {{8, 5, 5}, 0}, {{8, 5, 6}, 0}, {{8, 5, 7}, 0}, {{8, 5, 8}, 0}},
   {{{8, 6, 1}, 0}, {{8, 6, 2}, 0}, {{8, 6, 3}, -1}, {{8, 6, 4}, 0},
    {{8, 6, 5}, 0}, {{8, 6, 6}, 0}, {{8, 6, 7}, 0}, {{8, 6, 8}, 0}},
   {{{8, 7, 1}, 0}, {{8, 7, 2}, 1}, {{8, 7, 3}, 0}, {{8, 7, 4}, 0},
    {{8, 7, 5}, 0}, {{8, 7, 6}, 0}, {{8, 7, 7}, 0}, {{8, 7, 8}, 0}},
   {{{8, 8, 1}, 1}, {{8, 8, 2}, 0}, {{8, 8, 3}, 0}, {{8, 8, 4}, 0},
    {{8, 8, 5}, 0}, {{8, 8, 6}, 0}, {{8, 8, 7}, 0}, {{8, 8, 8}, 0}}}}
```

In[571]:=

```
(*FAτA=Sum[ E^A_a[hUSE][[A1,All]]*η_AB[[A1,A1]]*τ[A1-1],{A1,1,8}]//FullSimplify
   Dimensions[FAτA]*)
```

In[572]:=

```
(*E^A_a[hUSE]
    η_AB[[#,#]]*τ[#-1]&/@Range[8]
 Transpose[%%].%
 Dimensions[%]*)
```

In[573]:=

```
(*FAτA[[1]]
  Sum[ E^A_a[hUSE][[A1,1]]*η_AB[[A1,A1]]*τ[A1-1],{A1,1,8}]//FullSimplify*)
```

In[574]:=

```
(*mcab=Table[Sum[ E^A_a[hUSE][[A1,a1]]*η_AB[[A1,A1]]*τ[A1-1][[c,b]],{A1,1,8}],
  {a1,1,8},{b,1,8},{c,1,8}]*)
```

In[575]:=

```
(*m^c_ab=
 Table[Sum[ E^A_a[hUSE][[A1,a1]]*η_AB[[A1,A1]]*τ[A1-1][[c,b]],{A1,1,8}],{a1,1,8},{b,1,8},{c,1,8}]
  mCAB=( E^A_a[hUSE].(E^a_A[hUSE].%)).E^a_A[hUSE]*)
```

In[576]:=

```
(*mCAB[[1,2,3]]
 mCAB[[3,1,2]]
 mCAB[[2,1,3]]*)
```

In[ ]:= $(xw)w = xw^2$

is described as follows: If we let $\mathrm{Im}\,\mathbf{O} \subseteq \mathbf{O}$ be the hyperplane (through 0) orthogonal to $1 \in \mathbf{O}$, and we let $S^6 \subseteq \mathrm{Im}\,\mathbf{O}$ be the space of unit vectors, then right multiplication by $u \in S^6$ induces a linear transformation $R_u: \mathbf{O} \to \mathbf{O}$ which is orthogonal and satisfies $(R_u)^2 = -1$. Thus, associated to each $u \in S^6$ is a complex structure on $\mathbf{O}$ (induced by $J = R_u$) which is compatible with the natural inner product on $\mathbf{O}$. We denote by $\mathbf{O}_u$ the Hermitian vector space whose underlying real vector space (with inner product) is $\mathbf{O}$ and whose complex structure is given by R_u.

We let $S^6 = \{u \in \mathrm{Im}\,\mathbf{O} \mid \langle u, u\rangle = 1\}$. The elements of S^6 are called the *imaginary units* of $\mathbf{O}$. For any $u \in S^6$, we have $u = -\bar{u}$, so $u^2 = -u\bar{u} = -\langle u, u\rangle = -1$. We may use u to define a map $J_u: \mathbf{O} \to \mathbf{O}$ given by

$$J_u(x) = xu. \tag{1.15}$$

The identity (1.9) shows that $J_u^2(x) = (xu)u = xu^2 = -x$, so J_u defines a complex structure on $\mathbf{O}$. We write $\mathbf{O}_u$ to denote $\mathbf{O}$ endowed with the complex structure J_u. If $u \neq v$, then clearly $J_u \neq J_v$, so we actually have a six-sphere of *distinct* complex structures on $\mathbf{O}$. However, because S^6 is connected, we see that the orientation of $\mathbf{O}$ induced by the natural orientation of $\mathbf{O}_u$ as a complex vector space does not depend on u. We refer to this orientation as the natural orientation of $\mathbf{O}$.

Using (1.3), we see that if $u \in S$, then

$$\langle J_u(x), J_u(y)\rangle = \langle xu, yu\rangle = \langle x, y\rangle\langle u, u\rangle = \langle x, y\rangle$$

so J_u is an isometry for each $u \in S$. Moreover, it follows that $\mathbf{O}_u$ is endowed with a natural Hermitian structure with respect to the inner product $\langle\,,\rangle$. We denote the group of complex linear transformations of $\mathbf{O}_u$ by $GL(\mathbf{O}_u)$ and the special unitary transformations of $\mathbf{O}_u$ with its Hermitian metric by $SU(\mathbf{O}_u)$.

J_u

In[577]:=

```
Table[FullSimplify[
  Sum[ψ[A1] J[B1] η_AB[[A1, A1]] E^A_a[hUSE][[C]][[c]] τ[A1 - 1][[c]][[b]] E^a_A[hUSE][[b]][[B1]],
   {c, 1, 8}, {b, 1, 8}, {A1, 1, 8}, {B1, 1, 8}]], {C, 1, 8}]
```

Out[577]=

```
{J[1] ψ[1] - J[2] ψ[2] - J[3] ψ[3] - J[4] ψ[4] +
  J[5] ψ[5] + J[6] ψ[6] + J[7] ψ[7] + J[8] ψ[8], J[2] ψ[1] + J[1] ψ[2] -
  J[4] ψ[3] + J[3] ψ[4] + J[6] ψ[5] - J[5] ψ[6] - J[8] ψ[7] + J[7] ψ[8],
 J[3] ψ[1] + J[4] ψ[2] + J[1] ψ[3] - J[2] ψ[4] + J[7] ψ[5] + J[8] ψ[6] -
  J[5] ψ[7] - J[6] ψ[8], J[4] ψ[1] - J[3] ψ[2] + J[2] ψ[3] +
  J[1] ψ[4] + J[8] ψ[5] - J[7] ψ[6] + J[6] ψ[7] - J[5] ψ[8],
 J[5] ψ[1] + J[6] ψ[2] + J[7] ψ[3] + J[8] ψ[4] + J[1] ψ[5] - J[2] ψ[6] -
  J[3] ψ[7] - J[4] ψ[8], J[6] ψ[1] - J[5] ψ[2] + J[8] ψ[3] -
  J[7] ψ[4] + J[2] ψ[5] + J[1] ψ[6] + J[4] ψ[7] - J[3] ψ[8],
 J[7] ψ[1] - J[8] ψ[2] - J[5] ψ[3] + J[6] ψ[4] + J[3] ψ[5] - J[4] ψ[6] +
  J[1] ψ[7] + J[2] ψ[8], J[8] ψ[1] + J[7] ψ[2] - J[6] ψ[3] -
  J[5] ψ[4] + J[4] ψ[5] + J[3] ψ[6] - J[2] ψ[7] + J[1] ψ[8]}
```

In[578]:=

```
B = 1;
Grid[{EA, FullSimplify[ExpandAll[ η_AB[[B, B]] EA.E^A_a[h].τ[B - 1].E^a_A[h]]]}, Frame → All]
```

Out[578]=

```
Grid[{{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
  {eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}.𝕖^A_a[h].
   {{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0},
    {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0},
    {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 1}}.𝕖^a_A[h]}, Frame → All]
```

In[579]:=

```
AB = Array[AtB, {8, 8}];
productsAB =
 Table[Table[{{A1, B1}, Sum[mABC⟦A1, B1, C1⟧ * EA⟦C1⟧, {C1, 8}]}, {A1, 1, 8}], {B1, 1, 8}]
Partition[Flatten[%, 1], 8] === %
(*AB⟦#⟦1⟧,#⟦2⟧⟧=#⟦3⟧&/@Partition[%,8]*)
```

Out[580]=

```
{{{{1, 1}, eA[1]}, {{2, 1}, eA[2]}, {{3, 1}, eA[3]}, {{4, 1}, eA[4]},
  {{5, 1}, eA[5]}, {{6, 1}, eA[6]}, {{7, 1}, eA[7]}, {{8, 1}, eA[8]}},
 {{{1, 2}, eA[2]}, {{2, 2}, -eA[1]}, {{3, 2}, eA[4]}, {{4, 2}, -eA[3]},
  {{5, 2}, eA[6]}, {{6, 2}, -eA[5]}, {{7, 2}, -eA[8]}, {{8, 2}, eA[7]}},
 {{{1, 3}, eA[3]}, {{2, 3}, -eA[4]}, {{3, 3}, -eA[1]}, {{4, 3}, eA[2]},
  {{5, 3}, eA[7]}, {{6, 3}, eA[8]}, {{7, 3}, -eA[5]}, {{8, 3}, -eA[6]}},
 {{{1, 4}, eA[4]}, {{2, 4}, eA[3]}, {{3, 4}, -eA[2]}, {{4, 4}, -eA[1]},
  {{5, 4}, eA[8]}, {{6, 4}, -eA[7]}, {{7, 4}, eA[6]}, {{8, 4}, -eA[5]}},
 {{{1, 5}, eA[5]}, {{2, 5}, -eA[6]}, {{3, 5}, -eA[7]}, {{4, 5}, -eA[8]},
  {{5, 5}, eA[1]}, {{6, 5}, -eA[2]}, {{7, 5}, -eA[3]}, {{8, 5}, -eA[4]}},
 {{{1, 6}, eA[6]}, {{2, 6}, eA[5]}, {{3, 6}, -eA[8]}, {{4, 6}, eA[7]},
  {{5, 6}, eA[2]}, {{6, 6}, eA[1]}, {{7, 6}, eA[4]}, {{8, 6}, -eA[3]}},
 {{{1, 7}, eA[7]}, {{2, 7}, eA[8]}, {{3, 7}, eA[5]}, {{4, 7}, -eA[6]},
  {{5, 7}, eA[3]}, {{6, 7}, -eA[4]}, {{7, 7}, eA[1]}, {{8, 7}, eA[2]}},
 {{{1, 8}, eA[8]}, {{2, 8}, -eA[7]}, {{3, 8}, eA[6]}, {{4, 8}, eA[5]},
  {{5, 8}, eA[4]}, {{6, 8}, eA[3]}, {{7, 8}, -eA[2]}, {{8, 8}, eA[1]}}}
```

Out[581]=

```
True
```

In[582]:=

```
Table[AB⟦productsAB⟦A1, B1, 1⟧⟦1⟧, productsAB⟦A1, B1, 1⟧⟦2⟧⟧ = productsAB⟦A1, B1, 2⟧,
 {A1, 1, 8}, {B1, 1, 8}]
```

Out[582]=

```
{{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
 {eA[2], -eA[1], eA[4], -eA[3], eA[6], -eA[5], -eA[8], eA[7]},
 {eA[3], -eA[4], -eA[1], eA[2], eA[7], eA[8], -eA[5], -eA[6]},
 {eA[4], eA[3], -eA[2], -eA[1], eA[8], -eA[7], eA[6], -eA[5]},
 {eA[5], -eA[6], -eA[7], -eA[8], eA[1], -eA[2], -eA[3], -eA[4]},
 {eA[6], eA[5], -eA[8], eA[7], eA[2], eA[1], eA[4], -eA[3]},
 {eA[7], eA[8], eA[5], -eA[6], eA[3], -eA[4], eA[1], eA[2]},
 {eA[8], -eA[7], eA[6], eA[5], eA[4], eA[3], -eA[2], eA[1]}}
```

In[583]:=

```
gridx = Grid[{Flatten[{"A/B", EA}],
   Block[{A1 = 1}, Flatten[
     {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
   Block[{A1 = 2}, Flatten[
     {EA[[2]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
   Block[{A1 = 3}, Flatten[
     {EA[[3]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
   Block[{A1 = 4}, Flatten[
     {EA[[4]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
   Block[{A1 = 5}, Flatten[
     {EA[[5]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
   Block[{A1 = 6}, Flatten[
     {EA[[6]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
   Block[{A1 = 7}, Flatten[
     {EA[[7]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
   Block[{A1 = 8}, Flatten[
     {EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]
  }, Frame → All]
```

Out[583]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

1	i	j	k	ℓ	ℓi	ℓj	ℓk
i	-1	k	$-j$	$-\ell i$	ℓ	$-\ell k$	ℓj
j	$-k$	-1	i	$-\ell j$	ℓk	ℓ	$-\ell i$
k	j	$-i$	-1	$-\ell k$	$-\ell j$	ℓi	ℓ
ℓ	ℓi	ℓj	ℓk	1	i	j	k
ℓi	$-\ell$	$-\ell k$	ℓj	$-i$	1	k	$-j$
ℓj	ℓk	$-\ell$	$-\ell i$	$-j$	$-k$	1	i
ℓk	$-\ell j$	ℓi	$-\ell$	$-k$	j	$-i$	1

In[584]:=

```
{Flatten[{"A/B", EA}],
  Block[{A1 = 1}, Flatten[
    {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 2}, Flatten[
    {EA[[2]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 3}, Flatten[
    {EA[[3]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 4}, Flatten[
    {EA[[4]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 5}, Flatten[
    {EA[[5]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 6}, Flatten[
    {EA[[6]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 7}, Flatten[
    {EA[[7]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 8},
   Flatten[{EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]
 };
gridx = Grid[%, Frame → All]
```

Out[585]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

In[586]:=

```
{Flatten[{"A/B", EA}],
  Block[{A1 = 1}, Flatten[
    {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 2}, Flatten[
    {EA[[2]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 3}, Flatten[
    {EA[[3]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 4}, Flatten[
    {EA[[4]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 5}, Flatten[
    {EA[[5]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 6}, Flatten[
    {EA[[6]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 7}, Flatten[
    {EA[[7]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
  Block[{A1 = 8},
   Flatten[{EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]
 };
gridx = Grid[%, Frame → All]
```

Out[587]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

In[588]:=

```
{Flatten[{"A/B", EA}],
 Block[{A1 = 1},
  Flatten[{EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 2},
  Flatten[{EA[[2]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 3},
  Flatten[{EA[[3]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 4},
  Flatten[{EA[[4]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 5},
  Flatten[{EA[[5]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 6},
  Flatten[{EA[[6]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 7},
  Flatten[{EA[[7]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 8},
  Flatten[{EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]
}
Dimensions[%]
gridx = Grid[%%, Frame → All]
```

Out[588]=

```
{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
 {eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
 {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},
 {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},
 {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},
 {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},
 {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},
 {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]},
 {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}
```

Out[589]=

```
{9, 9}
```

Out[590]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

In[591]:=

```
{Flatten[{"A/B", EA}],
 Block[{A1 = 1},
  Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 2},
  Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 3},
  Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 4},
  Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 5},
  Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 6},
  Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 7},
  Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
 Block[{A1 = 8},
  Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]
}
Dimensions[%]
gridx = Grid[%%, Frame → All]
```

Out[591]=

```
{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
 {eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
 {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},
 {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},
 {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},
 {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},
 {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},
 {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]},
 {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}
```

Out[592]=

```
{9, 9}
```

Out[593]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

In[594]:=

```
Reap[For[A1 = 1, A1 < 9, A1++,
  Sow[Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]]]
```

Out[594]=

```
{Null, {{{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
   {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},
   {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},
   {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},
   {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},
   {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},
   {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]},
   {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}}
```

In[595]:=

```
Drop[Reap[For[A1 = 1, A1 < 9, A1++, Sow[Flatten[{EA[[A1]],
        Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]]], 1][[1]][[1]]
Dimensions[%]
```

Out[595]=

```
{{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
 {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},
 {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},
 {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},
 {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},
 {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},
 {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]},
 {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}
```

Out[596]=

```
{8, 9}
```

In[597]:=

```
Prepend[
 Drop[Reap[For[A1 = 1, A1 < 9, A1++, Sow[Flatten[{EA⟦A1⟧, Table[Sum[mABC⟦A1, B1, C1⟧ *
            EA⟦C1⟧, {C1, 1, 8}], {B1, 1, 8}]}]]]], 1]⟦1⟧⟦1⟧, Flatten[{"A/B", EA}]]
Dimensions[%]
gridx = Grid[%%, Frame → All]
```

Out[597]=

```
{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
 {eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
 {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},
 {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},
 {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},
 {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},
 {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},
 {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]},
 {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}
```

Out[598]=

```
{9, 9}
```

Out[599]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

In[600]:=

```
{Flatten[{"A/B", EA}],
  Reap[Do[Sow[Flatten[
      {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
    {A1, 1, 8}]]
 };
gridx = Grid[%, Frame → All]
```

Out[601]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
	{{{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, {eA[1], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, {eA[1], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, {eA[1], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, {eA[1], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, {eA[1], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, {eA[1], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[1], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}							

In[602]:=

```
{Flatten[{"A/B", EA}],
  Reap[Do[Sow[Flatten[
        {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]],
      {A1, 1, 8}]] // Drop[#, 1] &
 };
gridx = Grid[%, Frame → All]
```

Out[603]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{{{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, {eA[1], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, {eA[1], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, {eA[1], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, {eA[1], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, {eA[1], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, {eA[1], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[1], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}								

In[604]:=

```
{Flatten[{"A/B", EA}],
  Table[
   Flatten[{EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}],
   {A1, 1, 8}]
 };
gridx = Grid[%, Frame → All]
```

Out[605]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}	{eA[1], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}	{eA[1], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}	{eA[1], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}	{eA[1], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}	{eA[1], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}	{eA[1], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}	{eA[1], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}	

In[606]:=

```
{Flatten[{"A/B", EA}],
  Flatten[Reap[Do[Sow[
        Flatten[{EA⟦8⟧, Table[Sum[mABC⟦A1, B1, C1⟧ * EA⟦C1⟧, {C1, 1, 8}], {B1, 1, 8}]}
      ], {A1, 1, 8}]] // Drop[#, 1] &, 1]
 };
gridx = Grid[%, Frame → All]
```

Out[607]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{{eA[8], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},								
{eA[8], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},								
{eA[8], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},								
{eA[8], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},								
{eA[8], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},								
{eA[8], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},								
{eA[8], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]},								
{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}								

In[608]:=

```
{Flatten[{"A/B", EA}],
  Flatten[{Reap[Do[Sow[
        Block[{A1 = 8}, Flatten[
           {EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]
       ], {A1, 1, 8}]]}, 1]
 };
gridx = Grid[%, Frame → All]
```

Out[609]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
	{{{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}}							

In[610]:=

```
{Flatten[{"A/B", EA}],
  Reap[Do[Sow[
      Block[{A1 = 8}, Flatten[
        {EA[[8]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]
     ], {A1, 1, 8}]]
 };
gridx = Grid[%, Frame → All]
```

Out[611]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
	{{{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}							

In[612]:=

```
// Drop[#, 1] &
```

In[612]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],
   Flatten[Reap[Do[Sow[{EA⟦A1⟧, Table[Sum[mABC⟦A1, B1, C1⟧ * EA⟦C1⟧, {C1, 1, 8}],
            {B1, 1, 8}]}], {A1, 1, 8}]] // Drop[#, 1] &] // Partition[#, 1 + 8] &
  }, Frame → All]
```

Out[612]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}	{eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}	{eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}	{eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}	{eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}	{eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}	{eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}	{eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}	

In[613]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],
   Reap[Do[Sow[Flatten[{EA⟦A1⟧, Table[
          Sum[mABC⟦A1, B1, C1⟧ * EA⟦C1⟧, {C1, 1, 8}], {B1, 1, 8}]}]], {A1, 1, 8}]]
  }, Frame → All]
```

Out[613]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
	{{{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]}, {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]}, {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]}, {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]}, {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]}, {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[8], eA[8], eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}							

In[614]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],
   Reap[For[A1 = 1, A1 < 9, A1++,
      Sow[Flatten[{EA[[A1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}],
           {B1, 1, 8}]}]]]] // Drop[#, 1] &,
  }, Frame → All]
```

Out[614]=

```
Grid[{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
  {{{{eA[1], eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
     {eA[2], eA[2], -eA[1], -eA[4], eA[3], -eA[6], eA[5], eA[8], -eA[7]},
     {eA[3], eA[3], eA[4], -eA[1], -eA[2], -eA[7], -eA[8], eA[5], eA[6]},
     {eA[4], eA[4], -eA[3], eA[2], -eA[1], -eA[8], eA[7], -eA[6], eA[5]},
     {eA[5], eA[5], eA[6], eA[7], eA[8], eA[1], eA[2], eA[3], eA[4]},
     {eA[6], eA[6], -eA[5], eA[8], -eA[7], -eA[2], eA[1], -eA[4], eA[3]},
     {eA[7], eA[7], -eA[8], -eA[5], eA[6], -eA[3], eA[4], eA[1], -eA[2]}, {eA[8], eA[8],
      eA[7], -eA[6], -eA[5], -eA[4], -eA[3], eA[2], eA[1]}}}}, Null}, Frame → All]
```

Remove duplicates from a list, keeping original order :

In[615]:=

```
unsortedUnion[x_] := Reap[Sow[1, x], _, #1 &][[2]]
```

In[616]:=

```
unsortedUnion[x_] := Reap[Sow[1, x], _, #1 &]
```

In[617]:=

```
unsortedUnion[{b, b, c, a, c, a, b, d}]
```

Out[617]=

```
{1, {b, c, a, d}}
```

In[618]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],
   Reap[For[A1 = 1, A1 < 9, A1++, Sow[1, Flatten[{EA[[A1]], Table[
          Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]], _, #1 &][[2]],
  }, Frame → All]
```

Out[618]=

```
Grid[{{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]},
  {eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8], -eA[1], -eA[4],
   -eA[6], -eA[7], -eA[2], -eA[8], -eA[3], -eA[5]}, Null}, Frame → All]
```

In[619]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],
   For[A1 = 1, A1 < 9, A1++, Catch[Throw[Flatten[
        {EA[[1]], Table[Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]]],
  }, Frame → All]
```

Out[619]=

```
Grid[
 {{A/B, eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}, Null, Null}, Frame → All]
```

In[620]:=

```
gridxy = Grid[List[Flatten[{"A/B", EA}],
   Table[Flatten[
     {EA⟦A1⟧, Table[Sum[mABC⟦A1, B1, C1⟧ * EA⟦C1⟧, {C1, 1, 8}], {B1, 1, 8}]}]
    , {A1, 1, 8}]], Frame → All]
```

Out[620]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{eA[1],	{eA[2],	{eA[3],	{eA[4],	{eA[5],	{eA[6],	{eA[7],	{eA[8],	
eA[1],	eA[2],	eA[3],	eA[4],	eA[5],	eA[6],	eA[7],	eA[8],	
eA[2],	-eA[1],	eA[4],	-eA[3],	eA[6],	-eA[5],	-eA[8],	eA[7],	
eA[3],	-eA[4],	-eA[1],	eA[2],	eA[7],	eA[8],	-eA[5],	-eA[6],	
eA[4],	eA[3],	-eA[2],	-eA[1],	eA[8],	-eA[7],	eA[6],	-eA[5],	
eA[5],	-eA[6],	-eA[7],	-eA[8],	eA[1],	-eA[2],	-eA[3],	-eA[4],	
eA[6],	eA[5],	-eA[8],	eA[7],	eA[2],	eA[1],	eA[4],	-eA[3],	
eA[7],	eA[8],	eA[5],	-eA[6],	eA[3],	-eA[4],	eA[1],	eA[2],	
eA[8]}	-eA[7]}	eA[6]}	eA[5]}	eA[4]}	eA[3]}	-eA[2]}	eA[1]}	

In[621]:=

```
gridxy = Grid[{Flatten[{"A/B", EA}],
   Flatten[{Table[Flatten[
       {EA⟦A1⟧, Table[Sum[mABC⟦A1, B1, C1⟧ * EA⟦C1⟧, {C1, 1, 8}], {B1, 1, 8}]}]
      , {A1, 1, 8}]}, 1]}, Frame → All]
```

Out[621]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
{eA[1],	{eA[2],	{eA[3],	{eA[4],	{eA[5],	{eA[6],	{eA[7],	{eA[8],	
eA[1],	eA[2],	eA[3],	eA[4],	eA[5],	eA[6],	eA[7],	eA[8],	
eA[2],	-eA[1],	eA[4],	-eA[3],	eA[6],	-eA[5],	-eA[8],	eA[7],	
eA[3],	-eA[4],	-eA[1],	eA[2],	eA[7],	eA[8],	-eA[5],	-eA[6],	
eA[4],	eA[3],	-eA[2],	-eA[1],	eA[8],	-eA[7],	eA[6],	-eA[5],	
eA[5],	-eA[6],	-eA[7],	-eA[8],	eA[1],	-eA[2],	-eA[3],	-eA[4],	
eA[6],	eA[5],	-eA[8],	eA[7],	eA[2],	eA[1],	eA[4],	-eA[3],	
eA[7],	eA[8],	eA[5],	-eA[6],	eA[3],	-eA[4],	eA[1],	eA[2],	
eA[8]}	-eA[7]}	eA[6]}	eA[5]}	eA[4]}	eA[3]}	-eA[2]}	eA[1]}	

In[622]:=

```
(*Block[{A1=1,B1=2,C1=3},
 Sum[ E^A_a[hUSE]⟦C1,c⟧*(((η_AB⟦A1,A1⟧*τ[A1-1]⟦c,b⟧)*E^a_A[hUSE]⟦b,B1⟧)),{c,1,8},{b,1,8}]]*)
```

In[623]:=

```
(* E^A_a[hUSE].E^a_A[hUSE]===E^a_A[hUSE].E^A_a[hUSE]===ID8*)
```

In[624]:=

```
(*Symbolize[ m^C_AB ]*)
```

In[625]:=

```
(*m^C_AB=Table[Sum[ E^A_a[hUSE]⟦C1,c⟧*(((η_AB⟦A1,A1⟧*τ[A1-1]⟦c,b⟧)*E^a_A[hUSE]⟦b,B1⟧)),
   {c,1,8},{b,1,8}],{A1,1,8},{B1,1,8},{C1,1,8}]*)
```

In[626]:=

```
EA
```

Out[626]=

```
{eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}
```

In[627]:=

```
(*Table[E^A_a[hUSE].(η_AB[[A1,A1]]*τ[A1-1]).E^a_A[hUSE],{A1,1,8}]
 %[[1]]
 %%[[8]]
 %%%[[2]]*)
```

Spinor Lagrangian

In[628]:=

```
Protect[H, K]
```

Out[628]=

```
{}
```

In[629]:=

```
(*Clear[Lg];
 Lg[]:=√detgg *
    ( Transpose[Ψ16].σ16.Sum[FullSimplify[(T16^α[α1-1]/.sg),constraintVars].
          (D[ Ψ16,X[[α1]]]+(Q1/2)*Sum[ωmat[[α1,a,b]]*SAB[[a,b]].Ψ16,{a,1,8},{b,1,8}]),
        {α1,1,Length[X]}]+(H*M)*Transpose[Ψ16].σ16.
        Ψ16)//Simplify[#,constraintVars]&*)
```

In[630]:=

```
sx0x4
```

Out[630]=

$$\left\{x0 \to \frac{z}{6\,H},\ x4 \to \frac{t}{H}\right\}$$

In[631]:=

```
szt
```

Out[631]=

$$\{z \to 6\,H\,x0,\ t \to H\,x4\}$$

In[632]:=

```
Clear[La];
La[] :=
 (1/H Transpose[Ψ16].σ16.Sum[T16^A[α1 - 1].(D[ Ψ16, X[[α1]]]), {α1, 1, Length[X]}] +
    (2 * M/H) * Transpose[Ψ16].σ16.Ψ16) // Simplify[#, constraintVars] &
```

In[634]:=

```
La[];
- Sum[(D[Transpose[Ψ16], X[[α1]]]).σ16.T16^A[α1 - 1], {α1, 1, Length[X]}].Ψ16 +
   (2 * H * M) * Transpose[Ψ16].σ16.Ψ16 // FullSimplify[#, constraintVars] &;
ExpandAll[% - %%] // FullSimplify[#, constraintVars] &
```

Out[636]=

$$
\begin{aligned}
&-\frac{1}{H}(-1+H)\Big(4\,M\,f16[2][x0,x4]\times f16[6][x0,x4]+4\,H\,M\,f16[2][x0,x4]\times f16[6][x0,x4]+\\
&\quad 4\,M\,f16[3][x0,x4]\times f16[7][x0,x4]+4\,H\,M\,f16[3][x0,x4]\times f16[7][x0,x4]-\\
&\quad 4\,M\,f16[8][x0,x4]\times f16[12][x0,x4]-4\,H\,M\,f16[8][x0,x4]\times f16[12][x0,x4]-\\
&\quad 4\,M\,f16[9][x0,x4]\times f16[13][x0,x4]-4\,H\,M\,f16[9][x0,x4]\times f16[13][x0,x4]-\\
&\quad 4\,M\,f16[10][x0,x4]\times f16[14][x0,x4]-4\,H\,M\,f16[10][x0,x4]\times f16[14][x0,x4]-\\
&\quad 4\,M\,f16[11][x0,x4]\times f16[15][x0,x4]-4\,H\,M\,f16[11][x0,x4]\times f16[15][x0,x4]-\\
&\quad f16[9][x0,x4]\,f16[0]^{(0,1)}[x0,x4]+f16[8][x0,x4]\,f16[1]^{(0,1)}[x0,x4]+\\
&\quad f16[11][x0,x4]\,f16[2]^{(0,1)}[x0,x4]-f16[10][x0,x4]\,f16[3]^{(0,1)}[x0,x4]+\\
&\quad f16[13][x0,x4]\,f16[4]^{(0,1)}[x0,x4]-f16[12][x0,x4]\,f16[5]^{(0,1)}[x0,x4]-\\
&\quad f16[15][x0,x4]\,f16[6]^{(0,1)}[x0,x4]+f16[14][x0,x4]\,f16[7]^{(0,1)}[x0,x4]+\\
&\quad f16[3][x0,x4]\,f16[10]^{(0,1)}[x0,x4]-f16[2][x0,x4]\,f16[11]^{(0,1)}[x0,x4]-\\
&\quad f16[7][x0,x4]\,f16[14]^{(0,1)}[x0,x4]+f16[6][x0,x4]\,f16[15]^{(0,1)}[x0,x4]-\\
&\quad f16[12][x0,x4]\,f16[0]^{(1,0)}[x0,x4]-f16[13][x0,x4]\,f16[1]^{(1,0)}[x0,x4]-\\
&\quad f16[14][x0,x4]\,f16[2]^{(1,0)}[x0,x4]-f16[15][x0,x4]\,f16[3]^{(1,0)}[x0,x4]-\\
&\quad f16[8][x0,x4]\,f16[4]^{(1,0)}[x0,x4]-f16[9][x0,x4]\,f16[5]^{(1,0)}[x0,x4]-\\
&\quad f16[10][x0,x4]\,f16[6]^{(1,0)}[x0,x4]-f16[11][x0,x4]\,f16[7]^{(1,0)}[x0,x4]+\\
&\quad f16[4][x0,x4]\left(-f16[13]^{(0,1)}[x0,x4]+f16[8]^{(1,0)}[x0,x4]\right)+\\
&\quad f16[5][x0,x4]\left(f16[12]^{(0,1)}[x0,x4]+f16[9]^{(1,0)}[x0,x4]\right)+\\
&\quad f16[6][x0,x4]\,f16[10]^{(1,0)}[x0,x4]+f16[7][x0,x4]\,f16[11]^{(1,0)}[x0,x4]+\\
&\quad f16[0][x0,x4]\left(4\,(1+H)\,M\,f16[4][x0,x4]+f16[9]^{(0,1)}[x0,x4]+f16[12]^{(1,0)}[x0,x4]\right)+\\
&\quad f16[1][x0,x4]\left(4\,(1+H)\,M\,f16[5][x0,x4]-f16[8]^{(0,1)}[x0,x4]+f16[13]^{(1,0)}[x0,x4]\right)+\\
&\quad f16[2][x0,x4]\,f16[14]^{(1,0)}[x0,x4]+f16[3][x0,x4]\,f16[15]^{(1,0)}[x0,x4]\Big)
\end{aligned}
$$

In[637]:=

```
La[]
```

Out[637]=

$$\frac{1}{H}\Big(-4\,M\,(f16[0][x0,x4]\times f16[4][x0,x4]+f16[1][x0,x4]\times f16[5][x0,x4]+ f16[2][x0,x4]\times f16[6][x0,x4]+f16[3][x0,x4]\times f16[7][x0,x4]- f16[8][x0,x4]\times f16[12][x0,x4]-f16[9][x0,x4]\times f16[13][x0,x4]- f16[10][x0,x4]\times f16[14][x0,x4]-f16[11][x0,x4]\times f16[15][x0,x4])+ f16[12][x0,x4]\left(f16[5]^{(0,1)}[x0,x4]+f16[0]^{(1,0)}[x0,x4]\right)+ f16[13][x0,x4]\left(-f16[4]^{(0,1)}[x0,x4]+f16[1]^{(1,0)}[x0,x4]\right)+ f16[14][x0,x4]\left(-f16[7]^{(0,1)}[x0,x4]+f16[2]^{(1,0)}[x0,x4]\right)+ f16[15][x0,x4]\left(f16[6]^{(0,1)}[x0,x4]+f16[3]^{(1,0)}[x0,x4]\right)+ f16[8][x0,x4]\left(-f16[1]^{(0,1)}[x0,x4]+f16[4]^{(1,0)}[x0,x4]\right)+ f16[9][x0,x4]\left(f16[0]^{(0,1)}[x0,x4]+f16[5]^{(1,0)}[x0,x4]\right)+ f16[10][x0,x4]\left(f16[3]^{(0,1)}[x0,x4]+f16[6]^{(1,0)}[x0,x4]\right)+ f16[11][x0,x4]\left(-f16[2]^{(0,1)}[x0,x4]+f16[7]^{(1,0)}[x0,x4]\right)+ f16[4][x0,x4]\left(f16[13]^{(0,1)}[x0,x4]-f16[8]^{(1,0)}[x0,x4]\right)- f16[5][x0,x4]\left(f16[12]^{(0,1)}[x0,x4]+f16[9]^{(1,0)}[x0,x4]\right)- f16[6][x0,x4]\left(f16[15]^{(0,1)}[x0,x4]+f16[10]^{(1,0)}[x0,x4]\right)+ f16[7][x0,x4]\left(f16[14]^{(0,1)}[x0,x4]-f16[11]^{(1,0)}[x0,x4]\right)- f16[0][x0,x4]\left(f16[9]^{(0,1)}[x0,x4]+f16[12]^{(1,0)}[x0,x4]\right)+ f16[1][x0,x4]\left(f16[8]^{(0,1)}[x0,x4]-f16[13]^{(1,0)}[x0,x4]\right)+ f16[2][x0,x4]\left(f16[11]^{(0,1)}[x0,x4]-f16[14]^{(1,0)}[x0,x4]\right)- f16[3][x0,x4]\left(f16[10]^{(0,1)}[x0,x4]+f16[15]^{(1,0)}[x0,x4]\right)\Big)$$

In[638]:=

```
(*Clear[La];
 La[]:=useDSQRT *
    ( Transpose[Ψ16].σ16.Sum[useT16⟦α1⟧.(D[ Ψ16,X⟦α1⟧]+(Q1/2)*Sum[ωμIJ[α1]⟦A1,
                B1⟧*SAB⟦A1,B1⟧,{A1,1,8},{B1,1,8}].Ψ16),{α1,1,Length[X]}]+
      (H*M)*Transpose[Ψ16].σ16.Ψ16)//Simplify[#,constraintVars]&*)
```

In[639]:=

```
(Transpose[Ψ16].σ16.Sum[T16^A[α1 - 1].(D[Ψ16, X⟦α1⟧]), {α1, 1, Length[X]}] +
    (H * M) * Transpose[Ψ16].σ16.Ψ16) // FullSimplify[#, constraintVars] &;
- Sum[(D[Transpose[Ψ16], X⟦α1⟧]).σ16.T16^A[α1 - 1], {α1, 1, Length[X]}].Ψ16 +
   (H * M) * Transpose[Ψ16].σ16.Ψ16 // FullSimplify[#, constraintVars] &;
ExpandAll[% - %%] // FullSimplify[#, constraintVars] &
```

Out[641]=

0

In[642]:=

```
(*Clear[Lj];
  Lj[j_]:=useDSQRT *(((
        Transpose[Ψ16].σ16.useT16⟦1⟧.D[ Ψ16,X⟦1⟧]+

          Transpose[Ψ16].σ16.useT16⟦5⟧.D[ Ψ16,X⟦5⟧]+
        (K0/2)*Sum[(*useω⟦1,A1,B1⟧*)ωμIJ[1]⟦A1,B1⟧ Transpose[Ψ16].σ16.
            useT16⟦1⟧.SAB⟦A1,B1⟧.(base16⟦j,1⟧).Ψ16,{A1,1,8},{B1,1,8}])+
      Transpose[Ψ16].σ16.useT16⟦5⟧.D[ Ψ16,X⟦5⟧]+(K4/2)*
       Sum[(*useω⟦5,A1,B1⟧*) ωμIJ[5]⟦A1,B1⟧Transpose[Ψ16].σ16.
          useT16⟦5⟧.SAB⟦A1,B1⟧.(base16⟦j,1⟧).Ψ16,{A1,1,8},{B1,1,8}])+
     (H*M)*Transpose[Ψ16].σ16.Ψ16)//Simplify[#,constraintVars]&*)
```

In[643]:=

```
(*Lj[92]*)
```

In[644]:=

```
detgg
```

Out[644]=

```
detgg
```

In[645]:=

```
Clear[eL];
eL[Lagrangian_Symbol, detsqrt_] := Module[{L, t},
  L = Lagrangian[];
  t = Table[
    FullSimplify[1/detsqrt (D[L, f16[k][x0, x4]] - D[D[L, f16[k]^(1,0)[x0, x4]], x0] -
        D[D[L, f16[k]^(0,1)[x0, x4]], x4]), constraintVars], {k, 0, 15}];
  Return[t /. subsDefects];
 ]
```

---

eLa

In[647]:=

```
(*Get[
  "2025-12-14_Q2=0-WaveFunctionUniverse-4+4-Einstein-Lovelock-Nash-L9i-eLa.mx"];*)
```

In[648]:=

```
eLa = eL[La, 1];
```

In[649]:=

```
eLa
```

Out[649]=

$$\left\{-\frac{2\left(2\,M\,f16[4][x0,x4]+f16[9]^{(0,1)}[x0,x4]+f16[12]^{(1,0)}[x0,x4]\right)}{H},\right.$$
$$-\frac{2\left(2\,M\,f16[5][x0,x4]-f16[8]^{(0,1)}[x0,x4]+f16[13]^{(1,0)}[x0,x4]\right)}{H},$$
$$-\frac{2\left(2\,M\,f16[6][x0,x4]-f16[11]^{(0,1)}[x0,x4]+f16[14]^{(1,0)}[x0,x4]\right)}{H},$$
$$-\frac{2\left(2\,M\,f16[7][x0,x4]+f16[10]^{(0,1)}[x0,x4]+f16[15]^{(1,0)}[x0,x4]\right)}{H},$$
$$-\frac{2\left(2\,M\,f16[0][x0,x4]-f16[13]^{(0,1)}[x0,x4]+f16[8]^{(1,0)}[x0,x4]\right)}{H},$$
$$-\frac{2\left(2\,M\,f16[1][x0,x4]+f16[12]^{(0,1)}[x0,x4]+f16[9]^{(1,0)}[x0,x4]\right)}{H},$$
$$-\frac{2\left(2\,M\,f16[2][x0,x4]+f16[15]^{(0,1)}[x0,x4]+f16[10]^{(1,0)}[x0,x4]\right)}{H},$$
$$-\frac{2\left(2\,M\,f16[3][x0,x4]-f16[14]^{(0,1)}[x0,x4]+f16[11]^{(1,0)}[x0,x4]\right)}{H},$$
$$\frac{2\left(2\,M\,f16[12][x0,x4]-f16[1]^{(0,1)}[x0,x4]+f16[4]^{(1,0)}[x0,x4]\right)}{H},$$
$$\frac{2\left(2\,M\,f16[13][x0,x4]+f16[0]^{(0,1)}[x0,x4]+f16[5]^{(1,0)}[x0,x4]\right)}{H},$$
$$\frac{2\left(2\,M\,f16[14][x0,x4]+f16[3]^{(0,1)}[x0,x4]+f16[6]^{(1,0)}[x0,x4]\right)}{H},$$
$$\frac{2\left(2\,M\,f16[15][x0,x4]-f16[2]^{(0,1)}[x0,x4]+f16[7]^{(1,0)}[x0,x4]\right)}{H},$$
$$\frac{2\left(2\,M\,f16[8][x0,x4]+f16[5]^{(0,1)}[x0,x4]+f16[0]^{(1,0)}[x0,x4]\right)}{H},$$
$$\frac{2\left(2\,M\,f16[9][x0,x4]-f16[4]^{(0,1)}[x0,x4]+f16[1]^{(1,0)}[x0,x4]\right)}{H},$$
$$\frac{2\left(2\,M\,f16[10][x0,x4]-f16[7]^{(0,1)}[x0,x4]+f16[2]^{(1,0)}[x0,x4]\right)}{H},$$
$$\left.\frac{2\left(2\,M\,f16[11][x0,x4]+f16[6]^{(0,1)}[x0,x4]+f16[3]^{(1,0)}[x0,x4]\right)}{H}\right\}$$

In[650]:=

```
Length[eLa]
```

Out[650]=

16

In[651]:=

```
DumpSave[ToString[header <> "eLa.mx"], eLa];
```

In[652]:=

```
Print[ToString[header <> "eLa.mx"]]
```

```
Pre-gravity_Pre-Big_Bang_M6=3-Generations_of_Einstein-Rosen-2-Planes-eLa.mx
```

In[653]:=

```
(*Cases[
    eLa,
    h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], "f16"] :> n,
    Infinity,
    Heads → True
]*)
```

WRITE OUT THE VARIABLES THAT ARE COUPLED TO ANOTHER VARIABLE:

In[654]:=

```
eLaRawSets = Cases[
  #,
  h_Symbol[n_Integer] /; StringEndsQ[SymbolName[h], "f16"] :> n,
  Infinity,
  Heads → True
] & /@ eLa
```

Out[654]=

```
{{4, 9, 12}, {5, 8, 13}, {6, 11, 14}, {7, 10, 15},
 {0, 13, 8}, {1, 12, 9}, {2, 15, 10}, {3, 14, 11}, {12, 1, 4}, {13, 0, 5},
 {14, 3, 6}, {15, 2, 7}, {8, 5, 0}, {9, 4, 1}, {10, 7, 2}, {11, 6, 3}}
```

In[655]:=

```
Dimensions[eLaRawSets]
```

Out[655]=

```
{16, 3}
```

In[656]:=

```
rawSetsel16 = rawSets[eLa, "f16", 0]
% === eLaRawSets
```

Out[656]=

```
{{4, 9, 12}, {5, 8, 13}, {6, 11, 14}, {7, 10, 15},
 {0, 13, 8}, {1, 12, 9}, {2, 15, 10}, {3, 14, 11}, {12, 1, 4}, {13, 0, 5},
 {14, 3, 6}, {15, 2, 7}, {8, 5, 0}, {9, 4, 1}, {10, 7, 2}, {11, 6, 3}}
```

Out[657]=

```
True
```

Kähler. We show that such M^6 are foliated by 4-planes in $\mathbf{O}$ in a unique way.

In[658]:=

```
eLaCouplings = showCoupledEquations[eLaRawSets]
% // Flatten // Sort
% === Range[0, 15]
```

Out[658]=

```
{{0, 5, 8, 13}, {1, 4, 9, 12}, {2, 7, 10, 15}, {3, 6, 11, 14}}
```

Out[659]=

```
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
```

Out[660]=

```
True
```

In[661]:=

```
eLaztCouplings = eLaCouplings
```

Out[661]=

```
{{0, 5, 8, 13}, {1, 4, 9, 12}, {2, 7, 10, 15}, {3, 6, 11, 14}}
```

J. DIFFERENTIAL GEOMETRY
17 (1982) 185–232

SUBMANIFOLDS AND SPECIAL STRUCTURES ON THE OCTONIANS

ROBERT L. BRYANT

An inner product algebra over $\mathbf{R}$ is a vector space $\mathbf{A}$ over $\mathbf{R}$ which possesses a nondegenerate inner product $\langle\,,\rangle\colon \mathbf{A}\times\mathbf{A}\to\mathbf{R}$ and a multiplication $\mathbf{A}\times\mathbf{A}\to\mathbf{A}$ with unit $1\in\mathbf{A}$ so that for all $x, y\in\mathbf{A}$

$$\langle xy, xy\rangle = \langle x, x\rangle\langle y, y\rangle. \tag{1.1}$$

For convenience's sake we will identify $\mathbf{R}$ with the 1-dimensional subalgebra of $\mathbf{A}$ generated by $1\in\mathbf{A}$. By (1.1), we have $\langle 1,1\rangle = \langle 1,1\rangle^2$. If $\langle 1,1\rangle = 0$, then

$$x\times y = \tfrac{1}{2}(\bar{y}x - \bar{x}y). \tag{1.28}$$

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$\langle x, x\rangle = 0$ for all $x \in \mathbf{A}$, contradicting the nondegeneracy assumption. Hence $\langle 1, 1\rangle = 1$. We define the orthogonal compliment of 1 to be $\operatorname{Im}\mathbf{A} \subseteq \mathbf{A}$. It is a proper subspace and we have $\mathbf{A} = \mathbf{R} \oplus \operatorname{Im}\mathbf{A}$. Give $x \in \mathbf{A}$, we define $\bar{x} \in \mathbf{A}$,

(1.2) $$\bar{x} = 2\langle x, 1\rangle - x.$$

We denote $\langle x, 1\rangle$ by $\operatorname{Re} x$ and $(x - \operatorname{Re} x)$ by $\operatorname{Im} x$. Clearly $x \in \operatorname{Im}\mathbf{A}$ if and only if $x = -\bar{x}$ or $x = \operatorname{Im} x$ or $\operatorname{Re} x = 0$.

If we polarize (1.1) in the x-variable, we get the identity

(1.3) $$\langle xy, zy\rangle = \langle x, z\rangle\langle y, y\rangle.$$

If we expand $\langle x(1 + w), y(1 + w)\rangle$ in two ways and compare terms, we find

$$\langle xw, y\rangle = \langle x, y(2\langle w, 1\rangle - w)\rangle$$

or

(1.4) $$\langle xw, y\rangle = \langle x, y\bar{w}\rangle$$

for all $x, y, w \in \mathbf{A}$. In the same way, we get

(1.4′) $$\langle wx, y\rangle = \langle x, \bar{w}y\rangle.$$

Using (1.4) and (1.4′) repeatedly, we get

$$\begin{aligned}\langle w, \bar{y}\bar{x}\rangle &= \langle yw, \bar{x}\rangle = \langle y, \bar{x}\bar{w}\rangle = \langle xy, \bar{w}\rangle \\ &= \langle w(xy), 1\rangle = \langle w, \overline{(xy)}\rangle\end{aligned}$$

for all $x, y, w \in \mathbf{A}$. It follows that

(1.5) $$\overline{(xy)} = \bar{y}\bar{x}.$$

From (1.5), we conclude that $x\bar{x}$ is real for all $x \in A$. but then $\langle x, x\rangle = \langle x\bar{x}, 1\rangle = x\bar{x}$.

(1.6) $$x\bar{x} = \langle x, x\rangle = \bar{x}x.$$

Polarizing (1.6) we get

(1.7) $$\langle x, y\rangle = \tfrac{1}{2}(x\bar{y} + y\bar{x}).$$

We also compute

$$\langle (xw)\bar{w}, y\rangle = \langle xw, yw\rangle = \langle x, y\rangle\langle w, w\rangle = \langle x\langle w, w\rangle, y\rangle$$

so

(1.8) $$(xw)\bar{w} = x(w\bar{w}) \quad (= x\langle w, w\rangle).$$

by subtracting $2(xw)\langle w, 1\rangle$ from both sides of (1.8), we get

(1.9) $$(xw)w = xw^2$$

in spite of the fact that we have not assumed that $\mathbf{A}$ is associative. In a similar manner, we get

(1.8′) $$\bar{w}(wx) = (\bar{w}w)x,$$

(1.9′) $$w(wx) = w^2x.$$

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$$\bar{w}(wx) = (\bar{w}w)x, \tag{1.8'}$$

$$w(wx) = w^2x. \tag{1.9'}$$

By polarizing (1.8) and (1.8′), we get the identities

$$(xu)\bar{v} + (xv)\bar{u} = 2x\langle u, v\rangle, \tag{1.10}$$

$$u(\bar{v}x) + v(\bar{u}x) = 2x\langle u, v\rangle. \tag{1.11}$$

In particular, if $\langle u, v\rangle = 0$, then $(xu)\bar{v} = -(xv)\bar{u}$ and $u(\bar{v}x) = -v(\bar{u}x)$. We may use these facts to prove the following lemma (see [12]).

Lemma 1.1. *If* $\mathbf{B} \subseteq \mathbf{A}$ *is an inner product subalgebra and* $u \in \mathbf{A}$ *is orthogonal to* **B**, *then* $\mathbf{B}u \perp \mathbf{B}$ *and* $\mathbf{B} \oplus \mathbf{B}u$ *is a subalgebra of* **A** *which satisfies*

$$(a + bu)(c + du) = (ac - \langle u, u\rangle \bar{d}b) + (da + b\bar{c})u. \tag{1.12}$$

This lemma allows us to start with $\mathbf{B} = \mathbf{R}$ and "build up" to **A** by successively adding on orthogonal subspaces. Using this technique, one can show that if we assume that $\langle\, ,\rangle$ is positive definite, then there are only four inner product algebras over **R**, namely **R**, **C**, **H** (the quaternions) and **O** (the octonians).

Explicitly, we may regard **O** as the vector space $\mathbf{H} \oplus \mathbf{H}$. If we write 1 for $(1,0) \in \mathbf{O}$ and ε for $(0,1) \in \mathbf{O}$, the above lemma shows that the multiplication in **O** must be given by

$$(a + b\varepsilon), (c + d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon \tag{1.13}$$

where the inner product satisfies

$$\langle (a + b\varepsilon), (a + b\varepsilon)\rangle = a\bar{a} + b\bar{b} \tag{1.4}$$

whenever $a, b, c, d \in \mathbf{H}$.

We let $S^6 = \{u \in \operatorname{Im}\mathbf{O} \mid \langle u, u\rangle = 1\}$. The elements of S^6 are called the *imaginary units* of **O**. For any $u \in S^6$, we have $u = -\bar{u}$, so $u^2 = -u\bar{u} = -\langle u, u\rangle = -1$. We may use u to define a map $J_u\colon \mathbf{O} \to \mathbf{O}$ given by

$$J_u(x) = xu. \tag{1.15}$$

The identity (1.9) shows that $J_u^2(x) = (xu)u = xu^2 = -x$, so J_u defines a complex structure on **O**. We write $\mathbf{O}_u$ to denote **O** endowed with the complex structure J_u. If $u \neq v$, then clearly $J_u \neq J_v$, so we actually have a six-sphere of *distinct* complex structures on **O**. However, because S^6 is connected, we see that the orientation of **O** induced by the natural orientation of $\mathbf{O}_u$ as a complex vector space does not depend on u. We refer to this orientation as the natural orientation of **O**.

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Using (1.3), we see that if $u \in S$, then

$$\langle J_u(x), J_u(y)\rangle = \langle xu, yu\rangle = \langle x, y\rangle\langle u, u\rangle = \langle x, y\rangle$$

so J_u is an isometry for each $u \in S$. Moreover, it follows that $\mathbf{O}_u$ is endowed with a natural Hermitian structure with respect to the inner product $\langle\,,\rangle$. We denote the group of complex linear transformations of $\mathbf{O}_u$ by $GL(\mathbf{O}_u)$ and the special unitary transformations of $\mathbf{O}_u$ with its Hermitian metric by $SU(\mathbf{O}_u)$.

We let Spin(7) $\subseteq SO(8)$ denote the subgroup generated by the set $\{J_u \mid u \in S^6\} \subseteq SO(8)$. It is known (see [12]) that Spin(7) is a connected, simply connected, compact Lie group of real dimension 21. Its center is $\{\pm I_8\} \simeq \mathbf{Z}/2$ and Spin(7)$/\{\pm I_8\}$ is isomorphic to $SO(7)$, a simple group. We want to make explicit the structure equations of Spin(7) as a subgroup of $SO(8)$ in such a way that its relationship with the complex structures J_u is made clear.

Let $u \in S^6$ be an imaginary unit which is orthogonal to $\varepsilon \in \mathbf{O}$. For each $\lambda \in \mathbf{R}$, $(\cos\lambda\varepsilon + \sin\lambda u)$ is an imaginary unit. Hence $J_\varepsilon \circ J_{(\cos\lambda\varepsilon+\sin\lambda u)} = -\cos\lambda I + \sin\lambda J_\varepsilon \circ J_u$ is an element of Spin(7). We easily compute that $J_\varepsilon \circ J_u + J_u \circ J_\varepsilon = 0$ by using (1.10). Thus $(J_\varepsilon \circ J_u)^2 = J_\varepsilon \circ J_u \circ J_\varepsilon \circ J_u = -J_\varepsilon^2 \circ J_u^2 = -I$. It follows that

$$\exp(\lambda J_\varepsilon \circ J_u) = \cos\lambda I + \sin\lambda J_\varepsilon \circ J_u. \tag{1.16}$$

Thus, if spin(7) $\subseteq so(8)$ is the Lie algebra of Spin(7), we see that $J_\varepsilon \circ J_u \in$ spin(7) for all $u \in S^6$ with $\langle u, \varepsilon\rangle = 0$. Since spin(7) is a vector space, we see that $L \subseteq$ spin(7) where

$$L = \{J_\varepsilon \circ J_w \mid w \in \operatorname{Im}\mathbf{O}, \langle\varepsilon, w\rangle = 0\}. \tag{1.17}$$

Note that $\dim_{\mathbf{R}} L = 6$.

To go further, we will choose a basis and exhibit L as a vector space of matrices. In order to do this, let j and k be orthogonal imaginary units in **H**.[1] We define the *standard basis of* $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$, $(N, E, \overline{N}, \overline{E}) = (N, E_1, E_2, E_3, \overline{N}, \overline{E}_1, \overline{E}_2, \overline{E}_3)$ as follows: We set $N = \frac{1}{2}(1 - i\varepsilon)$, $\overline{N} = \frac{1}{2}(1 + i\varepsilon)$ and

$$\begin{aligned} E_1 &= jN, & \overline{E}_1 &= j\overline{N}, \\ E_2 &= kN, & \overline{E}_2 &= k\overline{N}, \\ E_3 &= (kj)N, & \overline{E}_3 &= (kj)\overline{N}. \end{aligned} \tag{1.18}$$

(Note that conjugation in $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$ occurs *only* in the **C**-factor.) By using the formulae in (1.13) and some elementary calculation, we see that if we set

[1] In all that follows, we *never* use i to denote a quaternion or an octonian. For us, $i \in \mathbf{C}$ and $\mathbf{C} \not\subseteq \mathbf{H}$.

$w = 2\mathbf{Re}(a^1E_1 + a^2E_2 + a^3E_3)$ where $a_i \in \mathbf{C}$ and $\mathbf{Re}\colon \mathbf{C} \otimes_{\mathbf{R}} \mathbf{O} \to \mathbf{O}$ is the real projection, then $w \in \operatorname{Im}\mathbf{O}$, $\langle\varepsilon, w\rangle = 0$ and

$$J_\varepsilon \circ J_w(N, E_1, E_2, E_3)$$

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$$(1.19)\qquad \begin{aligned} &J_\varepsilon \circ J_w(N, E_1, E_2, E_3) \\ &= (\bar{N}, \bar{E}_1, \bar{E}_2, \bar{E}_3)\begin{pmatrix} 0 & i\bar{a}^1 & i\bar{a}^2 & i\bar{a}^3 \\ -i\bar{a}^1 & 0 & ia^3 & -ia^2 \\ -i\bar{a}^2 & -ia^3 & 0 & ia^1 \\ -i\bar{a}^3 & ia^2 & -ia^1 & 0 \end{pmatrix}. \end{aligned}$$

To simplify matters, if $a = (a^i)$ is any column vector of height 3 (with complex entries), we define $[a]$ to be the 3×3 skew symmetric matrix

$$(1.20)\qquad [a] = \begin{pmatrix} 0 & a^3 & -a^2 \\ -a^3 & 0 & a^1 \\ a^2 & -a^1 & 0 \end{pmatrix}.$$

Note that $[a]$ is the matrix of the linear transformation from $\mathbf{C}^3$ to $\mathbf{C}^3$ determined by cross product with $a \in \mathbf{C}^3$. We will eventually need the following identities for $a, b \in \mathbf{C}^3$ and $A \in M_{3\times 3}(\mathbf{C})$.

$$(1.21)\qquad \begin{aligned} &[a]b + [b]a = 0, \\ &[Aa] = (\operatorname{tr} A)[a] - {}^tA[a] - [a]A, \\ &[a][b] = b\,{}^ta - {}^tab I_3 \end{aligned}$$

(I_3 is the 3×3 identity matrix).

We may now rewrite (1.19) in the more compact form

$$(1.19')\qquad J_\varepsilon \circ J_w(N, E) = (\bar{N}, \bar{E})\begin{pmatrix} 0 & i\,{}^t\bar{a} \\ -i\bar{a} & [ia] \end{pmatrix}$$

where $w = 2\,\mathbf{Re}(aE)$ (row by column multiplication is understood). It follows that, expressed in the full basis $(N, E, \bar{N}, \bar{E})$ we have

$$(1.19'')\qquad J_\varepsilon \circ J_w(N, E, \bar{N}, \bar{E}) = (N, E, \bar{N}, \bar{E})\begin{pmatrix} 0 & 0 & 0 & -i\,{}^ta \\ 0 & 0 & ia & [-i\bar{a}] \\ 0 & i\,{}^t\bar{a} & 0 & 0 \\ -i\bar{a} & [ia] & 0 & 0 \end{pmatrix}.$$

Thus, imbedding $\operatorname{End}(\mathbf{O}) \hookrightarrow M_{8\times 8}(\mathbf{C})$, the space of 8×8 complex matrices, via the standard basis, we get

$$(1.22)\qquad L = \left\{ \begin{pmatrix} 0 & 0 & 0 & -{}^ta \\ 0 & 0 & a & [\bar{a}] \\ 0 & -{}^t\bar{a} & 0 & 0 \\ \bar{a} & [a] & 0 & 0 \end{pmatrix} \middle| a \in \mathbf{C}^3 = M_{3\times 1}(\mathbf{C}) \right\}.$$

An easy computation, using (1.21), then shows

$$(1.23)\qquad [L, L] = \left\{ \begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix} \middle| \begin{array}{l} \kappa + {}^t\bar{\kappa} = 0 \text{ and } \operatorname{tr}\kappa = 0 \\ \text{and } \kappa \in M_{4\times 4}(\mathbf{C}) \end{array} \right\}.$$

Since $L \subseteq \operatorname{spin}(7)$, $[L, L] \subseteq \operatorname{spin}(7)$, and $L \cap [L, L] = 0$; and since

$$(1.24)\qquad \begin{aligned} &\dim_{\mathbf{R}} L = 6, \\ &\dim_{\mathbf{R}}[L, L] = 15, \\ &\dim_{\mathbf{R}} \operatorname{spin}(7) = 21, \end{aligned}$$

$$(1.22)\qquad L = \left\{ \begin{pmatrix} 0 & 0 & 0 & -{}^t\bar{a} \\ 0 & 0 & a & [\bar{a}] \\ 0 & -{}^t\bar{a} & 0 & 0 \\ \bar{a} & [a] & 0 & 0 \end{pmatrix} \middle| a \in \mathbf{C}^3 = M_{3\times 1}(\mathbf{C}) \right\}.$$

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Since $L \subseteq \operatorname{spin}(7)$, $[L, L] \subseteq \operatorname{spin}(7)$, and $L \cap [L, L] = 0$; and since

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we conclude that

$$(1.25)\qquad \operatorname{spin}(7) = L \oplus [L, L].$$

Finally, note that $[L, L] = su(\mathbf{O}_\varepsilon)$, the Lie algebra of $SU(\mathbf{O}_\varepsilon)$. If we note that

$$(1.26)\qquad gl(\mathbf{O}_\varepsilon) \cap \operatorname{spin}(7) = su(\mathbf{O}_\varepsilon)$$

and that Spin(7) is connected, we deduce that

$$(1.27)\qquad GL(\mathbf{O}_\varepsilon) \cap \operatorname{Spin}(7) = SU(\mathbf{O}_\varepsilon).$$

We record our main result so far:

Proposition 1.1. *Extend the elements of* $\operatorname{Spin}(7) \subseteq \operatorname{End}(\mathbf{O})$ *complex linearly so that* $\operatorname{Spin}(7) \subseteq \operatorname{End}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O})$. *If we use the standard basis* $(N, E, \bar{N}, \bar{E})$ *of* $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$ *to represent* $\operatorname{End}_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O})$ *as the* 8×8 *complex valued matrices, then*

$$\operatorname{spin}(7) = \left\{ \begin{pmatrix} ic & -{}^t\bar{b} & 0 & -{}^ta \\ b & d & a & [\bar{a}] \\ 0 & -{}^t\bar{a} & -ic & -{}^tb \\ \bar{a} & [a] & \bar{b} & \bar{d} \end{pmatrix} \middle| \begin{array}{l} a, b \in M_{3\times 1}(\mathbf{C}), \\ c \in \mathbf{R},\ d \in M_{3\times 3}(\mathbf{C}), \\ d + {}^t\bar{d} = 0, \\ \operatorname{tr} d + ic = 0. \end{array} \right\}.$$

As we will see below, Spin(7) actually satisfies $GL(\mathbf{O}_u) \cap \operatorname{Spin}(7) = SU(\mathbf{O}_u)$ for all $u \in S^6$.

For $x, y \in \mathbf{O}$, we define $x \times y$ by the formula

$$(1.28)\qquad x \times y = \tfrac{1}{2}(\bar{y}x - \bar{x}y).$$

$x \times y$ is called the cross product of x and y. Clearly $x \times y \in \operatorname{Im}\mathbf{O}$. We have the useful identities

$$(1.29)\qquad \langle x, y\rangle = 0 \Rightarrow x \times y = \bar{y}x = -\bar{x}y,$$

$$(1.30)\qquad E_i \times \bar{N} = N \times \bar{E}_i = 0.$$

For each $u \in S^6$, we let $r_u\colon \operatorname{Im}\mathbf{O} \to \operatorname{Im}\mathbf{O}$ be defined by $r_u(x) = \bar{u}(xu) = (\bar{u}x)u$ (this last association formula follows easily from (1.8) and (1.8′)). Using the Moufang identities (see Appendix B of [12]), one can verify that there

We will also have occasion to study the geometry of $\operatorname{Im}\mathbf{O}$ under a slightly smaller group than Spin(7). We get $G_2 \subseteq \operatorname{Spin}(7)$ be the subgroup which leaves $1 \in \mathbf{O}$ fixed. Thus G_2 is a compact subgroup of Spin(7). If we define $p\colon \operatorname{Spin}(7) \to S^7 \subseteq \mathbf{O}$ by setting $p(g) = n + \bar{n}$ where $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$, then clearly $p^{-1}(1) = G_2$. Computing the differential of p, we get

(1.30) $E_i \wedge N = N \wedge E_i = 0.$

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$$dp = i(n - \bar{n})\rho + f(\mathfrak{h} + \theta) + \bar{f}(\bar{\mathfrak{h}} + \bar{\theta}). \tag{2.14}$$

It follows that p has rank 7 and gives Spin(7) the structure of a G_2-bundle over S^7. The connectedness and simple-connectedness of Spin(7) and S^7 shows that G_2 must be connected and that the Lie algebra of G_2 is obtained from that of Spin(7) by setting $\rho = \mathfrak{h} + \theta = 0$.

For $g \in G_2$, we say an admissible basis of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}$, $(n, f, \bar{n}, \bar{f}) = (N, E, \bar{N}, \bar{E})g$ is G_2-admissible. Since $n + \bar{n} \equiv 1$ for such bases, we remove this information and set $u = i(n - \bar{n})$. We then have the following proposition whose proof is an easy computation and is omitted.

Proposition 2.3 (*The Structure Equations of* G_2). *The map* u: $G_2 \to S^6$ *makes* G_2 *into a principal right* $SU(3)$*-bundle over* S^6. *In fact, we have the structure equations*

$$du = f(-2i\theta) + \bar{f}(2i\bar{\theta}), \tag{2.15}$$

$$df = u(-i\,{}^t\bar{\theta}) + f\kappa + \bar{f}[\theta], \tag{2.16}$$

$$d\theta = -\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta}, \tag{2.17}$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\bar{\theta} - {}^t\theta \wedge \bar{\theta} I_3. \tag{2.18}$$

It follows that S^6 *possesses a unique nonintegrable almost complex structure so that a complex-valued* 1*-form* $\alpha \in \Omega^1_{\mathbf{C}}(S^6)$ *is of type* (1,0) *if and only if* $u^*(\alpha)$ *is a linear combination of* $\{\theta^1, \theta^2, \theta^3\}$.

prove that the almost complex structure on S^6 determines the metric structure of S^6 so that any invariant of the *local* almost complex structure is also a metric invariant (for a more precise statement, see Proposition 4.1 and its proof). (This is the compact-form analogue of Cartan's characterization of the split form of G_2 as the pseudo-group of a certain differential system on a five manifold.) This justifies our use of the metric structure on S^6 to study the almost complex structure of S^6.

multiplication. Since $G_2(\mathbf{C})$ acts as algebra automorphisms of $\mathbf{O}_{\mathbf{C}}$, this gives us another proof that $G_2(\mathbf{C})$ leaves $\mathfrak{L}$ invariant. (Cartan in [6], proves a striking

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multiplication. Since $G_2(\mathbf{C})$ acts as algebra automorphisms of $\mathbf{O}_{\mathbf{C}}$, this gives us another proof that $G_2(\mathbf{C})$ leaves $\mathfrak{L}$ invariant. (Cartan in [6], proves a striking

converse: The pseudo-group of bi-holomorphic transformations of $\tilde{G}(2, \operatorname{Im} \mathbf{O})$ which preserve $\mathfrak{L}$ is *exactly* the pseudo-group generated by the action of $G_2(\mathbf{C})$. This is the complex analogue of Proposition 4.1, but it is much harder to prove. We refer the interested reader to [6].)

[3] E. Cartan, *La théorie des groupes finis et continus et la geometrie differentielle*, Gauthier-Villars, Paris, 1937.

[4] _____, *Les systemes différentiel esterieurs*, Hermann, Paris, 1945.

[5] _____, *Sur les systemes en involution d'equations aux derivées partielles du second ordre à une fonction inconnue de trois variables indépendantes*, Oeuvres Complètes, Vol. 2, Pt. 2, pp. 1035–1125.

[6] _____, *Über die Einfachen Transformationsgruppen*, Oeuvres Complètes, Vol. 1, Pt. 1, pp. 107–132.

0. Introduction

Geometries associated to the exceptional groups and "exceptional" representations of classical groups often display interesting features closely related to (but distinctly different from) the more familiar features of the classical groups. This paper centers on the geometries in $\mathbf{E}^7$ and $\mathbf{E}^8$ whose groups of symmetries are $G_2 \subseteq SO(7)$ and $\operatorname{Spin}(7) \subseteq SO(8)$. Both of these groups are related to the octonians (sometimes called Cayley numbers) and may be defined in terms of octonionic multiplication. In particular, G_2, the compact exceptional group of (real) dimension 14, is the subgroup of algebra automorphisms of $\mathbf{O}$ (the octonians) and $\operatorname{Spin}(7) \subseteq SO(8)$ may be defined as the subgroup of $GL_{\mathbf{R}}(\mathbf{O})$ generated by right multiplication by unit octonians which are purely imaginary.

The geometry of the algebra $\mathbf{O}$ is closely related to the complex numbers. In §1, we develop some of the properties of $\mathbf{O}$ that we need for later sections. (Our presentation is essentially borrowed from Appendix A of [12], but any mistakes are, of course, due to the author.) A particularly interesting property

[5] ———, *Sur les systèmes en involution d'équations aux dérivées partielles du second ordre à une fonction inconnue de trois variables indépendantes*, Oeuvres Complètes, Vol. 2, Pt. 2, pp. 1035–1125.
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0. Introduction

Geometries associated to the exceptional groups and "exceptional" representations of classical groups often display interesting features closely related to (but distinctly different from) the more familiar features of the classical groups. This paper centers on the geometries in $\mathbf{E}^7$ and $\mathbf{E}^8$ whose groups of symmetries are $G_2 \subseteq SO(7)$ and $\mathrm{Spin}(7) \subseteq SO(8)$. Both of these groups are related to the octonians (sometimes called Cayley numbers) and may be defined in terms of octonionic multiplication. In particular, G_2, the compact exceptional group of (real) dimension 14, is the subgroup of algebra automorphisms of $\mathbf{O}$ (the octonians) and $\mathrm{Spin}(7) \subseteq SO(8)$ may be defined as the subgroup of $GL_{\mathbf{R}}(\mathbf{O})$ generated by right multiplication by unit octonians which are purely imaginary.

The geometry of the algebra $\mathbf{O}$ is closely related to the complex numbers. In §1, we develop some of the properties of $\mathbf{O}$ that we need for later sections. (Our presentation is essentially borrowed from Appendix A of [12], but any mistakes are, of course, due to the author.) A particularly interesting property is described as follows: If we let $\mathrm{Im}\,\mathbf{O} \subseteq \mathbf{O}$ be the hyperplane (through 0) orthogonal to $1 \in \mathbf{O}$, and we let $S^6 \subseteq \mathrm{Im}\,\mathbf{O}$ be the space of unit vectors, then right multiplication by $u \in S^6$ induces a linear transformation $R_u: \mathbf{O} \to \mathbf{O}$ which is orthogonal and satisfies $(R_u)^2 = -1$. Thus, associated to each $u \in S^6$ is a complex structure on $\mathbf{O}$ (induced by $J = R_u$) which is compatible with the natural inner product on $\mathbf{O}$. We denote by $\mathbf{O}_u$ the Hermitian vector space whose underlying real vector space (with inner product) is $\mathbf{O}$ and whose complex structure is given by R_u.

Classically, this observation was used to define an almost complex structure on S^6 as follows: If $u \in S^6$, then R_u preserves the 2-plane spanned by 1 and u and therefore preserves its orthogonal 6-plane, which may be identified with $T_u S^6 \subseteq \mathrm{Im}\,\mathbf{O}$ after translation to the origin. Thus R_u induces a complex

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particular, we show that Calabi's examples are exactly the M^6 with parallel asymptotic ruling which lie in the hyperplane $\operatorname{Im} \mathbf{O} \subseteq \mathbf{O}$. We then use Cartan's theory of differential systems in involution to show that the analytic non-Kähler but complex $M^6 \subseteq \mathbf{O}$ "depend" on 12 analytic functions of 1 (real) variable. (For a more precise statement, see §3).

We observe, as did Gray, that the canonical 2-form on $M^6 \subseteq \mathbf{O}$ is *always* co-closed. Finally, we show that any $M^6 \subseteq \mathbf{O}$ for which the canonical 2-form Ω is closed is necessarily Kähler (and therefore must be a complex hypersurface in $\mathbf{O}_u$ for some fixed $u \in S^6$). In particular, such $M^6 \subseteq \operatorname{Im} \mathbf{O}$ must be hyperplanes. This recovers a result of Gray (see [10]).

In the final section of the paper, we study the "complex curves" in S^6, i.e., those maps $\phi: M^2 \to S^6$ where M^2 is a Riemann surface and $d\phi$ is complex linear with respect to the almost complex structure on S^6 induced by the inclusion $S^6 \subseteq \operatorname{Im} \mathbf{O}$. This study is motivated by the fact that the cone on such a complex curve gives a 3-fold in $\operatorname{Im} \mathbf{O}$ which is *associative* in the sense of [12]. Such cones are absolutely mass minimizing and their singular structure reflects the singular structure of general associative varieties in $\mathbf{R}^7 \subseteq \operatorname{Im} \mathbf{O}$. We first prove that the almost complex structure on S^6 determines the metric structure of S^6 so that any invariant of the *local* almost complex structure is also a metric invariant (for a more precise statement, see Proposition 4.1 and its proof). (This is the compact-form analogue of Cartan's characterization of the split form of G_2 as the pseudo-group of a certain differential system on a five manifold.) This justifies our use of the metric structure on S^6 to study the almost complex structure of S^6.

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3. Oriented 6-manifolds in O

Let M^6 be an abstract oriented 6-manifold with a smooth differentiable structure. Let X: $M^6 \to \mathbf{O}$ be a smooth immersion of M^6 into $\mathbf{O}$. We say that an admissible frame $(y; n, f, \bar{n}, \bar{f}) \in \mathfrak{F}$ is *adapted* at $p \in M$ if $X(p) = y$ and if (f_1, f_2, f_3) is a (1,0) basis of $X_*(T_pM^6)$ with its induced orientation from M^6 and complex structure induced from right multiplication by $2in \times \bar{n}$. We let $\mathfrak{F}_X(M)$ denote the space of pairs $(p, (y; n, f, \bar{n}, \bar{f}))$, $p \in M^6$, $(y; n, f, \bar{n}, \bar{f}) \in \mathfrak{F}$ where $(y; n, f, \bar{n}, \bar{f})$ is adapted at p. We call $\mathfrak{F}_X(M)$ *the adapted frame bundle of the immersion* X: $M^6 \to \mathbf{O}$. We have a commutative diagram:

$$\begin{tikzcd} \mathfrak{F}(M) \arrow[r, "\hat{X}"] \arrow[d, "p"] & \mathfrak{F} \arrow[d, "x"] \\ M \arrow[r, "X"] & \mathbf{O} \end{tikzcd}$$

We see that p: $\mathfrak{F}_X(M) \to M$ is a right $U(3)$-bundle over M which may be regarded as a subbundle of the $GL(6, \mathbf{R})$ bundle of the tangential frames of M. We simply refer to this G-structure as the $U(3)$-structure on M induced by the immersion X: $M \to \mathbf{O}$. The reader should be aware that other authors have called such structures "almost hermitian".

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The forms on $\mathfrak{F}$ pullback under $\tilde{X}^*$ to give forms on $\mathfrak{F}_X(M)$ which we continue to denote by the same letters. The following basic theorem follows immediately from the theory of moving frames and the structure equations of $\mathbf{O}$ (see §1, (1.35), (1.36), (1.40), (1.41)).

Theorem 3.1. *Let X: $M^6 \to \mathbf{O}$ be an oriented immersion and let p: $\mathfrak{F}_X(M) \to M$ be the adapted frame bundle. Then M inherits a $U(3)$-structure where $\mathfrak{F}_X(M)$ is the bundle of unitary frames and whose features are described as follows:*

(i) $v = \bar{v} = 0$ *on* $\mathfrak{F}_X(M)$.

(ii) *A form* $\alpha \in \Omega^1_{\mathbf{C}}(M)$ *is of type* (1,0) *if and only if* $p^*(\alpha) \equiv 0 \bmod(\omega^1, \omega^2, \omega^3)$.

(iii) *A canonical 2-form,* Ω, *of type* (1,1) *is associated to the* $U(3)$*-structure and is characterized by the condition* $p^*(\Omega) = (i/2)^t\omega \wedge \bar{\omega}$.

(iv) *The metric g on M induced by X from* $\mathbf{O}$ *satisfies* $p^*(g) = {}^t\omega \circ \bar{\omega}$.

(v) *The structure equations hold:*

$$dx = f\omega + \bar{f}\bar{\omega}, \tag{3.1}$$

$$dn = ni\rho + f\mathfrak{h} + \bar{f}\bar{\theta}, \tag{3.2}$$

$$df = -n^t\mathfrak{h} + f\kappa - \bar{n}^t\bar{\theta} + \bar{f}\,[\theta], \tag{3.3}$$

(and the equations gotten from these by conjugation).

Of course, a $U(3)$-structure has many invariants and those $U(3)$-structures which satisfy extra conditions are of particular interest. Among these, the most important for us will be the following: A $U(3)$-structure on M will be said to be

(i) *complex* if the underlying almost complex structure is integrable to a complex structure (by the Newlander-Nirenberg theorem, this is equivalent to the condition $d\alpha \equiv 0 \bmod \Omega^{1,0}(M)$ for all $\alpha \in \Omega^{1,0}(M)$; see [7]);

(ii) *symplectic* if the canonical two-form Ω is closed;

(iii) *co-symplectic* if Ω is co-closed, i.e., $\delta\Omega = 0$ (this is equivalent to either of

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(ii) *symplectic* if the canonical two-form Ω is closed;

(iii) *co-symplectic* if Ω is co-closed, i.e., $\delta\Omega = 0$ (this is equivalent to either of the conditions $d\Omega^2 = 0$ or $d * \Omega = 0$);

(iv) *Kähler* if it is both complex and symplectic;

(v) *co-Kähler* if it is both complex and co-symplectic.

Note that symplectic implies co-symplectic, but not conversely (see below). Complex $U(3)$-structures are often called "Hermitian".[3]

Our analysis of $U(3)$-structures induced by oriented immersions $X\colon M^6 \to \mathbf{O}$ begins with the second fundamental form. If we differentiate the equation $\nu = 0$ on $\mathfrak{F}_X(M)$, the structure equations (1.41) give

$$^t\bar{\mathfrak{h}} \wedge \omega + {}^t\theta \wedge \bar{\omega} = 0. \tag{3.4}$$

Applying Cartan's Lemma, we conclude that there exist 3×3 matrices of functions, A, B, C on $\mathfrak{F}_X(M)$ (with complex values) satisfying

$$A = {}^tA, \quad C = {}^tC, \tag{3.5}$$

$$\begin{pmatrix} \mathfrak{h} \\ \theta \end{pmatrix} = \begin{pmatrix} \bar{B} & \bar{A} \\ {}^tB & \bar{C} \end{pmatrix} \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}. \tag{3.6}$$

Using these formulae, we easily compute the second fundamental form of $X\colon M^6 \to \mathbf{O}$ as an Euclidean immersion as

$$\mathrm{II} = -2\,\mathrm{Re}\{({}^t\bar{\mathfrak{h}} \circ \omega + {}^t\theta \circ \bar{\omega})n\}. \tag{3.7}$$

Classically, one views II as a linear map $\mathrm{II}\colon S^2(TM) \to NM$ where TM is the tangent bundle of the immersion X. Using the almost complex structure on M and the orientation of the 2-plane bundle N_xM, we have canonical splittings

$$\mathbf{C} \otimes_{\mathbf{R}} S^2(TM) = S^{2,0}_{\mathbf{C}}(M) \oplus S^{1,1}_{\mathbf{C}}(M) \oplus S^{0,2}_{\mathbf{C}}(M),$$
$$\mathbf{C} \otimes_{\mathbf{R}} NM = N^{1,0}M \oplus N^{0,1}M,$$

where the bundles on the right are complex vector bundles over M. For example, $S^{2,0}_{\mathbf{C},q}(M)$ for $q \in M$ is spanned by products of the form $e_1 \circ e_2$ where

e_1 and e_2 are (1,0) vectors in $T_{\mathbf{C},q}M$. If we extend II complex linearly to a map $\mathbf{C} \otimes_{\mathbf{R}} S^2(TM) \to \mathbf{C} \otimes_{\mathbf{R}} NM$, and split it into components via the above splittings, we see that II has three independent pieces, the rest being determined by symmetry and reality of II. These components are $\mathrm{II}^{2,0}\colon S^{2,0}(M) \to (M) \to N^{1,0}M$ given on $\mathfrak{F}_X(M)$ by

$$II^{2,0} = (-{}^t\omega \circ A\omega)n, \tag{3.7a}$$

$\mathrm{II}^{1,1}\colon S^{1,1}_{\mathbf{C}}(M) \to N^{1,0}M$ given by

$$\mathrm{II}^{1,1} = (-{}^t\bar{\omega} \circ {}^tB\omega - {}^t\omega \circ B\bar{\omega})n, \tag{3.7b}$$

and $\mathrm{II}^{0,2}\colon S^{0,2}_{\mathbf{C}}(M) \to N^{1,0}M$ given by

$$\mathrm{II}^{0,2} = -({}^t\bar{\omega} \circ \bar{C}\bar{\omega})n. \tag{3.7c}$$

From this, one easily computes the trace of II with respect to the first fundamental form $\mathrm{I} = {}^t\omega \circ \bar{\omega}$ as

$$\mathbf{C} \otimes_{\mathbf{R}} S^2(TM) = S_{\mathbf{C}}^{2,0}(M) \oplus S_{\mathbf{C}}^{1,1}(M) \oplus S_{\mathbf{C}}^{0,2}(M),$$
$$\mathbf{C} \otimes_{\mathbf{R}} NM = N^{1,0}M \oplus N^{0,1}M,$$

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$\mathrm{II}^{1,1}$: $S_{\mathbf{C}}^{1,1}(M) \to N^{1,0}M$ given by

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and $\mathrm{II}^{0,2}$: $S_{\mathbf{C}}^{0,2}(M) \to N^{1,0}M$ given by

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From this, one easily computes the trace of II with respect to the first fundamental form $\mathrm{I} = {}^t\omega \circ \overline{\omega}$ as

(3.8) $$H = \tfrac{1}{6}\mathrm{tr}_{\mathrm{I}}\,\mathrm{II} = -\tfrac{1}{3}(\mathrm{tr}\,Bn + \mathrm{tn}\,\overline{B}\overline{n}).$$

H is often called the mean curvature vector of the immersion X. The above discussion gives us a geometric interpretation of the components of II with respect to the $U(3)$-structure. We will now relate these components to the special conditions discussed above for $U(3)$-structures.

Theorem 3.2. *Let X: $M^6 \to \mathbf{O}$ be an immersion of the oriented manifold M^6. The induced $U(3)$-structure is complex if and only if $B = 0$.*

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eLazt

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$$\frac{-2 M Z[6][z,t] + H\left(Z[11]^{(0,1)}[z,t] - 6 Z[14]^{(1,0)}[z,t]\right)}{H^2},$$

$$-\frac{2 M Z[7][z,t] + H Z[10]^{(0,1)}[z,t] + 6 H Z[15]^{(1,0)}[z,t]}{H^2},$$

$$\frac{-2 M Z[0][z,t] + H\left(Z[13]^{(0,1)}[z,t] - 6 Z[8]^{(1,0)}[z,t]\right)}{H^2},$$

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$$-\frac{2 M Z[2][z,t] + H Z[15]^{(0,1)}[z,t] + 6 H Z[10]^{(1,0)}[z,t]}{H^2},$$

$$\frac{-2 M Z[3][z,t] + H\left(Z[14]^{(0,1)}[z,t] - 6 Z[11]^{(1,0)}[z,t]\right)}{H^2},$$

$$\frac{2 M Z[12][z,t] - H Z[1]^{(0,1)}[z,t] + 6 H Z[4]^{(1,0)}[z,t]}{H^2},$$

$$\frac{2 M Z[13][z,t] + H Z[0]^{(0,1)}[z,t] + 6 H Z[5]^{(1,0)}[z,t]}{H^2},$$

$$\frac{2 M Z[14][z,t] + H Z[3]^{(0,1)}[z,t] + 6 H Z[6]^{(1,0)}[z,t]}{H^2},$$

$$\frac{2 M Z[15][z,t] - H Z[2]^{(0,1)}[z,t] + 6 H Z[7]^{(1,0)}[z,t]}{H^2},$$

$$\frac{2 M Z[8][z,t] + H Z[5]^{(0,1)}[z,t] + 6 H Z[0]^{(1,0)}[z,t]}{H^2},$$

$$\frac{2 M Z[9][z,t] - H Z[4]^{(0,1)}[z,t] + 6 H Z[1]^{(1,0)}[z,t]}{H^2},$$

$$\frac{2 M Z[10][z,t] - H Z[7]^{(0,1)}[z,t] + 6 H Z[2]^{(1,0)}[z,t]}{H^2},$$

$$\left.\left.\frac{2 M Z[11][z,t] + H Z[6]^{(0,1)}[z,t] + 6 H Z[3]^{(1,0)}[z,t]}{H^2}\right\}\right\}$$

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$$\frac{-2 M Z[5][z,t]+H\left(Z[8]^{(0,1)}[z,t]-6 Z[13]^{(1,0)}[z,t]\right)}{H^2}$$

$$\frac{-2 M Z[6][z,t]+H\left(Z[11]^{(0,1)}[z,t]-6 Z[14]^{(1,0)}[z,t]\right)}{H^2}$$

$$-\frac{2 M Z[7][z,t]+H Z[10]^{(0,1)}[z,t]+6 H Z[15]^{(1,0)}[z,t]}{H^2}$$

$$\frac{-2 M Z[0][z,t]+H\left(Z[13]^{(0,1)}[z,t]-6 Z[8]^{(1,0)}[z,t]\right)}{H^2}$$

$$-\frac{2 M Z[1][z,t]+H Z[12]^{(0,1)}[z,t]+6 H Z[9]^{(1,0)}[z,t]}{H^2}$$

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$$\frac{2 M Z[15][z,t]-H Z[2]^{(0,1)}[z,t]+6 H Z[7]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2 M Z[8][z,t]+H Z[5]^{(0,1)}[z,t]+6 H Z[0]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2 M Z[9][z,t]-H Z[4]^{(0,1)}[z,t]+6 H Z[1]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2 M Z[10][z,t]-H Z[7]^{(0,1)}[z,t]+6 H Z[2]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2 M Z[11][z,t]+H Z[6]^{(0,1)}[z,t]+6 H Z[3]^{(1,0)}[z,t]}{H^2}$$

In[669]:=

```
varZ = Z[#] & /@ Range[0, 15]
```

Out[669]=

```
{Z[0], Z[1], Z[2], Z[3], Z[4], Z[5], Z[6],
 Z[7], Z[8], Z[9], Z[10], Z[11], Z[12], Z[13], Z[14], Z[15]}
```

In[670]:=

```
varZzt = #[z, t] & /@ varZ
```

Out[670]=

```
{Z[0][z, t], Z[1][z, t], Z[2][z, t], Z[3][z, t], Z[4][z, t],
 Z[5][z, t], Z[6][z, t], Z[7][z, t], Z[8][z, t], Z[9][z, t], Z[10][z, t],
 Z[11][z, t], Z[12][z, t], Z[13][z, t], Z[14][z, t], Z[15][z, t]}
```

In[671]:=

```
DzvarZ = D[varZzt, z]
```

Out[671]=

```
{Z[0]^(1,0)[z, t], Z[1]^(1,0)[z, t], Z[2]^(1,0)[z, t], Z[3]^(1,0)[z, t],
 Z[4]^(1,0)[z, t], Z[5]^(1,0)[z, t], Z[6]^(1,0)[z, t], Z[7]^(1,0)[z, t],
 Z[8]^(1,0)[z, t], Z[9]^(1,0)[z, t], Z[10]^(1,0)[z, t], Z[11]^(1,0)[z, t],
 Z[12]^(1,0)[z, t], Z[13]^(1,0)[z, t], Z[14]^(1,0)[z, t], Z[15]^(1,0)[z, t]}
```

In[672]:=

```
DtvarZ = D[varZzt, t]
```

Out[672]=

$$\{Z[0]^{(0,1)}[z,t], Z[1]^{(0,1)}[z,t], Z[2]^{(0,1)}[z,t], Z[3]^{(0,1)}[z,t], Z[4]^{(0,1)}[z,t], Z[5]^{(0,1)}[z,t], Z[6]^{(0,1)}[z,t], Z[7]^{(0,1)}[z,t], Z[8]^{(0,1)}[z,t], Z[9]^{(0,1)}[z,t], Z[10]^{(0,1)}[z,t], Z[11]^{(0,1)}[z,t], Z[12]^{(0,1)}[z,t], Z[13]^{(0,1)}[z,t], Z[14]^{(0,1)}[z,t], Z[15]^{(0,1)}[z,t]\}$$

In[673]:=

```
(DtvarZsubs = Solve[And @@ Thread[0 == eLazt], DtvarZ][[1]] //
    FullSimplify[#, constraintVars] &) // Column
DtvarZEQS = %[[1]] /. {Rule → Equal}
DtvarZrelations = DtvarZsubs /. {Rule → Subtract}
```

Out[673]=

$$Z[0]^{(0,1)}[z,t] \to -\frac{2MZ[13][z,t]}{H} - 6Z[5]^{(1,0)}[z,t]$$
$$Z[1]^{(0,1)}[z,t] \to \frac{2MZ[12][z,t]}{H} + 6Z[4]^{(1,0)}[z,t]$$
$$Z[2]^{(0,1)}[z,t] \to \frac{2MZ[15][z,t]}{H} + 6Z[7]^{(1,0)}[z,t]$$
$$Z[3]^{(0,1)}[z,t] \to -\frac{2MZ[14][z,t]}{H} - 6Z[6]^{(1,0)}[z,t]$$
$$Z[4]^{(0,1)}[z,t] \to \frac{2MZ[9][z,t]}{H} + 6Z[1]^{(1,0)}[z,t]$$
$$Z[5]^{(0,1)}[z,t] \to -\frac{2MZ[8][z,t]}{H} - 6Z[0]^{(1,0)}[z,t]$$
$$Z[6]^{(0,1)}[z,t] \to -\frac{2MZ[11][z,t]}{H} - 6Z[3]^{(1,0)}[z,t]$$
$$Z[7]^{(0,1)}[z,t] \to \frac{2MZ[10][z,t]}{H} + 6Z[2]^{(1,0)}[z,t]$$
$$Z[8]^{(0,1)}[z,t] \to \frac{2MZ[5][z,t]}{H} + 6Z[13]^{(1,0)}[z,t]$$
$$Z[9]^{(0,1)}[z,t] \to -\frac{2MZ[4][z,t]}{H} - 6Z[12]^{(1,0)}[z,t]$$
$$Z[10]^{(0,1)}[z,t] \to -\frac{2MZ[7][z,t]}{H} - 6Z[15]^{(1,0)}[z,t]$$
$$Z[11]^{(0,1)}[z,t] \to \frac{2MZ[6][z,t]}{H} + 6Z[14]^{(1,0)}[z,t]$$
$$Z[12]^{(0,1)}[z,t] \to -\frac{2MZ[1][z,t]}{H} - 6Z[9]^{(1,0)}[z,t]$$
$$Z[13]^{(0,1)}[z,t] \to \frac{2MZ[0][z,t]}{H} + 6Z[8]^{(1,0)}[z,t]$$
$$Z[14]^{(0,1)}[z,t] \to \frac{2MZ[3][z,t]}{H} + 6Z[11]^{(1,0)}[z,t]$$
$$Z[15]^{(0,1)}[z,t] \to -\frac{2MZ[2][z,t]}{H} - 6Z[10]^{(1,0)}[z,t]$$

Out[674]=

$$\left\{Z[0]^{(0,1)}[z,t] == -\frac{2\,M\,Z[13][z,t]}{H} - 6\,Z[5]^{(1,0)}[z,t],\right.$$

$$Z[1]^{(0,1)}[z,t] == \frac{2\,M\,Z[12][z,t]}{H} + 6\,Z[4]^{(1,0)}[z,t],$$

$$Z[2]^{(0,1)}[z,t] == \frac{2\,M\,Z[15][z,t]}{H} + 6\,Z[7]^{(1,0)}[z,t],$$

$$Z[3]^{(0,1)}[z,t] == -\frac{2\,M\,Z[14][z,t]}{H} - 6\,Z[6]^{(1,0)}[z,t],$$

$$Z[4]^{(0,1)}[z,t] == \frac{2\,M\,Z[9][z,t]}{H} + 6\,Z[1]^{(1,0)}[z,t],$$

$$Z[5]^{(0,1)}[z,t] == -\frac{2\,M\,Z[8][z,t]}{H} - 6\,Z[0]^{(1,0)}[z,t],$$

$$Z[6]^{(0,1)}[z,t] == -\frac{2\,M\,Z[11][z,t]}{H} - 6\,Z[3]^{(1,0)}[z,t],$$

$$Z[7]^{(0,1)}[z,t] == \frac{2\,M\,Z[10][z,t]}{H} + 6\,Z[2]^{(1,0)}[z,t],$$

$$Z[8]^{(0,1)}[z,t] == \frac{2\,M\,Z[5][z,t]}{H} + 6\,Z[13]^{(1,0)}[z,t],$$

$$Z[9]^{(0,1)}[z,t] == -\frac{2\,M\,Z[4][z,t]}{H} - 6\,Z[12]^{(1,0)}[z,t],$$

$$Z[10]^{(0,1)}[z,t] == -\frac{2\,M\,Z[7][z,t]}{H} - 6\,Z[15]^{(1,0)}[z,t],$$

$$Z[11]^{(0,1)}[z,t] == \frac{2\,M\,Z[6][z,t]}{H} + 6\,Z[14]^{(1,0)}[z,t],$$

$$Z[12]^{(0,1)}[z,t] == -\frac{2\,M\,Z[1][z,t]}{H} - 6\,Z[9]^{(1,0)}[z,t],$$

$$Z[13]^{(0,1)}[z,t] == \frac{2\,M\,Z[0][z,t]}{H} + 6\,Z[8]^{(1,0)}[z,t],$$

$$Z[14]^{(0,1)}[z,t] == \frac{2\,M\,Z[3][z,t]}{H} + 6\,Z[11]^{(1,0)}[z,t],$$

$$\left.Z[15]^{(0,1)}[z,t] == -\frac{2\,M\,Z[2][z,t]}{H} - 6\,Z[10]^{(1,0)}[z,t]\right\}$$

Out[675]=

$$\left\{\frac{2\,M\,Z[13][z,t]}{H}+Z[0]^{(0,1)}[z,t]+6\,Z[5]^{(1,0)}[z,t],\right.$$
$$-\frac{2\,M\,Z[12][z,t]}{H}+Z[1]^{(0,1)}[z,t]-6\,Z[4]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,Z[15][z,t]}{H}+Z[2]^{(0,1)}[z,t]-6\,Z[7]^{(1,0)}[z,t],$$
$$\frac{2\,M\,Z[14][z,t]}{H}+Z[3]^{(0,1)}[z,t]+6\,Z[6]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,Z[9][z,t]}{H}+Z[4]^{(0,1)}[z,t]-6\,Z[1]^{(1,0)}[z,t],$$
$$\frac{2\,M\,Z[8][z,t]}{H}+Z[5]^{(0,1)}[z,t]+6\,Z[0]^{(1,0)}[z,t],$$
$$\frac{2\,M\,Z[11][z,t]}{H}+Z[6]^{(0,1)}[z,t]+6\,Z[3]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,Z[10][z,t]}{H}+Z[7]^{(0,1)}[z,t]-6\,Z[2]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,Z[5][z,t]}{H}+Z[8]^{(0,1)}[z,t]-6\,Z[13]^{(1,0)}[z,t],$$
$$\frac{2\,M\,Z[4][z,t]}{H}+Z[9]^{(0,1)}[z,t]+6\,Z[12]^{(1,0)}[z,t],$$
$$\frac{2\,M\,Z[7][z,t]}{H}+Z[10]^{(0,1)}[z,t]+6\,Z[15]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,Z[6][z,t]}{H}+Z[11]^{(0,1)}[z,t]-6\,Z[14]^{(1,0)}[z,t],$$
$$\frac{2\,M\,Z[1][z,t]}{H}+Z[12]^{(0,1)}[z,t]+6\,Z[9]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,Z[0][z,t]}{H}+Z[13]^{(0,1)}[z,t]-6\,Z[8]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,Z[3][z,t]}{H}+Z[14]^{(0,1)}[z,t]-6\,Z[11]^{(1,0)}[z,t],$$
$$\left.\frac{2\,M\,Z[2][z,t]}{H}+Z[15]^{(0,1)}[z,t]+6\,Z[10]^{(1,0)}[z,t]\right\}$$

In[676]:=

```
(*ToString[FullForm[#]]&/@DtvarZsubs;
 StringReplace[#,"Rule"→ "Subtract"]&/@%;
 (DtvarZrelations=ToExpression[#]&/@%)//Column*)
```

WRITE OUT THE VARIABLES THAT ARE COUPLED TO ANOTHER VARIABLE :

In[677]:=

```
eLaztCouplings = eLaCouplings
```

Out[677]=

{{0, 5, 8, 13}, {1, 4, 9, 12}, {2, 7, 10, 15}, {3, 6, 11, 14}}

SEPARATE VARIABLES THAT ARE COUPLED TO EACH OTHER (reorder) :

In[678]:=

```
yZdef = yZ[#] & /@ Range[0, 15]
```

Out[678]=

```
{yZ[0], yZ[1], yZ[2], yZ[3], yZ[4], yZ[5], yZ[6], yZ[7],
 yZ[8], yZ[9], yZ[10], yZ[11], yZ[12], yZ[13], yZ[14], yZ[15]}
```

In[679]:=

```
Zdef = Z[#] & /@ Range[0, 15]
```

Out[679]=

```
{Z[0], Z[1], Z[2], Z[3], Z[4], Z[5], Z[6],
 Z[7], Z[8], Z[9], Z[10], Z[11], Z[12], Z[13], Z[14], Z[15]}
```

In[680]:=

```
sZtOyZ = Z[#[[1]]] → yZ[#[[2]]] & /@ Transpose[{Flatten[eLaztCouplings], Range[0, 15]}]
syZtoZ = Solve[And @@ (sZtOyZ /. {Rule → Equal}), (yZ[#] & /@ Range[0, 15])][[-1]]
zeroZyZforCaExpression = -sZtOyZ /. {Rule → Subtract}
zeroZyZEQS = sZtOyZ /. {Rule → Equal}
```

Out[680]=

```
{Z[0] → yZ[0], Z[5] → yZ[1], Z[8] → yZ[2], Z[13] → yZ[3], Z[1] → yZ[4], Z[4] → yZ[5],
 Z[9] → yZ[6], Z[12] → yZ[7], Z[2] → yZ[8], Z[7] → yZ[9], Z[10] → yZ[10],
 Z[15] → yZ[11], Z[3] → yZ[12], Z[6] → yZ[13], Z[11] → yZ[14], Z[14] → yZ[15]}
```

Out[681]=

```
{yZ[0] → Z[0], yZ[1] → Z[5], yZ[2] → Z[8], yZ[3] → Z[13], yZ[4] → Z[1], yZ[5] → Z[4],
 yZ[6] → Z[9], yZ[7] → Z[12], yZ[8] → Z[2], yZ[9] → Z[7], yZ[10] → Z[10],
 yZ[11] → Z[15], yZ[12] → Z[3], yZ[13] → Z[6], yZ[14] → Z[11], yZ[15] → Z[14]}
```

Out[682]=

```
{yZ[0] - Z[0], yZ[1] - Z[5], yZ[2] - Z[8], yZ[3] - Z[13], yZ[4] - Z[1],
 yZ[5] - Z[4], yZ[6] - Z[9], yZ[7] - Z[12], yZ[8] - Z[2], yZ[9] - Z[7], yZ[10] - Z[10],
 yZ[11] - Z[15], yZ[12] - Z[3], yZ[13] - Z[6], yZ[14] - Z[11], yZ[15] - Z[14]}
```

Out[683]=

```
{Z[0] == yZ[0], Z[5] == yZ[1], Z[8] == yZ[2], Z[13] == yZ[3], Z[1] == yZ[4], Z[4] == yZ[5],
 Z[9] == yZ[6], Z[12] == yZ[7], Z[2] == yZ[8], Z[7] == yZ[9], Z[10] == yZ[10],
 Z[15] == yZ[11], Z[3] == yZ[12], Z[6] == yZ[13], Z[11] == yZ[14], Z[14] == yZ[15]}
```

In[684]:=

```
(eLyZ = eLazt /. sZtOyZ // FullSimplify) // Column
```

Out[684]=

$$-\frac{2\,M\,yZ[5][z,t]+H\,yZ[6]^{(0,1)}[z,t]+6\,H\,yZ[7]^{(1,0)}[z,t]}{H^2}$$

$$\frac{-2\,M\,yZ[1][z,t]+H\left(yZ[2]^{(0,1)}[z,t]-6\,yZ[3]^{(1,0)}[z,t]\right)}{H^2}$$

$$\frac{-2\,M\,yZ[13][z,t]+H\left(yZ[14]^{(0,1)}[z,t]-6\,yZ[15]^{(1,0)}[z,t]\right)}{H^2}$$

$$-\frac{2\,M\,yZ[9][z,t]+H\,yZ[10]^{(0,1)}[z,t]+6\,H\,yZ[11]^{(1,0)}[z,t]}{H^2}$$

$$\frac{-2\,M\,yZ[0][z,t]+H\left(yZ[3]^{(0,1)}[z,t]-6\,yZ[2]^{(1,0)}[z,t]\right)}{H^2}$$

$$-\frac{2\,M\,yZ[4][z,t]+H\,yZ[7]^{(0,1)}[z,t]+6\,H\,yZ[6]^{(1,0)}[z,t]}{H^2}$$

$$-\frac{2\,M\,yZ[8][z,t]+H\,yZ[11]^{(0,1)}[z,t]+6\,H\,yZ[10]^{(1,0)}[z,t]}{H^2}$$

$$\frac{-2\,M\,yZ[12][z,t]+H\left(yZ[15]^{(0,1)}[z,t]-6\,yZ[14]^{(1,0)}[z,t]\right)}{H^2}$$

$$\frac{2\,M\,yZ[7][z,t]-H\,yZ[4]^{(0,1)}[z,t]+6\,H\,yZ[5]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2\,M\,yZ[3][z,t]+H\,yZ[0]^{(0,1)}[z,t]+6\,H\,yZ[1]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2\,M\,yZ[15][z,t]+H\,yZ[12]^{(0,1)}[z,t]+6\,H\,yZ[13]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2\,M\,yZ[11][z,t]-H\,yZ[8]^{(0,1)}[z,t]+6\,H\,yZ[9]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2\,M\,yZ[2][z,t]+H\,yZ[1]^{(0,1)}[z,t]+6\,H\,yZ[0]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2\,M\,yZ[6][z,t]-H\,yZ[5]^{(0,1)}[z,t]+6\,H\,yZ[4]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2\,M\,yZ[10][z,t]-H\,yZ[9]^{(0,1)}[z,t]+6\,H\,yZ[8]^{(1,0)}[z,t]}{H^2}$$

$$\frac{2\,M\,yZ[14][z,t]+H\,yZ[13]^{(0,1)}[z,t]+6\,H\,yZ[12]^{(1,0)}[z,t]}{H^2}$$

In[685]:=

```
(DtyZsubsA =
   Solve[And @@ Thread[0 == eLyZ], (D[(yZ[#][z, t] & /@ Range[0, 15]), t])][[1]] //
    FullSimplify[#, constraintVars] &) // Column
```

Out[685]=

$$yZ[0]^{(0,1)}[z,t] \to -\frac{2 M yZ[3][z,t]}{H} - 6\, yZ[1]^{(1,0)}[z,t]$$
$$yZ[1]^{(0,1)}[z,t] \to -\frac{2 M yZ[2][z,t]}{H} - 6\, yZ[0]^{(1,0)}[z,t]$$
$$yZ[2]^{(0,1)}[z,t] \to \frac{2 M yZ[1][z,t]}{H} + 6\, yZ[3]^{(1,0)}[z,t]$$
$$yZ[3]^{(0,1)}[z,t] \to \frac{2 M yZ[0][z,t]}{H} + 6\, yZ[2]^{(1,0)}[z,t]$$
$$yZ[4]^{(0,1)}[z,t] \to \frac{2 M yZ[7][z,t]}{H} + 6\, yZ[5]^{(1,0)}[z,t]$$
$$yZ[5]^{(0,1)}[z,t] \to \frac{2 M yZ[6][z,t]}{H} + 6\, yZ[4]^{(1,0)}[z,t]$$
$$yZ[6]^{(0,1)}[z,t] \to -\frac{2 M yZ[5][z,t]}{H} - 6\, yZ[7]^{(1,0)}[z,t]$$
$$yZ[7]^{(0,1)}[z,t] \to -\frac{2 M yZ[4][z,t]}{H} - 6\, yZ[6]^{(1,0)}[z,t]$$
$$yZ[8]^{(0,1)}[z,t] \to \frac{2 M yZ[11][z,t]}{H} + 6\, yZ[9]^{(1,0)}[z,t]$$
$$yZ[9]^{(0,1)}[z,t] \to \frac{2 M yZ[10][z,t]}{H} + 6\, yZ[8]^{(1,0)}[z,t]$$
$$yZ[10]^{(0,1)}[z,t] \to -\frac{2 M yZ[9][z,t]}{H} - 6\, yZ[11]^{(1,0)}[z,t]$$
$$yZ[11]^{(0,1)}[z,t] \to -\frac{2 M yZ[8][z,t]}{H} - 6\, yZ[10]^{(1,0)}[z,t]$$
$$yZ[12]^{(0,1)}[z,t] \to -\frac{2 M yZ[15][z,t]}{H} - 6\, yZ[13]^{(1,0)}[z,t]$$
$$yZ[13]^{(0,1)}[z,t] \to -\frac{2 M yZ[14][z,t]}{H} - 6\, yZ[12]^{(1,0)}[z,t]$$
$$yZ[14]^{(0,1)}[z,t] \to \frac{2 M yZ[13][z,t]}{H} + 6\, yZ[15]^{(1,0)}[z,t]$$
$$yZ[15]^{(0,1)}[z,t] \to \frac{2 M yZ[12][z,t]}{H} + 6\, yZ[14]^{(1,0)}[z,t]$$

In[686]:=

```
DtvaryZEQS = DtyZsubsA /. {Rule → Equal};
zeroDtyZeqs = DtyZsubsA /. {Rule → Subtract}
coupledyZeqs = Partition[DtvaryZEQS, 4]
```

Out[687]=

$$\left\{\frac{2\,M\,yZ[3][z,t]}{H}+yZ[0]^{(0,1)}[z,t]+6\,yZ[1]^{(1,0)}[z,t],\right.$$
$$\frac{2\,M\,yZ[2][z,t]}{H}+yZ[1]^{(0,1)}[z,t]+6\,yZ[0]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,yZ[1][z,t]}{H}+yZ[2]^{(0,1)}[z,t]-6\,yZ[3]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,yZ[0][z,t]}{H}+yZ[3]^{(0,1)}[z,t]-6\,yZ[2]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,yZ[7][z,t]}{H}+yZ[4]^{(0,1)}[z,t]-6\,yZ[5]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,yZ[6][z,t]}{H}+yZ[5]^{(0,1)}[z,t]-6\,yZ[4]^{(1,0)}[z,t],$$
$$\frac{2\,M\,yZ[5][z,t]}{H}+yZ[6]^{(0,1)}[z,t]+6\,yZ[7]^{(1,0)}[z,t],$$
$$\frac{2\,M\,yZ[4][z,t]}{H}+yZ[7]^{(0,1)}[z,t]+6\,yZ[6]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,yZ[11][z,t]}{H}+yZ[8]^{(0,1)}[z,t]-6\,yZ[9]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,yZ[10][z,t]}{H}+yZ[9]^{(0,1)}[z,t]-6\,yZ[8]^{(1,0)}[z,t],$$
$$\frac{2\,M\,yZ[9][z,t]}{H}+yZ[10]^{(0,1)}[z,t]+6\,yZ[11]^{(1,0)}[z,t],$$
$$\frac{2\,M\,yZ[8][z,t]}{H}+yZ[11]^{(0,1)}[z,t]+6\,yZ[10]^{(1,0)}[z,t],$$
$$\frac{2\,M\,yZ[15][z,t]}{H}+yZ[12]^{(0,1)}[z,t]+6\,yZ[13]^{(1,0)}[z,t],$$
$$\frac{2\,M\,yZ[14][z,t]}{H}+yZ[13]^{(0,1)}[z,t]+6\,yZ[12]^{(1,0)}[z,t],$$
$$-\frac{2\,M\,yZ[13][z,t]}{H}+yZ[14]^{(0,1)}[z,t]-6\,yZ[15]^{(1,0)}[z,t],$$
$$\left.-\frac{2\,M\,yZ[12][z,t]}{H}+yZ[15]^{(0,1)}[z,t]-6\,yZ[14]^{(1,0)}[z,t]\right\}$$

Out[688]=

$$\left\{\left\{yZ[0]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[3][z,t]}{H} - 6\,yZ[1]^{(1,0)}[z,t],\right.\right.$$
$$yZ[1]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[2][z,t]}{H} - 6\,yZ[0]^{(1,0)}[z,t],$$
$$yZ[2]^{(0,1)}[z,t] == \frac{2\,M\,yZ[1][z,t]}{H} + 6\,yZ[3]^{(1,0)}[z,t],$$
$$\left.yZ[3]^{(0,1)}[z,t] == \frac{2\,M\,yZ[0][z,t]}{H} + 6\,yZ[2]^{(1,0)}[z,t]\right\},$$
$$\left\{yZ[4]^{(0,1)}[z,t] == \frac{2\,M\,yZ[7][z,t]}{H} + 6\,yZ[5]^{(1,0)}[z,t],\right.$$
$$yZ[5]^{(0,1)}[z,t] == \frac{2\,M\,yZ[6][z,t]}{H} + 6\,yZ[4]^{(1,0)}[z,t],$$
$$yZ[6]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[5][z,t]}{H} - 6\,yZ[7]^{(1,0)}[z,t],$$
$$\left.yZ[7]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[4][z,t]}{H} - 6\,yZ[6]^{(1,0)}[z,t]\right\},$$
$$\left\{yZ[8]^{(0,1)}[z,t] == \frac{2\,M\,yZ[11][z,t]}{H} + 6\,yZ[9]^{(1,0)}[z,t],\right.$$
$$yZ[9]^{(0,1)}[z,t] == \frac{2\,M\,yZ[10][z,t]}{H} + 6\,yZ[8]^{(1,0)}[z,t],$$
$$yZ[10]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[9][z,t]}{H} - 6\,yZ[11]^{(1,0)}[z,t],$$
$$\left.yZ[11]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[8][z,t]}{H} - 6\,yZ[10]^{(1,0)}[z,t]\right\},$$
$$\left\{yZ[12]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[15][z,t]}{H} - 6\,yZ[13]^{(1,0)}[z,t],\right.$$
$$yZ[13]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[14][z,t]}{H} - 6\,yZ[12]^{(1,0)}[z,t],$$
$$yZ[14]^{(0,1)}[z,t] == \frac{2\,M\,yZ[13][z,t]}{H} + 6\,yZ[15]^{(1,0)}[z,t],$$
$$\left.\left.yZ[15]^{(0,1)}[z,t] == \frac{2\,M\,yZ[12][z,t]}{H} + 6\,yZ[14]^{(1,0)}[z,t]\right\}\right\}$$

In[689]:=

```
(*DtvaryZEQS=DtyZsubs/.{Rule→Equal};
 zeroDtyZeqs=DtyZsubs/.{Rule→Subtract}
 coupledyZeqs=Partition[DtvaryZEQS,4]*)
```

DSolve

In[690]:=

```
Clear[yZ, yZvar];
yZvar = (yZ[#][z, t] & /@ Range[0, 15])
```

Out[691]=

```
{yZ[0][z, t], yZ[1][z, t], yZ[2][z, t], yZ[3][z, t], yZ[4][z, t],
 yZ[5][z, t], yZ[6][z, t], yZ[7][z, t], yZ[8][z, t], yZ[9][z, t], yZ[10][z, t],
 yZ[11][z, t], yZ[12][z, t], yZ[13][z, t], yZ[14][z, t], yZ[15][z, t]}
```

In[692]:=

```
DSolve[coupledyZeqs[[1]], yZvar[[1 ;; 4]], {z, t}]
```

Out[692]=

$$\text{DSolve}\left[\left\{yZ[0]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[3][z,t]}{H} - 6\,yZ[1]^{(1,0)}[z,t],\right.\right.$$
$$yZ[1]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[2][z,t]}{H} - 6\,yZ[0]^{(1,0)}[z,t],$$
$$yZ[2]^{(0,1)}[z,t] == \frac{2\,M\,yZ[1][z,t]}{H} + 6\,yZ[3]^{(1,0)}[z,t],$$
$$\left.yZ[3]^{(0,1)}[z,t] == \frac{2\,M\,yZ[0][z,t]}{H} + 6\,yZ[2]^{(1,0)}[z,t]\right\},$$
$$\left.\{yZ[0][z,t], yZ[1][z,t], yZ[2][z,t], yZ[3][z,t]\}, \{z,t\}\right]$$

In[693]:=

```
DSolve[coupledyZeqs[[2]], yZvar[[5 ;; 8]], {z, t}]
```

Out[693]=

$$\text{DSolve}\left[\left\{yZ[4]^{(0,1)}[z,t] == \frac{2\,M\,yZ[7][z,t]}{H} + 6\,yZ[5]^{(1,0)}[z,t],\right.\right.$$
$$yZ[5]^{(0,1)}[z,t] == \frac{2\,M\,yZ[6][z,t]}{H} + 6\,yZ[4]^{(1,0)}[z,t],$$
$$yZ[6]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[5][z,t]}{H} - 6\,yZ[7]^{(1,0)}[z,t],$$
$$\left.yZ[7]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[4][z,t]}{H} - 6\,yZ[6]^{(1,0)}[z,t]\right\},$$
$$\left.\{yZ[4][z,t], yZ[5][z,t], yZ[6][z,t], yZ[7][z,t]\}, \{z,t\}\right]$$

In[694]:=

```
DSolve[coupledyZeqs[[3]], yZvar[[9 ;; 12]], {z, t}]
```

Out[694]=

$$\text{DSolve}\left[\left\{yZ[8]^{(0,1)}[z,t] == \frac{2\,M\,yZ[11][z,t]}{H} + 6\,yZ[9]^{(1,0)}[z,t],\right.\right.$$
$$yZ[9]^{(0,1)}[z,t] == \frac{2\,M\,yZ[10][z,t]}{H} + 6\,yZ[8]^{(1,0)}[z,t],$$
$$yZ[10]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[9][z,t]}{H} - 6\,yZ[11]^{(1,0)}[z,t],$$
$$\left.yZ[11]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[8][z,t]}{H} - 6\,yZ[10]^{(1,0)}[z,t]\right\},$$
$$\left.\{yZ[8][z,t], yZ[9][z,t], yZ[10][z,t], yZ[11][z,t]\}, \{z,t\}\right]$$

In[695]:=

```
DSolve[coupledyZeqs[[4]], yZvar[[13 ;; 16]], {z, t}]
```

Out[695]=

$$\text{DSolve}\left[\left\{yZ[12]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[15][z,t]}{H} - 6\,yZ[13]^{(1,0)}[z,t],\right.\right.$$
$$yZ[13]^{(0,1)}[z,t] == -\frac{2\,M\,yZ[14][z,t]}{H} - 6\,yZ[12]^{(1,0)}[z,t],$$
$$yZ[14]^{(0,1)}[z,t] == \frac{2\,M\,yZ[13][z,t]}{H} + 6\,yZ[15]^{(1,0)}[z,t],$$
$$\left.yZ[15]^{(0,1)}[z,t] == \frac{2\,M\,yZ[12][z,t]}{H} + 6\,yZ[14]^{(1,0)}[z,t]\right\},$$
$$\left.\{yZ[12][z,t], yZ[13][z,t], yZ[14][z,t], yZ[15][z,t]\}, \{z,t\}\right]$$

TRY MAPLE : MUST use ` List[Equal[Derivative[0,1][yZ8][z,t],Pl... `

In[696]:=

```
syZzt = yZ[#] → ToExpression["((yZ" <> ToString[#] <> "[#1,#2])&)"] & /@ Range[0, 15]
maplexyZeqs = FullForm[coupledyZeqs /. % // FullSimplify]
```

Out[696]=

```
{yZ[0] → (yZ0[#1, #2] &), yZ[1] → (yZ1[#1, #2] &),
 yZ[2] → (yZ2[#1, #2] &), yZ[3] → (yZ3[#1, #2] &), yZ[4] → (yZ4[#1, #2] &),
 yZ[5] → (yZ5[#1, #2] &), yZ[6] → (yZ6[#1, #2] &), yZ[7] → (yZ7[#1, #2] &),
 yZ[8] → (yZ8[#1, #2] &), yZ[9] → (yZ9[#1, #2] &), yZ[10] → (yZ10[#1, #2] &),
 yZ[11] → (yZ11[#1, #2] &), yZ[12] → (yZ12[#1, #2] &),
 yZ[13] → (yZ13[#1, #2] &), yZ[14] → (yZ14[#1, #2] &), yZ[15] → (yZ15[#1, #2] &)}
```

Out[697]//FullForm=

```
List[List[Equal[Plus[Times[2, Power[H, -1], M, yZ3[z, t]],
      Derivative[0, 1][yZ0][z, t], Times[6, Derivative[1, 0][yZ1][z, t]]], 0],
    Equal[Plus[Times[2, Power[H, -1], M, yZ2[z, t]], Derivative[0, 1][yZ1][z, t],
      Times[6, Derivative[1, 0][yZ0][z, t]]], 0], Equal[Derivative[0, 1][yZ2][z, t],
     Plus[Times[2, Power[H, -1], M, yZ1[z, t]], Times[6, Derivative[1, 0][yZ3][z, t]]]],
    Equal[Derivative[0, 1][yZ3][z, t],
     Plus[Times[2, Power[H, -1], M, yZ0[z, t]], Times[6, Derivative[1, 0][yZ2][z, t]]]]],
   List[Equal[Derivative[0, 1][yZ4][z, t], Plus[Times[2, Power[H, -1], M, yZ7[z, t]],
      Times[6, Derivative[1, 0][yZ5][z, t]]]], Equal[Derivative[0, 1][yZ5][z, t],
     Plus[Times[2, Power[H, -1], M, yZ6[z, t]], Times[6, Derivative[1, 0][yZ4][z, t]]]],
    Equal[Plus[Times[2, Power[H, -1], M, yZ5[z, t]], Derivative[0, 1][yZ6][z, t],
      Times[6, Derivative[1, 0][yZ7][z, t]]], 0],
    Equal[Plus[Times[2, Power[H, -1], M, yZ4[z, t]], Derivative[0, 1][yZ7][z, t],
      Times[6, Derivative[1, 0][yZ6][z, t]]], 0]],
   List[Equal[Derivative[0, 1][yZ8][z, t], Plus[Times[2, Power[H, -1], M, yZ11[z, t]],
      Times[6, Derivative[1, 0][yZ9][z, t]]]], Equal[Derivative[0, 1][yZ9][z, t],
     Plus[Times[2, Power[H, -1], M, yZ10[z, t]], Times[6, Derivative[1, 0][yZ8][z, t]]]],
    Equal[Plus[Times[2, Power[H, -1], M, yZ9[z, t]], Derivative[0, 1][yZ10][z, t],
      Times[6, Derivative[1, 0][yZ11][z, t]]], 0],
    Equal[Plus[Times[2, Power[H, -1], M, yZ8[z, t]], Derivative[0, 1][yZ11][z, t],
      Times[6, Derivative[1, 0][yZ10][z, t]]], 0]],
   List[Equal[Plus[Times[2, Power[H, -1], M, yZ15[z, t]], Derivative[0, 1][yZ12][z, t],
      Times[6, Derivative[1, 0][yZ13][z, t]]], 0],
    Equal[Plus[Times[2, Power[H, -1], M, yZ14[z, t]], Derivative[0, 1][yZ13][z, t],
      Times[6, Derivative[1, 0][yZ12][z, t]]], 0], Equal[Derivative[0, 1][yZ14][z, t],
     Plus[Times[2, Power[H, -1], M, yZ13[z, t]], Times[6, Derivative[1, 0][yZ15][z, t]]]],
    Equal[Derivative[0, 1][yZ15][z, t],
     Plus[Times[2, Power[H, -1], M, yZ12[z, t]], Times[6, Derivative[1, 0][yZ14][z, t]]]]]]
```

In[698]:=

```
mapleyZvars =
 FullForm[yZ[#][z, t] & /@ Range[4 * (# - 1), 4 * (# - 1) + 3] /. syZzt] & /@ Range[4]
```

Out[698]=

```
{List[yZ0[z, t], yZ1[z, t], yZ2[z, t], yZ3[z, t]],
 List[yZ4[z, t], yZ5[z, t], yZ6[z, t], yZ7[z, t]],
 List[yZ8[z, t], yZ9[z, t], yZ10[z, t], yZ11[z, t]],
 List[yZ12[z, t], yZ13[z, t], yZ14[z, t], yZ15[z, t]]}
```

FOR MAPLE : A set is an unordered sequence of distinct expressions enclosed in braces {}, representing a set in the mathematical sense.
A list is an ordered sequence of expressions enclosed in square brackets []. The ordering of the expressions is the ordering of es.

```
“maple-output.py”:
import textwrap
filename = "maple-output.txt"
maple_long _string = "paste maple string here"
# Wraps the text and joins with newlines
with open (filename, "w") as f :
    f . write ("\n" . join (textwrap . wrap (maple_long _string, width = 80)))
```

FOR MAPLE : A set is an unordered sequence of distinct expressions enclosed in braces {}, representing a set in the mathematical sense.
A list is an ordered sequence of expressions enclosed in square brackets []. The ordering of the expressions is the ordering of es.

"Solution 15feb" by Maple

In[699]:=

```
syZtoZ
```

Out[699]=

```
{yZ[0] → Z[0], yZ[1] → Z[5], yZ[2] → Z[8], yZ[3] → Z[13], yZ[4] → Z[1], yZ[5] → Z[4],
 yZ[6] → Z[9], yZ[7] → Z[12], yZ[8] → Z[2], yZ[9] → Z[7], yZ[10] → Z[10],
 yZ[11] → Z[15], yZ[12] → Z[3], yZ[13] → Z[6], yZ[14] → Z[11], yZ[15] → Z[14]}
```

In[700]:=

```
(*Z[#]&/@Range[0,15]*)
Table[Z[j] → ToExpression["((Z[" <> ToString[j] <> "][#1,#2])&)"], {j, 0, 15}];
ssyZtoZ = syZtoZ /. %
```

Out[701]=

```
{yZ[0] → (Z[0][#1, #2] &), yZ[1] → (Z[5][#1, #2] &),
 yZ[2] → (Z[8][#1, #2] &), yZ[3] → (Z[13][#1, #2] &), yZ[4] → (Z[1][#1, #2] &),
 yZ[5] → (Z[4][#1, #2] &), yZ[6] → (Z[9][#1, #2] &), yZ[7] → (Z[12][#1, #2] &),
 yZ[8] → (Z[2][#1, #2] &), yZ[9] → (Z[7][#1, #2] &), yZ[10] → (Z[10][#1, #2] &),
 yZ[11] → (Z[15][#1, #2] &), yZ[12] → (Z[3][#1, #2] &), yZ[13] → (Z[6][#1, #2] &),
 yZ[14] → (Z[11][#1, #2] &), yZ[15] → (Z[14][#1, #2] &)}
```

In[702]:=

```
Ψ16;
% /. sfψ16Aa /. sx0x4;
% /. sZtOyZ;
Thread[% == %%%];
Solve[And @@ %, yZ[#][z, t] & /@ Range[0, 15]][[1]]
% /. szt /. {Rule → Equal};
Solve[And @@ %, f16[#][x0, x4] & /@ Range[0, 15]][[1]];
% /. {Rule → Equal}
```

Out[706]=

```
{yZ[0][z, t] → f16[0][x0, x4], yZ[1][z, t] → f16[5][x0, x4],
 yZ[2][z, t] → f16[8][x0, x4], yZ[3][z, t] → f16[13][x0, x4],
 yZ[4][z, t] → f16[1][x0, x4], yZ[5][z, t] → f16[4][x0, x4], yZ[6][z, t] → f16[9][x0, x4],
 yZ[7][z, t] → f16[12][x0, x4], yZ[8][z, t] → f16[2][x0, x4], yZ[9][z, t] → f16[7][x0, x4],
 yZ[10][z, t] → f16[10][x0, x4], yZ[11][z, t] → f16[15][x0, x4],
 yZ[12][z, t] → f16[3][x0, x4], yZ[13][z, t] → f16[6][x0, x4],
 yZ[14][z, t] → f16[11][x0, x4], yZ[15][z, t] → f16[14][x0, x4]}
```

Out[709]=

```
{f16[0][x0, x4] == yZ[0][6 H x0, H x4], f16[1][x0, x4] == yZ[4][6 H x0, H x4],
 f16[2][x0, x4] == yZ[8][6 H x0, H x4], f16[3][x0, x4] == yZ[12][6 H x0, H x4],
 f16[4][x0, x4] == yZ[5][6 H x0, H x4], f16[5][x0, x4] == yZ[1][6 H x0, H x4],
 f16[6][x0, x4] == yZ[13][6 H x0, H x4], f16[7][x0, x4] == yZ[9][6 H x0, H x4],
 f16[8][x0, x4] == yZ[2][6 H x0, H x4], f16[9][x0, x4] == yZ[6][6 H x0, H x4],
 f16[10][x0, x4] == yZ[10][6 H x0, H x4], f16[11][x0, x4] == yZ[14][6 H x0, H x4],
 f16[12][x0, x4] == yZ[7][6 H x0, H x4], f16[13][x0, x4] == yZ[3][6 H x0, H x4],
 f16[14][x0, x4] == yZ[15][6 H x0, H x4], f16[15][x0, x4] == yZ[11][6 H x0, H x4]}
```

Load the Maple output strings. Get is a lexer and parer and ... , so use ReadString

In[710]:=

```
SetDirectory[NotebookDirectory[]];
```

In[711]:=

```
maplestringEQ1 = "{{yZ0(z, t) = exp((6*(z/6 + t)*H^2*C1^2 - 6*(-z/6 +
    t)*M^2)/(6*C1*H^2))*(H^2*(c11*c12 - c13*c14)*C1^2 - M^2*(c11*c12 +
    c13*c14))/(2*H*C1*M), yZ1(z, t) = -((H^2*(c11*c12 - c13*c14)*C1^2
    + M^2*(c11*c12 + c13*c14))*exp((6*(z/6 + t)*H^2*C1^2 - 6*(-z/6
    + t)*M^2)/(6*C1*H^2)))/(2*H*C1*M), yZ2(z, t) = c13*exp((6*(z/6
    + t)*H^2*C1^2 - 6*(-z/6 + t)*M^2)/(6*C1*H^2))*c14, yZ3(z, t) =
    c11*exp((6*(z/6 + t)*H^2*C1^2 - 6*(-z/6 + t)*M^2)/(6*C1*H^2))*c12}}";
```

In[712]:=

```
maplestringEQ2 = "{{yZ4(z, t) = -exp((6*(z/6 + t)*H^2*C2^2 - 6*(-z/6
    + t)*M^2)/(6*C2*H^2))*(H^2*(c21*c22 + c23*c24)*C2^2 - M^2*(c21*c22
    - c23*c24))/(2*H*C2*M), yZ5(z, t) = -((H^2*(c21*c22 + c23*c24)*C2^2
    + M^2*(c21*c22 - c23*c24))*exp((6*(z/6 + t)*H^2*C2^2 - 6*(-z/6
    + t)*M^2)/(6*C2*H^2)))/(2*H*C2*M), yZ6(z, t) = c23*exp((6*(z/6
    + t)*H^2*C2^2 - 6*(-z/6 + t)*M^2)/(6*C2*H^2))*c24, yZ7(z, t) =
    c21*exp((6*(z/6 + t)*H^2*C2^2 - 6*(-z/6 + t)*M^2)/(6*C2*H^2))*c22}}";
```

In[713]:=

```
maplestringEQ3 =
  "{{yZ10(z, t) = c33*exp((6*H^2*(z/6 + t)*C3^2 - 6*(-z/6 + t)*M^2)/(6*C3*H^2))*c34,
    yZ11(z, t) = c31*exp((6*H^2*(z/6 + t)*C3^2 - 6*(-z/6 +
    t)*M^2)/(6*C3*H^2))*c32, yZ8(z, t) = -exp((6*H^2*(z/6 + t)*C3^2
    - 6*(-z/6 + t)*M^2)/(6*C3*H^2))*(H^2*(c31*c32 + c33*c34)*C3^2 -
    M^2*(c31*c32 - c33*c34))/(2*H*C3*M), yZ9(z, t) = -((H^2*(c31*c32
    + c33*c34)*C3^2 + M^2*(c31*c32 - c33*c34))*exp((6*H^2*(z/6
    + t)*C3^2 - 6*(-z/6 + t)*M^2)/(6*C3*H^2)))/(2*H*C3*M)}}";
```

In[714]:=

```
maplestringEQ4 = "{{yZ12(z, t) = exp((6*H^2*(z/6 + t)*C4^2 - 6*(-z/6 +
    t)*M^2)/(6*C4*H^2))*(H^2*(c41*c42 - c43*c44)*C4^2 - M^2*(c41*c42 +
    c43*c44))/(2*H*C4*M), yZ13(z, t) = -((H^2*(c41*c42 - c43*c44)*C4^2
    + M^2*(c41*c42 + c43*c44))*exp((6*H^2*(z/6 + t)*C4^2 - 6*(-z/6 +
    t)*M^2)/(6*C4*H^2)))/(2*H*C4*M), yZ14(z, t) = c43*exp((6*H^2*(z/6
    + t)*C4^2 - 6*(-z/6 + t)*M^2)/(6*C4*H^2))*c44, yZ15(z, t) =
    c41*exp((6*H^2*(z/6 + t)*C4^2 - 6*(-z/6 + t)*M^2)/(6*C4*H^2))*c42}}";
```

In[715]:=

```
solved13FebEQ1 = ConvertMapleToMathematicaV2[maplestringEQ1]
```

Out[715]=

$$\left\{\left\{YZ0[z,t] == \frac{e^{\frac{-6M^2\left(t-\frac{z}{6}\right)+6C1^2H^2\left(t+\frac{z}{6}\right)}{6C1H^2}}\left(C1^2\,(c11\,c12-c13\,c14)\,H^2-(c11\,c12+c13\,c14)\,M^2\right)}{2\,C1\,H\,M},\right.\right.$$

$$YZ1[z,t] == -\frac{e^{\frac{-6M^2\left(t-\frac{z}{6}\right)+6C1^2H^2\left(t+\frac{z}{6}\right)}{6C1H^2}}\left(C1^2\,(c11\,c12-c13\,c14)\,H^2+(c11\,c12+c13\,c14)\,M^2\right)}{2\,C1\,H\,M},$$

$$\left.\left.YZ2[z,t] == c13\,c14\,e^{\frac{-6M^2\left(t-\frac{z}{6}\right)+6C1^2H^2\left(t+\frac{z}{6}\right)}{6C1H^2}},\ YZ3[z,t] == c11\,c12\,e^{\frac{-6M^2\left(t-\frac{z}{6}\right)+6C1^2H^2\left(t+\frac{z}{6}\right)}{6C1H^2}}\right\}\right\}$$

In[716]:=

```
solved13FebEQ2 = ConvertMapleToMathematicaV2[maplestringEQ2]
```

Out[716]=

$$\left\{\left\{YZ4[z,t] == -\frac{e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C2^2 H^2 \left(t+\frac{z}{6}\right)}{6 C2 H^2}} \left(C2^2 (c21\, c22 + c23\, c24) H^2 - (c21\, c22 - c23\, c24) M^2\right)}{2 C2 H M},\right.\right.$$

$$YZ5[z,t] == -\frac{e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C2^2 H^2 \left(t+\frac{z}{6}\right)}{6 C2 H^2}} \left(C2^2 (c21\, c22 + c23\, c24) H^2 + (c21\, c22 - c23\, c24) M^2\right)}{2 C2 H M},$$

$$\left.\left.YZ6[z,t] == c23\, c24\, e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C2^2 H^2 \left(t+\frac{z}{6}\right)}{6 C2 H^2}},\ YZ7[z,t] == c21\, c22\, e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C2^2 H^2 \left(t+\frac{z}{6}\right)}{6 C2 H^2}}\right\}\right\}$$

In[717]:=

```
solved13FebEQ3 = ConvertMapleToMathematicaV2[maplestringEQ3]
```

Out[717]=

$$\left\{\left\{YZ10[z,t] == c33\, c34\, e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C3^2 H^2 \left(t+\frac{z}{6}\right)}{6 C3 H^2}},\ YZ11[z,t] == c31\, c32\, e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C3^2 H^2 \left(t+\frac{z}{6}\right)}{6 C3 H^2}},\right.\right.$$

$$YZ8[z,t] == -\frac{e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C3^2 H^2 \left(t+\frac{z}{6}\right)}{6 C3 H^2}} \left(C3^2 (c31\, c32 + c33\, c34) H^2 - (c31\, c32 - c33\, c34) M^2\right)}{2 C3 H M},$$

$$\left.\left.YZ9[z,t] == -\frac{e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C3^2 H^2 \left(t+\frac{z}{6}\right)}{6 C3 H^2}} \left(C3^2 (c31\, c32 + c33\, c34) H^2 + (c31\, c32 - c33\, c34) M^2\right)}{2 C3 H M}\right\}\right\}$$

In[718]:=

```
solved13FebEQ4 = ConvertMapleToMathematicaV2[maplestringEQ4]
```

Out[718]=

$$\left\{\left\{YZ12[z,t] == \frac{e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C4^2 H^2 \left(t+\frac{z}{6}\right)}{6 C4 H^2}} \left(C4^2 (c41\, c42 - c43\, c44) H^2 - (c41\, c42 + c43\, c44) M^2\right)}{2 C4 H M},\right.\right.$$

$$YZ13[z,t] == -\frac{e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C4^2 H^2 \left(t+\frac{z}{6}\right)}{6 C4 H^2}} \left(C4^2 (c41\, c42 - c43\, c44) H^2 + (c41\, c42 + c43\, c44) M^2\right)}{2 C4 H M},$$

$$\left.\left.YZ14[z,t] == c43\, c44\, e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C4^2 H^2 \left(t+\frac{z}{6}\right)}{6 C4 H^2}},\ YZ15[z,t] == c41\, c42\, e^{\frac{-6 M^2 \left(t-\frac{z}{6}\right)+6 C4^2 H^2 \left(t+\frac{z}{6}\right)}{6 C4 H^2}}\right\}\right\}$$

In[719]:=

```
YZvar = ToExpression["YZ" <> ToString[#] <> "[z,t]"] & /@ Range[0, 15]
```

Out[719]=

```
{YZ0[z, t], YZ1[z, t], YZ2[z, t], YZ3[z, t], YZ4[z, t], YZ5[z, t], YZ6[z, t], YZ7[z, t],
 YZ8[z, t], YZ9[z, t], YZ10[z, t], YZ11[z, t], YZ12[z, t], YZ13[z, t], YZ14[z, t], YZ15[z, t]}
```

In[720]:=

```
sYZvar = Solve[And @@
    Flatten[{solved13FebEQ1, solved13FebEQ2, solved13FebEQ3, solved13FebEQ4}], YZvar][[1]]
```

Out[720]=

$$\left\{YZ0[z,t]\to\frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\,H^2}+C1\left(t+\frac{z}{6}\right)}\left(C1^2\,c11\,c12\,H^2-C1^2\,c13\,c14\,H^2-c11\,c12\,M^2-c13\,c14\,M^2\right)}{2\,C1\,H\,M},\right.$$

$$YZ1[z,t]\to-\frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\,H^2}+C1\left(t+\frac{z}{6}\right)}\left(C1^2\,c11\,c12\,H^2-C1^2\,c13\,c14\,H^2+c11\,c12\,M^2+c13\,c14\,M^2\right)}{2\,C1\,H\,M},$$

$$YZ2[z,t]\to c13\,c14\,e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\,H^2}+C1\left(t+\frac{z}{6}\right)},\ YZ3[z,t]\to c11\,c12\,e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\,H^2}+C1\left(t+\frac{z}{6}\right)},$$

$$YZ4[z,t]\to-\frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\,H^2}+C2\left(t+\frac{z}{6}\right)}\left(C2^2\,c21\,c22\,H^2+C2^2\,c23\,c24\,H^2-c21\,c22\,M^2+c23\,c24\,M^2\right)}{2\,C2\,H\,M},$$

$$YZ5[z,t]\to-\frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\,H^2}+C2\left(t+\frac{z}{6}\right)}\left(C2^2\,c21\,c22\,H^2+C2^2\,c23\,c24\,H^2+c21\,c22\,M^2-c23\,c24\,M^2\right)}{2\,C2\,H\,M},$$

$$YZ6[z,t]\to c23\,c24\,e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\,H^2}+C2\left(t+\frac{z}{6}\right)},\ YZ7[z,t]\to c21\,c22\,e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\,H^2}+C2\left(t+\frac{z}{6}\right)},$$

$$YZ8[z,t]\to-\frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\,H^2}+C3\left(t+\frac{z}{6}\right)}\left(C3^2\,c31\,c32\,H^2+C3^2\,c33\,c34\,H^2-c31\,c32\,M^2+c33\,c34\,M^2\right)}{2\,C3\,H\,M},$$

$$YZ9[z,t]\to-\frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\,H^2}+C3\left(t+\frac{z}{6}\right)}\left(C3^2\,c31\,c32\,H^2+C3^2\,c33\,c34\,H^2+c31\,c32\,M^2-c33\,c34\,M^2\right)}{2\,C3\,H\,M},$$

$$YZ10[z,t]\to c33\,c34\,e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\,H^2}+C3\left(t+\frac{z}{6}\right)},\ YZ11[z,t]\to c31\,c32\,e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\,H^2}+C3\left(t+\frac{z}{6}\right)},$$

$$YZ12[z,t]\to\frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\,H^2}+C4\left(t+\frac{z}{6}\right)}\left(C4^2\,c41\,c42\,H^2-C4^2\,c43\,c44\,H^2-c41\,c42\,M^2-c43\,c44\,M^2\right)}{2\,C4\,H\,M},$$

$$YZ13[z,t]\to-\frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\,H^2}+C4\left(t+\frac{z}{6}\right)}\left(C4^2\,c41\,c42\,H^2-C4^2\,c43\,c44\,H^2+c41\,c42\,M^2+c43\,c44\,M^2\right)}{2\,C4\,H\,M},$$

$$\left.YZ14[z,t]\to c43\,c44\,e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\,H^2}+C4\left(t+\frac{z}{6}\right)},\ YZ15[z,t]\to c41\,c42\,e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\,H^2}+C4\left(t+\frac{z}{6}\right)}\right\}$$

In[721]:=

```
yZ[#] & /@ Range[0, 15]
sYZvar[[All, 2]] /. {z → #1, t → #2};
Table[ToExpression["((" <> ToString[FullForm[%[[j]]]] <> ")&)"], {j, 1, Length[%]}];
ssyZ = Thread[%%% → %]
```

Out[721]=

```
{yZ[0], yZ[1], yZ[2], yZ[3], yZ[4], yZ[5], yZ[6], yZ[7],
 yZ[8], yZ[9], yZ[10], yZ[11], yZ[12], yZ[13], yZ[14], yZ[15]}
```

Out[724]=

$$\left\{yZ[0]\to\left(\frac{1}{C1\,H\,M}\,\text{Rational}[1,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\right.\right.$$

$$\left(C1^2\, c11\, c12\, H^2 - C1^2\, c13\, c14\, H^2 - c11\, c12\, M^2 - c13\, c14\, M^2\right)\ \&\Big),$$

$$yZ[1] \to \Big(\frac{1}{C1\, H\, M}\, \text{Rational}[-1,\ 2]\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C1\, H^2}+C1\,(\text{Rational}[1,6]\ \#1+\#2)}$$

$$\left(C1^2\, c11\, c12\, H^2 - C1^2\, c13\, c14\, H^2 + c11\, c12\, M^2 + c13\, c14\, M^2\right)\ \&\Big),$$

$$yZ[2] \to \left(c13\, c14\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C1\, H^2}+C1\,(\text{Rational}[1,6]\ \#1+\#2)}\ \&\right),$$

$$yZ[3] \to \left(c11\, c12\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C1\, H^2}+C1\,(\text{Rational}[1,6]\ \#1+\#2)}\ \&\right),$$

$$yZ[4] \to \Big(\frac{1}{C2\, H\, M}\, \text{Rational}[-1,\ 2]\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C2\, H^2}+C2\,(\text{Rational}[1,6]\ \#1+\#2)}$$

$$\left(C2^2\, c21\, c22\, H^2 + C2^2\, c23\, c24\, H^2 - c21\, c22\, M^2 + c23\, c24\, M^2\right)\ \&\Big),$$

$$yZ[5] \to \Big(\frac{1}{C2\, H\, M}\, \text{Rational}[-1,\ 2]\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C2\, H^2}+C2\,(\text{Rational}[1,6]\ \#1+\#2)}$$

$$\left(C2^2\, c21\, c22\, H^2 + C2^2\, c23\, c24\, H^2 + c21\, c22\, M^2 - c23\, c24\, M^2\right)\ \&\Big),$$

$$yZ[6] \to \left(c23\, c24\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C2\, H^2}+C2\,(\text{Rational}[1,6]\ \#1+\#2)}\ \&\right),$$

$$yZ[7] \to \left(c21\, c22\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C2\, H^2}+C2\,(\text{Rational}[1,6]\ \#1+\#2)}\ \&\right),$$

$$yZ[8] \to \Big(\frac{1}{C3\, H\, M}\, \text{Rational}[-1,\ 2]\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C3\, H^2}+C3\,(\text{Rational}[1,6]\ \#1+\#2)}$$

$$\left(C3^2\, c31\, c32\, H^2 + C3^2\, c33\, c34\, H^2 - c31\, c32\, M^2 + c33\, c34\, M^2\right)\ \&\Big),$$

$$yZ[9] \to \Big(\frac{1}{C3\, H\, M}\, \text{Rational}[-1,\ 2]\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C3\, H^2}+C3\,(\text{Rational}[1,6]\ \#1+\#2)}$$

$$\left(C3^2\, c31\, c32\, H^2 + C3^2\, c33\, c34\, H^2 + c31\, c32\, M^2 - c33\, c34\, M^2\right)\ \&\Big),$$

$$yZ[10] \to \left(c33\, c34\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C3\, H^2}+C3\,(\text{Rational}[1,6]\ \#1+\#2)}\ \&\right),$$

$$yZ[11] \to \left(c31\, c32\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C3\, H^2}+C3\,(\text{Rational}[1,6]\ \#1+\#2)}\ \&\right),$$

$$yZ[12] \to \Big(\frac{1}{C4\, H\, M}\, \text{Rational}[1,\ 2]\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C4\, H^2}+C4\,(\text{Rational}[1,6]\ \#1+\#2)}$$

$$\left(C4^2\, c41\, c42\, H^2 - C4^2\, c43\, c44\, H^2 - c41\, c42\, M^2 - c43\, c44\, M^2\right)\ \&\Big),$$

$$yZ[13] \to \Big(\frac{1}{C4\, H\, M}\, \text{Rational}[-1,\ 2]\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C4\, H^2}+C4\,(\text{Rational}[1,6]\ \#1+\#2)}$$

$$\left(C4^2\, c41\, c42\, H^2 - C4^2\, c43\, c44\, H^2 + c41\, c42\, M^2 + c43\, c44\, M^2\right)\ \&\Big),$$

$$yZ[14] \to \left(c43\, c44\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C4\, H^2}+C4\,(\text{Rational}[1,6]\ \#1+\#2)}\ \&\right),$$

$$yZ[15] \to \left(c41\, c42\ e^{-\frac{M^2\,(\text{Rational}[-1,6]\ \#1+\#2)}{C4\, H^2}+C4\,(\text{Rational}[1,6]\ \#1+\#2)}\ \&\right)\Big\}$$

CHECK MAPLE SOLUTIONS:

In[725]:=

```
FullSimplify[coupledyZeqs[[#]] /. ssyZ] & /@ Range[4]
```

Out[725]=

```
{{True, True, True, True}, {True, True, True, True},
 {True, True, True, True}, {True, True, True, True}}
```

In[726]:=

```
ssyZ
sZtOyZ
sZ = % /. %% // Sort
```

Out[726]=

$$\Big\{yZ[0] \to \Big(\frac{1}{C1\,H\,M}\,\text{Rational}[1,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\,\big(C1^2\,c11\,c12\,H^2-C1^2\,c13\,c14\,H^2-c11\,c12\,M^2-c13\,c14\,M^2\big)\ \&\Big),$$

$$yZ[1] \to \Big(\frac{1}{C1\,H\,M}\,\text{Rational}[-1,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\,\big(C1^2\,c11\,c12\,H^2-C1^2\,c13\,c14\,H^2+c11\,c12\,M^2+c13\,c14\,M^2\big)\ \&\Big),$$

$$yZ[2] \to \Big(c13\,c14\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\ \&\Big),$$

$$yZ[3] \to \Big(c11\,c12\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\ \&\Big),$$

$$yZ[4] \to \Big(\frac{1}{C2\,H\,M}\,\text{Rational}[-1,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C2\,H^2}+C2\,(\text{Rational}[1,6]\,\#1+\#2)}\,\big(C2^2\,c21\,c22\,H^2+C2^2\,c23\,c24\,H^2-c21\,c22\,M^2+c23\,c24\,M^2\big)\ \&\Big),$$

$$yZ[5] \to \Big(\frac{1}{C2\,H\,M}\,\text{Rational}[-1,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C2\,H^2}+C2\,(\text{Rational}[1,6]\,\#1+\#2)}\,\big(C2^2\,c21\,c22\,H^2+C2^2\,c23\,c24\,H^2+c21\,c22\,M^2-c23\,c24\,M^2\big)\ \&\Big),$$

$$yZ[6] \to \Big(c23\,c24\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C2\,H^2}+C2\,(\text{Rational}[1,6]\,\#1+\#2)}\ \&\Big),$$

$$yZ[7] \to \Big(c21\,c22\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C2\,H^2}+C2\,(\text{Rational}[1,6]\,\#1+\#2)}\ \&\Big),$$

$$yZ[8] \to \Big(\frac{1}{C3\,H\,M}\,\text{Rational}[-1,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C3\,H^2}+C3\,(\text{Rational}[1,6]\,\#1+\#2)}\,\big(C3^2\,c31\,c32\,H^2+C3^2\,c33\,c34\,H^2-c31\,c32\,M^2+c33\,c34\,M^2\big)\ \&\Big),$$

$$yZ[9] \to \Big(\frac{1}{C3\,H\,M}\,\text{Rational}[-1,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C3\,H^2}+C3\,(\text{Rational}[1,6]\,\#1+\#2)}\,\big(C3^2\,c31\,c32\,H^2+C3^2\,c33\,c34\,H^2+c31\,c32\,M^2-c33\,c34\,M^2\big)\ \&\Big),$$

$$yZ[10] \to \Big(c33\,c34\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C3\,H^2}+C3\,(\text{Rational}[1,6]\,\#1+\#2)}\ \&\Big),$$

$$yZ[11] \to \left(c31\, c32\, e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C3\,H^2}+C3\,(\text{Rational}[1,6]\,\#1+\#2)}\ \&\right),$$

$$yZ[12] \to \left(\frac{1}{C4\,H\,M}\text{Rational}[1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C4\,H^2}+C4\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C4^2\,c41\,c42\,H^2-C4^2\,c43\,c44\,H^2-c41\,c42\,M^2-c43\,c44\,M^2\right)\ \&\right),$$

$$yZ[13] \to \left(\frac{1}{C4\,H\,M}\text{Rational}[-1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C4\,H^2}+C4\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C4^2\,c41\,c42\,H^2-C4^2\,c43\,c44\,H^2+c41\,c42\,M^2+c43\,c44\,M^2\right)\ \&\right),$$

$$yZ[14] \to \left(c43\,c44\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C4\,H^2}+C4\,(\text{Rational}[1,6]\,\#1+\#2)}\ \&\right),$$

$$yZ[15] \to \left(c41\,c42\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C4\,H^2}+C4\,(\text{Rational}[1,6]\,\#1+\#2)}\ \&\right)\Big\}$$

Out[727]=

$$\{Z[0]\to yZ[0],\ Z[5]\to yZ[1],\ Z[8]\to yZ[2],\ Z[13]\to yZ[3],\ Z[1]\to yZ[4],\ Z[4]\to yZ[5],$$
$$Z[9]\to yZ[6],\ Z[12]\to yZ[7],\ Z[2]\to yZ[8],\ Z[7]\to yZ[9],\ Z[10]\to yZ[10],$$
$$Z[15]\to yZ[11],\ Z[3]\to yZ[12],\ Z[6]\to yZ[13],\ Z[11]\to yZ[14],\ Z[14]\to yZ[15]\}$$

Out[728]=

$$\Big\{Z[0] \to \left(\frac{1}{C1\,H\,M}\text{Rational}[1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C1^2\,c11\,c12\,H^2-C1^2\,c13\,c14\,H^2-c11\,c12\,M^2-c13\,c14\,M^2\right)\ \&\right),$$

$$Z[1] \to \left(\frac{1}{C2\,H\,M}\text{Rational}[-1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C2\,H^2}+C2\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C2^2\,c21\,c22\,H^2+C2^2\,c23\,c24\,H^2-c21\,c22\,M^2+c23\,c24\,M^2\right)\ \&\right),$$

$$Z[2] \to \left(\frac{1}{C3\,H\,M}\text{Rational}[-1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C3\,H^2}+C3\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C3^2\,c31\,c32\,H^2+C3^2\,c33\,c34\,H^2-c31\,c32\,M^2+c33\,c34\,M^2\right)\ \&\right),$$

$$Z[3] \to \left(\frac{1}{C4\,H\,M}\text{Rational}[1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C4\,H^2}+C4\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C4^2\,c41\,c42\,H^2-C4^2\,c43\,c44\,H^2-c41\,c42\,M^2-c43\,c44\,M^2\right)\ \&\right),$$

$$Z[4] \to \left(\frac{1}{C2\,H\,M}\text{Rational}[-1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C2\,H^2}+C2\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C2^2\,c21\,c22\,H^2+C2^2\,c23\,c24\,H^2+c21\,c22\,M^2-c23\,c24\,M^2\right)\ \&\right),$$

$$Z[5] \to \left(\frac{1}{C1\,H\,M}\text{Rational}[-1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C1^2\,c11\,c12\,H^2-C1^2\,c13\,c14\,H^2+c11\,c12\,M^2+c13\,c14\,M^2\right)\ \&\right),$$

$$Z[6] \to \left(\frac{1}{C4\,H\,M}\text{Rational}[-1,\,2]\,e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C4\,H^2}+C4\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C4^2\,c41\,c42\,H^2-C4^2\,c43\,c44\,H^2+c41\,c42\,M^2+c43\,c44\,M^2\right)\ \&\right),$$

$$Z[7] \to \left(\frac{1}{C3\,H\,M}\,\text{Rational}[-1,2]\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C3\,H^2}+C3\,(\text{Rational}[1,6]\,\#1+\#2)}\right.$$
$$\left.\left(C3^2\,c31\,c32\,H^2+C3^2\,c33\,c34\,H^2+c31\,c32\,M^2-c33\,c34\,M^2\right)\;\&\right),$$

$$Z[8] \to \left(c13\,c14\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\;\&\right),$$

$$Z[9] \to \left(c23\,c24\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C2\,H^2}+C2\,(\text{Rational}[1,6]\,\#1+\#2)}\;\&\right),$$

$$Z[10] \to \left(c33\,c34\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C3\,H^2}+C3\,(\text{Rational}[1,6]\,\#1+\#2)}\;\&\right),$$

$$Z[11] \to \left(c43\,c44\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C4\,H^2}+C4\,(\text{Rational}[1,6]\,\#1+\#2)}\;\&\right),$$

$$Z[12] \to \left(c21\,c22\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C2\,H^2}+C2\,(\text{Rational}[1,6]\,\#1+\#2)}\;\&\right),$$

$$Z[13] \to \left(c11\,c12\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C1\,H^2}+C1\,(\text{Rational}[1,6]\,\#1+\#2)}\;\&\right),$$

$$Z[14] \to \left(c41\,c42\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C4\,H^2}+C4\,(\text{Rational}[1,6]\,\#1+\#2)}\;\&\right),$$

$$Z[15] \to \left(c31\,c32\; e^{-\frac{M^2\,(\text{Rational}[-1,6]\,\#1+\#2)}{C3\,H^2}+C3\,(\text{Rational}[1,6]\,\#1+\#2)}\;\&\right)\Big\}$$

In[729]:=

```
(Ψ16a = Ψ16 /. sfψ16Aa /. sx0x4 /. sZ /. szt // FullSimplify) // Column
```

Out[729]=

$$\frac{e^{\frac{M^2 (x0-x4)}{C1\,H}+C1\,H\,(x0+x4)}\left(C1^2\,(c11\,c12-c13\,c14)\,H^2-(c11\,c12+c13\,c14)\,M^2\right)}{2\,C1\,H\,M}$$
$$-\frac{e^{\frac{M^2 (x0-x4)}{C2\,H}+C2\,H\,(x0+x4)}\left(C2^2\,(c21\,c22+c23\,c24)\,H^2+(-c21\,c22+c23\,c24)\,M^2\right)}{2\,C2\,H\,M}$$
$$-\frac{e^{\frac{M^2 (x0-x4)}{C3\,H}+C3\,H\,(x0+x4)}\left(C3^2\,(c31\,c32+c33\,c34)\,H^2+(-c31\,c32+c33\,c34)\,M^2\right)}{2\,C3\,H\,M}$$
$$\frac{e^{\frac{M^2 (x0-x4)}{C4\,H}+C4\,H\,(x0+x4)}\left(C4^2\,(c41\,c42-c43\,c44)\,H^2-(c41\,c42+c43\,c44)\,M^2\right)}{2\,C4\,H\,M}$$
$$-\frac{e^{\frac{M^2 (x0-x4)}{C2\,H}+C2\,H\,(x0+x4)}\left(C2^2\,(c21\,c22+c23\,c24)\,H^2+(c21\,c22-c23\,c24)\,M^2\right)}{2\,C2\,H\,M}$$
$$-\frac{e^{\frac{M^2 (x0-x4)}{C1\,H}+C1\,H\,(x0+x4)}\left(C1^2\,(c11\,c12-c13\,c14)\,H^2+(c11\,c12+c13\,c14)\,M^2\right)}{2\,C1\,H\,M}$$
$$-\frac{e^{\frac{M^2 (x0-x4)}{C4\,H}+C4\,H\,(x0+x4)}\left(C4^2\,(c41\,c42-c43\,c44)\,H^2+(c41\,c42+c43\,c44)\,M^2\right)}{2\,C4\,H\,M}$$
$$-\frac{e^{\frac{M^2 (x0-x4)}{C3\,H}+C3\,H\,(x0+x4)}\left(C3^2\,(c31\,c32+c33\,c34)\,H^2+(c31\,c32-c33\,c34)\,M^2\right)}{2\,C3\,H\,M}$$

$$c13\,c14\,e^{\frac{M^2 (x0-x4)}{C1\,H}+C1\,H\,(x0+x4)}$$
$$c23\,c24\,e^{\frac{M^2 (x0-x4)}{C2\,H}+C2\,H\,(x0+x4)}$$
$$c33\,c34\,e^{\frac{M^2 (x0-x4)}{C3\,H}+C3\,H\,(x0+x4)}$$
$$c43\,c44\,e^{\frac{M^2 (x0-x4)}{C4\,H}+C4\,H\,(x0+x4)}$$
$$c21\,c22\,e^{\frac{M^2 (x0-x4)}{C2\,H}+C2\,H\,(x0+x4)}$$
$$c11\,c12\,e^{\frac{M^2 (x0-x4)}{C1\,H}+C1\,H\,(x0+x4)}$$
$$c41\,c42\,e^{\frac{M^2 (x0-x4)}{C4\,H}+C4\,H\,(x0+x4)}$$
$$c31\,c32\,e^{\frac{M^2 (x0-x4)}{C3\,H}+C3\,H\,(x0+x4)}$$

In[851]:=

```
ξadv == x0 - x4 && ξret == x0 + x4
sξ = Solve[%, {ξadv, ξret}][[1]]
ssx0x4 = Solve[%%, {x0, x4}][[1]]
```

Out[851]=

```
ξadv == x0 - x4 && ξret == x0 + x4
```

Out[852]=

```
{ξadv → x0 - x4, ξret → x0 + x4}
```

Out[853]=

$$\left\{x0 \to \frac{\xi adv + \xi ret}{2},\ x4 \to \frac{1}{2}\,(-\xi adv + \xi ret)\right\}$$

In[854]:=

```
(*Clear[ssx0x4]; ssx0x4={}*)
```

```
ψ1 = Ψ16a[[1 ;; 8]] // FullSimplify; ψ2 = Ψ16a[[8 + 1 ;; 8 + 8]] // FullSimplify
```

Out[734]=

$$\left\{c13\, c14\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)},\ c23\, c24\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)},\ c33\, c34\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)},\ c43\, c44\, e^{\frac{M^2 (x0-x4)}{C4\, H}+C4\, H\, (x0+x4)},\ c21\, c22\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)},\ c11\, c12\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)},\ c41\, c42\, e^{\frac{M^2 (x0-x4)}{C4\, H}+C4\, H\, (x0+x4)},\ c31\, c32\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\right\}$$

In[735]:=

```
type1DIVBYtype2 = Ψ16a[[1 ;; 8]] / Ψ16a[[8 + 1 ;; 8 + 8]] // FullSimplify
```

Out[735]=

$$\left\{\frac{C1^2\, (c11\, c12 - c13\, c14)\, H^2 - (c11\, c12 + c13\, c14)\, M^2}{2\, C1\, c13\, c14\, H\, M},\ -\frac{C2^2\, (c21\, c22 + c23\, c24)\, H^2 + (-c21\, c22 + c23\, c24)\, M^2}{2\, C2\, c23\, c24\, H\, M},\ -\frac{C3^2\, (c31\, c32 + c33\, c34)\, H^2 + (-c31\, c32 + c33\, c34)\, M^2}{2\, C3\, c33\, c34\, H\, M},\ \frac{C4^2\, (c41\, c42 - c43\, c44)\, H^2 - (c41\, c42 + c43\, c44)\, M^2}{2\, C4\, c43\, c44\, H\, M},\ -\frac{C2^2\, (c21\, c22 + c23\, c24)\, H^2 + (c21\, c22 - c23\, c24)\, M^2}{2\, C2\, c21\, c22\, H\, M},\ -\frac{C1^2\, (c11\, c12 - c13\, c14)\, H^2 + (c11\, c12 + c13\, c14)\, M^2}{2\, C1\, c11\, c12\, H\, M},\ -\frac{C4^2\, (c41\, c42 - c43\, c44)\, H^2 + (c41\, c42 + c43\, c44)\, M^2}{2\, C4\, c41\, c42\, H\, M},\ -\frac{C3^2\, (c31\, c32 + c33\, c34)\, H^2 + (c31\, c32 - c33\, c34)\, M^2}{2\, C3\, c31\, c32\, H\, M}\right\}$$

In[736]:=

```
realPartOfType2 = unit.ψ2 // FullSimplify
```

Out[736]=

$$\frac{c13\, c14\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)} + c21\, c22\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}}{\sqrt{2}}$$

In[737]:=

```
realPartOfType1 = unit.ψ1 // FullSimplify
```

Out[737]=

$$\frac{\frac{e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)}\left(C1^2\, (c11\, c12-c13\, c14)\, H^2-(c11\, c12+c13\, c14)\, M^2\right)}{C1} - \frac{e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\left(C2^2\, (c21\, c22+c23\, c24)\, H^2+(c21\, c22-c23\, c24)\, M^2\right)}{C2}}{2\, \sqrt{2}\, H\, M}$$

$$\mathfrak{F}^a_A = \frac{1}{\sqrt{\tilde{u}\sigma u}} \tau_A{}^a{}_b u^b \tag{63}$$

$$\mathfrak{F}^A_a = \frac{1}{\sqrt{\tilde{u}\sigma u}} u^c \sigma_{cb} \overline{\tau}^{Ab}{}_a \tag{64}$$

In[738]:=

```
F^A_a.F^a_A === F^a_A.F^A_a === ID8
```

Out[738]=

```
True
```

In[739]:=

```
(vectorFormOfType2 = F^A_a.ψ2 // FullSimplify) // Column
```

Out[739]=

$$\frac{c13\ c14\ e^{\frac{M^2 (x0-x4)}{C1\ H}+C1\ H\ (x0+x4)}+c21\ c22\ e^{\frac{M^2 (x0-x4)}{C2\ H}+C2\ H\ (x0+x4)}}{\sqrt{2}}$$

$$-\frac{c31\ c32\ e^{\frac{M^2 (x0-x4)}{C3\ H}+C3\ H\ (x0+x4)}+c43\ c44\ e^{\frac{M^2 (x0-x4)}{C4\ H}+C4\ H\ (x0+x4)}}{\sqrt{2}}$$

$$\frac{c33\ c34\ e^{\frac{M^2 (x0-x4)}{C3\ H}+C3\ H\ (x0+x4)}+c41\ c42\ e^{\frac{M^2 (x0-x4)}{C4\ H}+C4\ H\ (x0+x4)}}{\sqrt{2}}$$

$$-\frac{c11\ c12\ e^{\frac{M^2 (x0-x4)}{C1\ H}+C1\ H\ (x0+x4)}+c23\ c24\ e^{\frac{M^2 (x0-x4)}{C2\ H}+C2\ H\ (x0+x4)}}{\sqrt{2}}$$

$$\frac{-c11\ c12\ e^{\frac{M^2 (x0-x4)}{C1\ H}+C1\ H\ (x0+x4)}+c23\ c24\ e^{\frac{M^2 (x0-x4)}{C2\ H}+C2\ H\ (x0+x4)}}{\sqrt{2}}$$

$$\frac{-c33\ c34\ e^{\frac{M^2 (x0-x4)}{C3\ H}+C3\ H\ (x0+x4)}+c41\ c42\ e^{\frac{M^2 (x0-x4)}{C4\ H}+C4\ H\ (x0+x4)}}{\sqrt{2}}$$

$$\frac{c31\ c32\ e^{\frac{M^2 (x0-x4)}{C3\ H}+C3\ H\ (x0+x4)}-c43\ c44\ e^{\frac{M^2 (x0-x4)}{C4\ H}+C4\ H\ (x0+x4)}}{\sqrt{2}}$$

$$\frac{c13\ c14\ e^{\frac{M^2 (x0-x4)}{C1\ H}+C1\ H\ (x0+x4)}-c21\ c22\ e^{\frac{M^2 (x0-x4)}{C2\ H}+C2\ H\ (x0+x4)}}{\sqrt{2}}$$

In[740]:=

```
η4488 // MatrixForm
```

Out[740]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[741]:=

```
(vectorFormOfType1 = (η4488.(Transpose[ψ1].σ.F^a_A)) // FullSimplify) // Column
```

Out[741]=

$$\frac{\frac{e^{\frac{M^2 (x0-x4)}{C1\,H}+C1\,H\,(x0+x4)}\left(C1^2\,(c11\,c12-c13\,c14)\,H^2-(c11\,c12+c13\,c14)\,M^2\right)}{C1}-\frac{e^{\frac{M^2 (x0-x4)}{C2\,H}+C2\,H\,(x0+x4)}\left(C2^2\,(c21\,c22+c23\,c24)\,H^2+(c21\,c22-c23\,c24)\,M^2\right)}{C2}}{2\sqrt{2}\,H\,M}$$

$$\frac{\frac{e^{\frac{M^2 (x0-x4)}{C3\,H}+C3\,H\,(x0+x4)}\left(C3^2\,(c31\,c32+c33\,c34)\,H^2+(c31\,c32-c33\,c34)\,M^2\right)}{C3}+\frac{e^{\frac{M^2 (x0-x4)}{C4\,H}+C4\,H\,(x0+x4)}\left(C4^2\,(-c41\,c42+c43\,c44)\,H^2+(c41\,c42+c43\,c44)\,M^2\right)}{C4}}{2\sqrt{2}\,H\,M}$$

$$\frac{-\frac{e^{\frac{M^2 (x0-x4)}{C3\,H}+C3\,H\,(x0+x4)}\left(C3^2\,(c31\,c32+c33\,c34)\,H^2+(-c31\,c32+c33\,c34)\,M^2\right)}{C3}-\frac{e^{\frac{M^2 (x0-x4)}{C4\,H}+C4\,H\,(x0+x4)}\left(C4^2\,(c41\,c42-c43\,c44)\,H^2+(c41\,c42+c43\,c44)\,M^2\right)}{C4}}{2\sqrt{2}\,H\,M}$$

$$\frac{\frac{e^{\frac{M^2 (x0-x4)}{C1\,H}+C1\,H\,(x0+x4)}\left(C1^2\,(c11\,c12-c13\,c14)\,H^2+(c11\,c12+c13\,c14)\,M^2\right)}{C1}+\frac{e^{\frac{M^2 (x0-x4)}{C2\,H}+C2\,H\,(x0+x4)}\left(C2^2\,(c21\,c22+c23\,c24)\,H^2+(-c21\,c22+c23\,c24)\,M^2\right)}{C2}}{2\sqrt{2}\,H\,M}$$

$$\frac{\frac{e^{\frac{M^2 (x0-x4)}{C1\,H}+C1\,H\,(x0+x4)}\left(C1^2\,(c11\,c12-c13\,c14)\,H^2+(c11\,c12+c13\,c14)\,M^2\right)}{C1}-\frac{e^{\frac{M^2 (x0-x4)}{C2\,H}+C2\,H\,(x0+x4)}\left(C2^2\,(c21\,c22+c23\,c24)\,H^2+(-c21\,c22+c23\,c24)\,M^2\right)}{C2}}{2\sqrt{2}\,H\,M}$$

$$\frac{\frac{e^{\frac{M^2 (x0-x4)}{C3\,H}+C3\,H\,(x0+x4)}\left(C3^2\,(c31\,c32+c33\,c34)\,H^2+(-c31\,c32+c33\,c34)\,M^2\right)}{C3}-\frac{e^{\frac{M^2 (x0-x4)}{C4\,H}+C4\,H\,(x0+x4)}\left(C4^2\,(c41\,c42-c43\,c44)\,H^2+(c41\,c42+c43\,c44)\,M^2\right)}{C4}}{2\sqrt{2}\,H\,M}$$

$$\frac{-\frac{e^{\frac{M^2 (x0-x4)}{C3\,H}+C3\,H\,(x0+x4)}\left(C3^2\,(c31\,c32+c33\,c34)\,H^2+(c31\,c32-c33\,c34)\,M^2\right)}{C3}+\frac{e^{\frac{M^2 (x0-x4)}{C4\,H}+C4\,H\,(x0+x4)}\left(C4^2\,(-c41\,c42+c43\,c44)\,H^2+(c41\,c42+c43\,c44)\,M^2\right)}{C4}}{2\sqrt{2}\,H\,M}$$

$$\frac{\frac{e^{\frac{M^2 (x0-x4)}{C1\,H}+C1\,H\,(x0+x4)}\left(C1^2\,(c11\,c12-c13\,c14)\,H^2-(c11\,c12+c13\,c14)\,M^2\right)}{C1}+\frac{e^{\frac{M^2 (x0-x4)}{C2\,H}+C2\,H\,(x0+x4)}\left(C2^2\,(c21\,c22+c23\,c24)\,H^2+(c21\,c22-c23\,c24)\,M^2\right)}{C2}}{2\sqrt{2}\,H\,M}$$

7 - plane example :

$\mathbb{O}_{4,4}$ = split octonion algebra ;

eA_1 = unit of $\mathbb{O}_{4,4}$; $\mathbb{R}$ = linear span of eA_1;

$\mathbb{O}_{4,4}$ vector and spinor bases [**Cartan's triality**] :

shortcut below : ψ reads as "type"; say "type" instead of "psi"

$o \in \mathbb{O}_{4,4}$; $o = eA_j\, o^j = ea\psi 1_a\, o\psi 1^a = ea\psi 2_a\, o\psi 2^a$

we employ $ea\psi 1_a = ea\psi 2_a = ea_a$, $a = 1, \ldots,$

8; $< ea_a, ea_b > = \sigma_{ab}$; also $< eA_A, eA_B > = \eta_{AB}$

Let type2Vector = vectorFormOfType2^A $eA_A \in \mathbb{O}_{4,4}$; $< eA_A,$

type2Vector $> = \eta_{AB}$ vectorFormOfType2^B

Let $z = Z^A\, eA_A \in \mathbb{O}_{4,4}$ be the 'position' vector

(split octonion) with 8 degrees of freedom; $< eA_A, z > = \eta_{AB}\, Z^B$

7 – plane with 7 degrees of

freedom : $Z^A \in \mathbb{R}^8 \ni 0 == (Z^A - 0^A)\, \eta_{AB}$ vectorFormOfType2^B , $0^A = 0$

In[1022]:=

```
Z = Array[z, 8];
sz8 = Solve[0 == vectorFormOfType2.Z, z[8]][[-1]] // FullSimplify
sevenPlane2 = Z /. sz8
```

Out[1023]=

$$\Big\{z[8] \to \Big(-c13\, c14\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)}\, z[1] - c21\, c22\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\, z[1] + c31\, c32\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\, z[2] - c33\, c34\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\, z[3] + c11\, c12\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)}\, z[4] + c23\, c24\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\, z[4] + c11\, c12\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)}\, z[5] - c23\, c24\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\, z[5] + c33\, c34\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\, z[6] - c31\, c32\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\, z[7] + e^{\frac{M^2 (x0-x4)}{C4\, H}+C4\, H\, (x0+x4)}\, (-c41\, c42\, (z[3]+z[6]) + c43\, c44\, (z[2]+z[7]))\Big) \Big/ \Big(c13\, c14\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)} - c21\, c22\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\Big)\Big\}$$

Out[1024]=

$$\Big\{z[1], z[2], z[3], z[4], z[5], z[6], z[7], \Big(-c13\, c14\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)}\, z[1] - c21\, c22\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\, z[1] + c31\, c32\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\, z[2] - c33\, c34\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\, z[3] + c11\, c12\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)}\, z[4] + c23\, c24\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\, z[4] + c11\, c12\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)}\, z[5] - c23\, c24\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\, z[5] + c33\, c34\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\, z[6] - c31\, c32\, e^{\frac{M^2 (x0-x4)}{C3\, H}+C3\, H\, (x0+x4)}\, z[7] + e^{\frac{M^2 (x0-x4)}{C4\, H}+C4\, H\, (x0+x4)}\, (-c41\, c42\, (z[3]+z[6]) + c43\, c44\, (z[2]+z[7]))\Big) \Big/ \Big(c13\, c14\, e^{\frac{M^2 (x0-x4)}{C1\, H}+C1\, H\, (x0+x4)} - c21\, c22\, e^{\frac{M^2 (x0-x4)}{C2\, H}+C2\, H\, (x0+x4)}\Big)\Big\}$$

7 - plane : 6-plane M^6

same split octonion algebra $\mathbb{O}_{4,4}$;

again i take shortcuts, and use same Z , but add more constraints

In[1009]:=

```
Symbolize[ M^6 ]
```

In[1048]:=

```
Solve[And @@ Thread[Z == Z], Z]
```

Out[1048]=

{{}}

In[1045]:=

```
sz7 = Solve[0 == vectorFormOfType1.Z, z[7]][[-1]] // (*Full*)Simplify;
sevenPlane1 = Z /. sz7;
sixPlane78 = Solve[And @@ Thread[sevenPlane1 == sevenPlane2], Z] // (*Full*)Simplify
```

Solve: Equations may not give solutions for all "solve" variables.

Out[1047]=

$$\Big\{\Big\{z[7] \to \Big(-\Big((c11\, c12 + c13\, c14)\, C2\, C3\, C4\, M^2 \Big(c31\, c32\, e^{\frac{(C1+C3)\, (M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4))}{C1\, C3\, H}}\, z[2] +$$

$$c43\, c44\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[2] - c33\, c34\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} z[3] -$$
$$c41\, c42\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[3] + c11\, c12\, e^{\frac{2 M^2 (x0-x4)}{C1\, H}+2\, C1\, H\, (x0+x4)} z[4] -$$
$$c13\, c14\, e^{\frac{2 M^2 (x0-x4)}{C1\, H}+2\, C1\, H\, (x0+x4)} z[4] + c23\, c24\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} z[4] -$$
$$c21\, c22\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} (2\, z[1] - z[4] - z[5]) + c11\, c12\, e^{\frac{2 M^2 (x0-x4)}{C1\, H}+2\, C1\, H\, (x0+x4)} z[5] -$$
$$c13\, c14\, e^{\frac{2 M^2 (x0-x4)}{C1\, H}+2\, C1\, H\, (x0+x4)} z[5] - c23\, c24\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} z[5] +$$
$$c33\, c34\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} z[6] - c41\, c42\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[6]\Big)\Big) +$$

$$C1^2\, (c11\, c12 - c13\, c14)\, C2\, C3\, C4\, H^2 \Big(c31\, c32\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} z[2] +$$
$$c43\, c44\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[2] - c33\, c34\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} z[3] -$$
$$c41\, c42\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[3] + c11\, c12\, e^{\frac{2 M^2 (x0-x4)}{C1\, H}+2\, C1\, H\, (x0+x4)} z[4] +$$
$$c13\, c14\, e^{\frac{2 M^2 (x0-x4)}{C1\, H}+2\, C1\, H\, (x0+x4)} z[4] + c23\, c24\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} z[4] +$$
$$c11\, c12\, e^{\frac{2 M^2 (x0-x4)}{C1\, H}+2\, C1\, H\, (x0+x4)} z[5] + c13\, c14\, e^{\frac{2 M^2 (x0-x4)}{C1\, H}+2\, C1\, H\, (x0+x4)} z[5] - c23\, c24$$
$$e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} z[5] - c21\, c22\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} (2\, z[1] + z[4] + z[5]) +$$
$$c33\, c34\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} z[6] - c41\, c42\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[6]\Big) +$$

$$C1 \Big((c21\, c22 - c23\, c24)\, C3\, C4\, M^2 \Big(c31\, c32\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[2] +$$
$$c43\, c44\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[2] - c33\, c34\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[3] -$$
$$c41\, c42\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[3] + c21\, c22\, e^{\frac{2 M^2 (x0-x4)}{C2\, H}+2\, C2\, H\, (x0+x4)} z[4] +$$
$$c23\, c24\, e^{\frac{2 M^2 (x0-x4)}{C2\, H}+2\, C2\, H\, (x0+x4)} z[4] + c11\, c12\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} z[4] -$$
$$c21\, c22\, e^{\frac{2 M^2 (x0-x4)}{C2\, H}+2\, C2\, H\, (x0+x4)} z[5] - c23\, c24\, e^{\frac{2 M^2 (x0-x4)}{C2\, H}+2\, C2\, H\, (x0+x4)} z[5] +$$
$$c11\, c12\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} z[5] + c33\, c34\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[6] -$$
$$c41\, c42\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[6]\Big) + C2^2\, (c21\, c22 + c23\, c24)\, C3\, C4\, H^2$$
$$\Big(c31\, c32\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[2] + c43\, c44\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[2] -$$
$$c33\, c34\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[3] - c41\, c42\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[3] -$$
$$c21\, c22\, e^{\frac{2 M^2 (x0-x4)}{C2\, H}+2\, C2\, H\, (x0+x4)} z[4] + c23\, c24\, e^{\frac{2 M^2 (x0-x4)}{C2\, H}+2\, C2\, H\, (x0+x4)} z[4] +$$
$$c11\, c12\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} z[4] + c21\, c22\, e^{\frac{2 M^2 (x0-x4)}{C2\, H}+2\, C2\, H\, (x0+x4)} z[5] -$$
$$c23\, c24\, e^{\frac{2 M^2 (x0-x4)}{C2\, H}+2\, C2\, H\, (x0+x4)} z[5] + c11\, c12\, e^{\frac{(C1+C2)\left(M^2 (x0-x4)+C1\, C2\, H^2 (x0+x4)\right)}{C1\, C2\, H}} z[5] +$$
$$c33\, c34\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[6] - c41\, c42\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[6]\Big) +$$

$$C2\, c21\, c22 \Big(-\Big((c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} M^2\, (z[2] + z[3] - z[6])\Big) -$$
$$C3^2\, (c31\, c32 + c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} H^2\, (z[2] - z[3] + z[6]) +$$
$$C3\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} \Big(-\Big((c41\, c42 + c43\, c44)\, M^2\, (z[2] - z[3] - z[6])\Big) +$$

$$C4^2\ (c41\ c42 - c43\ c44)\ H^2\ (z[2] + z[3] + z[6])\Big)\Big) -$$

$$c13\ c14\ \Big((c21\ c22 - c23\ c24)\ C3\ C4\ e^{\frac{(C1+C2)\ (M^2\ (x0-x4)+C1\ C2\ H^2\ (x0+x4))}{C1\ C2\ H}}\ M^2\ (2\ z[1] + z[4] - z[5]) +$$

$$C2^2\ (c21\ c22 + c23\ c24)\ C3\ C4\ e^{\frac{(C1+C2)\ (M^2\ (x0-x4)+C1\ C2\ H^2\ (x0+x4))}{C1\ C2\ H}}\ H^2\ (2\ z[1] - z[4] + z[5]) +$$

$$C2\ \Big(-\Big((c31\ c32 - c33\ c34)\ C4\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}}\ M^2\ (z[2] + z[3] - z[6])\Big) - C3^2$$

$$(c31\ c32 + c33\ c34)\ C4\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}}\ H^2\ (z[2] - z[3] + z[6]) + C3$$

$$e^{\frac{(C1+C4)\ (M^2\ (x0-x4)+C1\ C4\ H^2\ (x0+x4))}{C1\ C4\ H}}\ \Big(-\Big((c41\ c42 + c43\ c44)\ M^2\ (z[2] - z[3] - z[6])\Big) +$$

$$C4^2\ (c41\ c42 - c43\ c44)\ H^2\ (z[2] + z[3] + z[6])\Big)\Big)\Big)\Big)\Big)\Big/$$

$$\Big(C1^2\ (c11\ c12 - c13\ c14)\ C2\ C3\ C4\ \Big(c31\ c32\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}} - c43\ c44\ e^{\frac{(C1+C4)\ (M^2\ (x0-x4)+C1\ C4\ H^2\ (x0+x4))}{C1\ C4\ H}}\Big)$$

$$H^2 - (c11\ c12 + c13\ c14)$$

$$C2\ C3\ C4$$

$$\Big(c31\ c32\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}} - c43\ c44\ e^{\frac{(C1+C4)\ (M^2\ (x0-x4)+C1\ C4\ H^2\ (x0+x4))}{C1\ C4\ H}}\Big)$$

$$M^2 + C1\ \Big(C2^2\ (c21\ c22 + c23\ c24)\ C3\ C4$$

$$\Big(c31\ c32\ e^{\frac{(C2+C3)\ (M^2\ (x0-x4)+C2\ C3\ H^2\ (x0+x4))}{C2\ C3\ H}} - c43\ c44\ e^{\frac{(C2+C4)\ (M^2\ (x0-x4)+C2\ C4\ H^2\ (x0+x4))}{C2\ C4\ H}}\Big)\ H^2 +$$

$$(c21\ c22 - c23\ c24)\ C3\ C4\ \Big(c31\ c32\ e^{\frac{(C2+C3)\ (M^2\ (x0-x4)+C2\ C3\ H^2\ (x0+x4))}{C2\ C3\ H}} - c43\ c44\ e^{\frac{(C2+C4)\ (M^2\ (x0-x4)+C2\ C4\ H^2\ (x0+x4))}{C2\ C4\ H}}\Big)$$

$$M^2 + c13\ c14\ C2\ \Big(C3^2\ (c31\ c32 + c33\ c34)\ C4\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}}\ H^2 +$$

$$(c31\ c32 - c33\ c34)\ C4\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}}\ M^2 +$$

$$C3\ e^{\frac{(C1+C4)\ (M^2\ (x0-x4)+C1\ C4\ H^2\ (x0+x4))}{C1\ C4\ H}}\ \Big(C4^2\ (c41\ c42 - c43\ c44)\ H^2 - (c41\ c42 + c43\ c44)\ M^2\Big)\Big) -$$

$$C2\ c21\ c22\ \Big(C3^2\ (c31\ c32 + c33\ c34)\ C4\ e^{\frac{(C2+C3)\ (M^2\ (x0-x4)+C2\ C3\ H^2\ (x0+x4))}{C2\ C3\ H}}\ H^2 +$$

$$(c31\ c32 - c33\ c34)\ C4\ e^{\frac{(C2+C3)\ (M^2\ (x0-x4)+C2\ C3\ H^2\ (x0+x4))}{C2\ C3\ H}}\ M^2 +$$

$$C3\ e^{\frac{(C2+C4)\ (M^2\ (x0-x4)+C2\ C4\ H^2\ (x0+x4))}{C2\ C4\ H}}\ \Big(C4^2\ (c41\ c42 - c43\ c44)\ H^2 - (c41\ c42 + c43\ c44)\ M^2\Big)\Big)\Big)\Big),$$

$$z[8] \to \Big((c11\ c12 + c13\ c14)\ C2\ C3\ C4\ \Big(c31\ c32\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}} -$$

$$c43\ c44\ e^{\frac{(C1+C4)\ (M^2\ (x0-x4)+C1\ C4\ H^2\ (x0+x4))}{C1\ C4\ H}}\Big)$$

$$M^2\ (z[1] - z[4] - z[5]) - C1^2\ (c11\ c12 - c13\ c14)\ C2\ C3\ C4$$

$$\Big(c31\ c32\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}} - c43\ c44\ e^{\frac{(C1+C4)\ (M^2\ (x0-x4)+C1\ C4\ H^2\ (x0+x4))}{C1\ C4\ H}}\Big)$$

$$H^2\ (z[1] + z[4] + z[5]) +$$

$$C1\ \Big(-c13\ c14\ C2\ \Big(C3^2\ (c31\ c32 + c33\ c34)\ C4\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}}\ H^2 +$$

$$(c31\ c32 - c33\ c34)\ C4\ e^{\frac{(C1+C3)\ (M^2\ (x0-x4)+C1\ C3\ H^2\ (x0+x4))}{C1\ C3\ H}}\ M^2 +$$

$$C3\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} \left(C4^2 (c41\, c42 - c43\, c44)\, H^2 - (c41\, c42 + c43\, c44)\, M^2\right) z[1] +$$

$$(c21\, c22 - c23\, c24)\, C3\, C4 \left(c31\, c32\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} - c43\, c44\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}}\right)$$

$$M^2 (z[1] + z[4] - z[5]) + C2^2 (c21\, c22 + c23\, c24)\, C3\, C4$$

$$\left(c31\, c32\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} - c43\, c44\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}}\right) H^2 (z[1] - z[4] + z[5]) +$$

$$C2 \left(-c21\, c22 \left(C3^2 (c31\, c32 + c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} H^2 +\right.\right.$$

$$(c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} M^2 + C3\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}}$$

$$\left.\left(C4^2 (c41\, c42 - c43\, c44)\, H^2 - (c41\, c42 + c43\, c44)\, M^2\right)\right) z[1] +$$

$$(c31\, c32 - c33\, c34)\, C4\, M^2 \left(-c33\, c34\, e^{\frac{2 M^2 (x0-x4)}{C3\, H} + 2\, C3\, H\, (x0+x4)}\, z[3] - c41\, c42\right.$$

$$e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} z[3] + c11\, c12\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} z[4] + c23$$

$$c24\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[4] + c11\, c12\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}}$$

$$z[5] - c23\, c24\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[5] - c31\, c32\, e^{\frac{2 M^2 (x0-x4)}{C3\, H} + 2\, C3\, H\, (x0+x4)}$$

$$(z[3] - z[6]) + c43\, c44\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} (2\, z[2] + z[3] - z[6]) + c33$$

$$\left.c34\, e^{\frac{2 M^2 (x0-x4)}{C3\, H} + 2\, C3\, H\, (x0+x4)}\, z[6] - c41\, c42\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} z[6]\right) +$$

$$C3^2 (c31\, c32 + c33\, c34)\, C4\, H^2 \left(-c33\, c34\, e^{\frac{2 M^2 (x0-x4)}{C3\, H} + 2\, C3\, H\, (x0+x4)}\, z[3] - c41\, c42\right.$$

$$e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} z[3] + c11\, c12\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} z[4] + c23$$

$$c24\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[4] + c11\, c12\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}}$$

$$z[5] - c23\, c24\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} z[5] + c31\, c32\, e^{\frac{2 M^2 (x0-x4)}{C3\, H} + 2\, C3\, H\, (x0+x4)}$$

$$(z[3] - z[6]) + c33\, c34\, e^{\frac{2 M^2 (x0-x4)}{C3\, H} + 2\, C3\, H\, (x0+x4)}\, z[6] - c41\, c42\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}}$$

$$\left.z[6] + c43\, c44\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} (2\, z[2] - z[3] + z[6])\right) +$$

$$C3 \left((c41\, c42 + c43\, c44)\, M^2 \left(c33\, c34\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} z[3] -\right.\right.$$

$$c11\, c12\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[4] - c23\, c24\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[4] -$$

$$c11\, c12\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[5] + c23\, c24\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[5] - c33$$

$$c34\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} z[6] + c41\, c42\, e^{\frac{2 M^2 (x0-x4)}{C4\, H} + 2\, C4\, H\, (x0+x4)} (z[3] + z[6]) -$$

$$\left.c43\, c44\, e^{\frac{2 M^2 (x0-x4)}{C4\, H} + 2\, C4\, H\, (x0+x4)} (z[3] + z[6])\right) - C4^2 (c41\, c42 - c43\, c44)\, H^2$$

$$\left(c33\, c34\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} z[3] - c11\, c12\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[4] -\right.$$

$$c23\, c24\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[4] - c11\, c12\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} z[5] +$$

$$c23\, c24\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} z[5] - c33\, c34\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} z[6] +$$

$$c41\, c42\, e^{\frac{2 M^2 (x0-x4)}{C4\, H} + 2\, C4\, H\, (x0+x4)} (z[3] + z[6]) + c43\, c44\, e^{\frac{2 M^2 (x0-x4)}{C4\, H} + 2\, C4\, H\, (x0+x4)} (z[3] +$$

$$z[6])\Big) + c31\, c32\, e^{\frac{(C3+C4)\left(M^2 (x0-x4)+C3\, C4\, H^2 (x0+x4)\right)}{C3\, C4\, H}} \Big(-\Big((c41\, c42 + c43\, c44)\, M^2\, (2\, z[2] - z[3] - z[6])\Big) + C4^2\, (c41\, c42 - c43\, c44)\, H^2\, (2\, z[2] + z[3] + z[6])\Big)\Big)\Big)\Big)\Big)\Big) \Big/$$

$$\Big(C1^2\, (c11\, c12 - c13\, c14)\, C2\, C3\, C4 \Big(c31\, c32\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} - c43\, c44\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}}\Big) H^2 - (c11\, c12 + c13\, c14)\, C2\, C3\, C4 \Big(c31\, c32\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} - c43\, c44\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}}\Big) M^2 + C1 \Big(C2^2\, (c21\, c22 + c23\, c24)\, C3\, C4 \Big(c31\, c32\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} - c43\, c44\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}}\Big) H^2 + (c21\, c22 - c23\, c24)\, C3\, C4 \Big(c31\, c32\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} - c43\, c44\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}}\Big) M^2 + c13\, c14\, C2 \Big(C3^2\, (c31\, c32 + c33\, c34)\, C4\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} H^2 + (c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C1+C3)\left(M^2 (x0-x4)+C1\, C3\, H^2 (x0+x4)\right)}{C1\, C3\, H}} M^2 + C3\, e^{\frac{(C1+C4)\left(M^2 (x0-x4)+C1\, C4\, H^2 (x0+x4)\right)}{C1\, C4\, H}} \Big(C4^2\, (c41\, c42 - c43\, c44)\, H^2 - (c41\, c42 + c43\, c44)\, M^2\Big)\Big) - C2\, c21\, c22 \Big(C3^2\, (c31\, c32 + c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} H^2 + (c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2 (x0-x4)+C2\, C3\, H^2 (x0+x4)\right)}{C2\, C3\, H}} M^2 + C3\, e^{\frac{(C2+C4)\left(M^2 (x0-x4)+C2\, C4\, H^2 (x0+x4)\right)}{C2\, C4\, H}} \Big(C4^2\, (c41\, c42 - c43\, c44)\, H^2 - (c41\, c42 + c43\, c44)\, M^2\Big)\Big)\Big)\Big)\Big\}\Big\}$$

6 - plane = 7 - plane1 ∩ 7 - plane2 (example using z[7] and z[8]):

vectorFormOfType1 is dual to 7-plane 7_1 ;
vectorFormOfType2 is dual to 7-plane 7_2 ;
Assuming that 7_1 and 7_2 are not parallel,
M6 = $7_1 \cap 7_2$
M6 has 28 parameterizations; ℝ = linear span of eA[1]; skip ℝ and start with {z[8], z[7]} = ...

In[1049]:=

```
M^6 = Z /. sixPlane78 /. ssx0x4
```

Out[1049]=

$$\Big\{\Big\{z[1],\, z[2],\, z[3],\, z[4],\, z[5],\, z[6],$$

$$\Big(-\Big((c11\, c12 + c13\, c14)\, C2\, C3\, C4\, M^2 \Big(c31\, c32\, e^{\frac{(C1+C3)\left(M^2 \left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2 \left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} z[2] +$$

$$c43\, c44\, e^{\frac{(C1+C4)\left(M^2 \left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2 \left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} z[2] -$$

$$c33\, c34\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} z[3] -$$

$$c41\, c42\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} z[3] +$$

$$c11\, c12\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C1\, H}+2\, C1\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)} z[4] -$$

$$c13\, c14\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C1\, H}+2\, C1\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)} z[4] +$$

$$c23\, c24\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}} z[4] -$$

$$c21\, c22\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}} (2\, z[1] - z[4] - z[5]) +$$

$$c11\, c12\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C1\, H}+2\, C1\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)} z[5] -$$

$$c13\, c14\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C1\, H}+2\, C1\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)} z[5] -$$

$$c23\, c24\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}} z[5] +$$

$$c33\, c34\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} z[6] -$$

$$c41\, c42\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} z[6]\Bigg)\Bigg) +$$

$$C1^2\,(c11\, c12 - c13\, c14)\, C2\, C3\, C4\, H^2\Bigg(c31\, c32\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} z[2] +$$

$$c43\, c44\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} z[2] -$$

$$c33\, c34\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} z[3] -$$

$$c41\, c42\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} z[3] +$$

$$c11\, c12\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C1\, H}+2\, C1\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)} z[4] +$$

$$c13\, c14\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C1\, H}+2\, C1\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)} z[4] +$$

$$c23\, c24\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}} z[4] +$$

$$c11\, c12\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C1\, H}+2\, C1\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)} z[5] +$$

$$c13\, c14\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C1\, H}+2\, C1\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)} z[5] -$$

$$c23\, c24\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}} z[5] -$$

$$c21\, c22\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}} (2\, z[1] + z[4] + z[5]) +$$

$$c33\, c34\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} z[6] -$$

$$c41\, c42\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}}\, z[6]\Bigg) +$$

$$C1\Bigg((c21\, c22 - c23\, c24)\, C3\, C4\, M^2\Bigg(c31\, c32\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}}\, z[2] +$$

$$c43\, c44\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\, z[2] -$$

$$c33\, c34\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}}\, z[3] -$$

$$c41\, c42\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\, z[3] +$$

$$c21\, c22\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C2\, H}+2\, C2\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\, z[4] +$$

$$c23\, c24\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C2\, H}+2\, C2\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\, z[4] +$$

$$c11\, c12\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}}\, z[4] -$$

$$c21\, c22\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C2\, H}+2\, C2\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\, z[5] -$$

$$c23\, c24\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C2\, H}+2\, C2\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\, z[5] +$$

$$c11\, c12\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}}\, z[5] +$$

$$c33\, c34\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}}\, z[6] -$$

$$c41\, c42\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\, z[6]\Bigg) +$$

$$C2^2\, (c21\, c22 + c23\, c24)\, C3\, C4\, H^2\Bigg(c31\, c32\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}}\, z[2] +$$

$$c43\, c44\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\, z[2] -$$

$$c33\, c34\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}}\, z[3] -$$

$$c41\, c42\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\, z[3] -$$

$$c21\, c22\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C2\, H}+2\, C2\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\, z[4] +$$

$$c23\, c24\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C2\, H}+2\, C2\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\, z[4] +$$

$$c11\, c12\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}}\, z[4] +$$

$$c21\, c22\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C2\, H}+2\, C2\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\, z[5] -$$

$$c23\, c24\, e^{\frac{2 M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C2\, H}+2\, C2\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\, z[5] +$$

$$c11\, c12\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}}\, z[5] +$$

$$c33\, c34\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}}\, z[6] -$$

$$c41\, c42\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\, z[6]\Bigg) +$$

$$C2\, c21\, c22 \Bigg(-\Bigg((c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}}$$

$$M^2\, (z[2] + z[3] - z[6])\Bigg) - C3^2\, (c31\, c32 + c33\, c34)\, C4$$

$$e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}}\, H^2\, (z[2] - z[3] + z[6]) +$$

$$C3\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}} \Big(-\Big((c41\, c42 + c43\, c44)\, M^2$$

$$(z[2] - z[3] - z[6])\Big) + C4^2\, (c41\, c42 - c43\, c44)\, H^2\, (z[2] + z[3] + z[6])\Big)\Bigg) -$$

$$c13\, c14 \Bigg((c21\, c22 - c23\, c24)\, C3\, C4\, e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}}$$

$$M^2\, (2\, z[1] + z[4] - z[5]) + C2^2\, (c21\, c22 + c23\, c24)\, C3\, C4$$

$$e^{\frac{(C1+C2)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C2\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C2\, H}}\, H^2\, (2\, z[1] - z[4] + z[5]) +$$

$$C2 \Bigg(-\Bigg((c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}}$$

$$M^2\, (z[2] + z[3] - z[6])\Bigg) - C3^2\, (c31\, c32 + c33\, c34)\, C4$$

$$e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}}\, H^2\, (z[2] - z[3] + z[6]) +$$

$$C3\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} \Big(-\Big((c41\, c42 + c43\, c44)\, M^2\, (z[2] -$$

$$z[3] - z[6])\Big) + C4^2\, (c41\, c42 - c43\, c44)\, H^2\, (z[2] + z[3] + z[6])\Big)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg) \Big/$$

$$\Bigg(C1^2\, (c11\, c12 - c13\, c14)\, C2\, C3\, C4 \Bigg(c31\, c32\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} -$$

$$c43\, c44\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}}\Bigg)\, H^2 -$$

$$(c11\, c12 + c13\, c14)\, C2\, C3\, C4 \Bigg(c31\, c32\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} -$$

$$c43\, c44\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}}\Bigg)\, M^2 +$$

$$
\begin{aligned}
&C1 \Bigg(C2^2 \, (c21\, c22 + c23\, c24)\, C3\, C4 \Bigg(c31\, c32\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H} - \\
&\quad c43\, c44\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}} \Bigg) H^2 + \\
&\quad (c21\, c22 - c23\, c24)\, C3\, C4 \Bigg(c31\, c32\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H} - \\
&\quad c43\, c44\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}} \Bigg) M^2 + \\
&\quad c13\, c14\, C2 \Bigg(C3^2 \, (c31\, c32 + c33\, c34)\, C4\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} H^2 + \\
&\quad (c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} M^2 + \\
&\quad C3\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} \\
&\quad \left(C4^2 \, (c41\, c42 - c43\, c44)\, H^2 - (c41\, c42 + c43\, c44)\, M^2 \right) \Bigg) - \\
&\quad C2\, c21\, c22 \Bigg(C3^2 \, (c31\, c32 + c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}} H^2 + \\
&\quad (c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}} M^2 + \\
&\quad C3\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}} \\
&\quad \left(C4^2 \, (c41\, c42 - c43\, c44)\, H^2 - (c41\, c42 + c43\, c44)\, M^2 \right) \Bigg) \Bigg) \Bigg) \Bigg) ,
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg((c11\, c12 + c13\, c14)\, C2\, C3\, C4 \Bigg(c31\, c32\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} - \\
&\quad c43\, c44\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} \Bigg) \\
&\quad M^2 \, (z[1] - z[4] - z[5]) - C1^2 \, (c11\, c12 - c13\, c14) \\
&\quad C2 \\
&\quad C3 \\
&\quad C4 \\
&\quad \Bigg(c31\, c32\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} - \\
&\quad c43\, c44\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}} \Bigg) H^2 \, (z[1] + z[4] + z[5]) + \\
&C1 \Bigg(-c13\, c14\, C2 \Bigg(C3^2 \, (c31\, c32 + c33\, c34)\, C4\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} H^2 +
\end{aligned}
$$

$$
\begin{aligned}
&(c31\,c32 - c33\,c34)\,C4\,e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C3\,H}}\,M^2 + \\
&C3\,e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C4\,H}} \\
&\left(C4^2\,(c41\,c42 - c43\,c44)\,H^2 - (c41\,c42 + c43\,c44)\,M^2\right)\Bigg)\,z[1] + \\
&(c21\,c22 - c23\,c24)\,C3\,C4\,\Bigg(c31\,c32\,e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C3\,H}} - \\
&c43\,c44\,e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C4\,H}}\Bigg)\,M^2\,(z[1] + z[4] - z[5]) + \\
&C2^2\,(c21\,c22 + c23\,c24)\,C3\,C4\,\Bigg(c31\,c32\,e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C3\,H}} - \\
&c43\,c44\,e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C4\,H}}\Bigg)\,H^2\,(z[1] - z[4] + z[5]) + \\
&C2\,\Bigg(-c21\,c22\,\Bigg(C3^2\,(c31\,c32 + c33\,c34)\,C4\,e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C3\,H}}\,H^2 + \\
&(c31\,c32 - c33\,c34)\,C4\,e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C3\,H}}\,M^2 + \\
&C3\,e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C4\,H}} \\
&\left(C4^2\,(c41\,c42 - c43\,c44)\,H^2 - (c41\,c42 + c43\,c44)\,M^2\right)\Bigg)\,z[1] + \\
&(c31\,c32 - c33\,c34)\,C4\,M^2\,\Bigg(-c33\,c34\,e^{\frac{2\,M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C3\,H}+2\,C3\,H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,z[3] - \\
&c41\,c42\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}}\,z[3] + \\
&c11\,c12\,e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C3\,H}}\,z[4] + \\
&c23\,c24\,e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C3\,H}}\,z[4] + \\
&c11\,c12\,e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C3\,H}}\,z[5] - \\
&c23\,c24\,e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C3\,H}}\,z[5] - \\
&c31\,c32\,e^{\frac{2\,M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C3\,H}+2\,C3\,H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,(z[3] - z[6]) + \\
&c43\,c44\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}}\,(2\,z[2] + z[3] - z[6]) + \\
&c33\,c34\,e^{\frac{2\,M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C3\,H}+2\,C3\,H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,z[6] - \\
&c41\,c42\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}}\,z[6]\Bigg) +
\end{aligned}
$$

$$
\begin{aligned}
& C3^2\,(c31\,c32 + c33\,c34)\,C4\,H^2 \Bigg(-c33\,c34\,e^{\frac{2 M^2 \left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C3\,H}+2\,C3\,H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,z[3] - \\
& c41\,c42\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}}\,z[3] + \\
& c11\,c12\,e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C3\,H}}\,z[4] + \\
& c23\,c24\,e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C3\,H}}\,z[4] + \\
& c11\,c12\,e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C3\,H}}\,z[5] - \\
& c23\,c24\,e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C3\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C3\,H}}\,z[5] + \\
& c31\,c32\,e^{\frac{2 M^2 \left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C3\,H}+2\,C3\,H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,(z[3]-z[6]) + \\
& c33\,c34\,e^{\frac{2 M^2 \left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C3\,H}+2\,C3\,H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,z[6] - \\
& c41\,c42\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}}\,z[6] + \\
& c43\,c44\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}}\,(2\,z[2]-z[3]+z[6]) \Bigg) + \\
& C3 \Bigg((c41\,c42 + c43\,c44)\,M^2 \Bigg(c33\,c34\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}}\,z[3] - \\
& c11\,c12\,e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C4\,H}}\,z[4] - \\
& c23\,c24\,e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C4\,H}}\,z[4] - \\
& c11\,c12\,e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C4\,H}}\,z[5] + \\
& c23\,c24\,e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C4\,H}}\,z[5] - \\
& c33\,c34\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}}\,z[6] + \\
& c41\,c42\,e^{\frac{2 M^2 \left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C4\,H}+2\,C4\,H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,(z[3]+z[6]) - \\
& c43\,c44\,e^{\frac{2 M^2 \left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C4\,H}+2\,C4\,H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,(z[3]+z[6]) \Bigg) - \\
& C4^2\,(c41\,c42 - c43\,c44)\,H^2 \Bigg(c33\,c34\,e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\,C4\,H}} \\
& z[3] - c11\,c12\,e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C4\,H}}\,z[4] - \\
& c23\,c24\,e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\,C4\,H}}\,z[4] - \\
& c11\,c12\,e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\,C4\,H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\,C4\,H}}\,z[5] +
\end{aligned}
$$

$$c23\, c24\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\, z[5] -$$

$$c33\, c34\, e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\, C4\, H}}\, z[6] +$$

$$c41\, c42\, e^{\frac{2\, M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C4\, H}+2\, C4\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,(z[3]+z[6]) +$$

$$c43\, c44\, e^{\frac{2\, M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)}{C4\, H}+2\, C4\, H\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)}\,(z[3]+z[6])\Bigg) + c31\, c32$$

$$e^{\frac{(C3+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C3\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C3\, C4\, H}}\Big(-\big((c41\, c42+c43\, c44)\, M^2\,(2\, z[2]-z[$$

$$3]-z[6])\big)+C4^2\,(c41\, c42-c43\, c44)\, H^2\,(2\, z[2]+z[3]+z[6])\Big)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg/$$

$$\Bigg(C1^2\,(c11\, c12-c13\, c14)\, C2\, C3\, C4\Bigg(c31\, c32\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} -$$

$$c43\, c44\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}}\Bigg)$$

$$H^2-(c11\, c12+c13\, c14)\, C2\, C3$$

$$C4$$

$$\Bigg(c31\, c32\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}} -$$

$$c43\, c44\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}}\Bigg)\, M^2 +$$

$$C1\Bigg(C2^2\,(c21\, c22+c23\, c24)\, C3\, C4\Bigg(c31\, c32\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}} -$$

$$c43\, c44\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\Bigg)\, H^2 +$$

$$(c21\, c22-c23\, c24)\, C3\, C4\Bigg(c31\, c32\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}} -$$

$$c43\, c44\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}}\Bigg)\, M^2 +$$

$$c13\, c14\, C2\Bigg(C3^2\,(c31\, c32+c33\, c34)\, C4\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}}\, H^2 +$$

$$(c31\, c32-c33\, c34)\, C4\, e^{\frac{(C1+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C3\, H}}\, M^2 +$$

$$C3\, e^{\frac{(C1+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C1\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C1\, C4\, H}}$$

$$\left(C4^2\,(c41\, c42-c43\, c44)\, H^2-(c41\, c42+c43\, c44)\, M^2\right)\Bigg) -$$

$$C2\, c21\, c22 \left(C3^2 (c31\, c32 + c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}} H^2 + (c31\, c32 - c33\, c34)\, C4\, e^{\frac{(C2+C3)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C3\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C3\, H}} M^2 + C3\, e^{\frac{(C2+C4)\left(M^2\left(\frac{\xi adv-\xi ret}{2}+\frac{\xi adv+\xi ret}{2}\right)+C2\, C4\, H^2\left(\frac{1}{2}(-\xi adv+\xi ret)+\frac{\xi adv+\xi ret}{2}\right)\right)}{C2\, C4\, H}} \left(C4^2 (c41\, c42 - c43\, c44)\, H^2 - (c41\, c42 + c43\, c44)\, M^2\right)\right)\Bigg)\Bigg)\Bigg

$$c13\, c14\, C2 \left(C3^2 \left(c31\, c32 + c33\, c34\right) C4\, e^{\frac{(C1+C3)\left(M^2 \xi adv + C1\, C3\, H^2 \xi ret\right)}{C1\, C3\, H}} H^2 + \left(c31\, c32 - c33\, c34\right) C4\, e^{\frac{(C1+C3)\left(M^2 \xi adv + C1\, C3\, H^2 \xi ret\right)}{C1\, C3\, H}} M^2 + C3\, e^{\frac{(C1+C4)\left(M^2 \xi adv + C1\, C4\, H^2 \xi ret\right)}{C1\, C4\, H}} \left(C4^2 \left(c41\, c42 - c43\, c44\right) H^2 - \left(c41\, c42 + c43\, c44\right) M^2\right)\right) - C2\, c21\, c22 \left(C3^2 \left(c31\, c32 + c33\, c34\right) C4\, e^{\frac{(C2+C3)\left(M^2 \xi adv + C2\, C3\, H^2 \xi ret\right)}{C2\, C3\, H}} H^2 + \left(c31\, c32 - c33\, c34\right) C4\, e^{\frac{(C2+C3)\left(M^2 \xi adv + C2\, C3\, H^2 \xi ret\right)}{C2\, C3\, H}} M^2 + C3\, e^{\frac{(C2+C4)\left(M^2 \xi adv + C2\, C4\, H^2 \xi ret\right)}{C2\, C4\, H}} \left(C4^2 \left(c41\, c42 - c43\, c44\right) H^2 - \left(c41\, c42 + c43\, c44\right) M^2\right)\right)\Bigg)\Bigg) \Big/$$

$$\Bigg(C1^2 \left(c11\, c12 - c13\, c14\right) C2\, C3\, C4\, e^{\frac{M^2 \xi adv}{C1\, H} + C1\, H\, \xi ret} \left(c31\, c32\, e^{\frac{M^2 \xi adv}{C3\, H} + C3\, H\, \xi ret} - c43\, c44\, e^{\frac{M^2 \xi adv}{C4\, H} + C4\, H\, \xi ret}\right) H^2 - \left(c11\, c12 + c13\, c14\right) C2\, C3\, C4\, e^{\frac{M^2 \xi adv}{C1\, H} + C1\, H\, \xi ret} \left(c31\, c32\, e^{\frac{M^2 \xi adv}{C3\, H} + C3\, H\, \xi ret} - c43\, c44\, e^{\frac{M^2 \xi adv}{C4\, H} + C4\, H\, \xi ret}\right) M^2 + C1 \Bigg(C2^2 \left(c21\, c22 + c23\, c24\right) C3\, C4\, e^{\frac{M^2 \xi adv}{C2\, H} + C2\, H\, \xi ret} \left(c31\, c32\, e^{\frac{M^2 \xi adv}{C3\, H} + C3\, H\, \xi ret} - c43\, c44\, e^{\frac{M^2 \xi adv}{C4\, H} + C4\, H\, \xi ret}\right) H^2 + \left(c21\, c22 - c23\, c24\right) C3\, C4\, e^{\frac{M^2 \xi adv}{C2\, H} + C2\, H\, \xi ret} \left(c31\, c32\, e^{\frac{M^2 \xi adv}{C3\, H} + C3\, H\, \xi ret} - c43\, c44\, e^{\frac{M^2 \xi adv}{C4\, H} + C4\, H\, \xi ret}\right) M^2 + c13\, c14\, C2 \left(C3^2 \left(c31\, c32 + c33\, c34\right) C4\, e^{\frac{(C1+C3)\left(M^2 \xi adv + C1\, C3\, H^2 \xi ret\right)}{C1\, C3\, H}} H^2 + \left(c31\, c32 - c33\, c34\right) C4\, e^{\frac{(C1+C3)\left(M^2 \xi adv + C1\, C3\, H^2 \xi ret\right)}{C1\, C3\, H}} M^2 + C3\, e^{\frac{(C1+C4)\left(M^2 \xi adv + C1\, C4\, H^2 \xi ret\right)}{C1\, C4\, H}} \left(C4^2 \left(c41\, c42 - c43\, c44\right) H^2 - \left(c41\, c42 + c43\, c44\right) M^2\right)\right) - C2\, c21\, c22 \left(C3^2 \left(c31\, c32 + c33\, c34\right) C4\, e^{\frac{(C2+C3)\left(M^2 \xi adv + C2\, C3\, H^2 \xi ret\right)}{C2\, C3\, H}} H^2 + \left(c31\, c32 - c33\, c34\right) C4\, e^{\frac{(C2+C3)\left(M^2 \xi adv + C2\, C3\, H^2 \xi ret\right)}{C2\, C3\, H}} M^2 + C3\, e^{\frac{(C2+C4)\left(M^2 \xi adv + C2\, C4\, H^2 \xi ret\right)}{C2\, C4\, H}} \left(C4^2 \left(c41\, c42 - c43\, c44\right) H^2 - \left(c41\, c42 + c43\, c44\right) M^2\right)\right)\Bigg)\Bigg)\Bigg\}\Bigg\}$$

In[991]:=

```
E^A_a[hUSE].unit
%.EA
% === F^A_a.unit.EA
```

Out[991]=

```
{1, 0, 0, 0, 0, 0, 0, 0}
```

Out[992]=

```
eA[1]
```

Out[993]=

```
True
```

In[842]:=

```
Binomial[8, 2]
```

Out[842]=

```
28
```

vectorFormOfType1 is dual to 7-plane 7_1 ;

vectorFormOfType2 is dual to 7-plane 7_2 ;

Assuming that 7_1 and 7_2 are not parallel,
M6 = $7_1 \cap 7_2$
M6 has 28 parameterizations; ℝ = linear span of eA[1]; skip ℝ and start with {z[8], z[7]} = ...

In[855]:=

```
{sξ, ssx0x4}
```

Out[855]=

$$\left\{\{\xi\text{adv} \to \text{x0} - \text{x4},\ \xi\text{ret} \to \text{x0} + \text{x4}\},\ \left\{\text{x0} \to \frac{\xi\text{adv} + \xi\text{ret}}{2},\ \text{x4} \to \frac{1}{2}\ (-\xi\text{adv} + \xi\text{ret})\right\}\right\}$$

In[1050]:=

```
(*time1=Now
 timeA=AbsoluteTiming[
    M6=Solve[And@@Thread[0==Flatten[{vectorFormOfType1.Z,vectorFormOfType2.Z}]],
          {z[7],z[8]}][[-1]]/.ssx0x4//(*Full*)Simplify];
 Print[timeA,"; ",Now-time1];*)
```

(* ℝ = linear span of eA[1]; try solving for eA[1] and eA[7]*)

In[1028]:=

```
(*time1=Now
 timeA=AbsoluteTiming[M617=
     Solve[And@@Thread[0==Flatten[{vectorFormOfType1.EA,vectorFormOfType2.EA}]],
        {eA[1],eA[7]}]/.ssx0x4//(*Full*)Simplify];
 Print[timeA,"; ",Now-time1];*)
```

checks :

In[751]:=

$E^A{}_a[\text{hUSE}].E^a{}_A[\text{hUSE}] === E^a{}_A[\text{hUSE}].E^A{}_a[\text{hUSE}] === \text{ID8}$

Out[751]=

True

In[752]:=

$F^a_A.F^A_a === F^A_a.F^a_A === \text{ID8}$

Out[752]=

True

In[ ]:=

```
(*(EA/.M6).unit*)
```

Out[764]=

$$\frac{\text{eA}[1]}{\sqrt{2}} + \frac{\text{eA}[5]}{\sqrt{2}}$$

M617 $\subseteq \mathrm{Im}\,\mathbf{O}$, if 0 == eA[1] /. M617

In[1051]:=

```
(*eA[1]/.M617[[1]]//(*Full*)Simplify*)
```

M617 $\subseteq$ Im O, if 0 ==

In[1052]:=

```
(*(EA/.M617).unit//(*Full*)Simplify*)
```

try

In[814]:=

$$\text{EtoExp}\left[\text{YZ0}[z,t] \to \frac{e^{-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\,H^2}+C1\left(t+\frac{z}{6}\right)}\left(C1^2\,c11\,c12\,H^2-C1^2\,c13\,c14\,H^2-c11\,c12\,M^2-c13\,c14\,M^2\right)}{2\,C1\,H\,M}\right]$$

```
FullForm[%]
```

Out[814]=

$$\text{YZ0}[z,t] \to \frac{\left(C1^2\,c11\,c12\,H^2-C1^2\,c13\,c14\,H^2-c11\,c12\,M^2-c13\,c14\,M^2\right)\,\text{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\,H^2}+C1\left(t+\frac{z}{6}\right)\right]}{2\,C1\,H\,M}$$

Out[815]//FullForm=

```
Rule[YZ0[z, t], Times[Rational[1, 2], Power[C1, -1], Power[H, -1], Power[M, -1],
  Plus[Times[Power[C1, 2], c11, c12, Power[H, 2]], Times[-1, Power[C1, 2], c13, c14,
    Power[H, 2]], Times[-1, c11, c12, Power[M, 2]], Times[-1, c13, c14, Power[M, 2]]],
  Inactive[Exp][Plus[Times[-1, Power[C1, -1], Power[H, -2], Power[M, 2],
     Plus[t, Times[Rational[-1, 6], z]]], Times[C1, Plus[t, Times[Rational[1, 6], z]]]]]]]]
```

In[816]:=

```
EtoExp[#] & /@ sYZvar
# /. {Exp → Sin} & //@ %
```

Out[816]=

$$\left\{YZ0[z,t] \to \frac{\left(C1^2\, c11\, c12\, H^2 - C1^2\, c13\, c14\, H^2 - c11\, c12\, M^2 - c13\, c14\, M^2\right) \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\, H^2} + C1\left(t+\frac{z}{6}\right)\right]}{2\, C1\, H\, M},\right.$$

$$YZ1[z,t] \to -\frac{\left(C1^2\, c11\, c12\, H^2 - C1^2\, c13\, c14\, H^2 + c11\, c12\, M^2 + c13\, c14\, M^2\right) \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\, H^2} + C1\left(t+\frac{z}{6}\right)\right]}{2\, C1\, H\, M},$$

$$YZ2[z,t] \to c13\, c14 \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\, H^2} + C1\left(t+\frac{z}{6}\right)\right],$$

$$YZ3[z,t] \to c11\, c12 \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\, H^2} + C1\left(t+\frac{z}{6}\right)\right],$$

$$YZ4[z,t] \to -\frac{\left(C2^2\, c21\, c22\, H^2 + C2^2\, c23\, c24\, H^2 - c21\, c22\, M^2 + c23\, c24\, M^2\right) \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\, H^2} + C2\left(t+\frac{z}{6}\right)\right]}{2\, C2\, H\, M},$$

$$YZ5[z,t] \to -\frac{\left(C2^2\, c21\, c22\, H^2 + C2^2\, c23\, c24\, H^2 + c21\, c22\, M^2 - c23\, c24\, M^2\right) \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\, H^2} + C2\left(t+\frac{z}{6}\right)\right]}{2\, C2\, H\, M},$$

$$YZ6[z,t] \to c23\, c24 \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\, H^2} + C2\left(t+\frac{z}{6}\right)\right],$$

$$YZ7[z,t] \to c21\, c22 \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\, H^2} + C2\left(t+\frac{z}{6}\right)\right],$$

$$YZ8[z,t] \to -\frac{\left(C3^2\, c31\, c32\, H^2 + C3^2\, c33\, c34\, H^2 - c31\, c32\, M^2 + c33\, c34\, M^2\right) \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\, H^2} + C3\left(t+\frac{z}{6}\right)\right]}{2\, C3\, H\, M},$$

$$YZ9[z,t] \to -\frac{\left(C3^2\, c31\, c32\, H^2 + C3^2\, c33\, c34\, H^2 + c31\, c32\, M^2 - c33\, c34\, M^2\right) \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\, H^2} + C3\left(t+\frac{z}{6}\right)\right]}{2\, C3\, H\, M},$$

$$YZ10[z,t] \to c33\, c34 \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\, H^2} + C3\left(t+\frac{z}{6}\right)\right],$$

$$YZ11[z,t] \to c31\, c32 \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\, H^2} + C3\left(t+\frac{z}{6}\right)\right],$$

$$YZ12[z,t] \to \frac{\left(C4^2\, c41\, c42\, H^2 - C4^2\, c43\, c44\, H^2 - c41\, c42\, M^2 - c43\, c44\, M^2\right) \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\, H^2} + C4\left(t+\frac{z}{6}\right)\right]}{2\, C4\, H\, M},$$

$$YZ13[z,t] \to -\frac{\left(C4^2\, c41\, c42\, H^2 - C4^2\, c43\, c44\, H^2 + c41\, c42\, M^2 + c43\, c44\, M^2\right) \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\, H^2} + C4\left(t+\frac{z}{6}\right)\right]}{2\, C4\, H\, M},$$

$$YZ14[z,t] \to c43\, c44 \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\, H^2} + C4\left(t+\frac{z}{6}\right)\right],$$

$$\left.YZ15[z,t] \to c41\, c42 \operatorname{Exp}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\, H^2} + C4\left(t+\frac{z}{6}\right)\right]\right\}$$

Out[817]=

$$\left\{YZ0[z,t] \to \frac{\left(C1^2\, c11\, c12\, H^2 - C1^2\, c13\, c14\, H^2 - c11\, c12\, M^2 - c13\, c14\, M^2\right) \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\, H^2} + C1\left(t+\frac{z}{6}\right)\right]}{2\, C1\, H\, M},\right.$$

$$YZ1[z,t] \to -\frac{\left(C1^2\, c11\, c12\, H^2 - C1^2\, c13\, c14\, H^2 + c11\, c12\, M^2 + c13\, c14\, M^2\right) \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\, H^2} + C1\left(t+\frac{z}{6}\right)\right]}{2\, C1\, H\, M},$$

$$YZ2[z,t] \to c13\, c14\, \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\, H^2} + C1\left(t+\frac{z}{6}\right)\right],$$

$$YZ3[z,t] \to c11\, c12\, \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C1\, H^2} + C1\left(t+\frac{z}{6}\right)\right],$$

$$YZ4[z,t] \to -\frac{\left(C2^2\, c21\, c22\, H^2 + C2^2\, c23\, c24\, H^2 - c21\, c22\, M^2 + c23\, c24\, M^2\right) \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\, H^2} + C2\left(t+\frac{z}{6}\right)\right]}{2\, C2\, H\, M},$$

$$YZ5[z,t] \to -\frac{\left(C2^2\, c21\, c22\, H^2 + C2^2\, c23\, c24\, H^2 + c21\, c22\, M^2 - c23\, c24\, M^2\right) \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\, H^2} + C2\left(t+\frac{z}{6}\right)\right]}{2\, C2\, H\, M},$$

$$YZ6[z,t] \to c23\, c24\, \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\, H^2} + C2\left(t+\frac{z}{6}\right)\right],$$

$$YZ7[z,t] \to c21\, c22\, \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C2\, H^2} + C2\left(t+\frac{z}{6}\right)\right],$$

$$YZ8[z,t] \to -\frac{\left(C3^2\, c31\, c32\, H^2 + C3^2\, c33\, c34\, H^2 - c31\, c32\, M^2 + c33\, c34\, M^2\right) \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\, H^2} + C3\left(t+\frac{z}{6}\right)\right]}{2\, C3\, H\, M},$$

$$YZ9[z,t] \to -\frac{\left(C3^2\, c31\, c32\, H^2 + C3^2\, c33\, c34\, H^2 + c31\, c32\, M^2 - c33\, c34\, M^2\right) \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\, H^2} + C3\left(t+\frac{z}{6}\right)\right]}{2\, C3\, H\, M},$$

$$YZ10[z,t] \to c33\, c34\, \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\, H^2} + C3\left(t+\frac{z}{6}\right)\right],$$

$$YZ11[z,t] \to c31\, c32\, \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C3\, H^2} + C3\left(t+\frac{z}{6}\right)\right],$$

$$YZ12[z,t] \to \frac{\left(C4^2\, c41\, c42\, H^2 - C4^2\, c43\, c44\, H^2 - c41\, c42\, M^2 - c43\, c44\, M^2\right) \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\, H^2} + C4\left(t+\frac{z}{6}\right)\right]}{2\, C4\, H\, M},$$

$$YZ13[z,t] \to -\frac{\left(C4^2\, c41\, c42\, H^2 - C4^2\, c43\, c44\, H^2 + c41\, c42\, M^2 + c43\, c44\, M^2\right) \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\, H^2} + C4\left(t+\frac{z}{6}\right)\right]}{2\, C4\, H\, M},$$

$$YZ14[z,t] \to c43\, c44\, \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\, H^2} + C4\left(t+\frac{z}{6}\right)\right],$$

$$\left.YZ15[z,t] \to c41\, c42\, \mathrm{Sin}\left[-\frac{M^2\left(t-\frac{z}{6}\right)}{C4\, H^2} + C4\left(t+\frac{z}{6}\right)\right]\right\}$$

In[822]:=

```
EtoExp[ToString[#, InputForm]] & /@ sYZvar
```

Out[822]=

```
{YZ0[z, t] -> ((C1^2*c11*c12*H^2 -
   C1^2*c13*c14*H^2 - c11*c12*M^2 - c13*c14*M^2)*Inactive[Exp][-((M^2*(t
   - z/6))/(C1*H^2)) + C1*(t + z/6)])/(2*C1*H*M),
 YZ1[z, t] -> -1/2*((C1^2*c11*c12*H^2 - C1^2*c13*c14*H^2 + c11*c12*M^2 +
   c13*c14*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C1*H^2)) + C1*(t + z/6)])/(C1*H*M),
 YZ2[z, t] -> c13*c14*Inactive[Exp][-((M^2*(t - z/6))/(C1*H^2)) + C1*(t + z/6)],
 YZ3[z, t] -> c11*c12*Inactive[Exp][-((M^2*(t - z/6))/(C1*H^2)) + C1*(t + z/6)],
 YZ4[z, t] -> -1/2*((C2^2*c21*c22*H^2 + C2^2*c23*c24*H^2 - c21*c22*M^2 +
   c23*c24*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C2*H^2)) + C2*(t + z/6)])/(C2*H*M),
 YZ5[z, t] -> -1/2*((C2^2*c21*c22*H^2 + C2^2*c23*c24*H^2 + c21*c22*M^2 -
   c23*c24*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C2*H^2)) + C2*(t + z/6)])/(C2*H*M),
 YZ6[z, t] -> c23*c24*Inactive[Exp][-((M^2*(t - z/6))/(C2*H^2)) + C2*(t + z/6)],
 YZ7[z, t] -> c21*c22*Inactive[Exp][-((M^2*(t - z/6))/(C2*H^2)) + C2*(t + z/6)],
 YZ8[z, t] -> -1/2*((C3^2*c31*c32*H^2 + C3^2*c33*c34*H^2 - c31*c32*M^2 +
   c33*c34*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C3*H^2)) + C3*(t + z/6)])/(C3*H*M),
 YZ9[z, t] -> -1/2*((C3^2*c31*c32*H^2 + C3^2*c33*c34*H^2 + c31*c32*M^2 -
   c33*c34*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C3*H^2)) + C3*(t + z/6)])/(C3*H*M),
 YZ10[z, t] -> c33*c34*Inactive[Exp][-((M^2*(t - z/6))/(C3*H^2)) + C3*(t + z/6)],
 YZ11[z, t] -> c31*c32*Inactive[Exp][-((M^2*(t - z/6))/(C3*H^2)) + C3*(t + z/6)],
 YZ12[z, t] -> ((C4^2*c41*c42*H^2 - C4^2*c43*c44*H^2
   - c41*c42*M^2 - c43*c44*M^2)*Inactive[Exp][-((M^2*(t
   - z/6))/(C4*H^2)) + C4*(t + z/6)])/(2*C4*H*M),
 YZ13[z, t] -> -1/2*((C4^2*c41*c42*H^2 - C4^2*c43*c44*H^2 + c41*c42*M^2 +
   c43*c44*M^2)*Inactive[Exp][-((M^2*(t - z/6))/(C4*H^2)) + C4*(t + z/6)])/(C4*H*M),
 YZ14[z, t] -> c43*c44*Inactive[Exp][-((M^2*(t - z/6))/(C4*H^2)) + C4*(t + z/6)],
 YZ15[z, t] -> c41*c42*Inactive[Exp][-((M^2*(t - z/6))/(C4*H^2)) + C4*(t + z/6)]}
```

In[ ]:=

```
yZ[#] & /@ Range[0, 15]
sYZvar[[All, 2]] /. {z -> #1, t -> #2};
Table[ToExpression["((" <> ToString[FullForm[%[[j]]]] <> ")&)"], {j, 1, Length[%]}];
ssyZ = Thread[%%% -> %]
```

E

is the exponential constant E (base of natural logarithms), with numerical value ≈ 2.71828 .

In[982]:=

```
yZ[#][z, t] & /@ Range[0, 15];
% /. ssyZ;
EtoExp[#] & /@ %;
% /. {Exp → QF} // Activate;
% /. {z → #1, t → #2};
syZ = Table[
  ssyZ[[j, 1]] → ToExpression["((" <> ToString[%[[j]], InputForm] <> ")&)"], {j, 1, Length[%]}]
(*Thread[ssyZ[[All,1]]→%]/.{z→#1,t→#2}*)
```

Out[987]=

$$\left\{yZ[0] \to \left(\frac{\left(C1^2\, c11\, c12\, H^2 - C1^2\, c13\, c14\, H^2 - c11\, c12\, M^2 - c13\, c14\, M^2\right) QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C1\, H^2} + C1\left(\frac{\#1}{6}+\#2\right)\right]}{2\, C1\, H\, M}\ \&\right),\right.$$

$$yZ[1] \to \left(-\frac{\left(C1^2\, c11\, c12\, H^2 - C1^2\, c13\, c14\, H^2 + c11\, c12\, M^2 + c13\, c14\, M^2\right) QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C1\, H^2} + C1\left(\frac{\#1}{6}+\#2\right)\right]}{2\,(C1\, H\, M)}\ \&\right),$$

$$yZ[2] \to \left(c13\, c14\, QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C1\, H^2} + C1\left(\frac{\#1}{6}+\#2\right)\right]\ \&\right),$$

$$yZ[3] \to \left(c11\, c12\, QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C1\, H^2} + C1\left(\frac{\#1}{6}+\#2\right)\right]\ \&\right),\ yZ[4] \to$$

$$\left(-\frac{\left(C2^2\, c21\, c22\, H^2 + C2^2\, c23\, c24\, H^2 - c21\, c22\, M^2 + c23\, c24\, M^2\right) QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C2\, H^2} + C2\left(\frac{\#1}{6}+\#2\right)\right]}{2\,(C2\, H\, M)}\ \&\right),\ yZ[$$

$$5] \to \left(-\frac{\left(C2^2\, c21\, c22\, H^2 + C2^2\, c23\, c24\, H^2 + c21\, c22\, M^2 - c23\, c24\, M^2\right) QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C2\, H^2} + C2\left(\frac{\#1}{6}+\#2\right)\right]}{2\,(C2\, H\, M)}\ \&\right),$$

$$yZ[6] \to \left(c23\, c24\, QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C2\, H^2} + C2\left(\frac{\#1}{6}+\#2\right)\right]\ \&\right),$$

$$yZ[7] \to \left(c21\, c22\, QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C2\, H^2} + C2\left(\frac{\#1}{6}+\#2\right)\right]\ \&\right),\ yZ[8] \to$$

$$\left(-\frac{\left(C3^2\, c31\, c32\, H^2 + C3^2\, c33\, c34\, H^2 - c31\, c32\, M^2 + c33\, c34\, M^2\right) QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C3\, H^2} + C3\left(\frac{\#1}{6}+\#2\right)\right]}{2\,(C3\, H\, M)}\ \&\right),\ yZ[$$

$$9] \to \left(-\frac{\left(C3^2\, c31\, c32\, H^2 + C3^2\, c33\, c34\, H^2 + c31\, c32\, M^2 - c33\, c34\, M^2\right) QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C3\, H^2} + C3\left(\frac{\#1}{6}+\#2\right)\right]}{2\,(C3\, H\, M)}\ \&\right),$$

$$yZ[10] \to \left(c33\, c34\, QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C3\, H^2} + C3\left(\frac{\#1}{6}+\#2\right)\right]\ \&\right),$$

$$yZ[11] \to \left(c31\, c32\, QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C3\, H^2} + C3\left(\frac{\#1}{6}+\#2\right)\right]\ \&\right),\ yZ[12] \to$$

$$\left(\frac{\left(C4^2\, c41\, c42\, H^2 - C4^2\, c43\, c44\, H^2 - c41\, c42\, M^2 - c43\, c44\, M^2\right) QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C4\, H^2} + C4\left(\frac{\#1}{6}+\#2\right)\right]}{2\, C4\, H\, M}\ \&\right),$$

$$yZ[13] \to$$

$$\left(-\frac{\left(C4^2\,c41\,c42\,H^2-C4^2\,c43\,c44\,H^2+c41\,c42\,M^2+c43\,c44\,M^2\right)\,QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C4\,H^2}+C4\left(\frac{\#1}{6}+\#2\right)\right]}{2\,(C4\,H\,M)}\,\&\right),$$

$$yZ[14]\to\left(c43\,c44\,QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C4\,H^2}+C4\left(\frac{\#1}{6}+\#2\right)\right]\,\&\right),$$

$$yZ[15]\to\left(c41\,c42\,QF\left[-\frac{M^2\left(-\frac{\#1}{6}+\#2\right)}{C4\,H^2}+C4\left(\frac{\#1}{6}+\#2\right)\right]\,\&\right)\Big\}$$

Z to yZ mapping

In[771]:=

```
cayZ = CoefficientArrays[zeroZyZforCaExpression, yZdef]
```

Out[771]=

{SparseArray[Specified elements: 15 Dimensions: {15}], SparseArray[Specified elements: 15 Dimensions: {15, 15}]}

In[772]:=

```
caZ = CoefficientArrays[zeroZyZforCaExpression, Zdef]
```

Out[772]=

{SparseArray[Specified elements: 15 Dimensions: {15}], SparseArray[Specified elements: 15 Dimensions: {15, 15}]}

In[773]:=

```
zeroZyZforCaExpression === cayZ[[1]] + cayZ[[2]].yZdef // Expand
```

Out[773]=

True

In[774]:=

```
zeroZyZforCaExpression === caZ[[1]] + caZ[[2]].Zdef // Expand
```

Out[774]=

True

In[775]:=

```
(cayZ2 = cayZ[[2]] // Normal) === ID16
```

Out[775]=

True

In[776]:=

```
(cayZ1 = cayZ[[1]] // Normal)
```

Out[776]=

```
{-Z[0], -Z[5], -Z[8], -Z[13], -Z[1], -Z[4], -Z[9],
 -Z[12], -Z[2], -Z[7], -Z[10], -Z[15], -Z[3], -Z[6], -Z[11], -Z[14]}
```

In[777]:=

```
(caZ2 = caZ[[2]] // Normal) // MatrixForm
```

Out[777]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

In[778]:=

```
Transpose[caZ2].caZ2 === Id16
```

Out[778]=

```
True
```

In[779]:=

```
sZtOyZ
```

Out[779]=

```
{Z[0] → yZ[0], Z[5] → yZ[1], Z[8] → yZ[2], Z[13] → yZ[3], Z[1] → yZ[4], Z[4] → yZ[5],
 Z[9] → yZ[6], Z[12] → yZ[7], Z[2] → yZ[8], Z[7] → yZ[9], Z[10] → yZ[10],
 Z[15] → yZ[11], Z[3] → yZ[12], Z[6] → yZ[13], Z[11] → yZ[14], Z[14] → yZ[15]}
```

In[780]:=

```
Transpose[caZ2].caZ[[1]] + Transpose[caZ2].caZ[[2]].Zdef // Expand
sZyZ = Solve[And @@ Thread[0 == %], Zdef][[1]]
sZyZa = Thread[Zdef → -Transpose[caZ2].caZ[[1]]]
% == %%
```

Out[780]=

```
{-yZ[0] + Z[0], -yZ[4] + Z[1], -yZ[8] + Z[2], -yZ[12] + Z[3], -yZ[5] + Z[4], -yZ[1] + Z[5],
 -yZ[13] + Z[6], -yZ[9] + Z[7], -yZ[2] + Z[8], -yZ[6] + Z[9], -yZ[10] + Z[10],
 -yZ[14] + Z[11], -yZ[7] + Z[12], -yZ[3] + Z[13], -yZ[15] + Z[14], -yZ[11] + Z[15]}
```

Out[781]=

```
{Z[0] → yZ[0], Z[1] → yZ[4], Z[2] → yZ[8], Z[3] → yZ[12], Z[4] → yZ[5], Z[5] → yZ[1],
 Z[6] → yZ[13], Z[7] → yZ[9], Z[8] → yZ[2], Z[9] → yZ[6], Z[10] → yZ[10],
 Z[11] → yZ[14], Z[12] → yZ[7], Z[13] → yZ[3], Z[14] → yZ[15], Z[15] → yZ[11]}
```

Out[782]=

```
{Z[0] → yZ[0], Z[1] → yZ[4], Z[2] → yZ[8], Z[3] → yZ[12], Z[4] → yZ[5], Z[5] → yZ[1],
 Z[6] → yZ[13], Z[7] → yZ[9], Z[8] → yZ[2], Z[9] → yZ[6], Z[10] → yZ[10],
 Z[11] → yZ[14], Z[12] → yZ[7], Z[13] → yZ[3], Z[14] → yZ[15], Z[15] → yZ[11]}
```

Out[783]=

```
True
```

In[784]:=

```
almightyS = -Transpose[caZ2]
```

Out[784]=

```
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
 {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
 {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}}
```

$\overline{O(8)}$ not $\overline{O(4,4)}$:

In[785]:=

```
Transpose[almightyS].almightyS === ID16
```

Out[785]=

```
True
```

In[786]:=

```
Transpose[almightyS].σ16.almightyS === σ16
```

Out[786]=

```
False
```

In[787]:=

```
σ16 // MatrixForm
```

Out[787]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[788]:=

```
Transpose[almightyS].ArrayFlatten[{{0, σ}, {σ, 0}}].almightyS ===
 ArrayFlatten[{{0, σ}, {σ, 0}}]
```

Out[788]=

False

In[ ]:=

```
Clear[La];
La[] :=
 (Transpose[Ψ16].σ16.Sum[T16^A[α1 - 1].(D[Ψ16, X[[α1]]]), {α1, 1, Length[X]}] +
    (H * M) * Transpose[Ψ16].σ16.Ψ16) // Simplify[#, constraintVars] &
```

In[789]:=

```
σ16.T16^A[#] === -Transpose[σ16.T16^A[#]] & /@ Range[0, 7]
```

Out[789]=

{True, True, True, True, True, True, True, True}

In[790]:=

```
La[];
- Sum[(D[Transpose[Ψ16], X[[α1]]]).σ16.T16^A[α1 - 1], {α1, 1, Length[X]}].Ψ16 +
    (H * M) * Transpose[Ψ16].σ16.Ψ16 // FullSimplify[#, constraintVars] &;
ExpandAll[% - %%] // FullSimplify[#, constraintVars] &
```

Out[792]=

$$
\begin{aligned}
&\frac{1}{H}\Big(4\,M\,f16[2][x0,x4]\times f16[6][x0,x4] - \\
&\quad 2\,H^2\,M\,f16[2][x0,x4]\times f16[6][x0,x4] + 4\,M\,f16[3][x0,x4]\times f16[7][x0,x4] - \\
&\quad 2\,H^2\,M\,f16[3][x0,x4]\times f16[7][x0,x4] - 4\,M\,f16[8][x0,x4]\times f16[12][x0,x4] + \\
&\quad 2\,H^2\,M\,f16[8][x0,x4]\times f16[12][x0,x4] - 4\,M\,f16[9][x0,x4]\times f16[13][x0,x4] + \\
&\quad 2\,H^2\,M\,f16[9][x0,x4]\times f16[13][x0,x4] - 4\,M\,f16[10][x0,x4]\times f16[14][x0,x4] + \\
&\quad 2\,H^2\,M\,f16[10][x0,x4]\times f16[14][x0,x4] - 4\,M\,f16[11][x0,x4]\times f16[15][x0,x4] + \\
&\quad 2\,H^2\,M\,f16[11][x0,x4]\times f16[15][x0,x4] - f16[9][x0,x4]\,f16[0]^{(0,1)}[x0,x4] + \\
&\quad H\,f16[9][x0,x4]\,f16[0]^{(0,1)}[x0,x4] + f16[8][x0,x4]\,f16[1]^{(0,1)}[x0,x4] - \\
&\quad H\,f16[8][x0,x4]\,f16[1]^{(0,1)}[x0,x4] + f16[11][x0,x4]\,f16[2]^{(0,1)}[x0,x4] - \\
&\quad H\,f16[11][x0,x4]\,f16[2]^{(0,1)}[x0,x4] - f16[10][x0,x4]\,f16[3]^{(0,1)}[x0,x4] + \\
&\quad H\,f16[10][x0,x4]\,f16[3]^{(0,1)}[x0,x4] + f16[13][x0,x4]\,f16[4]^{(0,1)}[x0,x4] - \\
&\quad H\,f16[13][x0,x4]\,f16[4]^{(0,1)}[x0,x4] - f16[12][x0,x4]\,f16[5]^{(0,1)}[x0,x4] + \\
&\quad H\,f16[12][x0,x4]\,f16[5]^{(0,1)}[x0,x4] - f16[15][x0,x4]\,f16[6]^{(0,1)}[x0,x4] + \\
&\quad H\,f16[15][x0,x4]\,f16[6]^{(0,1)}[x0,x4] + f16[14][x0,x4]\,f16[7]^{(0,1)}[x0,x4] - \\
&\quad H\,f16[14][x0,x4]\,f16[7]^{(0,1)}[x0,x4] + f16[3][x0,x4]\,f16[10]^{(0,1)}[x0,x4] - \\
&\quad H\,f16[3][x0,x4]\,f16[10]^{(0,1)}[x0,x4] - f16[2][x0,x4]\,f16[11]^{(0,1)}[x0,x4] + \\
&\quad H\,f16[2][x0,x4]\,f16[11]^{(0,1)}[x0,x4] + f16[5][x0,x4]\,f16[12]^{(0,1)}[x0,x4] - \\
&\quad H\,f16[5][x0,x4]\,f16[12]^{(0,1)}[x0,x4] - f16[4][x0,x4]\,f16[13]^{(0,1)}[x0,x4] + \\
&\quad H\,f16[4][x0,x4]\,f16[13]^{(0,1)}[x0,x4] - f16[7][x0,x4]\,f16[14]^{(0,1)}[x0,x4] + \\
&\quad H\,f16[7][x0,x4]\,f16[14]^{(0,1)}[x0,x4] + f16[6][x0,x4]\,f16[15]^{(0,1)}[x0,x4] - \\
&\quad H\,f16[6][x0,x4]\,f16[15]^{(0,1)}[x0,x4] - f16[12][x0,x4]\,f16[0]^{(1,0)}[x0,x4] + \\
&\quad H\,f16[12][x0,x4]\,f16[0]^{(1,0)}[x0,x4] - f16[13][x0,x4]\,f16[1]^{(1,0)}[x0,x4] + \\
&\quad H\,f16[13][x0,x4]\,f16[1]^{(1,0)}[x0,x4] - f16[14][x0,x4]\,f16[2]^{(1,0)}[x0,x4] + \\
&\quad H\,f16[14][x0,x4]\,f16[2]^{(1,0)}[x0,x4] - f16[15][x0,x4]\,f16[3]^{(1,0)}[x0,x4] + \\
&\quad H\,f16[15][x0,x4]\,f16[3]^{(1,0)}[x0,x4] - f16[8][x0,x4]\,f16[4]^{(1,0)}[x0,x4] + \\
&\quad H\,f16[8][x0,x4]\,f16[4]^{(1,0)}[x0,x4] - f16[9][x0,x4]\,f16[5]^{(1,0)}[x0,x4] + \\
&\quad H\,f16[9][x0,x4]\,f16[5]^{(1,0)}[x0,x4] - f16[10][x0,x4]\,f16[6]^{(1,0)}[x0,x4] + \\
&\quad H\,f16[10][x0,x4]\,f16[6]^{(1,0)}[x0,x4] - f16[11][x0,x4]\,f16[7]^{(1,0)}[x0,x4] + \\
&\quad H\,f16[11][x0,x4]\,f16[7]^{(1,0)}[x0,x4] + f16[4][x0,x4]\,f16[8]^{(1,0)}[x0,x4] - \\
&\quad H\,f16[4][x0,x4]\,f16[8]^{(1,0)}[x0,x4] + f16[5][x0,x4]\,f16[9]^{(1,0)}[x0,x4] - \\
&\quad H\,f16[5][x0,x4]\,f16[9]^{(1,0)}[x0,x4] + f16[6][x0,x4]\,f16[10]^{(1,0)}[x0,x4] - \\
&\quad H\,f16[6][x0,x4]\,f16[10]^{(1,0)}[x0,x4] + f16[7][x0,x4]\,f16[11]^{(1,0)}[x0,x4] - \\
&\quad H\,f16[7][x0,x4]\,f16[11]^{(1,0)}[x0,x4] + f16[0][x0,x4]\,\Big(-2\,\big(-2+H^2\big)\,M\,f16[4][x0,x4] - \\
&\qquad (-1+H)\,\big(f16[9]^{(0,1)}[x0,x4] + f16[12]^{(1,0)}[x0,x4]\big)\Big) + f16[1][x0,x4] \\
&\qquad \Big(-2\,\big(-2+H^2\big)\,M\,f16[5][x0,x4] + (-1+H)\,\big(f16[8]^{(0,1)}[x0,x4] - f16[13]^{(1,0)}[x0,x4]\big)\Big) + \\
&\quad f16[2][x0,x4]\,f16[14]^{(1,0)}[x0,x4] - H\,f16[2][x0,x4]\,f16[14]^{(1,0)}[x0,x4] - \\
&\quad (-1+H)\,f16[3][x0,x4]\,f16[15]^{(1,0)}[x0,x4]\Big)
\end{aligned}
$$

In[793]:=

```
(Transpose[Ψ16].σ16.Sum[T16^A[α1 - 1].(D[Ψ16, X[[α1]]]), {α1, 1, Length[X]}] +
      (H * M) * Transpose[Ψ16].σ16.Ψ16) /. sfψ16Aa /.
   sx0x4 // FullSimplify[#, constraintVars] &;
- Sum[(D[Transpose[Ψ16], X[[α1]]]).σ16.T16^A[α1 - 1], {α1, 1, Length[X]}].Ψ16 +
     (H * M) * Transpose[Ψ16].σ16.Ψ16 /. sfψ16Aa /.
   sx0x4 // FullSimplify[#, constraintVars] &;
ExpandAll[% - %%] // FullSimplify[#, constraintVars] &
```

Out[795]=

```
0
```

In[796]:=

```
Thread[Zdef → -Transpose[caZ2].caZ[[1]]]
```

Out[796]=

```
{Z[0] → yZ[0], Z[1] → yZ[4], Z[2] → yZ[8], Z[3] → yZ[12], Z[4] → yZ[5], Z[5] → yZ[1],
 Z[6] → yZ[13], Z[7] → yZ[9], Z[8] → yZ[2], Z[9] → yZ[6], Z[10] → yZ[10],
 Z[11] → yZ[14], Z[12] → yZ[7], Z[13] → yZ[3], Z[14] → yZ[15], Z[15] → yZ[11]}
```

In[797]:=

```
zeroZyZforCaExpression === caZ[[1]] + caZ[[2]].Zdef // Expand
```

Out[797]=

```
True
```

In[798]:=

```
(caZ1 = caZ[[1]] // Normal) === yZdef
```

Out[798]=

```
True
```

In[799]:=

```
caZ[[1]] + caZ[[2]].Zdef // Expand
% /. sZtOyZ
```

Out[799]=

```
{yZ[0] - Z[0], yZ[1] - Z[5], yZ[2] - Z[8], yZ[3] - Z[13], yZ[4] - Z[1],
 yZ[5] - Z[4], yZ[6] - Z[9], yZ[7] - Z[12], yZ[8] - Z[2], yZ[9] - Z[7], yZ[10] - Z[10],
 yZ[11] - Z[15], yZ[12] - Z[3], yZ[13] - Z[6], yZ[14] - Z[11], yZ[15] - Z[14]}
```

Out[800]=

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

In[801]:=

```
almightyS.Zdef
% /. sZtOyZ
yZdef === (% // Sort)
```

Out[801]=

```
{Z[0], Z[4], Z[8], Z[12], Z[5], Z[1], Z[13],
 Z[9], Z[2], Z[6], Z[10], Z[14], Z[7], Z[3], Z[15], Z[11]}
```

Out[802]=

```
{yZ[0], yZ[5], yZ[2], yZ[7], yZ[1], yZ[4], yZ[3], yZ[6],
 yZ[8], yZ[13], yZ[10], yZ[15], yZ[9], yZ[12], yZ[11], yZ[14]}
```

Out[803]=

```
True
```

In[804]:=

```
cayZ2 === ID16
```

Out[804]=

```
True
```

In[805]:=

```
caZ2 // MatrixForm
```

Out[805]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

ORTHOGONAL O(16) SIMILARITY TRANSFORMATION :

In[806]:=

```
caZ2.Transpose[caZ2] === ID16
```

Out[806]=

```
True
```

BUT NOT ORTHOGONAL O(8,8) SIMILARITY TRANSFORMATION :

In[807]:=

```
caZ2.σ16.Transpose[caZ2] === σ16
```

Out[807]=

```
False
```

Also, Not a Direct Sum :

In[808]:=

```
caZ2 // MatrixForm
```

Out[808]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

In[809]:=

```
caZ2 === ArrayFlatten[{{caZ2[[1 ;; 8, 1 ;; 8]], caZ2[[1 ;; 8, 9 ;; 16]]},
   {caZ2[[9 ;; 16, 1 ;; 8]], caZ2[[9 ;; 16, 9 ;; 16]]}}]
```

Out[809]=

```
True
```

In[810]:=

```
caZ2 === ArrayFlatten[{{caZ2[[1 ;; 8, 1 ;; 8]], 0}, {0, caZ2[[9 ;; 16, 9 ;; 16]]}}]
```

Out[810]=

```
False
```

Notes

Scratch work of Author ;

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Scratch work only!!

Patrick L . Nash, Ph . D .

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Professor, UTSA Physics and Astronomy, Retired (UTSA)

Patrick299Nash at gmail ...

Do not read if you are irascible; I apologize for typos and duplications and everthing else .

git clone https : // github . com/43 d168f3e/Pre - Universe . git

We hope to employ the formalism developed in this notebook to study ideas put forward in "The Particle Problem in the General Theory of Relativity," published in Physical Review 48, 73–77 :

JULY 1, 1935 PHYSICAL REVIEW

The Particle Problem in the General Theory of Relativity

A. EINSTEIN AND N. ROSEN, *Institute for Advanced Study, Princeton*

(Received May 8, 1935)

If one tries to interpret the regular solution (5a) in the space of r, θ, ϕ, t, one arrives at the following conclusion. The four-dimensional space is described mathematically by two congruent parts or "sheets," corresponding to $u>0$ and $u<0$, which are joined by a hyperplane $r=2m$ or $u=0$ in which g vanishes.[2] We call such a connection between the two sheets a "bridge."

We see now in the given solution, free from singularities, the mathematical representation of an elementary particle (neutron or neutrino). Characteristic of the theory we are presenting is the description of space by means of two sheets. A bridge, spatially finite, which connects these sheets characterizes the presence of an electrically neutral elementary particle. With this conception one not only obtains the representation of an elementary particle by using only the field equations, that is, without introducing new field quantities to describe the density of matter; one

The origin of this calculation has its roots in "A Remarkable Representation of the 3+2 de Sitter Group" by P.A.M.Dirac, J.Math.Phys.4,901–909 (1963).

The term "reduced Brauer–Weyl generators", refers to a specific set of irreducible generators for certain complex Clifford algebras, particularly in the context of mathematical physics and the study of Dirac spinors. These generators were introduced by **E.A. Lord** (E. A. Lord. "The Dirac spinor in six dimensions". Mathematical Proceedings of the Cambridge Philosophical Society, vol. 64, pp. 765–778, 1968) and are described as "reduced" because they form an irreducible representation of the Clifford algebra C_{2n} from $C_{2(n-1)}$.

Here, we employ Lord's idea of reduced Brauer–Weyl generators to extend Dirac's 4×4 gamma matrices into this notebook's real tau8 8×8 and tau16 16×16 matrices,
which are employed to write down the 'Dirac Equation for the Pre-Universe'

[remember, this is before the Big Bang, and before the particles of the standard model exist]
formulated in terms of a 16-component $\overline{O(4,4)}$ spinor Ψ16
(Ψ16 AKA 'WAVE FUNCTION OF the UN-UNIVERSE,' which could possibly be named something more pompous).
This equation is used to investigate EINSTEIN AND ROSEN's idea

If one tries to interpret the regular solution (5a) in the space of r, θ, ϕ, t, one arrives at the following conclusion. The four-dimensional space is described mathematically by two congruent parts or "sheets," corresponding to $u>0$ and $u<0$, which are joined by a hyperplane $r=2m$ or $u=0$ in which g vanishes.[2] We call such a connection between the two sheets a "bridge."

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The four-dimensional space is
described mathematically by two congruent
parts or "sheets," corresponding to u>0 and
u <0, which are joined by a hyperplane r= 2m or
u=0 in which g vanishes .2 We call such a con-
nection between the two sheets a "bridge", which marketing departments now call a "wormhole".

THIS WORK IS PARTIALLY BASED ON :

Proc. Camb. Phil. Soc. (1968), **64**, 765
PCPS 64–96
Printed in Great Britain

The Dirac spinor in six dimensions

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 4, NUMBER 7 J

A Remarkable Representation of the 3 + 2 de Sitter Group

P. A. M. Dirac
Cambridge University, Cambridge, England and
*Institute for Advanced Study, Princeton, New Jersey**
(Received 20 February 1963)

Proc. Camb. Phil. Soc. (1968), **64**, 765
PCPS 64–96
Printed in Great Britain

765

The Dirac spinor in six dimensions

By E. A. LORD
Department of Mathematics, King's College, University of London

(*Received* 17 *November* 1966)

JOURNAL OF MATHEMATICAL PHYSICS **51**, 042501 (2010)

Second gravity

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(Received 29 July 2009; accepted 8 February 2010; published online 5 April 2010)

IL NUOVO CIMENTO Vol. 105 B, N. 1 Gennaio 1990

On the Structure of the Split Octonion Algebra.

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(ricevuto il 24 Luglio 1989)

A new spin-$\frac{1}{2}$ wave equation

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(ricevuto il 24 Luglio 1989)

Identities satisfied by the generators of the Dirac algebra

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(Received 26 July 1983; accepted for publication 23 September 1983)

A new spin-$\frac{1}{2}$ wave equation

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(Received 24 January 1984; accepted for publication 16 November 1984)

The split Octonion algebra carries basic 8-component representations of $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ (left-spinor, right-spinor and vector), which are equivalent [**Cartan's triality**].

In[836]:=

A. Representations of SO(8, ℂ)

Summarizing and applying a few facts that may be found in, for example, Boerner, *Representations of Groups*,[7] the Clifford algebra C_8 may be defined as the algebra generated by a set of eight elements e_j, $j=1,\ldots,8$, that anticommute with each other and have unit square $e_je_k+e_ke_j=2\delta_{jk}\mathbb{I}_{16\times16}$, $\mathbb{I}$=unit matrix. The scaled commutators $\frac{1}{4}(e_je_k-e_ke_j)$ computed from the 16-

FIG. 1. Dynkin diagram for D_4.

dimensional irreducible representation of the e_j are the infinitesimal generators of a reducible 16-dimensional representation of spin (8, ℂ), which is the universal double covering of the special orthogonal group SO(8, ℂ). This 16-dimensional representation is fully reducible to the direct sum of two inequivalent irreducible 8×8 spin representations of the infinitesimal generators of spin (8, ℂ), which leads to the identification of type 1 and type 2 spinors. The fundamental irreducible vector representation of SO(8, ℂ) is also 8×8. The Dynkin diagram for $D_4\cong\mathrm{SO}(8)$ is symmetrical and pictured in Fig. 1. The three outer nodes correspond to the vector representation (leftmost node), type 1 spinor and type 2 spinor representations of spin(8), and the central node corresponds

The canonical 2-1 homomorphism $\overline{\mathrm{SO}(4,4;\mathbb{R})}\to\mathrm{SO}(4,4;\mathbb{R}):g\mapsto L(g)$ is given by

$$8L^A{}_B=\mathrm{tr}(D_{(1)}^{-1}\bar{\tau}^A D_{(2)}\tau^C)\mathbb{G}_{CB}, \tag{23}$$

to the adjoint representation.

The canonical 2-1 homomorphism $\overline{\mathrm{SO}(4,4;\mathbb{R})} \to \mathrm{SO}(4,4;\mathbb{R}) : g \mapsto L(g)$ is given by

$$8L^A{}_B = \mathrm{tr}(D_{(1)}^{-1}\bar{\tau}^A D_{(2)}\tau^C)\mathbb{G}_{CB}, \tag{23}$$

where tr denotes the trace. Note that $D_{(1)}(g(\omega)) = D_{(2)}(g(\omega))$ when $\omega_{A8}=0$, i.e., when one restricts $\overline{\mathrm{SO}(4,4;\mathbb{R})}$ to

$$\overline{\mathrm{SO}(3,4;\mathbb{R})} = \left\{ g \in \overline{\mathrm{SO}(4,4;\mathbb{R})} \,\middle|\, g = \begin{pmatrix} \exp\left(\frac{1}{4}\omega_{AB}D_{(1)}{}^{AB}\right) & 0 \\ 0 & \exp\left(\frac{1}{4}\omega_{AB}D_{(2)}{}^{AB}\right) \end{pmatrix} \text{ and } \omega_{A8} = 0 \right\} \tag{24}$$

[one of the real forms of spin $(7,\mathbb{C})$].

$M_{4,4}$ = Minkowski spacetime, signature = {4, 4}

$\mathbb{O}_{4,4}$ = split octonion algebra

$M_{4,4}$ = Minkowski spacetime, signature = {4, 4}

$\mathbb{O}_{4,4}$ = split octonion algebra

with vector and spinor bases :

$o \in \mathbb{O}_{4,4}$; $o = eA_j\, o^j = ea\psi 1_j\, o\psi 1^j = ea\psi 2_j\, o\psi 2^j$ [**Cartan's triality**]

$= \psi 1_j\, ea^j = ea_j\, \psi 2^j$ [restrict to Spin (3, 4)]

$< ea_a, ea_b > = \sigma_{ab}$

$< eA_A, eA_B > = \eta_{AB}$

$\{X, EA\} \in M_{4,4} \otimes \mathbb{O}_{4,4}$

```
{X, EA}
{{x0, x1, x2, x3, x4, x5, x6, x7},
 {eA[1], eA[2], eA[3], eA[4], eA[5], eA[6], eA[7], eA[8]}}
```

In[839]:=

```
gridSplitOctonion =
 Grid[Prepend[Drop[Reap[For[A1 = 1, A1 < 9, A1++, Sow[Flatten[{EA[[A1]], Table[
            Sum[mABC[[A1, B1, C1]] * EA[[C1]], {C1, 1, 8}], {B1, 1, 8}]}]]]],
      1][[1]][[1]], Flatten[{"A/B", EA}]], Frame → All]
```

Out[839]=

A/B	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[1]	eA[1]	eA[2]	eA[3]	eA[4]	eA[5]	eA[6]	eA[7]	eA[8]
eA[2]	eA[2]	-eA[1]	-eA[4]	eA[3]	-eA[6]	eA[5]	eA[8]	-eA[7]
eA[3]	eA[3]	eA[4]	-eA[1]	-eA[2]	-eA[7]	-eA[8]	eA[5]	eA[6]
eA[4]	eA[4]	-eA[3]	eA[2]	-eA[1]	-eA[8]	eA[7]	-eA[6]	eA[5]
eA[5]	eA[5]	eA[6]	eA[7]	eA[8]	eA[1]	eA[2]	eA[3]	eA[4]
eA[6]	eA[6]	-eA[5]	eA[8]	-eA[7]	-eA[2]	eA[1]	-eA[4]	eA[3]
eA[7]	eA[7]	-eA[8]	-eA[5]	eA[6]	-eA[3]	eA[4]	eA[1]	-eA[2]
eA[8]	eA[8]	eA[7]	-eA[6]	-eA[5]	-eA[4]	-eA[3]	eA[2]	eA[1]

citations for the 1935 Einstein and Rosen article,
"The Particle Problem in the General Theory of Relativity," published in Physical Review 48, 73–77 :

APA / Chicago : Einstein, A., & Rosen,
N.(1935).The Particle Problem in the General Theory of Relativity.Physical Review,
48 (1), 73–77. https : // doi.org / 10.1103 / PhysRev .48 × .73

BibTeX :
@article {EinsteinRosen1935, author = {Einstein, A.and Rosen, N.},
title = {The Particle Problem in the General Theory of Relativity},
journal = {Phys.Rev.}, volume = {48}, issue = {1}, pages = {73 -- × 77}, year = {1935},
publisher = {American Physical Society}, doi = {10.1103 / PhysRev × .48 × .73}}

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@article{EinsteinRosen1935,
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volume = {48},
issue = {1},
pages = {73--77},
year = {1935},
publisher = {American Physical Society},
doi = {10.1103/PhysRev.48.73}
}
```


In the following we shall show that it is possible to do this in a natural way, that the question we are raising can be answered in the affirmative.

§1. A Special Kind of Singularity and Its Removal

The first step to the general theory of relativity was to be found in the so-called "Principle of Equivalence": If in a space free from gravitation a reference system is uniformly accelerated, the reference system can be treated as being "at rest," provided one interprets the condition of the space with respect to it as a homogeneous gravitational field. As is well known the latter is exactly described by the metric field[1]

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + \alpha^2 x_1^2 dx_4^2. \qquad (1)$$

The $g_{\mu\nu}$ of this field satisfy in general the equations

$$R^i{}_{klm} = 0, \qquad (2)$$

and hence the equations

$$R_{kl} = R^m{}_{klm} = 0. \qquad (3)$$

The $g_{\mu\nu}$ corresponding to (1) are regular for all finite points of space-time. Nevertheless one cannot assert that Eqs. (3) are satisfied by (1) for *all* finite values of $x_1, \cdots, x_4$. This is due to the fact that the determinant g of the $g_{\mu\nu}$ vanishes for $x_1 = 0$. The contravariant $g^{\mu\nu}$ therefore become infinite and the tensors $R^i{}_{klm}$ and R_{kl} take on the form $0/0$. From the standpoint of Eqs. (3) the hyperplane $x_1 = 0$ then represents a singularity of the field.

We now ask whether the field law of gravitation (and later on the field law of gravitation and electricity) could not be modified in a natural way without essential change so that the solution (1) would satisfy the field equations for all finite points, i.e., also for $x_1 = 0$. W. Mayer has called our attention to the fact that one can make $R^i{}_{klm}$ and R_{kl} into rational functions of the $g_{\mu\nu}$ and their first two derivatives by multiplying them by suitable powers of g. It is easy to show that in $g^2 R_{kl}$ there is no longer any denominator. If then we replace (3) by

$$R_{kl}^* = g^2 R_{kl} = 0, \qquad (3a)$$

this system of equations is satisfied by (1) at all finite points. This amounts to introducing in place of the $g^{\mu\nu}$ the cofactors $[g_{\mu\nu}]$ of the $g_{\mu\nu}$ in g in order to avoid the occurrence of denominators. One is therefore operating with tensor densities of a suitable weight instead of with tensors. In this way one succeeds in avoiding singularities of that special kind which is characterized by the vanishing of g.

The solution (1) naturally has no deeper physical significance insofar as it extends into spatial infinity. It allows one to see however to what extent the regularization of the hypersurfaces $g = 0$ leads to a theoretical representation of matter, regarded from the standpoint of the original theory. Thus, in the framework of the original theory one has the gravitational equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -T_{ik}, \qquad (4)$$

where T_{ik} is the tensor of mass or energy density. To interpret (1) in the framework of this theory we must approximate the line element by a slightly different one which avoids the singularity $g = 0$. Accordingly we introduce a small constant σ and let

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + (\alpha^2 x_1^2 + \sigma) dx_4^2; \qquad (1a)$$

the smaller $\sigma (>0)$ is chosen, the nearer does this gravitational field come to that of (1). If one calculates from this the (fictitious) energy tensor T_{ik} one obtains as nonvanishing components

$$T_{22} = T_{23} = \alpha^2/\sigma/(1 + \alpha^2 x_1^2/\sigma)^2.$$

We see then that the smaller one takes σ the more is the tensor concentrated in the neighborhood of the hypersurface $x_1 = 0$. From the standpoint of the original theory the solution (1) contains a singularity which corresponds to an energy or mass concentrated in the surface $x_1 = 0$; from the standpoint of the modified theory, however, (1) is a solution of (3a), free from singulari-

[1] It is worth pointing out that this metric field does not represent the whole Minkowski space but only part of it. Thus, the transformation that converts

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into (1) is

In the following we shall show that it is possible to do this in a natural way, that the question we are raising can be answered in the affirmative.

§1. A Special Kind of Singularity and Its Removal

The first step to the general theory of relativity was to be found in the so-called "Principle of Equivalence": If in a space free from gravitation a reference system is uniformly accelerated, the reference system can be treated as being "at rest," provided one interprets the condition of the space with respect to it as a homogeneous gravitational field. As is well known the latter is exactly described by the metric field[1]

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + \alpha^2 x_1^2 dx_4^2. \quad (1)$$

The $g_{\mu\nu}$ of this field satisfy in general the equations

$$R^i{}_{klm} = 0, \quad (2)$$

and hence the equations

$$R_{kl} = R^m{}_{klm} = 0. \quad (3)$$

The $g_{\mu\nu}$ corresponding to (1) are regular for all finite points of space-time. Nevertheless one cannot assert that Eqs. (3) are satisfied by (1) for *all* finite values of $x_1, \cdots, x_4$. This is due to the fact that the determinant g of the $g_{\mu\nu}$ vanishes for $x_1 = 0$. The contravariant $g^{\mu\nu}$ therefore become infinite and the tensors $R^i{}_{klm}$ and R_{kl} take on the form 0/0. From the standpoint of Eqs. (3) the hyperplane $x_1 = 0$ then represents a singularity of the field.

We now ask whether the field law of gravitation (and later on the field law of gravitation and electricity) could not be modified in a natural way without essential change so that the solution (1) would satisfy the field equations for all finite points, i.e., also for $x_1 = 0$. W. Mayer has called our attention to the fact that one can make $R^i{}_{klm}$ and R_{kl} into rational functions of the $g_{\mu\nu}$ and their first two derivatives by multiplying them by suitable powers of g. It is easy to show that in $g^2 R_{kl}$ there is no longer any denominator. If then we replace (3) by

$$R_{kl}^* = g^2 R_{kl} = 0, \quad (3a)$$

this system of equations is satisfied by (1) at all finite points. This amounts to introducing in place of the $g^{\mu\nu}$ the cofactors $[g_{\mu\nu}]$ of the $g_{\mu\nu}$ in g in order to avoid the occurrence of denominators. One is therefore operating with tensor densities of a suitable weight instead of with tensors. In this way one succeeds in avoiding singularities of that special kind which is characterized by the vanishing of g.

The solution (1) naturally has no deeper physical significance insofar as it extends into spatial infinity. It allows one to see however to what extent the regularization of the hypersurfaces $g = 0$ leads to a theoretical representation of matter, regarded from the standpoint of the original theory. Thus, in the framework of the original theory one has the gravitational equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -T_{ik}, \quad (4)$$

where T_{ik} is the tensor of mass or energy density. To interpret (1) in the framework of this theory we must approximate the line element by a slightly different one which avoids the singularity $g = 0$. Accordingly we introduce a small constant σ and let

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We see then that the smaller one takes σ the more is the tensor concentrated in the neighborhood of the hypersurface $x_1 = 0$. From the standpoint of the original theory the solution (1) contains a singularity which corresponds to an energy or mass concentrated in the surface $x_1 = 0$; from the standpoint of the modified theory, however, (1) is a solution of (3a), free from singularities, which describes the "field-producing mass," without requiring for this the introduction of any new field quantities.

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into (1) is

$$\xi_1 = x_1 \cosh \alpha x_4, \qquad \xi_3 = x_3,$$
$$\xi_2 = x_2, \qquad \xi_4 = x_1 \sinh \alpha x_4.$$

It follows that only those points for which $\xi_1^2 \geq \xi_4^2$ correspond to points for which (1) is the metric.

§2. The Schwarzschild Solution

As is well known, Schwarzschild found the spherically symmetric static solution of the gravitational equations

$$ds^2 = -\frac{1}{1 - 2m/r} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) + (1 - 2m/r) dt^2, \quad (5)$$

($r > 2m$, θ from 0 to π, ϕ from 0 to 2π); the variables x_1, x_2, x_3, x_4 are here r, θ, ϕ, t. The vanishing

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($r > 2m$, θ from 0 to π, ϕ from 0 to 2π); the variables x_1, x_2, x_3, x_4 are here r, θ, ϕ, t. The vanishing of the determinant of the $g_{\mu\nu}$ for $\theta = 0$ is unimportant, since the corresponding (spatial) direction is not preferred. On the other hand g_{11} for $r = 2m$ becomes infinite and hence we have there a singularity.

If one introduces in place of r a new variable according to the equation

$$u^2 = r - 2m,$$

one obtains for ds^2 the expression

$$ds^2 = -4(u^2+2m)du^2 - (u^2+2m)^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{u^2}{u^2+2m}dt^2. \quad (5a)$$

These new $g_{\mu\nu}$ are regular functions for all values of the variables. For $u = 0$, however, g_{44} vanishes, hence also the determinant g. This does not prevent the field equations (3a), which have no denominators, from being satisfied for all values of the independent variables. We are therefore dealing with a solution of the (new) field equations, which is free from singularities for all finite

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points. The hypersurface $u=0$ (or in the original variables, $r=2m$) plays here the same role as the hypersurface $x_1=0$ in the previous example.

As u varies from $-\infty$ to $+\infty$, r varies from $+\infty$ to $2m$ and then again from $2m$ to $+\infty$. If one tries to interpret the regular solution (5a) in the space of r, θ, ϕ, t, one arrives at the following conclusion. The four-dimensional space is described mathematically by two congruent parts or "sheets," corresponding to $u>0$ and $u<0$, which are joined by a hyperplane $r=2m$ or $u=0$ in which g vanishes.[2] We call such a connection between the two sheets a "bridge."

We see now in the given solution, free from singularities, the mathematical representation of an elementary particle (neutron or neutrino). Characteristic of the theory we are presenting is the description of space by means of two sheets. A bridge, spatially finite, which connects these sheets characterizes the presence of an electrically neutral elementary particle. With this conception one not only obtains the representation of an elementary particle by using only the field equations, that is, without introducing new field quantities to describe the density of matter; one is also able to understand the atomistic character of matter as well as the fact that there can be no particles of negative mass. The latter is made clear by the following considerations. If we had started from a Schwarzschild solution with negative m, we should not have been able to make the solution regular by introducing a new variable u instead of r; that is to say, no "bridge" is possible that corresponds to a particle of negative mass.

If we consider once more the solution (1) from the standpoint of the information we have acquired from the Schwarzschild solution, we see that there also the two congruent halves of the space for $x_1>0$ and $x_1<0$ can be interpreted as two sheets each corresponding to the same physical space. In this sense the example represents a gravitational field, independent of x_2 and x_3,

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0. Introduction

Geometries associated to the exceptional groups and "exceptional" representations of classical groups often display interesting features closely related to (but distinctly different from) the more familiar features of the classical groups. This paper centers on the geometries in $\mathbf{E}^7$ and $\mathbf{E}^8$ whose groups of symmetries are $G_2 \subseteq SO(7)$ and $\mathrm{Spin}(7) \subseteq SO(8)$. Both of these groups are related to the octonians (sometimes called Cayley numbers) and may be defined in terms of octonionic multiplication. In particular, G_2, the compact exceptional group of (real) dimension 14, is the subgroup of algebra automorphisms of $\mathbf{O}$ (the octonians) and $\mathrm{Spin}(7) \subseteq SO(8)$ may be defined as the subgroup of $GL_{\mathbf{R}}(\mathbf{O})$ generated by right multiplication by unit octonians which are purely imaginary.

The geometry of the algebra $\mathbf{O}$ is closely related to the complex numbers. In §1, we develop some of the properties of $\mathbf{O}$ that we need for later sections. (Our presentation is essentially borrowed from Appendix A of [12], but any mistakes are, of course, due to the author.) A particularly interesting property is described as follows: If we let $\mathrm{Im}\,\mathbf{O} \subseteq \mathbf{O}$ be the hyperplane (through 0) orthogonal to $1 \in \mathbf{O}$, and we let $S^6 \subseteq \mathrm{Im}\,\mathbf{O}$ be the space of unit vectors, then right multiplication by $u \in S^6$ induces a linear transformation $R_u\colon \mathbf{O} \to \mathbf{O}$ which is orthogonal and satisfies $(R_u)^2 = -1$. Thus, associated to each $u \in S^6$ is a complex structure on $\mathbf{O}$ (induced by $J = R_u$) which is compatible with the natural inner product on $\mathbf{O}$. We denote by $\mathbf{O}_u$ the Hermitian vector space whose underlying real vector space (with inner product) is $\mathbf{O}$ and whose complex structure is given by R_u.

Classically, this observation was used to define an almost complex structure on S^6 as follows: If $u \in S^6$, then R_u preserves the 2-plane spanned by 1 and u and therefore preserves its orthogonal 6-plane, which may be identified with $T_uS^6 \subseteq \mathrm{Im}\,\mathbf{O}$ after translation to the origin. Thus R_u induces a complex

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structure on T_uS^6 for each $u \in S^6$. This almost complex structure is not integrable (even locally) to a complex structure (see below). In [Ca], Calabi noticed that for *any* oriented $M^6 \subseteq \operatorname{Im}\mathbf{O}$, $R_{N(x)}$ induces a complex structure on T_xM^6 (where $N(x)$ is the oriented unit normal). Thus every oriented $M^6 \subseteq \operatorname{Im}\mathbf{O}$ inherits an almost complex structure. Moreover, M^6 inherits a metric from $\operatorname{Im}\mathbf{O}$, so we actually have a $U(3)$-structure on M^6. (Calabi calls these structures "almost Hermitian." He also proves that such M^6 possess a canonical $SU(3)$-substructure but we will not need this.) Calabi shows that the second fundamental form II of M decomposes with respect to the $U(3)$-structure into a piece $\mathrm{II}^{1,1}$ of type (1,1) and a piece $\mathrm{II}^{0,2}$ of type (0,2). He then shows that the almost complex structure of M is integrable if and only if $\mathrm{II}^{1,1} = 0$ and that the canonical 2-form of the $U(3)$-structure, say Ω, is closed if and only if $\mathrm{II}^{0,2} = 0$ and $\operatorname{tr}_1 \mathrm{II}^{1,1} = 0$. From this it follows that the $U(3)$-structure on M^6 is Kähler if and only if $\mathrm{II} \equiv 0$, so that M^6 is a hyperplane (or a union of pieces of hyperplanes). Calabi then constructs nontrivial examples of $M^6 \subseteq \operatorname{Im}\mathbf{O}$ for which the almost complex structure is integrable. His examples are of the form $S \times \mathbf{R}^4 \subseteq \operatorname{Im}\mathbf{O}$, where $S \subseteq \mathbf{R}^3$ is a minimal surface, $\mathbf{R}^3 \subseteq \operatorname{Im}\mathbf{O}$ is an *associative* 3-plane, and $\mathbf{R}^4 = (\mathbf{R}^3)^\perp$. Calabi leaves open the problem of determining whether or not there are nontrivial $M^6 \subseteq \operatorname{Im}\mathbf{O}$ for which the canonical 2-form is closed.

In [10], Gray generalized Calabi's construction somewhat by considering hypersurfaces in N^7 where T_xN^7 has a vector cross product modeled on $\operatorname{Im}\mathbf{O} \simeq \mathbf{R}^7$. In the case $N = \operatorname{Im}\mathbf{O}$, Gray observes that the canonical 2-form Ω on $M^6 \subseteq \operatorname{Im}\mathbf{O}$ is always co-closed, i.e., $\delta\Omega = 0$ (or equivalently $d\Omega^2 = 0$).

In the present paper, after some preliminary work establishing the structure equations of $\operatorname{Spin}(7) \subseteq SO(8)$, we study oriented manifolds $M^6 \subseteq \mathbf{O}$. As is pointed out in [12], every oriented 6-plane in $\mathbf{O} \simeq \mathbf{R}^8$ is a complex three-plane in $\mathbf{O}_u$ for a unique $u \in S^6$. Thus, every oriented six-manifold in $\mathbf{O}$ inherits a natural $U(3)$-structure generalizing the case where $M^6 \subseteq \operatorname{Im}\mathbf{O}$. In this case, we decompose the second fundamental form II of M into *three* pieces and prove the analogues of Calabi's theorems concerning when the $U(3)$-structure is complex integrable and when $d\Omega = 0$. In particular, we show that the induced $U(3)$-structure on $M^6 \subseteq \mathbf{O}$ is Kähler if and only if M^6 is a complex hypersurface in $\mathbf{O}_u$ for some fixed $u \in S^6$. We then go further in the study of those $M^6 \subseteq \mathbf{O}$ for which the $U(3)$-structure is complex integrable but which are not Kähler. We show that such M^6 are foliated by 4-planes in $\mathbf{O}$ in a unique way. We refer to this foliation as the *asymptotic ruling* of M^6. Using the moving frame, we prove that if the asymptotic ruling is parallel then M^6 is the product of a fixed 4-plane in $\mathbf{O}$ with a minimal surface in the orthogonal 4-plane. In

particular, we show that Calabi's examples are exactly the M^6 with parallel asymptotic ruling which lie in the hyperplane $\operatorname{Im}\mathbf{O} \subseteq \mathbf{O}$. We then use Cartan's theory of differential systems in involution to show that the analytic non-Kähler

but complex $M^6 \subseteq \mathbf{O}$ "depend" on 12 analytic functions of 1 (real) variable. (For a more precise statement, see §3).

We observe, as did Gray, that the canonical 2-form on $M^6 \subseteq \mathbf{O}$ is *always* co-closed. Finally, we show that any $M^6 \subseteq \mathbf{O}$ for which the canonical 2-form Ω is closed is necessarily Kähler (and therefore must be a complex hypersurface in $\mathbf{O}_u$ for some fixed $u \in S^6$). In particular, such $M^6 \subseteq \operatorname{Im}\mathbf{O}$ must be hyperplanes. This recovers a result of Gray (see [10]).

In the final section of the paper, we study the "complex curves" in S^6, i.e., those maps ϕ: $M^2 \to S^6$ where M^2 is a Riemann surface and $d\phi$ is complex linear with respect to the almost complex structure on S^6 induced by the inclusion $S^6 \subseteq \operatorname{Im}\mathbf{O}$. This study is motivated by the fact that the cone on such a complex curve gives a 3-fold in $\operatorname{Im}\mathbf{O}$ which is *associative* in the sense of [12]. Such cones are absolutely mass minimizing and their singular structure reflects the singular structure of general associative varieties in $\mathbf{R}^7 \subseteq \operatorname{Im}\mathbf{O}$. We first prove that the almost complex structure on S^6 determines the metric structure of S^6 so that any invariant of the *local* almost complex structure is also a metric invariant (for a more precise statement, see Proposition 4.1 and its proof). (This is the compact-form analogue of Cartan's characterization of the split form of G_2 as the pseudo-group of a certain differential system on a five manifold.) This justifies our use of the metric structure on S^6 to study the almost complex structure of S^6.

Since the generalized Cauchy-Riemann equations for local mappings of Riemann surfaces into an *almost* complex manifold form a determined elliptic system (which is first order, quasi-linear) we expect the local theory of complex curves in S^6 to be analogous to the local theory of complex curves in $\mathbf{C}^3$. (In the analytic category, this is certainly the case.) Along these lines, we develop a Frenet formalism for complex cuves in S^6 analogous to that developed for complex curves in $\mathbf{CP}^3$. We define the first, second and the third fundamental forms of ϕ: $M^2 \to S^6$ as holomorphic sections of line bundles over M^2. In particular, the third fundamental form III, analogous to the torsion of a space curve, plays a crucial role. The assumption that $\mathrm{III} \not\equiv 0$ places severe restrictions on the divisors of the three fundamental forms (see [11] for terminology concerning Riemann surfaces). We are able to prove, for example, that if $M^2 = \mathbf{P}^1$, then $\mathrm{III} \not\equiv 0$ is impossible. It seems likely that for fixed genus g, the space of complex curves ϕ: $M^2 \to S^6$ (where M^2 has genus g) with $\mathrm{III} \not\equiv 0$ is finite dimensional, but we have not proven this.

1. The octonians and Spin(7)

In this section, we give a brief description of the octonian algebra **O** and derive a few of its properties. We then go on to define the group Spin(7) $\subseteq SO(8)$ by octonian multiplication and to derive its Lie algebra and structure equations in a form suitable for our differential geometric investigations in the following sections. For more details on **O** and Spin(7), the reader is encouraged to consult Appendix A in [12] and the classical references listed in its bibliography.

An inner product algebra over **R** is a vector space **A** over **R** which possesses a nondegenerate inner product $\langle\, ,\rangle$: $\mathbf{A} \times \mathbf{A} \to \mathbf{R}$ and a multiplication $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$ with unit $1 \in \mathbf{A}$ so that for all $x, y \in \mathbf{A}$

$$\langle xy, xy\rangle = \langle x, x\rangle\langle y, y\rangle. \tag{1.1}$$

For convenience's sake we will identify **R** with the 1-dimensional subalgebra of **A** generated by $1 \in \mathbf{A}$. By (1.1), we have $\langle 1, 1\rangle = \langle 1, 1\rangle^2$. If $\langle 1, 1\rangle = 0$, then

show that if we assume that $\langle\, ,\rangle$ is positive definite, then there are only four inner product algebras over **R**, namely **R**, **C**, **H** (the quaternions) and **O** (the octonians).

Explicitly, we may regard **O** as the vector space $\mathbf{H} \oplus \mathbf{H}$. If we write 1 for $(1,0) \in \mathbf{O}$ and ε for $(0,1) \in \mathbf{O}$, the above lemma shows that the multiplication in **O** must be given by

$$(a + b\varepsilon), (c + d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon \tag{1.13}$$

where the inner product satisfies

$$\langle (a + b\varepsilon), (a + b\varepsilon)\rangle = a\bar{a} + b\bar{b} \tag{1.4}$$

whenever $a, b, c, d \in \mathbf{H}$.

We let $S^6 = \{u \in \operatorname{Im}\mathbf{O} \mid \langle u, u\rangle = 1\}$. The elements of S^6 are called the *imaginary units* of **O**. For any $u \in S^6$, we have $u = -\bar{u}$, so $u^2 = -u\bar{u} = -\langle u, u\rangle = -1$. We may use u to define a map J_u: $\mathbf{O} \to \mathbf{O}$ given by

$$J_u(x) = xu. \tag{1.15}$$

The identity (1.9) shows that $J_u^2(x) = (xu)u = xu^2 = -x$, so J_u defines a complex structure on **O**. We write $\mathbf{O}_u$ to denote **O** endowed with the complex structure J_u. If $u \neq v$, then clearly $J_u \neq J_v$, so we actually have a six-sphere of *distinct* complex structures on **O**. However, because S^6 is connected, we see that the orientation of **O** induced by the natural orientation of $\mathbf{O}_u$ as a complex vector space does not depend on u. We refer to this orientation as the natural orientation of **O**.