Science 2 Mid Sem

Total Marks: 40

Objective type Questions (4x2 = 8 marks)

1. Find the probability of decay of a nuclei in the next unit time, if the decay constant = $log_e(3)$.

   a. 0.67
   b. 0.33
   c. 1.67
   d. 1.33

2. Let S = $\{s_1, s_2, s_3, \ldots\ldots s_\infty\}$ be an infinitely long sequence generated by a Random Number Generator. Also assume that each s_i is between [0,1]. What is the arithmetic mean of the sequence?

   a. 0.25
   b. 1
   c. 0
   d. 0.5

3. Find a,b,c by balancing the following chemical equation: $aAl + bO_2 \rightarrow cAl_2O_3$. Which of the following is correct?

   a. (4, 2, 3)
   b. (4, 3, 2)
   c. (2, 3, 1)
   d. (2, 3, 2)

4. Consider the harmonic oscillator, where,

$$\dot{x} = v, \text{ and}$$
$$\dot{v} = -\omega^2 x$$

The orbits in the (x-v) plane can be captured by which of the following?

   a. Parabola
   b. Hyperbola
   c. Ellipse
   d. None of the above

Long Answer Type Questions (10+10+12 = 32 marks)

1. Balance the following chemical equations using a system of linear equations:

a. $C_{57}H_{110}O_6 + O_2 \rightarrow CO_2 + H_2O$ **(5 marks)**
b. $Pt + HNO_3 + HCl \rightarrow H_2PtCl_6 + NO_2 + H_2O$ **(5 marks)**

2. Consider the classic 'Rabbit vs Sheep' model. The equations are:

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

a) Find all the Fixed Points of the above system. **(2 marks)**
b) Construct the Jacobian around each of the fixed points. **(2 marks)**
c) Find the eigenvalues for the different fixed points and use these eigenvalues to classify the fixed points as Stable/Unstable/Saddle points. **(4 marks)**
d) State the Principle of Competitive Exclusion and verify it using the example of this system. **(2 marks)**

3. The simple random walk problem in one dimension. Let n_1 denote the number of steps to the right, n_2 the number of steps to the left, and N $(n_1 + n_2)$ the total number of steps. Assuming successive steps are independent of each other:

p = probability that the step is to the right
q = 1 - p = probability that the step is to the left

a) Calculate the probability $W(n_1)$ of taking (in a total of N steps) n_1 steps to the right and $n_2 = N - n_1$ steps to the left, in any order. **(3 marks)**
b) Verify the normalization of this probability: **(3 marks)**

$$\sum_{n_1=0}^{N} W(n_1) = 1$$

c) What is the mean number of steps to the right? **(6 marks)**

Answers

Objective:

1. a) $P = 1 - e^{-decay\_constant} = 1 - e^{-\ln(3)} = 1 - ⅓ = 0.67$.
2. d) 0.5 (As numbers are scaled between 0 and 1, equal probability for all).
3. b) $4\ Al + 3\ O_2 \rightarrow 2\ Al_2O_3$ is balanced.
4. c) $\omega^2 x^2 + v^2 = C$, which is an equation of an ellipse.

Long Type:

1. a. $2\ C_{57}H_{110}O_6 + 163\ O_2 \rightarrow 114\ CO_2 + 110\ H_2O$
   b. $Pt + 4\ HNO_3 + 6\ HCl \rightarrow H_2PtCl_6 + 4\ NO_2 + 4\ H_2O$

   [For each equation: 3 marks for correct equations for each atom, 2 marks for correct final answer]

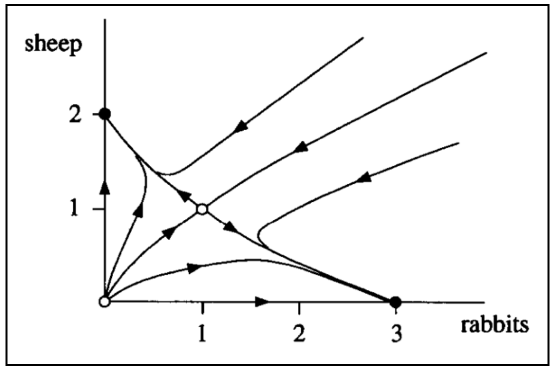
2. a) (0,0), (3,0), (0,2), (1,1) [Give 0.5 for each correct Fixed Point]

   b) $$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$
   [0.5 for each correct entry]

   c) i. (0, 0): $\boldsymbol{\lambda}_1 = 3$, $\boldsymbol{\lambda}_2 = 2$: Unstable
   ii. (3, 0): $\boldsymbol{\lambda}_1 = -3$, $\boldsymbol{\lambda}_2 = -1$: Stable
   iii. (0, 2): $\boldsymbol{\lambda}_1 = -1$, $\boldsymbol{\lambda}_2 = -2$: Stable
   iv. (1,1): $\boldsymbol{\lambda}_1 = -1 + \text{root}(2)$, $\boldsymbol{\lambda}_2 = -1 - \text{root}(2)$: Saddle

   [For each fixed point: 0.5 if eigenvalues are correct, 0.5 if Unstable/Stable/Saddle is correct]

   d) Principle: Two species competing for the same limited resource cannot coexist.



You need not draw this, but the idea is that as both rabbits and sheep are competing for grass (limited resource), the direction of the arrows ends up taking you to either the x-axis (no sheep) or the y-axis (no rabbits) from any point in the space. The two stable fixed points (3,0) -> no sheep and (0,2) -> no rabbit also reflect that only either of the two can remain.

[1 if you state the principle correctly. 1 mark if you explain either a) Grass/Food is the limited resource in this system or b) Only one will survive ultimately using stable Fixed points of the system]

3. All answers in Section 1.4 of Fundamentals of Statistical and Thermal Physics.

a)

Now, the probability of any *one* given sequence of n_1 steps to the right and n_2 steps to the left is given simply by multiplying the respective probabilities, i.e., by

$$\underbrace{p\,p\,\cdots\,p}_{n_1\text{ factors}}\;\underbrace{q\,q\,\cdots\,q}_{n_2\text{ factors}} = p^{n_1}q^{n_2} \tag{1·2·4}$$

But there are many different possible ways of taking N steps so that n_1 of them are to the right and n_2 are to the left (see illustration in Fig. **1·2·1**). Indeed, the number of distinct possibilities (as shown below) is given by

$$\frac{N!}{n_1!n_2!} \tag{1·2·5}$$

Hence the probability $W_N(n_1)$ of taking (in a total of N steps) n_1 steps to the right and $n_2 = N - n_1$ steps to the left, in any order, is obtained by multiplying the probability (1·2·4) of this sequence by the number (**1·2·5**) of possible sequences of such steps. This gives

▶ $$W_N(n_1) = \frac{N!}{n_1!n_2!}\,p^{n_1}q^{n_2} \tag{1·2·6}$$

b)

Let us first verify the normalization, i.e., the condition

$$\sum_{n_1=0}^{N} W(n_1) = 1 \tag{1·4·2}$$

which says that the probability of making any number of right steps between 0 and N must be unity. Substituting (1·4·1) into (1·4·2), we obtain

$$\begin{aligned} \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} &= (p+q)^N && \text{by the binomial theorem} \\ &= 1^N = 1 && \text{since } q \equiv 1-p \end{aligned}$$

which verifies the result.

c) 4 marks for the correct method (taking the derivative of $n_1 p^{n_1}$ and using $(p+q)^N$ formula. 2 marks for a finally correct answer.

What is the mean number $\bar{n}_1$ of steps to the right? By definition

$$\bar{n}_1 \equiv \sum_{n_1=0}^{N} W(n_1) n_1 = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} n_1 \tag{1·4·3}$$

If it were not for that extra factor of n_1 in each term of the last sum, this would again be the binomial expansion and hence trivial to sum. The factor n_1 spoils this lovely situation. But there is a very useful general procedure for handling such an extra factor so as to reduce the sum to simpler form. Let us consider the purely mathematical problem of evaluating the sum occurring in (1·4·3), where p and q are considered to be any two *arbitrary* parameters. Then one observes that the extra factor n_1 can be produced by differentiation so that

$$n_1 p^{n_1} = p \frac{\partial}{\partial p} (p^{n_1})$$

Hence the sum of interest can be written in the form

$$\begin{aligned} \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} n_1 &= \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} \left[p \frac{\partial}{\partial p} (p^{n_1}) \right] q^{N-n_1} \\ &= p \frac{\partial}{\partial p} \left[\sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} \right] && \text{by interchanging order of summation and differentiation} \\ &= p \frac{\partial}{\partial p} (p+q)^N && \text{by the binomial theorem} \\ &= pN(p+q)^{N-1} \end{aligned}$$

Since this result is true for arbitrary values of p and q, it must also be valid in our particular case of interest where p is some specified constant and $q \equiv 1 - p$. Then $p + q = 1$ so that (1·4·3) becomes simply

▶
$$\bar{n}_1 = Np \tag{1·4·4}$$

We could have guessed this result. Since p is the probability of making a right step, the mean number of right steps in a *total* of N steps is simply given

by $N \cdot p$. Clearly, the mean number of left steps is similarly equal to

$$\bar{n}_2 = Nq \tag{1·4·5}$$