Honors Independent Study Report

Quantum Mechanics and Partial Differential Equations

Nicholas Lu

A concise summary of the concepts,

exercises, and interpretations

Contents

Abstract

My topic is about understanding the theory of quantum mechanics and the mathematics behind it. Over the summer and the three semesters, I will be learning material equivalent to a term course on Partial Differential Equations and three term courses of Quantum Mechanics at an undergraduate level.

I became very interested in exploring the laws that govern everyday life ever since I learned calculus, a set of consistent logical tools designed, at the time, solely to explain gravity and planetary orbits. Exploring the realm of classical mechanics, I noticed the many inconsistencies. Why does an electric field cause a corresponding magnetic field and vice versa? How can an electron orbit a nucleus when an acceleration on a charged particle causes it to lose energy?

To fully understand how Quantum Mechanics works, I have to rely on mathematical tools such as Multivariable Calculus, Complex Analysis, and Fourier Analysis. The focus of this independent study is on partial differential equations, which have great utility in describing Quantum Mechanical systems.

This report will contain a concise set of concepts reduced from my notebook. And it will contain several exercises that I deem particular interesting, insightful, or significant. For my study on Quantum Mechanics, I will seek to explain the theory in layman's terms. I will also be demonstrating the double slit experiment live to demonstrate the real-life application of this theory.

1 Intro to Partial Differential Equations (Summer Work)

1.1 Summary

Differential Equations are used everywhere to describe complex systems. A pendulum is a system that can be modeled by a differential equation. A planet moving around a star can be modeled by a differential equation. A vibrating musical instruments can be modeled by a differential equation. Partial Differential Equations (PDEs) takes a look at these models in higher dimensions, in n dimensional space or $1 + n$ dimensional spacetime. Throughout this section of my study, I am familiarized with a few important PDEs with physical significance. In addition, I learned various different techniques of solving and assessing these PDEs.

1.2 Basics

To understand PDE's, I have included some groundwork, including some basic principles from past knowledge. These knowledge is important to understand PDEs. For example, understanding basic definitions of metric spaces is important to understand the scope of the question. Defining a set in a metric space is analogous to defining a segment of a number line. Similarly, understanding norms is important to measure how "large" a function is. With two numbers I can compare their value; with two functions I can compare their norm.

1.2.1 Definitions

A Partial Differential Equation PDE is defined as follows:

$$f(u, u_{x^1}, ..., u_{x^n}, u_{x^1x^1}, ..., u_{x^{i^1}}, u_{x^{i^N}}, .., x^1, ..., x^n) = 0$$

A Linear Differential Operator $\mathcal{L}$ follows: $\mathcal{L}(au+bv) = a\mathcal{L}(u) + b\mathcal{L}(v)$.

A PDE is homogenous if it can be written as: $f(x^1, x^2, ...) = 0$

A PDE is well-posed if the PDE and its data leads to unique solutions, and the solution is "continuous" with the data.

1.2.2 Einstein Summation Convention

$$J^\mu = T^{\mu\nu} X_\nu$$

is equivalent to:

$$\forall \mu \in n,\ J^\mu = \sum_{\nu=1}^{\nu=n} T^{\mu\nu} x_\nu$$

Matrix multiplication $AB = C$ can be written as,

$$A^\mu_\nu B^\nu_\gamma = C^\mu_\gamma$$

1.2.3 Tensor Transformation Laws

Under a coordinate transformation $x \to \bar{x}$, a contravariant vector transforms as

$$X^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \bar{X}^\alpha$$

A covariant vector transforms as:

$$X_\mu = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \bar{X}_\alpha$$

A tensor transforms as:

$$A^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m} = \frac{\partial x^{\mu_1}}{\partial \bar{x}^{\alpha_n}} \ldots \frac{\partial x^{\mu_1}}{\partial \bar{x}^{\alpha_n}} \frac{\partial \bar{x}^{\beta_1}}{\partial x^{\nu_1}} \ldots \frac{\partial \bar{x}^{\beta_m}}{\partial x^{\nu_m}} \bar{A}^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}$$

1.2.4 Metric Space Definitions

The ball $B_r(x) \triangleq \{y \in S \,|\, d(y,x) < r\}$, where $r > 0$ and $d(x,y)$ is a distance function.

A set S is open if $\forall x \in S$, $B_\delta(x) \subset S$ for some $\delta > 0$.

A set S is closed if its complement is open.

A set S is connected if there exists a polygonal path between every two points.

A set S is bound if $S \subset B_R(0)$ for some $R > 0$.

A set S is compact if it is both closed and bounded.

The closure $\bar{S}$ is the smallest closed set that contain S.

The interior $\mathrm{Int}(S)$ is the largest open set that is contained in S.

The boundary $\partial S = \bar{S} - \mathrm{Int}(S)$.

$x \in \partial S$ if $\forall r$, $\exists y \in S \cap B_r(x)$ and $\exists y \in S \cap B_r(x)^c$.

A domain Ω is an open and connected set.

A function $f(x)$ is compactly supported if there exist compact set C such that $f(x) = 0$ outside of C.

1.2.5 Norms

The C^k norm for $f \in C^k(\mathbb{R}^n)$ on domain Ω is:

$$||f(x)||_{C^k(\Omega)} = \sum_{a=0}^{k} \sup_{x \in \Omega} |f^{(a)}(x)|$$

The L^p norm for $f \in C^0(\mathbb{R}^n)$ on domain Ω is:

$$||f(x)||_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p d^n x \right)^{\frac{1}{p}}$$

1.2.6 Function Operators

The inner product of two real functions $f(x)$ and $g(x)$ defined in $\mathbb{R}^n$ is

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x)d^n x$$

The Hermitian product of two complex functions $f(x)$ and $g(x)$ defined in $\mathbb{R}^n$ is

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g^*(x)d^n x$$

The convolution of two real functions $f(x)$ and $g(x)$ defined in $\mathbb{R}^n$ is

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)d^n y = \int_{\mathbb{R}^n} f(y)g(x-y)d^n y$$

1.2.7 Common Functions

The Kirchhoff delta is defined as

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

With the following properties:

$$\int_{\mathbb{R}} \delta(x)dx = 1$$

$$[f * \delta](x) = \int_{\mathbb{R}} f(y)\delta(x-y)dy = f(x)$$

The Kronecker delta is defined as:

$$\delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

1.2.8 Identities

The Divergence Theorem for a domain Ω:

$$\int_{\Omega} \nabla \cdot \vec{F} d^n x = \int_{\partial\Omega} \vec{F} \cdot \hat{N}(\sigma) d\sigma$$

Integration by Parts:

$$\int_{\Omega} v(\nabla \cdot \vec{F}) d^n x = -\int_{\Omega} \nabla v \cdot \vec{F} d^n x + \int_{\partial\Omega} vF \cdot \hat{N}(\sigma) d\sigma$$

First Green's Identity

$$\int_{\Omega} v \Delta u d^n x = -\int_{\Omega} \nabla v \cdot \nabla u d^n x + \int_{\partial\Omega} v \partial_{\hat{N}(\sigma)} u d\sigma$$

Second Green's Identity

$$\int_{\Omega} (v\Delta u - u\Delta v) d^n x = \int_{\partial\Omega} (v \partial_{\hat{N}(\sigma)} u - u \partial_{\hat{N}(\sigma)} v) d\sigma$$

1.2.9 Exercise: Prove Green's Identity

Let $\vec{F} = u\nabla v - v\nabla u$. By the Divergence Theorem,

$$\int_{\Omega} \nabla \cdot (u\nabla v - v\nabla u) d^n x = \int_{\partial\Omega} (u\nabla v - v\nabla u) \cdot \hat{N}(\sigma) d\sigma$$

The left hand side becomes

$$\int_{\Omega} u\Delta v + \nabla u \cdot \nabla v - v\Delta u - \nabla v \cdot \nabla u \; d^n x =$$

$$= \int_{\Omega} (v\Delta u - u\Delta v) d^n x$$

Green's Identity follows.

$$\int_{\Omega}(v\Delta u - u\Delta v)d^n x = \int_{\partial\Omega}(v\partial_{\hat{N}(\sigma)}u - u\partial_{\hat{N}(\sigma)}v)d\sigma$$

□

1.2.10 Exercise: Prove Cauchy Schwartz Inequality and Triangle Inequality

Let u, v be defined in an inner-product space where $\langle u, v\rangle$ is defined. First, if $\langle x, y\rangle = 0$, the Pythagorean Theorem holds, where $\langle (x\pm y), (x\pm y)\rangle \triangleq ||x\pm y||^2 = ||x||^2 + ||y||^2$. Now,

$$\left\langle \frac{\langle v, u\rangle u}{||u||^2} - v, u \right\rangle = \langle v, u\rangle - \langle v, u\rangle = 0.$$

By the Pythagorean Theorem,

$$||-v||^2 = \left|\left|\frac{\langle v, u\rangle u}{||u||^2} - v\right|\right|^2 + \left|\left|-\frac{\langle v, u\rangle u}{||u||^2}\right|\right|^2$$

$$||v^2|| \geqslant \left|\left|-\frac{\langle v, u\rangle u}{||u||^2}\right|\right|^2$$

$$||v||\,||u|| \geqslant |\langle v, u\rangle|$$

Furthermore, the Triangle Inequality follows

$$||v+w||^2 = \langle v, v\rangle + \langle u, u\rangle + 2\langle u, v\rangle \leqslant ||u||^2 + ||v||^2 + 2||u||||v|| = (||u|| + ||v||)^2$$

$$||v+w|| \leqslant ||v|| + ||w||$$

In integral form, Cauchy-Schwartz states:

$$\left(\int_\Omega f(x)^2 dx\right)^{1/2} \left(\int_\Omega g(x)^2 dx\right)^{1/2} \geqslant \left(\int_\Omega f(x)g(x) dx\right)$$

□

1.3 The Heat Equation

The heat equation models the physical laws of heat flow. In layman's term, it states that

$$\text{Movement of heat} = D(\text{Temperature Difference}) + \text{heat from other sources}$$

where D is a constant. When you put your cold hand near a hot campfire, the heat moves from the hot air around your hand into your hand because the air is hotter than your hand. The movement of the heat, in this case, is proportional to the temperature difference.

Given sufficient information, the heat equation is unique. This means that there is only one physically possible ways for the temperature in a system to evolve. If I know everything about the campfire and your hand, I can determine the temperature of your hand any time later.

In the study of the Heat Equation, I learned many things. I learned the separation of variables technique, which may work for many other PDEs. I learned the Energy Method, a technique that can be generalized to many other PDEs to prove uniqueness. I learned many mathematical properties of heat flow.

1.3.1 Definition

$$\partial_t u - D\Delta u = f(t, x)$$

1.3.2 Spacetime Cylinder

A Spacetime Cylinder $Q_T \subset \mathbb{R}^{n+1}$ defines the scope of the heat equation. $Q_T = (0,T) \times \Omega$. Its boundary is $\partial_P Q_T = (\{0\} \times \bar{\Omega}) \cup ((0,T] \times \partial\Omega)$

1.3.3 Boundary Conditions

Cauchy Data defines the value of the function u on the Cylinder's base.

Dirichlet Boundary Conditions defines the value of the function u on the Cylinder's sides.

Neummann Boundary Conditions defines $\partial_{\hat{N}(\sigma)} u$ on the Cylinder's sides.

Robin Boundary Conditions defines the weighted average $-\alpha\partial_{\hat{N}(\sigma)} u + u$ on the Cylinder's sides

Mixed Boundary Conditions are where the side of the cylinder is partitioned into two sets, and each partition have either Dirichlet, Boundary, or Neumann boundary conditions.

1.3.4 Well-Posedness

The heat equation is well posed given Cauchy Data and either Dirichlet, Neumann, Robin, or Mixed boundary conditions.

1.3.5 Technique: Separation of variables

Separation of variables is a technique that can be used to solve a variety of PDEs. Here it is used to solve for the general modes of a data-less heat equation.

Suppose $\partial_t u - Du_{xx} = 0$ Let $u(x,t) = v(t)w(x)$. Then

$$\frac{v'(t)}{v(t)} = D\frac{w''(x)}{w(x)}$$

Since LHS only depend on t, and RHS only depend on x, they both must equal a constant. If the constant is positive or 0, the solution is trivial. Let that constant be $-\lambda^2$. Then $v(t) = e^{-\lambda^2 t}$ and $w(x) = e^{iD^{-1}\lambda x}$. Thus, $u_\lambda = e^{iD^{-\frac{1}{2}}\lambda x}e^{-\lambda^2 t}$, a exponentially decaying sinusoidal, is a mode of the heat equation.

1.3.6 Technique: Energy Method

The Energy Method is used to prove uniqueness for the Dirichlet boundary condition for the heat equation in $\mathbb{R}^{1+1}$, but can be used to generalize into proving many other things for a number of PDEs. This method involves finding a time invariant quantity for a PDE, called the energy.

Say u and v are solutions to the heat equation given Cauchy data and Dirichlet boundary Conditions:

$$\begin{cases} \partial_t u - D\partial_x^2 u = f(x,t), \qquad (t,x) \in [0,T] \times [0,L] \\ u(0,x) = g(x), \qquad x \in [0,L] \\ w(t,0) = h_0(x), \quad w(t,L) = h_L(x), \qquad t \in [0,T] \end{cases}$$

Then $w = u - v$ satisfies:

$$\begin{cases} \partial_t w - D\partial_x^2 w = 0, \qquad (t,x) \in [0,T] \times [0,L] \\ w(0,x) = 0, \qquad x \in [0,L] \\ w(t,0) = 0, \quad w(t,L) = 0, \qquad t \in [0,T] \end{cases}$$

Then define Energy

$$E(t) = \int_{[0,L]} w^2(t,x)dx$$

The time derivative of the Energy is

$$\frac{1}{2}\frac{d}{dt}E(t) = \int_{[0,L]} w\partial_t w\, dx = \int_{[0,L]} w\partial_x^2 w\, dx = -\int_{[0,L]} (\partial_x w)^2 dx \leqslant 0$$

Since $E(0) = 0$ and $E(t) \geqslant 0$, $E(t) = 0$ for all values of t. Thus $w = 0$ everywhere and $u = v$. Thus, the solution to the Heat Equation given Dirichlet Boundary Conditions is unique.

1.3.7 Weak Maximum Principle

Let $\partial_t w - D\Delta w = f$ on Q_T with $f \leqslant 0$. Then $\partial_t w$ achieves its maximum in $\bar{Q_T}$ on $\partial_P Q_T$.

$$\max_{\partial_P Q_T} w(t,x) = \max_{\bar{Q_T}} w(t,x)$$

1.3.8 Comparison and Stability

If $\partial_t v - D\partial_x^2 v = f$ and $\partial_t w - D\partial_x^2 w = g$, then

(Comparison) If $v \geqslant w$ on $\partial_p Q_T$ and $f \geqslant g$,then $v \geqslant w$ on all of Q_T.

(Stability) $\max_{\bar{Q_T}} |v - w| \leqslant \max_{\partial_p Q_T} |v - w| + T \max_{\bar{Q_T}} |f - g|$.

1.3.9 Fundamental Solution

$$\Gamma_D(t,x) \triangleq \frac{1}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}}$$

$\Gamma_D(t,x)$ solves the equation $\partial_t \Gamma_D - D\Delta\Gamma_D = 0$. In addition,

$$\lim_{t\downarrow 0} \Gamma_D(t,x) = \delta(x)$$

1.3.10 Solution to the Global Cauchy Problem

The solution to

$$\partial_t u - D\Delta u = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n$$

$$u(0,x) = g(x), \qquad x \in \mathbb{R}^n$$

is

$$u(t,x) = (\Gamma_D(t,\cdot) * g(\cdot))(x) = \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} g(y) e^{-\frac{|x-y|^2}{4Dt}} d^n y$$

Note: The solutions to the heat equation is smooth, even if the data is only continuous. Furthermore, this solution propagates with an infinite speed - even if $g(x)$ is compactly supported, every point, at any time, will be affected by $g(x)$.

1.3.11 Exercise: Duhamel's Principle

Let $\mathcal{L} = \partial_t + \tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}}$ is an linear differential operator that only operate on the spatial variable x. Show

$$v(t,x) = \int_0^t v_{(s)}(t-s,x)ds$$

Where

$$\mathcal{L}v_{(s)}(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n$$

$$v_{(s)}(0,x) = f(s,x), \qquad x \in \mathbb{R}^n$$

First, we will show v follows the PDE.

$$\mathcal{L}v = \partial_t \int_0^t v_{(s)}(t-s,x)\,ds + \tilde{\mathcal{L}} \int_0^t v_{(s)}(t-s,x)\,ds =$$

$$= v_{(t)}(0,x) + \int_0^t \partial_t v_{(s)}(t-s,x)\,ds + \int_0^t \tilde{\mathcal{L}} v_{(s)}(t-s,x)\,ds = f(t,x)$$

Then, we will show v follows the Cauchy Data.

$$v(0,x) = \int_0^0 v_{(s)}(t-s,x)\,ds = 0$$

Thus, v is the solution. $\square$

1.3.12 Exercise: Solution to the inhomogenous Global Cauchy Problem

Find the solution to:

$$\partial_t u - D\Delta u = f(t,x), \qquad (t,x) \in [0,\infty] \times \mathbb{R}^n$$

$$u(0,x) = g(x), \qquad x \in \mathbb{R}^n$$

Let $u = u_h + u_i$, where u_h solves the above equation when $f(t,x) = 0$, and u_i solves the above equation when $g(x) = 0$. Note that when u is defined as such, it solves the PDE. The solution to the homogeneous global Cauchy problem is:

$$u_h = (\Gamma_D(t,\cdot) * g(\cdot))(x)$$

From the previous exercise,

$$u_i = \int_0^t u_{(s)}(t-s,x)\,ds = \int_0^t (\Gamma_D(t-s,\cdot) * f(s,\cdot))(x)\,ds$$

Thus,

$$u = (\Gamma_D(t,\cdot) * g(\cdot))(x) + \int_0^t (\Gamma_D(t-s,\cdot) * f(s,\cdot))(x)\,ds$$

$\square$

1.4 Laplace's and Poisson's Equation

Laplace's and Poisson's equation comes up often in fluid dynamics and electricity and magnetism. For example,the voltage at any point in space, given a configuration of charges (a density function), conforms to the Poisson's equation. There are some very nice properties of functions that conform to Laplace's equation - these functions are called harmonic.

1.4.1 Definition

Laplace's Equation

$$\Delta u(x) = 0, \qquad x \in \Omega$$

A function that follows Laplace's Equation is *harmonic.* Note that all complex-differentiable functions that follow the Cauchy-Riemann condition is harmonic.

Poisson's Equation

$$\Delta u(x) = f(x), \qquad x \in \Omega$$

1.4.2 Wellposedness

If $\hat{N}$ is a normal vector to $\partial\Omega$. Then the Laplace's/Poisson's equation is well-posed under these conditions:

Dirichlet: $u(x)|_{\partial\Omega} = g(x)$

Neumann: $\nabla_{\hat{N}} u(x)\big|_{\partial\Omega} = h(x)$

Robin: $\nabla_{\hat{N}} u(x)\big|_{\partial\Omega} + \alpha u(x)|_{\partial\Omega} = h(x)$

Mixed: Any of the above conditions in each partition of Ω.

Conditions at infinity: Specify behavior of $u(x)$ as $x \to \infty$.

1.4.3 Mean Value Properties for harmonic functions.

Let $B_R(x) \subset \Omega$. Then,

$$u(x) = \frac{n}{\omega_n R^n} \int_{B_R(x)} u(y) d^n y$$

And

$$u(x) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x)} u(\sigma) d\sigma$$

1.4.4 Strong Maximum Principle

If u achieves its max or min at a point in Ω, then u is constant in Ω. If u is not constant,

$$u(x) < \max_{y \in \partial\Omega} u(y)$$

$$u(x) > \min_{y \in \partial\Omega} u(y)$$

1.4.5 Comparison and Stability Estimate

For the harmonic functions that solve the initial Cauchy data $u_f(x) = f(x)$ and $u_g = g(x)$.

Comparison If $f \geqslant g$ on $\partial\Omega$ and $f \not\equiv g$, $u_f > u_g$ on Ω.

Stability Estimate $|u_f(x) - u_g(x)| \leqslant \max_{y \in \partial\Omega} |f(y) - g(y)|$

1.4.6 Fundamental Solution

The fundamental solution $\Phi(x)$ to the Poisson's Equation is:

$$\Phi(x) \triangleq \frac{1}{2\pi} \ln |x|, \qquad n = 2$$

$$\Phi(x) \triangleq \frac{1}{\omega_n |x|^{n-2}}, \qquad n \geqslant 3$$

The solution to

$$\Delta u(x) = f(x), \qquad x \in \mathbb{R}^n$$

is

$$u(x) = f(x) * \Phi(x)$$

A concise proof of this, given the fact that $\Delta\Phi(x) = \delta(x)$:

$$\Delta(f * \Phi) = f * \Delta\Phi = f * \delta = f(x)$$

1.4.7 Green Functions

A green function in Ω, for each x, follows:

$$\Delta_y G(x, y) = \delta(x - y), \qquad y \in \Omega$$

$$G(x, \sigma) = 0, \qquad \sigma \in \partial\Omega$$

In addition, $G(x, y)$ can also be written as

$$G(x, y) = \Phi(x - y) - \phi(x, y)$$

with ϕ solving

$$\Delta_y \phi(x, y) = 0, \qquad y \in \Omega$$

$$\phi(x, \sigma) = \Phi(x - \sigma), \qquad \sigma \in \partial\Omega$$

1.4.8 Representation formulas for $u(x)$

For *any* $u \in C^2(\bar{\Omega})$, $u(x)$ can be represented as

$$u(x) = \int_\Omega \Phi(x-y)\Delta_y u(y)\, d^n y - \int_{\partial\Omega} \Phi(x-\sigma)\nabla_{\hat{N}(\sigma)} u(\sigma)\, d\sigma + \int_{\partial\Omega} u(\sigma)\nabla_{\hat{N}(\sigma)} \Phi(x-\sigma)\, d\sigma$$

1.4.9 Poisson's Kernel Formula

The solution to

$$\Delta u(x) = f(x), \qquad x \in \mathbb{R}^n$$

$$u(x) = g(x), \qquad x \in \Omega$$

can be written as

$$u(x) = -\int_{\Omega} \Phi(x-y) f(y)\, d^n y - \int_{\partial\Omega} g(\sigma) \nabla_{\hat{N}(\sigma)} u(\sigma)\, d\sigma$$

1.4.10 Poisson's Formula

The solution to

$$\Delta u(x) = 0, \qquad x \in \mathbb{R}^n$$

$$u(x) = f(x), \qquad x \in \Omega$$

is

$$u(x) = \frac{R^2 - |x-\rho|^2}{\omega_n R} \int_{\partial B_R(\rho)} \frac{f(\sigma)\, d\rho}{(x-\sigma)^n}$$

1.4.11 Harnack's Inequality

A harmonic function u that is non-negative in the ball $B_R(0)$ is bounded by

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}} u(0) \leqslant u(x) \leqslant \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}} u(0)$$

1.4.12 Exercise: Maximum of $|\nabla v|^2$ in torsion problem

Let $\Omega \in \mathbb{R}^n$ and $v \in C^1(\bar{\Omega}) \cap C^3(\Omega)$ solve

$$\Delta v = -2, \qquad \text{in } \Omega$$

$$v = 0, \qquad \text{on } \partial\Omega$$

Show $u(x) = |\nabla v|^2$ attains its maximum on $\partial\Omega$.

Let $p \in \Omega$. Since $\Delta v = -2$ and by the 1st Green's Identity

$$-2 \int_{B_R(p)} v dx = \int_{B_R(p)} v \Delta v\, dx = \int_{\partial B_R(p)} v \nabla_{\hat{N}(\sigma)} v\, d\sigma - \int_{B_R(p)} |\nabla v|^2 dx$$

Let $R \to 0$. By symmetry, the flux of any constant vector over a surface of a ball is 0, thus the first term on the right is 0. The above becomes

$$-2v(p) = -u(p)$$

Since every point in the closure is a limit point of Ω, note that $u(x) = 2v(x)$ on $\bar{\Omega}$ by continuity of $v(x)$. Since the function v is subharmonic it achieves its maximum on $\partial\Omega$, $u(x)$ achieves its maximum on $\partial\Omega$. $\square$

1.4.13 Exercise: $|u(x)| \leqslant \ln|x| + 1$

For a harmonic function $|u(x)| \leqslant \ln(|x| + 1)$ in $\mathbb{R}^3$, show $u(x) = 0$ for all x. Let $v(x) = u(x) + \ln(R + 1)$. Note that $v(x)$ is also a harmonic function. Since $v(x) \geqslant 0$ in $B_R(0)$, by Harnack's Inequality,

$$\frac{R(R - |x|)}{(R + |x|)^2} v(0) \leqslant v(x) \leqslant \frac{R(R + |x|)}{(R - |x|)^2} v(0)$$

As $R \to \infty$ and applying L'Hopital

$$\frac{2R - |x|}{2(R + |x|)}) u(0) \leqslant u(x) + \ln(|x| + 1) \leqslant \frac{2R + |x|}{2(R - |x|)} u(0)$$

The center diverges. This is a contradiction. $\square$

1.5 Wave Equation

The wave equation describes the propagation of waves. Water waves, sound waves, and light waves all conform to the wave equation. For example, understanding the wave equation allows us to understand how a ripple propagate through a pond. The method of spherical averages is a useful tool to evaluate the wave equation.

1.5.1 Definition

$$-\frac{1}{c^2}\partial_t^2 u + \Delta u = 0$$

1.5.2 Wellposedness

The wave equation is well posed for the Global Cauchy Problem.

$$-\partial_t^2 u + c^2 \Delta u = 0, \qquad (t, x) \in \mathbb{R}^{1+n}$$

$$u(0, x) = f(x), \qquad x \in \mathbb{R}^n$$

$$\partial_t u(0, x) = g(x), \qquad x \in \mathbb{R}^n$$

In addition, the $1 + 1$ dimension wave equation

$$-\partial_t^2 u + c^2 \partial_x^2 u = 0, \qquad (t, x) \in \mathbb{R} \times [0, L]$$

$$u(0, x) = f(x), \quad \partial_t u(0, x) = g(x), \qquad x \in [0, L]$$

is wellposed given either one of:

Dirichlet Boundary Conditions $u(t, 0) = a(t), \quad u(t, L) = b(t)$

Neumann Boundary Conditions $-\partial_x u(t, 0) = a(t), \quad \partial_x u(t, L) = b(t)$

Robin Boundary Conditions $\partial_x u(t, 0) = ku(t, 0), \quad \partial_x u(t, L) = -ku(t, L)$

Mixed Boundary Conditions, where one boundary condition given above is given at each boundary.

1.5.3 d'Alembert's formula and proof

Define Null Coordinates $q = t - x$ and $s = t + x$. Note that $\partial_q = \frac{1}{2}(\partial_t - \partial_x)$, $\partial_s = \frac{1}{2}(\partial_t + \partial_x)$, $\partial_t = \partial_q + \partial_s$, $\partial_x = \partial_s - \partial_q$. Note

$$\partial_s \partial_q u = \frac{1}{2}(\partial_t^2 - \partial_x^2)u = 0$$

Integrating w.r.t s

$$\partial_q u = H(q)$$

Note $(t, x) = (\tau, y)$ and $(0, y - \tau)$ have the same coordinates in q.

$$\partial_q u(\tau, y) = \partial_q u(0, y - \tau) = \frac{1}{2}(\partial_t - \partial_x)u\Big|_{(0,y-\tau)} = \frac{1}{2}(g(y - \tau) - f'(y - \tau)$$

With a similar process for s,

$$\partial_s u(\tau, y) = \frac{1}{2}(g(y + \tau) + f'(y + \tau))$$

Adding,

$$\partial_t u = (\partial_q + \partial_s)u = \frac{1}{2}(f'(x + t) - f'(x - t) + g(x + t) - g(x - t))$$

Integrating,

$$u(t, x) = \frac{1}{2}(f(x + t) - f(x) - f(x - t) + f(x)) + \int_0^t g(x + t) - g(x - t)dt$$

d'Alembert's formula follows:

$$u(t, x) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c}\int_{z=x-ct}^{z=x+ct} g(z)dz$$

□

1.5.4 Solution in 3 dimensions: Method of Spherical Averages

Given solutions $u(t,x)$, define a spherical average of u. For fixed x, a slight modification in $r = |x|$ will conform to the $1+1$ dimensional wave equation in (t,r)

Let $u(t,x)$ solve

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \qquad (t,x) \in \mathbb{R}^{n+1}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R}^3$$

$$\partial_t u(0,x) = g(x), \qquad x \in \mathbb{R}^3$$

For each $r > 0$, define

$$U(t,r;x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(t,\sigma)d\sigma = \frac{1}{4\pi}\int_{\partial B_r(x)} u(t,x+r\omega)d\omega$$

$$F(t,r;x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} f(\sigma)d\sigma, \quad G(t,r;x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} g(\sigma)d\sigma$$

Define modifications $\tilde{U} = rU$, $\tilde{F} = rF$, $\tilde{G} = rG$. Then $\tilde{U}$ solves

$$-\partial_t^2 \tilde{U}(t,x) + \partial_r^2 \tilde{U}(t,x) = 0, \qquad (r,x) \in \mathbb{R}^{1+1}$$

$$\tilde{U}(0,x) = \tilde{F}(x), \qquad x \in \mathbb{R}^3$$

$$\partial_t \tilde{U}(0,x) = \tilde{G}(x), \qquad x \in \mathbb{R}^3$$

With the fact that $\lim_{r\downarrow 0} \tilde{U}(t,r;x) = u(t,x)$, $u(t,x)$ then can be solved.

1.5.5 Kirchoff's formula

$$u(t,x) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(\sigma)d\sigma + \frac{1}{4\pi t}\int_{\partial B_t(x)} \nabla_{\hat{N}(\sigma)} f(\sigma)d\sigma + \frac{1}{4\pi t}\int_{t(x)} g(\sigma)d\sigma$$

Note: $u(x,t)$ only depend on the surface $\partial B_t(x)$. This is called the Sharp Huygens Principle. This principle only hold when $n \geqslant 3$ and is odd.

Note: There is a finite speed of propagation.

1.5.6 Exercise: Conservation of Energy under the Wave Equation

Let

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R}^n$$

$$\partial_t u(0,x) = g(x), \qquad x \in \mathbb{R}^n$$

Under the assumption that $f(x)$ and $g(x)$ is compactly supported, show

$$||\nabla_{t,x} u(t,x)||_{L^2(\mathbb{R}^n)} = ||\nabla_{t,x} u(0,x)||_{L^2(\mathbb{R}^n)} = \left(\int_{R^n} g(x)^2 + |\nabla f(x)|^2 \, d^n x\right)^{1/2}$$

Define the vectorfield

$$J = (J^0, ..., J^n) \triangleq (\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2, -\partial_1 u \partial_t u, ..., -\partial_n u \partial_t u)$$

The divergence of the vectorfield $\partial_\mu J^\mu = 0$:

$$\partial_\mu J^\mu = \partial_t J^0 + \partial_i J^i = [\partial_t u(\partial_t^2 u) + \sum_{i=1}^n \partial_i u(\partial_t \partial_i u)] + [-\sum_{i=1}^n \partial_i u(\partial_t \partial_i u) - \Delta u \partial_t u] = 0$$

Given any vector field $V = (V^0, ...V^n) = (1, \omega^1, ..., \omega^n)$ with $\sum_i (\omega_i)^2 \leqslant 1$.

$$\sum_{i=1}^n \partial_i u \partial_t u \omega_i \leqslant \left((\partial_t^2 u) \sum_{i=1}^n (\partial_i u)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n \omega_i^2\right)^{\frac{1}{2}} \leqslant (\partial_t u)|\nabla u|$$

Then,

$$V \cdot J = \sum_{\nu=0}^n V^\nu J^\nu = \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \sum_{i=1}^n \partial_i u \partial_t u \omega_i \geqslant$$

$$\geqslant \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - (\partial_t u)|\nabla u| = \frac{1}{2}(\partial_t u - |\nabla u|)^2 \geqslant 0$$

Define the Energy

$$E^2(t;R;p) = \int_{B_{R-t}(p)} J^0(t,x)d^n x = \frac{1}{2}\int_{B_{R-t}(p)} (\partial_t u)^2 + |\nabla u|^2 \, d^n x \triangleq$$

$$\triangleq \frac{1}{2}\int_{B_{R-t}(p)} |\nabla_{t,x} u|^2 d^n x$$

Define the cone $C_{t,p;R} = \{(\tau, y)\big||y-p| \leqslant R-\tau\}$. $\partial C_{t,p;R} = [\{0\} \times B_R(p)] \cup [[\tau] \times B_{R-\tau}(p)] \cup M_{t,p;R}$, where surface $M_{t,p;R}$ denotes the middle part of the cone. This surface can be seen as the kernel of function

$$f(t,x) = (x^1)^2 + ... + (x^n)^2 - (R-t)^2 = 0$$

Thus the normal vector to the surface is its gradient:

$$\nabla f(t,x) = (2(R-t), 2x^1, ..., 2x^n) \to \hat{N}(\sigma) = (1, \frac{x^1}{t-R}, ..., \frac{x^n}{t-R})$$

Applying the Divergence theorem on the vector field J,

$$\int_{C_{t,p;R}} \partial_\nu J^\nu d^{1+n}x = \int_{M_{t,p;R}} \hat{N}(\sigma) \cdot J d\sigma - \int_{B_R(p)} J^0 d^n x + \int_{B_{R-t}(p)} J^0 d^n x$$

The left hand side is 0, and since $\hat{N}(\sigma)$ is of the form $(1, \omega^1, ..., \omega^n)$ with $\sum_i \omega^i \leqslant 1$, the first term on the right is $\leqslant 0$. We see that

$$E^2(0;R;p) \geqslant E^2(t;R;p)$$

Furthermore, let $R \to \infty$. Under the assumption that $f(x)$ and $g(x)$ is compactly supported on a set $S \subset B_{R_0}(p)$, $u(x,t)$ is compactly supported on the set $B_{R_0+t}(p)$, evident from the method of spherical averages. J is also compactly supported on the same set as a result. For every t, chose $R \geqslant R_0 + t$. The term

$$\int_{M_{t,p;\infty}} \hat{N}(\sigma) \cdot J \, d\sigma = 0$$

and

$$E^2(0) = E^2(t)$$

or

$$||\nabla_{t,x} u(t,x)||_{L^2(\mathbb{R}^n)} = ||\nabla_{t,x} u(0,x)||_{L^2(\mathbb{R}^n)} = \left(\int_{R^n} g(x)^2 + |\nabla f(x)|^2 \, d^n x \right)^{1/2}$$

Which demonstrates that energy is conserved. $\square$

1.5.7 Exercise: Equipartition of energy under the Wave Equation

Let $f(x)$, $g(x)$ vanish outside of $[-R, R]$. Let

$$-\partial_t^2 u(t,x) + \partial_x^2 u(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R}$$

$$\partial_t u(0,x) = g(x), \qquad x \in \mathbb{R}$$

And define

$$P^2(t) = \int_{\mathbb{R}} (\partial_x u)^2 dx$$

$$P^2(t) = \int_R (\partial_t u)^2 dx$$

$$E^2(t) = P^2(t) + K^2(t)$$

Show $P^2(t) = K^2(t) = \frac{1}{2} E^2(t)$ for large t.

Define Null Coordinates $q = t - x$ and $s = t + x$ similar to the proof of d'Alembert's formula. Since $\partial_q = \frac{1}{2}(\partial_t - \partial_x)$ and $\partial_s = \frac{1}{2}(\partial_t + \partial_x)$, $\partial_s u \partial_q u = \frac{1}{4}(\partial_t^2 u - \partial_x^2 u)$. $\partial_q u$ and $\partial_s u$ can be written as:

$$\partial_q u(t,x) = \frac{1}{2}(g(x-t) - f'(x-t))$$

$$\partial_s u(t,x) = \frac{1}{2}(g(x+t) + f'(x+t))$$

Let $t > R$. If $\partial_s u(t,x)$ does not vanish, $x + t \in [-R, R]$. This implies $x - t \notin [-R, R]$ and $\partial_q u(t,x)$ vanishes. Thus, at least one of $\partial_q u(t,x)$ or $\partial_s u(t,x)$ vanishes. Integrating,

$$\int_{\mathbb{R}} \partial_s u(t,x) \partial_q u(t,x) dx = \frac{1}{4} \int_{\mathbb{R}} (\partial_t u(t,x))^2 - (\partial_x u(t,x))^2 dx = 0$$

$P^2(t) = K^2(t) = \frac{1}{2}E^2(t)$ follows. $\square$

1.6 Minkowski Spacetime and the Wave Equation

The Minkowski metric is a type of metric tensor, which measures the distance between two points in spacetime. The Minkowski metric is often used in Maxwell's equations - the fundamental laws that govern electricity and magnetism. It is also used to formulate Einstein's Special relativity. Understanding how to work with the Minkowski Spacetime mathematically is a tool towards understanding higher level physics.

1.6.1 The Minkowski Metric in 1+n Spacetime Dimensions

$$m_{\mu\nu} = \text{diag}(-1, 1, ..., 1)$$

$$(m^{-1})^{\mu\nu} = \text{diag}(-1, 1, ..., 1)$$

1.6.2 Vectors

The metric duel of a vector is defined

$$X_\mu = m_{\mu\alpha} X^\alpha$$

There are several qualities of vectors:

Timelike vectors: $m(X, X) \triangleq m_{\alpha\beta} X^\alpha X^\beta < 0$

Spacelike vectors: $m(X, X) > 0$

Null vectors: $m(X, X) = 0$

Casual vectors: $m(X, X) \leqslant 0$

Future directed: $X^0 > 0$

Past directed: $X^0 \leqslant 0$

1.6.3 Lorentz Transformations

A Lorentz transformation is a linear transformation that preserves the form of the Minkowski metric.

$$\Lambda_\mu^\alpha \Lambda_\nu^\beta m_{\alpha\beta} = m_{\mu\nu}$$

(In matrix form, this is equivalent to $\Lambda^T m \Lambda = m$)

If $\det \Lambda = 1$, Λ is orientation preserving and proper. Note that $m(\Lambda X, \Lambda Y) = m(X, Y)$, and that Λ preserves the quality of vectors.

1.6.4 Lorentz Group

The Lorentz group is a $\frac{1}{2}n(n+1)$ dimensional group generated by $\frac{1}{2}n(n-1)$ dimensional subgroup of spacial rotations and n dimensional subgroup of proper Lorentz boost, or spacetime rotations. An example of a spacial rotation in $\mathbb{R}^{1+3}$

is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An example of a spacetime rotation is

$$\begin{bmatrix} \cosh(\zeta) & -\sinh(\zeta) & 0 & 0 \\ \sinh(\zeta) & -\cosh(\zeta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\gamma = \frac{1}{\sqrt{1-v^2}}$ and $v \in (-1, 1)$

1.6.5 Null Frame

A Null frame is a basis $\{L, \underline{L}, e_{(1)}, ..., e_{(n-1)}\} \in \mathbb{R}^{1+n}$, where $m(L, \underline{L}) = -2$, $m(L, L) = m(\underline{L}, \underline{L}) = 0$, and $e_{(i)}$ are orthonormal vectors that span the complement of span$(L, \underline{L})$. So, $m(L, e_{(i)}) = m(\underline{L}, e_{(i)}) = 0$ and $m(e_{(i)}, e_{(j)}) = \delta_{ij}$.

Null Frame Decomposition: For any null frame, the Minkowski metric can be decomposed

$$m_{\mu\nu} = -\frac{1}{2}L_\mu \underline{L}_\nu - \frac{1}{2}\underline{L}_\mu L_\nu + \not{m}_{\mu\nu}$$

Where $\not{m}$ is positive definite on span$(e_{(1)}, ..., e_{n-1})$ and vanishes in span$(L, \underline{L})$. Also,

$$(m^{-1})^{\mu\nu} = -\frac{1}{2}L^\mu \underline{L}^\nu - \frac{1}{2}\underline{L}^\mu L^\nu + \not{m}^{\mu\nu}$$

1.6.6 Linear Wave Operator

The linear wave operator is defined as

$$\Box_m = (m^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta$$

The wave equation is

$$\Box_m\phi = 0$$

1.6.7 Energy Momentum Tensor for the Wave Equation

The energy momentum tensor for the wave equation is defined as

$$T_{\mu\nu} \triangleq \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_{\mu\nu}(m^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$$

Divergence: The divergence of the energy momentum tensor is 0:

$$\partial_\mu T^{\mu\nu} = (\Box_m\phi)(m^{-1})^{\nu\alpha}\partial_\alpha\phi = 0$$

Dominant energy condition: $T(X,Y) \geqslant 0$ if X, Y are both casual and future directed or casual and past-directed.

1.6.8 Compatible Current

Given vectorfield X_α, the compatible current is defined as

$$^{(X)}J^\nu = T^{\nu\alpha}X_\alpha$$

The divergence of the current is

$$\partial_\nu(^{(X)}J^\nu) = T^{\alpha\beta(X)}\pi_{\alpha\beta}$$

where

$$^{(X)}\pi_{\alpha\beta} = \frac{1}{2}(\partial_\nu X_\mu + \partial_\mu X_\nu)$$

1.6.9 Energy Estimates

Let

$$\Box_m \phi = 0, \in \mathbb{R}^{1+n}$$

$$\phi(0,x) = f(x), \qquad x \in \mathbb{R}^n$$

$$\partial_t \phi(0,x) = g(x), \qquad x \in \mathbb{R}^n$$

Let $R \in [0,\infty]$ and $X^\nu = -\delta_0^\nu$. If define energy

$$E^2[\phi](t) \triangleq \int_{B_{R-t}(p)} \hat{N}_\nu \, {}^{(X)}J^\nu[\phi(t,x)] d^n x = \int_{B_{R-t}(p)} |\nabla_{t,x}\phi(t,x)|^2 d^n x$$

where $\hat{N}_\nu = \delta_\nu^0$ is the past-pointing unit normal covector to $\{t\} \times B_{R-t}(p) \in \mathbb{R}^{1+n}$, then

$$E[\phi](t) \leqslant E[\phi](0)$$

1.6.10 Exercise: Timelike Vectors as a Linear Combination of Null Vectors

Let X and Y be future-directed timelike vectors in $1+n$-dimension Minkowski spacetime. Show that there exists a pair of future-directed null vectors L, $\underline{L}$ and positive constants a, b, c, d such that $X = aL + b\underline{L}$ and $Y = cL + d\underline{L}$.

We will first show, under a proper Lorentz transformation Λ, $\Lambda X = (\tilde{X}^0, 0, ..., 0)$ and $\Lambda Y = (\tilde{Y}^0, \tilde{Y^1}, 0, ..., 0)$ with $\tilde{X}^0 > 0$, $\tilde{X}^1 > 0$, and $\tilde{Y}^0 > 0$. Consider the orientation preserving spacial rotation R_n in $\mathbb{R}^n$ around the axis that is the orthogonal complement to the 2D-plane spanned by $(1, 0, ..., 0)$ and $(X^1, ..., X^n)$ that sends $(X^1, ..., X^n)$ onto $(c_2, 0, ..., 0)$ where $c_2 > 0$. R, defined as acting R_n on the spacial dimensions in $1+n$-dimensional Minkowski spacetime, is a Lorentz Transformation as it is a spacial rotation. Thus, $RX = (X^0, c_2, 0, ..., 0) \triangleq (c_1, c_2, 0, ..., 0)$ with $c_1, c_2 > 0$. We will show there is a Lorentz-boost B that sends RX to $(\tilde{X}^0, 0, ..., 0)$. In particular, consider the following, reduced to $\mathbb{R}^2$

$$B_2(RX)_2 \triangleq \begin{bmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \tilde{X}^0 \\ 0 \end{bmatrix}$$

where $\gamma = \sqrt{\frac{1}{1-v^2}}$. The second of these linear equations is $-\gamma v c_1 + \gamma c_2 = 0$, which means $v = \frac{c_2}{c_1}$. Since by the timelikeness of X, $m(X, X) = m(RX, RX) = -c_1 + c_2 > 0$. $v < 1$ and is positive, thus B is a proper Lorentz Transformation.

Similar to the spacial rotation R, there is an orientation-preserving spacial rotation $\tilde{R}$ that sends BRY to $(\tilde{Y}^0, \tilde{Y}^1, 0, ..., 0)$ with $\tilde{Y}^0, \tilde{Y}^1 > 0$. We conclude $\Lambda = \tilde{R}BR$. Λ is proper as it is consisted of two orientation-preserving rotations and a proper Lorentz boost.

Now, let $\tilde{L} = (1, -1, 0, ..., 0)$ and $\tilde{\underline{L}} = (1, 1, 0, ..., 0)$. Note $m(\tilde{L}, \tilde{L}) = m(\tilde{\underline{L}}, \tilde{\underline{L}}) = 0$ and $m(\tilde{L}, \tilde{\underline{L}}) = -2$. For $\tilde{X} = a\tilde{L} + b\tilde{\underline{L}}$ and $\tilde{Y} = c\tilde{L} + d\tilde{\underline{L}}$, solve for a, b, c, d:

$$a = \frac{\tilde{X}^0}{2}, \qquad b = \frac{\tilde{X}^0}{2}$$

$$c = \frac{\tilde{Y}^0 - \tilde{Y}^1}{2}, \qquad d = \frac{\tilde{Y}^0 + \tilde{Y}^1}{2}$$

By the time-likeness of $\tilde{Y}$, $(\tilde{Y}^1)^2 < (\tilde{Y}^0)^2$. By the time-likeness of $\tilde{Y}$, $\tilde{Y}^0 > 0$ and $\tilde{Y}^0 > \tilde{Y}^1$. With the fact that $\tilde{X}^0 > 0$, $a, b, c, d > 0$. Taking the inverse Lorentz transform,

$$X = \Lambda^{-1}\tilde{X} = \Lambda^{-1}(a\tilde{L} + b\tilde{\underline{L}}) \triangleq aL + b\underline{L}$$

$$Y = \Lambda^{-1}\tilde{Y} = \Lambda^{-1}(c\tilde{L} + d\tilde{\underline{L}}) \triangleq cL + d\underline{L}$$

□

1.6.11 Exercise: Proof of Dominant Energy Condition

Let X, Y be future directed and timelike. Show $T(X,Y) \triangleq T_{\mu\nu}X^\mu Y^\nu > 0$. Write $X = aL + b\underline{L}$ and $Y = cL + d\underline{L}$ with $a, b, c, d > 0$. Let $\{L, \underline{L}, e_{(1)}, ..., e_{(n-1)}\}$ be a null frame. By the null frame decomposition,

$$(m^{-1})^{\alpha\beta} = -\frac{1}{2}L^\alpha \underline{L}^\beta - \frac{1}{2}\underline{L}^\alpha L^\beta + \not{m}^{\alpha\beta}$$

Where $\not{m}$ is positive-definite on the span of $\{e_{(1)}, ..., e_{(n-1)}\}$ and vanishes on the span of $\{L, \underline{L}\}$.Then,

$$T_{\mu\nu}L^\mu L^\nu = \partial_\mu\phi\partial_\nu\phi L^\mu L^\nu - \frac{1}{2}\overbrace{(m_{\mu\nu}L^\mu L^\nu)}^{0}(m^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = (\nabla\phi \cdot L)^2 \geqslant 0$$

$$T_{\mu\nu}\underline{L}^\mu \underline{L}^\nu = \partial_\mu\phi\partial_\nu\phi \underline{L}^\mu \underline{L}^\nu - \frac{1}{2}\overbrace{(m_{\mu\nu}\underline{L}^\mu \underline{L}^\nu)}^{0}(m^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = (\nabla\phi \cdot \underline{L})^2 \geqslant 0$$

$$T_{\mu\nu}L^\mu \underline{L}^\nu = \partial_\mu\phi\partial_\nu\phi L^\mu \underline{L}^\nu - \frac{1}{2}\overbrace{(m_{\mu\nu}L^\mu \underline{L}^\nu)}^{-2}(-\frac{1}{2}L^\alpha \underline{L}^\beta - \frac{1}{2}\underline{L}^\alpha L^\beta + \not{m}^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi =$$

$$= (\nabla\phi \cdot L)(\nabla\phi \cdot \underline{L}) - \frac{1}{2}(\nabla\phi \cdot L)(\nabla\phi \cdot \underline{L}) - \frac{1}{2}(\nabla\phi \cdot \underline{L})(\nabla\phi \cdot L) + (\nabla\phi)^T \not{m}(\nabla\phi) =$$

$$= (\nabla\phi)^T \not{m}(\nabla\phi) \geqslant 0$$

If $(\nabla\phi)^T \not{m}(\nabla\phi) = 0$, then $\nabla\phi \in \text{span}(L, \underline{L})$, and $(\nabla\phi \cdot L)^2 > 0$. At least one of the above is strictly greater than 0. Then,

$$T(X,Y) = T_{\mu\nu}(aL^\mu + b\underline{L}^\mu)(cL^\nu + d\underline{L}^\nu) > 0$$

□

1.7 Generalized Second Order Linear PDE

The heat equation, Laplace's and Poisson's equation, and the wave equation are important equations. In general, second order linear PDEs can be generalized into these three categories and analyzed with similar tools associated with that classification.

Equation	Type	Wellposedness	Features
$\Delta u(x) = f(x)$	Elliptic	On all of $\mathbb{R}^n$ with conditions at ∞ or finite boundary conditions of either Dirichlet, Neumann, Robin, or mixed conditions.	mean value properties, maximum principle, Harnack's inequality.
$\partial_t u - \Delta u = f(t,x)$	Diffusive/ Parabolic	Initial value (Cauchy Data) at $t = 0$ and boundary conditions of either Dirichlet, Neumann, Robin, or mixed.	Infinite speed of propagation, maximum principle, $t^{-\frac{n}{2}}$ decay.
$\partial_t^2 u - \Delta u = f(t,x)$	Hyperbolic	Initial value (Cauchy Data) at $t = 0$ and boundary conditions of either Dirichlet, Neumann, Robin, or mixed.	Finite speed of propagation, domain of dependence and influence, energy identities, $t^{\frac{1-n}{2}}$ decay

1.7.1 Hadamard's Classification of Second Order Scalar PDEs

Let

$$A^{\alpha\beta}\partial_\alpha\partial_\beta u + B^\alpha\partial_\alpha u + Cu = 0$$

The equation is:

Elliptic: All of the eigenvalues of A have the same sign.

Hyperbolic: All but one eigenvalue have the same sign, and the remaining eigenvalue has the opposite sign.

Parabolic: All but one eigenvalue have the same sign, and the remaining eigenvalue is 0.

1.7.2 Representing Second Order Scalar PDEs with Change of Variables

If

$$\mathcal{L}u(x) = A^{\alpha\beta}\partial_\alpha\partial_\beta u + B^\alpha\partial_\alpha u + Cu = 0$$

Then there exists a change of variables $y^\mu = M^\mu_\alpha x^\alpha$ such that,

Elliptic: $\Delta_y u(y) + \tilde{B}^\alpha\partial_\alpha u(y) + Cu(y) = 0$

Hyperbolic: $\Box_y u(y) + \tilde{B}^\alpha\partial_\alpha u(y) + Cu(y) = 0$

Parabolic $\partial_0 u(y) + \sum_{i=1}^n \partial_i u(y) + \sum_{i=1}^n \tilde{B}^i\partial_i u(y) + Cu(y) = 0$

1.8 Fourier Transformation

The Fourier transformation is a significant tool to understand many ideas in both math and physics. The Fourier transformation takes a function and decomposes it into frequencies - like decomposing a musical note or a signal into its frequencies.

1.8.1 Definition

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x}d^n x$$

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi i\xi\cdot x}d^n \xi$$

1.8.2 Properties

Let τ_y denote translation by y, $\tau_y f(x) = f(x-y)$. Let $\partial_{\vec{\alpha}} = \prod_{i=0}^n \partial_i^{\alpha^i}$ and $x^{\vec{\alpha}} = \prod_{i=0}^n x^{\alpha^i}$. Then,

(a) $\widehat{(\tau_y f)}(\xi) = e^{-2\pi i\xi\cdot y}\hat{f}(\xi)$

(b) $\hat{h}(\xi) = \tau_\eta \hat{f}(\xi)$ if $h(x) = e^{2\pi i\eta\cdot x}f(x)$

(c) $\hat{h}(\eta) = t^n\hat{f}(t\eta)$ if $f(x) = f(t^{-1}(x)$

(d) $\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

(e) $\partial_\alpha\hat{f}(\eta) = [(-2\pi ix)^{\vec{\alpha}}f(x)]\hat{}(\xi)$

(f) $(\partial_{\vec{\alpha}}f)\hat{}(\xi) = (2\pi i\xi)^{\vec{\alpha}}f(\xi)$

(g) $\bar{\hat{f}}(\xi) = \bar{f}\check{}(\xi)$ and $\bar{\check{f}}(\xi) = \bar{f}\hat{}(\xi)$

Note: (e) shows if f decays very rapidly, $\hat{f}$ is very differentiable. (f) shows if f is very differentiable with rapidly decaying derivatives, $\hat{f}$ also rapidly decays.

1.8.3 Decay/Differentiability of Fourier Transform

Let $f \in C_c^\infty(\mathbb{R}^n$. Then $\hat{f}$ is smooth and rapidly decaying: $\forall N \geqslant 0$, $\exists C_N > 0$

$$|\hat{f}(\xi)| \leqslant C_N(1+|\xi|)^{-N}$$

Furthermore,

$$||\hat{f}(\xi)||_{L^1} \triangleq \int_{\mathbb{R}^\ltimes} |\hat{f}(\xi)|d^n\xi < \infty$$

These relations hold for all derivatives $\partial_{\vec{\beta}}\hat{f}$

1.8.4 Fourier Transform of a Gaussian

If $f(x) = e^{-\pi z|x|^2}$,

$$\hat{f}(\xi) = z^{-n/2}e^{-\pi|\xi|^2/z}$$

1.8.5 Plancherel's Theorem

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$$

and

$$||f||_{L^2} = ||\hat{f}||_{L^2}$$

1.8.6 Exercise: Solution to the Heat Equation using Fourier Transforms

Let $f : \mathbb{R}^n \to \mathbb{R}$, $f \in C_c^\infty$. Let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$-\partial_t u + \Delta u = 0, \qquad (t, x) \in (0, \infty) \times \mathbb{R}^n$$

$$u(0, x) = f(x), \qquad x \in \mathbb{R}^n$$

Show that

$$u(t, x) = [f(\cdot) * \Gamma(t, \cdot)](x)$$

with

$$\Gamma(t, x) \triangleq \frac{1}{\sqrt{4\pi t}}e^{-x^2/(4t)}$$

Taking the time derivative of the Fourier transform of the solution,

$$\partial_t \hat{u}(t, \xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\partial_t u(t, x)d^n x = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\sum_{i=1}^{n}\partial_i^2 u(t, x)d^n x =$$

$$= (2\pi i|\xi|)^2\hat{u}(t,\xi) = -4\pi^2|\xi|^2\hat{u}(t,\xi)$$

Note $\hat{u}(0,\xi) = \hat{f}(\xi)$. Integrating,

$$\hat{u}(t,\xi) = \hat{f}(\xi)e^{-4\pi^2|\xi|^2 t}$$

Let

$$g(\xi) = e^{-4\pi^2|\xi|^2 t}$$

Then

$$\check{g}(x) = \int_{\mathbb{R}^n} e^{-4\pi^2|\xi|^2 t}e^{i2\pi\xi\cdot x}d\xi = \int_{\mathbb{R}^n} e^{-4\pi^2 t(\xi-\frac{ix}{4\pi t})(\xi-\frac{ix}{4\pi t})}e^{-\frac{|x|^2}{4t}}d\xi =$$

$$= e^{-|x|^2/4t}\frac{\pi^{n/2}}{(4\pi^2 t)^{n/2}} = \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/4t} = \Gamma(t,x)$$

Taking the inverse Fourier transform of $\hat{u}$ gives

$$u(t,x) = [f * \check{g}](x) = [f(\cdot) * \Gamma(t,\cdot)](x)$$

□

1.8.7 Exercise: Version of the Heisenberg's Uncertainty principle

Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and compactly supported. Let x_0 and ξ_0 be any value. Using integration by parts,

$$\int_{\mathbb{R}} f(x)^2 dx = -\int_{\mathbb{R}} 2(x-x_0)f(x)f'(x)dx$$

Note that the boundary term vanishes by the compactly supportedness of f. By Cauchy-Schwartz,

$$||f||_{L^2}^2 \leqslant 2||xf||_{L^2}||f'(x)||_{L^2}$$

Multiplying by a phase does not change the over all norm:

$$||f||_{L^2}^2 \leqslant 2||xf||_{L^2}||e^{2\pi i\xi_0 x}f'(x)||_{L^2}$$

Since $\widehat{e^{2\pi i\xi_0 x}f'}(x) = 2\pi i\xi\hat{f}(\xi - \xi_0)$ and by Plancherel's Theorem,

$$||f||_{L^2}^2 \leqslant 4\pi||(x - x_0)f(x)||_{L^2}||(\xi - \xi_0)\hat{f}(\xi)||_{L^2}$$

This shows that $f(x)$ and $\hat{f}(\xi)$ cannot be both simultaneously centered at the any points x_0 and ξ_0, otherwise, the right hand side becomes too small. $\square$

1.9 Schrodinger's Equation

The Schrodinger's equation describes how a configuration of small particles change over time. More precisely, Quantum mechanics is governed by probability, and the Schrodinger's equation governs how the probability distribution of where the particle is changes over time. This will be covered in more detail in the next course on Quantum Mechanics.

1.9.1 Definition

$$i\partial_t\Psi(t,x) + \frac{1}{2}\Delta\Psi(t,x) = V(t,x)\Psi(t,x)$$

1.9.2 Free waves

$$\Psi(t,x) = e^{i(\omega t - \xi\cdot x)}$$

where ω is the frequency and ξ is the wave vector.

1.9.3 Dispersion Relation

Plugging free waves into the Schrodinger's equation,

$$-(\omega + \frac{|\xi|^2}{2})e^{i(\omega t + \xi \cdot x)} = 0$$

$$\Rightarrow \omega = \frac{|\xi|^2}{2}$$

Also, the phase velocity is $\frac{\omega}{|\xi|} = -\frac{|\xi|}{2}$, which means the speed of the plan wave depends on $|\xi|$. Physically,

$$\omega = \frac{\hbar |\xi|^2}{2m}$$

where $\hbar$ is the reduced Planck's constant and m is the mass of the particle.

1.9.4 Global Cauchy Problem and Fundamental Solution

Let Ψ solve

$$i\partial_t \Psi + \frac{1}{2}\Delta\Psi = 0$$

$$\Psi(0, x) = \phi(x)$$

Then

$$\Psi(t, x) = (K(t, \cdot) * \phi(\cdot))(x)$$

with

$$K(t, x) = \frac{1}{(2\pi i t)^{n/2}} e^{i|x|^2/2t}$$

1.9.5 Dispersive Estimate

If $\phi(x)$ is smooth and compactly supported ($\phi \in C_c^\infty(\mathbb{R}^n)$),

$$||\Psi(t,\cdot)||_{C^0} \triangleq \max_{x\in\mathbb{R}^n} |\Psi(t,x)| \leqslant \frac{C}{t^{n/2}}||\phi(\cdot)||_{L^1} \triangleq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n} |\phi(x)| d^n x$$

1.9.6 Conservation of L^2 norm

$$||\Psi(t,\cdot)||_{L^2} = ||\phi||_{L^2}$$

1.9.7 Exercise: Solution to the Inhomogenous Schrodinger's Equation

Let $\phi : \mathbb{R}^n \to \mathbb{C}$ and Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ with

$$i\partial_t \Psi + \frac{1}{2}\Delta\Psi = f(t,x), \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n$$

$$\Psi(0,x) = \phi(x), \qquad x \in \mathbb{R}^n$$

Show that

$$\Psi(t,x) = (K(t,\cdot) * \phi(\cdot))(x) - i\int_{s=0}^{t} (K(t-s,\cdot) * f(s,\cdot))(x) ds$$

First note that

$$\hat{K}(\xi) = e^{-i2\pi^2 t|\xi|^2}$$

For each fixed ξ,

$$\partial_t (e^{-2\pi^2 t|\xi|^2}\hat{\Psi}(t,\xi)) = e^{-2\pi^2 t|\xi|^2} \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x} (\partial_t \Psi + \overbrace{i2\pi^2|\xi|^2\Psi}^{\frac{1}{2i}\Delta\Psi}) d^n x =$$

$$= -ie^{-2\pi^2 t|\xi|^2} \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x} (i\partial_t \Psi + \frac{1}{2}\Delta\Psi) d^n x = -ie^{-2\pi^2 t|\xi|^2} \hat{f}(t,\xi)$$

and note that

$$\Psi(\hat{0}, \xi) = \hat{\phi}(\xi)$$

Integrating,

$$e^{i2\pi^2 t|\xi|^2}\hat{\Psi}(t,\xi) = -i\int_0^t e^{-2\pi^2 t|\xi|^2}\hat{f}(t,\xi)dt + \hat{\phi}(\xi)$$

Solving for Ψ and taking the inverse Fourier transform of both sides,

$$\Psi(t,x) = (K(t,\cdot) * \phi(\cdot))(x) - i\int_{s=0}^t (K(t-s,\cdot) * f(s,\cdot))(x)ds$$

$\square$

1.10 Lagrangian Mechanics

An idea central to Lagrangian Mechanics is the Lagrangian, which has units of energy and describes a system. This is a convenient way to describe some systems, such as the wave equation.

1.10.1 Lagrangians

A Lagrangian $\mathcal{L}$ is a function of ϕ and the partial derivatives $\nabla\phi$.

$$\mathcal{L}(\phi, \nabla\phi)$$

1.10.2 Action

Let $\mathcal{R} \subset \mathbb{R}^{1+n}$ be a compact subset of spacetime. define the action $\mathcal{A}$ of ϕ on $\mathcal{R}$ to be:

$$\mathcal{A}[\phi;\mathcal{R}] \triangleq \int_{\mathcal{R}} \mathcal{L}(\phi(x), \nabla\phi(x))d^{1+n}x$$

1.10.3 Variation

Given $\mathcal{R}$, $\psi \in C_c^\infty(\mathcal{R})$ is a variation. Given variation ψ and ϵ, define

$$\phi_\epsilon = \phi + \epsilon\psi$$

1.10.4 Stationary Point

The stationary point is analogous to the critical point in calculus. ϕ is a stationary point of the action if for all $\psi \in C_c^\infty(\mathcal{R})$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} A[\phi_\epsilon; \mathcal{R}] = 0$$

1.10.5 Euler-Lagrange formula for Stationary Points

ϕ is a stationary point iff

$$\nabla_\alpha \left(\frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial(\nabla_\alpha \phi)} \right) = \frac{\partial \mathcal{L}(\phi, \nabla\phi, x)}{\partial \phi}$$

1.10.6 Autonomous Systems and Diffemorphisms

An autonomous system is a system that does not depend on the independent variable, in this case, ϵ. Let $Y(x) = (Y^0(x), ..., Y^n(x))$ be a smooth vectorfield on $\mathbb{R}^{1+n}$. Assume $|\nabla_\mu Y^\nu(x)|$ is bounded for $0 \leqslant \mu, \nu \leqslant n$. Consider the autonomous system, given data x^μ

$$\frac{d}{d\epsilon}\tilde{x}^\mu(\epsilon) = Y^\mu(\tilde{x})$$

$$\tilde{x}^\mu(0) = x^\mu$$

Then there exists $\epsilon_0 > 0$ such that the above has a unique solution for $\epsilon \in [-\epsilon_0, \epsilon_0]$ that is smooth in ϵ. On $[-\epsilon_0, \epsilon_0]$, denote the flow map $x \to F_\epsilon(x) \triangleq \tilde{x}$.

This is a smooth bijective map with smooth inverses $F_{-\epsilon}$. This is diffeomorphism of $\mathbb{R}^{1+n}$, a smooth homeomorphism. Furthermore, if ϵ is sufficiently small,

$$F_\epsilon^\mu(x) = x^\mu + \epsilon Y^\mu(x) + \epsilon^2 \mathcal{R}^\mu(\epsilon, x)$$

$$M_\nu^\mu \triangleq \frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \delta_\nu^\mu + \epsilon \nabla_\nu Y^\mu(x) + \epsilon^2 \nabla_\nu \mathcal{R}^\mu(\epsilon, x)$$

$$(M^{-1})_\nu^\mu \triangleq \frac{\partial x^\mu}{\partial \tilde{x}^\nu} = \delta_\nu^\mu - \epsilon \nabla_\nu Y^\mu(x) + \epsilon^2 \mathcal{S}_\nu^\mu(\epsilon, x)$$

$$\det M^{-1} = 1 - \epsilon \nabla_\alpha Y_\alpha + \epsilon^2 S(\epsilon, x)$$

where $\mathcal{R}^\mu$, $\nabla_\nu \mathcal{R}^\mu$, $\mathcal{S}_\nu^\mu$, and S are smooth functions of $(\epsilon, x) \in [-\epsilon_0, \epsilon_0] \times \mathbb{R}^{1+n}$

1.10.7 Transformation properties of fields

Let $m(x)$ be a metric. The following quantities transform according to the transformation laws for covariant and contravariant tensors.

$$\tilde{\phi}(\tilde{x}) \triangleq \phi(x(\tilde{x}))$$

$$\tilde{\nabla}_\mu \tilde{\phi}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \nabla_\alpha \phi(x(\tilde{x}))$$

$$\tilde{m}_{\mu\nu}(\tilde{x}) \triangleq \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} m_{\alpha\beta}(x(\tilde{x}))$$

$$(\tilde{m}^{-1})^{\mu\nu}(\tilde{x}) \triangleq \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} (m^{-1})^{\alpha\beta}(x(\tilde{x}))$$

1.10.8 Coordinate Invariant Lagrangians

$\mathcal{L}$ is coordinate invariant if for all spacetime diffeomorphisms $x \to \tilde{x}$,

$$\mathcal{L}(\phi(x), \nabla\phi(x), m(x)) = \mathcal{L}(\tilde{\phi}(x), \tilde{\nabla}\tilde{\phi}(\tilde{x}), \tilde{m}(\tilde{x}))$$

1.10.9 Derivatives with respect to the flow parameter ϵ

$$\partial_\epsilon|_{\epsilon=0}\,\tilde{\phi} = -Y^\alpha \nabla_\alpha \phi$$

$$\partial_\epsilon|_{\epsilon=0}\,\tilde{\nabla}_\mu \tilde{\phi} = -\nabla_\mu Y^\alpha \nabla_\alpha \phi - Y^\alpha \nabla_\alpha \nabla_\mu \phi = -\nabla_\mu (Y^\alpha \nabla_\alpha \phi)$$

$$\partial_\epsilon|_{\epsilon=0}\,m_{\mu\nu} = -m_{\nu\alpha} \nabla_\mu Y^\alpha - m_{\mu\alpha} \nabla_\nu Y^\alpha - Y^\alpha \nabla_\alpha m_{\mu\nu}$$

$$\partial_\epsilon|_{\epsilon=0}\,(m^{-1})^{\mu\nu} = (m^{-1})^{\alpha\nu} \nabla_\alpha Y^\nu + (m^{-1})^{\mu\alpha} \nabla_\alpha Y^\nu - Y^\alpha \nabla_\alpha (m^{-1})^{\mu\nu}$$

$$\partial_\epsilon|_{\epsilon=0}\,\det(M^{-1}) = -\nabla_\alpha Y^\alpha$$

1.10.10 $\partial_\epsilon \mathcal{L}$

$$\begin{aligned}\partial_\epsilon|_{\epsilon=0}\,\mathcal{L}(\tilde{\phi}, \tilde{\nabla}\tilde{\phi}, \tilde{m}) = &-\frac{\partial \mathcal{L}(\phi, \nabla\phi, m)}{\partial \phi} Y^\alpha \nabla_\alpha \phi \\ &-\frac{\partial \mathcal{L}(\phi, \nabla\phi, m)}{\partial (\nabla\phi)} \nabla_\mu (Y^\alpha \nabla_\alpha \phi) \\ &-\frac{\partial \mathcal{L}(\phi, \nabla\phi, m)}{\partial m_{\mu\nu}} \Big\{ m_{\alpha\nu} \nabla_\mu Y^\alpha + m_{\mu\alpha} \nabla_\nu Y^\alpha + Y^\alpha \nabla_\alpha m_{\mu\nu} \Big\}\end{aligned}$$

This relation follows from the chain rule.

1.10.11 Energy Momentum Tensor

Let $\mathcal{L}$ be a coordinate invariant Lagrangian. Define

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu} \mathcal{L}$$

If ϕ verifies the Euler-Lagrange formula, i.e. it is a stationary point, then

$$\nabla_\mu T^{\mu\nu} = 0$$

Example: For the wave equation,

$$\mathcal{L} = -\frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$$

Then,

$$T^{\mu\nu} = (m^{-1})^{\mu\alpha}(m^{-1})^{\nu\beta}\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$$

1.10.12 Exercise: Inhomogenous Wave Equation and Conservation Law

Define the potential $V : \mathbb{R} \to \mathbb{R}$ and Lagrangian

$$\mathcal{L} = -\frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)$$

This Lagrangian is coordinate-invariant:

$$\tilde{\mathcal{L}} = -\frac{1}{2}(\tilde{m}^{-1})^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\phi}\tilde{\nabla}_\beta\tilde{\phi} - V(\tilde{\phi}) =$$

$$= -\frac{1}{2}\underbrace{(\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\frac{\partial x^\nu}{\partial\tilde{x}^\beta})}_{1}(m^{-1})^{\mu\nu}\nabla_\nu\phi\nabla_\mu\phi - V(\phi) = \mathcal{L}$$

The Euler-Lagrange condition is

$$\nabla_\alpha\left(\frac{\partial\mathcal{L}(\phi,\nabla\phi,m)}{\partial(\nabla_\alpha\phi)}\right) = \frac{\partial\mathcal{L}(\phi,\nabla\phi,m)}{\partial\phi}$$

$$\nabla_\alpha\Big(-(m^{-1})^{\mu\alpha}\nabla_\mu\phi\Big) = -(m^{-1})^{\mu\alpha}\nabla_\alpha\nabla_\mu\phi = -V'(\phi)$$

$$\Box_m\phi = V'(\phi)$$

which is the inhomogenous wave equation. Now, the energy momentum tensor is

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu}\mathcal{L} =$$

$$= 2(-\frac{1}{2})(-(m^{-1})^{\alpha\nu}(m^{-1})^{\beta\nu})\nabla_\alpha\phi\nabla_\beta\phi + (m^{-1})^{\mu\nu}(-\frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)) =$$

$$= (m^{-1})^{\alpha\mu}(m^{-1})^{\beta\nu}\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + (m^{-1})^{\mu\nu}V(\phi)$$

The divergence of the energy momentum tensor is 0:

$$\nabla_\mu T^{\mu\nu} = \Box_m\phi(m^{-1})^{\nu\beta}\nabla_\beta\phi + (m^{-1})^{\mu\alpha}(m^{-1})^{\nu\beta}\nabla_\mu\nabla_\beta\phi\nabla_\alpha\phi$$

$$-\frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{\alpha\beta}\nabla_\mu\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\mu\nabla_\beta\phi$$

$$-\nabla_\mu V(\phi)(m^{-1})^{\mu\nu} =$$

$$= \frac{\partial V}{\partial\phi}\frac{\partial\phi}{\partial x^\beta}(m^{-1})^{\nu\beta} - \frac{\partial V}{\partial x^\mu}(m^{-1})^{\mu\nu} = 0$$

Now, let $Y : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be a smooth vectorfield. For sufficiently small ϵ, let x transform as follows:

$$\frac{d}{d\epsilon}\tilde{x}^\mu(\epsilon) = Y^\mu(\tilde{x})$$

$$\tilde{x}^\mu(0) = x^\mu$$

Assume the Minkowski metric is invariant under the transformation $\tilde{x} \to x$: $(\tilde{m}^{-1})^{\mu\nu} = (m^{-1})^{\mu\nu}$. Note that

$$(\tilde{m}^{-1})^{\mu\nu} = (m^{-1})^{\mu\nu} = \frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial\tilde{x}^\nu}{\partial x^\beta}(m^{-1})^{\alpha\beta}$$

Taking the derivative w.r.t to ϵ at $\epsilon = 0$ for both sides,

$$0 = \frac{\partial}{\partial \epsilon}\left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\right)\underbrace{\frac{\partial \tilde{x}^\nu}{\partial x^\beta}}_{\delta^\nu_\beta}(m^{-1})^{\alpha\beta} + \underbrace{\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}}_{\delta^\mu_\alpha}\frac{\partial}{\partial \epsilon}\left(\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\right)(m^{-1})^{\alpha\beta}$$

Commutating derivative operations,

$$(m^{-1})^{\alpha\nu}\nabla_\alpha Y^\mu + (m^{-1})^{\mu\alpha}\nabla_\alpha Y^\nu = 0$$

Now, let $\mathcal{R} \subset \mathbb{R}^{1+n}$ be compact and have a smooth boundary. Note that $\nabla_\mu(T^{\mu\nu}Y_\nu) = \nabla_\nu(T^{\nu\mu}Y_\mu) = \nabla_\nu(T^{\mu\nu}Y_\mu)$ by the symmetry of $T^{\mu\nu}$. Then,

$$\int_{\mathcal{R}} \nabla_\mu(T^{\mu\nu}Y_\nu)d^{1+n}x = \frac{1}{2}\int_{\mathcal{R}} \nabla_\mu(T^{\mu\nu}Y_\nu) + \nabla_\nu(T^{\mu\nu}Y_\mu)d^{1+n}x$$

Since $\nabla_\mu T^{\mu\nu} = \nabla_\nu T^{\mu\nu} = 0$, raising the indices on Y.,

$$= \frac{1}{2}\int_{\mathcal{R}} T^{\mu\nu}(m_{\alpha\nu}\nabla_\mu Y^\alpha + m_{\mu\alpha}\nabla_\nu Y^\alpha)d^{1+n}x = 0$$

Where the last equality follows from a result taken from a lecture. In the case that $Y^\mu = (-1, 0, ..., 0)$, assume $\phi(t, \cdot)$ is compactly supported. Let $B_{R_t}(0)$ contain the support of $\phi(t, \cdot)$ and let $\mathcal{R}_t = [0, t] \times B_{R_t}(0)$. By the divergence theorem,

$$0 = \int_{\mathcal{R}} \nabla_\mu(T^{\mu\nu}Y_\nu)d^{1+n}x = -\int_{B_{R_0}(0)} T^{00}Y_0\Big|_t d^n x + \int_{B_{R_t}(0)} T^{00}Y_0\Big|_t d^n x$$

Since

$$T^{00}Y_0 = -V(\phi) - \frac{1}{2}(-(\partial_t\phi)^2 + |\nabla_x\phi|^2) - (\partial_t\phi)^2 = -(|\nabla\phi|^2 + V(\phi))$$

The conservation law holds:

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla\phi(t,x)|^2 + V(\phi(t,x))d^n x = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla\phi(0,x)|^2 + V(\phi(0,x))d^n x$$

□

2 Quantum Mechanics (Summer and Fall)

2.1 Summary

2.2 Experimental Results

2.3 Schrodinger's Equation

2.4 Harmonic Oscillator

References