Honors Independent Study Report Sample

Quantum Mechanics and Partial Differential Equations

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A concise summary of the concepts,

exercises, and interpretations

Note: This is a small sample of a currently 50 page report of an ongoing Senior independent study; only summer work for Partial Differential Equations is shown.

Contents

Abstract

Over the summer and the three semesters, I will be learning material equivalent to an undergraduate-level term course on Partial Differential Equations and three term courses of Quantum Mechanics at an undergraduate level. This report will contain a complete and concise set of concepts reduced from my notebook. And it will contain several exercises that I deem particular interesting, insightful, or significant.

I became very interested in exploring the laws that govern everyday life ever since I learned Calculus, a set of consistent logical tools designed, at the time, solely to explain planetary orbits. Exploring the realm of Newtonian Mechanics, I was left with questions. Why does an magnetic field generate an electrical field and vice versa? How can an electron orbit a nucleus without losing energy via Synchrotron radiation?

To fully understand how Quantum Mechanics works, I have to rely on mathematical tools such as Multivariable Calculus, Complex Analysis, and Fourier Analysis. The focus of this independent study is on Partial Differential Equations, which have great utility in describing Quantum systems.

1 Partial Differential Equations (Summer Work)

1.1 Definition: Fourier Transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} d^n x$$

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi \cdot x} d^n \xi$$

1.2 Exercise: Proof of the Solution to the Heat Equation

Let $f : \mathbb{R}^n \to \mathbb{R}$, $f \in C_c^\infty$. Let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$-\partial_t u + \Delta u = 0, \qquad (t, x) \in (0, \infty) \times \mathbb{R}^n$$

$$u(0, x) = f(x), \qquad x \in \mathbb{R}^n$$

Show that

$$u(t, x) = [f(\cdot) * \Gamma(t, \cdot)](x)$$

with

$$\Gamma(t, x) \triangleq \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

Taking the time derivative of the Fourier transform of the solution,

$$\partial_t \hat{u}(t, \xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \partial_t u(t, x) d^n x = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \sum_{i=1}^{n} \partial_i^2 u(t, x) d^n x =$$

$$= (2\pi i |\xi|)^2 \hat{u}(t, \xi) = -4\pi^2 |\xi|^2 \hat{u}(t, \xi)$$

Note $\hat{u}(0, \xi) = \hat{f}(\xi)$. Integrating,

$$\hat{u}(t, \xi) = \hat{f}(\xi) e^{-4\pi^2 |\xi|^2 t}$$

Let

$$g(\xi) = e^{-4\pi^2|\xi|^2 t}$$

Then

$$\check{g}(x) = \int_{\mathbb{R}^n} e^{-4\pi^2|\xi|^2 t} e^{i2\pi\xi\cdot x} d\xi = \int_{\mathbb{R}^n} e^{-4\pi^2 t(\xi - \frac{ix}{4\pi t})(\xi - \frac{ix}{4\pi t})} e^{-\frac{|x|^2}{4t}} d\xi =$$

$$= e^{-|x|^2/4t} \frac{\pi^{n/2}}{(4\pi^2 t)^{n/2}} = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} = \Gamma(t,x)$$

Taking the inverse Fourier transform of $\hat{u}$ gives

$$u(t,x) = [f * \check{g}](x) = [f(\cdot) * \Gamma(t,\cdot)](x)$$

$\square$

1.3 Definition: Energy Momentum Tensor

Let $\mathcal{L}$ be a coordinate invariant Lagrangian. Define

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu}\mathcal{L}$$

1.4 Exercise: Proof of Conservation Law for the Inhomogenous Wave Equation

Define the potential $V : \mathbb{R} \to \mathbb{R}$ and Lagrangian

$$\mathcal{L} = -\frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)$$

This Lagrangian is coordinate-invariant:

$$\tilde{\mathcal{L}} = -\frac{1}{2}(\tilde{m}^{-1})^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\phi}\tilde{\nabla}_\beta\tilde{\phi} - V(\tilde{\phi}) =$$

$$= -\frac{1}{2}(\underbrace{\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta}}_{1})(m^{-1})^{\mu\nu}\nabla_\nu\phi\nabla_\mu\phi - V(\phi) = \mathcal{L}$$

The Euler-Lagrange condition reduces to the inhomogenous wave equation.

$$\nabla_\alpha\left(\frac{\partial \mathcal{L}(\phi, \nabla\phi, m)}{\partial(\nabla_\alpha\phi)}\right) = \frac{\partial \mathcal{L}(\phi, \nabla\phi, m)}{\partial \phi}$$

$$\nabla_\alpha\big(-(m^{-1})^{\mu\alpha}\nabla_\mu\phi\big) = -(m^{-1})^{\mu\alpha}\nabla_\alpha\nabla_\mu\phi = -V'(\phi)$$

$$\Box_m\phi = V'(\phi)$$

Now, the energy momentum tensor $T^{\mu\nu}$ is

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial m_{\mu\nu}} + (m^{-1})^{\mu\nu}\mathcal{L} =$$

$$= 2(-\frac{1}{2})(-(m^{-1})^{\alpha\nu}(m^{-1})^{\beta\nu})\nabla_\alpha\phi\nabla_\beta\phi + (m^{-1})^{\mu\nu}(-\frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)) =$$

$$= (m^{-1})^{\alpha\mu}(m^{-1})^{\beta\nu}\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + (m^{-1})^{\mu\nu}V(\phi)$$

The divergence of the energy momentum tensor is zero:

$$\nabla_\mu T^{\mu\nu} = \Box_m\phi(m^{-1})^{\nu\beta}\nabla_\beta\phi + (m^{-1})^{\mu\alpha}(m^{-1})^{\nu\beta}\nabla_\mu\nabla_\beta\phi\nabla_\alpha\phi$$

$$-\frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{\alpha\beta}\nabla_\mu\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(m^{-1})^{\mu\nu}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\mu\nabla_\beta\phi$$

$$-\nabla_\mu V(\phi)(m^{-1})^{\mu\nu} =$$

$$= \frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x^\beta}(m^{-1})^{\nu\beta} - \frac{\partial V}{\partial x^\mu}(m^{-1})^{\mu\nu} = 0$$

Now, let $Y : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be a smooth vectorfield. For sufficiently small ϵ, let x transform as follows:

$$\frac{d}{d\epsilon}\tilde{x}^\mu(\epsilon) = Y^\mu(\tilde{x})$$

$$\tilde{x}^{\mu}(0) = x^{\mu}$$

Assume the Minkowski metric is invariant under the transformation $x \to \tilde{x}$: $(\tilde{m}^{-1})^{\mu\nu} = (m^{-1})^{\mu\nu}$. Note that

$$(\tilde{m}^{-1})^{\mu\nu} = (m^{-1})^{\mu\nu} = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}} (m^{-1})^{\alpha\beta}$$

Taking the derivative w.r.t to ϵ at $\epsilon = 0$ for both sides,

$$0 = \frac{\partial}{\partial \epsilon} \left(\frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}} \right) \underbrace{\frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}}}_{\delta^{\nu}_{\beta}} (m^{-1})^{\alpha\beta} + \underbrace{\frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}}}_{\delta^{\mu}_{\alpha}} \frac{\partial}{\partial \epsilon} \left(\frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}} \right) (m^{-1})^{\alpha\beta}$$

Commutating derivative operations,

$$(m^{-1})^{\alpha\nu} \nabla_{\alpha} Y^{\mu} + (m^{-1})^{\mu\alpha} \nabla_{\alpha} Y^{\nu} = 0$$

Now, let $\mathcal{R} \subset \mathbb{R}^{1+n}$ be compact and have a smooth boundary. Note that $\nabla_{\mu}(T^{\mu\nu} Y_{\nu}) = \nabla_{\nu}(T^{\nu\mu} Y_{\mu}) = \nabla_{\nu}(T^{\mu\nu} Y_{\mu})$ by the symmetry of $T^{\mu\nu}$. Then,

$$\int_{\mathcal{R}} \nabla_{\mu}(T^{\mu\nu} Y_{\nu}) d^{1+n}x = \frac{1}{2} \int_{\mathcal{R}} \nabla_{\mu}(T^{\mu\nu} Y_{\nu}) + \nabla_{\nu}(T^{\mu\nu} Y_{\mu}) d^{1+n}x$$

Since $\nabla_{\mu} T^{\mu\nu} = \nabla_{\nu} T^{\mu\nu} = 0$, raising the indices on $Y.$,

$$= \frac{1}{2} \int_{\mathcal{R}} T^{\mu\nu} (m_{\alpha\nu} \nabla_{\mu} Y^{\alpha} + m_{\mu\alpha} \nabla_{\nu} Y^{\alpha}) d^{1+n}x = 0$$

Where the last equality follows from a result taken from a lecture. In the case that $Y^{\mu} = (-1, 0, ..., 0)$, assume $\phi(t, \cdot)$ is compactly supported. Let $B_{R_t}(0)$ contain the support of $\phi(t, \cdot)$ and let $\mathcal{R}_t = [0, t] \times B_{R_t}(0)$. By the divergence theorem,

$$0 = \int_{\mathcal{R}_t} \nabla_\mu (T^{\mu\nu} Y_\nu) d^{1+n}x = -\int_{B_{R_0}(0)} T^{00} Y_0 \Big|_0 d^n x + \int_{B_{R_t}(0)} T^{00} Y_0 \Big|_t d^n x$$

Since

$$T^{00} Y_0 = -V(\phi) - \frac{1}{2}(-(\partial_t \phi)^2 + |\nabla_x \phi|^2) - (\partial_t \phi)^2 = -(|\nabla \phi|^2 + V(\phi))$$

The conservation law holds:

$$\int_{\mathbb{R}^n} \frac{1}{2} |\nabla \phi(t,x)|^2 + V(\phi(t,x)) d^n x = \int_{\mathbb{R}^n} \frac{1}{2} |\nabla \phi(0,x)|^2 + V(\phi(0,x)) d^n x$$

□