CHAPTER 13
Neurodynamics

Problem 13.1

The equilibrium state $\mathbf{x}(0)$ is (asymptotically) stable if in a small neighborhood around $\mathbf{x}(0)$, there exists a positive definite function $V(\mathrm{x})$ such that its derivative with respect to time is negative definite in that region.

Problem 13.3

Consider the symem of coupled nonlinear differential equations:

$$\frac{dx_j}{dt} = \varphi_j(\mathbf{W}, \mathbf{i}, \mathbf{x}), \qquad j = 1, 2, ..., N$$

where $\mathbf{W}$ is the weight matrix, $\mathbf{i}$ is the bias vector, and $\mathbf{x}$ is the state vector with its jth element denoted by x_j.

(a) With the bias vector $\mathbf{i}$ treated as input and with fixed initial condition $\mathbf{x}(0)$, let $\mathbf{x}(\infty)$ denote the final state vector of the system. Then,

$$0 = \varphi_j(\mathbf{W}, \mathbf{i}, \mathbf{x}(\infty)), \qquad j = 1, 2, ..., N$$

For a given matrix $\mathbf{W}$ and input vector $\mathbf{i}$, the set of initial points $\mathbf{x}(0)$ evolves to a fixed point. The fixed points are functions of $\mathbf{W}$ and $\mathbf{i}$. Thus, the system acts as a "mapper" with $\mathbf{i}$ as input and $\mathbf{x}(\infty)$ as output, as shown in Fig. 1(a):

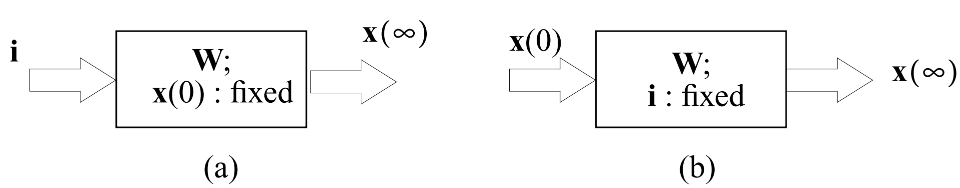

Figure 1: Problem 13.3

(b) With the initial state vector $\mathbf{x}(0)$ treated as input, and the bias vector $\mathbf{i}$ being fixed, let $\mathbf{x}(\infty)$ denote the final state vector of the system. We may then write

$$0 = \varphi_j(\mathbf{W}, \mathbf{i}\text{:fixed}, \mathbf{x}(\infty)), \qquad j = 1, 2, .., N$$

Thus with $\mathbf{x}(0)$ acting as input and $\mathbf{x}(\infty)$ acting as output, the dynamic system behaves like a pattern associator, as shown in Fig. 1b.

Problem 13.4

(a) We are given the fundamental memories:

$$\xi_1 = [+1,\ +1,\ +1,\ +1,\ +1]^T$$
$$\xi_2 = [+1,\ -1,\ -1,\ +1,\ -1]^T$$
$$\xi_3 = [1-,\ +1,\ -1,\ +1,\ +1]^T$$

The weight matrix of the Hopfield network (with $N = 25$ and $p = 3$) is therefore

$$\mathbf{W} = \frac{1}{N}\sum_{i=1}^{p} \xi_i \xi_i^T - \frac{P}{N}\mathbf{I}$$
$$= \frac{1}{5}\begin{bmatrix} 0 & -1 & +1 & +1 & -1 \\ -1 & 0 & +1 & +1 & +3 \\ +1 & +1 & 0 & -1 & +1 \\ +1 & +1 & -1 & 0 & +1 \\ -1 & +3 & +1 & +1 & 0 \end{bmatrix}$$

(b) According to the alignment condition, we write

$$\xi_i = \operatorname{sgn}(\mathbf{W}\xi_i), \quad i = 1, 2, 3$$

Consider first ξ_1, for which we have

$$\operatorname{sgn}(\mathbf{W}\xi_1) = \operatorname{sgn}\left(\frac{1}{5}\begin{bmatrix} 0 & -1 & +1 & +1 & -1 \\ -1 & 0 & +1 & +1 & +3 \\ +1 & +1 & 0 & -1 & +1 \\ +1 & +1 & -1 & 0 & +1 \\ -1 & +3 & +1 & +1 & 0 \end{bmatrix}\begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \\ +1 \end{bmatrix}\right)$$

$$= \text{sgn}\left(\frac{1}{5}\begin{bmatrix} 0 \\ +4 \\ +2 \\ +2 \\ +4 \end{bmatrix}\right) = \begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \\ +1 \end{bmatrix} = \xi_1$$

$$\text{sgn}(\mathbf{W}\xi_2) = \text{sgn}\left(\frac{1}{5}\begin{bmatrix} 0 & -1 & +1 & +1 & -1 \\ -1 & 0 & +1 & +1 & +3 \\ +1 & +1 & 0 & -1 & +1 \\ +1 & +1 & -1 & 0 & +1 \\ -1 & +3 & +1 & +1 & 0 \end{bmatrix}\begin{bmatrix} +1 \\ -1 \\ -1 \\ +1 \\ -1 \end{bmatrix}\right)$$

$$= \text{sgn}\left(\frac{1}{5}\begin{bmatrix} +2 \\ -4 \\ -2 \\ 0 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} +1 \\ -1 \\ -1 \\ +1 \\ -1 \end{bmatrix} = \xi_2$$

$$\text{sgn}(\mathbf{W}\xi_3) = \text{sgn}\left(\frac{1}{5}\begin{bmatrix} 0 & -1 & +1 & +1 & -1 \\ -1 & 0 & +1 & +1 & +3 \\ +1 & +1 & 0 & -1 & +1 \\ +1 & +1 & -1 & 0 & +1 \\ -1 & +3 & +1 & +1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \\ +1 \end{bmatrix}\right)$$

$$= \text{sgn}\left(\frac{1}{5}\begin{bmatrix} -2 \\ +4 \\ 0 \\ +2 \\ +4 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \\ +1 \end{bmatrix} = \xi_2$$

Thus all three fundamental memories satisfy the alignment condition.

Note: Wherever a particular element of the product $\mathbf{W}\xi_i$ is zero, the neuron in question is left in its previous state.

(c) Consider the noisy probe:

$$\mathbf{x} = \left[+1,\ -1,\ +1,\ +1,\ +1\right]^T$$

which is the fundamental memory with its second element reversed in polarity. We write

$$\mathbf{W}\mathbf{x} = \frac{1}{5}\begin{bmatrix} 0 & -1 & +1 & +1 & -1 \\ -1 & 0 & +1 & +1 & +3 \\ +1 & +1 & 0 & -1 & +1 \\ +1 & +1 & -1 & 0 & +1 \\ -1 & +3 & +1 & +1 & 0 \end{bmatrix}\begin{bmatrix} +1 \\ -1 \\ +1 \\ +1 \\ +1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} +2 \\ +4 \\ 0 \\ 0 \\ -2 \end{bmatrix} \tag{1}$$

Therefore,

$$\text{sgn}(\mathbf{W}\mathbf{x}) = \begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \\ -1 \end{bmatrix}$$

Thus, neurons 2 and 5 want to change their states. We therefore have 2 options:

- Neuron 5 is chosen for a state change, which yields the result

  $$\mathbf{x} = \left[+1,\ +1,\ +1,\ +1,\ +1\right]^T$$

  This vector is recognized as the fundamental memory ξ_1, and the computation is thereby terminated.

- Neuron 2 is chosen to change its state, yielding the vector

  $$\mathbf{x} = \left[+1,\ -1,\ +1,\ +1,\ -1\right]^T$$

  Next, we go on to compute

$$\mathbf{Wx} = \frac{1}{5}\begin{bmatrix} 0 & -1 & +1 & +1 & -1 \\ -1 & 0 & +1 & +1 & +3 \\ +1 & +1 & 0 & -1 & +1 \\ +1 & +1 & -1 & 0 & +1 \\ -1 & +3 & +1 & +1 & 0 \end{bmatrix}\begin{bmatrix} +1 \\ -1 \\ +1 \\ +1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} +4 \\ -2 \\ -2 \\ -2 \\ -2 \end{bmatrix}$$

$$\text{sgn}(\mathbf{Wx}) = \begin{bmatrix} +1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Hence, neurons 3 and 4 want to change their states:

- If we permit neuron 3 to change its state from +1 to -1, we get
  $$\mathbf{x} = [+1,\ -1,\ -1,\ +1,\ -1]^T$$
  which is recognized as the fundamental memory ξ_2.

- If we permit neuron 4 to change its state from +1 to -1, we get
  $$\mathbf{x} = [+1,\ -1,\ +1,\ -1,\ -1]$$
  which is recognized as the negative of the third fundamental memory ξ_3.

In both cases, the new state would satisfy the alignment condition and the computation is then terminated.

Thus, when the noisy version of ξ_1 is applied to the network, with its second element changed in polarity, one of 2 things can happen with equal likelihood:

1. The original ξ_1 is recovered after 1 iteration.
2. The second fundamental memory ξ_2 or the negative of the third fundamental memory ξ_3 is recovered after 2 iterations, which, of course, is in error.

Problem 13.5

Given the probe vector

$$\mathbf{x} = [+1, -1, +1, +1, +1]^T$$

and the weight matrix of (1) Problem 13.4, we find that

$$\mathbf{W}\mathbf{x} = \frac{1}{5}\begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

and

$$\text{sgn}(\mathbf{W}\mathbf{x}) = \begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \\ -1 \end{bmatrix}$$

According to this result, neurons 2 and 5 have changed their states. In synchronous updating, this is permitted. Thus, with the new state vector

$$\mathbf{x} = \begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \\ -1 \end{bmatrix}$$

on the next iteration,we compute

$$\mathbf{W}\mathbf{x} = \frac{1}{5}\begin{bmatrix} 0 & -1 & +1 & +1 & -1 \\ -1 & 0 & +1 & +1 & +3 \\ +1 & +1 & 0 & -1 & +1 \\ +1 & +1 & -1 & 0 & +1 \\ -1 & +3 & +1 & +1 & 0 \end{bmatrix}\begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} +2 \\ -2 \\ 0 \\ 0 \\ +4 \end{bmatrix}$$

Hence,

$$\operatorname{sgn}(\mathbf{W}\mathbf{x}) = \begin{bmatrix} +1 \\ -1 \\ +1 \\ +1 \\ +1 \end{bmatrix}$$

The new state vector is therefore

$$\mathbf{x} = \begin{bmatrix} +1 \\ -1 \\ +1 \\ +1 \\ +1 \end{bmatrix}$$

which is recognized as the original probe. In this problem, we thus find that the network experiences a limit cycle of duration 2.

Problem 13.6

(a) The vectors

$$\xi_1 = [-1, -1, -1, +1, -1]^T$$
$$\xi_2 = [+1, +1, +1, -1, +1]^T$$
$$\xi_3 = [+1, -1, +1, -1, -1]^T$$

are simply the negatives of the three fundamental memories considered in Problem 13.4, respectively. These 3 vectors are therefore also fundamental memories of the Hopfield network.

(b) Consider the vector

$$\mathbf{x} = [0, +1, +1, +1, +1]^T$$

which is the result of masking the first element of the fundamental memory ξ_1 of Problem 13.4. According to our notation, a neuron of the Hopfield network is in either state +1 or -1. We therefore have the choice of setting the zero element of **x** to +1 or -1. The first option restores the vector **x** to its original form: fundamental memory ξ_1, which satisfies the alignment condition. Alternatively, we may set the zero element equal to -1, obtaining

$$\mathbf{x} = [-1, +1, +1, +1, +1]^T$$

In this latter case, the alignment condition is not satisfied. The obvious choice is therefore the former one.

Problem 13.7

We are given

$$\mathbf{W} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

(a) For state $\mathbf{s}_2$ we have

$$\mathbf{W}\mathbf{s}_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

which yields

$$\text{sgn}(\mathbf{W}\mathbf{s}_2) = \begin{bmatrix} -1 \\ +1 \end{bmatrix} = \mathbf{s}_2$$

Next for state $\mathbf{s}_4$, we have

$$\mathbf{W}\mathbf{s}_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

which yields

$$\text{sgn}(\mathbf{W}\mathbf{s}_4) = \begin{bmatrix} +1 \\ -1 \end{bmatrix} = \mathbf{s}_4$$

Thus, both states $\mathbf{s}_2$ and $\mathbf{s}_4$ satisfy the alignment condition and are therefore stable.

Consider next the state $\mathbf{s}_1$, for which we write

$$\mathbf{W}\mathbf{s}_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

which yields

$$\text{sgn}(\mathbf{W}\mathbf{s}_1) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \mathbf{s}_1$$

Thus, both neurons want to change; suppose we pick neuron 1 to change its state, yielding the new state vector $[-1, +1]^T$. This is a stable vector as it satisfies the alignment condition. If, however, we permit neuron 2 to change its state, we get a state vector equal to $\mathbf{s}_4$. Similarly, we may show that the state vector $\mathbf{s}_3 = [-1, -1]^T$ is also unstable. The resulting state-transition diagram of the network is thus as depicted in Fig. 1.

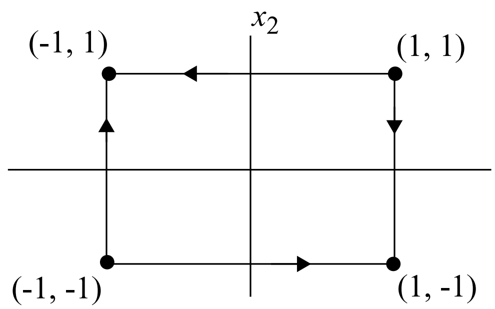

Figure 1: Problem 13.7

The results depicted in Fig. 1 assume the use of asynchronous updating. If, however, we use synchronous updating, we find that in the case of $\mathbf{s}_1$:

$$\mathrm{sgn}(\mathbf{W}\mathbf{s}_1) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Permitting both neurons to change state, we get the new state vector $[-1, -1]^T$. This is recognized to be stable state $\mathbf{s}_3$. Now, we find that

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

which takes back to state $\mathbf{s}_1$.

Thus, in the synchronous updating case, the states $\mathbf{s}_1$ and $\mathbf{s}_3$ represent a limit cycle with length 2.

Returning to the normal operation of the Hopfield network, we note that the energy function of the network is

$$\begin{aligned} E &= -\frac{1}{2}\sum_{i}\sum_{\substack{j \\ i \neq j}} w_{ji} s_i s_j \\ &= -\frac{1}{2} w_{12} s_1 s_2 - \frac{1}{2} w_{21} s_2 s_1 \\ &= -w_{12} s_1 s_2 \quad \text{since } w_{12} = w_{21} \\ &= s_1 s_2 \end{aligned} \tag{1}$$

Evaluating (1) for all possible states of the network, we get the following table:

State	Energy
[+1, +1]	+1
[-1, +1]	-1
[-1. -1]	+1
[+1. -1]	-1

Thus, states $\mathbf{s}_1$ and $\mathbf{s}_3$ represent global minima and are therefore stable.

Problem 13.8

The energy function of the Hopfield network is

$$E = -\frac{1}{2}\sum_i \sum_j w_{ji} s_j s_i \tag{1}$$

The overlap m_v is defined by

$$m_v = \frac{1}{N}\sum_j s_j \xi_{v,\,j} \tag{2}$$

and the weight w_{ji} is itself defined by

$$w_{ji} = \frac{1}{N}\sum_v \xi_{v,\,j}\xi_{v,\,i} \tag{3}$$

Substituting (3) into (1) yields

$$\begin{aligned} E &= -\frac{1}{2N}\sum_i \sum_j \sum_v \xi_{v,\,j}\xi_{v,\,i} s_j s_i \\ &= -\frac{1}{2N}\sum_v \left(\sum_i s_i \xi_{v,\,i}\right)\left(\sum_j s_j \xi_{v,\,j}\right) \\ &= -\frac{1}{2N}\sum_v (m_v N)(m_v N) \\ &= -\frac{N}{2}\sum_v m_v^2 \end{aligned}$$

where, in the third line, we made use of (2).

Problem 13.11

We start with the function (see (13.48) of the text)

$$E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ji}\varphi_i(u_i)\varphi_j(u_j) - \sum_{j=1}^{N}\int_0^{u_j} b_j(\lambda)\varphi'_j(\lambda)d\lambda \tag{1}$$

where $\varphi'_j(\cdot)$ is the derivative of the function $\varphi_j(\cdot)$ with respect to its argument. We now differentiate the function E with respect to time t and note the following relations:

1. $C_{ji} = C_{ij}$

2. $$\frac{\partial}{\partial t}\varphi_j(u_j) = \frac{\partial u_j}{\partial t}\frac{\partial}{u_j}\varphi_j(u_j)$$
$$= \frac{\partial u_j}{\partial t}\varphi'_j(u_j)$$

3. $$\frac{\partial}{\partial t}\int_0^{u_j} b_j(\lambda)\varphi'_j(\lambda)d\lambda = \frac{\partial u_j}{\partial t}\frac{\partial}{u_j}\int_0^{u_j} b_j(\lambda)\varphi'_j(\lambda)d\lambda$$
$$= \frac{\partial u_j}{\partial t} b_j(u_j)\varphi'_j(u_j)$$

Accordingly, we may use (1) to express the derivative $\partial E/\partial t$ as follows:

$$\frac{\partial E}{\partial t} = \frac{\partial u_j}{\partial t}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ji}\varphi'_j(u_j) - \sum_{j=1}^{N} b_j(u_j)\varphi'_j(u_j)\right) \tag{2}$$

From Eq. (13.47) in the text, we have

$$\frac{\partial u_j}{\partial t} = a_j(u_j)\left(b_j(u_j) - \sum_{j=1}^{N} c_{ji}\varphi_i(u_i)\right), \qquad j = 1, 2, ..., N \tag{3}$$

Hence using (3) in (2) and collecting terms, we get the final result

$$\frac{\partial E}{\partial t} = -\sum_{j=1}^{N} a_j(u_j)\varphi'_j(u_j)\left(b_j(u_j) - \sum_{i=1}^{N} c_{ji}\varphi_i(u_i)\right)^2 \tag{4}$$

Provided that the coefficient $a_j(u_j)$ satisfies the nonnegativity condition

$a_j(u_j) > 0 \quad$ for all u_j

and the function $\varphi'_j(u_j)$ satisfies the monotonicity condition

$\varphi'_j(u_j) \geq 0 \quad$ for all u_j,

we then immediately see from (4) that

$$\frac{\partial E}{\partial t} \leq 0 \quad \text{for all } t$$

In words, the function E defined in(1) is the Lyapunov function for the coupled system of nonlinear differential equations (3).

Problem 13.12

From (13.61) of the text:

$$\frac{d}{dt}v_j(t) = -v_j(t) + \sum_{i=1}^{N} c_{ji}\varphi_i(v_i), \qquad j = 1, 2, ..., N \tag{1}$$

where

$$c_{ji} = \delta_{ji} + \beta w_{ji}$$

where δ_{ji} is a Kronecker delta. According to the Cohen-Grossberg theorem of (13.47) in the text, we have

$$\frac{d}{dt}u_j(t) = -a_j(u_j)\left[b_j(u_j) - \sum_{i=1}^{N} c_{ji}\varphi_i(u_i)\right] \tag{2}$$

Comparison of (1) and (2) yields the following correspondences between the Cohen-Grossberg theorem and the brain-in-state-box (BSB) model:

Cohen-Grossberg Theorem	**BSB Model**
u_j	v_j
$a_j(u_j)$	1
$b_j(u_j)$	$-v_j$
c_{ji}	$-c_{ji}$
$\varphi_i(u_i)$	$\varphi(v_i)$

Therefore, using these correspondences in (13.48) of thetext:

$$E = \frac{1}{2}\sum_i \sum_j c_{ji}\varphi_i(u_i)\varphi_j(u_j) - \sum_j \int^{u_j} b_j(\lambda)\varphi'_j(\lambda)d\lambda,$$

we get the following Liapunov function for the BSB model:

$$E = -\frac{1}{2}\sum_i \sum_j c_{ji}\varphi(v_i)\varphi(v_j) + \sum_j \int_0^{v_j} \lambda\varphi'(\lambda)d\lambda \tag{3}$$

From (13.55) in the text, we note that

$$\varphi(y_j) = \begin{cases} +1 \text{ if } y_j > 1 \\ y_j \text{ if } -1 \le y_j \le 1 \\ -1 \text{ if } y_j \le -1 \end{cases}$$

We therefore have

$$\varphi'(y_j) = \begin{cases} 0, & |y_j| > 1 \\ 1, & |y_j| \le 1 \end{cases}$$

Hence, the second term of (3) is given by

$$\begin{aligned}\sum_j \int_0^{v_j} \lambda\varphi'(\lambda)d\lambda &= \sum_j \int_0^{v_j} \lambda d\lambda = \frac{1}{2}\sum_j v_j^2 \\ &= \frac{1}{2}\sum_j x_j^2 \quad \text{inside the linear region}\end{aligned} \tag{4}$$

The first term of (3) is given by

$$\begin{aligned}-\frac{1}{2}\sum_j \sum_i c_{ji}\varphi(v_i)\varphi(v_j) &= -\frac{1}{2}\sum_j \sum_i (\delta_{ji} + \beta w_{ji})\varphi(v_i)\varphi(v_j) \\ &= -\frac{\beta}{2}\sum_j \sum_i w_{ji}x_j x_i - \frac{1}{2}\sum_j \varphi^2(v_j) \\ &= -\frac{\beta}{2}\sum_j \sum_i w_{ji}x_j x_i - \frac{1}{2}\sum_j x_j^2\end{aligned} \tag{5}$$

Finally, substituting (4) and (5) into (3), we obtain

$$E = -\frac{\beta}{2}\sum_j \sum_i w_{ji} x_j x_i - \frac{\beta}{2}\mathbf{x}^T\mathbf{W}\mathbf{x}$$

which is the desired result

Problem 13.13

The activation function $\varphi(v)$ of Fig. P13.13 is a nonmonotonic function of the argument *v*; that is, $\partial\varphi / \partial v$ assumes both positive and negative values. It therefore violates the monotonicity condition required by the Cohen-Grossberg theorem; see Eq. (4) of Problem 13.11. This means that the cohen-Grossberg theorem is not applicable to an associative memory like a Hopfield network that uses the activation function of Fig. P14.15.