Name________ Student ID________ School________ Major/Class________ Seat No.________

(DONNOT WRITE YOUR ANSWER IN THIS AREA)

Seal Line Seal Line Seal Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

Final Exam

Probability & Statistics Exam Paper A （2020-2021-2）

Notice: **1. Make sure that you have filled the form on the left side of seal line.**
2. Write your answers on the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts 120 minutes.

Question No.	I	II	III	IV	V	VI	VII	VIII	Sum
Score									

The possible useful critical values：

$Z_{0.05} = 1.64$； $Z_{0.025} = 1.96$；

$t_{0.05,9} = 1.833$； $t_{0.05,8} = 1.86$； $t_{0.025,9} = 2.262$； $t_{0.025,8} = 2.306$；

I. **Single choice.** Choose the **one** alternative that best completes the statement or answers the question. (6 questions, 3 points per question, 18 points in total)

1. Suppose A, B and C are two events. Which of the following statements is true? (　　)

A. $P(A|A\cup B) \le P(A)$

B. $P(A\cup B|C\cup B) = \frac{P(B)+P(AC)}{P(C\cup B)}$

C. $P(A\cap B') - P(A'\cap B) = P(A) - P(B) - 2P(A\cap B)$

D. $P(A\cup B) = P(A) + P(B\cap A')$

2. Suppose there are 10 questions in an exam. It is known that students will get 10 points if they answer one question correctly, and 5 points will be deducted for a wrong or no answer. If the probability that a certain student answers any a question correctly is 80%, his resulting expected points is (　　)

A. 60　　B. 70　　C. 75　　D. 80

3. Let X be a continuous rv with pdf $f_X(x)$. Then the pdf of $\mathrm{Y} = 2\mathrm{X}$, denoted by $f_Y(y)$, is (　　)

A. $f_X(2y)$　　B. $f_X\left(\frac{y}{2}\right)$　　C. $2f_X(2y)$　　D. $\frac{1}{2}f_X\left(\frac{y}{2}\right)$

4. Suppose pulses arrive at a counter according to a Passion process with a rate of 6 per minute. Let X be the number of pulses received in a 30-sec interval and Y be the time gap in minute between two successive pulses. Which of the following statements is **Not** true? (　　)

A. $P(X=0) = e^{-180}$　　B. $P(X \ge 1) = 1 - \mathrm{e}^{-3}$

C. $P(Y < 1) = 1 - e^{-6}$　　D. $P(Y > 2) = e^{-12}$

5. Let $X_1, X_2, \ldots, X_{10}$ be a random sample from the normal population $N(\mu, \sigma^2)$, $\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i$ and $S^2 = \frac{1}{9}\sum(X_i - \bar{X})^2$. Which of the following statements is true? ( )

A. $\frac{\bar{X}-\mu}{\sigma/10} \sim N(0,1)$

B. $\frac{\bar{X}-\mu}{S/\sqrt{10}} \sim N(0,1)$

C. $\frac{\bar{X}-\mu}{S/\sqrt{10}}$ has t distribution with 9 df

D. $\frac{\bar{X}-\mu}{S/\sqrt{10}}$ has t distribution with 10 df.

6. Suppose $X \sim Bin\,(n, p)$. Which of the following statements is true? ( )

A. If $np > 10$ and $n(1-p) > 10$, X has approximately Poisson distribution.

B. If $n > 50$ and $np < 5$, X has approximately Poisson distribution with parameter np.

C. $P(X = k)$ reaches its maximum value when $k = 0$.

D. X has approximately Normal distribution if and only if n is big and p is small.

II. Fill-in-the-blanks（6 questions, 3 points per question, 18 points in total）

1. A university warehouse has received 3 printers, of which 2 are laser printers and 1 is inkjet models. Suppose that the probability that a laser printer requires service while under warranty is 20%, whereas the probability that an inkjet printer needs such service is 10%. Let $X =$ the number among the 3 printers that will need warranty service. Then $P(X = 2) =$________________.

2. Let X and Y be two independent random variables with distribution functions $F_X(x)$ and $F_Y(y)$ respectively. Then $P(\min(X, Y) \leq z) =$______________________, where z is a given constant.

3. If X is a random variable with a uniform distribution on the interval $[-1, 2]$, then $P(|X - \mu_X| > \frac{\sqrt{3}}{2}\sigma_X) =$__________.

4. Suppose both X and Y have normal distribution $N(0, 2)$ and $\rho_{X,Y} = 0.4$. Then $\mathrm{E}[(\mathrm{X}+\mathrm{Y})^2] =$______

5. Consider a random sample X_1, X_2, X_3 from the normal population $N(\mu, \sigma^2)$. For $a =$______, $\hat{\mu} = \frac{1}{6}X_1 + \frac{1}{3}X_2 + aX_3$ is an unbiased estimator of μ.

6. Let X_1, X_2, X_3 be a random sample from the a discrete population X with pmf $P(X = 1) = \theta^2, P(X = 2) = 2\theta(1-\theta), P(X = 3) = (1-\theta)^2$, where θ is a parameter. Given the observed sample values $x_1 = 1, x_2 = 1, x_3 = 2$, the maximum likelihood estimate of θ is ________.

III.（10 points）It is known that 2% of sick people with fever (发烧病人) in a neighborhood have virus (病毒), while the other 98% are sick with other diseases. Now the sick people can get a free test in a few hours and know if they have virus or not. However this test is not 100% reliable. The true positive rate (when a person who has the virus is tested positive) is 0.98, while the true negative rate (when a person who does not have the virus is tested negative) is 0.95.

a. A fever patient has been tested for the first time. If the result is positive, what is the probability that he has the virus？ (Round to 3 decimal place)

b. Based on the first test result, we update the probability that the fever patient has and has no virus. If the result of the second test is still positive, what is the probability that the fever patient has the virus？ (Round to 3 decimal place)

IV. (10 points) Suppose A and B are two events. $P(A') = \frac{1}{2}, P(A'|B) = \frac{2}{3}$, $P(B|A) = \frac{1}{4}$. Let

$$X = \begin{cases} 1, & A \text{ occurs} \\ -1, & A \text{ does not occur} \end{cases}, \qquad Y = \begin{cases} 1, & B \text{ occurs} \\ -1, & B \text{ does not occur} \end{cases}.$$

a. Display the joint pmf of X and Y in a joint probability table (in fractions).

b. Compute the correlation coefficient ρ for X and Y.

V. (12 points) Suppose X and Y are independent and that each has an exponential distribution with parameter $\lambda = 1$ and $\lambda = 4$ respectively. Let Z denote the number among 3 independent observations that $\{X < Y\}$ occurs.

a. Find $P(X < Y)$.

b. Find $P(Z \leq 1)$.

c. What are the expected value and standard deviation of $Y - X + 1$?

VI. (12 points) Suppose the pdf of X is $f(x;\theta) = \begin{cases} \frac{ax}{\theta^3}(\theta - x), & 0 < x < \theta \\ 0 & , \text{otherwise} \end{cases}$, where $\theta > 0$ is an unknown parameter, a is a constant to be determined and $X_1, X_2, \ldots, X_n$ are a random sample of X.

a. What is the value of a?

b. Use the method of moments to obtain an estimator of θ. Let's denote it by $\hat{\theta}$.

c. What is the variance of $\hat{\theta}$.

VII. (8 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal population $N(\mu, 100)$. It is known that $P(-1 < Z < 1) = 0.6526$, where $Z \sim N(0,1)$.

a. If the sample size $n = 25$, compute $P(|\bar{X} - \mu| < 2)$?

b. What sample size is necessary to ensure that the resulting 95% CI have a width of (at most) 2?

VIII.（12 points）Consider the following sample of melting point of $n = 9$ randomly selected a certain palm oil：

27 30 30.5 32 29.5 31.5 32.5 35 31

Suppose the melting point is normally distributed. Let μ denote the true melting point.

a. Suppose the standard deviation σ is unknown. Please find a 90% confidence interval for μ. (Round to 3 decimal place)

b. Assume that $\sigma = 2$. Does the data support that $\mu > 30$? Carry out a test with a significance level of 0.05.

c. For the test used in b, what is the probability of a type II error if $\mu = 31.0933$.