Lesson 2

FROM ZERO TO ONE

Digital Logic

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Bits, Bytes, Nibbles…

- Bits

$$\underbrace{1}_{\text{most significant bit}}001011\underbrace{0}_{\text{least significant bit}}$$

- Bytes & Nibbles

$$\overbrace{1001\underbrace{0110}_{\text{nibble}}}^{\text{byte}}$$

- Bytes

$$\underbrace{\text{CE}}_{\text{most significant byte}}\text{BF9A}\underbrace{\text{D7}}_{\text{least significant byte}}$$

Large Powers of Two

- 2^{10} = 1 kilo ≈ 1000 (1024)
- 2^{20} = 1 mega ≈ 1 million (1,048,576)
- 2^{30} = 1 giga ≈ 1 billion (1,073,741,824)

Estimating Powers of Two

- What is the value of 2^{24}?

- How many values can a 32-bit variable represent?

Estimating Powers of Two

- What is the value of 2^{24}?

  - **$2^4 \times 2^{20} \approx 16$ million**

- How many values can a 32-bit variable represent?

  - **$2^2 \times 2^{30} \approx 4$ billion**

Addition

- Decimal

```
   11     ← carries
   3734
+  5168
  ------
   8902
```

- Binary

```
   11     ← carries
   1011
+  0011
  ------
   1110
```

Most Significant digit

Least significant digit

Binary Addition Examples

- Add the following 4-bit binary numbers

$$\begin{array}{r} 1001 \\ +\ 0101 \\ \hline \end{array}$$

- Add the following 4-bit binary numbers

$$\begin{array}{r} 1011 \\ +\ 0110 \\ \hline \end{array}$$

Binary Addition Examples

- Add the following 4-bit binary numbers

$$\begin{array}{r} {\color{blue}1} \\ 1001 \\ +\ \ 0101 \\ \hline {\color{blue}1110} \end{array}$$

- Add the following 4-bit binary numbers

$$\begin{array}{r} {\color{blue}111} \\ 1011 \\ +\ \ 0110 \\ \hline {\color{blue}1\boxed{0001}} \end{array}$$

Overflow!

Overflow

- Digital systems operate on a **fixed number of bits**
- Overflow: when result is too big to fit in the available number of bits
- See previous example of 11 + 6

Signed Binary Numbers

- Sign/Magnitude Numbers
- Two's Complement Numbers

Sign/Magnitude Numbers

- 1 sign bit, *N*-1 magnitude bits
- Sign bit is the most significant (left-most) bit
  - Positive number: sign bit = 0
  - Negative number: sign bit = 1

$$A: \{a_{N-1}, a_{N-2}, \cdots a_2, a_1, a_0\}$$

$$A = (-1)^{a_{n-1}} \sum_{i=0}^{n-2} a_i 2^i$$

- Example, 4-bit sign/mag representations of ± 6:

  +6 =

  - 6 =

- Range of an *N*-bit sign/magnitude number:

Sign/Magnitude Numbers

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- Example, 4-bit sign/mag representations of ± 6:

  +6 = **0110**

  - 6 = **1110**

- Range of an *N*-bit sign/magnitude number:

  [-(2^N-1^-1), 2^N-1^-1]

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Sign/Magnitude Numbers

- Problems:
  - Addition doesn't work, for example -6 + 6:

$$\begin{array}{r} 1110 \\ +\ 0110 \\ \hline 10100 \end{array}$$ (wrong!)

  - Two representations of 0 (± 0):

1000

0000

Two's Complement Numbers

- Don't have same problems as sign/magnitude numbers:
  - Addition works
  - Single representation for 0

Two's Complement Numbers

- Msb has value of -2^{N-1}

$$A = a_{n-1}\left(-2^{n-1}\right) + \sum_{i=0}^{n-2} a_i 2^i$$

- Most positive 4-bit number:
- Most negative 4-bit number:
- The most significant bit still indicates the sign (0 = positive, 1 = negative)
- Range of an *N*-bit two's comp number:

Two's Complement Numbers

- Msb has value of -2^{N-1}

$$A = a_{n-1}\left(-2^{n-1}\right) + \sum_{i=0}^{n-2} a_i 2^i$$

- Most positive 4-bit number: **0111**
- Most negative 4-bit number: **1000**
- The most significant bit still indicates the sign (0 = positive, 1 = negative)
- Range of an *N*-bit two's comp number:

$[-(2^{N-1}), 2^{N-1}-1]$

"Taking the Two's Complement"

- Flip the sign of a two's complement number
- Method (*invert & add-one*):
    1. **Invert the bits**
    2. **Add 1**
- Example: Flip the sign of $3_{10} = 0011_2$

"Taking the Two's Complement"

- Flip the sign of a two's complement number
- Method:
  1. **Invert the bits**
  2. **Add 1**
- Example: Flip the sign of $3_{10} = 0011_2$
  1. **1100**
  2. **+ 1**

  1101 = -3_{10}

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Two’s Complement Examples

- Take the two’s complement of $6_{10} = 0110_2$

- What is the decimal value of the two’s complement number 1001_2?

Two's Complement Examples

- Take the two's complement of $6_{10} = 0110_2$
  1. 1001
  2. + 1

  $1010_2 = -6_{10}$

- What is the decimal value of the two's complement number 1001_2?
  1. 0110
  2. + 1

  $0111_2 = 7_{10}$, so $1001_2 = -7_{10}$

Two's Complement Addition

- Add 6 + (-6) using two's complement numbers

$$\begin{array}{r} 0110 \\ +\ \ 1010 \\ \hline \end{array}$$

- Add -2 + 3 using two's complement numbers

$$\begin{array}{r} 1110 \\ +\ \ 0011 \\ \hline \end{array}$$

Two's Complement Addition

- Add 6 + (-6) using two's complement numbers

```
   111
   0110
+  1010
-------
  10000
```

Omitted

- Add -2 + 3 using two's complement numbers

```
   111
   1110
+  0011
-------
  10001
```

Omitted

Increasing Bit Width

- **Extend number from *N* to *M* bits (*M* > *N*) :**
  - Sign-extension
  - Zero-extension

Sign-Extension

- Sign bit copied to msb's
- Number value is same

- **Example 1:**
  - 4-bit representation of 3 = 0011
  - 8-bit sign-extended value: 00000011
- **Example 2:**
  - 4-bit representation of -5 = 1011
  - 8-bit sign-extended value: 11111011

Zero-Extension

- Zeros copied to msb's
- Value changes for negative numbers

- **Example 1:**
  - 4-bit value = $0011_2 = 3_{10}$
  - 8-bit zero-extended value: $\color{red}{0000}0011 = 3_{10}$
- **Example 2:**
  - 4-bit value = $1011 = -5_{10}$
  - 8-bit zero-extended value: $\color{red}{0000}1011 = 11_{10}$

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Number System Comparison

Number System	Range
Unsigned	$[0, 2^N-1]$
Sign/Magnitude	$[-(2^{N-1}-1), 2^{N-1}-1]$
Two's Complement	$[-2^{N-1}, 2^{N-1}-1]$

For example, 4-bit representation:

	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Unsigned									0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
Two's Complement	1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111								
Sign/Magnitude		1111	1110	1101	1100	1011	1010	1001	0000 1000	0001	0010	0011	0100	0101	0110	0111								