Logo

Discrete Mathematics

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Logo

Chapter 4. Graphs

Euler and Hamilton Paths

Section 4.5

Contents

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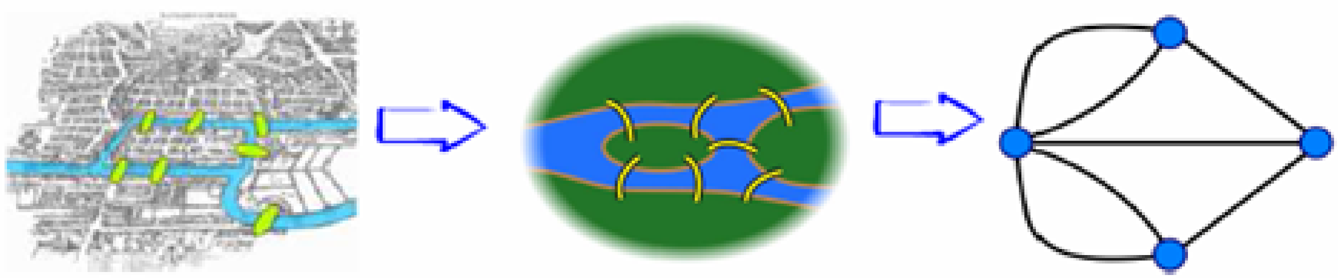
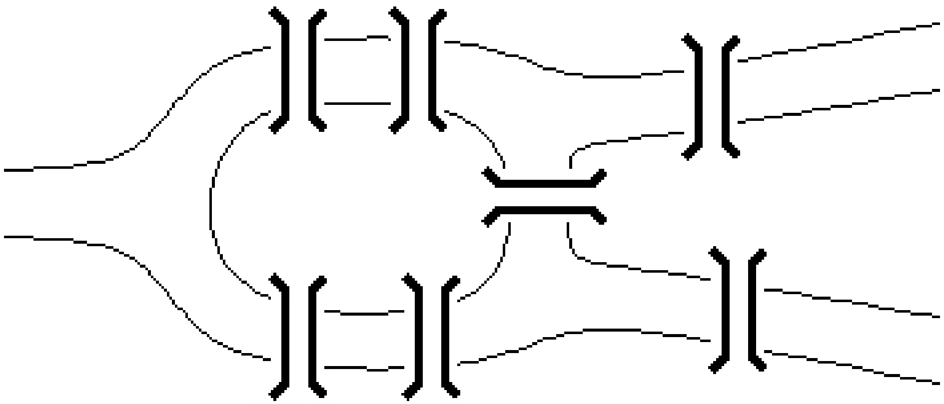
Euler Paths and Circuit

Konigsberg Seven Bridges Problem

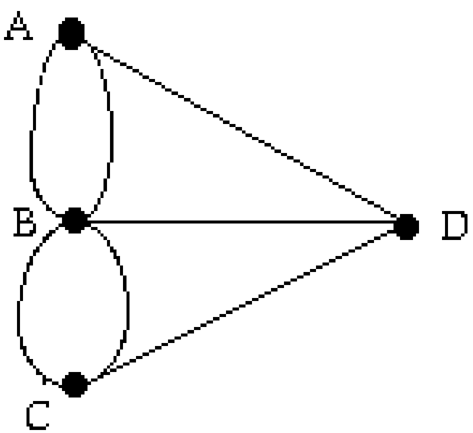
- The city of Konigsberg was divided into 4 sections by the Pregel River. The 4 sections are connected by 7 bridges.
- Is it possible to start at some location, travel across all the bridges without crossing any bridge twice, and return to the starting point?



Konigsberg Seven Bridges Problem



The original problem



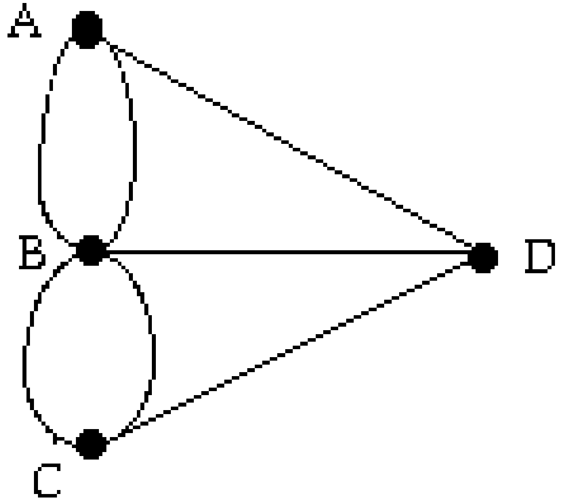
Equivalent multigraph

- The Swiss mathematician Leonhard Euler solved this problem. His solution, published in 1736, may be the first use of graph theory.
- Euler studied the problem by using a multiple graph.

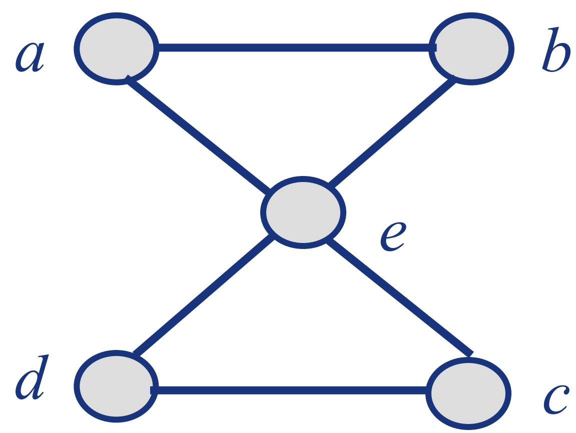
Euler circuit

- Definition 1:
- An Euler circuit in a graph G is simple circuit containing every edge of G.
- An Euler path in G is a simple path containing every edge of G.

No solution!!



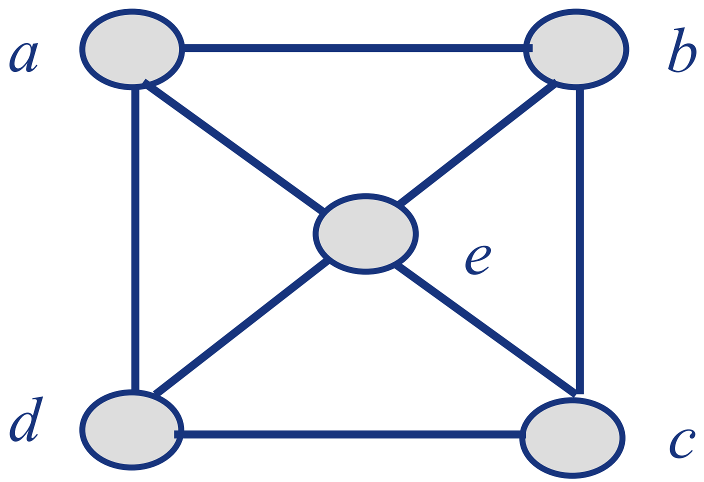
Example 1



G1

- The graph G1 has an Euler circuit, for example a, e, c, d, e, b, a.

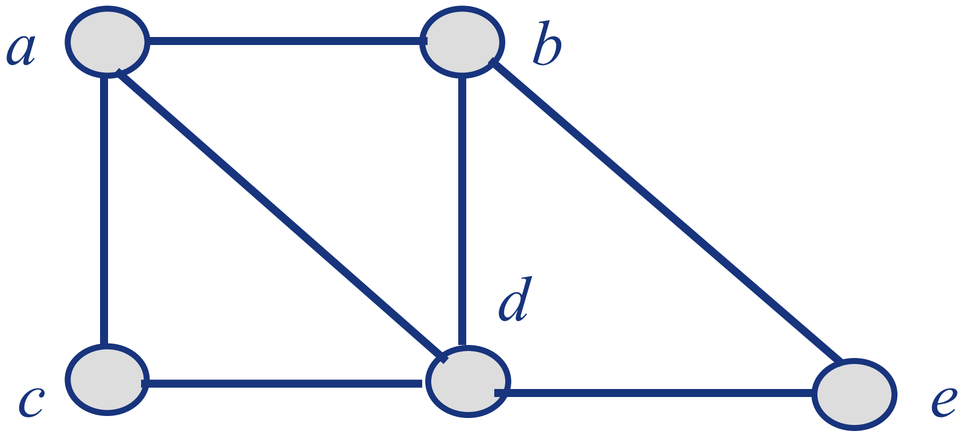
Example 1



G2

- **G2 does not have an Euler circuit.**
- **G2 does not have an Euler path, either.**

Example 1



G3

- **G3 does not have an Euler circuit.**
- **G3 has an Euler path:** ***a*, *c*, *d*, *e*, *b*, *d*, *a*, *b*.**

Necessary and Sufficient Conditions

- **There are simple criteria for determining whether a multigraph has an Euler circuit or an Euler path.**
- **An Euler circuit begins with a vertex a and continues with an edge incident to a, say $\{a, b\}$.**
- **The edge $\{a, b\}$ contributes one to $\deg(a)$.**
- **Each time the circuit passes through a vertex it contributes two to the vertex's degree, since the circuit enters via an edge incident with this vertex and leaves via another such edge.**

Necessary and Sufficient Conditions

- Therefore, deg(a) must be even, because the circuit contributes two every time when it passes through a.
- We conclude that if a connected graph has an Euler circuit, then every vertex must have even degree.
- Therefore, if a connected graph has an Euler circuit, then every vertex must have even degree.

Necessary condition!!!

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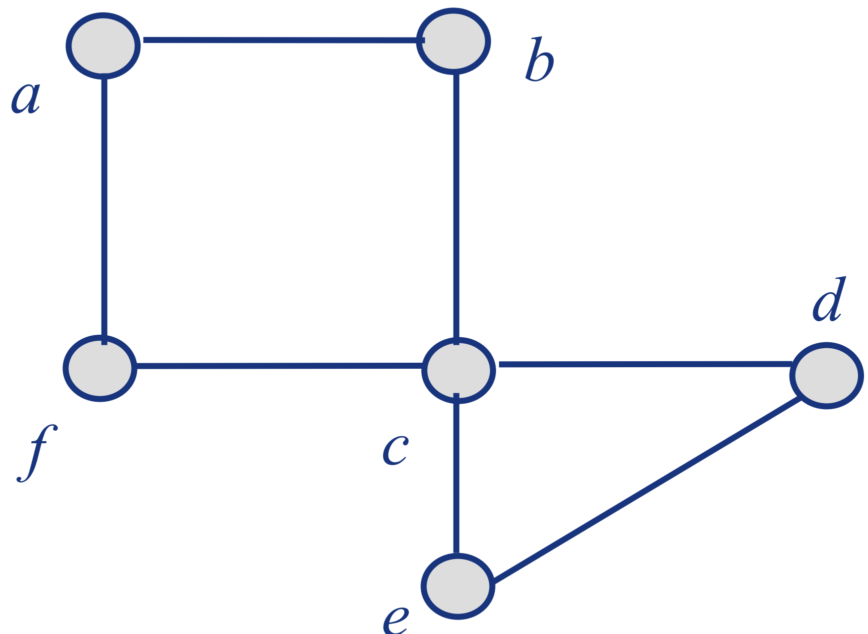
Necessary and Sufficient Conditions

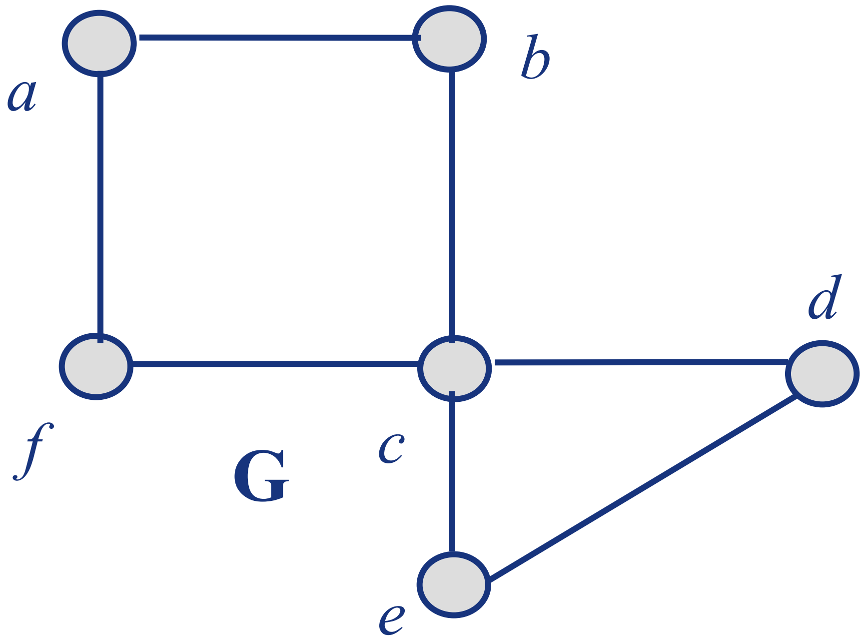
- Is this necessary condition for the existence of an Euler circuit also sufficient?
- An Euler circuit exists in a connected multigraph if all vertices have even degree?
- The proof is given with a construction.
- Suppose that the degree of every vertex of G is even.
- We can build a simple path $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$ where $x_0 = a$ is an arbitrary vertex.

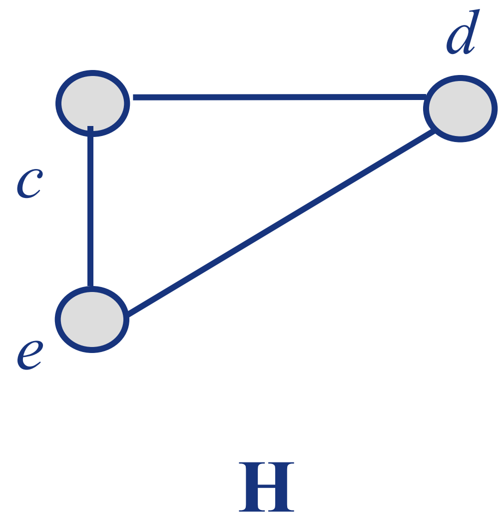


- **It begins at a with an edge of the form {a, x}, and it terminates at a with an edge of the form {y, a}.**
- **Each time the path goes through a vertex with even degree, it uses only one edge to enter this vertex,**
- **So that at least one edge remains for the path to leave the vertex.**

Logo

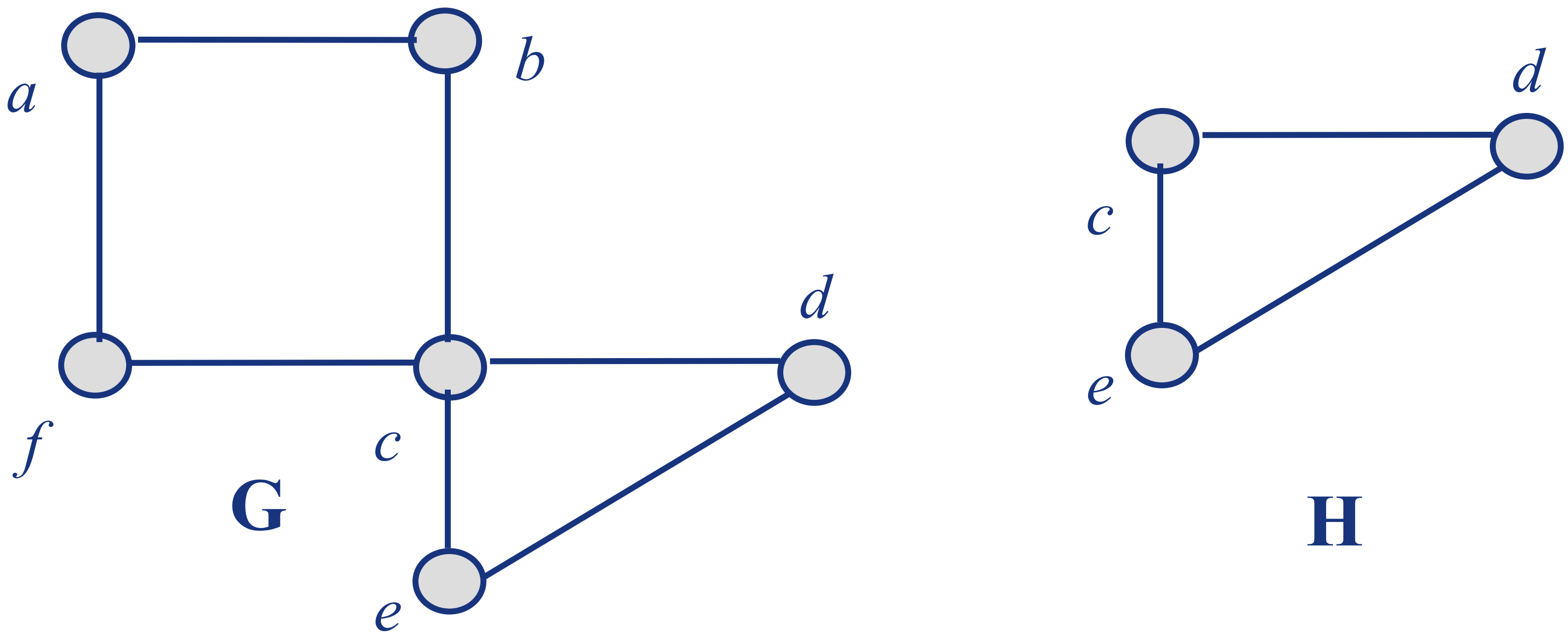
- **An Euler circuit has been constructed if all the edges have been used.**
- **Otherwise, consider the subgraph H obtained from G by deleting the edges already used and vertices that are not incident with any edges.**
- **When we delete the circuit a, f, c, b, a from the graph, we obtain the subgraph labeled as H.**





- **Since G is connected, H has at least one vertex in common with the circuit that has been deleted. Let *w* be such a vertex.(In our example, *c* is the vertex)**

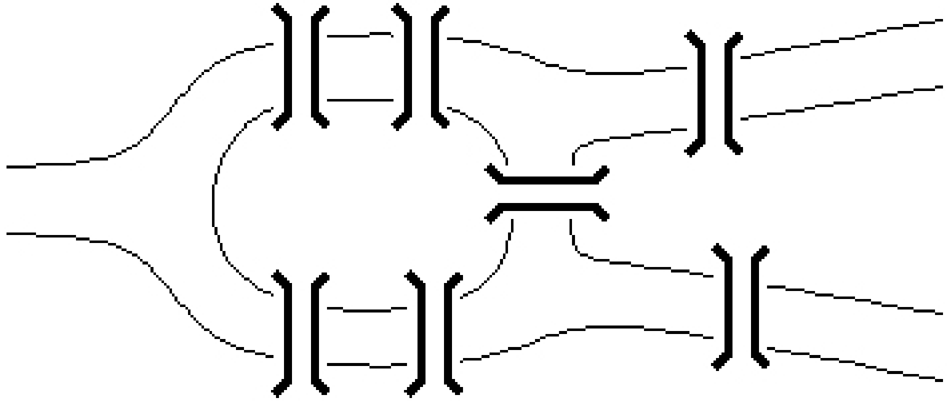
- Every vertex in H has even degree.
- H may not be connected.
- Beginning at w, construct a simple path in H by choosing edges as long as possible, as was done in G.
- This path must terminate at w.
- For instance, c, d, e, c is a path in H. Form a circuit in G by splicing the circuit in H with the original circuit in G.

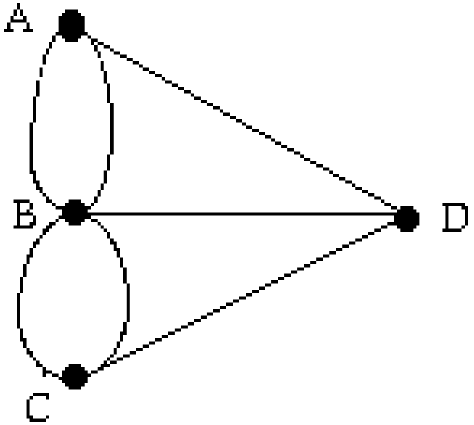


- Form a circuit in G by splicing the circuit in H with the original circuit in G. This can be done since w is one of the vertices in this circuit.
- When this is done in the graph G, we obtain the circuit a, f, c, d, e, c, b, a.

- Continue this process until all edges have been used.
- The process must terminate since there are only a finite number of edges in the graph.
- This produces an Euler circuit .
- If the vertices of a connected multigraph all have even degree, then the graph has an Euler circuit.

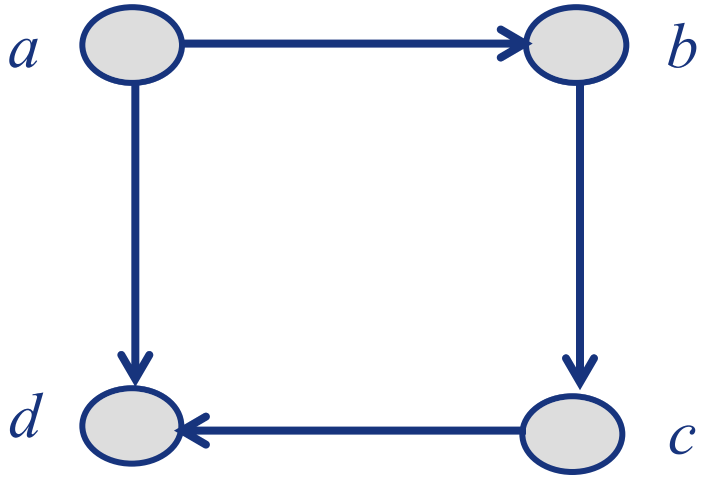
- Theorem 1: A connected multigraph has an Euler circuit if and only if each of its vertices has even degree.





- 4 vertices of odd degree, it does not have an Euler circuit.

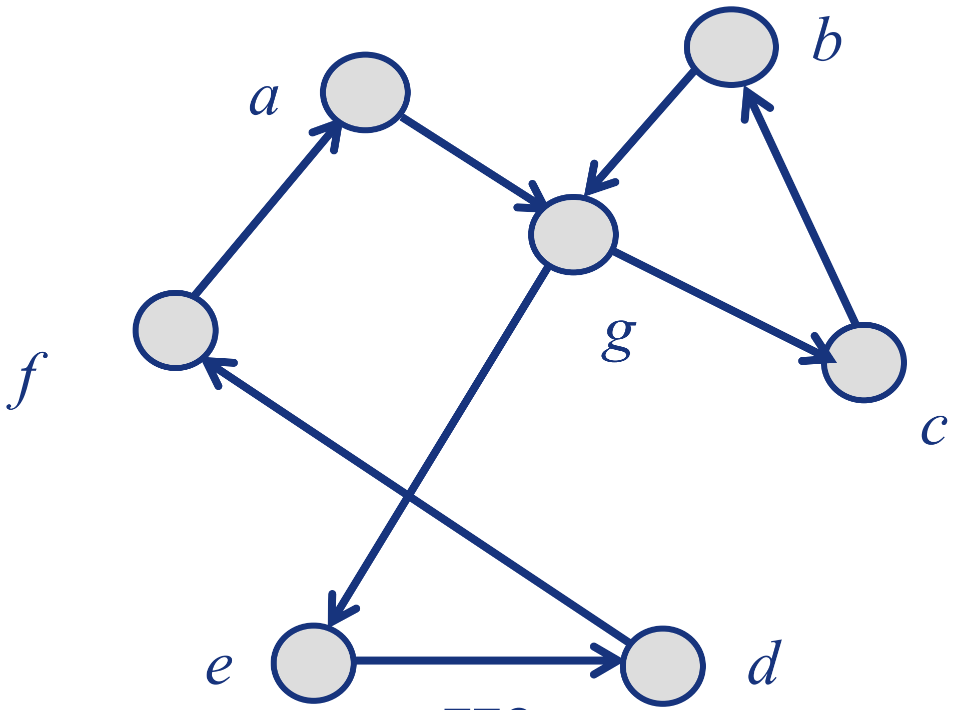
Example 2



H1

- H1 does not have an Euler circuit.
- H1 does not have an Euler path, either.

Example 2

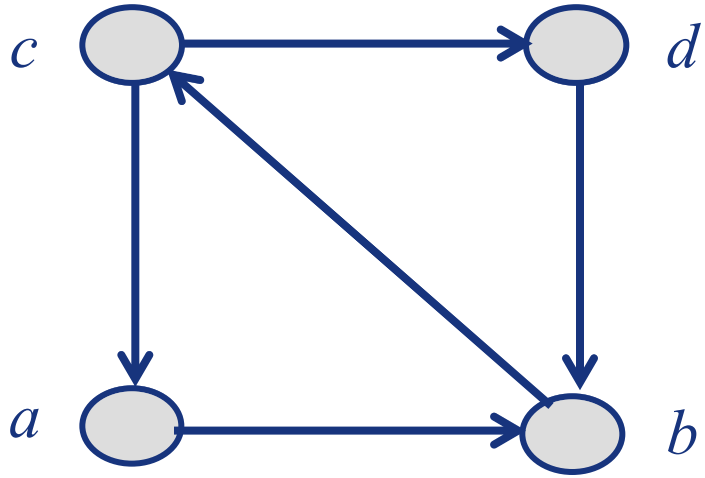


H2

- **H2 has an Euler circuit:**

  a, g, c, b, g, e, d, f, a

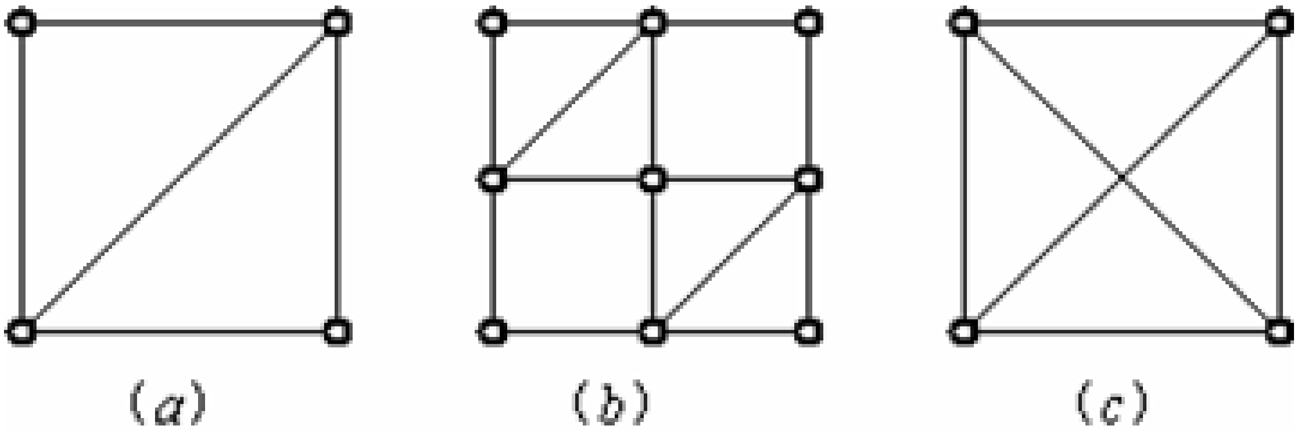
Example 2



H3

- **H3 does not have an Euler circuit.**
- **H3 has an Euler path: c, a, b, c, d, b**

More examples



- (a) and (c) does not have an Euler circuit.
- (c) does not have an Euler path.
- (b) has both an Euler circuit and an Euler path.

Algorithm

- Algorithm 1 gives the constructive procedure for finding Euler circuits given in the discussion preceding Theorem 1.

- The algorithm specifies the steps of the procedure more precisely.

Algorithm 1 Constructing Euler Circuit

- Procedure Euler (G: connected multigraph with all vertices of even degree)
- 1: circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex.
- 2: H := G with the edges of this circuit removed

Algorithm 1 Constructing Euler Circuit

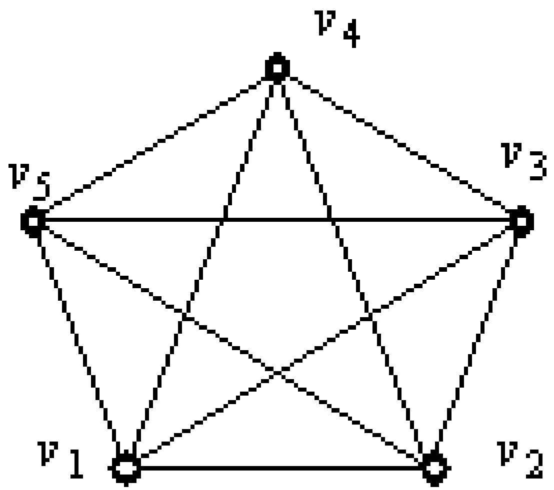
- 3: while H has edges
- Begin
- subcircuit := a circuit in H beginning at a vertex in H that also is the endpoint of an edge of circuit

  H := H with edges of subcircuit and all isolated vertices removed

  circuit := circuit with subcircuit inserted at the appropriate vertex

End {the circuit is an Euler circuit.}

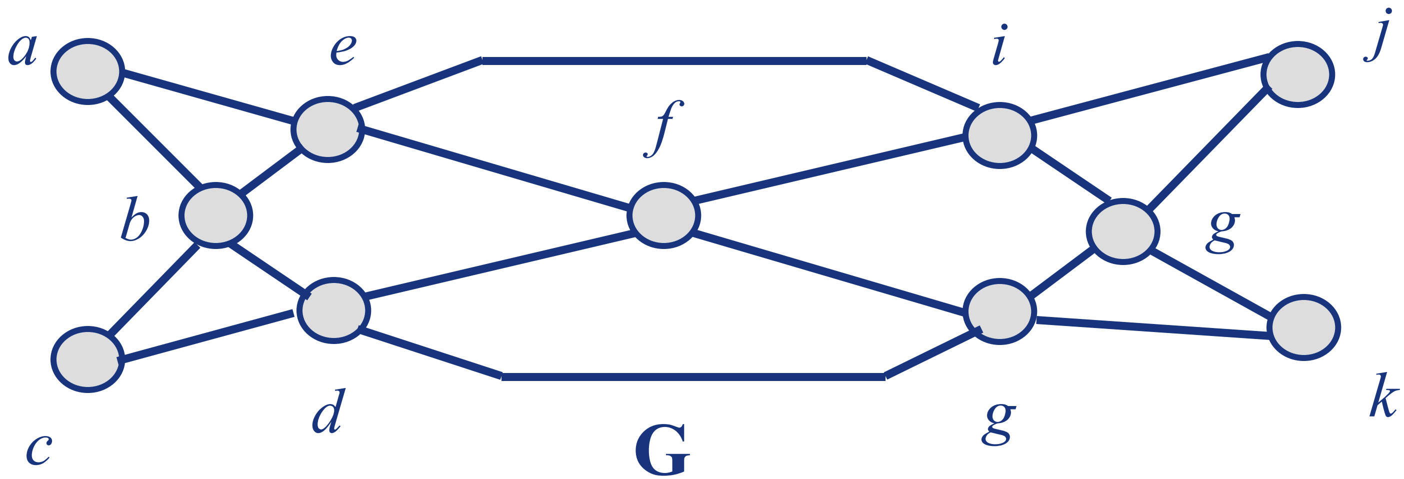
Example



- $C1$ in G: $v_1v_2v_3v_4v_5v_1$
- $C2$ *in* $G - C1$: $v_1v_3v_5v_2v_4v_1$
- $v1$ is the shared vertex of C1 and C2.
- The Euler circuit is : $v_1v_2v_3v_4v_5v_1v_3v_5v_2v_4v_1$

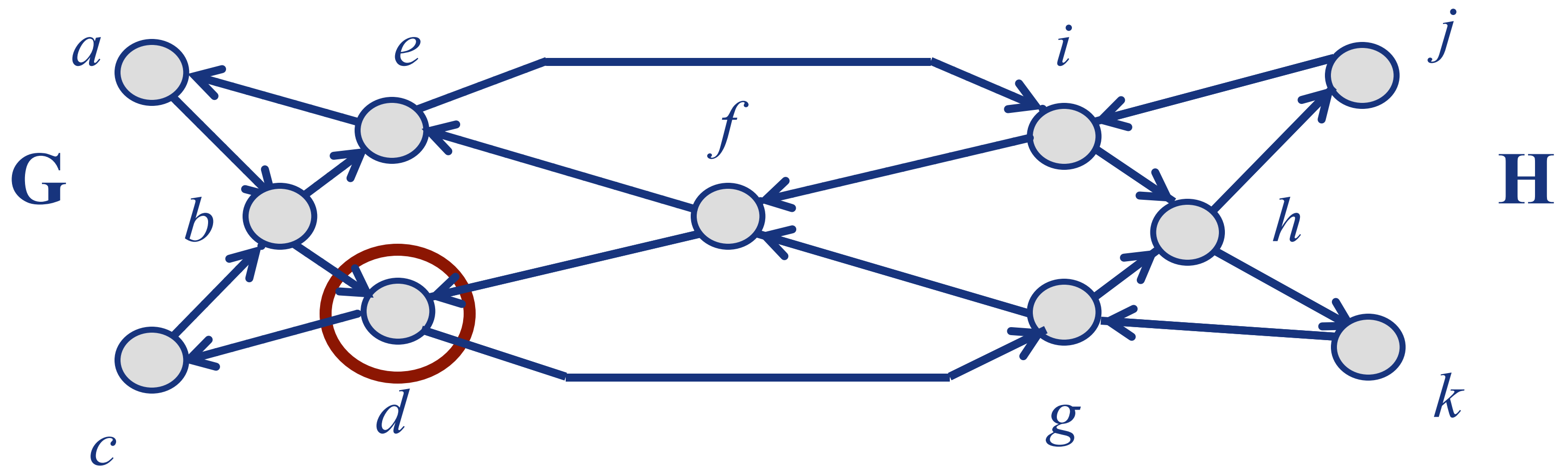
Example 3

- Mohammed s scimitars



- Every vertex has even degree.
- Therefore, we can use algorithm 1 to solve its Euler circuit.

Example 3



1、 a, b, d, c, b, e, i, f, e, a in G

2、 d, g, h, j, i, h, k, g, f, d in H

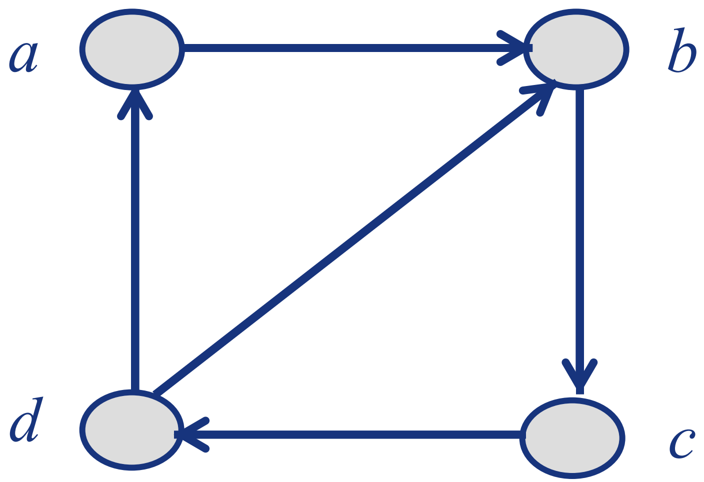
3、 a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a

Logo

- Theorem 2: A connected multigraph has an Euler path but not an Euler circuit if and if only if it has exactly two vertices of odd degree.

- Main idea of the proof :
- Every time the path goes through a there is path contributes one to the degree of a.
- The terminal and the beginning vertices have odd degree.

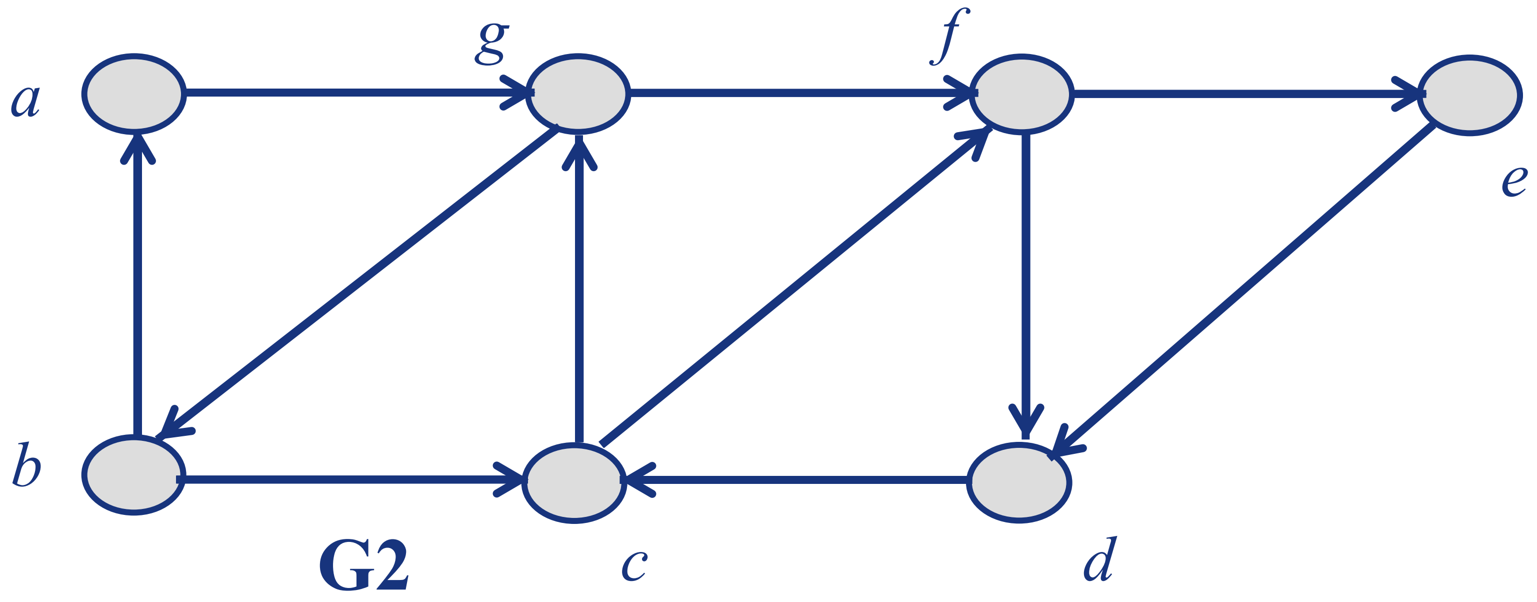
Example 4



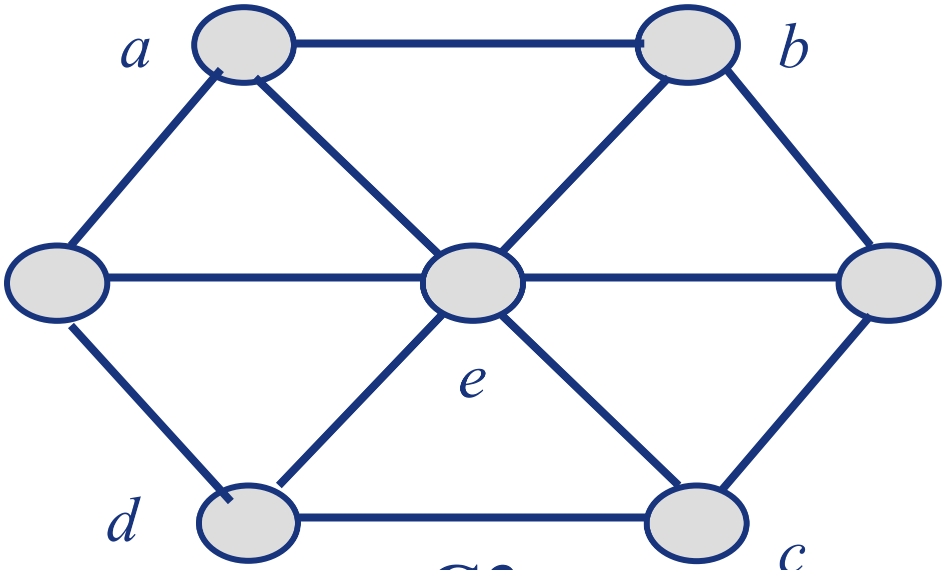
G1

- **G1 contains exactly two vertices of odd degree, namely, b and d.**
- **Euler path: d, a, b, c, d, b**

Example 4

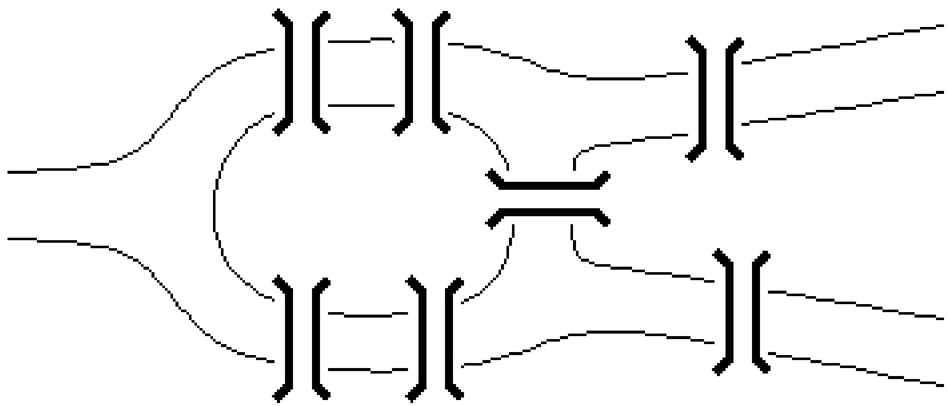


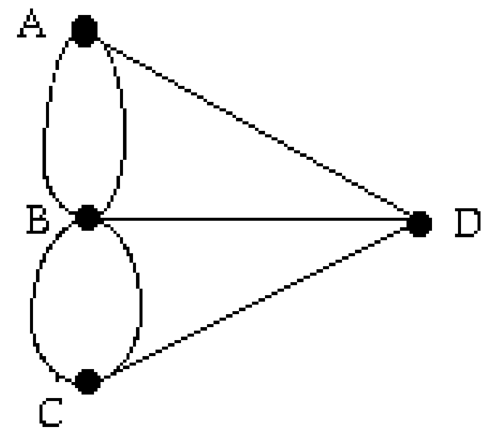
- G2 contains exactly two vertices of odd degree, namely, b and d.
- Euler path: b, a, g, f, e, d, c, g, b, c, f, d



G3

- **G3 has no Euler path since it six vertices of odd degree.**





- 4 vertices of odd degree, it does not have an Euler path, such trip is impossible.

Application of Euler path

- Chinese postman problem
- If a postman can find an Euler path in the graph that represents the streets the postman needs to cover, this path produces a route that traverses each street of the route exactly one.
- If no Euler path exists, some streets will have to be traversed more than once.

L o g o

Hamilton Paths and Circuit

Hamilton Paths

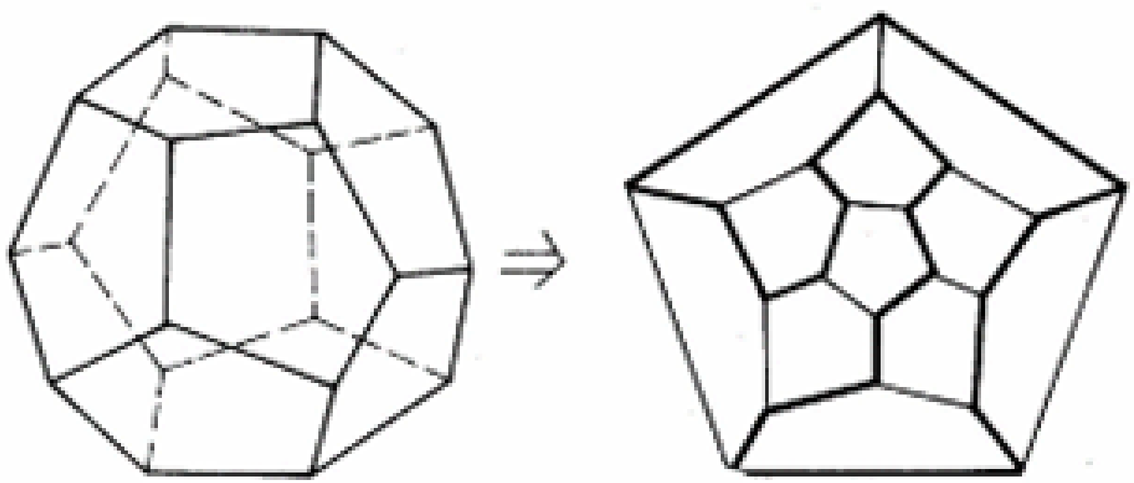
- An *Euler circuit* in a graph *G* is a simple circuit containing **every edge** of *G*.
- An *Euler path* in *G* is a simple path containing **every edge** of *G*.
- A *Hamilton circuit* is a circuit that traverses **each vertex** in *G* exactly once.
- A *Hamilton path* is a path that traverses **each vertex** in *G* exactly once.)

Hamilton Paths

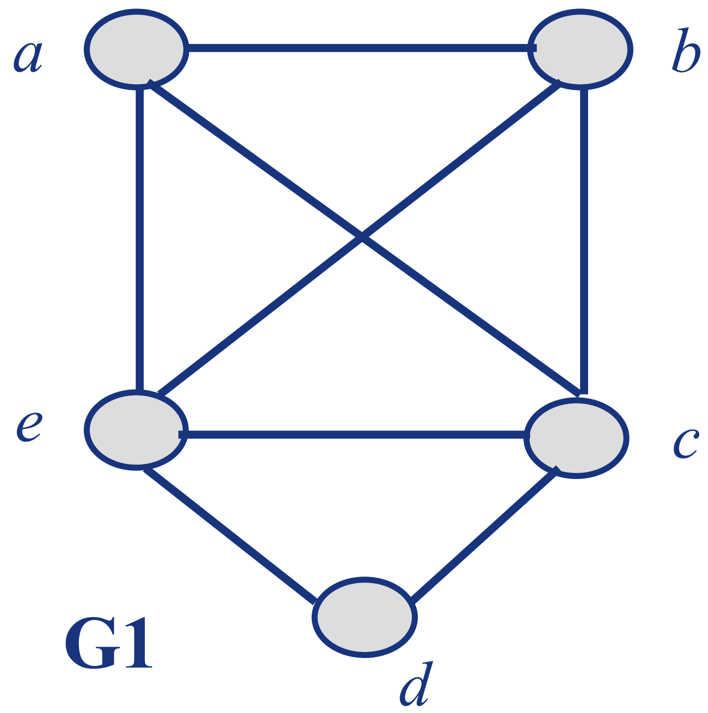
- Definition 2:
- A simple path $x_0, x_1, \dots, x_{n-1}, x_n$ in the graph G=(V,E) is called Hamilton path if V= $(x_0, x_1, \dots, x_{n-1}, x_n)$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$
- A simple circuit $x_0, x_1, \dots, x_{n-1}, x_n, x_0$ (with n>0) in the graph G=(V,E) is called Hamilton circuit if $x_0, x_1, \dots, x_{n-1}, x_n$ is a Hamilton path.

Source of the terminology

- Hamilton path comes from a game, called Voyage Round the word puzzle, invented by Irish mathematician Sir William Rowan Hamilton.
- The puzzle was to start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and back for the first city.

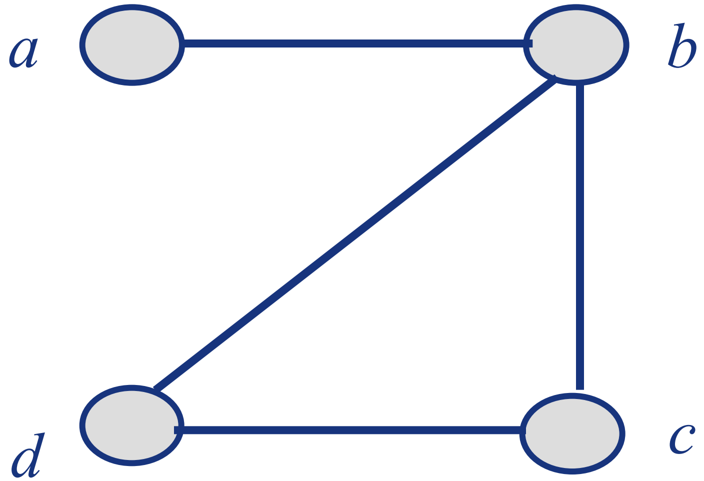


Example 5



- G1 has a Hamilton circuit: a, b, c, d, e, a.

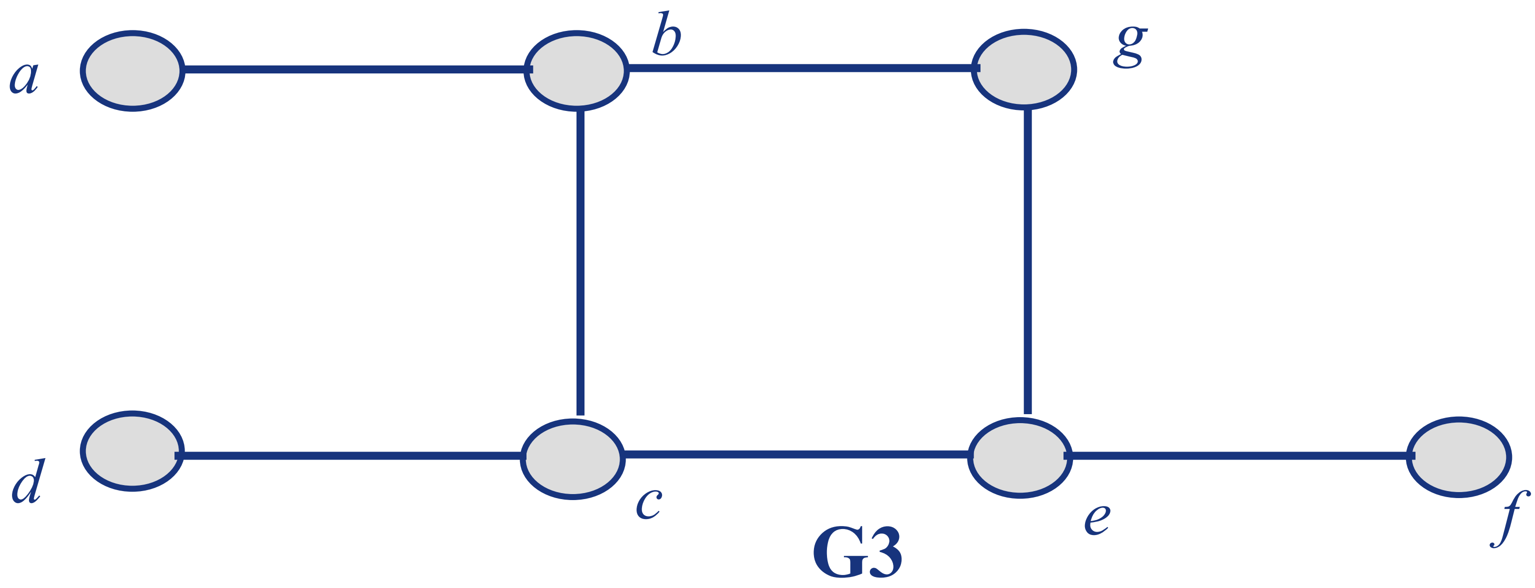
Example 5



G2

- G2 does not have a Hamilton circuit.
- G2 has a Hamilton path: a, b, c, d.

Example 5



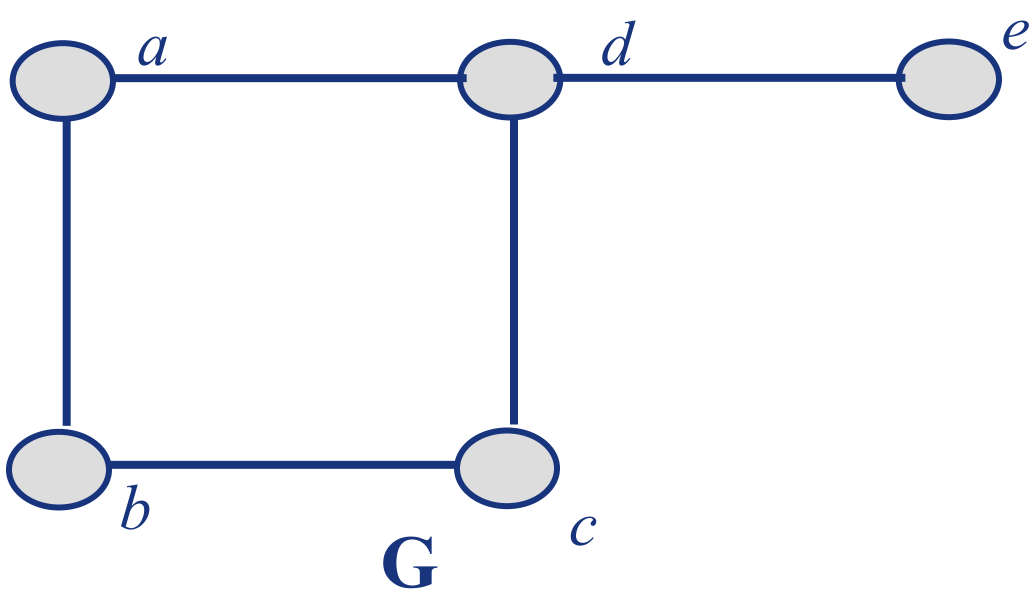
- G3 does not have a Hamilton circuit.
- G3 does not have a Hamilton path.

- Surprisingly, there are no known simple necessary and sufficient criteria for the existence of Hamilton circuits.
- However, some sufficient criteria are known.

- 1. a graph with a vertex of degree one cannot have a Hamilton circuit.
- 2. a Hamilton circuit cannot contain a smaller circuit within it.

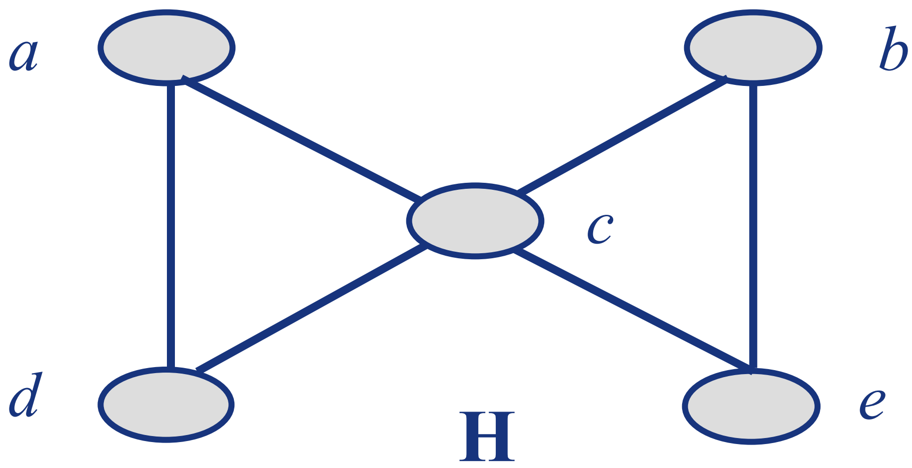
Logo

Example 6



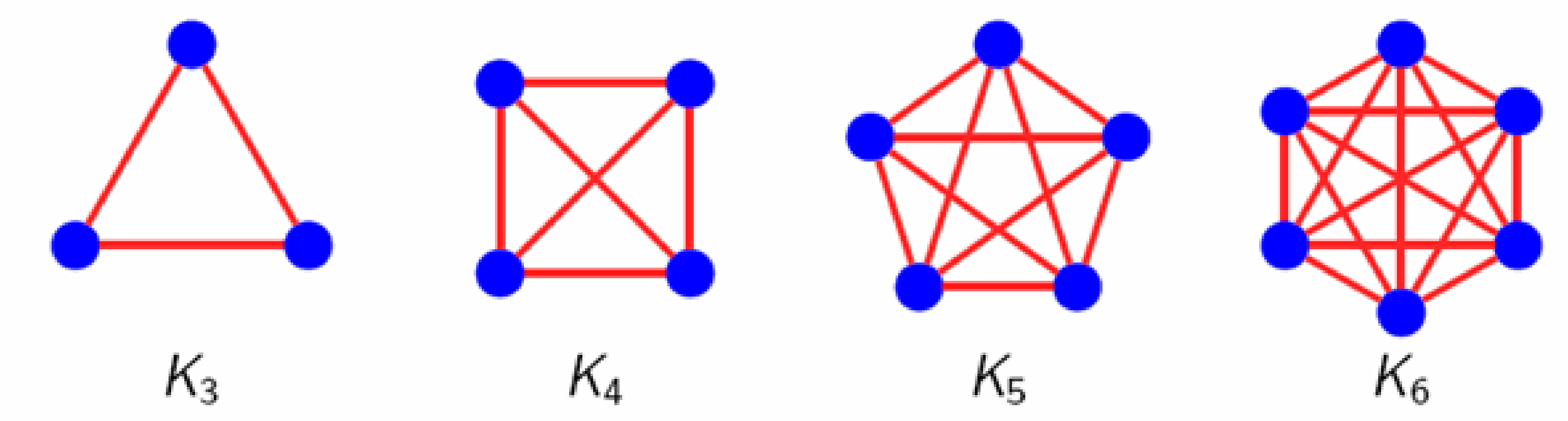
- There is no Hamilton circuit in G
- since G has a vertex of degree one.

Example 6



- All vertices have two degree.
- But H has no Hamilton circuit, for any possible Hamilton circuit have to pass c twice.

Example 7



- **We can form a Hamilton circuit in K_n beginning at any vertex.**
- **Such a circuit can be built by visiting vertices in any order we choose, which is possible since there are edges in Kn between any two vertices.**

Hamiltonian Path Theorems

- **Dirac's theorem: If (but <u>not</u> only if) $G=(V,E)$ is a simple graph, has $n \geq 3$ vertices, and $\forall v \in V$, $\deg(v) \geq n/2$, then G has a Hamilton circuit.**

- **Ore's theorem : $G=(V,E)$ is a simple graph , has $n \geq 3$ nodes, and $\deg(u)+\deg(v) \geq n$ for every pair u,v of non-adjacent nodes, then G has a Hamilton circuit.**

Logo

Proof

Ore's theorem： $G=<V, E>$是n阶简单图，$n\geq 3$，若对G中两不相邻结点a、$b\in V$，均有$d(a)+d(b)\geq n$（*），则G中存在哈密尔顿回路。

证明思路：反证法：

1. 假设存在满足（*）的图G，而且假设G中不存在哈密顿回路。

2. 构造一个具有更多边的图G'，而且G'中也不存在哈密顿回路。

3. 证明G'存在哈密顿通路。

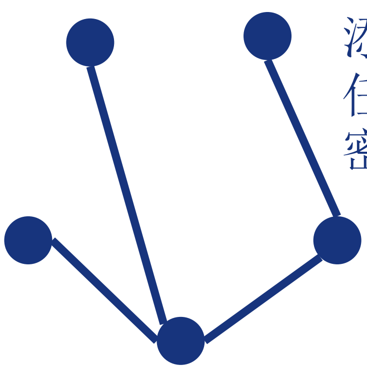
4. 在该哈密顿通路中推导出矛盾，从而假设1不成立，原命题得证。

Proof

1. 假设存在满足（*）的图G，而且假设G中不存在哈密顿<u>回路</u>。(前提假设)

2. 构造一个具有更多边的图G’，而且G’中也不存在哈密顿<u>回路</u>。

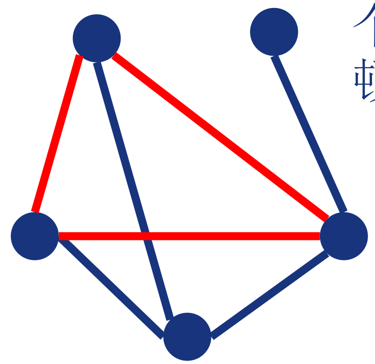
假设G不存在哈密顿回路，则可以构造一个与G具有相同顶点的图G’，G’通过往G的每一个顶点加入不产生哈密顿回路的尽可能多的边来构造：不断地加若干条边到G，<u>直到再加入一条边就产生哈密顿回路为止（1）</u>。另外G’依然满足（*），因为对于$\forall v \in V, d_G(v) \le d_{G'}(v)$。



图G

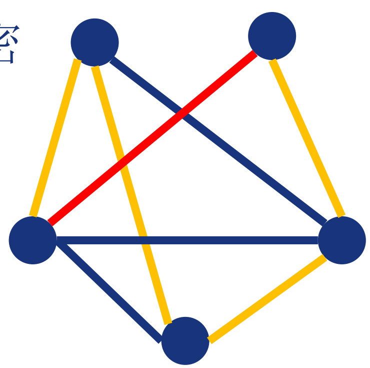
重复在G上任意地添加边，直到再加任意边就会出现哈密顿回路

(添加边的顺序不影响证明的结果，只需要最终G’满足（1）)



图G’

在G’上增加任意一个边都会出现哈密顿回路

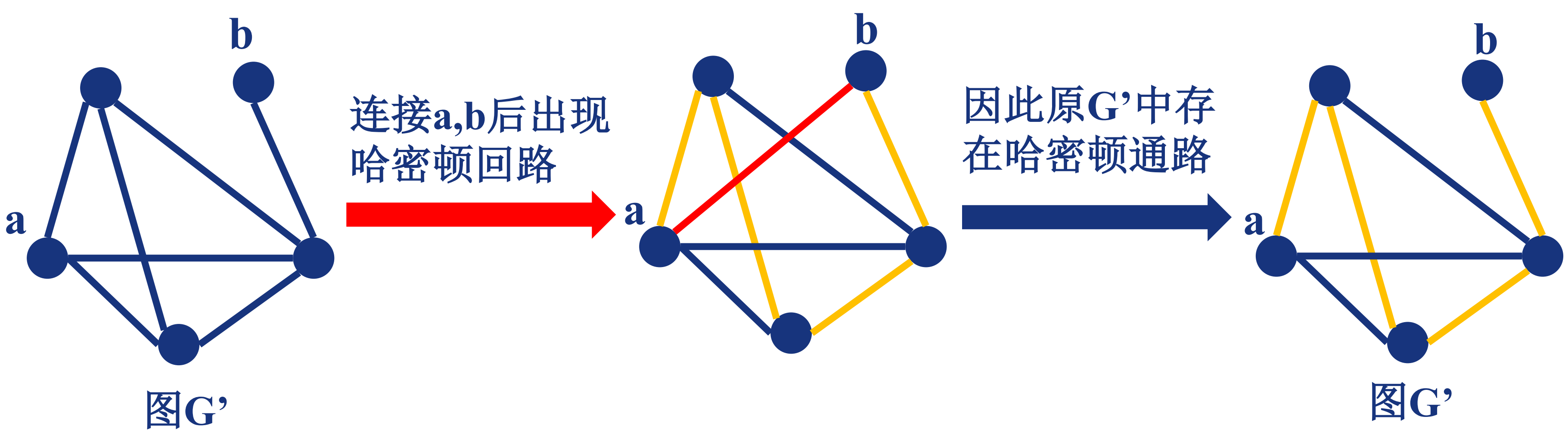


Proof

3. 证明G'存在哈密顿通路。

命题：对G'中任意两个不相邻的节点a，b，都能存在一条以a为起点，b为终点的哈密顿通路。

证明：根据G'的定义，在G'中添加任意一个边都会出现哈密顿回路。因此在G'中添加连接a，b的边就会出现哈密顿回路，那么该哈密顿回路除去边(a,b)后就是一条以a为起点，b为终点的哈密顿通路。



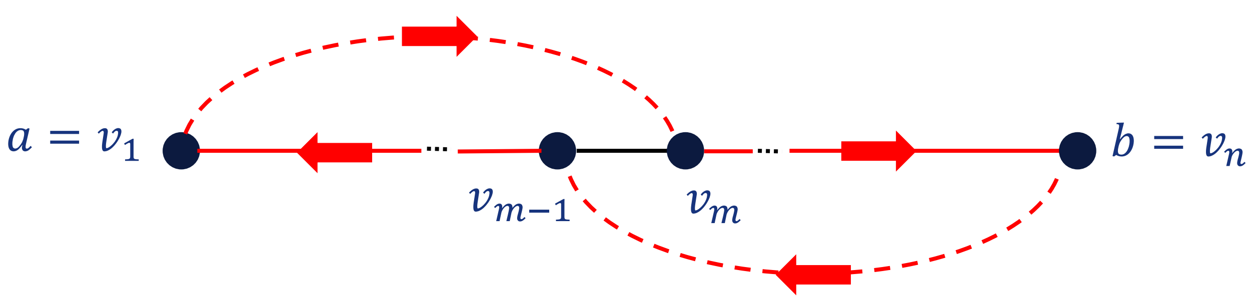
Proof

4. 证明该哈密顿通路中存在矛盾。

对G’中任意两个不相邻的节点a，b，令以a为起点、b为终点的哈密顿通路为$v_1, v_2 \dots v_n$，其中$a = v_1, b = v_n$。

命题：如果在G’中a与$v_{i_1}, v_{i_2}, \dots, v_{i_k}$相邻，则b与$v_{i_1-1}, v_{i_2-1}, \dots, v_{i_k-1}$都不相邻。

证明：根据G’的定义，G’中不存在哈密顿回路。因此如果b与$v_{i_1-1}, v_{i_2-1}, \dots, v_{i_k-1}$中的一个节点相邻，设与a，b相邻的节点分别为$v_m$和$v_{m-1}$，那么G’就会出现一个哈密顿回路：$a \to v_m \to \cdots \to b \to v_{m-1} \to \cdots \to a$，违反了G’的定义。因此b与$v_{i_1-1}, v_{i_2-1}, \dots, v_{i_k-1}$都不相邻。



Proof

4. 证明该哈密顿通路中存在矛盾。

引理：如果在G'中a与$v_{i_1}, v_{i_2}, \dots, v_{i_k}$相邻，则b与$v_{i_1-1}, v_{i_2-1}, \dots, v_{i_k-1}$都不相邻。

如果$d(a)$=k，

那么有：

$d(b)$=n-1-k; (b与a和$v_{i_1-1}, v_{i_2-1}, \dots, v_{i_k-1}$ 都不相邻)

因此有：

$d(a) + d(b) = k + [n - 1 - k] = n - 1$;

$d_G(a) + d_G(b) \le d_{G'}(u) + d_{G'}(v) < n$

而假设（*）为$d_G(a) + d_G(b) \ge n$，矛盾。

因此，原假设不成立，不存在一个既满足（*），又不是哈密尔顿图的图，证毕。

- Both Ore’s Theorem and Diac’s Theorem provide sufficient conditions for a connected simple graph to have a Hamilton circuit.
- However, the theorems do not provide necessary conditions for the existence of a Hamilton circuit.

Logo

- The best algorithm known for finding a Hamilton circuit in a graph or determining that no such circuit exist have exponential worst-case time complexity.
- Finding an algorithm that solves this problem with polynomial worst-case time complexity would be a major accomplishment because it has been proved to be NP-complete.

Application

- Hamilton circuits and paths can be used to solve practical problems.

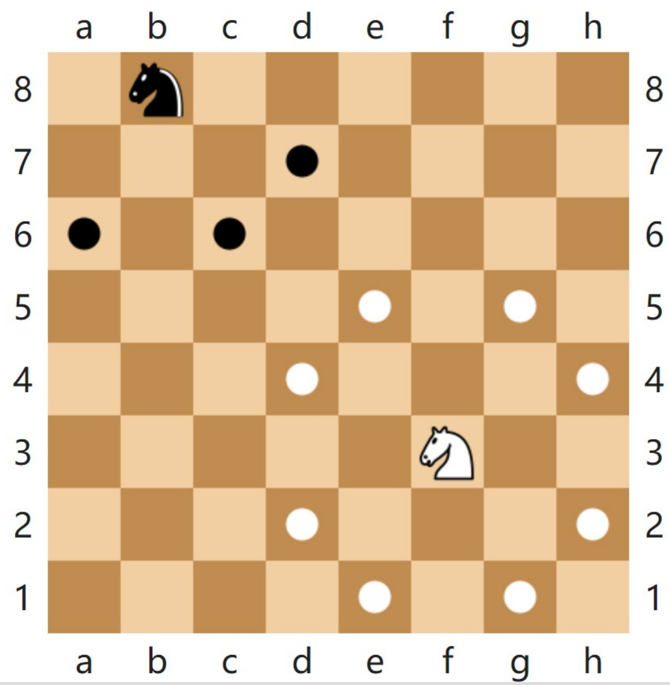
- Traveling salesman problem asks for the shortest route a traveling salesman should take to visit a set of cities.
- The problem reduces to finding a Hamilton circuit in a complete graph such that the total weight of its edges is as small as possible.

Logo

Applications

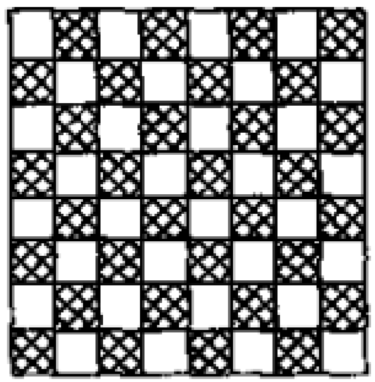
Applications

1. 在国际象棋中马是这样一种棋子：它的移动可以是水平两格和垂直一格，或者是水平一格和垂直两格。如图所示，白色马有8个合法的移动落点（用白色点表示）。在8*8黑白相同的棋盘上跳动一只马，不论跳动方向如何，要使这只马完成每一种可能的跳动恰好一次（即不产生重复的跳动），请问有可能吗？



Applications

解 图 4.4-3(a)给出了一张 8×8 黑白方格的棋盘，将棋盘上的一个方格对应一个结点，



(a)

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

(b)

图 4.4-3

两个结点之间有边当且仅当马可从一个结点跳到另一个结点，可得一张跳马图 G，将 G 中各结点的度数写在对应的方格中，如图 4.4-3(b)所示。可见，图 G 中有八个结点的度数为 3，其余都是偶度数结点。因此 G 中既无欧拉回路，也无欧拉通路。也就是说，要使马在 8×8 棋盘上完成所有可能的跳动仅一次是不可能的。

Logo

Applications

2. 11个学生打算这几天都在一张圆桌上共进午餐，并且希望每次午餐时每个学生两旁所坐的人都不同，问这11个人共进午餐最多能有多少天？（提示: 无向完全图K_n有多少条无公共边的汉密尔顿回路?）

Applications

2. 11个学生打算这几天都在一张圆桌上共进午餐，并且希望每次午餐时每个学生两旁所坐的人都不同，问这11个人共进午餐最多能有多少天？（提示无向完全图K_n有$\lfloor\frac{n-1}{2}\rfloor$条无公共边的哈密顿回路）

解 将11个学生分别用结点表示，由于任意两个学生都可能是邻座，因此每两个结点之间都连一条边，得到无向完全图K_{11}，每次午餐时学生都按一条汉密尔顿回路沿桌而坐，若两条汉密尔顿回路有公共边，则公共边端点上的两个学生是相邻的，从而上述问题转化为求K_{11}有多少条无公共边的汉密尔顿回路问题。由第11题知，K_{11}中无公共边的汉密尔顿回路共有$\frac{n-1}{2}=5$条，故这11个人共进午餐最多能有5天。

K_N有$\frac{N(N-1)}{2}$条边，每个哈密尔顿回路有N条边，因此边不重复的哈密尔顿回路最多有$\left\lfloor\frac{N-1}{2}\right\rfloor$条

Logo

Applications

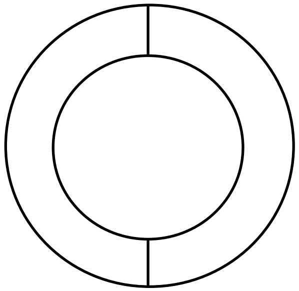
3. Seven people who don't know each other go to the dating party (交友会). The organizer uses round tables to ensure that everyone can communicate with people on both sides. To promote mutual understanding among participants, the organizer will rearrange their seats at set intervals (每隔一段时间), so that everyone has the chance to know unfamiliar participants.
To ensure that everyone can communicate with all members, how many times does the organizer need to arrange their seats of the people **at least**?
If there are *n* (n is a prime number where $n \geq 3$) people, how many times does the organizer need to arrange their seats of the people **at least**?

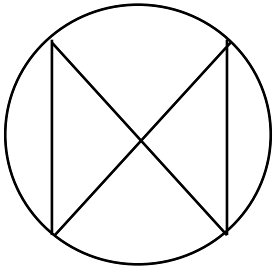
至少需要3次

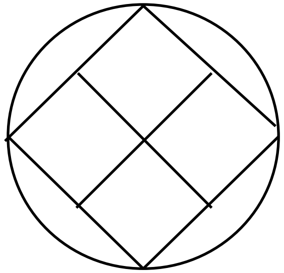
人数为n时至少需要$\left\lceil\frac{n-1}{2}\right\rceil$次。由第2题可知有$\left\lfloor\frac{n-1}{2}\right\rfloor$条哈密尔顿回路，当n为奇数时，需要安排$\frac{n-1}{2}$次；当n为偶数时，需要安排$\left\lfloor\frac{n-1}{2}\right\rfloor + 1$次。

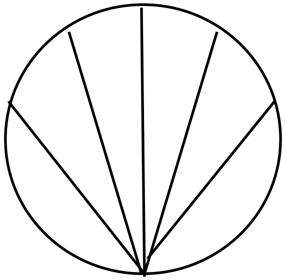
Exercises

- 1. Which of the given graphs has an Euler circuit? (B)

A) 

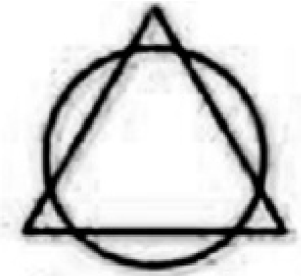
B) 

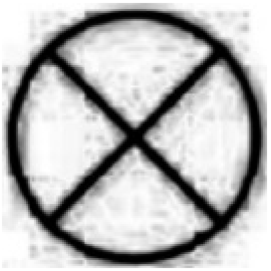
C) 

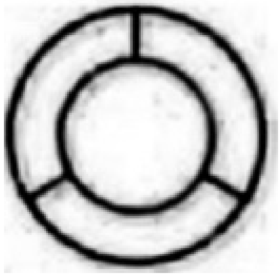
D) 

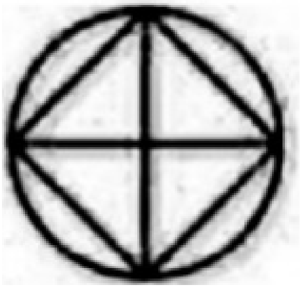
Exercises

- 2. Which of the given graphs has an Euler circuit? (A)

A) 

B) 

C) 

D) 

Exercises

- 5. Which statement is wrong? （ D ）
- A. If a graph has an Euler circuit, it must be a strongly connected graph.
- B. A graph with cut edge cannot have an Euler circuit.
- C. A graph with cut vertex cannot have a Hamilton circuit.
- D. If a directed graph is strongly connected, it must have an Euler circuit.

欧拉图中的欧拉回路删除任意一条边之后仍是一条path，如果图中存在割边，那么删除这条边之后，会形成两个不连通的图。哈密尔顿图同理

Exercises

- 6. Select the **true** statement from the following statements about graph theory. ___C___
- A) A Hamilton graph must be an Euler graph.
- B) An undirected complete graph K_n (n>=3) must be a Euler graph.
- C) A connected undirected graph, with 0 or 2 vertexes having odd degrees, has an Euler path.
- D) A Hamilton graph is a planar graph.

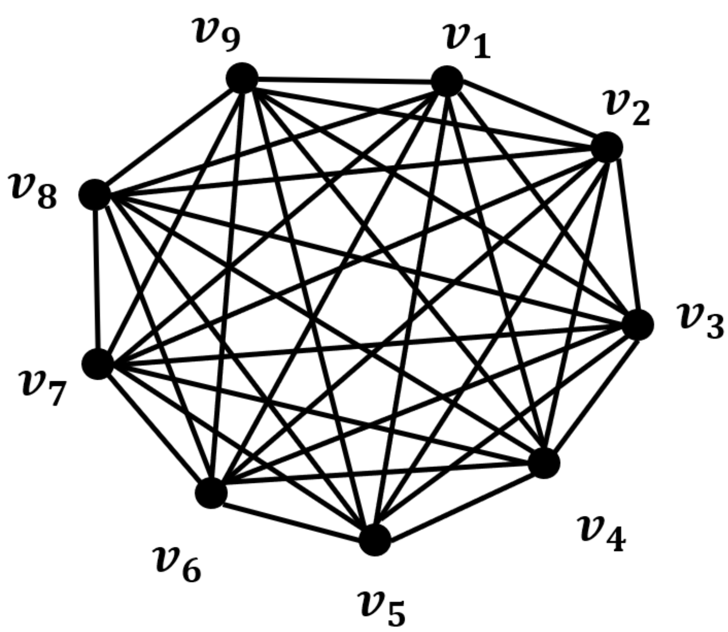
Logo

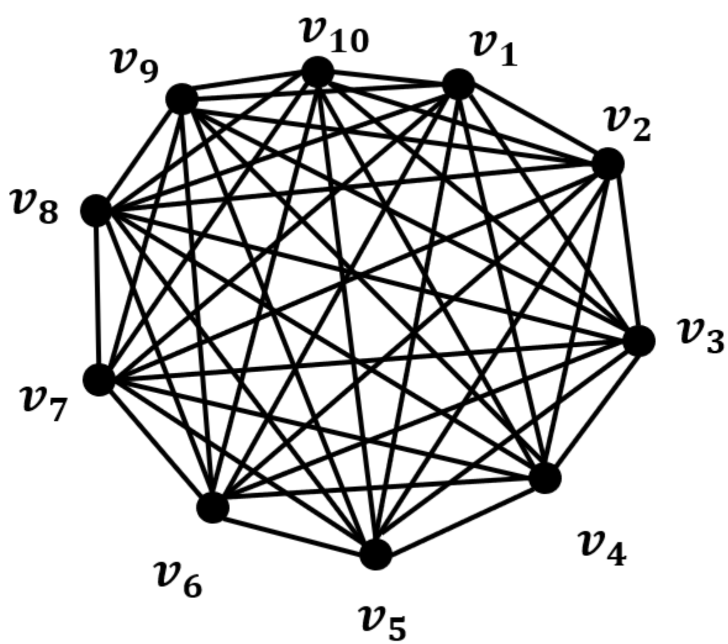
Exercises

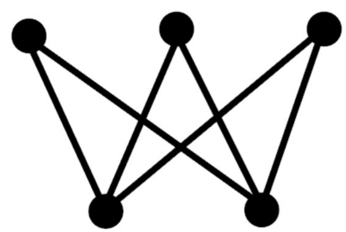
- 7. A connected and nontrivial undirected graph G has an Euler circuit if and only if G （ D ）
- A) has only one odd degree node
- B) has only two odd degree nodes
- C) has only three odd degree nodes
- D) has no odd degree node

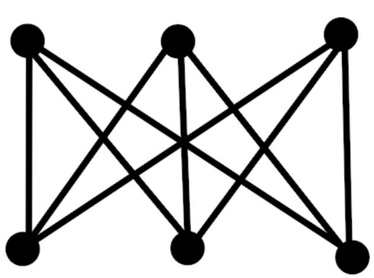
Exercises

- 8. Which graph has both Euler and Hamilton circuits? (A)
- A) $K_{9.}$ B) K_{10}. C) $K_{2,3}$. D) $K_{3,3}$.



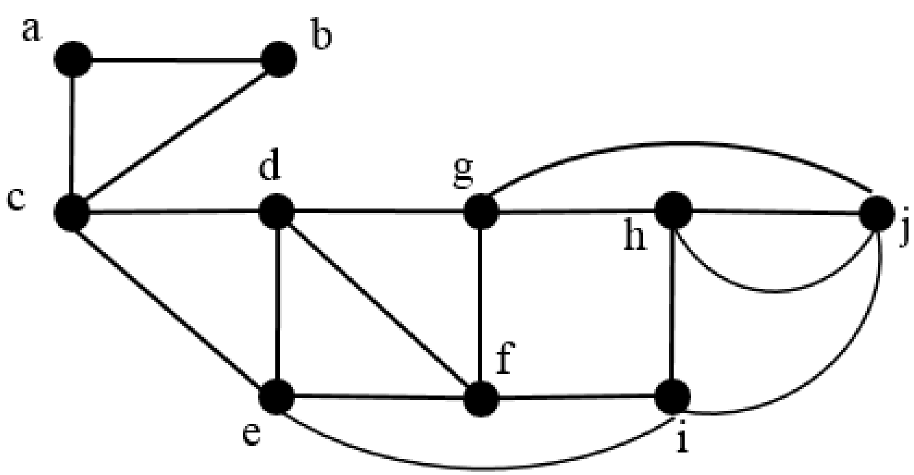






Exercises

- 10. One Euler circuit is ______________ in Figure

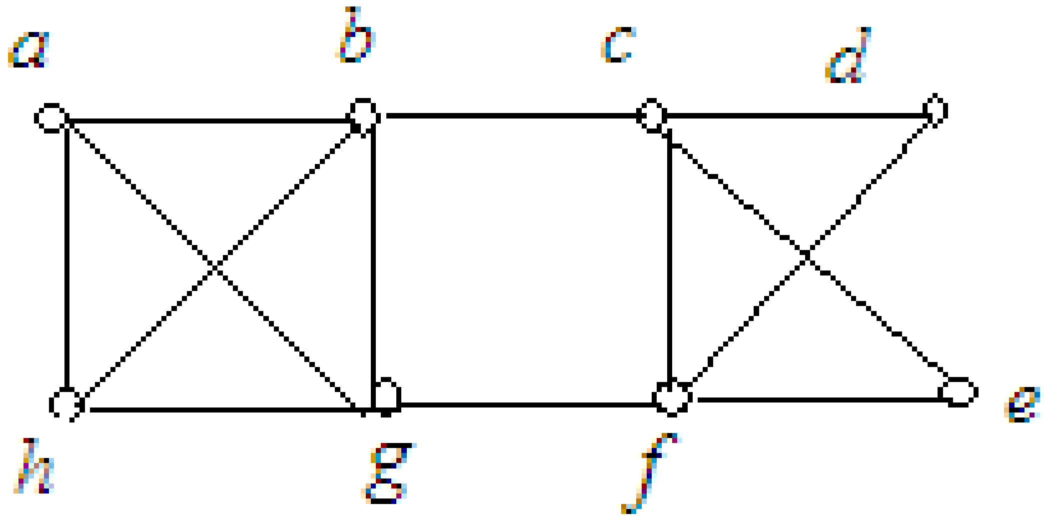


a,c,e,i,j,h,i,f,d,e,f,g,h,j,g,d,c,b,a

Logo

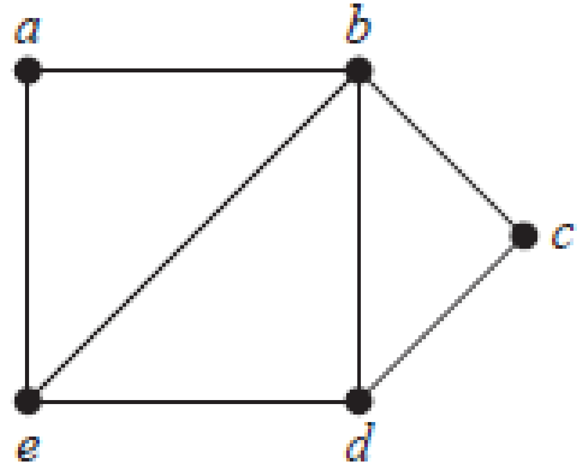
Exercises

- 11. Please find an Euler path in Figure below: ___ *h,a,g,h,b,g,f,e,c,d,f,c,b,a* ___



Exercises

12. Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.



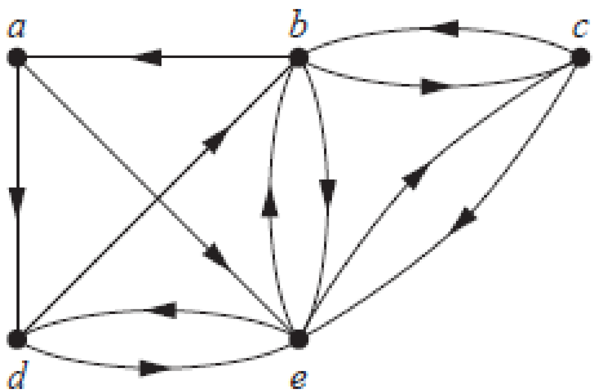
Yes, it does, a, b, c, d, e, a is a Hamilton circuit.

Exercises

- 13. Complete undirected graph K_n is an Euler graph when n is __odd__(odd/even)

Exercises

- 14. Determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path.

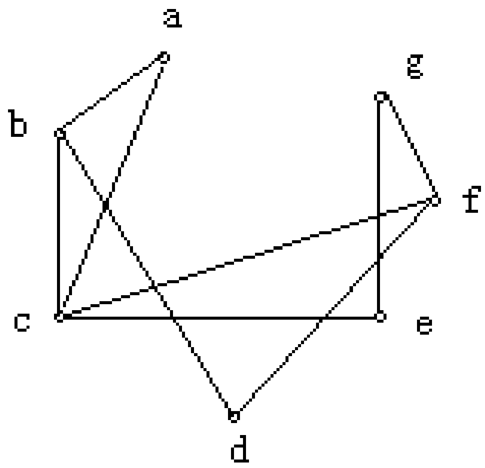


No Euler circuit;， Euler path: a, d, e, d, b, a, e, c, e, b, c, b, e

Exercises

- 15. There are seven people a, b, c, d, e, f, g. The languages they will speak are as follows: a: English, b: Chinese, English, c: English, Spanish, Russian, d: Japanese, Chinese, e: Germany, Spain, f: France, Japan, Russia, g: France, Germany, can you arrange the seats of these seven people at the round table so that everyone can talk to the people next to him?

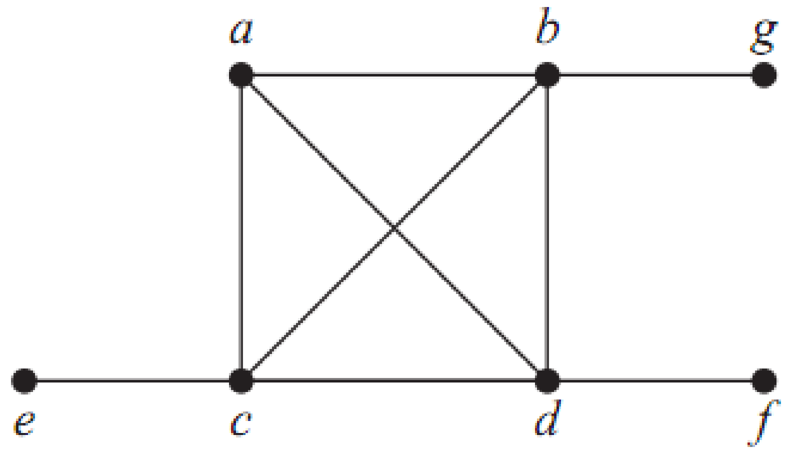
Hamilton circuit: a b d f g e c a



Logo

Exercises

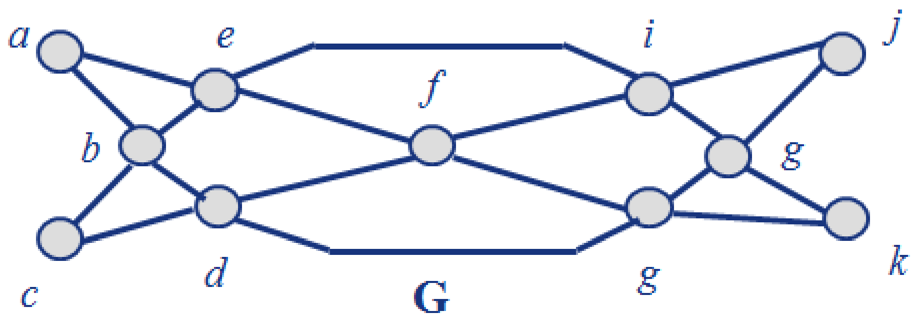
- 16. Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.



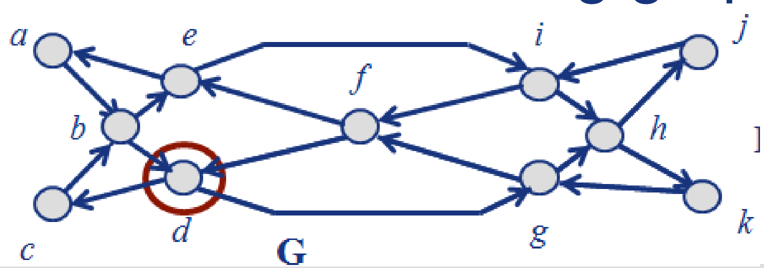
No Hamilton circuit exists, because once a purported circuit has reached e it would have nowhere to go

Exercises

- 17. Can the following graph G be drawn in one stroke? Why? Please present the order of edges if G can be drawn in one stroke.



A connected multigraph has an Euler circuit if and only if each of its vertices has even degree. Thus, G can be drawn in one stroke as the following graph show.



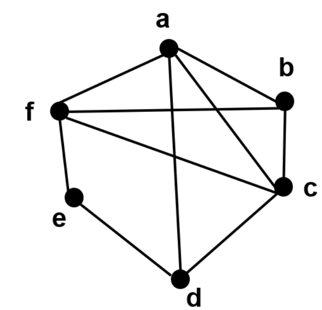
a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a

Exercises

- 18. If M_A is the matrix of an undirected graph. Please draw the graph and judge whether the graph is ① an Euler circuit, ② an Euler path, and ③ a Hamilton circuit. Please give your reason.

M

①It does not have an Euler circuit , because it has 2 vertices of odd degree.
②It has an Euler path, because it has exactly 2 vertices of odd degree.
③It has a Hamilton circuit.



Exercises

- 19. 6 people are supposed to accomplish 3 tasks in groups (2 people in one group). The people in the same group should cooperate with each other to accomplish the task. We now know each person could cooperate with at least other 3 people. Is that possible that all the tasks could be accomplished?
  - Yes. All the tasks could be accomplished.
  - Denote the people by vertices and their cooperative relationships by undirected edges. Hence, we obtain a undirected graph G=<V,E>.
  - For each $v_i \in V$, $d(v_i) \geq n/2$, therefore, G is a Hamilton graph which contains a Hamilton circuit.
  - In the Hamilton circuit, the adjacent vertices can be put into a group to accomplish the task.

Exercises

- 20. For n≥1, the *hypercube* $\mathrm{H_n}$ is a graph $(\mathrm{V_n},\mathrm{E_n})$ with 2^n vertices, constructed as follows. Label the vertices with the integers from 0 to $2^n - 1$; then, if i, j ∈ $\mathrm{V_n}$, then (i, j)∈$\mathrm{E_n}$ if and only if the expansions of i and j in base two differ exactly in one single bit (note that each i ∈ V can be written in base two using exactly n bits). For example, if n = 3, then V = {0, 1, 2, 3, 4, 5, 6, 7}, and (5, 7) is an edge of $\mathrm{H_3}$, because 5 = $(101)_2$ and 7 =$(111)_2$ differ exactly in one bit; similarly, (5, 6) is not an edge of $\mathrm{H_3}$, because 5 = $(101)_2$ and 6 =$(110)_2$ differ in two bits.
- (a) How many edges are there in $\mathrm{H_n}$? $n2^{n-1}$
- (b) Find an n which enables $\mathrm{H_n}$ to have an Euler circuit.

  There is an Euler circuit if and only if n is even.

Exercises

- 21. Suppose G is an undirected simple graph with n vertices and m edges, and $m = \frac{1}{2}(n-1)(n-2)+2$. Prove that G is a Hamilton graph.

证明：证G中任何不相邻两结点度数之和不小于n。
反证法：若存在两结点u，v不相邻且d(u)+d(v)≤n-1,设G1为G删去u,v两结点以及所有与u或v相连的边后的图。则G1为具有n-2个结点的简单图，它的边数m’≥ m – (n-1) = (1/2) * (n-1) * (n-2) +2- (n-1)，所以m’≥(1/2)*(n-2)*(n-3)+1，这与G1是n-2个结点的简单图的题设矛盾，因而G中任何两个相邻的结点度数和不少于n。
所以G为Hamilton图。

Logo

Exercises

- 22. There are 12 people attending a meeting, and each of them has at least 6 friends. These 12 people are surrounded by a round table. Does it make two neighbors of each person are friends? Please answer the question and explain the reason.

- The requirements of the title can be met. The reasons are as follows: 12 nodes are used to represent 12 people on the plane. If the two are friends, then one edge is connected between the corresponding two nodes, and the obtained graph is G. The degree of each node in G is ≥6, so the sum of each pair of nodes in G is ≥12. According to Ore's theorem, G is a Hamilton graph, that is, there is a Hamilton circuit in G, so when 12 people form a table, each person can be friends with two neighbors.

Logo

End of Section 4.5