Logo

Discrete Mathematics

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L o g o

Chapter 4. Graphs

Connectivity

Section 4.4

Logo

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1 Paths

2 Connectedness

3 Paths and isomorphism

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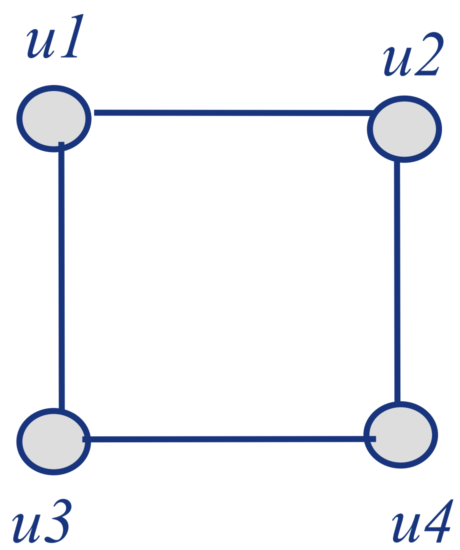
Paths

Paths

- A path is a sequence of edges that begins at a vertex of a graph and travels along edges of the graph, always connecting pairs of adjacent vertices.



Definition 1

- **Let n be a nonnegative integer and G an undirected graph.**
- **A path of length n from u to v in G is sequence of n edges $e_1, \ldots, e_n$ of G**
- **such that $f(e_1) = \{x_0, x_1\}, f(e_2)=\{x_1, x_2\}, \ldots, f(e_n) = \{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n= v$.**

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Definition 1

- When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$.
- The list of the vertices uniquely determines the path.
- The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.

Definition 1

- The path of circuit is said to pass through the vertices $x_1, x_2, \dots, x_{n-1}$ or traverse the edges $e_1, e_2, \dots, e_n$.
- A path or circuit is simple if it does not contain the same edge more than once.
- A path is denoted as $e_1, e_2, \dots, e_n$, where $f(e_i) = \{x_{i-1}, x_i\}$ for $i = 1, 2, \dots, n$, which is not necessary for the multiple edges.
- A path of length zero consists of a single vertex.

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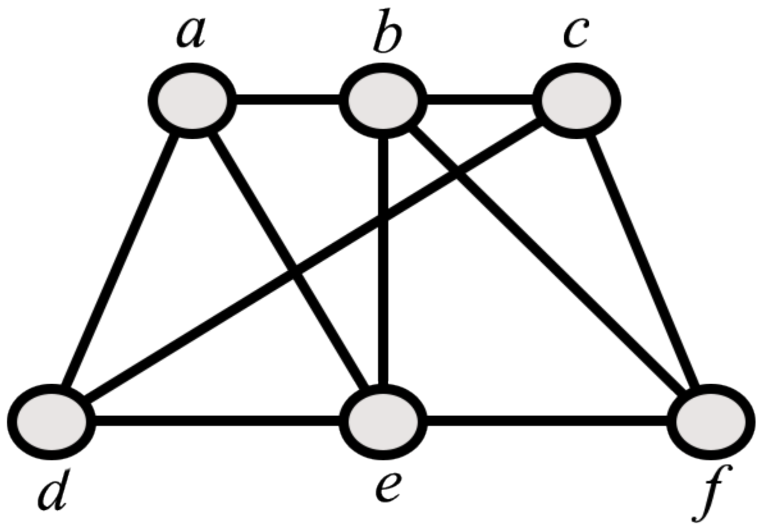
Example 1



- **a,d,c,f,e is a simple path of length 4.**
- **d,e,c,a is not a path.**
- **b,c,f,e,b is a circuit of length 4.**
- **a,b,e,d,a,b is of length 5 but not simple.**

Definition 2

- Let n be a nonnegative integer and G an directed multigraph.
- A path of length n from u to v in G is sequence of n edges $e_1, e_2, \ldots, e_n$ of G
- such that $f(e_1) = (x_0, x_1), f(e_2) = (x_1, x_2), \ldots, f(e_n) = (x_{n-1}, x_n)$, where $x_0 = u$ and $x_n = v$.

Definition 2

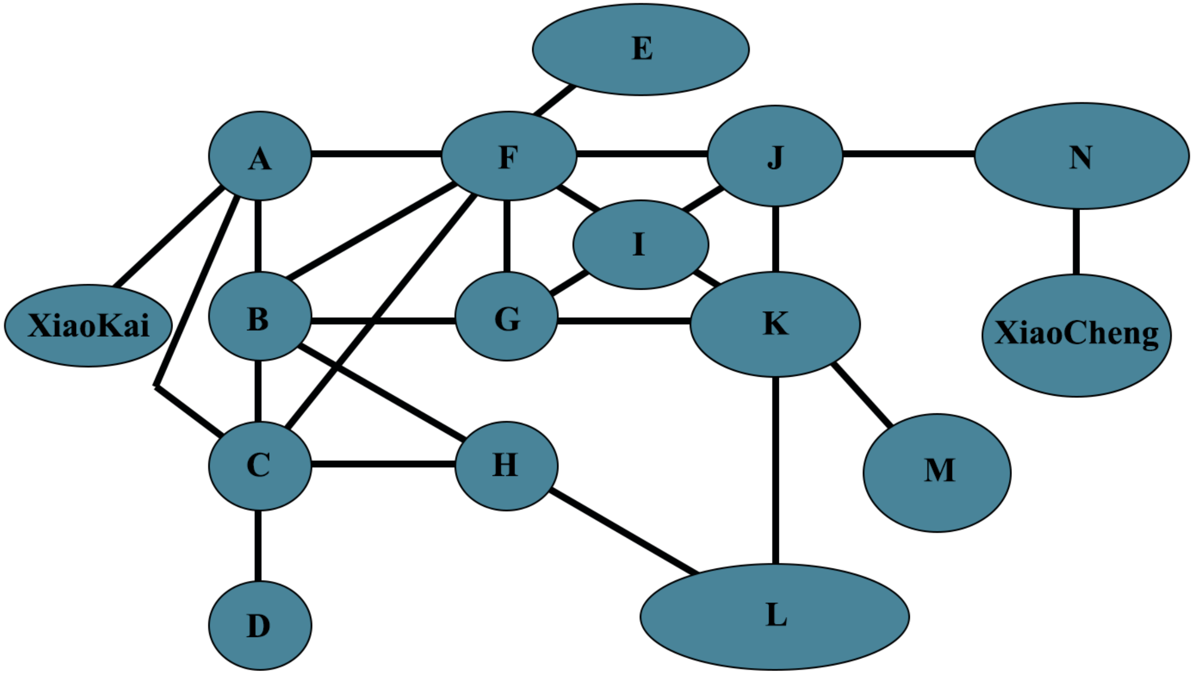
- When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, \ldots, x_n$.
- The path is a circuit or cycle if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.
- A path or a circuit is called simple if it does not contain the same edge more than once.

Definition 2

- The terminal vertex of an edge in a path is the initial vertex of the next edges in the path.
- When it is not necessary to distinguish between multiple edges, we will denote a path $e_1, e_2, \ldots, e_n$ where $f(e_i) = (x_{i-1}, x_i)$ for $i = 1, 2, \ldots, n$ by its vertex sequence $x_0, x_1, \ldots, x_n$.
- The notation identifies a path only up to the vertices it passes through.
- There may be more than one path that passes through this sequence of vertices.

Example 2

- Paths in Acquaintanceship Graphs



- There is a chain of six people linking XiaoKai and XiaoCheng.

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- Many social scientists have conjectured that almost every pair of people in the world are linked by a small chain of people, perhaps contains just 5 or fewer people.
- This would mean that almost every pair of vertices in the acquaintanceship graph containing all people in the world is linked by a path of length not exceeding four.
- John Guare : Six Degree of Separation

Connectedness

An important question

- When is there always a path between two vertices in the graph?
- Definition 3: An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.
- Any two computers in the network can communicate if and only if the graph of this network is connected.

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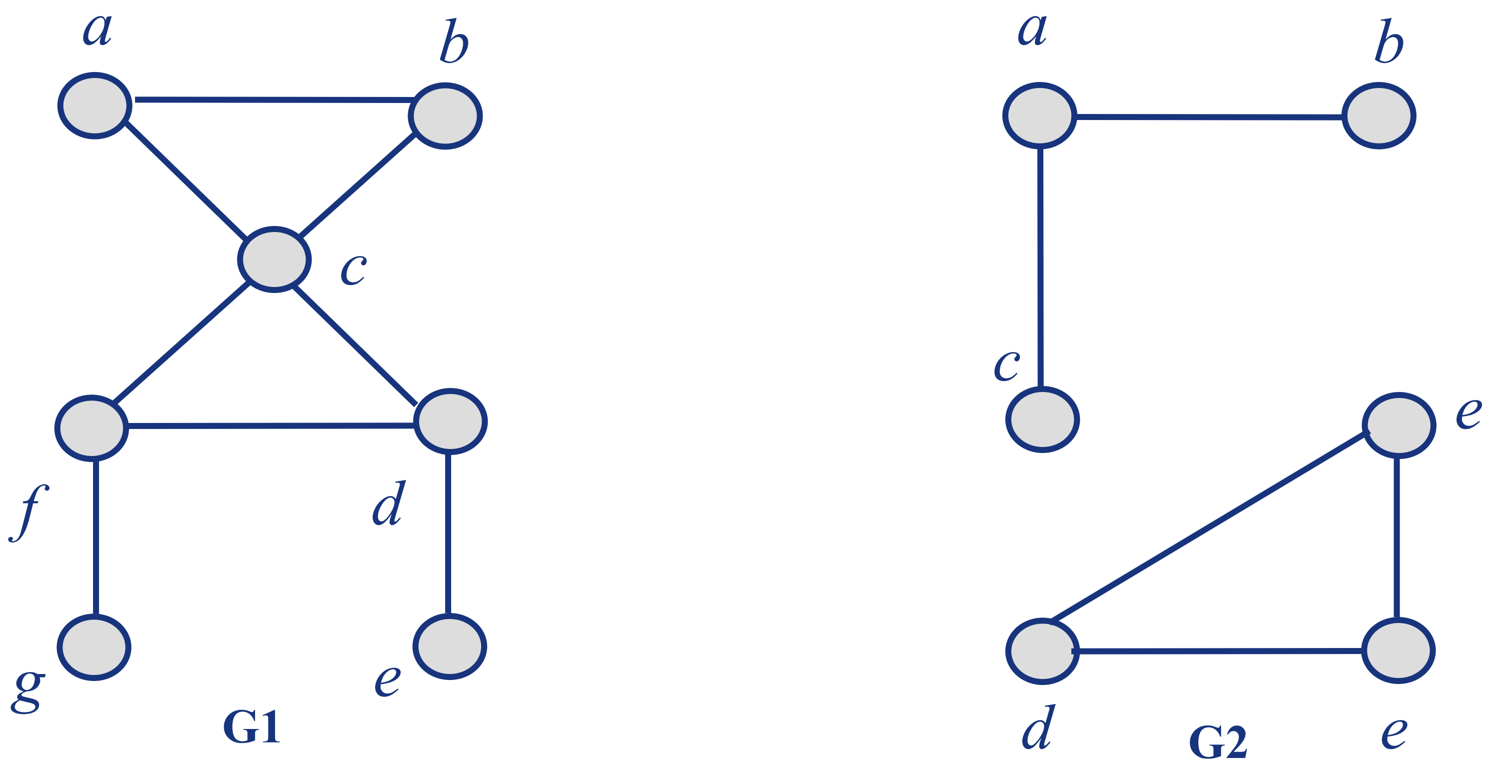
Example 5



- G1 is connected, since every pair of distinct vertices there is a path between them.
- G2 is not connected for there is no path in G2 between a and d for instance.

Theorem 1

- There is a simple path between every pair of distinct vertices of a connected undirected graph.
- Proof: Let u and v be distinct vertices of the connected undirected graph G=<V,E>.
- Since G is connected, there is at least one path between u and v.
- Let $x_0, x_1, \dots, x_n$, where $x_0 = u$ and $x_n = v$, be the vertex sequence of a path of least length.

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- Now we will prove that the path of least length is simple.
- Suppose it is not simple. Then $x_i=x_j$ for some i and j with $0 \leq i<j$.
- This means that there is a path from u to v of shorter length with vertex sequence $x_0,x_1,\ldots x_{i-1},x_j,\ldots,x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i,\ldots,x_{j-1}$.

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- A graph that is not connected is the union of two or more connected subgraphs, each pair of which has no vertex in common.
- These disjoint connected subgraphs are called the connected components of the graph.

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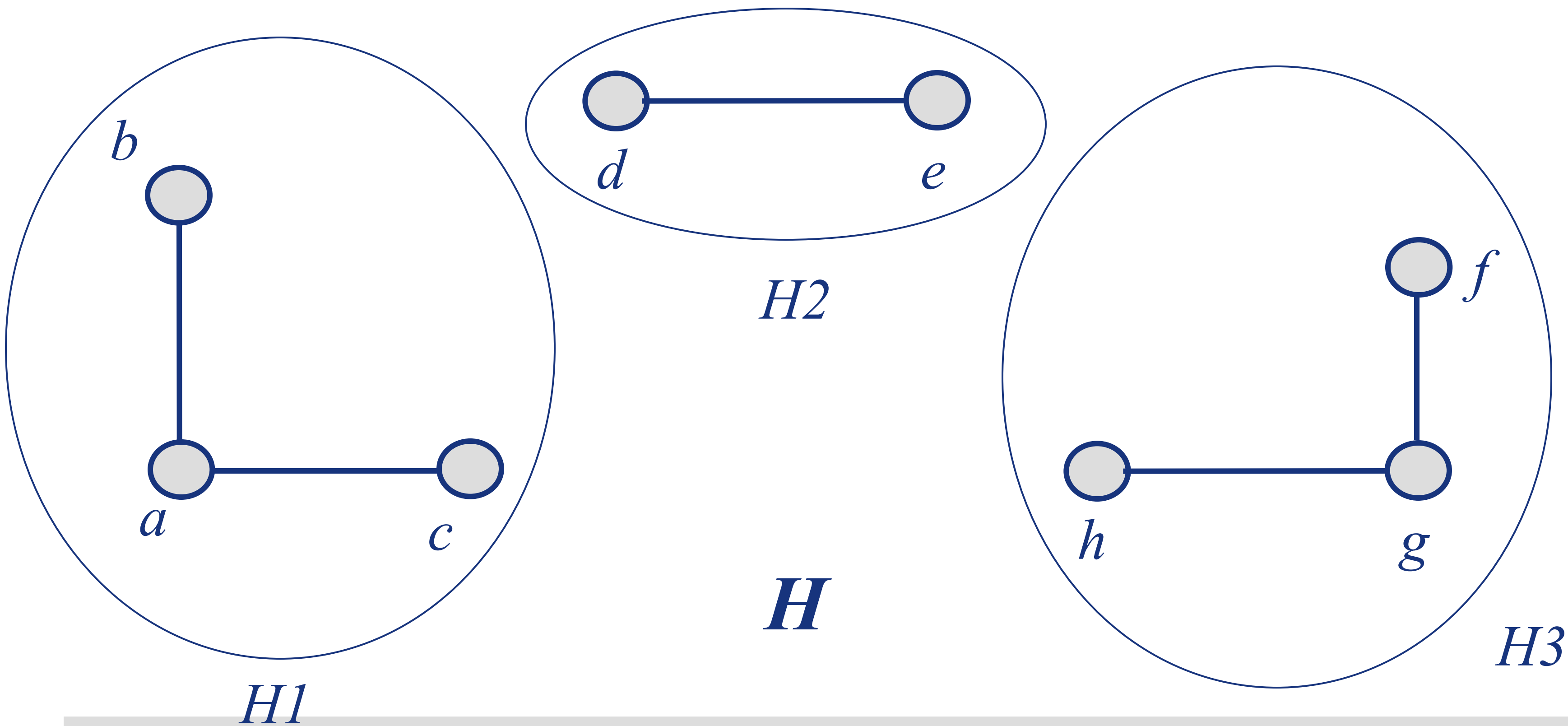
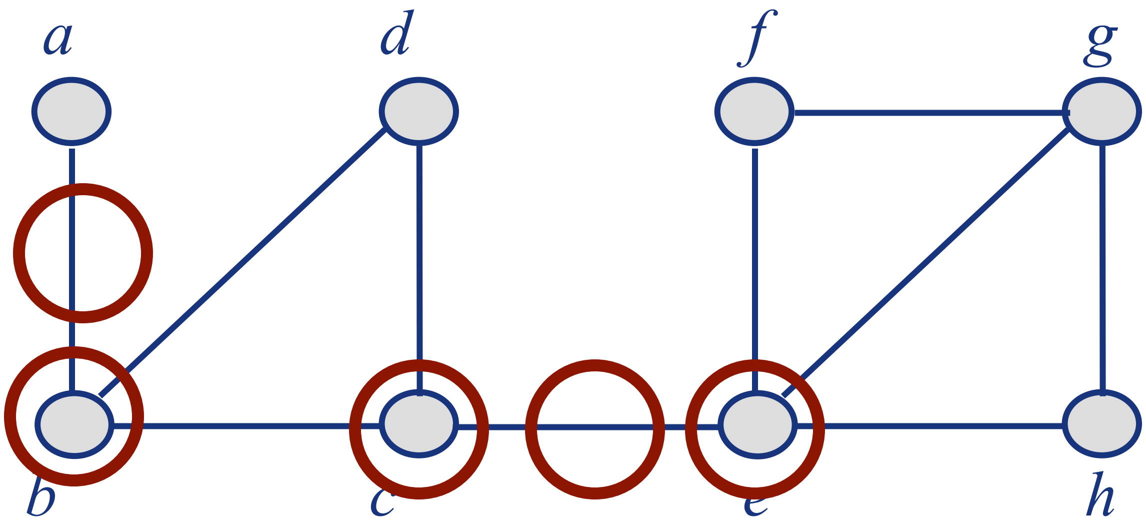
Example 6

- Subgraphs H1, H2 and H3 are the connected components of H.



- The removal of a vertex and all edges incident with it produces a subgraph with more connected components than in the original graph.
- Such vertices are called cut vertices.
- The removal of a cut vertex from a connected graph produces a subgraph that is not connected.

- An edge whose removal produces a graph with more connected components than in the original graph is called a cut edge or bridge.



- The cut vertices of G are b, c and e.
- The cut edges {a, b} and {c, e}.

Connectedness in Directed Graphs

- **When is there always a path between two vertices in the directed graph?**

- **Definition 4: A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.**

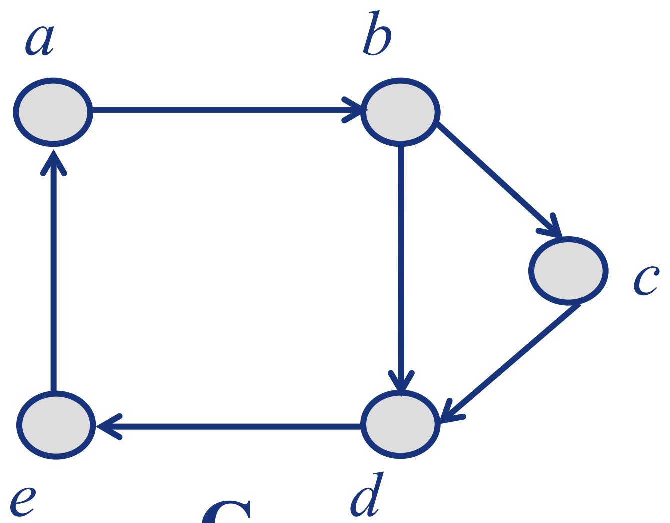
- **A directed graph can fail to be strongly connected but still be in “one piece”.**
- **Definition 5 makes this notion precise.**

- **Definition 5: A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.**

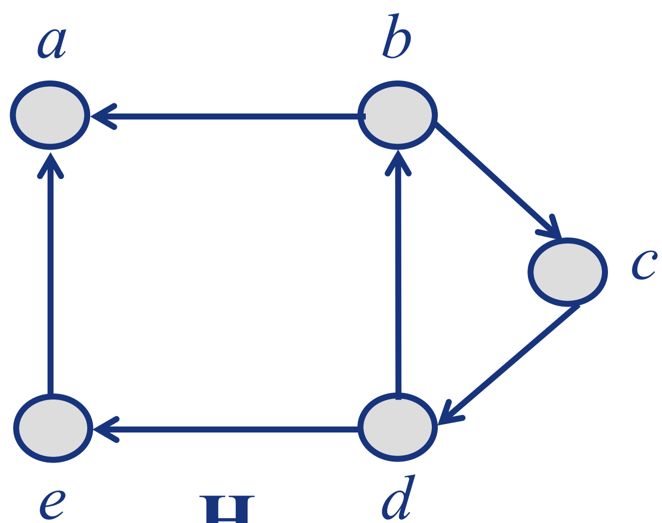
Strong connected and weakly connected

- A directed graph is weakly connected if and only if there is always a path between two vertices when the directions of the edges are disregarded.
- Any strongly connected directed graph is also weakly connected.

Example 9



G



H

- G is strongly connected because there is a path between any two vertices in this directed graph.
- Hence, G is also weakly connected.

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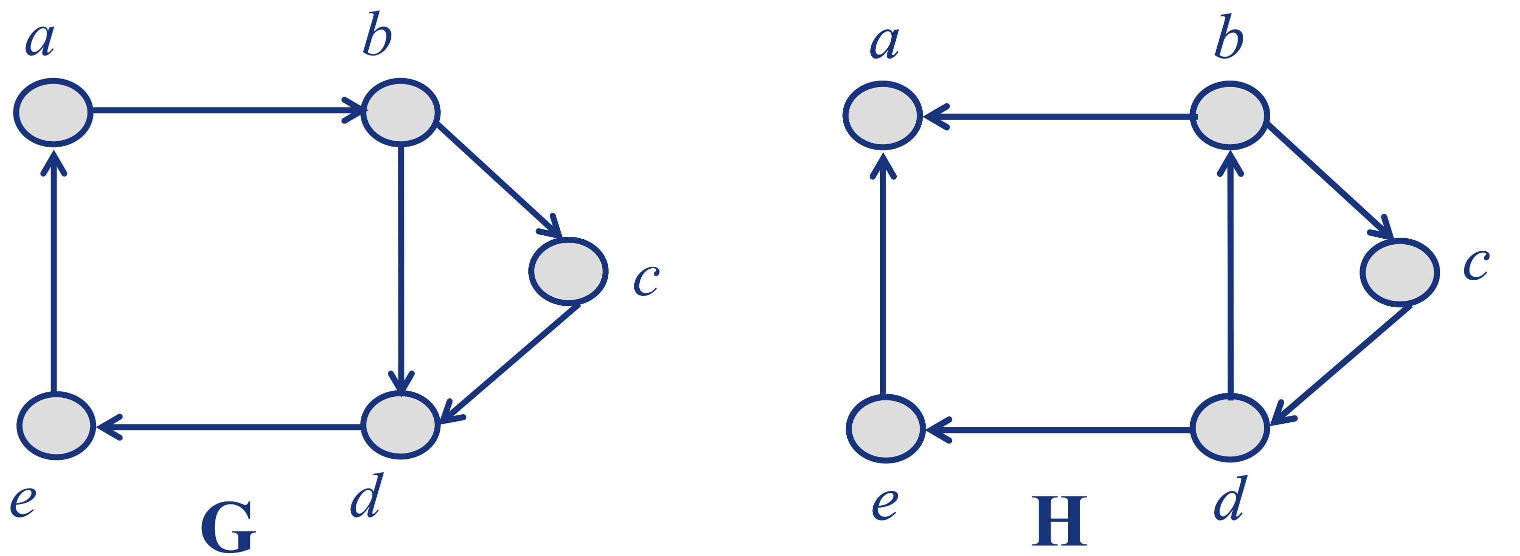
Example 9



- The graph H is not strongly connected. There is no directed path from a to b in this graph.
- However, H is weakly connected, since there is a path between any two vertices in the underlying undirected graph of H.

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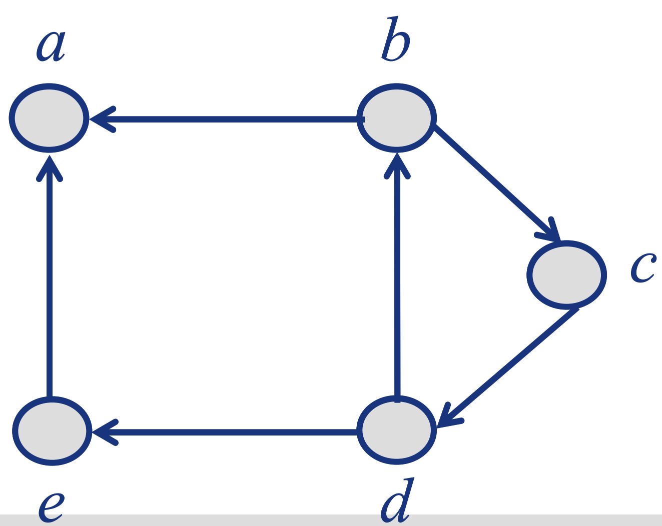
- **The subgraphs of a directed graph G that are strongly connected but not contained in larger strongly connected subgraph, that is,**
- **The maximal strongly connected subgraphs, are called the strongly connected components or strong component of G.**



1、Vertex a

2、Vertex e

3、Graph consisting of vertices b, c and d and edges (b, c), (c, d) and (d, b)

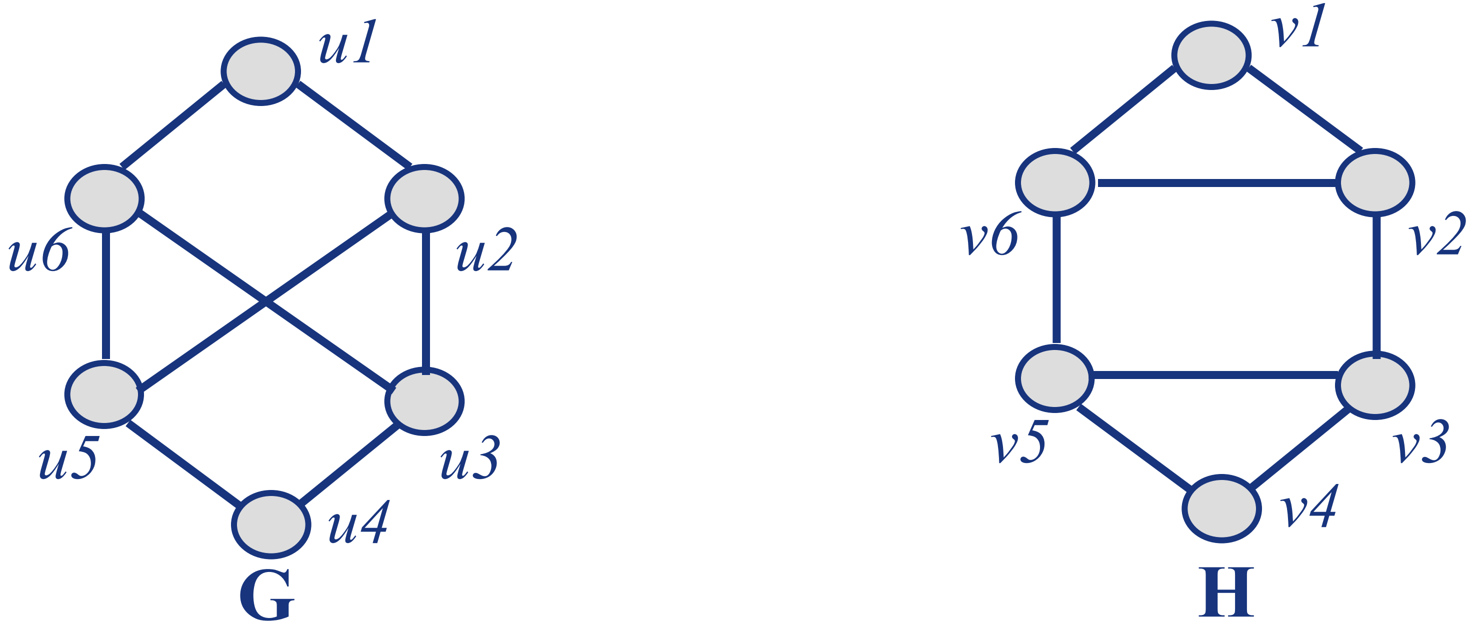
Paths and Isomorphism

- **There are several ways that paths and circuits can help determine whether two graphs are isomorphic.**
- **For example, the existence of a simple circuit of a particular length is a useful invariant that can be used to show two graphs are not isomorphic.**
- **Paths can be used to construct mappings that may be isomorphic.**

- As we mentioned, a useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer greater than 2.

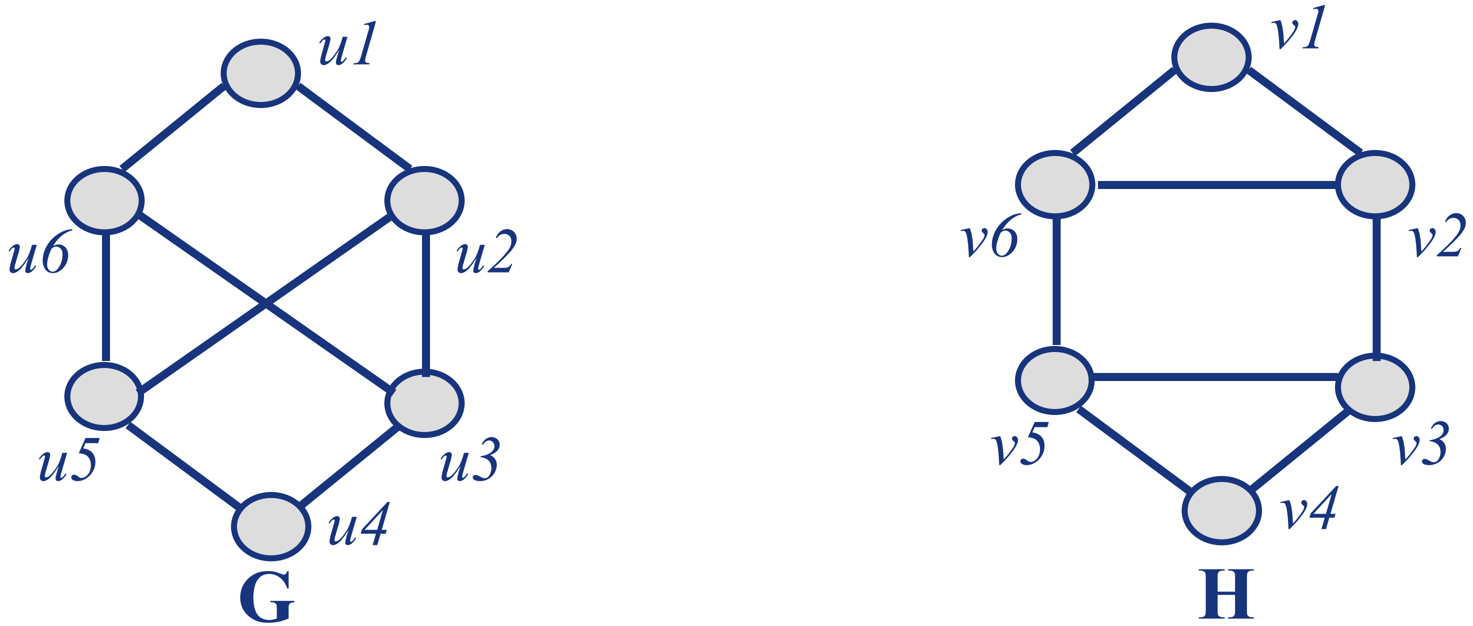
- Example 12 illustrates how this invariant can be used to show that two graphs are not isomorphic.

Example 12



- Six vertices, and eight edges.
- Four vertices of degree three, and two vertices of degree two.
- Three invariants - all agree for two graphs.

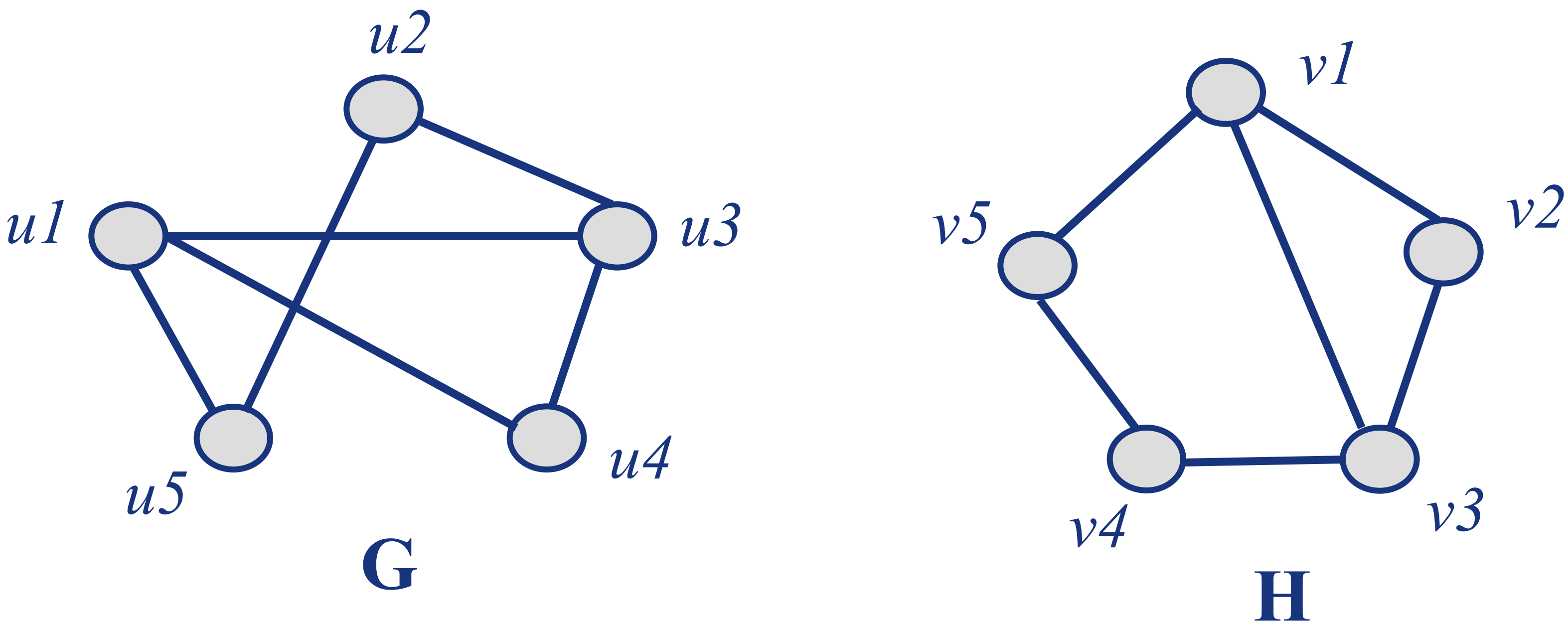
Example 12



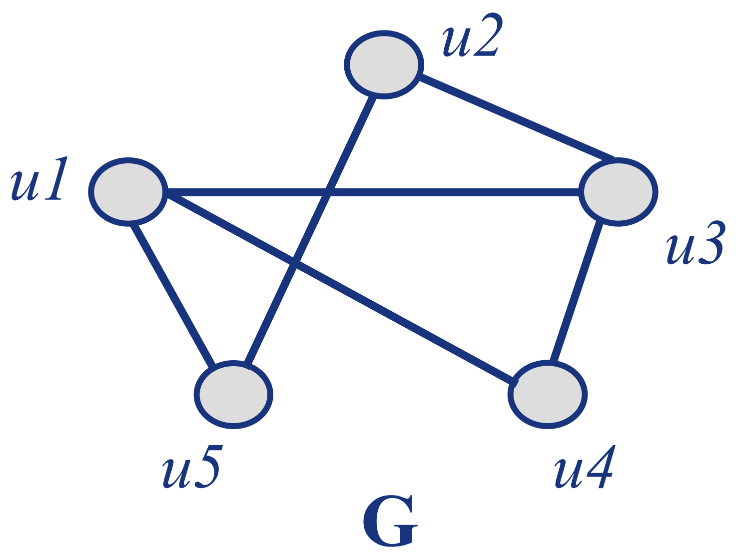
- However, H has a simple circuit of length three, namely, v1,v2,v6,v1
- whereas G has no simple circuit of length three, as can be determined by inspection.

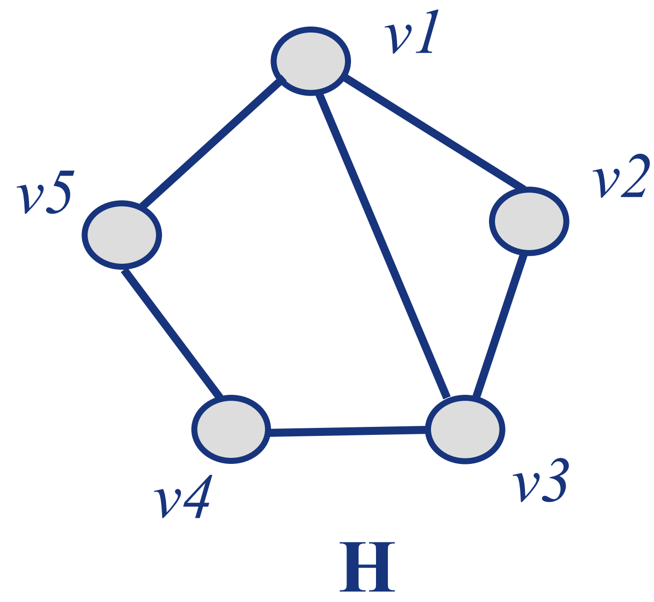
Example

- **We can also use paths to find mappings that are potential isomorphsms.**



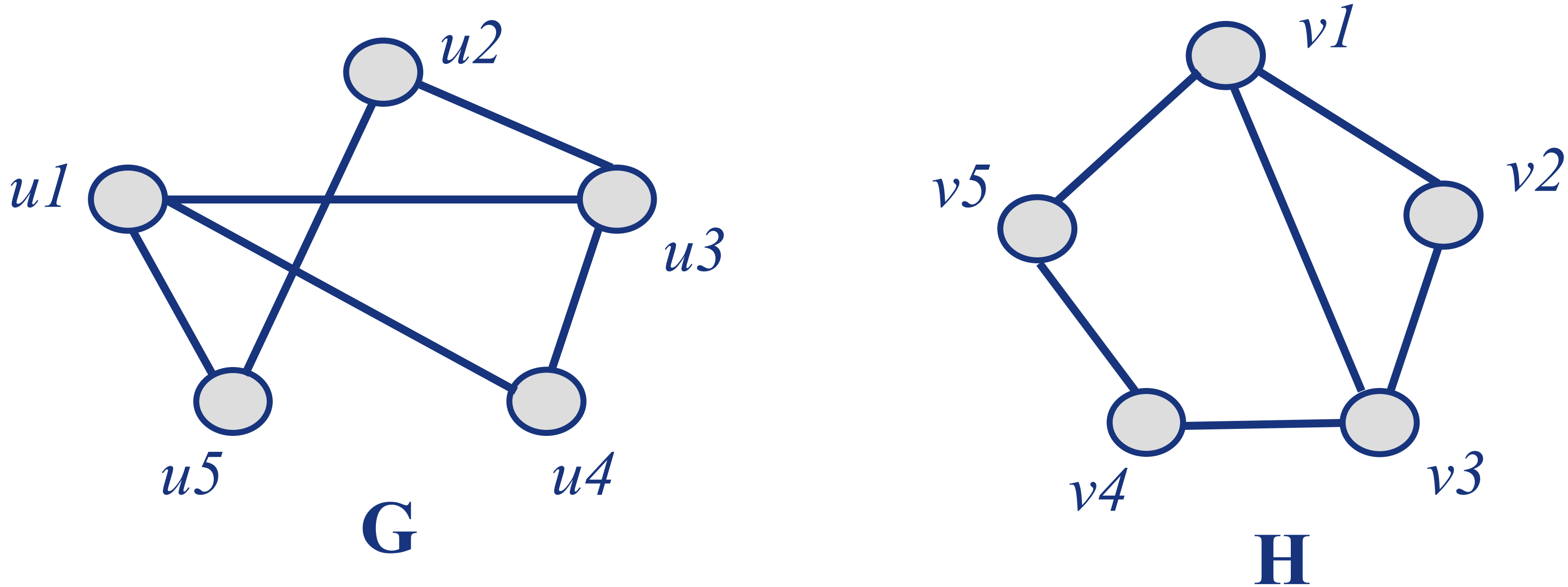
Example 13





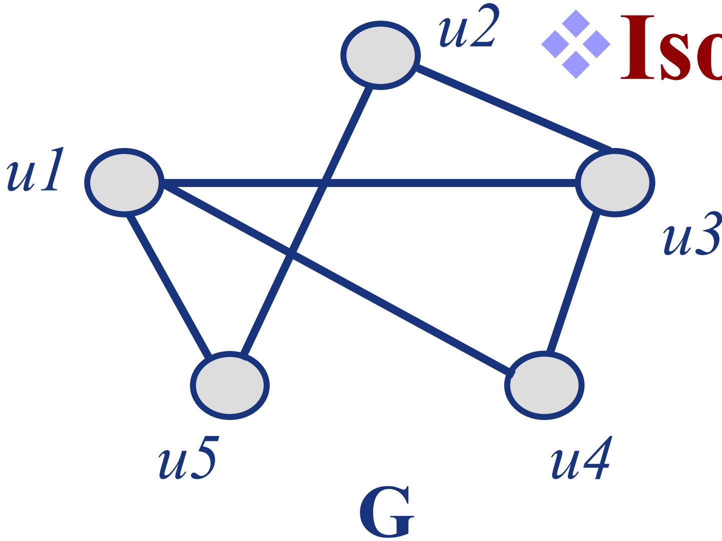
- 5 vertices and 6 edges.
- 2 vertices of degree 3 and 3 vertices of degree 2
- 1 simple circuit of length 3, 1 simple circuit of length 4, and simple circuit of length 5.

Example 13

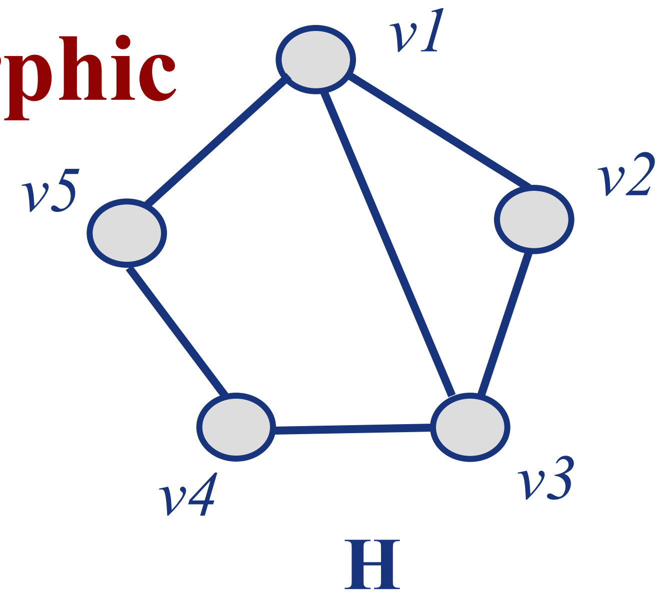


- We follow paths that go through all vertices so that the corresponding vertices in 2 graphs have the same degree.
- u1,u4,u3,u2,u5 in G – v3,v2,v1,v5,v4 in H
- The paths both go trough every vertex in the graph.

❖Isomorphic



G



H

- u1,u4,u3,u2,u5 in G
- v3,v2,v1,v5,v4 in H
- **Degree: 3 2 3 2 2**

$$f(u_1)=v_3, f(u_4)=v_2, f(u_3)=v_1, f(u_2)=v_5, f(u_5)=v_4$$

Count paths between vertices

- The number of paths between two vertices in a graph can be determined using its adjacency matrix.
- Theorem 2: Let G be a graph with adjacency matrix A with respect to the ordering v1,v2,…,vn.
- The number of different paths of length r from vi to vj, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

Logo

- Proof. The number of paths from vi to vj of length 1 is the (i, j)th entry of A, since this entry is the number of edges from vi to vj.

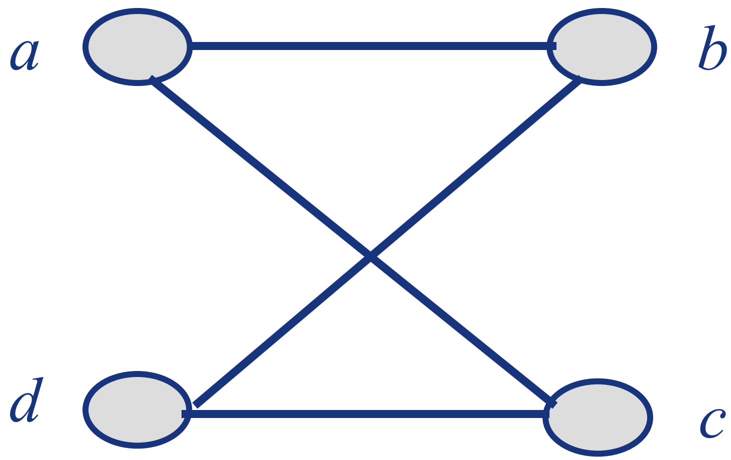
$$\mathbf{A}^{r+1} = \mathbf{A}^{r}\mathbf{A}$$

- The entry of $\mathbf{A}^{r+1}$ equals

$$a_{ij}^{(r+1)} = a_{i1}^{(r)}a_{1j} + a_{i2}^{(r)}a_{2j} + \ldots + a_{in}^{(r)}a_{nj}$$

- $a_{in}^{(r)}$ is the number of the paths of length r from v_i to v_k .

Example 14



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

- **a,b,a,b,d; a,b,a,c,d;**
- **a,b,d,b,d;a,b,d,c,d;**
- **a,c,a,b,d;a,c,a,c,d;**
- **a,c,d,b,d;a,c,d,c,d;**

Applications

Logo

鸡鸣三省大桥

建于鸡鸣三省（云南省镇雄县、四川省叙永县、贵州省毕节市七星关区交界处）的一座桥梁。赤水河和渭河相汇于此，三省分居于悬崖的三侧，此地都是地理死角，交通闭塞，来往极度困难。路线全长1041.3米

Logo

鸡鸣三省大桥

社会意义：

两省人民的百年大梦，建成之后就让两个国家级贫困县告别千百年来隔河相望的历史，加深两省人民的来往。

把鸡鸣三省一带独特的峡谷自然风光和红色旅游资源贯通起来，相互之间合作沟通将更加紧密，大家共同迈向发展新阶段。

川藏铁路

是中国境内一条连接四川省与西藏自治区的快速铁路，呈东西走向，东起四川省成都市、西至西藏自治区拉萨市，是中国国内第二条进藏铁路。川藏铁路东起四川省成都市、西至西藏自治区拉萨市，线路全长1838千米（约1550千米)

川藏铁路

川藏铁路之藏木特大桥

川藏铁路之雅鲁布江特大桥

社会意义：

1. 建设川藏铁路，是促进民族团结、维护中国国家统一、巩固边疆稳定的需要，是促进西藏经济社会发展的需要，是贯彻落实党中央治藏方略的重大举措。*（中共中央总书记、国家主席、中央军委主席习近平 评）*

2. 是西藏自治区对外运输通道的重要组成部分；对于完善西藏铁路网结构、改善沿线交通基础设施条件、促进西藏经济社会发展、增进中华民族团结具有重要意义。

3. 川藏铁路成蒲段作为成都中心城区连接西部县市区的快速铁路通道，增强川西地区交通基础设施建设，促进四川西部、青藏高原东部地区交通不便的城镇和四川省内甘孜、阿坝等少数民族自治州经济社会发展具有十分重要的意义。*（中国铁路总公司 评）*

川藏铁路有多难修？比青藏铁路难修5倍，为什么还要建设？_哔哩哔哩_bilibili

❖北盘江大桥

连接云南省曲靖市宣威市普立乡与贵州省六盘水市水城区都格镇的特大桥。

社会意义：

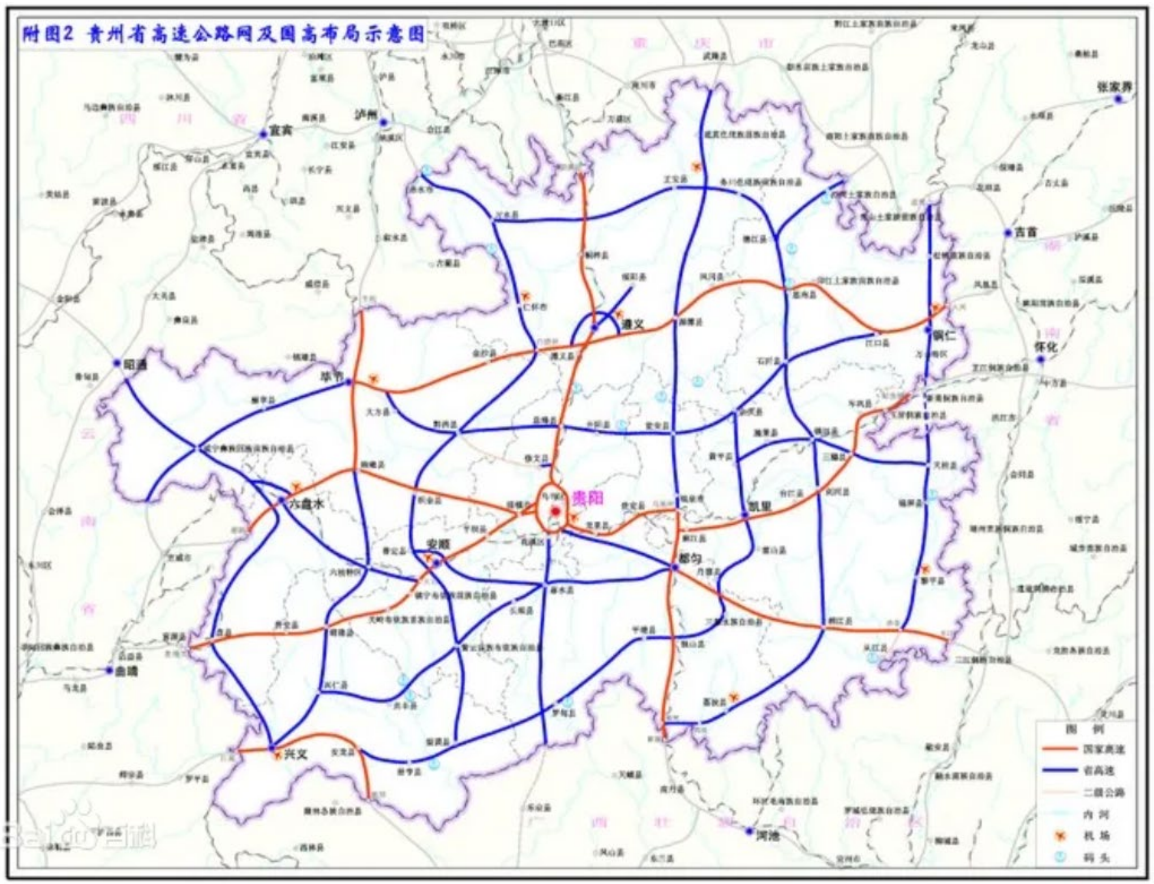
1. 大桥的建成结束了宣威与水城不通高速的历史，两地行车时间从4个多小时缩短至1小时之内。
2. 该桥有效改善云、贵、川、渝等地与外界的交通状况、提高区域路网服务水平、充分发挥高速公路辐射带动效应、促进地方社会经济发展，为中国国家“一带一路”战略添上了浓墨重彩的一笔。*（宣威市人民政府 评）*

播放视频

❖贵州省县县通高速公路

贵州省是西南地区的交通枢纽。地形特殊，主要有高原、山地、丘陵，以及盆地四种地形，其中山地和丘陵占据了全省总面积的92.5%。

“十三五”贵州省高速公路总里程将超过7000公里，基本建成“678”高速公路网



❖贵州省县县通高速公路

中心聚集、多级辐射、互联互通、覆盖广泛、能力充分、衔接顺畅

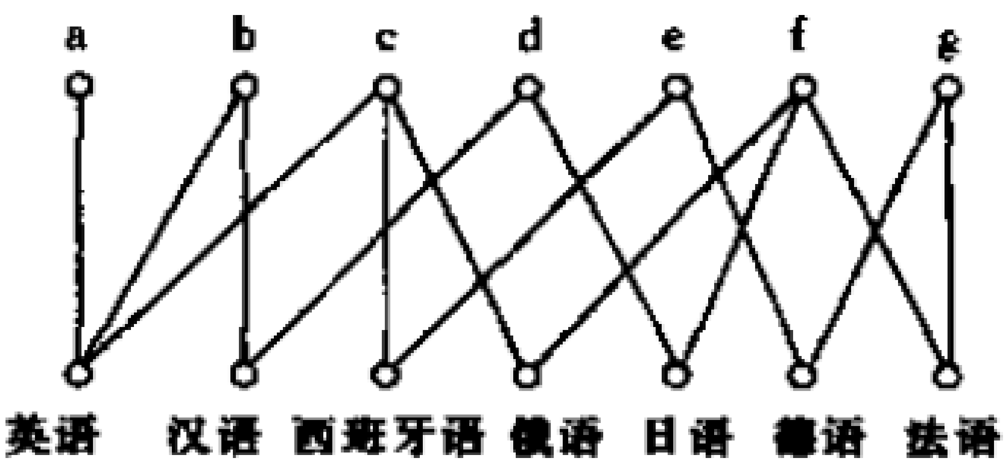
播放视频

Applications

1. 设有a,b,c,d,e,f,g七个人，他们分别会讲的语言如下：a会讲英语；b会讲汉语和英语；c会讲英语、西班牙语和俄语；d会讲日语和汉语；e会讲德语和西班牙语；f会讲法语、日语和俄语；g会讲法语和德语。试问这七个人中，是否任意两个都能交谈（必要时可借助其他人的翻译）

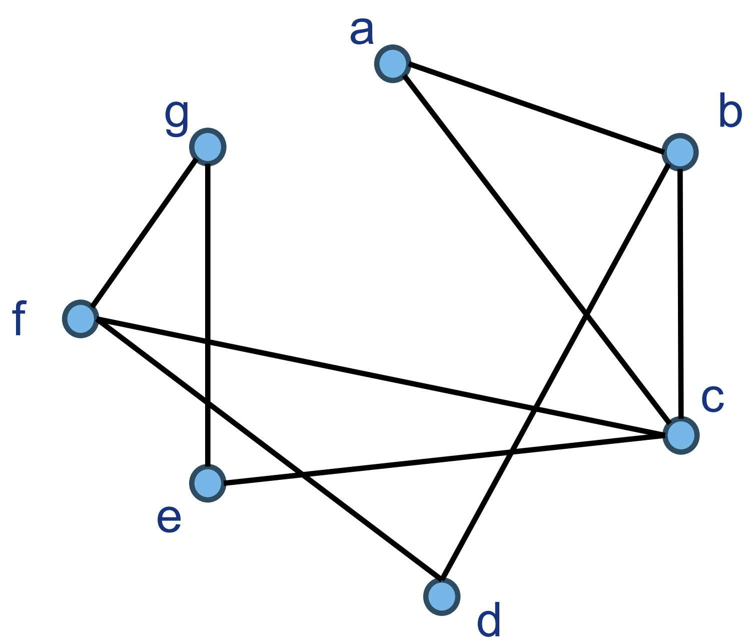
Applications

解 我们分别用结点表示将七个人和七种语言，若某人会讲某种语言，则用一条无向边将它们连接起来，则上述问题就转化为判断图 4.2-2 所示的无向图是否为连通图。显然，该为连通图，故他们七个人中任意两个都能交谈。



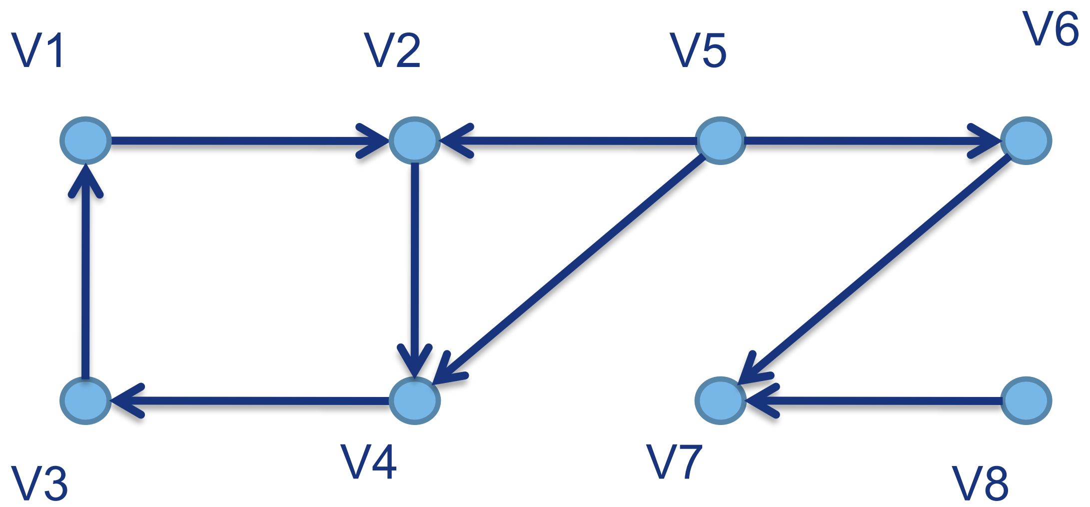
Applications

解法2：若两人之间存在共同语言，则用一条无向边将它们连接起来，如下图所示。由于该图为连通图，因此他们七个人中任意两个都能交谈。



Applications

2.求下图G最大的强连通分支和最大弱连通分支



解：由结点集合｛v1,v2,v3,v4｝所导出的子图为该图的最大强连通分支，图G自身就是该图最大弱连通分支。

Exercises

1. How many cut edges does the complete bipartite graph $K_{6,10}$ have ? (A).

A) 0. B) 6. C) 10. D) 16.

Exercises

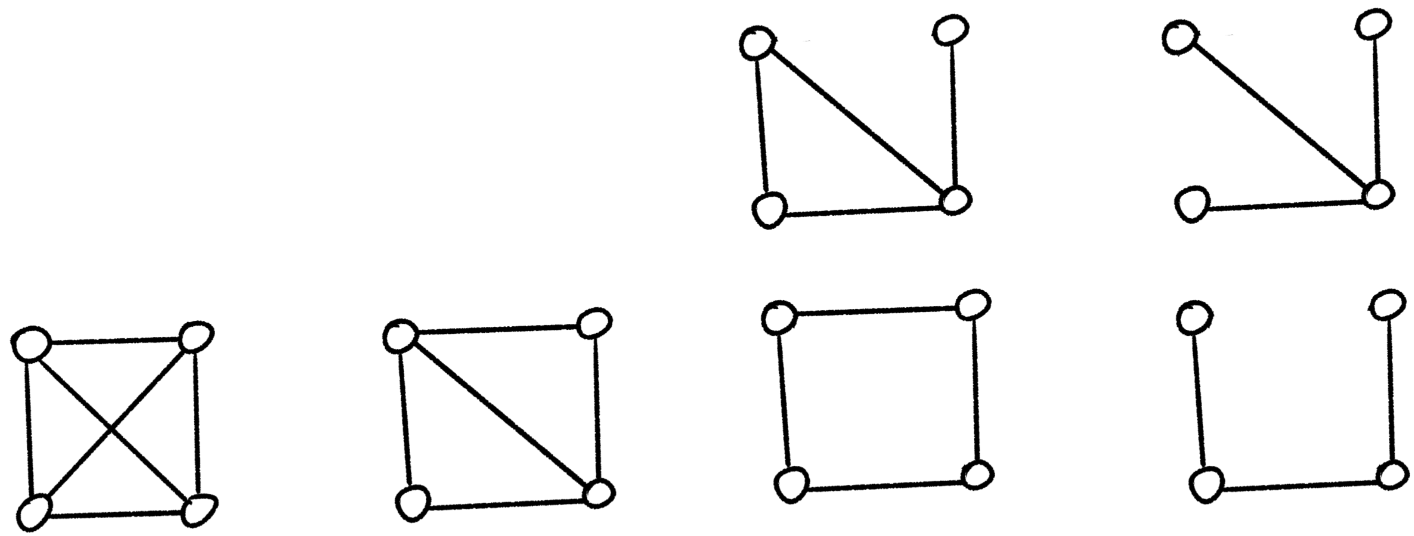
- 2. Find the number of paths of length n between two different vertices in K_4 if n is 4
- 20

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{bmatrix}$$

Exercises

- 3. How many nonisomorphic connected simple graphs with 4 vertices? （ B ）
- A. 1　　B. 6　　C. 21　　D. 2



$|E|=6$　　$|E|=5$　　$|E|=4$　　$|E|=3$

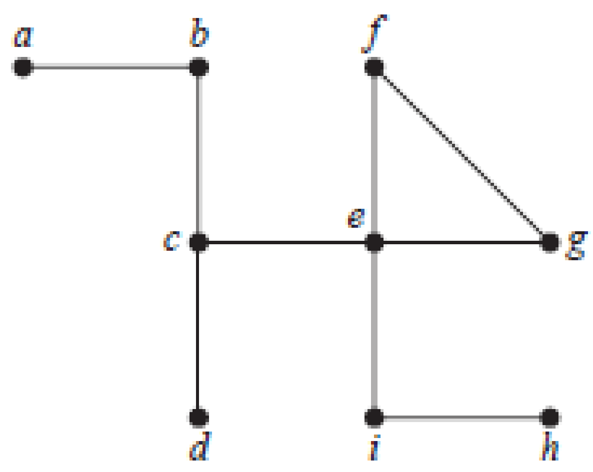
全连接

Logo

Exercises

4. Which is not the cut vertice of the given graph.(D)

A. b B. e C. i D. f



Exercises

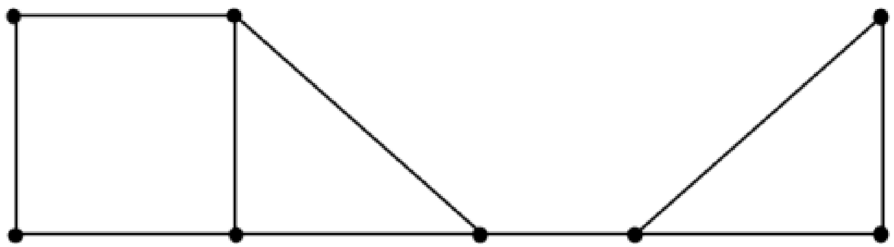
5. How many cut vertices does the graph K_5 has?(A)

A. 0　　B. 1　　C. 2　　D. 3

Exercises

6. How many cut edges are there in Figure 1?(B)

A. 0 B. 1 C. 2 D. 3



Logo

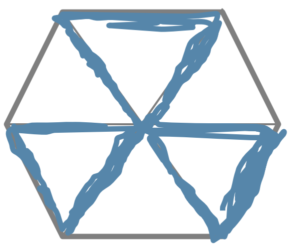
Exercises

- 7. The length of the longest simple circuit in Wn is $\left\lceil \frac{3n-1}{2} \right\rceil$

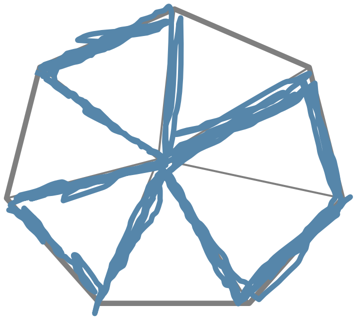
解析：首先分奇偶讨论，对于n为偶数的情况存在一种方案是由n/2个三角形组成的回路，此时边数为3n/2。对于n为奇数（n＞3）的情况，则存在一种方案是由（（n-1）/2）-1个三角形与一个四边形组成的回路，此时边数为$\frac{3n-1}{2}$。因此合并得$\left\lceil \frac{3n-1}{2} \right\rceil$。

对于上述方案即为最长simple circuit的证明思路如下：

首先对于simple circuit 而言，其一定是一个欧拉回路（在后续图论知识中将有介绍），因此simple circuit 中每个点的度一定为偶数。因此只需证明如果Wn中存在一条路径长大于$\left\lceil \frac{3n-1}{2} \right\rceil$，则这条路径中一定会有奇数度即可。



W6 最长回路方案（蓝色）



W7 最长回路方案（蓝色）

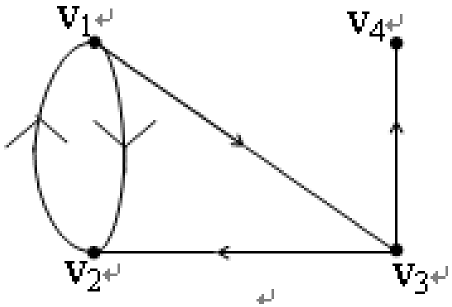
Exercises

8. Given a directed graph D on the right side, please find out:

① the adjacency matrix A of D

② How many paths of length 2 in D ?

③ How many simple circuits in D ?



Solution: ① $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

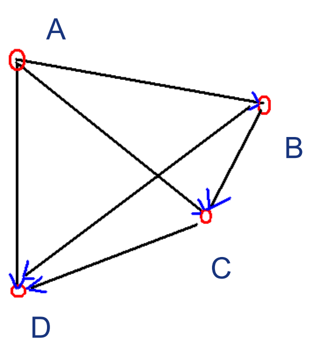
②$A^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 6 paths of length 2.

③ 2 circuits.

Exercises

- 9. Assume that four teams A, B, C, D attend a football match, please use directed graph (G) to record the match result. All the following questions are based on the directed graph G.
- 1) Team A won Team B. 2) Team A won Team C. 3) Team A won Team D. 4) Team B won Team C. 5) Team B won Team D. 6) Team C won Team D.
- a. Please give the directed graph.
- b. Please give the adjacency matrix of directed graph G, and calculate the number of paths with less or equal 6 edges. (Not include cycle.)
- (Note: The winner is the start vertex of the directed graph G, while the loser is the end of directed graph G)

a



b
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

通路长度小于或等于6的条数为: 11

长度为1的通路：AB, AC, AD, BC, BD, CD
长度为2的通路：ABC, ABD, ACD, BCD
长度为3的通路：ABCD

Exercises

10. Prove that the undirected graph is a connected graph or its complementary graph is a connected graph.

- 如果图G(V,E)不连通的话，它的顶点可以分为两个非空集合A,B，其中对于任意在A中的点P和任意在B中的点Q都没有PQ这条边。
- 这样的话，取其补图G'，则对于任意在A中的点P和任意在B中的点Q都有PQ这条边。这样的话，对于任意两点P,Q，如果它们分别处于A,B的话，它们之间就有边相连；否则，不失一般性设它们都在A中，由于B非空，我们可以在B中任取一点R，我们知道PR和QR这两条边都是存在的，所以P,Q是连在一起的。
- 综上，知G'连通。

Logo

Exercises

11. If a undirected graph G only has two nodes with odd degree, these two nodes must be connected. Please prove the proposition above.

- 若无向图G中只有两个奇数度结点，则这两个结点一定连通。
- 证明：设G中两奇数度结点分别为u 和v，若 u，v不连通，则G至少有两个连通分支G_1、G_2，使得u和v分别属于G_1和G_2，于是G_1和G_2中各含有1个奇数度结点，这与图论基本定理矛盾，因而u，v一定连通。

Logo

Exercises

12. n cities are connected by k roads. A road is incident with only two cities, which is defined as an edge between two vertices (cities). A property of the roads and cities is $k>(n-1)(n-2)/2$. The question is whether people can travel between any two cities through the roads.(there is a road between two cities at most)

1. Supposed that a graph of the given n cities and k roads is G, the question can be considered to be the proof that G is connected.
2. Given simple graph $G=(V,E)$, $|V|=n$ cities and $|E|=k$ roads, please prove that G is connected when $k>(n-1)(n-2)/2$.
3. Supposed that G is unconnected, G has at least 2 connected component, signed as $G1=(V1, E1)$ and $G2=(V2, E2)$, where $|V1|=n1$, $|V2|=n2$ and $n1+n2=n$.
4. Because G is a simple graph, $|E1|<=n1(n1-1)/2$ and $|E2|<=n2(n2-1)/2$.
5. Therefore, $k<=n1(n1-1)/2+n2(n2-1)/2$.
6. Since $n1<=n-1$ and $n2<=n-1$, $k<=(n-1)(n1-1+n2-1)/2=(n-1)(n-2)/2$.
7. This is contradictory to the condition that $k>(n-1)(n-2)/2$.
8. Thus, G is connected, so people can travel between any two cities through the roads.

Logo

End of Section 4.4