Logo

Discrete Mathematics

Dr. Han Huang

South China University of Technology

Logo

Chapter 3. Relations

Partial Orderings

Section 3.6

Logo

Contents

Logo

Introduction

Logo

Definition 1

- A relation **R** on a set **S** is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive.
- A set **S** together with a partial ordering **R** is called a partial ordered set, or poset, and is denoted by **(S,R)**.

Logo

Examples

- 1、 “greater than or equal” relation (≥) is partial ordering on the set of integer.
- 2、 “divisibility relation” (|) is a partial ordering on the set of positive integer.
- 3、”inclusion relation” ($\subseteq$) is a partial ordering on the power set of a set S.

Examples

调度问题是偏序关系应用的经典实例，一般性的调度问题可以描述如下：

给定有穷的任务集T和m台相同的机器，T上存在偏序关系$\prec$，如果$t_1 \prec t_2$，那么任务t_1完成以后t_2才能开始工作。$\forall t \in T, l(t)$表示完成任务t所需要的时间，$d(t)$表示任务t的截止时间，$l(t), d(t) \in Z^+$。设开始时间为$0, \sigma: T \to \{0,1,\cdots\}$表示对任务集$T$的一个调度方案，其中$\sigma(t)$表示任务$t$的开始时间。$D = \max\{\sigma(t) + l(t) \mid t \in T\}$表示完成所有任务的最终时间。假设每项任务都可以安排在任何一台机器上进行加工，如果σ满足下述三个条件，则称T为可行调度。

（1） $\forall t \in T, \sigma(t) + l(t) \le d(t)$

（2） $\forall i, 0 \le i \le D, |\{t \in T \mid \sigma(t) \le i < \sigma(t) + l(t)\}| \le m$

（3） $\forall t, t' \in T, t \prec t' \Rightarrow \sigma(t) + l(t) \le \sigma(t')$

Logo

Definition 2

- The elements a and b of a poset $(S, \preccurlyeq)$ are called comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
- When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called incomparable.

- In poset (Z+, |) are the integers 3 and 9 comparable? Are 5 and 7 comparable ?

Logo

Definition 3

- If (S, $\preccurlyeq$) is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order.
- A totally ordered set is also called a chain.
- (Z,$\leq$) is totally ordered, but (Z^+, $|$) is not.

Definition 4

- (S, $\preccurlyeq$) is well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and such that every nonempty subset of S has a least element.

- ($Z+$, $\leq$) is well ordered.

Theorem 1

- The principle of well-ordered induction

  Suppose that S is well-ordered set. Then P(x) is true for all $x \in S$, if:

BASIS STEP: P(x0) is true for the least element of S, and

INDUCTION STEP: For every $y \in S$ if P(x) is true for all $x < y$, then P(y) is true.

Lexicographic Order and Hasse Diagrams

Logo

The lexicographic ordering

- $(a1, a2) \prec (b1, b2)$
- Either if $a1 < b1$
- or if both $a1 = b1$ and $a2 < b2$

- $(3,5) \prec (4,8)$ $(3,8) \prec (4,5)$ $(4,9) \prec (4,11)$
- $(1,2,3,5) \prec (1,2,4,3)$
- discreet $\prec$ discrete
- discreet $\prec$ discreetness

Logo

Lexicographic Order

- (1,7) (2,7) (3,7) (4,7) (5,7) (6,7) (7,7)
- (1,6) (2,6) (3,6) (4,6) (5,6) (6,6) (7,6)
- (1,5) (2,5) (3,5) (4,5) (5,5) (6,5) (7,5)
- (1,4) (2,4) (3,4) (4,4) (5,4) (6,4) (7,4)
- (1,3) (2,3) (3,3) (4,3) (5,3) (6,3) (7,3)
- (1,2) (2,2) (3,2) (4,2) (5,2) (6,2) (7,2)
- (1,1) (2,1) (3,1) (4,1) (5,1) (6,1) (7,1)

Logo

Hasse Diagrams

- For instance, if (a,b) and (b,c) are in the partial ordering, remove (a,c).
- Furthermore, if (c,d) is also in the partial ordering, remove the edge (a,d).
- Arrange each edge so that its initial vertex is below its terminal vertex.
- Remove all the arrows on the directed edges, since all edges point “upward” their terminal vertex.

Logo

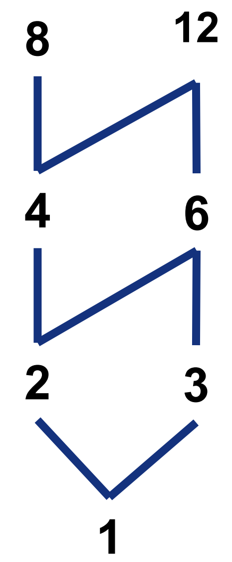
Examples

- The partial ordering { (a,b) | a divides b }
- { 1,2,3,4,6,8,12 }

Examples

- The partial ordering { (a,b) | a divides b }
- { 1,2,3,4,6,8,12 }

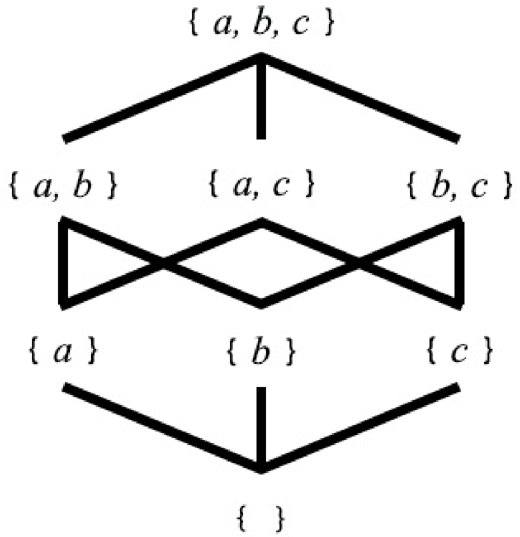
- (1,2) (1,3) ~~(1,4)~~ ~~(1,6)~~ ~~(1,8)~~ ~~(1,12)~~
- (2,4) (2,6) ~~(2,8)~~ ~~(2,12)~~
- (3,6) ~~(3,12)~~
- (4,8) (4,12)
- (6,12)



Logo

Examples

- The partial ordering { (A,B)| A $\subseteq$ B } on the power set P(S) where S={ a,b,c }
- ~~(Ø,{a,b})~~ ~~(Ø,{a,c})~~ ~~(Ø,{b,c})~~ ~~(Ø,{a,b,c})~~
- ~~({a},{a,b,c})~~ ~~({b},{a,b,c})~~ ~~({c},{a,b,c})~~



Examples

Logo

Maximal and Minimal Elements

Maximal and Minimal

- a is **maximal** in the poset $(S,\preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$.
- a is **minimal** of there is no element $b \in S$ such that $b \prec a$.
- They are the “top” and “bottom” elements in the diagram.

Examples

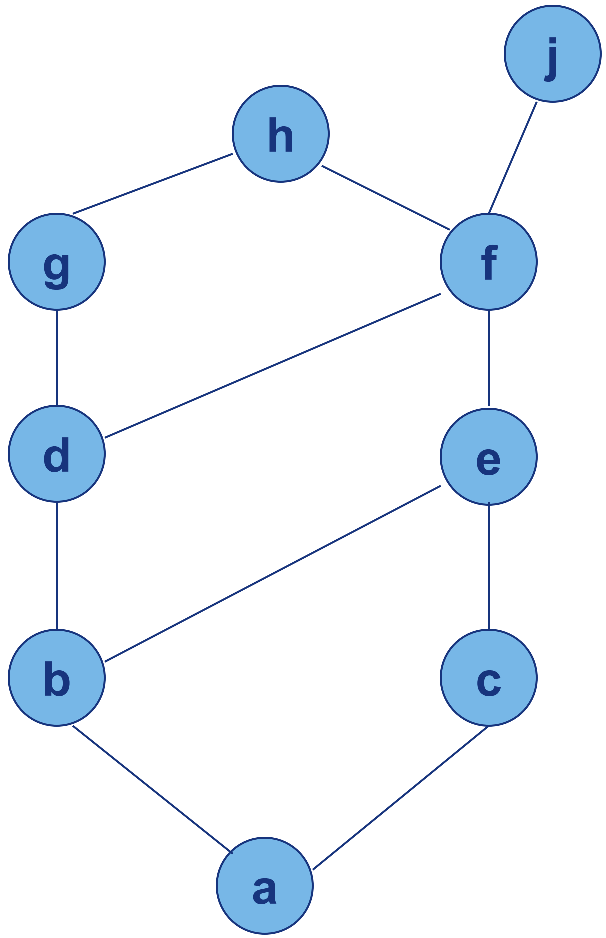
- 1、Which element of the poset ({2,4,5,10,12,20,25},|) are maximal, and minimal?
- 12,20 and 25;   2 and 5.
- 2、What is the greatest and least element in the poset (P(S), $\subseteq$) ?
- S and Ø.
- 3、Is there a greatest element and a least element in the poset (Z+,|)?
- no one and 1.

Logo

Upper bound and Lower bound

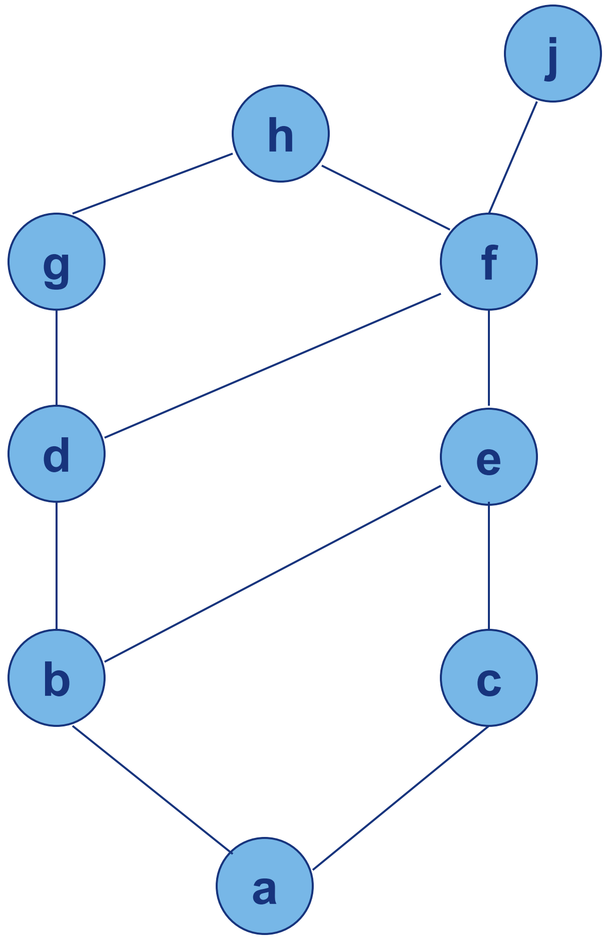
- A is a subset of a poset $(S, \preccurlyeq)$.
- If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called as an upper bound of A.
- If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called as an lower bound of A.

Example 17 Upper bound



- The upper bounds of { a,b,c } are e, f, j and h.
- There is no upper bounds of { j, h }.
- The upper bounds of { a,c,d,f } are f, h, and j.

Example 17 Lower bound

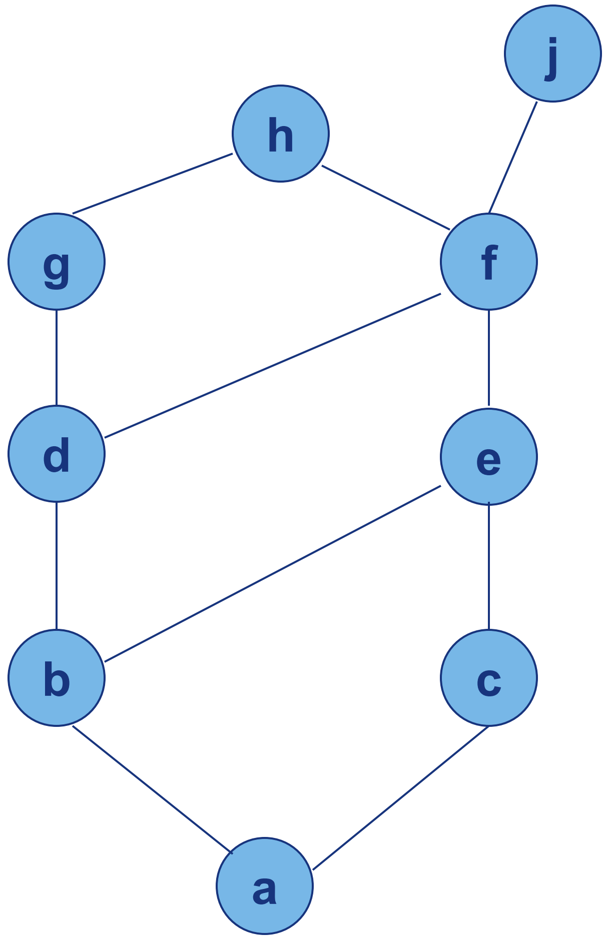


- The lower bound of { a,b,c } is a.
- The lower bounds of { j, h } are a,b,c, d, e and f.
- The lower bounds of { a,c,d,f } is a.

- The element x is called **the least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A.

- The element y is called **the greatest lower bound** of A if y is a lower bound of A and $z \preccurlyeq y$ whenever y is a lower bound of A.

Example 18



- The least upper bounds of { b,d,g } is g.
- The greatest lower bounds of { b,d,g } is b.

Logo

Lattices and Topological Sorting

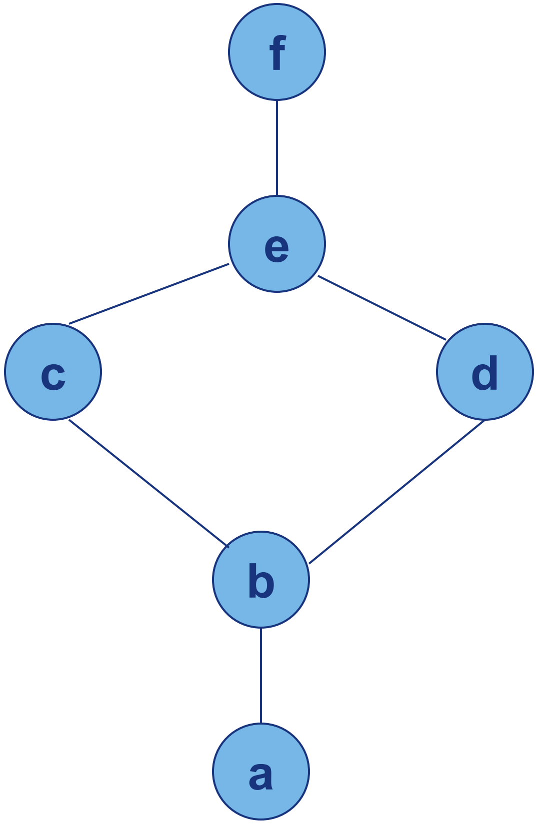
Logo

Lattices

- A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

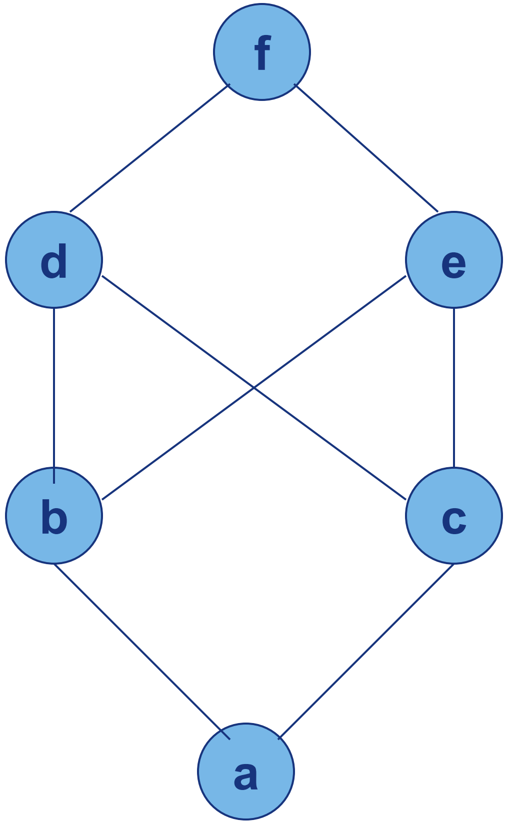
- The poset { $Z+$, | } is a lattice.
- ({ 1,2,3,4,5 }, |) is not a lattice.
- ({ 1,2,4,8,16 }, |) is a lattice.
- ($P(S)$, $\subseteq$) is a lattice where S is a Set.

Example 20



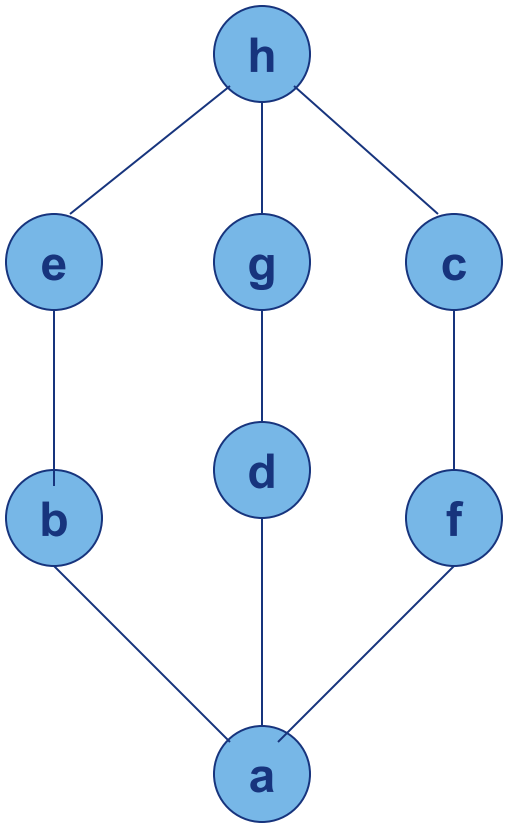
- Lattices.
- Because in the poset every pair of elements has both a least upper bound and a greatest lower bound.

Example 20



- Not a lattices.
- Because elements b and c have no least upper bound.

Example 20



- Lattices.
- Because in the poset every pair of elements has both a least upper bound and a greatest lower bound.

Topological Sorting

- A total ordering $\preccurlyeq$ is said to be compatible with partial ordering R if a $\preccurlyeq$ b whenever a R b.

- Constructing a compatible total ordering from a partial ordering is called **topological sorting**.

Lemma 1

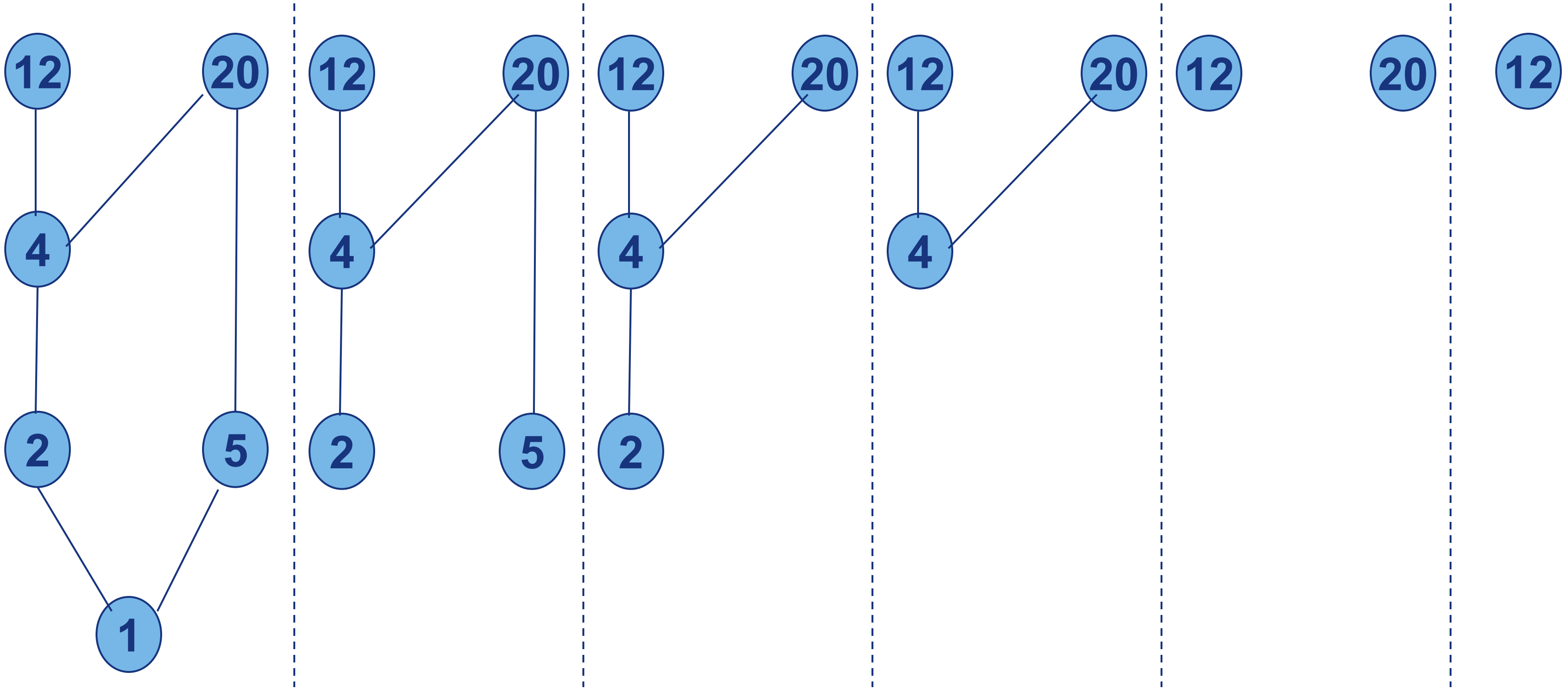
- Every finite nonempty poset $(S, \preccurlyeq)$ has a minimal element.
- Proof can be found directedly.
- Therefore, every finite nonempty poset can be sorted “topologically” by an algorithm.

Algorithm: Topological Sorting

- Procedure topological sort (S: finite poset)
- k:=1
- while S ≠ Ø
- begin
-     a_k:=a minimal element of S {Lemma 1}
-     S:=S - {a_k}
-     k:=k+1
- end {$a_1, a_2, \ldots, a_n$ is a compatible total ordering of S}

Example 25

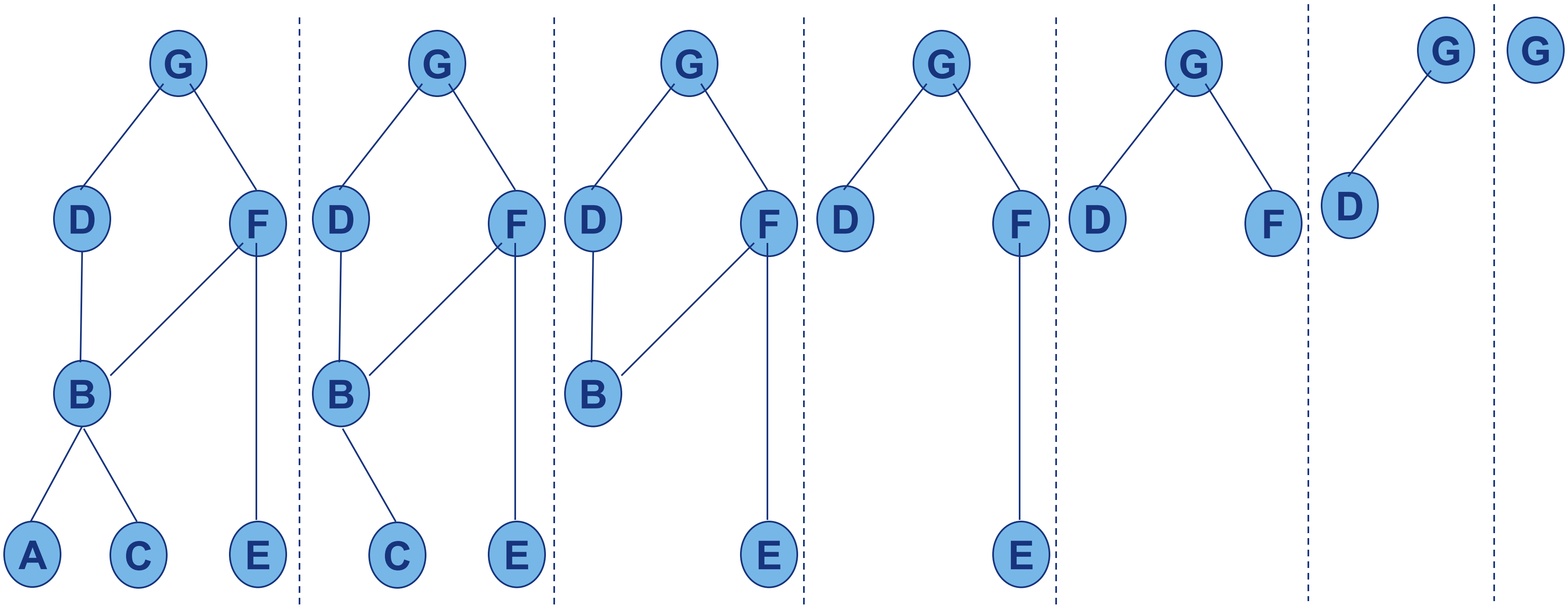
- Find a compatible total ordering for the poset ({1,2,4,5,12,20},|)



$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$$

Example 26

- Find topological sort of the task.



$$A \prec C \prec B \prec E \prec F \prec D \prec G$$

Logo

总结

- 偏序关系：自反，反对称，传递
- 可比性、总序（链）、良序（非空子集有最小元）
- 应用工具：lexiorder，排序，可列
- Hasse Graph⟹极大元，极小元，上界，下界，最大元，最小元，上确界，下确界

Exercises

Logo

Exercises

1. Which of these arguments is true? (**A**)

A. (Z^+,|) is a poset.

B. (Z^+,|) is totally ordered.

C. (P(S), subset of) is a poset and also a total order.

D. (N, $\geq$) is well-ordered.

Logo

Exercises

2. Select the relationship R which is **not** a partial ordering. (**A**)

A. The "greater than" relation $>$ on the set of integers.

B. The "less than or equal" relation $\leq$ on the set of integers.

C. The inclusion relation $\subseteq$ on the power set of a set S.

D. The divisibility relation $|$ on the set of positive integers.

Exercises

3. Which of the following matrices represents a partial ordering relation R such that R is a total order? (**D**)

A) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

B) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

D) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Logo

Exercises

4. Select the posets which is a lattice. (D)

A) $(\{1,3,6,9,12\}, |)$

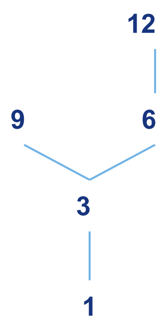
B) $(\{2,3,4,5,6\}, |)$

C) $(\{1,2,3,12,18,36\}, |)$

D) $(\{1,4,5,10,20\}, |)$

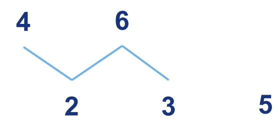
Exercises

A)



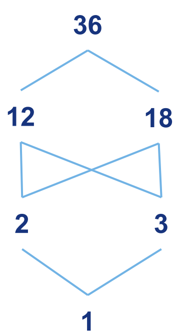
（9,12）没有最小上界

B)



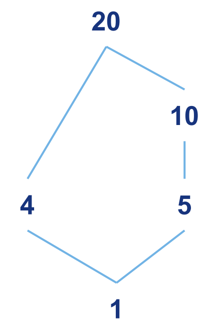
（2,3）没有最大下界

C)



（12,18）没有最大下界

D)



√

Exercises

5. For the set S={a, b, c, d}, which relation is a partial order? (**D**)

A) {(a,a), (a,b), (b,a), (b,b), (c,c), (d,d)}.

B) {(a,a), (a,c), (b,b) , (b,c), (c,c), (c,d), (d,d)}.

C) {(a,b), (a,c), (b,a), (b,d), (c,a), (c,d), (d,c), (d,b)}.

D) {(a,a), (a,b), (a,d), (b,b), (b,d), (c,c), (d,d)}

Exercises

6. For the set S={0, 1, 2, 3}, which relation is a partial order? (**D**)

A) {(0,0), (0,1), (1,0), (1,1), (2,2), (3,3)}.
B) {(0,0), (0,2), (1,1), (1,2), (2,2), (2,3), (3,3)}.
C) {(0,1), (0,2), (1,0), (1,3), (2,0), (2,3), (3,2), (3,1)}.
D) {(0,0), (0,1), (0,3), (1,1), (1,3), (2,2), (3,3)}

Exercises

7. Mark each of the following T (TRUE) or F (FALSE).

Let (A, S) is a partial ordered set. Therefore, there isn’t always a greatest element, but must be maximal elements in A.（ **F** ）

Logo

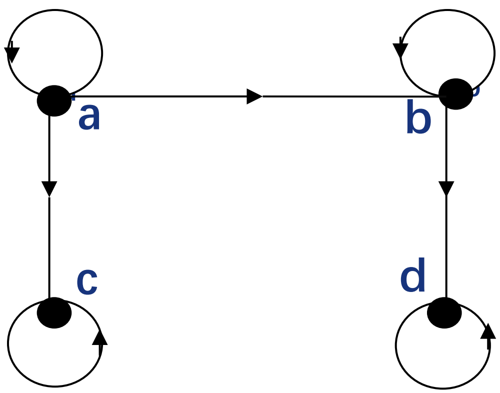
Exercises

8. For the poset $(\{2, 4, 6, 12, 16, 32, 48\}, |)$, the lower bounds of $\{4, 12\}$ are _______.

{2,4}

Exercises

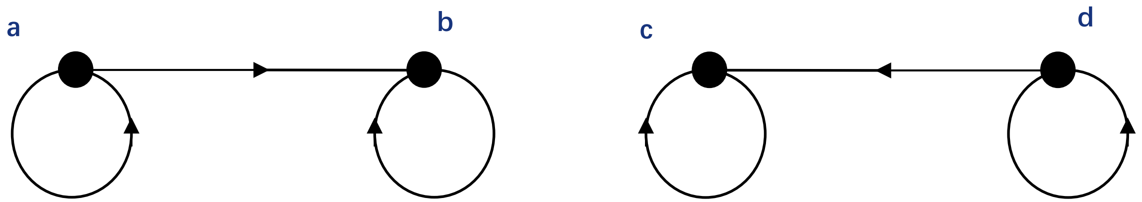
9. Whether the relation with the directed graph shown is a partial order. (Yes or No)________



No

Exercises

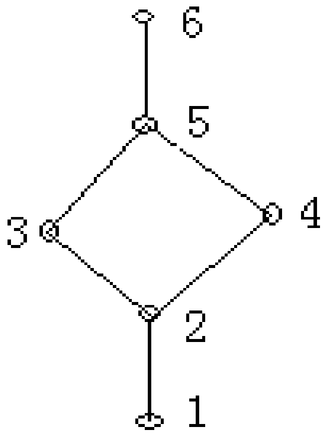
10. Whether the relation with the directed graph shown is a partial order. (Yes or No)______



Yes

Exercises

11. The Hasse graph of a partially ordered set is as follows, and the greatest lower bound of {3,4,5} is_______



2

Exercises

12. For the poset ({2, 6, 7, 14, 18, 42, 126 }, |), the minimal elements are __________, and the upper bounds of {6, 14} are __________

2,7

42,126

Exercises

13. For the poset ({2, 4, 5, 10, 12, 20, 25}, |), the minimal elements are __________, and the upper bounds of {4, 10} are __________

{2,5}
{20}

Exercises

14. For the poset ({3, 6, 8, 9, 15, 24, 45}, ∣), the minimal elements are __________, and the maximal elements are __________

{3,8}
{24,45}

Exercises

15. Supposed A = { 1， 2， 3， 4， 6， 9， 24， 54 }, R is a division relation on A, and we have a poset <A， R>. Please answer the following three questions.

① Draw Hasse graph to describe the partial ordering R.

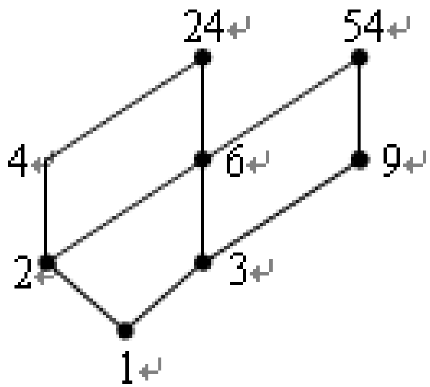
② Find out the maximal element, minimal element, greatest element and least element of the poset.

③ Find out the upper bound, lower bound, greatest lower bound and least upper bound of the subset B = { 4， 6， 9 }.

Exercises

15. Supposed A = { 1， 2， 3， 4， 6， 9， 24，54 }, R is a division relation on A, and we have a poset <A， R>. Please answer the following three questions.

① **Draw Hasse graph to describe the partial ordering R.**



Logo

Exercises

15. Supposed A = { 1， 2， 3， 4， 6， 9， 24， 54 }, R is a division relation on A, and we have a poset <A， R>. Please answer the following three questions.

② Find out the maximal element, minimal element, greatest element and least element of the poset.

- **maximal elements are 24 and 54,and no greatest element**
- **minimal elements is 1, and least element is 1**

Logo

Exercises

15. Supposed A = { 1， 2， 3， 4， 6， 9， 24， 54 }, R is a division relation on A, and we have a poset <A， R>. Please answer the following three questions.

③ Find out the upper bound, lower bound, greatest lower bound and least upper bound of the subset B = { 4 ， 6， 9 }.

- **no upper bound and least upper bound, 1 is lower bound and greatest lower bound**

Logo

Exercises

13. Suppose a relation $R = \{\langle a_1, a_2\rangle | a_1, a_2 \in A, a_1 | a_2\}$ on the set A={1,2,3,4,6,8,12,24}

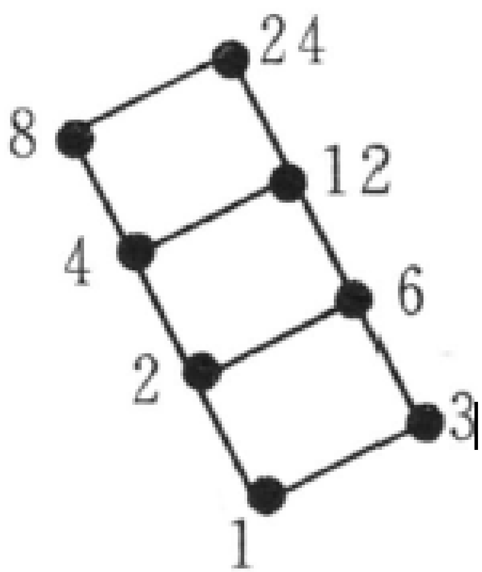
1、 Draw the Hasse Diagram of R.

2、 Find the maximal element, minimal element, greatest element and least element of this poset.

Exercises

13. Suppose a relation $R = \{\langle a_1, a_2\rangle \mid a_1, a_2 \in A, a_1 | a_2\}$ on the set A={1,2,3,4,6,8,12,24}

① Draw the Hasse Diagram of R.



Exercises

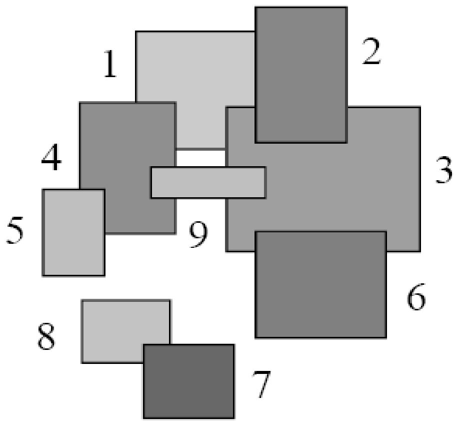
13. Suppose a relation $R = \{\langle a_1, a_2 \rangle | a_1, a_2 \in A, a_1 | a_2\}$ on the set A={1,2,3,4,6,8,12,24}

② Find the maximal element, minimal element, greatest element and least element of this poset.

- 极小元、最小元是**1**
- 极大元、 最大元是**24**

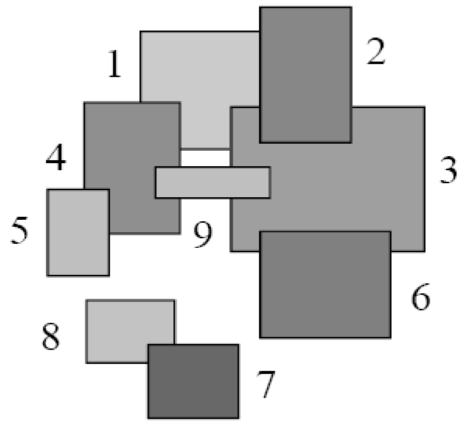
Exercises

17. Consider the rectangles T={1,…,9} and the relation R such that iRj = "i is more distant than j from the viewer." Then R is a partial order on the set of rectangles. Please show an order of topological sorting according to the following figure.

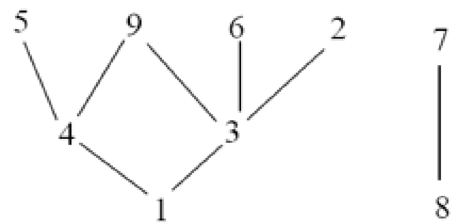


Exercises

17. Consider the rectangles T={1,…,9} and the relation R such that iRj = “i is more distant than j from the viewer.” Then R is a partial order on the set of rectangles. Please show an order of topological sorting according to the following figure.



Then 1R2, 1R4, 1R3, 4R9, 4R5, 3R2, 3R9, 3R6, 8R7.
The Hasse diagram for R is



An order of topological sorting is “1 4 3 8 5 9 6 2 7”.

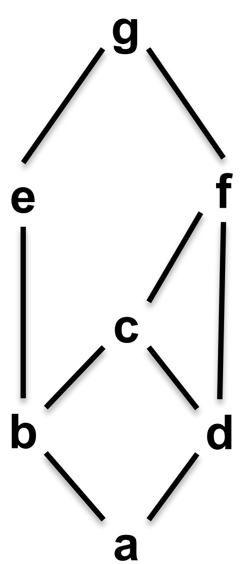
Exercises

18. It takes seven different operations to produce a product. Some operations require that all prerequired operations should be finished before its procession. The prerequired operations of the seven operations are showed in the following table. Please illustrate the Hasse Graph of these operations and determine a legitimate operation sequence.

Operations	Prerequired operations
a	none
b	{a}
c	{b, d}
d	{a}
e	{b}
f	{c, d}
g	{e, f}

Exercises

18.



Operations	Prerequired operations
a	none
b	{a}
c	{b, d}
d	{a}
e	{b}
f	{c, d}
g	{e, f}

参考序列：a b d c e f g
其他可选序列：
a d b c e f g
a b e d c f g

Logo

End of Section 3.6