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# Discrete Mathematics

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Chapter 4. Graphs

# Graph Terminology

**Section 4.2**

# Contents

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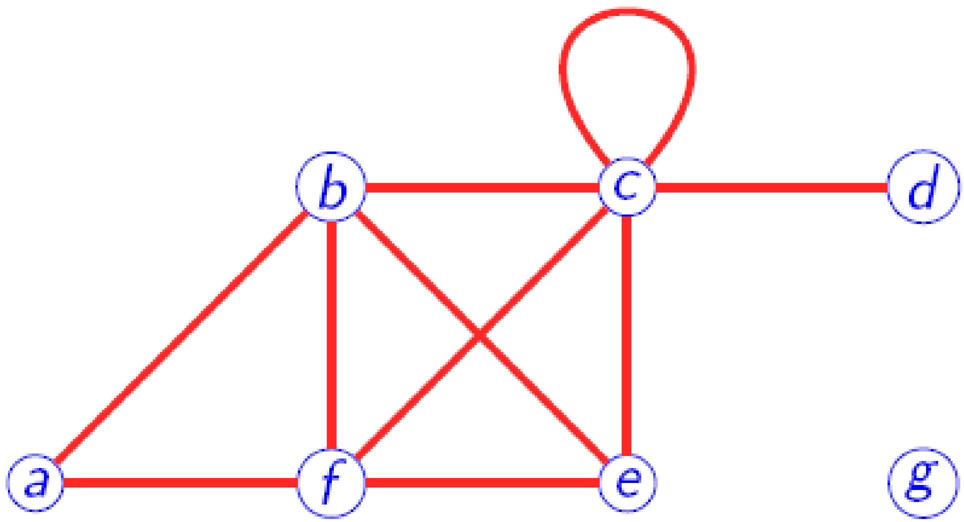
# Basic Terminology

**Definition 2.1** For an edge $e = (u, v)$ in an undirected graph $G$, we say that

- the vertices $u$ and $v$ are adjacent.
- the vertices $u$ and $v$ are the endpoints(端點) of the edge $e$
- the edge $e$ is incident with $u$ and $v$
- the edge $e$ connects $u$ and $v$

**Definition 2.2** In an undirected graph, the degree of a vertex $v$, denoted by $deg(v)$, is the number of edges incident with $v$. A self-loop at a vertex $v$ contributes twice to the degree of $v$.

**Example 2.1** What are the degrees of the vertices in the graph displayed below.




**Solution**

$deg(a) = 2$; $deg(b) = 4$; $deg(c) = 6$; $deg(d) = 1$; $deg(e) = 3$; $deg(f) = 4$; $deg(g) = 0$.

# Handshaking Theorem

**Theorem 2.1** Let $G = (V, E)$ be an undirected graph. Then

$$\sum_{v \in V} deg(v) = 2|E|$$

**Example 2.3** There are 30 edges in a graph with ten vertices and each of degree six.

**Theorem 2.2** An undirected graph has an even number of vertices of odd degree. (degree 是奇數的節點有偶數個)

說明: For an undirected graph $G = (V, E)$

- Let $V_1$ be the set of vertices of even degree.
  $deg(v_1)$ is even for $v_1 \in V_1$.

- Let $V_2$ be the set of vertices of odd degree.
  $deg(v_2)$ is odd for $v_2 \in V_2$.

- $$\underbrace{2|E|}_{even} = \underbrace{\sum_{v \in V_1} deg(v_1)}_{even} + \underbrace{\sum_{v \in V_2} deg(v_2)}_{even}$$

- Since the second summation is even and all terms in this summation is odd, there must be odd number of such terms.

**Definition 2.3** Let $(u, v)$ be an edge of the directed graph. Then we say that

- $u$ is adjacent to $v$
- $v$ is adjacent from $u$
- $u$ is the initial vertex of $(u, v)$
- $v$ is the terminal or end vertex of $(u, v)$

Note: The initial vertex and terminal vertex of a loop are the same.

# In-degree and Out-degree

**Definition 2.4** Let $v$ be a vertex of the directed graph.

- The in-degree of $v$, denoted by $deg^{-}(v)$, is the number of edges with $v$ as their terminal vertex.
- The out-degree of $v$, denoted by $deg^{+}(v)$, is the number of edges with $v$ as their initial vertex.

**Theorem 2.3** Let $G = (V, E)$ be a directed graph. Then

$$\sum_{v \in V} deg^{+}(v) = \sum_{v \in V} deg^{-}(v) = |E|$$

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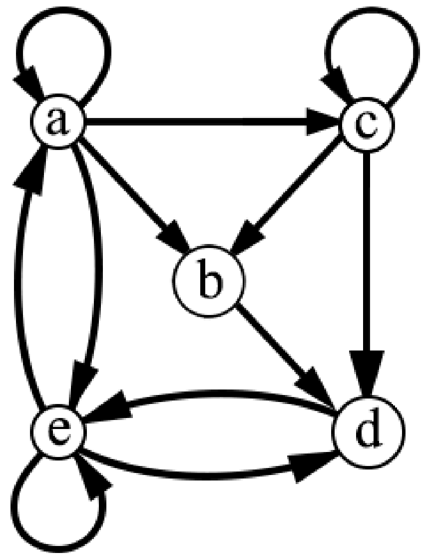
**Example 2.4** Find the in-degree and out-degree of each vertex in the directed graph $G$.




| Vertex | a | b | c | d | e |
|---|---|---|---|---|---|
| In-degree | 2 | 2 | 2 | 3 | 3 |
| Out-degree | 4 | 1 | 3 | 1 | 3 |

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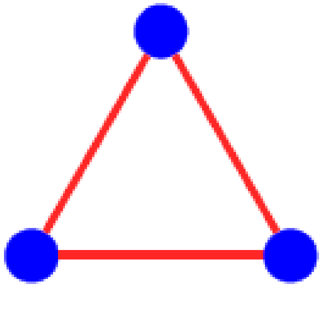
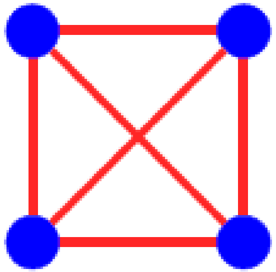
# Some Special Simple Graphs

# Complete Graphs

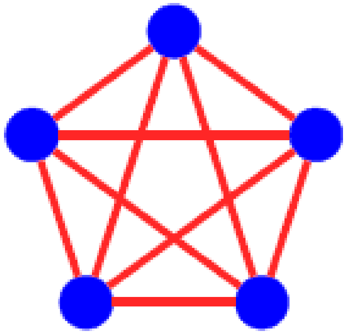
**Example 2.5** The complete graph on $n$ vertices, denoted by $K_n$, is the simple graph in which every vertex is adjacent to every other vertex.



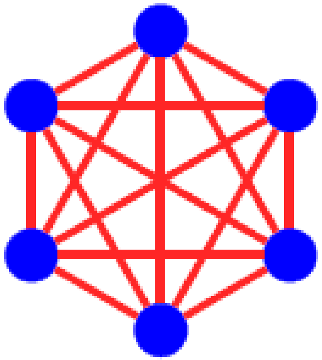
$K_3$



$K_4$



$K_5$

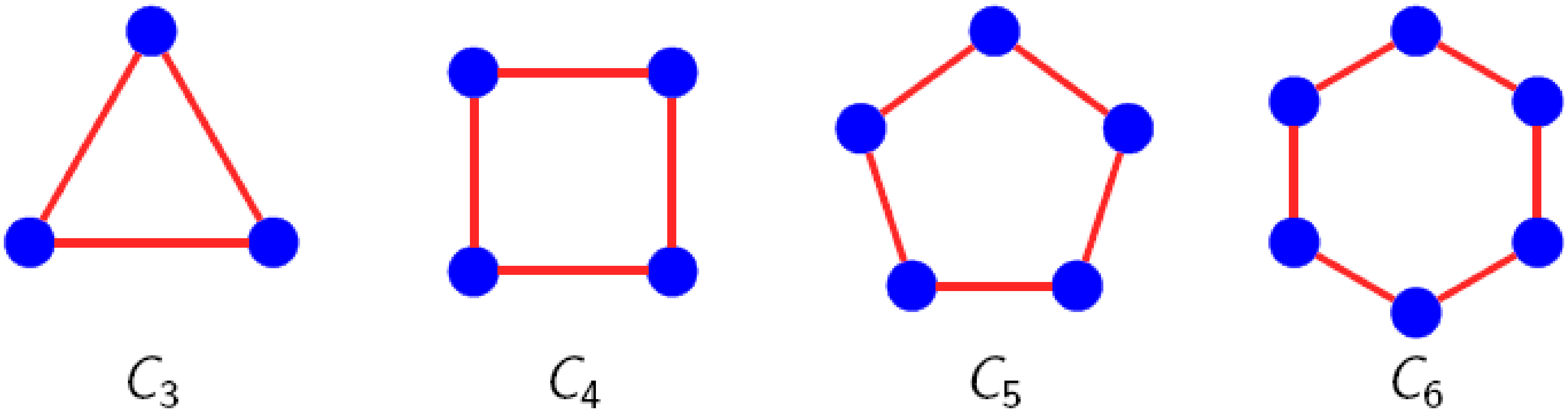


$K_6$

# Cycles

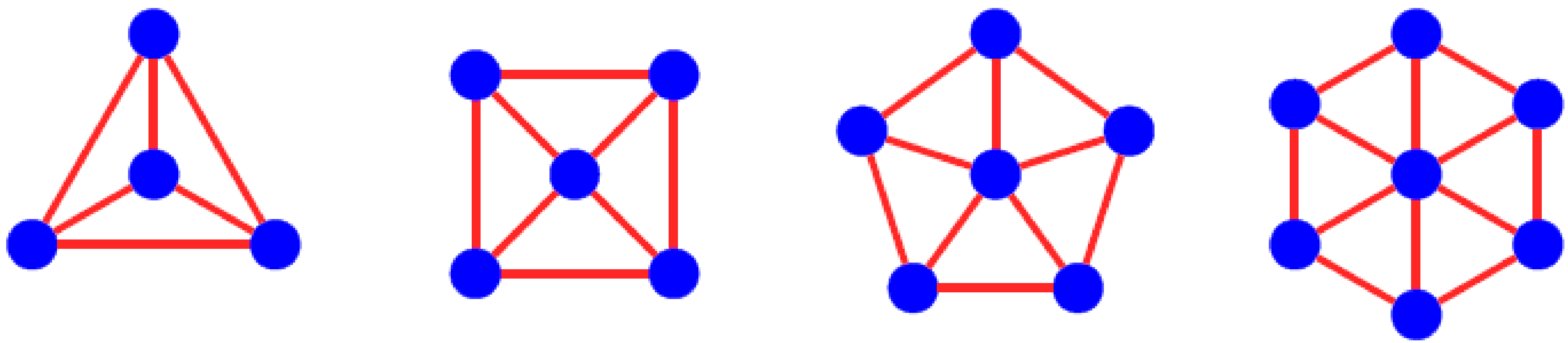
**Example 2.6** The cycle $C_n$ with $n$ vertices $v_1, v_2, \ldots, v_n$ has the edges given by

$$(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$$



# Wheels

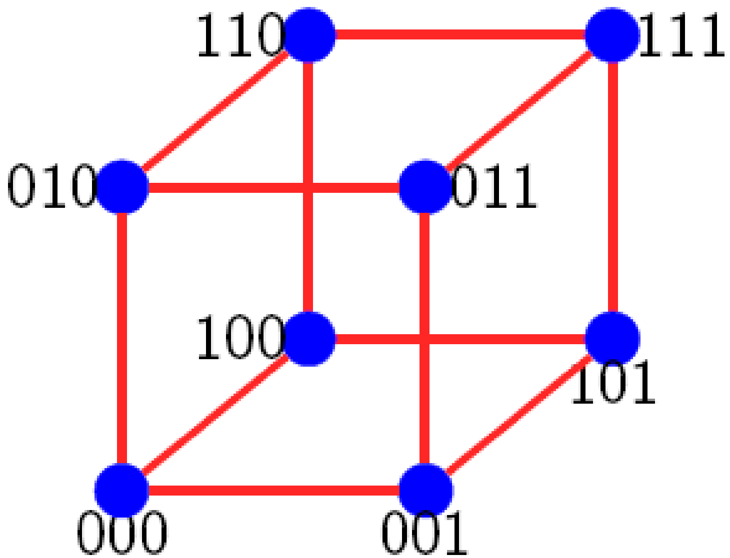
**Example 2.7** The wheel $W_n$ is a cycle $C_n$ together with an additional vertex that is adjacent to every vertex of $C_n$.



$W_3$ $W_4$ $W_5$ $W_6$

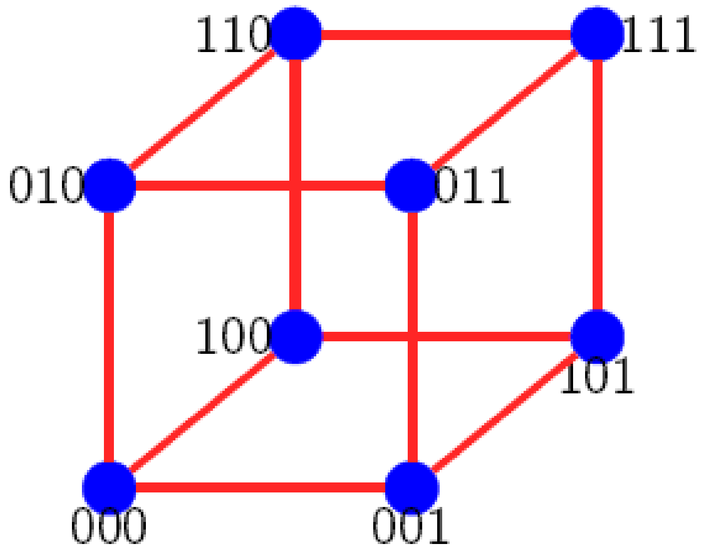
# n-Cubes

**Example 2.8** The $n$-cubes, denoted by $Q_n$, has $2^n$ vertices which are labeled by bit strings of length $n$ representing $0, 1, 2, \ldots, 2^n - 1$. Two vertices are joined when their bit strings differ exactly one bit position.

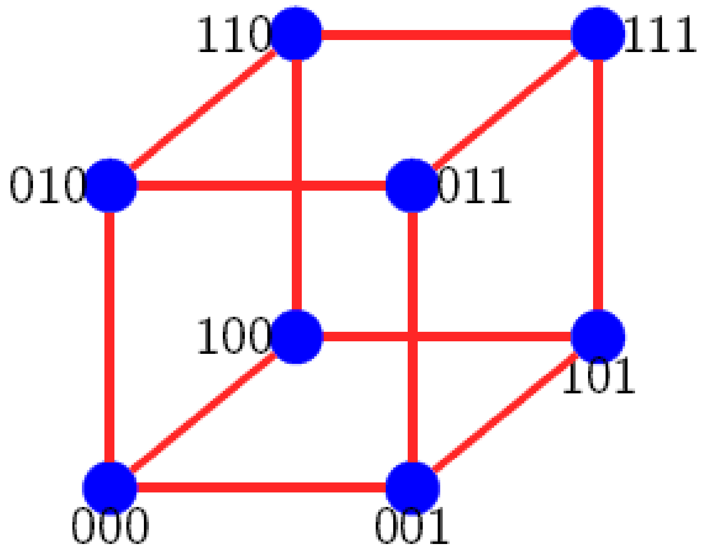


# Construct n+1-Cubes from n-Cubes

- Make a new copy, named $Q_{n,new}$, of $Q_n$, named $Q_{n,old}$.



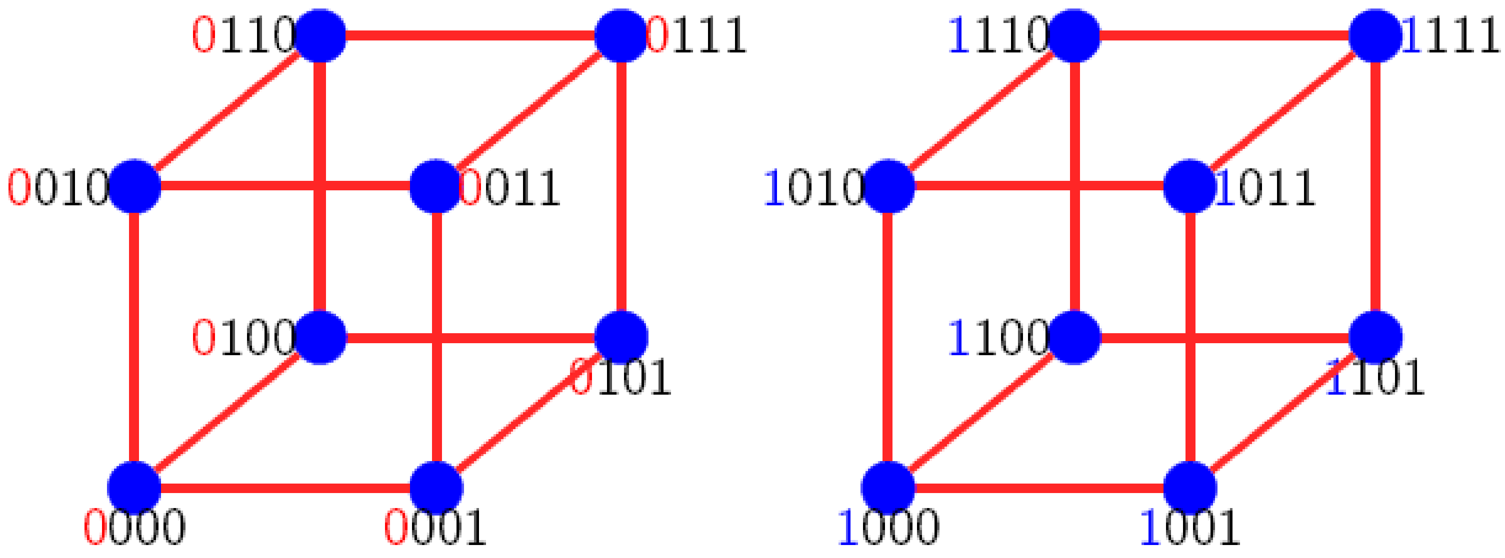

$Q_{n,old}$




$Q_{n,new}$

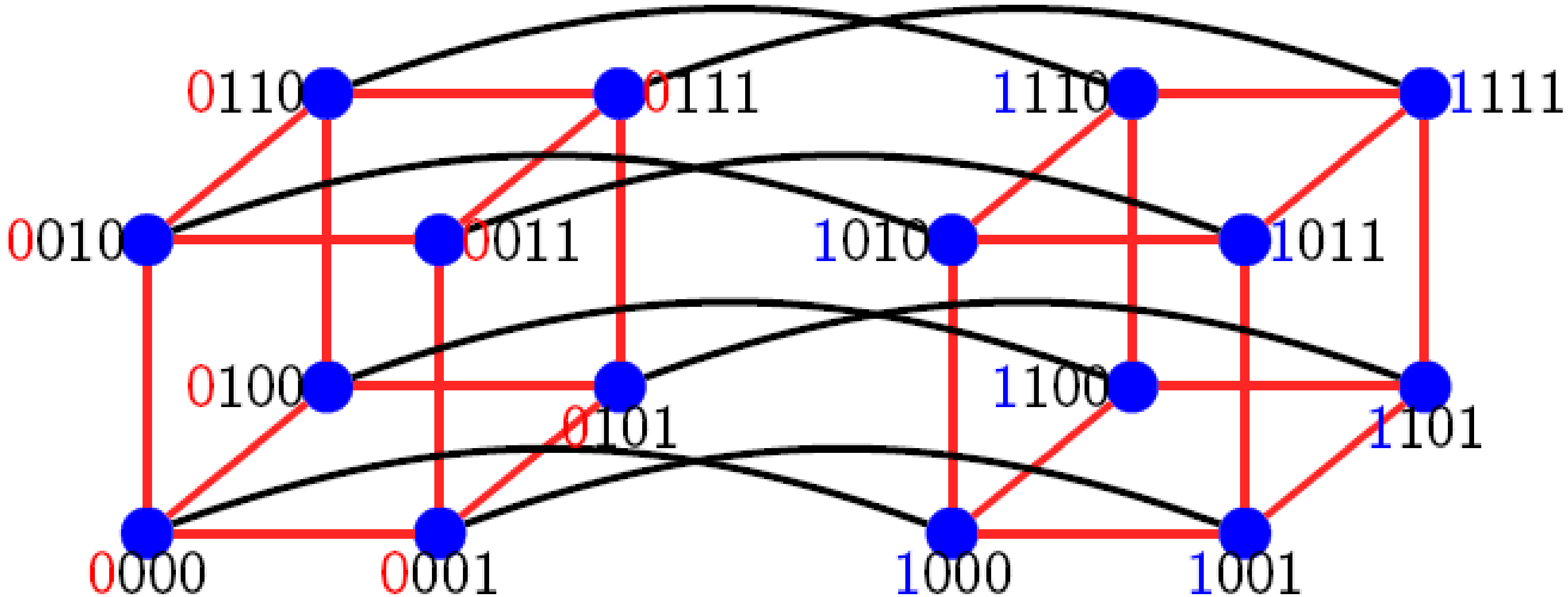
# Construct n+1-Cubes from n-Cubes

- Preface the labels on the vertices in $Q_{n,old}$ with a "0", and with a "1" in $Q_{n,new}$



# Construct n+1-Cubes from n-Cubes

- Add edges that connect two vertices that have labels differing in the first bit



# Bipartite Graphs

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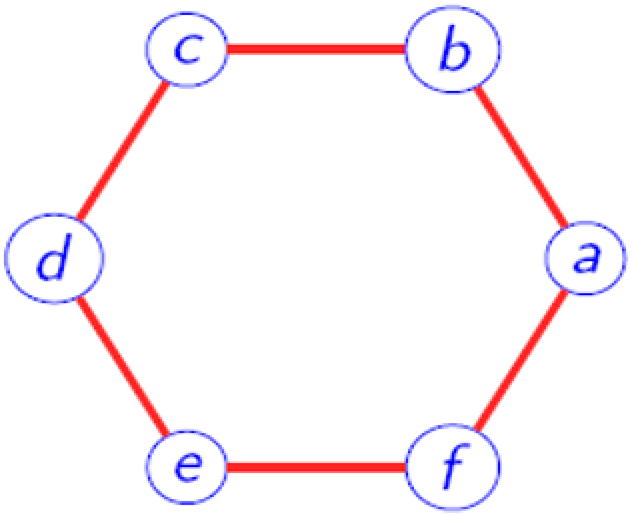
# Bipartite Graphs

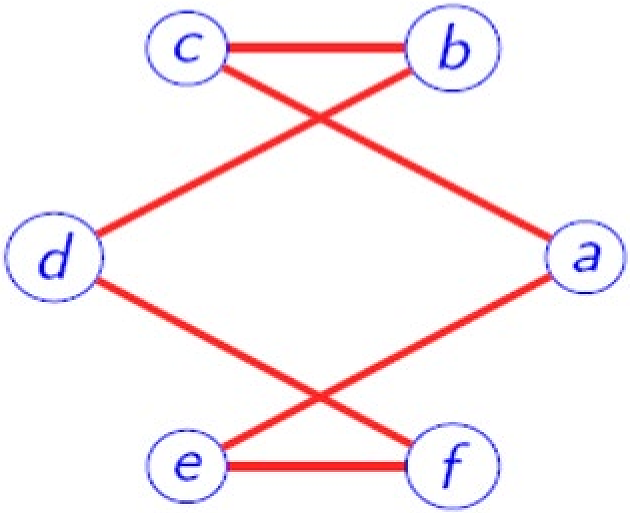
**Definition 2.5** A graph $G$ is a bipartite graph if its vertices are partitioned into two disjoint sets $V_1$ and $V_2$, called a bipartition, such that every edge join a vertex in $V_1$ with a vertex in $V_2$.

Example 2.9 $C_6$ is bipartite.



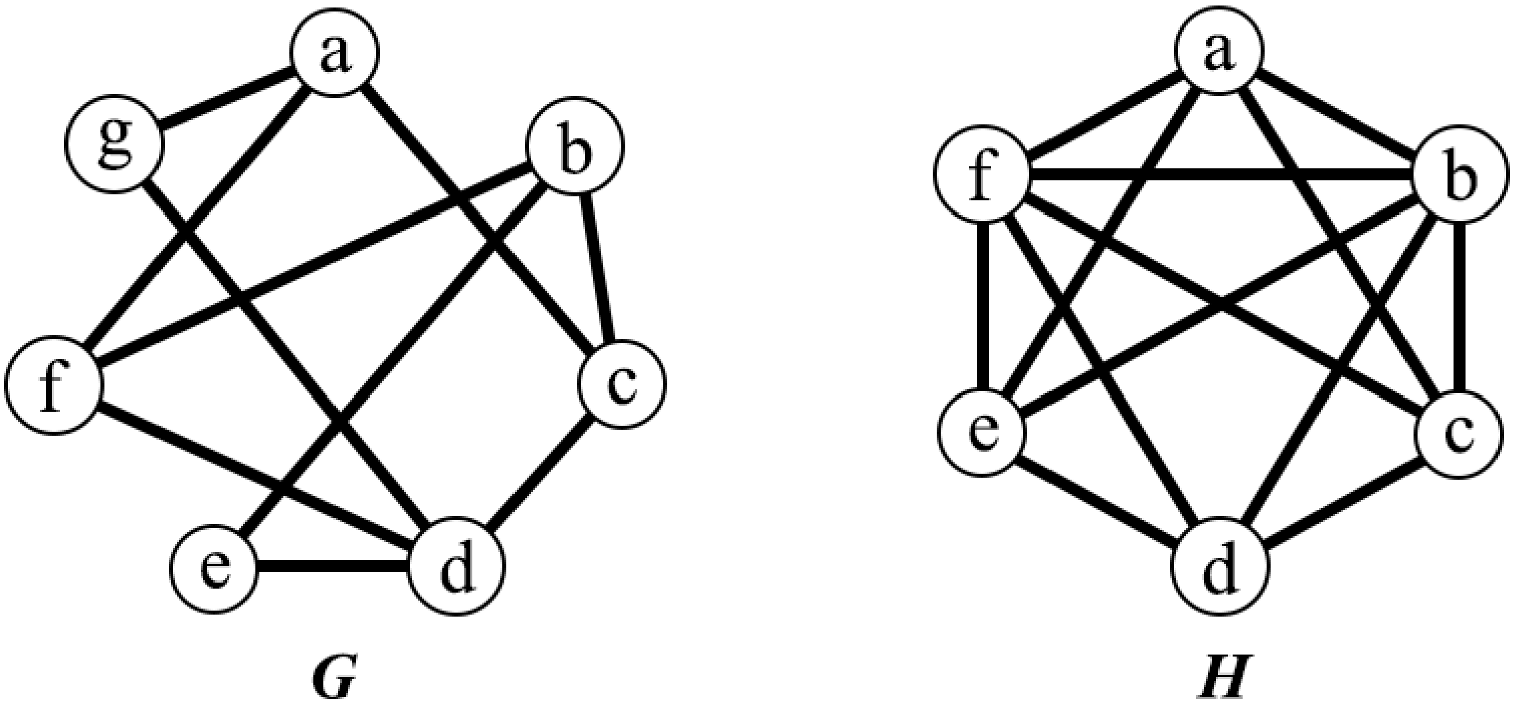

$C_6$

Example 2.10 K is bipartite.



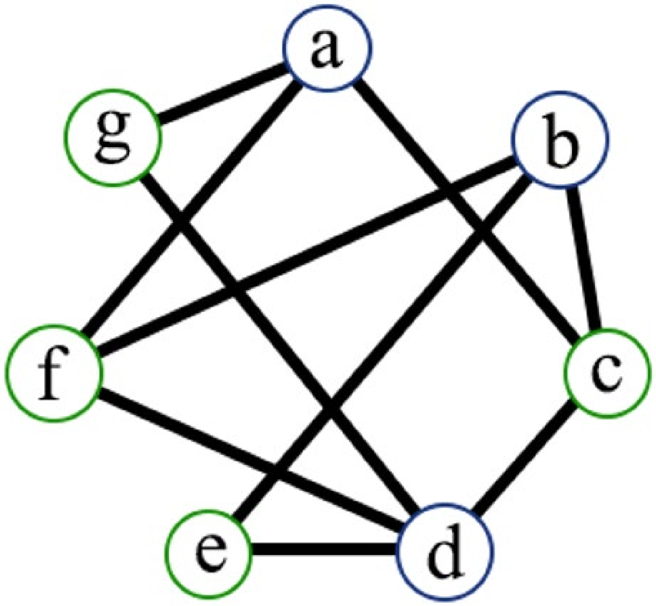

K

**Example 2.11** $G$ is bipartite. $H$ is not bipartite.



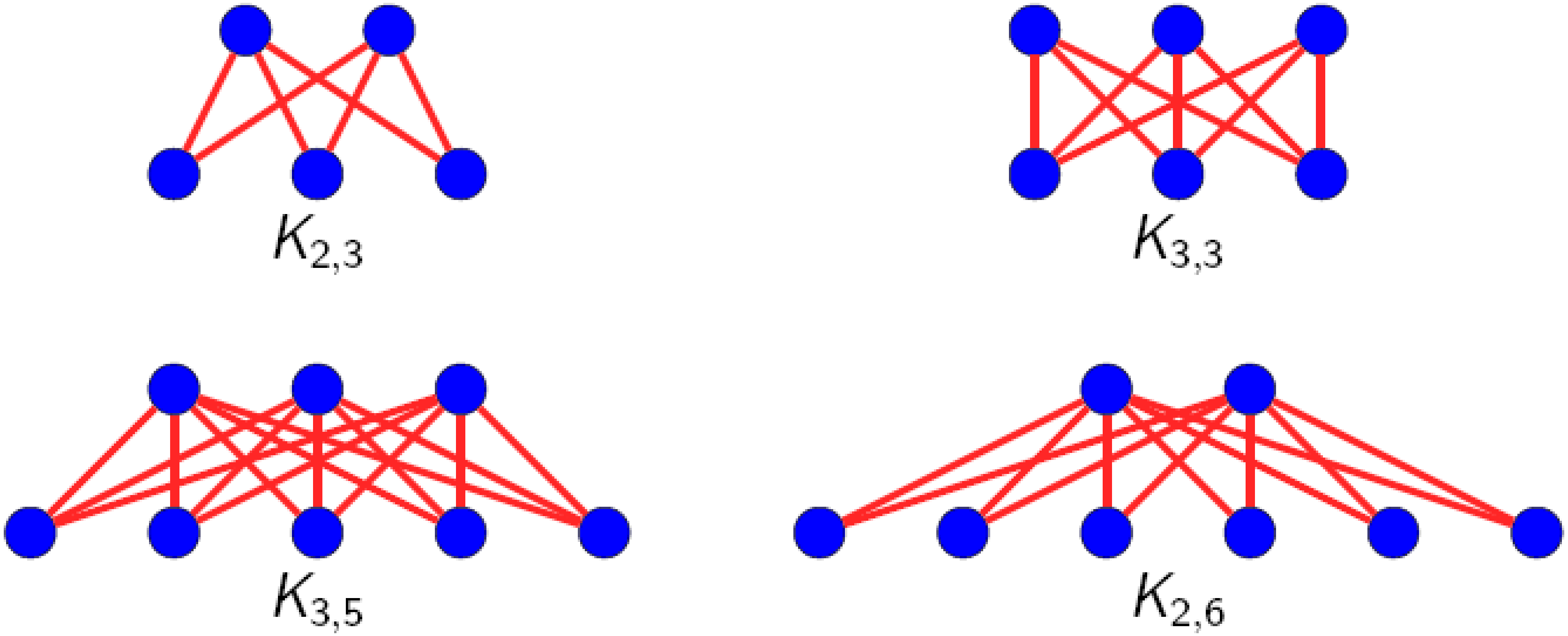
**Theorem 2.4** A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no adjacent vertices are assigned the same color.

**Example 2.12** $G$ is bipartite.



# Complete Bipartite Graphs

**Example 2.13** The complete bipartite graph $K_{m,n}$ is a bipartite graph which vertices are partitioned into two disjoint subsets of $m$ and $n$ vertices, respectively. Every vertex in one subset must be adjacent to every vertex in the other subset.



$K_{2,3}$ $K_{3,3}$ $K_{3,5}$ $K_{2,6}$

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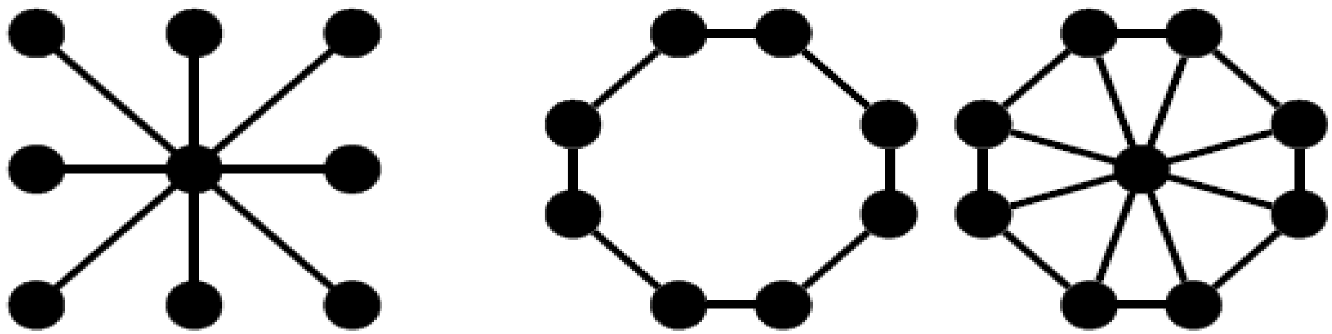
# Some Application of Special Types

L o g o

# Network Topology

**Example 2.15**

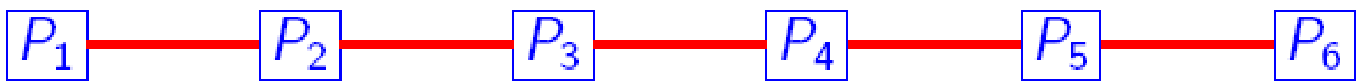


- $G_1$: star topology.
  All devices are connected to a center control device.
- $G_2$: ring topology.
  Messages are sent around the ring.
- $G_3$: hybrid topology.
  Messages are sent around the ring, or through a center control device.

# Parallel Algorithms

**Example 2.16** Parallel algorithms break a problem into many subproblems solved by using a computer with many processors. Here introduces three common parallel architectures.

- Linear array



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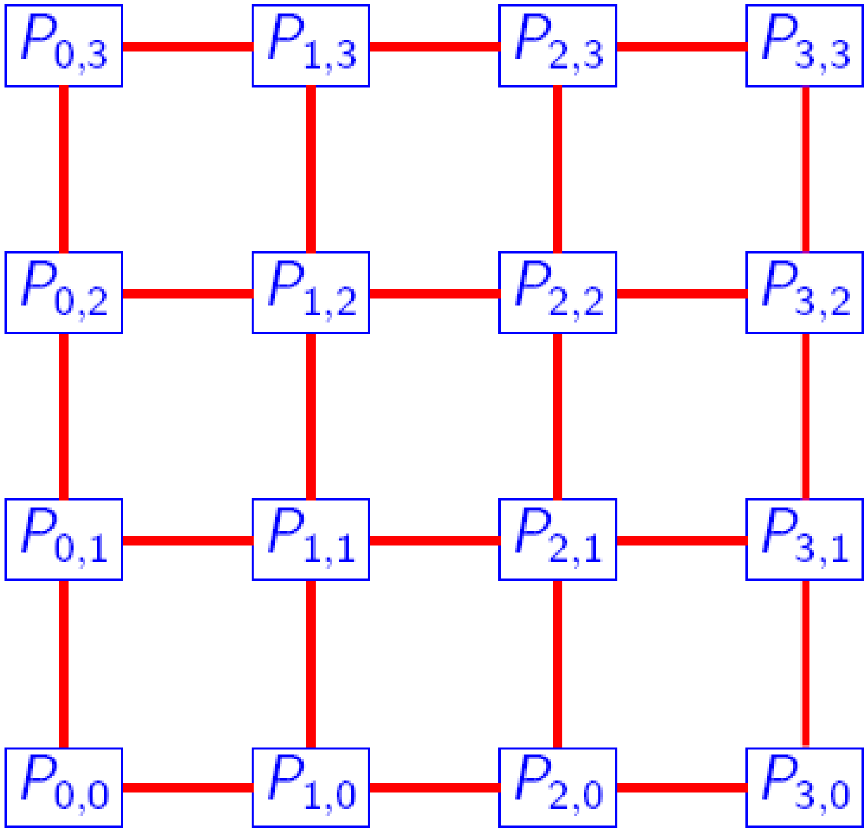
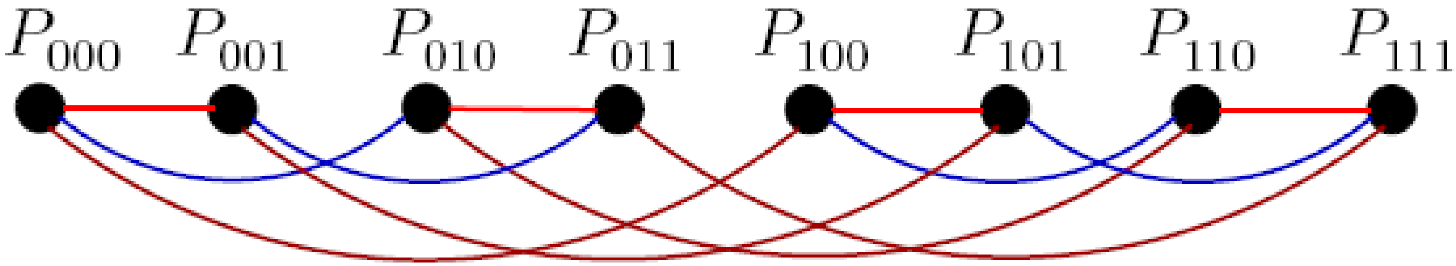
# Parallel Algorithms

- Mesh Network



# Parallel Algorithms

- Hypercube



**Definition 2.6** The graph $H = (W, F)$ is a subgraph of a graph $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$.

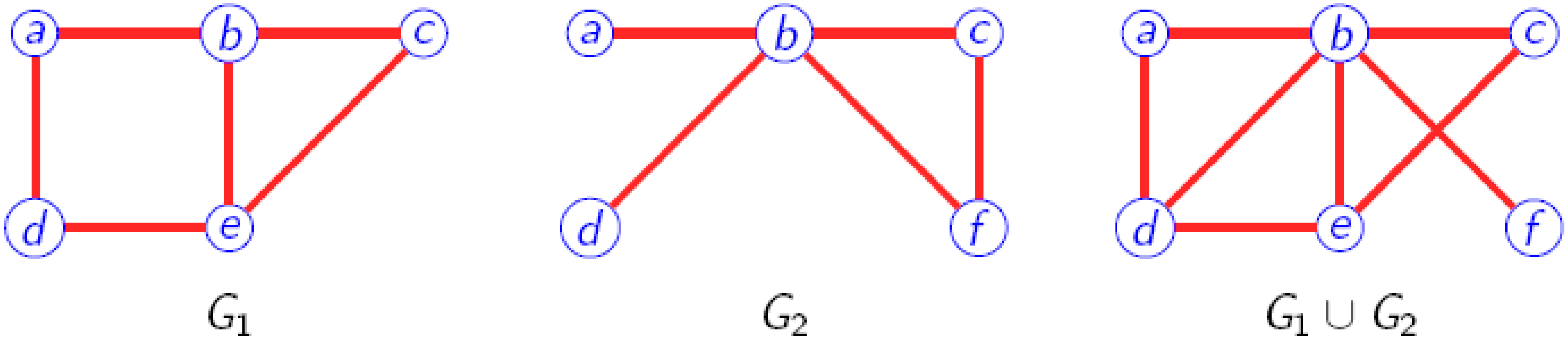
## Example 2.17


$K_5$ $\qquad$ $G$

$G$ is a subgraph of $K_5$

**Definition 2.7** The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

**Example 2.18**




The graph $G_1 \cup G_2$ is the union of $G_1$ and $G_2$.

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# Applications

# Applications

1. Solve the following problems.
1) Graph S, constructed by some cities and the roads among them. The graph contains 16 edges (roads). And the degree of each vertex (cities) is 2. How many cities?

2) Graph T, constructed by some harbors and the waterways. The graph contains 21 edges (waterways). 3 of the vertices (harbors) have the degree of 4, and the others have the degree of 3. How many harbors?

# Applications

1. Solve the following problems.

1) Graph S, constructed by some cities and the roads among them. The graph contains 16 edges (roads). And the degree of each vertex (cities) is 2. How many cities?

Solution:
There are 16 cities. Suppose there are x vertices. Hence,
the sum of degree= 2*|E|
$2x = 2*16$
$x=16$.

2) Graph T, constructed by some harbors and the waterways. The graph contains 21 edges (waterways). 3 of the vertices (harbors) have the degree of 4, and the others have the degree of 3. How many harbors?

Solution:
There are 13 harbors. Suppose there are x vertices. Hence,
the sum of degree= 2*|E|
$3*4+(x-3)*3 = 2*21$
$x=13$.

# Applications

2. How many edges in the following graphs?

1) $K_n$ 1+2+…+n=n(n-1)/2
2) $C_n$ n
3) $W_n$ 2n
4) $K_{m,n}$ mn

# Exercises

# Exercises

- 1. Select an integer degree sequence which can formulate a simple graph. (　D　)
- A、1，2，2，3，4，5
- B、2，3，3，4，4，5
- C、2，2，3，4，5，6
- D、1，2，2，3，3，5

解析：无向图奇数度的节点有偶数个，A和B不符；简单图节点度数小于节点数，C不符

# Exercises

- 2. Suppose that there are eight vertices in a simple directed graph, then the edge number of the graph is impossible to be

( D )

A) 1 B) 55 C) 34 D) 57

解析：简单有向图是1)不存在重复边;2)不存在顶点到自身的边;3)有向。题干中提到8个顶点，因此边数 ≤ 8*7

# Exercises

- 3. The vertices in an undirected graph G are of degree 4 or 5. Suppose that the number of edges in the graph G can be divided by 59, what is the minimum possible number of vertices in the graph G? ( D )

A) 6. B) 12. C) 18. D) 24

解析：设度数为4的节点有x个，总结点数为n个，由握手定理可得$4x+5(n-x)\geq 59*2 \Longrightarrow 5n-x\geq 118 \Longrightarrow 5n\geq 118$，由n取整数可得$n\geq 24$

**Theorem 2.1** Let $G = (V, E)$ be an undirected graph. Then

$$\sum_{v\in V} deg(v) = 2|E|$$

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# Exercises

- 4. The graph contains 21 edges. Three of the vertices have the degree of four, and the others have the degree of three. How many vertices does the graph have? (　B )

A) 10　　　B) 13

C) 28　　　D) 42

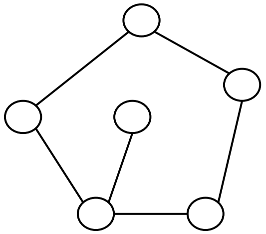
解析：设总结点数为x，由握手定理得21*2=3*4+3*(x-3)，解得x=13

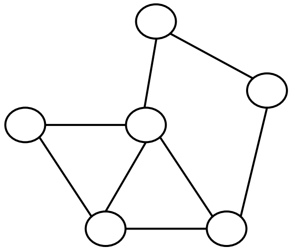
**Theorem 2.1** Let $G = (V, E)$ be an undirected graph. Then

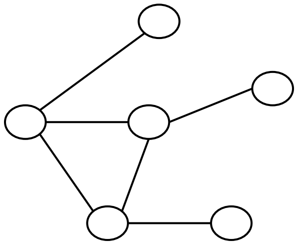
$$\sum_{v \in V} deg(v) = 2|E|$$

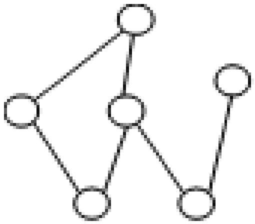
# Exercises

5. Which graph is the bipartite graph from the following undirected graphs? ( D )

A) 

B) 

C) 

D) 

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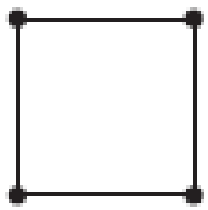
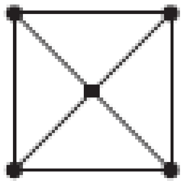
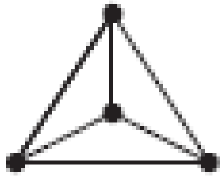
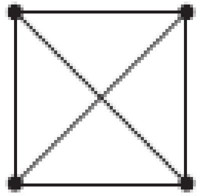
# Exercises

6. Which graph is $W_4$? ( B )

A) 

B) 

C) 

D) 

# Exercises

7. Which one is NOT true? （ C ）

A)

解析：**n立方体图**记作$Q_n$，是用顶点表示$2^n$个长度为n的位串的图。沿cube-n图中的一条边移动相当于将节点的二进制串修改一位。要沿回路回到起点，那么必对原二进制串修改了偶数次。因此，当$n \geq 1$时，$Q_n$是二分图。

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# Exercises

8. Supposed that there are seven vertices in a simple graph, the maximum number of edges in the graph is ____21____.

$$C_7^2 = \frac{7 \times 6}{2 \times 1} = 21$$

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# Exercises

9. There are ___8___ vertices, ___15___ edges in the complete bipartite graph $K_{3,5}$.

# Exercises

10. Graph $W_n$ have ________n+1________ vertices and ________2n________ edges.

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# Exercises

11. How many edges are there in $Q_n$?

$n2^{n-1}$

解析： $Q_n$有$2^n$个节点，每个节点有n条边，每条边计算了2次，所以边数为$\frac{n2^n}{2} = n2^{n-1}$

# Exercises

12. The graph contains 32 edges. Six of the vertices have the degree of four, and the others have the degree of five. The graph has ___14___ vertices.

解析：32*2=6*4+5(x-6)　解得x=14

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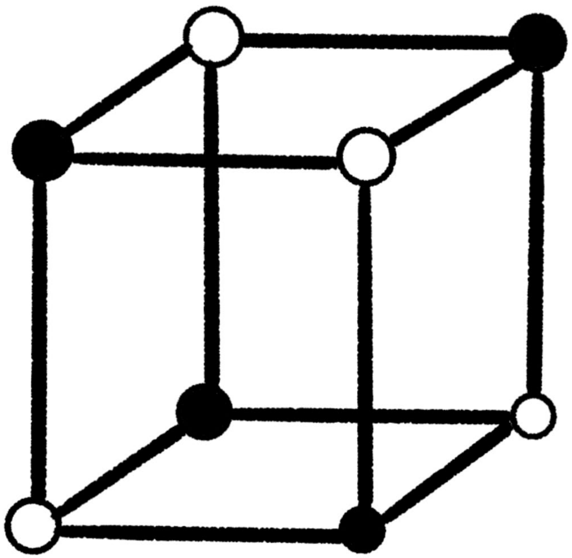
# Exercises

13. Is undirected graph $Q_n$ ($n \geq 1$) a bipartite graph?



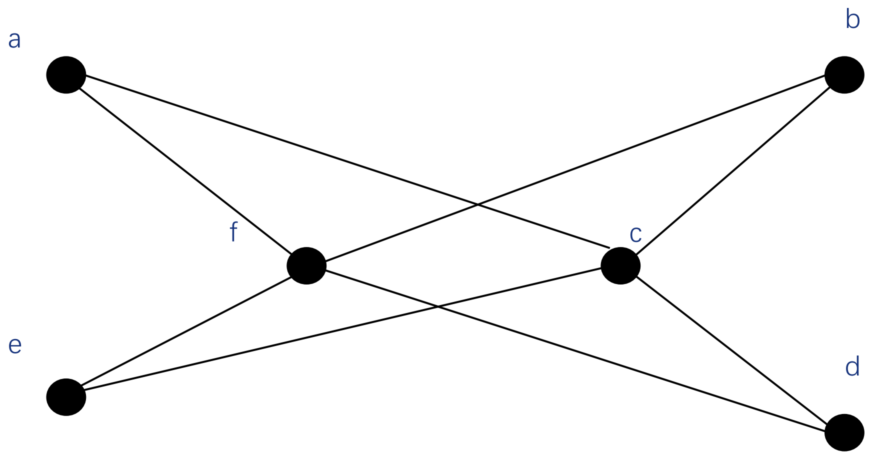
$Q_3$

引理：无向图G是二分图的充要条件是G的所有回路的长度为偶数。

思路：沿cube-n图中的一条边移动相当于将节点的二进制串修改一位。要沿回路回到起点，那么必对原二进制串修改了偶数次。
因此，当$n \geq 1$时，$Q_n$是二分图。

# Exercises

- 14. Determine whether the graph is bipartite. (Yes or No)____yes____



# Exercises

- 15. If an undirected graph *G* has *n* vertices and *m* edges (*m*=*n*+1), prove that *G* has an vertex *v* whose degree d(*v*)>=3.

  - 设n阶m条边的无向图G中，m=n+1，证明G中存在顶点v：d(v)≥3。
  - 证：用反证法，假设不存在顶点度数大于等于3，则$\forall v \in V(G)$,均有$d(v) \leq 2$，由握手定理有：$2m = 2(n+1) = 2n + 2 = \sum d(v_i) \leq 2n$，矛盾！所以G中存在顶点v：d(v)≥3

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# Exercises

- 16. Is there a graph with 102 vertices, such that exactly 49 vertices have degree 5, and the remaining 53 vertices have degree 6? Explain the reason.

  - No, because we know that for any graph with n vertices and e edges, $deg(1) + \cdot \cdot \cdot + deg(n) = 2e$. In our case this formula would reduce to $49 \cdot 5 + 53 \cdot 6 = 2e$, which is impossible since the sum on the left is an odd number, whereas 2e is in any case an even number.

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# End of Section 4.2