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Discrete Mathematics

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Chapter 3. Relations

Representing Relations

Section 3.3

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Contents

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Representing Relations Using Matrices

§ 7.3: Representing Relations

- Before saying more about the n-th power of a relation, let’s talk about representations
- Some ways to represent *n*-ary relations:
  - With a list of n-tuples.
  - With a function from the (n-ary) domain to {T,F}.
- Special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

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- Why bother with alternative representations? Is one not enough?
- One reason: some calculations are easier using one representation, some things are easier using another
- There are even some basic ideas that are suggested by a particular representation

It’s often worth playing around with different representations!

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Using Zero-One Matrices

- To represent a binary relation $R{:}A\times B$ by an $|A|\times|B|$ 0-1 matrix $M_R = [m_{ij}]$, let $m_{ij} = 1$ iff $(a_i,b_j)\in R$.
- *E.g.*, Suppose 小白 likes 小黄 and 小黑, 小红 likes 小黑, and 小绿 likes 小蓝.
- Then the 0-1 matrix representation of the relation Likes:Boys×Girls relation is:

$$
\begin{array}{c c}
 & \begin{array}{ccc} 小黄 & 小黑 & 小蓝 \end{array} \\
\begin{array}{c} 小白 \\ 小红 \\ 小绿 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

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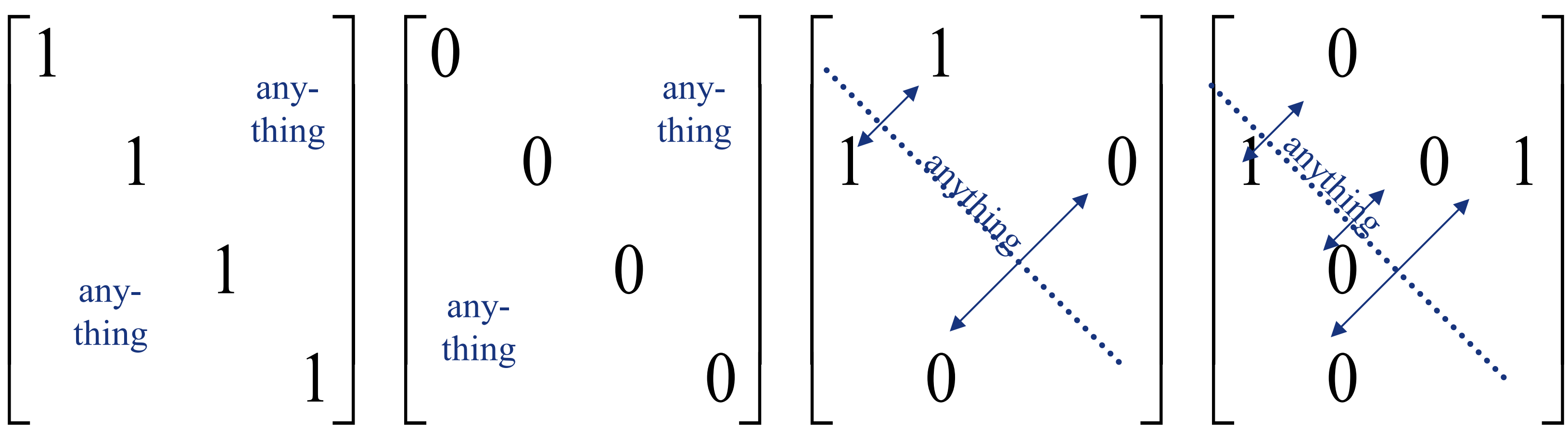
- **Special case 1-0 matrices for a relation on A (that is, $R{:}A \times A$)**
- ***Convention*: rows and columns list elements in the same order**
- **This where 1-0 matrices come into their own!**

Zero-One Reflexive, Symmetric

- Recall: ***Reflexive***, ***irreflexive***, ***symmetric, and asymmetric*** relations.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



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Zero-One Reflexive, Symmetric

- Recall: *Reflexive*, *irreflexive*, *symmetric, and asymmetric* relations.
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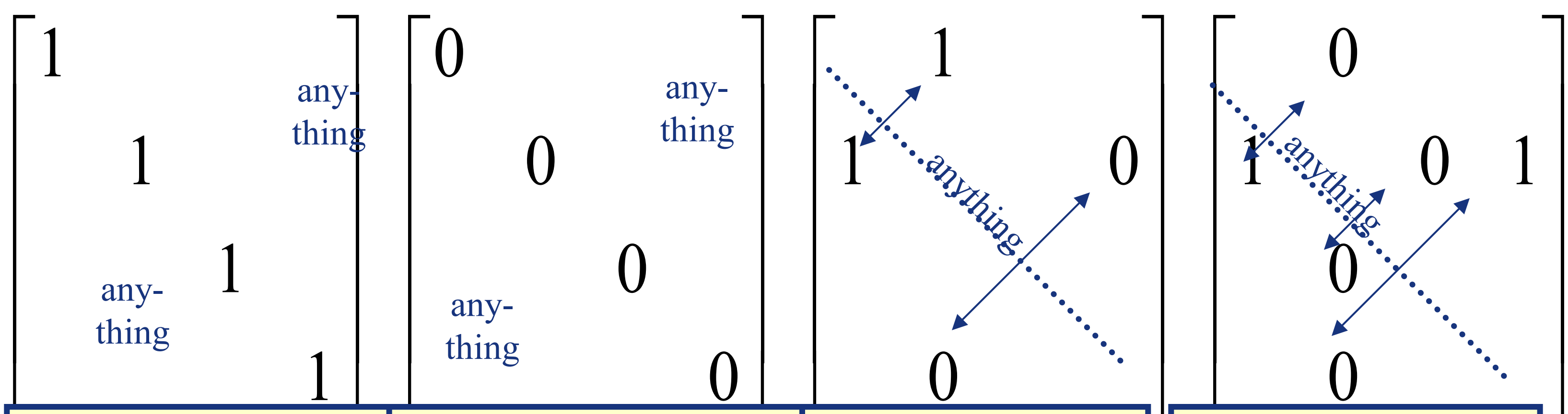

***Reflexive*:** **only 1's on diagon**

***Irreflexive*:** **only 0's on diagonal**

***Symmetric*:** **all identical across diagonal**

***Asymmetric*:** **all 1's are across from 0's**

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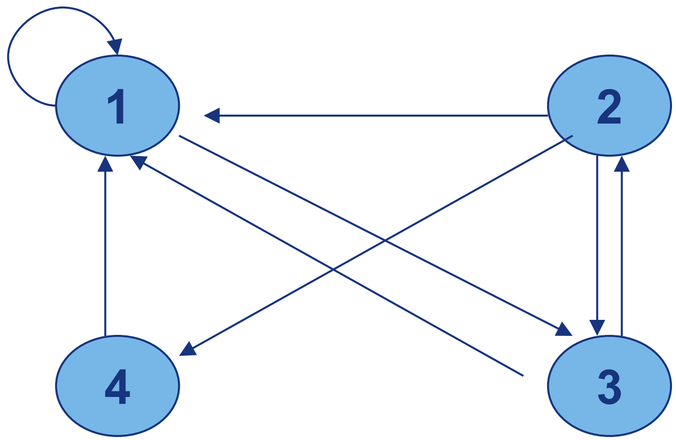
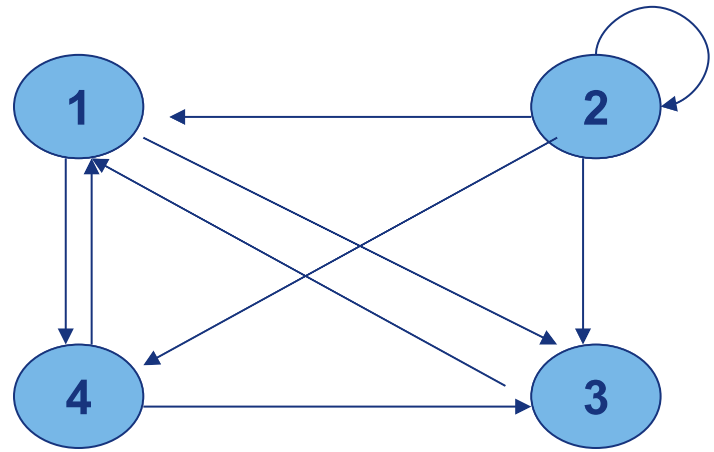
Representing Relations Using Digraphs

Definition

- A directed graph, or digraph, consist of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a,b), and the vertex b is called the terminal vertex of this edge.





- **R1 = { (1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1) }**
- **R2 = { (1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(4,1),(4,3) }**

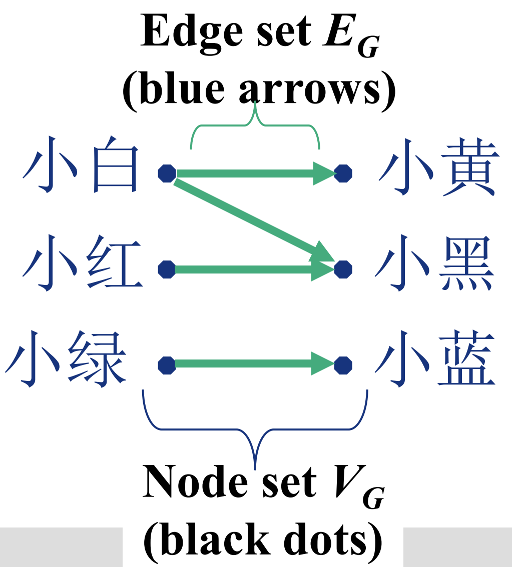
Using Directed Graphs

- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs)*. Visually represented using dots for nodes, and arrows for edges. A relation $R{:}A \times B$ can be represented as a graph $G_R=(V_G=A \cup B,\ E_G=R)$.

Matrix representation $\mathbf{M}_R$:

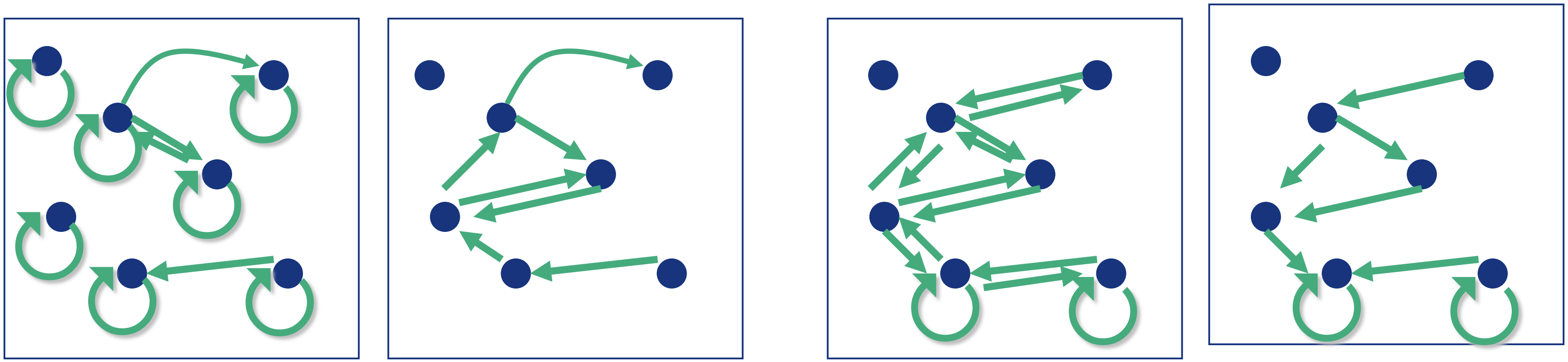
$$
\begin{array}{c}
\\ \text{小白} \\ \text{小红} \\ \text{小绿}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{小黄} & \text{小黑} & \text{小蓝} \end{array} \\
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

Graph rep. G_R:



Digraph Reflexive, Symmetric

Many properties of a relation are easily determined by inspection of its graph.



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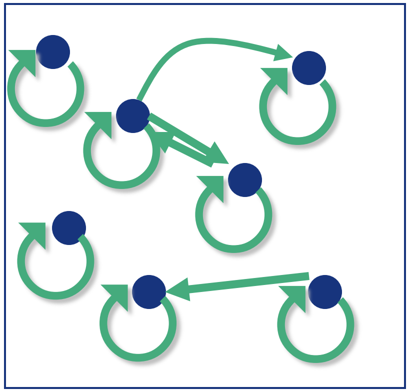
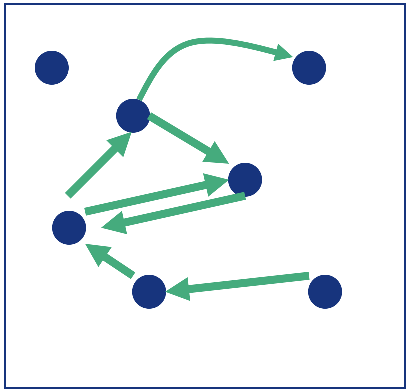
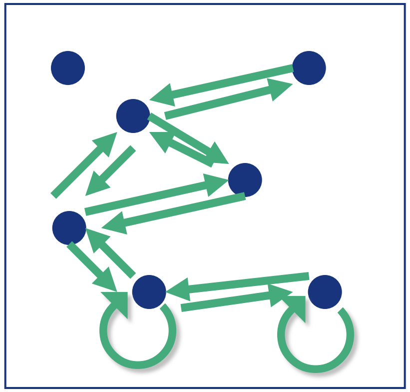


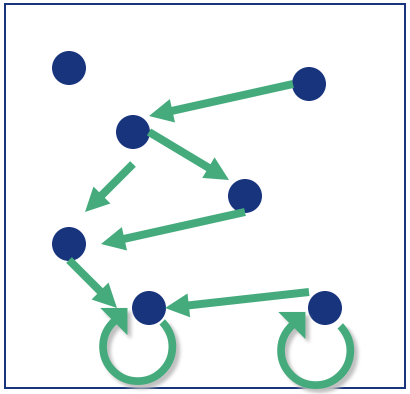
Reflexive: Every node has a self-loop



Irreflexive: No node links to itself



Symmetric: Every link is bidirectional



Asymmetric: No link is bidirectional

These are not symmetric & not asymmetric

These are non-reflexive & non-irreflexive

Particularly easy with a graph

- Properties that are somehow ‘local’ to a given element, e.g.,
  - “does the relation contain any elements that are unconnected to any others?”
- Properties that involve combinations of pairs, e.g.,
  - “does the relation contain any cycles?”
  - things to do with the composition of relations (e.g. the n-th power of R)

Boolean Operation

Example 4

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Example 5

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 6

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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Exercises

Exercises

1. Supposed a binary relation R (Figure 1) on the set A = { 1, 2, 3 }, R is (**B**)

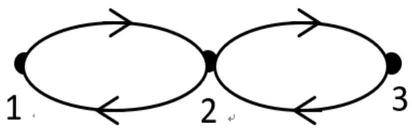
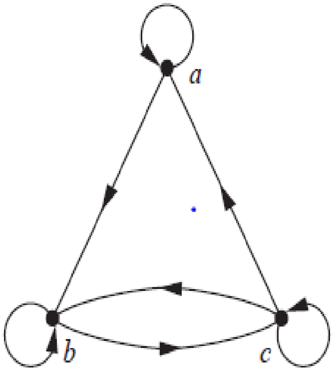


Figure 1.

A. reflexive, antisymmetric, transitive
B. irreflexive, symmetric, non-transitive
C. irreflexive, antisymmetric, transitive
D. reflexive, antisymmetric, non-transitive

Exercises

2. Supposed a binary relation R (Figure 1) on the set A = { a, b, c }, R is (**D**)



A. reflexive, antisymmetric, transitive

B. reflexive, not antisymmetric, transitive

C. not reflexive, symmetric, transitive

D. reflexive, not antisymmetric, non-transitive

There exist (a,b) and (b,c), but there not exists (a,c)

Exercises

3. For the relation {(a, c), (a, d), (b, c), (b, d), (c, a), (c, d)} on the set {a, b, c, d}, which is its property? (**D**)

A. reflexive

B. transitive

C. symmetric

D. none of these properties above

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Exercises

4. Which of the following matrices represents an anti-symmetric relation （ **D** ）?

A) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

B) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

D) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Exercises

5. List the ordered pairs in the relations on {1, 2, 3} corresponding to the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)

Exercises

6. Determine whether the relations represented by the matrix are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Reflexive, symmetric, transitive

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Exercises

7. Determine whether the relations represented by the matrix are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Symmetric

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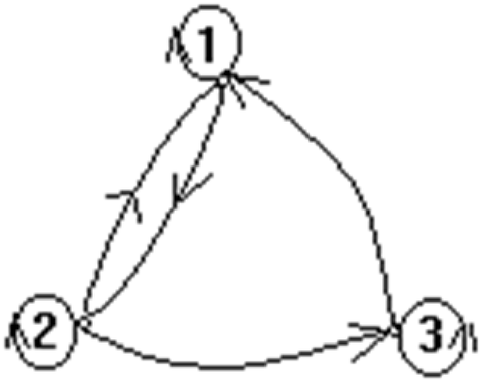
Exercises

8. Set $S = \{1, \quad 2, 3\}$, Fig. 2-(2) shows that R has a relation on S, R is ___A___ ?

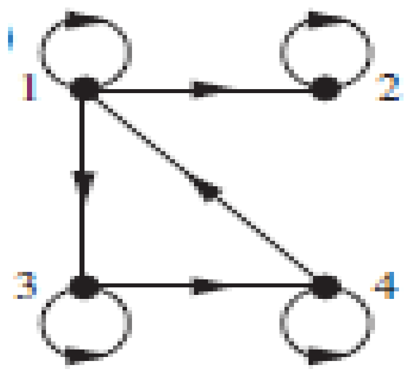
A. reflexive B. symmetric C. antisymmetric D. transitive



Exercises

9. Given the following matrix for a relation, draw the directed graph with vertices 1,2,3,4.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$



Exercises

10. The matrix representing R is $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. The matrix for M_{R^2} is ____.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercises

11. Let R be the relation represented by the matrix $M_R = \begin{bmatrix} 010 \\ 001 \\ 110 \end{bmatrix}$. Please write down the matrix M_{R^4}.

$$M_R=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow M_{R^2}=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow M_{R^4}=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercises

12. $R = \{(a, b) \mid a \neq b\}$. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, . . . , 99, 100\}$ have ?.

100*100-100=9900

Exercises

13. Suppose that the relation R on a set is represented by the matrix $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, which of the following properties does R have?

A. reflexive B. symmetric C. transitive

D. none of these properties above

Exercises

13. Suppose that the relation R on a set is represented by the matrix $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, which of the following properties does R have? (C)

A. reflexive B. symmetric C. transitive

D. none of these properties above

Exercises

14. If relation R on the set S={1,2,…,n} is antisymmetric and reflexive, how many none-zero elements in the matrix of $R \cap R^{-1}$?

Because R is antisymmetric, $\boldsymbol{a_{ij} \neq a_{ji} (i \neq j)}$ and $\boldsymbol{a_{ij} = a_{ji} (i = j)}$.
R^{-1} is the transposition of R, so $\boldsymbol{b_{ij} = a_{ij}}$. $\boldsymbol{b_{ij} = 0\ when\ a_{ji} = 0\ and\ b_{ij} = 1\ when\ a_{ji} = 1 (i \neq j)}$. $\boldsymbol{b_{ii} = 0\ when\ a_{ii} = 0\ and\ b_{ii} = 1\ when\ a_{ii} = 0 (i = j)}$. Sign $\boldsymbol{M_{R \cap R^{-1}}}$ as the matrix of $R \cap \boldsymbol{R^{-1}}$: $\boldsymbol{M_{R \cap R^{-1}} = M_R \wedge M_{R^{-1}}}$
$\boldsymbol{a_{ij} \wedge b_{ij} = 0 (i \neq j)}$.

Therefore, none-zero elements in the matrix of $R \cap \boldsymbol{R^{-1}}$ equals to the number of the none-zero diagonal elements in R. Because R on the set S={1,2,⋯,n} is reflexive, the number of the none-zero diagonal elements in R equals to n.

Exercises

15.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find R^{-1} **and** $\overline{R}$

Exercises

15. Solution:

$$\mathbf{M}_{R^{-1}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{M}_{\overline{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

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End of Section 7.3