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Discrete Mathematics

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Chapter 4. Graphs

Representing Graphs and Graph Isomorphism

Section 4.3

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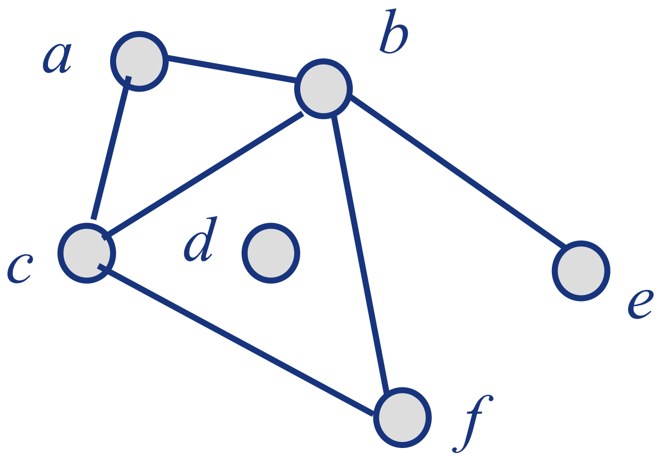
1 Adjacency and Incidence Matrices

2 Isomorphism of Graphs

Representing Graphs

Adjacency Lists of Simple Graphs

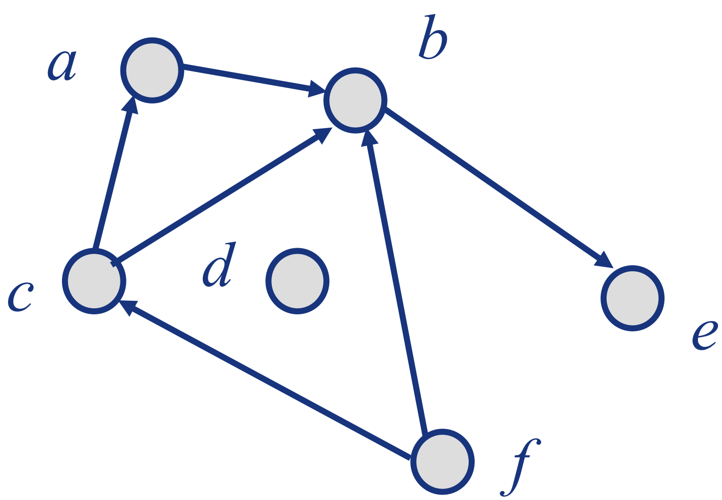
- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	Adjacent Vertices
a	b, c
b	a, c, e, f
c	a, b, f
d	
e	b
f	c, b

Adjacency Lists of Directed Graphs

- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	Adjacent Vertices
a	b
b	e
c	a, b
d	
e	
f	c, b

Adjacency Matrix

- Suppose that $G=<V, E>$ is a simple graph where $|V|=n$.
- Suppose that the vertices of G are listed arbitrarily as $V=\{v_1, v_2, \ldots, v_n\}$.
- The adjacency matrix A (or A_G) of G is the n x n zero-one matrix with 1 as its (i,j)th entry when v_i and v_j are adjacency matrix is A=[a_{ij}].

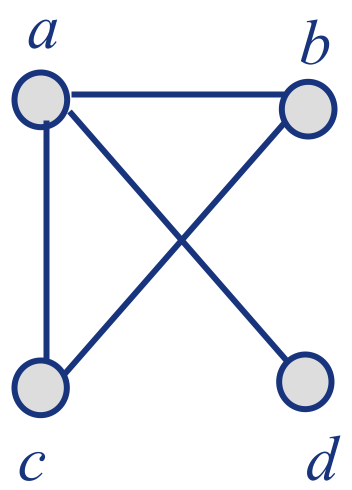
$$a_{ij}=\begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

Some properties of adjacency matrix

- The adjacency matrix of a simple graph is symmetric, that is, $a_{ij} = a_{ji}$, since both of these entries are 1 then v_i **and** v_j are adjacent, and both are 0 otherwise.
- Furthermore, since a simple graph has no loops, each entry a_{ii}, i=1,2,3,…,n is 0.
- There are relatively few edges in a graph, the adjacency matrix is a sparse matrix, that is, a matrix with few nonzero entries.

Example 3

- Use an adjacency matrix to represent the graph.

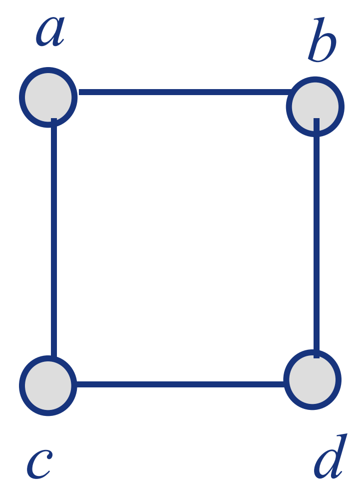


$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Example 4

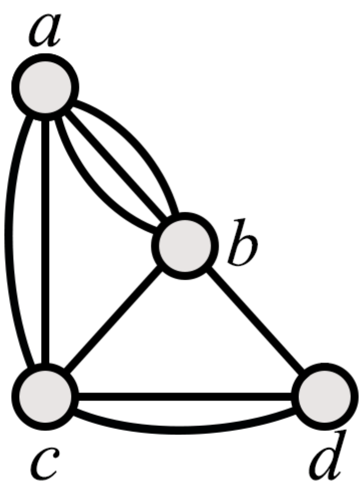
- Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Example 5

- When multiple edges are present, the adjacency matrix is no longer a zero-one matrix, since the (i, j)th entry of this matrix equals the number of edges that are associated to $\{a_i, a_j\}$.



$$\begin{bmatrix} 0 & 3 & 2 & 0 \\ 3 & 0 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

- The matrix for a directed graph ***G*＝<*V*, *E*>** has a 1 in its (i, j)th position if there is an edge from v_i to v_j, where $v_1, v_2, \ldots, v_n$ is an arbitrary listing of the vertices of the directed graph.

$$a_{ij}=\begin{cases} 1 & if\,(v_i, v_j) \quad is \quad an \quad edge \quad of \quad G \\ 0 & \text{otherwise} \end{cases}$$

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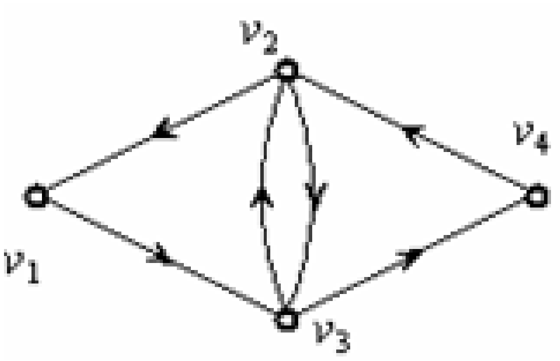
$$a_{ij}=\begin{cases}1 & if\,(v_i, v_j) \quad is \quad an \quad edge \quad of \quad G\\ 0 & \text{otherwise}\end{cases}$$

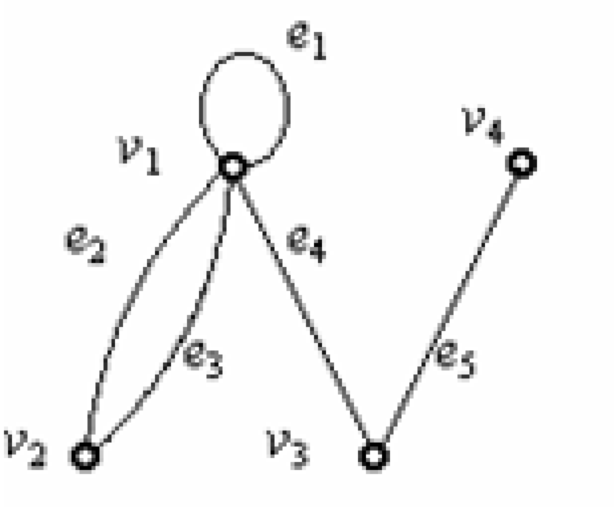


$$A=\begin{pmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix}$$

Incidence Matrices

- Let ***G*=<*V*, *E*>** be an undirected graph. Suppose that $v_1, v_2, \dots, v_n$ are vertices and $e_1, e_2, \dots, e_n$ are the edges of G.
- Then the incidence matrix with respect to this ordering of V and E is the n x m matrix **M = [m_{ij}]**, where

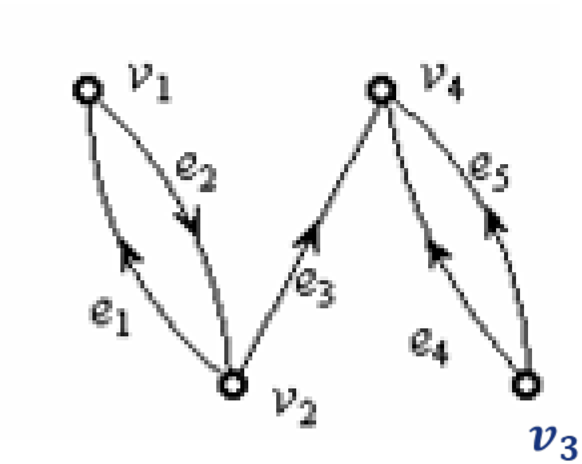
$$m_{ij} = \begin{cases} 1 & when \quad edge \quad e_j \quad is \quad incident \quad with \quad v_i \\ 0 & \text{otherwise} \end{cases}$$



$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Directed Graphs

$$m_{ij}=\begin{cases} 1 & v_i \ \ is \ \ the \ \ begining \ \ of \ \ e_j \\ 0 & v_i \ \ is \ \ not \ \ incident \ \ with \ \ e_j \\ -1 & v_i \ \ is \ \ the \ \ ending \ \ of \ \ e_j \end{cases}$$



$$M_2=\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

Composite information matrix

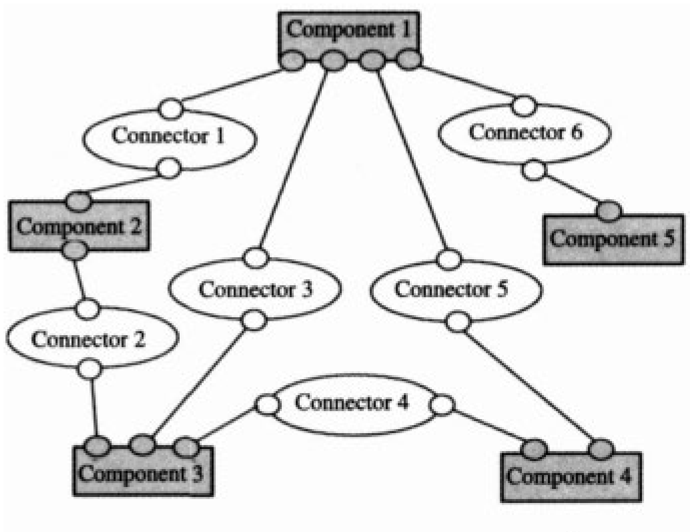


图 3　SA 模型示例

$$a_{ij}=\begin{cases}-1, \text{当连接件}i\text{被构件}j\text{指向时} \\ 1, \text{当连接件}i\text{指向构件}j\text{时} \\ x, \text{当连接件}i\text{与构件}j\text{互相指向时} \\ 0, \text{其他情况}\end{cases}$$

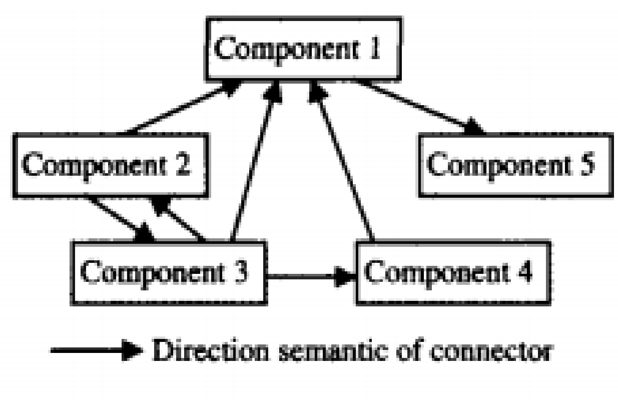


图 4　简化模型

图 3 和图 4 的 SA 结构可以由复合信息矩阵 A_1 表示：

$$A_1=\begin{bmatrix} 1 & -1 & & & \\ & x & x & & \\ 1 & & -1 & & \\ & & -1 & 1 & \\ 1 & & & -1 & \\ -1 & & & & 1 \end{bmatrix}$$

Isomorphism of Graphs

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What is isomorphic?

- The simple graphs $G_1=(V_1, E_1)$ and $G_2 =(V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from V_1 to V_2.
- with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2, for all a and b in V_1.
- Such a function is called an isomorphism.

Graph Isomorphism

- **Formal definition:**
- **Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* iff $\exists$ a bijection $f:V_1 \rightarrow V_2$ such that $\forall$ $a,b \in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.**
- ***f* is the "renaming" function between the two node sets that makes the two graphs identical.**
- **This definition can be extended to other types of graphs.**

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An equivalence relation between graphs

- Unlike ordinary pictures, we can say precisely when two graphs are similar
- Obviously, a given graph (V,E) may be drawn in different ways.
- But even (V,E) and (V',E') (where V<>V' and E<>E') may in some sense be equivalent:
- Graph **isomorphism** (informal):
  - Two graphs are isomorphic iff they are identical except for their node names.

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- Graphs that are isomorphic share all their ‘important’ properties, e.g.,
  - The number of nodes and edges
  - The degrees of all their nodes
  - Whether they are bipartite or not, etc.
- How would you define graph isomorphism formally?
  - For simplicity: focus on simple graphs
  - Hint: use the notion of a bijection

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Graph Isomorphism

- How can we tell whether two graphs are isomorphic?
- The best algorithms that are known to solve this problem have exponential worst-case time complexity. (Faster solutions may be possible.)
- In practice, a few tests go a long way …

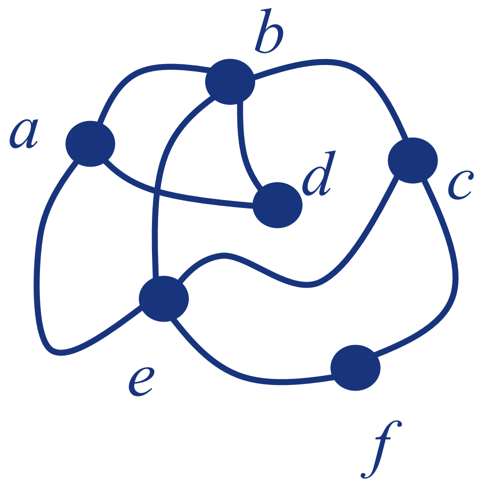
Graph Invariants under Isomorphism

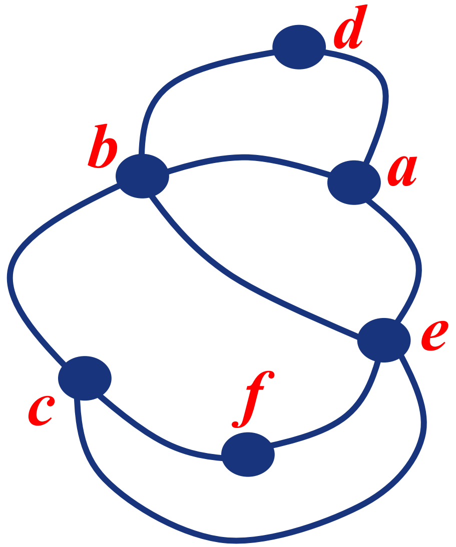
Necessary **but not** ***sufficient*** **conditions for** $G_1=(V_1, E_1)$ **to be isomorphic to** $G_2=(V_2, E_2)$:

- $|V_1|=|V_2|$ **and** $|E_1|=|E_2|$.
- **The number of vertices with degree** n **is the same in both graphs.**
- **For every proper subgraph** g **of one graph, there is a proper subgraph of the other graph that is isomorphic to** g.

Isomorphism Example

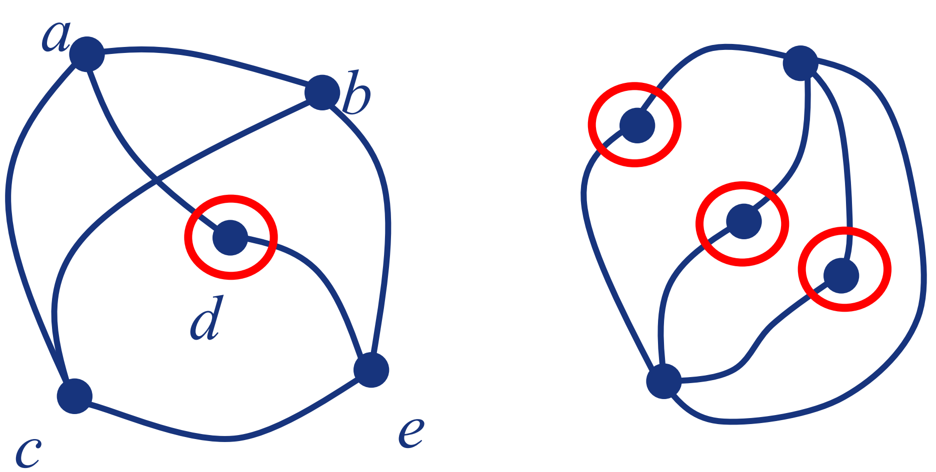
- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.





Are These Isomorphic?

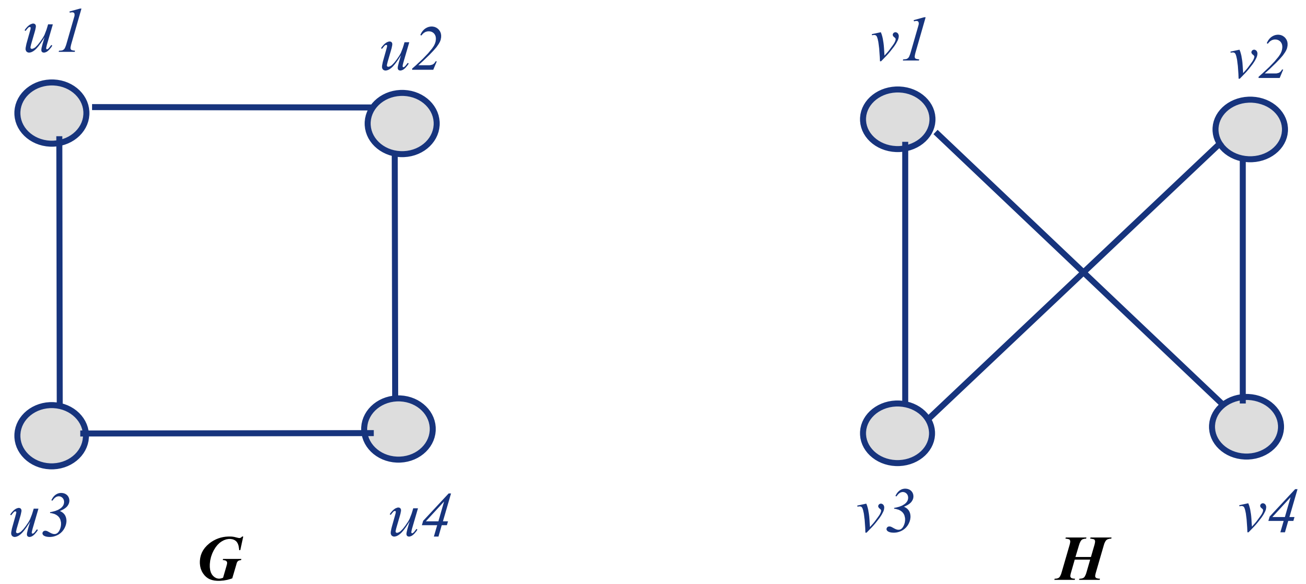
- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



- ***Same number of vertices***
- ***Same number of edges***
- ***Different number of verts of degree 2!***
  (1 vs 3)

Isomorphism of Graphs

- **Example 8: G=(V, E) and H=(W, F)**



$$f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3, f(u_4) = v_2$$

$$(u_1, u_2), (u_1, u_3), (u_2, u_4), (u_4, u_3)$$

$$(v_1, v_4), (v_1, v_3), (v_4, v_2), (v_2, v_3)$$

Time Complexity

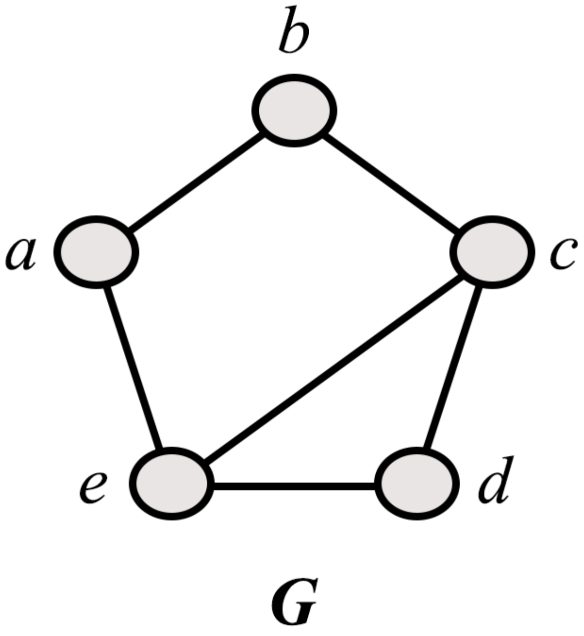
- The are n! possible one-to-one correspondences between the vertex sets of two simple graphs with n vertices.
- However, we can often show that two simple graphs are not isomorphic.
- The properties of isomorphic graphs are two invariants:

  (1) same number of vertices

  (2) same degrees of v and f(v)

Isomorphism of Graphs

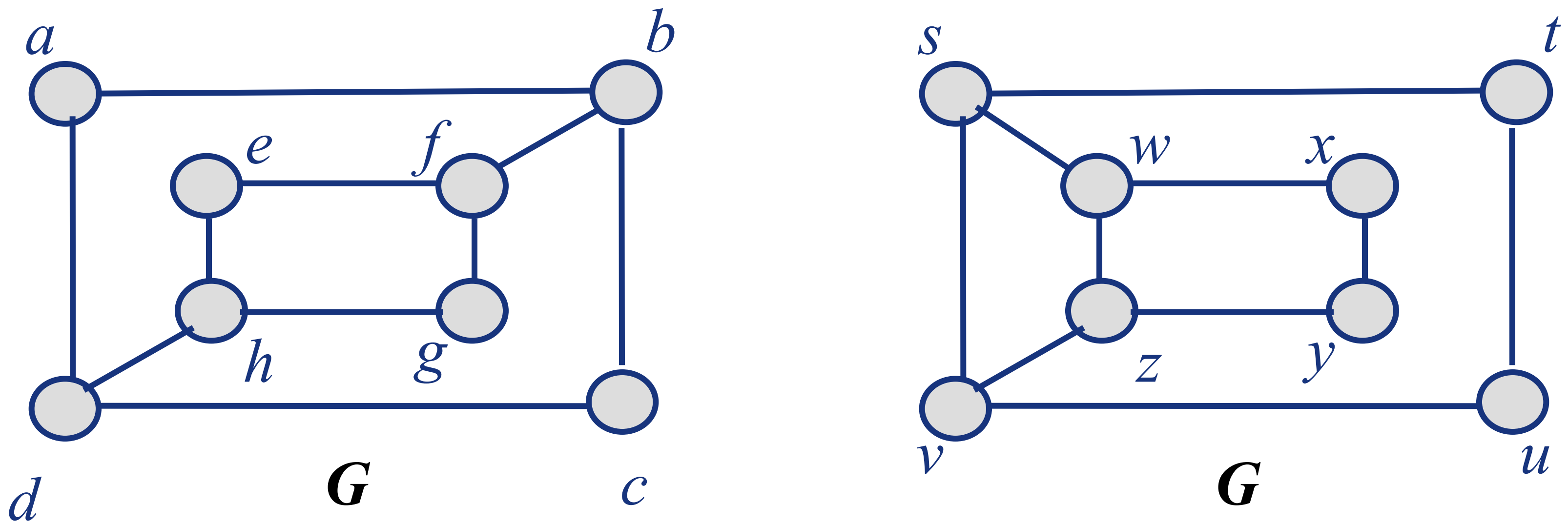
- Example 9: G and H



- G has no vertices of degree one, but H has one.

Isomorphism of Graphs

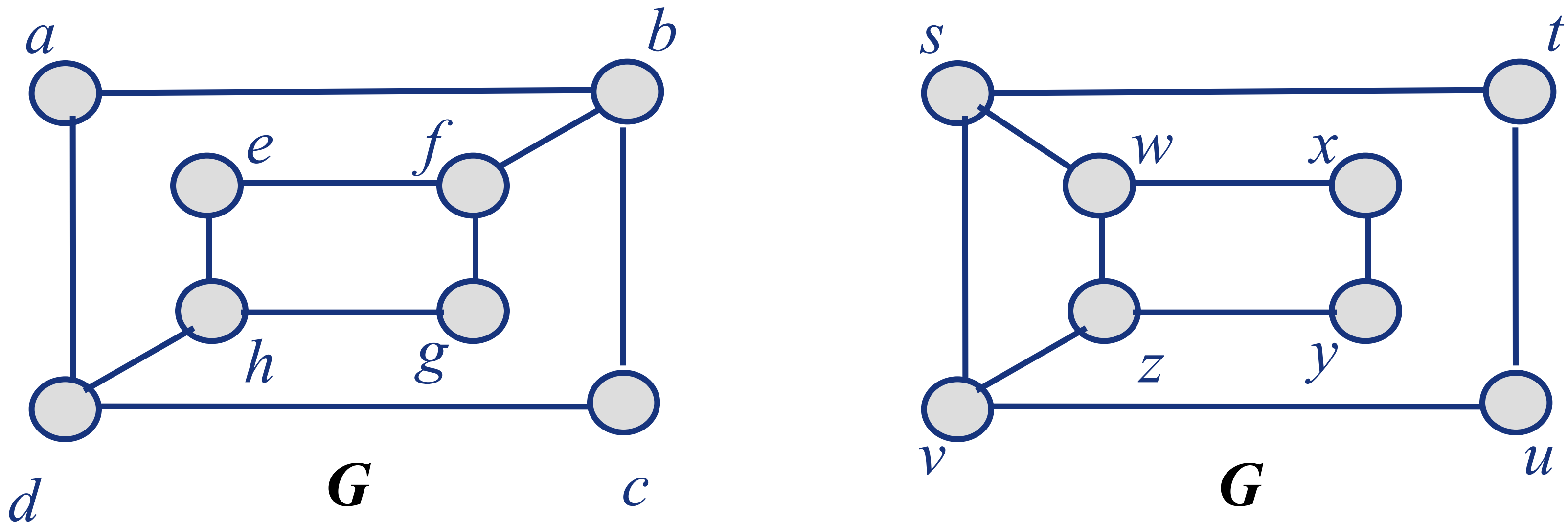
- Example 10: G and H



- They also both have four vertices of degree two and four of degree three. Are they isomorphism?

- **Conceivable!!**

Isomorphism of Graphs



$$f(a)=t\,?\ f(a)=x\,?\ f(a)=u\,?\ f(a)=y\,?$$

- **However, their adjacent vertices are different for their degrees are different.**

- **Not isomorphism!!**

Not isomorphism example

- Another way to see that G and H are not isomorphic is to note that
- the subgraphs of G and H made up of vertices of degree three and the edges connecting them must be isomorphic
- if these two graphs are isomorphic.

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- To show that a function f from the vertex set of a graph G to the vertex set of a graph H is an isomorphism, we need to show that f preserves edges.
- One helpful way to do this is to use adjacency matrices.
- If f is isomorphism, we can show that the adjacency matrix of G is the same as the adjacency matrix of H with corresponding rows and columns.

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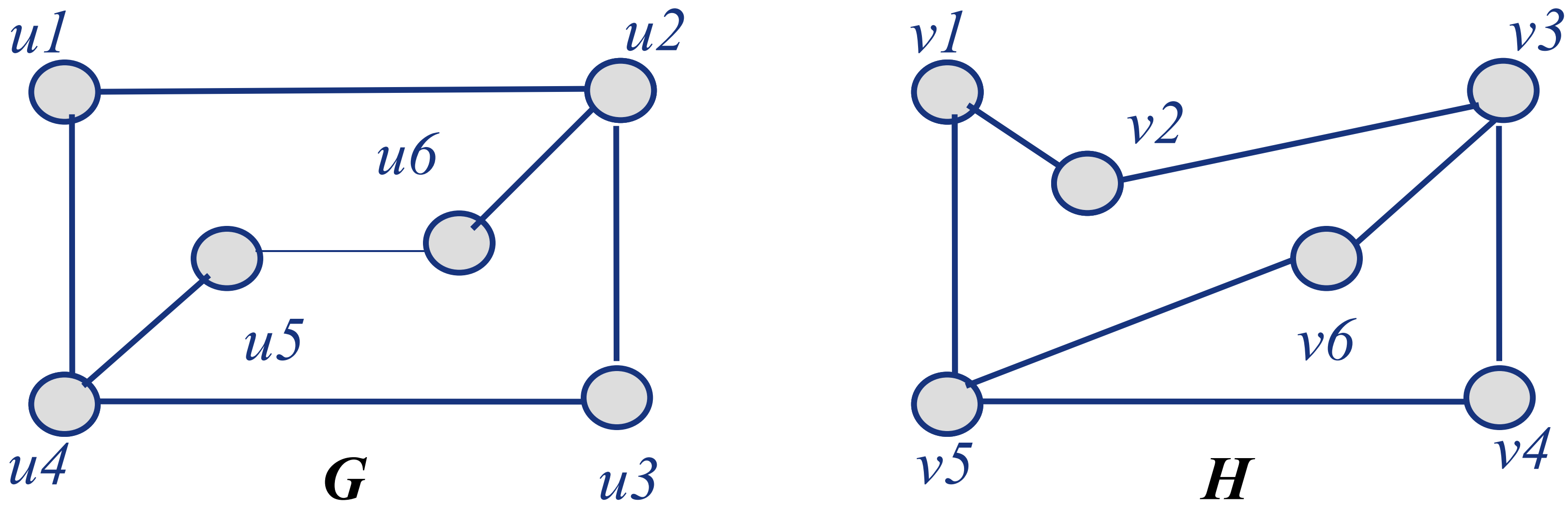
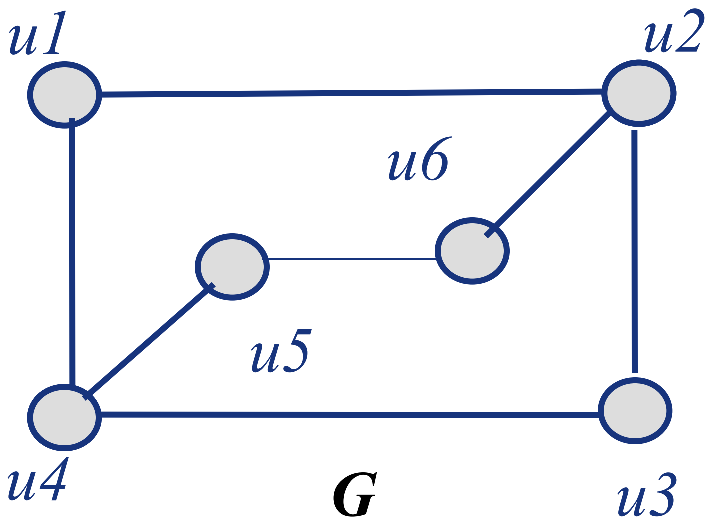
Isomorphism of Graphs



- G and H have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three.
- The subgraphs of G and H consisting of all vertices of degree two and the edges connecting them are isomorphic.

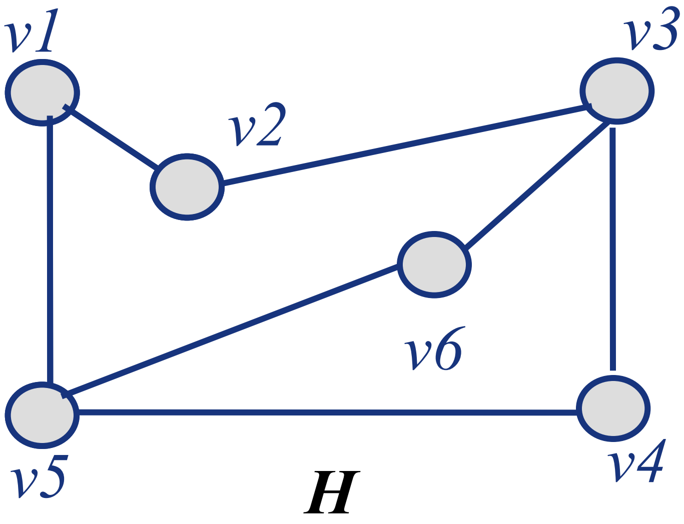
- Conceivable!!

Adjacency Matrix of G



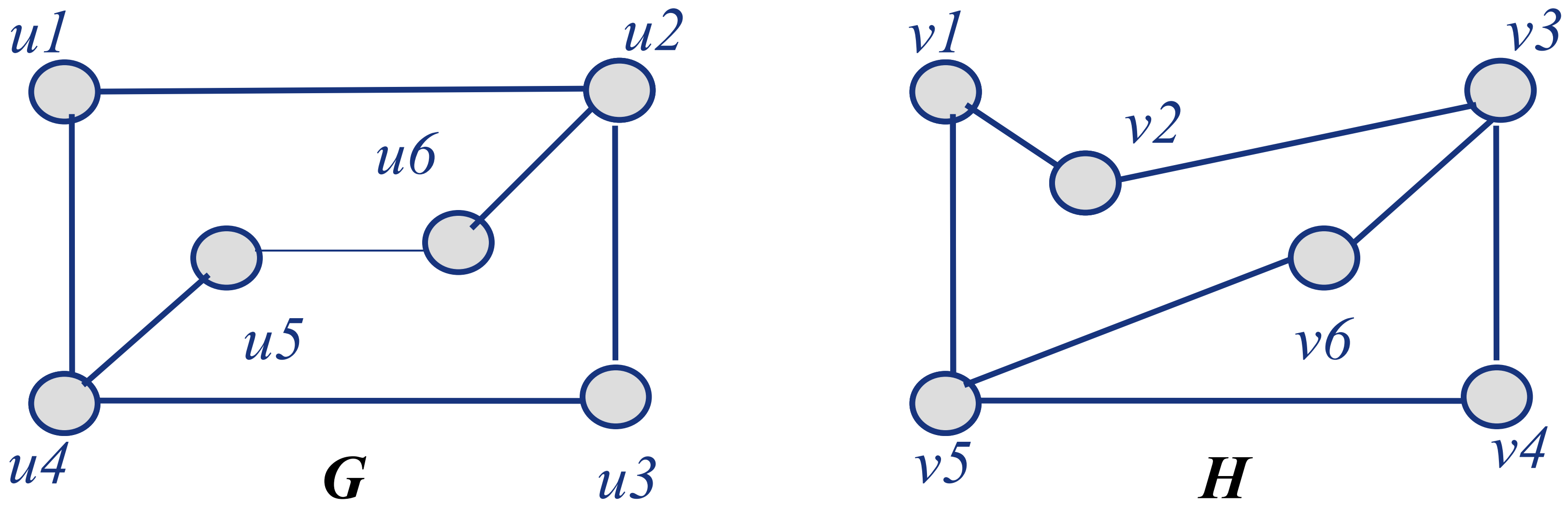
$$
\begin{array}{c|c}
 & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\
\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} &
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

Adjacency Matrix of H



$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 0 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 & 1 & 0 \\ v_5 & 1 & 0 & 0 & 1 & 0 & 1 \\ v_6 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Isomorphism of Graphs



- u1’s adjacency vertices (u2 and u4) have 3 degree.
- v6 or v4.
- Try f(u1)=v6, f(u2)=v3, f(u3)=v4, f(u4)=v5
  f(u5)=v1, f(u6)=v2

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f(u1)=v6, f(u2)=v3, f(u3)=v4, f(u4)=v5, f(u5)=v1, f(u6)=v2

$$\begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c c} & \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{array}{c} v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

- The best algorithms known for determining whether two graphs are isomorphic have exponential worst-case time complexity.
- Linear average-case time complexity for some.
- Polynomial worst-case time complexity is hoped.
- The best algorithm is called NAUTY, which uses less than one second in determining two 100-vertex graphs are isomorphic.

necessary condition

- Two isomorphic graphs have:
- 1、same number of vertices
- 2、same number of edges
- 3、same number of same-degree vertices

- However, there has been no sufficient condition for determining whether two graphs are isomorphic.

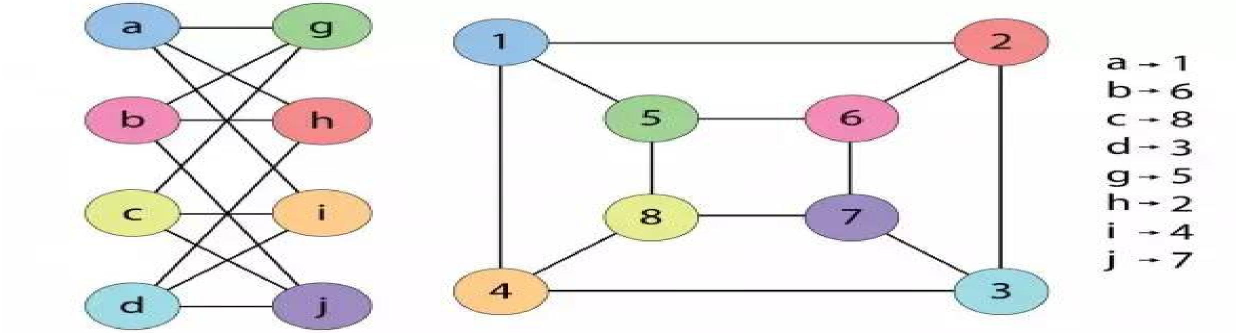
Applications

Applications-isomorphic

芝加哥大学的数学家与计算机科学家拉斯洛·鲍鲍伊（László Babai）提出了一种数学方法，可以将一个原本被认为属于NP的问题，即判断两个网络是不是完全相同，变成了较为容易的P问题。不管这些网络有多复杂或缠结。

László Babai

问题原型：无论是传染流感的人群，还是与生物体发生相互作用的蛋白质，都可以抽象为一系列的点（计算机专业术语称为“节点”，node），它们之间的相互关系则用连接点的直线（称为“边”，edge）来表示。由于节点可以被任意地拖来拖去，所以即使是两个看起来完全不同的图，其连接方式也可能是完全相同的（如下）



Applications-isomorphic

于芝加哥大学主办的“Combinatorics and Theoretical Computer Science seminar”会议上，鲍鲍伊对他的工作进行了汇报：

- He outlines an algorithm that solves the Graph Isomorphism (GI) problem and the related problems of String Isomorphism (SI) and Coset Intersection (CI) in quasipolynomial (exp(polylogn)) time.

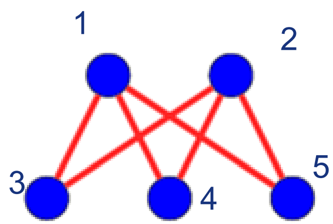
- The best previous bound for GI was exp($\sqrt{n\log n}$), where n is the number of vertices (Luks, 1983). For SI and CI the best previous bound was similar, exp($\sqrt{n}(\log n)^c$), where n is the size of the permutation domain (the speaker, 1983)

"如果这一方法是对的，它可能会成为计算机理论领域和复杂理论十年来最重要的成果。并给互联网领域带来新的曙光。因为新的突破与网络之间的比较有关，正可以应用于人与人之间的互联网联结，目前很多难以解决的问题最终都可以归结到比较两个网络是否相同这一任务上。"

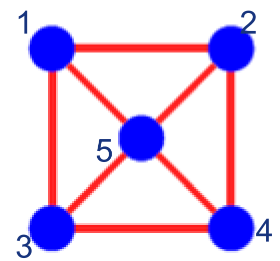
--麻省理工学院的计算机科学家、博客作者斯科特·阿伦森（Scott Aaronson）

Applications

1.Represent each of these graphs with an adjacency matrix.



$K_{2,3}$



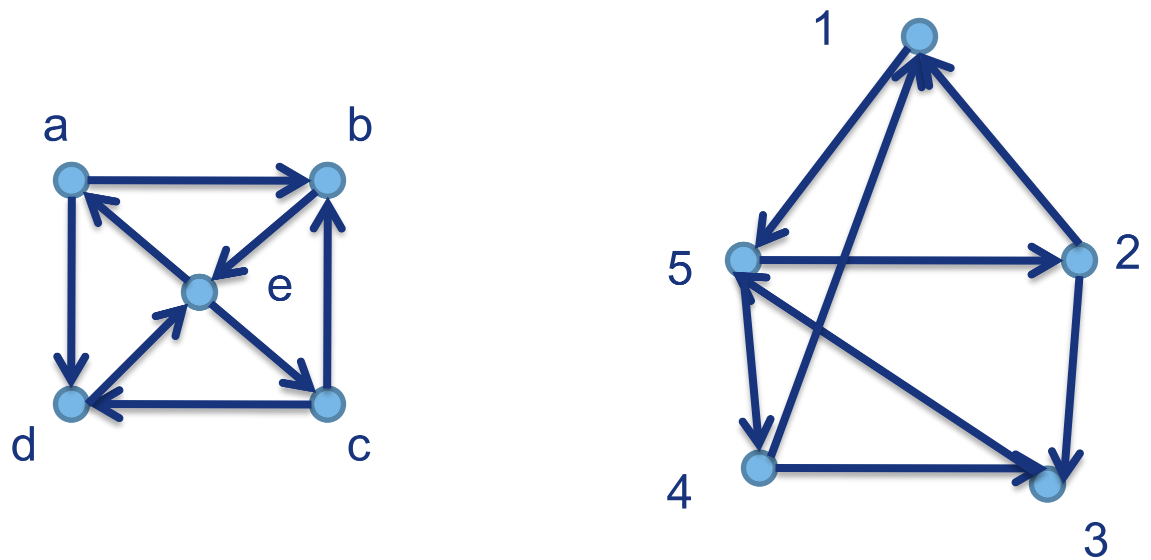
W_4

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

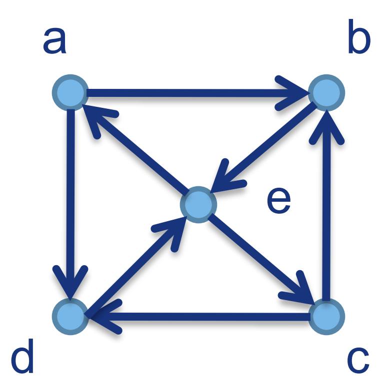
Applications

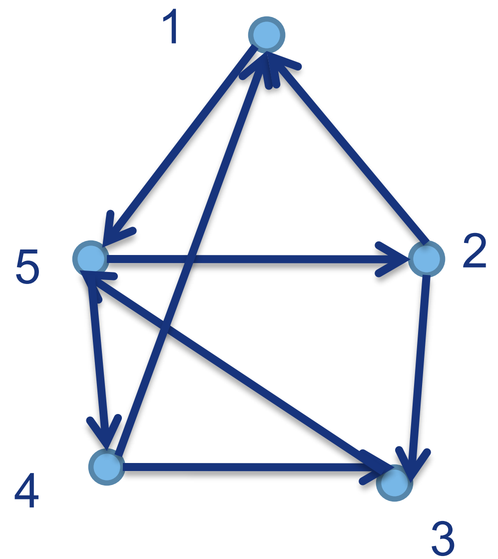
2. Determine whether the two graphs are isomorphism.



Applications

2. Determine whether the two graphs are isomorphism.





Define the function: $f(a)=2, f(b)=3, f(c)=4, f(d)=1, f(e)=5$

They are isomorphism.

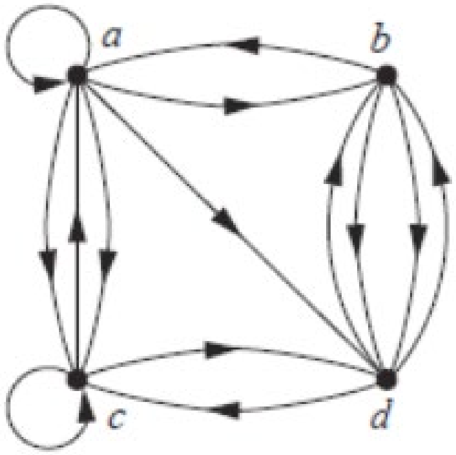
Exercises

1. A directed graph $D=<V,E>$, $V=\{1,2,3,4\}$, $E=\{<1,2>,<1,4>,<2,3>,<3,4>,<2,4>,<3,1>\}$, the adjacency matrix of D is.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercises

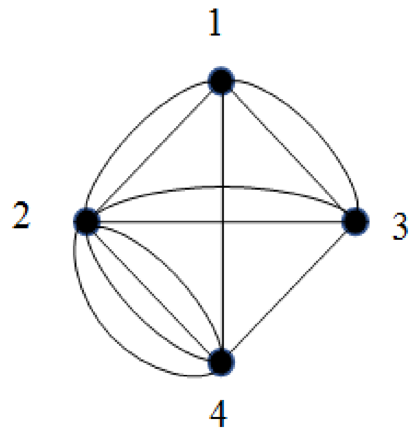
2. With an adjacency matrix ______________________ to represent the following directed multigraph.



$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Exercises

3. With an adjacency matrix ______________________ to represent the multigraph below



$$\begin{bmatrix} 0221 \\ 2024 \\ 2201 \\ 1410 \end{bmatrix}$$

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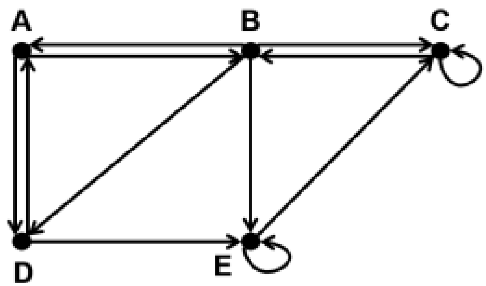
Exercises

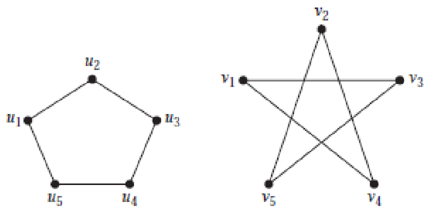
4. Use an adjacency matrix _______________ to represent the graph below

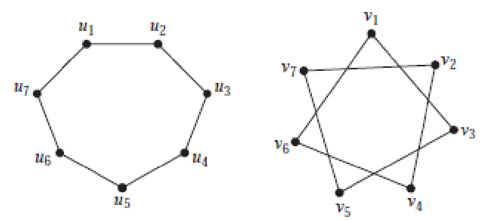


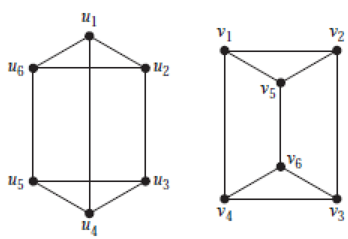
$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

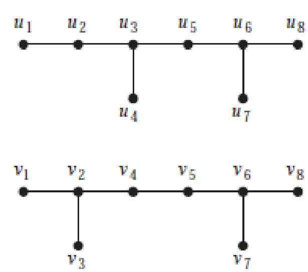
Exercises

5. Which of the given pair of graphs is NOT isomorphic? (D).

A 

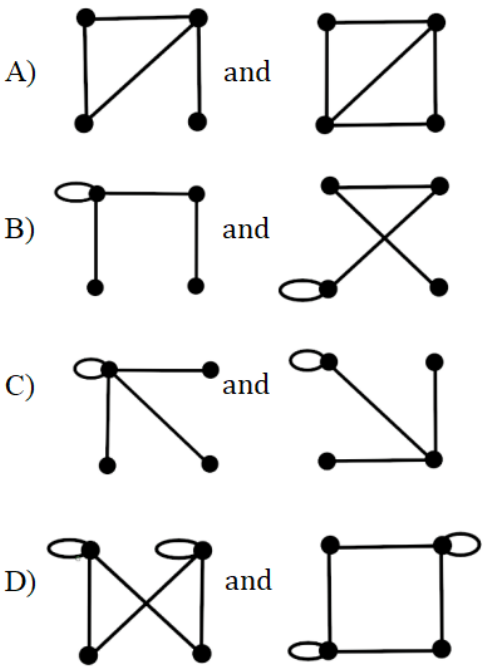
B 

C 

D 

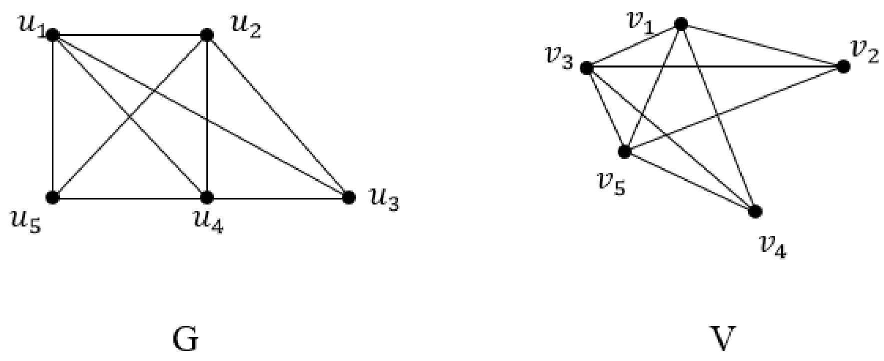
Exercises

6. Which of the given pairs of graphs is isomorphic.(D)



Exercises

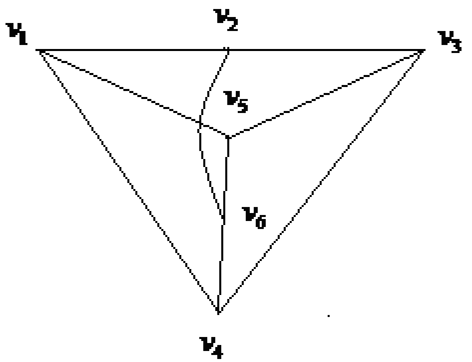
9. Determine whether graphs G and V are isomorphic. If isomorphic, show the isomorphic function, else identify their difference.

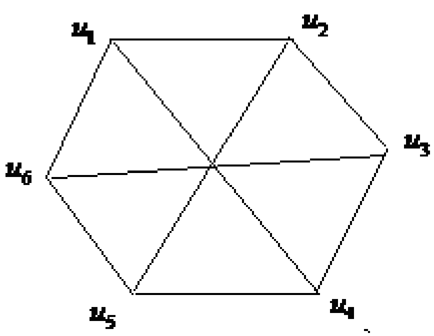


$$f(u_1) = v_3, f(u_2) = v_1, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_4$$

Exercises

10. Whether the graph shown in Figure 1 is isomorphic to Figs? Please give the reason.



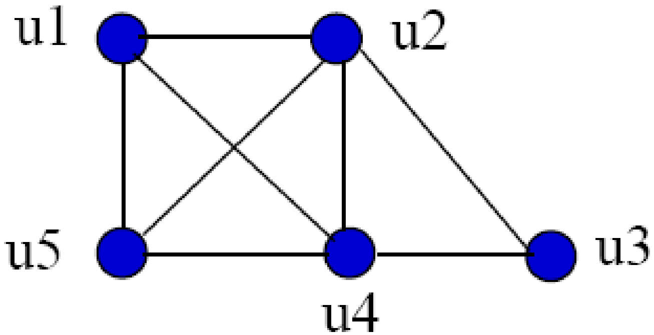


The above two figures are isomorphic. According to the relationship between points and edges, get$h: V_1 \rightarrow V_2, h(v_i) = u_i, i = 1,2,3,4$ $h(v_5) = u_6, h(v_6) = u_5$

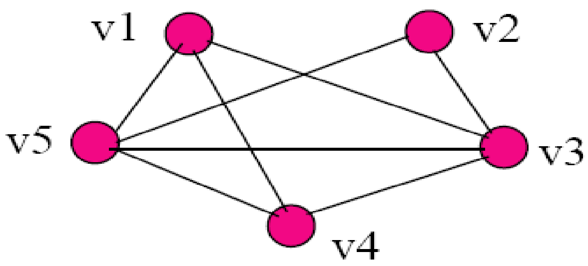
Obviously h is bijective and satisfies the definition of isomorphism.

Exercises

11. Find out whether G and H are isomorphic. No matter what the judgment is, please give your explanation and argument.



G



H

Solution:
deg(u3) = deg(v2) = 2 so f(u3) = v2 is our only choice.
deg(u1) = deg(u5) = deg(v1) = deg(v4) = 3 so we must have either f(u1) = v1 and f(u5) = v4 or f(u1) = v4 and f(u5) = v1.
Finally since deg(u2) = deg(u4) = deg(v3) =deg(v5) = 4 we must have either f(u2) = v3 and f(u4) = v5 or f(u2) = v5 and f(u4) = v3.
We first try the relabeling using in each case to get the function f(u3)=v2, f(u1)=v1, f(u5)=v4, f(u2)=v3, f(u4)=v5. Thus, G and H are isomorphic.

Logo

End of Section 4.3