Logo

Discrete Mathematics

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Chapter 3. Relations

Equivalence Relations

Section 3.5

Contents

Equivalence Relations

§ 7.5: Equivalence Relations

- **Definition: An *equivalence relation* on a set *A* is any binary relation on *A* that is *reflexive, symmetric, and transitive.***
  - ***E.g.*, = is an equivalence relation.**
  - **But many other relations follow this pattern too**

§ 7.5: Equivalence Relations

- Definition: An *equivalence relation* on a set A is any binary relation on A that is *reflexive, symmetric, and transitive.*
- *E.g.*, = is an equivalence relation.
- For any function $f:A\rightarrow B$, the relation "have the same f value", or $=_f :\equiv \{(a_1,a_2) \mid f(a_1)=f(a_2)\}$
is an equivalence relation,
- *e.g.*, let m="mother of" then $=m$ = "have the same mother" is an equivalence relation

Equivalence Relation Examples

- "Strings *a* and *b* are the same length."
- "Integers *a* and *b* have the same absolute value."

Let's talk about relations between functions:

1. How about: $R(f,g) \Leftrightarrow f(2)=g(2)$?
2. How about: $R(f,g) \Leftrightarrow f(1)=g(1) \vee f(2)=g(2)$?

Equivalence Relation Examples

1. How about: $R(f,g) \Leftrightarrow f(2)=g(2)$?

⑩ Yes. **Reflexivity**: f(2)=f(2), for all f
Sym: f(2)=g(2) implies g(2)=f(2)
Trans: f(2)=g(2) and g(2)=h(2) implies f(2)=h(2).

1. How about: $R(f,g) \Leftrightarrow f(1)=g(1) \vee f(2)=g(2)$?

Equivalence Relation Examples

How about $R(f,g) \Leftrightarrow f(1)=g(1) \vee f(2)=g(2)$?

- No. Counterexample against transitivity:

$f(1)=\mathbf{a}$, $f(2)=b$

$g(1)=\mathbf{a}$, $g(2)=\mathbf{c}$

$h(1)=b$, $h(2)=\mathbf{c}$

Example 4

- Congruence Modulo m
- Let m be a positive integer with m>1. Show that the relation
- $R = \{ (a,b) \mid a \equiv b \pmod{m} \}$
- is an equivalence relation on the set of integers.

Example 5

设 $A=\{T_1,T_2,T_3,T_4,T_5,T_6\}$ 是某台微机上 6 项任务的集合，有五个子程序 S_1，S_2，S_3，S_4 和 S_5 供它们选择调用，下表列出了它们调用子程序的情况。

任务名称	调用的子程序
T_1	S_1,S_2
T_2	S_2,S_3
T_3	S_3,S_1
T_4	S_5
T_5	S_4
T_6	S_5

定义 A 上的关系 $\varphi=\{(x,y)\mid x,y\in A\text{且}x\text{与}y\text{调用了相同的子程序}\}$，$\varphi$ 是一个等价关系。

$$\varphi=\{(T_1,T_1),(T_1,T_2),(T_2,T_1),(T_2,T_2),(T_1,T_3),(T_3,T_1),(T_2,T_3),(T_3,T_2),(T_3,T_3),(T_4,T_4),(T_4,T_6),$$
$$(T_6,T_4),(T_6,T_6),(T_5,T_5)\}$$

Applications

设R表示S × S上的二元关系，当且仅当xy=uv时，便有<x,y>R<u,v>，试证明R是S× S上的等价关系

Applications

证明

（1）对任意<x,y>∈S× S，由xy=xy，所以<x,y>R<x,y>。所以R是自反的。

（2）对任意 $<x,y>,<u,v>\in S\times S,$

$$<x,y>R<u,v>\Rightarrow xy=uv$$

$$\Rightarrow uv=xy$$

$$\Rightarrow<u,v>R<x,y>$$

所以R是对称的。

Applications

证明

（3）对任意$<x,y>,<u,v>,<w,t>\in S\times S$,

$$<x,y>R<u,v>\wedge<u,v>R<w,t>\Rightarrow(xy=uv)\wedge(uv=wt)$$

$$\Rightarrow xy=wt$$

$$\Rightarrow<x,y>R<w,t>$$

所以R是传递的。

由（1）（2）（3）知，R是等价关系。

Equivalence Class

Definition 2

- Let **R** be an equivalence relation on a set **A**.
- The set of all elements that are related to an element **a** of **A** is called the equivalence class of **a**.
- The equivalence class of a with respect to R is denoted by $[a]_R$.
- When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class.

- In other words, if R is an equivalence on a set A, the equivalence class of the element a is
- $[a]_R = \{ s \mid (a,s) \in R \}$
- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of this class.
- $[0]=\{\ldots,-8,-4,0,4,8,\ldots\}$

  $\{ (a,0) \mid a \equiv 0 \pmod{4} \}$

Equivalence Classes

- Why can we talk so loosely about elements being equivalent to each other (as if the relation didn't have a direction)?
- In some sense, it does not matter which representative of an equivalence class you take as your starting point:

  If aRb then $\{ x \mid aRx \} = \{ x \mid bRx \}$

Equivalence Classes

If aRb then aRx $\Leftrightarrow$ bRx

Proof:

1. Suppose aRb while bRx.
Then aRx follows directly by *transitivity.*

2. Suppose aRb while aRx.

aRb implies bRa *(symmetry).* But
bRa and aRx imply bRx by *transitivity*

Equivalence Classes

We now know that

If aRb then { x | aRx } = { x | bRx }

Equally,

If aRb then { x | xRa } = { x | xRb }

(due to symmetry)

In other words, an equivalence class based on R is simply a maximal set of things related by R

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Equivalence Class Examples

- "(Strings *a* and *b*) *have the same length*."
  - Suppose a has length 3. Then [*a*] = the set of all strings of length 3.
- "(Integers *a* and *b*) *have the same absolute value*."
  - [*a*] = the set {*a*, −*a*}

Equivalence and Partitions

Partitions

- A *partition* of a set *A* is a collection of **disjoint** nonempty **subsets** of A that have **A as their union**.

- Intuitively: a partition of A divides A into separate parts (in such a way that there is no remainder).

Partitions and equivalence classes

- Consider a *partition* of a set A into $A_1, ..A_n$
  - The A_i's are all disjoint : For all x and for all i, j $\in[1,n]$, if $x\in A_i$ and $x\in A_j$ then $A_i = A_j$
  - The union of the A_i's = A

Partitions and equivalence classes

- A *partition* of a set *A* can be viewed as the set of all the equivalence classes $\{A_1, A_2, \ldots\}$ for some equivalence relation on *A*.
- For example, consider the set A={1,2,3,4,5,6} and its partition {{1,2,3},{4},{5,6}}
- R = { (1,1),(2,2),(3,3),(1,2),(1,3),(2,3),(2,1),(3,1),(3,2) ,(4,4), (5,5),(6,6),(5,6),(6,5) }

Partitions and equivalence classes

Theorem

给定集合 A 的一个划分 $\pi = \{S_1, S_2, \cdots, S_m\}$, 则由该划分确定的关系 $R = (S_1 \times S_1) \cup (S_2 \times S_2) \cup \cdots \cup (S_m \times S_m)$ 是 A 上的等价关系。

Proof.

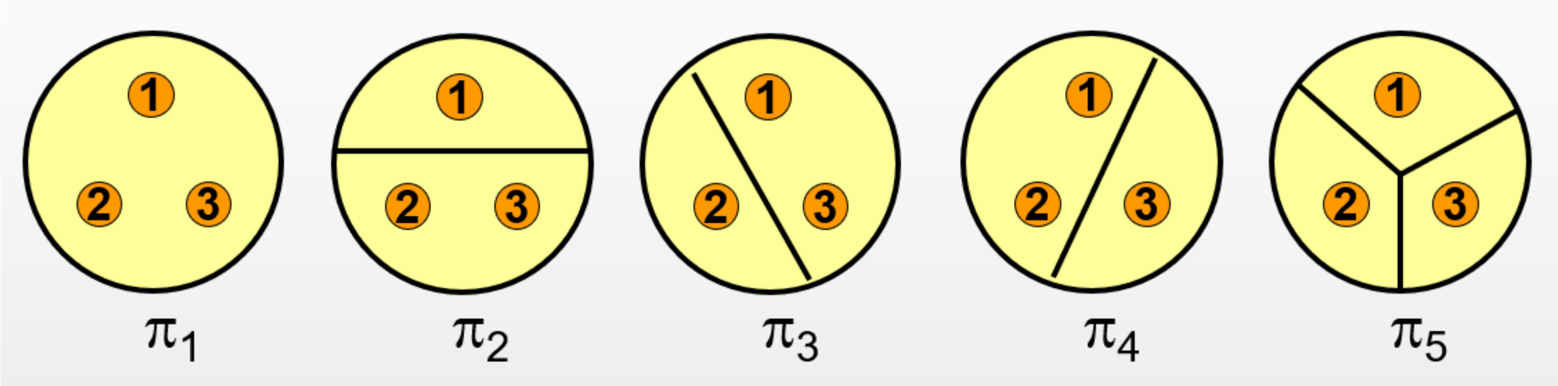
- 对 $\forall x \in A$, 必 $\exists i > 0$, 使得 $x \in S_i$, 所以 $< x, x > \in S_i \times S_i$, 即 $< x, x > \in R$, 因此 R 是自反的.
- 对 $\forall x, y \in A$, 如果 $< x, y > \in R$, 必 $\exists j > 0$, 使得 $< x, y > \in S_j \times S_j$, 从而 $< y, x > \in S_j \times S_j$, 即 $< y, x > \in R$, 因此 R 是对称的。
- 对 $\forall x, y, z \in A$, 如果 $< x, y > \in R, < y, z > \in R$, 必 $\exists i, j > 0$, 使得 $< x, y > \in S_i \times S_i$, $< y, z > \in S_j \times S_j$, 即 $x, y \in S_i$ 且 $y, z \in S_j$, 从而 $y \in S_i \cap S_j$, 由集合划分定义, 必有 $S_i = S_j$, 因此 x 和 z 同属于集合 A 的一个划分块 S_i, 从而 $< x, z > \in R$, 所以 R 是传递的. $\square$

Partitions and equivalence classes

- We sometimes say:
  - A partition of A *induces* an equivalence relation on A
  - An equivalence relation on A *induces* a partition of A
  - One to one correspondence(一一对应) between a partition of A and an equivalence relation on A

Partitions and equivalence classes

- A={1,2,3}，How many equivalence relation on the set A（include every element in A）?
- There are 5 ways to partition set A



Properties of Partitions

- **Theorem 1** Let R be an equivalence relation on a set A. These statements are equivalent:
- (i) a R b
- (ii) [a] = [b]
- (iii) $[a] \cap [b] \neq \emptyset$

Properties of Partitions

- **Theorem 2** Let R be an equivalence relation on a set S.
- Then the equivalence classes of R form a partition $\{ A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i (i \in I)$, as its equivalence classes.

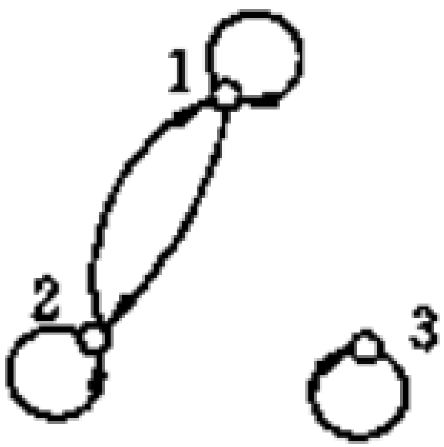
Example 8

- List the ordered pairs in the equivalence relation R produced by the partition

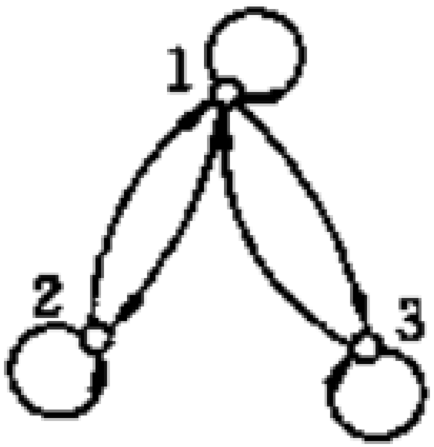
  $A_1 = \{1,2,3\}$, $A_2=\{4,5\}$, and $A_3=\{6\}$ of $S=\{1,2,3,4,5,6\}$, given in Example 7.

Applications

令 $A=\{1,2,3\}$，在A上的关系如图所示，判断他们是否等价关系

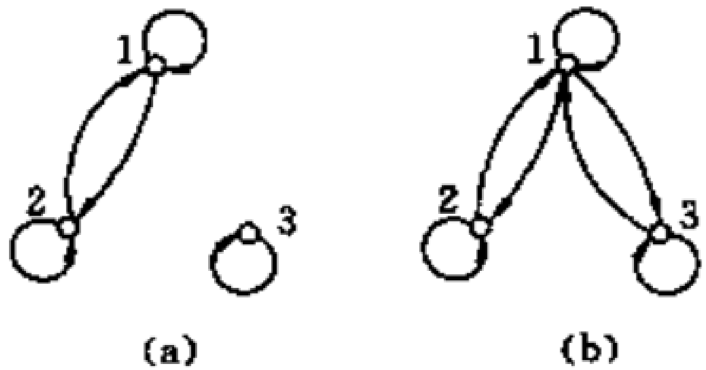


(a)



(b)

Applications



Solution.

a)关系是自反、对称、传递的。所以(a)图是等价的。

b)关系是自反、对称、但不传递。所以(b)图不是等价关系。

Exercises

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Exercises

1. How many equivalence relation on the set { a， b， c } （include every element in the set） (　　　)

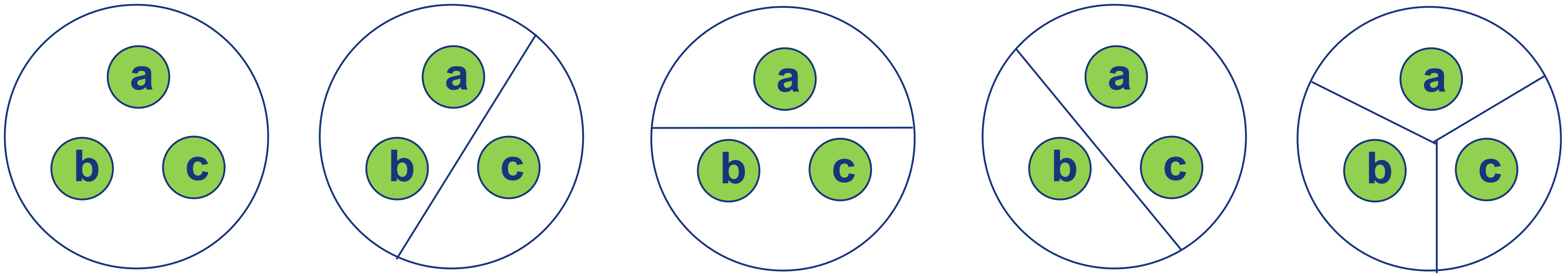
A． 4　　　B． 5　　　C． 6　　　D． 7

L o g o

Exercises

1. How many equivalence relation on the set { a， b， c }（include every element in the set）
(**B**)

A. 4 B. 5 C. 6 D. 7



Exercises

2. Which of the following relations is an equivalence relation? (**A**)

A. $\{(f, g) \mid f(1) = g(1)\}$

B. $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$

C. $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in Z\}$

D. $\{(f, g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$

Exercises

3. Which of these relations on {0, 1, 2, 3} are equivalence relations? (**B**)

A. {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

B. {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

C. {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

D. {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

Exercises

4. For the set S={a, b, c, d}, which is an equivalence relation? (**D**)

A. {(a,b), (a,c), (b,a), (b,d), (c,a), (c,d), (d,c), (d,b)}.

B. {(a,b), (b,a), (c,c), (c,d), (d,c), (d,d)}.

C. {(a,c), (a,b), (b,b), (c,c), (c,a), (d,b)}.

D. {(a,a), (a,b), (b,a), (b,b), (c,c), (d,d)}.

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Exercises

5. Select the relationship R which is **not** an equivalence relation. (B)

A) Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$.

B) Let R be the relation on the set of real numbers such that aRb if and only if $a + b$ is an integer.

C) Let R be the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x.

D) Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$.

Exercises

6 Assume the set is $A = \{1,2,3\}$, select the relationship R which is **not** an equivalence relation (C)

A. $R = \{<1,1>, <2,2>, <3,3>\}$

B. $R = \{<1,1>, <2,2>, <3,3>, <3,2>, <2,3>\}$

C. $R = \{<1,1>, <2,2>, <1,2>, <2,1>, <1,3>, <3,3>\}$

D. $R = \{<1,1>, <2,2>, <1,2>, <2,1>, <1,3>, <3,1>, <3,3>, <3,2>, <2,3>\}$

Exercises

- 7. Which of the following relations on {a,b,c,d} are equivalence relations?

A) {(a,a),(b,b),(c,a),(c,c),(c,d),(d,c),(d,d)}
B) {(a,a),(b,b),(b,c),(c,b),(c,c),(d,d)}
C) {(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,c),(d,d)}
D) {(a,a),(b,b),(b,d),(c,c),(c,d),(d,a),(d,c),(d,d)}

Exercises

- 7. Which of the following relations on {a,b,c,d} are equivalence relations? (B)

A) {(a,a),(b,b),(c,a),(c,c),(c,d),(d,c),(d,d)}

B) {(a,a),(b,b),(b,c),(c,b),(c,c),(d,d)}

C) {(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,c),(d,d)}

D) {(a,a),(b,b),(b,d),(c,c),(c,d),(d,a),(d,c),(d,d)}

Exercises

- 8. For the set A={a,b,c,d,e}, how many equivalence relations that contain (a,b) can be obtained in A （include every element in the set）? **(D)** 解析见下页
- A) 5
- B) 8
- C) 10
- D) 15

Exercises

等价关系中包含(a,b)说明a和b必须在一个划分块中

①首先考虑a和b单独在一个划分块中，即{a,b}，剩下的c,d,e有5种划分

{c,d,e} {{c},{d,e}} {{d},{c,e}} {{e},{c,d}} {{c},{d},{e}}

②考虑a,b和{c,d,e}中的某一个构成一个划分块，有3种情况，剩下的两个元素有2种划分方式，共3*2=6种划分

③考虑a, b和{c, d, e}中的某两个构成一个划分块，有3种情况，剩下的一个元素有1种划分方式，共3种划分

④{a, b, c, d, e}　　　　1种划分

5+6+3+1=15

Exercises

9\. The smallest equivalence relation on the set {1,2,3,4} containing the relation {(1,2),(1,4),(3,3),(4,1)}（include every element in the set）is R, then the equivalence class induced from R is _______

{{1,2,4},{3}}

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Exercises

10. The smallest equivalence relation on the set {a,b,c,d,e} containing the relation {(a,b),(a,c),(d,e)}（include every element in the set） is ______

{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b), (b,a),(a,c),(c,a),(b,c),(c,b),(d,e),(e,d)}

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Exercises

12. Set $\mathrm{A} = \mathrm{Z}$, suppose R is a relation on $\mathrm{A} \times \mathrm{A}$, where $((x, y), (u, v)) \in R \Leftrightarrow x+y = u+v$. Prove: R is an equivalence relation on $\mathrm{A} \times \mathrm{A}$.

(1) reflexivity: **Any** $< x,y > \in A \times A$

$$x + y = x + y \Leftrightarrow << x,y >,< x,y >> \in R$$

(2) symmetry: **Any** $<< x,y >,< u,v >> \in R$

$$\Leftrightarrow x + y = u + v \Leftrightarrow u + v = x + y \Leftrightarrow << u,v >,< x,y >> \in R$$

(3) transitivity: **Any** $<< x,y >,< u,v >> \in R \wedge << u,v >,< r,s >> \in R$

$$\Leftrightarrow x + y = u + v \wedge u + v = r + s \Leftrightarrow x + y = r + s \Leftrightarrow << x,y >,< r,s >> \in R$$

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Exercises

13 Suppose R is relation on A, where $S = \{\langle a,b\rangle | \exists c(\langle a,c\rangle \in R \wedge \langle c,b\rangle \in R)\}$. Please prove: if R is an equivalence relation, S is an equivalence relation.

(1) Reflexive $\forall x,$

$$x \in A \Rightarrow \; <x,x> \in R \Rightarrow \exists x\,(<x,x> \in R \wedge <x,x> \in R) \Rightarrow \; <x,x> \in S$$

(2) Symmetric $\forall <x,y>,$

$$<x,y> \in S \Rightarrow \exists c(<x,c> \in R \wedge <c,y> \in R)$$

$$\Rightarrow \exists c\,(<c,x> \in R \wedge <y,c> \in R) \Rightarrow <y,x> \in S$$

(3) Transitive $\forall <x,y>, <y,z>,$

$$<x,y> \in S \wedge \; <y,z> \in S$$

$$\Rightarrow \exists c\,(<x,c> \in R \wedge <c,y> \in R) \wedge \exists d\,(<y,d> \in R \wedge <d,z> \in R)$$

$$\Rightarrow \; <x,y> \in R \wedge <y,z> \in R \Rightarrow \; <x,z> \in S$$

Exercises

14 Suppose R is a reflexive and transitive relation on A. Prove $R \cap R^{-1}$ is an equivalence relation on A.

(1) Reflexive $\forall x$,

$$x \in A \Rightarrow \langle x, x \rangle \in R \Rightarrow \langle x, x \rangle \in R \wedge \langle x, x \rangle \in R^{-1} \Rightarrow \langle x, x \rangle \in R \cap R^{-1}$$

(2) Symmetric $\forall \langle x, y \rangle$,

$$\langle x, y \rangle \in R \cap R^{-1} \Rightarrow \langle x, y \rangle \in R \wedge \langle x, y \rangle \in R^{-1}$$
$$\Rightarrow \langle y, x \rangle \in R^{-1} \wedge \langle y, x \rangle \in R \Rightarrow \langle y, x \rangle \in R \cap R^{-1}$$

(3) Transitive $\forall \langle x, y \rangle, \langle y, z \rangle$,

$$\langle x, y \rangle \in R \cap R^{-1} \wedge \langle y, z \rangle \in R \cap R^{-1}$$
$$\Rightarrow \langle x, y \rangle \in R \wedge \langle x, y \rangle \in R^{-1} \wedge \langle y, z \rangle \in R \wedge \langle y, z \rangle \in R^{-1}$$
$$\Rightarrow (\langle x, y \rangle \in R \wedge \langle y, z \rangle \in R) \wedge (\langle x, y \rangle \in R^{-1} \wedge \langle y, z \rangle \in R^{-1})$$
$$\Rightarrow \langle x, z \rangle \in R \wedge \langle x, z \rangle \in R^{-1} \Rightarrow \langle x, z \rangle \in R \cap R^{-1}$$

Exercises

15. Suppose R is a reflexive and transitive relation on A. T is also a relation on A, such that: $\langle a, b\rangle \in T \Leftrightarrow \langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$

Prove that T is an equivalence relation.

(1) Since R is reflexive, $\langle a,a\rangle \in R$ and $\langle a,a\rangle \in R \Leftrightarrow \langle a, a\rangle \in T$. T is reflexive.
(2) Since $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R \Leftrightarrow \langle a, b\rangle \in T$ and $\langle b, a\rangle \in T$. T is symmetric.
(3) If $\langle a, b\rangle \in T$ and $\langle b, c\rangle \in T$, then $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$. Since R is transitive, $\langle a,c\rangle \in R$. Also, $\langle b, a\rangle \in R$ and $\langle c, b\rangle \in R$, imply $\langle c,a\rangle \in R$. Therefore, $\langle a,c\rangle \in T$, transitive.

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Exercises

16. Given $S = \{ <x , y> | x , y \in R, (x-y)/3$ is integer }, prove the relation S is an equivalence relations.

(1) (x–x)/3 = 0 is integer，so x S x，S is reflexive.
(2) If x S y that (x–y)/3 is integer, (y–x)/3 = -(x–y)/3 is integer. Hence, y S x, so S is symmetric.
(3) If x S y and y S z, (x–y)/3 and (y–z)/3 are integer. Thus, (x–z)/3 = (x–y)/3 + (y–z)/3 is integer, so x S z. S is transitive.

Exercises

17. Which is these collections of subsets are partitions of {a,b,c,d,e,f,g}?

A) {a, b, c}, {c, d, e}, {f, g}
B) {a, b}, {c, d}, {e, f}, {g}
C) {a, b, c, d, e}, {e, f, g}
D) {a, c}, {e, f, g}

Exercises

17. Which is these collections of subsets are partitions of {a,b,c,d,e,f,g}? (B)

A) {a, b, c}, {c, d, e}, {f, g}
B) {a, b}, {c, d}, {e, f}, {g}
C) {a, b, c, d, e}, {e, f, g}
D) {a, c}, {e, f, g}

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End of Section 3.5