Catmull-Rom Spline

CSC 418/2504: Computer Graphics – Winter 2011

Department of Computer Science

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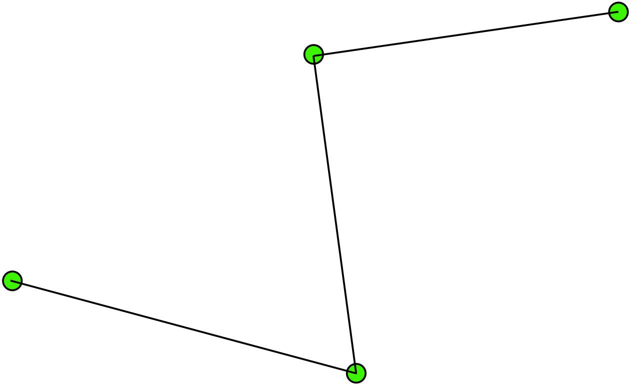
What is curve interpolation?

- **Interpolation** - Force a curve to pass through a set of *control points*.



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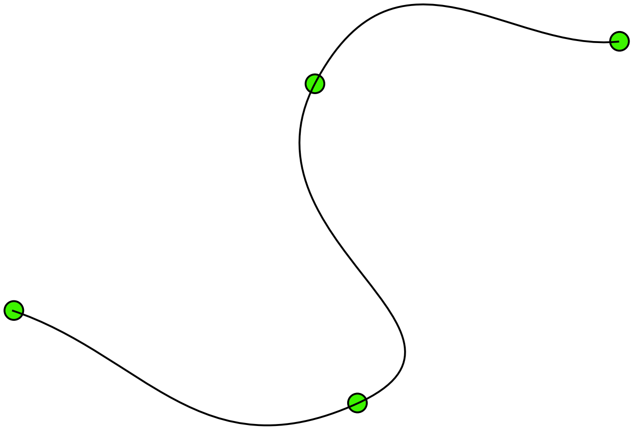
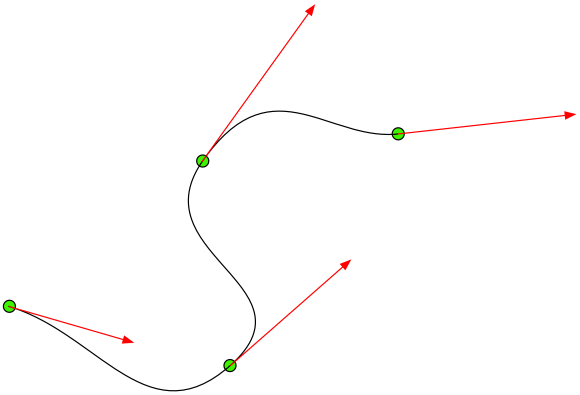


What are we looking for?

- Easy to work with - intuitive.
- Used in key-frame animation - smooth.
- Easy to represent - compact representation.
- Easy to generate - computationally cheap.

Cubic Hermite Spline

- A C^1 (continuous up to its 1^{st}derivative) curve, with control points of position and first derivative.
- 4 constraints $\Rightarrow$ cubic function (4^{th} degree polynomial).



Derivation

- We want to calculate $\bar{c}(t) = (x(t), y(t))^T; t \in [0,1]$ **s.t.** $\bar{c}(0) = \bar{p}_0, \bar{c}'(0) = \bar{p}_0', \bar{c}(1) = \bar{p}_1, \bar{c}(1) = \bar{p}_1'$
- $x(t) = (1, t, t^2, t^3) \cdot (a_0, a_1, a_2, a_3)^T$
- We need to solve $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} c_{0,x} \\ c_{0,x}' \\ c_{1,x} \\ c_{1,x}' \end{pmatrix} \Rightarrow$
  $M \cdot \bar{a} = \bar{b} \Rightarrow \bar{a} = M^{-1} \cdot \bar{b}$
- $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_{0,x} \\ c_{0,x}' \\ c_{1,x} \\ c_{1,x}' \end{pmatrix}$

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Catmull-Rom Spline

Cubic Hermite Spline - Unit Interval

$$\bar{c}(t) = \left(2t^3 - 3t^2 + 1\right)\bar{p}_0 + \left(t^3 - 2t^2 + t\right)\bar{p}_0' + \left(-2t^3 + 3t^2\right)\bar{p}_1 + \left(t^3 - t^2\right)\bar{p}_1'$$

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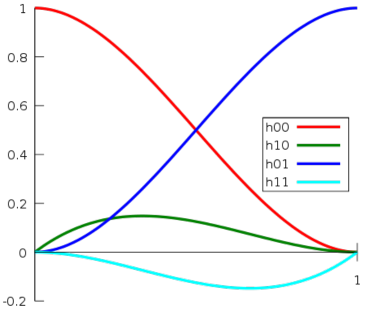
- Catmull-Rom spline defines the derivatives as:
  - $\bar{p}_j' = \kappa\left(\bar{p}_{j+1} - \bar{p}_{j-1}\right)$
  - $\bar{p}_0' = \kappa\left(\bar{p}_1 - \bar{p}_0\right)$
  - $\bar{p}_N' = \kappa\left(\bar{p}_N - \bar{p}_{N-1}\right)$
- NOTE: the formula is correct for unit interval only!



Catmull-Rom Spline for Non-unit Interval

- $t = \frac{x - x_j}{x_{j+1} - x_j} \Rightarrow \bar{p}'_j(t) = \frac{d\bar{p}_j}{dt} = \frac{d\bar{p}_j}{dx}\frac{dx}{dt} = \bar{p}'_j(x)(x_{j+1} - x_j)$
- Where $\bar{p}_j(t = 0) \equiv \bar{p}_j(x = x_j); \bar{p}'_j(t = 0) \equiv \bar{p}'_j(x = x_{j+1})$