Automorphisms of direct products of circulant graphs

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All graphs in this presentation are **finite**, **simple** and **undirected**.

An **automorphism** of a graph $X = (V, E)$ is a permutation of V which preserves E.

$\mathrm{Aut}(X)$ is the **automorphism group** of X.

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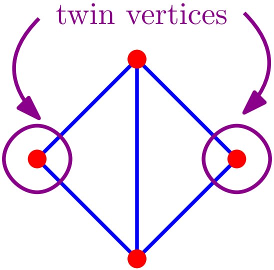
Two vertices are **twins** if they have exactly the same neighbours. A graph is **twin-free** if it contains no twins.



Definition (Direct product $\times$)

$$V(X \times Y) = \{(x, y) \mid x \in V(X), y \in V(Y)\}$$

$$(x_1, y_1) \sim_{X \times Y} (x_2, y_2) \iff x_1 \sim_X x_2 \text{ and } y_1 \sim_Y y_2$$

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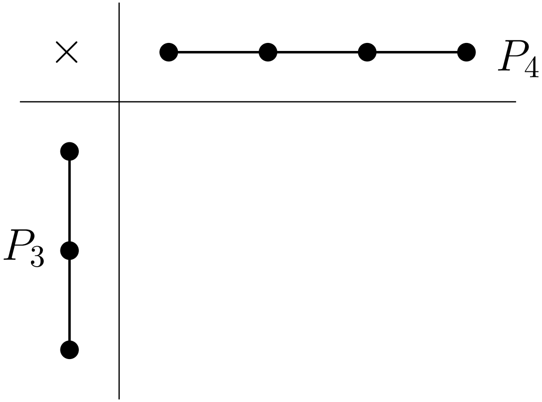

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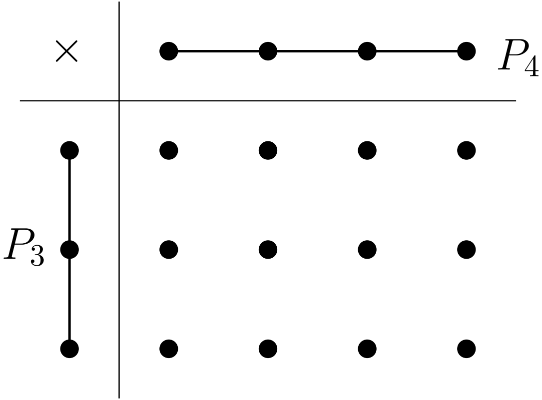

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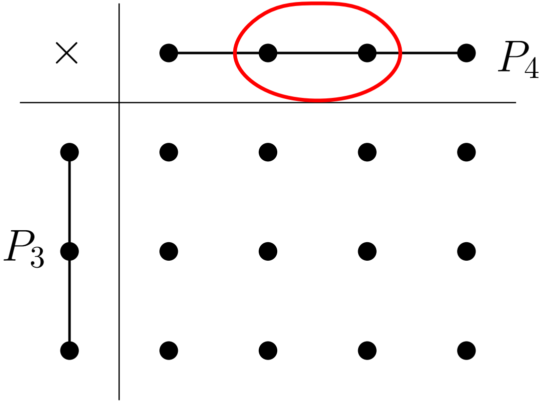

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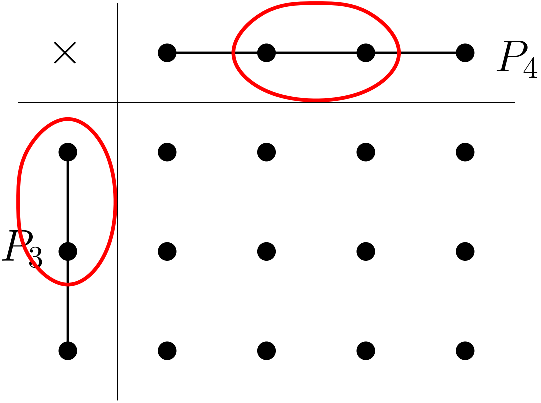

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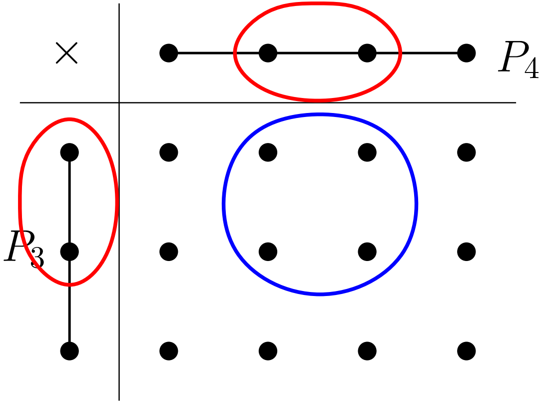

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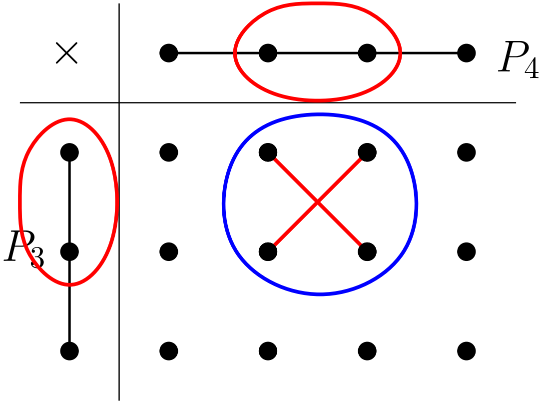

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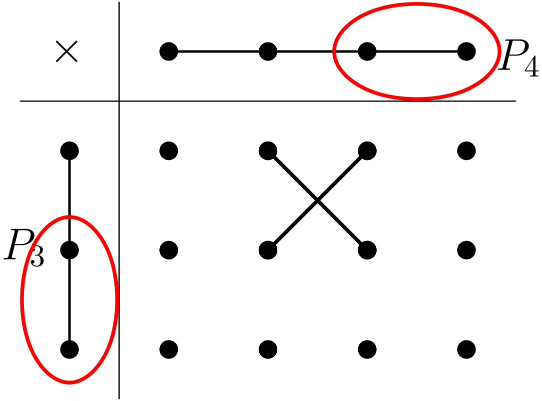

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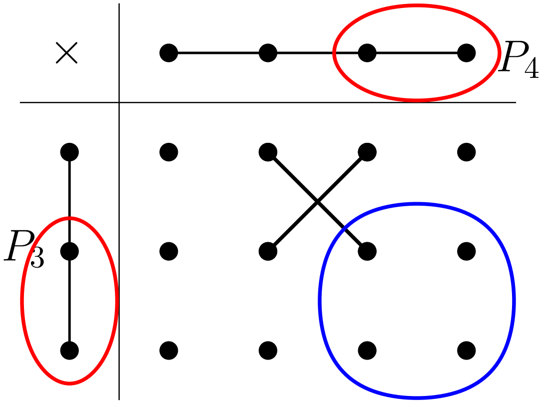

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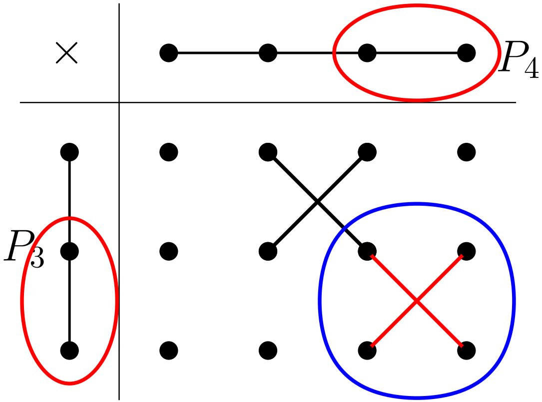

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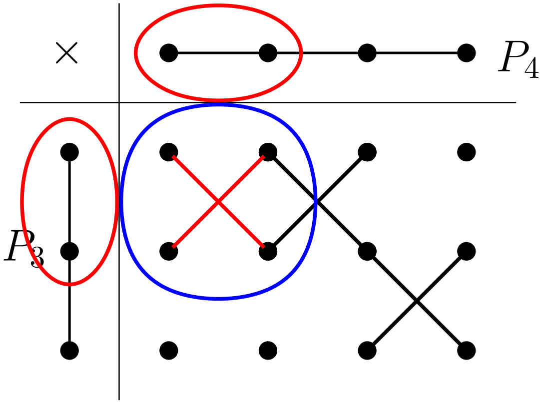

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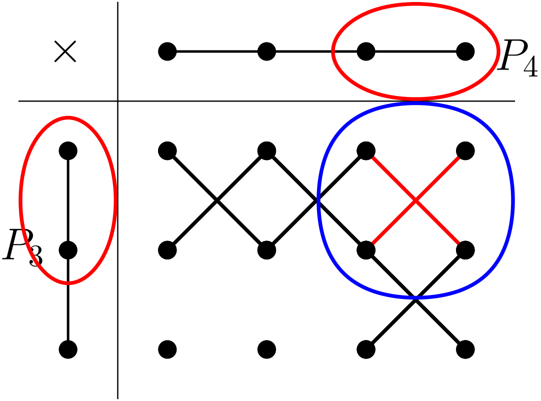

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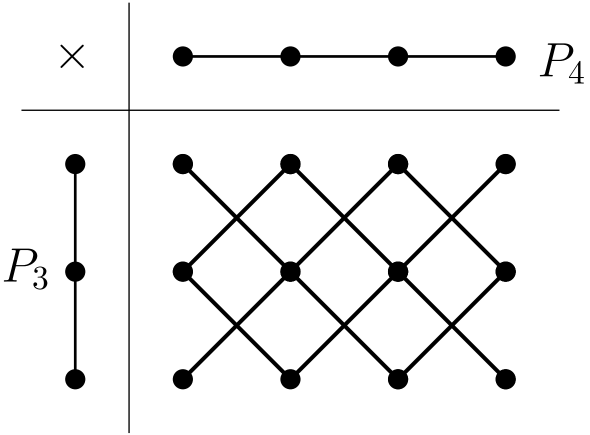

What is $\mathsf{Aut}(X \times Y)$?

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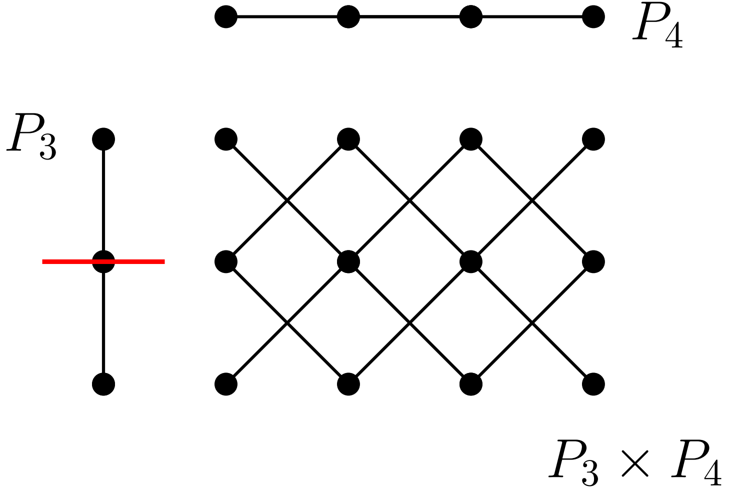

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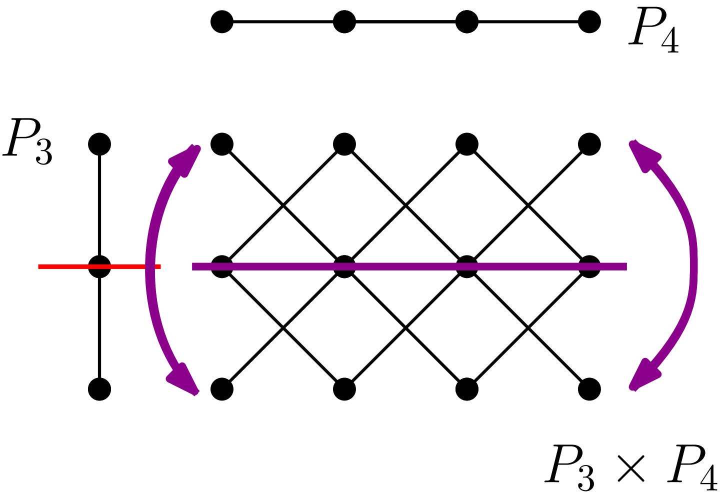

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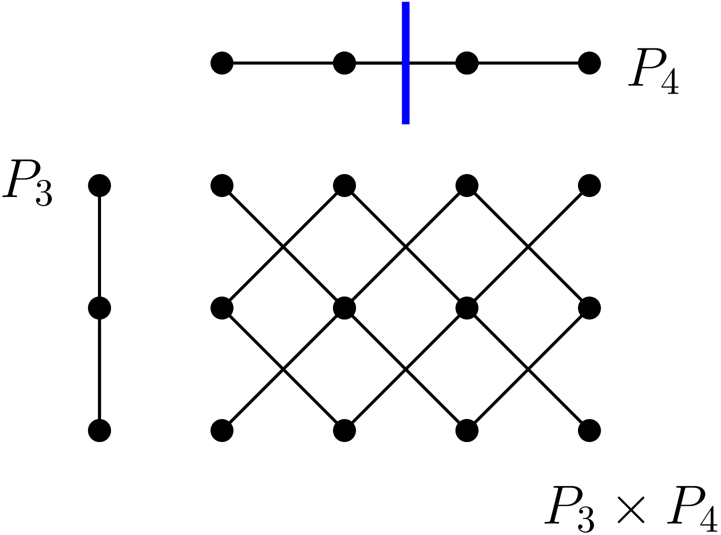

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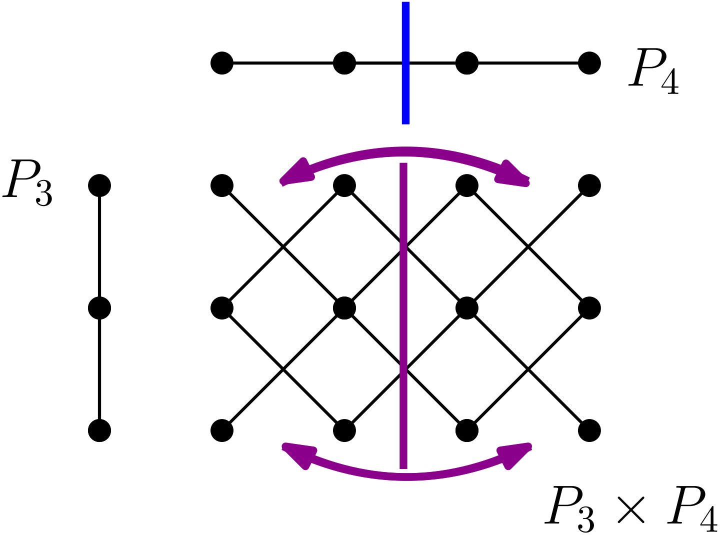

When does $\mathrm{Aut}(X \times Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$?

When does $\mathrm{Aut}(X \times Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$?

What else can $\mathrm{Aut}(X \times Y)$ contain?

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Theorem (Dörfler 1974)

Let X and Y be a connected, non-bipartite, twin-free graphs with unique prime decompositions $X = X_1 \times \ldots \times X_n$ and $Y = Y_1 \times \ldots \times Y_m$. Then $\mathrm{Aut}(X \times Y)$ *is generated by*

1. ***automorphisms of X***,
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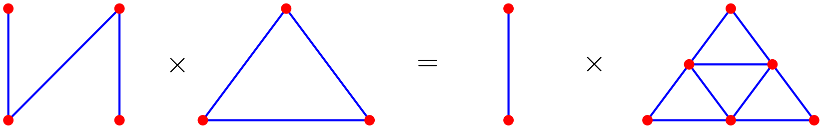
Corollary

Let X and Y be a connected, non-bipartite, twin-free graphs. Then $\mathrm{Aut}(X \times Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ if and only if X and Y are $\times$**-coprime**.

Dörfler's theorem does not hold for bipartite graphs!



When does $\mathrm{Aut}(X \times Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$?

(X is non-bipartite and Y is bipartite)

Reduction to the case $Y = K_2$

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Lemma (folklore?)

Let X be a connected, non-bipartite, twin-free graph such that

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Lemma (folklore?)

Let X be a connected, non-bipartite, twin-free graph such that

$$\mathrm{Aut}(X \times K_2) = \mathrm{Aut}(X) \times \mathrm{Aut}(K_2).$$

Then for a connected, bipartite, twin-free graph Y (satisfying mild technical conditions)

$$\mathrm{Aut}(X \times Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y).$$

Canonical double cover
$X \times K_2$

Canonical double cover of a graph X

$$BX := X \times K_2$$

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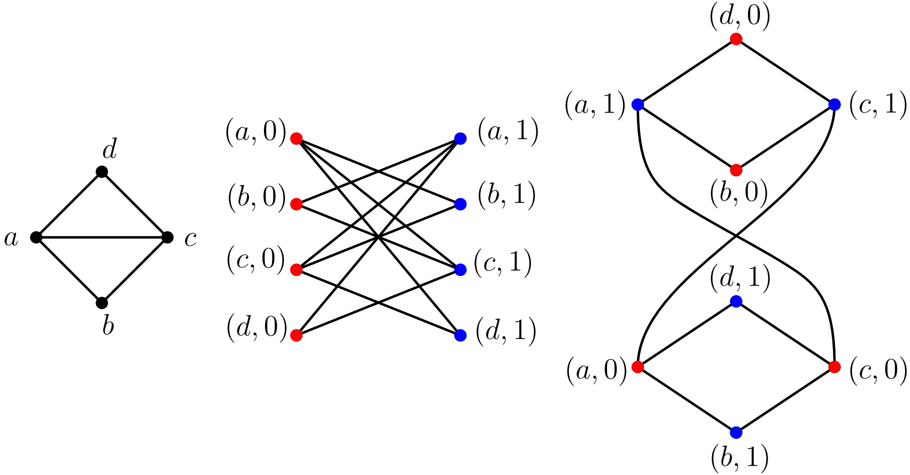

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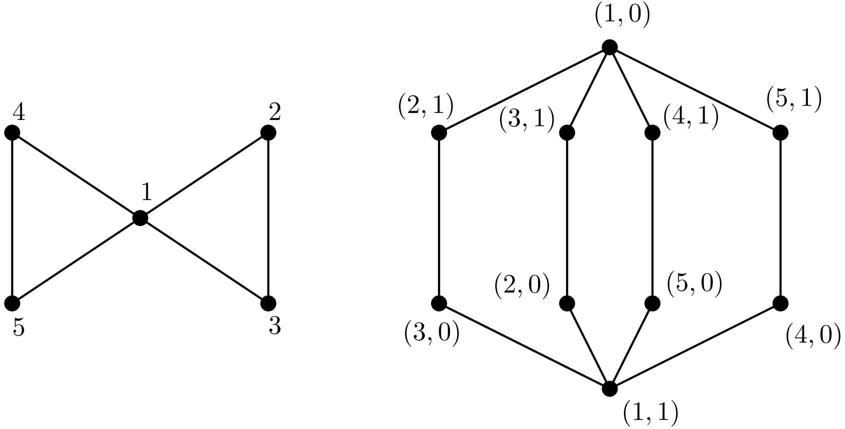
The equality does not always hold!

Definition

A graph X is **non-trivially unstable** if it is connected, non-bipartite and twin-free, and

$$\mathrm{Aut}(BX) \neq \mathrm{Aut}(X) \times \mathrm{Aut}(K_2).$$

The smallest non-trivially unstable graph



What circulant graphs are (non-trivially) unstable?

(Wilson in 2008)

Unstable circulants and Wilson conditions

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A circulant graph $\mathrm{Circ}(n, S)$ is a Cayley graph of a cyclic groups.

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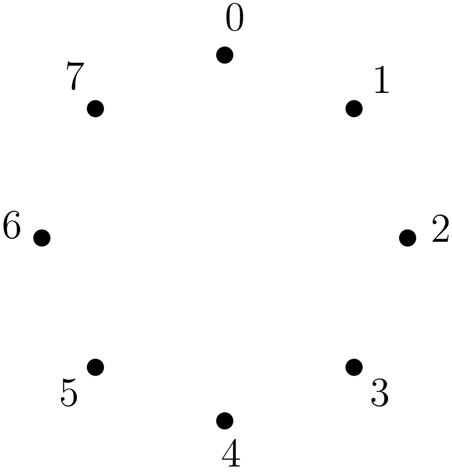

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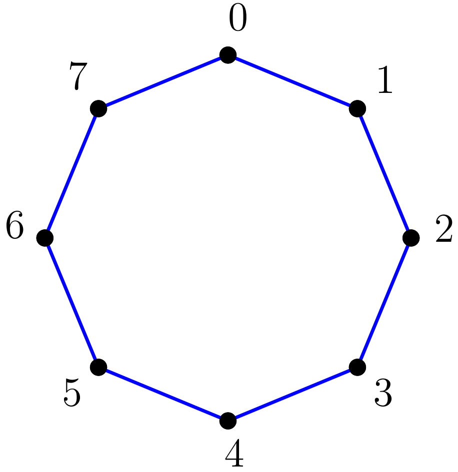

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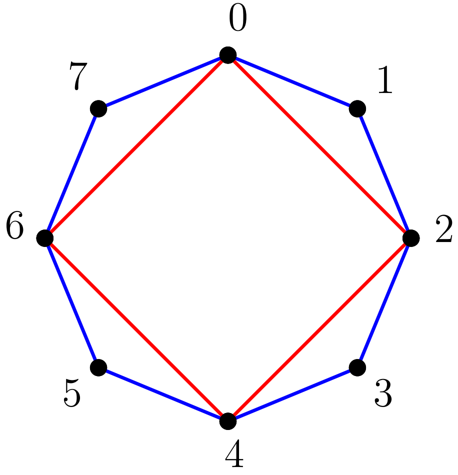

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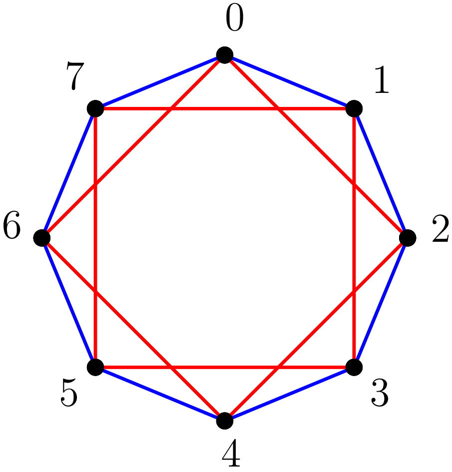

$\mathrm{Circ}(8, \{\pm 1, \pm 2\})$

Wilson conditions

$=$

sufficient conditions for a circulant graph to be unstable

(introduced by Wilson in 2008)

Example of a Wilson condition

Proposition (Wilson condition (C.4))

If there exists $m \in \mathbb{Z}_n^\times$ such that

$$(n/2) + mS = S, \text{ then } \mathrm{Circ}(n, S) \text{ is unstable.}$$

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An unexpected automorphism of the double cover of $\mathrm{Circ}(n, S)$

$$\phi\colon (x, i) \mapsto \begin{cases} (mx, 0), & \text{if } i = 0 \\ (mx + (n/2), 1), & \text{if } i = 1 \end{cases}$$

Wilson conditions - Corrections

- **Qin-Xia-Zhou (2019)** updated Wilson condition $(C.2)$ to $(C.2')$.
- **Hujdurović-Morris-Mitrović (2021)** updated Wilson condition $(C.3)$ to $(C.3')$.

Wilson's conjecture

Wilson's conjecture

Every **non-trivially unstable circulant graph** satisfies at least one of the **Wilson conditions**.

Circulants of odd order

Theorem (Fernandez-Hujdurović 2022)

There are no non-trivially unstable circulants of odd order.

Circulants of order 2p

Let p be a prime.

Theorem (Hujdurović-Morris-Mitrović 2021)

Every non-trivially unstable circulant of order $2p$ satisfies Wilson condition $(C.4)$.

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Reminder: Wilson condition (C.4)

$$(n/2) + mS = S, m \in \mathbb{Z}_n^\times$$

Circulants of low valency

Theorem (Hujdurović-Morris-Mitrović 2022+)

Every non-trivially unstable circulant of ***valency at most 7*** *satisfies* ***at least one Wilson condition****.*

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1. For each valency, we provide a **complete list of connections sets**.
2. For each graph, we find a **Wilson condition** it satisfies.

Example of a classification result

Theorem (Hujdurović, Morris, Mitrović 2022+)

A 5-valent circulant is unstable if and only if either it is trivially unstable, or it is one of the following:

1. $\mathrm{Circ}(12k, \{\pm s, \pm 2k, 6k\})$ *with s odd, satisfying Wilson condition $(C.1)$.*
2. $\mathrm{Circ}(8, \{\pm 1, \pm 3, 4\})$ *satisfying Wilson condition $(C.3')$.*

Non-trivially unstable circulants of low valency

- **valency** ≤ 3: **none**
- **valency** 4: **two** infinite families satisfying $(C.4)$.
- **valency** 5: **one** infinite family $(C.1)$; **one** sporadic example $(C.3')$.
- **valency** 6: **seven** infinite families $(C.1) - (C.4)$.
- **valency** 7: **six** infinite families $(C.1) - (C.3')$.

Theorem (Hujdurović-Morris-Mitrović 2022+)

Every non-trivially unstable circulant of ***valency at most 7*** *satiesfies* ***at least one Wilson condition****.*

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*Every non-trivially unstable circulant of **valency at most 7** satiesfies **at least one Wilson condition**.*

This bound is sharp!

Counterexample of minimal valency

The following graph is an 8-valent, non-trivially unstable circulant satisfying no Wilson condition.

$$\mathrm{Circ}(48, \{\pm 3, \pm 6, \pm 4, \pm 21\})$$

Generalizations of Wilson conditions

Generalized Wilson condition (C.4)

Theorem (Hujdurović-Morris-Mitrović 2021)

If $X = \mathrm{Circ}(n, S) \cong \mathrm{Circ}(n, (n/2) + S)$*, then* X *is unstable.*

New families of counterexamples to Wilson's conjecture

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If $X = \mathrm{Circ}(n, S) \cong \mathrm{Circ}(n, (n/2) + S)$, then X is unstable.

For $\ell \geq 4$, set $n := 3 \cdot 2^{\ell}$.

$$S := \{\pm 3, \pm 6, \pm n/12, n/2 \pm 3\}$$

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For $\ell \geq 4$, set $n := 3 \cdot 2^{\ell}$.

$$S := \{\pm 3, \pm 6, \pm n/12, n/2 \pm 3\}$$

Let $X := \mathrm{Circ}(n, S)$.

1. X is 8-valent and is non-trivially unstable.
2. X satisfies the **generalized** Wilson condition $(C.4)$.
3. X does not satisfy any of Wilson conditions.

Other generalizations

$X = \mathrm{Circ}(n, S)$, $H, K \leq \mathbb{Z}_n$ be non-trivial, such that $|K|$ is even, $K_o = K \setminus 2K$.

Theorem (Hujdurović-Morris-Mitrović 2021)

If either

1. $S + H \subseteq S \cup (K_o + H)$ *and* $H \cap K_o = \varnothing$, *or*
2. $(S \setminus K_o) + H \subseteq S \cup K_o$ *and either* $|H| \neq 2$ *or* $|K|$ *is divisible by* 4

then X *is unstable.*

Other generalizations

This result generalizes Wilson conditions $(C.1), (C.2')$ and $(C.3')$.

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Computational results

Newly introduced generalizations can explain instability of all non-trivially unstable circulants up to order 50.

Recent developments

Analogues of Wilson's conjecture for other families of graphs turned out to be true!

1. Generalized Petersen graphs - Qin, Xia, Zhou in 2020.
2. Toroidal grids and Triangular grids - Witte Morris in 2023.
3. Rose-Window graphs - Ahanjideh, Kovács, Kutnar in 2023.

Thank you for your attention!

- $X \times K_2$ plays a major role in understanding $\mathrm{Aut}(X \times Y)$ with X non-bipartite and Y bipartite.
- X is unstable if $\mathrm{Aut}(X) \times \mathrm{Aut}(K_2) \neq \mathrm{Aut}(X \times K_2)$.
- X is non-trivially unstable if it is unstable, connected, non-bipartite and twin-free.
- Wilson's conjecture: Every non-trivially unstable circulant graph satisfies at least one Wilson condition.

Results

- Generalizations of Wilson conditions
- New infinite families of counterexamples to Wilson's conjecture.
- Wilson's conjecture is true for
  - circulants of order $2p$, and
  - circulants with valency at most 7.

Additional slides

Definition (Marušič-Scapellato-Zagaglia 1989)

X is **unstable** if

$$\mathrm{Aut}(BX) \neq \mathrm{Aut}(X) \times \mathrm{Aut}(K_2).$$

Trivially unstable graphs

Fact

BX is connected if and only if X is connected and non-bipartite.

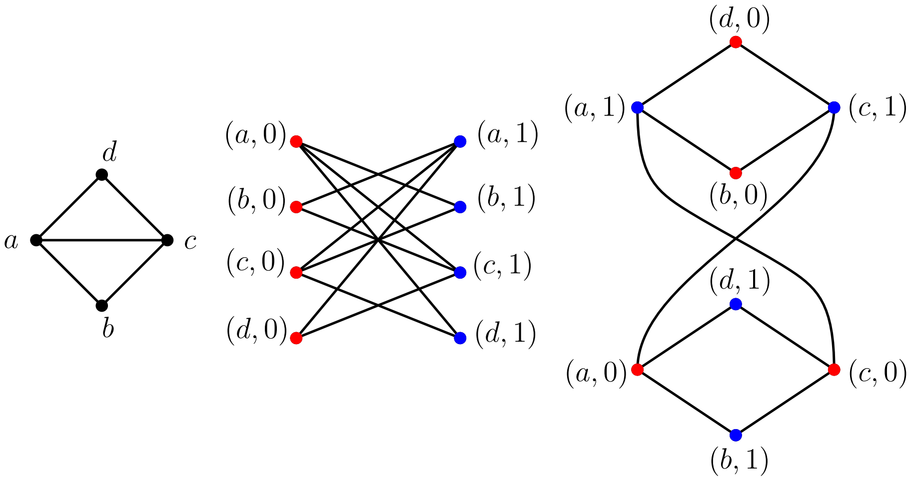
Trivially unstable graphs include

- disconnected graphs,
- bipartite graphs (with non-trivial automorphism group),
- graphs with twin-vertices.

Canonical double covers of graphs with twin-vertices

Twin vertices

$x, y \in V(X)$ are twins if $\{\text{neighbours of } x\} = \{\text{neighbours of } y\}$.



$X = \mathrm{Circ}(n, S)$, n is even. Let $S_e = S \cap 2\mathbb{Z}_n$ and $S_o = S \setminus S_e$.

Wilson conditions

1. There is a non-zero element h of $2\mathbb{Z}_n$, such that $h + S_e = S_e$.
2. n is divisible by 4, and there exists $h \in 1 + 2\mathbb{Z}_n$, such that
   - $2h + S_o = S_o$, and
   - **for each $s \in S$, such that $s \equiv 0$ or $-h \pmod 4$, we have $s + h \in S$.**
3. There is a subgroup H of $\mathbb{Z}_n$, such that the set

   $$R = \{s \in S \mid s + H \not\subseteq S\},$$

   is non-empty and has the property that if we let $d = \gcd(R \cup \{n\})$, then n/d is even, r/d is odd for every $r \in R$, and **either $H \not\subseteq d\mathbb{Z}_n$ or $H \subseteq 2d\mathbb{Z}_n$.**
4. There exists $m \in \mathbb{Z}_n^{\times}$, such that $(n/2) + mS = S$.

Theorem (Valency 4)

A circulant graph $\mathrm{Circ}(n, \{\pm a, \pm b\})$ *of valency* 4 *is unstable if and only if either it is trivially unstable, or one of the following conditions is satisfied (perhaps after interchanging a and b):*

1. $n \equiv 2 \pmod 4$, $\gcd(a, n) = 1$, *and* $b = ma + (n/2)$, *for some* $m \in \mathbb{Z}_n^{\times}$, *such that* $m^2 \equiv \pm 1 \pmod n$, *or*
2. n *is divisible by* 8 *and* $\gcd(|a|, |b|) = 4$.

In both of these cases, X satisfies Wilson condition $(C.4)$.

Theorem (Valency 5)

A circulant graph $\mathrm{Circ}(n, S)$ *of valency* 5 *is unstable if and only if either it is trivially unstable, or it is one of the following:*

1. $\mathrm{Circ}(12k, \{\pm s, \pm 2k, 6k\})$ *with* s *odd, satisfying Wilson condition* $(C.1)$.
2. $\mathrm{Circ}(8, \{\pm 1, \pm 3, 4\})$, *satisfying Wilson condition* $(C.3')$.

Theorem (Valency 6)

A circulant graph $X = \mathrm{Circ}(n, \{\pm a, \pm b, \pm c\})$ *of valency* 6 *is unstable if and only if either it is trivially unstable, or it is one of the following*

1) $\mathrm{Circ}(8k, \{\pm a, \pm b, \pm 2k\})$, *where a and b are odd, satisfying Wilson condition* $(C.1)$.
2) $\mathrm{Circ}(4k, \{\pm a, \pm b, \pm b + 2k\})$, *where a is odd and b is even, satisfying Wilson condition* $(C.1)$.
3) $\mathrm{Circ}\big(4k, \big\{\pm a, \pm(a+k), \pm(a-k)\big\}\big)$, *where* $a \equiv 0 \pmod 4$ *and k is odd, satisfying Wilson condition* $(C.2')$.
4) $\mathrm{Circ}(8k, \{\pm a, \pm b, \pm b + 4k\})$, *where a is even and* $|a|$ *is divisible by* 4, *satisfying Wilson condition* $(C.3')$.
5 $\mathrm{Circ}(8k, \{\pm a, \pm k, \pm 3k\})$, *where* $a \equiv 0 \pmod 4$ *and k is odd, satisfying Wilson condition* $(C.3')$.

6) $\mathrm{Circ}(4k, \{\pm a, \pm b, \pm mb + 2k\})$, where

$$\gcd(m, 4k) = 1, \quad (m-1)a \equiv 2k \pmod{4k}, \quad \text{and}$$

$$\text{either} \;\; m^2 \equiv 1 \pmod{4k} \;\; \text{or} \;\; (m^2+1)b \equiv 0 \pmod{4k},$$

satisfying Wilson condition $(C.4)$.

7) $\mathrm{Circ}(8k, \{\pm a, \pm b, \pm c\})$, where there exists $m \in \mathbb{Z}$, such that

$$\gcd(m, 8k) = 1, \quad m^2 \equiv 1 \pmod{8k}, \quad \text{and}$$

$$(m-1)a \equiv (m+1)b \equiv (m+1)c \equiv 4k \pmod{8k},$$

satisfying Wilson condition $(C.4)$.

Theorem (Valency 7)

A circulant graph $\mathrm{Circ}(n, S)$ *of valency* 7 *is unstable if and only if either it is trivially unstable, or it is one of the following:*

1) $\mathrm{Circ}(6k, \{\pm 2t, \pm 2(k-t), \pm 2(k+t), 3k\})$, *with* k *odd, satisfying Wilson condition* $(C.1)$.
2) $\mathrm{Circ}(12k, \{\pm 2k, \pm b, \pm c, 6k\})$, *with* b *and* c *odd, satisfying Wilson condition* $(C.1)$.
3) $\mathrm{Circ}(20k, \{\pm t, \pm 2k, \pm 6k, 10k\})$, *with* t *odd, satisfying Wilson condition* $(C.1)$.
4) $\mathrm{Circ}(4k, \{\pm t, \pm(k-t), 2k \pm t, 2k\})$, *with* k *odd and* $t \equiv k \pmod 4$, *satisfying Wilson condition* $(C.2')$.

5) $\mathrm{Circ}(8k, \{\pm 4t, \pm k, \pm 3k, 4k\})$, with k and t odd, satisfying Wilson condition $(C.3')$.
6) $\mathrm{Circ}(12k, \{\pm t, \pm(4k - t), \pm(4k + t), 6k\})$, with t odd, satisfying Wilson condition $(C.3')$.