Proof of the Clustered Hadwiger Conjecture

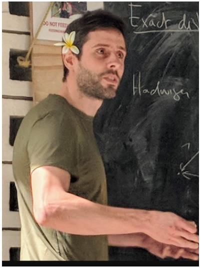
David Wood

Monash University

Melbourne, Australia

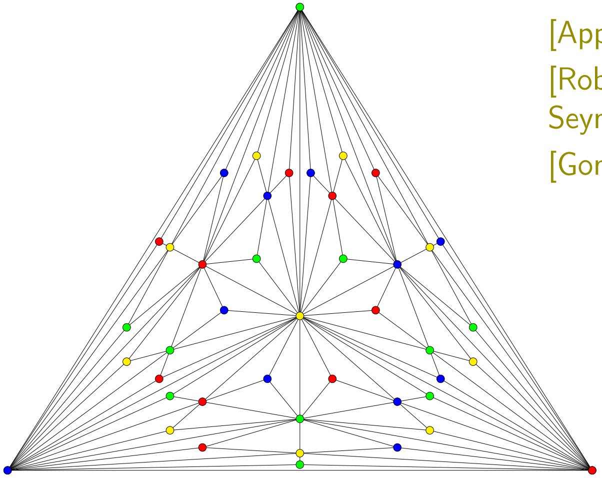
joint work with

Vida Dujmović | Louis Esperet | Pat Morin



arXiv:2306.06224 and FOCS 2023

4-colour theorem: every planar graph is 4-colourable



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[Robertson, Sanders, Seymour & Thomas '97]

[Gonthier '08]

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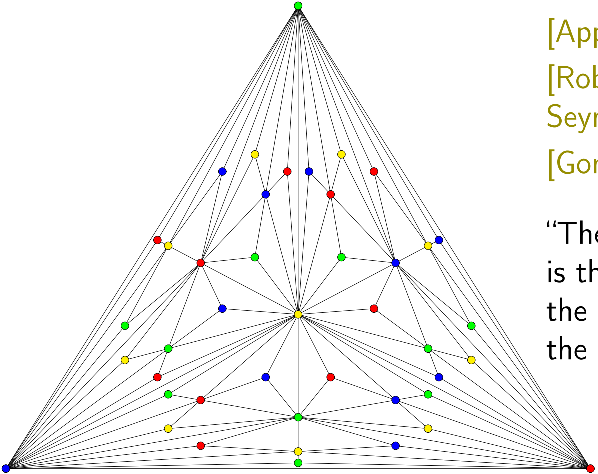

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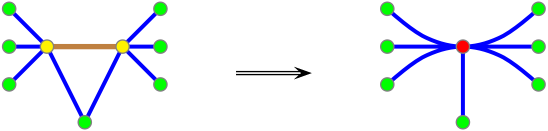
"The 4-colour theorem
is the tip of the iceberg,
the thin end of the wedge,
the first cuckoo of Spring"

[Tutte '78]

a graph H is a minor of a graph G if
H can be obtained from a subgraph of G by contracting edges



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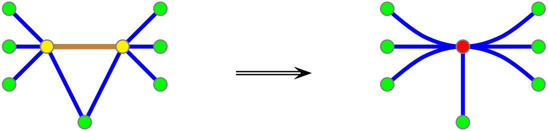
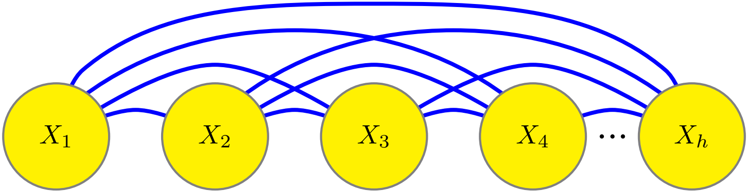


complete K_h minor $\equiv$
h pairwise-disjoint pairwise-adjacent connected subgraphs



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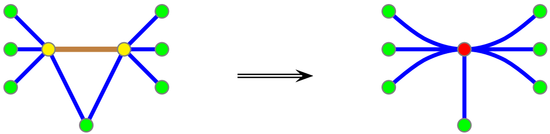

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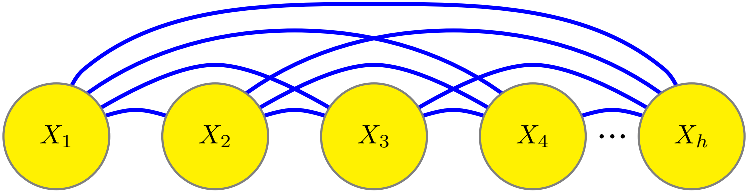

planar graphs are 4-colourable and are K_5-minor-free

excluded minor	graph family	colourable	
K_1	no vertices	0	
K_2	no edges	1	
K_3	forests	2	
K_4	series parallel	3	[Hadwiger '43, Dirac '52]

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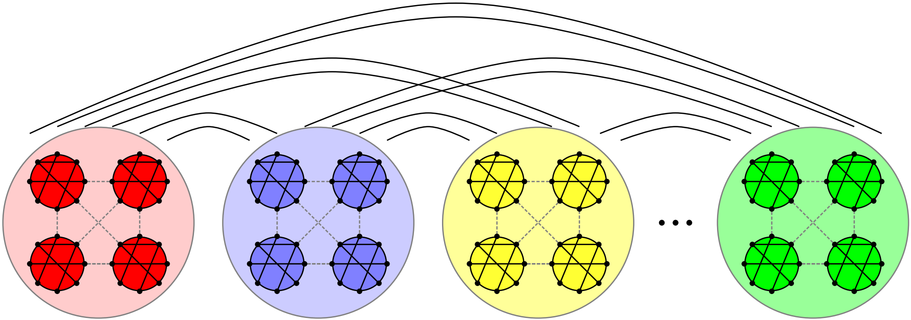
- $O(h \log \log h)$ colours [Delcourt–Postle '21]

open problem: are K_h-minor-free graphs $O(h)$-colourable?

clustered colouring

k-colouring with clustering c:

- each v is assigned one of k colours
- each monochromatic component has $\leqslant c$ vertices



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or clustering 2 [Cowen, Cowen, Woodall '86]

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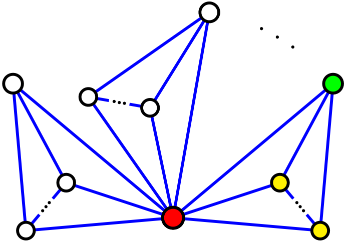
but planar graphs are not 3-colourable with bounded clustering



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clustered chromatic number

$\chi^{\text{CLUS}}(\mathcal{G}) :=$ minimum k such that $\exists c$ and every graph in $\mathcal{G}$ is k-colourable with clustering c

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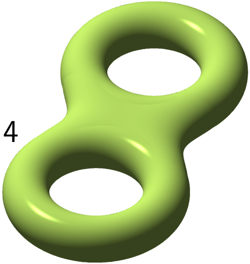
$\chi^{\text{CLUS}}(\text{planar graphs}) = 4$

$\chi^{\text{CLUS}}(\text{graphs embeddable on any fixed surface}) = 4$
[Dvořák & Norin '17]



clustered colouring of K_h-minor-free graphs

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lower bound

$$\chi^{\mathsf{CLUS}}(K_h\text{-minor-free graphs}) \geqslant h-1$$

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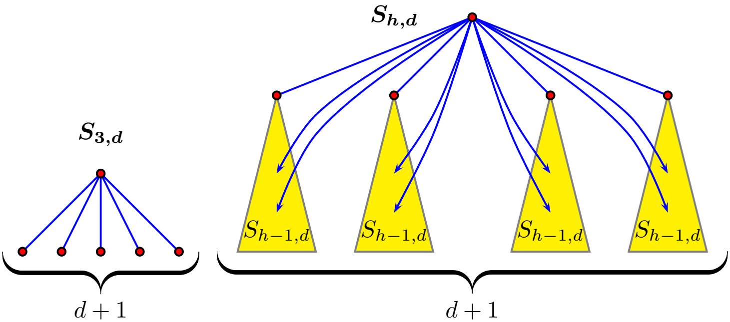

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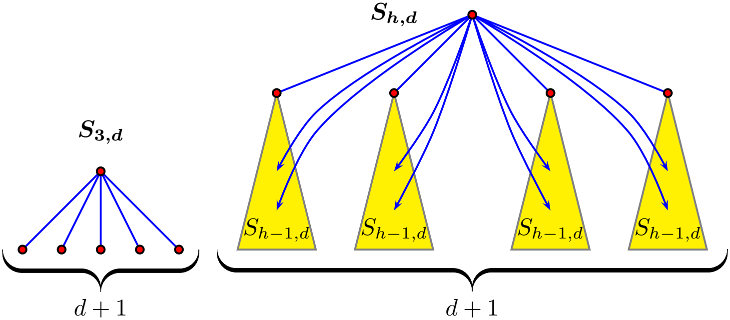

$S_{h,d}$ is K_h-minor-free and has no $(h-2)$-colouring with clustering $\leqslant d$

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upper bounds

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upper bounds

	# colours	clustering
[Kawarabayashi & Mohar '07]	$\lceil \frac{31}{2}h \rceil$	$c(h)$
[W. '10]*	$\lceil \frac{7h-3}{2} \rceil$	$c(h)$
[Edwards, Kang, Kim, Oum, Seymour '14]	$4h-4$	$c(h)$
[Liu & Oum '15]	$3h-3$	$c(h)$
[Norin '15]	$2h-2$	$c(h)$
[van den Heuvel & W. '17]	$2h-2$	$\lceil \frac{h-2}{2} \rceil$
[Liu & W. '19]	h	$c(h)$

clustered colouring of K_h-minor-free graphs

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upper bounds

clustered Hadwiger theorem [Dujmović, Esperet, Morin, W. '23]

- K_h-minor-free graphs are $(h-1)$-colourable with clustering $c(h)$
- $\chi^{\mathsf{CLUS}}(K_h\text{-minor-free graphs}) = h-1$

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(announced by Dvořák & Norin '17)

$K_{s,t}$-minor-free graphs ($s \leqslant t$)

Hadwiger's Conjecture $K_{s,t}$-minor-free graphs are $(s+t-1)$-colourable

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upper bounds

	# colours	clustering
[van den Heuvel & W. '17]	$3s$	$O(t^s)$
[Dvořák & Norin '17]	$2s+2$	$c(s,t)$
[Liu & W. '19]	$s+2$	$c(s,t)$

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theorem [Dujmović, Esperet, Morin, W. '23]

- $K_{s,t}$-minor-free graphs are $(s+1)$-colourable with clustering $c(s,t)$
- $\chi^{\text{CLUS}}(K_{s,t}\text{-minor-free graphs}) = s+1$ for $t \geqslant \max\{s,3\}$

unifying K_h-minor-free and $K_{s,t}$-minor-free results

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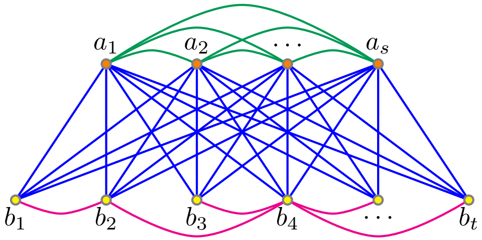
let $\mathcal{J}_{s,t}$ be the class of all graphs with



s-clique

all $a_i b_j$ edges

t-vertex tree

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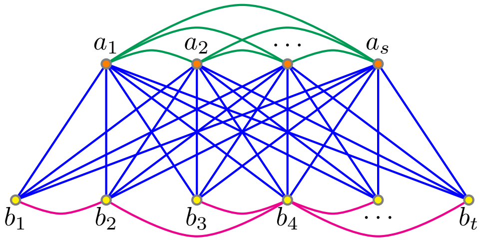

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main theorem [Dujmović, Esperet, Morin, W. '22]
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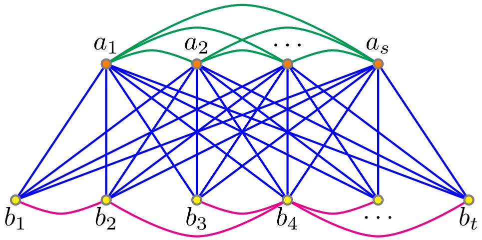

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implies previous theorems since $\mathcal{J}_{h-2,2} = \{K_h\}$

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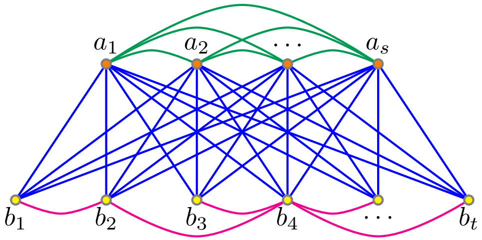

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proof tools

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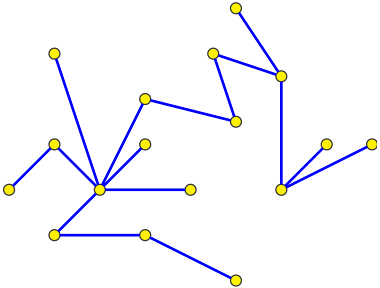
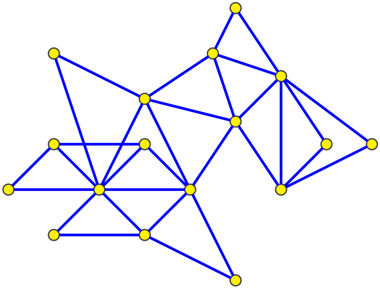
1. treewidth

measures how similar a graph is to a tree



treewidth 1



treewidth 2

proof tools

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2. graph product structure theory

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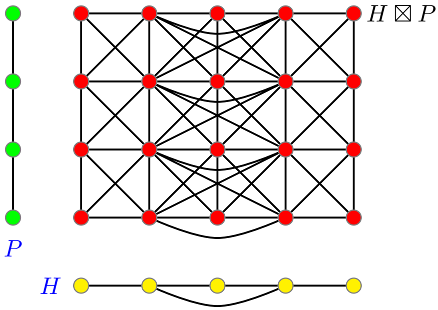
> **theorem** [Dujmović, Joret, Micek, Morin, Ueckerdt, W. '19]
> every planar graph G is a subgraph of $H \boxtimes P$
> for some graph H with treewidth $\leqslant 8$ and some path P



proof tools

1. treewidth
2. graph product structure theory
3. graph minor structure theorem

theorem [Robertson & Seymour '80s]
graphs excluding a fixed minor can be constructed from graphs embedded on surfaces, vortices, apex vertices, and clique-sums

proof tools

1. treewidth
2. graph product structure theory
3. graph minor structure theorem
4. islands, curtains, drapes, etc.

conclusion

K_h-minor-free	K_h-subdivision-free
Hadwiger Conjecture: $h-1$ colours suffice	Hajós' Conjecture: $h-1$ colours suffice

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true for clustered colouring [Dujmović, Esperet, Morin, W. '23]	almost true for clustered colouring $4h-9$ colours [Liu W. '19]

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so do not think of the Clustered Hadwiger Theorem
as evidence for Hadwiger's Conjecture