Some lower bounds on conditionally decomposable polytopes

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The Minkowski sum of two polytopes Q, R is defined by

$$Q + R = \{q + r : \forall q \in Q, r \in R\}.$$

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(it's time to use the white board)

Suppose we have two polytopes combinatorially equivalent, but one decomposable, the other one indecomposable, such polytopes are called conditionally decomposable.

[1] source: https://polyhedr.com/cuboctahedron2.html

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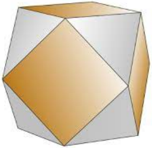

Figure 1: Meyer's example [5], a cuboctahedron C [1]

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1982 Kallay [3]

Theorem (Kallay, 1982)

P is decomposable iff the geometric graph $G(P) = (V(P), E(P))$ is decomposable, i.e. there is an isomorphism ϕ between $G(P)$ and $G(Q)$ such that $\phi(p_i) - \phi(p_j) = \lambda(p_i - p_j), \lambda > 0$.

Theorem (Kallay, 1982)

Let A_1, A_2 be indecomposable graphs. Then if there are two disjoint edges connecting A_1 and A_2, and the lines containing these two edges are skew, then $A_1 \cup A_2$ is indecomposable.

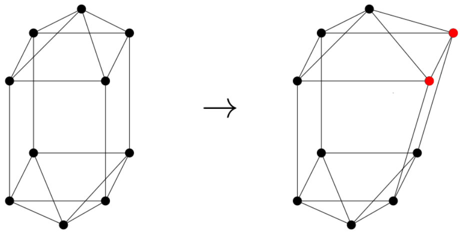

Figure 2: Kallay's example [3]

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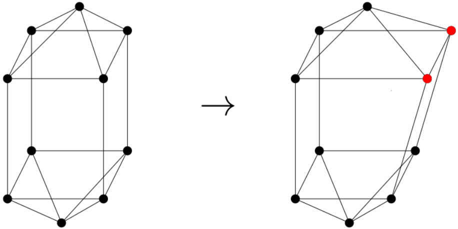

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Conditionally decomposable 3-polytopes with V vertices and F facets exist if and only if $V \leq F \leq 2V - 8$.

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Some other work done by Yost [16], and Przesławski and Yost [10].

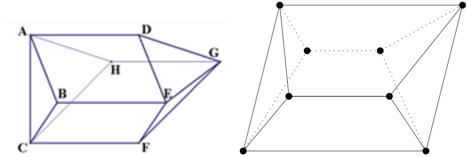

Figure 3: F288 and F282

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Let P be a d-polytope, defined by intersection of closed half-spaces, $P = \bigcap_{i \in I} f_i(x) \geq 0$, where f_i is an affine function. Let F be a facet of P defined by $f_1 = 0$. Then the wedge of P at F is $W(P, F) = \bigcap_{i \in I \cup \{j_1, j_2\}} f_i(x) \geq 0$, where $f_{j_1} = x_{d+1}$, $f_{j_2} = f_1 - x_{d+1}$.

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Theorem 2

Let P be a d-polytope and F a nonempty face. Then,
(1) P is decomposable if and only if $W(P, F)$ is decomposable; if P is conditionally decomposable, then $W(P, F)$ is conditionally decomposable.
(2) For $d \geq 4$, there is a conditionally decomposable d-polytope with $4d - 4$ vertices and onwards.

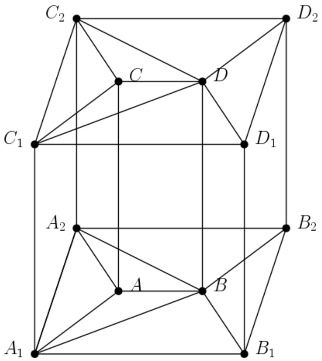

Figure 4: sketch of one of the examples

Let P be a d-polytope, $[0, a]$ be a line segment. Then $P_1 = P + [0, a]$ and $P_2 = P + [0, ka]$ are combinatorially equivalent, $k > 0$.

Theorem 3

Let P be a d-polytope with no more than $4d - 5$ vertices that has a line segment for a summand. Then P is combinatorially decomposable.

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Theorem 3

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sketch of proof:
There is a partition of the vertices $V = V_1 \cup V_2$ that *w.l.o.g* we may assume that $|V_1| \leq 2d - 3 = 2(d - 1) - 1$.
Recall that a a d-polytope with no more than $2d - 1$ vertices has a strongly connected triangular chain of faces [15].

For a decomposable polytope with

- $< 2d$ vertices $\Rightarrow$ indecomposable [15]
- $2d$ vertices $\Rightarrow$ it is a simplicial prism [2], [11]
- $2d+1$ vertices $\Rightarrow$ it is either Σ_3, $\Delta_{2,2}$ or a pentasm or a capped prism [6]
- $2d+2$ vertices $\Rightarrow$ when $d \geq 6$, they are either 2-capped prism, or have a line segment for a summand, whose cross section is a $(d-1)$-polytope with $d+1$ vertices

Lemma 4

Let P be a d-dimensional polytope having a line segment for a summand, and with $\leq 3d-4$ vertices. Let V_1, V_2 be the partition of vertices of P such that edges in between V_1 and V_2 are parallel to that line segment. Then the subgraph of P induced by V_1 (and V_2, resp.) is indecomposable.

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Let P be a decomposable d-polytope with $[2d, 3d-4]$ vertices, then P has a line segment for a summand w.r.t combinatorial equivalence.

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Theorem 5

Let P be a decomposable d-polytope with $[2d, 3d-4]$ vertices, then P has a line segment for a summand w.r.t combinatorial equivalence.

Corollary 6

There is no conditionally decomposable polytope with $[2d, 3d-4]$ vertices.

A small application

Grünbaum's lower bound theorem says that any d-polytope with $d + k \leq 2d$ vertices has at least

$$\binom{d}{2} - \binom{k}{2} + kd$$

edges, which was proved independently by Pineda-Villavicencio, Ugon and Yost [7], and by Xue [13] for any k-faces.

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It is conjectured [8] that for d-polytopes with $2d + k \in [2d + 3, 3d - 6]$ vertices, the lower bound of the number of edges is attained at a $(d - k - 1)$-fold pyramid over a $(k + 1)$-simplicial prism, which is $d(d + k) - \binom{k+1}{2}$.

Theorem 7

Let P be a decomposable d-polytope with $2d + k$ vertices, where $k \in [0, d-4]$. Then the minimum number of edges of P is $d(d+k) - \binom{k+1}{2}$.

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sketch of proof: Assume that the number of edges in between V_1, V_2 is $d+m$, and there are $d+m+i$ vertices in V_1, $d+k-m-i$ vertices in V_2. Then by Grünbaum's lower bound theorem,

$$\begin{aligned} f_1(P) &\geq (d+m) + \binom{d}{2} - \binom{m+i+1}{2} + (m+i+1)d - (d+m) \\ &\quad + \binom{d}{2} - \binom{k-m-i+1}{2} + (k-m-i+1)d - (d+m) \\ &= d^2 + kd - \binom{m+i}{2} - \binom{k-m-i+1}{2} - 2m - i. \\ &\geq d^2 + kd - \binom{k+1}{2}. \end{aligned}$$

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The characterization from McMullen [4] that, P is decomposable if and only if in the Gale diagram of P°, $\dim(\cap S_i) > 0$, where S_i ranges from all the cofacets of P°. Refer also to the book of Jesús, Jörg and Francisco [1].

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Proposition 1

There is no conditionally decomposable d-polytopes with $d + 3$ facets.

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There is no conditionally decomposable 4-polytope with 8 facets.

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Proposition 2

There is no conditionally decomposable 4-polytope with 8 facets.

The existence of a conditionally decomposable 5-polytope with 9 facets can be shown by a suitable arrangement in the gale diagram of P°, and hence the existence of a conditionally decomposable d-polytope with $d+4$ facets.

J. W. and David Yost, Some lower bounds of conditionally decomposable polytopes, 2024+

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Thank you for your attention!