'Segre-type' theorems: combinatorial characterisations for algebraic objects

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ACC 2023, Perth

...this talk is about certain point sets in $\mathrm{PG}(n, q)$

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- $\mathrm{PG}(n, q)$: n-dimensional projective space over $\mathbb{F}_q$
- points have $n + 1$ homogeneous coordinates over the finite field $\mathbb{F}_q$, $q = p^h$, p prime

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  - There are $\frac{q^3-1}{q-1}$ points in $\mathrm{PG}(2, q)$.
- the projective space contains points, lines, planes, solids,... and hyperplanes.

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  - The point $(1, 1, 2)$ lies on the line $x + y - z = 0$.
- A *conic* in a projective plane is a set of points whose coordinates (x_0, y_0, z_0) satisfy a homogeneous quadratic equation.

EXAMPLE

The set of points (x, y, z) with $y^2 = xz$ is a (non-degenerate) conic in $\mathrm{PG}(2, q)$.

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$$\{(1, t, t^2) : t \in \mathbb{F}_q\} \cup \{(0, 0, 1)\}$$

- Every non-degenerate conic in $\mathrm{PG}(2, q)$ has $q + 1$ points.
- Every line meets a non-degenerate conic in either 0, 1 or 2 points, that is, no three of its points are collinear.

OVALS IN $\mathrm{PG}(2, q)$

DEFINITION

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A set of points in $\mathrm{PG}(2, q)$ with no three collinear points has size at most $q + 2$.

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A set of points in $\mathrm{PG}(2, q)$ with no three collinear points has size at most $q + 2$.

Lemma (Bose (1947))

Let $\mathcal{A}$ be a set of points in $\mathrm{PG}(2, q)$, q odd, such that no three points are collinear, then

$$|\mathcal{A}| \leq q + 1.$$

Ovals in $\mathrm{PG}(2, q)$

Theorem (Qvist 1952)

Every oval in $\mathrm{PG}(2, q)$, q even, can be extended to a set of $q + 2$ points, no three collinear (a hyperoval).

Every non-degenerate conic is an oval, but...

Question

Is every oval in $\mathrm{PG}(2, q)$ a conic?

OVALS IN PG(2, q)

MR0054979 (14,1008d) Reviewed

Järnefelt, G.; Kustaanheimo, Paul

An observation on finite geometries. *Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949,* pp. 166–182. *Johan Grundt Tanums Forlag, Oslo,* 1952.

48.0X

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Citations
From References: 4
From Reviews: 3

In a geometry with coordinates from a field with a prime number of elements, p, the axioms of incidence will of course be satisfied. It is observed here that the quadratic form $x^2 - ky^2$ with k a quadratic non-residue may be used to define a metric. Certain axioms of congruence are satisfied if this metric is used. It is conjectured that in a plane with $p^2 + p + 1$ points a set of $p + 1$ points, no three on a line, will form a quadric. The reviewer finds this conjecture implausible.

Reviewed by Marshall Hall Jr.

Theorem (B. Segre 1955)

Every set of $q+1$ points in $\mathrm{PG}(2, q)$, q odd, such that no three are collinear, is the set of points on a conic.

MR0071034 (17,72g) Reviewed

Segre, Beniamino

Ovals in a finite projective plane.

Canadian J. Math. 7 (1955), 414–416.

48.0X

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In a finite projective plane with $n+1$ points on a line there can be at most $n+2$ points with the property that no three are on a line, and if n is odd there can be at most $n+1$ with this property. If n is even and we have $n+1$ points, no three on a line, then there exists a further point which can be adjoined to these giving $n+2$ points, no three on a line. In a Desarguesian plane a non-degenerate conic contains $n+1$ points, no three on a line. If, when n is odd, we call $n+1$ points, no three on a line, an oval, then it was conjectured by Järnefelt and Kustaanheimo [Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, Tanum, 1952, pp. 166–182; MR0054979] that in a Desarguesian plane of odd order n, an oval is necessarily a conic. This conjecture is shown to be true in this paper. The method of proof is ingenious. We may take three points of the oval to be $A_1:(1,0,0)$, $A_2:(0,1,0)$, and $A_3:(0,0,1)$ and if $P(a_1,a_2,a_3)$ is a further point on the oval and $x_2=\lambda_1x_3$, $x_3=\lambda_2x_1$, $x_1=\lambda_3x_2$ are the three secants PA_1, PA_2, PA_3, then immediately $\lambda_1\lambda_2\lambda_3=1$. Since the product of all non-zero elements in the field is -1, it will follow that for the tangents at A_1, A_2, A_3 that $x_2=k_1x_3$, $x_3=k_2x_1$, $x_1=k_3x_2$ we will have $k_1k_2k_3=-1$. From this the inscribed triangle and its circumscribed triangle are perspective with respect to the center $(1, k_1k_2, -k_2)$. It follows generally that every inscribed triangle and its circumscribed triangle are perspective. Using this relation on the triangles formed from P, A_1, A_2, and A_3, we find that the coordinates of P satisfy a quadratic equation which becomes $x_2x_3+x_3x_1+x_1x_2=0$ if we take C as $(1,1,1)$, as we may. [The fact that this conjecture seemed implausible to the reviewer seems to have been at least a partial incentive to the author to undertake this work. It would be very gratifying if further expressions of doubt were as fruitful.]

Reviewed by Marshall Hall Jr.

Recall:

Definition

A set of $q+2$ points in $\mathrm{PG}(2,q)$, no three of which are collinear, is called a hyperoval.

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Example

The set

$$\{(1,t,t^2) : t \in \mathbb{F}_{2^h}\} \cup \{(0,0,1)\} \cup \{(0,1,0)\}$$

is a hyperoval.

What about q even?

Recall:

Definition

A set of $q+2$ points in $\mathrm{PG}(2,q)$, no three of which are collinear, is called a hyperoval.

Every line meets a hyperoval in 0 or 2 points.

Example

The set

$$\{(1,t,t^2) : t \in \mathbb{F}_{2^h}\} \cup \{(0,0,1)\} \cup \{(0,1,0)\}$$

is a hyperoval.

More generally, every conic has a nucleus in $\mathrm{PG}(2,q)$ and hence gives rise to a hyperoval. These hyperovals are the regular hyperovals.

Bill Cherowitzo's Hyperoval Page

Introduction | Table of Contents | Bibliography | Table of Known Hyperovals | Open Problems | Glossary | Search Index | Exit

Known Hyperovals in $PG(2,2^h)$

Name	O-Polynomial	Field Restriction	Section Comments	Properties
Hyperconic	$f(x) = x^2$	None	Section 2	Available
Translation	$f(x) = x^{2^i}$ $(i,h) = 1$	None	Section 2	
Segre	$f(x) = x^6$	h odd	Section 2	
Glynn I	$f(x) = x^{3\sigma + 4}$	h odd	Section 2	
Glynn II	$f(x) = x^{\sigma + \gamma}$	h odd	Section 2	
Payne	$f(x) = x^{1/6}+x^{1/2}+x^{5/6}$	h odd	Section 3	
Cherowitzo	$f(x) = x^{\sigma}+ x^{\sigma+2} + x^{3\sigma+4}$	h odd	Section 3	
Subiaco	see comments	None	Section 3	
Adelaide	see comments	h even	Section 3	
Penttila-O'Keefe	see comments	h = 5	Section 4	

$$\gamma^4 \equiv \sigma^2 \equiv 2 \bmod (2^h-1)$$

eturn to Research Section of Bill Cherowitzo's Home Page. Page established October 1, 1999 *Last Updated June 8, 2004*.

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Subiaco Oval

From Wikipedia, the free encyclopedia

Coordinates: 31°56′40″S 115°49′48″E

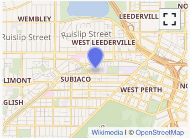
Subiaco Oval (/suːbiˈækoʊ/; nicknamed **Subi**) was a sports stadium in Perth, Western Australia, located in the suburb of Subiaco. It was opened in 1908 and closed in 2017 after the completion of the new Perth Stadium in Burswood, Western Australia.

Subiaco Oval was the highest capacity stadium in Western Australia and one of the main stadiums in Australia, with a final capacity of 43,500 people. It began as the home ground for the Subiaco Football Club and from the 1930s onward was the home of Australian rules football in Western Australia. It hosted the annual grand final of the West Australian Football League (WAFL), with the ground record attendance of 52,781 set at the 1979 Grand Final. It later served as the home ground of the West Coast Eagles and the Fremantle Football Club, the two Perth teams in the Australian Football League (AFL). Other events included Socceroos International Friendly Game in 2005, Perth Glory soccer games (including two National Soccer League grand finals), Western Force rugby games, International rules football matches, and rock concerts. Under naming rights the stadium was known as **Patersons Stadium** (2011–2014) and **Domain Stadium** (2015–2017) in its final years.

The demolition of the stadium was completed in November 2019, though the oval playing surface was retained as part of the school grounds of Bob Hawke College.[2][3] The refurbished oval was opened to the general public in June 2020.[4]

Contents [hide]

Subiaco Oval



Former names	Mueller Park, Patersons Stadium, Domain Stadium
Location	Roberts Road, Subiaco, Western Australia, Australia
Coordinates	31°56′40″S 115°49′48″E
Owner	Western Australian Government
Operator	West Australian Football Commission
Capacity	43,082[1]
Record attendance	**Concerts**: 65,000 (Adele Live 2017) **Sports**: 52,781 (1979 WANFL Grand Final)

Hyperovals: summary

- If a hyperoval exists, necessarily q is even
- For all even q, there is a regular hyperoval
- Other examples are known
- The classification seems hopeless

General 'Segre-type' problem:

(A) Start with a 'nice' point set defined by an algebraic property

BACK TO SEGRE'S RESULT

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- Question: Is every set satisfying (b) of the form (a)?

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(A) Start with a 'nice' point set defined by an algebraic property *(e.g. a non-degenerate conic)*

(B) Determine its combinatorial properties *(every line meets this conic in* $0, 1$ *or* 2 *points)*

- Question: Is every set satisfying (b) of the form (a)? If no, under which extra assumptions can we draw the conclusion?

Where to go from here?

'Segre-type theorems' in different settings:

- higher dimension: *quasi-quadrics*
- other polarity: *unitals* (or *quasi-Hermitian varieties*)
- sets with few intersection numbers

Hyperoval: every line intersects in 0 or 2 points.

- What if we ask for 0 and d?
- Or 1 and d?
- Or m and n?
- (And why stop at 2 different intersection numbers?)

Definition

A set of points in $\mathrm{PG}(2, q)$ such that every line meets it in 0 or d points necessarily has $(q+1)(d-1)+1$ points, and is called a maximal arc.

- A hyperoval is a maximal arc of degree $d = 2$.
- Trivial examples: $d = 1$ (one point), $d = q$ (plane with line removed)

Sets with two intersection numbers: 0 and d

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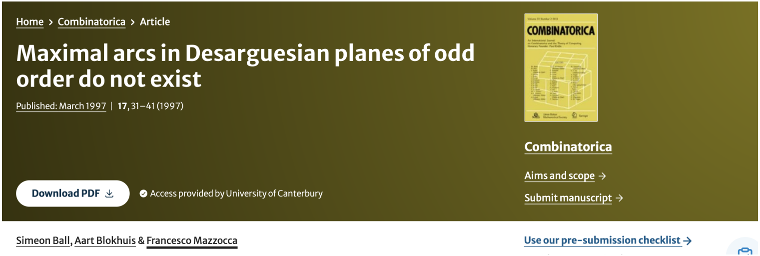
Home > Combinatorica > Article

Maximal arcs in Desarguesian planes of odd order do not exist

Published: March 1997 | 17, 31–41 (1997)

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Access provided by University of Canterbury

Simeon Ball, Aart Blokhuis & Francesco Mazzocca

SETS WITH TWO INTERSECTION NUMBERS: 0 AND d

- If d is a maximal arc of degree d, necessarily $d|q$, and q is even.
- Denniston constructed maximal arcs of degree 2^i for all i
- Other examples are known
- The classification seems hopeless

SETS WITH TWO INTERSECTION NUMBERS: 1 AND d

THEOREM (TALLINI SCAFATI 1966)

Let $\mathcal{S}$ be a set in $\mathrm{PG}(2, q)$ with intersection numbers $(1, d)$, $1 < d < q + 1$. Then q is a square, $d = \sqrt{q} + 1$, and $\mathcal{S}$ is a Baer subplane or a unital.

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Let $\mathcal{S}$ be a set in $\mathrm{PG}(2, q)$ with intersection numbers $(1, d)$, $1 < d < q + 1$. Then q is a square, $d = \sqrt{q} + 1$, and $\mathcal{S}$ is a Baer subplane or a unital.

A *Baer subplane* is a subplane of $\mathrm{PG}(2, q)$ of order $\sqrt{q}$ and has size $q + \sqrt{q} + 1$; a unital has $q\sqrt{q} + 1$ points.

What about set of points with two intersection numbers (m, n)?

What about set of points with two intersection numbers (m, n)?

- (folklore, and Calderbank-Kantor 1986:) two-intersection sets give rise to strongly regular graphs and to two-weight codes
- Many examples are known
- The classification seems hopeless

A conic is a set of point such that every line meets it in $0, 1, 2$ points.

- What if we replace '2' by d?

THEOREM (UEBERBERG 1993)

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- *A line*
- *A maximal arc*
- *A Baer subplane*
- *A* unital

[In each of those cases, only two of the three values of $\{0,1,d\}$ occurs!]

DEFINITION

$X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} + Z^{\sqrt{q}+1} = 0$ in $\mathrm{PG}(2, q)$, q square, defines a *Hermitian curve* $\mathcal{U}$.

(These are precisely the set of absolute points of a unitary polarity).

- Every line meets $\mathcal{U}$ in 1 or $\sqrt{q}+1$ points.
- The number of points of $\mathcal{U}$ is $q\sqrt{q}+1$.

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- The number of points of $\mathcal{U}$ is $q\sqrt{q} + 1$.

DEFINITION

A unital in $\mathrm{PG}(2, q)$, q square, is a set of $q\sqrt{q} + 1$ points meeting every line in 1 or $\sqrt{q} + 1$ points.

Unitals

Is every unital a Hermitian curve?

- The answer is NO!
- ‘One’ other family is known: Buekenhout-Metz unitals and Buekenhout-Tits unitals
- constructed from cones over quasi-quadrics

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Unitals in the news

Mathematicians discovered a solution to a century-old problem that's perfect for your next party

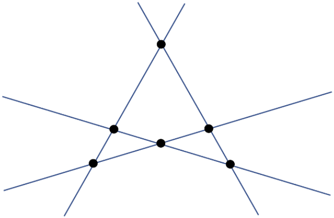
by **Shawn Johnson** — November 4, 2023 in **Innovation**

Mathematicians discovered a solution to a century-old problem that's perfect for your next party

by Shawn Johnson — November 4, 2023 in Innovation

Instead, Mattheus remembered a strange object called a Hermitian unital, something that finite geometers tend to be very familiar with, but that a mathematician working in combinatorics was unlikely to ever encounter (Qanta).

The construction by Mattheus-Verstraëte in their work on $R(4, t)$ used the fact that a Hermitian unital does not contain an O'Nan configuration.



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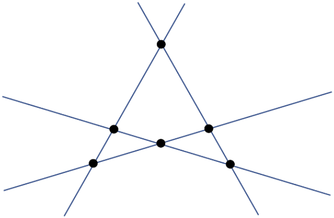

CONJECTURE

A unital in $\mathrm{PG}(2, q^2)$ is a Hermitian unital if and only if it does not contain an O'Nan configuration.

The feet of a point P, not on a unital, are the points F of $\mathcal{H}$ such that PF is a tangent line to the unital.

CLASSICAL RESULTS

- For a Hermitian curve $\mathcal{H}$ for every point, the *feet* are collinear.

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CLASSICAL RESULTS

- For a Hermitian curve $\mathcal{H}$ for every point, the *feet* are collinear.
- If the feet are collinear for every point, then the unital is a Hermitian curve (J.A. Thas 1992)

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CLASSICAL RESULTS

- For a Hermitian curve $\mathcal{H}$ for every point, the *feet* are collinear.
- If the feet are collinear for every point, then the unital is a Hermitian curve (J.A. Thas 1992)
- For a non-classical Buekenhout-Metz unital: feet are collinear if and only if $P \in \ell_\infty$ (G. Ebert. 2004)

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Classical results

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- If the feet are collinear for every point, then the unital is a Hermitian curve (J.A. Thas 1992)
- For a non-classical Buekenhout-Metz unital: feet are collinear if and only if $P \in \ell_\infty$ (G. Ebert. 2004)

Open problem/conjecture

If the feet are collinear for all $P \in \ell_\infty$, is the unital Buekenhout-Metz?

QUESTION (EBERT)

If the feet of a point are not collinear, which configurations are possible?

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THEOREM (ABARZUA, POMAREDA, VEGA 2018)

A line meets the feet set of a Buekenhout-Metz unital in $0, 1, 2, 4$ or $\sqrt{q}+1$ points; if not collinear, the feet set of a point form two arcs.

THEOREM (FAULKNER-VDV 2023)

A line meets the feet set of the *Buekenhout-Tits unital in* $0, 1, 2, 3, 4$ *or* $\sqrt{q} + 1$ *points. (and* 3 *occurs).*

Recall: a point set such that every line meets it in $0, 1$, or d points necessarily only has two intersection sizes.

- What about $0, 2, d$?

Math. Proc. Camb. Phil. Soc. (1990), **108**, 445 445
Printed in Great Britain

On $(q+t)$-arcs of type $(0, 2, t)$ in a desarguesian plane of order q

BY GÁBOR KORCHMÁROS
Department of Mathematics, University of Basilicata, 85100 Potenza, Italy

AND FRANCESCO MAZZOCCA
Department of Mathematics and its Applications, University of Napoli, via Mezzocannone 8, 80134 Napoli, Italy

(*Received* 13 *December* 1989; *revised* 2 *March* 1990)

THEOREM (KORCHMÁROS-MAZZOCCA 1990, GÁCS-WEINER 2003)

If $\mathcal{A}$ is a KM-arc of type t in $\mathrm{PG}(2, q)$, $2 \leq t < q$, then

- q is even;
- t is a divisor of q.

Theorem (Korchmáros-Mazzocca 1990, Gács-Weiner 2003)

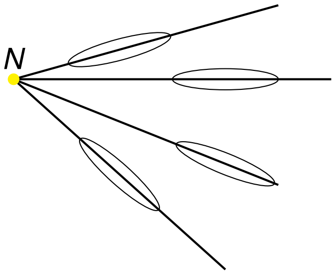
If $\mathcal{A}$ is a KM-arc of type t in $\mathrm{PG}(2, q)$, $2 \leq t < q$, then

- q is even;
- t is a divisor of q.

If $t > 2$, then

- there are $\frac{q}{t} + 1$ different t-secants to $\mathcal{A}$, and they are concurrent.

The common point of the t-secants is called the t-nucleus.



OVERVIEW: INFINITE FAMILIES OF KM-ARCS OF TYPE 2^i IN $\mathrm{PG}(2, 2^h)$ FOR

(A) $h - i \mid h$ (Korchmáros–Mazzocca 1990, Gács–Weiner 2003)

(B) $h - i + 1 \mid h$ (Gács–Weiner 2003)

(C) $i = h - 2$ (Vandendriessche, De Boeck-VdV 2015)

(D) $i = h - 3$ (De Boeck-VdV 2017)

(E) $i = h - 4$ for some h (De Boeck-VdV 2017)

(F) $i = 1$ Hyperovals

THEOREM (DE BOECK–VDV 2015)

Translation KM-arcs of type 2^i in $\mathrm{PG}(2, 2^h)$ and i-clubs of rank h in $\mathrm{PG}(1, 2^h)$ are equivalent objects.

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- Via i-clubs: examples of type $q/2$, $q/4$, 2^i with $h - i | h$, $h - i + 1 | h$.
- No 2-club in $\mathrm{PG}(2, 32)$, but there is a KM-arc of type 4 in $\mathrm{PG}(2, 32)$ and $\mathrm{PG}(2, 64)$.
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- Weaker than translation: elation
- Still no infinite family for KM-arcs of type 4.
- The classification seems hopeless (except for type $q/2$!)

QUADRICS

Conics in $\mathrm{PG}(2, q) \rightarrow$ quadrics in $\mathrm{PG}(n, q)$

- Conics and Hermitian curves are polar spaces in a projective plane
- Higher-dimensional analogues (quadrics and hermitian varieties) have points, lines, planes,etc..fully contained in them.

(Characteristic $\neq 2$ here)

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Conic in $\mathrm{PG}(2, q)$: points $X = (x, y, z)$ with $XAX^t = 0$, $A = A^t$:

$$[x, y, z] \begin{bmatrix} a & f & e \\ f & b & d \\ e & d & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\leftrightarrow ax^2 + by^2 + cz^2 + 2dyz + 2exz + 2fxy = 0.$$

Set of points $X = (x_0, x_1, \ldots, x_r)$ in $\mathrm{PG}(r, q)$ with $XAX^t = 0$, where $A = A^T$:

- In $\mathrm{PG}(2n+1, q)$: comes in elliptic or hyperbolic type
- In $\mathrm{PG}(2n, q)$: parabolic type

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- polar space: Points of those sets + subspaces fully contained in them

SEGRE-TYPE PROBLEM

If a point set in $\mathrm{PG}(n, q)$ has the same intersection sizes with respect to hyperplanes as a non-degenerate quadric, a quadric?

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FIRST STEP

How does a hyperplane H intersect a non-singular quadric?

Segre-type problem

If a point set in $\mathrm{PG}(n, q)$ has the same intersection sizes with respect to hyperplanes as a non-degenerate quadric, a quadric?

First step

How does a hyperplane H intersect a non-singular quadric?

- non-singular quadric in H or
- cone with vertex a point and base a non-singular quadric of the same type

Depending on the type/dimension this gives us two or three intersection numbers.

DEFINITION

A *quasi-quadric* is a point set that has the same intersection numbers with respect to hyperplanes as a non-degenerate quadric.

DEFINITION

An elliptic quadric in $\mathrm{PG}(3, q)$ is a point set satisfying an equation of the form

$$X_0X_1 + f(X_2, X_3) = 0$$

where f is an irreducible polynomial of degree 2.

- Every plane meets an elliptic quadric in 1 or $q + 1$ points.

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DEFINITION

An ovoid in $\mathrm{PG}(3, q)$ is a point set of size $q^2 + 1$ such that every plane meets it in 1 or $q + 1$ points.

Elliptic in dimension 3: ovoids

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Theorem (Barlotti/Panella 1956)

If q is odd, then every ovoid is an elliptic quadric.

Elliptic in dimension 3: ovoids

- Elliptic quadrics are examples of ovoids.
- If $q = 2^{2e+1}$, one other example is known: (Suzuki) Tits-ovoid.
- The classification seems hopeless

BACK TO THE SEGRE-TYPE PROBLEM

Apart from those lower-dimensional exceptions, is every quasi-quadric a quadric?

Back to the Segre-type problem

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Theorem (De Winter- Schillewaert 2010)

A quasi-quadric that has the same intersection numbers with co-dimension two spaces as a non-degenerate quadric, is a quadric.

BACK TO THE SEGRE-TYPE PROBLEM

Apart from those lower-dimensional exceptions, is every quasi-quadric a quadric?

THEOREM (DE WINTER- SCHILLEWAERT 2010)

A quasi-quadric that has the same intersection numbers with co-dimension two spaces as a non-degenerate quadric, is a quadric.

Do we need the co-dimension two spaces?

De Clerck, Hamilton, O'Keefe, Penttila 2000

Quasi-quadrics and related structures

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Abstract

In a projective space $\mathrm{PG}(n,q)$ a *quasi-quadric* is a set of points that has the same intersection numbers with respect to hyperplanes as a non-degenerate quadric in that space. Of course, non-degenerate quadrics themselves are examples of quasi-quadrics, but many other examples exist. In the case that n is odd, quasi-quadrics have two sizes of intersections with hyperplanes and so are *two-character sets*. These sets are

De Clerck, Hamilton, O'Keefe, Penttila

- Elliptic and hyperbolic quasi-quadrics are two-intersection sets with respect to hyperplanes.
- So they give rise to strongly regular graphs
- (Schillewaert-VdV 2021) ‘Switching is pivoting’
- Many other constructions are known
- The classification *is* hopeless

THREE-INTERSECTION SETS IN HIGHER DIMENSION?

OBSERVATION

A cone in $\mathrm{PG}(3,q)$ with vertex a point and base a two-intersection set B w.r.t. lines of type (m,n) is a 3-intersection set with respect to planes with intersection sizes $|B|, qm+1, qn+1$.

OBSERVATION

A cone in $\mathrm{PG}(3, q)$ with vertex a point and base a two-intersection set B w.r.t. lines of type (m, n) is a 3-intersection set with respect to planes with intersection sizes $|B|, qm + 1, qn + 1$.

THEOREM (ZUANNI-INNAMORATI 2020)

A blocking set *with respect to lines in* $\mathrm{PG}(3, q)$ *such that every plane meets it as it would meet a unital cone (resp. Baer cone) is a unital cone (Baer cone).*

- D. Jena 2022: arbitrary dimension and base hyperovals, Baer subgeometries, unitals, maximal arcs.

Three-intersection sets in higher dimension?

The case of an ovoidal cone in $\mathrm{PG}(4, q)$ was left open

Three-intersection sets in higher dimension?

The case of an ovoidal cone in $\mathrm{PG}(4, q)$ was left open

- This is the Segre-type problem for certain singular quadrics in $\mathrm{PG}(4, q)$: quadratic cones.
- If q is odd: every ovoidal cone is a quadratic cone.
- Recall: there exist non-singular quasi-quadrics in $\mathrm{PG}(4, q)$ that are not quadrics.

A plane meets a quadratic cone in $\mathrm{PG}(4, q)$ in 1, $q+1$, or $2q+1$ points.

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THEOREM (DE BRUYN-VDV 20??)

A set of points in $\mathrm{PG}(4, q)$ *with the same intersection numbers with respect to planes as a quadratic cone is either:*

- *an ovoidal cone (and has* $q^3 + q + 1$ *points);*

A plane meets a quadratic cone in $\mathrm{PG}(4, q)$ in 1, $q+1$, or $2q+1$ points.

Theorem (De Bruyn-VdV 20??)

A set of points in $\mathrm{PG}(4, q)$ with the same intersection numbers with respect to planes as a quadratic cone is either:

- *an ovoidal cone (and has q^3+q+1 points);*
- *a parabolic quasi-quadric $\mathcal{Q}(4, q)$ (and has q^3+q^2+q+1 points);*

THREE-INTERSECTION SETS IN HIGHER DIMENSION?

A plane meets a quadratic cone in $\mathrm{PG}(4, q)$ in $1, q+1$, or $2q+1$ points.

THEOREM (DE BRUYN-VDV 20??)

A set of points in $\mathrm{PG}(4, q)$ with the same intersection numbers with respect to planes as a quadratic cone is either:

- *an ovoidal cone (and has q^3+q+1 points);*
- *a parabolic quasi-quadric $\mathcal{Q}(4, q)$ (and has q^3+q^2+q+1 points);*
- *a sporadic example of size 55 with automorphism group M_{11} for $q=3$.*

A solid meets a quadratic cone in $\mathrm{PG}(4, q)$ in $q + 1$, $q^2 + 1$, or $q^2 + q + 1$ points.

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THEOREM (DE BRUYN-VDV 20??)

A set of points in $\mathrm{PG}(4,q)$ *with the same intersection numbers with respect to solids as a quadratic cone* and blocks all planes *is either:*

- *a plane (and has q^2+q+1 points),*
- *an ovoidal cone (and has q^3+q+1 points).*

Three-intersection sets in higher dimension?

Theorem (De Bruyn-VdV 20??)

A set of points in $\mathrm{PG}(4, q)$ with the same intersection numbers with respect to solids as a quadratic cone is either:

- *a plane (and has $q^2 + q + 1$ points),*

THEOREM (DE BRUYN-VDV 20??)

A set of points in $\mathrm{PG}(4, q)$ *with the same intersection numbers with respect to solids as a quadratic cone is either:*

- *a plane (and has* $q^2 + q + 1$ *points),*
- *the union of a cone with base a partial ovoid of size* q^2 *and a line disjoint from this set (and has* $q^3 + q + 1$ *points).*

THREE-INTERSECTION SETS IN HIGHER DIMENSION?

THEOREM (DE BRUYN-VDV 20??)

A set of points in $\mathrm{PG}(4, q)$ with the same intersection numbers with respect to solids as a quadratic cone is either:

- *a plane (and has $q^2 + q + 1$ points),*
- *the union of a cone with base a partial ovoid of size q^2 and a line disjoint from this set (and has $q^3 + q + 1$ points).*
- *a sporadic example of size 11 in $\mathrm{PG}(4, 2)$.*

CONCLUSION

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- Point sets with few intersection numbers play a central role in finite geometry
- These sets form a premium supplier of ‘nice’ objects for constructions in graph theory
- The difficulty of characterising and classifying them ranges from trivial to impossible

Combinatorics in Christchurch

Tuesday 4 June 2024– Thursday 6 June 2024
University of Canterbury, Christchurch, New Zealand

Speakers:

- Bill Martin (keynote)
- Carmen Amarra, John Bamberg, Gary Greaves, Anita Liebenau, Sho Suda

Organisers: Jesse Lansdown and Geertrui Van de Voorde

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