# Spherical designs and the $D_4$ lattice

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Joint works with

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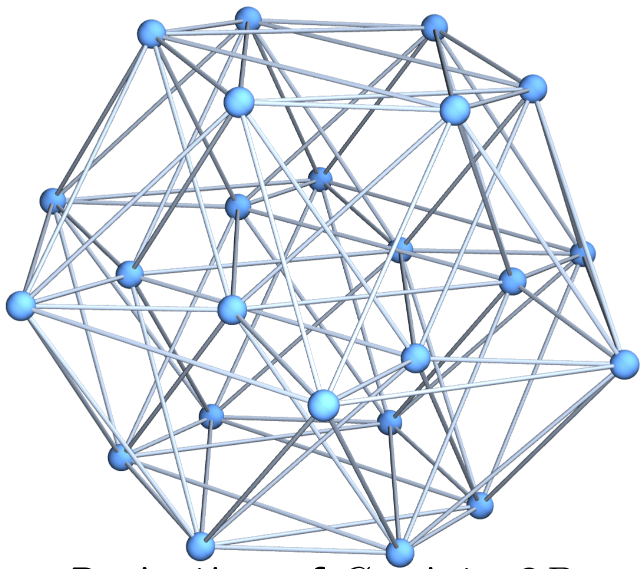

Projection of $C_{24}$ into $3D$

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# Today's talk

MH, H. Nozaki, K. Tasaka, Spherical designs and modular forms of the $D_4$ lattice, Res. Number Theory, 9, (2023), 77 (arXiv: 2303.09000v2) + recent results

• The $D_4$ root system $\mathbf{D}_4$ (i.e., vertices of 24-cell $C_{24}$) is
a unique tight antipodal $\{10, 4, 2\}$-design of $\mathbb{S}^3$

† Each shell of the $D_4$ lattice can be decomposed into orthogonal transformations of the $D_4$ root system

† Partial results on Lehmer's conjecture for $D_4$, i.e., each $m$-shell $(D_4)_{2m}$ is **not** a spherical 6-design

• $\mathbf{D}_4 \cup \mathbf{D}_4^*$ is a unique tight antipodal $\{14, 10, 6, 4, 2\}$-design of $\mathbb{S}^3$

Today we will especially talk about • and •

# Spherical $T$-design

- (usual) spherical design: Delsarte-Goethals-Seidel(1977)

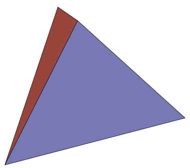
$\boxed{\text{Def.}}$ (Delsarte-Seidel, 1989) $T \subset \mathbb{N}$
$X \subset \mathbb{S}^{d-1}$ $(|X| < \infty)$: $T$-design

$$\iff \quad \sum_{\boldsymbol{x} \in X} f(\boldsymbol{x}) = 0, \quad \forall f \in \mathrm{Harm}_\ell(\mathbb{R}^d), \quad \forall \ell \in T$$

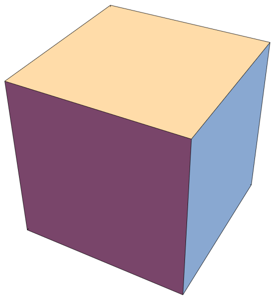
$$\left( \iff \quad \frac{1}{|X|} \sum_{\boldsymbol{x} \in X} f(\boldsymbol{x}) = \int_{\mathbb{S}^{d-1}} f(\boldsymbol{x})\, d\sigma(\boldsymbol{x}), \quad \forall f \in \mathrm{Harm}_\ell(\mathbb{R}^d), \quad \forall \ell \in T \right)$$

$\boxed{\text{Rem.}}$ For $T = \{t, t-1, \ldots, 1\}$, a $T$-design is an usual $t$-design.

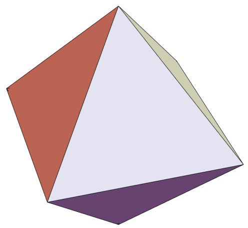
$\boxed{\text{Ex.}}$ ($T$-design on $\mathbb{S}^2$, vertices of regular polyhedrons).



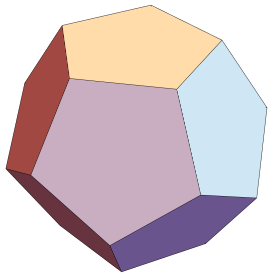
$T = \{5, 2, 1\}$
$t = 2$



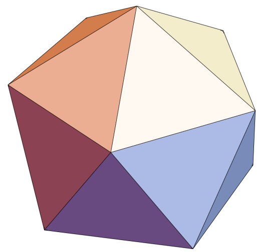
$T = \{2\} \cup 2\mathbb{N} + 1$
$t = 3$



$T = \{2\} \cup 2\mathbb{N} + 1$
$t = 3$



$T = \{14, 8, 4, 2\} \cup 2\mathbb{N} + 1$
$t = 5$

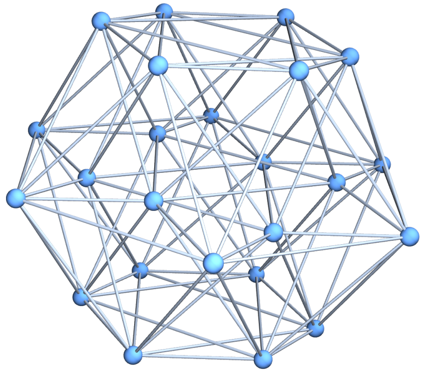


$T = \{14, 8, 4, 2\} \cup 2\mathbb{N} + 1$
$t = 5$

## $C_{24}$: 24-cell

$\boxed{\text{Rem.}}$

- $C_{24}$ is a 5-design, and
$T = \{10, 4, 2\} \cup 2\mathbb{N} + 1$-design (cf. Pache(2005))
- any half set of $C_{24}$ is a $T = \{10, 4, 2\}$-design



Projection of $C_{24}$ into 3D

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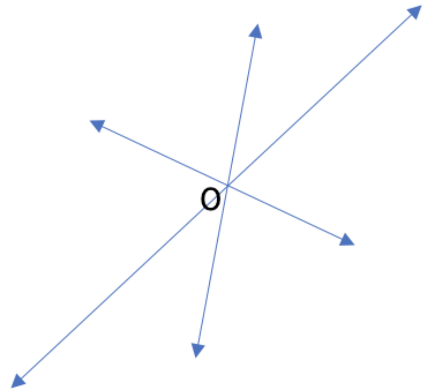
$X$: antipodal $\iff -X = X$

($*$ $X$: antipodal $\Rightarrow T \supset 2\mathbb{N} + 1$)

$Y \subset X$: half set of $X$

$\iff X = Y \cup (-Y)$, $Y \cap Y' = \emptyset$



# Tight design & LP bound

$\boxed{\text{Thm.}}$ (Delsarte et al.,1977). $X \subset \mathbb{S}^{d-1}$: $T$-design

$\exists F(x) = \sum_i f_i Q_i(x)$, s.t., ($Q_i(x)$: Gegenbauer poly. of deg. $i$)

(LP1) $\forall x \in [-1, 1]$, $F(x) \geq 0$, $F(1) > 0$; (LP2) $\forall i \notin T$, $f_i \leq 0$, $f_0 > 0$

then

$$|X| \geq \frac{F(1)}{f_0}$$

If "=" is attained,

then $A(X) := \{\langle \boldsymbol{x}, \boldsymbol{y} \rangle \mid \boldsymbol{x}, \boldsymbol{y} \in X, \boldsymbol{x} \neq \boldsymbol{y}\} \subset \{x \mid F(x) = 0\}$.

$\boxed{\text{Def.}}$ $X$: tight $T$-design

$\Longleftrightarrow$ $\exists F(x)$ satisfying (LP1) & (LP2) for $T$, s.t., $|X| = F(1)/f_0$

$\boxed{\text{Def.}}$ $X$: tight antipodal $T$-design $\Longleftrightarrow$ $Y$: tight $T$-design

a half of set $X$

# $D_4$ lattice and its root system $\mathbf{D}_4$

the $D_4$ lattice:

$$D_4 = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv 0 \pmod 2\}$$

the $m$-shell of the $D_4$ lattice:

$$(D_4)_{2m} = \{\boldsymbol{x} \in D_4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2m\}$$

the $D_4$ root system: $\mathbf{D}_4 = (D_4)_2 = (\pm 1, \pm 1, 0, 0)^P$, $|\mathbf{D}_4| = 24$

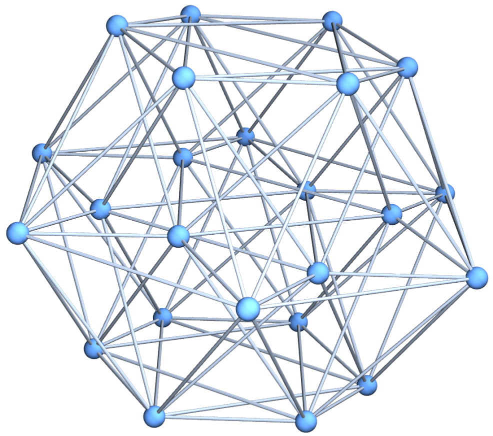
$\boxed{\text{Rem.}}$ • $(D_4)_{2m-1} = \emptyset, \quad \forall m \in \mathbb{N}$

• From Jacobi's four-square theorem,

$$|(D_4)_{2m}| = 24 \sum_{\substack{d \mid 2m \\ d>0:\text{odd}}} d$$

$\therefore |\mathbf{D}_4| = |(D_4)_2| = 24,$

$$|(D_4)_4| = 96,\ |(D_4)_8| = 96, \ldots$$



Projection of $C_{24}$ into 3D

# The LP bound for $\mathbf{D}_4$

$Y$: half set of $\mathbf{D}_4$, s.t., $Y \cup -Y = \mathbf{D}_4$ and $|Y| = 12$.

$\boxed{\text{Thm.}}$ (HNT23). A half set of $\mathbf{D}_4$ is a tight $\{10, 4, 2\}$-design. Equivalently, $\mathbf{D}_4$ is a tight antipodal $\{10, 4, 2\}$-design.

Proof)
$$\begin{aligned} F(x) &= \frac{1}{11264} Q_{10}(x) + \frac{1}{2560} Q_4(x) + \frac{1}{768} Q_2(x) + \frac{3}{1024} \\ &= \frac{1}{16} x^2 \left(x + \frac{1}{2}\right)^2 \left(x - \frac{1}{2}\right)^2 (16x^4 - 28x^2 + 13) \end{aligned}$$

satisfies the conditions of the LP bound, and

$$|Y| \geq \frac{F(1)}{f_0} = \frac{9/256}{3/1024} = 12.$$

- $\mathbf{D}_4$ is not a tight 5-des. but a tight antipodal $\{10, 4, 2\}$-des.

# Uniqueness of tight antipodal $\{10, 4, 2\}$-design

$\boxed{\text{Thm.}}$ (HNT23).

$\mathbf{D}_4$ is unique as a tight antipodal $\{10, 4, 2\}$-design.

The proof is similar to the method of $C_{600}$ (Boyvalenkov-Danev,2001)

$X := \frac{1}{\sqrt{2}}\mathbf{D}_4$

- $A(X) = \{-1, -1/2, 0, 1/2\}$
(distance dist.)

- For given $\boldsymbol{x}_0 \in X$,

$X_0 = \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x}_0, \boldsymbol{x} \rangle = 0\} \subset \mathbb{R}^3$, $|X_0| = 6$

is the regular octahedron (tight 3-des. of $\mathbb{S}^2$)

$X$ is uniquely onstructed only from $X_0$
(derived code)

$$
\begin{aligned}
X = \{&(1, 0, 0, 0), \\
&(-1, 0, 0, 0), \\
&(0, \pm 1, 0, 0), \\
&(0, 0, \pm 1, 0), \\
&(0, 0, 0, \pm 1), \\
&\tfrac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)\}
\end{aligned}
$$

# $\mathbf{D}_4 \cup \mathbf{D}_4^*$ (H.-Nozaki-Tasaka, 2023+)

$\mathbf{D}_4^*$: the minimum vectors of $D_4^* = \{\boldsymbol{x} \in \mathbb{R}^4 \mid \forall \boldsymbol{y} \in D_4, \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{Z}\}$

$$\frac{1}{\sqrt{2}}(\mathbf{D}_4 \cup \mathbf{D}_4^*) = (\pm 1, 0, 0, 0)^P \cup \frac{1}{\sqrt{2}}(1, 1, 0, 0)^P \cup \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)^P$$

$\boxed{\text{Thm.}}$ (HNT23+). A half set of $\mathbf{D}_4 \cup \mathbf{D}_4^*$ is a tight $\{14, 10, 6, 4, 2\}$-des. Equivelntly, $\mathbf{D}_4 \cup \mathbf{D}_4^*$ is a tight antipodal $\{14, 10, 6, 4, 2\}$-des.

Proof) We obtain the following test function

$$\begin{aligned} F(x) &= \frac{1}{3072} x^2(-1+2x)^2(1+2x)^2(-1+2x^2)^2(37-84x^2+48x^4) \\ &= \frac{1}{245760} Q_{14}(x) + \frac{1}{135168} Q_{10}(x) + \frac{1}{114688} Q_6(x) \\ &\quad + \frac{1}{49152} Q_4(x) + \frac{1}{147456} Q_2(x) + \frac{1}{8192}, \end{aligned}$$

which holds $\frac{F(1)}{f_0} = 24 = \frac{|\mathbf{D}_4 \cup \mathbf{D}_4^*|}{2}$

$\boxed{\text{Thm.}}$ (HNT23+). $\mathbf{D}_4 \cup \mathbf{D}_4^*$ is unique as a tight antipodal $\{14, 10, 6, 4, 2\}$-des.

# Known uniqueness designs

| $X$ | $\lvert X\rvert$ | $t$ | $T$ |
|---|---|---|---|
| $\frac{1}{\sqrt{2}}\mathbf{E}_8$ | 240 | 7 | $\{10, 6, 4, 2\}$ |
| $\frac{1}{2}\Lambda_{24}$ | 196560 | 11 | $\{14, 10, 8, 6, 4, 2\}$ |
| $C_{600}$ | 120 | 11 | $\{58, 46, 38, 34, 28, 26, 22, 18, 16, 14, 10, 8, 6, 4, 2\}$ |

$\boxed{\text{Prop.}}$ (HNT23+). (i) A half set of $\frac{1}{\sqrt{2}}\mathbf{E}_8$ is a tight $\{10, 6, 4, 2\}$-des.

$$
\begin{aligned}
F(x) &= \frac{1}{292864}Q_{10}(x) + \frac{3(187-4\sqrt{759})}{7884800}Q_6(x) + \frac{11131-252\sqrt{759}}{39424000}Q_4(x) \\
&\quad + \frac{3568-81\sqrt{759}}{4928000}Q_2(x) + \frac{9(661-12\sqrt{759})}{2816000} \\
&= \frac{1}{4400}x^2(x-\frac{1}{2})^2(x+\frac{1}{2})^2(4400x^4-6050x^2+3633-36\sqrt{759}).
\end{aligned}
$$

(ii) A half set of $\frac{1}{2}\Lambda_{24}$ is a tight $\{14, 10, 8, 6, 4, 2\}$-design.

$$\begin{aligned}
F(x) &= \frac{1}{73030041600}Q_{14}(x) + \frac{529+6\sqrt{12259}}{455707459584}Q_{10}(x) + \frac{21353+224\sqrt{12259}}{1822829838336}Q_8(x) \\
&\quad + \frac{1776821+18092\sqrt{12259}}{29165277413376}Q_6(x) + \frac{116957+1164\sqrt{12259}}{511671533568}Q_4(x) \\
&\quad + \frac{5(119431+1140\sqrt{12259})}{810146594816}Q_2(x) + \frac{5(1477+12\sqrt{12259})}{2508193792} \\
&= \frac{1}{17644}x^2(x-\frac{1}{2})^2(x+\frac{1}{2})^2(x-\frac{1}{4})^2(x+\frac{1}{4})^2(17664x^4-22448x^2+15123+84\sqrt{12259}).
\end{aligned}$$

(iii) A half set of the vertices of $C_{600}$ is a tight $\{18, 16, 14, 10, 8, 6, 4, 2\}$-design.

$$\begin{aligned}
F(x) &= \frac{1}{4980736}Q_{18}(x) + \frac{3353+540\sqrt{30}}{18075353088}Q_{16}(x) + \frac{-3169+1188\sqrt{30}}{15948840960}Q_{14}(x) \\
&\quad + \frac{9(545-84\sqrt{30})}{1949302784}Q_{10}(x) + \frac{11719-1836\sqrt{30}}{1594884096}Q_8(x) + \frac{7225-1188\sqrt{30}}{531628032}Q_6(x) \\
&\quad + \frac{39121-6372\sqrt{30}}{1772093440}Q_4(x) + \frac{104503-16092\sqrt{30}}{3189768192}Q_2(x) + \frac{5(749-108\sqrt{30})}{88604672} \\
&= \frac{1}{16224}x^2(x-\frac{\sqrt{5}-1}{4})^2(x+\frac{\sqrt{5}-1}{4})^2(x-\frac{\sqrt{5}+1}{4})^2(x+\frac{\sqrt{5}+1}{4})^2 \\
&\quad \times \{16224x^4+(-33151+540\sqrt{30})x^2+17676-648\sqrt{30}\}
\end{aligned}$$

# Conclusion and future tasks

- The $D_4$ root system (vertices of regular 24-cell) is
  a unique tight antipodal $\{10, 4, 2\}$-design of $\mathbb{S}^3$.

  † Each shell of the $D_4$ lattice can be decomposed into orthogonal transformations of the $D_4$ root system.

  † Partial results on Lehmer's conjecture for $D_4$.

- $\mathbf{D}_4 \cup \mathbf{D}_4^*$ is a unique tight antipodal $\{14, 10, 6, 4, 2\}$-design of $\mathbb{S}^3$

- Find other tight $T$-designs

- Find similar decompositions of shells of other lattices

## Thank you for your attention!

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# Appendix. $\mathbf{D}_4$-decompose of $(D_4)_{2m}$

- $W(F_4) \subset O(4)$ acts on the $D_4$ lattice. $|W(F_4)| = 1152$.
- the $m$-shell $(D_4)_{2m}$ is decomposed by orbits of $W(F_4)$.
- There exists a subgr. $N$ of $W(F_4)$ whose harmonic Molien series is

$$\sum_{i \geq 0} \dim \mathrm{Harm}_i(\mathbb{R}^d)^N t^i = 1 + 7t^6 + 9t^8 + 26t^{12} + \cdots .$$

Moreover, $|N| = 24$ and $-I \in N$.

- Any orbit of $N$ is an antipodal $\{10, 4, 2\}$-design.
- From $|N| = 24$, any orbit of $N$ is an orthogonal trans. of $\mathbf{D}_4$.

Thm. (HNT23). Each $m$-shell $(D_4)_{2m}$ can be decomposed by orthogonal transformations of $\mathbf{D}_4$.