On a coloring of a δ-complement graph

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Introduction

In 2022, Pai et al. [2] introduced a δ-complement graph with the concept of a complement graph by complementing the subgraphs consisting of the vertices of the same degree.

Definition (Pai et al.,2022)

For a graph G, the δ-complement graph of G, denoted by G_δ, is a a graph in which $V(G_\delta) = V(G)$ and $uv \in E(G_\delta)$ if either

- $uv \in E(G)$ and $\deg(u) \neq \deg(v)$, or
- $uv \notin E(G)$ and $\deg(u) = \deg(v)$.

For example,

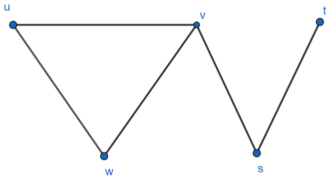

Figure: G

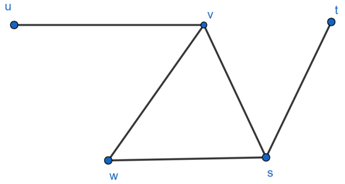

Figure: G_δ

Note $G \ncong (G_\delta)_\delta$ and $\overline{G} \ncong G_\delta$.

Introduction

Application:

1. collaboration's graph [2],
2. network of data centers [5].

Introduction

Network of data centers:

- each vertex in the network G represents a data center;
- edge appears when two data centers are sharing information at a specific time;
- in each center, the number of centers that it is sharing information with is the degree of that center;
- to avoid a problem of losing information due to a malfunction of a center, if two centers of the same rank have already shared information, then we try to find another center with the same rank and both centers have yet communicated;
- however, we do not allow a new information sharing if two centers of different ranks have never directly communicated before;
- the chromatic number is the minimum number of security keys needed at a given time.

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Structure of a δ-complement graph

Several structural properties of the δ-complement graph had been given in the work of Pai et al. [2].

Theorem (Pai et al., 2022)

A graph G_δ is a complete graph if and only if G is a complete multipartite graph with the partition of the point set $\{V_1, V_2, \ldots, V_k\}$ with $|V_i| \neq |V_j|$ for $i \neq j$.

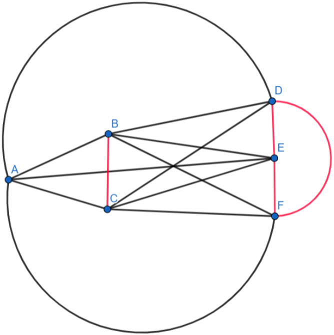

Figure: G_δ

Structure of a δ-complement graph

Result's from Pai et al. [2]

- neccessary and sufficient condition on the degree of vertices for a graph so that G is $G \cong G_\delta$ or $\overline{G} \cong G_\delta$,
- Vertex-degree preservation property,
- a sufficient condition for an Eulerian G_δ graph
- sufficient conditions for a Hamiltonian G_δ graph
- sufficeint condition for a graph G_δ to be disconnected.

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Chromatic number of a δ-complement graph

Definition

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- $\overline{\chi} = \chi(\overline{G})$,
- $\chi_\delta = \chi(G_\delta)$.

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We denoted χ_δ by a δ-chromatic number of G.

Chromatic number of a δ-complement graph

In a study of a relation between the chromatic numbers of a graph G and its complement $\overline{G}$, one of the well-known relation is the Nordhaus-Gaddum type bounds [1].

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Theorem (Nordhaus-Gaddum, 1956)

Let G be a graph with n vertices. Then

$$2\sqrt{n} \leq \chi + \overline{\chi} \leq n + 1 \tag{1}$$

and

$$n \leq \chi \cdot \overline{\chi} \leq \left(\frac{n+1}{2}\right)^2. \tag{2}$$

In 2023, P. Vichitkunakorn, R. Maungchang and W. Tangjai [5] investigated a Nordhaus-Gaddum type relation between the chromatic numbers of a graph and that of its δ-complement graph.

Theorem (Vichitkunakorn et al., 2023)

For $n \geq 4$, let G be a graph with n vertices. Let $d_1, \ldots, d_m$ be all the distinct values of the degrees of the vertices in G. Partition $V(G)$ into non-empty sets $V_{d_1}, V_{d_2}, \ldots, V_{d_m}$. We have that

$$2 \cdot \sqrt{\max_{1 \leq i \leq m} \{|V_{d_i}|\}} \leq \chi + \chi_\delta \leq m + n, \tag{3}$$

and

$$\max_{1 \leq i \leq m} \{|V_{d_i}|\} \leq \chi \cdot \chi_\delta \leq \left(\frac{m+n}{2}\right)^2. \tag{4}$$

The bounds are sharp and there infinite number of non-regular graphs satisfied such bounds.

Proof.

Consider G with its χ-coloring. In each V_{d_i}, we list the number of vertices with the same color, say $n_1 \geq n_2 \geq \ldots \geq n_\chi \geq 0$. We note that $n_1 + n_2 + \cdots + n_\chi = |V_{d_i}|$ and $n_1 \geq \frac{|V_{d_i}|}{\chi}$.

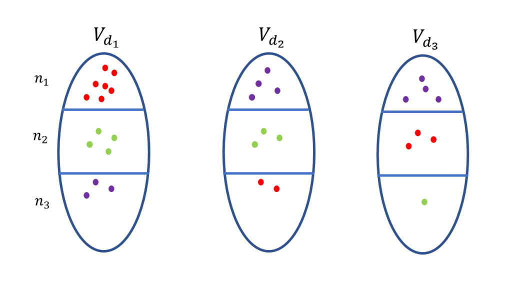

Figure: Vertex partition

Hence, $\chi_\delta \geq n_1 \geq |V_{d_i}|/\chi$, implying that $\chi \cdot \chi_\delta \geq |V_{d_i}|$. □

Proof.

Thus $\max_{1\le i\le m}\{|V_{d_i}|\} \le \chi \cdot \chi_\delta$. Since

$$0 \le (\chi - \chi_\delta)^2 \text{ and } \max_{1\le i\le m}\{|V_{d_i}|\} \le \chi \cdot \chi_\delta,$$

we have

$$2 \cdot \sqrt{\max_{1\le i\le m}\{|V_{d_i}|\}} \le \chi + \chi_\delta.$$

Proof.

Thus $\max_{1\le i\le m}\{|V_{d_i}|\} \le \chi\cdot\chi_\delta$. Since

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we have

$$2\cdot\sqrt{\max_{1\le i\le m}\{|V_{d_i}|\}} \le \chi + \chi_\delta.$$

Next, we investigate the upper bound on $\chi\cdot\chi_\delta$ and $\chi+\chi_\delta$. Let $G_i = G[V_{d_i}]$ be the subgraph of G induced by V_{d_i} and let $\chi_i = \chi(G_i)$ for $i = 1, \ldots, m$. We have

$$\chi \le \sum_{i=1}^{m} \chi_i. \tag{5}$$

□

Proof.

The graph G_δ consists of $\overline{G}_1, \ldots, \overline{G}_m$, and an edge in G_δ, if any, appears between distinct pair of G_i and G_j for $i, j \in \{1, \ldots, m\}$. Let $\overline{\chi}_i = \chi(\overline{G}_i)$ and $n_i = |V(G_i)|$. Similar to (5), we also have

$$\chi_\delta \leq \sum_{i=1}^{m} \overline{\chi}_i. \tag{6}$$

By Theorem 4, we have $\chi_i + \overline{\chi}_i \leq n_i + 1$. Therefore, by (5) and (6),

$$\chi + \chi_\delta \leq \sum_{i=1}^{m} (\chi_i + \overline{\chi}_i) \leq \left(\sum_{i=1}^{m} n_i \right) + m = n + m. \tag{7}$$

Since $4\chi \cdot \chi_\delta \leq (\chi + \chi_\delta)^2$, we get $\chi \cdot \chi_\delta \leq \left(\frac{m+n}{2}\right)^2$. □

The graphs achieving the bounds will be given.

Let us recall operation on graphs.

Definition

Let G and H be graphs. The *Cartesian product graph* of G and H is a graph $G \Box H$ where $V(G \Box H) = V(G) \times V(H)$ and and $uv \in E(G \Box H)$ if either $x = x'$ and $yy' \in E(H)$ or $y = y'$ and $xx' \in E(G)$ for $u = (x, y)$ and $v = (x', y')$.

Let P_n be a path with n vertices.

Examples

$P_3 \Box P_4$

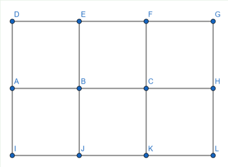

Figure: $P_3 \Box P_4$

Definition

A *join* of the graphs G_1 and G_2, denoted $G_1 \vee G_2$, is a graph whose the vertex set $V(G_1 \vee G_2)$ is the disjoint union $V(G_1) \sqcup V(G_2)$, and each pair of $u, v \in V(G_1 \vee G_2)$ is adjacent if and only if $uv \in E(G_1) \cup E(G_2)$ or $(u, v) \in V(G_1) \times V(G_2)$.

Examples

$P_2 \vee P_3$

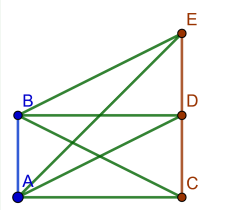

Figure: $P_2 \vee P_3$

m and n	G	sharpness of
$n \geq 4$	$P_2 \Box P_n$	lower bound on $\chi \cdot \chi_\delta$
$n \geq 3$	$P_{n+2} \Box K_n$	lower bound on $\chi + \chi_\delta$
$1 \leq n_1 < n_2 < \cdots < n_m$	$K_{n_1,\ldots,n_m}$	upper bound on $\chi + \chi_\delta$
$1 < n_1 < \cdots < n_{m-1}$ $n_m = n_1 + \cdots + n_{m-1}$ $-m + 2$	$K_{n_1,\ldots,n_{m-1}} \vee K_{n_m}$	upper bound on $\chi \cdot \chi_\delta$

Table: Sharpness [Vichitkunakorn et al., 2023]

Theorem (Vichitkunakorn et al., 2023)

Let G be a graph with n vertices and $m = |\{\deg(v) : v \in V(G)\}|$. Then $\chi \cdot \chi_\delta = \left(\frac{m+n}{2}\right)^2$ if and only if $\chi = \chi_\delta = \frac{m+n}{2}$.

Theorem (Vichitkunakorn et al., 2023)

Let G be a graph with n vertices where $n > 1$. Then

$$\chi \cdot \chi_\delta \leq \begin{cases} n(n-1) & \text{if } n = 2, 3, \\ 9 & \text{if } n = 4, \\ n(n-2) & \text{if } n \geq 5, \end{cases} \tag{8}$$

and

$$\chi + \chi_\delta \leq \begin{cases} 2n-1 & \text{if } n = 2, 3, \\ 2(n-1) & \text{if } n \geq 4. \end{cases} \tag{9}$$

Theorem (Vichitkunakorn et al., 2023)

Let G be a graph with n vertices, and let $\omega = \omega(G)$ be the clique number of G. If $2 \leq \omega \leq n-2$, then $\chi_\delta \leq \min\{\omega, n-\omega\} + n - \omega$.

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Cartesian Product

Later, W. Tangjai, W. Pho-on and V. Vichitkunakorn [4] investigates the δ-chromatic number of the Cartesian product of graphs.

Theorem

For graphs G and H, we have $(G\Box H)_\delta = (V, E)$ where $V = V(G\Box H)$ and $E = E(G_\delta \Box H_\delta) \cup S$ where $S = \{uv : u = (u_1, u_2) \in V(G\Box H)$ and $v = (v_1, v_2) \in V(G\Box H)$ where $u_1 \neq v_1, u_2 \neq v_2$ and $d_{G\Box H}(u) = d_{G\Box H}(v)\}$.

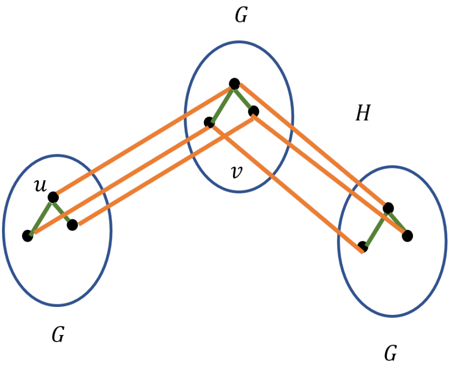

Figure: $G\Box H$

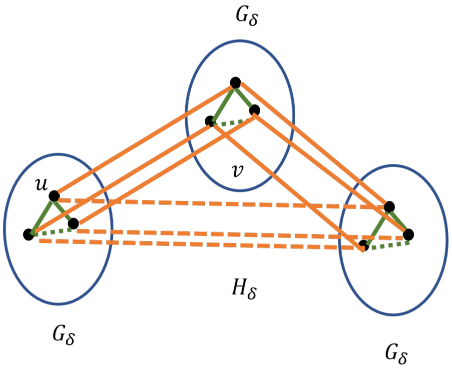

Figure: $G_\delta \Box H_\delta$

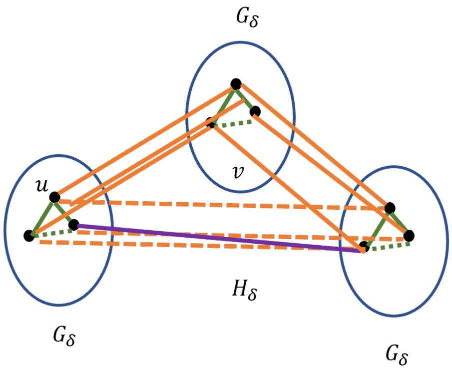

Figure: The purple edge is an edge in S

Theorem

For graphs $G_1, \ldots, G_k$*, we have* $(G_1 \square \cdots \square G_k)_\delta = (V, E)$ *where* $V = V(G_1 \square \cdots \square G_k)$ *and* $E = E((G_1)_\delta \square \cdots \square (G_k)_\delta) \cup S$ *such that* S *is the set of* uv *where* $u = (u_1, \ldots, u_k) \in V$, $v = (v_1, \ldots, v_k) \in V$*, there are at least two indices* i *that* $u_i \neq v_i$*, and* $d_{G_1 \square \cdots \square G_k}(u) = d_{G_1 \square \cdots \square G_k}(v)$.

Theorem

$(G_1 \square \cdots \square G_k)_\delta = (G_1)_\delta \square \cdots \square (G_k)_\delta$ *if and only if there are at most one i such that $G_i \neq K_1$.*

The following theorem gave the chromatic number of the Cartesian product graph.

Theorem (Sabidussi [3], 1957)

Let G and H be graphs. We have $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$.

Theorem

Let $G_1, \ldots, G_k$ be graphs. We have

$$\max\{\chi_\delta(G_1), \ldots, \chi_\delta(G_k)\} \leq \chi_\delta(G_1 \square \cdots \square G_k).$$

Theorem

Let G and H be graphs. If any positive degree difference of vertices in G is not equal to that of in H, then

$$\chi_\delta(G \square H) \leq n_{\max}(H) \cdot \max(\chi_\delta(G), m(H))$$

where $n_{\max}(H)$ denotes the maximum number of vertices of the same degree in H and $m(H)$ is the number of distinct degrees in H. Furthermore, the bound is sharp.

Proof.

Since any positive degree difference of vertices in G is not equal to that of in H, the edges in S are uv where $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that $u_1 \neq v_1$, $u_2 \neq v_2$, $d_G(u_1) = d_G(v_1)$ and $d_H(u_2) = d_H(v_2)$.

Proof.

Since any positive degree difference of vertices in G is not equal to that of in H, the edges in S are uv where $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that $u_1 \neq v_1$, $u_2 \neq v_2$, $d_G(u_1) = d_G(v_1)$ and $d_H(u_2) = d_H(v_2)$. We partition $V(H)$ according to vertex degree into $W_1, W_2, \ldots, W_{m(H)}$. Write $W_j = \{h_{j,1}, h_{j,2}, \ldots, h_{j,n_j}\}$ for $1 \leq j \leq m(H)$.

Proof.

Since any positive degree difference of vertices in G is not equal to that of in H, the edges in S are uv where $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that $u_1 \neq v_1$, $u_2 \neq v_2$, $d_G(u_1) = d_G(v_1)$ and $d_H(u_2) = d_H(v_2)$. We partition $V(H)$ according to vertex degree into $W_1, W_2, \ldots, W_{m(H)}$. Write $W_j = \{h_{j,1}, h_{j,2}, \ldots, h_{j,n_j}\}$ for $1 \leq j \leq m(H)$.

□

Proof.

Define $p = \max(\chi_\delta(G), m(H))$. Let $c_0 : V(G) \to \{1, 2, \ldots, \chi_\delta(G)\}$ be a proper coloring of G_δ. We define a coloring $c : V(G) \times V(H) \to \{1, 2, \ldots, n_{\max}(H) \cdot p\}$ as

$$c(g, h_{j,k}) = f(g, j) + (k-1)p,$$

for $k = 1, \ldots, n_j$, where $f(g, j) \in \{1, 2, \ldots, p\}$ and $f(g, j) \equiv c_0(g) + j - 1 \pmod{p}$.

	$h_{1,1}$	$h_{1,2}$	$h_{2,1}$	$h_{3,1}$	$h_{3,2}$	$h_{4,1}$	$h_{4,2}$
g_1	1	5	2	3	7	4	8
g_2	3	7	4	1	5	2	6
g_3	1	5	2	3	7	4	8
g_4	2	6	3	4	8	1	5
g_5	3	7	4	1	5	2	6

Figure: An example of a coloring c_0 where $\chi_\delta(G) = 3$, $m(H) = 4$ and $n_{\max}(H) = 2$.

Proof.

We see that the vertices in the same copy of G received a coloring equivalent to c_0 and a cyclic permutation modulo p up to an additive constant $(k-1)p$ for some $k = 1, \ldots, n_j$. For a fixed $g \in V(G)$, the vertices in the same copy of H, written in the form $(g, h_{j,k})$ where $1 \leq j \leq m(H)$ and $1 \leq k \leq n_j$, received distinct colors because $j \leq p$ and $k \leq n_{\max}(H)$.

Lastly, any endpoints of an edge in S are of the form $(g, h_{j,k})$ and $(g', h_{j,k'})$ where $g \neq g'$ and $k \neq k'$, which received different colors as $k \neq k'$. □

The sharpness will be given in the next theorem.

Cartesian product

Theorem

For $n \geq 5$, we have $\chi_\delta(C_n \square P_3) = 2\chi_\delta(C_n) = 2\left\lceil \frac{n}{2} \right\rceil$.

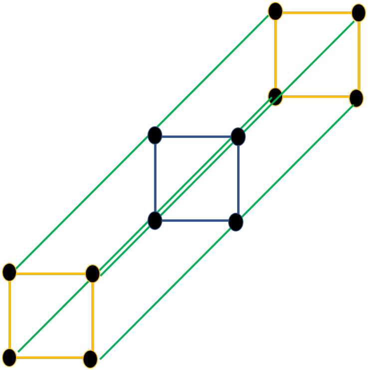

Figure: $C_4 \square P_3$

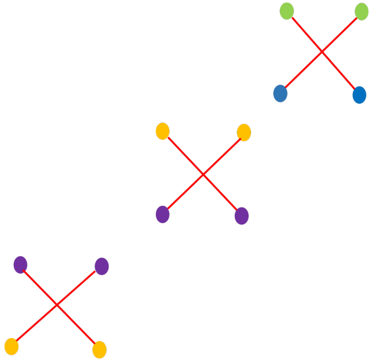

Figure: Coloring

Theorem

Let H be a k-regular graph. Let $G = \{u\} \vee H$ be the join of a singleton and H. Suppose $|V(H)| \geq 3$ and $\chi_\delta(H) \geq 2$. If $|V(H)| > k + 2$, then $\chi_\delta(G \square P_3) \leq 2\chi_\delta(H)$.

The following are the lists of the computed δ-chromatic number of a Cartesian product of special classes of graphs.

- $\chi_\delta(C_n \square P_n) = 2\left\lceil \frac{n}{2} \right\rceil$ for $n \geq 5$ (sharpness),
- $\chi_\delta(S_{1,m} \square S_{1,n}) = mn$ for $m, n \geq 3$,
- $\chi_\delta(S_{1,m} \square P_n) = m\left\lceil \frac{n-2}{2} \right\rceil$ for $m \geq 3$ and $n \geq 3$,
- $\chi_\delta(P_n \square P_k) = \left\lceil \frac{(n-2)(k-2)}{2} \right\rceil$ for $6 \leq n \leq k$.

Cartesian product

$\chi_\delta(S_{1,m} \square S_{1,n}) = mn$ for $m, n \geq 3$

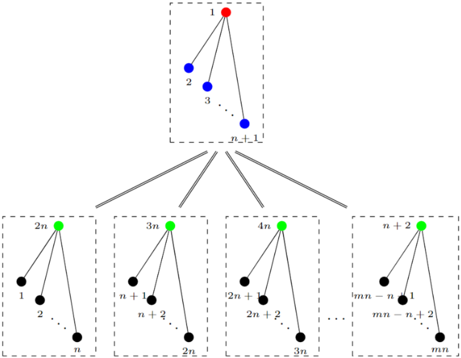

Figure: $S_{1,m} \square S_{1,n}$

Cartesian product

$\chi_\delta(S_{1,m}\Box P_n) = m\left\lceil \frac{n-2}{2} \right\rceil$ for $m \geq 3$ and $n \geq 3$

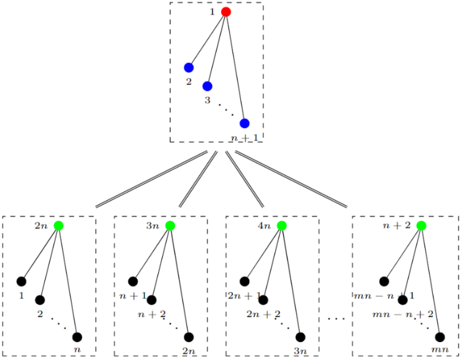

Figure: $S_{1,m}\Box P_n$

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