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Some properties of q-perfect matroid designs

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Matroid is a generalization of linear independence.

- **Matroid**
  - Combinatorial structure based on the concept of the linear independence.
- **q-Matroid**
  - q-analogue of a matroid
  - R. Jurrius, R. Pellikaan established. (2018)

q-analogue: the way of generalization

	normal	q-analogue
finite set	$[0, 1, \cdots, n]$	$\mathbb{F}_q^n$
size of X	$\lvert X \rvert$	$\dim X$
subset	subset	subspace
union	$X \cup Y$	$X + Y$
intersection	$X \cap Y$	$X \cap Y$

Preliminaries

- q : a prime power
- $E(=\mathbb{F}_q^n)$: an n-dimensional vector space over $\mathbb{F}_q$
- $\mathcal{L}(X)$: the collection of all subspaces of a vector space X
- $\begin{bmatrix} X \\ k \end{bmatrix}_q$: the collection of all k-dimensional subspaces of a vector space X

Definition [q-matroid]

A q-*matroid* is a pair (E, r) satisfying (qR1), (qR2) and (qR3):

	q-matroid	matroid
ground space E	$\mathbb{F}_q^n$	$[n] = \{1, 2, \cdots n\}$
rank function r	$\mathcal{L}(E) \to \mathbb{Z}_{\geq 0}$	$2^E \to \mathbb{Z}_{\geq 0}$
1st axiom	(qR1) $0 \leq r(A) \leq \dim A$	(R1) $0 \leq r(A) \leq \lvert A \rvert$
2nd axiom	(qR2) $A \subseteq B \Rightarrow r(A) \leq r(B)$	(R2) $A \subseteq B \Rightarrow r(A) \leq r(B)$
3rd axiom	(qR3) $r(A+B) + r(A \cap B) \leq r(A) + r(B)$	(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

- integer $k : 0 \leq k \leq n$

Example [The uniform q-matroid]

$r(X) := \min\{\dim X, k\}$.
Then, the pair (E, r) is a q-matroid.

Remark. This q-matroid is called a ***uniform*** q-***matroid*** $U_{k,n}[\mathbb{F}_q]$

- $M = (E, r)$: q-matroid

Definition [Flat of a q-matroid]

$F \leq E$ is a *flat* of M if and only if

$$x \in \begin{bmatrix} E \\ 1 \end{bmatrix}_q \setminus \begin{bmatrix} F \\ 1 \end{bmatrix}_q \Rightarrow r(F + x) = r(F) + 1$$

holds.

- $M = (E, r)$: matroid

Definition [Flat of a matroid]

$F \subseteq E$ is a *flat* of M if and only if

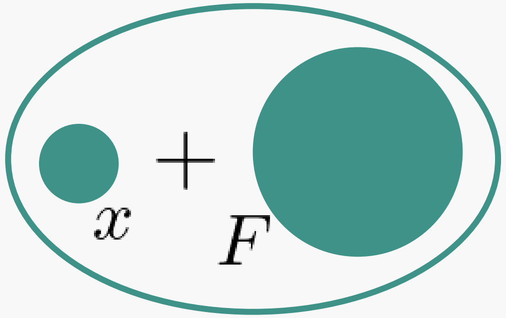
$$x \in E - F \Rightarrow r(F \cup x) = r(F) + 1$$

holds.

Remark

We denote $\mathcal{F}$ as a collection of all flats in M.

- If a flat F satisfies $r(F) = i$, F is called $\boldsymbol{i}$**-flat.**
- The collection of all i-flats is denoted by $\boldsymbol{\mathcal{F}_i}$.



$r(F) + 1$

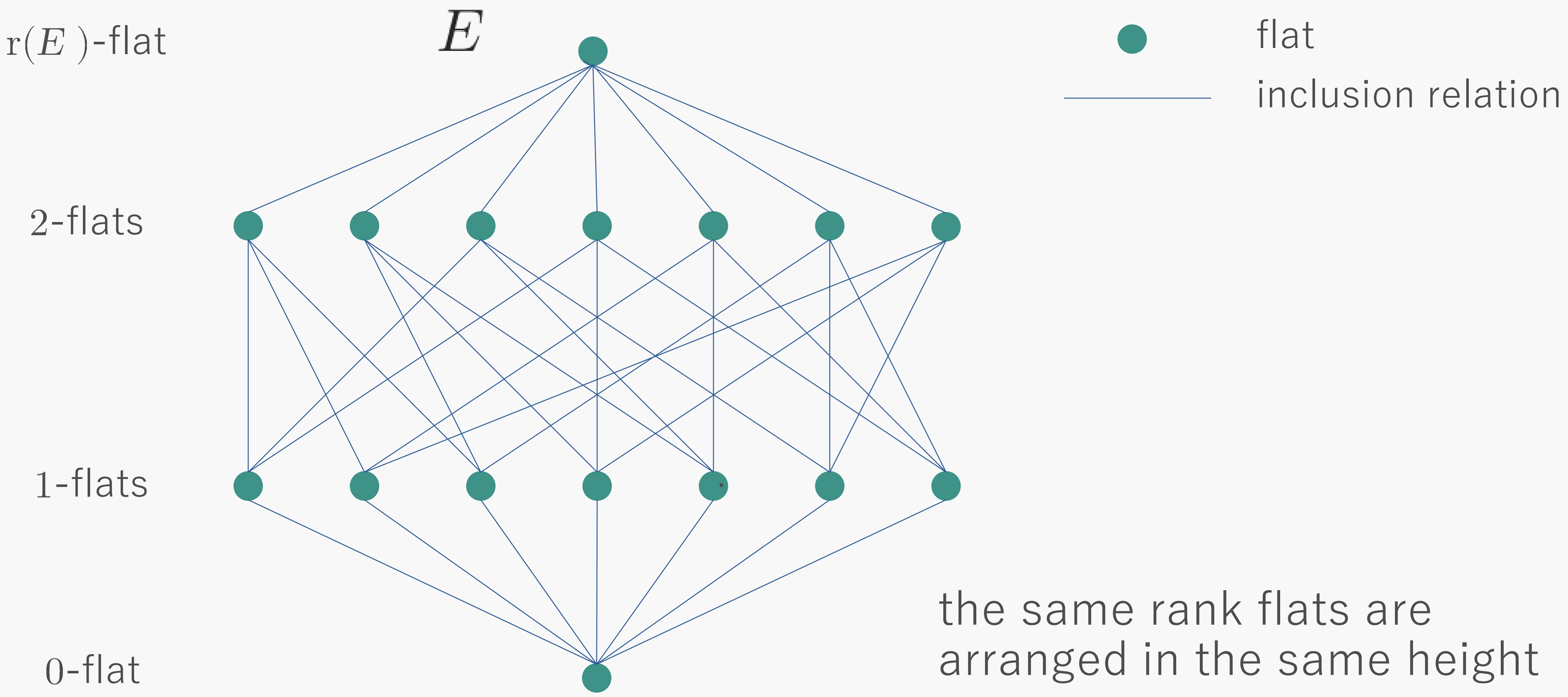
Proposition [axiom of flats of q-matroid] (E. Byrn et al. 2022 +)

$$\mathcal{F} \text{ is a collection of all flats of } q\text{-matroid} \\ \Rightarrow \mathcal{F} \text{ satisfies } (q\mathrm{F}1),\ (q\mathrm{F}2) \text{ and } (q\mathrm{F}3)$$

- $F_1, F_2, F \in \mathcal{F}$

(E, r)	q-matroid	matroid
1st axiom	$(q\mathrm{F}1)\ \ E \in \mathcal{F}$	$(\mathrm{F}1)\ \ E \in \mathcal{F}$
2nd axiom	$(q\mathrm{F}2)\ \ F_1 \cap F_2 \in \mathcal{F}$	$(\mathrm{F}2)\ \ F_1 \cap F_2 \in \mathcal{F}$
3rd axiom	$(q\mathrm{F}3)\ \ \exists! F' \in \mathcal{F}_{r(F)+1}\ s.t.\ F + x \leq F'$ $\left(\forall x \in \begin{bmatrix} E \\ 1 \end{bmatrix}_q \setminus \begin{bmatrix} F \\ 1 \end{bmatrix}_q\right)$	$(\mathrm{F}3)\ \ \exists! F' \in \mathcal{F}_{r(F)+1}\ s.t.\ F \cup x \subseteq F'$ $(\forall x \in E - F)$

Hasse diagram of flats



the same rank flats are arranged in the same height

t-dimensional subspace included in λ blocks

- $\mathcal{B} \subseteq \begin{bmatrix} E \\ k \end{bmatrix}_q$

Definition [subspace design]

A t-$(n, k, \lambda; q)$ subspace design is a pair $(E, \mathcal{B})$ with the property that every t-dimensional subspace of E is contained in exactly λ elements of $\mathcal{B}$.

Remark

- A member of $\mathcal{B}$ is called a ***block***.
- A subspace design t-$(n, k, 1; q)$ is called a q-***Steiner system*** denoted by $\mathcal{S}(t, k, n; q)$.
- Subspace designs have been actively studied because of their application to random network coding.

Perfect matroid designs have a lot of t-designs.

- **Perfect matroid design (PMD)**
  - A matroid whose flats of the same rank all are the same cardinality.
  - U.S.R. Murty, P. Young and J. Edmonds established(1970).
  - M. Deza and N.M Singhi studied some properties of PMD and the PMD of rank 4 .
  - There are many kinds of blocks of t-design in PMD.
    - flats. bases, circuit
  - Steiner systems induce PMDs
- **q-perfect matroid design (q-PMD)**
  - q-analogue of PMDs
  - E. Byrne, M. Ceria, S. Ionica and R. Jurius (2022)
  - q-Seiner systems induce q-PMDs

Motivation: flats of q-PMDs and subspace designs.

	PMD	q-PMD
How to construct non trivial (q-)PMD	• Projective geometries • Affine geometries • Affine triple systems • Steiner system	• Steiner system • ??? • ??? • ???
α-sequence	the cardinalities of i-flats	the dimensions of i-flats
t-funtion	the number of j-flats between an i-flat and a k-flat	**???**
flats and design	If flats have all of the subsets whose cardinalities are less than t-1,m-flats are t-design ($m \geq t$)	**???** (main result)

Definition of q-PMDs and α-sequences

	PMD	q-PMD
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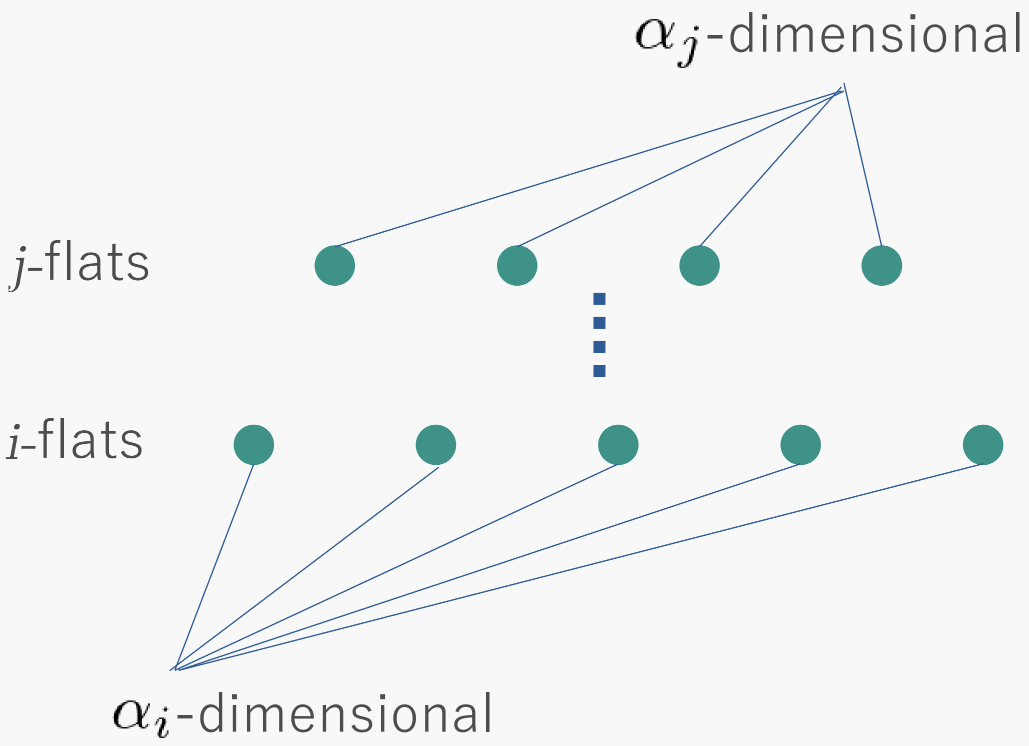
Definition [q-PMD]

A q-perfect matroid design (q-PMD) is a q-matroid with the property that any two flats of the same rank have the same dimension.

Definition [α-sequence]

- α_i : the dimension of the i-flats of a q-PMD
- $\{\alpha_i\}_{i=0}^{r(E)}$: is called an α -***sequence*** of the q-PMD



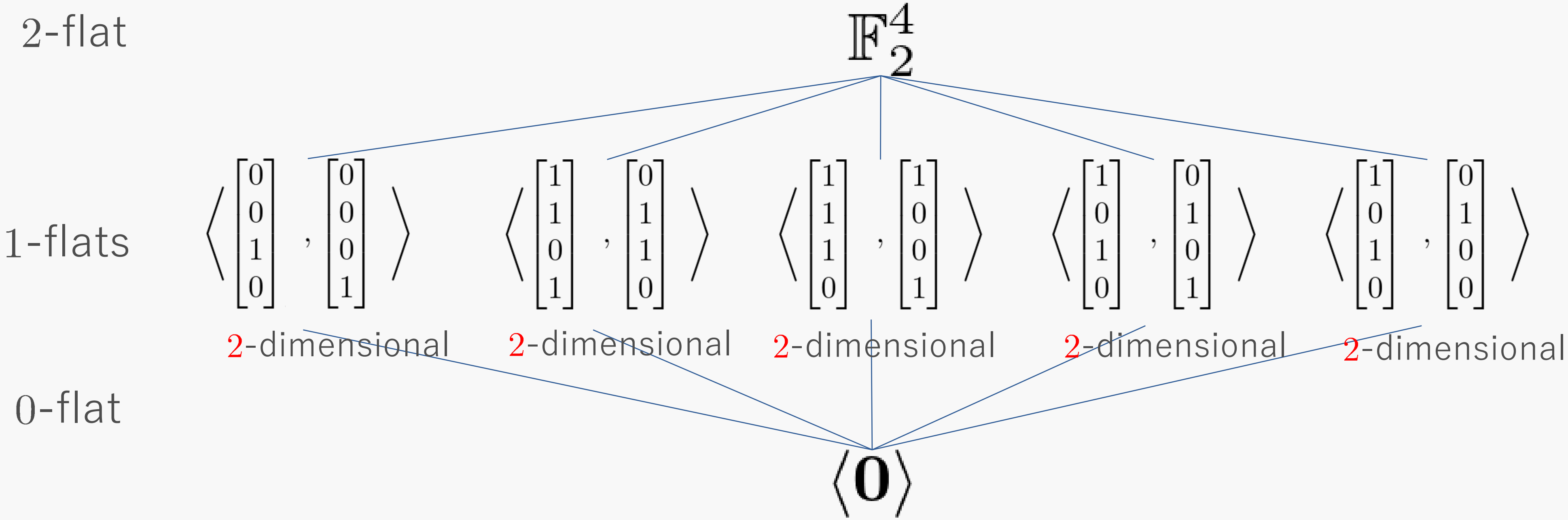
Example 1

$U_{k,n}[\mathbb{F}_q]$ is a q-PMD with an α-sequence $(0, 1, \cdots, k-1, n)$.

$\because$ All of i-dimensional subspaces are i-flats of $U_{k,n}[\mathbb{F}_q]$ $(i \leq k-1)$. The ground space is the k-flat.

Example 2

If the ground set $\mathbb{F}_q^{lm}$ is partitioned into $\mathbb{F}_q^{m}$, the partition induces a q-PMD with an α-sequence $(0, m, lm)$.



$q = 2, l = 2, m = 2$

$\alpha = (0, 2, 4)$

Definition of q-PMDs and α-sequences

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A t-function of a q-PMD.

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- $F_i \in \mathcal{F}_i, F_k \in \mathcal{F}_k$ with $F_i \leq F_k$

Proposition

The number $|\mathcal{F}_j(F_i, F_k)|$ of j-flat F_j with $F_i \leq F_j \leq F_k$ is independent of the choice of F_i and F_k.

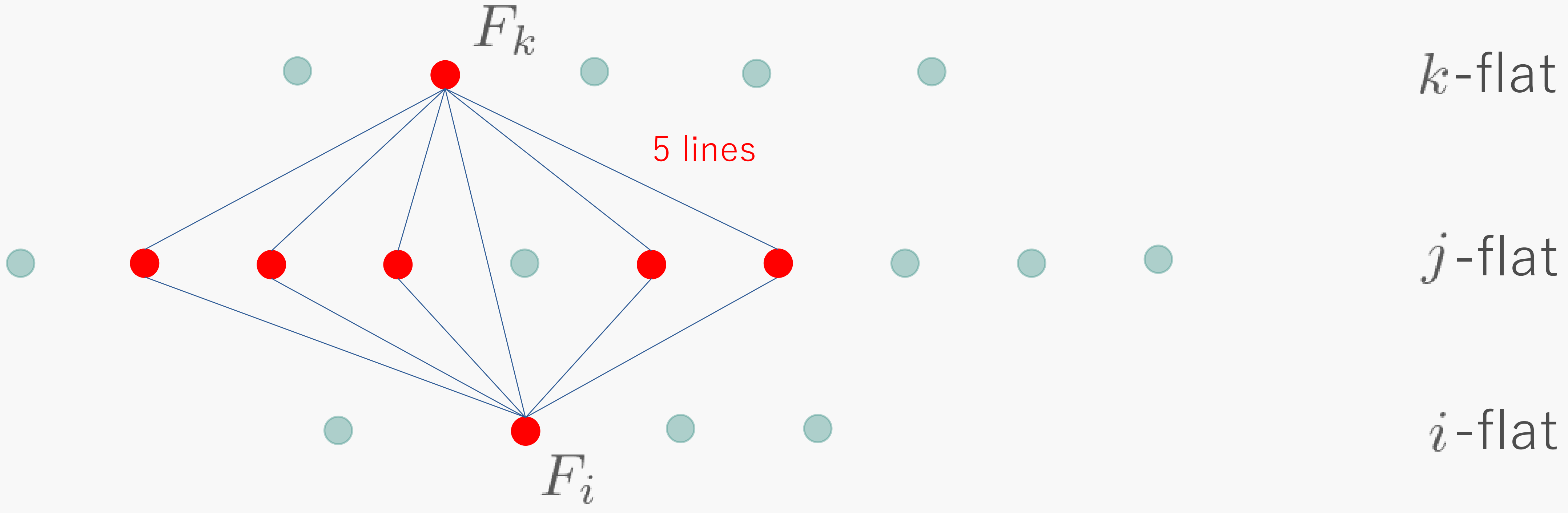
Definition [t-function of q-PMD]

We define *t-function* of M as follows:

$$t_M(i, j, k) := |\mathcal{F}_j(F_i, F_k)|.$$

Proposition

The number $|\mathcal{F}_j(F_i, F_k)|$ of j-flat F_j with $F_i \leq F_j \leq F_k$ is independent of the choice of F_i and F_k.



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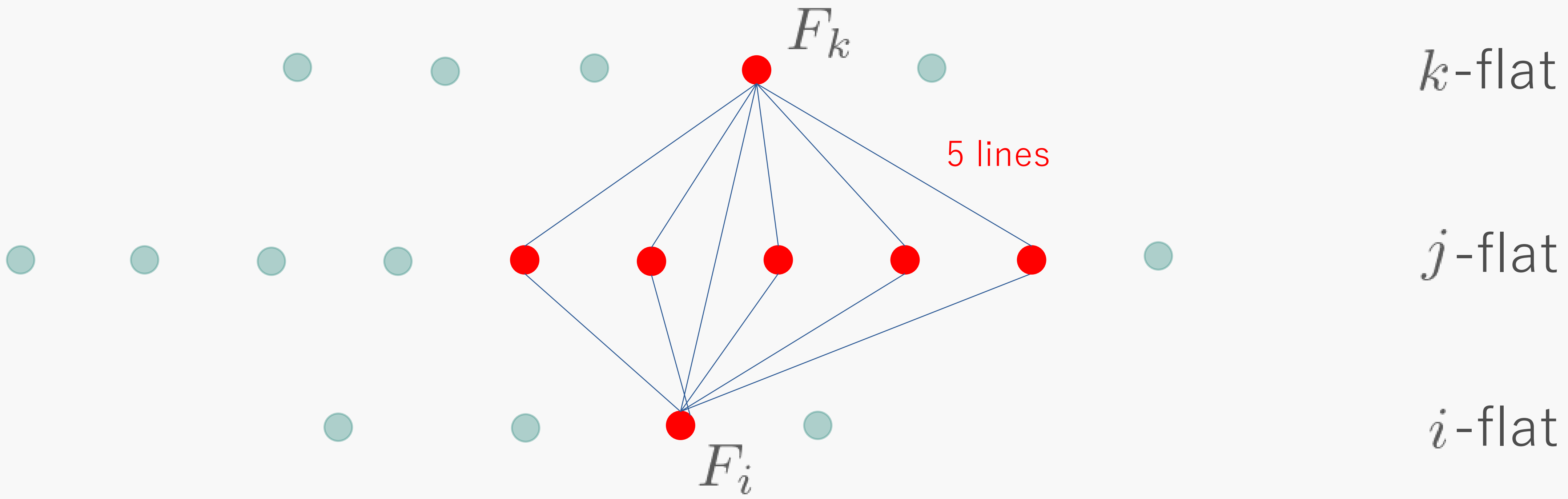

t-function is calculated by α-sequence

Proposition

Let $M=(E,r)$ be a q-PMD with α-sequence $\{\alpha_i\}_{i=0}^{i=r(E)}$ and t-function t_M. Then,

$$t_M(i,j,k)=\prod_{l=0}^{j-i-1}\frac{q^{\alpha_k}-q^{\alpha_{i+l}}}{q^{\alpha_j}-q^{\alpha_{i+l}}}$$

holds.

A t-function of a q-PMD.

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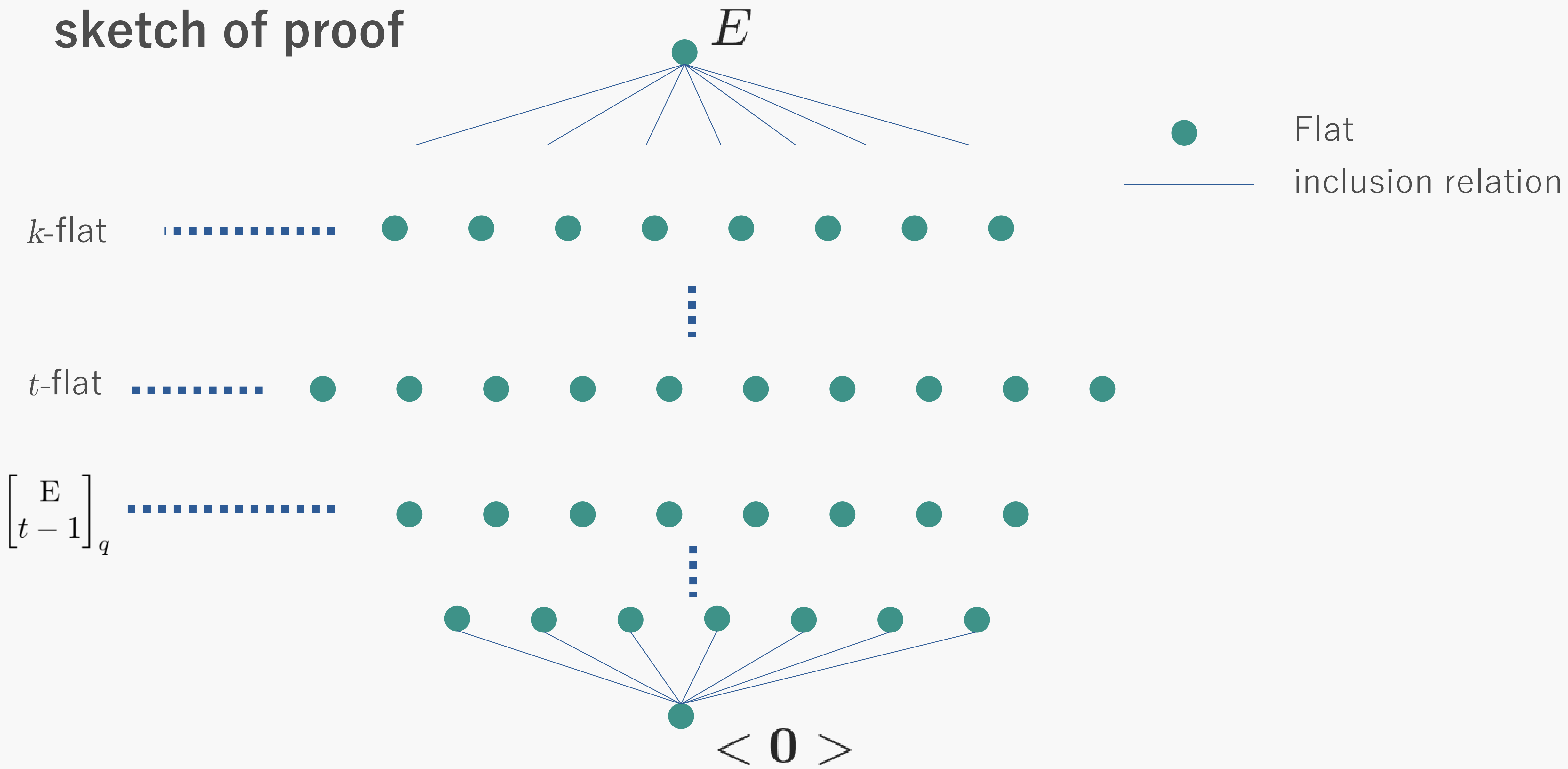
i-Flats of the q-PMD are the blocks of a t-design.

- integer t : $0 \leq t \leq n$

Theorem

If $\begin{bmatrix} E \\ i \end{bmatrix}_q \subset \mathcal{F}$ for all integers i satisfying $0 \leq i \leq t-1$, then $(E, \mathcal{F}_k)$ is a subspace design t-$(n, \alpha_k, t_M(t, k, r(E)))$ for all integers k satisfying $t \leq k \leq r(E)$

sketch of proof



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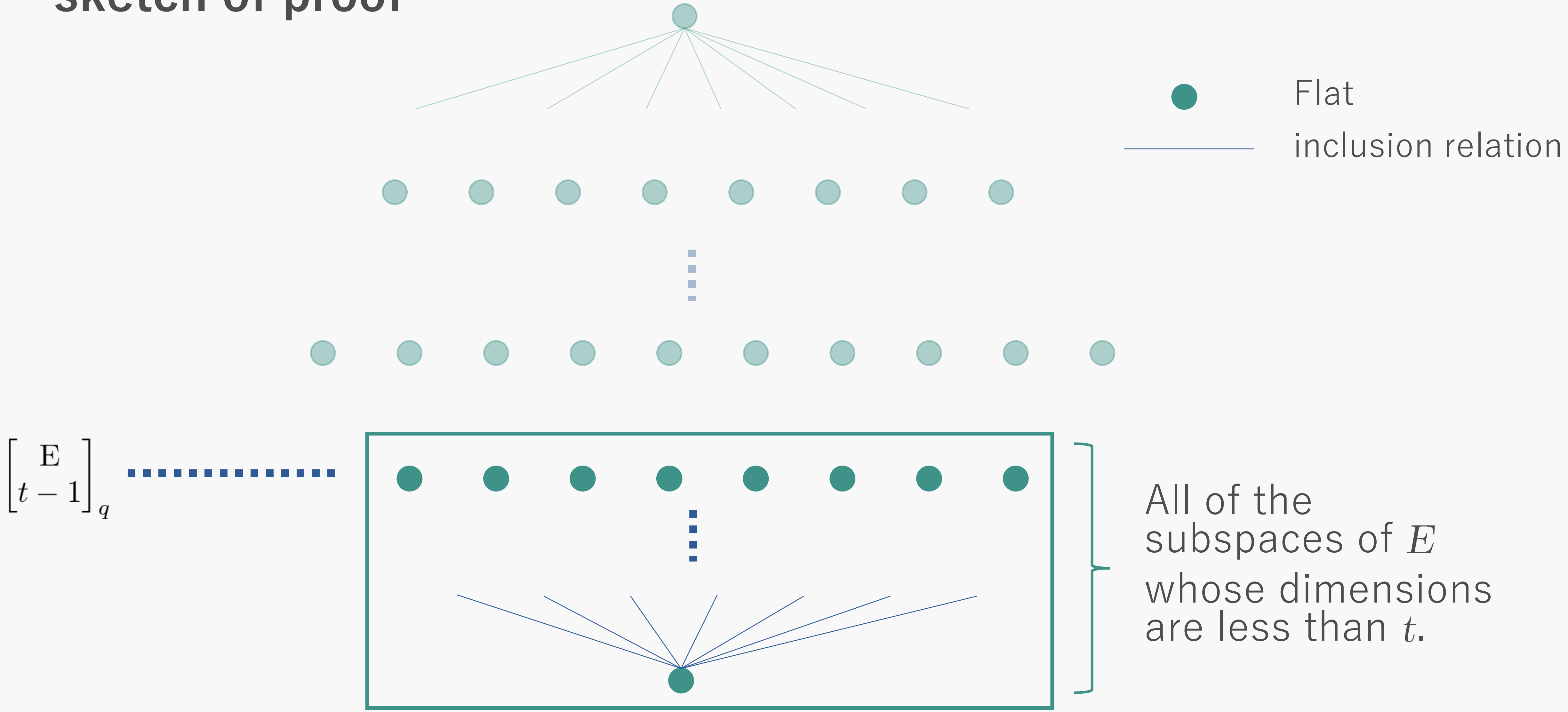

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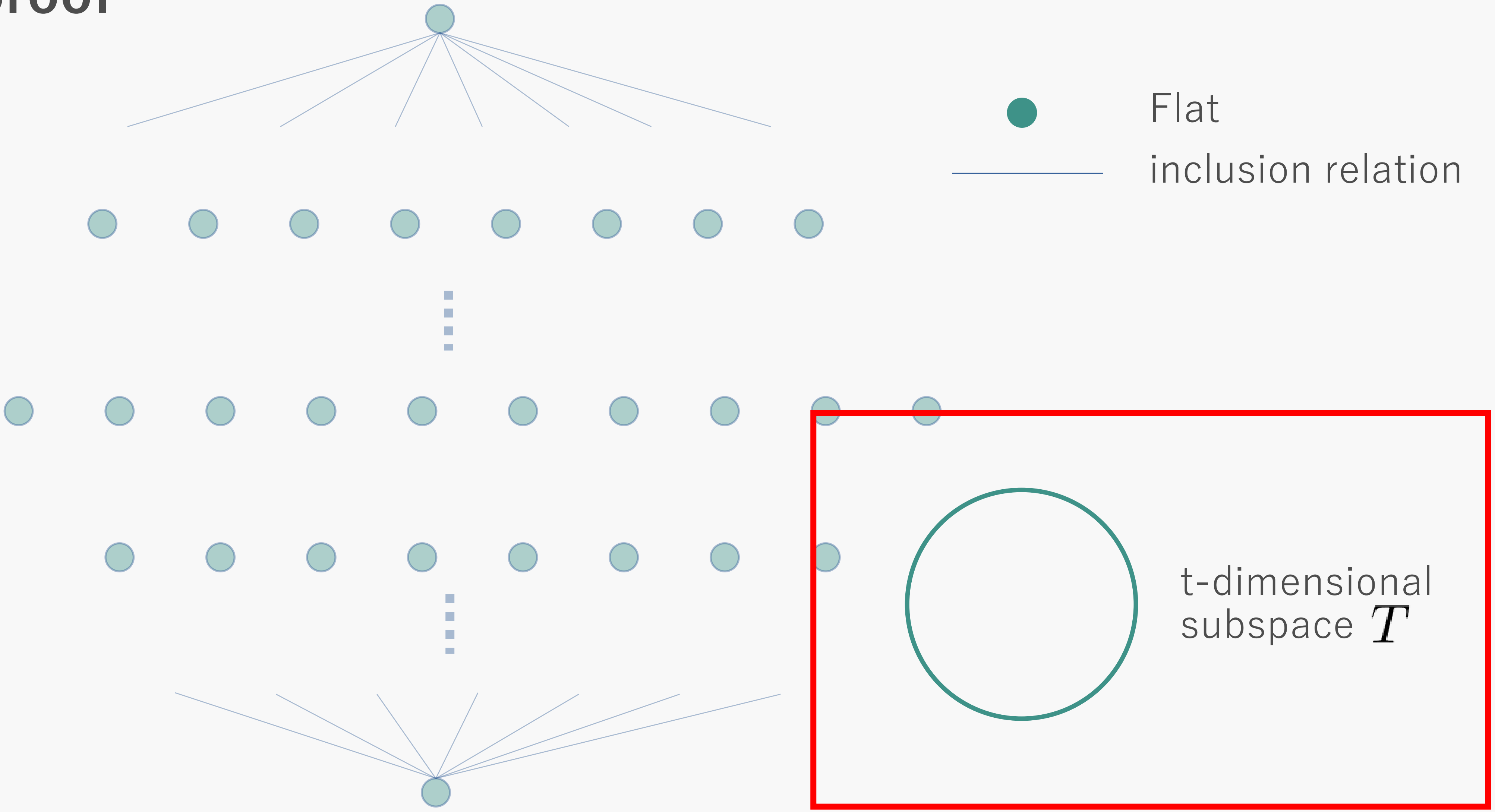

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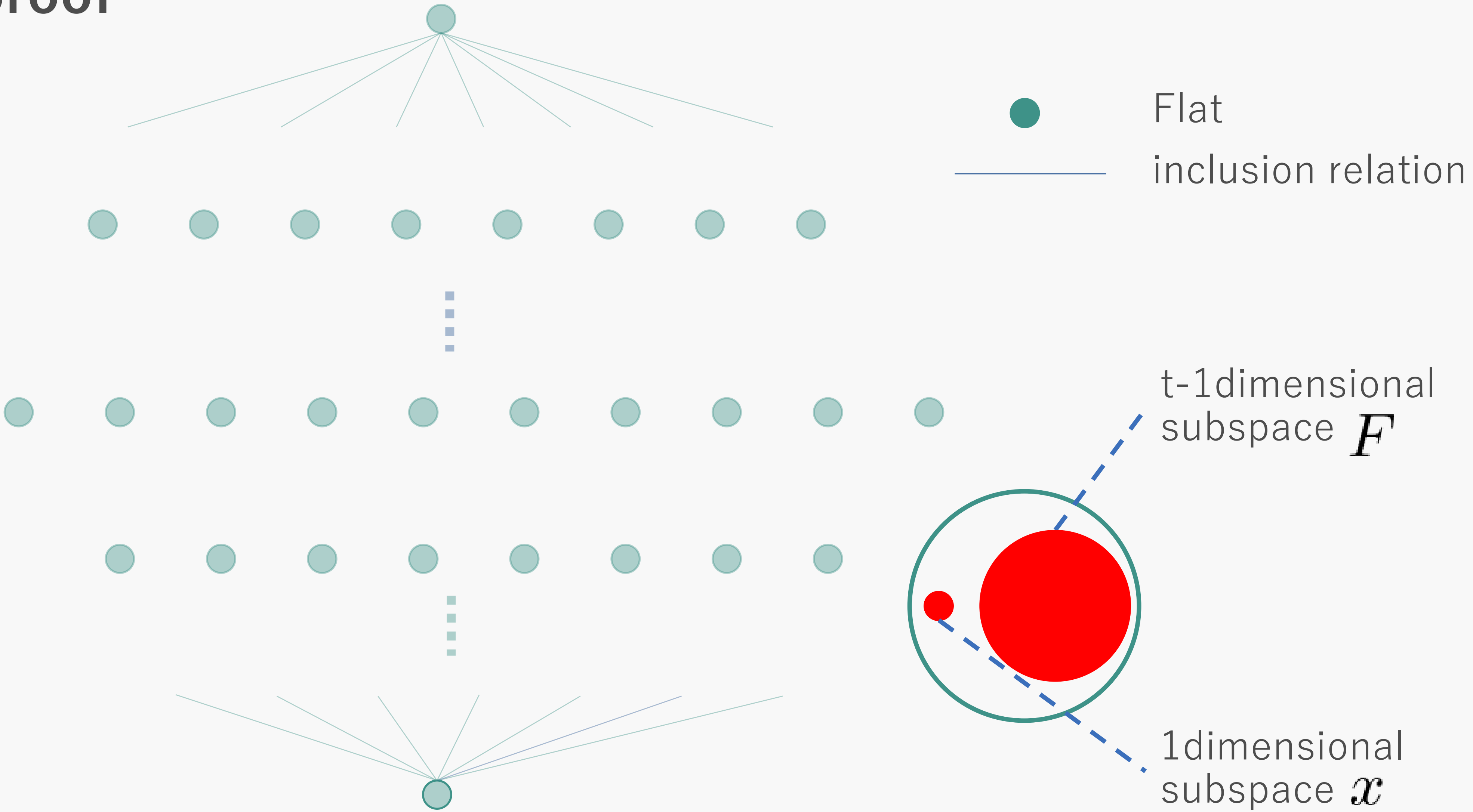

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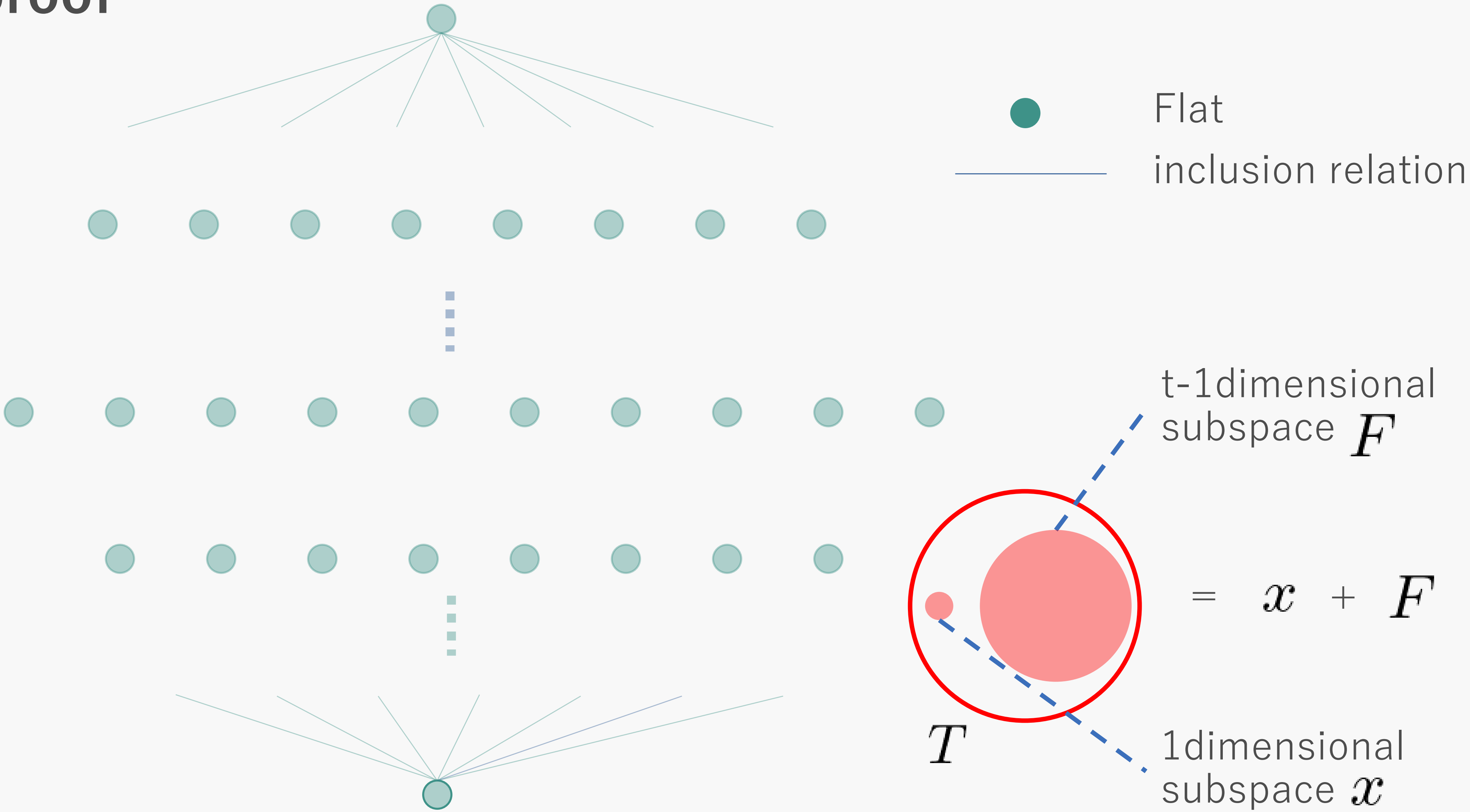

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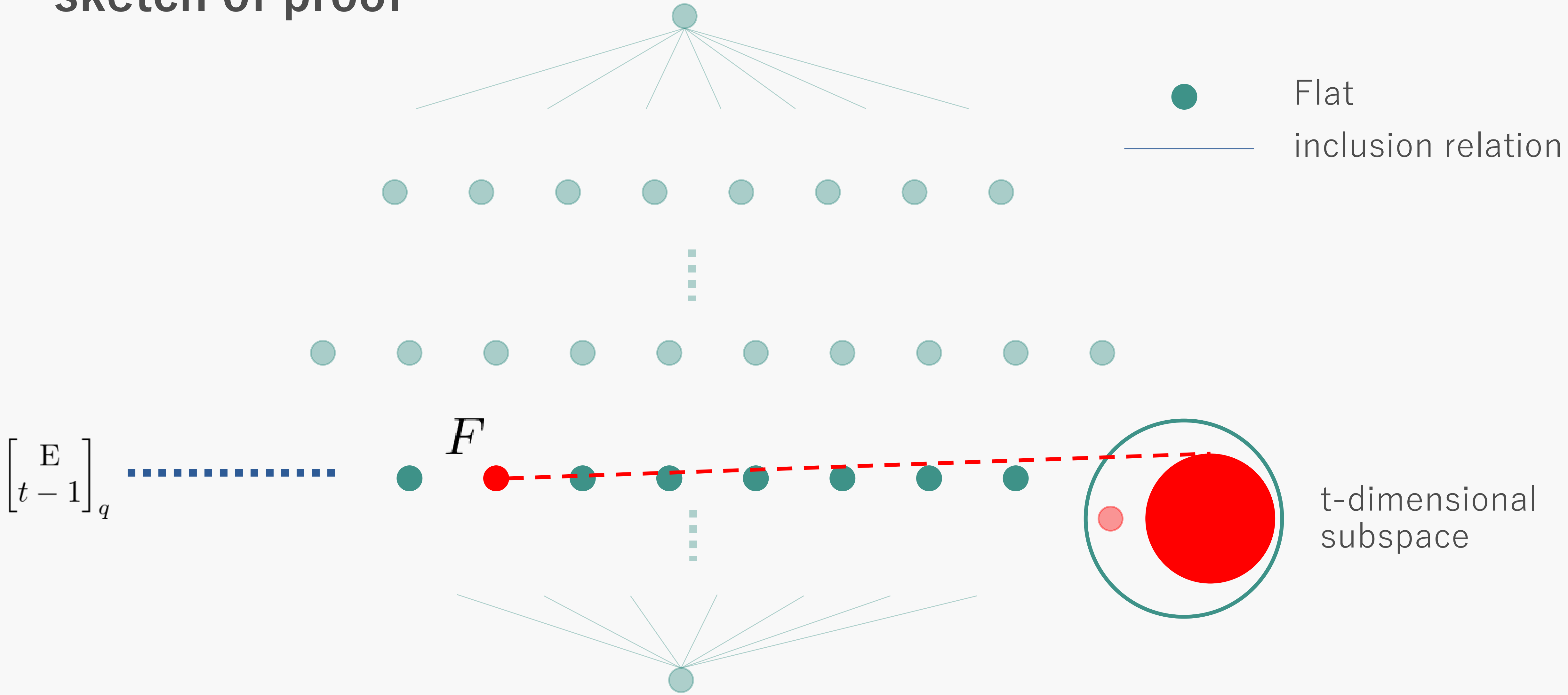
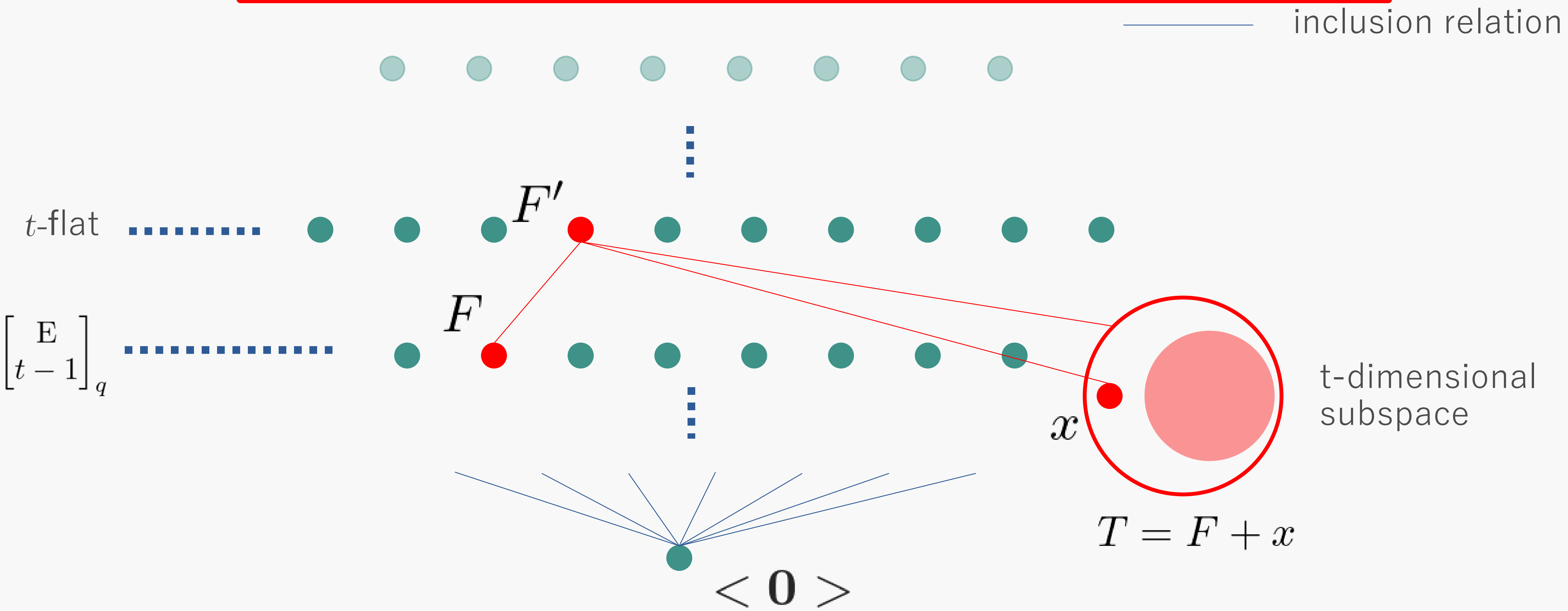
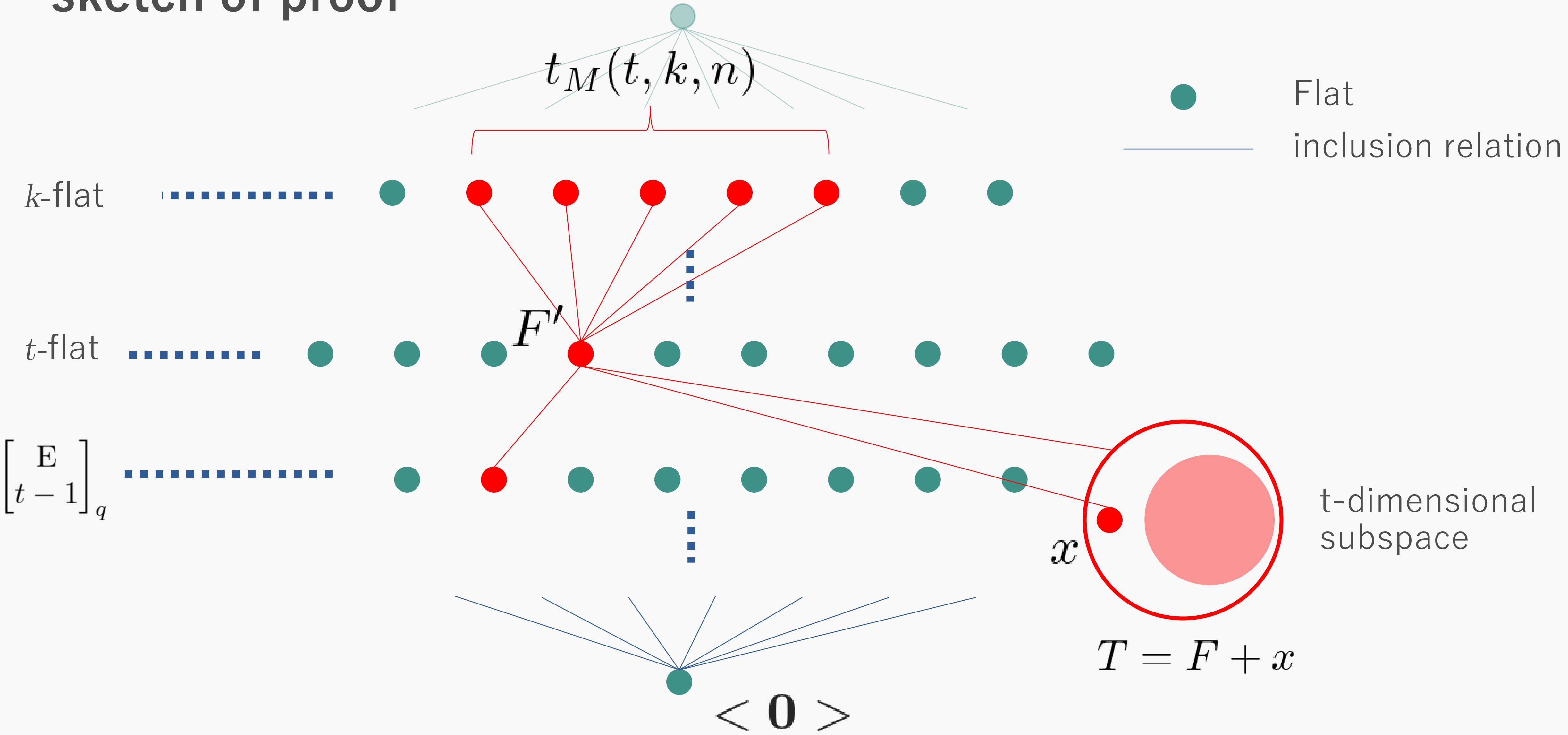


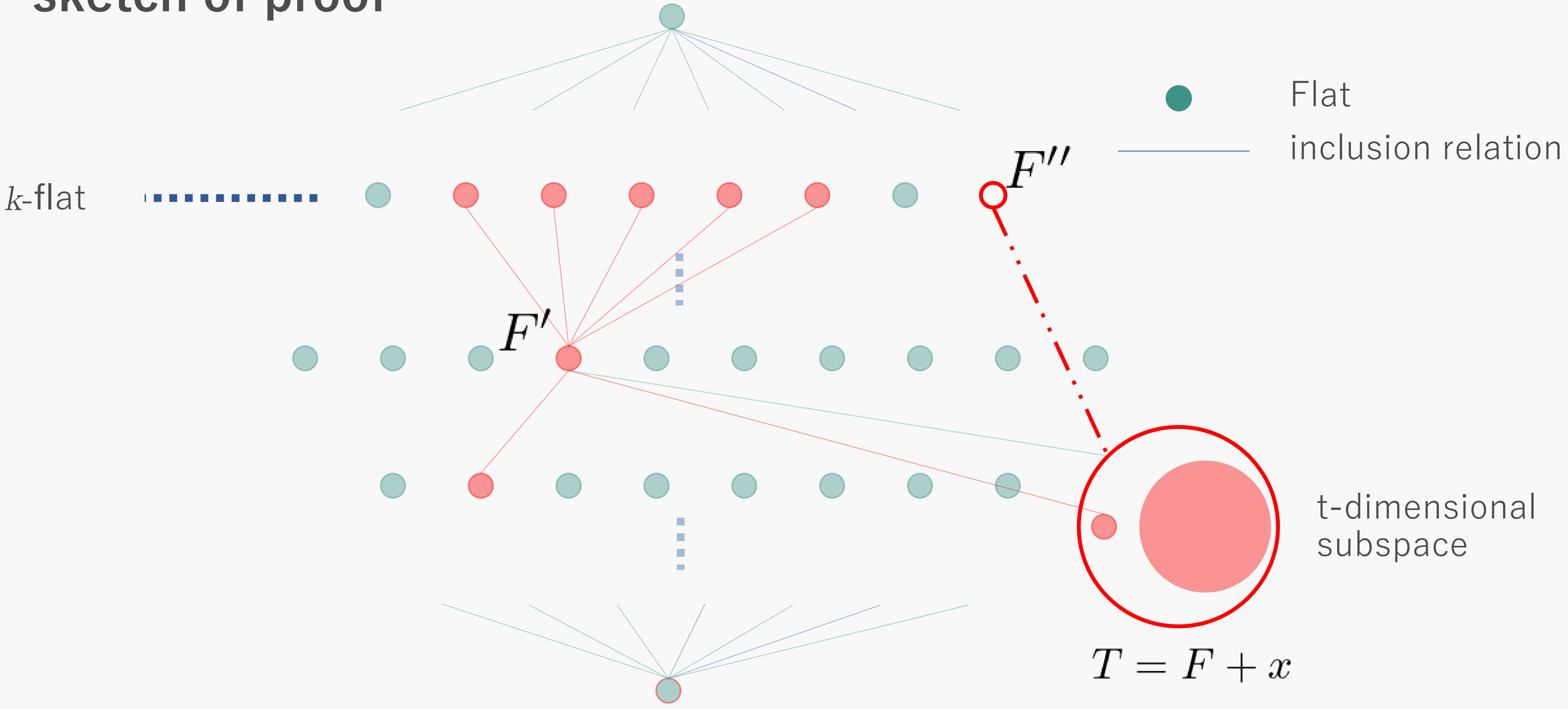
$$\text{(qF3)}\ \ \forall F \in \mathcal{F},\ \forall x \in \begin{bmatrix} E \\ 1 \end{bmatrix}_q \setminus \begin{bmatrix} F \\ 1 \end{bmatrix}_q,\ \exists! F' \in \mathcal{F} \text{ with } r(F') = r(F) + 1$$
$$s.t.\ F + x \subseteq F'.$$



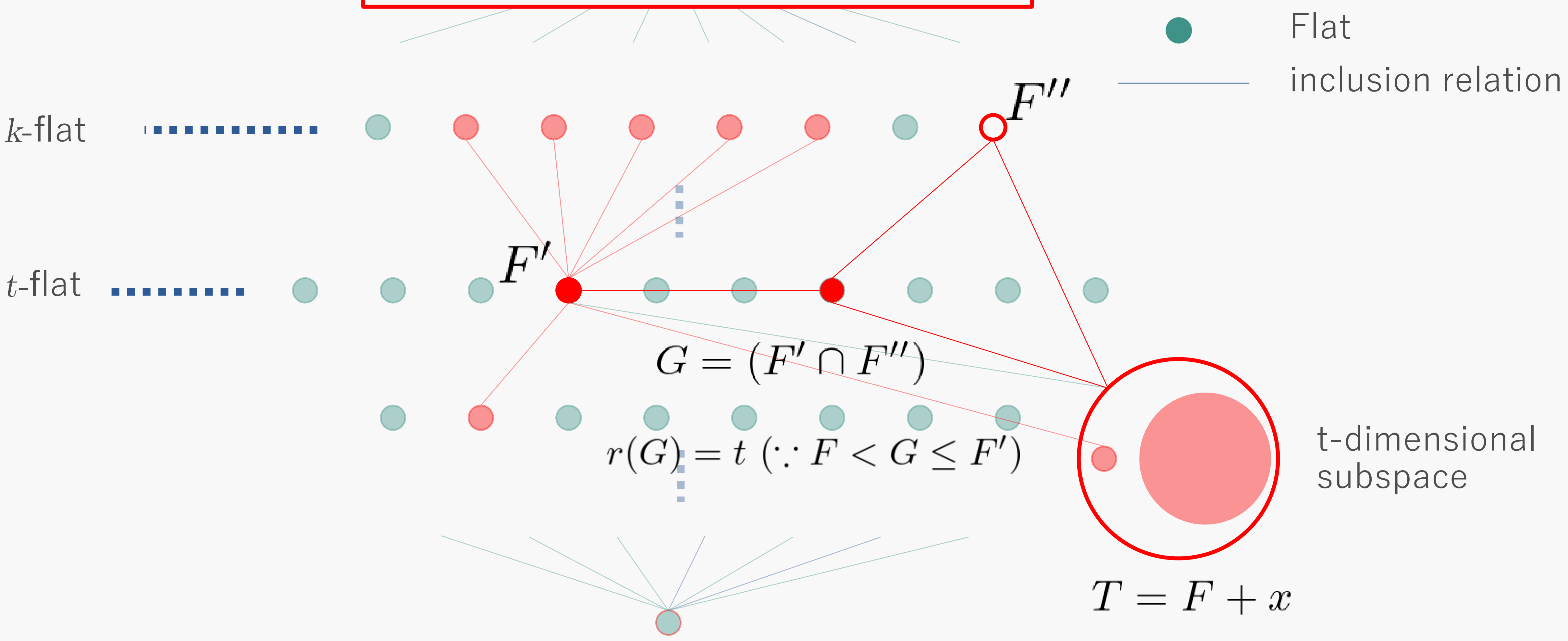
sketch of proof



sketch of proof



$$\text{(qF2)}\ \ \forall F_1, F_2 \in \mathcal{F},\ F_1 \cap F_2 \in \mathcal{F}.$$



$$\text{(qF3)}\ \ \forall F \in \mathcal{F},\ \forall x \in \begin{bmatrix} E \\ 1 \end{bmatrix}_q \setminus \begin{bmatrix} F \\ 1 \end{bmatrix}_q,\ \exists! F' \in \mathcal{F} \text{ with } r(F') = r(F) + 1$$
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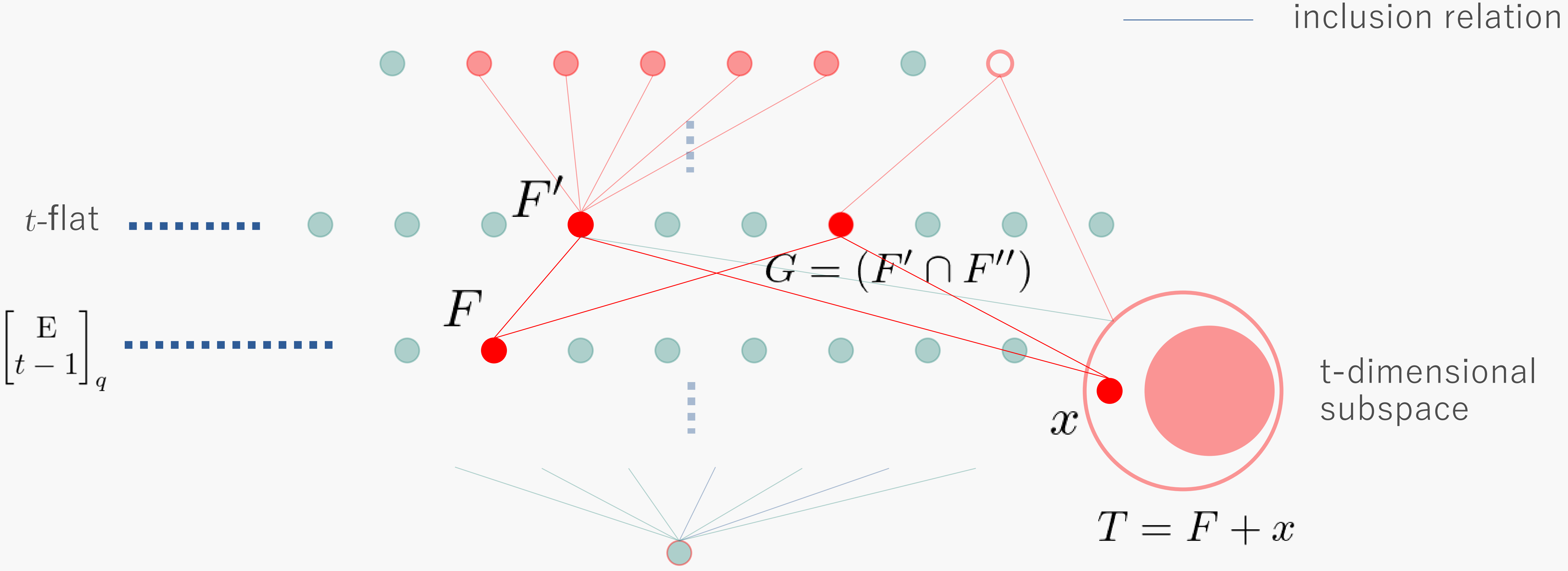

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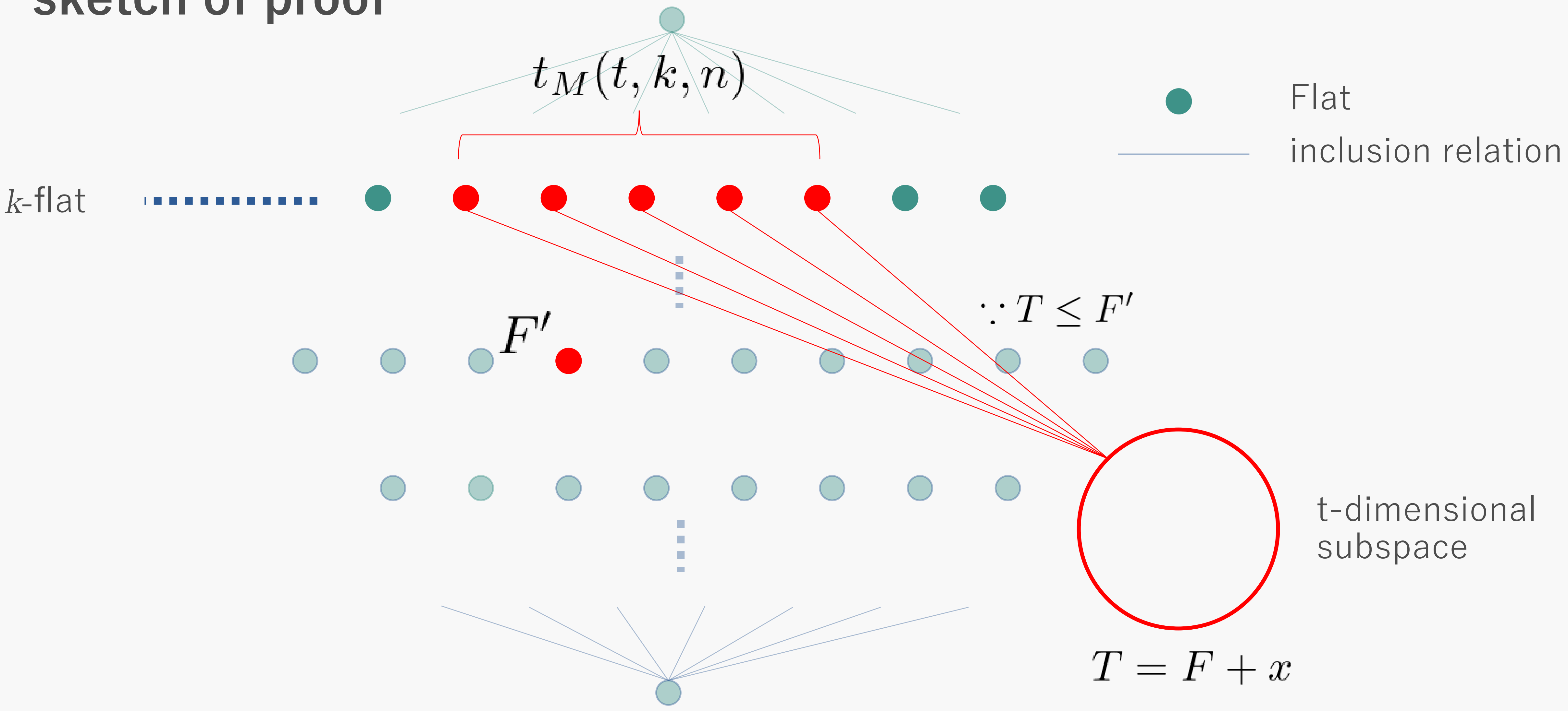

Theorem

If $\begin{bmatrix} E \\ i \end{bmatrix}_q \subset \mathcal{F}$ for all integers i satisfying $0 \leq i \leq t-1$, then $(E, \mathcal{F}_k)$ is a subspace design t-$(n, \alpha_k, \boxed{t_M(t, k, r(E))})$ for all integers k satisfying $t \leq k \leq r(E)$

if $k = 1, t_M(t, t, r(E)) = 1$

Corollary

If $\begin{bmatrix} E \\ i \end{bmatrix}_q \subset \mathcal{F}$ for all integers i satisfying $0 \leq i \leq t-1$, then $(E, \mathcal{F}_t)$ is a q-Steiner system $\mathcal{S}(t, \alpha_t, n; q)$

Theorem

M is induced by q-Stienr system $\mathcal{S}(t, k, n)$

E. Byrn and others 2022+

$\Longrightarrow$

$\Longleftarrow$ **Corollary**

an α-sequence of M is $(0, 1, \cdots, t-1, k, n)$

Theorem

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$\Longrightarrow$ / $\Longleftarrow$ Corollary

an α-sequence of M is $(0,1,\cdots,t-1,k,n)$

sketch of proof

$$|\mathcal{F}_j| = t_M(0,j,r(E)) = \prod_{l=0}^{j-1}\frac{q^n-q^l}{q^j-q^l} = \begin{bmatrix} n \\ j \end{bmatrix}_q \qquad \left(\because t_M(i,j,k) = \prod_{l=0}^{j-i-1}\frac{q^{\alpha_k}-q^{\alpha_{i+l}}}{q^{\alpha_j}-q^{\alpha_{i+l}}}\right)$$

$$\therefore \quad \mathcal{F}_j = \begin{bmatrix} E \\ j \end{bmatrix}_q \qquad (0 \le j \le t-1)$$

$\Rightarrow$ $(E,\mathcal{F}_t)$ is a q-Steiner system $\mathcal{S}(t,k,n;q)$ $\quad(\because$ Corollary$)$

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flats and design	If flats have all of the subsets whose cardinalities are less than t-1,m-flats are t-design ($m \geq t$)	If flats have all of the subsets whose dimensions are less than t-1, m-flats are t-subspace design ($m \geq t$)

Problems for q-PMD

- **Are there any other non-trivial q-Steiner systems?**
  - A q-Steiner system induce a q-PMD[1]
  - The only known q-Steiner system is $\mathcal{S}(2,3,13;2)$ [3]
- **Are there q-PMDs not induced by q-Steiner systems?**
  - There are some PMDs not induced by Steiner system
    - Projective geometries
    - Affine geometries
    - Affine triple systems

[1] E. Byrne et al. : Constructions of new Mtroids and Designs over $\mathbb{F}_q$, (2022)
[2] M. Deza
[3] M. Braun et al.: EXISTENCE OF q-ANALOGS OF STEINER SYSTEMS (2013)

Thank you for your kind attention

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Matroid is a pair (E, r) satisfies (R1), (R2) and (R3).

Definition 1.1. [matroid]

A q-matroid M is a pair (E, r) where r is an inteer-valued function defined on $\mathcal{L}(E)$ with the following properties:

(R1) $\forall A \subseteq E, 0 \leq r(A) \leq |A|$.

(R2) $\forall A, B \subseteq E$ with $A \subseteq B$, $0 \leq r(A) \leq |A|$.

(R3) $\forall A, B \subseteq E$, $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

Remark 1.2. the function r is called **rank function**.

- M : q-PMD

$$M \text{ is induced by } q\text{-Stienr system } \mathcal{S}(t,k,n)$$

E. Byrn and others 2022+ $\Downarrow$ $\Uparrow$ **Theorem**

$$\alpha\text{-sequence of } M \text{ is } (0,\ 1,\ \cdots t-1, k, n)$$

sketch of proof

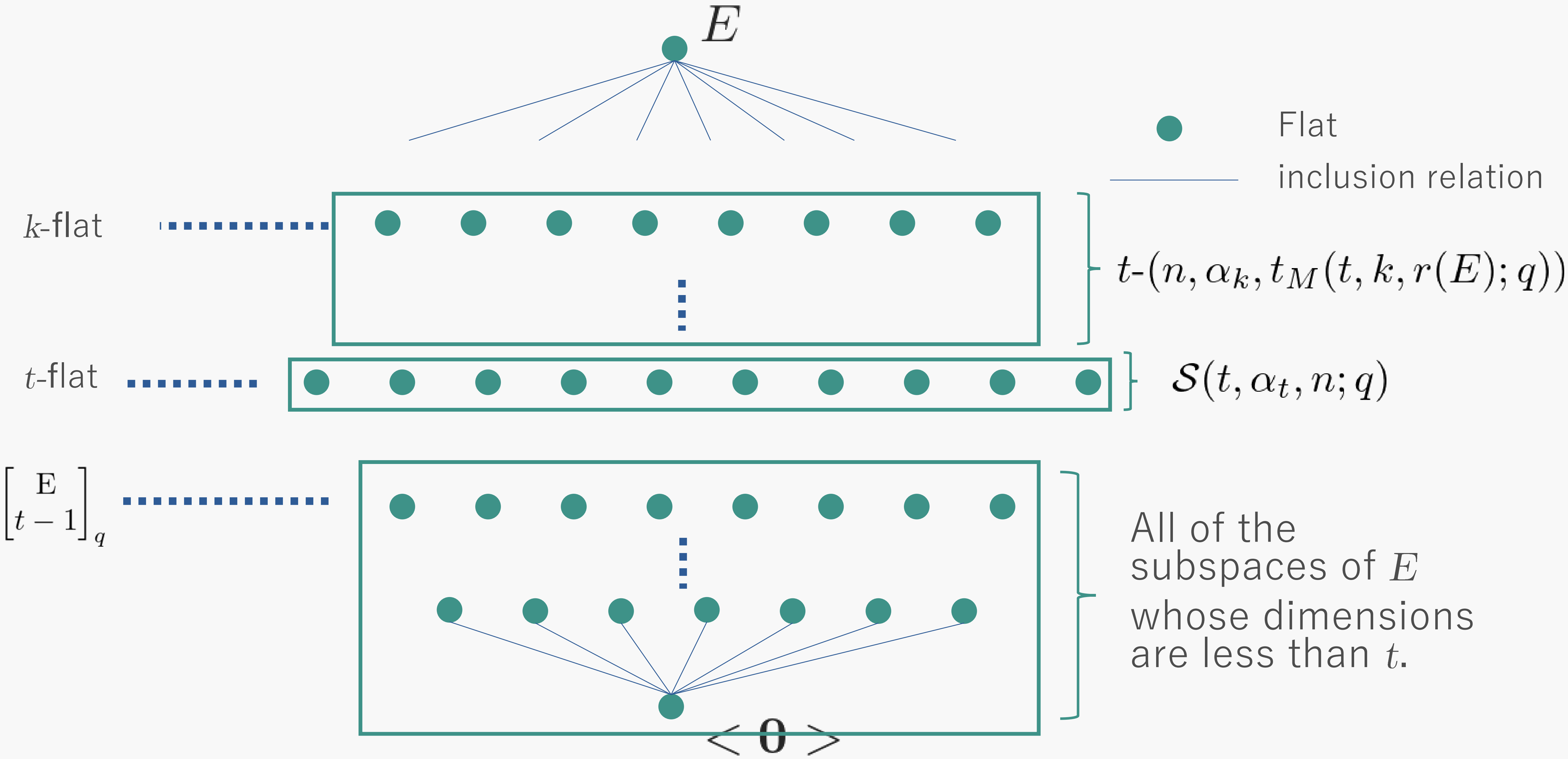
$$|\mathcal{F}_j| = t_M(0,j,r(E)) = \prod_{l=0}^{j-1} \frac{q^n - q^l}{q^j - q^l} = \begin{bmatrix} n \\ j \end{bmatrix}_q \qquad \left(\because t_M(i,j,k) = \prod_{l=0}^{j-i-1} \frac{q^{\alpha_k} - q^{\alpha_{i+l}}}{q^{\alpha_j} - q^{\alpha_{i+l}}} \right)$$

$$\therefore \quad \mathcal{F}_j = \begin{bmatrix} E \\ j \end{bmatrix}_q \qquad (0 \le j \le t-1)$$

Thank you for your kind attention.

- **PMD**
  - **α-sequences**
    - represents the numbers of elements in one i-flat
  - **t-functions**
    - determine the number of i-flats of q-PMD
  - **flats and design**
    - in some conditions, flats are the block of t-design
- **q-PMD**
  - **α-sequences**
    - represents the dimension of one i-flat
  - **t-functions**
    - determine the number of i-flats of q-PMD
  - **flats and t-subspace design**
    - in some conditions, flats are the blocks of a t-subspace design

classification of i-flats



q-**Matroid is a pair $(\boldsymbol{E}, \boldsymbol{r})$ satisfies (qR1), (qR2) and (qR3).**

Definition 1.1. [q-matroid]

(qR1) $\forall A \in \mathcal{L}(E),\ 0 \le r(A) \le \dim A.$

(qR2) $\forall A, B \in \mathcal{L}(E)$ with $A \subseteq B,\ r(A) \le r(B).$

(qR3) $\forall A, B \in \mathcal{L}(E), r(A+B) + r(A \cap B) \le r(A) + r(B)$

Remark 1.5. The function $\boldsymbol{r}$ is called **rank function** of M.

Remark 1.5.

We denote $\mathcal{F}$ as the set of flats.

- If a flat F satisfies $r(F) = i$, F is called $\boldsymbol{i}$**-flat.**
- The collection of i -flat is denoted by $\boldsymbol{\mathcal{F}_i}$.

Theorem (E. Byrn et al. 2022+)

- $(E, \mathcal{B})$: q-Steiner system $\mathcal{S}(t, k, n)$

Let $\mathcal{F}$ be defined as follows:

$$\mathcal{F} := \left\{ \bigcap_{B \in S} B \mid S \subseteq \mathcal{B} \right\}.$$

Then, there exists a q-PMD whose ground set is E and the flats are $\mathcal{F}$.

Remark

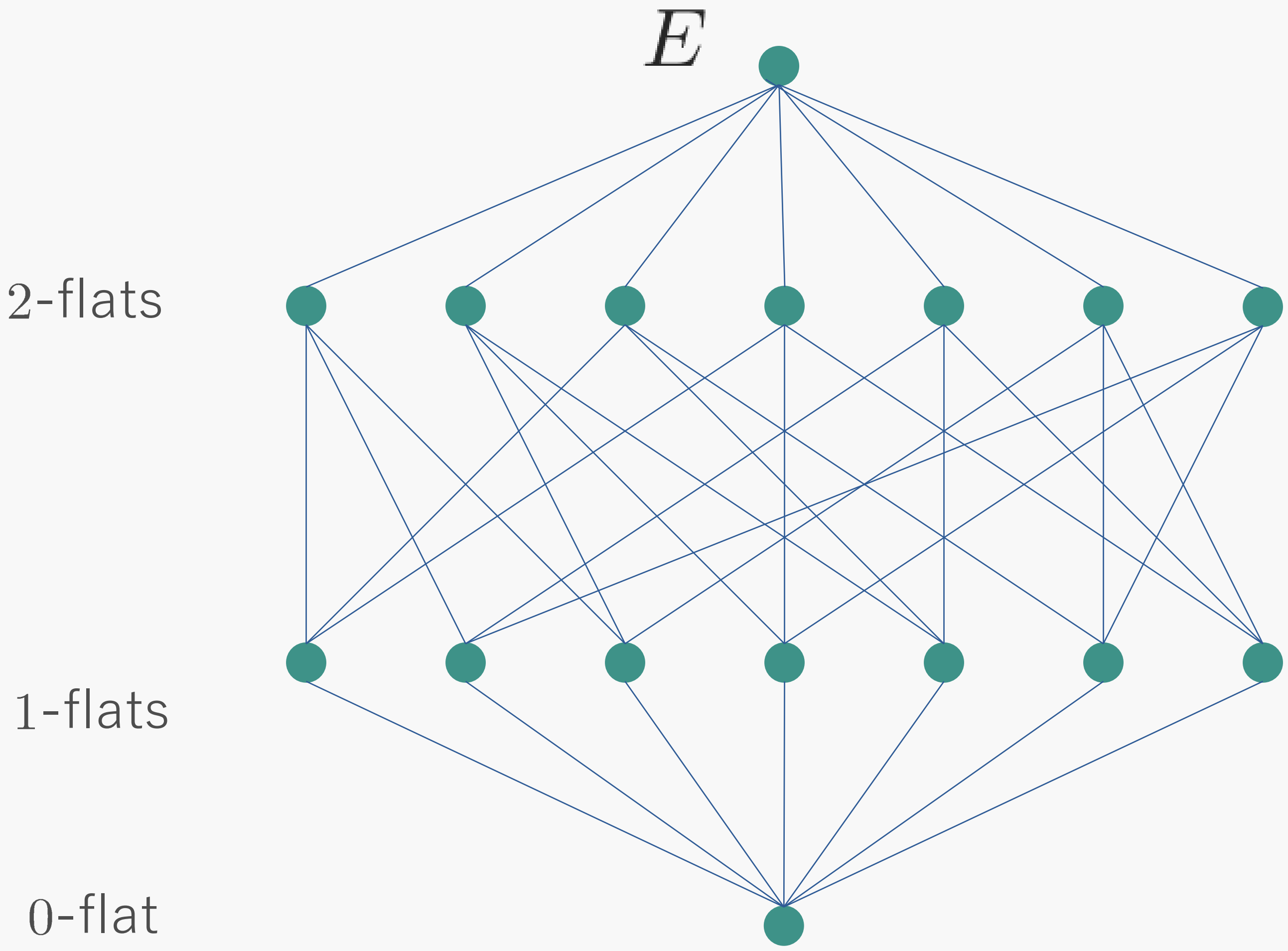
- The α-sequence is $(0, 1, \cdots t-1, k, n)$.

The flats of q-matroid satisfy (qF1), (qF2) and (qF3)

	q-matroid	matroid
1st axiom	$(\mathrm{qF1})\ \ E \in \mathcal{F}$	$(\mathrm{qF1})\ \ E \in \mathcal{F}$
2nd axiom	$(\mathrm{qF2})\ \ F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$	$(\mathrm{qF2})\ \ F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$
3rd axiom	$(\mathrm{qF3})\ \ F \in F, x \in \begin{bmatrix} E \\ 1 \end{bmatrix}_q \setminus \begin{bmatrix} F \\ 1 \end{bmatrix}_q$ $\Rightarrow \exists! F' \in \mathcal{F}_{r(F)+1} \quad s.t. \quad F + x \subseteq F'$	$(\mathrm{qF3})\ \ F \in F, x \in E - X$ $\Rightarrow \exists! F' \in \mathcal{F}_{r(F)+1} \quad s.t. \quad F \cup x \subseteq F'$

[1] E. Byrne, M.Ceria, S. Ionica, R. Jurrius : Constructions of new Mtroids and Designs over $\mathbb{F}_q$, (2022)

Example3



proof

By (qF3), for all 1 dimennal subspace x of F_k, there are unique $i+1$ flat that includs x and F_i. Now, we count the number of the pair of $i+1$-flat and 1 dimensional subspace of them by 2 ways.

We can prove these identities by following the way P. Young and J. Edmond did for normal PMD.

Lemma [Properties of t-function]

Let $M=(E,r)$ be a q-PMD with the t-function t_M. The followings hold:

(T0) $t_M(i,i,k)=1$, for $0\le i\le k\le r(E)$.

(T1) $t_M(0,1,i+1)>t_M(0,1,i)$, for $0\le i\le r(E)-1$.

(T2) $t_M(i,i+1,k)=\frac{t_M(0,1,k)-t_M(0,1,i)}{t_M(0,1,i+1)-t_M(0,1,i)}$, for $0\le i<k\le r(E)$.

(T3) $t_M(i,j,k)=\frac{t_M(i,l,k)t_M(l,j,k)}{t_M(i,l,j)}$, for $0\le i\le l\le j\le k\le r(E)$.

proof

By (qF3), for all 1 dimennal subspace x of F_k, there are unique $i+1$ flat that includs x and F_i. Now, we count the number of the pair of $i+1$-flat and 1 dimensional subspace of them. We define

$$C := \{(F_{i+1}, a) \mid F_{i+1} \in \mathcal{F}_{i+1}(F_i, F_k) \wedge a \in \begin{bmatrix} F_{i+1} \\ 1 \end{bmatrix}_q \}.$$

Let $t = |\mathcal{F}_{i+1}(F_i, F_k)|$. then $|C| = t \begin{bmatrix} \alpha_k \\ 1 \end{bmatrix}_q$

$r(X) = \min(|X|, k)$ is an example of rank function.

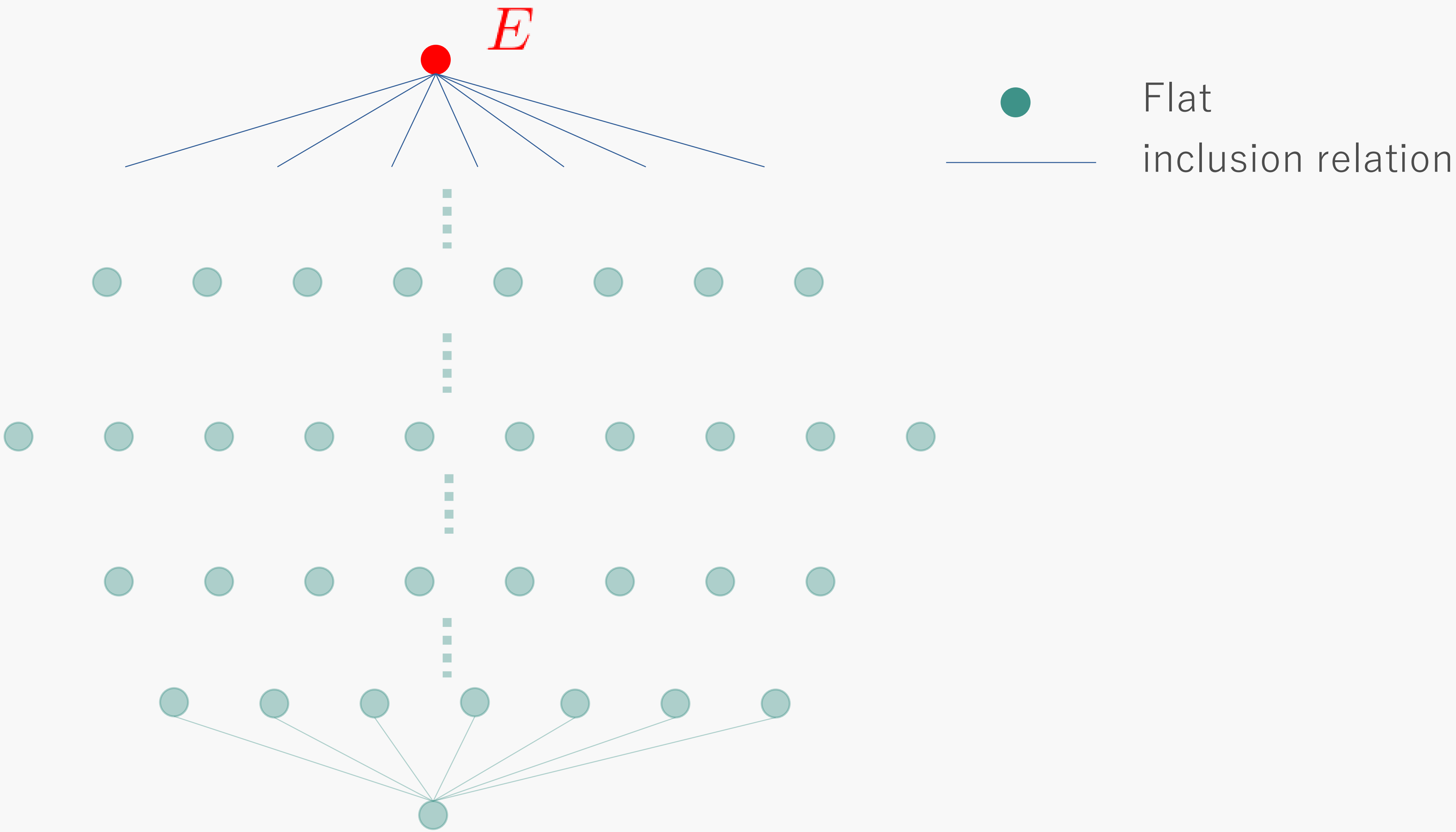
Example1.3. [The uniform matroid]

Let E be a finite set with n elements and k be a integer satisfying $0 \leq k \leq n$.We define the rank function r as follow:

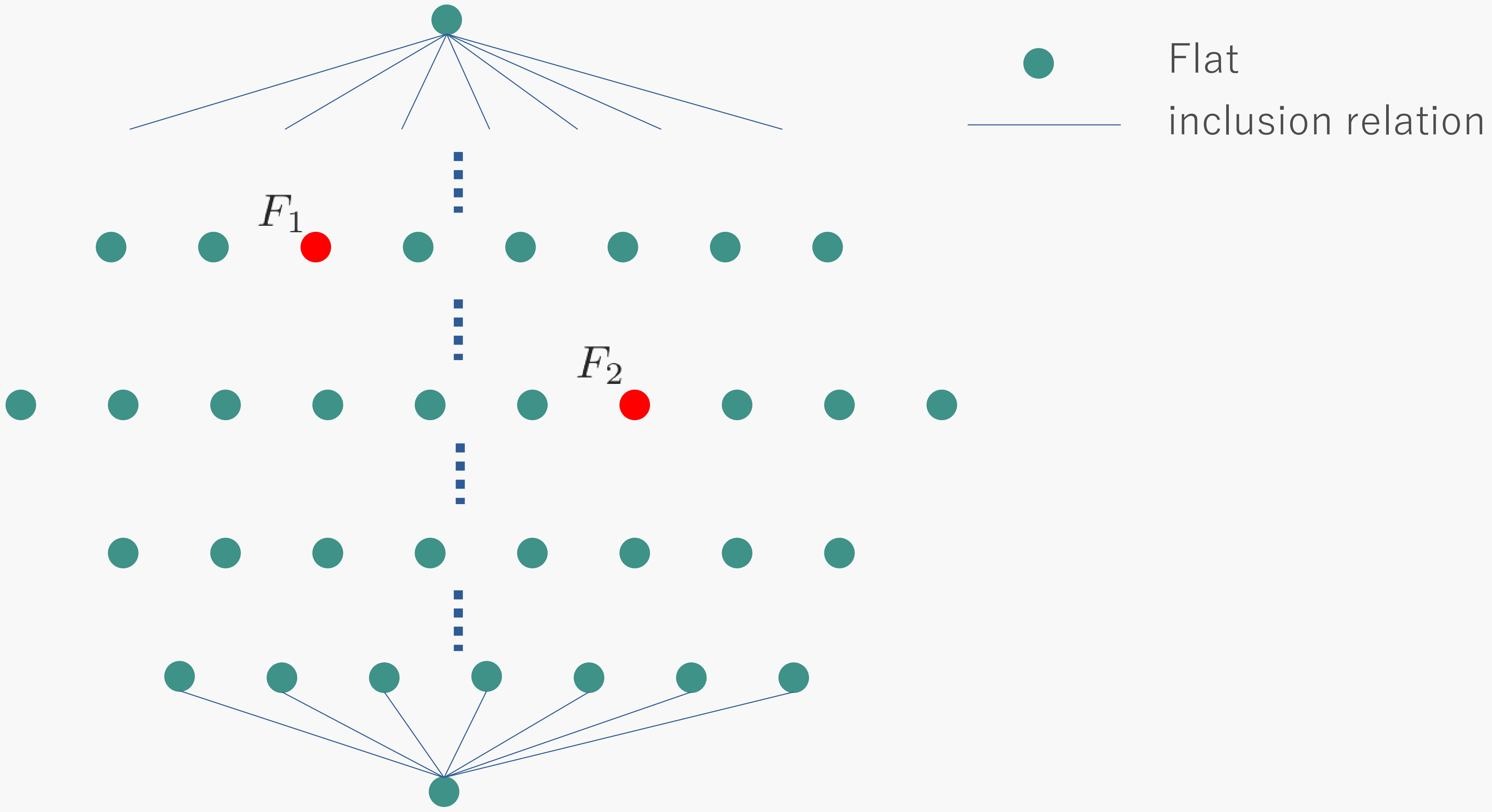
$$r(X) = \min(|X|, k).$$

Then, the pair (E, r) is a matroid.

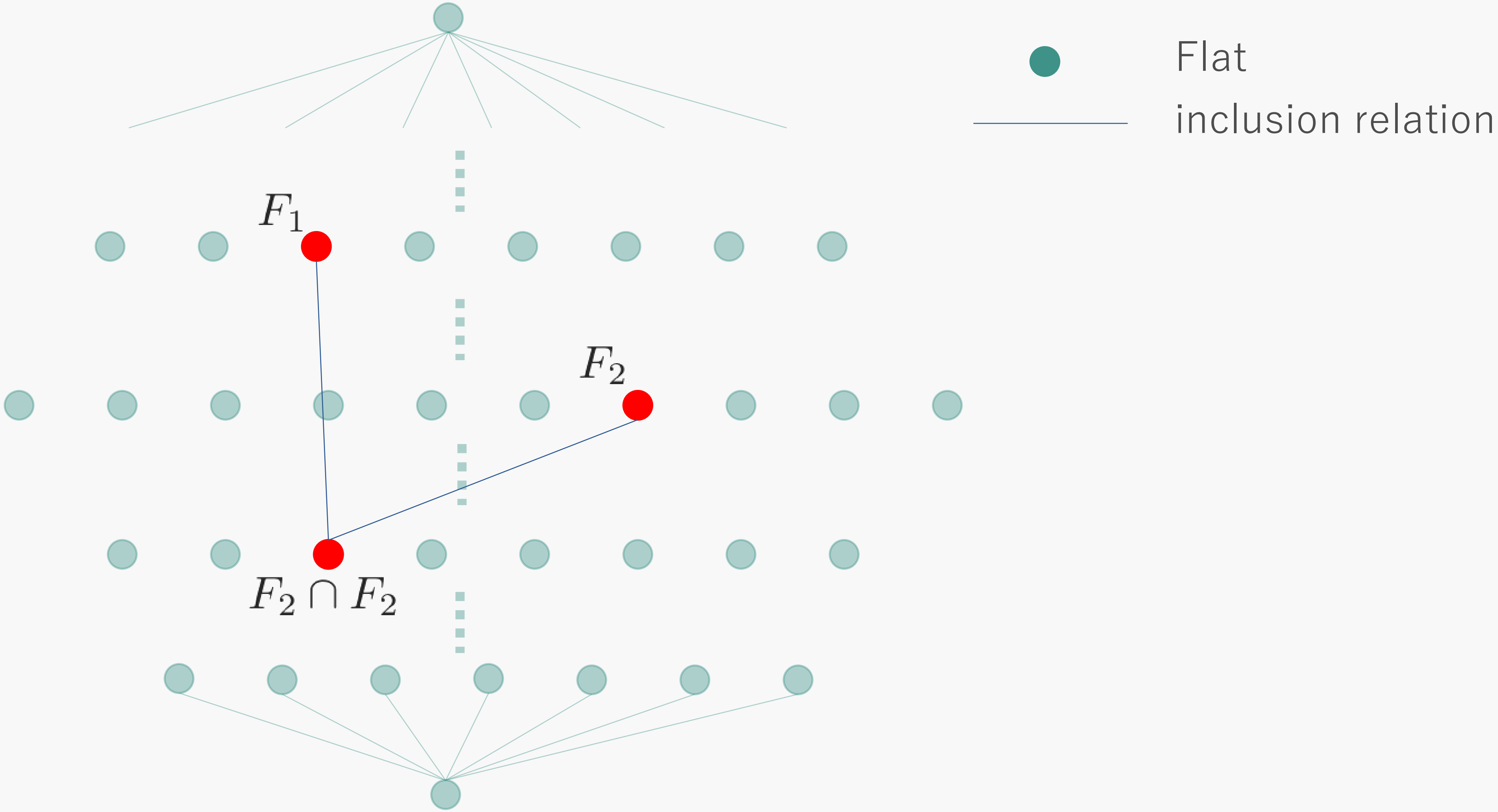
$(qF1)$



$(q\mathrm{F2})$

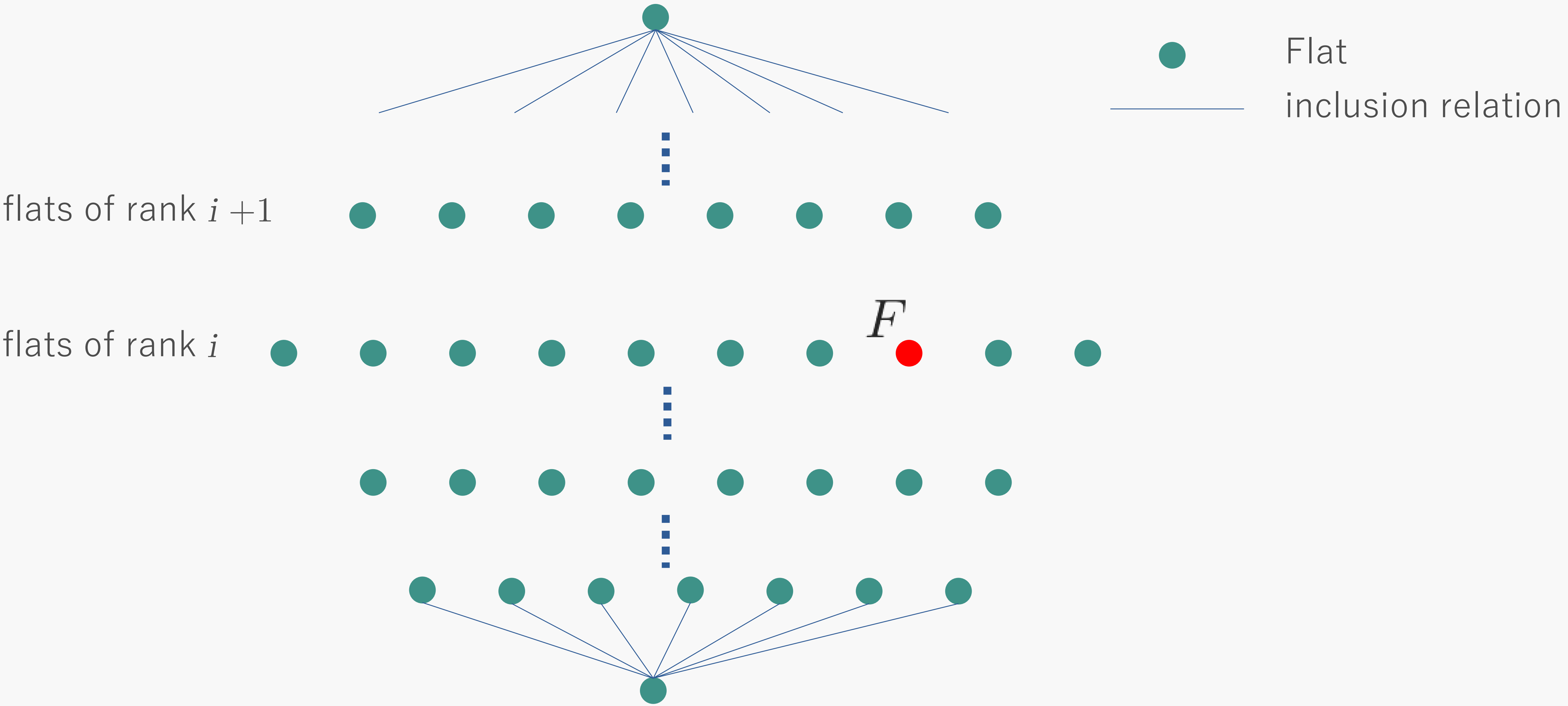


$(q\mathrm{F}2)$

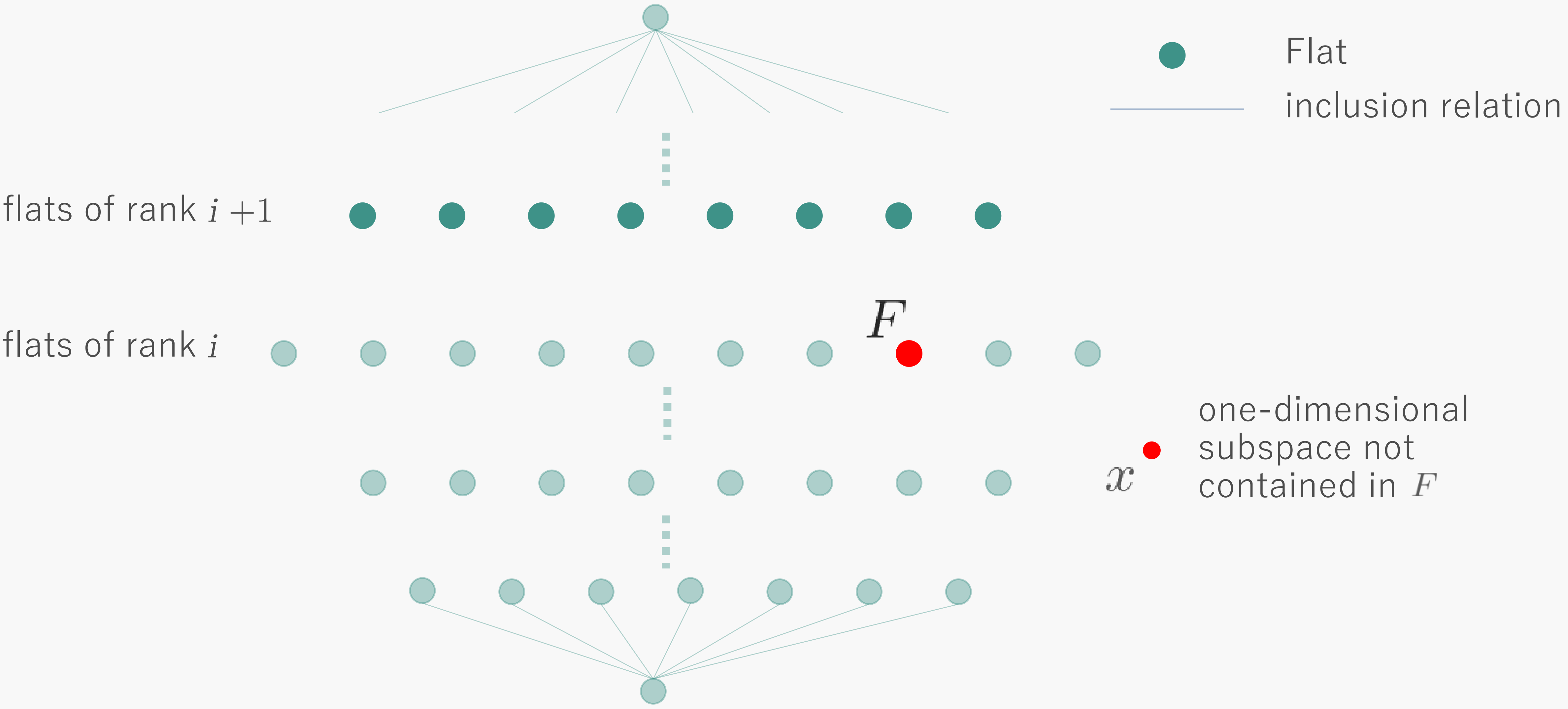


Lattice of flats

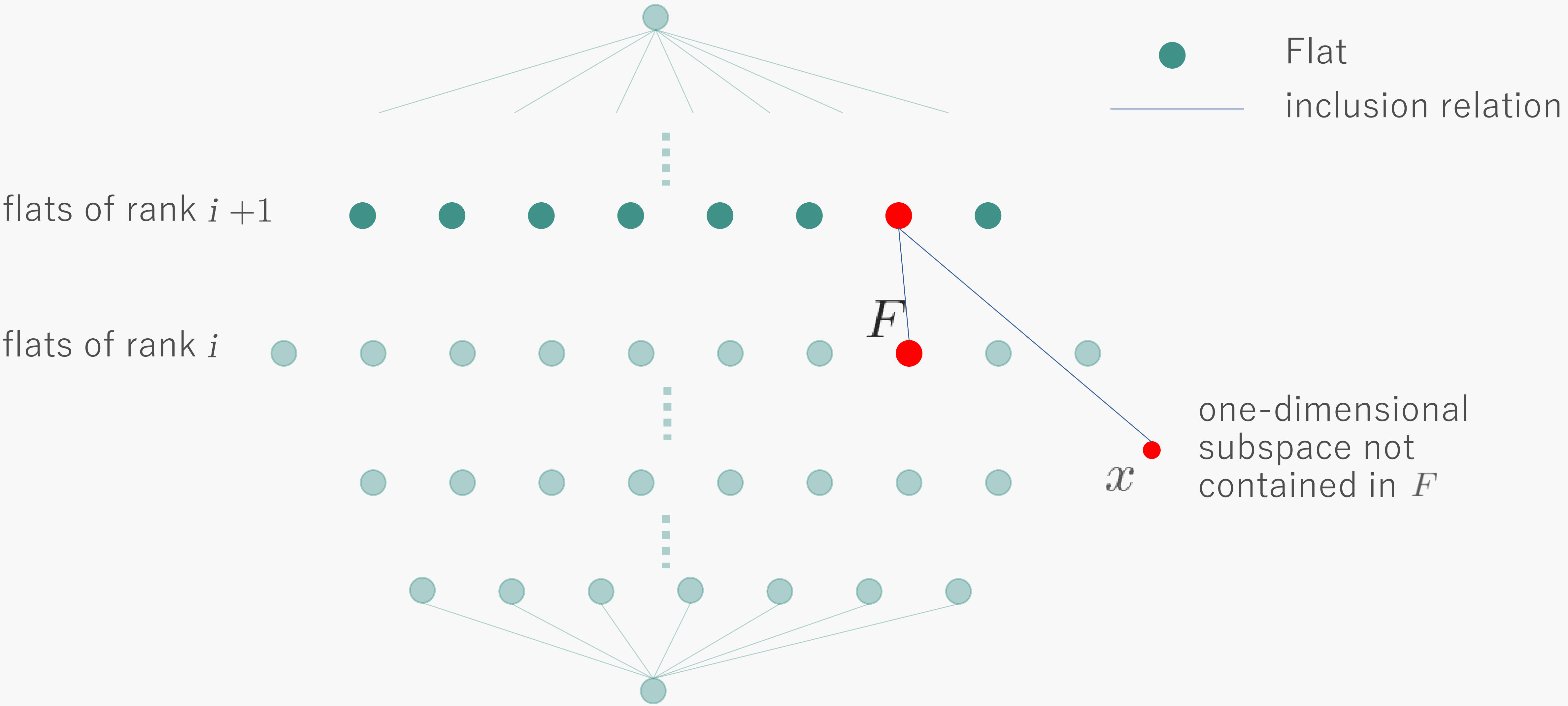
$(q\mathrm{F}3)$



$$(q\mathrm{F}3)$$


Flat
inclusion relation
flats of rank $i+1$
flats of rank i
F
x
one-dimensional
subspace not
contained in F

$(q\mathrm{F}3)$


Flat
inclusion relation
flats of rank $i+1$
F
flats of rank i
one-dimensional subspace not contained in F
x

proof

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)\text{ .}} \left.\vphantom{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}\right\} \text{1-dimensional subspaces of } F_k\text{ .}$$

proof

$$
\begin{array}{cl}
 & \qquad\quad F_{i+1} \\
\begin{array}{c} \\ \\ \\ x \\ \\ \\ \\ \\ \\ \\ \\ \end{array} & \left.\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)\text{ .}}\right\} \text{1-dimensional subspaces of } F_k\text{ .}
\end{array}
$$

$$1 \cdots x \subseteq F_{i+1}$$

$$0 \cdots x \not\subseteq F_{i+1}$$

proof

$$
\underbrace{\left.\begin{array}{c|ccccc|}
 & & F_{i+1} & & & \\
 & 1 & 1 & \cdots & 1 & 1 \\
 & 1 & 1 & \cdots & 1 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 1 & 1 & \cdots & 1 & 1 \\
 & 1 & 0 & \cdots & 0 & 0 \\
x & 1 & 0 & \cdots & 0 & 0 \\
 & 0 & 1 & \cdots & 0 & 0 \\
 & 0 & 1 & \cdots & 0 & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & \cdots & 0 & 1 \\
 & 0 & 0 & \cdots & 0 & 1
\end{array}\right\}\text{1-dimensional subspaces of } F_k\,.}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)\,.}
$$

$$1 \cdots x \subseteq F_{i+1}$$

$$0 \cdots x \nsubseteq F_{i+1}$$

proof

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)}$$

The number of 1 ⋯⋯

proof

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)}$$

The number of 1 ⋯⋯ $\boxed{\begin{bmatrix} \alpha_{i+1} \\ 1 \end{bmatrix}_q}$

proof

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)}$$

The number of 1 ⋯⋯ $\boxed{|\mathcal{F}_{i+1}(F_i, F_k)| \begin{bmatrix} \alpha_{i+1} \\ 1 \end{bmatrix}_q}$

proof

$$
\underbrace{\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)}
\left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{matrix}}\right\}
\text{1 dimensional subspace of } F_k
$$

proof

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
$$

1 dimensional
subspace of F_k

The number of 1 $\cdots\cdots$ $|\mathcal{F}_{i+1}(F_i, F_k)| \times \begin{bmatrix}\alpha_i \\ 1\end{bmatrix}_q +$

The flats of q-PMD are easily counted.

proof

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

1 dimensional subspace of F_k

The number of 1 ······ $|\mathcal{F}_{i+1}(F_i, F_k)| \times \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q + \begin{bmatrix} \alpha_k \\ 1 \end{bmatrix}_q - \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q$

Corollary [t -function]

Let M be a q-PMD with flats $\mathcal{F}$ and F_i, F_j are an i-flat and a k-flat respectively with $F_i \subseteq F_j$.

We define t ***-function*** of M as follows:

$$t_M(i,j,k) := |\mathcal{F}_j(F_i, F_k)|\,.$$

Flat axiom

Definition 1.7.

Let E be a finite dimension vectorspace over $\mathbb{F}_q$ and $\mathcal{F} \subseteq \mathcal{L}(E)$. We define flat axioms as follows:

(qF1) $E \in \mathcal{F}$.

(qF2) $\forall F_1, F_2 \in \mathcal{F},\ F_1 \cap F_2 \in \mathcal{F}$.

(qF3) $\forall F \in \mathcal{F}, \forall x \in \begin{bmatrix} E \\ 1 \end{bmatrix}_q \setminus \begin{bmatrix} F \\ 1 \end{bmatrix}_q, \exists! F' \in \mathcal{F}$ with $r(F') = r(F) + 1$ $s.t.\ F + x \subseteq F'$.

.

proof

$$\underbrace{\left[\begin{array}{ccccc} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{array}\right]}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)} \begin{array}{l} \\ \\ \} F_i \\ \\ \\ \\ \} x \\ \\ \\ \\ \\ \end{array}$$

The number of 1 …… $|\mathcal{F}_{i+1}(F_i, F_k)| \times \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q + \begin{bmatrix} \alpha_k \\ 1 \end{bmatrix}_q - \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q$

proof

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \boxed{1} & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)}$$

F_i

x

The number of 1 $\cdots\cdots$ $|\mathcal{F}_{i+1}(F_i, F_k)| \times \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q + \begin{bmatrix} \alpha_k \\ 1 \end{bmatrix}_q - \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q$

proof

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{\text{i+1 flat in } \mathcal{F}_{i+1}(F_i, F_k)}$$

F_i

x

The number of 1 ······ $|\mathcal{F}_{i+1}(F_i, F_k)| \times \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q + \begin{bmatrix} \alpha_k \\ 1 \end{bmatrix}_q - \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q$

The flats of q-PMD are easily counted.

proof

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
$$

1 dimensional subspace of

The number of 1 ⋯⋯ $\boxed{|\mathcal{F}_{i+1}(F_i, F_k)| \times \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q + \begin{bmatrix} \alpha_k \\ 1 \end{bmatrix}_q - \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q}$

proof

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\boxed{|\mathcal{F}_{i+1}(F_i, F_k)| \times \begin{bmatrix} \alpha_{i+1} \\ 1 \end{bmatrix}_q} = \boxed{|\mathcal{F}_{i+1}(F_i, F_k)| \times \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q + \begin{bmatrix} \alpha_k \\ 1 \end{bmatrix}_q - \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}_q}$$

$$\therefore |\mathcal{F}_{i+1}(F_i, F_k)| = \frac{q^{\alpha_k} - q^{\alpha_i}}{q^{\alpha_{i+1}} - q^{\alpha_i}}$$

We can prove these identities by following the way P. Young and J. Edmond did for normal PMD.

Lemma [Properties of t-function]

Let $M = (E, r)$ be a q-PMD with the t-function t_M. The followings hold:

(T0) $t_M(i, i, k) = 1$, for $0 \leq i \leq k \leq r(E)$.

(T1) $t_M(0, 1, i+1) > t_M(0, 1, i)$, for $0 \leq i \leq r(E) - 1$.

(T2) $t_M(i, i+1, k) = \frac{t_M(0,1,k) - t_M(0,1,i)}{t_M(0,1,i+1) - t_M(0,1,i)}$, for $0 \leq i < k \leq r(E)$.

(T3) $t_M(i, j, k) = \frac{t_M(i,l,k)\, t_M(l,j,k)}{t_M(i,l,j)}$, for $0 \leq i \leq l \leq j \leq k \leq r(E)$.

(T3) $t_M(i,j,k) = \frac{t_M(i,l,k)t_M(l,j,k)}{t_M(i,l,j)}$, for $0 \leq i \leq l \leq j \leq k \leq r(E)$.

$$t_M(i,j,k) = \frac{t_M(i,i+1,k)}{t_M(i,i+1,j)} t_M(i+1,j,k)$$

(T3) $t_M(i,j,k) = \frac{t_M(i,l,k)t_M(l,j,k)}{t_M(i,l,j)}$, for $0 \le i \le l \le j \le k \le r(E)$.

$$\begin{aligned}
t_M(i,j,k) &= \frac{t_M(i,i+1,k)}{t_M(i,i+1,j)} t_M(i+1,j,k) \\
&= \frac{t_M(i,i+1,k)}{t_M(i,i+1,j)} \cdot \frac{t_M(i+1,i+2,k)}{t_M(i+1,i+2,j)} \cdot t_M(i+2,j,k) \\
&\vdots \\
&= \prod_{l=0}^{j-i-1} \frac{t_M(i+l,i+l+1,k)}{t_M(i+l,i+l+1,j)}
\end{aligned}$$

by Lemma $$t_M(i,i+1,k) := |\mathcal{F}_{i+1}(F_i,F_k)| = \frac{q^{\alpha_k} - q^{\alpha_i}}{q^{\alpha_{i+1}} - q^{\alpha_i}}$$

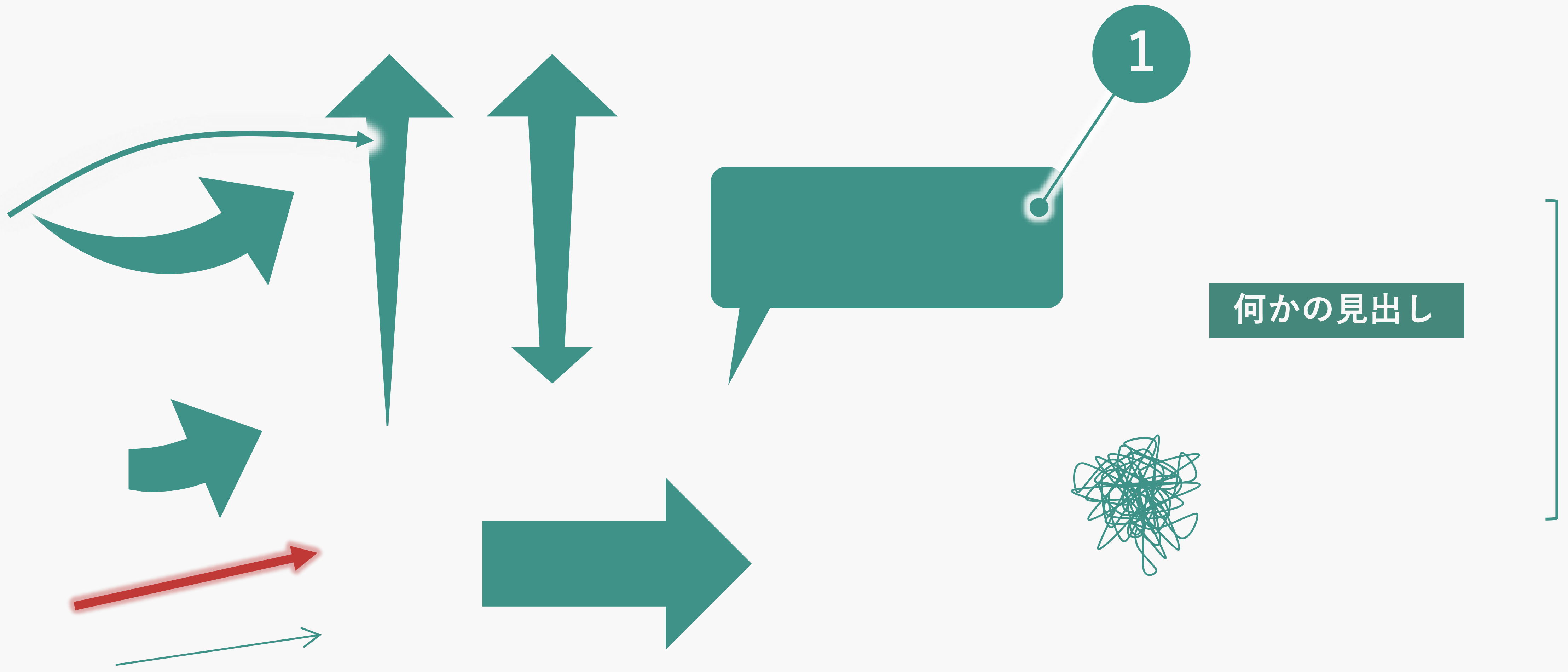
$$\begin{aligned}
t_M(i,j,k) &= \frac{t_M(i,i+1,k)}{t_M(i,i+1,j)} t_M(i+1,j,k) \\
&= \frac{t_M(i,i+1,k)}{t_M(i,i+1,j)} \cdot \frac{t_M(i+1,i+2,k)}{t_M(i+1,i+2,j)} \cdot t_M(i+2,j,k) \\
&\vdots \\
&= \prod_{l=0}^{j-i-1} \frac{t_M(i+l,i+l+1,k)}{t_M(i+l,i+l+1,j)} \\
&= \prod_{l=0}^{j-i-1} \frac{q^{\alpha_k} - q^{\alpha_{i+l}}}{q^{\alpha_{i+j+1}} - q^{\alpha_{i+l}}} \cdot \frac{q^{\alpha_{i+j+1}} - q^{\alpha_{i+l}}}{q^{\alpha_j} - q^{\alpha_{i+l}}} = \prod_{l=0}^{j-i-1} \frac{q^{\alpha_k} - q^{\alpha_{i+l}}}{q^{\alpha_j} - q^{\alpha_{i+l}}}
\end{aligned}$$

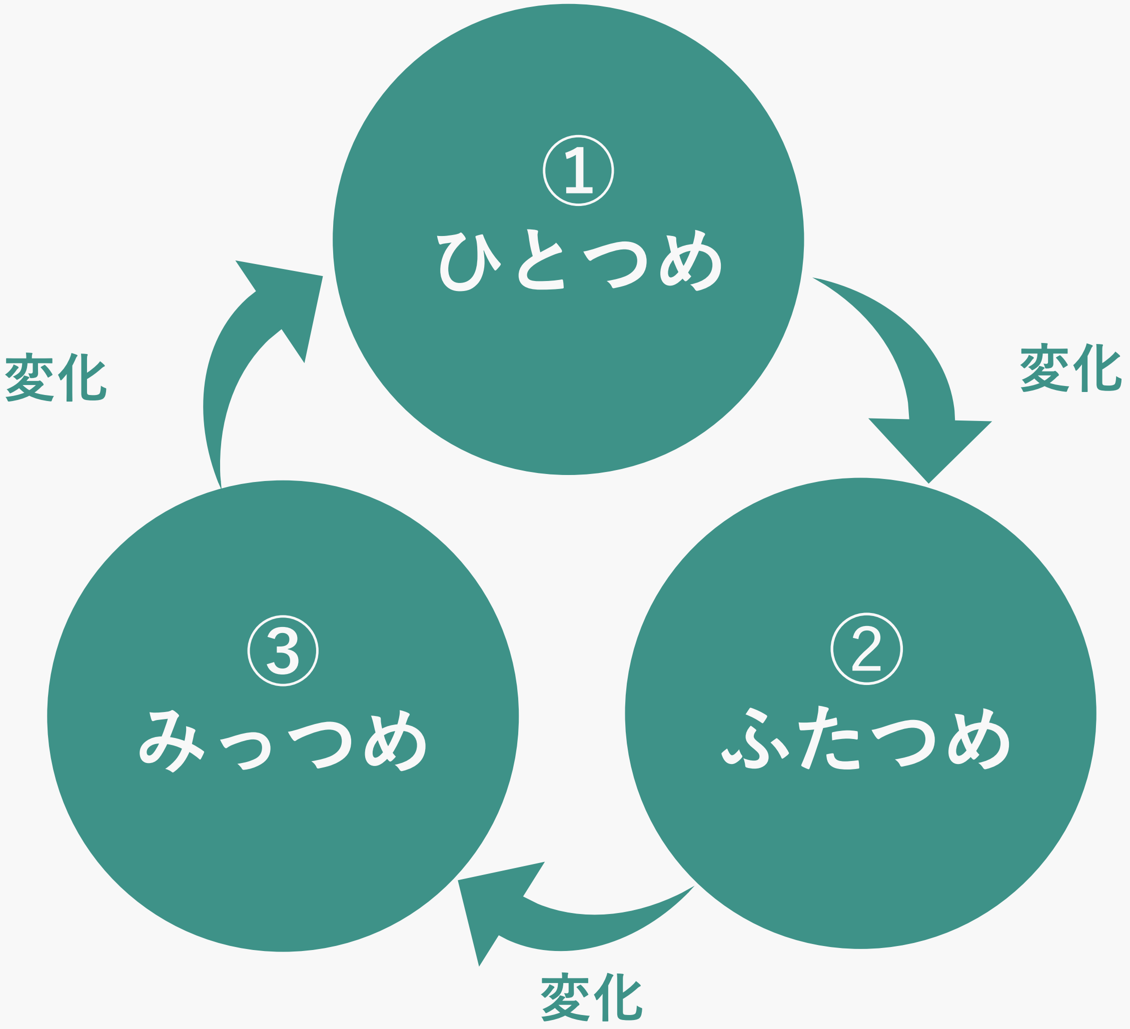
Known q-PMDs.

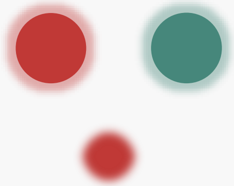
α -sequence	$\mathcal{F}_{r(E)-1}$	description
$(0, 1, \cdots, k-1, n)$	$\mathcal{S}(k-1, k-1, n; q)$	uniform q-matroid $U_{k,n}[\mathbb{F}_q]$
$(0, m, lm)$	$\mathcal{S}(1, m, lm; q)$	split of vector space
$(0,\ 1,\ 3,\ 13)$	$\mathcal{S}(2, 3, 13; 2)$	only known non-trivial q-PMD [1]

$\mathbb{F}_q$

寸法や色は場合に応じて調整しましょう




①
ひとつめ
②
ふたつめ
③
みっつめ
変化
変化
変化




何か
メッセージ

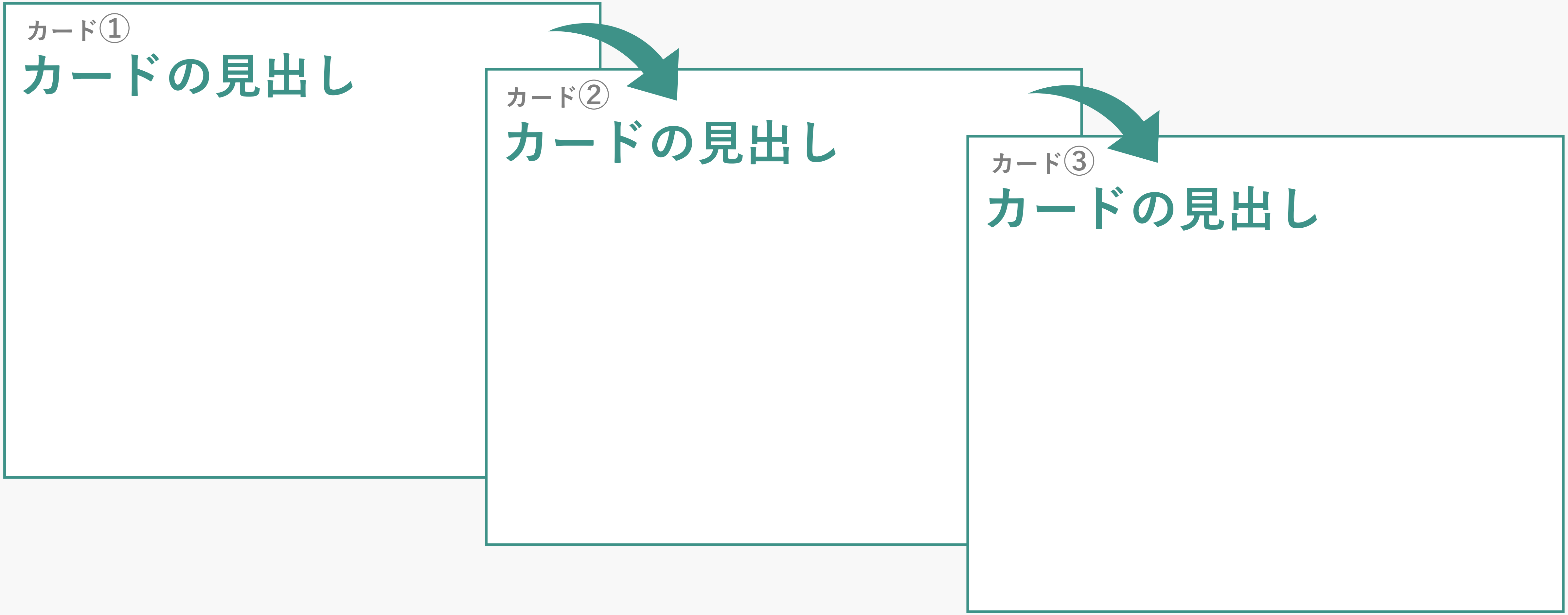
３つあることが一目でわかる

カード①
カードの見出し

カード②
カードの見出し

カード③
カードの見出し

順序関係がある場合は並列よりこちらが良い



カードの間隔やアスペクト比は適宜調整

カード①
カードの見出し

カード③
カードの見出し

カード②
カードの見出し

カード④
カードの見出し

循環がある場合はこんな感じが良い

カード①
カードの見出し

カード③
カードの見出し

カード②
カードの見出し

カード④
カードの見出し