Association Schemes on Triples from Two-transitive Groups

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Contents

Introduction

$$
\begin{aligned}
R_0 &= \{(0,0),(1,1),(2,2),(3,3)\}\\
R_1 &= \{(0,1),(1,2),(2,3),(3,0)\}\\
R_2 &= \{(0,2),(1,3),(2,0),(3,1)\}\\
R_3 &= \{(0,3),(1,0),(2,1),(3,2)\}
\end{aligned}
$$

$$
\begin{tikzcd}
0 \arrow[loop left] & 1 \arrow[loop right] \\
3 \arrow[loop left] & 2 \arrow[loop right]
\end{tikzcd}
\quad
\begin{tikzcd}
0 \arrow[r] & 1 \arrow[d] \\
3 \arrow[u] & 2 \arrow[l]
\end{tikzcd}
\quad
\begin{tikzcd}
0 \arrow[dr, leftrightarrow] & 1 \arrow[dl, leftrightarrow] \\
3 & 2
\end{tikzcd}
\quad
\begin{tikzcd}
0 \arrow[d] & 1 \arrow[l] \\
3 \arrow[r] & 2 \arrow[u]
\end{tikzcd}
$$

$$
A_0 := \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ A_1 := \begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0 \end{pmatrix}
$$

$$
A_2 := \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix},\ A_3 := \begin{pmatrix} 0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix}
$$

Introduction

Association Schemes on Triples [MB90]

- **Higher Dimensional** Object
- **Hypermatrices**
- **Ternary** Algebras

Relation Form

Adjacency Relations

$$R_0 = \{(1,1,1),(2,2,2),(3,3,3)\}$$
$$R_1 = \{(1,2,2),(1,3,3),(2,1,1),(2,3,3),(3,1,1),(3,2,2)\}$$
$$R_2 = \{(2,1,2),(3,1,3),(1,2,1),(3,2,3),(1,3,1),(2,3,2)\}$$
$$R_3 = \{(2,2,1),(3,3,1),(1,1,2),(3,3,2),(1,1,3),(2,2,3)\}$$
$$R_4 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$$

1. The R_i **partition** $\Omega \times \Omega \times \Omega$
2. **Switching** coordinates of R_i yields an R_j.
3. **(Trivial Relations)**
   R_0, R_1, R_2, R_3: relations with triples with **identical** elements

Relation Form

Adjacency Relations

$$R_0 = \{(1,1,1),(2,2,2),(3,3,3)\}$$
$$R_1 = \{(\mathbf{1},2,2),(1,3,3),(2,1,1),(2,3,3),(3,1,1),(3,2,2)\}$$
$$R_2 = \{(2,1,2),(3,1,3),(1,2,\mathbf{1}),(3,2,3),(1,3,1),(2,3,2)\}$$
$$R_3 = \{(2,2,1),(3,3,1),(1,\mathbf{1},2),(3,3,2),(1,1,3),(2,2,3)\}$$
$$R_4 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$$

4. (Third valencies)

$x \neq y$ then $|\{z : (x,y,z) \in R_i\}| = n_i^{(3)}$

$$n_1^{(3)} = 1$$

5. (Intersection numbers)

$(x,y,z) \in R_l$, then

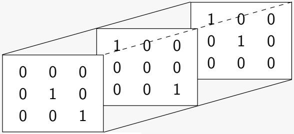
$$|\{w \in \Omega : (w,y,z) \in R_i, (x,w,z) \in R_j, (x,y,w) \in R_k\}| = p^l_{ijk}$$

$$p^1_{132} = 1$$

ASTs as ternary algebras

Hypermatrix Form

1. $R_1 = \{(1,2,2),(1,3,3),(2,1,1),(2,3,3),(3,1,1),(3,2,2)\}$



A_1

2. **Ternary multiplication**: $(ABC)_{ijk} = \sum_w A_{wjk} B_{iwk} C_{ijw}$

3. **Adjacency hypermatrices** satisfy $A_i A_j A_k = \sum_{l=0}^{m} p_{ijk}^{l} A_l$

4. $Span_{\mathbb{C}} \{A_i\}_{i=0}^{m}$ is a **ternary algebra**;
   neither associative nor commutative

TWO-TRANSITIVE GROUPS AND ASTS

ASTs from Two-transitive groups

Two-transitive Groups

G a group acts two transitively on Ω

$a \neq b$ and $x \neq y$

$$(\exists g \in G)\,((g \cdot a, g \cdot b) = (x, y))$$

Two-transitive Actions yield ASTs [MB90]

G a group acting two-transitively on Ω

$\rightarrow$ **Induced action** $\quad g \cdot (x, y, z) := (g \cdot x, g \cdot y, g \cdot z)$

$\rightarrow$ **Orbits** of G on $\Omega \times \Omega \times \Omega$ forms an **AST**

Example: $A\Gamma L(1,8)$

Orbits of $A\Gamma L(1,8)_{0,1}$: $\{\mathfrak{a}, \mathfrak{a}^2, \mathfrak{a}^4\}$, $\{\mathfrak{a}^3, \mathfrak{a}^5, \mathfrak{a}^6\}$

$R_4 = \{(0,1,\mathfrak{a}), (0,1,\mathfrak{a}^2), (0,1,\mathfrak{a}^4) \ldots\}$

$R_5 = \{(0,1,\mathfrak{a}^3), (0,1,\mathfrak{a}^5), (0,1,\mathfrak{a}^6), \ldots\}$

1. Nontrivial R_i has **representative** with form $(0,1,x)$
2. **Other** representatives $(0,1,y) \in R_i$:

$$\{(0,1,y) : y \in A\Gamma L(1,8)_{0,1}(x)\}$$

3.

$$n_4^{(3)} = \left|\left\{\mathfrak{a}, \mathfrak{a}^2, \mathfrak{a}^4\right\}\right| = 3$$

$$n_5^{(3)} = \left|\left\{\mathfrak{a}^3, \mathfrak{a}^5, \mathfrak{a}^6\right\}\right| = 3$$

Sizes of ASTs from Groups

In general [MB90, BB22a]

$$\# \text{ of } \textbf{nontrivial relations} = \# \text{ of } \textbf{orbits}$$

$$\textbf{third valencies} = \textbf{sizes} \text{ of orbits}$$

AST Parameters

Projective Semilinear Group

Projective Space

$PG(2, n)$

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_h : (x_1, x_2, x_3) \neq (0, 0, 0),\ x_i \in GF(n) \right\}$$

$$(\forall \kappa \neq 0) \left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_h = \begin{bmatrix} \kappa x_1 \\ \kappa x_2 \\ \kappa x_3 \end{bmatrix}_h \right)$$

$P\Gamma L(3, n)$

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_h \mapsto A \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \\ \phi(x_3) \end{bmatrix}_h : A \in GL(3, n), \phi \in Gal(GF(n)) \right\}$$

Orbits of two-point stabilizer

$$P\Gamma L(3,n)_{\begin{bmatrix}1\\0\\0\end{bmatrix}_h, \begin{bmatrix}0\\1\\0\end{bmatrix}_h} = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}_h \mapsto \begin{bmatrix}a & 0 & c\\0 & b & d\\0 & 0 & e\end{bmatrix} \begin{bmatrix}\phi(x_1)\\\phi(x_2)\\\phi(x_3)\end{bmatrix}_h : a,b,e \neq 0, \phi \in Gal(GF(n)) \right\}$$

Orbits

Ideal Points: $\left\{ \begin{bmatrix}1\\c\\0\end{bmatrix}_h : c \neq 0 \right\}$

Affine points: $\left\{ \begin{bmatrix}c\\d\\e\end{bmatrix}_h : e \neq 0 \right\}$

Sizes: $n-1,\ n^2$

AST from Projective Groups

Nontrivial Relations

$$R_4 = \left[\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_h, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_h, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_h \right) \right]$$

Third valency: $n-1$

$$R_5 = \left[\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_h, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_h, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_h \right) \right]$$

Third valency: n^2

Computing p^l_{ijk} for $P\Gamma L(3,n)$ AST

Nontrivial relations

$$R_4 = \left[\left(\begin{bmatrix}1\\0\\0\end{bmatrix}_h, \begin{bmatrix}0\\1\\0\end{bmatrix}_h, \begin{bmatrix}1\\1\\0\end{bmatrix}_h\right)\right]$$

Example: $\mathbf{p^4_{444}} = n-2$

$$\left|\left\{\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}_h : \left(\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}_h, \begin{bmatrix}0\\1\\0\end{bmatrix}_h, \begin{bmatrix}1\\1\\0\end{bmatrix}_h\right), \left(\begin{bmatrix}1\\0\\0\end{bmatrix}_h, \begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}_h, \begin{bmatrix}1\\1\\0\end{bmatrix}_h\right), \left(\begin{bmatrix}1\\0\\0\end{bmatrix}_h, \begin{bmatrix}0\\1\\0\end{bmatrix}_h, \begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}_h\right) \in R_4\right\}\right|$$

Computing $p^4_{444} = n - 2$ for $P\Gamma L(3, n)$ AST

First inclusion

$$\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}_h, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_h, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_h \right) \in R_4 = \left[\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_h, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_h, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_h \right) \right]$$

Computing $p^4_{444} = n - 2$ for $P\Gamma L(3, n)$ AST

$$A\left(\begin{bmatrix}1\\0\\0\end{bmatrix}_h, \begin{bmatrix}0\\1\\0\end{bmatrix}_h, \begin{bmatrix}1\\1\\0\end{bmatrix}_h\right) = \left(\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}_h, \begin{bmatrix}0\\1\\0\end{bmatrix}_h, \begin{bmatrix}1\\1\\0\end{bmatrix}_h\right)$$

Column 1 of A: $\begin{bmatrix}az_1\\az_2\\az_3\end{bmatrix}$ **Column 2** of A: $\begin{bmatrix}0\\b\\0\end{bmatrix}$

Proceed

$$\begin{bmatrix}az_1\\az_2+b\\az_3\end{bmatrix} = \begin{bmatrix}d\\d\\0\end{bmatrix} \implies z_3 = 0, \quad az_1 = az_2 + b$$

$$\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}_h \in \left\{\begin{bmatrix}1\\c\\0\end{bmatrix}_h : c \neq 0, 1\right\}$$

Intersection Numbers

Commutative AST from $P\Gamma L(k,n)$

1. $A_4 A_4 A_4 = (n-2)A_4$
2. $A_4 A_4 A_5 = A_4 A_5 A_4 = A_5 A_4 A_4 = 0$
3. $A_4 A_5 A_5 = A_5 A_4 A_5 = A_5 A_5 A_4 = (n-1)A_5$
4. $A_5 A_5 A_5 = \dfrac{n^2(n^{k-2}-1)}{n-1} A_4 + (\dfrac{n^k - 1}{n-1} - 3n)A_5$

Note

$\mathbf{A_4}$ generates **subalgebra**

ASTs from $PGL(k,n)$, $PSL(k,n)$, and $P\Gamma L(k,n)$ **equal** for $k \geq 3$

Affine Semilinear Group

Affine Semilinear Group

$\mathbf{A\Gamma L(2, p^\alpha)}$

Action on $V = (GF(p^\alpha))^2$

$A \in GL(2, n), \quad \phi \in Gal(GF(n)), \quad v \in V$

$$(v, A, \phi) \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := A \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \end{bmatrix} + v$$

Two-point stabilizer

$$A\Gamma L(2, n)_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \end{bmatrix} : b \neq 0, \phi \in Gal(GF(n)) \right\}$$

Orbits of two-point stabilizer

Type 1:

Correspond to **Galois conjugacy classes**

$$\left\{ \begin{bmatrix} a^{p^{\mu}} \\ 0 \end{bmatrix} : 0 \leq \mu < \alpha \right\}$$

(**Size** $\deg_{\mathbb{Z}_p}(a)$)

Type 2:

Corresponds to vectors **linearly independent** from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\left\{ \begin{bmatrix} c \\ d \end{bmatrix} \notin Span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right\}$$

(**Size** $(p^{\alpha})^2 - p^{\alpha}$)

Intersection Numbers

Hypermatrices from Type 1 orbits form subalgebra

$a, b, c \in GF(q) \setminus \{0, 1\}$
T a **transversal** of the orbits of $A\Gamma L(1, p^\alpha)_{0,1}$

$$A^a A^b A^c = \sum_{\ell \in T} p_\ell A^\ell,$$

$$p_\ell = \left| \left\{ c^{p^\mu} : (\exists \kappa, \lambda)(1 - c^{p^\mu}) a^{p^\kappa} + c^{p^\mu} = c^{p^\mu} b^{p^\lambda} = \ell \right\} \right|$$

Other intersection numbers found **similarly**

Computed via equations involving **Galois conjugates**

Sporadic Groups

Sporadic Groups

Computed through GAP

Group	AST Size	$\mathbf{n_i^{(3)}}$	Group	AST Size	$\mathbf{n_i^{(3)}}$
$M(11)$	5	9	$M(11)$ (degree 12)	5	10
$M(12)$	5	10	$PSL(2, 11)$ (degree 11)	6	3, 6
$M(22)$	5	20	A_7 (degree 15)	6	1, 12
$M(23)$	5	21	HS	7	12, 72, 90
$M(24)$	5	22	Co_3	6	112, 162

The ASTs are commutative

Intersection numbers in manuscript [BB22a]

Other Groups

Sizes, third valencies, and intersection numbers

- S_n and A_n
- $PSL(2, n)$
- Other subgroups of $A\Gamma L(k, n)$

Sizes and third valencies

- $PGU(3, q)$ and $PSU(3, q)$
- $Sp(2\ell, 2)$
- $Sz(q)$ and $Ree(q)$

Other Groups

Sizes and third valencies

1. $PGU(3,q)$ and $PSU(3,q)$ on **isotropic** lines
   - **Orbits**: **solutions** of $r + r^q = 1$ and $s + s^q = 0$
2. $Sp(2\ell, 2)$ on **quadratic forms**
   - **Orbits**: **isotropic** vectors orthogonal (or not) to a fixed vector
3. $Sz(q)$ and $Ree(q)$
   - **Orbits**: **solutions** to some lengthy equations

Research Directions

1. **Intersection numbers:**

$$PGU(3,q), PSU(3,q), Sp(2\ell,2), Sz(k), Ree(k),$$

   other subgroups of $A\Gamma L(k,q)$ and $P\Gamma U(k,q)$

2. **Classification** of ASTs over small vertices [BB22c]
3. **Identity** pairs and **inverse** pairs [MB94]
4. Algebraic/Combinatorial AST **structure theory** [Lis71]
5. **Spectral theory** of hypermatrices [GF20]
6. Other **types** and **constructions** of ASTs [PB21, BB22b]

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