On the minimal 2-blocking sets in PG(5,2)

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ABSTRACT

An n-set B in PG(r , q) is a k-blocking set
if every $(r-k)$-space in PG(r , q) meets B in at least one point.

Bono et al.(2021) proved that there are exactly six non-trivial minimal 2-blocking sets in PG(4 , 2) up to projective equivalence.

ABSTRACT

We consider the non-trivial minimal 2-blocking sets in $PG(5,2)$ and their generalizations.

INTRODUCTION

We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension r over the field of q elements $\mathbb{F}_q$.

INTRODUCTION

A j-space is a projective subspace of dimension j in $PG(r, q)$

The 0-*spaces*, 1-*spaces*, 2-*spaces*, 3-*spaces* and $(r-1)$-*spaces* are respectively called points, lines, planes, solids and hyperplanes.

INTRODUCTION

A set of points in PG (r, q) meeting every $(r-k)$- $space$ is called a k-blocking set.

In this presentation, we call this set k-block simply.

INTRODUCTION

A k-*space* in $PG(r,q)$ is the smallest k-block and a k-block containing a k-*space* in $PG(r,q)$ is called trivial.

INTRODUCTION

Let $P_{1,}\ P_{2,} \dots P_{r+1}$ be $r+1$ points of PG (r , 2) in general position.

We call the ($r+2$)-set $\{P_1, P_2, \dots, P_{r+1}, \sum_{i=1}^{r+1} P_i\}$ a skeleton in PG (r , 2).
An elliptic quadric in PG(3,2) is a skeleton.

INTRODUCTION

Denote by $\mathrm{Cone}(\Pi_k, \beta)$ a cone with vertex a k-space Π_k and base β in an s-space Δ skew to Π_k.

INTRODUCTION

Let Q be a point of a minimal k-block B in PG (r, q).
An $(r-k)$ – space Π is called a tangent of B at Q
if $\Pi \cap B = \{Q\}$.
B is minimal if every point of B has a tangent.

INTRODUCTION

Govaerts and Storme proved the following.

Theorem 1

(1) Any smallest non-trivial 1-block in $\mathrm{PG}(r,2)$, $r \geq 3$, is an elliptic quadric in a solid in $\mathrm{PG}(r,2)$.

(2) Every non-trivial minimal 2-block in $\mathrm{PG}(3,2)$ is the complement of an elliptic quadric.

INTRODUCTION

(3) Any smallest non-trivial k-block in $\mathrm{PG}(r, 2)$, $r \geq 3$, with $2 \leq k \leq r-1$ is $\mathrm{Cone}(\Pi_{k-3}, \mathrm{T})$ where T is the set of 10 points consisting of the complement of an elliptic quadric in a solid Δ.

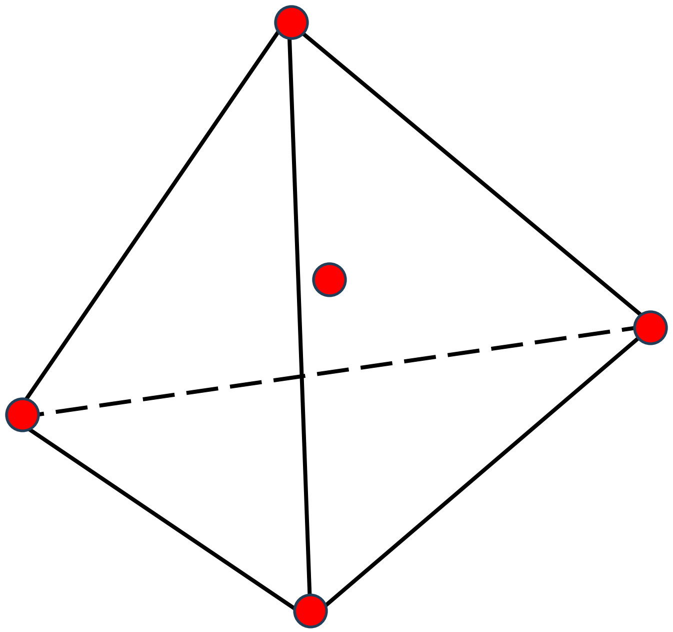
INTRODUCTION

Six non-trivial minimal 2-blocking sets in PG(4, 2) proved by Bono et al.(2021) are the followings.

INTRODUCTION

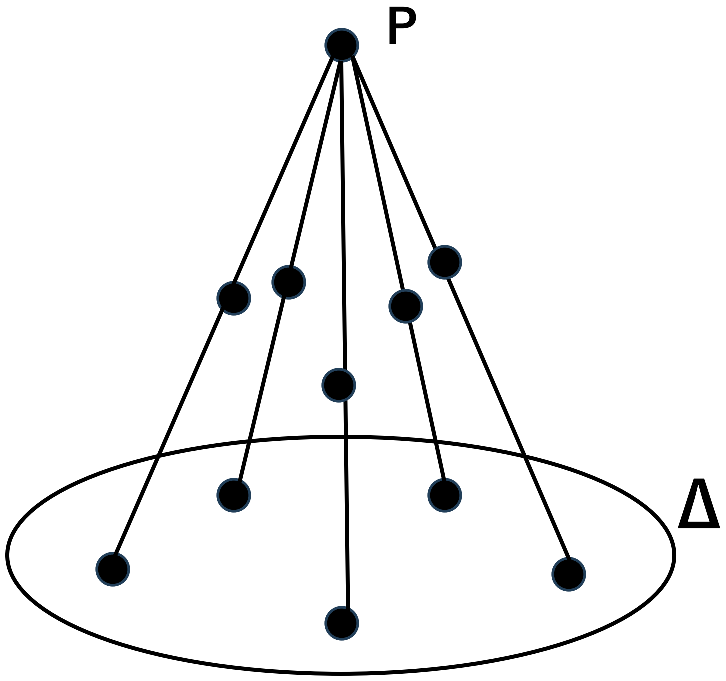
(a) $S_{10} = \Delta \setminus \mathcal{E}_3$ with some solid Δ and a skeleton $\mathcal{E}_3$ in Δ.

● $\notin S_{10}$



INTRODUCTION

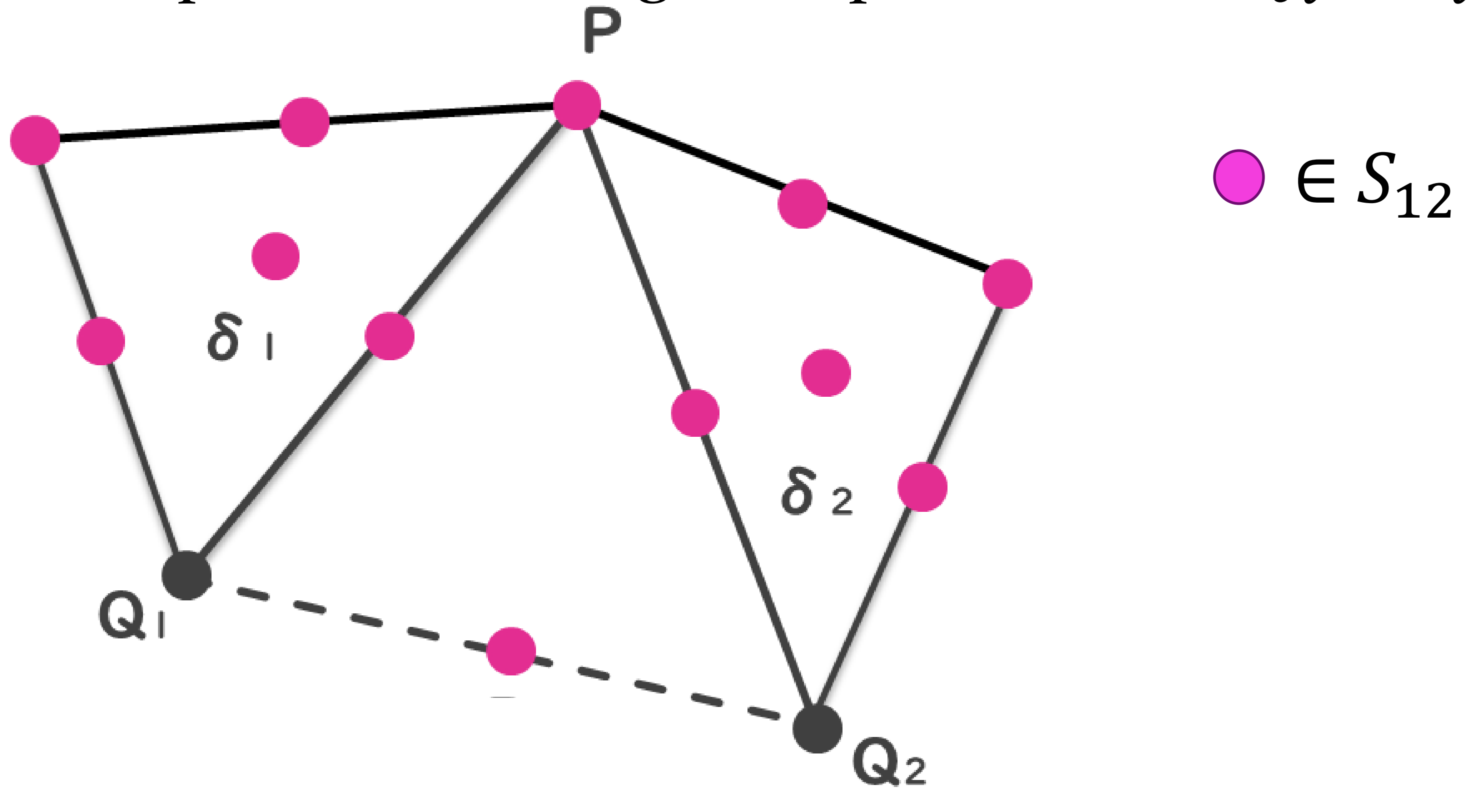
(b) S_{11}: the cone with vertex a point P and base $\mathcal{E}_3$ in Δ not containing P.



●$\in S_{11}$

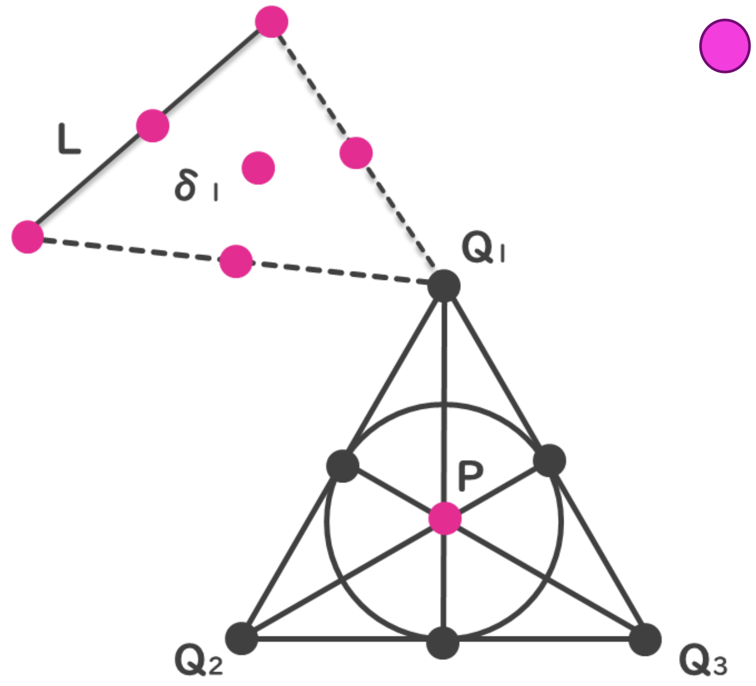
INTRODUCTION

(c) $S_{12} = (\delta_1 \setminus \{Q_1\}) \cup (\delta_2 \setminus \{Q_2\}) \cup \{Q_1 + Q_2\}$ where δ_1, δ_2 are planes meeting in a point P and $Q_i \in \delta_i \setminus \{P\}$ for $i = 1,2$.



INTRODUCTION

(d) $S_{13} = (\{P\} \cup \delta_1 \cup \delta_2 \cup \delta_3) \setminus \{Q_1, Q_2, Q_3\}$ where Q_1, Q_2, Q_3 are not on a line, L is a line which is skew to $\delta = \langle\, Q_1, Q_2, Q_3 \,\rangle$, $\delta_i = \langle Q_i\,, L\rangle$ and $P = Q_1 + Q_2 + Q_3$.

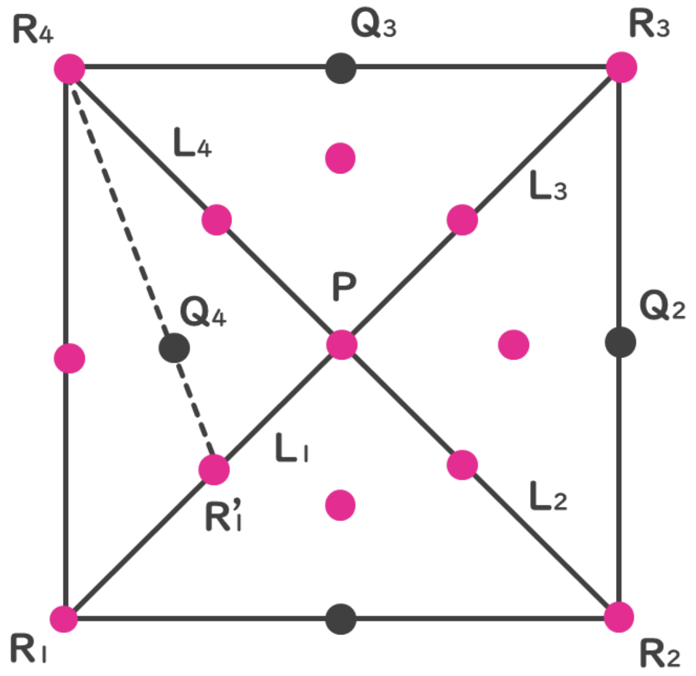


$\in S_{13}$

INTRODUCTION

(e) $S'_{13} = \bigcup_{i=1}^{4}(L_i \cup \{P+Q_i\})$ where Δ is a solid, R_1 is a point out of Δ, and $\{Q_1, Q_2, Q_3, Q_4, P\}$ is a skeleton in Δ.

The lines L_j 's are defined as $L_1 = \{P, R_1, R'_1 = P + R_1\}$ and $L_j = \{P, R_j = R_{j-1} + Q_{j-1}, R'_j = R'_{j-1} + Q_{j-1}\}$ for $j = 2,3,4$.



$\in S_{13}$

INTRODUCTION

(f) $S_{15} = \mathcal{P}_4$ is a parabolic quadric.

On the minimal 2-blocking sets in PG(5,2)

For a given set B, a line l is called an i-line of B if $|B \cap l| = i$.

An i-plane, i-solid and so on are defined similarly.

On the minimal 2-blocking sets in PG(5,2)

We want to find a set B satisfying the following conditions.

・ $|B| = n$
・ $B_1 \cap \Delta \neq \emptyset$ for any solid Δ (2- block)
・ There exists a 1-solid Δ of P for any $P \in B$ (minimal)
・ B contains no plane. (non-trivial)
・ Rank B = 6

On the minimal 2-blocking sets in PG(5,2)

Let s, t, u, δ, Δ, a_i, b_j, c_k and H be the followings.

$s = \ max\{\, s \mid \exists H\!:\ s-$hyperplane$\}$.
$t = \ max\{\, t \mid \exists\Delta\!:\ t-$solid$\}$.
$u = max\{\, u \mid \exists\delta\!:\ u-$plane$\}$.

δ: u- plane, Δ : t- solid, H : s- hyperplane.

$a_i = \#\{\, i - hps\,\}$, $b_j = \#\{\, j - solids\,\}$, $c_k = \#\{\, k - planes\,\}$.

And $\forall \mathrm{P} \in \mathrm{B} \cap \Delta$, $\Delta \subset H_i$ $(H_1, H_2, H_3$: hyperplanes)

On the minimal 2-blocking sets in PG(5,2)

First, we consider range of n.

For $n \leq 12$, Koji Imamura did an exhaustive computer search and there is no 2-block in PG(5,2).
Considering the 0-planes in 1-solds, we get $n \leq 21$.
From the above, $13 \leq n \leq 21$.

On the minimal 2-blocking sets in PG(5,2)

Next, we consider range of t.

Since B is minimal and since two different solids meet in a line or plane in $PG(5,2)$,
for $\forall P \in B \cap \Delta$, there is a 1-line through P in Δ.

On the minimal 2-blocking sets in PG(5,2)

Since B is non-trivial, $t \leq 12$.

For $t = 12$, $\Delta \setminus B = line$ and there is no 1-line through P in Δ, a contradiction.

For $t = 10$ *or* 11, if $\Delta \setminus B$ contains a $line$, then it is same with $t = 12$.

On the minimal 2-blocking sets in PG(5,2)

For $t = 10$, assume $\Delta \setminus B$ contains no *line*.

Then B is complement of skeleton in Δ and rank B is 4.

Therefore $t \leq 9$.

On the minimal 2-blocking sets in PG(5,2)

For $t = 4$, there is no $[7,5,3]_2$code, so $s \leq 6$ and then $n \leq (6-4) \times 3 + 4 = 10$.
For $t = 5$, there is no $[8,5,3]_2$code, so $s \leq 7$ and then $n \leq (7-5) \times 3 + 5 = 11$.
These two are not satisfying $13 \leq n$, thus $6 \leq t$.

On the minimal 2-blocking sets in PG(5,2)

From the above,

$13 \leq n \leq 15 \Rightarrow 6 \leq t \leq 9.$

$16 \leq n \leq 21 \Rightarrow 7 \leq t \leq 9.$

On the minimal 2-blocking sets in PG(5,2)

There are four different 6-solids.

$(a1)$ $\delta_5 \cup \{P\}$ $(5-$plane δ_5, P $\notin \delta_5)$.

$(a2)$ $\Delta \setminus B = \delta \cup \{P_1, P_2\}$ (0-plane δ, 0-pts P_1,P_2).

$(a3)$ skeleton $\cup$ {P} (3-line is only one through P).

$(a4)$ $l_1 \cup l_2$ ($l_1 \cap l_2 = \emptyset$).

On the minimal 2-blocking sets in PG(5,2)

There are five different 7-solids.

$(b1)$ $\delta_6 \cup \{\mathrm{P}\}$(6-plane $\delta_6, \mathrm{P} \notin \delta_6$).

$(b2)$ $\Delta \setminus B = \delta \cup \{\mathrm{P}\}$(0-plane δ, 0-pt $\mathrm{P} \notin \delta$).

$(b3)$ $\delta_5 \cup \{P_1, P_2\}$ (5-plane δ_5,

line $\langle P_1, P_2\rangle$ meets δ_5 at 0-pt.)

$(b4)$ 3 lines meeting in a point and each line are

not coplanar.

$(b5)$ $l_1 \cup l_2 \cup \{\mathrm{P}\}$ (l_1, l_2: skew lines, $\mathrm{P} = P_1 + P_2$, $P_i \in l_i$).

On the minimal 2-blocking sets in PG(5,2)

There are two different 8-solids.

$(c1)$ $\Delta \setminus B = l_1 \cup l_2 \cup \{\mathrm{P}\}$(complement of (b5)).

$(c2)$ 6-plane and 5-plane meeting in a 3-line.

On the minimal 2-blocking sets in PG(5,2)

There are two different 9-solids.

($d1$) $\Delta \setminus B = l_1 \cup l_2$ (l_1, l_2: skew lines).

($d2$) Two 6-planes meeting in a 3-line.

On the minimal 2-blocking sets in PG(5,2)

We explain how to add $(n-t)$ points to t-solid Δ and find B with the aid of a computer.

We assume $H_1 = [0,0,0,0,1,0]$, $H_2 = [0,0,0,0,0,1]$, $H_3 = [0,0,0,0,1,1]$.

Let n_i be the number of points of B selected from $H_i \setminus \Delta$. We may assume $n_1 \geq n_2 \geq n_3$.

On the minimal 2-blocking sets in PG(5,2)

For example, for $n = 13$ and $|B \cap \Delta| = 6$

We need to add $(n - t) = 7$ points with $n_1 \geq n_2 \geq n_3$.

(n_1, n_2, n_3)=(3,3,1),(3,2,2)

(4,3,0),(4,2,1),

(5,2,0),(5,1,1),

(6,1,0),

We need to check 7 $\times$ 4 (different 6-solids)= 28 casess.

On the minimal 2-blocking sets in PG(5,2)

This table shows the number of non-trivial minimal 2-blocks in PG(5,2).

size	(s,t,u)	#
13	(9,7,6)	3
14	(10,8,6)	2
15	(10,7,5)	1
	(11,7,5)	1
	(10,8,6)	1
	(11,8,6)	6
	(11,9,6)	12

size	(s,t,u)	#
16	(10,7,5)	1
	(11,8,5)	1
	(12,8,5)	1
	(12,9,6)	30
17	(12,8,5)	3
	(13,8,5)	2
	(13,9,5)	4

size	(s,t,u)	#
18	(12,8,5)	6
	(12,8,6)	2
	(12,9,6)	1
	(14,9,5)	2
	(14,9,6)	1
19	(13,9,5)	3
20	(12,8,5)	1

$s = max\{ s \mid \exists H\colon s-\text{hyperplane}\}$, $t = max\{ t \mid \exists \Delta\colon \text{t}-\text{solid}\}$, $u = max\{ u \mid \exists \delta\colon \text{u}-\text{plane}\}$

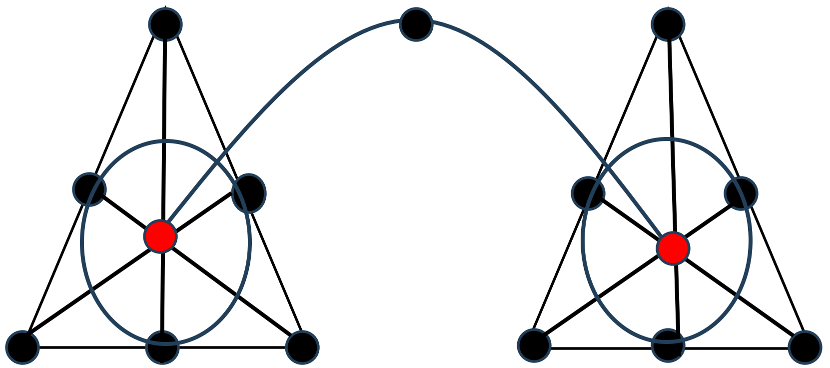
GENERALIZATIONS

For n = 13, there are three different 2-blocks in $PG(5,2)$.

The followings are figures of the 2-blocks B of size 13.

- $\in B$
- $\notin B$

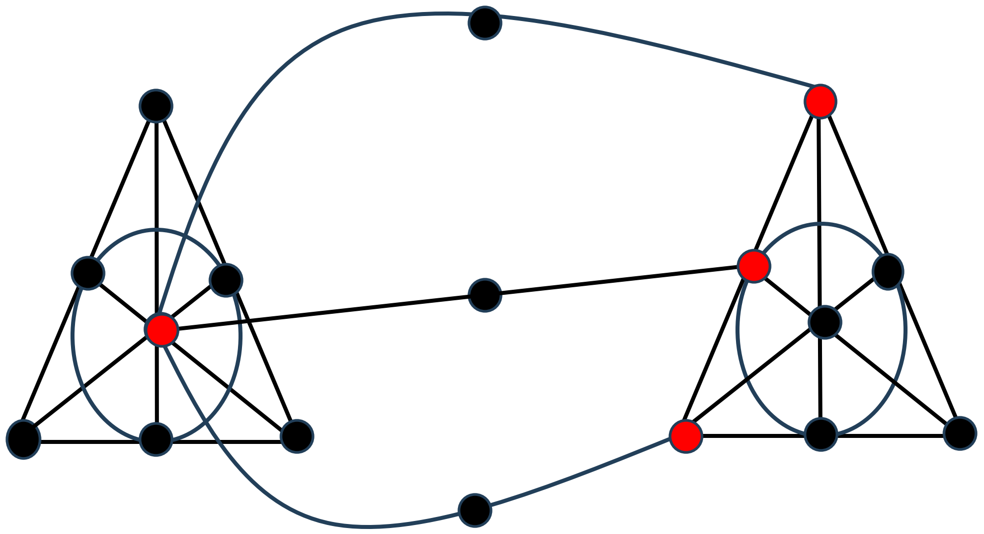
(a)



GENERALIZATIONS

(b) ● $\in B$

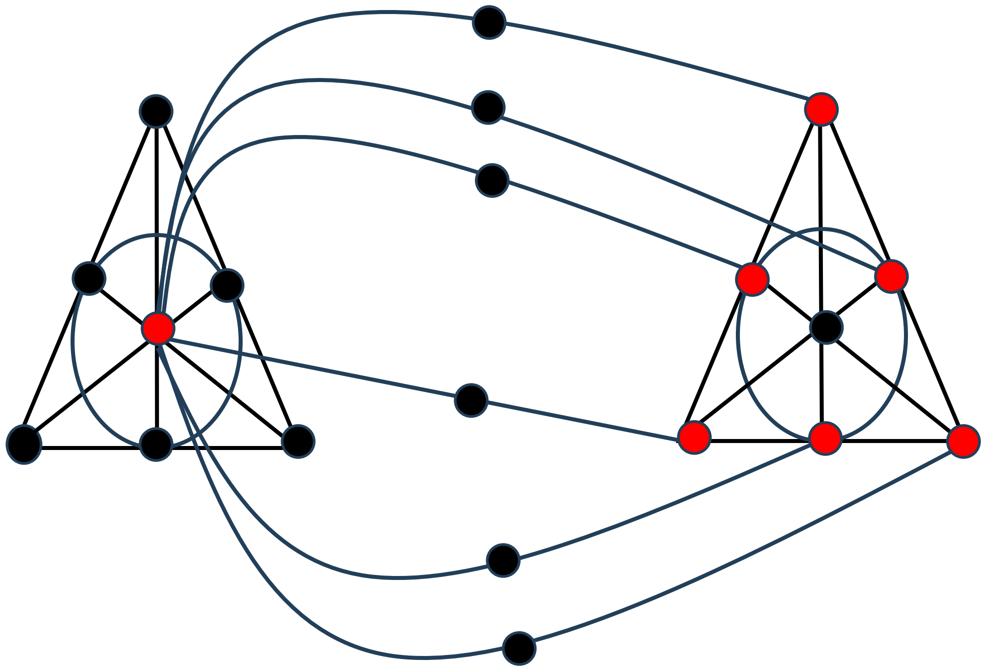
● $\notin B$



GENERALIZATIONS

(c) $\bullet \in B$

$\bullet \notin B$



GENERALIZATIONS

We construct a minimal k-block B in $\Sigma = \mathrm{PG}(r, 2)$ from two disjoint k-spaces B_1 *and* B_2.
For a point P_1 *of* B_1 and $T \subset B_2$,
we denote by $(B_1; P_1) + (B_2; T)$ the set

$$(B_1 \setminus \{P_1\}) \cup (B_2 \setminus T) \cup \{P_1 + R \mid R \in T\}$$

GENERALIZATIONS

Theorem 2

Let B_1 , B_2 be k-spaces in $\Sigma = \mathrm{PG}\,(\,r\,,2\,)$ with $B_1 \cap B_2 = \emptyset$,
$r = 2k + 1.$

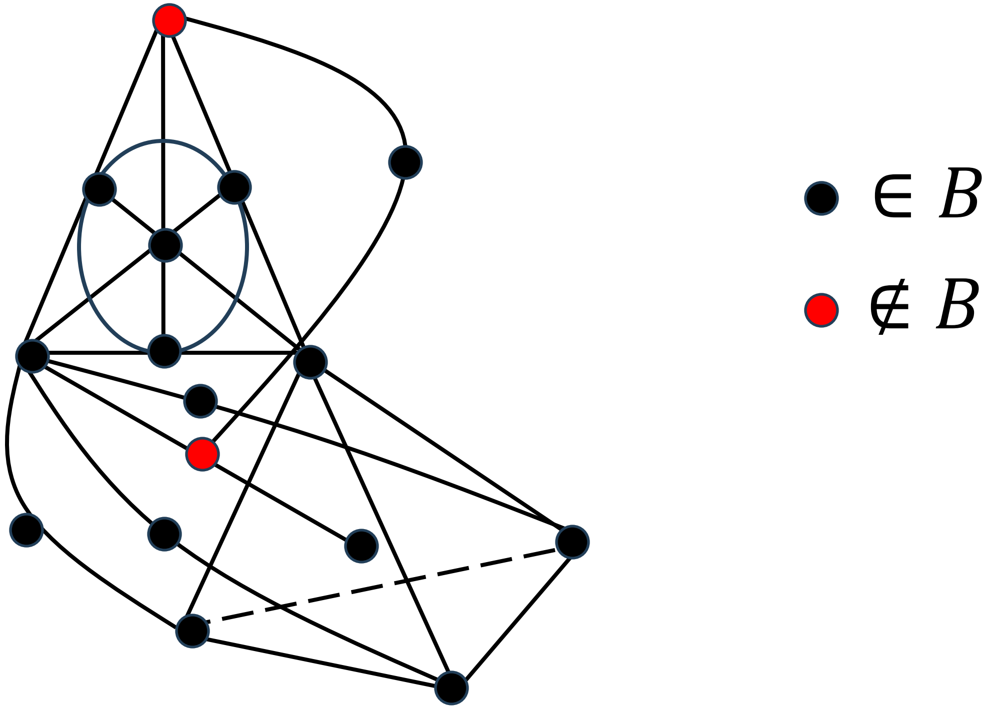
Let P_1 be a point of B_1 and T be a subset of B_2 , T $\neq B_2$.

Then , B $= (\,B_1\,;\,P_1\,) + (\,B_2\,;\,T\,)$

is a non-trivial minimal k-block in Σ.

FUTURE WORK

The figure of the 2-block of size 14 is more complicated than the figures of the three 2-blocks of size 13.



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[2] N. Bono, T. Maruta, K. Shiromoto, On the non-trivial minimal blocking sets in binary projective spaces, Finite Fields Appl. 72(2021), 101814

[3] P. Govaerts, L. Storme, The classication of the smallest nontrivial blocking sets in PG(n, 2), J. Combin. Theory Ser. A 113 (2006) 1543-1548.

- (b1) 6-plane $\delta_6 + P$ (P $\notin \delta_6$)
- (b2) $\Delta \setminus B = \delta + P$ (0-plane δ, 0-pt P $\notin \delta$)
- (b3) 5-plane $\delta_5 + P_1 + P_2$ (line $\langle P_1 , P_2 \rangle$ meet δ_5 at 0-pt.
- (b4) 1点を共有する同一plane上にないline3本
- (b5) $l_1 + l_2 +$ P (l_1, l_2: skew lines, P $= P_1 + P_2$, $P_i \in l_i$)
- (b4) 3 lines having same one point and each line are not on same planes.

We explain how to add (n-t)points to t-solid Δ and find B.

We define H1＝[0,0,0,0,1,0], H2=[0,0,0,0,0,1], H3=[0,0,0,0,1,1], and Let ni be points added to t-solid Δ from Hi. Then we can consider as $n_1 \geqq n_2 \geqq n_3$.

- We consider projective transformation like $(x_1, x_2, x_3, x_4, x_5, x_6) \to (x_1, x_2, x_3, x_4, x_6, x_5)$ and
- $(x_1, x_2, x_3, x_4, x_5, x_6) \to (x_1, x_2, x_3, x_4, x_5, x_5 + x_6)$
- [0,0,0,0,0,1] [0,0,0,0,1,1] [0,0,0,0,1,0]

コピペ用

$(a) \in \{P_1 + R \mid R \in T\} \subset \mathrm{B}\ f : \mathrm{B}_1 \to \mathrm{B}_2\ \mathbb{F}_q\ \ S_{10}$

$(\, r \,, 2\,)\ \mathrm{B} = (\, B_1; P_1) + (\, B_2; T)\ (\, B_1 \setminus \{\, P_1 \,\}\,) \cup (\, B_2 \setminus T\,)$

$$\Sigma = PG\,(\, r \,, 2\,)\ j - space\ \langle P_1 \rangle$$

$$\in\ \ \notin B$$

Σ

$r = 2k + 1$

$B_1 \cap B_2 = \emptyset$

$\neq$

$\mathrm{B} = (\, B_1; P_1) + (\, B_2; T)$

Q

$r - k = k + 1$

游明朝

コピペ用

- ・$|B|=n$
- ・$B \cap \Delta \neq \emptyset$ for any solid Δ （2-block）
- ・there exists a 1-solid Δ of P for any $P \in B$ （minimal）
- ・B contains no plane (non-trivial)
- ・rank B =6

$s = max\{ s \mid \exists H\colon\ s-\text{hyperplane}\}.$
$t = max\{ t \mid \exists \Delta\colon\ \text{t}-\text{solid}\}.$
$u = max\{ u \mid \exists \delta\colon\ \text{u}-\text{plane}\}.$

Lemma 1

Let Π and Π' be disjoint subspace of $PG(r, 2)$.
For any subset S of Π' and a point P of Cone (Π, S) with
$P \notin \Pi \cup S$, there is a unique line through P in the cone.

コピペ用

- 13≦ n ≦15のとき6≦ t ≦9となり、 n ≧16のときは７≦ t ≦9となる。
- (c1) Δ＼B = l_1+l_2＋P ((b5)の補集合) $\Delta \setminus B = l_1 + l_2 + P$ (complement of (b5))
- (c2)　3-lineを共有する 6-plane と 5-plane
- 6-plane and 5-plane having same one 3-line.
- (d1) $\Delta \setminus B = l_1 + l_2$ (l_1, l_2: skew lines)
- (d2) Two 6-plane having same one 3-line.
- {P} ∪

Sakoda Lemma

Let Π_s and Π_s' be disjoint s-space of PG (r, q).
Then there is a s-space in $(\Pi_s \vee \Pi_s') \setminus (\Pi_s \cup \Pi_s)$.

minimal

Let Q be a point of a minimal k-block B in Σ =PG $(r, 2)$.
An $(r-k)$ – space Π is called a tangent of B at Q
if $\Pi \cap B = \{Q\}$.
B is minimal if every point of B has a tangent.

proof

Since $r-k=k+1$, every $(r-k)$-space of Σ meets B_1 *and* B_2 in at least one point.

Hence , B is a k-block in Σ since $\{ P_1 + R \mid R \in T \} \subset$ B.

(k-block)

It is obvious that B contains no k-space.

(non-trivial)

For any point Q of B ,
we shall show the existence of a tangent of B at Q.

For an isomorphism f : $\mathrm{B}_1 \to \mathrm{B}_2$,
let $B_1 + f(B_1) := \{P + f(P) \mid P \in \mathrm{B}_1\}$.