Group divisible designs with block size three and two group sizes

Yudhistira Andersen Bunjamin

joint work with Oden Petersen

and under the supervision of
Catherine Greenhill, Diana Combe and Julian Abel

45th Australasian Combinatorics Conference

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and why 2 is the devil

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1 2	3 4	5 6

$\{1, 3, 5\}$

$\{1, 4, 6\}$

$\{2, 3, 6\}$

$\{2, 4, 5\}$

Group divisible designs

Here is an example of a 3-GDD (i.e. a GDD with **block size** 3).
The **points** are $1, 2, 3, 4, 5, 6$.
The **groups** of size 2 are $\{1, 2\}, \{3, 4\}, \{5, 6\}$.
The **blocks** are

$$\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}.$$

An **incidence matrix** for this 3-GDD is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hdashline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

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Group types

Definition

The *group type* (or *type*) of a k-GDD is the multiset $\{|G| : G \in \mathcal{G}\}$ of group sizes.

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We usually use "exponential" notation for the groups.

Example

A 4-GDD with group sizes $\{2, 2, 2, 5, 5, 5, 5\}$ is of type $2^3 5^4$
(i.e. it has three groups of size 2 and four groups of size 5).

Necessary conditions on GDDs

Theorem (Colbourn 1993)

Suppose there exists a 3-GDD of type $\{g_1, g_2, \ldots, g_m\}$ and $g_i > 0$ for $i = 1, 2, \ldots, m$. Set $v = \sum_{i=1}^{m} g_i$ to be the number of points in the 3-GDD. Then

1. $m \geq 3$;
2. $v \equiv g_i \pmod 2$ *for* $i = 1, 2, \ldots, m$;
3. $\binom{v}{2} \equiv \sum_{i=1}^{m} \binom{g_i}{2} \pmod 3$;
4. $g_1 \leq \sum_{i=3}^{m} g_i$;
5. *whenever* $\alpha_i \in \{0, 1\}$ *for* $1 \leq i \leq m$ *and* $v_0 = \sum_{i=1}^{m} \alpha_i g_i$, *we have* $$[\text{some really complicated inequality}].$$
6. $2g_2g_3 \geq g_1(g_2 + g_3 - \sum_{i=4}^{m} g_i)$; *and*
7. *if* $g_1 = \sum_{i=3}^{m} g_i$, *then* $2g_3g_4 \geq (g_1 - g_2)(g_3 + g_4 - \sum_{i=5}^{m} g_i)$.

Known families of 3-GDDs

The necessary conditions are known to be sufficient for these families of 3-GDDs:

- $v \leq 60$ (Colbourn 1993)
- g^p (Wilson 1972)
- $g^p n^1$ (Colbourn, Hoffman and Rees 1992)
- $g^p 1^q$ (Colbourn, Cusack and Kreher 1995)
- $g^1 n^1 1^q$ (Bryant and Horsley 2006)
- $g^3 n^2$ (Colbourn, Keranen and Kreher 2016)

Construction of GDDs

There are two general classes of techniques for constructing GDDs:

- Direct construction
- Recursive construction

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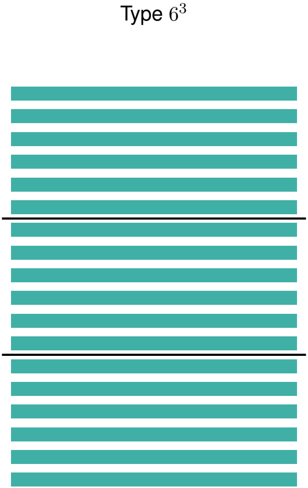
There are two general classes of techniques for constructing GDDs:

- Direct construction
  - by computer search
  - by assuming an automorphism
- Recursive construction
  - fill-in construction
  - Wilson's fundamental GDD construction
  - etc.

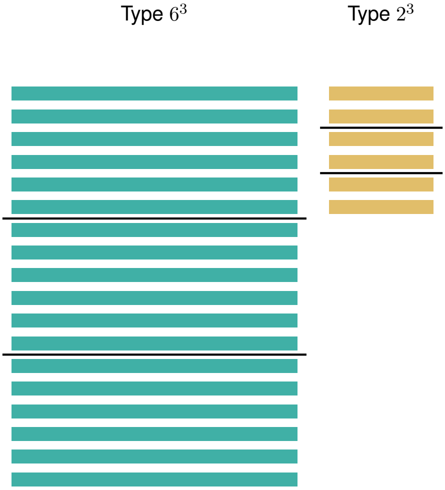
Recursive "fill-in" construction

Start with a 3-GDD of type 6^3.



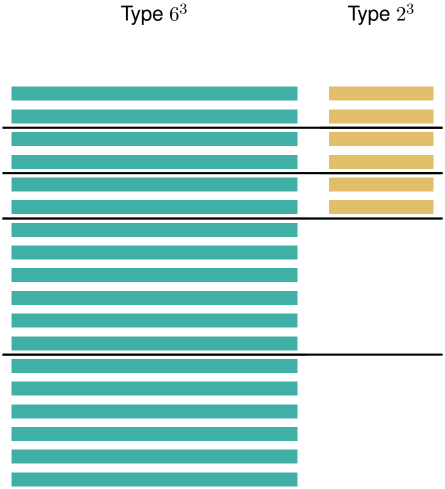
Recursive "fill-in" construction

Fill in a group of size 6 with a 3-GDD of type 2^3.



Recursive “fill-in” construction

This gives a 3-GDD of type $2^3 6^2$.



3-GDDs of type $g^t(gh)^s$

The general goal is to construct 3-GDDs with two group sizes where one group size is a multiple of the other, that is,

3-GDDs of type $g^t(gh)^s$.

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3-GDDs of type $g^t(gh)^s$.

To start with, we will consider 3-GDDs of type $6^t(6h)^s$.

3-GDDs of type $6^t(6h)^s$

$$6^2 \ 18^{11}$$

3-GDDs of type $6^t(6h)^s$

$6^2\ 18^{11}$

$\downarrow$ $\quad 6^2\ (6^3)18^{10}$

3-GDDs of type $6^t(6h)^s$

$$6^2\ 18^{11}$$

$$\downarrow \qquad 6^2\ (6^3)18^{10}$$

$$6^5\ 18^{10}$$

3-GDDs of type $6^t(6h)^s$

$$\begin{array}{cc} 6^2\ 18^{11} & \\ \downarrow & 6^2\ (6^3)18^{10} \\ 6^5\ 18^{10} & \\ \downarrow & 6^5\ (6^3)18^9 \\ 6^8\ 18^9 & \end{array}$$

3-GDDs of type $6^t(6h)^s$

$$\begin{array}{cl}
6^2\ 18^{11} & \\
\downarrow & 6^2\ (6^3)18^{10} \\
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6^{33}18^{1} & \\
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6^{35} &
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3-GDDs of type $6^t(6h)^s$ where $t < h$

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2. If $t \geq 3$, then fill in the group of size $6t$ with a 3-GDD of type 6^t.

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    - if $t = 2$, then ...

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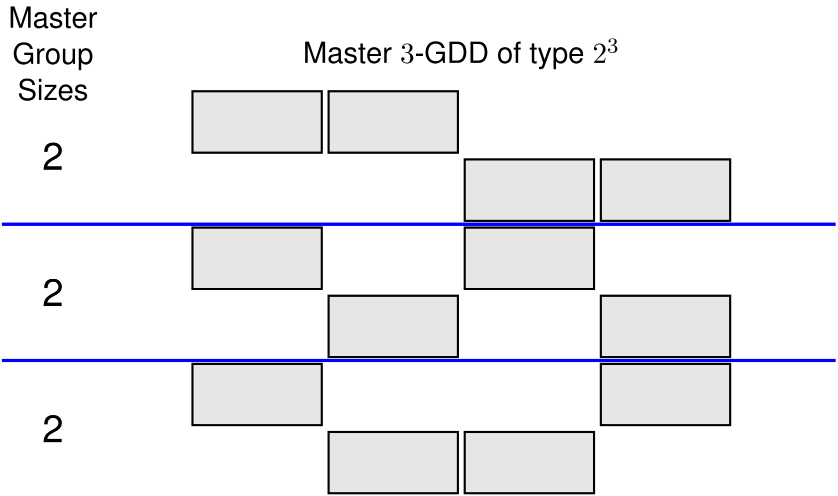
   - if $t = 1$, then we started with a 3-GDD of type $(6h)^s 6^1$,
   - if $t = 2$, then ... [can't fill in type $(6h)^s 12^1$ with type 6^2]

Wilson's Fundamental GDD Construction

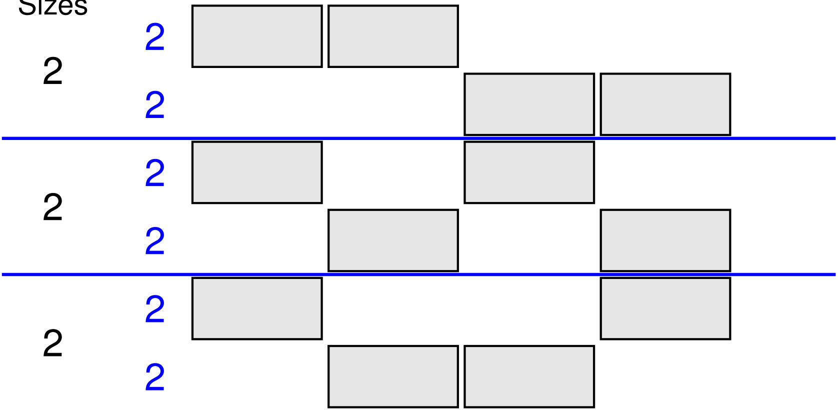


Wilson’s Fundamental GDD Construction

Master Group Sizes

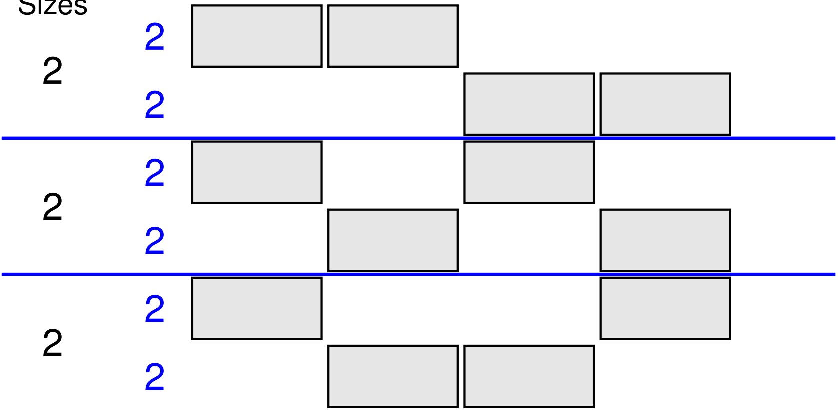
Weight

Master 3-GDD of type 2^3



Wilson's Fundamental GDD Construction

Master Group Sizes | Weight | Master 3-GDD of type 2^3

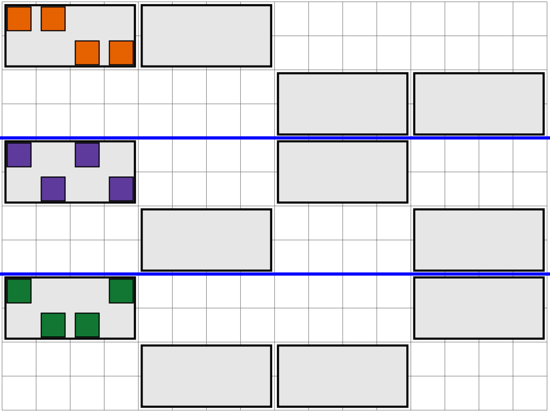


Block Weights: $\{2, 2, 2\}$

Wilson’s Fundamental GDD Construction

Master Group Sizes | Weight | Master 3-GDD of type 2^3

Master Group Sizes	Weight
2	2
	2
2	2
	2
2	2
	2



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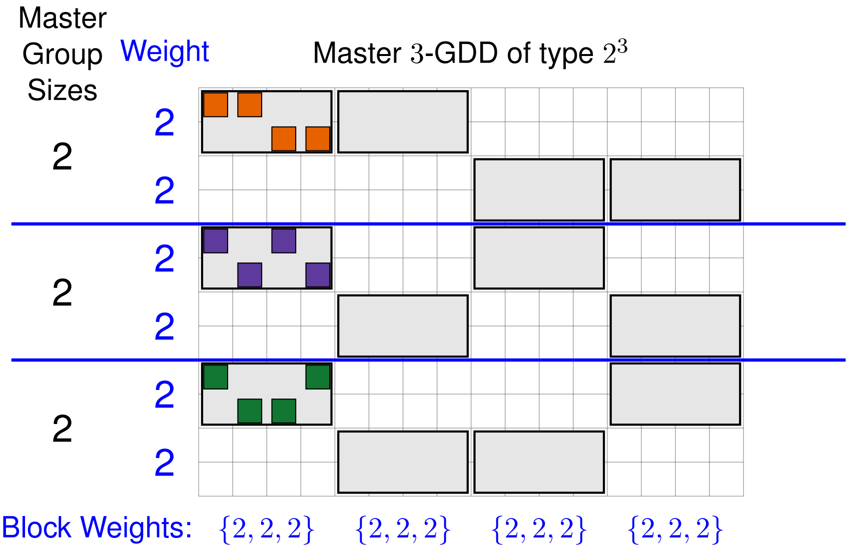

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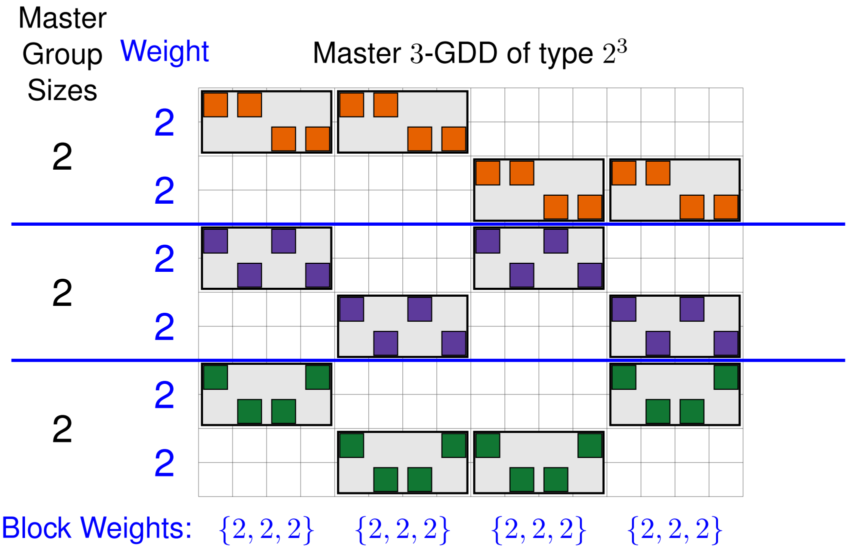
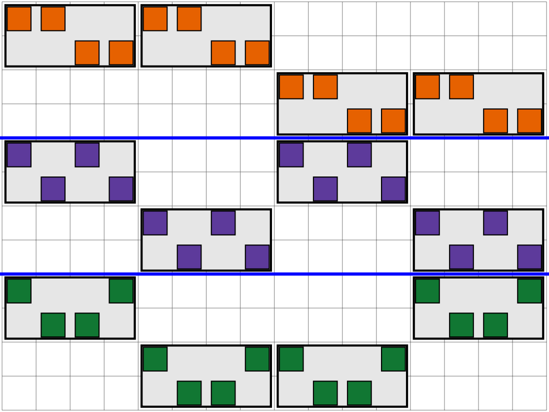


Wilson's Fundamental GDD Construction

Master Group Sizes | Weight | Master 3-GDD of type 2^3 | Target Group Sizes

Master Group Sizes	Weight	Target Group Sizes
2	2, 2	4
2	2, 2	4
2	2, 2	4

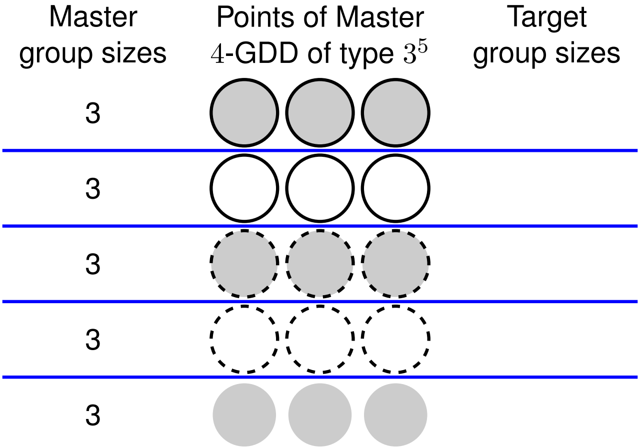


Block Weights: $\{2, 2, 2\}$ $\{2, 2, 2\}$ $\{2, 2, 2\}$ $\{2, 2, 2\}$

3-GDDs of type $6^2(6h)^s$

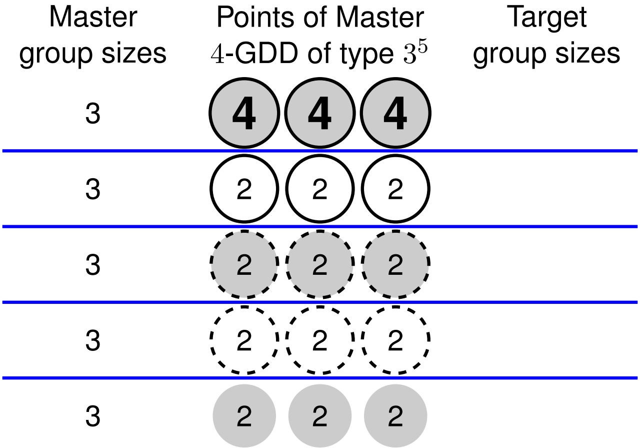
To construct the 3-GDDs of type $6^2(6h)^s$, we use k-GDDs where the block size $k > 3$.

Wilson's GDD construction



Required ingredient 3-GDD types:

Wilson's GDD construction



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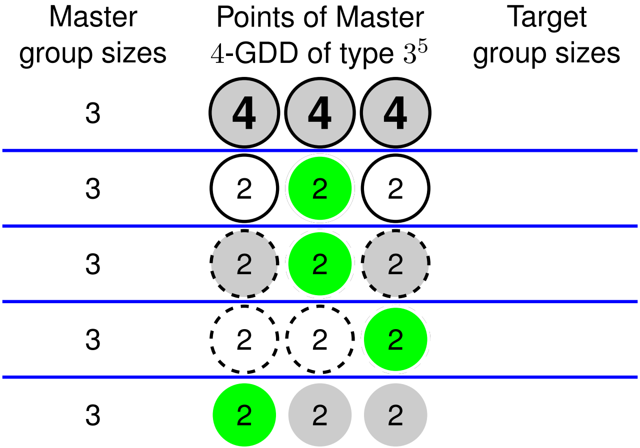

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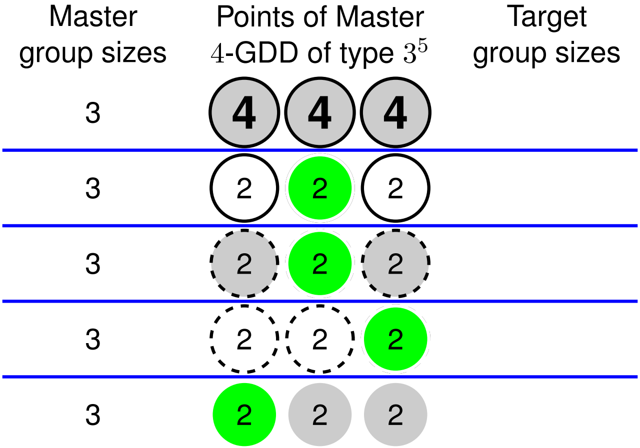


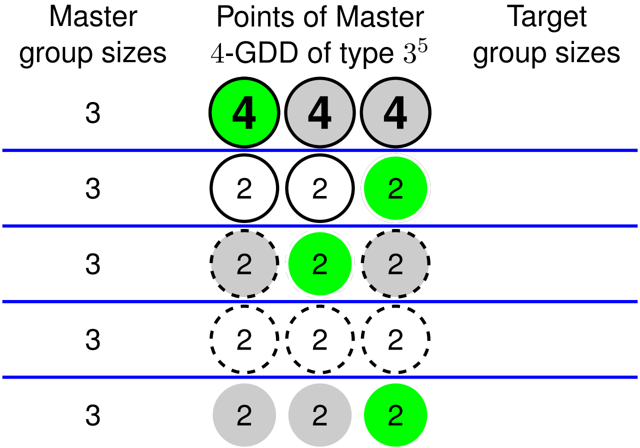
Required ingredient 3-GDD types: $\{2, 2, 2, 2\}$ or 2^4

Wilson's GDD construction

Master group sizes	Points of Master 4-GDD of type 3^5			Target group sizes
3	**4**	**4**	**4**	
3	2	2	2	
3	2	2	2	
3	2	2	2	
3	2	2	2	

Required ingredient 3-GDD types: $\{2,2,2,2\}$ or 2^4

Wilson’s GDD construction



Required ingredient 3-GDD types: $\{2, 2, 2, 2\}$ or 2^4, $\{4, 2, 2, 2\}$ or $2^3 4^1$

Wilson's GDD construction

Master group sizes	Points of Master 4-GDD of type 3^5			Target group sizes
3	**4**	**4**	**4**	12
3	2	2	2	6
3	2	2	2	6
3	2	2	2	6
3	2	2	2	6

Required ingredient 3-GDD types: $\{2, 2, 2, 2\}$ or 2^4, $\{4, 2, 2, 2\}$ or $2^3 4^1$

3-GDDs of type $6^2(6h)^s$

To construct the 3-GDDs of type $6^2(6h)^s$, we use k-GDDs where the block size $k > 3$.

3-GDDs of type $6^2(6h)^s$

To construct the 3-GDDs of type $6^2(6h)^s$, we use k-GDDs where the block size $k > 3$.

We apply Wilson's GDD construction to a combination of

- k-GDDs of type g^k where $k \geq 5$ and
- 4-GDDs of type $g^p n^1$.

3-GDDs of type $6^t(6h)^s$

$$
\begin{array}{ccl}
6^2\ 18^{11} & & \\
\downarrow & & 6^2\ (6^3)18^{10} \\
6^5\ 18^{10} & & \\
\downarrow & & 6^5\ (6^3)18^{9} \\
6^8\ 18^{9} & & \\
\downarrow & & 6^8\ (6^3)18^{8} \\
\vdots & & \\
\downarrow & & 6^{30}(6^3)18^{2} \\
6^{33}18^{1} & & \\
\downarrow & & 6^{33}(6^3)18^{1} \\
6^{35} & &
\end{array}
$$

3-GDDs of type $6^t(6h)^s$

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\begin{array}{cl}
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When $h = 2$

We cannot apply the fill-in construction to type $6^t 12^s$ because there is no 3-GDD of type 6^2.

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Instead, we apply Wilson's construction to

$$4\text{-GDDs of type } 3^t 6^s$$

to get

$$3\text{-GDDs of type } 6^t 12^s.$$

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We cannot apply the fill-in construction to type $6^t 12^s$ because there is no 3-GDD of type 6^2.

Instead, we apply Wilson's construction to

4-GDDs of type $3^t 6^s$

to get

3-GDDs of type $6^t 12^s$.

Theorem (Abel-B.-Combe 2020)

The necessary conditions for the existence of a 4-GDD of type $3^t 6^s$ are sufficient except possibly for a list of twelve group types.

3-GDDs of type $6^t(6h)^s$

Theorem

Suppose that $h \geq 2$.
Then there exists a 3-GDD of type $6^t(6h)^s$ if and only if

1. *either $t \geq 3$ or $s \geq 3$ and*
2. *whenever $v = 6t + 6hs < 18h$, either*
   - $s = 0$ *and* $t \geq 3$, *or*
   - $s = 1$ *and* $t \geq h + 1$.

3-GDDs of type $g^t(gh)^s$

Lemma

Suppose that there exists a 3-GDD of type $g^t(gh)^s$ where $h \geq 2$. Then all of these conditions hold:

1. *either $t \geq 3$ or $s \geq 3$,*
2. *either $g \equiv 0 \pmod 2$ or $t + s \equiv h \equiv 1 \pmod 2$,*
3. *either $g \equiv 0 \pmod 3$ or one of the following holds:*
   - [some long list of mutually-exclusive conditions].

Since we have Wilson's GDD construction, to construct all 3-GDDs of type $g^t(gh)^s$, it suffices to construct all 3-GDDs of types

	$g \not\equiv 0 \pmod 3$	$g \equiv 0 \pmod 3$
$g \not\equiv 0 \pmod 2$	$1^t h^s$	$3^t(3h)^s$
$g \equiv 0 \pmod 2$	$2^t(2h)^s$	$6^t(6h)^s$

3-GDDs of type $2^t(2h)^s$ and $3^t(3h)^s$

Theorem

Suppose that $h \geq 3$. If there exists a 3-GDD of type $3^t(3h)^s$ then the following necessary conditions hold:

1. [a long list of necessary conditions].

These necessary conditions are sufficient except possibly when:

- $t - h \equiv 8 \pmod{12}$ *where* $h \leq t < 2h$ *and* $s = 2$, *or*
- $t \equiv 0, 2, 4, 6$ *or* $10 \pmod{12}$ *where* $0 < t < h$ *and* $s \in \{5, 7\}$, *or*
- $t \equiv 8 \pmod{48}$ *where* $t < h$ *and* $s \in \{3, 5, 7, 9, 11, 23, 25, 27\}$, *or*
- [a few more conditions like the one above,

3-GDDs of type $2^t(2h)^s$ and $3^t(3h)^s$

Theorem

Suppose that $h \geq 3$. If there exists a 3-GDD of type $3^t(3h)^s$ then the following necessary conditions hold:

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These necessary conditions are sufficient except possibly when:

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- $t \equiv 0, 2, 4, 6$ *or* $10 \pmod{12}$ *where* $0 < t < h$ *and* $s \in \{5, 7\}$, *or*
- $t \equiv 8 \pmod{48}$ *where* $t < h$ *and* $s \in \{3, 5, 7, 9, 11, 23, 25, 27\}$, *or*
- [a few more conditions like the one above, involving values of t where $t \equiv 2 \pmod{6}$].

Direct constructions

We used *Stinson's hill-climbing algorithm* to directly construct some remaining 3-GDDs of types $2^t(2h)^s$ and $3^t(3h)^s$.

3-GDDs of type $3^t(3h)^s$ where $h \equiv 3 \pmod 6$

h	Types to construct directly					
3		$3^2 9^5$	$3^2 9^7$			
9	$3^{17}27^2$	$3^2 27^5$	$3^2 27^7$	$3^8 27^3$	$3^8 27^5$	$3^8 27^7$
15	$3^{23}45^2$	$3^2 45^5$	$3^2 45^7$	$3^8 45^3$	$3^8 45^5$	$3^8 45^7$
		$3^{14}45^5$	$3^{14}45^7$			
21	$3^{29}63^2$	$3^2 63^5$	$3^2 63^7$	$3^8 63^3$	$3^8 63^5$	$3^8 63^7$
	$3^{41}63^2$	$3^{14}63^5$	$3^{14}63^7$	$3^{20}63^3$	$3^{20}63^5$	$3^{20}63^7$
27	$3^{35}81^2$	$3^2 81^5$	$3^2 81^7$	$3^8 81^3$	$3^8 81^5$	$3^8 81^7$
	$3^{47}81^2$	$3^{14}81^5$	$3^{14}81^7$	$3^{20}81^3$	$3^{20}81^5$	$3^{20}81^7$
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		$3^{14} 45^5$	$3^{14} 45^7$			
21	$3^{29} 63^2$	$3^2 63^5$	$3^2 63^7$	$3^8 63^3$	$3^8 63^5$	$3^8 63^7$
	$3^{41} 63^2$	$3^{14} 63^5$	$3^{14} 63^7$	$3^{20} 63^3$	$3^{20} 63^5$	$3^{20} 63^7$
27	$3^{35} 81^2$	$3^2 81^5$	$3^2 81^7$	$3^8 81^3$	$3^8 81^5$	$3^8 81^7$
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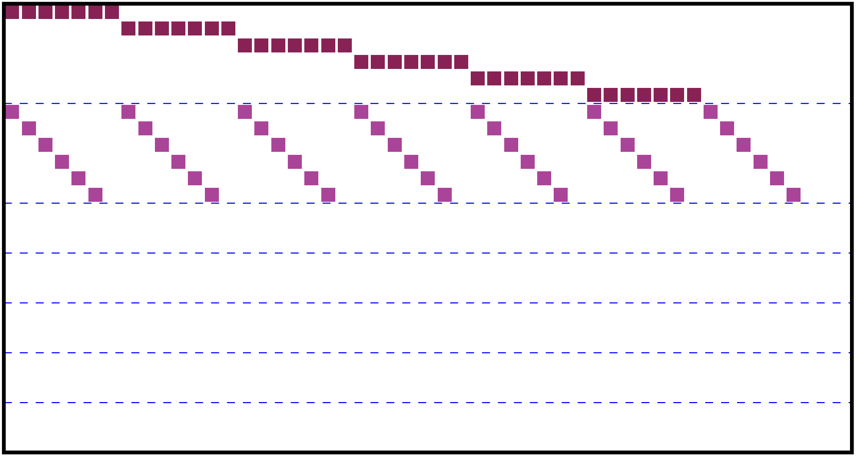
Using the M-Edge approach



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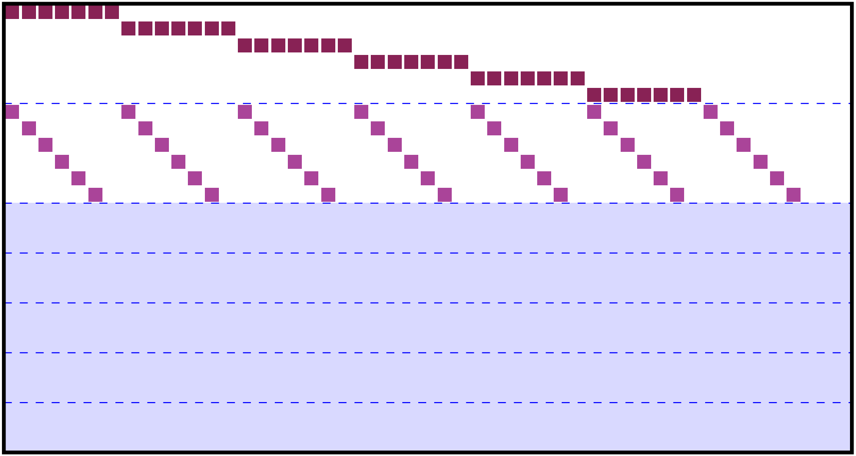

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		$3^{26}81^5$	$3^{26}81^7$			
$\vdots$	$\vdots$					

3-GDDs of type $g^t(gh)^s$

Theorem (B.-Petersen)

For each $g \geq 1$ and $h \geq 2$, the necessary conditions for the existence of a 3-GDD of type $g^t(gh)^s$ are sufficient except possibly for a finite list of group types.

2 is the devil

~~2 is the devil~~

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2 is the joy of being a mathematician

Thank you

References

R.J.R. Abel, Y.A. Bunjamin and D. Combe, Some new group divisible designs with block size 4 and two or three group sizes, *J. Combin. Des.* **28** (2020), 614–628.

D. Bryant and D. Horsley, Steiner triple systems with two disjoint subsystems, *J. Combin. Designs*, **14** (2006).

C. J. Colbourn, Small group divisible designs with block size three, *J. Combin. Math. Combin. Comput.* **14** (1993).

C. J. Colbourn, C. A. Cusack and D. L. Kreher, Partial Steiner triple systems with equal-sized holes. *J. Combin. Theory A*, **70** (1995).

C. J. Colbourn, D. G. Hoffman and R. Rees, A new class of group divisible designs with block size three. *J. Combin. Theory A*, **59** (1992).

C. J. Colbourn, M. S. Keranen and D. L. Kreher, The 3-GDDs of type g^3u^2. *J. Algebra Comb. Discrete Struct. Appl.*, **3** (2016).

R. M. Wilson, An existence theory for pairwise balanced designs I. Composition theorems and morphisms. *J. Combin. Theory A*, **13** (1972).

R. M. Wilson, An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures. *J. Combin. Theory A*, **13** (1972).

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Group divisible designs

Definition

A k-GDD, or *group divisible design* with block size k, is a triple $(X, \mathcal{G}, \mathcal{B})$ where

- X is a set of *points*,
- $\mathcal{G}$ is a partition of X into subsets (called *groups*) and
- $\mathcal{B}$ is a collection of k-element subsets of X (called *blocks*)

such that

- no two distinct points from any group appear together in any block and
- any two points from distinct groups appear together in exactly one block.

Known families of 3-GDDs

Theorem (Colbourn, Hoffman and Rees 1992)

Suppose that $g \geq 1$ and $n \geq 1$ where $g \neq n$. Then there exists a 3-GDD of type $g^p n^1$ if and only if all the following conditions hold:

1. $p \geq 3$;
2. $n \leq g(p-1)$;
3. $g(p-1) + n \equiv 0 \pmod 2$;
4. $gp \equiv 0 \pmod 2$; *and*
5. $gp(g(p-1)/2 + n) \equiv 0 \pmod 3$.

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Theorem

Suppose that $g \equiv 0 \pmod 6$. Then there exists a 3-GDD of type g^p if and only if $p \geq 3$.