Explicit $K_{3,3}$-subdivisions of Markoff mod p graphs

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Joint work with Yoshinori Yamasaki (Ehime University)

Markoff equation & Vieta operations

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- $\mathcal{M}(\mathbb{Z}_{\geq 0}) := \{(x, y, z) \in (\mathbb{Z}_{\geq 0})^3 \mid x^2 + y^2 + z^2 - xyz = 0\}$

- $\mathcal{M}^*(\mathbb{Z}_{\geq 0}) := \mathcal{M}(\mathbb{Z}_{\geq 0}) \setminus \{(0,0,0)\}$

Markoff equation & Vieta operations

- Vieta operation: an involution $R_i : \mathcal{M}^*(\mathbb{Z}_{\geq 0}) \to \mathcal{M}^*(\mathbb{Z}_{\geq 0})$ $(i = 1,2,3)$ s.t.

$$R_1(x, y, z) \coloneqq (yz - x, y, z)$$

$$R_2(x, y, z) \coloneqq (x, xz - y, z)$$

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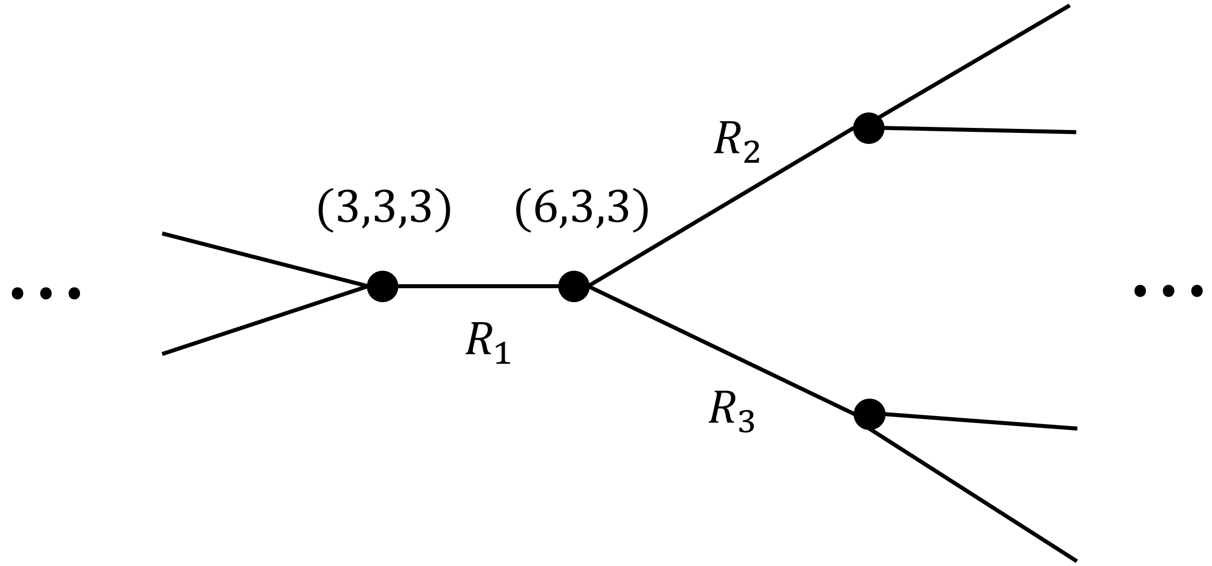

- The above infinite 3-regular tree has vertex set $\mathcal{M}^*(\mathbb{Z}_{\geq 0})$. (Markoff 1879, 1880)

Markoff mod p graph G_p

- $p > 3$: prime
- $\mathcal{M}(\mathbb{F}_p) := \{(x, y, z) \in \mathbb{F}_p^3 \mid x^2 + y^2 + z^2 - xyz = 0\}$
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- $\mathcal{M}^*(\mathbb{F}_p) \coloneqq \mathcal{M}(\mathbb{F}_p) \setminus \{(0,0,0)\}$
- $V(G_p) \coloneqq \mathcal{M}^*(\mathbb{F}_p)$
- $E(G_p) \coloneqq \{(X, Y) \in \left(\mathcal{M}^*(\mathbb{F}_p)\right)^2 \mid R_i(X) = Y \text{ for some } i = 1,2,3\}$

Recall:

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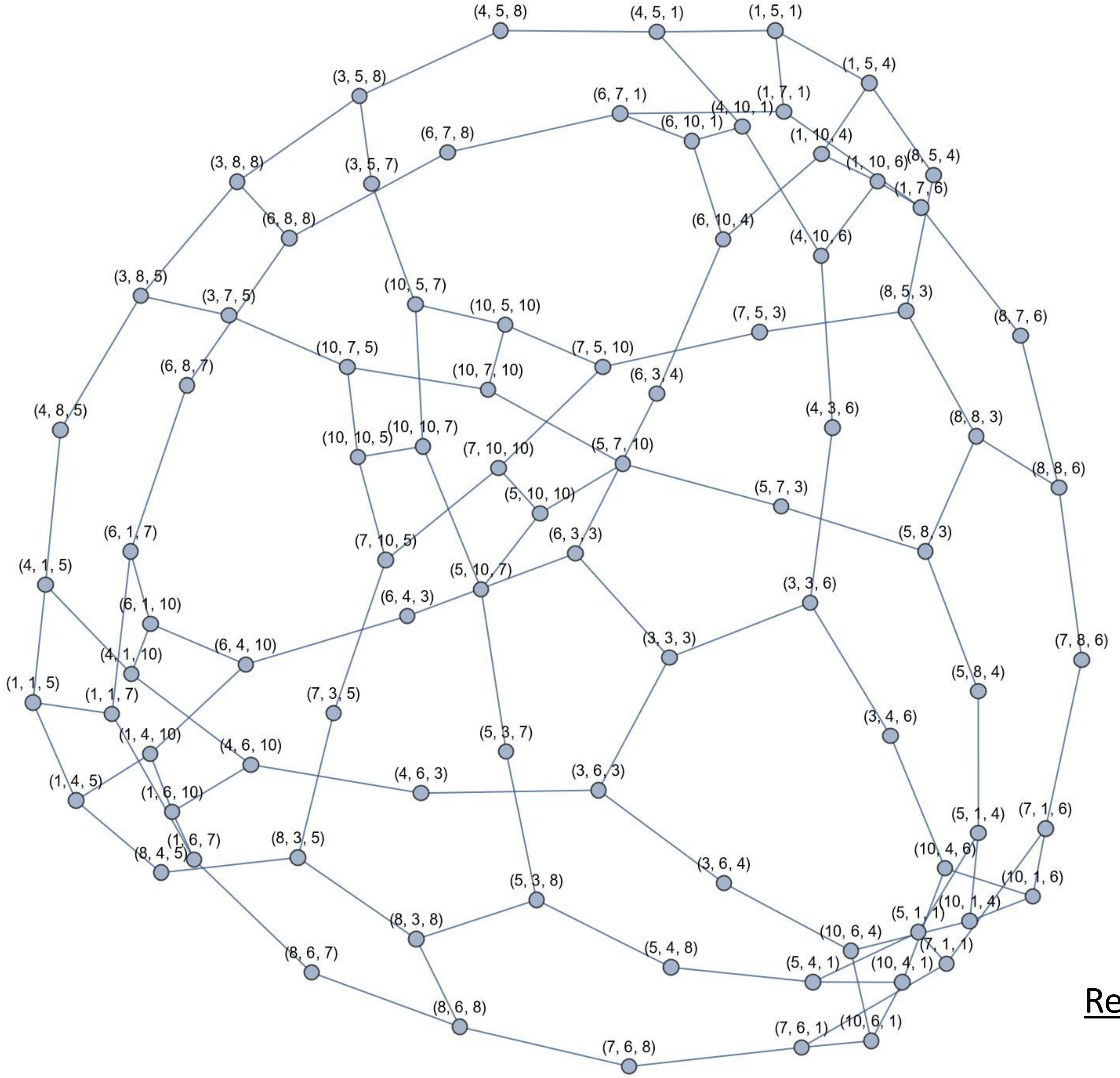
G_{11}



<u>Rem</u> Vertex with degree $= 2$ has one loop.

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<u>Conjecture</u> (Bourgain-Gamburd-Sarnak 2016)

$\{G_p\}_{p>3:\mathrm{prime}}$ is an expander (i.e. the spectral gap is a positive constant).

Non-planarity of G_p

<u>Theorem</u> (Courcy-Ireland 2021)

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- For small p: construct explicit $K_{3,3}$-subdivisions.
  (systematic constructions are exhibited for p that either $p \equiv 1 \bmod 4$ or $\sqrt{-7} \in \mathbb{F}_p$)
- There are infinitely many primes p that there is NO known systematic & explicit constructions of $K_{3,3}$-subdivisions in G_p.
  E.g. $p = 19$ (Courcy-Ireland found a $K_{3,3}$-subdivision by trial and error.)

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- Our thm holds for:

$$p \equiv 6, 11, 14, 19, 24, 26, 29, 34, 44, 54, 56, 69, 71, 76, 79, 89, 94, 96, 99, 101, 104, 106, 109, 111, 116, 126, 129, 134, 136, 149, 151, 161, 171, 176, 179, 181, 186, 191, 194, 199 \pmod{205}$$

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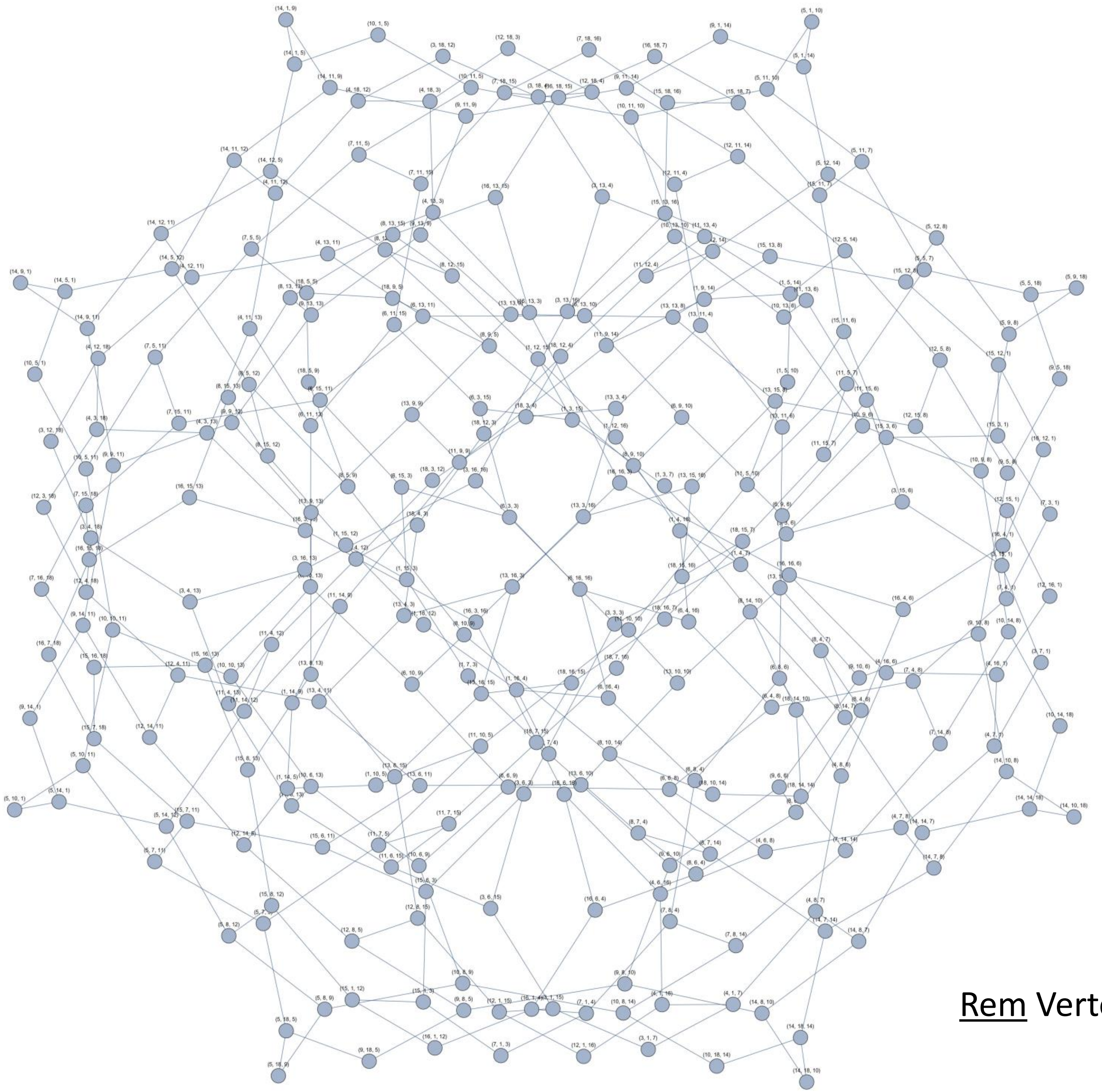
- Uncovered families of primes in Courcy-Ireland's thm:

$$p \equiv 19, 69, 71, 89, 99, 101, 111, \cdots \pmod{5740}$$

G_{19}



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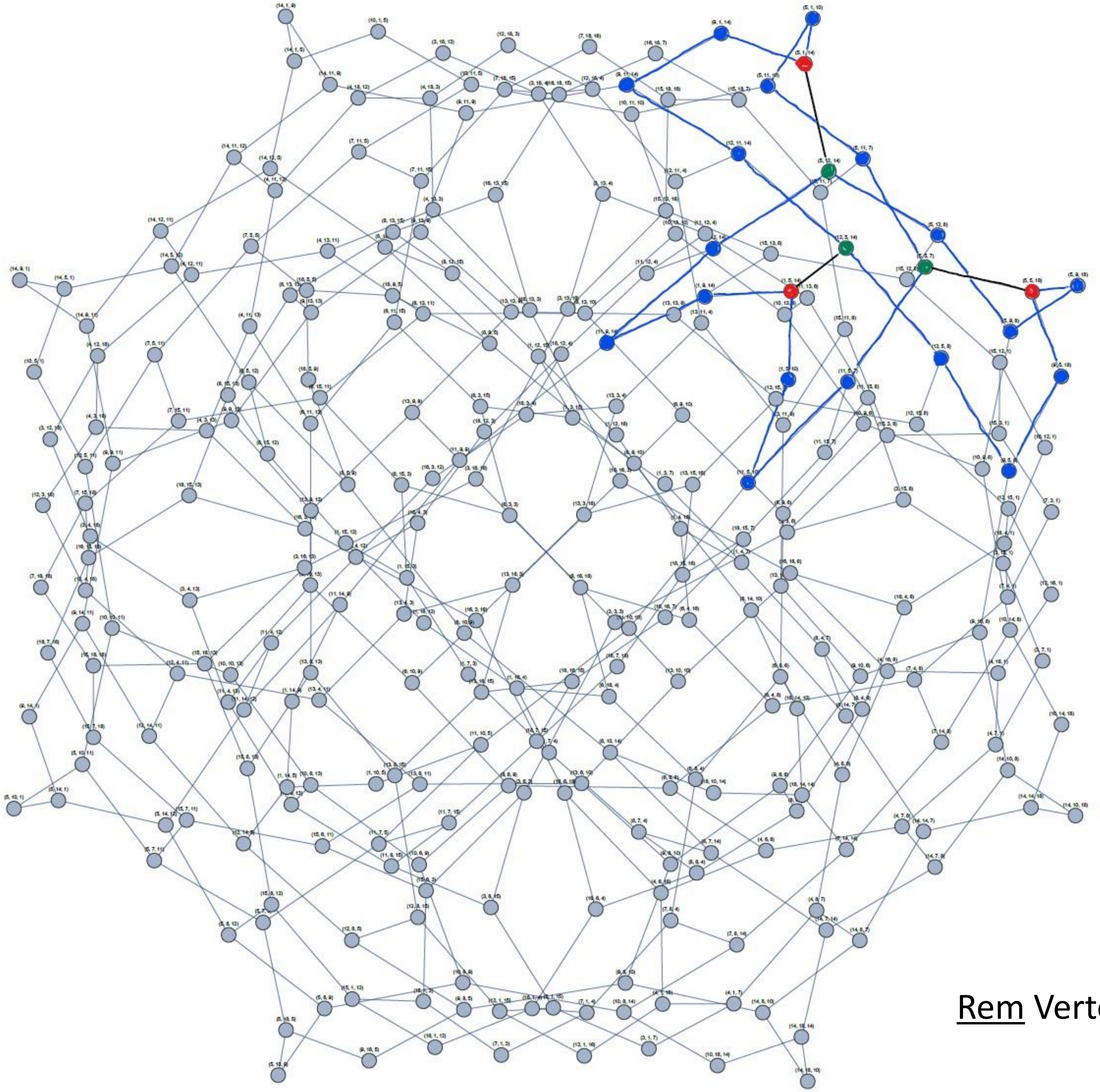

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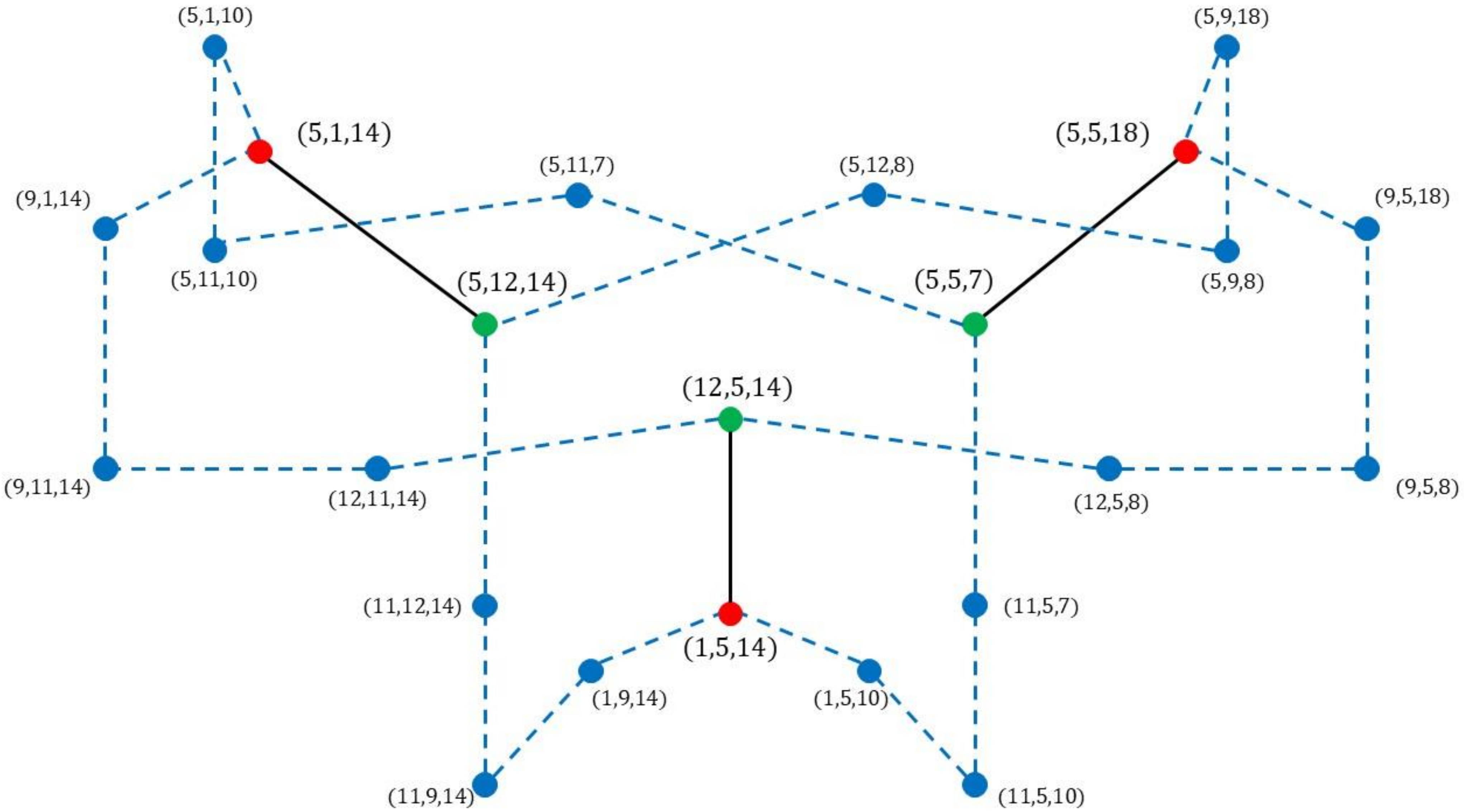
An explicit $K_{3,3}$-subdivision in G_{19}



Our construction (proof)

- Suppose: $\alpha \coloneqq \sqrt{5} \in \mathbb{F}_p$ & $\beta \coloneqq \sqrt{-34 - 10\sqrt{5}} \in \mathbb{F}_p$ ･･･(∗)

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- $X_1 := \left(\frac{-3-\alpha-\beta}{4}, \frac{1+\alpha}{2}, \frac{-1-\alpha}{2}\right), X_2 := \left(\frac{1+\alpha}{2}, \frac{-3-\alpha-\beta}{4}, \frac{-1-\alpha}{2}\right), X_3 := \left(\frac{1+\alpha}{2}, \frac{1+\alpha}{2}, \frac{3+\alpha+\beta}{4}\right)$

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Thank you! & Time for lunch…