On linear-algebraic notions of expansion

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Centre for Quantum Software and Information
University of Technology Sydney

45th Australasian Combinatorics Conference, December 2023

For more details, please refer to ArXiv:2212.13154.

Outline

- Introduction about graph-theoretic expansion and a classical result.
- Introduction about linear-algebraic expansion and some previous results.
- An overview of our main results.
  1. Dimension expansion $\not\Rightarrow$ Quantum expansion.
  2. Quantum expansion $\Rightarrow$ Dimension expansion.
  3. Linear-algebraic expansion properly generalizes graph-theoretic expansion.

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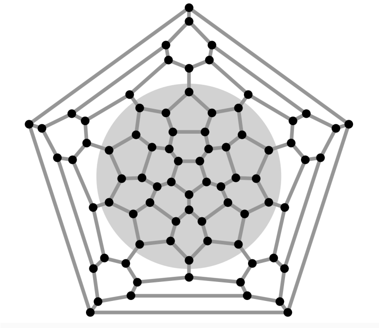
What are expanders?

- Expanders are graphs that are simultaneously sparse and highly connected.



- Expanders are graphs for which a random walk converges to its limiting distribution as rapidly as possible.

Figure source: `https://www.ams.org/notices/200407/what-is.pdf`

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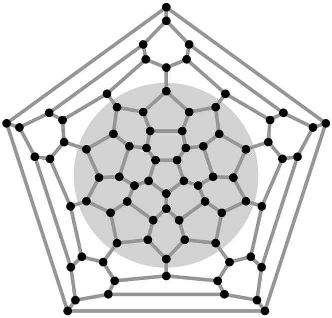

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Definition of graph-theoretic expansion

- Let $G = ([n], E)$ be a d-regular graph.
- The **spectral expansion** of G:

$$\lambda(G) := \text{the second-largest absolute value over all eigenvalues of } A,$$

where A is the adjacency matrix of G.

- The largest absolute value over all eigenvalues of A is 1!
- The larger the spectral gap $1 - \lambda(G)$ is, the better the expansion is.

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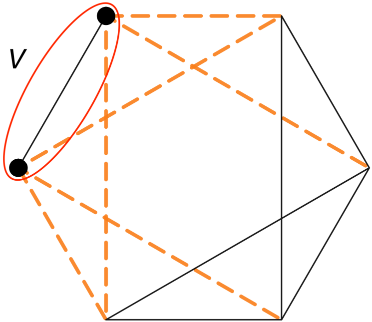
Examples of graph expansion



Edge expansion for vertex subset $V = \frac{6}{2} = 3$

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- The **edge expansion** of G:

$$h(G) := \min_{\substack{V \subseteq [n] \\ 1 \leq |V| \leq \frac{n}{2}}} \frac{|\partial(V)|}{|V|},$$

where $\partial(V) := \{\{i,j\} \in E : i \in V, j \in [n] \setminus V\}$.

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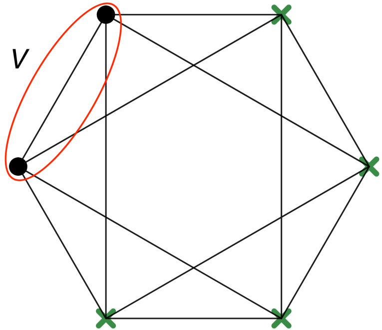

Vertex expansion for vertex subset $V = \frac{4}{2} = 2$

Definition of graph-theoretic expansion

- Let $G = ([n], E)$ be a d-regular graph.
- The **vertex expansion** of G:

$$\mu(G) := \min_{\substack{V \subseteq [n] \\ 1 \leq |V| \leq \frac{n}{2}}} \frac{|\partial_{\text{out}}(V)|}{|V|},$$

where $\partial_{\text{out}}(V) := \{j \in [n] \setminus V : \exists i \in V, \text{ s.t. } \{i, j\} \in E\}$.

A classical result of their relationship

Recall that

- λ: spectral expansion
- h: edge expansion
- μ: vertex expansion

For any d-regular graph G, the three notions of expansion are all equivalent, in the sense that

- $\frac{\mu(G)}{d} \leq h(G) \leq \mu(G)$ (By definition);
- $\frac{1-\lambda(G)}{2} \leq h(G) \leq \sqrt{2(1-\lambda(G))}$ (discrete Cheeger's inequality)

  [Dodziuk'84, Alon-Milman'85, Alon'86]

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Definition of some linear-algebraic expansion

- Given a matrix tuple $\mathbf{B} = (B_1, \ldots, B_d) \in \mathrm{M}(n, \mathbb{C})^d$.
- $\mathbf{B}$ is a *doubly stochastic matrix tuple* if $\sum_{i=1}^d B_i B_i^* = \sum_{i=1}^d B_i^* B_i = dI_n$.
- The *associated quantum operator* is the linear map $\Phi_{\mathbf{B}} : \mathrm{M}(n, \mathbb{C}) \to \mathrm{M}(n, \mathbb{C})$ defined by

$$\Phi_{\mathbf{B}}(X) := \frac{1}{d} \sum_{i=1}^{d} B_i X B_i^*.$$

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Definition of some linear-algebraic expansion

- The **quantum expansion** of $\Phi_{\mathbf{B}}$ [Ben–Aroya-Ta–shma'07, Hastings'07]:

  $\lambda(\mathbf{B}) :=$ the second-largest absolute value over all eigenvalues of $\Phi_{\mathbf{B}}$.

- The **quantum edge expansion** of $\Phi_{\mathbf{B}}$ [Hastings'07]:

$$h_Q(\mathbf{B}) := \min_{\substack{V \le \mathbb{C}^n \\ 1 \le \dim(V) \le \frac{n}{2}}} \frac{\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V) \rangle}{\dim(V)},$$

  where P_V is the orthogonal projection to the subspace $V \le \mathbb{C}^n$.

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Intuition of the definition

edge expansion **quantum edge expansion**

$$\min_{\substack{V\subseteq[n]\\1\leq|V|\leq\frac{n}{2}}} \frac{|\partial(V)|}{|V|} \quad ===\Rightarrow \quad \min_{\substack{V\leq\mathbb{C}^n\\1\leq\dim(V)\leq\frac{n}{2}}} \frac{\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V)\rangle}{\dim(V)}$$

- $\Phi_{\mathbf{B}}$ is an analogue of the normalized adjacency matrix A of a graph G.
- Consider that all the B_i's are permutation matrices.
- Consider that V is a coordinate subspace. (So P_V is diagonal of 0 and 1!)
- Then $I_n - P_V$ can be treated as an indicator vector x, and $\Phi_{\mathbf{B}}(P_V) = Ax$.
- So $\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V)\rangle$ counts the edges between a set and its complement.

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Definition of some linear-algebraic expansion

- Given a matrix tuple $\mathbf{B} = (B_1, \ldots, B_d) \in \mathrm{M}(n, \mathbb{C})^d$.
- The **dimension expansion** of $\mathbf{B}$ [Barak-Impagliazzo-Shipilka-Wigderson'04]:

$$\mu(\mathbf{B}) := \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\dim(V + \mathbf{B}(V)) - \dim(V)}{\dim(V)},$$

  where $\mathbf{B}(V) := \left\langle \cup_{i \in [d]} \{B_i v : v \in V\} \right\rangle$.
- Given the **vertex expansion** $\mu(G)$ and treat G as a tuple of permutations $(P_1, \ldots, P_d)$ acting on $[n]$:

$$\forall V \subseteq [n], \left| V \cup \bigcup_{i=1}^{d} P_i(V) \right| \geq (1 + \mu(G))|V|$$

- Change the permutation action and underlying object to be more general.

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where $V^\perp$ means the orthogonal complement of V, and the columns of T_V form an orthonormal basis of V.

- Let $\dim(V) = r$ and $U = \begin{bmatrix} T_V & T_{V^\perp} \end{bmatrix}$ be an $n \times n$ unitary matrix.

$$U^* B_i U = \begin{bmatrix} * & * \\ T_{V^\perp}^* B_i T_V & * \end{bmatrix},$$

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  - the matrix tuple consisting of permutation matrices only and;
  - the minimum to coordinate subspaces only,

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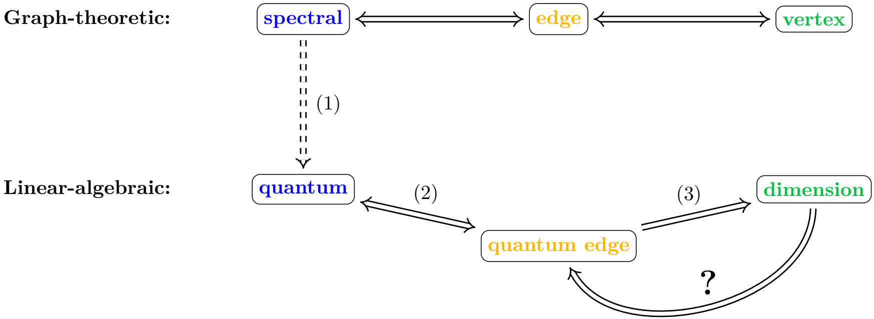
Overview of previous results



- (1): [Bannink-Briët-Labib-Maassen'20]
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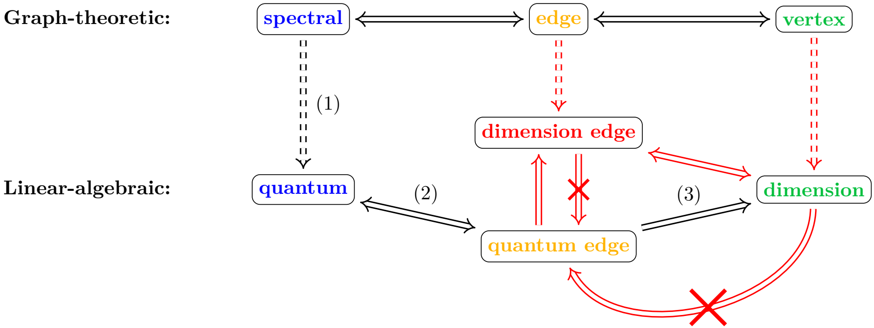
Overview of our main results



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Question and Answer

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Main result (1)

- Dimension expansion $\not\Rightarrow$ Quantum expansion.
- For any matrix tuple $\mathbf{B}_n = (B_1, \ldots, B_d)$ consisting of unitary matrices, $\mathbf{B}_n^s = (B_1^s, \ldots, B_d^s)$ satisfies that

$$\mu(\mathbf{B}_n^s) \geq \frac{\mu(\mathbf{B}_n)}{d} \text{ and } \lim_{n\to\infty} \lambda(\mathbf{B}_n^s) = 1$$

  for some sufficiently small power $s > 0$.
- This proof requires some advanced techniques, such as a number of compactness arguments.

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Main result (2)

- Quantum expansion ⇒ Dimension expansion.
- For any doubly stochastic matrix tuple $\mathbf{B}$,

$$\frac{1-\lambda(\mathbf{B})}{2d} \leq \frac{h_Q(\mathbf{B})}{d} \leq h_D(\mathbf{B}) \leq \mu(\mathbf{B}).$$

- It follows that if there is a spectral gap $1-\lambda(\mathbf{B}) > 0$, then $\mu(\mathbf{B}) > 0$.
- In case $\mathbf{B}$ consists of unitary matrices only, we can make a stronger bound,

$$\frac{1-\lambda(\mathbf{B})}{2} \leq h_Q(\mathbf{B}) \leq h_D(\mathbf{B}) \leq \mu(\mathbf{B}).$$

- This improves the result of [Lubotzky-Zelmanov'08]:

$$\frac{1-\lambda(\mathbf{B})}{6} \leq \mu(\mathbf{B}).$$

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- This improves the result of [Lubotzky-Zelmanov'08]:

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Main result (2)

- Quantum expansion $\Rightarrow$ Dimension expansion.
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- The *graphical matrix tuple* associated to a d-regular graph $G = ([n], E)$ is defined as

$$\mathbf{B}_G := (\sqrt{n} \cdot \mathrm{E}_{i,j} : \{i, j\} \in E),$$

where $\mathrm{E}_{i,j}$ is the elementary matrix with a 1 in position (i, j) and zeros in all other entries.

- For example, when $n = 3$,

$$\mathrm{E}_{2,3} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

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dimension expansion

$$\min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\dim(V + \mathbf{B}(V)) - \dim(V)}{\dim(V)}$$

dimension edge expansion

$$\min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\sum_{i=1}^{d} \operatorname{rank}(T_{V^\perp}^t B_i T_V)}{\dim(V)}$$

- A quick example about the relationship between them:

$$B_i(V) = \begin{bmatrix} * & * & * \\ * & * & * \\ \boldsymbol{b}_{31} & \boldsymbol{b}_{32} & * \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \\ \boldsymbol{b}_{31} & \boldsymbol{b}_{32} \end{bmatrix}$$

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Thank you so much!