EEEN3006J

Wireless Systems

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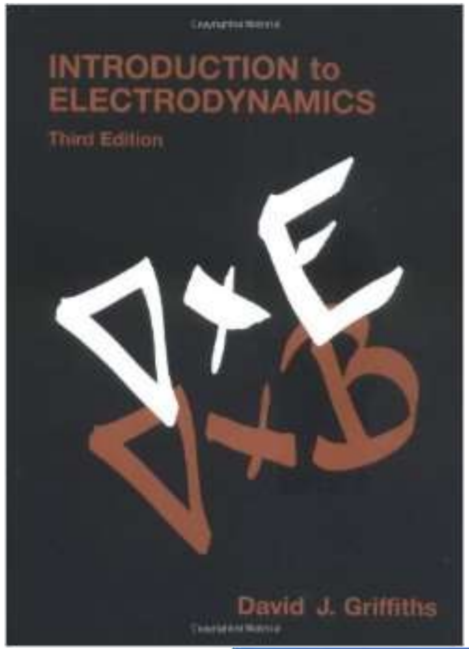
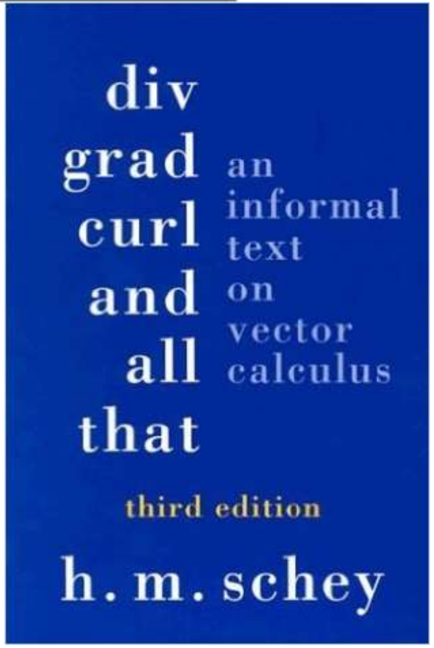
Additional reading

- If you would like to know more, I recommend Griffiths.



- Schey is also famous as a gentle introduction to the subject.



- **Neither book is required.**

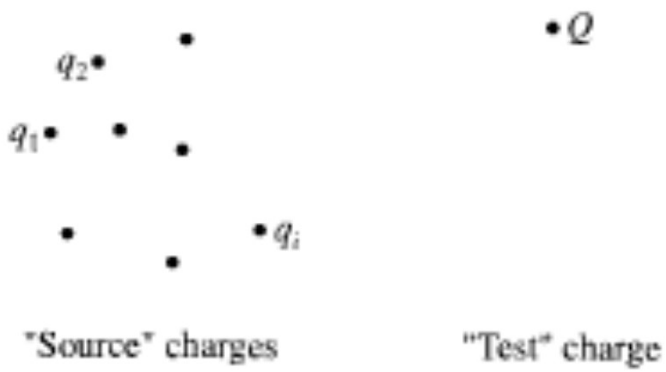
- Four kinds of forces: strong, electromagnetic, weak, gravitational.
- We are only really interested in electromagnetics, which is the best understood, and the most often used by our technology.
- It is formulated as a field theory. If we have some collection of charges in some distribution, what force would they exert on a charge at some other point?

Charge

- Charge is quantised.
  - The charge on any object is a multiple of e, the charge on a proton.
- Two varieties: positive and negative.
  - An example of negative charge is the electron, which has charge −e.
- Charge is conserved: it cannot be created or destroyed, only moved.
- Unit of charge is the Coulomb (C).

Electrostatics

- The electric field describes the force exerted on a test charge by some distribution of source charges.



- We will initially assume all the source charges are stationary. The resulting theory is called electrostatics.

Coulomb's law

- Coulomb's law gives the force exerted on a charge Q from a single source charge q.

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{\left|\vec{\vartheta}\right|^2} \hat{\vartheta}$$

  - The permittivity of free space, $\varepsilon_0 = 8.85 \times 10^{-12}$ $C^2\ N^{-1}\ m^{-2}$.
  - The unit vector in the direction of $\vec{\vartheta}$ is $\hat{\vartheta}$. $\vec{\vartheta} = \vec{r} - \overrightarrow{r'}$. $\vec{r}$ is the location of Q, and $\overrightarrow{r'}$ of q.
  - The force exerted is repulsive if Q and q have the same sign, and attractive otherwise.

Electric field

- If we have a collection of source charges, the force exerted on a test charge is the vector sum of the forces given by Coulomb's law for each of the source charges.

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_i \frac{q_i}{\left|\vec{\vartheta_i}\right|^2}\hat{\vartheta}_i$$

- The Electric field is given by $\mathrm{Q}\vec{E} = \vec{F}$. It is a function of the position vector.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{\left|\vec{\vartheta_i}\right|^2}\hat{\vartheta}_i$$

Electric field

- Now let us consider a situation where we replace the discrete points in space with an even distribution of charge over a region of space.
- For such continuous distributions of charge, the sum becomes an integral over the charge density function.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho(r')}{\left|\vec{\vartheta_i}\right|^2}\hat{\vartheta}_i dq$$

Maxwell’s Equations

A little joke:

And God said

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

and there was light.

James Clerk Maxwell
(1831 – 1879)

- Scottish scientist.
- Credited for the classical theory of electromagnetic radiation. This unified the theories of
  - electricity,
  - magnetism,
  - and light.

- Predicts radio waves, microwave radiation.
- Leads on to other unifications of physical theory: QED, electroweak, etc.

GAUSS’S LAW FOR ELECTRIC FIELDS

- In Maxwell's Equations, we find two kinds of electric field.
  - The electrostatic field produced by electric charge.
  - The induced electric field produced by a changing magnetic field.
- Gauss's law deals with the electrostatic field.
  - It relates the spatial behaviour of the electrostatic field to the charge distribution that produces it.

Integral form of Gauss's law

Electric field is a vector

The normal vector to the surface.

Enclosed charge

Integral is over a closed surface

$$\oint_S \underbrace{\vec{E} \circ \hat{n}}\, da = \frac{q_{\text{enc}}}{\varepsilon_0}$$

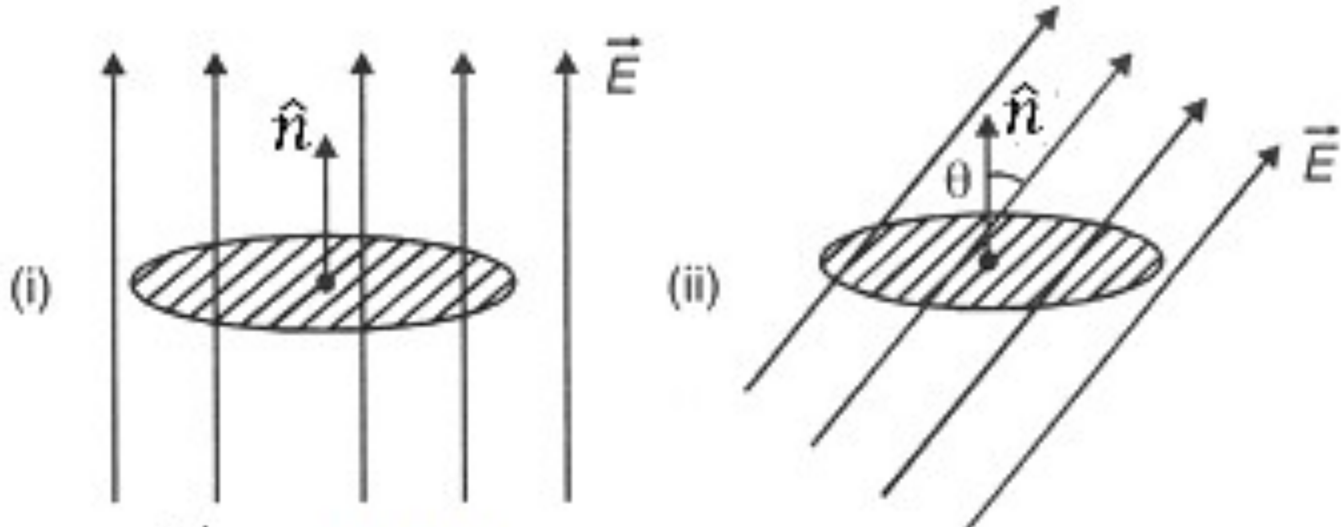
Permittivity of free space

The dot product of these two gives the component of the electric field normal to the surface.

- The left side is simply a mathematical description of electric flux passing through a closed surface, S.
  - Electric flux is the number of electric field lines.

Dot product and flux of a field

Consider, for example a field which defines the velocity of a fluid. The flux is the amount of the fluid flows through a surface.



$\phi = \vec{E} \circ \hat{n} = |E||n|$ $\qquad$ $\phi = \vec{E} \circ \hat{n}$

The dot product allows us to calculate the component of the field normal to the surface: it accounts for the angle of the field relative to the surface.

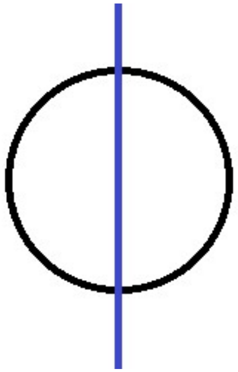
Some basic electric fields

Point charge (charge *q* C, distance *r* m)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$
Conducting sphere (charge *Q* C)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside, *r* m from centre) $\vec{E} = 0$ (inside)
Uniformly charged insulating sphere (radius r_0 m)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside) $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{r_0^3}\hat{r}$(inside, *r* m from centre)
Infinite line charge (linear charge density λ C/m)	$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{r}$ (*r* m from line)
Infinite flat plane (surface charge density σ C/m^2)	$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

- There are two basic problems you can solve with Gauss's law for electric fields.

1. Given information about a distribution of electric charge, you can find the electric flux through a surface enclosing that charge.
2. Given information about the electric flux through a surface, you can find the total charge enclosed by the surface.

- **Example:** A line with charge density 10^{-12} C/m passes through the centre of a sphere. If the flux through the surface of the sphere is 1.13×10^{-13} Vm, find the radius of the sphere.



Solution:

- If the sphere is of radius *r*, there is $2 \times r$ m of the line inside it.
- The total enclosed charge is therefore $2 \times r \times 10^{-12}$ C.
- The flux is the enclosed charge divided by the permittivity of free space, so

$$1.13 \times 10^{-3} = \frac{2 \times r \times 10^{-12}}{8.85 \times 10^{-12}}$$

- Solving for *r*, we get *r* = 5 x 10^{-3} m

- **Example:** The electric field at a distance r from an infinite line charge with line density λ is given by

$$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{r}$$

- Use this expression to find the electric flux through a cylinder surrounding a finite portion of length h of an infinite line charge.
- Use Gauss's law to verify the enclosed charge is in fact λh.

Solution:

- We want to evaluate the expression $\oint_S \vec{E} \circ \hat{n}\, da$ over a cylinder. It is simplest to split the problem into the top, bottom and sides.
- However, the electric field is normal to the infinite line of charge, and therefore perpendicular to the normal to the top and bottom of the cylinder. The remaining term is:

$$\Phi = \int_S \overbrace{\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}}^{\text{E field, given.}} \hat{r}\circ\hat{n}\, da$$

The surface in question in this integral is the curved part of the cylinder.

$$= \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\int_S da$$

r and n vectors are in the same direction.

$$= \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\underbrace{(2\pi r h)}$$

Area of curved surface of a cylinder.

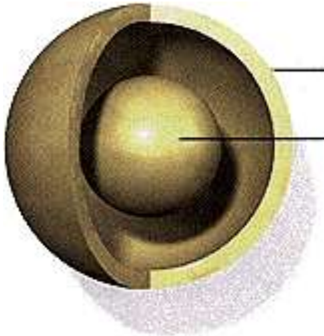
$$= \frac{\lambda h}{\varepsilon_0}.$$

- There are some circumstances in which we can find $\vec{E}$ from the charge distribution.
- These circumstances always involve particular symmetries that allow us to take the electric field term outside the integral in Gauss's law for electric fields.
- Example: Determine the electric field surrounding a sphere of radius a m and uniform charge density ρ C/m^3.

Solution:

- Due to the symmetries of the set-up, the electric field must be radial. (The problem is identical viewed from any angle.)
- We need to choose our surface for the integral carefully.
  - The electric field must be parallel or perpendicular to the normal at all points.
  - The electric field must be uniform at all points.
  - Such a surface is called a *special Gaussian surface*.
  - The only such surfaces here are spheres concentric to the charged sphere.

- The electric flux is given by



$$\Phi = \oint_S |\vec{E}|\, da$$

$$= |\vec{E}| \oint_S da$$

$$= |\vec{E}|\, \underbrace{4\pi r^2}$$

Surface area of a sphere of radius *r*.

- Gauss's law gives

$$\Phi = \frac{q}{\varepsilon_0}$$

$$|\vec{E}|\, 4\pi r^2 = \overbrace{\frac{4}{3}\pi a^3}\rho/\varepsilon_0$$

Volume of a sphere of radius *a*.

$$|\vec{E}| = \frac{a^3 \rho}{3\varepsilon_0 r^2}$$

Differential form of Gauss's law

Electric field is
a vector

Charge density

Del

$$\vec{\nabla} \circ \vec{E} = \frac{\rho}{\varepsilon_0}$$

The dot product turns the del
operator into **divergence**.

Permittivity of
free space

- Deals with the divergence of the field and the charge density at <u>individual points</u> rather than over a surface.
- Useful in different situations to the integral form.

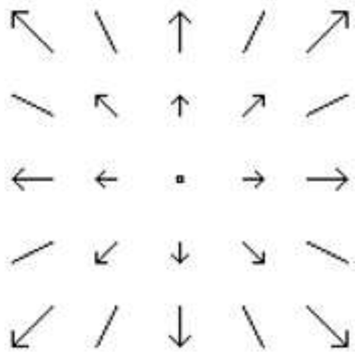
Nabla – del operator

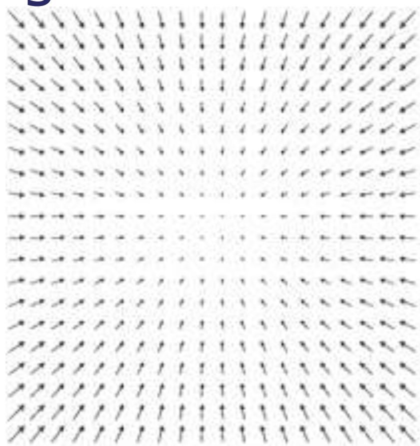
- Vector differential operator.
- Instruction to take derivatives.
  - $\vec{\nabla}\circ$ is the divergence
  - $\vec{\nabla}\times$ is the curl
  - $\vec{\nabla}$ is the gradient
- Specifically

$$\vec{\nabla} \equiv \hat{\imath}\frac{\delta}{\delta x} + \hat{\jmath}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}.$$

Divergence

- Important in many areas of physics.
- While flux and divergence deal with flow of a field, flux is defined over an area, while divergence deals with points.
  - Positive divergence -> sources
  - Negative divergence -> sinks





Divergence

- Can think of as ratio of flux to volume in the limit.

$$\text{div}(A) = \vec{\nabla} \circ \vec{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \vec{A} \circ \hat{n}\, da$$

- More useful form:

$$\vec{\nabla} \circ \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

- *Scalar* quantity: no direction.

Divergence in other co-ordinate systems

- Spherical co-ordinates:

$$\vec{\nabla}\circ\vec{A}$$

$$= \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)$$

- Cylindrical co-ordinates:

$$\vec{\nabla}\circ\vec{A} = \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\right)$$

Example:

- Given the following electric field, calculate the charge density at x = 2 m and at x = 5 m.

$$\vec{E}(x) = \begin{cases} ax^2\hat{\imath}\ V/m, & x < 3\text{ m} \\ b\hat{\imath}\ V/m, & x \geq 3\text{ m} \end{cases}$$

Solution:

- (i) x = 2 m

$$\vec{\nabla} \circ \vec{E} = \frac{\rho}{\varepsilon_0}$$

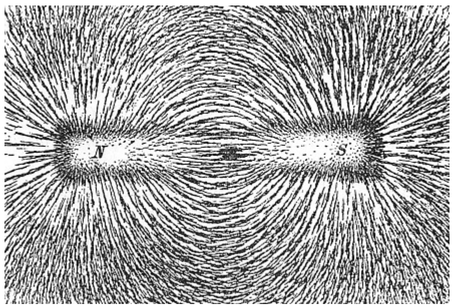
$$\frac{\partial}{\partial x}(ax^2) = \frac{\rho}{\varepsilon_0}$$

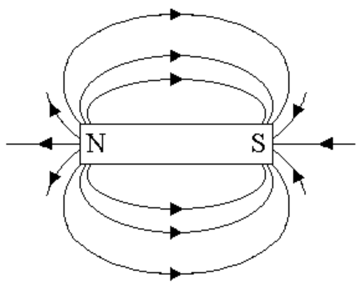
$$\rho = 2ax\varepsilon_0 = 4a\varepsilon_0 \text{ C/m}$$

- (ii) x = 5 m
- Electric field is constant, so all derivatives are 0. Therefore $\rho = 0$ C/m.

GAUSS'S LAW FOR MAGNETIC FIELDS

- Similar to law for electric fields in form, but the differences are significant.
- Can be stated as: The total magnetic flux passing through a closed surface is zero.
- Consequence: no magnetic monopoles. (They are theorised, but haven't been observed.)
- Magnetic field lines always form loops.





Integral form of Gauss's law

Magnetic field is a vector

The normal vector to the surface.

Integral is over a closed surface

$$\oint_S \underbrace{\vec{B} \circ \hat{n}}\, da = 0$$

The dot product of these two gives the component of the magnetic field normal to the surface.

- The left side is simply a mathematical description of magnetic flux passing through a closed surface, S.
  - Magnetic flux is the number of magnetic field lines.

Some basic magnetic fields

Infinite straight wire carrying current I (at distance r)	$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$
Segment of straight wire carrying current I (at distance r)	$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{L} \times \hat{r}}{r^2}$ Biot-Savart law
Circular loop of radius R carrying current I (loop in yz plane, distance x along x-axis)	$\vec{B} = \frac{\mu_0 I R^2}{2(x^2 + R^2)^{3/2}}\hat{x}$
Solenoid with N turns and length L carrying current I.	$\vec{B} = \frac{\mu_0 N I}{L}\hat{x}$ (inside)
Torus with N turns and radius r carrying current I.	$\vec{B} = \frac{\mu_0 N I}{2\pi r}\hat{\varphi}$ (inside)

μ_0 is the permeability of free space.

Example:

- A closed cylinder is placed in a magnetic field given by $\vec{B} = B_0(\hat{j} - \hat{k})$. If the axis of the cylinder is aligned along the k-axis, find the flux through
  - (i) the top and bottom surfaces and
  - (ii) the curved surface of the cylinder.

Solution:

- The key is to note that $\Phi_{B,\text{top}} + \Phi_{B,\text{bottom}} + \Phi_{B,\text{curvedsurface}} = 0$
- For the top, $\hat{n} = \hat{k}$, so

$$\Phi_{B,\text{top}} = \int_S -B_0\, da = -B_0 \pi r^2$$

- Similarly, for the bottom, $\hat{n} = -\hat{k}$, so

$$\Phi_{B,\text{bottom}} = \int_S B_0\, da = B_0 \pi r^2$$

- Consequently, $\Phi_{B,\text{curvedsurface}} = 0$

Differential form of Gauss's law for magnetic fields

Magnetic field
is a vector

Del

$$\vec{\nabla} \circ \vec{B} = 0$$

The dot product turns the del
operator into divergence.

- No sources, no sinks.

Example:

- A magnetic field is given by $\vec{B} = axz\hat{\imath} + byz\hat{\jmath} + c\hat{k}$. Show that it is a consequence of Gauss's law for magnetic fields that *a* and *b* cannot have independent values.

Solution:

$$\vec{\nabla} \circ \vec{B} = 0$$

$$\frac{\partial}{\partial x}(axz) + \frac{\partial}{\partial y}(byz) + \frac{\partial}{\partial z}(c) = 0$$

$$az + bz + 0 = 0$$

$$a = -b$$

FARADAY’S LAW

Michael Faraday (1791-1867)

- English scientist.
- Major contribution to electromagnetic theory.
- Farad named in his honour.
- Poor mathematician but great experimentalist.
- Maxwell built on his work.

Faraday’s law

- In 1831, Faraday performed a number of important experiments.
- He demonstrated that an electric current could be induced in a circuit by a time-varying magnetic field.
- Such induced electric fields are quite different from the electrostatic fields we dealt with in the previous lecture.

Integral form of Faraday's law 1

Rate of change of time

Magnetic flux through surface S bounded by C

Lenz's law.

$$\oint_C \vec{E} \circ d\vec{l} = \underbrace{- \frac{d}{dt} \overbrace{\int_S \vec{B} \circ \hat{n}\, da}}$$

Line integral, not surface or volume.

Dot product obtains part of field along path C

Induced EMF, Flux rule

S is <u>not necessarily</u> a closed surface. (Gauss's law doesn't apply.)

- Strictly only correct when $\vec{E}$ represents the electric field in the rest frame of each segment *dl*.

Integral form of Faraday’s law 2

Rate of change of time

Magnetic flux through surface S bounded by C

Lenz’s law.

$$\oint_C \vec{E}\circ d\vec{l} = -\underbrace{\int_S \frac{\partial \vec{B}}{\partial t}\circ \hat{n}\, da}$$

Flux of time derivative of magnetic field.

- Alternative formulation.
- Time derivative operates on magnetic field, not flux.
- Slightly different E – measured in lab reference frame.

- Can solve problems of two types:

1. Given information about the changing magnetic flux, find the induced emf.
2. Given induced emf along a given path, can find the time derivative of the magnetic field, the direction or the area enclosed by the path.

- Highly symmetric cases offer additional possibilities as for previous equations.

Induced electric field

- No sources or sinks.
- Field lines form loops.

Circulation of a field - emf

- For field A, $\oint_C \vec{A} \circ d\vec{l}$ is the circulation.
- Circulation of electric field is called electro-motive force.
- Work done moving unit charge around closed loop C.

Lenz’s Law

- Currents induced by changing magnetic fields always oppose the change in flux.
- Applies even if no conducting path exists
  - no current flows, but electric field still circulates.

Example:

- Given an expression for the magnetic field as a function of time, find the emf induced in a loop of specified size.

$$\vec{B}(y,t) = B_0\left(\frac{t}{t_0}\right)\frac{y}{y_0}\hat{z}$$

- Square loop, side L m lying in xy plane, with one corner at the origin.

Solution:

- Use the flux law,

- emf $= -\frac{d}{dt}\int_S \ \vec{B}\circ\hat{n}\,\mathrm{d}a$
- Here, $\hat{n} = \hat{z}$, and $\mathrm{d}a = dx\,dy$. Hence,

- emf $= -\frac{d}{dt}\int_{y=0}^{L}\int_{x=0}^{L} B_0\left(\frac{t}{t_0}\right)\frac{y}{y_0}\hat{z}\circ\hat{z}\,\mathrm{d}x\,\mathrm{d}y$

$$= -\frac{\boldsymbol{d}}{\boldsymbol{dt}}\left[B_0\left(\frac{t}{t_0}\right)\frac{L^3}{2y_0}\right]$$

$$= -B_0\frac{L^3}{2t_0y_0}$$

Differential form of Faraday's law

Electric field is a vector

Rate of change of magnetic field

Del

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The cross product turns the del operator into **curl**.

- A circulating electric field is produced by a time-varying magnetic field.

Curl

- Can be defined as:

$$\mathrm{curl}(A) = (\vec{\nabla}\times\vec{A})\circ\hat{n} \equiv \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_C \vec{A}\circ d\vec{l}$$

- More usefully written as a determinant:

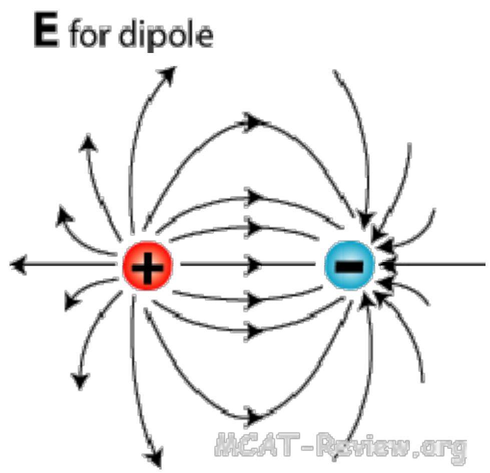
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

Each component indicates the tendency of the field to rotate in one of the co-ordinate planes.

Curl of the electric field

- Field lines can’t turn back on themselves.
- So to integrate around a loop gives 0.
  - Hence dipole field, like all electrostatic fields, has no curl.



Example:

- The magnetic field in a certain region is given by

$$\vec{B}(t) = B_0 \cos(kz - \omega t)\,\hat{j}$$

a) Find the curl of the induced electric field.

b) The z-component of the field is known to be zero. Find the x-component of the field.

Solution:

a) By Faraday's Law:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = \quad -\frac{\partial}{\partial t}\left(B_0 \cos(kz - \omega t)\,\hat{j}\right)$$
$$= -\omega B_0 \sin(kz - \omega t)\,\hat{j}$$

b) From the definition of curl,

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{j} = -\omega B_0 \sin(kz - \omega t)\,\hat{j}$$

$$\frac{\partial E_x}{\partial z} = -\omega B_0 \sin(kz - \omega t)$$

As $E_z = 0$

$$E_x = \frac{\omega}{k} B_0 \cos(kz - \omega t) + c$$

Example:

- Find the rate of change with time of the magnetic field at a location at which the induced electric field is given by the following.

$$\vec{E}(x,y,z) = E_0\left[\left(\frac{z}{z_0}\right)^2 \hat{\imath} + \left(\frac{x}{x_0}\right)^2 \hat{\jmath} + \left(\frac{y}{y_0}\right)^2 \hat{k}\right]$$

Solution:

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

$$= -E_0 \left[\left(\frac{2y}{y_0^2} \right) \hat{\imath} + \left(\frac{2z}{z_0^2} \right) \hat{\jmath} + \left(\frac{2x}{x_0^2} \right) \hat{k} \right]$$

THE AMPERE-MAXWELL LAW

- For thousands of years, the only known sources of magnetic fields were the earth’s, that of certain iron ores and that of other materials that had been accidentally or deliberately magnetised.
- In 1820, Hans Christian Oersted showed that an electric current could deflect a compass needle.
- Within a week, Ampere had begun quantifying the relationship between E and M fields.

- Ampere’s law was well known by the time Maxwell began his work in the field (1850s).
- However, it only applied to static situations involving steady currents.
- Maxwell added another source term – a changing electric flux – and extended the applicability of Ampere’s law to time-varying fields.

Integral form of the Ampere-Maxwell law

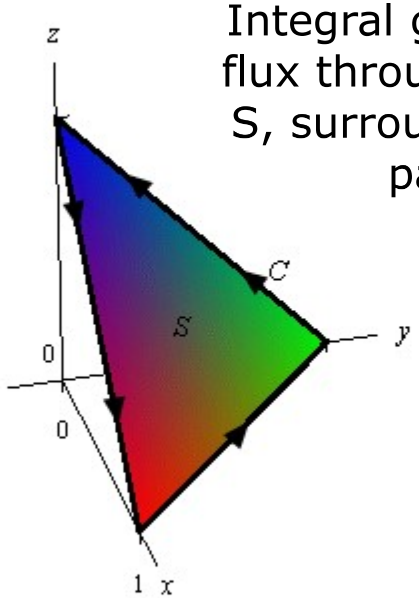
The rate of change with time

$$\oint_C \vec{B} \circ d\vec{l} = \mu_0 \left(I_{\text{enc}} + \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \circ \hat{n}\, da \right)$$

Integral gives the circulation of the magnetic field around a path, C

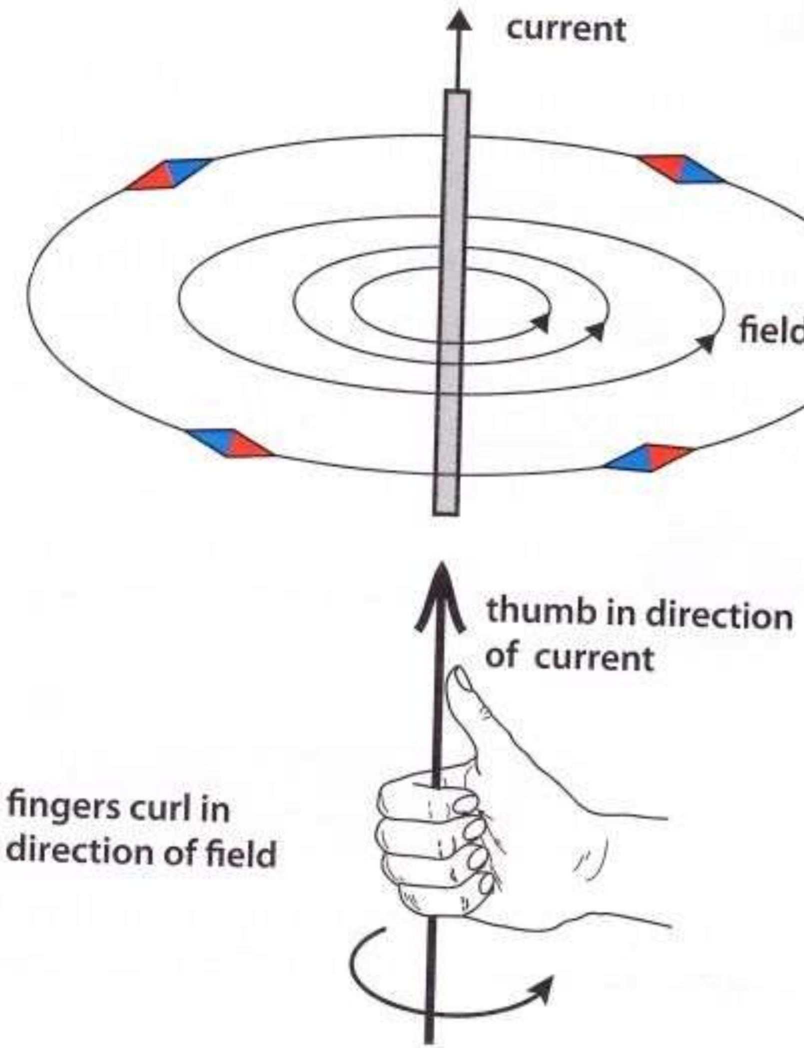
Steady conduction current

Integral gives Electric flux through a surface S, surrounded by the path C



Magnetic field circulation

$$\oint_C \vec{B} \circ d\vec{l}$$



- Add vector components of B along some closed path C.
- Can define special Amperian loop to extract B, just like special Gaussian surface.

Differential form of Ampere-Maxwell law

The electrical current density (A/m^2)

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Curl of the magnetic field, its circulation.

Displacement current

- A circulating magnetic field is produced by an electric current and a time-varying electric field.

Curl of the magnetic field

- All magnetic field lines form loops.
- Any such non-zero field must have regions where the path integral of the field is non-zero.
- Recall that the magnetic field due to an infinite line charge is

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$$

- The curl of this field is easy to calculate in cylindrical coordinates. It is left as an exercise. The result may surprise you.

Curl Cylindrical\spherical coordinates

(cylindrical)

$$\nabla \times \vec{A} = \left(\frac{1}{r} \frac{\partial Az}{\partial \varphi} - \frac{\partial A\varphi}{\partial z} \right) \vec{r} + \left(\frac{\partial Ar}{\partial z} - \frac{\partial Az}{\partial r} \right) \vec{\varphi} + \frac{1}{r} \left(\frac{\partial \left(rA\varphi \right)}{\partial r} - \frac{\partial Ar}{\partial \varphi} \right) \vec{z}$$

$$\nabla \times \vec{A} = \left(\frac{1}{r \sin\theta} \frac{\partial A\varphi \sin\theta}{\partial \theta} - \frac{\partial A\theta}{\partial \varphi} \right) \vec{r}$$ (spherical)

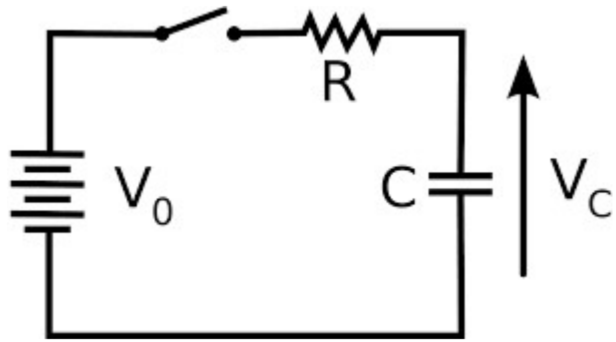
$$+ \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial Ar}{\partial \varphi} - \frac{\partial rA\varphi}{\partial r} \right) \vec{\theta}$$

$$+ \frac{1}{r} \left(\frac{\partial \left(rA\theta \right)}{\partial r} - \frac{\partial Ar}{\partial \theta} \right) \vec{\varphi}$$

Electric current density

- Vector current flowing through a unit cross-sectional area perpendicular to the field.
- Units are A/m^2
- Complexity of the relationship between current and current density depends on geometry of the situation.
- For uniform J, $I = JA$. A is cross-sectional area.
- More generally, current is the flux of current density.

$$I = \oint_S \vec{J} \circ \hat{n} \, da$$



Example:

- A capacitor, C is charged using the circuit above. Given its magnetic field (below), find the displacement current density between the plates of the capacitor.

$$\vec{B} = \frac{\mu_0 V_0}{2\pi R} e^{-t/RC} \left(\frac{r}{r_0^2}\right) \hat{\varphi}$$

Solution

- Use curl of B in cylindrical co-ordinates.

$\vec{\nabla} \times \vec{B}$

$$= \left(\frac{1}{r}\frac{\partial B_z}{\partial \varphi} - \frac{\partial B_\varphi}{\partial z}\right)\hat{r} + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\varphi} + \frac{1}{r}\left(\frac{\partial B_\varphi}{\partial r} - \frac{\partial B_r}{\partial \varphi}\right)\hat{z}$$

$$\vec{\nabla} \times \vec{B} = \left[\frac{\mu_0 \Delta V}{R} e^{-t/RC}\left(\frac{1}{\pi r_0^2}\right)\right]\hat{z}$$

- No conduction current, so

$$\vec{\nabla} \times \vec{B} = \mu_0\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

- Hence

$$\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \left[\frac{\Delta V}{R} e^{-t/RC}\left(\frac{1}{\pi r_0^2}\right)\right]\hat{z}$$

FROM MAXWELL’S EQUATIONS TO THE WAVE EQUATION

Maxwell's Equations

Integral form	Differential form	
$\oint_S \vec{E}\circ\hat{n}\,da = \frac{q_{\text{enc}}}{\varepsilon_0}$	$\vec{\nabla}\circ\vec{E} = \frac{\rho}{\varepsilon_0}$	Gauss's Law for E-fields
$\oint_S \vec{B}\circ\hat{n}\,da = 0$	$\vec{\nabla}\circ\vec{B} = 0$	Gauss's Law for B-fields (no monopoles)
$\oint_C \vec{E}\circ d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\circ\hat{n}\,da$	$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	Faraday's Law
$\oint_C \vec{B}\circ d\vec{l} = \mu_0\left(I_{\text{enc}} + \varepsilon_0\frac{d}{dt}\int_S \vec{E}\circ\hat{n}\,da\right)$	$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$	Ampere-Maxwell Law

- Maxwell’s achievement went beyond the synthesis of the equations or the addition of the displacement current term to Ampere’s law.
- It was by considering these equations in combination that he developed a comprehensive theory of electromagnetism.
- In this section, we’ll show how to move from these equations to the wave equation.

Divergence (or Gauss's) theorem

Divergence
of a field, A

$$\oint_S \vec{A} \circ \hat{n}\, da = \oint_V \left(\nabla \circ \vec{A}\right) dV$$

Flux of a
field, A

- This only applied when the field and its first derivative are both continuous.
- Relates integral and differential forms of Gauss's laws.

Stokes theorem

Line integral over C

Surface integral over S, bounded by C

Circulation of vector field A

$$\oint_C \vec{A}\circ d\vec{l} = \int_S \underline{\underline{(\vec{\nabla}\times\vec{A})}\circ\hat{n}}\, da$$

Normal component of the curl of the field.

- This only applied when the field and its first derivative are both continuous.
- Relates the integral and differential forms of Faraday's law.
- Relates the integral and differential forms of the Ampere-Maxwell law.

Gradient

- Third kind of operation involving del.
- Applies to scalar fields.

$$\vec{\nabla}\psi = \hat{\imath}\frac{\partial\psi}{\partial x} + \hat{\jmath}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}$$

- Direction and steepness of slope at each point in the field.

Some useful relationships

- Curl of grad is always zero.

$$\vec{\nabla} \times \vec{\nabla}\psi = 0$$

- Laplacian

$$\vec{\nabla} \circ \vec{\nabla}\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

- We can use these, e.g. to obtain Poisson's equation.
- Grad of div less Laplacian is curl of curl.

$$\vec{\nabla}\left(\vec{\nabla} \circ \psi\right) - \nabla^2\psi = \vec{\nabla} \times \left(\vec{\nabla} \times \psi\right)$$

The wave equation

- Recall Faraday's Law:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

- Take the curl of both sides.

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial\left(\vec{\nabla}\times\vec{B}\right)}{\partial t}$$

$$\vec{\nabla}\left(\vec{\nabla}\circ\vec{E}\right) - \nabla^2\vec{E} = -\frac{\partial\left(\vec{\nabla}\times\vec{B}\right)}{\partial t}$$

The wave equation

- Ampere-Maxwell and Gauss's law for E-fields give

$$\vec{\nabla}\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2\vec{E} = -\mu_0\frac{\partial\vec{J}}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

- In a charge- and current-free region, $\rho = 0$ and $\vec{J} = 0$.

$$\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

- This linear, second order, homogeneous partial differential equation describes the propagation of a wave.

- Can do same starting with Ampere-Maxwell, obtaining

$$\nabla^2\vec{B} = \mu_0\varepsilon_0 \frac{\partial^2\vec{B}}{\partial t^2}$$

- In general, wave equation is $\nabla^2\vec{A} = \frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2}$, so we can calculate velocity.

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$$

- This agreement with the experimentally determined velocity of light caused Maxwell to declare it an e-m wave.