Chapter 1

Fourier transforms

Some commonly used Fourier transforms are listed below:

time domain	frequency domain
$\delta(t)$	1
1	$\delta(f)$
$e^{j2\pi f_c t}$	$\delta(f - f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}\left[\delta(f - f_c) - \delta(f + f_c)\right]$
$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}\left(\frac{t}{T}\right)$	$T\,\mathrm{rect}(fT)$
$\sum_{n=-\infty}^{\infty}\delta(t - nT)$	$\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T}\right)$

Table 1.1: Useful Fourier Transforms.

In the above table we have the following definitions[1]:

$$
\begin{aligned}
\mathrm{sinc}(x) &\triangleq \frac{\sin(\pi x)}{\pi x} \quad \text{and} \\
\mathrm{rect}(x) &\triangleq \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}
\end{aligned}
$$

1.1 Properties of the Fourier Transform

Here we denote the FTs of $x(t)$ and $y(t)$ by $X(f)$ and $Y(f)$ respectively. Some useful properties of the FT are then as follows.

- Linearity: $\mathcal{F}\{\alpha x(t) + \beta y(t)\} = \alpha X(f) + \beta Y(f)$

[1]Note that some texts use an alternative definition $\mathrm{sinc}(x) \triangleq \frac{\sin(x)}{x}$, but we will NOT be using that here unless otherwise stated.

- Time scaling: if $a \in \mathbb{R}$, $a \neq 0$,

$$\mathcal{F}\left\{x\left(at\right)\right\} = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

- Duality: if $\mathcal{F}\left\{g\left(t\right)\right\} = G\left(f\right)$, then $\mathcal{F}\left\{G\left(t\right)\right\} = g\left(-f\right)$

- Multiplication in the time domain corresponds to convolution in the frequency domain

$$\mathcal{F}\left\{x\left(t\right)y\left(t\right)\right\} = \int_{-\infty}^{\infty} X\left(\nu\right)Y\left(f-\nu\right)\,d\nu$$

- Multiplication in the frequency domain corresponds to convolution in the time domain

$$\mathcal{F}^{-1}\left\{X\left(f\right)Y\left(f\right)\right\} = \int_{-\infty}^{\infty} x\left(\tau\right)y\left(t-\tau\right)\,d\tau$$

- Time shift:

$$\mathcal{F}\left\{x\left(t-t_0\right)\right\} = e^{-j2\pi f t_0} X\left(f\right)$$

- Frequency shift:

$$\mathcal{F}\left\{e^{j2\pi f_c t}x\left(t\right)\right\} = X\left(f-f_c\right) \tag{1.1.1}$$

- ‘Rayleigh’s energy theorem’ (or ‘Parseval’s Theorem’)

$$\int_{-\infty}^{\infty} \left|x\left(t\right)\right|^2\,dt = \int_{-\infty}^{\infty} \left|X\left(f\right)\right|^2\,df$$

1.2 Fourier Transform Symmetry Properties

$$
\begin{aligned}
g(t) \text{ real} &\implies G(-f) = G^{*}(f) \\
g(t) \text{ even} &\implies G(-f) = G(f) \\
g(t) \text{ odd} &\implies G(-f) = -G(f) \\
g(t) \text{ real and even} &\implies G(f) \text{ real and even} \\
g(t) \text{ real and odd} &\implies G(f) \text{ imaginary and odd}
\end{aligned}
$$

Note in particular that if $g(t)$ is real, then $|G(f)|$ has even symmetry, and $\angle G(f)$ has odd symmetry.

Chapter 2

Energy Signals and Power Signals

Let $x(t)$ be a real signal. The instantaneous power in $x(t)$ is given by $x^2(t)$. An *energy signal* is a real signal which has nonzero but finite energy, i.e.

$$E_x = \int_{-\infty}^{\infty} x^2(t)\ dt \qquad 0 < E_x < \infty$$

A power signal is a real signal which has nonzero but finite average power, i.e.

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\ dt \qquad 0 < P_x < \infty$$

Note that power signals have infinite energy, and energy signals have zero average power. Therefore, a signal can be an energy signal or a power signal, but not both.

2.1 Real Energy Signals

The autocorrelation function of a real energy signal is given by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\ dt$$

This function provides an indication of how closely related $x(t)$ is to a time-shifted version of itself. The autocorrelation function has the properties:

1. $R_x(0) = E_x$ is the energy in the signal

2. $R_x(\tau)$ is an even function, i.e. $R_x(-\tau) = R_x(\tau)$

For a real energy signal $x(t)$, the *energy spectral density* (ESD) is given by

$$S_x(f) = |X(f)|^2$$

This function has the properties:

1. $S_x(f)$ is real
2. $S_x(f)$ is an even function, i.e. $S_x(-f) = S_x(f)$

It may be shown (proof omitted) that the autocorrelation function and the energy spectral density are a Fourier transform pair, i.e.

$$\begin{aligned} S_x(f) &= \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \\ R_x(\tau) &= \int_{-\infty}^{\infty} S_x(f)\, e^{j2\pi f\tau}\, df \end{aligned}$$

This is known as the *Wiener-Khintchine theorem*.

2.2 Real Power Signals

The autocorrelation function of a real power signal is given by

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t-\tau)\, dt$$

This function has the properties:

1. $R_x(0) = P_x$ is the power in the signal
2. $R_x(\tau)$ is an even function, i.e. $R_x(-\tau) = R_x(\tau)$

For a real power signal $x(t)$, the *power spectral density* (PSD) is given by

$$S_x(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2$$

where $X_T(f)$ is the FT of the truncated time signal

$$x_T(t) = \begin{cases} x(t) & \text{if } -\frac{T}{2} < t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

It may also be shown (proof omitted) that the autocorrelation function and the power spectral density are a Fourier transform pair, i.e.

$$
\begin{aligned}
S_x(f) &= \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \\
R_x(\tau) &= \int_{-\infty}^{\infty} S_x(f)\, e^{j2\pi f\tau}\, df
\end{aligned}
$$

This is known as the *Wiener-Khintchine theorem*.