EEEN3006J – Wireless Systems

Some preliminaries before we study Maxwell's equations

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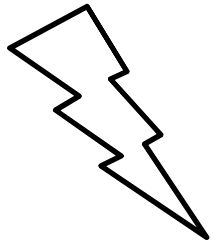
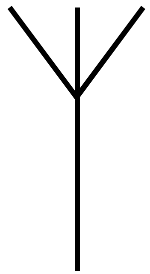
Purpose of this lecture

- In this lecture, we will review some mathematics used in electromagnetic theory.
- This will help you understand Maxwell’s equations over the next few lectures.



- From Maxwell’s equations, we will ultimately describe how antennas work.



Vector algebra

- Electromagnetic fields are in 3D space. You should review vector algebra.
- Can you
  - Add or subtract two vectors?
  - Multiply a vector by a scalar?
  - Calculate the dot product of two vectors?
  - Calculate the cross product of two vectors?
- Do you understand triple products?
- Given two points as vectors, can you find the difference vector between them?

Dot product

- The dot product (or scalar product, or inner product) of two vectors A and B is given by

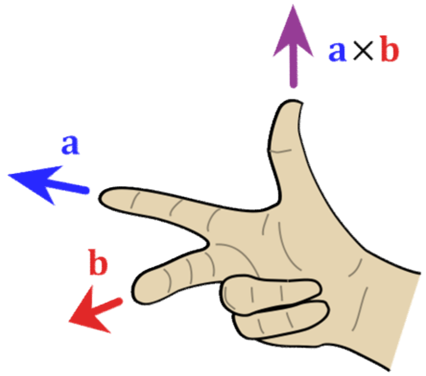
$$\mathbf{A}\cdot\mathbf{B}=\sum_{i=1}^{n}A_iB_i=A_1B_1+A_2B_2+\cdots+A_nB_n$$

- or

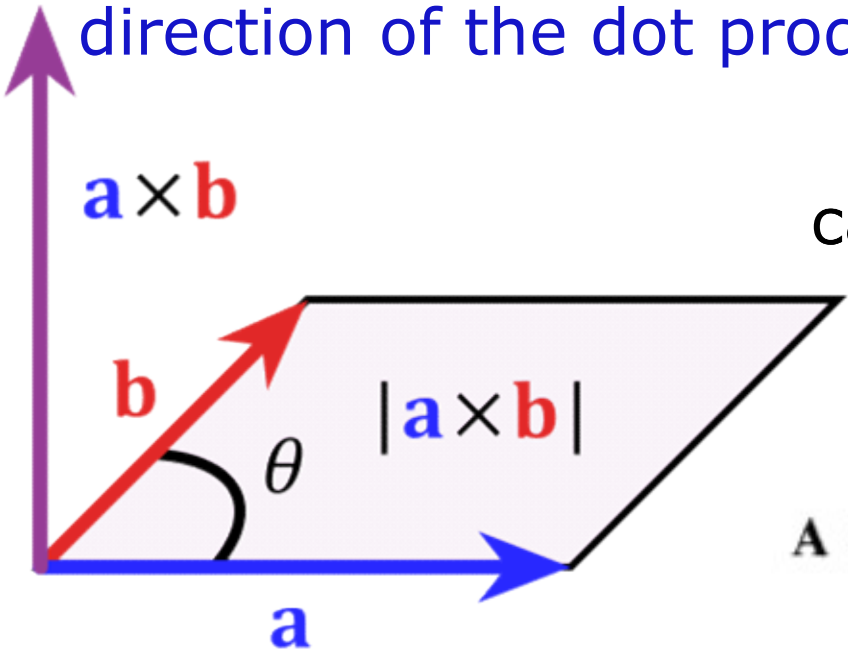
$$\mathbf{A}\cdot\mathbf{B}=\|\mathbf{A}\|\,\|\mathbf{B}\|\cos\theta,$$

Cross product

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \; \mathbf{n}$$



- You can use the right hand rule to obtain the direction of the dot product.

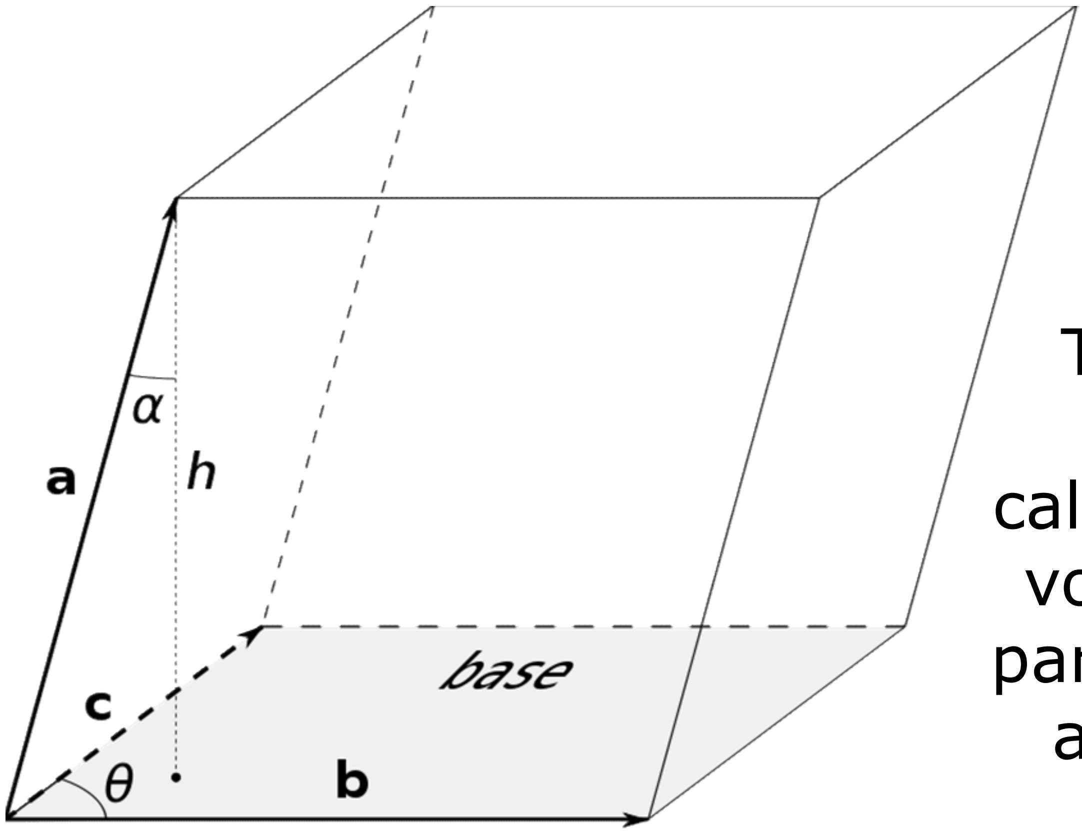


The cross product calculates the area of a parallelogram as shown.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

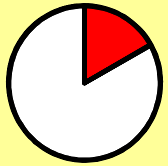
Triple product

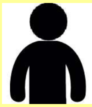
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$



The triple product calculates the volume of a parallelepiped as shown.

In-class Revision Exercise

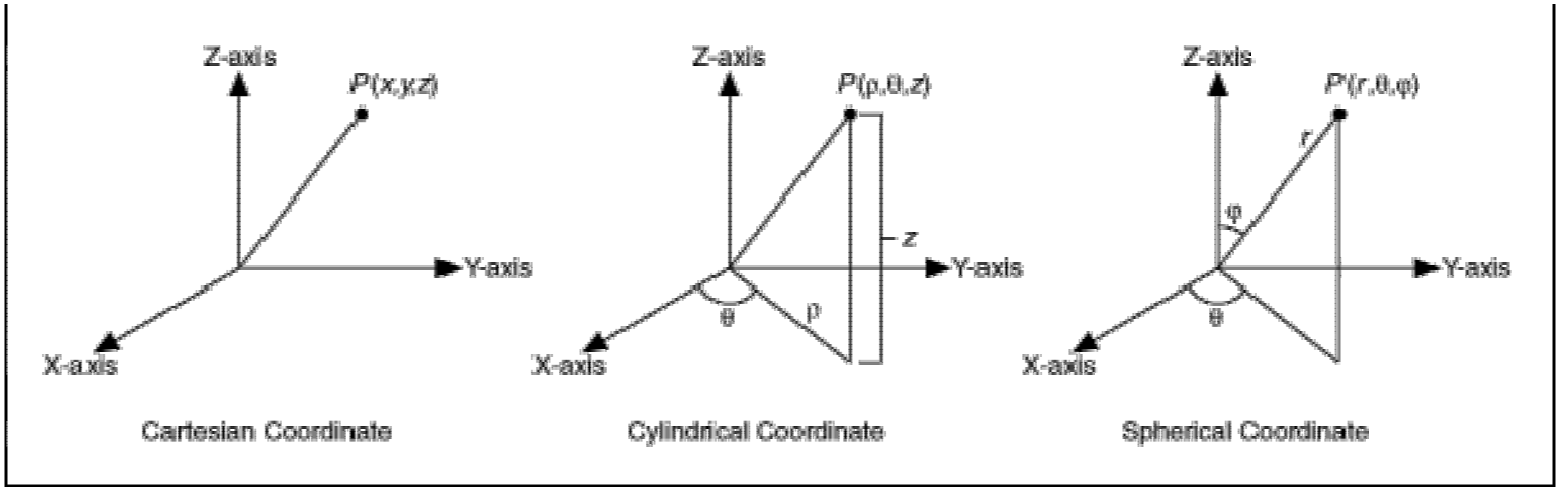
Time:
10 minutes 

- Take out a pen or pencil.
- Work individually. 
- Answer the questions on the vector algebra revision sheet.

- This is an exercise, <u>not</u> a test. It is more important to learn what you do and do not remember than to get the right answer. Do not confer with your classmates.
- The numbers in this exercise are simple. You will not require a calculator.
- Don't forget to write your UCD student number at the top of the paper.

Co-ordinate systems

- You are familiar with Cartesian co-ordinates.
- In this course, it will sometimes be necessary to switch to spherical polar co-ordinates or cylindrical co-ordinates in order to solve a problem.



Vector calculus

When dealing with electromagnetics, we prefer to use vector calculus.

There are three important operations

- Divergence
- Gradient and
- Curl

Gradient

- Applies to scalar fields.

$$\vec{\nabla}\psi = \hat{\imath}\frac{\partial\psi}{\partial x} + \hat{\jmath}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}$$

- Direction and steepness of slope at each point in the field.

Example

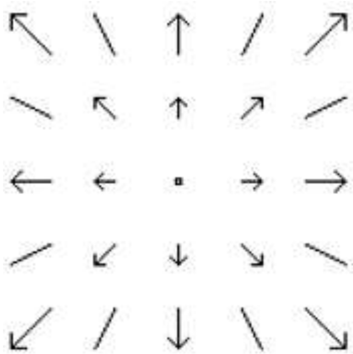
- Height of a mountain in m near a village is described by function

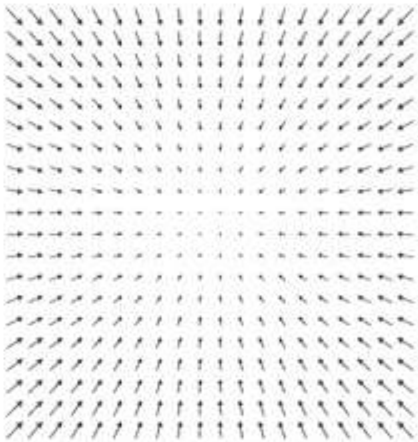
  $$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

- x is the distance in km east of the village, and y is the distance north.
  a) Find the top of the mountain.
  b) How high is the mountain?
  c) I want to build a house 1 km north and 1 km east of the village. How steep is the mountain here?
  d) At this point, in what direction does the slope run?

Divergence

- Important in many areas of physics.
- Later we will discuss electric flux. Both flux and divergence deal with flow of a field, but flux is defined over an area, while divergence deals with points.



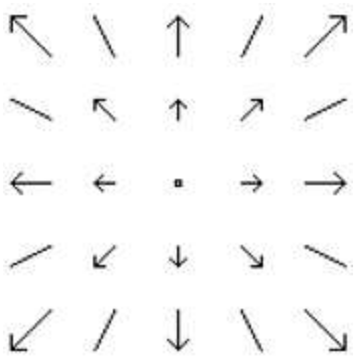


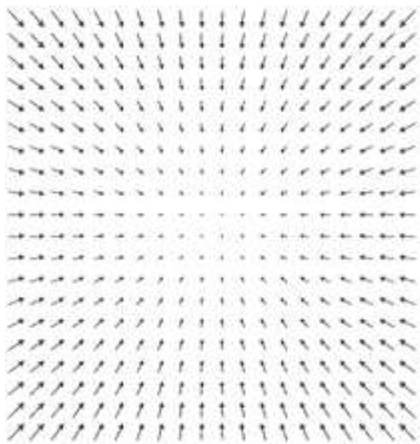
Divergence

- Given by

$$\vec{\nabla}\circ\vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

- *Scalar* quantity: no direction.
  - Positive divergence -> sources
  - Negative divergence -> sinks





Divergence

- Spherical co-ordinates:

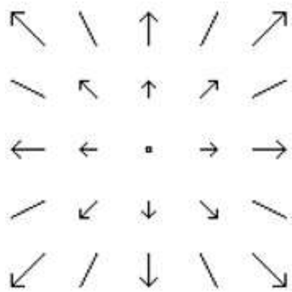
$$\vec{\nabla}\circ\vec{A} = \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)$$

- Cylindrical co-ordinates:

$$\vec{\nabla}\circ\vec{A} = \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\right)$$

Example

- Calculate the divergence of the following function.



$$f = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{\nabla}\circ\vec{f} = \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right)$$

$$\vec{\nabla}\circ\vec{f} = \left(\frac{\partial}{\partial x}(x) + \frac{\partial f_y}{\partial y}(y) + \frac{\partial f_z}{\partial z}(z)\right)$$

$$\vec{\nabla}\circ\vec{f} = (1 + 1 + 1) = 3$$

Curl

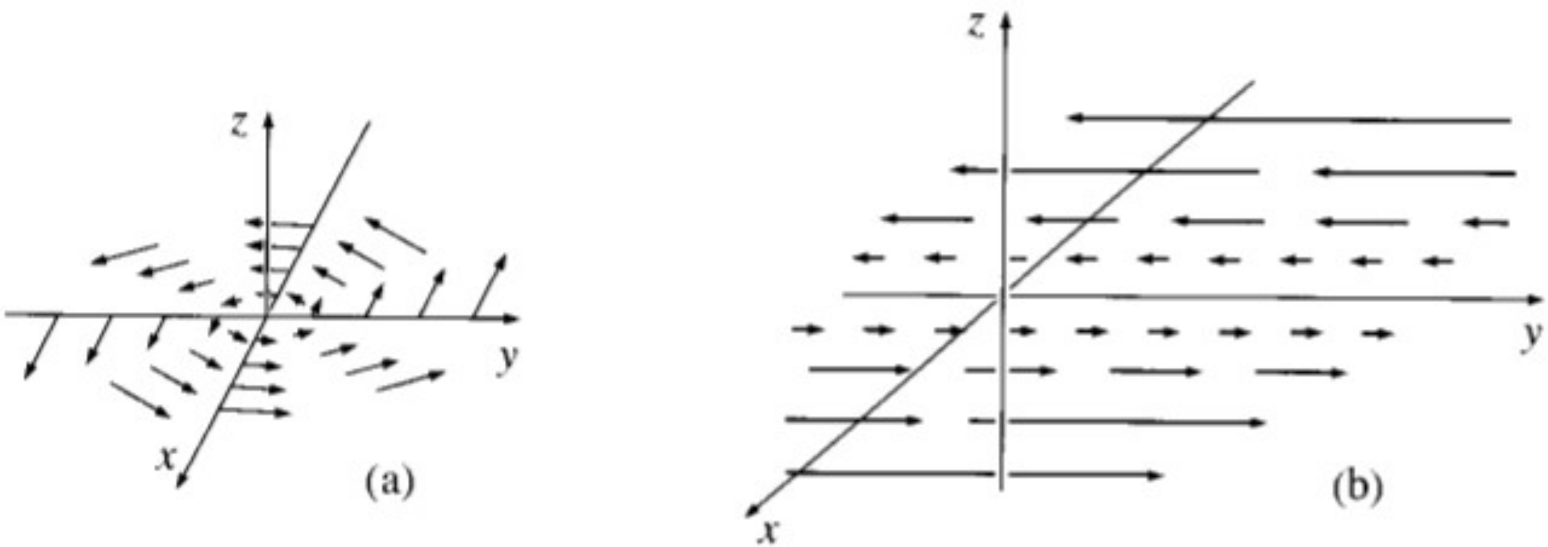
- Can be defined as:

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

$$=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{k}$$

Curl

- Each component of the curl indicates the tendency of the field to rotate in one of the co-ordinate planes.



(a) (b)

Example

- Calculate the curl of

$$f = -y\hat{x} + x\hat{y} \quad \text{and} \quad g = x\hat{y}$$

$$\vec{\nabla}\times\vec{\mathrm{f}} = \begin{vmatrix} \hat{\mathrm{x}} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 0\hat{x} + 0\hat{y} + 2\hat{z} = 2\hat{z}$$

$$\vec{\nabla}\times\vec{g} = \begin{vmatrix} \hat{\mathrm{x}} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = 0\hat{x} + 0\hat{y} + \hat{z} = \hat{z}$$

Product rules

- Curl of grad is always zero.

$$\vec{\nabla} \times \vec{\nabla}\psi = 0$$

- Laplacian (i.e. div of grad)

$$\vec{\nabla}\circ\vec{\nabla}\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

- We can use these, e.g. to obtain Poisson's equation. (See later in the course.)
- Grad of div less Laplacian is curl of curl.

$$\vec{\nabla}\left(\vec{\nabla}\circ\psi\right) - \nabla^2\psi = \vec{\nabla} \times \left(\vec{\nabla} \times \psi\right)$$

Helmholtz theorem

- If we know the div and curl of a field, can you uniquely define the field?
  - No, we also require boundary conditions.
- The Helmholtz Theorem tells us that if we require the fields to go to zero at infinity, we can uniquely define the field given its div and curl.

Potentials

- If a field has zero curl everywhere (many fields we will encounter do), then the field may be written as the gradient of a scalar potential.

$$\nabla \times \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = -\nabla V$$

Integral calculus

- Line, surface and volume integrals
- Fundamental theorem of calculus
- Fundamental theorem for gradients
- Fundamental theorem for divergences
- Fundamental theorem for curls
- Integration by parts

Charge and current

- Moving charge.
- Unit of current is the Ampere (A).