Beijing Dublin International College

EEEN3006J Wireless Systems

Problem set: Electromagnetic Fields and Waves

From – A students guide to Maxwell's Eqns - Fleisch

Problem 1: Calculate the electric flux through the surface of a cube that contains 12 protons and 7 electrons.

Does the size of the cube alter the result? No

Flux = q_{enc}/E_0

q_{enc} = 12*(1.6x10^{-19} C) + 7*(-1.6x10^{-19} C)

q_{enc} = 1.92x10^{-18}+1.12x10^{-18} = 8x10^{-19}

Flux = q_{enc}/E_0 = 8x10^{-19}/8.85x10^{-12} = 9.04x10^{-8}

Problem 2: A cube of side L m contains a flat plate with variable surface charge density of $\boldsymbol{\sigma}$ = -3$\boldsymbol{xy}$ **C/m^2**. If the plate extends from $\boldsymbol{x}$ = 0 to $\boldsymbol{x} = \boldsymbol{L}$ and from $\boldsymbol{y}$ = 0 to $\boldsymbol{y} = \boldsymbol{L}$, what is the total electric flux through the walls of the cube?

Use Gaus's Law. Flux = q_{enc}/ε_0

Find the charge on the plate:

integrate over the area of the plate using the formula for charge density

Qenc = $\int_{y=0}^{L}\int_{x=0}^{L} \sigma\ dxdy$

Qenc = $\int_{y=0}^{L}\int_{x=0}^{L} -3xy\ dxdy$

Qenc = $-\frac{3L^4}{4}$

Flux = $-\frac{3L^4}{4\varepsilon 0}$

Problem 3: What is the flux through an arbitrary closed surface surrounding a charged sphere of radius a_0 with volume charge density of $\boldsymbol{\rho} = \boldsymbol{\rho_0(r/a_0)}$, where $\boldsymbol{r}$ is the distance from the centre of the sphere?

Use Gauss's Law. Flux = q_{enc}/ε_0

First get the charge of the sphere. Integrate charge density over the sphere.
Note that the charge density changes with respect to radius r!
Use spherical co-ordinates

Qenc = $\int_V \rho dV$

Integrate over the volume of a sphere:

=> $\int_r \int_\phi \int_\theta (r^2)dr\, d\theta\, (sin\theta)d\phi$

= $\int_r \int_\phi \int_\theta r^2.sin\theta\, drd\theta d\phi$

qenc = $\int_V \rho 0 \left(\frac{r}{a0}\right) dV$

qenc = $\int_{r=0}^{a0} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \rho 0 \left(\frac{r}{a0}\right)(r^2)(sin\theta)drd\theta d\phi$

= $\rho 0 a 0^3 \pi$

Flux = q_{enc}/ε_0

= $\rho 0 a 0^3 \pi/\varepsilon 0$

Problem 4: Use a special Gaussian surface around an infinite line charge to find the electric field of the line charge as a function of distance.

Logic suggests that for a line charge the electric field must be radial (pointing toward or away from the charge on the line), and the field must be the same all along the line (since the line is infinitely long, and no section of the line is any different from any other section). Hence, a cylinder with its axis along the line is a good choice for your special Gaussian surface in this case.

You can now use Gauss's law to find the electric field:

$$\oint_S \vec{E} \circ \hat{n}\, da = q_{enc}/\varepsilon_0$$

For the top and bottom surfaces of the cylinder, $\vec{E} \circ \hat{n} = 0$, since $\vec{E}$ is perpendicular to the surface normals. For the curved side of the cylinder, $\vec{E}$ is parallel to $\hat{n}$, so $\vec{E} \circ \hat{n} = |\vec{E}|$. And since the electric field must be constant over the curved surface, you can pull $|\vec{E}|$ outside the integral.

The integral then becomes the area of the curved surfac of the cylindar:

$$\oint_S \bar{E} \circ \hat{n}\, da = \oint_S |\bar{E}| da = |\bar{E}| \oint_S da = |\bar{E}|(2\pi\, rL)$$

Qenc for a line charge of length of wire L (the wire inside the cylinder) and charge density λ is:

= λL

Thus:

$$E(2\pi rL) = \frac{\lambda L}{\varepsilon 0}$$

$$E = \frac{\lambda}{\varepsilon 0(2\pi r)}$$

Problem 5: Find the divergence of the field given by $\vec{E} = \frac{1}{r}\hat{r}$ in spherical coordinates.

The divergence in spherical coordinates is given by:

$$\bar{\nabla} \circ \bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$Div.E = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(0.\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(0)$$

Note: no Field in E_θ or E_ϕ coordinates.

$$Div.E = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r}\right)$$

$$Div.E = \frac{1}{r^2}\frac{\partial}{\partial r}(r)$$

$$Div.E = \frac{1}{r^2}$$

Problem 6 Find the charge density in a region for which the electric field in cylindrical coordinates is given by

$$\vec{E} = \frac{az}{r}\hat{r} + br\hat{\phi} + cr^2 z^2 \hat{z}$$

We are looking for charge density at a point given an E field….
We can use differential form of Gauss's Law.

$$\bar{\nabla} \circ \bar{E} = \rho / \varepsilon_0$$

Div in cylindrical coordinates:

$$\bar{\nabla} \circ \bar{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

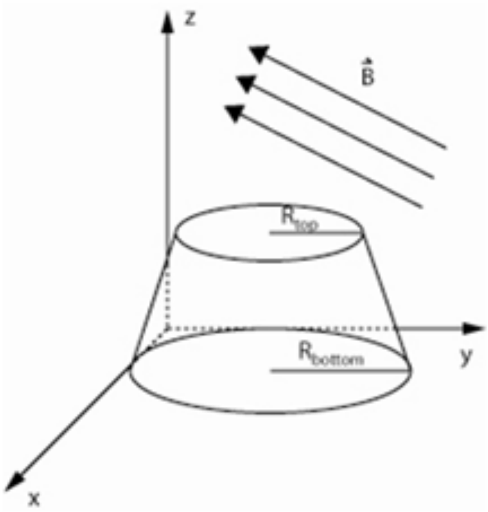
$$Div.E = \frac{1}{\mathrm{r}}\frac{\partial}{\partial r}\left(r\frac{az}{r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}(br) + \frac{\partial}{\partial z}(cr^2 z^2)$$

$$Div.E = \frac{1}{\mathrm{r}}0 + \frac{1}{r}0 + (2cr^2 z)$$

$$(2cr^2 z) = \frac{\rho}{\varepsilon_0}$$

$$(2cr^2 z)\varepsilon_0 = \rho$$

Problem 7: Find the magnetic flux produced by the magnetic field $\vec{B} = 5\hat{i} - 3\hat{j} + 4\hat{k}$ T through the top, bottom, and side surfaces of the flared cylinder shown in the figure.



Gauss's law for magnetic field tells you that the magnetic flux through any closed surface must be zero

$$\Phi_B = \oint_S \vec{B} \circ \hat{n}\ da = 0$$

We can say that:

$$\Phi_{B,top} + \Phi_{B,botttom} + \Phi_{B,sides} = 0$$

So, one approach to this problem is to calculate the flux through the top and bottom surfaces of the flared cylinder, and then to use that value to determine the flux through the curved side.

$$\Phi_{top} = \int_{top} B \circ \hat{n}\ da$$

$$\Phi_{bottom} = \int_{bottom} B \circ \hat{n}\ da$$

and we can also say:

$$\hat{n}_{top} = \hat{k} \quad \text{and} \quad \hat{n}_{bottom} = -\hat{k}$$

so:

$$\Phi_{top} = \int_{top} (5\hat{\imath} - 3\hat{\jmath} + 4\hat{k}) \circ \hat{k}\ dA_{top}$$

only the 4k component part of the B field will go through the surface.

$$\Phi_{top} = \int_{top} 4\ dA_{top}$$

$$\Phi_{top} = 4\ \pi R_{top}^2$$

$$\Phi_{bottom} = \int_{bottom} (5\hat{\imath} - 3\hat{\jmath} + 4\hat{k}) \circ -\hat{k}\ dA_{bottom}$$

$$\Phi_{bottom} = -4\ \pi R_{bottom}^2$$

$$\Phi_{sides} = -(\Phi_{bottom} + \Phi_{top})$$

$$\Phi_{sides} = 4\pi(R_{bottom} - R_{top})$$

Problem 8: A square loop lies in a plane perpendicular to the following magnetic field.

$$\vec{B}(t) = B_0 e^{-5t/t_0}\,\hat{i}$$

Find the emf induced in the loop by the field.

Looking to find the EMF of a changing B field over time:

Faradays law.

$$emf = -\frac{d}{dt}\int_S \bar{B} \circ \hat{n}\, da$$

Since the loop in this problem lies in the yz-plane, you can take the unit normal as $\hat{n} = \hat{i}$, which means

$$emf = -\frac{d}{dt}\int_S B_0 e^{-5t/t_0}\,\hat{i} \circ \hat{i}\, da$$

$$= -\frac{d}{dt}[B_0 e^{-5t/t_0}\int_S da]$$

$$= -\frac{d}{dt}[B_0 e^{-5t/t_0}(a^2)]$$

$$= -a^2 B_0 \frac{d}{dt}[e^{-5t/t_0}]$$

$$= \frac{5a^2 B_0}{t_0}\, e^{-5t/t_0} \quad \text{(answer)}$$

Problem 9: The current in a long solenoid varies as $I(t) = I_0 \sin(\omega t)$. Use Faraday's law to find the induced electric field as a function of r both inside and outside the solenoid, where r is the distance from the axis of the solenoid.

For this problem it's important to remember that Faraday's law indicates that a circulating electric field will be induced by a changing magnetic field even if no physical loop is present. Thus

$$\oint_C \bar{E} \circ d\bar{l} = -\frac{d}{dt}\int_S \bar{B} \circ \hat{n}\, da$$

To find the induced electric field as a function of r , the distance from the solenoid axis, consider an imaginary loop of radius r with normal along the solenoid axis (so that the magnetic field of the solenoid is perpendicular to the loop area). If r Â is less than the radius of the solenoid, then

$$\oint_C \bar{E} \circ d\bar{l} = |\bar{E}| \int_C |d\bar{l}| = |\bar{E}|(2\pi r)$$

And

$$-\frac{d}{dt}\int_S \bar{B} \circ \hat{n}\, da = -\frac{d}{dt}\left(\frac{\mu_0 N I}{l}\pi r^2\right)$$

So

$$|\bar{E}|(2\pi r) = -\frac{\mu_0 N(\pi r^2)}{l}\frac{dI}{dt}$$

Since

$$I = I_0 \sin(\omega t), \quad \frac{dI}{dt} = \omega I_0 \cos(\omega t),$$

And

$$|\bar{E}|(2\pi r) = -\frac{\mu_0 N(\pi r^2)}{l}[\omega I_0 \cos(\omega t)]$$

Thus

$$\begin{aligned} |\bar{E}| &= -\frac{\mu_0 N(\pi r^2)\omega I_0}{2\pi r l}\cos(\omega t) \\ &= -\frac{\mu_0 N r \omega I_0}{2l}\cos(\omega t) \qquad \text{(answer)} \end{aligned}$$

Here the minus sign indicates that the direction of the induced electric field is so as to drive a current that opposes the change in magnetic flux.

Outside the solenoid, where the loop radius (r) is greater than the solenoid radius (R), you have

$$\oint_C \bar{E} \circ d\bar{l} = -\frac{d}{dt}\int_S \bar{B} \circ \hat{n}\, da = -\frac{d}{dt}\left[\frac{\mu_0 N I}{l}\pi R^2\right]$$

So

$$|\bar{E}|(2\pi r) = -\frac{\mu_0 N(\pi R^2)}{l}[\omega I_0 \cos(\omega t)]$$

And

$$|\bar{E}| = -\frac{\mu_0 N R^2 \omega I_0}{2 r l}\cos(\omega t) \quad \text{(answer)}$$

Notice that inside the solenoid, the induced electric field increases with distance from the axis, while outside the solenoid the induced electric field decreases as $1/r$.

Problem 10: Find the displacement current produced between the plates of a discharging capacitor for which the charge varies as

$$Q(t) = Q_0 e^{-t/RC}$$

where Q_0 is the initial charge, C is the capacitance of the capacitor, and R is the resistance of the circuit through which the capacitor is discharging.

the displacement current is:

$$I_d = \varepsilon_0 \frac{d}{dt}\int_S \bar{E} \circ \hat{n}\ da$$

To evaluate this, remember that the magnitude of the electric field between the two conducting plates is

$$\left|\bar{E}\right| = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

where Q is the magnitude of the charge on each plate and A is the area of each plate.

Thus

$$I_d = \varepsilon_0 \frac{d}{dt}\int_S \frac{Q}{\varepsilon_0 A}\ da = \varepsilon_0 \frac{d}{dt}\left[\frac{Q}{\varepsilon_0 A}\int_S da\right] = \frac{dQ}{dt}$$

In this case,

$Q = Q_0\, e^{-t/RC}$, so

$$I_d = \frac{dQ}{dt} = \frac{d}{dt}\left[Q_0\, e^{-t/RC}\right] = Q_0 \frac{d}{dt}\left[e^{-t/RC}\right]$$

$$I_d = -\frac{Q_0}{RC}\, e^{-t/RC} \quad \text{(answer)}$$

Problem 12: A cell phone is 1 km from a receiver. It transmits at a frequency of 1.9 GHz and has a transmit power of 2 W. Its antenna has a gain of 0dB. The cell tower's antenna has a receiver gain of 17 dB. Receiver noise is 2 dBm. Calculate the received SNR.

$r = 1000$ m
$f = 1.5$ GHz
$\lambda = c / f = 3 \times 10^8/1.5 \times 10^9 = .2$ m
$P_{tx} = 2$ W
$G_{tx} = 0$dB
$G_{rx} = 17$dB => $10\log_{10}$(linear gain) = 50.11
Noise = 2 dBm => $10\log_{10}$(noise (mW)) = 1.58 mW

SNR = $P_{rx}/1.58$ mW

$P_{rx} = G_{tx}P_{tx}G_{rx}\lambda^2/(4\pi)^2r^2$

$P_{rx} = 17 * 2 * 50.11 * (.2)^2 / ((4 * \pi)^2 * 1000^2)$

$P_{rx} = 4.3 \times 10^{-7}$ W

SNR = $4.3 \times 10^{-5}/1.58 = 2.7 \times 10^{-5}$

Problem 13: A Bluetooth transceiver operates on the 2.4 GHz ISM band, with transmit power 1 mW and receiver sensitivity -90 dBm, using the onboard antenna with gain -0.5 dB. Calculate the maximum distance at which two of these transceivers could communicate, assuming ideal free-space propagation conditions.

$f = 2.4$ GHz
$\lambda = c / f = 3 \times 10^8/2.4 \times 10^9 = .125$ m
$P_{tx} = 1$ mW
$G_{rx} = G_{tx} = -0.5$ dB = $10\log_{10}$(linear Gain) = 0.89
$P_{rx} > -90$dBm = $10\log_{10}$(Receive power (mW)) = 1×10^{-9} mW

$P_{rx} = G_{tx}P_{tx}G_{rx}\lambda^2/(4\pi)^2r^2$

$r < \sqrt{(G_{tx}P_{tx}G_{rx}\lambda^2/(4\pi)^2P_{rx})}$

$r < \sqrt{(.89^2 * 1 * .125^2 / ((4 * \pi)^{2} * 1 \times 10^{-9}))}$
$r < 280$ m

Problem 14: A radio transmitter is being designed to operate at frequencies from 950 MHz to 1.03 GHz. The transmit antenna gain is 1.5 dB at these frequencies. The receive antenna gain is 12 dB. The receiver sensitivity is -95 dBm. The required range is 50 km over a clear line of sight path. What transmit power is required?