Chapter 1

Course outline

1.1 Part I

1. Sampling Theorem
2. Aliasing
3. Discrete Time Fourier Transform (DTFT)
4. Discrete Fourier Transform (DFT)
5. Fast Fourier Transform (FFT)
6. Discrete Convolution (direct and periodic)
7. Parseval's Theorem
8. Spectrum Estimation (Welch's method)

1.2 Part II

1. The z transform
2. Discrete Linear systems
   (Transfer functions, impulse response, pole/zero plots)
3. Digital Filters - Intro
4. Finite Impulse Response (FIR) digital filters
   (Phase linearity condition, design Methods)

5. Window design method of FIR Filters

6. Stability of Digital Filters

7. Infinite Impulse Response Digital Filters (design methods)

This will be followed by an end-of-term exam.

1.3 Grading

	Component	%
1.	Matlab assignments / Labs	15%
2.	End-of-term exam	85%

Table 1.1: Overall grade breakdown.

1.4 Recommended texts

There two main books I recommend:

1. "Digital Signal Processing", by John G. Proakis & Dimitris G. Manolakis, and

2. (advanced text) "Digital signal Analysis" by Samuel D. Stearns & Don R. Hush

The first book is by Proakis who is very well known in DSP and communications, I highly recommend this text for this module and as an every day DSP handbook. The other text is more technical, harder to follow, but arguably better in some detailed derivations. Material from both books was used in the development of this module.

Chapter 2

Impulse sampling and Discrete Signals

In this chapter, we will learn what continuous and discrete signals are. DSP is the study of how we process (analyse and change) discrete signals. Real-world signals are often continuous, but computers can only deal with discrete signals, so the foundation of all our analyses is to understand how continuous and discrete signals are related to each other. To that end, this chapter will introduce the Dirac delta function, its definition and properties.

2.1 Continuous signals

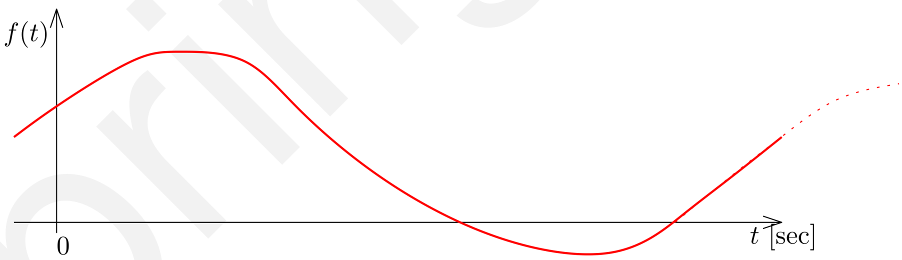

Figure 2.1.1: Continuous time analog signal

A *continuous signal*, $f(t)$, has the following properties:

- It is continuous in t.
- It is defined for all t, i.e. $-\infty \to +\infty$.
- It can obtain any amplitude value.

$f(t)$ could be a voltage, a current, a light intensity, a stock price, or anything we care to measure and plot. The independent variable t could be time, distance or some other quantity. A continuous signal is also known as an *analog signal*. An example of a *continuous signal* is shown in Figure 2.1.1. $f(t)$ can be real or complex, depending on what the signal is.

2.2 The Dirac delta function $\delta(t)$

2.2.1 Definition

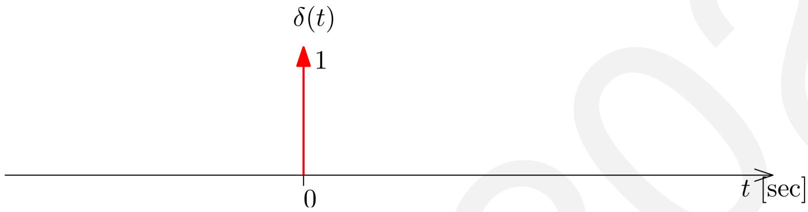

Figure 2.2.1: The Dirac delta function $\delta(t)$. Note the weight marked on the figure is “1”, i.e. the signal is $1.\delta(t)$.

A *Dirac delta function*, $\delta(t)$, otherwise know as an *impulse* like the one shown in Figure 2.2.1, is defined by how it behaves within an integral:

$$\int_{\mathcal{D}} \delta(t)\, dt \triangleq \begin{cases} 1 & \text{if } \mathcal{D} \text{ contains orgin } (t=0) \\ 0 & \text{otherwise} \end{cases}$$

The first part says the area under the function in any arbitrarily small, non-zero region about $t = 0$ is always $= 1$ irrespective of the size of the integration range; i.e. we say $\delta(t)$ has *weight* 1. (Note that the function has infinite amplitude at $t = 0$). The second part says that the delta function is zero everywhere except at $t = 0$ as shown in Figure 2.2.1.

A Dirac delta function is often simply referred to as an *impulse.*

We can think of an impulse as being an impossibility narrow rectangular pulse with unit area centered on $t = 0$:

$$\begin{aligned} \delta\left(t\right) &= \lim_{\varepsilon \to 0}\left\{\mathrm{rect}\left(\varepsilon;\, t\right)\right\} \\ \text{Where} \quad \mathrm{rect}\left(\varepsilon;\, t\right) &\triangleq \begin{cases} \frac{1}{\varepsilon} & \text{if } |t| < \frac{\varepsilon}{2} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

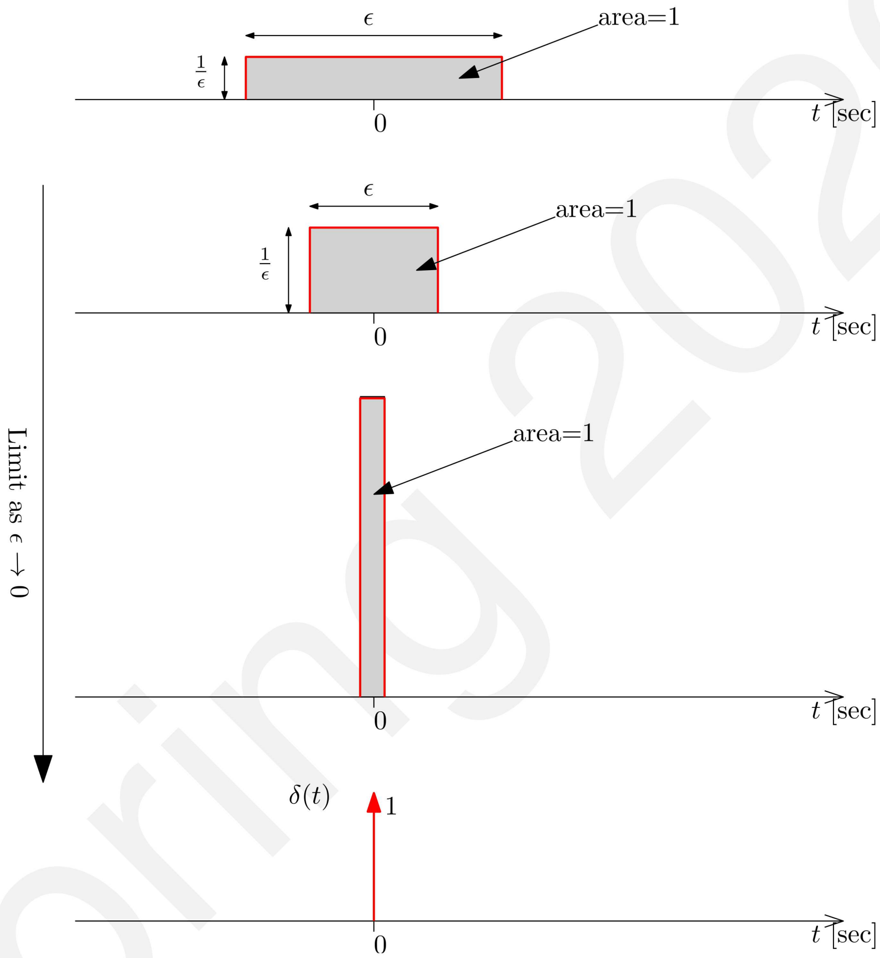

Figure 2.2.2: The Dirac delta function can be thought of as the limit of a rectangle function of area 1 and width ϵ in the limit as $\epsilon \to 0$.

2.2.2 The delayed impulse

An impulse that is delayed in time by t_o seconds is written $\delta\left(t - t_o\right)$. It is illustrated in Figure 2.2.3. If t represents a distance, we sometimes call this property *shifting*.

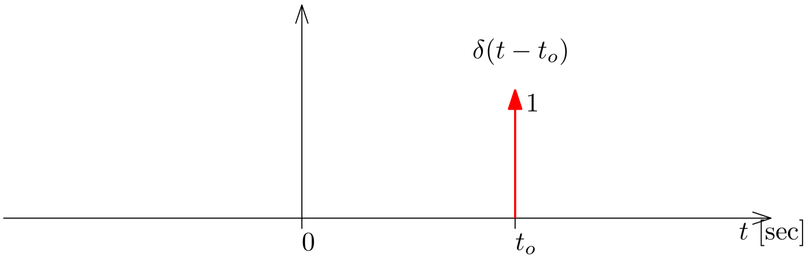

Figure 2.2.3: The Dirac delta function, delayed by t_0 s. Note that t_0 can be positive or negative. delayed by t_o.

2.2.3 The weighted impulse

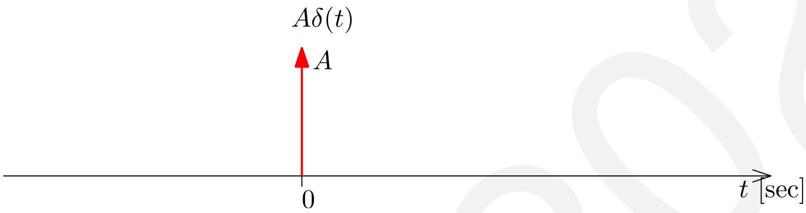

Figure 2.2.4: The Dirac delta function weighted by A.

An impulse with weight A is simply written $A\delta(t)$. It is illustrated in Figure 2.2.4.

2.2.4 Sifting property

Perhaps the most important property of the delta function is that of the sifting[1] property. Simply put it states that integral of any function $f(t)$ times a delta function $\delta(t-t_o)$ is just just the original function evaluated at $t=t_o$, i.e.:

$$\int_D f(t)\,\delta(t-t_o)\,dt \quad = \quad f(t_o)$$

provided the integration range D includes t_o.

Some textbooks actually take this as the definition of the delta function, and prove everything else from that starting point, i.e. it's the delta function's behavior in an integral that makes it so important.

[1] Some students confuse sifting and shifting. Please make sure you remember the difference!

2.2.5 Scaling property

Care must be taking when changing variables involving the delta function. We have the following property:

$$\delta(\lambda x) = \frac{1}{|\lambda|}\delta(x)$$

for any real valued scalar λ.

This is particularly an issue when we have a delta function in the frequency domain f and we do a change of variable to $\omega = 2\pi f$. In that case, we have:

$$\delta(f) = 2\pi\delta(\omega)$$

As a consequence, we note the following Fourier transform pairs:

$$\begin{aligned} e^{j2\pi f_o t} &\leftarrow \mathcal{F} \rightarrow \delta(f - f_o) \\ &\text{and} \\ e^{j\omega_o t} &\leftarrow \mathcal{F} \rightarrow 2\pi\delta(\omega - \omega_o) \end{aligned} \tag{2.2.1}$$

Exercise:

Prove the scaling property. (Hint: start by computing $\int_{-\infty}^{+\infty} f(x)\,\delta(\lambda x)\,dx$).

Solution:

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)\,\delta(\lambda x)\,dx &= \begin{cases} \frac{1}{\lambda}\int_{-\infty}^{+\infty} f\left(\frac{1}{\lambda}u\right)\delta(u)\,du & \text{for } \lambda > 0 \\ \frac{1}{\lambda}\int_{+\infty}^{-\infty} f\left(\frac{1}{\lambda}u\right)\delta(u)\,du & \text{for } \lambda < 0 \text{ (note sign of limits)} \end{cases} \\ &= \frac{1}{|\lambda|}\int_{-\infty}^{+\infty} f\left(\frac{1}{\lambda}u\right)\delta(u)\,du \\ &= \frac{1}{|\lambda|}f(0) \\ &= \frac{1}{|\lambda|}\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx \\ \Rightarrow \delta(\lambda x) &= \frac{1}{|\lambda|}\delta(x) \end{aligned}$$

2.2.6 The impulse train $\delta_T(t)$

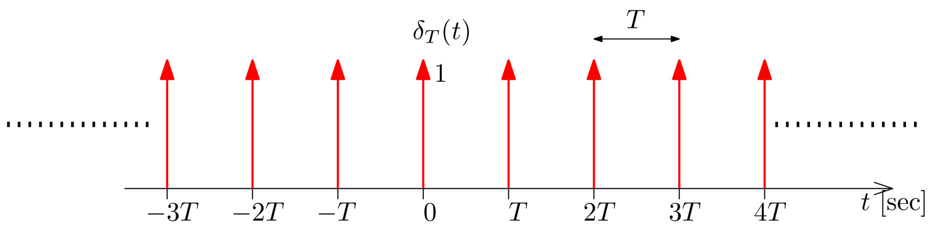

Figure 2.2.5: A train (sequence) of uniformly spaced impulses.

A train (sequence) of uniformly spaced impulses, $\delta_T(t)$, is illustrated in Figure 2.2.5 and defined as:

$$\delta_T(t) \triangleq \sum_{n=-\infty}^{+\infty} \delta(t-nT).$$

2.2.7 Fourier of series $\delta_T(t)$

We will use the delta train to sample an analog signal, so we will call T the sampling period or interval, and $f_s \triangleq \frac{1}{T}$ the sampling rate or frequency. The units of T are seconds, and the units of f_s are Hertz.

As $\delta_T(t)$ is a periodic function with frequency f_s, it can be written as a Fourier Series, i.e. an infinite weighted sum of complex sinusoids having frequencies equal to integer multiples of f_s, as follows.

$$\delta_T(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn2\pi f_s t} = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_s t}, \tag{2.2.2}$$

where we define, for convenience, $\omega_s \triangleq 2\pi f_s$, meaning the angular sampling frequency (units = rads/sec). We will see later that DSP engineers, for good reason, don't like Hz as a measure of frequency, but rather they often use this angular frequency instead.

The C_n are the Fourier coefficients which are given by:

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \delta_T(t)\, e^{-jn\omega_s t} dt \quad n = -\infty, \cdots, 0, \cdots, +\infty.$$

As the range of integration only contains one delta function (see Figure 2.2.6), we can replace the train $\delta_T(t)$ by just $\delta(t)$, yielding:

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \delta(t)\, e^{-jn\omega_s t} dt \quad n = -\infty, \cdots, 0, \cdots, +\infty$$

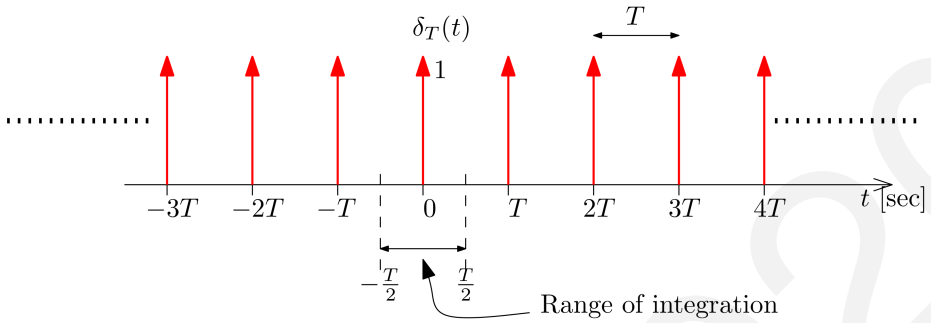

Figure 2.2.6: Range of integration in the Fourier coefficients computation.

By the sifting property of the delta function (see Section 2.2.4) we have:

$$C_n = \frac{1}{T} e^{-jn\omega_s 0} = \frac{1}{T} \quad \forall n,$$

i.e. all the Fourier coefficients have the same value $\frac{1}{T}$. Putting this into equation 2.2.2 gives us the Fourier series of the impulse train:

$$\boxed{\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} e^{jn\omega_s t}.} \tag{2.2.3}$$

We know from equation (2.2.1) that a complex exponential in the time domain is a delta function in the frequency domain, so we have the following Fourier transform:

$$\begin{aligned} \delta_T(t) &\leftarrow \mathcal{F} \rightarrow \frac{1}{T}\delta_{f_s}(f) = f_s \delta_{f_s}(f) \\ &\text{or} \\ \delta_T(t) &\leftarrow \mathcal{F} \rightarrow \frac{2\pi}{T}\delta_{\omega_s}(\omega) = \omega_s \delta_{\omega_s}(\omega) \end{aligned} \tag{2.2.4}$$

This is illustrated in Figure (2.2.7).

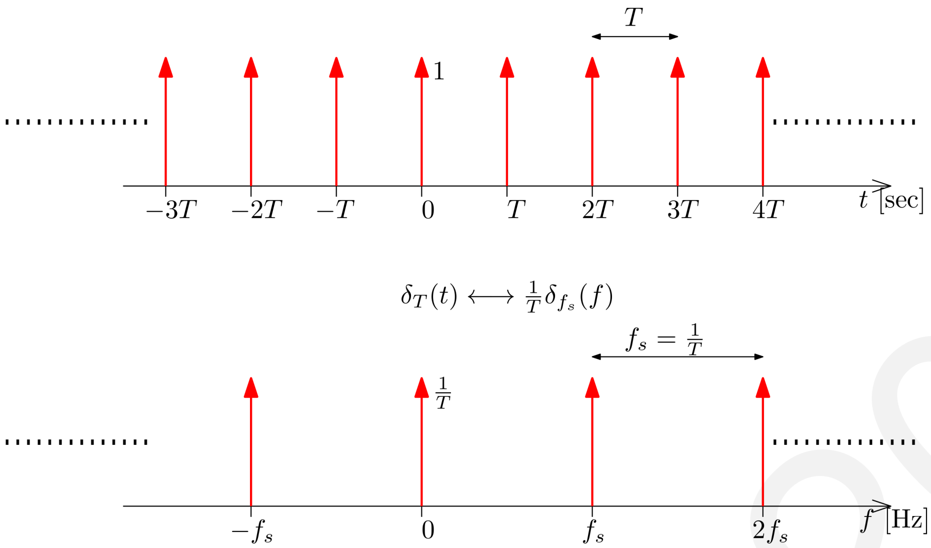

Figure 2.2.7: Fourier Transform of a train of delta function is another train of delta functions with inverse spacing and weight $\frac{1}{T} = f_s$. (Note that if plotted against ω instead the weights would be $\frac{2\pi}{T} = \omega_s$).

2.3 Discrete Signals

Mathematically, a *discrete*[2] signal can be obtained from a continuous one by multiplying the continuous signal by train of impulses as shown in Figure 2.3.1. This is called impulse sampling.

[2]Some students confusing the spelling discrete (in separate pieces) with discreet (prudent silence). Once again, please be careful about this.

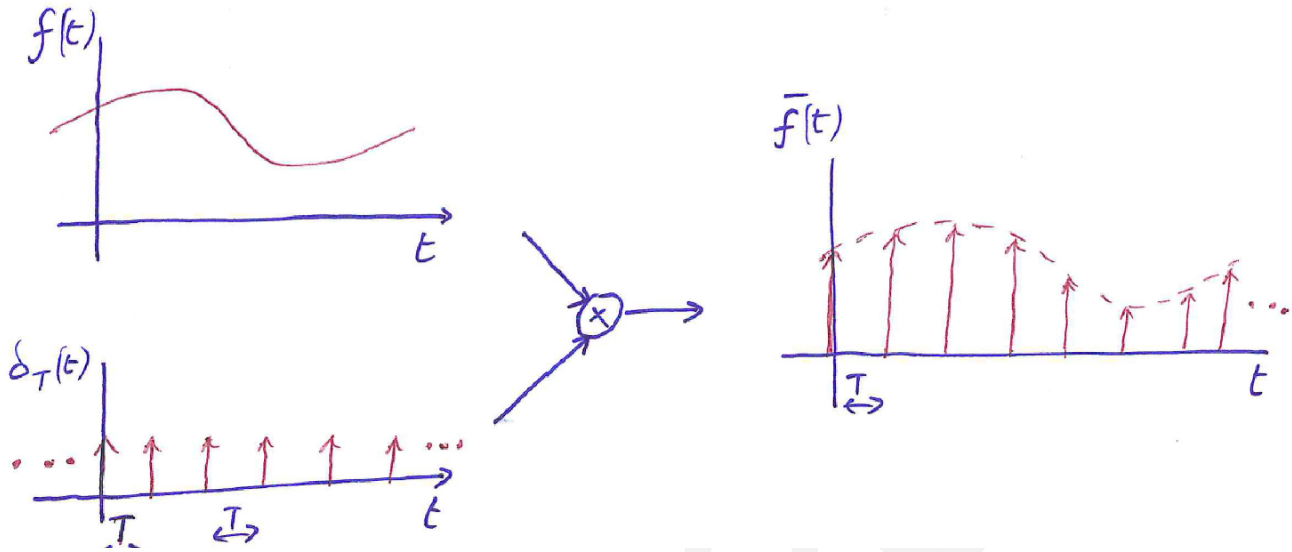

Figure 2.3.1: Impulse sampling.

We can write this discrete signal, $\bar{f}(t)$, as follows.

$$\begin{aligned}\bar{f}(t) &\triangleq f(t)\,\delta_T(t)\\ &= f(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)\\ &= \sum_{n=-\infty}^{+\infty} f(nT)\,\delta(t-nT),\end{aligned}$$

where T is sampling interval. The sampling frequency is $f_s = \frac{1}{T}$ Hz.

Notation:

It is usual to use the following notation

$$f(t)|_{t=nT} = f(nT) \triangleq f_n$$

Here f_n represents the sample at ANY time instant $t = nT$. We use the notation $\{f_n\}$ to represent the complete set of samples, i.e. ALL samples of a discrete signal.

Using this notation we have:

$$\bar{f}(t) = \sum_{n=-\infty}^{+\infty} f_n \delta(t - nT)$$

2.3.1 Properties of $\bar{f}(t)$

$\bar{f}(t)$ has the following basic properties:

- $\bar{f}(t)$ is a train of impulses, defined for all t.
- the weights f_n are the samples of the signal taken at time $t = nT$
- the amplitudes $\bar{f}(nT)$ are all infinite (except for the case where weight is zero).

Sometimes we are only interested in the values of the samples, $\{f_n\}$. Then, we plot the discrete signal using a *stem diagram*, as depicted in Figure 2.3.2.

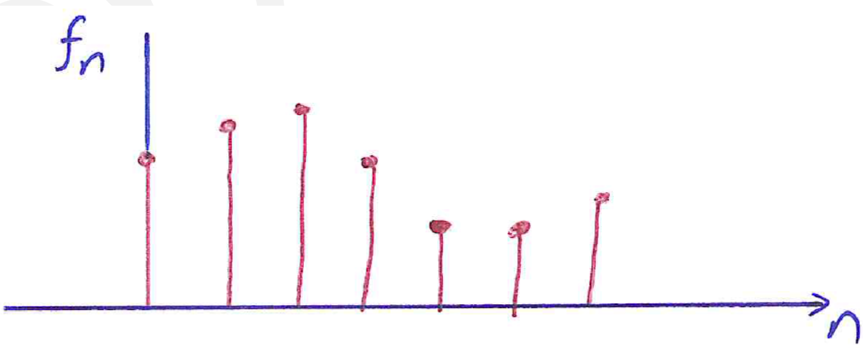

Figure 2.3.2: Illustration of a sampled signal using a stem diagram.

Discrete signals can be real or complex. Discrete signals can be multidimensional, e.g. a still image is 2-D (x and y), video is 3-D (x, y, and t). The pixels on the screen in your phone or laptop are very small, but they are samples of an image.

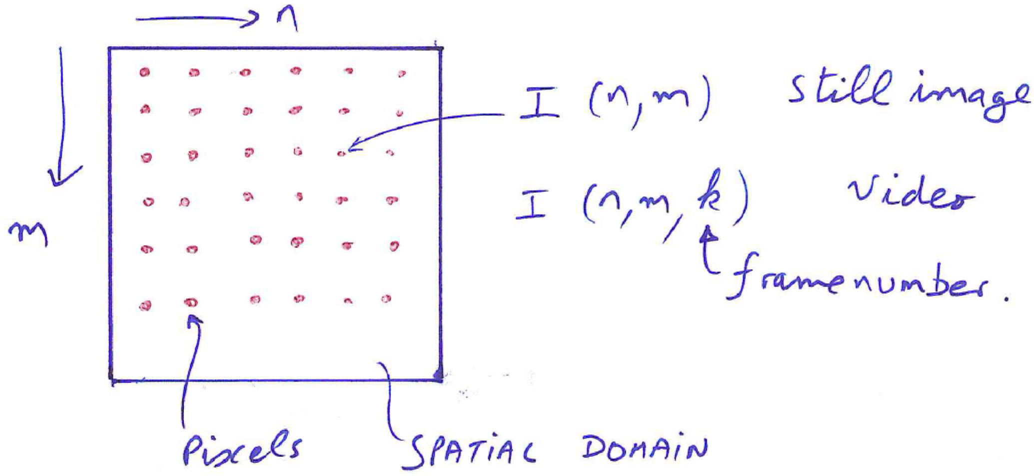

Figure 2.3.3: Illustration of 2D sampling (e.g. in the spatial domain).

If a sequence (discrete signal), f_n, is represented by numbers stored in a machine then we have a *digital signal*. Discrete signals can be represented in other ways, e.g. by charges on a collection of capacitors, but digital signals are very common.

Chapter 3

Nyquist's Sampling Theorem

3.1 Introduction

An important question in digital systems is 'What is the correct sampling rate?' How well represented is an analog signal by its samples? In other words, how accurately could we reconstruct a signal from its samples? Intuitively we might feel that the sampling interval should be as small as possible, $T \to 0$, in order to have perfect reconstruction. However, to aid storage / transmission of digital data we would like to use the smallest number of samples possible, i.e. a large T.

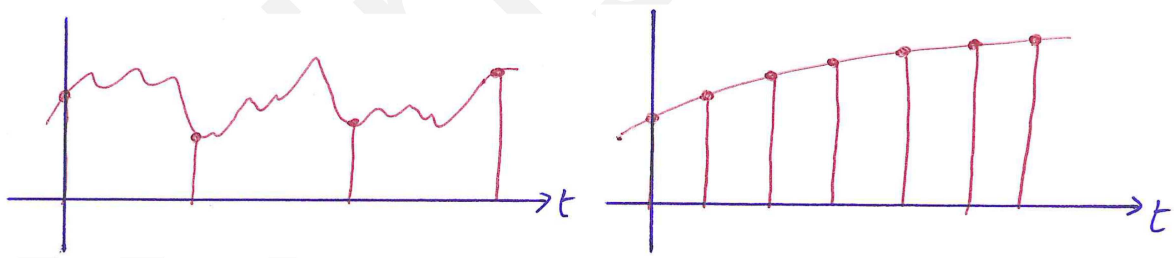

Figure 3.1.1: (left) If the signal changes quickly, is the sample rate too low to represent the signal? (right) If the signal changes slowly, is the sampling rate too high, wasting storage/bandwidth?

We have a problem, but Nyquist's sampling theorem comes to rescue!

3.2 Nyquist's sampling theorem

We start by deriving the spectrum of a sampled signal and then graphically derive the sampling theorem.

3.2.1 Spectrum of sampled signal

Consider a signal, $f(t)$, that is sampled by a train of impulses:

$$\bar{f}(t) = f(t)\,\delta_T(t) \tag{3.2.1}$$

We know, from equation (2.2.3), the Fourier series of an impulse train, which by substitution yields:

$$\bar{f}(t) = \frac{1}{T} f(t) \sum_{n=-\infty}^{+\infty} e^{jn\omega_s t}$$

Taking the Fourier transform of both sides:

$$\begin{aligned}
\mathcal{F}\left[\bar{f}(t)\right] &= \frac{1}{T}\mathcal{F}\left[f(t)\sum_{n=-\infty}^{+\infty} e^{jn\omega_s t}\right] \\
\Rightarrow \bar{F}(j\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty} \mathcal{F}\left[f(t)\,e^{jn\omega_s t}\right] \\
&= \frac{1}{T}\sum_{n=-\infty}^{+\infty} F\left(j\left(\omega - n\omega_s\right)\right)
\end{aligned}$$

where ω is the angular frequency in units of rads/sec.

In the last step we used the shift theorem of the Fourier transform [1].

In summary, the spectrum of the sampled signal is:

$$\bar{F}(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} F\left(j\left(\omega - n\omega_s\right)\right)$$

Where:

$F(j\omega) =$ complex spectrum of the original analog signal.

$\bar{F}(j\omega) =$ complex spectrum of the sampled signal (sampled by impulses).

[1] The shift theorem states that if $G(f)$ is the Fourier transform of $g(t)$, then multiplying (in the time domain) by a complex exponential, i.e. $g(t)\,e^{-j2\pi f_o t}$ results in a shifted Fourier transform $G(f - f_o)$.

We see that when a signal is sampled by impulses, two things happen to the spectrum:

1. The spectrum is scaled by a factor $\frac{1}{T}$.
2. An infinite number of copies of the (scaled) original spectrum, each shifted by a unique integer multiple of the sampling frequency, are added together.

3.2.2 Graphical development of the sampling theorem

We will consider two cases:

- band limited signals, and
- non-band limited signals

3.2.2.1 Band-limited signal case

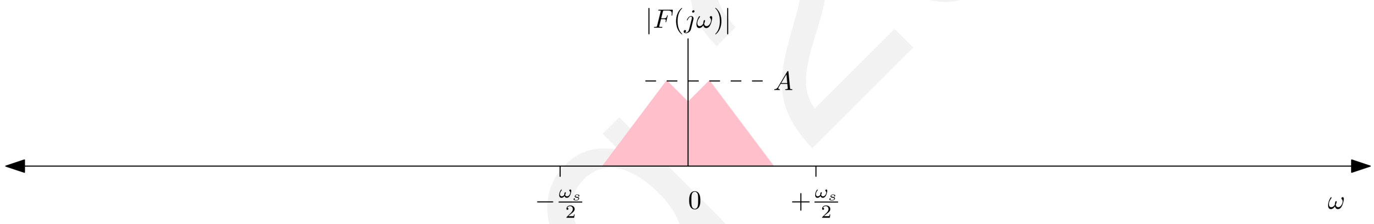

Figure 3.2.1: Spectrum of a band-limited analog signal.

Here the spectrum is strictly band limited to within the range $-\frac{\omega_s}{2} < \omega < +\frac{\omega_s}{2}$. $\frac{\omega_s}{2}$ is the half angular sampling frequency. If we apply what we learned about the spectrum of a sampled signal, we see that $\left|\bar{F}\,(j\omega)\right|$ looks like this:

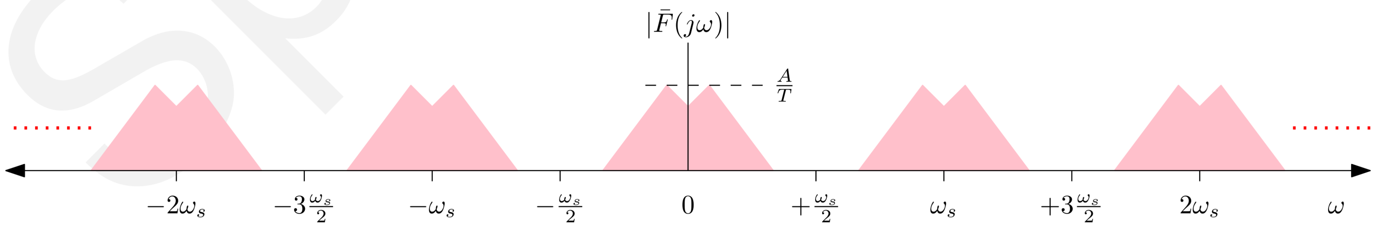

Figure 3.2.2: Spectrum of a band-limited signal after sampling.

We see that the spectrum of the original signal exists unchanged (apart from scaling) centered on $\omega = 0$. So we *can*, at least in theory, have perfect reconstruction, i.e. it *should* be possible reconstruct the original signal by low pass filtering.

3.2.2.2 Perfect Reconstruction

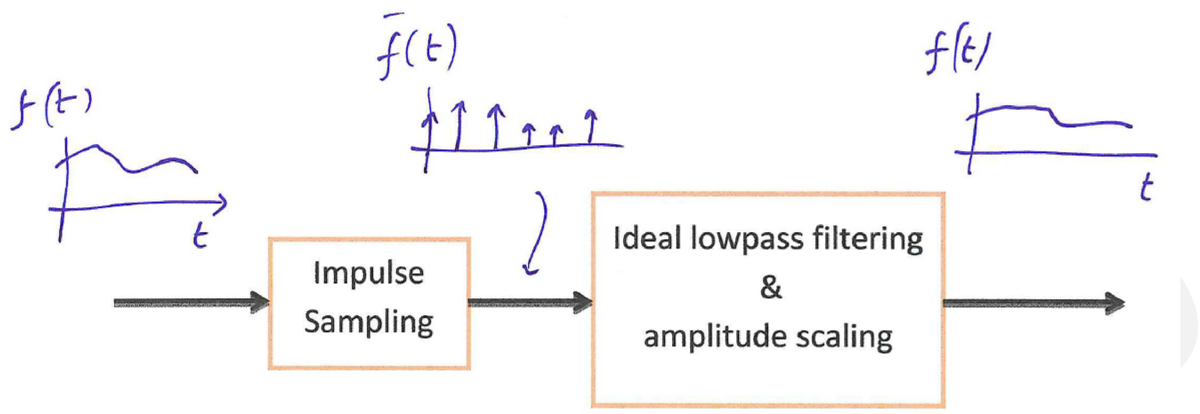

Figure 3.2.3: Perfect Reconstruction of an analog signal from its samples using a low pass filter.

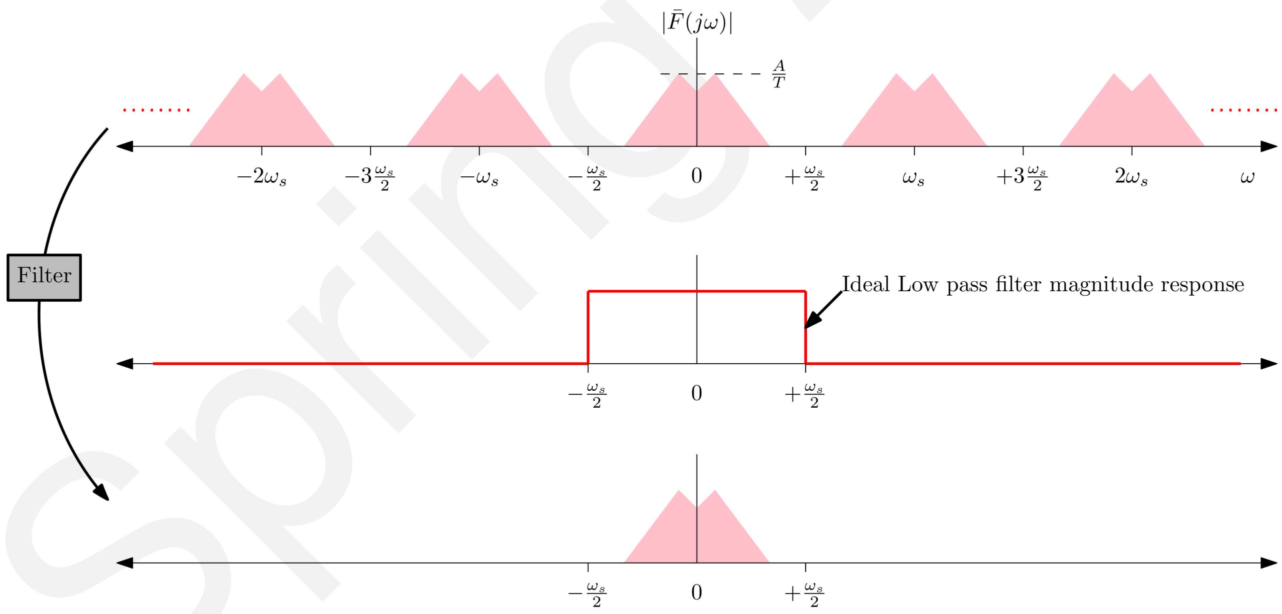

Figure 3.2.4: Perfect Reconstruction of an analog signal from its samples using a low pass filter.

3.2.2.3 Non-band-limited signal case

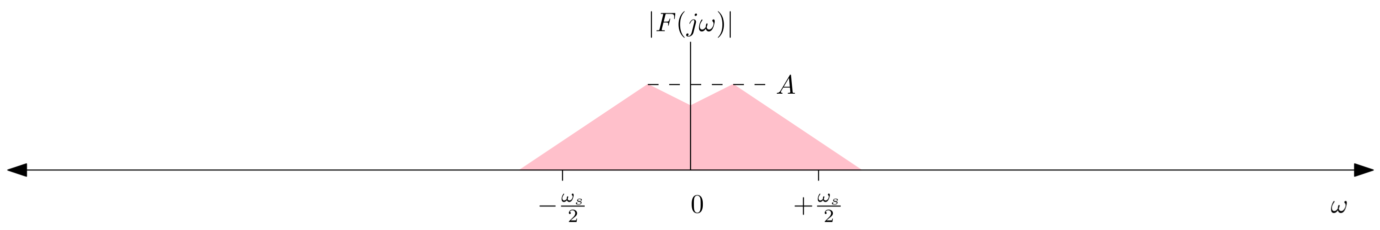

Figure 3.2.5: Spectrum of a non-band-limited analog signal.

The spectrum of the original analog signal is NOT band limited to within $-\frac{\omega_s}{2} < \omega < +\frac{\omega_s}{2}$.

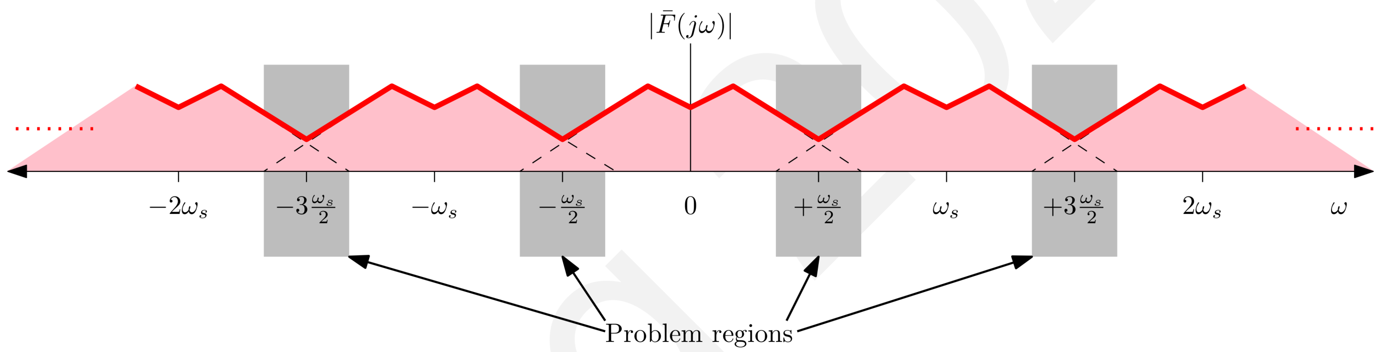

Figure 3.2.6: Spectrum of a non-band-limited signal after sampling.

At frequencies where the shifted spectra overlap, they add. There is a consequent loss of information, and so the signal cannot be reconstructed from its samples.

3.3 In summary

The *Nyquist Sampling Theorem* states that a signal can be exactly reconstructed from its uniformly spaced samples if and only if the sampling frequency, $f_s = \frac{1}{T}$, is greater than twice the highest frequency component of the signal.
The reconstruction is done using an ideal low-pass filter with cut-off frequency at $\frac{1}{2}f_s$.

Aside: The above statement of the sampling theorem is for low-pass signals, there is a more general form for band-pass signals, which we will not consider in this module. However, it is very useful when discussing digital radios, where we may use a slow analogue-to-digital converter to convert a continuous band-pass signal into a baseband digital signal.