Set Theory

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Odd Sem 2021

Sets and Subsets: Definitions and Properties

Set: Well-defined collection of distinct objects

(Ex: $S = \{4, 9, 16 \ldots, 81, 100\} = \{x^2 \mid x \text{ is integer and } 1 < x \leq 10\}$)

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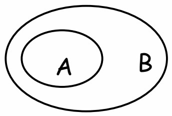
Null Set: Set containing NO element, denoted using φ or $\{\}$
(Ex: $Q = \{z \mid x + y = z \text{ and all } x, y, z \text{ are odd}\} = \varphi$)

Subset

Definition: Given two sets A and B, we say A is a **subset** of B, denoted by $A \subseteq B$, if every element of A is also an element of B.



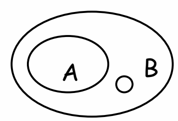
not a subset

- If A={4, 8, 12, 16} and B={2, 4, 6, 8, 10, 12, 14, 16}, then $A \subseteq B$ but $B \not\subseteq A$
- $A \subseteq A$ because every element in A is an element of A.
- $\emptyset \subseteq A$ for any A because the empty set has no elements.
- If A is the set of prime numbers and B is the set of odd numbers, then $A \not\subseteq B$

Fact: If $A \subseteq B$, then |A| <= |B|.

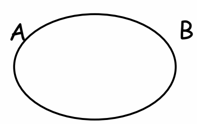
Proper Subset, Equality

Definition: Given two sets A and B, we say A is a **proper subset** of B, denoted by $A \subset B$, if every element of A is an element of B,
But there is an element in B that is not contained in A.



Fact: If $A \subset B$, then |A| < |B|.

Definition: Given two sets A and B, we say A = B if $A \subseteq B$ and $B \subseteq A$.



Fact: If A = B, then |A| = |B|.

Frequently-Used Set Examples and Notations

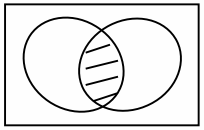
Popular Set Examples:

- $\mathbb{N}$ = Set of Non-negative natural numbers = $\{0, 1, 2, \ldots\}$
- $\mathbb{Z}$ = Set of Integers = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Z}^+$ = Set of Positive Integers = $\{x \in \mathbb{Z} \mid x > 0\}$
- $\mathbb{Q}$ = Set of Rational Numbers $\bar{b}$
- $\mathbb{R}$ = Set of Real Numbers
- $\mathbb{R}^+$ = Set of Positive Real Numbers
- $\mathbb{R}^*$ = Set of Non-zero Real Numbers
- $\mathbb{C}$ = Set of Complex Numbers = $\{a + ib \mid a, b \in \mathbb{R}, i^2 = -1\}$
- $\mathbb{C}^*$ = Set of Non-zero Complex Numbers = $\{c \in \mathbb{C} \mid c \neq 0\}$

Basic Operations on Sets

Let A, B be two subsets of a *universal* set U
(depending on the context U could be R, Z, or other sets).

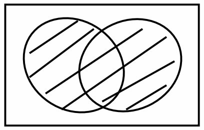
intersection: $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$



Defintion: Two sets are said to be **disjoint** if their intersection is an empty set.

e.g. Let A be the set of odd numbers, and B be the set of even numbers. Then A and B are disjoint.

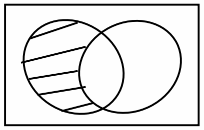
union: $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$



Fact: $|A \cup B| = |A| + |B| - |A \cap B|$

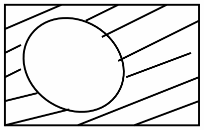
Basic Operations on Sets

differenc e: $A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$



Fact: $|A - B| = |A| - |A \cap B|$

complement : $\overline{A} = A^c = \{x \in U \mid x \notin A\}$



e.g. Let U = Z and A be the set of odd numbers. Then $\overline{A}$ is the set of even numbers.

Fact: If $A \subseteq B$, ther $\overline{B} \subseteq \overline{A}$

Examples

$A = \{1, 3, 6, 8, 10\}$ $B = \{2, 4, 6, 7, 10\}$

$A \cap B = \{6, 10\}$, $A \cup B = \{1, 2, 3, 4, 6, 7, 8, 10\}$ $A - B = \{1, 3, 8\}$

Let $U = \{ x \in Z \mid 1 <= x <= 100\}$.

$A = \{ x \in U \mid x$ is divisible by $3\}$, $B = \{ x \in U \mid x$ is divisible by $5\}$

$A \cap B = \{ x \in U \mid x$ is divisible by $15\}$

$A \cup B = \{ x \in U \mid x$ is divisible by 3 or is divisible by 5 (or both)$\}$

$A - B = \{ x \in U \mid x$ is divisible by 3 but is not divisible by 5 $\}$

Exercise: compute $|A|$, $|B|$, $|A \cap B|$, $|A \cup B|$, $|A - B|$.

Partitions of Sets

Two sets are disjoint if their intersection is empty.

A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A if and only if

$$A = A_1 \cup A_2 \cup \cdots \cup A_n$$

$A_1, A_2, \ldots, A_n$ are mutually disjoint (or pairwise disjoint).

e.g. Let A be the set of integers.

Let A_1 be the set of negative integers.

Let A_2 be the set of positive integers.

Then $\{A_1, A_2\}$ is not a partition of A, because $A \neq A_1 \cup A_2$

as 0 is contained in A but not contained in $A_1 \cup A_2$

Partitions of Sets

e.g. Let A be the set of integers divisible by 6.

A_1 be the set of integers divisible by 2.

A_2 be the set of integers divisible by 3.

Then $\{A_1, A_2\}$ is not a partition of A, because A_1 and A_2 are not disjoint, and also $A \subset A_1 \cup A_2$ (so both conditions are not satisfied).

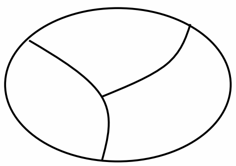
e.g. Let A be the set of integers.

$A_1 = \{x \in A \mid x = 3k+1 \text{ for some integer } k\}$

$A_2 = \{x \in A \mid x = 3k+2 \text{ for some integer } k\}$

$A_3 = \{x \in A \mid x = 3k \text{ for some integer } k\}$

Then $\{A_1, A_2, A_3\}$ is a partition of A



S = all possible outcomes where the white die and the black die have different values?

T ≡ set of outcomes where dice agree.

= { <1,1>, <2,2>, <3,3>,<4,4>,<5,5>,<6,6>}

$| S \cup T |$ = # of outcomes = 36

$|S| + |T| = 36$

$|T| = 6$

$|S| = 36 – 6 = 30$

S = all possible outcomes where the white die and the black die have different values?

A_i = Set of outcomes where black die says i and white die says something else.

$$|S| = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30$$

Let S = All possible outcomes where the black die shows the smaller number than white die.

A_i = Set of outcomes where black die is i, smaller than white die

$$|S| = \sum_{i=1}^{6} |A_i| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5|$$
$$= 5 + 4 + 3 + 2 + 1 = 15$$

Power Set and Set Properties

Power Set: Set of all possible subsets of a set (A), denoted as $P(A)$ or 2^A

(Ex: Let $A = \{1, 2, 3\}$,

Thus, $P(A) = \{\varphi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$)

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Cardinality: $|P(A)| = 2^{|A|}$ (Why?)

Cartesian Products

Definition: Given two sets A and B, the **Cartesian product** A x B is the set of all **ordered** pairs (a,b), where a is in A and b is in B. Formally,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Ordered pairs means the ordering is important, e.g. (1,2) ≠ (2,1)

e.g. Let A be the set of letters, i.e. {a,b,c,…,x,y,z}.

Let B be the set of digits, i.e. {0,1,…,9}.

AxA is just the set of strings with two letters.

BxB is just the set of strings with two digits.

AxB is the set of strings where the first character is a letter and the second character is a digit.

Cartesian Products

The definition can be generalized to any number of sets, e.g.

$$A \times B \times C = \{(a,b,c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\}$$

Using the above examples, AxAxA is the set of strings with three letters.

An ID card number has one letter and then six digits,
so the set of ID card numbers is the set AxBxBxBxBxBxB.

Fact: If $|A| = n$ and $|B| = m$, then $|A \times B| = mn$.

Fact: If $|A| = n$ and $|B| = m$ and $|C| = l$, then $|A \times B \times C| = mnl$.

Fact: $|A_1 \times A_2 \times \ldots \times A_k| = |A_1| \times |A_2| \times \ldots \times |A_k|$.

Exercises

1. Let A be the set of prime numbers, and let B be the set of even numbers. What is $A \cap B$ and $|A \cap B|$?

2. Is $|A \cup B| > |A| > |A \cap B|$ always true?

3. Let A be the set of all n-bit binary strings, A_i be the set of all n-bit binary strings with i ones. Is $(A_1, A_2, \ldots, A_i, \ldots, A_n)$ a partition of A?

4. Let $A = \{x,y\}$. What is pow(A)xpow(A) and |pow(A)xpow(A)|?

Set Identities

Some basic properties of sets, which are true for all sets.

$$A \cap B \subseteq A$$

$$A \subseteq A \cup B$$

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C$$

$$A \cap \overline{A} = \emptyset$$

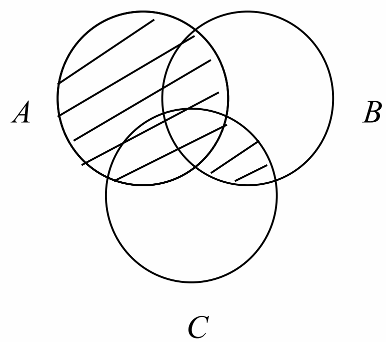
$$\overline{\overline{A}} = A$$

$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

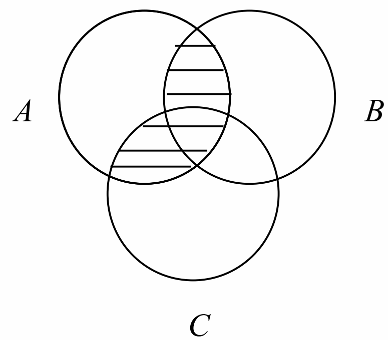
Set Identities

Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (1)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (2)



(1)

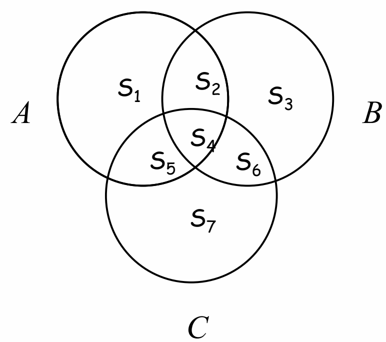


(2)

Set Identities

Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

We can also verify this law more carefully



L.H.S

$A = S_1 \cup S_2 \cup S_4 \cup S_5$

$B \cap C = S_4 \cup S_6$

$A \cup (B \cap C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$

R.H.S.

$(A \cup B) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$

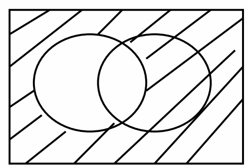
$(A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6 \cup S_7$

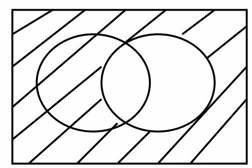
$(A \cup B) \cap (A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$

Set Identities

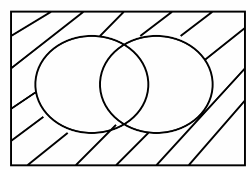
De Morgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A}$





$\overline{B}$



$\overline{A \cup B} = \overline{A} \cap \overline{B}$

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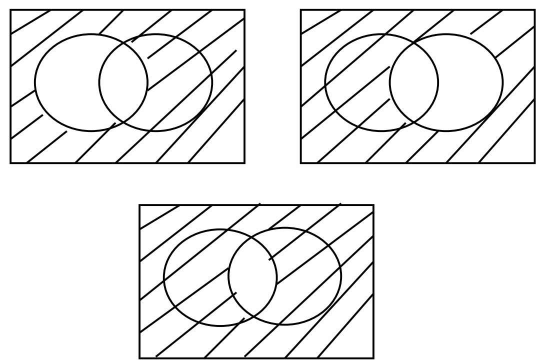
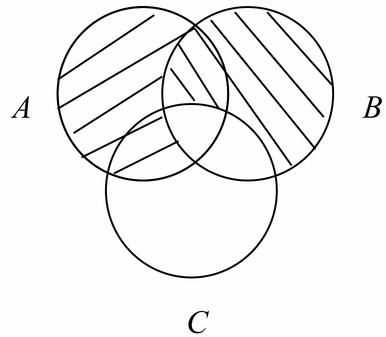


$\overline{B}$

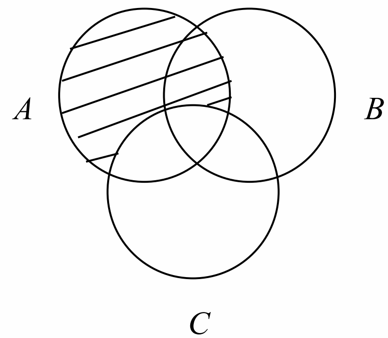
$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Disproof

$$(A - B) \cup (B - C) = A - C?$$



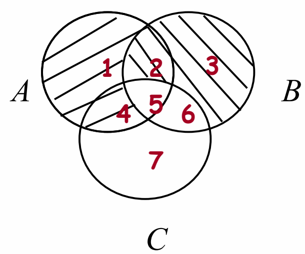
L.H.S

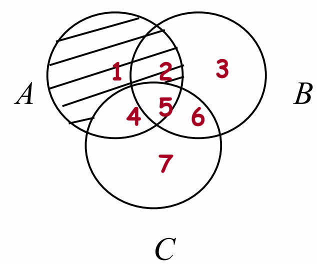


R.H.S

Disproof

$$(A - B) \cup (B - C) = A - C?$$





We can easily construct a **counterexample** to the equality, by putting a number in each region in the figure.

Let A = {1,2,4,5}, B = {2,3,5,6}, C = {4,5,6,7}.

Then we see that L.H.S = {1,2,3,4} and R.H.S = {1,2}.

Algebraic Proof

Sometimes when we know some rules, we can use them to prove new rules without drawing figures.

e.g. we can prove $\overline{(\overline{A} \cap \overline{B})} = A \cup B$ without drawing figures.

$$\overline{(\overline{A} \cap \overline{B})} = \overline{\overline{A}} \cup \overline{\overline{B}}$$ by using DeMorgan's rule on $\overline{A}$ and $\overline{B}$

$$= A \cup B$$

Algebraic Proof

$$\overline{((A \cup C) \cap (B \cup C))} = (\overline{A} \cup \overline{B}) \cup \overline{C}?$$

$$\overline{((A \cup C) \cap (B \cup C))}$$

$= \overline{(A \cup C)} \cup \overline{(B \cup C)}$ by DeMorgan's law on A U C and B U C

$= (\overline{A} \cap \overline{C}) \cup \overline{(B \cup C)}$ by DeMorgan's law on the first half

$= (\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C})$ by DeMorgan's law on the second half

$= (\overline{A} \cup \overline{B}) \cap \overline{C}$ by distributive law

$\neq (\overline{A} \cup \overline{B}) \cup \overline{C}$

Exercises

$$A - (A \cap B) = A - B?$$

$$(A \cup B) - C = (A - C) \cup (B - C)?$$

$$\overline{(A \cup B \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}?$$