1) $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L \quad a_n = n!$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = L$$

$$\lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

since $L < 1$

the series $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ is convergent.

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2) $\sum n^{1-n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$\lim_{n\to\infty} \frac{(n+1)^{-n}}{n^{1-n}} = L$$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \times \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{n^n}{(n+1)^n n}$$

$$\lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n \times \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{ne} = 0$$

$\therefore L < 1$

$\therefore$ the series $\sum_{n=1}^{\infty} n^{1-n}$ is convergent

3) $\sum_{n=1}^{\infty} \frac{n^p}{n!}$ $\quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

$$\lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \frac{n!}{n^p} = L$$

$$\lim_{n\to\infty} \frac{1}{(n+1)} \left(\frac{n+1}{n}\right)^p$$

$\boxed{\text{If } p=1}$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \therefore \sum_{n=1}^{\infty} \frac{n^p}{n!} \text{ is converging for } \boxed{p=1}$$

$\boxed{\text{If } p>1}$

$$\lim_{n\to\infty} \frac{(n+1)^{p-1}}{n^p}$$

Since the power of n in numerator is less than that of denom the limit value at infinity $= 0$

$\therefore$ series sum is converging for $\boxed{p>1}$

If $p<1$

for $p=0$ $\sum \frac{1}{n!}$ is converging

$$\lim_{n\to\infty} \frac{n^{-p}}{(n+1)^{1-p}}$$

Since p is negative for $p<0$

let $-p = t$ $\quad t>0$

$$\lim_{n\to\infty} \frac{n^t}{(n+1)^{1+t}}$$

since power of n in denom is greater than that of numerator the limit value at $n\to\infty = 0$

$\therefore$ series is converging for all value of p.

4) $\sum \frac{n^2}{2^n}$

$$\lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2}$$

$$\lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2$$

$$\lim_{n\to\infty} \frac{1}{2}\left(\frac{1+\frac{1}{n}}{1}\right)^2$$

$\frac{1}{2} \times 1 = \frac{1}{2}$ $\quad L<1$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{2^n}$ is converging

$\sum \frac{1}{1+t^n}$

5) $\sum \frac{a^n}{x^n+a^n} \quad \left(\frac{x}{a}>1\right)$

$\lim_{n\to\infty} \left(\frac{a^{n+1}}{x^{n+1}+a^{n+1}}\right)\left(\frac{x^n+a^n}{a^n}\right)$

$\lim_{n\to\infty} a \left(\frac{1+\left(\frac{a}{x}\right)^n}{x+\left(\frac{a}{x}\right)^n \times a}\right)$

$\lim_{n\to\infty} \frac{a}{x} \left(\frac{1+\left(\frac{a}{x}\right)^n}{1+\left(\frac{a}{x}\right)^{n+1}}\right)$

$\frac{a}{x} \quad \frac{\left(\frac{a}{x}\right)^n \log\left(\frac{a}{x}\right)}{\left(\frac{a}{x}\right)^{n+1} \log\left(\frac{a}{x}\right)}$

$= \frac{a}{x} \times \frac{x}{a}$

$= 1$

test fails.

(A) $\sum \frac{a^n}{x^n+a^n} \quad \left(\frac{x}{a}>1\right)$

$\sum \frac{1}{1+\left(\frac{x}{a}\right)^n}$

$\left(\frac{x}{a}\right)^n < \left(\frac{x}{a}\right)^n + 1$

$\frac{1}{\left(\frac{x}{a}\right)^n + 1} < \frac{1}{\left(\frac{x}{a}\right)^n} \quad \text{—} (1)$

$\lim_{n\to\infty} \frac{1}{\left(\frac{x}{a}\right)^n}$

$= 0$

$\therefore \sum \frac{1}{\left(\frac{x}{a}\right)^n}$ is converging $\to (2)$

from (1) & (2)

$\sum \frac{1}{1+\left(\frac{x}{a}\right)^n}$ is also converging.

c) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \dots$

$\rightarrow \sum_{n\geq 1} \frac{x^n}{n(n+1)}$

Ratio test:

$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^n}$

$\lim_{n\to\infty} \frac{n x}{n+2}$

$x \lim_{n\to\infty} \frac{1}{1+2/n}$

$= x$

if $x > 1$

series sum is divergent

if $x < 1$

series sum is convergent.

if $x = 1$

Ratio test fails.

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1$

$\therefore$ at $x=1$ series sum is converging.

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7) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

→ Ratio test

$\lim_{n\to\infty} \frac{2^{n+1}+5}{3^{n+1}} \times \frac{3^n}{2^n+5}$

$\lim_{n\to\infty} \frac{1}{3} \left(\frac{2^{n+1}+5}{2^n+5}\right)$

$\lim_{n\to\infty} \frac{1}{3} \frac{2^{n+1}\log 2}{2^n \log 2}$

$= \frac{2}{3} < 1$

∴ series sum is converging

8) $\sum_{n=1} \frac{(2n)!}{n!\,n!}$

Ratio test

$\lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!\,(n+1)!} \cdot \frac{n!\,n!}{(2n)!}$

$\lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2}$

$\lim_{n\to\infty} \frac{2(2n+1)}{n+1}$

$\lim_{n\to\infty} 2\left(\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\right)$

$= 2 \times 2$

$= 4 > 1$

∴ series sum is diverging.

$$\lim_{n\to\infty} 2\left(\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\right)$$

$= 2 \times 2$

$= 4 > 1$

$\therefore$ series sum is diverging.

x) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots$

$$\sum_{n=0}^{\infty} \frac{x^n}{n^2+1}$$

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^2+1}\left(\frac{n^2+1}{x^n}\right)$$

$$\lim_{n\to\infty} x\left(\frac{n^2+1}{n^2+2n+1+1}\right)$$

$$\lim_{n\to\infty} x\left(\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}\right)$$

$= x$

if $x > 1$ series is diverging.

if $x < 1$ series is converging.

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ix) $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

$\lim_{n\to\infty} \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{4^n n! n!} = L$

$\lim_{n\to\infty} \frac{4(n+1)}{2(2n+1)}$

$= \frac{2}{2} = 1 = L$

$\therefore L = 1$ test fails.

But if we notice.

$\frac{2n+2}{2n+1}$ is always greater than 1 $\forall\, n \geq 1$

$\therefore$ the series diverges.

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