Physics 107 Problem 5.1 O. A. Pringle

Which of the wave functions in Fig. 5.14 cannot have physical significance in the interval shown? Why not?

Recall that $\int_{-\infty}^{\infty} \left(\left| \Phi \right| \right)^2 dV$

must be finite, and that Φ must be single-valued and continuous with finite, single-valued, and continuous first derivatives.

The function in figure 5.14(a) appears to satisfy all of the conditions over the interval shown, so it can be a wave function. If you want to be picky, its zero slope everywhere means the momentum is zero everywhere, which is not allowed by the uncertainty principle. I will not be that tricky on the exam.

The function in figure 5.14(b) is not single-valued, so it cannot be a wave function.

The function in figure 5.14(c) does not satisfy the condition for a continuous first derivative, so it cannot be a wave function.

The function in figure 5.14(d) does not satisfy the condition for a continuous first derivative, so it cannot be a wave function. In addition, the function is not continuous. Beiser in his answer implies the function goes to infinity, which is not obvious from the figure as it is drawn.

The function in figure 5.14(e) appears to satisfy all of the conditions, so it can be a wave function.

The function in figure 5.14(f) is not continuous, so it cannot be a wave function. This is tricky, because it is possible that the derivative could be continuous and finite.

Physics 107 Problem 5.2 O. A. Pringle

Which of the wave functions in Fig. 5.15 cannot have physical significance in the interval shown? Why not?

"YES" means allowed, "NO" means not allowed.

(a) YES.

(b) YES if you assume the "interval shown" is only where the function is defined. NO if you assume the "interval shown" means the entire shown part of the positive x-axis.

(c) NO. The first derivative is discontinuous in the middle. Beiser also probably intends to imply that the function goes to infinity at the middle.

(d) YES

(d) YES.

(e) NO. The function is not single valued.

(f) YES. The function in figure 5.15(f) appears to satisfy all of the conditions, so it can be a wave function, although it is not clear how the amplitude can decrease with increasing x without the wavelength changing.

Physics 107 Problem 5.3 O. A. Pringle

Which of the following wave functions cannot be solutions of Schrodinger's equation for all values of x? Why not?

I will plot the functions so we can better see what is going on.

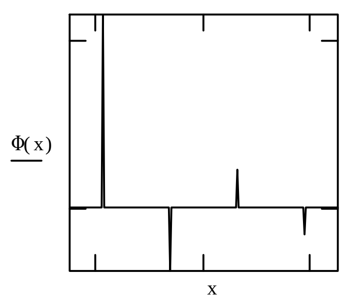
$$A := 1$$

(a) The secant is 1/cosine. Let's plot the secant function for small values of x.

$$\Phi(x) := \frac{1}{\cos(x)}$$

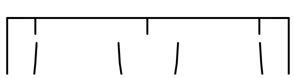
$$x := -2\cdot\pi, \left(-2\cdot\pi + \frac{2\cdot\pi}{100}\right) .. 2\cdot\pi$$

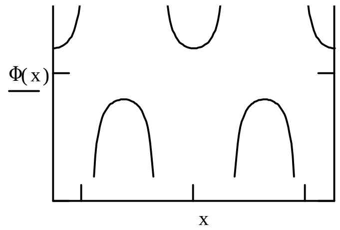
First plot $\Phi(x)$, letting Mathcad choose the plot scale.



There are singularities here, which seem to scale differently. Because of our choice of values of x, the function does not always "hit" the singularity at exactly the same position.

We can see the function better if we choose the plot scale ourselves.

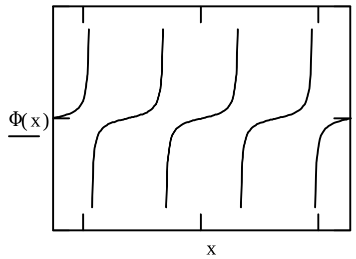




There are singularities wherever cos(x)=0, so this is not an allowed wave function if its range includes an x-value where there is a singularity.

(b) $\Phi(x) := A \cdot \tan(x)$

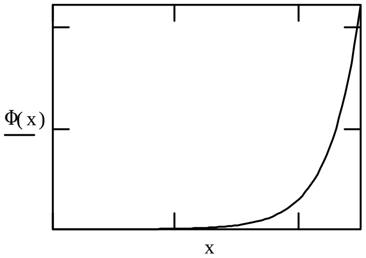
$x := -2\cdot\pi, \left(-2\cdot\pi + \frac{2\cdot\pi}{100}\right) .. 2\cdot\pi$



This function has singularities at π/2, 3π/2, etc..

(d) $\Phi(x) := A \cdot \left(e^{x}\right)^{2}$

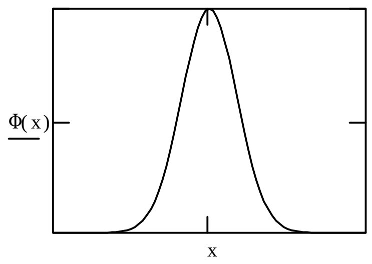
$x := 0, .05 .. 5$



This function cannot be normalized because of the divergence as x goes to ∞

(e) $\Phi(x) := A \cdot e^{-x^{2}}$

$x := -4, -3.9 .. 4$



This function is allowed, because the function and all of its derivatives are continuous and integrable

Physics 107 Problem 5.4 O. A. Pringle

Find the value of the normalization constant A for the wave function

$$\Phi(x) := A \cdot x \cdot e^{-\frac{x^2}{2}}$$

The DOS version of Mathcad is a real pain for symbolic manipulation. Let's try it anyway. To normalize, find A such that...

$$1 := \int_{-\infty}^{\infty} A^2 \cdot x^2 \cdot \left(e^{\frac{-x^2}{2}}\right)^2 dx$$

$$1 := A^2 \cdot \int_{-\infty}^{\infty} x^2 \cdot \left(e^{\frac{-x^2}{2}}\right)^2 dx$$

$$1 := A^2 \cdot \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2}\, dx$$

$$1 := A^2 \cdot 2 \cdot \int_{0}^{\infty} x^2 \cdot e^{-x^2}\, dx$$

$$1 := A^2 \cdot 2 \cdot \left(\frac{\sqrt{\pi}}{4}\right)$$

Solve for A:

$$A^2 := \frac{2}{\sqrt{\pi}} \quad := \sqrt{\frac{4}{\pi}} \qquad \text{or} \qquad A := \left(\frac{4}{\pi}\right)^{\frac{1}{4}}$$

Everything from here on down is optional, for info only.

Let's double-check and see if it works.

$$\int_{-\infty}^{\infty} A^2 \cdot x^2 \cdot \left(e^{\frac{-x^2}{2}}\right)^2 dx =$$

overflow

Notice Mathcad says "overflow" because it tries to calculate the integrand as x becomes infinite, and the x^2 part blows up. Of course the exponential part goes to zero faster than x^2 blows up.

To get around the overflow, we can try reducing the range of integrating from ∞ to something very large.

$$\int_{-1}^{1} A^2 \cdot x^2 \cdot \left(e^{\frac{-x^2}{2}}\right)^2 dx = 0.428$$

Obviously -1 to 1 is not "large enough"; we have missed lots of the area under the integral.

$$\int^{10}$$

$$\int_{-10}^{10} A^2 \cdot x^2 \cdot \left(e^{\frac{-x^2}{2}}\right)^2 dx = 0.999999917507$$

It looks like -10 to 10 is nearly "infinity".

Physics 107 Problem 5.5 O. A. Pringle

The wave function of a certain particle is ϕ=A $\cos^2$(x) for $-\pi/2 <= x <= \pi/2$.

(a) Find the value of A.

$$1 := \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(A \cdot \cos(x)^2\right)^2 dx$$

$$1 := A^2 \cdot \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x)^4 dx$$

You could look the integral up in a table and evaluate it at its upper and lower limits. I see from the result in the table that it will be easier to evaluate if I do this:

$$1 := 2 \cdot A^2 \cdot \int_0^{\frac{\pi}{2}} \cos(x)^4 dx$$

I can do this because $\cos^4$ is symmetric about the origin.

From tables

$$\int \cos(x)^4 dx := \frac{3 \cdot x}{8} + \frac{\sin(2 \cdot x)}{4} + \frac{\sin(4 \cdot x)}{32}$$

And all three terms are zero when x=0, so that

$$1 := 2 \cdot A^2 \cdot \left(\frac{3}{8} \cdot \frac{\pi}{2} + \frac{\sin(\pi)}{4} + \frac{\sin(2 \cdot \pi)}{32}\right)$$

Both sine terms are zero, leaving

$$1 := 2\cdot A^2\cdot\left(\frac{3}{8}\cdot\frac{\pi}{2}\right)$$

$$A^2 := \frac{16}{6\cdot\pi}$$

$$A := \sqrt{\frac{8}{3\cdot\pi}}$$

(b) Find the probability that the particle can be found between x=0 and x=π/4.

$$P := \int_0^{\frac{\pi}{4}} \left(A\cdot\cos(x)^2\right)^2\, dx$$

$P = 0.462$

That was easy with Mathcad. Analytically, here's what you would do:

$$P := \frac{8}{3\cdot\pi}\cdot\int_0^{\frac{\pi}{4}} \cos(x)^4\, dx$$

$$P := \frac{8}{3\cdot\pi}\cdot\left(\frac{3}{8}\cdot\frac{\pi}{4} + \frac{\sin\left(\frac{\pi}{2}\right)}{4} + \frac{\sin(\pi)}{32}\right)$$

$P = 0.462$

Physics 107 Problem 5.11 O. A. Pringle

One of the possible wave functions of a particle in the potential well of Fig. 5.16 is sketched there. Explain why the wavelength and amplitude of ϕ vary as they do.

This potential well is an infinite potential well (because the potential is infinite beyond the boundaries) with a constant slope bottom (as opposed to parabolic or periodic).

Do the following on a photocopy of figure 5.16, or else do it in your head:

Draw a horizontal line across the well, somewhere about the level of the "V", it doesn't matter exactly where, to represent the constant total energy, E = K + V.

The amplitude of the wave function is zero outside the well due to the infinite potential energy there.

The kinetic energy is the difference between your drawn horizontal line and the potential V, which is the slopeing line at the bottom of the well. The kinetic energy is greater at the left of the well, so the wavelength is shorter there. Remember, K=p^2/2m, and λ=h/p. That's why the wiggles in ϕ are closer together at the left of the well.

Because the kinetic energy is greater at the left, the velocity is also greater. This means that the particle remains in any interval δx for a shorter time on the left than on the right. In probability terms, there is less probability for finding the particle on the left half of the well than there is on the right half. That's why the amplitude of ϕ increases on going from left to right.

This problem can be solved using the program EXAM3 which can be found in the Physics 107 coursework file on the network. The wavefunction shown has quantum number n=6. How do I know? (Hint: count the bumps.)

Physics 107 Problem 5.12 O. A. Pringle

In Sec. 5.6 a box was considered that extends from x=0 to x=L. Suppose the box extends instead from x=x.0 to x=x.0+L, where x.0 is not zero. Would the expression for the wave functions of a particle in this box be any different from those in the box that extends from x=0 to x=L? Would the energy levels be different?

The wave functions we derived are

$$\phi(x) := \sqrt{\frac{2}{L}}\cdot\sin\left(\frac{n\cdot\pi\cdot x}{L}\right)$$

To get the wave functions for this problem, simply make a change of variables:

$y := x + x_0$

$x := y - x_0$

$\phi\left(y - x_0\right) := \sqrt{\frac{2}{L}} \cdot \sin\left[\frac{n \cdot \pi \cdot \left(y - x_0\right)}{L}\right]$

ϕ is just a name, so let's call it ϕprime, and y is just a dummy variable, so let's call it x.

$\phi\text{prime}(x) := \sqrt{\frac{2}{L}} \cdot \sin\left[\frac{n \cdot \pi \cdot \left(x - x_0\right)}{L}\right]$

Double check. Invent some values and calculate ϕprime.

$L := 1$ $\qquad x_0 := 1$

$\phi\text{prime}(x, n) := \sqrt{\frac{2}{L}} \cdot \sin\left[\frac{n \cdot \pi \cdot \left(x - x_0\right)}{L}\right]$

$\phi\text{prime}\left(x_0, 1\right) = 0$ as required

$\phi\text{prime}(L, 1) = 0$ as required

$\phi\text{prime}\left(x_0, 2\right) = 0$

$\phi\text{prime}(L, 2) = 0$

Your intuition ought to suggest that moving the box along the x-axis should not change the energy levels. We can verify this by plugging ϕ back into Schrodinger's eqn.

Verify that

$$\frac{d}{dx}\left(\frac{d}{dx}\phi\text{prime}\right) + \frac{2 \cdot m}{\text{hbar}^2} \cdot E \cdot \phi\text{prime} := 0$$

The second derivative of ϕprime with respect to x is

$$-\sqrt{\frac{2}{L}} \cdot \left(\frac{n \cdot \pi}{L}\right)^2 \cdot \sin\left(\left(n \cdot \pi \cdot \frac{x - x_0}{L}\right)\right)$$

which is equal to

$$-\left(\frac{n \cdot \pi}{L}\right)^2 \cdot \phi\text{prime}(x)$$

Plugging the second derivative into Schrodinger's equation gives

$$-\left(\frac{n\cdot\pi}{L}\right)^2\cdot\phi prime+\frac{2\cdot m}{hbar^2}\cdot E\cdot\phi prime := 0$$

$$-\left(\frac{n\cdot\pi}{L}\right)^2+\frac{2\cdot m}{hbar^2}\cdot E := 0$$

Solve for E

$$E := \frac{n^2\cdot\pi^2\cdot hbar^2}{2\cdot m\cdot L^2}$$

Which is the same as eq. 5.27.

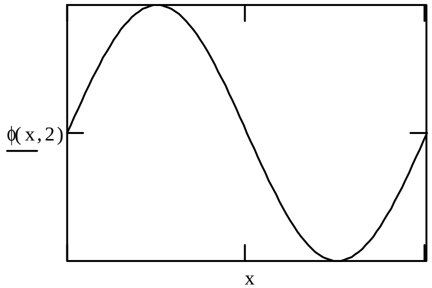
Physics 107 Problem 5.15 O. A. Pringle

As shown in the text, the expectation value <x> of a particle trapped in a box L wide is L/2, which means that its average position is the middle of the box. Find the expectation value <x^2>.

Set the width of the box: $L := 1$

The nth wavefunction is: $\phi(x,n) := \sqrt{\frac{2}{L}}\cdot\sin\left(n\cdot\pi\cdot\frac{x}{L}\right)$ $x := 0, .01\cdot L .. L$

Just for kicks, plot the n=2 wavefunction:



Everything looks OK so far.

Calculate <x²>:

$$\int_0^L \phi(x,1)\cdot x^2\cdot\phi(x,1)\,dx = 0.283\cdot L$$

Does this numerical answer make sense? Compare with textbook answer

$$x2bar(n) := \frac{L^2}{3} - \frac{L^2}{2\cdot n^2\cdot\pi^2}$$

This analytical answer is obtained by substituting in the general expression for the nth wave function, and carrying out the integral.

$$x2bar(1) = 0.283\cdot L$$

So our numerical answer is the same as the analytical answer.

Let's run some double-checks:

Is the wave function normalized?

$$\int_0^L \phi(x,2)\cdot\phi(x,2)\,dx = 1$$

So the wave function is properly normalized.

Check the expectation value of <x>:

$$\int_0^L \phi(x,2)\cdot x\cdot\phi(x,2)\,dx = 0.5\cdot L$$

This is L/2, as expected.

We did this for the n=1 wave function. Beiser's analytic answer is true in general. Let's check the n=2 and n=3 cases.

For n=2:

$$\int_0^L \phi(x,2)\cdot x^2\cdot\phi(x,2)\,dx = 0.321\cdot L$$

$$x2bar(2) = 0.321\cdot L$$

For n=3:

$$\int_0^L \phi(x,3)\cdot x^2\cdot\phi(x,3)\,dx = 0.328\cdot L$$

$$x2bar(3) = 0.328\cdot L$$

Our numerical answers using Mathcad are in agreement with the analytical answers.

See the document 5-15a.mcd for an analytical solution to this problem

Physics 107 Problem 5.15a O. A. Pringle

As shown in the text, the expectation value <x> of a particle trapped in a box L wide is L/2, which means that its average position is the middle of the box. Find the expectation value $\langle x^2 \rangle$.

This is an alternative, analytical solution to the problem,

$$\phi(x,n) := \sqrt{\frac{2}{L}}\cdot\sin\left(n\cdot\pi\cdot\frac{x}{L}\right)$$

Calculate $\langle x^2 \rangle$:

$$x2\_expect := \int_0^L \phi(x,1)\cdot x^2\cdot\phi(x,1)\,dx$$

$$x2\_expect := \int_0^L \sqrt{\frac{2}{L}}\cdot\sin\left(n\cdot\frac{\pi}{L}\cdot x\right)\cdot x^2\cdot\sqrt{\frac{2}{L}}\cdot\sin\left(n\cdot\frac{\pi}{L}\cdot x\right)dx$$

$$x2\_expect := \left(\frac{2}{L}\right)\cdot\left[\int_0^L x^2\cdot\sin\left(n\cdot\frac{\pi}{L}\cdot x\right)^2 dx\right] \quad \text{------(1)}$$

From table of integrals:

$$\int x^2 \cdot \sin(ax)^2 \, dx := \frac{x^3}{6} - \left(\frac{x^2}{4 \cdot a} - \frac{1}{8 \cdot a^3}\right) \cdot \sin(2 \cdot a \cdot x) - \frac{x \cdot \cos(2 \cdot a \cdot x)}{4 \cdot a^2}$$

Here, $a=n\pi/L$.

When ------(1) is evaluated at the lower limit, everything has either a factor of x or sin(2ax), so everything goes to zero.

When ------(1) is evaluated at the upper limit, you get

$$\frac{L^3}{6} - \left[\frac{L^2}{4 \cdot \left(n \cdot \frac{\pi}{L}\right)} - \frac{1}{8 \cdot \left(n \cdot \frac{\pi}{L}\right)^3}\right] \cdot \sin\left[2 \cdot \left(n \cdot \frac{\pi}{L}\right) \cdot L\right] - \frac{L \cdot \cos\left[2 \cdot \left(n \cdot \frac{\pi}{L}\right) \cdot L\right]}{4 \cdot \left(n \cdot \frac{\pi}{L}\right)^2}$$

The middle term is zero because $\sin(2\pi)=0$, so you are left with

$$\frac{L^3}{6} - \frac{L \cdot \cos(2 \cdot n \cdot \pi)}{4 \cdot \left(n \cdot \frac{\pi}{L}\right)^2}$$

Because $\cos(2n\pi)=1$, we get, after simplifying

$$\frac{L^3}{6} - \frac{L^3}{4 \cdot n^2 \cdot \pi^2}$$

The above constant is just the value of the integral.
Put this back into ------(1) to give

$$x2\_expect := \left(\frac{2}{L}\right) \cdot \left[\left(\frac{L^3}{6}\right) - \left(\frac{L^3}{4 \cdot n^2 \cdot \pi^2}\right)\right]$$

Multiply it out to get

$$x2\_expect := \left(\frac{L^2}{3}\right) - \left(\frac{L^2}{2 \cdot n^2 \cdot \pi^2}\right)$$

Which is the answer given on page 595 of Beiser.

See the document 5-11.mcd for a numerical solution to this problem.

Physics 107 Problem 5.17 O. A. Pringle

Find the probability that a particle in a box L wide can be found between x=0 and x=L/n when it is in the nth state.

$$\phi(x) := \sqrt{\frac{2}{L}}\cdot\sin\left(\frac{n\cdot\pi\cdot x}{L}\right)$$

$$P := \int_{x1}^{x2} \overline{\phi(x)}\cdot\phi(x)\,dx$$

$$P := \int_{0}^{\frac{L}{n}} \left(\frac{2}{L}\right)\cdot\sin\left(\frac{n\cdot\pi\cdot x}{L}\right)^{2}\,dx$$

$$P := \left(\frac{2}{L}\right)\cdot\int_{0}^{\frac{L}{n}} \sin\left(\frac{n\cdot\pi\cdot x}{L}\right)^{2}\,dx$$

From table of integrals:

$$\int \sin(a\cdot x)^{2}\,dx := \frac{x}{2} - \frac{1}{4\cdot a}\cdot\sin(2\cdot a\cdot x)$$

Thus

$$\int \sin\left(\frac{n\cdot\pi\cdot x}{L}\right)^{2}\,dx := \frac{x}{2} - \frac{L}{4\cdot n\cdot\pi}\cdot\sin\left(\frac{2\cdot n\cdot\pi\cdot x}{L}\right)$$

The result is zero when evaluated at x=0, so

$$P := \frac{2}{L}\cdot\left(\frac{x}{2} - \frac{L}{4\cdot n\cdot\pi}\cdot\sin\left(\frac{2\cdot n\cdot\pi\cdot x}{L}\right)\right)$$

evaluated at x=L/n.

The result is

$$P := \frac{2}{L}\cdot\left(\frac{L}{2\cdot n} - \frac{L}{4\cdot n\cdot\pi}\cdot\sin\left(\frac{2\cdot n\cdot\pi\cdot\frac{L}{n}}{L}\right)\right)$$

$$P := \frac{2}{L}\cdot\left(\frac{L}{2\cdot n} - \frac{L}{4\cdot n\cdot\pi}\cdot\sin(2\cdot\pi)\right)$$

$$P := \frac{1}{n}$$

On a test or quiz, I usually give numerical problems rather than symbolic ones like this, but the technique is the same.

Physics 107 Problem 5.22 O. A. Pringle

An electron and a proton with the same energy E approach a potential barrier whose height U is greater than E. Do they have the same probability of getting through? If not, which has the greater probability?

At its simplest level, this question can be answered by inspecting equation 5.69, the approximate transmission probability:

$$T := \exp\left(-2\cdot k_2\cdot L\right)$$

$$k_2 := \frac{\sqrt{2\cdot m\cdot(E - V)}}{hbar}$$

Larger mass means larger k_2, larger negative value of $2*k_2*L$, and smaller value of T, so larger mass means smaller transmission probability.

This question can be answered in much more detail by looking at the exact transmission probability.

Here are the transmission equations:

$$hbar := 1.055\cdot10^{-34} \qquad i := \sqrt{-1}$$

$$k1(E,m) := \frac{\sqrt{2\cdot m\cdot E\cdot1.602\cdot10^{-19}}}{hbar}$$

Equation 5.45; E is in units of eV and k1 is in mks units

$$k2(E,m,V) := \frac{\sqrt{2\cdot m\cdot(V - E)\cdot1.602\cdot10^{-19}}}{hbar}$$

Equation 5.54; V and E are in units of eV and k2 is in mks units

Equation 5.63 gives the ratio A/F.

$$A\_F(E,m,V,L) := \left[\left[\frac{1}{2} + \frac{i}{4}\cdot\left(\frac{k2(E,m,V)}{k1(E,m)} - \frac{k1(E,m)}{k2(E,m,V)}\right)\right]\cdot e^{(i\cdot k1(E,m) - k2(E,m,V))\cdot L} \ldots\right]$$

$$+\left[\frac{1}{2}-\frac{i}{4}\cdot\left(\frac{k2(E,m,V)}{k1(E,m)}-\frac{k1(E,m)}{k2(E,m,V)}\right)\right]\cdot e^{(i\cdot k1(E,m)+k2(E,m,V))\cdot L}$$

The transmission probability is F*F/A*A.

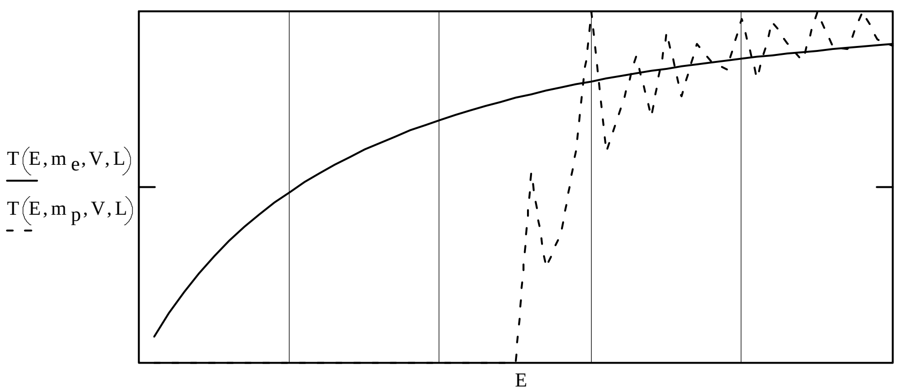
$$T(E,m,V,L) := \frac{1}{A\_F(E,m,V,L)\cdot\overline{A\_F(E,m,V,L)}}$$

Let's put some numbers in: $E := 4$ $V := 5$ $L := 0.1\cdot 10^{-9}$ $m_e := 9.11\cdot 10^{-31}$ $m_p := 1.672\cdot 10^{-27}$

$T(E,m_e,V,L) = 0.691$ $T(E,m_p,V,L) = 0$

The transmission probability for the proton is much much smaller. You can experiment with E and V to see that the transmission probability for the proton is comparable with that for the electron only when the incoming energy is much larger than the barrier height. It then makes sense that the more massive proton ought to find it easier to get past the smaller barrier.

$E := 0.2, 0.4 .. 10$ $V = 5$



The bumps in the electron's transmission probability are interference effects which result because the electron's wave function is more likely to reflect off both the first and second walls of the barrier. The proton is less likely to get through the first wall of the barrier and experience interference effects.

Physics 107 Problem 5.23 O. A. Pringle

A beam of electrons is incident on a barrier 6.00 eV high and 0.200 nm wide. Use Eq. 5.69 to find the energy they should have if 1.00 percent of them are to get through the barrier.

$m_{electron} := 9.11 \cdot 10^{-31}$ $e := 1.60 \cdot 10^{-19}$ $L := 0.200 \cdot 10^{-9}$ $U := 6.00 \cdot e$

$h := 6.63 \cdot 10^{-34}$ $hbar := \frac{h}{2 \cdot \pi}$ $T := 1.00 \cdot 10^{-2}$

Equation 5.69:

$$T := \exp(-2 \cdot k_2 \cdot L)$$

Solve Eq. 5.69 for E

$$-2 \cdot k_2 \cdot L := \ln(T)$$

$$k_2 := -\frac{\ln(T)}{2 \cdot L}$$

$$\frac{\sqrt{2 \cdot m_{electron} \cdot (U - E)}}{hbar} := -\frac{\ln(T)}{2 \cdot L}$$

$$\left[2 \cdot m_{electron} \cdot (U - E)\right] := \left(\frac{\ln(T) \cdot hbar}{2 \cdot L}\right)^2$$

$$U - E := \frac{1}{2 \cdot m_{electron}} \cdot \left(\frac{\ln(T) \cdot hbar}{2 \cdot L}\right)^2$$

$$E := U - \frac{1}{2 \cdot m_{electron}} \cdot \left(\frac{\ln(T) \cdot hbar}{2 \cdot L}\right)^2$$

$E = 0$ joules (The value is smaller than the zero tolerance, so it looks like zero.)

$\frac{E}{e} = 0.937$ eV

e

Beiser's solution manual gives an answer of 0.95 eV, which I get if I use e=1.602x10[-19] and h=6.626x10[-34].
On the exam, use the values of the constants which I give on the first page.

Physics 107 Problem 5.24 O. A. Pringle

Electrons with energies of 0.400 eV are incident on a barrier 3.00 eV high and 0.100 nm wide. Find the approximate probability for these electrons to penetrate the barrier.

$m_{electron} := 9.11\cdot 10^{-31}$ $\quad e := 1.60\cdot 10^{-19}$ $\quad L := 0.100\cdot 10^{-9}$ $\quad U := 3.00\cdot e$

$E := 0.400\cdot e$ $\quad h := 6.63\cdot 10^{-34}$ $\quad hbar := \dfrac{h}{2\cdot\pi}$

The problem asks for the approximate probability, so use Equation 5.69:

$$k_2 := \frac{\sqrt{2\cdot m_{electron}\cdot(U-E)}}{hbar}$$

$$T(k_2, L) := \exp(-2\cdot k_2\cdot L)$$

$$T(k_2, L) = 0.192$$

That's all you need to do to solve the problem. Below are some extra "goodies".

Is the barrier "high" enough for the approximation to be valid?

$$k_1 := \frac{\sqrt{2\cdot m_{electron}\cdot E}}{hbar} \qquad k_1 = 3.236\cdot 10^{9}$$

$$\frac{k_1}{k_2} = 0.392 \qquad \frac{k_2}{k_1} = 2.55$$

k2/k1 is greater than k1/k2 by a factor of about 6.5, so "high" is a marginally good approximation.

Is the barrier "wide" enough?

$k_2 \cdot L = 0.825$ This looks even less promising.

For a wide barrier,

$\exp(k_2 \cdot L) = 2.282$ >> $\exp(-k_2 \cdot L) = 0.438$

They differ by a factor of 5.2, which is again marginal.

If we use the "better" version of the transmission probability, we get

$$T := \left[\frac{16}{4 + \left(\frac{k_2}{k_1}\right)^2}\right] \cdot \exp(-2 \cdot k_2 \cdot L)$$

$T = 0.293$ And because the barrier is marginally high and wide, we expect this result to be somewhat off from the full, correct expression.

Physics 107 Problem 5.25 O. A. Pringle

Consider a beam of particles of kinetic energy E incident on a potential step at x = 0 that is U high, where E > U (Fig. 5.18).

(a) Explain why the solution D*exp(-jk.2x) (in the notation of Sec. 5.8) has no physical meaning in this situation, so that D = 0.

This term represents a wave travelling to the left (-x direction). There is no boundary from which to reflect such a wave beyond x = 0. Therefore D = 0.

(b) Show that the transmission probability here is T = CC*/AA* and is equal to 4k.1^2/(k.1+k')^2.

Set the wave function amplitudes equal at x = 0:

$$A \cdot e^{j \cdot k_1 \cdot x} + B \cdot e^{-(j \cdot k_1 \cdot x)} := C \cdot e^{j \cdot k_2 \cdot x}$$

which gives... (1) $A + B := C$

Set the wave function first derivatives equal at x = 0:

$$j \cdot k_1 \cdot A \cdot e^{j \cdot k_1 \cdot x} - j \cdot k_1 \cdot B \cdot e^{-(j \cdot k_1 \cdot x)} := j \cdot k_2 \cdot e^{j \cdot k_2 \cdot x}$$

which gives... (2) $(A - B) := \left(\frac{k_2}{k_1}\right) \cdot C$

Adding (2) to (1) and arranging terms gives:

(3) $$\frac{C}{A} := 2 \cdot \frac{k_1}{k_1 + k_2}$$

The transmission probability is found by taking the ratio of the probability CURRENTS. Probability CURRENTS are found by multiplying probability densities by their respective particle velocities. Since the velocity is proportional to k (v = hbar*k/m), ratios of velocities are simply ratios of k.

$$k_1 := \sqrt{2 \cdot \frac{m}{hbar^2}} \cdot \sqrt{E} \qquad k_2 := \sqrt{2 \cdot \frac{m}{hbar^2}} \cdot \sqrt{E - V}$$

Transmission probability:

$$T := \frac{C \cdot \overline{C}}{A \cdot \overline{A}} \cdot \left(\frac{k_2}{k_1}\right) \qquad T := 4 \cdot \frac{k_1^{\,2}}{(k_1 + k_2)^2} \cdot \frac{k_2}{k_1}$$

For part (c), I am going to re-write the transmission equation in terms of the particle velocities...

Simplifying and keeping only the square roots:

$$T := 4 \cdot \frac{k_1 \cdot k_2}{(k_1 + k_2)^2}$$

Notice that T = 1 when k.1 = k.2, as when V = 0. This equation is also valid when substituting v.1 and v.2 for k.1 and k.2.

$$T(E,V) := 4\cdot\frac{\sqrt{E}\cdot\sqrt{E-V}}{\left(\sqrt{E}+\sqrt{E-V}\right)^2}$$

Likewise, T = 1 when V = 0, or when the energy is well above the barrier, i.e., when E >> V.

$$T(v_1,v_2) := 4\cdot\frac{v_1\cdot v_2}{\left(v_1+v_2\right)^2}$$

c) A 1.00 ma beam of electrons moving at 2*10^6 m/s enters a region with a sharply defined boundary in which the electron speeds are reduced to 1*10^6 m/s by a difference in potential. Find the transmitted and reflected currents. Note that the powers of ten cancel in this equation. Relative currents are all that are required.

$$T(2,1) = 0.889$$

88.9% are transmitted.
current =0.889 ma

$$R(v_1,v_2) := 1 - T(v_1,v_2)$$

$$R(2,1) = 0.111$$

11.1% are reflected.
current =0.111 ma

Physics 107 Problem 5.29 O. A. Pringle

Show that for the n=0 state of a harmonic oscillator whose classical amplitude of motion is A, y=1 at x=A, where y is the quantity defined by Eq. 5.76.

A classical amplitude of motion of A means that A is the largest value that x takes on.
We can calculate the total energy from this largest value:

$$E := \frac{1}{2}\cdot k\cdot A^2$$

In the ground state

$$E := \frac{1}{2}\cdot h\cdot f$$

Equate the two energies:

$$\frac{1}{2}\cdot k\cdot A^2 := \frac{1}{2}\cdot h\cdot f$$

$$A := \sqrt{\frac{h \cdot f}{k}}$$

This value of A is the largest value that x takes on (classically). The value of y corresponding to this x is

$$y := \sqrt{\frac{2 \cdot \pi \cdot m \cdot f}{hbar}} \cdot x$$

$$y := \sqrt{\frac{2 \cdot \pi \cdot m \cdot f}{hbar}} \cdot \sqrt{\frac{h \cdot f}{k}}$$

$$y := \sqrt{\frac{4 \cdot \pi^2 \cdot m \cdot h \cdot f^2}{h \cdot k}}$$

$$y := 2 \cdot \pi \cdot \sqrt{\frac{m \cdot h \cdot f^2}{h \cdot k}}$$

But, from Eq. 5.73, $f := \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{k}{m}}$

So $y := 2 \cdot \pi \cdot \sqrt{\frac{m \cdot h \cdot k}{(h \cdot k) \cdot (2 \cdot \pi)^2 \cdot m}}$

$$y := 1$$

Physics 107 Problem 5.30 O. A. Pringle

Find the probability density ϕ.0^2 dx at x=0 and at x=+-A of a harmonic oscillator in its n=0 state.

In the ground state,

$$\phi_0(x) := \left(\frac{2\cdot m\cdot f}{hbar}\right)^{\frac{1}{4}}\cdot e^{-\frac{y^2}{2}}$$

$$\left(\left|\phi_0(x)\right|\right)^2\cdot dx := \left(\frac{2\cdot m\cdot f}{hbar}\right)^{\frac{1}{2}}\cdot e^{-y^2}\cdot dx$$

Eq. 5.76 relates x and y. At x=0, y=0 and

$$\left(\left|\phi_0(0)\right|\right)^2\cdot dx := \left(\frac{2\cdot m\cdot f}{hbar}\right)^{\frac{1}{2}}\cdot e^{-0^2}\cdot dx$$

$$\left(\left|\phi_0(0)\right|\right)^2\cdot dx := \left(\frac{2\cdot m\cdot f}{hbar}\right)^{\frac{1}{2}}\cdot dx$$

At x=+-A, as we showed in problem 5.29, y=1, so that

$$\left(\left|\phi_0(A)\right|\right)^2\cdot dx := \left(\frac{2\cdot m\cdot f}{hbar}\right)^{\frac{1}{2}}\cdot e^{-1^2}\cdot dx$$

$$\left(\left|\phi_0(A)\right|\right)^2\cdot dx := \left(\frac{2\cdot m\cdot f}{hbar}\right)^{\frac{1}{2}}\cdot\frac{1}{e}\cdot dx$$

Physics 107 Problem 5.31 O. A. Pringle

Find the expectation values <x> and <x^2> for the first two states of a harmonic oscillator.

This is a nice physics problem. Lots of fun math. Probably not a good physics 107 problem.

I will post a partial solution in the Wordperfect for Windows file 5-31.wp6 in the 107l lecture notes subdirectory. You can also look at the Mathcad 2.5 file harmosc.mcd in the 107 homework subdirectory.

For the exam, I would expect you to be able to show that <x>=0 for the ground state of a harmonic oscillator, and to compare that result with what you would expect from classical physics.

Physics 107 Problem 5.33 O. A. Pringle

A pendulum with a 1.00 g bob has a massless string 250 mm long. The period of the pendulum is 1.00 s.

$h := 6.63 \cdot 10^{-34}$ $\quad$ $hbar := \dfrac{h}{2 \cdot \pi}$ $\quad$ $e := 1.6 \cdot 10^{-19}$

(a) What is its zero-point energy? Would you expect the zero-point oscillations to be detectable?

Just as a quick double-check, let's see if the period Beiser gives agrees with the classical pendulum period

$$T := 2 \cdot \pi \cdot \sqrt{\frac{.25}{9.8}} \qquad T = 1.004$$ seconds close enough

The frequency is 1/T

$$T := 1$$

$$f := \frac{1}{T}$$

The zero-point energy is

$$E_0 := \left(0 + \frac{1}{2}\right) \cdot h \cdot f$$

$$E_0 = 0$$ joules

Even expressed in eV, the energy is very small:

$$\frac{E_0}{e} = 2.072 \cdot 10^{-15}$$ eV

With such a small energy, zero-point oscillations will be undetectable.

(b) The pendulum swings with a very small amplitude such that its bob rises a maximum of

1.00 mm above its equilibrium position. What is the corresponding quantum number?

The height of the rise gives us the maximum potential energy of the pendulum: we assume the pendulum is on the earth and calculate the corresponding gravitational potential energy. Then we can set the energy equal to the maximum potential energy and calculate the quantum number.

$m := 1 \cdot 10^{-3}$ $\qquad$ $x_{max} := 1 \cdot 10^{-3}$

$U_{max} := m \cdot 9.8 \cdot x_{max}$

$U_{max} = 9.8 \cdot 10^{-6}$ $\qquad$ A considerably larger number than the zero-point energy.

$$n := \frac{U_{max}}{h \cdot f} - \frac{1}{2}$$

$n = 1.478 \cdot 10^{28}$ $\qquad$ A variation on this problem would be a mass on a spring; to calculate the quantum number given x.max, you would first calculate k and then use U=kx.max^2/2.