Inverse Z-Transforms (Grewal) book.

① Inspection (Direct Inversion Method):-

Ex: ① find u_n if $U(z) = 1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2} + \frac{1}{8} z^{-3} + \cdots$

sol:-

$$U(z) = \sum_{n=0}^{\infty} u_n z^{-n} = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2}\right)^n}_{u_n} \cdot z^{-n}$$

$$u_n = \left(\frac{1}{2}\right)^n.$$

(2) Find ~~U(z)~~ u_n where $U(z) = \dfrac{z^3}{(z-1)^3}$

Sol:

$$U(z) = \left(1 - \frac{1}{z}\right)^{-3} \qquad (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \cdots$$

$$= 1 + \frac{3}{z} + \frac{6}{z^2} + \frac{10}{z^3} + \cdots$$

$$U(z) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} z^{-n}$$

$\longrightarrow u_n$

$$\begin{aligned} &1, 2, 3, 4, \underline{5} \\ &2, 4, 6, 8, \underline{10} \\ &1, 4, 7, 10, \underline{13} \end{aligned}$$

H.W.

(3) $u_n \rightarrow U(z) = 3 + \dfrac{2z}{z-1} - \dfrac{z}{2z-1}$

② Direct division Method:-

① Find the inverse z-Transform of $\dfrac{z}{z^2-3z+2}$.

Sol:-

$$
\begin{array}{r l}
 & z^{-1} + 3z^{-2} + 7z^{-3} + \cdots \\
z^2-3z+2 \big) & z \\
 & z - 3 + 2z^{-1} \\
 & - \quad + \quad - \\
\hline
 & 3 - 2z^{-1} \\
 & 3 - 9z^{-1} + 6z^{-2} \\
 & - \quad + \quad - \\
\hline
 & 7z^{-1} - 6z^{-2} \\
 & 7z^{-1} - 21z^{-2} + 14z^{-3} \\
 & - \quad + \quad - \\
\hline
 & 15z^{-2} - 14z^{-3}
\end{array}
$$

--- continue

$$U(z) = z^{-1} + 3z^{-2} + 7z^{-3} + \cdots$$

$0, 1, 3, 7, \cdots$

$2^0-1,\ 2^1-1,\ 2^2-1,\ 2^3-1$

$$U(z) = \sum_{n=0}^{\infty} (2^n - 1) \cdot z^{-n}$$

$$\underline{\underline{u_n = 2^n - 1}}$$

H.W.

② Find the inverse z-Transform of $\dfrac{4z^2+2z}{2z^2-3z+1}$

③ ~~pract~~ partial fraction method:-

$$U(z) = \frac{z}{z^2} \Rightarrow \frac{U(z)}{z} = \left(\frac{z^4+2}{z^4+z^2+2}\right) \rightarrow \text{partial fra}$$

Ex:

① Find the inverse z-transform of $\dfrac{z}{6z^2-5z+1}$

Sol:- $U(z) = \dfrac{z}{6z^2-5z+1}$

$$\frac{U(z)}{z} = \frac{1}{(2z-1)(3z-1)}$$

$$\frac{U(z)}{z} = \frac{2}{2z-1} - \frac{3}{3z-1}$$

$$U(z) = \frac{2.z}{2z-1} - \frac{3.z}{3z-1}$$

$$U(z) = \frac{z}{z-1/2} - \frac{z}{z-1/3}$$

$$z\{a^n\} = \frac{z}{z-a} \Rightarrow z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$$

$$U_n = (1/2)^n - (1/3)^n$$ //

H-W.

2) Find the inverse z-Transform of $\dfrac{4z^2+2z}{2z^2-3z+1}$

$$U(z) = \frac{-8z}{2z-1} + \frac{6z}{z-1} \rightarrow \text{Hint}$$

Method of Residues (inverse integral):-

$\cancel{U_n = \frac{\oint}{2\pi i}}$ $\quad U_n = \dfrac{1}{2\pi i}\oint_C U(z)\, z^{n-1}\, dz =$ Sum of Residues of $U(z)$

① Find the inverse z-Transform of $\dfrac{z}{z^2+7z+10} = U(z)$

Sol:
$$U_n = \frac{1}{2\pi i}\oint_C U(z)\, z^{n-1}\, dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{z}{z^2+7z+10} \cdot z^{n-1}\, dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{z^n}{(z+2)(z+5)}\, dz$$

$z = -2$, & $z = -5$.

Residue at $z = -2$

$$\lim_{z \to -2} \frac{\cancel{(z+2)}\, z^n}{\cancel{(z+2)}(z+5)} = \frac{(-2)^n}{3}$$

Residue at $z = -5$

$$\lim_{z \to -5} \frac{\cancel{(z+5)} \cdot z^n}{(z+2)\cancel{(z+5)}} = \frac{(-5)^n}{-3}$$

U_n = Sum of residue.

$$= \frac{(-2)^n}{3} - \frac{(-5)^n}{3}$$

② $\dfrac{z(z+1)}{(z-1)^3} = U(z)$

Sol:- $U_n = \dfrac{1}{2\pi i} \oint_C \dfrac{z(z+1)}{(z-1)^3} \cdot z^{n-1}\, dz$

$U_n = \dfrac{1}{2\pi i} \oint_C \dfrac{z^n (z+1)}{(z-1)^3}\, dz$

$z = 1 \longrightarrow$ order 3

$$\boxed{\begin{array}{l} z = a \rightarrow \text{order } n \\ \text{Residue at } z = a \\ \dfrac{1}{(n-1)!} \lim_{z\to a} \dfrac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \cdot f(z)\right] \end{array}}$$

Residue at $z=1$

$\dfrac{1}{2!} \lim_{z\to 1} \dfrac{d^2}{dz^2} \left[\dfrac{\cancel{(z-1)^3}\, z^n (z+1)}{\cancel{(z-1)^3}}\right]$

$= \dfrac{1}{2!} \lim_{z\to 1} \dfrac{d^2}{dz^2} \left[z^n (z+1)\right]$

$= \dfrac{1}{2!} \lim_{z\to 1} \left[(n+1)n z^{n-1} + n(n-1) z^{n-2}\right]$

$= n^2 \Rightarrow U_n = n^2.$

Power Series Method

① $U(z) = \log \frac{z}{z+1}$

Sol:- $U(z) = \log\left(\frac{z+1}{z}\right)^{-1} = -\log\left(1+\frac{1}{z}\right) =$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$U(z) = \frac{-1}{z} + \frac{1}{2z^2} - \frac{1}{3z^3} + \frac{1}{4z^4} - \cdots$

$U(z) = 0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n} \Rightarrow u_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{(-1)^n}{n} & \text{otherwise} \end{cases}$

Convolution theorem method:-

~~Convo~~ If $U(z) = Z\{u_n\}$ & $V(z) = Z\{v_n\}$

$Z\{u_n * v_n\} = U(z) \cdot V(z)$ $\Rightarrow$ $Z^{-1}[U(z) \cdot V(z)] = u_n * v_n$

Ex: (1) $\dfrac{z^2}{(z-1)(2z-1)}$

Sol:- $U(z) = \dfrac{z}{z-1}$, $V(z) = \dfrac{z}{2z-1}$

$u_n = (1)^n$, $v_n = \frac{1}{2}\left(\frac{1}{2}\right)^n$

$Z^{-1}[U(z) \cdot V(z)] = u_n * v_n$

$= (1)^n * \frac{1}{2}\left(\frac{1}{2}\right)^n$

$u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$

$= \sum_{m=0}^{n} (1)^m \cdot \left(\frac{1}{2}\right)^{n+1-m}$

$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n\right]$

$S_n = \dfrac{a}{1-r}(1-r^n)$

$u_n = 1 - \left(\frac{1}{2}\right)^{n+1}$

partial fraction, Convolution, Residue

Solution of Difference equation Using z-Transform

$$z(y_{k+n}) = z^n \left[z[y_k] - y_0 - \frac{y_1}{z} - \cdots - \frac{y_{n-1}}{z^{n-1}} \right]$$

$$z[y_{k+1}] = z\left[z[y_k] - y_0 \right]$$

$$z[y_{k+2}] = z^2 \left[z[y_k] - y_0 - \frac{y_1}{z} \right]$$

$$\vdots$$

$$z[y_{k-n}] = z^{-n} \cdot z(y_k)$$

① Solve the difference equation

$6\,y_{k+2} - y_{k+1} - y_k = 0$, $y(0) = 0$, $y(1) = 1$ by z-Transform

Sol:-

$$6\,z[y_{k+2}] - z[y_{k+1}] - z[y_k] = 0$$

$$6\left[z^2 y(z) - z^2 y(0) - z\,y(1)\right] - \left[z\,y(z) - y(0)\right] - y(z) = 0$$

$$y(z) = \frac{6z}{(6z^2 - z - 1)} = \frac{6/5}{1 - \frac{z^{-1}}{2}} - \frac{\frac{6}{5}}{1 + \frac{z^{-1}}{3}}$$

$$y_k = \frac{6}{5}\left(\frac{1}{2}\right)^k - \frac{6}{5}\left(\frac{-1}{3}\right)^k$$

(2) $y_{k+1} + \frac{1}{4} y_k = \left(\frac{1}{4}\right)^k, \quad k \geq 0, \quad y(0) = 0$

Sol:- $z[y_{k+1}] + \frac{1}{4} z[y_k] = z\left[\left(\frac{1}{4}\right)^k\right]$

$$z\,y(z) - z\,y(0) + \frac{1}{4} y(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$y(z) = \frac{1}{z + \frac{1}{4}} \times \frac{1}{1 - \frac{1}{4} z^{-1}} = \frac{z^{-1}}{1 + \frac{1}{4} z^{-1}} \times \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$= \frac{-2}{1 + \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{4} z^{-1}} \Rightarrow y_k = -2\left(\frac{-1}{4}\right)^k + 2\left(\frac{1}{4}\right)^k$$

H.W.

(3) $y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = u(k)$, ~~u(k)~~ unit fun

$y(0) = y(1) = y(2) = 0.$