Frobenius method :-

Let

$$x^2 y'' + x\, b(x)\, y' + c(x)\, y = 0 \quad \text{— (1)}$$

$x^2 = 0, \Rightarrow x = 0$ RSP.

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{m+n}, \quad a_0 \neq 0$$

$$y'(x) = \sum_{n=0}^{\infty} (m+n)\, a_n x^{m+n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (m+n)(m+n-1)\, a_n x^{m+n-2}$$

$$y(x) = A\,u(x) + B\,\boxed{v(x)}$$

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

$$y = x^m \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \right]$$

$$y' = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} (m+n)\, a_n x^{m+n-1}$$

$$x^2 \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} + x(b_0 + b_1 x + b_2 x^2 + \cdots) \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} + (c_0 + c_1 x + c_2 x^2 + \cdots) \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n} + (b_0 + b_1 x + b_2 x^2 + \cdots) \sum_{n=0}^{\infty} (m+n) a_n x^{m+n} + (c_0 + c_1 x + c_2 x^2 + \cdots) \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

x^m

$$m(m-1) a_0 + b_0 m a_0 + c_0 a_0 = 0$$

$$[m(m-1) + b_0 m + c_0] a_0 = 0$$

$m(m-1) + b_0 m + c_0 = 0$ – Indicial Equation

Say $\boxed{m = r_1 \text{ and } r_2}$

x^{m+n} for the recurrence relation

Case i Indicial roots are distinct and do not differ by an integer

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} A_n x^n$$

Case ii Indicial roots are Equal

$$r_1 = r_2 = r$$

$$y_2 = y_1(x) \ln(x) + x^{r} \sum_{n=1}^{\infty} A_n x^n$$

Case iii Indicial roots are distinct and differ by an integer

r_1 and r_2 $\quad r_1 > r_2$

$$y_2(x) = K y_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} A_n x^n.$$

Solve $2x^2y'' + xy' - (x^2+1)y = 0 \quad -(1)$

$x=0$ Singular point

$$y'' + \frac{1}{2x}y' - \left(\frac{x^2+1}{2x^2}\right)y = 0$$

$$\lim_{x\to 0} (x-0)\cdot\frac{1}{2x} = \frac{1}{2}$$

$$\lim_{x\to 0} (x-0)^2 \cdot \frac{-(x^2+1)}{2x^2} = -\frac{1}{2}$$

$x=0$ RSP.

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}, \quad a_0 \neq 0$$

$$y' = \sum_{n=0}^{\infty} (m+n)\, a_n x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1)\, a_n x^{m+n-2}$$

$$2x^2 \sum_{n=0}^{\infty} (m+n)(m+n-1)\, a_n x^{m+n-2} + x \sum_{n=0}^{\infty} (m+n)\, a_n x^{m+n-1} - (x^2+1) \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$2 \sum_{n=0}^{\infty} \underset{(1)}{\underline{(m+n)}(m+n-1)\, a_n x^{m+n}} + \sum_{n=0}^{\infty} \underset{(2)}{\underline{(m+n)}\, a_n x^{m+n}} - \sum_{n=0}^{\infty} a_n x^{m+n+2} - \sum_{n=0}^{\infty} \underset{(4)}{a_n x^{m+n}} = 0$$

$$\sum_{n=0}^{\infty} \Big[(2m+2n-1)(m+n) - 1 \Big] a_n x^{m+n} - \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$$

$\uparrow$ $n=0$ $\qquad\qquad$ $\nearrow$ $n=2$ $\qquad$ $n \to (n-2)$

$$\sum_{n=0}^{\infty}\Big[(m+n)(2m+2n-1)-1\Big]a_n\,\underline{x}^{m+n} - + \sum_{n=2}^{\infty} a_{n-2}\,\underline{x}^{m+n} = 0$$

$$\Big[m(2m-1)-1\Big]a_0\,x^{m} + \Big[(m+1)(2m+1)-1\Big]a_1\,x^{m+1} + \sum_{n=2}^{\infty}\Big[\big\{(m+n)(2m+2n-1)-1\big\}a_n - a_{n-2}\Big]x^{m+n} = 0$$

Equate coefficient of lowest power of x is zero for indicial equation

$\underline{\underline{x^m}}$

$$\Big[m(2m-1)-1\Big]a_0 = 0\,, \quad a_0 \neq 0$$

$$m(2m-1)-1 = 0 \Rightarrow 2m^2 - m - 1 = 0$$

$$m = 1,\ -1/2$$

$\underline{\underline{x^{m+1}}}$

$$\Big[(m+1)(2m+1)-1\Big]a_1 = 0 \Rightarrow a_1 = 0$$

Recurrence relation:

$$a_n = \frac{a_{n-2}}{\left[(m+n)(2m+2n-1) - 1\right]}, \quad n \geq 2$$

Indicial roots

$$m = -\tfrac{1}{2}, 1$$

$m = \frac{-1}{2}$

$$a_n = \frac{a_{n-2}}{(n-\frac{1}{2})(2n-2) - 1} \qquad n \geq 2$$

$n=2 \Rightarrow a_2 = \frac{a_0}{2}$

$n=3 \Rightarrow a_3 = \frac{a_1}{?} = 0$

$n=4 \Rightarrow a_4 = \frac{a_2}{20} = \frac{a_0}{40}$.

.....

$m = 1$

$$a_n = \frac{a_{n-2}}{(n+1)(2n+1) - 1} \qquad n \geq 2$$

$n=2 \Rightarrow a_2 = \frac{a_0}{14}$

$n=3 \Rightarrow a_3 = 0$

$n=4 \Rightarrow a_4 = \frac{a_2}{44} = \frac{a_0}{616}$

General soln of (1)

$$y(x) = A\, y_1(x) + B\, y_2(x)$$

where $y_1(x) = y_m(x)\Big|_{m=m_1=-1/2}$

$$y_2(x) = y_m(x)\Big|_{m=m_2=1}$$

$$y_m(x) = \sum_{n=0}^{\infty} a_n x^{m+n} = x^m \left[a_0 + \cancel{a_1 x}^{\,0} + a_2 x^2 + \cancel{a_3 x^3}^{\,0} + a_4 x^4 + \cdots \right]$$

$$y_1(x) = y_m\Big|_{m=-1/2} = a_0 x^{-1/2} \left[1 + \frac{x^2}{2} + \frac{x^4}{40} + \cdots \right]$$

$$y_2(x) = y_m\Big|_{m=1} = x^m\left[a_0 + \cancel{a_1 x}^{0} + a_2x^2 + \cancel{a_3x^3}^{0} + a_4x^4 + \cdots\right]$$

$$y_2 = a_0 x\left[1 + \frac{x^2}{14} + \frac{x^4}{616} + \cdots\right]$$

complete soln

$$y(x) = A^* y_1(x) + B^* y_2(x)$$, where $A^* = Aa_0$, $B^* = Ba_0$

$$y_1(x) = x^{-1/2}\left[1 + \frac{x^2}{2} + \frac{x^4}{40} + \cdots\right]$$

$$y_2(x) = x\left[1 + \frac{x^2}{14} + \frac{x^4}{616} + \cdots\right]$$

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Solve:- $xy'' + y' - xy = 0$