The Foundations: Logic and Proofs

Chapter 1, Part II: Predicate Logic

With Question/Answer Animations

Summary

- Predicate Logic (First-Order Logic (FOL), Predicate Calculus)
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
  - Translation from Predicate Logic to English
  - Translation from English to Predicate Logic

Predicates and Quantifiers

Section 1.4

Section Summary

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan's Laws for Quantifiers
- Translating English to Logic

Extending Propositional Logic

- Can statements with variables, e.g. $0 \leq x < 4$ implies $\sqrt{x} < 2$ be turned into (formal) *propositions*?

  Need propositional functions (based on predicates)

- If "All men are mortal" and "Socrates is a man"

  does it follow that "Socrates is mortal?"

Need to learn how to create and work with propositions expressed using ALL, EVERY, SOME, etc.

We will also discuss some rules of inference with quantifies.

Introducing Predicate Logic

- Predicate logic uses the following new features:

    - Variables: e.g.   *x, y, z*
    - Predicates (*P*): e.g. *is mortal,   can not be sent,   equals 10,   is green.*
    - Propositional functions: *P(x),   Q(x,y),   M(z)*
    - Quantifiers (see later)

- *Propositional functions* combine variables and predicates e.g.

$$P(x)   =   x \text{ is mortal}$$
$$\text{predicate } P$$
$$Q(y)   =   y \text{ is greater than } 4   (\text{ that is, } y>4)$$
$$\text{predicate } Q$$

    - *Propositional function* becomes a *proposition* when variables are replaced by elements from their domain (denoted by U), e.g.

$$P(\text{Socrates}) = \text{Socrates is mortal}$$
$$Q(5)   = 5 \text{ is greater than } 4$$
$$Q(2)   = 2 \text{ is greater than } 4$$

propositions

Propositional Functions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or when *bound* by a quantifier, see later).
- The statement $P(x)$ is said to be the value of the propositional function P at x.

> E.g., let $P(x)$ denote "$x > 0$" and the domain to be the integers. Then:
>
> > P(-3)   is false.
> > P(0)   is false.
> > P(3)   is true.

- Often the domain is denoted by U.
  So, in the example above U is the integers.

Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  R(3,4,7)

  R(x, 3, z)

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

  Q(2,-1,3)

  Q(3,4,7)

  Q(x, 3, z)

Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)
  > **Solution: F**

  R(3,4,7)
  > **Solution: T**

  R(x, 3, z)
  > **Solution: Not a Proposition**

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

  Q(2,-1,3)
  > **Solution: T**

  Q(3,4,7)
  > **Solution: F**
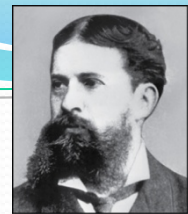
  Q(x, 3, z)
  > **Solution: Not a Proposition**

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If *P(x)* denotes "*x* > 0," find these truth values:

    P(3) ∨ P(-1)

    P(3) ∧ P(-1)

    P(3) → P(-1)

    P(-3) → P(-1)

- Expressions with variables are not propositions and therefore do not have truth values. For example,

    P(3) ∧ P(*y*)

    P(*x*) → P(*y*)

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If *P(x)* denotes "*x* > 0," find these truth values:

  P(3) ∨ P(-1)   **Solution**: T

  P(3) ∧ P(-1)   **Solution**: F

  P(3) → P(-1)   **Solution**: F

  P(-3) → P(-1)   **Solution**: T

- Expressions with variables are not propositions and therefore do not have truth values. For example,

  P(3) ∧ P(*y*)

  P(*x*) → P(*y*)

NOTE: instead of specifying values of variables, one can convert
a propositional function into a proposition using quantifiers (see next slide)

Quantifiers

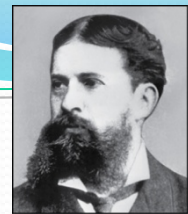Charles Peirce (1839-1914)

- an alternative way to convert propositional functions into propositions

Example: propositional function $P(x)$: x is mortal

<div align="center"><i>variable   predicate P</i></div>

Proposition(s):   Socrates is mortal   (specifying value of variable x)
P(Socrates)

All men are mortal   (quantifier "for all")
For all x, P(x)

Some men are mortal   (quantifier "for some")
For some x, P(x)

Quantifiers

Charles Peirce (1839-1914)

- an alternative way to convert propositional functions into propositions

Example: propositional function $P(x)$: x is mortal
                                                                variable   predicate P

Proposition(s):   Socrates is mortal   (specifying value of variable x)
                            P(Socrates)

                        All men are mortal   (quantifier $\forall$ - "for all")
                        $\forall x \ P(x)$

                        Some men are mortal   (quantifier $\exists$ - "for some")
                        $\exists x \ P(x)$

Quantifiers

Charles Peirce (1839-1914)

- an alternative way to convert propositional functions into propositions

Example: propositional function $Q(y)$ = y > 4

variable   *predicate P*

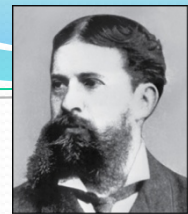Proposition(s):   5 > 4   (specifying value of variable y)
Q(5)

$\forall$y y>4   (quantifier $\forall$ - "for all")
for all y, Q(y)

$\exists$y y>4   (quantifier $\exists$ - "for some")
for some y, Q(y)

Quantifiers

Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier,* "For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x \, P(x)$ and $\exists x \, P(x)$.
- $\forall x \, P(x)$ asserts $P(x)$ for <u>every</u> x in the *domain*.
- $\exists x \, P(x)$ asserts $P(x)$ for <u>some</u> x in the *domain*.
- The quantifiers are said to *bind* the variable x in these expressions.

Universal Quantifier $\forall$

- $\forall x\, P(x)$ is read as "for all x, P(x)" or "for every x, P(x)"

Examples:

1)   If $P(x)$ denotes "$x > 0$" and U is the integers, then proposition $\forall x\, P(x)$ is false.

2)   If $P(x)$ denotes "$x > 0$" and U is the positive integers, then proposition $\forall x\, P(x)$ is true.

3)   If $P(x)$ denotes "x is even" and U is the integers, then proposition $\forall\, x\, P(x)$ is false.

Existential Quantifier $\exists$

- $\exists x\, P(x)$ is read as "for some x, P(x)", or "there is an x such that P(x)," or "there exists x such that P(x)," or "for at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.

2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then proposition $\exists x\, P(x)$ is false.

3. If $P(x)$ denotes "x is even" and U is the integers, then proposition $\exists x\, P(x)$ is true.

Thinking about Quantifiers

- When the domain (of variable) is finite, we can think of quantification as looping through the elements of the domain.

- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.

- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x where $P(x)$ is true, then $\exists x\, P(x)$ is false.

- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

- **Examples**:
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is ?, and $\forall x\, P(x)$ is ?.

  2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is ?, and $\forall x\, P(x)$ is ?.

  3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then $\exists x\, P(x)$ is ?, and $\forall x\, P(x)$ is ? . But if $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is ?, and $\forall x\, P(x)$ is ?

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

- **Examples**:
  1. If U is all integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  1. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  2. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$

- $\forall x\, (P(x) \vee Q(x))$ means something different.

- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\ J(x).$

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\ (S(x) \rightarrow J(x)).$

$\forall x\ (S(x) \land J(x))$ is not correct. What does it mean?

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if U is all people, then translate as

$$\exists x\, (S(x) \land J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Returning to the Socrates Example

- Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal." Specify the domain U as all people.

- The two premises are:
$$\forall x \ Man(x) \rightarrow Mortal(x)$$
$$Man(Socrates)$$

- The conclusion is:
$$Mortal(Socrates)$$

NOTE: This is a valid argument form ("universal modus ponens") combining "universal instantiation" (Table 2 on p.76) $\forall x \ P(x) \ then \ P(c)$ (for any c in U) and "modus ponens" (see previous slides) $if \ p \ and \ p \rightarrow q \ then \ q$

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value for...

  - every predicate substituted into these statements

    and

  - every domain used for the variables in the expressions.

- The notation $S \equiv T$ indicates that S and T are logically equivalent.

- **Example**: $\quad \forall x \, \neg\neg S(x) \equiv \forall x \, S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and

  the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

  Symbolically $\neg \forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions (*continued*)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2	De Morgan's Laws for Quantifiers.		
Negation	**Equivalent Statement**	**When Is Negation True?**	**When False?**
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will use these.

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

2. "Every student in this class has visited Canada or Mexico."

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x \; (S(x) \wedge M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

   $$\forall x \; (S(x) \rightarrow (M(x) \vee C(x)))$$

Some Fun with Translating from English into Logical Expressions

- U = {fleegles, snurds, thingamabobs}

    F(x): *x* is a fleegle

    S(x): *x* is a snurd

    T(x): x is a thingamabob

  Translate "Everything is a fleegle"

Some Fun with Translating from English into Logical Expressions

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  Translate "Everything is a fleegle"

  Solution: $\forall x\, F(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

  Translate "Nothing is a snurd."

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "Nothing is a snurd."

  Solution: $\neg \exists x \, S(x)$ What is this equivalent to?

  Solution: $\forall x \neg \, S(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

    $F(x)$: x is a fleegle

    $S(x)$: x is a snurd

    $T(x)$: x is a thingamabob

  Translate "All fleegles are snurds."

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

"All fleegles are snurds."

Solution: $\forall x \, (F(x) \rightarrow S(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

  Translate "Some fleegles are thingamabobs."

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

 "Some fleegles are thingamabobs."

 Solution: $\exists x \, (F(x) \wedge T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

  Translate "No snurd is a thingamabob."

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

"No snurd is a thingamabob."

Solution: $\neg \exists x \, (S(x) \land T(x))$ What is this equivalent to?

Solution: $\forall x \, (\neg S(x) \lor \neg T(x))$

Another equivalent solution $\forall x \, (S(x) \rightarrow \neg T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): x is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

  Translate "If any fleegle is a snurd then it is also a thingamabob."

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

    F(x): x is a fleegle

    S(x): x is a snurd

    T(x): x is a thingamabob

 "If any fleegle is a snurd then it is also a thingamabob."

 Solution: $\forall x \, ((F(x) \wedge S(x)) \rightarrow T(x))$

System Specification Example

- Predicate logic is used for specifying properties that systems must satisfy.

- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."

- Decide on predicates and domains (left implicit here) for the variables:
  - Let $L(m, y)$ be "Mail message m is larger than y megabytes."
  - Let $C(m)$ denote "Mail message m will be compressed."
  - Let $A(u)$ represent "User u is active."
  - Let $S(n, x)$ represent "Network link n is state x.

- Now we have: $\forall m (L(m, 1) \rightarrow C(m))$

$$\exists u\, A(u) \rightarrow \exists n\, S(n, available)$$

Lewis Carroll Example

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

- Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.

1. $\forall x\ (P(x) \rightarrow Q(x))$
2. $\exists x\ (P(x) \wedge \neg R(x))$
3. $\exists x\ (Q(x) \wedge \neg R(x))$

- Later we will see how to prove that 3 (the conclusion) follows from 1 and 2 (the premises).

Nested Quantifiers

Section 1.5

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $$\forall x \exists y (x + y = 0)$$

  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

  $\forall x \exists y (x + y = 0)$ can be viewed as $\forall x \, Q(x)$ where $Q(x)$ is $\exists y \, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x :
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.

  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

    $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x \; \forall y P(x,y)$ and $\forall y \; \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y P(x,y)$ is ?, but $\exists y \; \forall x P(x,y)$ is ?.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\, \forall y P(x,y)$ and $\forall y\, \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x\, \exists y P(x,y)$ is true, but $\exists y\, \forall x P(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let *U* be the real numbers,

Define *P(x,y)* : *x* · *y* = 0

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

2. $\forall x \exists y P(x,y)$

3. $\exists x \forall y\ P(x,y)$

4. $\exists x \exists y\ P(x,y)$

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y \, P(x,y)$

   Answer: True

4. $\exists x \, \exists y \, P(x,y)$

   Answer: True

Questions on Order of Quantifiers

Example 2: Let U be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

2. $\forall x \exists y P(x,y)$

3. $\exists x \forall y P(x,y)$

4. $\exists x \exists y P(x,y)$

Questions on Order of Quantifiers

Example 2: Let U be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: False

3. $\exists x \forall y P(x,y)$

   Answer: False

4. $\exists x \exists y P(x,y)$

   Answer: True

Quantifications of Two Variables

Statement A	When A is true?	When A is false? (¬A is true)
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$$

where C(x) is "*x has a computer*," and *F*(*x*,*y*) is "*x and y are friends*," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \ \forall y \ \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Quantifiers and domains should be made explicit as in:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce variables x and y:

   "For every integers x and y, if $x>0$ and $y>0$, then $x+y>0$."

3. Using symbols for connectives and quantifiers we get

$$\forall x \, \forall y \, ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$

   where the domain for x and y are the integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on airline a."

2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.

3. Then the statement can be expressed as:

$$\exists w \, \forall a \, \exists f \, (P(w,f) \land Q(f,a))$$

order of quantifies is very important!   Try changing it and see what that means

Negating Nested Quantifiers

$$\exists w\, \forall a\, \exists f\, (P(w,f) \wedge Q(f,a))$$

Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

$$\neg \exists w\, \forall a\, \exists f\, (P(w,f) \wedge Q(f,a))$$

Now use De Morgan's Laws to move the negation as far inwards as possible.

$\neg \exists w\, \forall a\, \exists f\, (P(w,f) \wedge Q(f,a))$

1. $\forall w \neg\, \forall a\, \exists f\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
2. $\forall w \exists a \neg\, \exists f\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
3. $\forall w \exists a \forall f \neg\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
4. $\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Can you translate the result back into English?

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.
Example 1: "Brothers are siblings."

Example 2: "Siblinghood is symmetric."

Example 3: "Everybody loves somebody."

Example 4: "There is someone who is loved by everyone."

Example 5: "There is someone who loves someone."

Example 6: "Everyone loves himself"

Questions on Translation from English

B(x,y): x and y are brothers
S(x,y): x and y are siblings
L(x,y): x loves y

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

      Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

      Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

      Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

      Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

      Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: "Everyone loves himself"

      Solution: $\forall x\ L(x,x)$

Some Questions about Quantifiers

- **Can you switch the order of quantifiers?** **Depends**
  - Is this a valid equivalence? $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$

    Solution: Yes! The left and the right side will always have the same truth value. The order in which *x* and *y* are picked does not matter.
  - Is this a valid equivalence? $\forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y)$

    Solution: No! The left and the right side may have different truth values for some propositional functions for *P*. Try "x + y = 0" for *P(x,y)* with *U* being the integers. The order in which the values of *x* and *y* are picked does matter.

- **Can you distribute quantifiers over logical connectives?** **Depends**
  - Is this a valid equivalence? $\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$

    Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.
  - Is this a valid equivalence? $\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

    Solution: No! The left and the right side may have different truth values. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.