Laplace Transforms of Derivatives :-

If $f'(t)$ be continuous & $L\{f(t)\} = f(s)$ then

$$L\{f'(t)\} = s.f(s) - f(0)$$

Pf :-

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s).\, e^{-st} f(t)\,dt$$

$$= \underbrace{\underset{s\to\infty}{Lt}\, e^{-st}.\, f(t)} - f(0) + s.\underbrace{\int_0^\infty e^{-st} f(t)dt}_{L\{f(t)\}}$$

Since $f(t)$ is exponential order

$$\underset{s\to\infty}{Lt}\, e^{-st} f(t) = 0$$

$$= 0 - f(0) + s.\, L\{f(t)\}$$

$$\boxed{L\{f'(t)\} = s.f(s) - f(0)}$$

$$L\{f''(t)\} = s.\, L\{f'(t)\} - f'(0)$$

$$L\{f''(t)\} = s^2 f(s) - s.f(0) - f'(0)$$

$$\boxed{L\{f^{(n)}(t)\} = s^{n} L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)}$$

Ex:- ① e^{at} → using the theorem on transforms of derivatives find L.T.

Sol:- $f(t) = e^{at}$ & $f'(t) = a.e^{at}$ & $f(0) = 1$.

$$L\{f'(t)\} = s.L\{f(t)\} - f(0)$$

$$L\{a e^{at}\} = s L\{e^{at}\} - 1$$

$$(a-s) L\{e^{at}\} = -1 \Rightarrow L\{e^{at}\} = \frac{1}{s-a}.$$

Laplace Transforms on Integrals:-

If $L\{f(t)\} = f(s)$ then $L\left\{\int_0^t f(x)\,dx\right\} = \frac{f(s)}{s}$

Pf:- $g(t) = \int_0^t f(x)\,dx \Rightarrow g'(t) = \frac{d}{dt}\int_0^t f(x)\,dx = f(t)$

$g(0) = 0.$

$\Rightarrow g'(t) = f(t)$

$\Rightarrow L\{g'(t)\} = L\{f(t)\}$

$s \cdot L\{g(t)\} - g(0) = L\{f(t)\}$

$s \cdot L\{g(t)\} = f(s)$

$L\{g(t)\} = \frac{f(s)}{s}$

$$L\left\{\int_0^t f(x)\,dx\right\} = \frac{f(s)}{s}$$

Ex: (1) Find L.T. $\int_0^t \sin at\, dt$

Sol: $L\{\sin at\} = \frac{a}{s^2+a^2} = f(s)$

$$L\left\{\int_0^t \sin at\, dt\right\} = \frac{f(s)}{s} = \frac{a}{s(s^2+a^2)}$$

(2) $\int_0^t \frac{\sin t}{t}\, dt$

Sol: $L\left\{\frac{\sin t}{t}\right\} \Longrightarrow$ Division by 't'

$$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}(s)$$

$$L\left\{\int_0^t \frac{\sin t}{t}\, dt\right\} = \frac{\cot^{-1}(s)}{s}$$

H.W. (3) L.T. of $e^{-t}\int_0^t \frac{\sin t}{t}\, dt$

Hint: $L\left\{\frac{\sin t}{t}\right\} \Rightarrow$ Division by 't'

$L\left\{\int_0^t \frac{\sin t}{t}\, dt\right\} \rightarrow$ previous p

$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\, dt\right\} \Rightarrow$ First shifting thm.

Laplace Transform of periodic funcs :-

If $f(t)$ is a periodic function with period 'a'

i.e., $f(t+a) = f(a)$ then $L\{f(t)\} = \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)\,dt$.

Ex: $\sin x \Rightarrow \sin(2\pi + x) = \sin x \rightarrow 2\pi$.

① A function $f(t)$ is periodic in $(0, 2b)$ and is defined as

$f(t) = \begin{cases} 1 & \text{if } 0 < t < b \\ -1 & \text{if } b < t < 2b \end{cases}$. Find it's Laplace Transform

Sol:- $L\{f(t)\} = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2bs}} \left[\int_0^b 1 \cdot e^{-st}\,dt + \int_b^{2b} (-1) e^{-st}\,dt \right]$

$L\{f(t)\} = \frac{1}{s(1-e^{-2bs})} \left[1 - 2e^{-sb} + e^{-2bs} \right]$

H-W.

(2) Find the L.T. of the function $f(t) = \begin{cases} \sin\omega t & \text{if } 0 < t < \frac{\pi}{\omega} \\ 0 & \text{if } \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$

where $f(t)$ has period $\frac{2\pi}{\omega}$.

L.T. of unit step function (or) Heaviside unit function:-

(1) The unit step function is defined as $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

& $L\{u(t-a)\} = \frac{e^{-as}}{s}$. $\Leftarrow \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt$

L.T. of Dirac delta func^n:-

The Dirac Delta func^n or Unit impulse func^n

$$f_\varepsilon(t) = \begin{cases} 1/\varepsilon & 0 \le t \le \varepsilon \\ 0 & t > \varepsilon \end{cases}$$ and $$\delta(t) = \lim_{\varepsilon \to 0} f_\varepsilon(t)$$

So, $$L\{f_\varepsilon(t)\} = \frac{1 - e^{-s\varepsilon}}{s\varepsilon}$$ & $$L\{\delta(t)\} = 1$$

① $$\int_0^\infty \delta(t)\,dt = 0$$

② $$\int_0^\infty \delta(t)\,G(t)\,dt = G(0)$$ | $G(t)$ is Continuous func^n

③ $$\int_0^\infty \delta(t-a)\,G(t)\,dt = G(a)$$

④ $$\int_0^\infty G(t)\,\delta'(t-a) = -G'(a)$$

① prove that $L\{\delta(t-a)\} = e^{-as}$

Sol. from First shifting theorem.

$L\{f(t)\} = F(s)$

$L\{f(t-a)\} = e^{-as} L\{f(t)\}$

$$L\{\delta(t-a)\} = e^{-as} L\{\delta(t)\} = e^{-as} \cdot 1$$

② Evaluate $\int_0^\infty \cos 2t \, \delta(t - \pi/3)\, dt$.

Sol: $G(t) = \cos 2t$ [$\because$ property No-3]

$$\int_0^\infty \cos 2t \, \delta(t-\pi/3)\, dt = G(a) = \cos \frac{2\pi}{3} = \frac{-1}{2}$$

fillup

Inverse Laplace Transform :-

If $f(s)$ is the Laplace Transform of a function $f(t)$.

$\Rightarrow$ $L\{f(t)\} = f(s)$. Then $f(t)$ is called the inverse Laplace transform of $f(s)$ and is written as

$f(t) = L^{-1}\{f(s)\}$

$\therefore$ L^{-1} is called the inverse L.T. operator.

L.T.	I.L.T.
$L\{f(t)\} = f(s)$	$L^{-1}\{f(s)\} = f(t)$
$L\{1\} = 1/s$	$L^{-1}\{1/s\} = 1$
$L\{e^{at}\} = \frac{1}{s-a}$	$L^{-1}\{\frac{1}{s-a}\} = e^{at}$
$L\{e^{-at}\} = \frac{1}{s+a}$	$L^{-1}\{\frac{1}{s+a}\} = e^{-at}$
$L\{\sin at\} = \frac{a}{s^2+a^2}$	$L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a}\sin at$
$L\{\cos at\} = \frac{s}{s^2+a^2}$	$L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$

H.W.

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

t^n, $\sinh at$, $\cosh at$, $e^{at}\sin bt$, $e^{at}\cos bt$, $e^{at}\sinh bt$, $e^{at}\cosh bt$, $e^{-at}\sinh bt$, $e^{-at}\cosh bt$,

application L.T

$\left\{ y'' + y' + y = 0 \quad \begin{matrix} y(0) = 1 \\ y'(0) = 0 \end{matrix} \right\}$

P.D.E., Int.