Z - Transforms

Definition:- The z-Transform of a sequence u_n defined for discrete values $n = 0, 1, 2, 3, \ldots$ and ($u_n = 0$, for $n < 0$) is defined as $z\{u_n\} = \sum_{n=0}^{\infty} u_n . z^{-n}$. z-Transform of the sequence u_n, i.e., $z\{u_n\}$ is a function of z and may be denoted by $U(z)$.

* z-Transform exists only when the infinite series $\sum_{n=0}^{\infty} u_n z^{-n}$ is convergent.

* $z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$ is termed as one sided transform and for two sided z-transform $z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n . z^{-n}$.

Z - Transforms of standard sequences :-

1) $z\{a^n\} = \dfrac{z}{z-a}$

Sol:- $z\{a^n\} = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$

$= 1 + \dfrac{a}{z} + \dfrac{a^2}{z^2} + \cdots + \dfrac{a^n}{z^n} + \cdots$

$= \dfrac{1}{1-\left(\frac{a}{z}\right)}, \quad \left|\dfrac{a}{z}\right| < 1$

$z\{a^n\} = \dfrac{z}{z-a}$.

② $z\{1\}$:

put $a=1 \Rightarrow z\{1\} = \dfrac{z}{z-1}$

H.W.

③ $z\{(-1)^n\}$ put $a=-1$

H.W.

④ $z\{k\} = \dfrac{kz}{z-1}$.

① $Z\{n^p\} = -z \dfrac{d}{dz} Z\{n^{p-1}\}$ $\longrightarrow$ $Z\{n^p\}$ & $Z\{n^{p-1}\}$

② Multiplication by n :-

$Z\{n u_n\} = -z \dfrac{d}{dz} Z\{u_n\}$ $\longrightarrow$ $Z\{n u_n\} = \sum_{n=0}^{\infty} n u_n z^{-n}$

$-z \sum_{n=0}^{\infty} (-n) u_n z^{-n-1}$

Ex:- $Z\{n\} = \dfrac{z}{(z-1)^2}$

Sol:- $Z\{n^p\} = -z \cdot \dfrac{d}{dz} Z\{n^{p-1}\}$

$Z\{n\} = -z \cdot \dfrac{d}{dz} Z\{n^0\}$

$= -z \cdot \dfrac{d}{dz}\left(\dfrac{z}{z-1}\right)$

$$\boxed{Z\{n\} = \dfrac{z}{(z-1)^2}}$$

H-W

② $Z\{n^2\} = \dfrac{z^2+z}{(z-1)^3}$

③

③ $z\{u(n)\} = \frac{z}{z-1}$ where $u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$ unit step func

Sol

Sol: $z\{u(n)\} = \sum_{n=0}^{\infty} u(n)\, z^{-n} = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = z\{1\} = \frac{z}{z-1}$

④ $z\{\delta(n)\} = 1$ where $\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$ unit impulse sequence

Sol: $z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n)\, z^{-n}$

$= 1 + 0 + \cdots = 1$

properties :-

① Linearity : $z\{a u_n + b\cdot v_n\} = a\, z\{u_n\} + b\cdot z\{v_n\}$

② Change of scale (Damping rule)

$z\{u_n\} = U(z)$, then $z\{a^{-n} u_n\} = U(az)$ & $z\{a^n u_n\} = U\left(\frac{z}{a}\right)$

Ex: $z\{a^n \cdot n\}$

sol:- $z\{n\} = \dfrac{z}{(z-1)^2} = U(z)$

$z\{a^n \cdot n\} = U\left(\frac{z}{a}\right)$

$= \dfrac{z/a}{(z/a - 1)^2} = \dfrac{az}{(z-a)^2}$ //

H.W.

② $z\{a^n n^2\}$

③ $Z\{\cos n\theta\}$ & $Z\{\sin n\theta\}$

Sol: $Z\{e^{-in\theta}\} = Z\{(e^{i\theta})^{-n} \cdot 1\}$.

$Z\{1\} = \frac{z}{z-1} = U(z)$

~~$Z\{(e^{-i\theta})^{-1} \cdot 1\} = U(\ldots)$~~

$Z\{u_n\} = U(z)$ then

$Z\{a^{-n} \cdot u_n\} = U(az)$

$Z\{(e^{i\theta})^{-n} \cdot 1\} = U(e^{i\theta} \cdot z)$

$= \frac{z e^{i\theta}}{z e^{i\theta} - 1}$

$= \frac{z}{z - e^{-i\theta}} \times \frac{z - e^{i\theta}}{z - e^{i\theta}}$

$= \frac{z(z - \cos\theta - i\sin\theta)}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1}$

$= \frac{z(z - \cos\theta - i\sin\theta)}{z^2 - 2z\cos\theta + 1}$

$Z\{e^{-in\theta}\} = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} - i \cdot \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

$Z\{\cos n\theta\} = \int$ $Z\{\sin n\theta\} = \int$

H.W.

④ $Z\{a^n \cos n\theta\}$ & $Z\{a^n \sin n\theta\}$.

Right shifting property:-

For $n \geq k$, $Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$, k is positive integer

Left shifting property:-

If k is any +ve integer ~~$Z\{u_{n+k}\} = z^{k}\{Z(u$~~

$$Z\{u_{n+k}\} = z^{k}\left\{Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots - \frac{u_{k-1}}{z^{k-1}}\right\}$$

Initial value theorem:-

If $Z\{u_n\} = U(z)$ then

$$u_0 = \lim_{z \to \infty} U(z)$$

$$u_1 = \lim_{z \to \infty} z\left[U(z) - u_0\right]$$

$$u_2 = \lim_{z \to \infty} z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$$

$\vdots$

Final value theorem:

If $Z\{u_n\} = U(z)$ then $\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)\,U(z)$

Convolution theorem

If $U(z) = Z\{u_n\}$, $V(z) = Z\{v_n\}$ then $Z\{u_n * v_n\} = U(z).V(z)$.

Ex: ① H.W. $Z\{2n + 3\sin\frac{n\pi}{4} - 5a^4\}$.

Sol: $2Z\{n\} + 3Z\{\sin\frac{n\pi}{4}\} - 5.Z\{a^4\}$.

② H.W. $Z\{(n+1)^2\}$

Sol: $Z\{n^2+2n+1\} = Z\{n^2\} + 2Z\{n\} + Z\{1\}$.

(3) $z\{{}^nC_r\} = \sum_{r=0}^{n} {}^nC_r z^{-r} \qquad 0 \le r \le n$

$= 1 + {}^nC_1 z^{-1} + {}^nC_2 z^{-2} + \cdots + {}^nC_n z^{-n}$

$= (1+z^{-1})^n.$

(4) $z\{\frac{1}{n!}\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \cdots$

$= 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$

$= e^{1/z} \quad \left\{ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right.$

(5) $z\left\{\frac{1}{(n+r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+r)!} z^{-n}$

We know $z\left\{\frac{1}{n!}\right\} = e^{1/z}$ $\quad u_n = \frac{1}{n!}$, $\quad u_0 = 1, u_1 = \frac{1}{1!}, u_2 = \frac{1}{2!}$ ---

By left shifting property ~~zd~~

$z\{u_{n+k}\} = z^k\left\{ z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots - \frac{u_{k-1}}{z^{k-1}} \right\}$

$z\left\{\frac{1}{(n+r)!}\right\} = z^r\left\{ e^{1/z} - 1 - \frac{1}{z} - \frac{1}{2!}\frac{1}{z^2} - \cdots - \frac{1}{(r-1)!\, z^{r-1}} \right\}$

(5) $z\{u_n\} = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$ then find $z\{u_{n+2}\}$

Sol:- From Left shifting property

$$z\{u_{n+2}\} = z^2\left\{ z\{u_n\} - u_0 - \frac{u_1}{z} \right\}.$$

From initial value theorem $u_0 = \lim_{z\to\infty} U(z)$

$$u_0 = \lim_{z\to\infty} \frac{z}{z-1} + \frac{z}{z^2+1} = \lim_{z\to\infty}\left(\frac{1}{1-1/z} + \frac{1/z}{1+1/z^2}\right)$$

$$u_0 = 1+0 = 1$$

$$u_1 = \lim_{z\to\infty} z\left[U(z) - u_0\right] = 2.$$

$$z\{u_{n+2}\} = z^2\left\{ \frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z} \right\} //.$$