Math 215 HW #7 Solutions

1. Problem 3.3.8. If P is the projection matrix onto a k-dimensional subspace $\mathbf{S}$ of the whole space $\mathbb{R}^n$, what is the column space of P and what is its rank?

   Answer: The column space of P is $\mathbf{S}$. To see this, notice that, if $\vec{x} \in \mathbb{R}^n$, then $P\vec{x} \in \mathbf{S}$ since P projects $\vec{x}$ to $\mathbf{S}$. Therefore, $\text{col}(P) \subset \mathbf{S}$. On the other hand, if $\vec{b} \in \mathbf{S}$, then $P\vec{b} = \vec{b}$, so $\mathbf{S} \subset \text{col}(P)$. Since containment goes both ways, we see that $\text{col}(P) = \mathbf{S}$.

   Therefore, since the rank of P is equal to the dimension of $\text{col}(P) = \mathbf{S}$ and since $\mathbf{S}$ is k-dimensional, we see that the rank of P is k.

2. Problem 3.3.12. If $\mathbf{V}$ is the subspace spanned by $(1, 1, 0, 1)$ and $(0, 0, 1, 0)$, find

   (a) a basis for the orthogonal complement $\mathbf{V}^\perp$.

   Answer: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

   By construction, the row space of A is equal to $\mathbf{V}$. Therefore, since the nullspace of any matrix is the orthogonal complement of the row space, it must be the case that $\mathbf{V}^\perp = \text{nul}(A)$. The matrix A is already in reduced echelon form, so we can see that the homogeneous equation $A\vec{x} = \vec{0}$ is equivalent to

$$\begin{aligned} x_1 &= -x_2 - x_4 \\ x_3 &= 0. \end{aligned}$$

   Therefore, the solutions of the homogeneous equation are of the form

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

   so the following is a basis for $\text{nul}(A) = \mathbf{V}^\perp$:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   (b) the projection matrix P onto $\mathbf{V}$.

   Answer: From part (a), we have that $\mathbf{V}$ is the row space of A or, equivalently, $\mathbf{V}$ is the column space of

$$B = A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the projection matrix P onto $\mathbf{V} = \text{col}(B)$ is

$$P = B(B^TB)^{-1}B^T = A^T(AA^T)^{-1}A.$$

Now,

$$B^TB = AA^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

so

$$(AA^T)^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P &= A^T(AA^T)^{-1}A \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{aligned}$$

(c) the vector in $\mathbf{V}$ closest to the vector $\vec{b} = (0, 1, 0, -1)$ in $\mathbf{V}^{\perp}$.

Answer: The closest vector to $\vec{b}$ in $\mathbf{V}$ will necessarily be the projection of $\vec{b}$ onto $\mathbf{V}$. Since $\vec{b}$ is perpendicular to $\mathbf{V}$, we know this will be the zero vector. We can also double-check this since the projection of $\vec{b}$ onto $\mathbf{V}$ is

$$P\vec{b} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

3. Problem 3.3.22. Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Answer: If the above data points actually lay on a straight line $C + Dt$, we would have

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Call the matrix A and the vector on the right-hand side $\vec{b}$. Of course this system is inconsistent, but we want to find $\widehat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$ such that $A\widehat{x}$ is as close as possible to $\vec{b}$. As we've seen, the correct choice of $\widehat{x}$ is given by

$$\widehat{x} = (A^T A)^{-1} A^T \vec{b}.$$

To compute this, first note that

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}.$$

Therefore,

$$(A^T A)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{bmatrix}$$

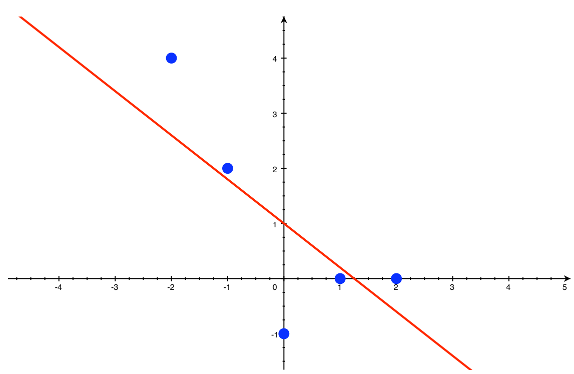
and so

$$\begin{aligned} \widehat{x} &= (A^T A)^{-1} A^T \vec{b} \\ &= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix} \end{aligned}$$

Therefore, the best-fit line for the data is

$$1 - \frac{4}{5}t.$$

Here are the data points and the best-fit line on the same graph:



4. Problem 3.3.24. Find the best straight-line fit to the following measurements, and sketch your solution:

$$\begin{array}{llllllll} y = & 2 & \text{at} & t = & -1, & y = & 0 & \text{at} \quad t = \ 0, \\ y = & -3 & \text{at} & t = & 1, & y = & -5 & \text{at} \quad t = \ 2. \end{array}$$

Answer: As in Problem 3, if the data actually lay on a straight line $y = C + Dt$, we would have

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}.$$

Again, this system is not solvable, but, if A is the matrix and $\vec{b}$ is the vector on the right-hand side, then we want to find $\widehat{x}$ such that $A\widehat{x}$ is as close as possible to $\vec{b}$. This will happen when

$$\widehat{x} = (A^T A)^{-1} A^T \vec{b}.$$

Now,

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}.$$

To find $(A^T A)^{-1}$, we want to perform row operations on the augmented matrix

$$\begin{bmatrix} 4 & 2 & & 1 & 0 \\ 2 & 6 & & 0 & 1 \end{bmatrix}$$

so that the 2×2 identity matrix appears on the left. To that end, scale the first row by $\frac{1}{4}$ and subtract 2 times the result from row 2:

$$\begin{bmatrix} 1 & 1/2 & & 1/4 & 0 \\ 0 & 5 & & -1/2 & 1 \end{bmatrix}.$$

Now, scale row 2 by $\frac{1}{5}$ and subtract half the result from row 1:

$$\begin{bmatrix} 1 & 0 & & 3/10 & -1/10 \\ 0 & 1 & & -1/10 & 1/5 \end{bmatrix}.$$

Therefore,

$$(A^T A)^{-1} = \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{bmatrix}$$

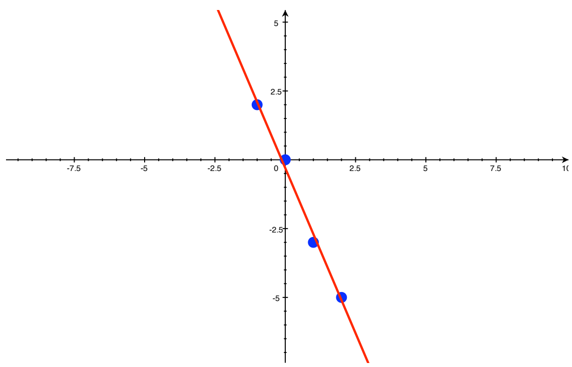
and so

$$
\begin{aligned}
\widehat{x} &= (A^T A)^{-1} A^T \vec{b} \\
&= \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix} \\
&= \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{bmatrix} \begin{bmatrix} -6 \\ -15 \end{bmatrix} \\
&= \begin{bmatrix} -3/10 \\ -12/5 \end{bmatrix}.
\end{aligned}
$$

Therefore, the best-fit line for the data is

$$y = -\frac{3}{10} - \frac{12}{5}t.$$

Here's a plot of both the data and the best-fit line:



5. Problem 3.3.25. Suppose that instead of a straight line, we fit the data in Problem 24 (i.e #3 above) by a parabola $y = C + Dt + Et^2$. In the inconsistent system $A\vec{x} = \vec{b}$ that comes from the four measurements, what are the coefficient matrix A, the unknown vector $\vec{x}$, and the data vector $\vec{b}$? For extra credit, actually determine the best-fit parabola.

   Answer: Since the data hasn't changed, the data vector $\vec{b}$ will be the same as in the previous problem. If the data were to lie on a parabola $C + Dt + Et^2$, then we would have that

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix},$$

   so A is the matrix above and $\vec{x}$ is the vector next to A on the left-hand side.

   To actually determine the best-fit parabola, we just need to find $\widehat{x}$ such that $A\widehat{x}$ is as close as possible to $\vec{b}$. This will be the vector

$$\widehat{x} = (A^T A)^{-1} A^T \vec{b}.$$

Now,

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

To find $(A^T A)^{-1}$, we want to use row operations to convert the left-hand side of this augmented matrix to I:

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 6 & 1 & 0 & 0 \\ 2 & 6 & 8 & 0 & 1 & 0 \\ 6 & 8 & 18 & 0 & 0 & 1 \end{array}\right].$$

First, scale row 1 by $\frac{1}{4}$ and subtract twice the result from row 2 and six times the result from row 3:

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/4 & 0 & 0 \\ 0 & 5 & 5 & -1/2 & 1 & 0 \\ 0 & 5 & 9 & -3/2 & 0 & 1 \end{array}\right].$$

Next, subtract row 2 from row 3, scale row 2 by $\frac{1}{5}$ and subtract half the result from row 1:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3/10 & -1/10 & 0 \\ 0 & 1 & 1 & -1/10 & 1/5 & 0 \\ 0 & 0 & 4 & -1 & -1 & 1 \end{array}\right].$$

Finally, scale row 3 by $\frac{1}{4}$ and subtract the result from rows 1 and 2:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/20 & 3/20 & -1/4 \\ 0 & 1 & 0 & 3/20 & 9/20 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 1/4 \end{array}\right].$$

Therefore,

$$(A^T A)^{-1} = \begin{bmatrix} 11/20 & 3/20 & -1/4 \\ 3/20 & 9/20 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{bmatrix}$$

and so

$$\begin{aligned} \widehat{x} &= (A^T A)^{-1} A^T \vec{b} \\ &= \begin{bmatrix} 11/20 & 3/20 & -1/4 \\ 3/20 & 9/20 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 11/20 & 3/20 & -1/4 \\ 3/20 & 9/20 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -6 \\ -15 \\ -21 \end{bmatrix} \\ &= \begin{bmatrix} -3/10 \\ -12/5 \\ 0 \end{bmatrix}. \end{aligned}$$

Thus, the best-fit parabola is

$$y = -\frac{3}{10} - \frac{12}{5}t + 0t^2 = -\frac{3}{10} - \frac{12}{5}t,$$

which is the same as the best-fit line!

6. Problem 3.4.4. If Q_1 and Q_2 are orthogonal matrices, so that $Q^TQ = I$, show that Q_1Q_2 is also orthogonal. If Q_1 is rotation through θ and Q_2 is rotation through ϕ, what is Q_1Q_2? Can you find the trigonometric identities for $\sin(\theta+\phi)$ and $\cos(\theta+\phi)$ in the matrix multiplication Q_1Q_2?

   Answer: Note that

$$(Q_1Q_2)^T(Q_1Q_2) = Q_2^TQ_1^TQ_1Q_2 = Q_2^TIQ_2 = Q_2^TQ_2 = I,$$

   since both Q_1 and Q_2 are orthogonal matrices. Therefore, the columns of Q_1Q_2 are orthonormal. Moreover, since both Q_1 and Q_2 are square and must be the same size for Q_1Q_2 to make sense, it must be the case that Q_1Q_2 is square. Therefore, since Q_1Q_2 is square and has orthonormal columns, it is an orthogonal matrix.

   If Q_1 is rotation through and angle θ, then, as we've seen,

$$Q_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

   Likewise, if Q_2 is rotation through and angle ϕ, then

$$Q_2 = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}.$$

   With these choices of Q_1 and Q_2, if $\vec{x}$ is any vector in the plane $\mathbb{R}^2$, we see that

$$Q_1Q_2\vec{x} = Q_1(Q_2\vec{x}),$$

   meaning that $\vec{x}$ is first rotated by an angle ϕ, then the result is rotated by an angle θ. Of course, this is the same as rotating $\vec{x}$ by an angle $\theta+\phi$, so Q_1Q_2 is precisely the matrix of the transformation which rotates the plane through an angle of $\theta+\phi$. On the one hand, we know that

$$Q_1Q_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}.$$

   On the other hand, the matrix which rotates the plane through an angle of $\theta+\phi$ is precisely

$$\begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}.$$

   Hence, it must be the case that

$$\begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}.$$

   This implies the following trigonometric identities:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$$

7. Problem 3.4.6. Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \\ 1/\sqrt{3} & 2/\sqrt{14} & \\ 1/\sqrt{3} & -3/\sqrt{14} & \end{bmatrix}$$

is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

Answer: Let $\vec{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ and $\vec{q}_2 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}$. If $\vec{q}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\|\vec{q}_3\| = 1$, $\langle \vec{q}_1, \vec{q}_3 \rangle = 0$ and $\langle \vec{q}_2, \vec{q}_3 \rangle = 0$ then we have that

$$\begin{aligned} 1 &= \|\vec{q}_3\|^2 = \langle \vec{q}_3, \vec{q}_3 \rangle = a^2 + b^2 + c^2 \\ 0 &= \langle \vec{q}_1, \vec{q}_3 \rangle = a/\sqrt{3} + b/\sqrt{3} + c/\sqrt{3} \\ 0 &= \langle \vec{q}_2, \vec{q}_3 \rangle = a/\sqrt{14} + 2b/\sqrt{14} - 3c/\sqrt{14}. \end{aligned}$$

Multiplying the second line by $\sqrt{3}$ and the third line by $\sqrt{14}$, we get the equivalent system

$$\begin{aligned} 1 &= a^2 + b^2 + c^2 \\ 0 &= a + b + c \\ 0 &= a + 2b - 3c \end{aligned}$$

From the second line we have that $b = -a - c$ and so, from the third line,

$$a = -2b + 3c = -2(-a - c) + 3c = 2a + 5c.$$

Thus $a = -5c$, meaning that $b = -a - c = -(-5c) - c = 4c$. Therefore

$$1 = a^2 + b^2 + c^2 = (-5c)^2 + (4c)^2 + c^2 = 42c^2,$$

meaning that $c = \pm 1/\sqrt{42}$. Thus, we see that

$$\vec{q}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5c \\ 4c \\ c \end{bmatrix} = \pm \begin{bmatrix} -5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}.$$

Therefore, there are two possible choices; one of them gives the following orthogonal matrix:

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & -5/\sqrt{42} \\ 1/\sqrt{3} & 2/\sqrt{14} & 4/\sqrt{42} \\ 1/\sqrt{3} & -3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix}.$$

It is straightforward to check that each row has length 1 and is perpendicular to the other rows.

8. Problem 3.4.12. What multiple of $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be subtracted from $\vec{a}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to make the result orthogonal to $\vec{a}_1$? Factor $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ into QR with orthonormal vectors in Q.

   Answer: Let's do Gram-Schmidt on $\{\vec{a}_1, \vec{a}_2\}$. First, we let

$$\vec{v}_1 = \frac{\vec{a}_1}{\|\vec{a}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

   Next,

$$\vec{w}_2 = \vec{a}_2 - \langle \vec{v}_1, \vec{a}_2 \rangle \vec{v}_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - 4/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

   By construction, $\vec{w}_2$ is orthogonal to $\vec{a}_1$, so we see that we needed to subtract 2 times $\vec{a}_1$ from $\vec{a}_2$ to get a vector perpendicular to $\vec{a}_1$.

   Now, continuing with Gram-Schmidt, we get that

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

   Therefore, if $A = [\vec{a}_1 \ \vec{a}_2]$ and $Q = [\vec{v}_1 \ \vec{v}_2]$, then

$$A = QR$$

   where

$$R = \begin{bmatrix} \langle \vec{a}_1, \vec{v}_1 \rangle & \langle \vec{a}_2, \vec{v}_1 \rangle \\ 0 & \langle \vec{a}_2, \vec{v}_2 \rangle \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}.$$

9. Problem 3.4.18. If $A = QR$, find a simple formula for the projection matrix P onto the column space of A.

   Answer: If $A = QR$, then

$$A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T I R = R^T R,$$

   since Q is an orthogonal matrix (meaning $Q^T Q = I$). Thus, the projection matrix P onto the column space of A is given by

$$P = A(A^T A)^{-1} A^T = QR(R^T R)^{-1} (QR)^T = QRR^{-1} (R^T)^{-1} R^T Q^T = QQ^T$$

   (provided, of course, that R is invertible).

10. Problem 3.4.32.

   (a) Find a basis for the subspace $\mathbf{S}$ in $\mathbb{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

Answer: The solutions of the given equation are, equivalently, solutions of the matrix equation

$$[1\ 1\ 1\ -1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0,$$

so $\mathbf{S}$ is the nullspace of the 1×4 matrix $A = [1\ 1\ 1\ -1]$. Since A is already in reduced echelon form, we can read off that the solutions to the above matrix equation are the vectors of the form

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, a basis for $\mathrm{nul}(A) = \mathbf{S}$ is given by

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a basis for the orthogonal complement $\mathbf{S}^\perp$.

Answer: Since $\mathbf{S} = \mathrm{nul}(A)$, it must be the case that $\mathbf{S}^\perp$ is the row space of A. Hence, the one row of A gives a basis for $\mathbf{S}^\perp$, meaning that the following is a basis for $\mathbf{S}^\perp$:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(c) Find $\vec{b}_1$ in $\mathbf{S}$ and $\vec{b}_2$ in $\mathbf{S}^\perp$ so that $\vec{b}_1 + \vec{b}_2 = \vec{b} = (1, 1, 1, 1)$.

Answer: For any $\vec{b}_1 \in \mathbf{S}$, we know that $\vec{b}_1$ is a linear combination of elements of the basis for $\mathbf{S}$ that we found in part (a). In other words,

$$\vec{b}_1 = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for some choice of $a, b, c \in \mathbb{R}$. Also, if $\vec{b}_2 \in \mathbf{S}^\perp$, then $\vec{b}_2$ is a multiple of the basis vector for $\mathbf{S}^\perp$ we found in part (b). Thus,

$$\vec{b}_2 = d \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

for some $d \in \mathbb{R}$. Therefore,

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

To solve this matrix equation, we just do elimination on the augmented matrix

$$\left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right].$$

Add row 1 to row 2:

$$\left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right].$$

Next, add row 2 to row 3:

$$\left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right].$$

Finally, subtract row 3 from row 4:

$$\left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -4 & -2 \end{array}\right].$$

Therefore, $-4d = -2$, so $d = \frac{1}{2}$. Hence,

$$3 = c + 3d = c + \frac{3}{2},$$

so $c = \frac{3}{2}$. In turn,

$$2 = -b + c + 2d = -b + \frac{3}{2} + 1 = -b + \frac{5}{2},$$

meaning $b = \frac{1}{2}$. Finally,

$$1 = -a - b + c + d = -a - \frac{1}{2} + \frac{3}{2} + \frac{1}{2} = -a + \frac{3}{2},$$

so $a = -\frac{1}{2}$.

Therefore,

$$\vec{b}_1 = -\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$$

and

$$\vec{b}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

11. (**Bonus Problem**) Problem 3.4.24. Find the fourth Legendre polynomial. It is a cubic $x^3 + ax^2 + bx + c$ that is orthogonal to 1, x, and $x^2 - \frac{1}{3}$ over the interval $-1 \le x \le 1$.

Answer: We can find the fourth Legendre polynomial in the same style as Strang finds the third Legendre polynomial on p. 185:

$$v_4 = x^3 - \frac{\langle 1, x^3 \rangle}{\langle 1, 1 \rangle}1 - \frac{\langle x, x^3 \rangle}{\langle x, x \rangle}x - \frac{\left\langle x^2 - \frac{1}{3}, x^3 \right\rangle}{\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \right\rangle}\left(x^2 - \frac{1}{3}\right). \tag{1}$$

Now, we just compute each of the inner products in turn:

$$\begin{aligned}
\langle 1, x^3 \rangle &= \int_{-1}^{1} x^3 dx = 0 \\
\langle 1, 1 \rangle &= \int_{-1}^{1} 1 dx = 2 \\
\langle x, x^3 \rangle &= \int_{-1}^{1} x^4 dx = \frac{2}{5} \\
\langle x, x \rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} \\
\left\langle x^2 - \frac{1}{3}, x^3 \right\rangle &= \int_{-1}^{1} \left(x^5 - \frac{x^3}{3}\right) dx = 0 \\
\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \right\rangle &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \frac{8}{45}.
\end{aligned}$$

Therefore, (1) becomes

$$v_4 = x^3 - 0 \cdot 1 - \frac{2/5}{2/3}x - 0 \cdot \left(x^2 - \frac{1}{3}\right) = x^3 - \frac{3}{5}x.$$

Therefore, the fourth Legendre polynomial is $x^3 - \frac{3}{5}x$.