Series solution

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n(x)\,y = 0 \quad \text{—} (1)$$

$a_0, a_1, a_2 \,..\, a_n$ are constants.

$y = y_{C.F} + y_{P.I}$

$x = e^z$

e^{ax}

$\sin bx$

$\cos bx$

x^k.

$a_0, a_1, a_2 \,..\, a_n$ functions of x ?

$y'' + x^2 y = 0$

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \quad \text{—} (1)$$

Ordinary point :-

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = 0 \quad \text{--- (1)}$$

A point $x = x_0$ for which $a_0(x_0) \neq 0$ called ordinary point

$a_1(x)$; $a_2(x)$ are analytic ✓

otherwise singular point.

$$(1-x^2)\, y'' - 2x\, y' + 5y = 0$$

$1 - x^2 = 0$

✓ $x = \pm 1$

ordinary points $\mathbb{R} - \{\pm 1\}$

singular points $x = -1, +1$

ordinary points $\mathbb{R} - \{\pm 1\}$

Singular points

Regular Singular point

Irregular Singular point.

Regular Singular points

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

$$① \Rightarrow \quad y'' + \frac{a_1(x)}{a_0(x)}y' + \frac{a_2(x)}{a_0(x)}y = 0$$

where $P(x) = \dfrac{a_1(x)}{a_0(x)}$

$$y'' + P(x)y' + Q(x)y = 0$$

$Q(x) = \dfrac{a_2(x)}{a_0(x)}$

$x = x_0$

A singular point $x = x_0$ of eq ① is called R.S.P if

(i) $\lim\limits_{x \to x_0} (x - x_0)P(x)$ Exists

(ii) $\lim\limits_{x \to x_0} (x - x_0)^2 Q(x)$

otherwise $x = x_0$ Irregular singular point.

(i) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

(ii) $x^2 y'' + axy' + by = 0$

Soln:

$1 - x^2 = 0$

$x = \pm 1$

ordinary points $\mathbb{R} - \{\pm 1\}$

Singular points -1 and 1

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

At $x = -1$

(i) $\lim_{x \to -1} (x+1) \cdot \frac{-2x}{1-x^2} = 1$

(ii) $\lim_{x \to -1} (x+1)^2 \frac{n(n+1)}{1-x^2} = 0$

$x = -1$ RSP.

At $x = 1$

$\lim_{x \to 1} (x-1) \frac{-2x}{1-x^2}$

$\lim_{x \to 1} (x-1)^2 \frac{n(n+1)}{1-x^2}$

$x = 1$ RSP.

Theorem When $x = x_0$ is an ordinary point of ① its every solution can be expressed in the form

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

$$y = \sum_{n=0}^{\infty} a_m (x - x_0)^n$$

Theorem When $x = x_0$ is a Reguler Singular point of ① atleast one of the solution can be expressed as

$$y = (x - x_0)^m \left[a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \right]$$

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{m+n}$$

① $y'' + y = 0$

ordinary points $\underline{\underline{\mathbb{R}}}$

At $x = 0$

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n .$$

$$\underline{y = \sum_{n=0}^{\infty} a_n x^n}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = 2a_2 + 6a_3 x + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$y = e^{mx}$

$y' = m e^{mx}$

$y'' = m^2 e^{mx}$

$(m^2 + 1) e^{mx} = 0$

$m^2 + 1 = 0$

$m = \pm i$

$$\boxed{y = c_1 \cos x + c_2 \sin x}$$

$$y'' + y = 0 \quad — (1)$$

$x=0$ ordinary point

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} .$$

$$\underbrace{\sum_{\underline{n=2}}^{\infty} n(n-1) a_n \underline{x}^{n-2}}_{(1)} + \underbrace{\sum_{\underline{n=0}}^{\infty} a_n \underline{x}^n}_{(2)} = 0$$

$$n \longrightarrow n+2$$

$$\sum_{n+\cancel{2}=\cancel{2}}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + a_n \right] x^n = 0$$

$$(n+2)(n+1)\, a_{n+2} + a_n = 0 \,, \quad n \geq 0$$

$$\checkmark \quad \boxed{a_{n+2} = - \frac{a_n}{(n+2)(n+1)}} \quad n \geq 0$$

Recurrence relation

$n=0 \Rightarrow \quad \underline{a_2} = \dfrac{-a_0}{2!}$

$n=1 \Rightarrow \quad a_3 = \dfrac{-a_1}{3!}$

$n=2 \Rightarrow a_4 = \dfrac{-a_2}{4 \cdot 3} = \dfrac{a_0}{4!}$

$n=3 \Rightarrow a_5 = \dfrac{-a_3}{5 \cdot 4} = \dfrac{a_1}{5!}$

$n=4 \Rightarrow a_6 = -\dfrac{a_0}{6!}$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$$

$$= a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right]$$

$$\boxed{y(x) = a_0 \cos x + a_1 \sin x}$$

Solve $y'' + xy = 0$