Inverse Laplace Transforms (I.L.T.)

(1) Find the I.L.T. of $\dfrac{s^2 - 3s + 4}{s^3}$

Sol:
$$L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = L^{-1}\left\{\frac{1}{s} - 3\frac{1}{s^2} + 4\cdot\frac{1}{s^3}\right\}$$

$$\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$$

$$= L^{-1}\{1/s\} - 3\, L^{-1}\left\{\frac{1}{s^2}\right\} + 4\, L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 - 3\cdot\frac{t^1}{1!} + 4\cdot\frac{t^2}{2!} = 1 - 3t + 2t^2$$

② $L^{-1}\left\{\dfrac{s+2}{s^2-4s+13}\right\}$

Sol:- $L^{-1}\left\{\dfrac{s+2}{(s-2)^2+9}\right\} = L^{-1}\left\{\dfrac{s-2+4}{(s-2)^2+3^2}\right\}$

$= L^{-1}\left\{\dfrac{s-2}{(s-2)^2+3^2}\right\} + L^{-1}\left\{\dfrac{4}{(s-2)^2+3^2}\right\}$

$$\left\{\begin{array}{l} L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2} \Rightarrow L^{-1}\left\{\dfrac{1}{(s-a)^2+b^2}\right\} = \dfrac{1}{b}e^{at}\sin bt \\ L\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2} \Rightarrow L^{-1}\left\{\dfrac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt. \end{array}\right\}$$

$= e^{2t}\cos 3t + 4\,\frac{1}{3}\,e^{2t}\sin 3t$ //

③ $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$

Sol:- $2L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5L^{-1}\left\{\frac{1}{s^2-4}\right\} = 2\cosh 2t - 5\frac{1}{2}\sinh 2t.$

$$\left.\begin{aligned} L\{\cosh at\} &= \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at \\ L\{\sinh at\} &= \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at \end{aligned}\right\}$$

④ $L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$

Sol:- $L^{-1}\left\{\frac{s+s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{\cancel{s}}{\cancel{s}(s+1)}\right\} + L^{-1}\left\{\frac{\cancel{s+1}}{s\cancel{(s+1)}}\right\}$

$= L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s}\right\} = e^{-t}+1 = 1+e^{-t}.$

① $L^{-1}\left\{ \dfrac{3s+7}{s^2-2s-3} \right\}$

Sol:- $\dfrac{3s+7}{s^2-2s-3} = \dfrac{A}{s+1} + \dfrac{B}{s-3}$

$A(s-3) + B(s+1) = 3s+7$

Put $s=3 \Rightarrow B = 4$

$s=-1 \Rightarrow A = -1$

$L^{-1}\left\{ \dfrac{3s+7}{s^2-2s-3} \right\} = L^{-1}\left\{ \dfrac{-1}{s+1} \right\} + L^{-1}\left\{ \dfrac{4}{s-3} \right\} = -e^{-t} + 4e^{3t}$

$= 4e^{3t} - e^{-t}.$

first shifting theorem:-

If $L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\{f(s-a)\} = e^{at} f(t)$

① $L^{-1}\left\{\dfrac{1}{(s+2)^2+16}\right\} = e^{-2t}\left\{\dfrac{1}{s^2+16}\right\} = e^{-2t}\,\dfrac{1}{4}\sin 4t$

In problem $L^{-1}\{f(s-a)\} = e^{at}\cdot L^{-1}\{f(s)\} = e^{at}\cdot f(t)$.

② $L^{-1}\left\{\dfrac{s+3}{s^2-10s+29}\right\}$

Sol:- $L^{-1}\left\{\dfrac{s+3}{(s-5)^2+2^2}\right\} = L^{-1}\left\{\dfrac{(s-5)+8}{(s-5)^2+2^2}\right\} =$

$= e^{5t}L^{-1}\left\{\dfrac{s+8}{s^2+2^2}\right\} = e^{5t}L^{-1}\left\{\dfrac{s}{s^2+2^2}\right\} + e^{5t}L^{-1}\left\{\dfrac{8}{s^2+2^2}\right\}$

$= e^{5t}\cos 2t + e^{5t}\dfrac{8}{2}\sin 2t.$

H.W. ① $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$ ② $L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$

H.W.

③ $L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}$

Sol.:

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$A = 0, \quad C = 0, \quad B = \frac{-1}{2}, \quad D = \frac{1}{2}$$

$$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} = L^{-1}\left\{\frac{-1}{2(s+1)^2}\right\} + L^{-1}\left\{\frac{1}{2(s^2+1)}\right\}$$

$$= \frac{-1}{2} L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{-1}{2} e^{-t} L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\sin t = \frac{-1}{2} e^{-t}\, t + \frac{1}{2}\sin t$$

⑤ $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

Sol^n:

$$s^4+4a^4 = (s^2)^2+(2a^2)^2+4a^2s^2-4a^2s^2$$

$$= (s^2+2a^2)^2-(2as)^2$$

$$= (s^2+2a^2+2as)(s^2+2a^2-2as)$$

$$\frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2+2a^2-2as}$$

$$A=0,\quad C=0,\quad B=\frac{-1}{4a},\quad D=\frac{1}{4a}$$

$$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{-1}{4a}L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\} + \frac{1}{4a}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$

$$= \frac{-1}{4a}e^{-at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} + \frac{1}{4a}e^{at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{-1}{4a}e^{-at}\frac{1}{a}\sin at$$

$$+\frac{1}{4a}\frac{1}{a}e^{at}\sin at$$

$$= \frac{1}{4a^2} \sin at \left(e^{at} - e^{-at} \right) = \frac{1}{4a^2} \sin at \cdot 2 \sinh at = \frac{1}{2a^2} \sin at \cdot \sinh at.$$

H.W.

① $L^{-1}\left\{ \frac{s^2 - 3s + 4}{s^3} \right\}$ ② $L^{-1}\left\{ \frac{3(s^2-2)^2}{2s^5} \right\}$

③ $L^{-1}\left\{ \frac{4}{(s+1)(s+2)} \right\}$ ④ $L^{-1}\left\{ \frac{s^2 + s - 2}{s(s+3)(s-2)} \right\}$

Second shifting theorem:-

If $L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\{e^{-as} f(s)\} = G(t)$, $G(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

Ex: ① $L^{-1}\left\{ \dfrac{1+e^{-\pi s}}{s^2+1} \right\}$.

Sol: $L^{-1}\left\{ \dfrac{1}{s^2+1} \right\} + L^{-1}\left\{ \dfrac{e^{-\pi s}}{s^2+1} \right\}$

$L^{-1}\left\{ \dfrac{1}{s^2+1} \right\} = \sin t = f(t)$.

By second shifting thm, we have

$$L^{-1}\left\{ \frac{e^{-\pi s}}{s^2+1} \right\} = \begin{cases} \sin(t-\pi) & t > \pi \\ 0, & t < \pi \end{cases}$$

$= \sin(t-\pi) \cdot H(t-\pi)$.

$= -\sin t \cdot H(t-\pi)$.

$L^{-1}\left\{ \dfrac{1+e^{-\pi s}}{s^2+1} \right\}$

$= \sin t - \sin t \cdot H(t-\pi)$

$= \sin t \{1 - H(t-\pi)\}$

H.W. ② $L^{-1}\left\{ \dfrac{e^{-3s}}{(s-4)^2} \right\}$. //

Change of scale property :-

If $L^{-1}\{f(s)\} = f(t)$ Then $L^{-1}\{f(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$, $a>0$.

$\{L\{f(t)\} = f(s)\}$

Ex: ① If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t$, find $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$ by C.O.P.

Sol:

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t$$

$$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{2}\frac{1}{a}\frac{t}{a} \cdot \sin\left(\frac{t}{a}\right) \quad \text{by change of scale property}$$

$a=2 \Rightarrow$ $$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{1}{2}\frac{1}{2}\frac{t}{2} \sin\left(\frac{t}{2}\right) \Rightarrow L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = \frac{t}{2} \sin\frac{t}{2}$$

I.L.T. of derivatives :-

If $L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\{f^{(n)}(s)\} = (-1)^n t^n f(t)$, where $f^{(n)}(s) = \frac{d^n}{ds^n}[f(s)]$

Ex:- ① Find $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$

Sol:- $L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = f(t) \rightarrow ①$

$L\{f(t)\} = \log\frac{s+1}{s-1}$

$L\{t f(t)\} = \frac{-d}{ds}\left\{\log\frac{s+1}{s-1}\right\}$

$= \frac{-d}{ds}\left[\log s+1 - \log s-1\right]$

$= \frac{-1}{s+1} + \frac{1}{s-1}$

$L\{t f(t)\} = \frac{1}{s-1} - \frac{1}{s+1}$

$t f(t) = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$

$t f(t) = e^{-t} + e^{t} = 2\sinh t$

$f(t) = \frac{2}{t}\sinh t$

$L^{-1}\left\{\log\frac{(s+1)}{(s-1)}\right\} = \frac{2}{t}\sinh t$

H.W: ② Find $L^{-1}\{\cot^{-1} s\}$

I.L.T. of integrals :-

$L^{-1}\{f(s)\} = f(t)$, then $L^{-1}\left\{\int_s^\infty f(s)\,ds\right\} = \frac{f(t)}{t}$

Ex: (1) $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$, then $L^{-1}\left\{\int_s^\infty (s+1)/(s^2+2s+2)^2\,ds\right\}$

Sol:- $L^{-1}\left\{\frac{s+1}{[(s+1)^2+1]^2}\right\} = e^{-t}\, L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = e^{-t}\cdot\frac{t}{2}\sin t$

$L^{-1}\left\{\int_s^\infty \frac{s+1}{(s^2+2s+2)^2}\,ds\right\} = \frac{1}{2}\frac{\cancel{t}\,e^{-t}\sin t}{\cancel{t}} = \frac{1}{2}e^{-t}\sin t$