Worked Examples

3.1 A We are given three different vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$. Construct a matrix so that the equations $A\boldsymbol{x} = \boldsymbol{b}_1$ and $A\boldsymbol{x} = \boldsymbol{b}_2$ are solvable, but $A\boldsymbol{x} = \boldsymbol{b}_3$ is not solvable. How can you decide if this is possible? How could you construct A?

Solution We want to have $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ in the column space of A. Then $A\boldsymbol{x} = \boldsymbol{b}_1$ and $A\boldsymbol{x} = \boldsymbol{b}_2$ will be solvable. *The quickest way is to make* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *the two columns of* A. Then the solutions are $\boldsymbol{x} = (1, 0)$ and $\boldsymbol{x} = (0, 1)$.

Also, we don't want $A\boldsymbol{x} = \boldsymbol{b}_3$ to be solvable. So don't make the column space any larger! Keeping only the columns $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, the question is:

$$\text{Is } A\boldsymbol{x} = \begin{bmatrix} & & \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{b}_3 \text{ solvable?} \qquad \text{Is } \boldsymbol{b}_3 \text{ a combination of } \boldsymbol{b}_1 \text{ and } \boldsymbol{b}_2\text{?}$$

If the answer is *no*, we have the desired matrix A. If the answer is *yes*, then it is *not possible* to construct A. When the column space contains $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, it will have to contain all their linear combinations. So $\boldsymbol{b}_3$ would necessarily be in that column space and $A\boldsymbol{x} = \boldsymbol{b}_3$ would necessarily be solvable.

3.1 B Describe a subspace $\mathbf{S}$ of each vector space $\mathbf{V}$, and then a subspace $\mathbf{SS}$ of $\mathbf{S}$.

$$\begin{aligned}
\mathbf{V}_1 &= \text{all combinations of } (1,1,0,0) \text{ and } (1,1,1,0) \text{ and } (1,1,1,1) \\
\mathbf{V}_2 &= \text{all vectors perpendicular to } \boldsymbol{u} = (1,2,1), \text{ so } \boldsymbol{u}\cdot\boldsymbol{v} = 0 \\
\mathbf{V}_3 &= \text{all symmetric 2 by 2 matrices (a subspace of } \mathbf{M}) \\
\mathbf{V}_4 &= \text{all solutions to the equation } d^4y/dx^4 = 0 \text{ (a subspace of } \mathbf{F})
\end{aligned}$$

Describe each $\mathbf{V}$ two ways: "*All combinations of* ..." "*All solutions of the equations...*"

Solution $\mathbf{V}_1$ starts with three vectors. A subspace $\mathbf{S}$ comes from all combinations of the first two vectors $(1,1,0,0)$ and $(1,1,1,0)$. A subspace $\mathbf{SS}$ of $\mathbf{S}$ comes from all multiples $(c,c,0,0)$ of the first vector. So many possibilities.

A subspace $\mathbf{S}$ of $\mathbf{V}_2$ is the line through $(1,-1,1)$. This line is perpendicular to $\boldsymbol{u}$. The vector $\boldsymbol{x} = (0,0,0)$ is in $\mathbf{S}$ and all its multiples $c\boldsymbol{x}$ give the smallest subspace $\mathbf{SS} = \mathbf{Z}$.

The diagonal matrices are a subspace $\mathbf{S}$ of the symmetric matrices. The multiples cI are a subspace $\mathbf{SS}$ of the diagonal matrices.

$\mathbf{V}_4$ contains all cubic polynomials $y = a + bx + cx^2 + dx^3$, with $d^4y/dx^4 = 0$. The quadratic polynomials give a subspace $\mathbf{S}$. The linear polynomials are one choice of $\mathbf{SS}$. The constants could be $\mathbf{SSS}$.

In all four parts we could take $\mathbf{S} = \mathbf{V}$ itself, and $\mathbf{SS} =$ the zero subspace $\mathbf{Z}$.

Each $\mathbf{V}$ can be described as *all combinations of* .... and as *all solutions of* ....:

$$\begin{array}{ll}
\mathbf{V}_1 = \text{all combinations of the 3 vectors} & \mathbf{V}_1 = \text{all solutions of } v_1 - v_2 = 0 \\
\mathbf{V}_2 = \text{all combinations of } (1,0,-1) \text{ and } (1,-1,1) & \mathbf{V}_2 = \text{all solutions of } \boldsymbol{u}\cdot\boldsymbol{v} = 0. \\
\mathbf{V}_3 = \text{all combinations of } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. & \mathbf{V}_3 = \text{ all solutions } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ of } b = c \\
\mathbf{V}_4 = \text{all combinations of } 1, x, x^2, x^3 & \mathbf{V}_4 = \text{all solutions to } d^4y/dx^4 = 0.
\end{array}$$