Complex Analysis

Power series:- A series in which non-negative powers of z are taken are known as power series.

i.e. $\sum_{n=1}^{\infty} a_n z^n \rightarrow$ power series in z about '0'.

If power series $\sum_{n=1}^{\infty} a_n z^n$ converges absolutely when $|z| \le R$ & its absolute term diverges if $|z| > R$ and ~~ab~~ absolute term either converges or diverges when $|z| = R$. Then R is said to be Radius of Convergence of the power series.

→ $R = \infty$, then the power series is convergent everywhere

→ $R = 0$, " " nowhere except $z = 0$

* $\sum_{n=0}^{\infty} a_n z^n \Rightarrow \boxed{R = \dfrac{1}{\overline{\lim} |a_n|^{1/n}}}$ {Cauchy's Root test}

$$\boxed{R = \dfrac{1}{\lim \left|\dfrac{a_{n+1}}{a_n}\right|}}$$ {D'Alembert's Ratio test}

① $\sum_{n=0}^{\infty} \frac{1}{(1+i)^{3n}} z^n$

sol: $a_n = \frac{1}{(1+i)^{3n}}$

$|a_n| = \frac{1}{|1+i|^{3n}}$

$|a_n|^{1/n} = \frac{1}{|1+i|^3}$

$= \frac{1}{(\sqrt{2})^3} = \frac{1}{2\sqrt{2}}$

$R = \frac{1}{|a_n|^{1/n}} = \frac{1}{1/2\sqrt{2}} = 2\sqrt{2}$

H.W

② $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} z^n$.

Partial Differential equations

First order PDE:-

Linear PDE:- $P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = [R(x,y)]z + S(x,y)$

Semi linear PDE:- $P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = R(x,y,z)$

Ex:- ① $xy^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy\,z^2 \longrightarrow$ semi linear

② $x^2y\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = xyz \longrightarrow$ linear & semi linear.

Quasi linear:- $P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)$.

which is not above all is known as non-linear.

Classification of 2^{nd} order PDE:

Linear PDE:

$$\Rightarrow R(x,y)\, r + S(x,y)\, s + T(x,y)\, t + P(x,y)\, p + Q(x,y)\, q$$
$$+ A(x,y)\, z + B(x,y) = 0.$$

$r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

* Semi Linear PDE:

$$\left\{ \begin{aligned} & R(x,y)\, r + S(x,y)\, s + T(x,y)\, t \\ & \quad + f(x,y,z,p,q) = 0. \end{aligned} \right\}$$

Quasi linear

$$R(x,y,z,p,q)\, r + S(x,y,z,p,q)\, s + T(x,y,z,p,q)\, t$$
$$+ f(x,y,z,p,q) = 0$$

Consider 2^{nd} order semi linear P.D.E.

$$R(x,y)\, r + S(x,y)\, s + T(x,y)\, t + f(x,y,z,p,q) = 0 \longrightarrow (1)$$

* If $S^2 - 4RT > 0$ then eqn (1) Hyperbolic
* If $S^2 - 4RT = 0$ " parabolic
* If $S^2 - 4RT < 0$ " Elliptic.

$S^2 - 4RT \rightarrow$ discriminant.

① $r - t = 0$

Sol:- $R = 1, \; S = 0, \; T = -1$

$$S^2 - 4RT = 0 - 4(1)(-1) = 4 > 0$$

Hyperbolic

② $r + q = 0$

Sol:- $R = 1, \; S = 0, \; T = 0$

$S^2 - 4RT = 0$ parabolic

③ $r + t = 0$

Sol:- $R = 1, \; S = 0, \; T = 1$

$$S^2 - 4RT = -4 < 0$$

Elliptic

H.W. ④ $xy\,r - (x^2 - y^2)\,s - xy\,t = 2x^2$

Classification of 2^{nd} order linear PDE in three variables

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu + g = 0$$

where $a_{ij} = a_{ji}$, b_i, c, g are fun of x_1, x_2, x_3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ real & Symmetric

(1) ~~Eigen values~~

① Eigen values of A non-zero & have same sign except precisely one of them. Ex: 2, 2, -1 (or) -1, -3, 4

Then given eqn is called "Hyperbolic."

② If any of the Eigen values is zero. Then given eqn "parabolic".

③ If all the eigen values are non-zero & have same sign then given eqn is "elliptic".

① $U_{xx} + U_{yy} + U_{zz} + 2U_{yz} = 0.$

$U_{yz} + U_{zy}$

Sol:-

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & 1 & 0 & 0 \\ y & 0 & 1 & 1 \\ z & 0 & 1 & 1 \end{array}$$

Eigen values = 2, 1, 0

parabolic

H.W.

② $U_{xx} + U_{yy} - U_{zz} = 0$

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & -1 \end{array}$$

1, 1, -1 → Eigen values

Hyperbolic

(3) $u_{xx} + u_{yy} + u_{zz} + 2x^2 u_{xz} = 0$

$$\begin{bmatrix} 1 & 0 & x^2 \\ 0 & 1 & 0 \\ x^2 & 0 & 1 \end{bmatrix} \Rightarrow 1,\ 1+x^2,\ 1-x^2$$

If $|x| = 1$ (or) $x = \pm 1$ parabolic.

$|x| > 1 \Rightarrow$ Hyperbolic

$|x| < 1 \Rightarrow$ Elliptic.