Laplace Transforms

e^{at}, t^n, $\sin at$, $\cos at$, $\sinh at$, $\cosh at$, - - -

properties of L.T.:-

① **Linearity property:-**

(i) $L\{c\cdot f(t)\} = c\cdot L\{f(t)\}$ (ii) $L\{f(t)+g(t)\} = L\{f(t)\}+L\{g(t)\}$

② **change of scale property:-**

If $L\{f(t)\} = f(s)$ then $L\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Pf:- $L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt = \bar{f}(s)$

put $at = u \Rightarrow dt = \frac{du}{a}$

$t \to \infty \Rightarrow u \to \infty$ & $t = 0 \Rightarrow u = 0$

$L\{f(at)\} = \int_0^\infty e^{-\frac{su}{a}} f(u) \cdot \frac{du}{a}$

$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a} \cdot u} f(u)\, du$

$= \frac{1}{a} f\left(\frac{s}{a}\right)$

examples

① $L\{\sinh 3t\}$

sol:- $L\{\sinh t\} = \frac{1}{s^2 - 1} = f(s)$

$L\{\sinh 3t\} = \frac{1}{3} f\left(\frac{s}{3}\right)$

$= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$ //

② $L\{\cos 7t\}$

sol:- $L\{\cos t\} = \frac{s}{s^2 + 1} = f(s)$

$L\{\cos 7t\} = \frac{1}{7} f\left(\frac{s}{7}\right) = \frac{1}{7} \frac{s/7}{\frac{s^2}{7^2} + 1}$

③ First shifting property :- If $L\{f(t)\} = f(s)$ ~~then L{~~

then $L\{e^{at} \cdot f(t)\} = f(s-a)$

Pf:

$$L\{e^{at} \cdot f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)\cdot t} f(t)\,dt = \underline{f(s-a)}$$

Examples:-

① $L\{e^{at} \cdot t^n\}$

Sol:- $L\{t^n\} = \frac{n!}{s^{n+1}} = f(s)$

$L\{e^{at} \cdot t^n\} = f(s-a)$

$= \frac{n!}{(s-a)^{n+1}}$

② $L\{e^{3t} \cdot \sin 4t\}$

Sol: $L\{\sin 4t\} = \frac{4}{s^2+4^2} = f(s)$

$L\{e^{3t} \cdot \sin 4t\} = f(s-3)$

$= \frac{4}{(s-3)^2+4^2}$ //

H.W.

③ $L\{e^{at} \cos bt\}$

④ $L\{e^{at} \sinh bt\}$

⑤ $L\{e^{at} \cosh bt\}$

H.W.

(8) $L\{e^{2t}\cos^2 t\}$

Sol: $\cos^2 t = \dfrac{1+\cos 2t}{2}$

First shifting

$$L\{e^{2t}\cos^2 t\} = L\left\{e^{2t}\left(\frac{1+\cos 2t}{2}\right)\right\} = \frac{1}{2}L\{e^{2t}\} + \frac{1}{2}L\{e^{2t}\cdot\cos 2t\}$$

Second shifting (translation) theorem:

If $L\{f(t)\} = f(s)$ & $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$ then $L\{g(t)\} = e^{-as}\cdot f(s)$

Pf: $$L\{g(t)\} = \int_0^\infty e^{-st}\cdot g(t)\,dt = \int_0^a e^{-st}\,0\cdot\cancel{f(t-a)}\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

let $t-a=u \Rightarrow dt=du$ & $u=0$ when $t=a$, $u=\infty$ when $t=\infty$

$$L\{g(t)\} = \int_0^\infty e^{-s(a+u)}\cdot f(u)\,du = e^{-as}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\cdot f(s)$$

$$= e^{-as}\cdot f(s)$$

Another form of second shifting theorem:-

If $L\{f(t)\} = f(s)$ ~~then g(t)~~ and $a>0$ then

$L\{f(t-a)\,H(t-a)\} = e^{-as}\cdot f(s)$, where $H(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$ and

$H(t)$ is called Heaviside unit step function.

Examples:- (1) $L\{g(t)\}$ where $g(t) = \begin{cases} \cos(t-\pi/3), & t>\pi/3 \\ 0, & t<\pi/3 \end{cases}$

Sol:- $f(t) = \cos t$

$$L\{f(t)\} = \frac{s}{s^2+1} = f(s)$$

$$g(t) = \begin{cases} f(t-\pi/3) = \cos(t-\pi/3), & t>\pi/3 \\ 0, & t<\pi/3 \end{cases}$$

second shifting thm

$$L\{g(t)\} = e^{-as}\cdot f(s)$$

$$= e^{-\pi s/3}\,\frac{s}{s^2+1}.$$

② $(t-2)^3 \, H(t-2) \rightarrow u(t-2)$

Sol: $f(t-a)\,H(t-a)$ $\quad a=2, \; f(t) = t^3$

$L\{f(t)\} = L\{t^3\} = \dfrac{3!}{s^4} = \dfrac{6}{s^4} = F(s).$

$L\{f(t-a)\,H(t-a)\} = e^{-as}.F(s) = e^{-2s}.\dfrac{6}{s^4}.$ //

<u>Multiplication by 't'</u>:- If $L\{f(t)\} = F(s)$ then $L\{t f(t)\} = \dfrac{-d}{ds}F(s).$

Pf: $F(s) = \displaystyle\int_0^\infty e^{-st} f(t)\,dt$

$\dfrac{d}{ds}F(s) = \dfrac{d}{ds}\displaystyle\int_0^\infty e^{-st} f(t)\,dt$

$\dfrac{dF(s)}{ds} = \displaystyle\int_0^\infty \dfrac{\partial}{\partial s} e^{-st} f(t)\,dt = \int_0^\infty -t.e^{-st}.f(t)\,dt.$

$\dfrac{dF(s)}{ds} = -\displaystyle\int_0^\infty e^{-st}(t f(t))\,dt = -L\{t f(t)\}$

$\Rightarrow L\{t f(t)\} = -\dfrac{d}{ds}F(s).$

Thus $\boxed{L\{t\,f(t)\} = -\frac{d}{ds} f(s) \quad \& \quad L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)}$

Examples:-

① $L\{t \cos at\}$

Sol:- $L\{\cos at\} = \frac{s}{s^2+a^2}$

$$L\{t \cos at\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\left[\frac{(s^2+a^2)\cdot 1 - s\cdot(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

② $L\{t^2 \sin at\}$

Sol:- $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$L\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$$

H-W

Ans:- $\frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

③ $L\{t e^{3t} \sin 3t\}$

Sol: $L\{\sin 3t\} = \frac{3}{s^2+9} = f(s)$

$L\{e^{3t} \sin 3t\} = f(s-3)$

(First shifting thm)

$= \frac{3}{(s-3)^2+9}$

$L\{t e^{3t} \sin 3t\} = -\frac{d}{ds} \frac{3}{(s-3)^2+9}$

H-W.

④ $L\{t e^{-t} \sin 3t\}$.

⑤ $L\{(1+t e^{-t})^2\}$

⑥ $L\{t^3 e^{-3t}\}$

First shifting

multiplication by 't'.

⑦ $f(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & 0 < t < 1 \end{cases}$

Sol: $L\{f(t)\} = \int_0^1 0 \cdot dt + \int_1^\infty e^{-st} (t-1)^2 dt$.

integrate

Division by 't' :- If $L\{f(t)\} = f(s)$ then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty f(s)\,ds$

Pf :- $$f(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Integrate both sides w.r.to 's' from s to ∞, one obtains

$$\int_s^\infty f(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^\infty \int_s^\infty f(t)\, e^{-st}\,ds\,dt \quad \text{[change of order of int]}$$

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-st}\,ds\right] dt$$

$$= \int_0^\infty f(t) \left(\frac{e^{-st}}{-t}\right)_s^\infty dt$$

$$= \int_0^\infty e^{-st}\, \frac{f(t)}{t}\,dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

$$\boxed{L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty f(s)\,ds}$$

Example:-

(1) $L\left\{\frac{\sin t}{t}\right\}$

Sol:- $L\{\sin t\} = \frac{1}{s^2+1} = F(s)$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(s)\,ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \tan^{-1} s \Big|_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \pi/2 - \tan^{-1}s = \cot^{-1}s$$

H.W (2) $L\left\{\frac{\sin at}{t}\right\}$

H.W. (3) $L\left\{\frac{1-\cos at}{t}\right\}$

Hint! $L\{1-\cos at\} = L\{1\} - L\{\cos at\}$

$$= \frac{1}{s} - \frac{s}{s^2+a^2}$$

$$L\left\{\frac{1-\cos at}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right)ds \quad \text{solve}$$

H.W. (4) $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

H.W (5) $L\left\{\frac{1-\cos at}{t^2}\right\} = L\left\{\frac{1}{t}\cdot\left(\frac{1-\cos at}{t}\right)\right\}$