Roll No								

National Institute of Technology Goa

Programme Name: B.Tech

Online Mid Semester Examinations, March 2022

Course Name: **Mathematics-IV**
Date: 11/03/2022
Duration: 90 Minutes

Course Code: **MA250**
Time: 10:00AM - 11:30 AM
Max. Marks: 50

---

1. Answer any TEN Questions from the following.
2. **No marks will be given if the explanation of your answer is missing**.
3. The question paper consists of **Four** pages.

---

1. A random variable X has pdf:

$$f_X(x) = \begin{cases} c(1-x^2), & -1 \leq x \leq 1. \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Find c and plot the pdf.

   (b) Plot the cdf of X.

   (c) Find $P[X=0]$, $P[0<X<0.5]$, and $P[|X-0.5|<0.25]$.

   (d) Find $E[X]$ and $E[X^2]$. [5M]

2. Let $Y = 3X + 2$

   (a) Find the mean and variance of Y in terms of the mean and variance of X.

   (b) Evaluate the mean and variance of Y if X is Laplacian.

   (c) Evaluate the mean and variance of Y if X is an arbitrary Gaussian random variable.

   (d) Evaluate the mean and variance of Y if $X = b\ cos(2\pi U)$ where U is a uniform random variable in the unit interval. [5M]

3. Let $Y = e^X$.

   (a) Find the cdf and pdf of Y in terms of the cdf and pdf of X.

   (b) Find the pdf of Y when X is a Gaussian random variable. In this case Y is said to be a lognormal random variable. [5M]

4. Two chips are being considered for use in a certain system. The lifetime of chip 1 is modeled by a Gaussian random variable with mean 20,000 hours and standard deviation 5000 hours. (The probability of negative lifetime is negligible.) The lifetime of chip 2 is also a Gaussian random variable but with mean 22,000 hours and standard deviation 1000 hours. Which chip is preferred if the target lifetime of the system is 20,000 hours? 24,000 hours? [5M]

5. A Chip manufacturing process produces a mix of good memory chips and bad memory chips. The lifetime of good chips follows the exponential law, with a rate of failure α. The lifetime of bad chips also follows the exponential distribution, but the rate of failure is 1000 α. Suppose that the fraction of good chips is and of bad chips, p. (i) Find the probability that a randomly selected chip is still functioning after t seconds (ii) Suppose that in order to weed out the bad chips, every chip is tested for t seconds prior to leaving the factory. The chips that fail are discarded and the remaining chips are sent out to customers. Find the value of t for which 95% of the chips sent out to customers are good, for $p = 0.10$ and $1/\alpha = 10000$ hours. [5M]

6. (a) Five balls are placed at random in five buckets. What is the probability that each bucket has a ball?

   (b) A computer manufacturer uses chips from three sources. Chips from sources A, B, and C are defective with probabilities 0.005, 0.001, and 0.010, respectively. If a randomly selected chip is found to be defective, find the probability that the manufacturer was A; that the manufacturer was C. Assume that the proportions of chips from A, B, and C are 0.5, 0.1, and 0.4, respectively. [5M]

7. (a) Let A and B be two finite sets, with $\mid A \mid= m$ and $\mid B \mid= n$.

      i. How many distinct functions (mappings) can you define from set A to set B ie., $f : A \rightarrow B$?

      ii. How many distinct one-to-one functions (mappings) can you define from set A to set B ie., $f : A \rightarrow B$?

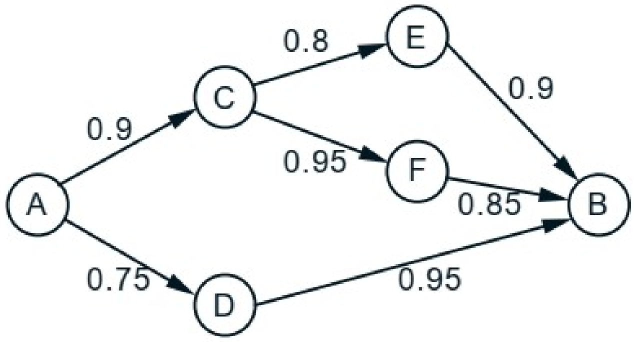
   (b) A class consisting of 4 graduate and 12 undergraduate students is randomly divided into four groups of 4. What is the probability that each group includes a graduate student? [5M]

8. A student has five different t-shirts and three pairs of jeans (brand new, broken in, and perfect).

   (a) How many days can the student dress without repeating the combination of jeans and t-shirt?

   (b) How many days can the student dress without repeating the combination of jeans and t-shirt and without wearing the same t-shirt on two consecutive days? [5M]

9. A computer network connects two nodes A and B through intermediate nodes $C, D, E, F,$ as shown in Fig. below. For every pair of directly connected nodes, say i and j, there is a given probability p_{ij} that the link from i to j is up. We assume that link failures are independent of each other.



What is the probability that there is a path connecting A and B in which all links are up?

10. An urn initially contains two black balls and two white balls. The following experiment is repeated indefinitely: A ball is drawn from the urn; if the color of the ball is the same as the majority of balls remaining in the urn, then the ball is put back in the urn. Otherwise the ball is left out.

    (a) Draw the trellis diagram for this experiment and label the branches by the transition probabilities.

    (b) Find the probabilities for all sequences of outcomes of length 2 and length 3.

    (c) Find the probability that the urn contains no black balls after three draws; no white balls after three draws.

    (d) Find the probability that the urn contains two black balls after n trials; two white balls after n trials. [5M]

11. Let X be a random variable with pmf $p_k = c/k^2$ for $k = 1, 2, 3, ...$

    (a) Estimate the value of c numerically. Note that the series converges.

    (b) Find $P[X > 4]$

    (c) Find $P[6 \leq X \leq 8]$.

    (d) Find $E[X]$ and $Var[X]$. [5M]

12. Eight numbers are selected at random from the unit interval.

    (a) Find the probability that the first four numbers are less than 0.25 and the last four are greater than 0.25.

    (b) Find the probability that four numbers are less than 0.25 and four are greater than 0.25.

    (c) Find the probability that the first three numbers are less than 0.25, the next two are between 0.25 and 0.75, and the last three are greater than 0.75.

    (d) Find the probability that three numbers are less than 0.25, two are between 0.25 and 0.75, and three are greater than 0.75.

    (e) Find the probability that the first four numbers are less than 0.25 and the last four are greater than 0.75. [5M]

13. At a given time, the number of households connected to the Internet is a Poisson random variable with mean 50. Suppose that the transmission bit rate available for the household is 20 Megabits per second.

    (a) Find the probability of the distribution of the transmission bit rate per user.

    (b) Find the transmission bit rate that is available to a user with probability 90% or higher.

    (c) What is the probability that a user has a share of 1 Megabit per second or higher? [5M]

14. Three types of customers arrive at a service station. The time required to service type 1 customers is an exponential random variable with mean 2. Type 2 customers have a Pareto distribution with $\alpha = 3$ and $x_m = 1$. Type 3 customers require a constant service time of 2 seconds. Suppose that the proportion of type 1, 2, and 3 customers is 1/2, 1/8, and 3/8, respectively. Find the probability that an arbitrary customer requires more than 15 seconds of service time. Compare the above probability to the bound provided by the Markov inequality. [5M]

15. The lifetime X of a light bulb is a random variable with $P[X > t] = 2/(2 + t)$ for $t > 0$. Suppose three new light bulbs are installed at time $t = 0$. At time $t = 1$ all three light bulbs are still working. Find the probability that at least one light bulb is still working at time $t = 9$. [5M]

$* * *$ALL THE BEST $* **$