Power Series

Our last subject will be *power series*. We've seen one power series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad (|x| < 1).$$

This is our geometric series, with x in place of a. We'll now see why the sum should equal $\frac{1}{1-x}$.

Suppose that:

$$1 + x + x^2 + x^3 + \cdots = S$$

for some number S. Multiply both sides of this equation by x:

$$x + x^2 + x^3 + x^4 + \cdots = Sx.$$

Now subtract the two equations.

$$\begin{array}{ccccccccccc} 1 & + & x & + & x^2 & + & x^3 & + & \cdots & = & S \\ & & x & + & x^2 & + & x^3 & + & \cdots & = & Sx \\ \hline 1 & + & 0 & + & 0 & + & 0 & + & \cdots & = & S - Sx \end{array}$$

Lots of terms cancel! Continuing, we get:

$$\begin{aligned} 1 &= S - Sx \\ 1 &= S(1-x) \\ \frac{1}{1-x} &= S. \end{aligned}$$

There is a flaw in this reasoning — the argument only works if S exists. For example, if $x = 1$ this technique tells us that $\infty - \infty = \infty - \infty$. This is not a useful result.

This line of reasoning leads to a correct answer exactly when the series converges; in other words, when $|x| < 1$.