Surface Area of an Ellipsoid

Next we'll find the surface area of the surface formed by revolving our elliptical curve:

$$\begin{aligned} x &= 2\sin t \\ y &= \cos t \end{aligned}$$

about the y-axis.

Remember that our surface area element dA is the area of a thin circular ribbon with width ds. The radius of this circle is $x = 2\sin t$, which is the distance between the ribbon and the y-axis.

$$dA = 2\pi \underbrace{(2\sin t)}_{x} \underbrace{\sqrt{4\cos^2 t + \sin^2 t}dt}_{ds=\text{arc length}}.$$

To find the surface area we need to integrate dA between certain limits; what are they?

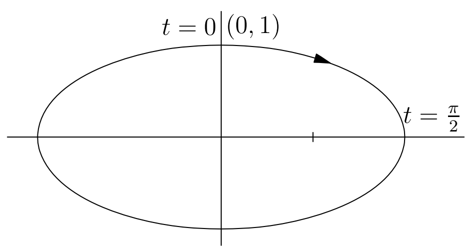

Figure 1: Elliptical path described by $x = 2\sin t$, $y = \cos t$.

By looking at Figure 1 we can see that we need to integrate from 0 to π. Remember that we only need to go from the top to the bottom of the ellipse to trace the right hand side; including the left hand side of the ellipse would double our result and give the wrong answer.

$$A = \int_0^\pi 2\pi(2\sin t)\sqrt{4\cos^2 t + \sin^2 t}dt.$$

Notice that we're integrating from the top of the ellipse to the bottom; if we think in terms of the y-variable we tend to think of going the opposite way.

This integral turns out to be do-able but long. Start by using the substitution $u = \cos t$, $du = -\sin t\, dt$.

$$\begin{aligned} A &= \int_0^\pi 2\pi(2\sin t)\sqrt{4\cos^2 t + \sin^2 t}dt \\ &= \int_{u=1}^{u=-1} -4\pi\sqrt{3u^2 + 1}du. \end{aligned}$$

Next would be another trigonometric substitution to deal with the square root, and so on.