Differential Equations and Slope, Part 2

Find the curves that are perpendicular to the parabolas $y = ax^2$ from the previous example.

$\frac{dy}{dx} = \frac{d}{dx} ax^2 = 2ax = \frac{2y}{x}$

We get a new differential equation from the one in the last example by using the fact that if a line has slope m, a line perpendicular to it will have slope $-\frac{1}{m}$. So:

$$\begin{aligned}
\text{slope of curve} &= \frac{dy}{dx} \\
&= -\frac{1}{\text{slope of parabola}} \\
&= -\frac{1}{\frac{2y}{x}} \\
\frac{dy}{dx} &= \frac{-x}{2y}
\end{aligned}$$

Separate variables:

$$2y\,dy = -x\,dx$$

Take the antiderivative:

$$\begin{aligned}
\int 2y\,dy &= \int -x\,dx \\
y^2 &= -\frac{x^2}{2} + c
\end{aligned}$$

So the general solution to this differential equation is:

$$y^2 + \frac{x^2}{2} = c.$$

This describes a family of ellipses. The y-semi-minor axis of these ellipses has length $\sqrt{c}$ and the x-semi-major axis has length $\sqrt{2c}$; the ratio of the x-semi-major axis to the y-semi-minor axis is $\sqrt{2}$ (see Fig. 1).

Unlike last time, this solution only works when $c > 0$. For some problems your constant parameter can be any real value; for some it can't.

Separation of variables leads to implicit formulas for y, but in this case you can solve for y.

$$y = \pm\sqrt{c - \frac{x^2}{2}}$$

Writing the solution in this form brings an important point to our attention — the equation of an ellipse does not describe a function! The explicit solution gives you functions that describe the top and bottom halves of the ellipses

The explicit solution also suggests that there's a problem when $y = 0$ and $x = \pm\sqrt{2c}$. Here the ellipse has a vertical tangent line; also the explicit solution isn't defined for $|x| > \sqrt{2c}$. This makes sense when we consider the fact that $\frac{dy}{dx} = \frac{-x}{2y}$. When $y = 0$ the slope of the tangent line to the curve should be infinite.

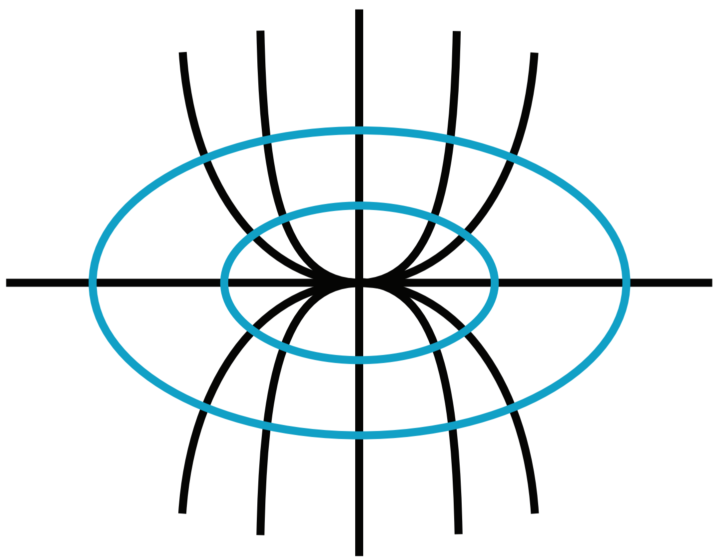

Figure 1: The curves perpendicular to the parabolas are ellipses.