Substitution When u' Changes Sign

We've been told that changing variables of integration only works if $u(x)$ is either always increasing or always decreasing on the interval of integration. Let's see what goes wrong by trying to calculate $\int_{-1}^{1} x^2\,dx$.

We'll try plugging in $u(x) = x^2$; then we get:

$$\begin{aligned} du &= 2x\,dx \\ dx &= \frac{1}{2x}\,du = \frac{1}{2\sqrt{u}}\,du \\ u_1 &= (-1)^2 \quad \text{and} \end{aligned}$$

$$u_2 = (-1)^2$$

Thus:

$$\int_{-1}^{1} x^2\,dx = \int_{1}^{1} u\frac{1}{2\sqrt{u}}\,du = 0.$$

But we know that $\int_{-1}^{1} x^2\,dx$ is not zero; it's the area under a parabola. Our conclusion is **not true**.

The reason for this is that $u'(x) = 2x$ is negative when $x < 0$ and positive when $x > 0$; the sign change causes us trouble. If we break the integral into two halves so that u' has a consistent sign on each half, we'll be able to compute the integral without difficulty.

We could actually have caught this early; there is a mistake in our calculation of the expression for dx. In fact, when we wrote:

$$\frac{1}{2x}\,du = \frac{1}{2\sqrt{u}}\,du$$

we should have noticed that in fact:

$$\frac{1}{2x}\,du = \frac{1}{\pm 2\sqrt{u}}\,du.$$

It's possible to use this formula to get the correct answer, but not recommended. Instead, just split your integral into intervals over which u' is always either positive or negative.