**Example:** $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$

We've been told that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. We can't compute the exact value of this integral, but *can* use a simple comparison to check that the value is finite.

We start by using the fact that this is an even function, symmetric about the $y$-axis, to rewrite the integral as:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{0}^{\infty} e^{-x^2}\,dx.$$

The function $f(x) = e^{-x^2}$ goes to zero so quickly that we can't find a function $g(x)$ that's comparable to $f(x)$ for a limit comparison, so we'll have to use an ordinary comparison to determine whether this improper integral converges.

Because $x^2 \geq x$ when $x \geq 1$, we know that $-x^2 \leq -x$ and $e^{-x^2} \leq e^{-x}$ for $x \geq 1$. To show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges we split the integral again between $x > 1$ and $x < 1$. We compare integrals using our understanding that increasing the integrand increases the value of the integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2}\,dx &= 2\int_{0}^{\infty} e^{-x^2}\,dx \\
&= 2\int_{0}^{1} e^{-x^2}\,dx + 2\int_{1}^{\infty} e^{-x^2}\,dx \\
&\leq 2\int_{0}^{1} e^{-x^2}\,dx + 2\int_{1}^{\infty} e^{-x}\,dx \quad \text{(larger integrand)}
\end{aligned}$$

Since $\int_0^1 e^{-x^2}\,dx$ is finite and $\int_1^{\infty} e^{-x}\,dx$ converges, we conclude that $\int_0^{\infty} e^{-x^2}\,dx$ converges.

Ordinary comparison is a good tool for proving the convergence of integrals whose integrands decay very rapidly.