Example: Cumulative Debt

In this example we see an integral that represents a cumulative sum, rather than an area.

Let $t =$ time in years
and $f(t) =$ dollars/year; $f(t)$ is a borrowing rate.

Notice the units in this problem; they are one of the reasons we include a differential like dx in all of our integrals. This notation to be consistent with units, helps in changing variables, and allows us to develop meaningful formulas which are consistent across the board.

Suppose you're borrowing money every day; then $\Delta t = \frac{1}{365}$ years. In terms of years, this is a nearly infinitesimal interval of time. Your borrowing rate varies over the year; you borrow more when you need more, and less when you need less. How much do you borrow?

On Day 45, which is at $t = 45/365$, you borrowed $f\left(\frac{45}{365}\right)\Delta t = f\left(\frac{45}{365}\right)\frac{1}{365}$. Here $f(t)$ is measured in dollars per year and Δt is measured in years, so $f\left(\frac{45}{365}\right)\frac{1}{365}$ is a number of dollars; in fact it's the amount that you actually borrow on the 45^{th} day.

The total amount borrowed in an entire year is:

$$\sum_{i}^{365} f\left(\frac{i}{365}\right)\Delta t.$$

This is a messy sum, but your bank knows how to keep track of it. However, when we're modeling trading strategies of course and trying to cleverly optimize how much you borrow, how much you spend, and how much you invest you will want to replace it by $\int_0^1 f(t)\,dt$. If $\Delta t = \frac{1}{365}$, this is probably a good enough approximation.

$t = \frac{i}{365}$

But we're not done yet; it's equally important to model how much you owe, in addition to how much you borrowed. Since we assumed that we could approximate the Riemann sum by an integral, we'll also assume that the interest on our debt is compounded continuously. If we start with a debt of P, then after time t you owe Pe^{rt}, where r is the interest rate. Let's assume you're borrowing at a 5% interest rate; then $r = 0.05$ per year.

Over the course of the year you borrowed the amounts $f\left(\frac{i}{365}\right)\Delta t$. When you borrow this amount, the amount of time left in the year is $T = 1 - \frac{i}{365}$, which is the amount of time this incremental debt will accumulate interest. So a debt of $f\left(\frac{i}{365}\right)\Delta t$ on day i increases to a debt of:

$$\left(f\left(\frac{i}{365}\right)\Delta t\right)e^{r\left(1-\frac{i}{365}\right)}$$

at the end of the year. This is the term you sum to get your total debt at the

$g(t) = f(t)e^{r(1-t)} \quad \left(t = \frac{i}{365}\right)$

$\Rightarrow g(t)\Delta t$

$\sum_{t=\frac{1}{365}}^{\frac{365}{365}} g(t)\Delta t \to \int_0^1 g(t)\,dt$

end of the year:

$$\sum_{1}^{365} \left(f\left(\frac{i}{365}\right) \Delta t \right) e^{r\left(1-\frac{i}{365}\right)} \longrightarrow \int_0^1 e^{r(1-t)} f(t)\, dt.$$

If you're trying to decide a borrowing strategy, you're faced with integrals of this type.