

Example: Volume of a Cauldron

In our next, Halloween themed, example we'll compute the volume of the region shown below.

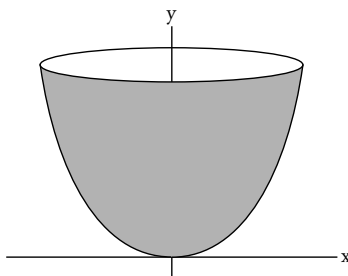


Figure 1: $y = x^2$ rotated around the y -axis.

We could use the method of disks to calculate this volume, but instead we will use the other standard method of finding volumes — the *method of shells*.

A cross section of the cauldron has boundaries $y = x^2$ and $y = a$. We revolve this cross section around the y -axis to get our cauldron. (Note that we can revolve shapes around the y -axis as well as the x -axis.) Our “shell” will be the result of revolving a thin rectangle with its base on $y = x^2$ and its top at $y = a$, as shown in Figure 2. This “shell” shape might also be described as a cylinder. Make your own by rolling a piece of paper into a tube!

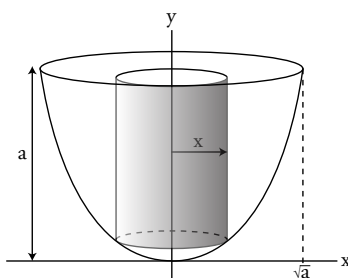


Figure 2: Cylindrical shell with radius x , thickness dx and height $a - y = a - x^2$.

We need to compute the volume of this shell. It's thickness is dx , and its height is $y_{top} - y_{bottom} = a - x^2$. Since the shell is very thin, we get a good approximation of its volume by “unrolling” it like a piece of paper and computing the volume of a rectangular slab with thickness dx , height $a - x^2$ and width equal to the circumference of the shell. To compute the circumference we multiply the radius r by 2π ; our estimate of the shell's volume is then:

$$dV = (2\pi x)(a - x^2) dx = 2\pi(ax - x^3) dx$$

$$\begin{aligned} \Delta V &= \pi[(x + \Delta x)^2 - x^2](a - x^2) & \pi[(x + dx)^2 - x^2] &= \pi(2x dx + dx^2) \\ &= \pi(a - x^2)(2x \Delta x + \Delta x^2) \end{aligned}$$

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$$\frac{\Delta V}{\Delta x} = \pi(a - x^2)(2x + \Delta x)$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta V}{\Delta x} = \frac{dV}{dx} = \pi(a - x^2)(2x + 0)$$

(We can roughly check our work by noting that we're multiplying three lengths here, so the units do match up on both sides of the equation.)

To compute the volume of the cauldron we'll integrate this; all that's left is to find the limits. The entire volume of the cauldron is swept out by the right side of the parabola as it spins about the y -axis, so our limits of integration start at 0 (not $-\sqrt{a}$). In other words, if we just rotate the right half of this it covers the left half; if we counted the volume swept out by the left half we would be counting the volume twice.

The upper limit of integration is at the farthest rightmost spot, where $y = a$ and $y = x^2$ simultaneously; in other words, at $x = \sqrt{a}$. Getting the correct limits is as important as getting the right integrand.

So the volume of the cauldron is:

$$\begin{aligned} V &= 2\pi \int_0^{\sqrt{a}} (ax - x^3) dx \\ &= 2\pi \left(a\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^{\sqrt{a}} \\ &= 2\pi \left(\frac{a^2}{2} - \frac{a^2}{4} \right) = 2\pi \left(\frac{a^2}{4} \right) \\ V &= \frac{\pi}{2} a^2. \end{aligned}$$

Question: How do you know whether the rectangle should be vertical or horizontal?

Answer: You can always set it up both ways. One way may be a difficult calculation and one way may be an easier calculation. In the example of $y = x^2$ and $y = x - 2$ the horizontal and the vertical calculations were quite different in character. One of them was really a mess, and one of them was a little easier. This is often the case, and once in a while, one of them is impossible and the other one is possible. By choosing your method carefully you can save yourself a lot of work.

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