Properties of Integrals

The symbol $\int$ originated as a stylized letter S; in French, they call integrals sums. We know from our discussion of Riemann sums that definite integrals are just limits of sums. Because of this, it's not surprising that:

1. The integral of a sum is the sum of the integrals:

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2. We can factor out a constant multiple:

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad (c \text{ constant})$$

(don't try to factor out a non-constant function!)

3. We can combine definite integrals. If $a < b < c$ then:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

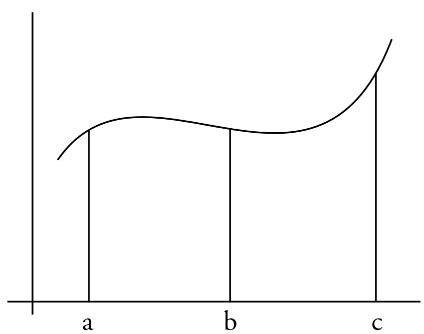

Figure 1: Combining two areas under a curve

4. $\int_a^a f(x)\,dx = 0$

5. This statement gives us some freedom in choosing limits of integration and allows us to remove the condition that $a < b < c$ from property (3):

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

This makes sense; $F(b) - F(a) = -(F(a) - F(B))$.

6. (Estimation) If $f(x) \leq g(x)$ and $a < b$, then:

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

In other words, if I'm going more slowly than you then you go further than I do. Caution: this only works if $a < b$.

7. (Change of Variables or "Substitution") In indefinite integrals, if $u = u(x)$ then $du = u'(x)\,dx$ and $\int g(u)du = \int g(u(x))u'(x)\,dx$. To adapt this to definite integrals we need to know what happens to our limits of integration; it turns out that the answer is very simple.

$$\int_{u_1}^{u_2} g(u)du = \int_{x_1}^{x_2} g(u(x))u'(x)\,dx,$$

where $u_1 = u(x_1)$ and $u_2 = u(x_2)$. This is true if u is always increasing or always decreasing on $x_1 < x < x_2$; in other words, if u' does not change sign. (If u' does change sign you must break the integral into pieces; we'll see an example of this later.)