

Approximation of $\ln e$

Here's an example of the power of linear approximation, and of what quadratic approximation can do for us that linear approximation cannot.

Recall that when we discussed exponential and logarithmic functions we said that:

$$a_k = \left(1 + \frac{1}{k}\right)^k$$

tends to e as k goes to infinity. We did that by taking the logarithm of both sides:

$$\ln a_k = k \ln \left(1 + \frac{1}{k}\right),$$

and then analyzing the limit on the right hand side. That was a fairly difficult calculation which is made much easier by linear approximation. Since the linear approximation of $\ln(1+x)$ is just x ,

$$\ln a_k = k \ln \left(1 + \frac{1}{k}\right) \approx k(1/k) = 1.$$

Can we conclude that $\ln a_k = 1$ so that $a_k = e$? (We used an approximation “ \approx ” in the above, not “ $=$ ”.) The linear approximation only works when x is near 0, but as k goes to infinity $\frac{1}{k}$ is indeed near 0. So as k approaches infinity, the linear approximation gets closer and closer to the exact value of the function, and $\ln a_k$ approaches 1. Linear approximation is often used in this way to evaluate limits.

Now if we want to find the *rate* of convergence — if we want to find out how fast the value of $k \ln \left(1 + \frac{1}{k}\right)$ approaches 1 — we need to look at the size of $\ln a_k - 1$ for large values of k . To do this you'll use quadratic approximation; the formula for the quadratic approximation of the natural log function was given above:

$$\ln(1+x) \approx x - \frac{1}{2}x^2 \quad (\text{for } x \text{ near } 0).$$

You need the next higher order term to get a more detailed understanding of $\lim_{k \rightarrow \infty} \ln a_k$; this question will be on the problem set.

Question: How do we know when to use a linear approximation and when to use a quadratic one?

Answer: This is a very good question. For now the questions you get will specify whether to use a linear or a quadratic approximation. As time goes on you should try to get a feel for when you can get away with a linear approximation. In general, only use a quadratic approximation if a linear approximation won't work, because quadratic approximations are much more complicated (as you'll see in the next example).

In real life when you're faced with a problem like this — for example, determining the effects of gravity on an orbiting satellite — nobody is going to tell you anything. You won't even be told whether a linear approximation is relevant; you're on your own.

$$\begin{aligned} \ln a_k - 1 &= k \ln \left(1 + \frac{1}{k}\right) - 1 \\ &\approx k \left(\frac{1}{k} - \frac{1}{2} \cdot \frac{1}{k^2} \right) - 1 \\ &= -\frac{1}{2k} \end{aligned}$$

$$\begin{aligned} k \ln \left(1 + \frac{1}{k}\right) & \quad (k \rightarrow \infty) \\ &= \frac{1}{x} \ln(1+x) \quad (x = \frac{1}{k}) \\ &\approx \frac{1}{x} \left(x - \frac{1}{2}x^2 \right) \quad (x \rightarrow 0) \\ &= 1 - \frac{1}{2x} \end{aligned}$$

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