Alternate Definition of Natural Log

The second fundamental theorem of calculus says that the derivative of an integral gives you the function back again:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We saw a few examples of how to use this to solve differential equations. In particular, we can solve $y' = \frac{1}{x}$ to get

$$L(x) = \int_1^x \frac{dt}{t}.$$

We could use this formula to define the logarithm function and derive all of its properties.

The first property of the function L is that $L'(x) = \frac{1}{x}$. (We defined it that way.)

The other piece of information that we need in order to completely describe the function is its output at one value; knowing its derivative only tells us its value up to a constant.

$$L(1) = \int_1^1 \frac{dt}{t} = 0$$

This will be the case with all definite integrals; if we evaluate them at their starting place, we'll get 0.

Together these two properties uniquely describe $L(x)$.

Next we want to understand the properties of the function; we'll start by graphing it. We know that the derivative of $L(x)$ is $\frac{1}{x}$; when the function is given as an integral it's easy to compute its first derivative! The derivative of L "blows up" as x approaches 0. To avoid this problem we'll consider only positive values of x.

The second derivative of L is $-\frac{1}{x^2}$. From it we learn that the graph of $L(x)$ is concave down. We know that $L(1) = 0$ and $L'(1) = 1$, which gives us a point on the graph and its slope at that point. We also know that $L'(x) > 0$ when x is positive, so we know that the function is increasing — the graph rises as we move to the right.

Knowing that $L(x)$ is increasing when x is positive allows us to work backwards from this definition to the one we used previously. If we draw the line $y = 1$ it intersects the graph of $L(x)$ at some point (we could confirm this using a Riemann sum if we had to). We'll define the number e so that this point of intersection is $(e, 1)$. In other words, e is the unique value for which $L(e) = 1$. We know there's only one such value because the graph can never "dip down" to cross the line $y = 1$ again.

Since we know $L(x)$ is increasing, we know there are no critical points; the only other interesting thing is the ends. It turns out that the limit as x

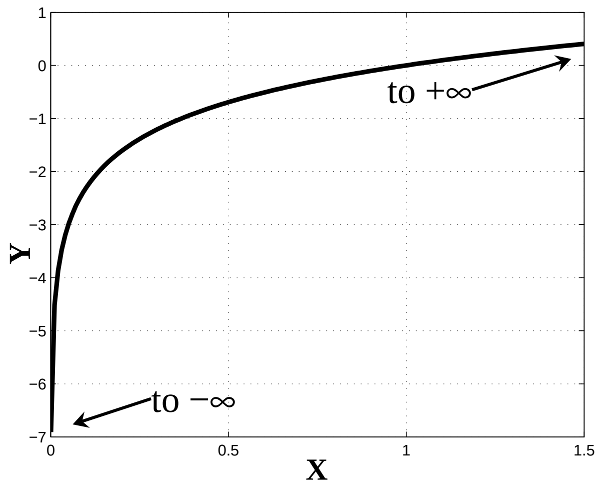

Figure 1: Graph of $y = \ln(x)$.

approaches 0 is minus infinity. As x approaches positive infinity, the limit is positive infinity (there's no horizontal asymptote). We won't get into those details today.

Instead, we'll look at one more qualitative feature of the graph; the fact that $L(x) < 0$ for $x < 1$. Why is this true?

- $L(1) = 0$ and L is increasing — if L increases to 0 as x goes toward 1 it must be negative before then.

- $L(x) = \int_1^x \frac{dt}{t} = -\int_x^1 \frac{dt}{t} < 0$ when $0 < x < 1$ because $\int_x^1 \frac{dt}{t}$ describes a positive area.