## Finding a Formula for the Best Degree $n$ Approximation

Linear approximation uses the values of a function and its first derivative at $x_0$ — or equivalently the slope of the tangent line and the point of tangency — to find the equation of a degree 1 polynomial approximating the function. The formula for the linear approximation of a function $f(x)$ near the value $x = x_0$ is:

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0).$$

The formula for quadratic approximation:

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

describes the best degree 2 approximation of the function $f(x)$ for $x \approx x_0$. We could also describe a degree 0 approximation: $f(x) \approx f(x_0)$ when $x$ is sufficiently close to $x_0$. As the degree of the approximation increases we add new terms, but the lower degree terms stay the same.

Can we define higher degree approximations, and if so, what terms should we add to do so? We'll focus on finding approximations near the value $x_0 = 0$; the calculations for the general case are very similar.

a) Find the best third degree approximation of a function $f(x)$ near $x = 0$:

$$\cancel{A}f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3. = A(x)$$

In other words, find values for the $a_i$ so that the first, second, and third derivatives of $f(x)$ are the same as the first, second and third derivatives of $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ when $x = 0$. (You may wish to refer to your notes on the derivation of the formula for the quadratic approximation.)

b) Use the fact that the $n^{\text{th}}$ derivative of $x^m$ is 0 for $m < n$ to find the $n^{\text{th}}$ derivative of:

$$A(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n.$$

c) Use your answer to (b) to give a formula for an $n^{\text{th}}$ degree polynomial approximation of a function $f(x)$ near $x = 0$.

### Solution

a) Find the best third degree approximation of a function $f(x)$ near $x = 0$:

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

In other words, find values for the $a_i$ so that the first, second, and third derivatives of $f(x)$ are the same as the first, second and third derivatives of $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ when $x = 0$. (You may wish to refer to your notes on the derivation of the formula for the quadratic approximation.)



Use $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ to approximate $f(x)$ when $x \approx 0$

$\Rightarrow$ for $x \to 0$ : $A(x) = f(x)$ $\quad A^{(3)}(x) = f^{(3)}(x)$

$A'(x) = f'(x)$ $\quad \vdots$

$A''(x) = f''(x)$ $\quad A^{(n)}(x) = f^{(n)}(x)$ [for $n$ degree approximation]

Known $f(x)$, Unknown $A(x)$, Find $a_0, a_1, \ldots, a_n$ such that $A(x)$ approximate $f(x)$

We start by taking derivatives of the approximation function $A(x)$:

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ A'(x) &= a_1 + 2a_2 x + 3a_3 x^2 \\ A''(x) &= 2a_2 + 6a_3 x \\ A^{(3)}(x) &= 6a_3 \end{aligned}$$

Next, we set the $n^{\text{th}}$ derivative of $A(x)$ equal to the $n^{\text{th}}$ derivative to $f(x)$ and evaluate at $x = 0$ to solve for $a_n$.

$$\begin{aligned} f(0) = A(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 \quad &\Longrightarrow \quad a_0 = f(0) \\ f'(0) = A'(0) = a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0^2 = b \quad &\Longrightarrow \quad a_1 = f'(0) \\ f''(0) = A''(0) = 2a_2 + 6a_3 \cdot 0 \quad &\Longrightarrow \quad a_2 = \frac{f''(0)}{2} \\ f^{(3)}(0) = A^{(3)}(0) = 6a_3 \quad &\Longrightarrow \quad a_3 = \frac{f^{(3)}0)}{6} \end{aligned}$$

Now that we know the coefficients of our approximation function $A(x)$ we can say that:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6} \quad (x \approx 0).$$

b) Use the fact that the $n^{\text{th}}$ derivative of $x^m$ is 0 for $m < n$ to find the $n^{\text{th}}$ derivative of:

$$A(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n.$$

As we take derivatives of a polynomial, the lower degree terms vanish and we're left with the derivatives of the highest degree terms. If we take $n$ derivatives of $A(x)$ we'll be left with just the $n^{\text{th}}$ derivative of $a_n x^n$.

To find the $n^{\text{th}}$ derivative of $a_n x^n$ we start taking derivatives and look for a pattern:

$$\begin{aligned} \frac{d}{dx} a_n x^n &= n a_n x^{n-1} \\ \frac{d^2}{dx^2} a_n x^n &= (n-1) n a_n x^{n-2} \\ \frac{d^3}{dx^3} a_n x^n &= (n-2)(n-1) n a_n x^{n-3} \\ \frac{d^4}{dx^4} a_n x^n &= (n-3)(n-2)(n-1) n a_n x^{n-4} \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
\frac{d^{n-2}}{dx^{n-2}} a_n x^n &= (n-(n-2-1))\cdots(n-2)(n-1)n a_n x^{n-(n-2)} \\
\frac{d^{n-1}}{dx^{n-1}} a_n x^n &= (n-(n-2))\cdot 3\cdots(n-2)(n-1)n a_n x^1 \\
\frac{d^{n}}{dx^{n}} a_n x^n &= 1\cdot 2\cdot 3\cdots(n-2)(n-1)n a_n = n! a_n
\end{aligned}
$$

c) Use your answer to (b) to give a formula for an $n^{\text{th}}$ degree polynomial approximation of a function $f(x)$ near $x = 0$.

Our goal is to have the $n^{\text{th}}$ derivatives of $f(x)$ and $A(x)$ agree when $x = 0$: $f^{(n)}(0) = A^{(n)}(0)$. We know that $A^{(n)}(x) = \frac{d^n}{dx^n} a_n x^n = n! a_n$, so:

$$
f^{(n)}(0) = n! a_n \qquad \Longrightarrow \qquad a_n = \frac{f^{(n)}(0)}{n!}.
$$

This agrees with our previous calculation that $a_3 = \frac{f^{(n)}(0)}{6}$.

We conclude that:

$$
f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(0)}{n!}x^n
$$

when $x$ is close to 0.

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