Another Example of Logarithmic Differentiation

This example could be done equally well by converting to base e, but we're going to do it using logarithmic differentiation. Recall that the rule we use for logarithmic differentiation is $(\ln u)' = u'/u$.

Here we have a "moving" (non-constant) exponent and a moving base.

Example: Let $v = x^x$. Find v'.

First, we take the natural log of both sides to see that $\ln v = \ln(x^x) = x \ln x$.

Next, we differentiate both sides of the equation, using the product rule and the rule for the derivative of $\ln x$ on the right hand side:

$$(\ln v)' = \ln x + x \cdot \frac{1}{x}.$$

Now apply the formula $(\ln u)' = u'/u$. to get:

$$v'/v = 1 + \ln x$$

Plugging in x^x for v and solving for v', we get:

$$\begin{aligned} \frac{v'}{x^x} &= 1 + \ln x \\ v' &= x^x(1 + \ln x) \\ \frac{d}{dx} x^x &= x^x(1 + \ln x) \end{aligned}$$

$$x^x = \left(e^{\ln x}\right)^x$$
$$= e^{x \ln x}$$
$$\frac{d}{dx} x^x = \frac{d}{dx} e^{x \ln x}$$
$$= e^{x \ln x} \cdot (\ln x + 1)$$

$$\ln x^x = x \ln x$$
$$\frac{d}{dx} \ln x^x = \frac{d}{dx} x \ln x$$
$$\frac{1}{x^x} \cdot \frac{d}{dx} x^x = \ln x + 1$$