The Fundamental Theorem and the Mean Value Theorem

Our goal is to use information about F' to derive information about F. Our first example of this process will be to compare the first fundamental theorem to the Mean Value Theorem.

We'll use the notation $\Delta F = F(b) - F(a)$ and $\Delta x = b - a$. The first fundamental theorem then tells us that:

$$\Delta F = \int_a^b f(x)\,dx.$$

If we divide both sides by Δx we get:

$$\frac{\Delta F}{\Delta x} = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{Average(f)}$$

the expression on the right is the average value of the function $f(x)$ on the interval $[a, b]$.

Why is this the average of f and not of F? Consider the following Riemann sum:

$$\int_0^n f(x)\,dx \approx f(1) + f(2) + \cdots + f(n).$$

This is a cumulative sum of values of $f(x)$. The quantity:

$$\frac{\int_0^n f(x)\,dx}{n} \approx \frac{f(1) + f(2) + \cdots + f(n)}{n}$$

is an average of values of $f(x)$; in the limit, the average value of $f(x)$ on the interval $[a, b]$ is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$.

We'll rewrite the first fundamental theorem one more time as:

$$\Delta F = \text{Average}(F')\Delta x.$$

In other words, the change in F is the average of the infinitesimal change times the amount of time elapsed. We can now use inequalities to compare this to the mean value theorem, which says that $\frac{F(b)-F(a)}{b-a} = F'(c)$ for some c between a and b. We can rewrite this as:

$$\Delta F = F'(c)\Delta x.$$

The value of Average(F') in the first fundamental theorem is very specific, but the $F'(c)$ from the mean value theorem is not; all we know about c is that it's somewhere between a and b.

Even if we don't know exactly what c is, we know for sure that it's less than the maximum value of F' on the interval from a to b, and that it's greater than the minimum value of F' on that interval:

$$\left(\min_{a<x<b} F'(x)\right)\Delta x \leq \Delta F = F'(c)\Delta x \leq \left(\max_{a<x<b} F'(x)\right)\Delta x.$$

The first fundamental theorem of calculus gives us a much more specific value — Average(F') — from which we can draw the same conclusion.

$$\left(\min_{a<x<b} F'(x)\right) \Delta x \leq \Delta F = \text{Average} F' \Delta x \leq \left(\max_{a<x<b} F'(x)\right) \Delta x.$$

The fundamental theorem of calculus is much stronger than the mean value theorem; as soon as we have integrals, we can abandon the mean value theorem. We get the same conclusion from the fundamental theorem that we got from the mean value theorem: the average is always bigger than the minimum and smaller than the maximum. Either theorem gives us the same conclusion about the change in F:

$$\left(\min_{a<x<b} F'(x)\right) \Delta x \leq \Delta F \leq \left(\max_{a<x<b} F'(x)\right) \Delta x.$$