Smoothing a Piecewise Polynomial

For each of the following, find all values of a and b for which $f(x)$ is differentiable.

a) $f(x) = \begin{cases} ax^2 + bx + 6, & x \le 0; \\ 2x^5 + 3x^4 + 4x^2 + 5x + 6, & x > 0. \end{cases}$

b) $f(x) = \begin{cases} ax^2 + bx + 6, & x \le 1; \\ 2x^5 + 3x^4 + 4x^2 + 5x + 6, & x > 1. \end{cases}$

Solution

a) $f(x) = \begin{cases} ax^2 + bx + 6, & x \le 0; \\ 2x^5 + 3x^4 + 4x^2 + 5x + 6, & x > 0. \end{cases}$

This problem is similar to one we have already seen. The piecewise function $f(x)$ is made up of two polynomial functions, each of which is continuous and differentiable. The only point at which the derivative of $f(x)$ might not be defined is at $x = 0$.

Our task is to find values of a and b that ensure that the limit:

$$\lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$$

is well defined, no matter how the limit is computed.

First, we find values of a and b for which $f(x)$ is continuous:

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} ax^2 + bx + 6 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} 2x^5 + 3x^4 + 4x^2 + 5x + 6 \\ &= 6 \end{aligned}$$

For all values of a and b, $f(x)$ is continuous at $x = 0$.

Next we compute the derivative of $f(x)$ where it is defined:

$$f'(x) = \begin{cases} 2ax + b, & x < 0; \\ 10x^4 + 12x^3 + 8x + 5, & x > 0. \end{cases}$$

Finally, we determine the values of a and b that ensure that the slopes of the two parts of $f(x)$ match at $x = 0$:

$$\lim_{\Delta x \to 0^-} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{x \to 0^-} f'(x) \ ?$$

from intuitive

$$\begin{aligned} \lim_{x \to 0^-} f'(x) &= \lim_{x \to 0^-} 2ax + b \\ &= b \end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f'(x) &= \lim_{x\to 0^+} 10x^4+12x^3+8x+5\\ &= 5\end{aligned}$$

We conclude that when $b=5$ and a is any real number, $f(x)$ is differentiable. This makes sense; the two parts of the graph of $f(x)$ always touch at $(0,6)$ and the constraint $b=5$ is all that's needed to ensure that their tangent lines have the same slopes at that point.

b) $f(x) = \begin{cases} ax^2+bx+6, & x \le 1;\\ 2x^5+3x^4+4x^2+5x+6, & x > 1.\end{cases}$

The procedure for this part of the problem is the same as for the previous part. Because the definition of $f(x)$ changes at $x=1$, the requirement that $f(x)$ be continuous now affects the values of a and b:

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} ax^2+bx+6\\ &= a+b+6\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} 2x^5+3x^4+4x^2+5x+6\\ &= 20.\end{aligned}$$

We conclude that $f(x)$ is continuous whenever $a+b+6=20$, or when $a+b=14$.

To determine where f is differentiable, we start by finding the slope of the tangent line to the graph of $f(x)$ at every point $x \ne 1$:

$$f'(x) = \begin{cases} 2ax+b, & x < 1;\\ 10x^4+12x^3+8x+5, & x > 1.\end{cases}$$

We could now substitute $14-b$ for a. We choose not to for the sake of simplicity.

$$\begin{aligned}\lim_{x\to 1^-} f'(x) &= \lim_{x\to 1^-} 2ax+b\\ &= 2a+b\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+} f'(x) &= \lim_{x\to 1^+} 10x^4+12x^3+8x+5\\ &= 35\end{aligned}$$

In order for $f(x)$ to be differentiable, it must be continuous and the left hand and right hand limits of its derivative must match at $x=1$. In other words:

$$\begin{aligned}a+b &= 14\\ 2a+b &= 35\end{aligned}$$

Subtracting the first equation from the second, we find that $a = 21$ and $b = -7$. (We could get the same result by substituting $14 - b$ into the second equation for a and then solving for b.)

We conclude that $f(x)$ is differentiable when $a = 21$ and $b = -7$. We could check our work by graphing.