

Indefinite Integrals over Singularities

When computing $\int_0^\infty \frac{dx}{\sqrt{x^2+10}}$ we had to take an extra step to avoid the integral $\int_0^1 \frac{dx}{x}$. We'll now go back and discuss integration near singular points.

Integrals like $\int_0^1 \frac{dx}{x}$ are known as *improper integrals of the second type*. Examples include:

$$\int_0^1 \frac{dx}{\sqrt{x}}, \quad \int_0^1 \frac{dx}{x}, \quad \text{and} \quad \int_0^1 \frac{dx}{x^2}.$$

These integrals turn out to be fairly straightforward to calculate:

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x}} &= \int_0^1 x^{-1/2} dx \\ &= \left. \frac{1}{1/2} x^{1/2} \right|_0^1 \\ &= \left. 2x^{1/2} \right|_0^1 \\ &= 2 \cdot 1^{1/2} - 2 \cdot 0^{1/2} \\ &= 2. \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{dx}{x} &= \ln x \Big|_0^1 \\ &= \ln 1 - \ln 0 \quad (\text{diverges.}) \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{dx}{x^2} &= \left. -x^{-1} \right|_0^1 \\ &= -\frac{1}{1} - \left(-\frac{1}{0} \right) \quad (\text{diverges.}) \end{aligned}$$

However, you can get into trouble if you're not careful. Consider the following calculation:

$$\begin{aligned} \int_{-1}^1 \frac{dx}{x^2} &= \left. -x^{-1} \right|_{-1}^1 \\ &= -(1^{-1}) - (-(-1)^{-1}) \\ &= -1 - 1 \\ &= -2. \end{aligned}$$

This is ridiculous! As we see from Figure 1, $\frac{1}{x^2}$ is always positive. The area under the graph of $y = \frac{1}{x^2}$ between -1 and 1 is clearly greater than 2 ; in particular it cannot be a negative number.

In fact, the area under the graph of $y = \frac{1}{x^2}$ between -1 and 1 is infinite, not -2 . The calculation above is nonsense.

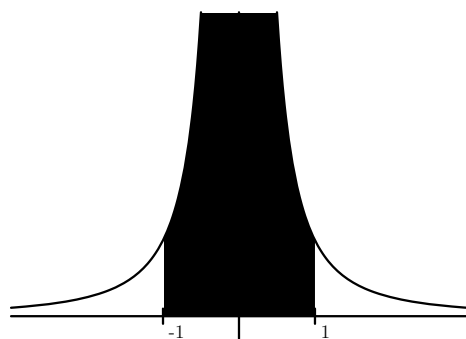


Figure 1: Graph of $y = \frac{1}{x^2}$.

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