Log of a Product

Claim: $L(ab) = L(a) + L(b)$, where $L(x) = \int_1^x \frac{dt}{t}$ is an alternately defined natural log function.

To prove this, we just plug in the formula and see what happens. On the left hand side we have:

$$L(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$$

By definition, $\int_1^a \frac{dt}{t} = L(a)$. If we could show that $\int_a^{ab} \frac{dt}{t} = L(b)$, we'd be done with the proof.

It turns out that we can prove this by using a change of variables. We start with $\int_a^{ab} \frac{dt}{t} = L(b)$, and substitute $t = au$ (so $dt = a\,du$). The limits of integration are from $u = 1$ to $u = b$. If we plug these into $\int_a^{ab} \frac{dt}{t}$, we get:

$$\int_a^{ab} \frac{dt}{t} = \int_{u=1}^{u=b} \frac{a\,du}{au} = \int_1^{b} \frac{du}{u} = L(b).$$

We can now conclude that:

$$L(ab) = L(a) + L(b)$$