Derivative of the Inverse of a Function

One very important application of implicit differentiation is to finding derivatives of inverse functions.

We start with a simple example. We might simplify the equation $y = \sqrt{x}$ ($x > 0$) by squaring both sides to get $y^2 = x$. We could use function notation here to say that $y = f(x) = \sqrt{x}$ and $x = g(y) = y^2$.

In general, we look for functions $y = f(x)$ and $g(y) = x$ for which $g(f(x)) = x$. If this is the case, then g is the inverse of f (we write $g = f^{-1}$) and f is the inverse of g (we write $f = g^{-1}$).

How are the graphs of a function and its inverse related? We start by graphing $f(x) = \sqrt{x}$. Next we want to graph the inverse of f, which is $g(y) = x$. But this is exactly the graph we just drew. To compare the graphs of the functions f and f^{-1} we have to exchange x and y in the equation for f^{-1}. So to compare $f(x) = \sqrt{x}$ to its inverse we replace y's by x's and graph $g(x) = x^2$.

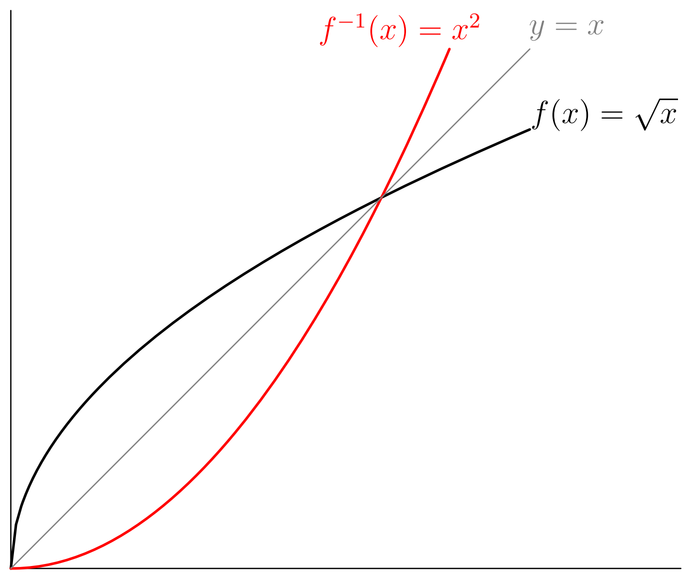

Figure 1: The graph of f^{-1} is the reflection of the graph of f across the line $y = x$

In general, if you have the graph of a function f you can find the graph of f^{-1} by exchanging the x- and y-coordinates of all the points on the graph. In other words, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

This suggests that if $\frac{dy}{dx}$ is the slope of a line tangent to the graph of f, then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

is the slope of a line tangent to the graph of f^{-1}. We could use the definition of the derivative and properties of inverse functions to turn this suggestion into a proof, but it's easier to prove using implicit differentiation.

Let's use implicit differentiation to find the derivative of the inverse function:

$$\begin{aligned} y &= f(x) \\ f^{-1}(y) &= x \\ \frac{d}{dx}(f^{-1}(y)) &= \frac{d}{dx}(x) = 1 \end{aligned}$$

By the chain rule:

$$\frac{d}{dy}(f^{-1}(y))\frac{dy}{dx} = 1$$

so

$$\frac{d}{dy}(f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}.$$

Implicit differentiation allows us to find the derivative of the inverse function $x = f^{-1}(y)$ whenever we know the derivative of the original function $y = f(x)$.