Example: $\int x \tan^{-1}(x)\, dx$

This is a slightly harder problem than any on the test, but something like this might appear on the final. How should we approach this?

Student: Integration by parts.

Great! Because $\tan^{-1}(x)$ is begging to be differentiated to be made simpler. So we choose:

$$\begin{aligned} u &= \tan -1(x), & v' &= x, \\ u' &= \frac{1}{1+x^2}, & v &= \frac{x^2}{2}. \end{aligned}$$

Integration by parts then gives us:

$$\int x \tan^{-1}(x)\, dx = \frac{x^2}{2} \tan^{-1}(x) - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2}\, dx$$

We're not done yet! We still have to integrate:

$$-\frac{1}{2} \int \frac{x^2}{1+x^2}\, dx.$$

What do we do know?

Student: Trig substitution.

Trig substitution will work, but that's not what I had in mind.

Student: Add and subtract one in the numerator.

That's a good idea. This is an example of a rational expression in which the numerator and denominator have the same degree, so you could use long division to simplify the "improper fraction". An equivalent shortcut is:

$$\begin{aligned} \frac{x^2}{1+x^2} &= \frac{x^2+1-1}{1+x^2} \\ &= \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \\ &= 1 - \frac{1}{1+x^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} -\frac{1}{2} \int \frac{x^2}{1+x^2}\, dx &= -\frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= -\frac{1}{2}x + \frac{1}{2} \tan^{-1} x + C. \end{aligned}$$

The answer to the original question is then:

$$\begin{aligned} \int x \tan^{-1}(x)\, dx &= \frac{x^2}{2} \tan^{-1}(x) - \int \frac{x^2}{2} \frac{1}{1+x^2}\, dx \\ &= \frac{x^2}{2} \tan^{-1}(x) + \frac{1}{2}x - \frac{1}{2} \tan^{-1} x + c. \end{aligned}$$