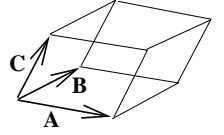
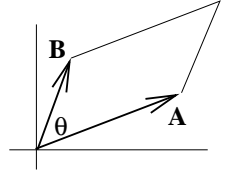


Determinants 2. Area and Volume

Area and volume interpretation of the determinant:

$$(1) \quad \pm \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \text{area of parallelogram with edges } \mathbf{A} = (a_1, a_2), \mathbf{B} = (b_1, b_2).$$

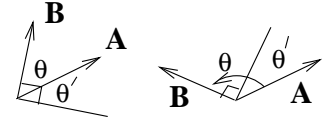
$$(2) \quad \pm \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{volume of parallelepiped with edges row-vectors } \mathbf{A}, \mathbf{B}, \mathbf{C}.$$



In each case, choose the sign which makes the left side non-negative.

Proof of (1). We begin with two preliminary observations.

Let θ be the positive angle from \mathbf{A} to \mathbf{B} ; we assume it is $< \pi$, so that \mathbf{A} and \mathbf{B} have the general positions illustrated.

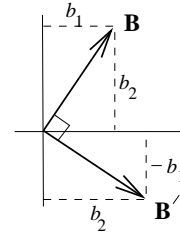


Let $\theta' = \pi/2 - \theta$, as illustrated. Then $\cos \theta' = \sin \theta$.

Draw the vector \mathbf{B}' obtained by rotating \mathbf{B} to the right by $\pi/2$. The picture shows that $\mathbf{B}' = (b_2, -b_1)$, and $|\mathbf{B}'| = |\mathbf{B}|$.

To prove (1) now, we have a standard formula of Euclidean geometry,

$$\begin{aligned} \text{area of parallelogram} &= |\mathbf{A}||\mathbf{B}| \sin \theta \\ &= |\mathbf{A}||\mathbf{B}'| \cos \theta', && \text{by the above observations} \\ &= \mathbf{A} \cdot \mathbf{B}', && \text{by the geometric definition of dot product} \\ &= a_1 b_2 - a_2 b_1 && \text{by the formula for } \mathbf{B}' \end{aligned}$$



This proves the area interpretation (1) if \mathbf{A} and \mathbf{B} have the position shown. **If their positions are reversed, then the area is the same, but the sign of the determinant is changed**, so the formula has to read,

$$\text{area of parallelogram} = \pm \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \text{whichever sign makes the right side } \geq 0.$$

The proof of the analogous volume formula (2) will be made when we study the scalar triple product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$.

Generalizing (1) and (2), $n \times n$ determinants can be interpreted as the **hypervolume in n -space of a n -dimensional parallelotope**.

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