Non-independent Variables

3. Abstract partial differentiation; rules relating partial derivatives

Often in applications, the function w is not given explicitly, nor are the equations connecting the variables. Thus you need to be able to work with functions and equations just given abstractly. The previous ideas work perfectly well, as we will illustrate. However, we will need (as in section 2) to distinguish between

formal partial derivatives, written here $f_x,\ f_y, \ldots$ (calculated as if all the variables were independent), and

actual partial derivatives, written $\partial f/\partial x, \ldots$, which take account of any relations between the variables.

Example 5. If $f(x,y,z) = xy^2z^4$, where $z = 2x+3y$, the three formal derivatives are

$$f_x = y^2z^4, \qquad f_y = 2xyz^4, \qquad f_z = 4xy^2z^3,$$

while three of the many possible actual partial derivatives are (we use the chain rule)

$$\begin{aligned}
\left(\frac{\partial f}{\partial x}\right)_y &= f_x + f_z\left(\frac{\partial z}{\partial x}\right)_y = y^2z^4 + 8xy^2z^3;\\
\left(\frac{\partial f}{\partial y}\right)_x &= f_y + f_z\left(\frac{\partial z}{\partial y}\right)_x = 2xyz^4 + 12xy^2z^3;\\
\left(\frac{\partial f}{\partial z}\right)_x &= f_y\left(\frac{\partial y}{\partial z}\right)_x + f_z = \tfrac{2}{3}xyz^4 + 4xy^2z^3.
\end{aligned}$$

Rules connecting partial derivatives. These rules are widely used in the applications, especially in thermodynamics. Here we will use them as an excuse for further practice with the chain rule and differentials.

With an eye to thermodynamics, we assume a set of variables $t, u, v, w, x, y, z, \ldots$ connected by several equations in such a way that

- any *two* are independent;
- any *three* are connected by an equation.

Thus, one can choose any two of them to be the independent variables, and then each of the other variables can be expressed in terms of these two.

We give each rule in two forms—the second form is the one ordinarily used, while the first is easier to remember. (The first two rules are fairly simple in either form.)

(8a,b) $$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial x}\right)_z = 1 \qquad \left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{(\partial y/\partial x)_z} \qquad \textbf{reciprocal rule}$$

(9a,b) $$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial t}\right)_z = \left(\frac{\partial x}{\partial t}\right)_z \qquad \left(\frac{\partial x}{\partial y}\right)_z = \frac{(\partial x/\partial t)_z}{(\partial y/\partial t)_z}, \qquad \textbf{chain rule}$$

(10a,b) $$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1 \qquad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial x/\partial z)_y}{(\partial y/\partial z)_x}, \qquad \textbf{cyclic rule}$$

Note how the successive factors in the cyclic rule are formed: the variables are used in the successive orders $x, y, z;\ \ y, z, x;\ \ z, x, y;$ one says they are permuted cyclically, and this explains the name.

Proof of the rules. The first two rules are simple: since z is being held fixed throughout, each variable becomes a function of just one other variable, and (9) is just the one-variable chain rule. Then (8) is just the special case of (9) where $x = t$.

The cyclic rule is less obvious — on the right side it looks almost like the chain rule, but different variables are being held constant in each of the differentiations, and this changes it entirely. To prove it, we suppose $f(x, y, z) = 0$ is the equation satisfied by x, y, z; taking y and z as the independent variables and differentiating $f(x, y, z) = 0$ with respect to y gives:

$df = f_x\,dx + f_y\,dy + f_z\,dz = 0$

diff to y, ind z

$\Rightarrow$ fix $dz = 0$

$\Rightarrow f_x\,dx + f_y\,dy = 0$

$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$

$\Rightarrow f_x\left(\frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz\right) + f_y\,dy = 0$

$\Rightarrow f_x\frac{\partial x}{\partial y}dy + f_y\,dy = 0$

$$f_x \left(\frac{\partial x}{\partial y}\right)_z + f_y = 0; \qquad \text{therefore} \quad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{f_y}{f_x}\,. \tag{11}$$

Permuting the variables in (11) and multiplying the resulting three equations gives (10a):

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_y}{f_x} \cdot -\frac{f_z}{f_y} \cdot -\frac{f_x}{f_z} = -1.$$

Example 6. Suppose $w = w(x, r)$, with $r = r(x, \theta)$. Give an expression for $\left(\frac{\partial w}{\partial r}\right)_\theta$ in terms of formal partial derivatives of w and r.

Solution. Evidently the independent variables are to be r and θ, since these are the ones that occur in the lower part of the partial derivative, with x dependent on them. Since θ is viewed as a constant, the chain rule gives

$$\left(\frac{\partial w}{\partial r}\right)_\theta = w_x \left(\frac{\partial x}{\partial r}\right)_\theta + w_r\,;$$
$$\left(\frac{\partial x}{\partial r}\right)_\theta = \frac{1}{(\partial r/\partial x)_\theta},$$

by the reciprocal rule (8). and therefore finally,

$$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{w_x}{r_x} + w_r\,.$$