Velocity, speed and arc length

Speed

Velocity, being a vector, has a magnitude and a direction. The direction is tangent to the curve traced out by $\mathbf{r}(t)$. The magnitude of its velocity is the speed.

$$\text{speed } = |\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right|.$$

Speed is in units of distance per unit time. It reflects how fast our moving point is moving.

Example: A point goes one time around a circle of radius 1 unit in 3 seconds. What is its average velocity and average speed.

Answer: The distance the point traveled equals the circumference of the circle, 2π. Its net displacement is $\mathbf{0}$, since it ends where it started. Thus, its average speed = distance/time $= 2\pi/3$ and its average velocity = displacement/time = $\mathbf{0}$.

If you look carefully, we've used a boldface $\mathbf{0}$ because velocity is a vector.

Our usual symbol for distance traveled is s. For a point moving along a curve the distance traveled is the length of the curve. Because of this we also refer to s as *arc length.*

Notation and nomenclature summary:

Since we will use a variety of notations, we'll collect them here. The unit tangent vector will be explained below. As you should expect, we will also be able to view everything from a geometric perspective.

$d\vec{r} = \langle dx, dy, dz \rangle$ $\quad |d\vec{r}| = \pm ds$

$|d\vec{r}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$\frac{|d\vec{r}|}{dt} = \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(dt)^2}}$

can replace "d" to "Δ" to help understanding

$\left|\frac{d\vec{r}}{dt}\right| = \left|\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

$\Rightarrow \frac{|d\vec{r}|}{dt} = \left|\frac{d\vec{r}}{dt}\right|$

$\mathbf{r}(t)$ = position.

In the plane $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = \langle x, y \rangle$

In space $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

$\frac{d\mathbf{r}}{dt} = \mathbf{v}(t)$ = velocity = tangent vector.

In the plane $\mathbf{v} = x'(t)\mathbf{i} + y'(t)\mathbf{j} = \langle x', y' \rangle$

In space $\mathbf{v} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} = \langle x', y', z' \rangle$.

$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ = unit tangent vector.

s = arclength, speed $= \frac{ds}{dt} = |\mathbf{v}|$. $\Leftarrow\!= \left|\frac{d\vec{r}}{dt}\right| \Leftarrow\!= \frac{|d\vec{r}|}{dt}$

speed $= \frac{ds}{dt} = \frac{|d\vec{r}|}{dt} \stackrel{?}{=} \left|\frac{d\vec{r}}{dt}\right| = |\vec{v}|$ ✓

constant $\cdot |\vec{A}| = |\text{constant} \cdot \vec{A}|$

In the plane $\frac{ds}{dt} = \sqrt{(x')^2 + (y')^2}$.

In space $\frac{ds}{dt} = \sqrt{(x')^2 + (y')^2 + (z')^2}$.

$\mathbf{v} = \frac{ds}{dt}\mathbf{T}, \quad \mathbf{T} = \frac{\mathbf{v}}{ds/dt}$

$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$ = acceleration.

In the plane $\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} = \langle x'', y'' \rangle$

In space $\mathbf{a} = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k} = \langle x'', y'', z'' \rangle$.

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \underbrace{\frac{d^2y}{dx^2}}_{= f^{(2)}(x)} \neq \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \cdot \frac{dy}{dx}$

a confusing notation

Unit tangent vector

As its name implies, the *unit tangent vector* is a unit vector in the same direction as the tangent vector. We usually denote it $\mathbf{T}$. We compute it by dividing the tangent vector by its length. Here are several ways of writing this.

$= \frac{d\vec{r}}{ds}$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{v}}{ds/dt}.$$

Multiply $\mathbf{T}$ by ds/dt gives the formula

$$\mathbf{v} = \frac{ds}{dt}\mathbf{T},$$

which expresses velocity as a magnitute, ds/dt and a direction $\mathbf{T}$.

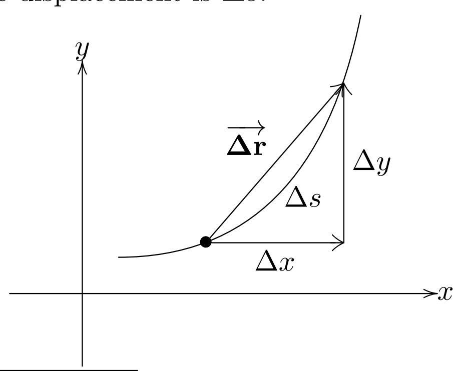
Geometric considerations

Here we'll offer a mathematical justification for our statement that

$$\text{speed} = \frac{ds}{dt} = |\mathbf{v}|.$$

We'll work in two dimensions. The extension to 3D is straightforward.

The figure below shows a curve, and a small displacement $\Delta\mathbf{r}$. The length along the curve from the start to end of the displacement is Δs.



We see $\Delta s \approx |\Delta\mathbf{r}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Dividing by Δt gives

$$\frac{\Delta s}{\Delta t} \approx \left|\frac{\Delta\mathbf{r}}{\Delta t}\right| = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

Taking the limit as $\Delta t \to 0$ gives

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$