

Parametric Curves

General parametric equations

We have seen parametric equations for lines. Now we will look at parametric equations of more general trajectories. Repeating what was said earlier, a parametric curve is simply the idea that a point moving in the space traces out a path.

We can use a parameter to describe this motion. Quite often we will use t as the parameter and think of it as time. Since the position of the point depends on t we write

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

to indicate that x , y and z are functions of t . We call t the parameter and the equations for x , y and z are called *parametric equations*.

It is not always necessary to think of the parameter as representing time. We will see cases where it is more convenient to express the position as a function of some other variable.

The position vector

In order to use vector techniques we define the *position vector*

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \langle x(t), y(t), z(t) \rangle.$$

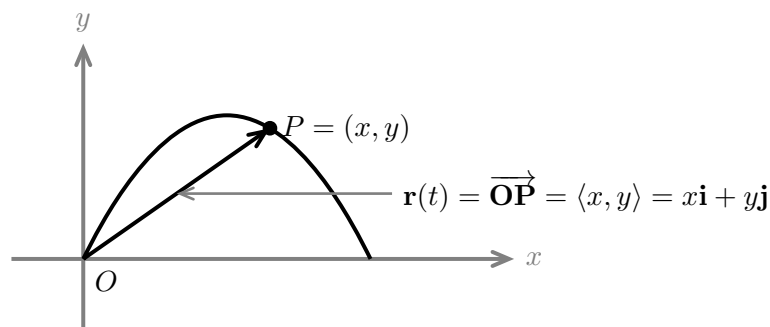
This is just the vector from the origin to the moving point. As the point moves so does the position vector –see the figure with example 1.

Example 1: Thomas Pynchon fires a rocket from the origin. Its initial x -velocity is $v_{0,x}$ and its initial y -velocity is $v_{0,y}$.

You’ve probably seen this, but in any case, physics tells us that the parametric equations for its parabolic trajectory are

$$x(t) = v_{0,x}t, \quad y(t) = -\frac{1}{2}gt^2 + v_{0,y}t.$$

At time t the rocket is at point $P = (x(t), y(t))$. The position vector can be written in many different ways: $\mathbf{r}(t) = \overrightarrow{\mathbf{OP}} = x(t)\mathbf{i} + y(t)\mathbf{j} = \langle x, y \rangle$.



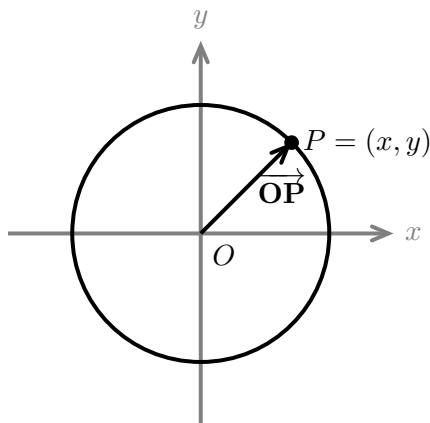
Next we will give a series of examples of parametrized curves. The most important are circles and lines. The last one is the *cycloid*. It is an important example which combines lines and circles.

Circles and ellipses

Consider the parametric curve in the plane

$$x(t) = a \cos t, \quad y(t) = a \sin t.$$

Easily we get the relation $x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2$. Therefore the trajectory is on a circle of radius a centered at O .



We will call $x(t) = a \cos t$, $y(t) = a \sin t$ the *parametric form* of the curve and $x^2 + y^2 = a^2$ the *symmetric form*.

Note, a different parametrization, say

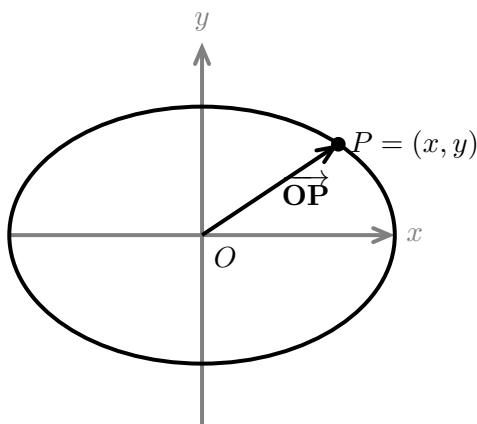
$$x(t) = a \cos(3t), \quad y(t) = a \sin(3t)$$

results in the same path, i.e. the circle $x^2 + y^2 = a^2$, but the two trajectories differ by how fast they travel around the circle.

The circle is easily changed to an ellipse by

parametric form: $x(t) = a \cos t, \quad y(t) = b \sin t$

symmetric form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Lines

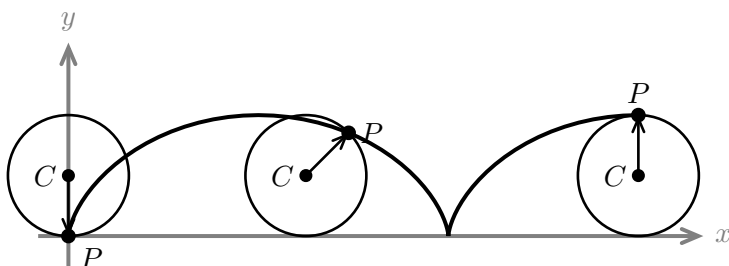
We review parametric equations of lines by writing the equation of a general line in the plane. We know we can parametrize the line through (x_0, y_0) parallel to $\langle b_1, b_2 \rangle$ by

$$x(t) = x_0 + tb_1, \quad y(t) = y_0 + tb_2 \Leftrightarrow \mathbf{r}(t) = \langle x, y \rangle = \langle x_0 + tb_1, y_0 + tb_2 \rangle = \langle x_0, y_0 \rangle + t\langle b_1, b_2 \rangle.$$

The cycloid

The cycloid has a long and storied history and comes up surprisingly often in physical problems. For us it is a curve that has no simple symmetric form, so we will only work with it in its parametric form.

The cycloid is the trajectory of a point on a circle that is rolling without slipping along the x -axis. To be specific, we'll follow the point P that starts at the origin.



The natural parameter to use is the angle θ that the wheel has turned. We'll use vector methods to find the position vector for P as a function of θ .

Our strategy is to break the motion up into translation of the center and rotation about the center. The figure shows the wheel after it has turned through a small θ . We see the position vector

$$\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}.$$

We'll compute each piece separately.

After turning θ radians the wheel has rolled a distance $a\theta$, so the center of the circle is at $(a\theta, a)$, i.e.,

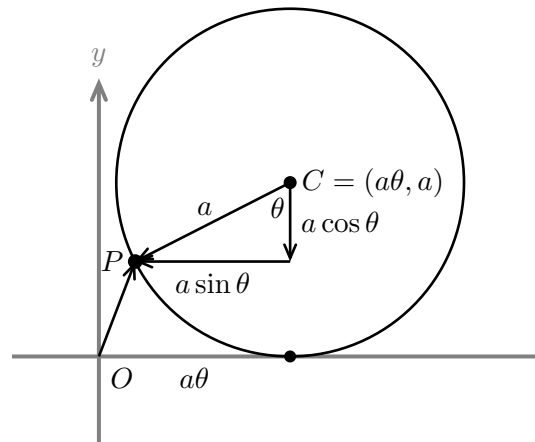
$$\overrightarrow{OC} = \langle a\theta, a \rangle.$$

The figure also shows that

$$\overrightarrow{CP} = \langle -a \sin \theta, -a \cos \theta \rangle.$$

Putting the pieces together we get parametric equations for the cycloid

$$\begin{aligned} \overrightarrow{OP} &= \langle a\theta - a \sin \theta, a - a \cos \theta \rangle \\ \Leftrightarrow \quad x(\theta) &= a\theta - a \sin \theta, \quad y(\theta) = a - a \cos \theta. \end{aligned}$$

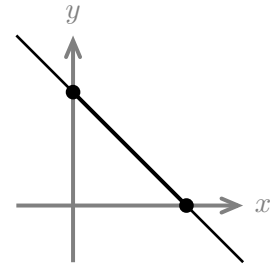


Example 2: (Where the symmetric form loses information.)

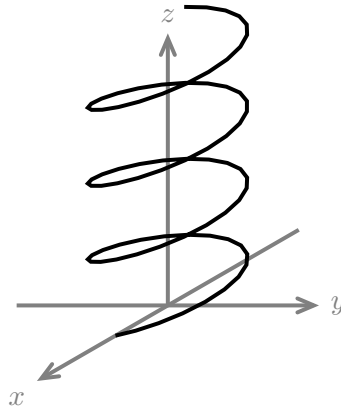
Find the symmetric form for $x = 3 \cos^2 t$, $y = 3 \sin^2 t$.

Easily we get: $x + y = 3$, with x, y non-negative.

The symmetric form shows a line, but the parametric trajectory only traces out a part of the line. In fact, it goes back and forth over the part of the line in the first quadrant.



Example 3: The curve $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + at \mathbf{k}$ is a helix winding around the z -axis.



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