Gradient: definition and properties

Definition of the gradient

If $w = f(x, y)$, then $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ are the rates of change of w in the $\mathbf{i}$ and $\mathbf{j}$ directions.

It will be quite useful to put these two derivatives together in a vector called the *gradient* of w.

$$\operatorname{grad} w = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\rangle.$$

We will also use the symbol $\boldsymbol{\nabla} w$ to denote the gradient. (You read this as 'gradient of w' or 'grad w'.)

Of course, if we specify a point $P_0 = (x_0, y_0)$, we can evaluate the gradient at that point. We will use several notations for this

$$\operatorname{grad} w(x_0, y_0) = \boldsymbol{\nabla}\, w|_{P_0} = \boldsymbol{\nabla}\, w|_o = \left\langle \left.\frac{\partial w}{\partial x}\right|_o, \left.\frac{\partial w}{\partial y}\right|_o \right\rangle.$$

Note well the following: (as we look more deeply into properties of the gradient these can be points of confusion).

1. The gradient takes a scalar function $f(x, y)$ and produces a vector $\boldsymbol{\nabla} f$.
2. The vector $\boldsymbol{\nabla} f(x, y)$ lies in the plane.

For functions $w = f(x, y, z)$ we have the gradient

$$\operatorname{grad} w = \boldsymbol{\nabla} w = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle.$$

That is, the gradient takes a scalar function of three variables and produces a three dimensional vector.

The gradient has many geometric properties. In the next session we will prove that for $w = f(x, y)$ the gradient is perpendicular to the level curves $f(x, y) = c$. We can show this by direct computation in the following example.

Example 1: Compute the gradient of $w = (x^2 + y^2)/3$ and show that the gradient at $(x_0, y_0) = (1, 2)$ is perpendicular to the level curve through that point.

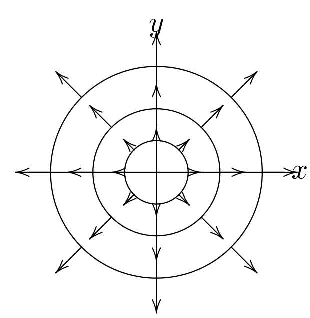
Answer: The gradient is easily computed

$$\boldsymbol{\nabla} w = \langle 2x/3, 2y/3 \rangle = \frac{2}{3}\langle x, y \rangle.$$

At $(1, 2)$ we get $\boldsymbol{\nabla} w(1, 2) = \frac{2}{3}\langle 1, 2 \rangle$. The level curve through $(1, 2)$ is

$$(x^2 + y^2)/3 = 5/3,$$

which is identical to $x^2 + y^2 = 5$. That is, it is a circle of radius $\sqrt{5}$ centered at the origin. Since the gradient at (1,2) is a multiple of $\langle 1, 2 \rangle$, it points radially outward and hence is perpendicular to the circle. Below is a figure showing the gradient field and the level curves.



$z = (x^2 + y^2)/3$

Example 2: Consider the graph of $y = \mathrm{e}^x$. Find a vector perpendicular to the tangent to $y = \mathrm{e}^x$ at the point $(1, \mathrm{e})$.

Old method: Find the slope take the negative reciprocal and make the vector.

New method: This graph is the level curve of $w = y - \mathrm{e}^x = 0$.

$\boldsymbol{\nabla} w = \langle -\mathrm{e}^x, 1 \rangle \Rightarrow$ (at $x = 1$) $\boldsymbol{\nabla} w(1, e) = \langle -\mathrm{e}, 1 \rangle$ is perpendicular to the tangent vector to the graph, $\mathbf{v} = \langle 1, \mathrm{e} \rangle$.

Higher dimensions

Similarly, for $w = f(x, y, z)$ we get level surfaces $f(x, y, z) = c$. The gradient is perpendicular to the level surfaces.

Example 3: Find the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point $P = (1, 1, 1)$.

Answer: Introduce a new variable

$$w = x^2 + 2y^2 + 3z^2.$$

Our surface is the level surface $w = 6$. Saying the gradient is perpendicular to the surface means exactly the same thing as saying it is normal to the tangent plane. Computing

$$\boldsymbol{\nabla} w = \langle 2x, 4y, 6z \rangle \Rightarrow \boldsymbol{\nabla} w|_P = \langle 2, 4, 6 \rangle.$$

Using point normal form we get the equation of the tangent plane is

$$2(x-1) + 4(y-1) + 6(z-1) = 0, \quad \text{or} \quad 2x + 4y + 6z = 12.$$