Directional Derivatives

Directional derivative

Like all derivatives the *directional derivative* can be thought of as a ratio. Fix a unit vector $\mathbf{u}$ and a point P_0 in the *plane*. The **directional derivative** of w at P_0 in the direction $\mathbf{u}$ is defined as

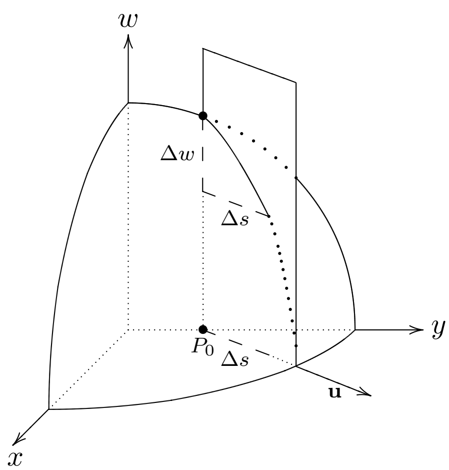
$$\left.\frac{dw}{ds}\right|_{P_0,\mathbf{u}} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s}.$$

Here Δw is the change in w caused by a step of length Δs in the direction of $\mathbf{u}$ (all in the xy-plane).

Below we will show that

$$\left.\frac{dw}{ds}\right|_{P_0,\mathbf{u}} = \boldsymbol{\nabla} w(P_0) \cdot \mathbf{u}. \tag{1}$$

We illustrate this with a figure showing the graph of $w = f(x, y)$. Notice that Δs is measured in the plane and Δw is the change of w on the graph.



$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y$$

$$\langle \Delta x, \Delta y\rangle \xrightarrow{\text{Unit}} \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s}\right\rangle \quad \left(\Delta s = \sqrt{\Delta x^2 + \Delta y^2}\right)$$

$$\Delta w \approx \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right\rangle \cdot \langle \Delta x, \Delta y\rangle$$

$$= \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right\rangle \cdot \frac{\langle \Delta x, \Delta y\rangle}{|\langle \Delta x, \Delta y\rangle|} \cdot |\langle \Delta x, \Delta y\rangle|$$

$$= \vec{\nabla w} \cdot \frac{\vec{r}}{|\vec{r}|} \cdot |\vec{r}|$$

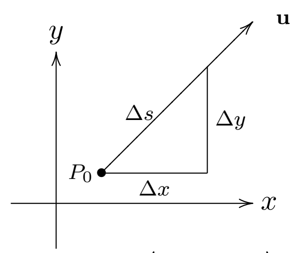
$$= \vec{\nabla w} \cdot \vec{u} \cdot |\vec{r}|$$

$$\Rightarrow \frac{\Delta w}{|\vec{r}|} = \frac{\Delta w}{\Delta s} \approx \vec{\nabla w} \cdot \vec{u}$$

$$\Rightarrow \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s} = \frac{dw}{ds} = \vec{\nabla w} \cdot \vec{u}$$

Proof of equation 1

The figure below represents the change in position from P_0 resulting from taking a step of size Δs in the $\mathbf{u}$ direction.



Since $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$ we have that $\left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s}\right\rangle$. is a unit vector, so

$$\mathbf{u} = \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s}\right\rangle.$$

The tangent plane approximation at P_0 is

$$\Delta w \approx \left.\frac{\partial w}{\partial x}\right|_{P_0} \Delta x + \left.\frac{\partial w}{\partial y}\right|_{P_0} \Delta y$$

Dividing this approximation by Δs gives

$$\frac{\Delta w}{\Delta s} \approx \left.\frac{\partial w}{\partial x}\right|_{P_0} \frac{\Delta x}{\Delta s} + \left.\frac{\partial w}{\partial y}\right|_{P_0} \frac{\Delta y}{\Delta s}.$$

We can rewrite this as a dot product

$$\frac{\Delta w}{\Delta s} \approx \left\langle \left.\frac{\partial w}{\partial x}\right|_{P_0}, \left.\frac{\partial w}{\partial y}\right|_{P_0} \right\rangle \cdot \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s} \right\rangle.$$

In the dot product the first term is $\nabla w|_{P_0}$ and the second is just $\mathbf{u}$, so,

$$\frac{\Delta w}{\Delta s} \approx \nabla w|_{P_0} \cdot \mathbf{u}.$$

Now taking the limit we get equation (1).

Example: (Algebraic example) Let $w = x^3 + 3y^2$.
Compute $\frac{dw}{ds}$ at $P_0 = (1, 2)$ in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.
<u>**Answer:**</u> We compute all the necessary pieces:
i) $\nabla w = \langle 3x^2, 6y\rangle \;\Rightarrow\; \nabla w|_{(1,2)} = \langle 3, 12\rangle$.
ii) $\mathbf{u}$ must be a unit vector, so $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \langle \frac{3}{5}, \frac{4}{5}\rangle$.
iii) $\left.\frac{dw}{ds}\right|_{P_0,\mathbf{u}} = \nabla w|_{(1,2)} \cdot \mathbf{u} = \langle 3, 12\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \boxed{\frac{57}{5}.}$

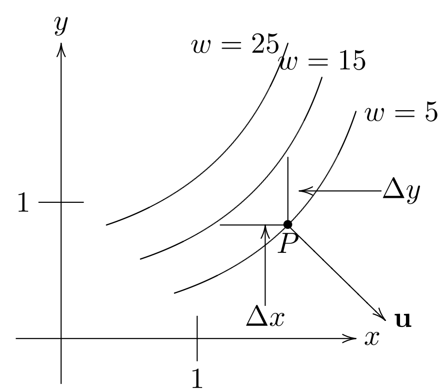
Example: (Geometric example) Let $\mathbf{u}$ be the direction of $\langle 1, -1\rangle$.
Using the picture at right estimate $\left.\frac{\partial w}{\partial x}\right|_{P}$, $\left.\frac{\partial w}{\partial y}\right|_{p}$, and $\left.\frac{dw}{ds}\right|_{P,\mathbf{u}}$.
By measuring from P to the next in level curve in the x direction we see that $\Delta x \approx -.5$.

$\Rightarrow \boxed{\left.\frac{\partial w}{\partial x}\right|_{P} \approx \frac{\Delta w}{\Delta x} \approx \frac{10}{-.5} = -20.}$

Similarly, we get $\boxed{\left.\frac{\partial w}{\partial y}\right|_{P} \approx 20.}$

Measuring in the $\mathbf{u}$ direction we get $\Delta s \approx -.3$

$\Rightarrow \boxed{\left.\frac{dw}{ds}\right|_{P,\mathbf{u}} \approx \frac{\Delta w}{\Delta s} \approx \frac{10}{.3} = -33.3.}$



Direction of maximum change:
The direction that gives the maximum rate of change is in the same direction as ∇w. The proof of this uses equation (1). Let θ be the angle between ∇w and $\mathbf{u}$. Then the geometric form of the dot product says

$$\left.\frac{dw}{ds}\right|_{\mathbf{u}} = \nabla w \cdot \mathbf{u} = |\nabla w||\mathbf{u}|\cos\theta = |\nabla w|\cos\theta.$$

(In the last equation we dropped the $|\mathbf{u}|$ because it equals 1.) Now it is obvious that this is greatest when $\theta = 0$. That is, when $\boldsymbol{\nabla} w$ and $\mathbf{u}$ are in the same direction.