Identifying Gradient Fields and Exact Differentials

1. Compute the curl of the tangential vector field $\mathbf{F} = \left\langle -\frac{y}{r^2}, \frac{x}{r^2} \right\rangle$.

Answer: We know that if $\mathbf{F} = \langle M, N \rangle$ then $\text{curl}\mathbf{F} = N_x - M_y$. In this case, $M = -\frac{y}{r^2}$ and $N = \frac{x}{r^2}$. Applying the chain rule and differentiating $r^2 = x^2 + y^2$ as needed, we get $N_x = \dfrac{y^2 - x^2}{r^4}$ and $M_y = \dfrac{y^2 - x^2}{r^4}$. Thus, $\text{curl}\mathbf{F} = 0$.

2. Show that $\mathbf{F}$ is not conservative by computing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the unit circle.

Answer: Note: since $\mathbf{F}$ is not defined at $(0, 0)$, $\text{curl}\mathbf{F} = 0$ does not necessarily mean $\mathbf{F}$ is conservative.

We parametrize C by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Then $dx = -\sin t\, dt$ and $dy = \cos t\, dt$.

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C M\, dx + N\, dy \\ &= \int_0^{2\pi} -\frac{\sin t}{1^2}(-\sin t)\, dt + \frac{\cos t}{1^2} \cos t\, dt \\ &= 2\pi \end{aligned}$$

If $\mathbf{F}$ were conservative its line integral over a simple, closed curve (like the unit circle) would be zero. Since this is not the case, $\mathbf{F}$ must not be conservative.

3. Why do you think we refer to $\mathbf{F}$ as a "tangential" vector field?

Answer: Every vector in $\mathbf{F}$ is tangential to some circle centered at the origin. You can see this because $\mathbf{F}$ is clearly orthogonal to the "radial" vector field $\langle x, y \rangle$.

4 In polar coordinates, $\theta(x, y) = \tan^{-1} y/x$. Show that $\mathbf{F} = \nabla\theta$.

Answer: We wish to show that $M = \theta_x$ and $N = \theta_y$.

$$\theta_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{-y}{x^2} = -\frac{y}{r^2} = M.$$

$$\theta_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{x}{r^2} = N.$$