Meaning of Matrix Multiplication

1. In this problem we will show that multiplication by the matrix

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

acts by rotating vectors 45° counterclockwise. As usual, we write the vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ as a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

a) Show that the length of $A\mathbf{v}$ is the same as the length of $\mathbf{v}$.

b) Use the dot product to show the angle between $\mathbf{v}$ and $A\mathbf{v}$ is $\pi/4$ radians.

c) Use the cross product to show $A\mathbf{v}$ is $\pi/4$ radians counterclockwise from $\mathbf{v}$.

Answer: a)

$$A\mathbf{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x-y}{\sqrt{2}} \\ \frac{x+y}{\sqrt{2}} \end{pmatrix}.$$

This has length $\sqrt{\dfrac{(x-y)^2}{2} + \dfrac{(x+y)^2}{2}} = \sqrt{x^2+y^2}$. That is, we have shown $|A\mathbf{v}| = |\mathbf{v}|$ as required.

b) Using the expression for $A\mathbf{v}$ found in part (a) we compute the dot product

$$A\mathbf{v} \cdot \mathbf{v} = \left\langle \frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right\rangle \cdot \langle x, y \rangle = \frac{(x^2+y^2)}{\sqrt{2}}.$$

By part (a) we know $|A\mathbf{v}| = |\mathbf{v}| = \sqrt{x^2+y^2}$. So the cosine of the angle between the two vectors is

$$\frac{A\mathbf{v} \cdot \mathbf{v}}{|A\mathbf{v}||\mathbf{v}|} = \frac{1}{\sqrt{2}} = \cos(\pi/4).$$

c) We compute the cross product

$$\mathbf{v} \times A\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ (x-y)/\sqrt{2} & (x+y)/\sqrt{2} & 0 \end{vmatrix} = \frac{x^2+y^2}{\sqrt{2}}\mathbf{k}.$$

Since the coefficient of $\mathbf{k}$ is positive the right hand rule tells us $A\mathbf{v}$ is counterclockwise from $\mathbf{v}$.