Non-independent Variables

1. We give a worked example here. A fuller explanation will be given in the next session.

Let
$$w = x^3y^2 + x^2y^3 + y$$
and assume x and y satisfy the relation
$$x^2 + y^2 = 1.$$
We consider x to be the independent variable, then, because y depends on x we have w is ultimately a function of the single variable x.

a) Compute $\dfrac{dw}{dx}$ using implicit differentiation.

b) Compute $\dfrac{dw}{dx}$ using total differentials.

Answer:

a) Implicit differentiation means remembering that y is a function of x, e.g., $\dfrac{dy^2}{dx} = 2y\dfrac{dy}{dx}$.

Thus,
$$\frac{dw}{dx} = 3x^2y^2 + 2x^3y\frac{dy}{dx} + 2xy^3 + 3x^2y^2\frac{dy}{dx} + \frac{dy}{dx}.$$
Now we differentiate the constraint to find $\dfrac{dy}{dx}$.
$$x^2 + y^2 = 1 \;\Rightarrow\; 2x + 2y\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = -\frac{x}{y}.$$
Substituting this in the equation for $\dfrac{dw}{dx}$ gives
$$\frac{dw}{dx} = 3x^2y^2 - 2x^3y\frac{x}{y} + 2xy^3 - 3x^2y^2\frac{x}{y} - \frac{x}{y} = 3x^2y^2 - 2x^4 + 2xy^3 - 3x^3y - \frac{x}{y}.$$

b) Taking total differentials of both w and the constraint equation gives
$$\begin{aligned} dw &= 3x^2y^2\,dx + 2x^3y\,dy + 2xy^3\,dx + 3x^2y^2\,dy + dy \\ &= (3x^2y^2 + 2xy^3)\,dx + (2x^3y + 3x^2y^2 + 1)\,dy \end{aligned}$$
$$2x\,dx + 2y\,dy = 0.$$
We can solve the second equation for dy and substitute in the equation for dw.
$$\begin{aligned} dy &= -\frac{x}{y}\,dx \;\Rightarrow \\ dw &= (3x^2y^2 + 2xy^3)\,dx + (2x^3y + 3x^2y^2 + 1)\left(-\frac{x}{y}\right)dx \\ &= (3x^2y^2 - 2x^4 + 2xy^3 - 3x^3y - \frac{x}{y})\,dx \end{aligned}$$
Thus,
$$\frac{dw}{dx} = 3x^2y^2 - 2x^4 + 2xy^3 - 3x^3y - \frac{x}{y}.$$