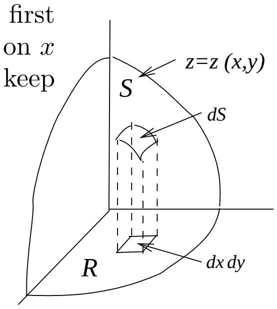
V9.3-4 Surface Integrals

3. Flux through general surfaces.

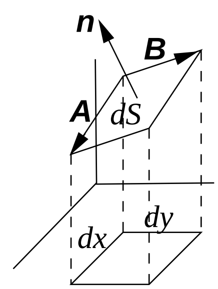
For a general surface, we will use xyz-coordinates. It turns out that here it is simpler to calculate the infinitesimal vector $d\mathbf{S} = \mathbf{n}\, dS$ directly, rather than calculate $\mathbf{n}$ and dS separately and multiply them, as we did in the previous section. Below are the two standard forms for the equation of a surface, and the corresponding expressions for $d\mathbf{S}$. In the first we use z both for the dependent variable and the function which gives its dependence on x and y; you can use $f(x, y)$ for the function if you prefer, but that's one more letter to keep track of.



(11a) $\qquad z = z(x, y), \qquad d\mathbf{S} = (-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \mathbf{k})\, dx\, dy \qquad$ ($\mathbf{n}$ points "up")

(11b) $\qquad F(x, y, z) = c, \qquad d\mathbf{S} = \pm\dfrac{\nabla F}{F_z}\, dx\, dy \qquad$ (choose the right sign);

Derivation of formulas for $d\mathbf{S}$.



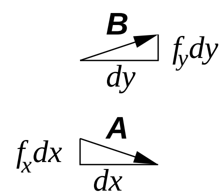
Refer to the pictures at the right. The surface S lies over its projection R, a region in the xy-plane. We divide up R into infinitesimal rectangles having area $dx\, dy$ and sides parallel to the xy-axes — one of these is shown. Over it lies a piece dS of the surface, which is approximately a parallelogram, since its sides are approximately parallel.

The infinitesimal vector $d\mathbf{S} = \mathbf{n}\, dS$ we are looking for has

direction: perpendicular to the surface, in the "up" direction;
magnitude: the area dS of the infinitesimal parallelogram.

This shows our infinitesimal vector is the cross-product

$$d\mathbf{S} = \mathbf{A} \times \mathbf{B}$$



where $\mathbf{A}$ and $\mathbf{B}$ are the two infinitesimal vectors forming adjacent sides of the parallelogram. To calculate these vectors, from the definition of the partial derivative, we have

$\mathbf{A}$ lies over the vector $dx\,\mathbf{i}$ and has slope f_x in the $\mathbf{i}$ direction, so $\mathbf{A} = dx\,\mathbf{i} + f_x\, dx\,\mathbf{k}$;
$\mathbf{B}$ lies over the vector $dy\,\mathbf{j}$ and has slope f_y in the $\mathbf{j}$ direction, so $\mathbf{B} = dy\,\mathbf{j} + f_y\, dy\,\mathbf{k}$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k})\, dx\, dy\ ,$$

which is (11a).

To get (11b) from (11a), , our surface is given by

(12) $$F(x, y, z) = c, \qquad z = z(x, y)$$

where the right-hand equation is the result of solving $F(x, y, z) = c$ for z in terms of the independent variables x and y. We differentiate the left-hand equation in (12) with respect to the independent variables x and y, using the chain rule and remembering that $z = z(x, y)$:

$$F(x, y, z) = c \quad \Rightarrow \quad F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad F_x + F_z\frac{\partial z}{\partial x} = 0$$

from which we get

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \text{and similarly,} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}\ .$$

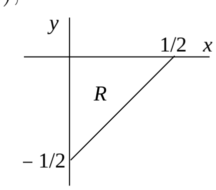
Therefore by (11a),

$$d\mathbf{S} \;=\; \left(-\frac{\partial z}{\partial x}\,\mathbf{i} - \frac{\partial z}{\partial y}\,\mathbf{j} + 1\right) dx\,dy \;=\; \left(\frac{F_x}{F_z}\,\mathbf{i} + \frac{F_y}{F_z}\,\mathbf{j} + 1\right) dx\,dy \;=\; \frac{\nabla F}{F_z}\,dx\,dy\ ,$$

which is (11b).

Example 3. The portion of the plane $2x - 2y + z = 1$ lying in the first octant forms a triangle S. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through S; take the positive side of S as the one where the normal points "up".

Solution. Writing the plane in the form $z = 1 - 2x + 2y$, we get using (11a),



$$\begin{aligned} d\mathbf{S} \;&=\; (2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}\,)\,dx\,dy\ , \qquad \text{so} \\ \iint_S \mathbf{F}\cdot d\mathbf{S} \;&=\; \iint_S (2x - 2y + z)\,dy\,dx \\ &=\; \iint_R \big(2x - 2y + (1 - 2x + 2y)\big)\,dy\,dx\ , \end{aligned}$$

where R is the region in the xy-plane over which S lies. (Note that since the integration is to be in terms of x and y, we had to express z in terms of x and y for this last step.) To see what R is explicitly, the plane intersects the three coordinate axes respectively at $x = 1/2,\ y = -1/2,\ z = 1$. So R is the region pictured; our integral has integrand 1, so its value is the area of R, which is 1/8.

Remark. When we write $z = f(x, y)$ or $z = z(x, y)$, we are agreeing to parametrize our surface using x and y as parameters. Thus the flux integral will be reduced to a double integral over a region R in the xy-plane, involving only x and y. Therefore you must *get rid of* z *by using the relation* $z = z(x, y)$ after you have calculated the flux integral using (11a). Then determine the region R (the projection of S onto the xy-plane), and supply the limits for the iterated integral over R.

Example 4. Set up a double integral in the xy-plane which gives the flux of the field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through that portion of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ lying in the first octant; take $\mathbf{n}$ in the "up" direction.

Solution. Using (11b), we have $d\,\mathbf{S} = \dfrac{\langle 8x, 2y, 8z\rangle}{8z}\,dx\,dy$. Therefore

$$\iint_S \mathbf{F}\cdot d\,\mathbf{S} = \iint_S \frac{8x^2 + 2y^2 + 8z^2}{8z}\,dx\,dy = \iint_S \frac{1}{z}\,dx\,dy = \iint_R \frac{dx\,dy}{\sqrt{1 - x^2 - (y/2)^2}}\ ,$$

where R is the portion of the ellipse $4x^2 + y^2 = 4$ lying the the first quadrant.

The double integral would be most simply evaluated by making the change of variable $u = y/2$, which would convert it to a double integral over a quarter circle in the xu-plane easily evaluated by a change to polar coordinates.

4. General surface integrals.* The surface integral $\iint_S f(x,y,z)\,dS$ that we introduced at the beginning can be used to calculate things other than flux.

a) **Surface area.** We let the function $f(x,y,z)=1$. Then the area of $S = \iint_S dS$.

b) **Mass, moments, charge.** If S is a thin shell of material, of uniform thickness, and with density (in gms/unit area) given by $\delta(x,y,z)$, then

$$\text{mass of } S = \iint_S \delta(x,y,z)\,dS, \tag{13}$$

$$x\text{-component of center of mass} = \overline{x} = \frac{1}{\text{mass } S}\iint_S x\cdot\delta\,dS \tag{14}$$

with the y- and z-components of the center of mass defined similarly. If $\delta(x,y,z)$ represents an electric charge density, then the surface integral (13) will give the total charge on S.

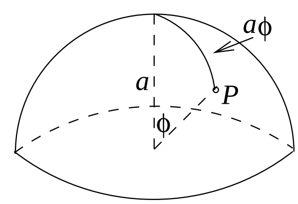
c) **Average value.** The average value of a function $f(x,y,z)$ over the surface S can be calculated by a surface integral:

$$\text{average value of } f \text{ on } S = \frac{1}{\text{area } S}\iint_S f(x,y,z)\,dS\ . \tag{15}$$

Calculating general surface integrals; finding dS.

To evaluate general surface integrals we need to know dS for the surface. For a sphere or cylinder, we can use the methods in section 2 of this chapter.

Example 5. Find the average distance along the earth of the points in the northern hemisphere from the North Pole. (Assume the earth is a sphere of radius a.)



Solution. — We use (15) and spherical coordinates, choosing the coordinates so the North Pole is at $z=a$ on the z-axis. The distance of the point (a,ϕ,θ) from $(a,0,0)$ is $a\phi$, measured along the great circle, i.e., the longitude line — see the picture). We want to find the average of this function over the upper hemisphere S. Integrating, and using (9), we get

$$\iint_S a\phi\,dS = \int_0^{2\pi}\int_0^{\pi/2} a\phi a^2\sin\phi\,d\phi\,d\theta = 2\pi a^3\int_0^{\pi/2}\phi\,\sin\phi\,d\phi = 2\pi a^3\ .$$

(The last integral used integration by parts.) Since the area of $S=2\pi a^2$, we get using (15) the striking answer: average distance $= a$.

For more general surfaces given in xyz-coordinates, since $d\mathbf{S}=\mathbf{n}\,dS$, the area element dS is the magnitude of $d\mathbf{S}$. Using (11a) and (11b), this tells us

$$z=z(x,y), \qquad dS=\sqrt{z_x^2+z_y^2+1}\;dx\,dy \tag{16a}$$

$$F(x,y,z)=c, \qquad dS=\frac{|\nabla F|}{|F_z|}\;dx\,dy \tag{16b}$$

Example 6. The area of the piece S of $z=xy$ lying over the unit circle R in the xy-plane is calculated by (a) above and (16a) to be:

$$\iint_S dS = \iint_R \sqrt{y^2+x^2+1}\,dx\,dy = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\,r\,dr\,d\theta = 2\pi\cdot\frac{1}{3}(r^2+1)^{3/2}\Big]_0^1 = \frac{2\pi}{3}(2\sqrt{2}-1).$$