Proof of Lagrange Multipliers

Here we will give two arguments, one geometric and one analytic for why Lagrange multipliers work.

Critical points

For the function $w = f(x, y, z)$ constrained by $g(x, y, z) = c$ (c a constant) the critical points are defined as those points, which satisfy the constraint and where ∇f is parallel to ∇g. In equations:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad and \quad g(x, y, z) = c.$$

Statement of Lagrange multipliers

For the constrained system local maxima and minima (collectively extrema) occur at the critical points.

Geometric proof for Lagrange

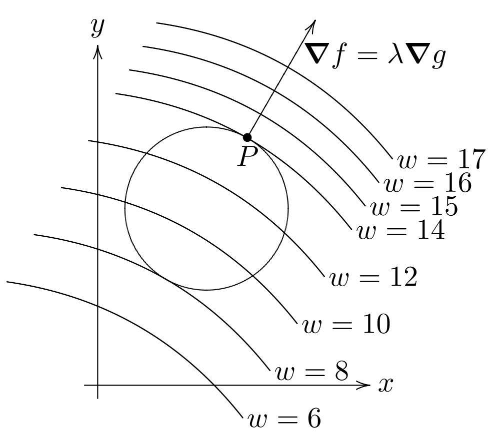
(We only consider the two dimensional case, $w = f(x, y)$ with constraint $g(x, y) = c$.)

For concreteness, we've drawn the constraint curve, $g(x, y) = c$, as a circle and some level curves for $w = f(x, y) = c$ with explicit (made up) values. Geometrically, we are looking for the point on the circle where w takes its maximum or minimum values.

Now, start at the level curve with $w = 17$, which has no points on the circle. So, clearly, the maximum value of w on the constraint circle is less than 17. Move down the level curves until they first touch the circle when $w = 14$. Call the point where the first touch P. It is clear that P gives a local maximum for w on $g = c$, because if you move away from P in either direction on the circle you'll be on a level curve with a smaller value.

Since the circle is a level curve for g, we know ∇g is perpendicular to it. We also know ∇f is perpendicular to the level curve $w = 14$, since the curves themselves are tangent, these two gradients must be parallel.

Likewise, if you keep moving down the level curves, the last one to touch the circle will give a local minimum and the same argument will apply.



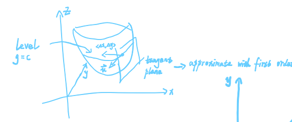
$Z^2 = x^2 + y^2$ $\frac{dz}{dx} = \frac{dz}{dx}\Big|_{y=y_0}$

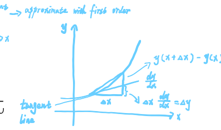
$\frac{\partial Z^2}{\partial x} = 2x$

$\frac{dZ^2}{dZ} \cdot \frac{\partial z}{\partial x} = 2x$

$2Z \cdot \frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial x} = \frac{x}{z}$





Analytic proof for Lagrange (in three dimensions)

Suppose f has a local maximum at P on the constraint surface.

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be an arbitrary parametrized curve which lies on the constraint surface and has $(x(0), y(0), z(0)) = P$. Finally, let $h(t) = f(x(t), y(t), z(t))$. The setup guarantees that $h(t)$ has a maximum at $t = 0$.

$\Delta f = \nabla f \cdot \vec{u}\, \Delta s$ first order $\Delta g = \nabla g \cdot \vec{u}\, \Delta s$

at critical point $\Delta f \approx 0$ at level slice $g = c \Rightarrow \Delta g \approx 0$

$\Rightarrow \nabla f \cdot \vec{u}\, \Delta s \approx 0$ $\Rightarrow \nabla g \cdot \vec{u}\, \Delta s \approx 0$

constrain $g = c$ level

Taking a derivative using the chain rule in vector form gives

$$h'(t) = \nabla f|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t).$$

Since $t = 0$ is a local maximum, we have

$$h'(0) = \nabla f|_P \cdot \mathbf{r}'(0) = 0.$$

$\vec{u} = \langle \Delta x, \Delta y \rangle$

when $\Delta s \to 0$ then $df = \nabla f \cdot \vec{u}\, ds$ and $dg = \nabla g \cdot \vec{u}\, ds$

$\Rightarrow \frac{df}{ds} = \nabla f \cdot \vec{u} = 0$ $\Rightarrow \frac{dg}{ds} = \nabla g \cdot \vec{u} = 0$

Thus, $\nabla f|_P$ is perpendicular to any curve on the constraint surface through P.

This implies $\nabla f|_P$ is perpendicular to the surface. Since $\nabla g|_P$ is also perpendicular to the surface we have proved $\nabla f|_P$ is parallel to $\nabla g|_P$. QED