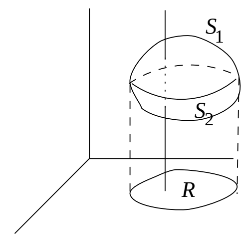
V10.2 The Divergence Theorem

2. Proof of the divergence theorem.

We give an argument assuming first that the vector field $\mathbf{F}$ has only a $\mathbf{k}$-component: $\mathbf{F} = P(x,y,z)\,\mathbf{k}$. The theorem then says

$$\iint_S P\,\mathbf{k}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial P}{\partial z}\,dV\ . \tag{4}$$

The closed surface S projects into a region R in the xy-plane. We assume S is vertically simple, i.e., that each vertical line over the interior of R intersects S just twice. (S can have vertical sides, however — a cylinder would be an example.) S is then described by two equations:



(5) $\quad z = g(x,y)$ *(lower surface);* $\qquad z = h(x,y)$ *(upper surface)*

The strategy of the proof of (4) will be to reduce each side of (4) to a double integral over R; the two double integrals will then turn out to be the same.

We do this first for the triple integral on the right of (4). Evaluating it by iteration, we get as the first step in the iteration,

$$\begin{aligned}\iiint_D \frac{\partial P}{\partial z}\,dV &= \iint_R \int_{g(x,y)}^{h(x,y)} \frac{\partial P}{\partial z}\,dz\,dx\,dy \\ &= \iint_R \big(P(x,y,h) - P(x,y,g)\big)\,dx\,dy \end{aligned} \tag{6}$$

To calculate the surface integral on the left of (4), we use the formula for the surface area element $d\mathbf{S}$ given in V9, (13):

$$d\mathbf{S} \;=\; \pm(-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \vec{k})\,dx\,dy,$$

where we use the $+$ sign if the normal vector to S has a positive k-component, i.e., points generally upwards (as on the upper surface here), and the $-$ sign if it points generally downwards (as it does for the lower surface here).

This gives for the flux of the field $P\,\mathbf{k}$ across the upper surface S_2, on which $z = h(x,y)$,

$$\iint_{S_2} P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R P(x,y,z)\,dx\,dy \;=\; \iint_R P(x,y,h(x,y))\,dx\,dy\ ,$$

while for the flux across the lower surface S_1, where $z = g(x,y)$ and we use the $-$ sign as described above, we get

$$\iint_{S_1} P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R -P(x,y,z)\,dx\,dy \;=\; \iint_R -P(x,y,g(x,y))\,dx\,dy\ ;$$

adding up the two fluxes to get the total flux across S, we have

$$\iint_S P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R P(x,y,h)\,dx\,dy \;-\; \iint_R P(x,y,g)\,dx\,dy$$

which is the same as the double integral in (6). This proves (4). □

In the same way, if $\mathbf{F} = M(x,y,z)\,\mathbf{i}$ and the surface is simple in the $\mathbf{i}$ direction, we can prove

$$(4') \qquad \iint_S M\,\mathbf{i}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial M}{\partial x}\,dV$$

while if $\mathbf{F} = N(x,y,z)\,\mathbf{j}$ and the surface is simple in the $\mathbf{j}$ direction,

$$(4'') \qquad \iint_S N\,\mathbf{j}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial N}{\partial y}\,dV\ .$$

Finally, for a general field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ and a closed surface S which is simple in all three directions, we have only to add up (4), (4′), and (4″). and we get the divergence theorem.

If the domain D is not bounded by a closed surface which is simple in all three directions, it can usually be divided up into smaller domains D_i which are bounded by such surfaces S_i. Adding these up gives the divergence theorem for D and S, since the surface integrals over the new faces introduced by cutting up D each occur twice, with the opposite normal vectors $\mathbf{n}$, so that they cancel out; after addition, one ends up just with the surface integral over the original S.