Cross Product

The cross product is another way of multiplying two vectors. (The name comes from the symbol used to indicate the product.) Because the result of this multiplication is *another vector* it is also called the *vector product*.

As usual, there is an algebraic and a geometric way to describe the cross product. We'll define it algebraically and then move to the geometric description.

Determinant definition for cross product

For the vectors $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$ we define the cross product by the following formula

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
&= (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} \\
&= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle.
\end{aligned}$$

The bottom three equations above are easily seen to be equivalent and should be taken as the definition of the cross product. The top line is technically flawed because we are not really allowed to use vectors as entries in a determinant. Nonetheless is is an excellent way to remember how to compute the cross product.

Example: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \mathbf{k} = -13\,\mathbf{k}$

Example: Compute $\mathbf{i} \times \mathbf{j}$.

<u>**Answer:**</u> $\mathbf{i} = \langle 1, 0, 0 \rangle$ and $\mathbf{j} = \langle 0, 1, 0 \rangle$ therefore

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(1) = \mathbf{k}.$$

Algebraic facts: (these follow easily from properties of determinant).

1. $\mathbf{A} \times \mathbf{A} = \mathbf{0}$
2. Anti-commutivity: $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$
3. Distributive law: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
4. Non-associativity: $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ (example in a moment).

For the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Example: (non-associativity) $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = -\mathbf{i}$ but $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = 0$.

Example: It is possible to compute a cross product using the algebraic facts and the known products of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. For example,

$$(2\mathbf{i}+3\mathbf{j})\times(3\mathbf{i}-2\mathbf{j}) = (6\mathbf{i}\times\mathbf{i}) - (4\mathbf{i}\times\mathbf{j}) + (9\mathbf{j}\times\mathbf{i}) - (6\mathbf{j}\times\mathbf{j}) = -13\mathbf{k}.$$

The first equation follows from the distributive law. In the second, we used $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = 0$ (algebraic fact 1), $\mathbf{i}\times\mathbf{j} = \mathbf{k}$ (computed above) and $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$ (anti-commutivity).

Geometric description

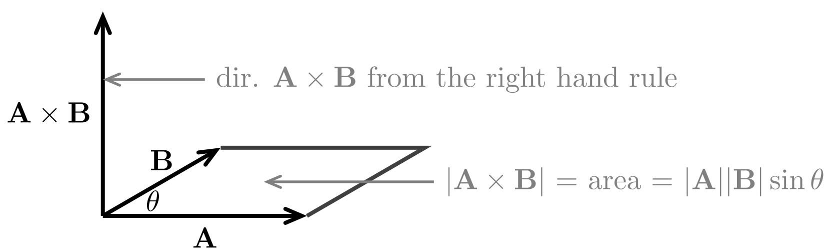
To describe the cross product geometrically we need to describe its magnitude and direction. This is done in the following theorem.

Theorem: The magnitude of $\mathbf{A}\times\mathbf{B}$ is

$$\begin{aligned} |\mathbf{A}\times\mathbf{B}| &= |\mathbf{A}||\mathbf{B}|\sin\theta, \text{ where } \theta \text{ is the angle between them} \\ &= \text{area of the parallelogram spanned by } \mathbf{A} \text{ and } \mathbf{B}. \end{aligned}$$

The direction of $\mathbf{A}\times\mathbf{B}$ is determined as follows.

$\mathbf{A}\times\mathbf{B}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. In the figure below there are two directions perpendicular to the plane –up and down. The choice is made by the *right hand rule.* This rule says to take your right hand and point your fingers in the direction of $\mathbf{A}$ so that they curl towards $\mathbf{B}$; then your thumb points in the direction of $\mathbf{A}\times\mathbf{B}$.

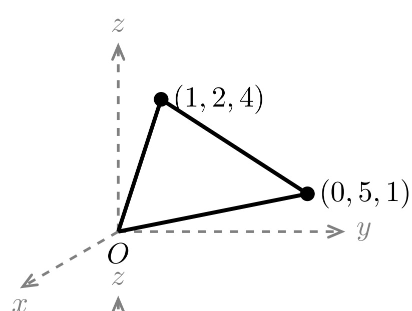


We will not go through the proof of this theorem. It makes use of the Lagrange identity

$$|\mathbf{A}\times\mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A}\cdot\mathbf{B})^2.$$

This identity is easily show by expanding both sides using components.

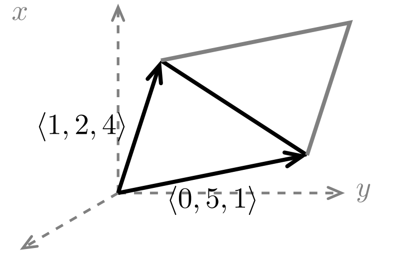
Example: Find the area of the triangle shown.



<u>**Answer:**</u>

The area of the triangle is half the area of the parallelogram (see figure).

So, area triangle $= \dfrac{1}{2}|\langle 1,2,4\rangle \times \langle 0,5,1\rangle|$.



$$\langle 1,2,4\rangle \times \langle 0,5,1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 4 \\ 0 & 5 & 1 \end{vmatrix} = \mathbf{i}(-18) - \mathbf{j} + 5\mathbf{k}.$$

Area triangle $= \dfrac{1}{2}\sqrt{18^2+1^2+5^2} = \dfrac{1}{2}\sqrt{350}.$

DON'T FORGET THE GEOMETRY -it will be used to solve problems.

18.02SC Multivariable Calculus
Fall 2010