

The Tangent approximation

4. Critique of the approximation formula.

First of all, the approximation formula for functions of two or three variables

$$(6) \quad \Delta w \approx \left(\frac{\partial w}{\partial x} \right)_0 \Delta x + \left(\frac{\partial w}{\partial y} \right)_0 \Delta y, \quad \text{if } \Delta x \approx 0, \Delta y \approx 0.$$

$$(7) \quad \Delta w \approx \left(\frac{\partial w}{\partial x} \right)_0 \Delta x + \left(\frac{\partial w}{\partial y} \right)_0 \Delta y + \left(\frac{\partial w}{\partial z} \right)_0 \Delta z, \quad \text{if } \Delta x, \Delta y, \Delta z \approx 0.$$

is not a precise mathematical statement, since the symbol \approx does not specify exactly how close the quantities on either side of the formula are to each other. To fix this up, one would have to specify the error in the approximation. (This can be done, but it is not often used.)

A more fundamental objection is that our discussion of approximations was based on the assumption that the tangent plane is a good approximation to the surface at (x_0, y_0, w_0) . Is this really so?

Look at it this way. The tangent plane was determined as the plane which has the same slope as the surface in the **i** and **j** directions. This means the approximation (6) will be good if you move away from (x_0, y_0) in the **i** direction (by taking $\Delta y = 0$), or in the **j** direction (putting $\Delta x = 0$). **But does the tangent plane have the same slope as the surface in all the other directions as well?**

Intuitively, we should expect that this will be so if the graph of $f(x, y)$ is a “smooth” surface at (x_0, y_0) — it doesn’t have any sharp points, folds, or look peculiar. Here is the mathematical hypothesis which guarantees this.

Smoothness hypothesis. We say $f(x, y)$ is **smooth** at (x_0, y_0) if

$$(8) \quad f_x \text{ and } f_y \text{ are continuous in some rectangle centered at } (x_0, y_0).$$

If (8) holds, the approximation formula (6) will be valid.

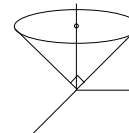
Though pathological examples can be constructed, in general the normal way a function fails to be smooth (and in turn (6) fails to hold) is that one or both partial derivatives fail to exist at (x_0, y_0) . This means of course that you won’t even be able to write the formula (6), unless you’re sleepy. Here is a simple example.

Example 3. Where is $w = \sqrt{x^2 + y^2}$ smooth? Discuss.

Solution. Calculating formally, we get

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

These are continuous at all points except $(0, 0)$, where they are undefined. So the function is smooth except at the origin; the approximation formula (6) should be valid everywhere except at the origin.



Indeed, investigating the graph of this function, since $w = \sqrt{x^2 + y^2}$ says that

height of graph over $(x, y) =$ distance of (x, y) from w -axis,

the graph is a right circular cone, with vertex at $(0, 0)$, axis along the w -axis, and vertex angle a right angle. Geometrically the graph has a sharp point at the origin, so there should be no tangent plane there, and no valid approximation formula (6) — there is no linear function which approximates a cone at its vertex.

A non-geometrical argument for the approximation formula

We promised earlier a non-geometrical approach to the approximation formula (6) that would generalize to higher-dimensions, in particular to the 3-variable formula (7). This approach will also show why the hypothesis (8) of *smoothness* is needed. The argument is still imprecise, since it uses the symbol \approx , but it can be refined to a proof (which you will find in your book, though it's not easy reading).

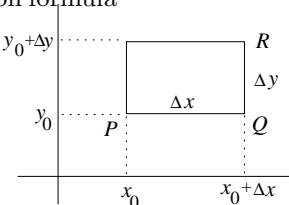
It uses the one-variable approximation formula for a differentiable function $w = f(u)$:

$$(9) \quad \Delta w \approx f'(u_0) \Delta u, \quad \text{if } \Delta u \approx 0.$$

We wish to justify — without using reasoning based on 3-space — the approximation formula

$$(6) \quad \Delta w \approx \left(\frac{\partial w}{\partial x} \right)_0 \Delta x + \left(\frac{\partial w}{\partial y} \right)_0 \Delta y, \quad \text{if } \Delta x \approx 0, \Delta y \approx 0.$$

We are trying to calculate the change in w as we go from P to R in the picture, where $P = (x_0, y_0)$, $R = (x_0 + \Delta x, y_0 + \Delta y)$. This change can be thought of as taking place in two steps:



$$(10) \quad \Delta w = \Delta w_1 + \Delta w_2,$$

the first being the change in w as you move from P to Q , the second the change as you move from Q to R . Using the one-variable approximation formula (9) :

$$(11) \quad \Delta w_1 \approx \left. \frac{d}{dx} f(x, y_0) \right|_{x_0} \cdot \Delta x = f_x(x_0, y_0) \Delta x;$$

similarly,

$$(12) \quad \begin{aligned} \Delta w_2 &\approx \left. \frac{d}{dy} f(x_0 + \Delta x, y) \right|_{y_0} \cdot \Delta y = f_y(x_0 + \Delta x, y_0) \Delta y \\ &\approx f_y(x_0, y_0) \Delta y, \end{aligned}$$

if we assume that f_y is continuous (i.e., f is smooth), since the difference between the two terms on the right in the last two lines will then be like $\epsilon \Delta y$, which is negligible compared with either term itself. Substituting the two approximate values (11) and (12) into (10) gives us the approximation formula (6). \square

To make this a proof, the error terms in the approximations have to be analyzed, or more simply, one has to replace the \approx symbol by equalities based on the Mean-Value Theorem of one-variable calculus.

This argument readily generalizes to the higher-dimensional approximation formulas, such as (7); again the essential hypothesis would be smoothness: the three partial derivatives w_x, w_y, w_z should be continuous in a neighborhood of the point (x_0, y_0, z_0) .

For $f(x)$ continuous
in $a < x < b$:

Mean-Value Theorem : $\frac{f(a) - f(b)}{a - b} = f'(c)$, $a < c < b$

$$\Rightarrow f(a) - f(b) = (a - b) f'(c)$$

$$(a - b) f'_{\min}(c) \leq$$

$$\leq (a - b) f'_{\max}(c)$$

Fundamental Theorem of Calculus:
for $F(x) = f(x)$

$$F(a) - F(b) = \int_b^a f(x) dx$$

$$\int_b^a f_{\min}(x) dx \leq \int_b^a F(x) dx \leq \int_b^a f_{\max}(x) dx$$

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