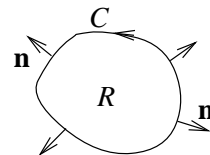


V4.1-2 Green's Theorem in Normal Form

1. Green's theorem for flux.

Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ represent a two-dimensional flow field, and C a simple closed curve, positively oriented, with interior R .



According to the previous section,

$$(1) \quad \text{flux of } \mathbf{F} \text{ across } C = \oint_C M dy - N dx .$$

Notice that since the normal vector points outwards, away from R , the flux is positive where the flow is out of R ; flow into R counts as negative flux.

We now apply Green's theorem to the line integral in (1); first we write the integral in standard form (dx first, then dy):

$$\oint_C M dy - N dx = \oint_C -N dx + M dy = \iint_R (M_x - (-N)_y) dA .$$

This gives us **Green's theorem in the normal form**

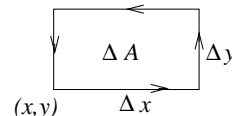
$$(2) \quad \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA .$$

Mathematically this is the same theorem as the tangential form of Green's theorem — all we have done is to juggle the symbols M and N around, changing the sign of one of them. What is different is the physical interpretation. The left side represents the flux of \mathbf{F} across the closed curve C . **What does the right side represent?**

2. The two-dimensional divergence.

Once again, let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. We give a name to and a notation for the integrand of the double integral on the right of (2):

$$(3) \quad \text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}, \quad \text{the } \mathbf{divergence} \text{ of } \mathbf{F} .$$



Evidently $\text{div } \mathbf{F}$ is a scalar function of two variables. To get at its physical meaning, look at the small rectangle pictured. If \mathbf{F} is continuously differentiable, then $\text{div } \mathbf{F}$ is a continuous function, which is therefore approximately constant if the rectangle is small enough. We apply (2) to the rectangle; the double integral is approximated by a product, since the integrand is approximately constant:

$$(4) \quad \text{flux across sides of rectangle} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta A , \quad \Delta A = \text{area of rectangle} .$$

Because of the importance of this approximate relation, we give a more direct derivation of it which doesn't use Green's theorem. The reasoning which follows is widely used in mathematical modeling of physical problems.

Consider the small rectangle shown; we calculate approximately the flux over each side.

$$\text{flux across top} \approx (\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j}) \Delta x = N(x, y + \Delta y) \Delta x$$

$$\text{flux across bottom} \approx (\mathbf{F}(x, y) \cdot -\mathbf{j}) \Delta x = -N(x, y) \Delta x ;$$

adding these up,

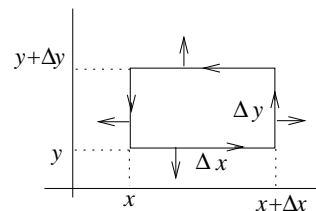
$$\begin{aligned} \text{total flux across} \\ \text{top and bottom} \end{aligned} \approx (N(x, y + \Delta y) - N(x, y)) \Delta x \approx \left(\frac{\partial N}{\partial y} \Delta y \right) \Delta x.$$

By similar reasoning applied to the two sides,

$$\begin{aligned} \text{total flux across} \\ \text{left and right sides} \end{aligned} \approx \left(M(x + \Delta x, y) - M(x, y) \right) \Delta y \approx \left(\frac{\partial M}{\partial x} \Delta x \right) \Delta y.$$

Adding up the flux over the four sides, we get (4) again:

$$\begin{aligned} \text{total flux over four} \\ \text{sides of the rectangle} \end{aligned} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y.$$



Approximate Dividibly

Continuing our search for a physical meaning for the divergence, if the total flux over the sides of the small rectangle is positive, this means there is a net flow *out* of the rectangle. According to conservation of matter, the only way this can happen is if there is a *source* adding fluid directly to the rectangle. If the flow is taking place in a shallow tank of uniform depth, such a source can be visualized as someone standing over the tank, pouring fluid directly into the rectangle. Similarly, a net flow *into* the rectangle implies there is a *sink* withdrawing fluid from the rectangle. It is best to think of such a sink as a “negative source”. The net rate (positive or negative) at which fluid is added directly to the rectangle from above may be called the “source rate” for the rectangle. Thus, since matter is conserved,

$$\text{flux over sides of rectangle} = \text{source rate for the rectangle};$$

combining this with (4) shows that

$$(5) \quad \text{source rate for the rectangle} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta A.$$

We now divide by ΔA and pass to the limit, getting by definition

$$(6) \quad \text{the source rate at } (x, y) = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) = \text{div } \mathbf{F}.$$

The definition of the double integral as the limit of a sum shows in the usual way now that

$$(7) \quad \text{source rate for } R = \iint_R \text{div } \mathbf{F} dA. \quad \text{Summation of source rate or flux}$$

These two relations (6) and (7) interpret the divergence physically, for a flow field, and they interpret also Green's theorem in the normal form:

$$\begin{aligned} \text{total flux across } C &= \text{source rate for } R \\ \oint_C M dy - N dx &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA \end{aligned}$$

Since Green's theorem is a mathematical theorem, one might think we have "proved" the law of conservation of matter. This is not so, since this law was needed for our interpretation of $\operatorname{div} \mathbf{F}$ as the source rate at (x, y) .

We give side-by-side the two forms of Green's theorem, first in the vector form, then in the differential form used when calculations are to be done.

Tangential form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, dA$$

$$\oint_C M \, dx + N \, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

work by \mathbf{F}
around C

Normal form

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \operatorname{div} \mathbf{F} \, dA$$

$$\oint_C M \, dy - N \, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

flux of \mathbf{F}
across C

source rate
for R

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