Matrices 2. Solving Square Systems of Linear Equations; Inverse Matrices

Solving square systems of linear equations; inverse matrices.

Linear algebra is essentially about solving systems of linear equations, an important application of mathematics to real-world problems in engineering, business, and science, especially the social sciences. Here we will just stick to the most important case, where the system is *square*, i.e., there are as many variables as there are equations. In low dimensions such systems look as follows (we give a 2×2 system and a 3×3 system):

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \tag{7}$$

In these systems, the a_{ij} and b_i are given, and we want to solve for the x_i.

As a simple mathematical example, consider the linear change of coordinates given by the equations

$$\begin{aligned} x_1 &= a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ x_2 &= a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \\ x_3 &= a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \end{aligned}$$

If we know the y-coordinates of a point, then these equations tell us its x-coordinates immediately. But if instead we are given the x-coordinates, to find the y-coordinates we must solve a system of equations like (7) above, with the y_i as the unknowns.

Using matrix multiplication, we can abbreviate the system on the right in (7) by

$$A\mathbf{x} = \mathbf{b}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{8}$$

where A is the square matrix of coefficients $(\, a_{ij}\,)$. (The 2×2 system and the $n \times n$ system would be written analogously; all of them are abbreviated by the same equation $A\mathbf{x} = \mathbf{b}$, notice.)

You have had experience with solving small systems like (7) by *elimination*: multiplying the equations by constants and subtracting them from each other, the purpose being to eliminate all the variables but one. When elimination is done systematically, it is an efficient method. Here however we want to talk about another method more compatible with hand-held calculators and MatLab, and which leads more rapidly to certain key ideas and results in linear algebra.

Inverse matrices.

Referring to the system (8), suppose we can find a square matrix M, the same size as A, such that

$$MA = I \qquad \text{(the identity matrix).} \tag{9}$$

We can then solve (8) by matrix multiplication, using the successive steps,

$$
\begin{aligned}
A\,\mathbf{x} &= \mathbf{b} \\
M(A\,\mathbf{x}) &= M\,\mathbf{b} \\
\mathbf{x} &= M\,\mathbf{b};
\end{aligned}
\tag{10}
$$

where the step $M(A\mathbf{x}) = \mathbf{x}$ is justified by

$$
\begin{aligned}
M(A\,\mathbf{x}) &= (MA)\mathbf{x}, && \text{by associative law;} \\
&= I\,\mathbf{x}, && \text{by (9);} \\
&= \mathbf{x}, && \text{because } I \text{ is the identity matrix .}
\end{aligned}
$$

Moreover, the solution is unique, since (10) gives an explicit formula for it.

The same procedure solves the problem of determining the inverse to the linear change of coordinates $\mathbf{x} = A\mathbf{y}$, as the next example illustrates.

Example 2.1 Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $M = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$. Verify that M satisfies (9) above, and use it to solve the first system below for x_i and the second for the y_i in terms of the x_i:

$$
\begin{aligned}
x_1 + 2x_2 &= -1 \qquad & x_1 &= y_1 + 2y_2 \\
2x_1 + 3x_2 &= 4 \qquad & x_2 &= 2y_1 + 3y_2
\end{aligned}
$$

Solution. We have $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, by matrix multiplication. To solve the first system, we have by (10), $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -6 \end{pmatrix}$, so the solution is $x_1 = 11, x_2 = -6$. By reasoning similar to that used above in going from $A\mathbf{x} = \mathbf{b}$ to $\mathbf{x} = M\mathbf{b}$, the solution to $\mathbf{x} = A\mathbf{y}$ is $\mathbf{y} = M\mathbf{x}$, so that we get

$$
\begin{aligned}
y_1 &= -3x_1 + 2x_2 \\
y_2 &= 2x_1 - x_2
\end{aligned}
$$

as the expression for the y_i in terms of the x_i.

Our problem now is: how do we get the matrix M? In practice, you mostly press a key on the calculator, or type a Matlab command. But we need to be able to work abstractly with the matrix — i.e., with symbols, not just numbers, and for this some theoretical ideas are important. The first is that M doesn't always exist.

$$
M \text{ exists} \quad \Leftrightarrow \quad |A| \neq 0.
$$

The implication $\Rightarrow$ follows immediately from the law **M-5** in section M.1 ($\det(AB) = \det(A)\det(B)$) , since

$$
MA = I \quad \Rightarrow \quad |M||A| = |I| = 1 \quad \Rightarrow \quad |A| \neq 0.
$$

The implication in the other direction requires more; for the low-dimensional cases, we will produce a formula for M. Let's go to the formal definition first, and give M its proper name, A^{-1}:

Definition. Let A be an $n \times n$ matrix, with $|A| \neq 0$. Then the **inverse** of A is an $n \times n$ matrix, written A^{-1}, such that

$$A^{-1}A = I_n, \qquad AA^{-1} = I_n \tag{11}$$

(It is actually enough to verify either equation; the other follows automatically — see the exercises.)

Using the above notation, our previous reasoning (9) - (10) shows that

$$|A| \neq 0 \quad \Rightarrow \quad \text{the unique solution of } \ A\,\mathbf{x} = \mathbf{b} \ \text{ is } \ \mathbf{x} = A^{-1}\mathbf{b}; \tag{12}$$

$$|A| \neq 0 \quad \Rightarrow \quad \text{the solution of } \ \mathbf{x} = A\,\mathbf{y} \ \text{ for the } y_i \text{ is } \ \mathbf{y} = A^{-1}\mathbf{x}. \tag{12}$$

Calculating the inverse of a 3×3 matrix

Let A be the matrix. The formulas for its **inverse** A^{-1} and for an auxiliary matrix $\operatorname{adj} A$ called the **adjoint** of A (or in some books the **adjugate** of A) are

$$A^{-1} = \frac{1}{|A|}\operatorname{adj} A = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T. \tag{13}$$

In the formula, A_{ij} is the cofactor of the element a_{ij} in the matrix, i.e., its minor with its sign changed by the checkerboard rule (see section 1 on determinants).

Formula (13) shows that the steps in calculating the inverse matrix are:

1. Calculate the matrix of minors.
2. Change the signs of the entries according to the checkerboard rule. → Cofactors
3. Transpose the resulting matrix; this gives $\operatorname{adj} A$.
4. Divide every entry by $|A|$.

(If inconvenient, for example if it would produce a matrix having fractions for every entry, you can just leave the $1/|A|$ factor outside, as in the formula. Note that step 4 can only be taken if $|A| \neq 0$, so if you haven't checked this before, you'll be reminded of it now.)

> The notation A_{ij} for a cofactor makes it look like a matrix, rather than a signed determinant; this isn't good, but we can live with it.

Example 2.2 Find the inverse to $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

Solution. We calculate that $|A| = 2$. Then the steps are (T means transpose):

$$\underset{\text{matrix } A}{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \to \underset{\text{cofactor matrix}}{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}} \underset{\text{T}}{\to} \underset{\text{adj } A}{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}} \to \underset{\text{inverse of } A}{\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}}$$

To get practice in matrix multiplication, check that $A \cdot A^{-1} = I$, or to avoid the fractions, check that $A \cdot \text{adj}\ (A) = 2I$.

The same procedure works for calculating the inverse of a 2×2 matrix A. We do it for a general matrix, since it will save you time in differential equations if you can learn the resulting formula.

$$\underset{\text{matrix } A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \to \underset{\text{cofactors}}{\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}} \underset{\text{T}}{\to} \underset{\text{adj } A}{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} \to \underset{\text{inverse of } A}{\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Example 2.3 Find the inverses to: a) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

Solution. a) Use the formula: $|A| = 2$, so $A^{-1} = \dfrac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

b) Follow the previous scheme:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} -5 & -3 & 7 \\ 2 & 0 & -1 \\ 4 & 3 & -5 \end{pmatrix} \to \begin{pmatrix} -5 & 2 & 4 \\ -3 & 0 & 3 \\ 7 & -1 & -5 \end{pmatrix} \to \frac{1}{3}\begin{pmatrix} -5 & 2 & 4 \\ -3 & 0 & 3 \\ 7 & -1 & -5 \end{pmatrix} = A^{-1}.$$

Both solutions should be checked by multiplying the answer by the respective A.

Proof of formula (13) for the inverse matrix.

We want to show $A \cdot A^{-1} = I$, or equivalently, $A \cdot \text{adj}\ A = |A|I$; when this last is written out using (13) (remembering to transpose the matrix on the right there), it becomes

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}. \tag{14}$$

To prove (14), it will be enough to look at two typical entries in the matrix on the right — say the first two in the top row. According to the rule for multiplying the two matrices on the left, what we have to show is that

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|; \tag{15}$$

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0 \tag{16}$$

These two equations are both evaluating determinants by Laplace expansions: the first equation (15) evaluates the determinant on the left below by the cofactors of the first row; the second equation (16) evaluates the determinant on the right below by the cofactors of the second row (notice that the cofactors of the second row don't care what's actually in the second row, since to calculate them you only need to know the other two rows).

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The two equations (15) and (16) now follow, since the determinant on the left is just $|A|$, while the determinant on the right is 0, since two of its rows are the same. $\square$

The procedure we have given for calculating an inverse works for $n \times n$ matrices, but gets to be too cumbersome if $n > 3$, and other methods are used. The calculation of A^{-1} for reasonable-sized A is a standard package in computer algebra programs and MatLab. Unfortunately, social scientists often want the inverses of very large matrices, and for this special techniques have had to be devised, which produce approximate but acceptable results.