The gradient is perpendicular to the level curves

1. Here is a challenging problem. Use the chain rule to show the slope of the gradient is the negative reciprocal of the slope of the level curves. (This is another way of saying the gradient is perpendicular to the level curves.)

Note, this problem is strictly about 2D functions $w = f(x, y)$ and their gradients and level curves. Also note, for a 2D vector $\langle a, b\rangle$ the slope is b/a.

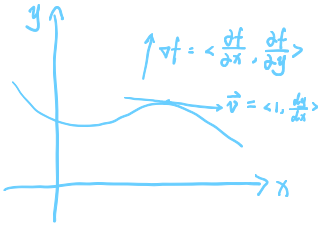
Answer: Suppose $w = f(x, y)$ and we have a level curve $f(x, y) = c$. Implicitly this gives a relation between x and y, which means y can be thought of as a function of x, say $y = y(x)$. We then rewrite the equation of the level curve as

$$f(x, y(x)) = c.$$

The chain rule gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

This last expression is the slope of the level curve.



Now, $\boldsymbol{\nabla} f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$, which has slope

$$\frac{\partial f/\partial y}{\partial f/\partial x}.$$

$$\nabla f \cdot \vec{v} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \left(- \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \right) = 0$$

This slope is clearly the negative reciprocal of the slope of the level curve computed above.

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