Changing Variables in Multiple Integrals

1. Changing variables.

Double integrals in x, y coordinates which are taken over circular regions, or have integrands involving the combination $x^2 + y^2$, are often better done in polar coordinates:

$$(1) \qquad \iint_R f(x,y)\, dA = \iint_R g(r,\theta)\, r\, dr d\theta \ .$$

This involves introducing the new variables r and θ, together with the equations relating them to x, y in both the forward and backward directions:

$$(2) \qquad r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}(y/x); \qquad\qquad x = r\cos\theta, \qquad y = r\sin\theta \ .$$

Changing the integral to polar coordinates then requires three steps:

A. Changing the integrand $f(x,y)$ to $g(r,\theta)$, by using (2);

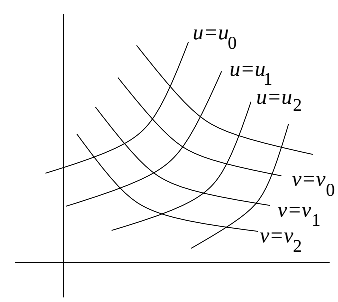
B. Supplying the area element in the r,θ system: $dA = \ r\, dr\, d\theta$;

C. Using the region R to determine the limits of integration in the r,θ system.

In the same way, double integrals involving other types of regions or integrands can sometimes be simplified by changing the coordinate system from x, y to one better adapted to the region or integrand. Let's call the new coordinates u and v; then there will be equations introducing the new coordinates, going in both directions:

$$(3) \qquad u = u(x,y), \qquad v = v(x,y); \qquad\qquad x = x(u,v), \qquad y = y(u,v)$$

(often one will only get or use the equations in one of these directions). To change the integral to u, v-coordinates, we then have to carry out the three steps **A**, **B**, **C** above. A first step is to picture the new coordinate system; for this we use the same idea as for polar coordinates, namely, we consider the grid formed by the level curves of the new coordinate functions:



$$(4) \qquad u(x,y) = u_0, \qquad v(x,y) = v_0 \ .$$

Once we have this, algebraic and geometric intuition will usually handle steps **A** and **C**, but for **B** we will need a formula: it uses a determinant called the **Jacobian**, whose notation and definition are

$$(5) \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \ .$$

Using it, the formula for the area element in the u, v-system is

$$(6) \qquad dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \ ,$$

so the change of variable formula is

$$\iint_R f(x,y)\,dx\,dy \;=\; \iint_R g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\ , \tag{7}$$

where $g(u,v)$ is obtained from $f(x,y)$ by substitution, using the equations (3).

We will derive the formula (5) for the new area element in the next section; for now let's check that it works for polar coordinates.

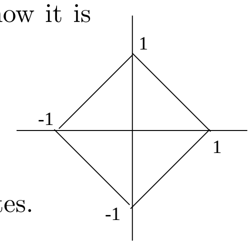
Example 1. Verify (1) using the general formulas (5) and (6).

Solution. Using (2), we calculate:

$$\frac{\partial(x,y)}{\partial(r,\theta)} \;=\; \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} \;=\; \begin{vmatrix} \cos\theta & -r\,\sin\theta \\ \sin\theta & r\,\cos\theta \end{vmatrix} \;=\; r(\cos^2\theta + \sin^2\theta) \;=\; r\ ,$$

so that $dA \;=\; r\,dr\,d\theta$, according to (5) and (6); note that we can omit the absolute value, since by convention, in integration problems we always assume $r \ge 0$, as is implied already by the equations (2).

We now work an example illustrating why the general formula is needed and how it is used; it illustrates step **C** also — putting in the new limits of integration.



Example 2. Evaluate $\displaystyle\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy$ over the region R pictured.

Solution. This would be a painful integral to work out in rectangular coordinates. But the region is bounded by the lines

$$x + y = \pm 1, \qquad x - y = \pm 1 \tag{8}$$

and the integrand also contains the combinations $x-y$ and $x+y$. These powerfully suggest that the integral will be simplified by the change of variable (we give it also in the inverse direction, by solving the first pair of equations for x and y):

$$u = x + y, \qquad v = x - y; \qquad\qquad x = \frac{u+v}{2}, \qquad y = \frac{u-v}{2}\ . \tag{9}$$

We will also need the new area element; using (5) and (9) above. we get

$$\frac{\partial(x,y)}{\partial(u,v)} \;=\; \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} \;=\; -\frac{1}{2}\ ; \tag{10}$$

note that it is the second pair of equations in (9) that were used, not the ones introducing u and v. Thus the new area element is (this time we do need the absolute value sign in (6))

$$dA \;=\; \frac{1}{2}\,du\,dv\ . \tag{11}$$

We now combine steps **A** and **B** to get the new double integral; substituting into the integrand by using the first pair of equations in (9), we get

$$\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy \;=\; \iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\,du\,dv\ . \tag{12}$$

In uv-coordinates, the boundaries (8) of the region are simply $u = \pm 1$, $v = \pm 1$, so the integral (12) becomes

$$\iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv \;=\; \int_{-1}^{1}\int_{-1}^{1} \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv$$

We have

$$\text{inner integral} \;=\; -\frac{v^2}{2(u+2)}\Big]_{u=-1}^{u=1} \;=\; \frac{v^2}{3}\;; \qquad \text{outer integral} \;=\; \frac{v^3}{9}\Big]_{-1}^{1} \;=\; \frac{2}{9}\;.$$