

Example

Let's apply what we just learned to a specific example. First, recall the basics. For the real homogeneous constant coefficient linear DE with sinusoidal input

$$p(D)x = B \cos(\omega t)$$

we have the unique real periodic solution

$$x_p = B \operatorname{Re} \left(\frac{e^{i\omega t}}{p(i\omega)} \right) = \frac{B}{|p(i\omega)|} \cos(\omega t - \phi)$$

where $\phi = \operatorname{Arg}(p(i\omega))$. In this case the complex gain is $\frac{1}{p(i\omega)}$, and the phase lag is $\phi = \operatorname{Arg}(p(i\omega))$.

Example. Find the periodic solution to

$$x'' + x' + 2x = \cos t.$$

Solution. $p(s) = s^2 + s + 2$, $\omega = 1$, $B = 1$.

$$p(i\omega) = p(i) = i^2 + i + 2 = -1 + i + 2 = 1 + i \quad \left[1 + i | e^{i\phi} = \sqrt{2} e^{i\frac{\pi}{4}} \right],$$

since $\phi = \operatorname{Arg}(1 + i) = \tan^{-1}(1/1) = \frac{\pi}{4}$.

$$\text{Thus, Complex gain} = \frac{1}{p(i)} = \frac{1}{1 + i}.$$

$$\text{Gain} = \frac{1}{|p(i)|} = \frac{1}{\sqrt{2}}.$$

$$\text{Phase lag} = \phi = \operatorname{Arg}(p(i)) = \frac{\pi}{4}.$$

$$\text{Periodic solution} = x_p = \frac{1}{\sqrt{2}} \cos\left(t - \frac{\pi}{4}\right).$$

Looking at the output x_p in relation to the input signal we see $q(t) = \cos t$.

The amplitude of $x_p = \frac{1}{\sqrt{2}} \times \text{amplitude of } q$ so the gain is $\frac{1}{\sqrt{2}}$. We also see that x_p lags *behind* q by $\pi/4$ radians, so the phase lag $\phi = \frac{\pi}{4}$.

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