# Domain of $F(s)$

## 1. Complex $s$ and region of convergence

We will allow $s$ to be complex, using as needed the properties of the complex exponential we learned in unit 1.

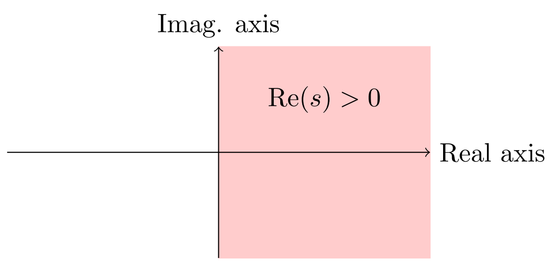
**Example 1.** In the previous note we saw that $\mathcal{L}(1) = 1/s$, valid for all $s > 0$. Let's recompute $\mathcal{L}(1)(s)$ for complex $s$. Let $s = \alpha + i\beta$.

$$\begin{aligned} \mathcal{L}(1)(s) &= \int_0^\infty e^{-st}\,dt \\ &= \lim_{T\to\infty} \left.\frac{e^{-st}}{-s}\right]_0^T \\ &= \lim_{T\to\infty} \left.\frac{e^{-\alpha t}(\cos(\beta t) + i\sin(\beta t))}{-s}\right]_0^T \end{aligned}$$

This converges if $\alpha > 0$ and diverges if $\alpha < 0$. Since $\alpha = \text{Re}(s)$ we have

$$\mathcal{L}(1) = 1/s, \quad \text{for } \text{Re}(s) > 0.$$

The region $\text{Re}(s) > 0$ is called the **region of convergence** of the transform. It is a right half-plane.




Region of convergence: right half-plane $\text{Re}(s) > 0$.

**Frequency:** The Laplace transform variable $s$ can be thought of as complex frequency. It will take us a while to understand this, but we can begin here. Euler's formula says $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and we call $\omega$ the angular frequency. By analogy for any complex number exponent we call $s$ the **complex frequency** in $e^{st}$. If $s = a + i\omega$ then $s$ is the complex frequency and its imaginary part $\omega$ is an actual frequency of a sinusoidal oscillation.

## 2. Piecewise continuous functions and functions of exponential order

If the integral fails to converge for any $s$ then the function does not have a Laplace transform.

$\int_0^\infty e^{t^2} e^{-st}\,dt$
$= \int_0^\infty e^{t(t-s)}\,dt$

**Example.** It is easy to see that $f(t) = e^{t^2}$ has no Laplace transform.

The problem is the $e^{t^2}$ grows too fast as $t$ gets large. Fortunately, all of the functions we are interested in do have Laplace transforms valid for $\text{Re}(s) > a$ for some value $a$.

$t-s>0$ must happen as $t$ grows
$\Rightarrow \lim_{t\to\infty} e^{t(t-s)} \to \infty$ not converge

**Functions of Exponential Order**
The class of functions that do have Laplace transforms are those of *exponential order*. Fortunately for us, all the functions we use in 18.03 are of this type.

A function is said to be of **exponential order** if there are numbers $a$ and $M$ such that $|f(t)| < Me^{at}$. In this case, we say that $f$ has exponential order $a$.

order 1

**Examples.** 1, $\cos(\omega t)$, $\sin(\omega t)$, $t^n$ all have exponential order 0.
$e^{at}$ has exponential order $a$.

A function $f(t)$ is **piecewise continuous** if it is continuous everywhere except at a finite number of points in any finite interval and if at thesepoints it has a jump discontinuity (i.e. a jump of finite height).

**Example.** The square wave is piecewise continuous.

**Theorem:** If $f(t)$ is piecewise continuous and of exponential order $a$ then the Laplace transform $\mathcal{L}f(s)$ converges for all $s$ with $\text{Re}(s) > a$.

$\text{Re}(s) > a$
$\Rightarrow \text{Re}(s) = a+\alpha$ with $\alpha > 0$
$\Rightarrow s = (a+\alpha) + ib$

**Proof:** Suppose $\text{Re}(s) > a$ and $|f(t)| < Me^{at}$. Then we can write $s = (a+\alpha) + ib$, where $\alpha > 0$. Then, since $|e^{-ibt}| = 1$,

$$|f(t)e^{-st}| = |f(t)e^{-(a+\alpha)t}e^{-ibt}| = |f(t)e^{-(a+\alpha)t)}| < Me^{-\alpha t},$$

Since $\int_0^\infty Me^{-\alpha t}\,dt$ converges for $\alpha > 0$, the Laplace transform integral also converges.

**Domain of** $F(s)$**:** For $f(t)$ we have $F(s) = 1/s$ with region of convergence $\text{Re}(s) > 0$. But, the function $1/s$ is well defined for all $s \neq 0$. The process of extending the domain of $F(s)$ from the region of convergence is called *analytic continuation*. In this class analytic continuation will always consist of extending $F(s)$ to the complex plane minus the zeros of the denominator. ?

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