Poles and Amplitude Response

We started the session by considering the poles of functions $F(s)$, and saw that, by definition, the graph of $|F(s)|$ went off to infinity at the poles. Since it tells us where $|F(s)|$ is infinite, the pole diagram provides a crude graph of $|F(s)|$: roughly speaking, $|F(s)|$ will be large for values of s near the poles. In this note we show how this basic fact provides a useful graphical tool for spotting resonant or near-resonant frequencies for LTI systems.

Example 1. Figure 1 shows the pole diagram of a function $F(s)$. At which of the points A, B, C on the diagram would you guess $|F(s)|$ is largest?

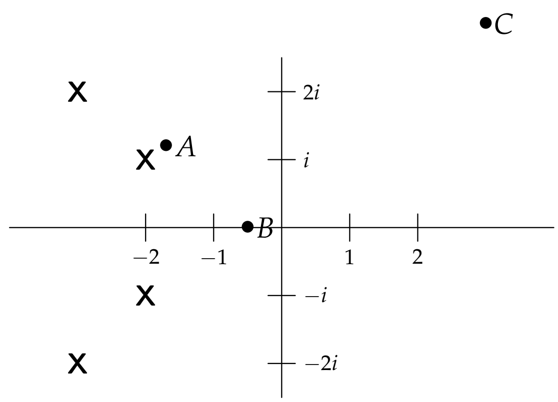

Figure 2: Pole diagram for example 1.

Solution. Point A is close to a pole and B and C are both far from poles so we would guess point $|F(s)|$ is largest at point A.

Example 2. The pole diagram of a function $F(s)$ is shown in Figure 2. At what point s on the positive imaginary axis would you guess that $|F(s)|$ is largest?

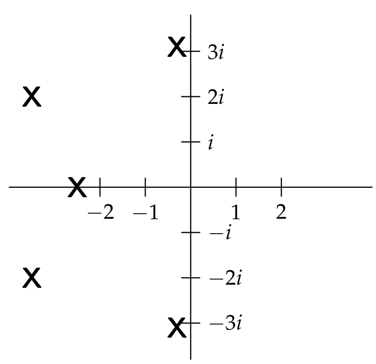

Figure 2: Pole diagram for example 2.

Solution. We would guess that s should be close to $3\,i$, which is near a pole. There is not enough information in the pole diagram to determine the exact location of the maximum, but it is most likely to be near the pole.

1. Amplitude Response and the System Function

Consider the system

$$p(D)x = f(t). \tag{1}$$

where we take $f(t)$ to be the input and $x(t)$ to be the output. The transfer function of this system is

$$W(s) = \frac{1}{p(s)}. \tag{2}$$

When $f(t) = B\cos(\omega t)$ the Exponential Response Formula from unit 2 gives the following periodic solution to (1)

$$x_p(t) = \frac{B\cos(\omega t - \phi)}{|p(i\omega)|}, \quad \text{where } \phi = \text{Arg}(p(i\omega)). \tag{3}$$

If the system is stable, then all solutions are asymptotic to the periodic solution in (3). In this case, we saw in the session on Frequency Response in unit 2 that the amplitude response of the system as a function of ω is

$$g(\omega) = \frac{1}{|p(i\omega)|.} \tag{4}$$

Comparing (2) and (4), we see that for a stable system the amplitude response is related to the transfer function by

$$g(\omega) = |W(i\omega)|. \tag{5}$$

Note: The relation (5) holds for all stable LTI systems.

Using equation (5) and the language of amplitude response we will now re-do example 2 to illustrate how to use the pole diagram to estimate the practical resonant frequencies of a stable system.

Example 3. Figure 3 shows the pole diagram of a stable LTI system. At approximately what frequency will the system have the biggest response?

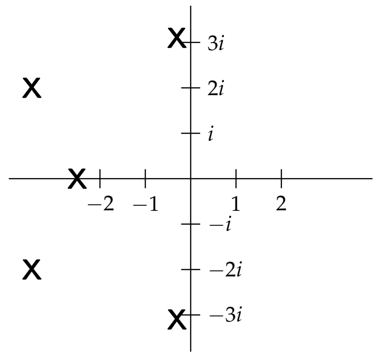

Figure 3: Pole diagram for example 3 (same as Figure 2).

Solution. Let the transfer function be $W(s)$. Equation (5) says the amplitude response $g(\omega) = |W(i\omega)|$. Since $i\omega$ is on the positive imaginary axis, the amplitude response $g(\omega)$ will be largest at the point $i\omega$ on the imaginary axis where $|W(i\omega)|$ is largest. This is exactly the point found in example 2. Thus, we choose $i\omega \approx 3i$, i.e. the practical resonant frequency is approximately $\omega = 3$.

Note: Rephrasing this in graphical terms: we can graph the magnitude of the system function $|W(s)|$ as a surface over the s-plane. The amplitude response of the system $g(\omega) = |W(i\omega)|$ is given by the part of the system function graph that lies above the imaginary axis. This is all illustrated beautifully by the applet `Amplitude: Pole Diagram` explored in the next note in this session.