Example

Let's apply what we just learned to a specific example. First, recall the basics. For the real homogeneous constant coefficient linear DE with sinusoidal input

$$p(D)x = B\cos(\omega t)$$

we have the unique real periodic solution

$$x_p = B\,\mathrm{Re}\left(\frac{e^{i\omega t}}{p(i\omega)}\right) = \frac{B}{|p(i\omega)|}\cos(\omega t - \phi)$$

where $\phi = \mathrm{Arg}(p(i\omega))$. In this case the complex gain is $\frac{1}{p(i\omega)}$, and the phase lag is $\phi = \mathrm{Arg}(p(i\omega))$.

Example. Find the periodic solution to

$$x'' + x' + 2x = \cos t.$$

Solution. $p(s) = s^2 + s + 2,\ \omega = 1,\ B = 1.$
$p(i\omega) = p(i) = i^2 + i + 2 = -1 + i + 2 = 1 + i\,\left[|1+i|e^{i\phi} = \sqrt{2}e^{\frac{i\pi}{4}}\right],$
since $\phi = \mathrm{Arg}(1+i) = \tan^{-1}(1/1) = \dfrac{\pi}{4}.$
Thus, Complex gain $= \dfrac{1}{p(i)} = \dfrac{1}{1+i}.$
Gain $= \dfrac{1}{|p(i)|} = \dfrac{1}{\sqrt{2}}.$
Phase lag $= \phi = \mathrm{Arg}(p(i)) = \dfrac{\pi}{4}.$
Periodic solution $= x_p = \dfrac{1}{\sqrt{2}}\cos(t - \dfrac{\pi}{4}).$

Looking at the output x_p in relation to the input signal we see $q(t) = \cos t$. The amplitude of $x_p = \frac{1}{\sqrt{2}} \times$ amplitude of q so the gain is $\frac{1}{\sqrt{2}}$. We also see that x_p lags *behind* q by $\pi/4$ radians, so the phase lag $\phi = \frac{\pi}{4}$.