Part I Problems and Solutions

Problem 1: *Find a particular solution to the DE*

$$\ddot{x} + x = t^2 + \cos(2t - 1)$$

Solution: $x_p = x_{p,1} + x_{p,2}$ where $p(D)x_{p,1} = t^2$ and $p(D)x_{p,2} = \cos(2t-1)$, by superposition. Here $p(D) = D^2 + 1$.

$x_{p,1}$: try a solution of form $x_{p,1} = At^2 + Bt + C$. $p(D)x_{p,1} = \ddot{x}_{p,1} + x_{p,1} = 2A + (At^2 + Bt + C) = t^2 \to A = 1, B = 0, C = -2$. Thus, $x_{p,1} = t^2 - 2$.

$x_{p,2}$: try solution of the form $x_{p,2} = A\cos(2t-1) + B\sin(2t-1)$. Then $p(D)x_{p,2} = \ddot{x}_{p,2} + x_{p,2} = \cos(2t-1) \to$
$(-4A\cos(2t-1) - 4B\sin(2t-1) + A\cos(2t-1) + B\sin(2t-1)) =$
$\cos(2t-1) \to A = -\frac{1}{3}$, $B = 0$

Thus, $x_{p,2} = -\frac{1}{3}\cos(2t-1)$. Combining, we get
$x_p = x_{p,1} + x_{p,2} = (t^2 - 2) - \frac{1}{3}\cos(2t-1)$.

Problem 2: *Find the general solution to*

$$y'' + y' + y = 2xe^x$$

Solution: Characteristic equation: $p(s) = s^2 + s + 1 = 0 \to$ roots $r = \frac{-1 \pm \sqrt{-3}}{2}$ so the solution to the homogeneous equation is

$$y_h = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

To find the particular solution, try

$$\begin{aligned}
\boxed{y_p = a_1 xe^x + a_2 e^x} \\
y_p' &= a_1 e^x(x+1) + a_2 e^x \\
y_p'' &= a_1 e^x(x+2) + a_2 e^x \\
2xe^x &= 3a_1 xe^x + (3a_1 + 3a_2)e^x
\end{aligned}$$

So $a_1 = \frac{2}{3}, a_2 = -\frac{2}{3}$.Thus the general solution is

$$y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) + \frac{2}{3}e^x(x-1)$$

Problem 3: *Find a particular solution to the DE* ?

$$y^{(4)} - 2y'' + y = xe^x$$

Solution: $p(s) = s^4 - 2s^2 + 1 = (s^2-1)^2$ so $p(1) = 0$ repeated root (order 2), so try $y_p = x^2(Ax+B)e^x$. Use the exponential shift rule to get $A = \frac{1}{24}$, $B = -\frac{3}{24}$, and so

$$y_p = \frac{x^2e^x}{24}(x-3)$$