18.03SC Practice Problems 22

Fourier Series

Solution suggestions

1. *Graph the function $f(t)$ which is even, periodic of period 2π, and such that $f(t) = 2$ for $0 < t < \frac{\pi}{2}$ and $f(t) = 0$ for $\frac{\pi}{2} < t < \pi$.*

Here is the graph of $f(t)$. Note that there is only one way to extend the definition of f over all real t since f is specified to be even and periodic.

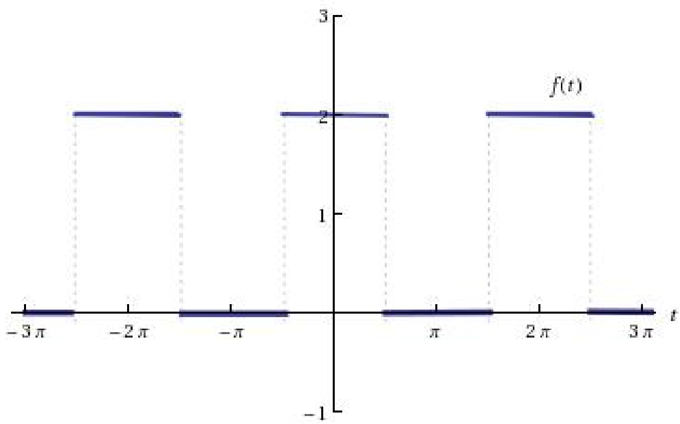

Figure 1: Graph of $f(t)$ over three periods.

Find its Fourier series in two ways:

(a) *Use the integral expressions for the Fourier coefficients. (Is the function even or odd? What can you say right off about the coefficients?)*

The function $f(t)$ is even, so $b_n = 0$ for all $n > 0$.

So the only nonzero coefficients are the a_n's. Compute a_0 first.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 2dt = 2.$$

Now compute a_n for $n > 0$.

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \\
&= \frac{2}{\pi}\left(\int_0^{\pi/2} 2\cos(nt)dt + \int_{\pi/2}^{\pi} 0dt\right) \\
&= \frac{4}{n\pi}\sin(nt)|_0^{\pi/2} \\
&= \frac{4}{n\pi}\sin(n\pi/2)
\end{aligned}$$

If n is even, this is always zero. If n is odd, then this alternates between $+\frac{4}{n\pi}$ when n of the form $4k+1$ and $-\frac{4}{n\pi}$ when n is of the form $4k+3$.

The Fourier series is then

$$f(t) = 1 + \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \frac{4}{7\pi}\cos(7t) + \dots$$

(b) *Express $f(t)$ in terms of* $\mathrm{sq}(t)$*, substitute the Fourier series for* $\mathrm{sq}(t)$ *and use some trig identities.*

First we see that f can be expressed in terms of the standard square wave as

$$f(t) = 1 + \mathrm{sq}(t + \pi/2).$$

Now, as given in the introduction to this problem session, the Fourier series for $\mathrm{sq}(t)$ is

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \dots\right),$$

so we can substitute this in to get the Fourier series for $f(t)$ directly.

$$\begin{aligned} f(t) &= 1 + \frac{4}{\pi}\left(\sin(t+\pi/2) + \frac{1}{3}\sin(3t + 3\pi/2) + \frac{1}{5}\sin(5t + 5\pi/2) + \dots\right) \\ &= 1 + \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \dots \end{aligned}.$$

This coincides with the answer we got for Part (a).

(c) *Now find the Fourier series for $f(t) - 1$.*

The Fourier series of $f(t) - 1$ has 1 subtracted from the constant term $a_0/2$ in the Fourier series for $f(t)$, so we get

$$f(t) - 1 = \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \frac{4}{7\pi}\cos(7t) + \dots$$

2. *What is the Fourier series for* $\sin^2 t$*?*

We could compute the Fourier coefficients directly from the formulas, but instead we use a trig identity. By the double angle formula, $\cos(2t) = 1 - 2\sin^2 t$, so

$$\sin^2 t = \frac{1}{2} - \frac{1}{2}\cos(2t).$$

The right hand side is a Fourier series; it happens to be finite here. That is, the Fourier series for $\sin^2 t$ has only two nonzero coefficients. When we regard $\sin^2 t$ as having period 2π, its series has Fourier coefficients $a_0 = 1$ and $a_2 = -1/2$.

This answer makes sense for two reasons. First, $\sin^2 t$ is an even function, and here all the b_n's are zero. Second, we expect polynomial functions of sine and cosine to have short Fourier series.

A remark from the point of view of material to be introduced later: This function has minimal period π, so it might be more natural to speak about its Fourier series for period π. This would be the same series, but the coefficients would be indexed differently. (If we thought of this Fourier series as having period π, a_0 and a_1 would be the nonzero coefficients.)

3. *Graph the odd function $g(x)$ which is periodic of period π and such that $g(x) = 1$ for $0 < x < \frac{\pi}{2}$. 2π is also a period of $g(x)$, so it has a Fourier series of period 2π as above. Find it by expressing $g(x)$ in terms of the standard squarewave.*

Here is the graph of $g(x)$.

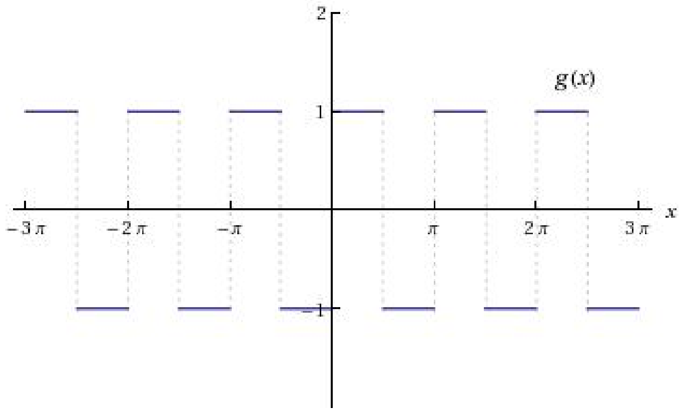

Figure 2: Graph of $g(x)$ over six periods.

We observe that $g(x) = \text{sq}(2x)$, so it has the Fourier series

$$g(x) = \frac{4}{\pi}\sin(2x) + \frac{4}{3\pi}\sin(6x) + \frac{4}{5\pi}\sin(10x) + \frac{4}{7\pi}\sin(14x) + \ldots.$$

Once again, as in the remark at the end of Problem 2, note that here if we regard g as being of period 2π, the nonzero coefficients would be indexed $b_2, b_6, \ldots$, while if we regarded g as being of period π (which is its minimal period), the nonzero coefficients would be indexed $b_1, b_3, \ldots$.

4. *Graph the function $h(t)$ which is odd and periodic of period 2π and such that $h(t) = t$ for $0 < t < \frac{\pi}{2}$ and $h(t) = \pi - t$ for $\frac{\pi}{2} < t < \pi$. Find its Fourier series, starting with your solution to **1(c)**.*

The graph of $h(t)$ is a zigzag wave.

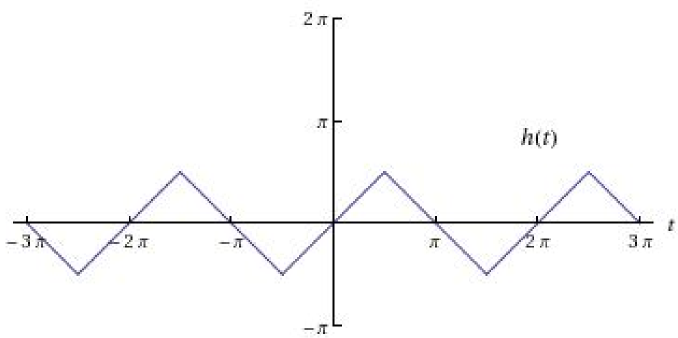

Figure 3: Graph of $h(t)$ over three periods.

We observe that the function $h(t)$ has derivative $f(t) - 1$, the function from 1(c). The Fourier series for $f(t) - 1$ has zero constant term, so we can integrate it term by term to get the Fourier series for $h(t)$, up to a constant shift. Since $h(t)$ is odd, the constant of integration here is 0. The rest of the series is computed below.

$$\begin{aligned} h(t) &= \int f(t) - 1 dt = \int \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \frac{4}{7\pi}\cos(7t) + \ldots dt \\ &= \frac{4}{\pi}\sin t - \frac{4}{9\pi}\sin(3t) + \frac{4}{25\pi}\sin(5t) - \frac{4}{49\pi}\sin(7t) + \ldots \end{aligned}$$

5. *Explain why any function $F(x)$ is a sum of an even function and an odd function in just one way. What is the even part of e^x? What is the odd part?*

This is a standard question to ask, and an important method to know.

An easy way to make an even function from an arbitrary $F(x)$ is to take the sum $F(x) + F(-x)$. (Why is this even?)

Similarly, subtracting $F(x) - F(-x)$ gives an odd function. (Check this is odd.)

Adding the two together would give $2F(x)$, so we go back and divide by this factor of two:

$$F(x) = \frac{F(x) + F(-x)}{2} + \frac{F(x) - F(-x)}{2}$$

To show that this decomposition is unique, we suppose we have another decomposition $F_{even}(x) + F_{odd}(x) = F(x)$, where $F_{even}(x)$ is even and $F_{odd}(x)$ is odd.

We are assuming that $F_{even}(x) + F_{odd}(x) = F(x) = \frac{F(x)+F(-x)}{2} + \frac{F(x)-F(-x)}{2}$. Rearranging terms, this means that

$$F_{even}(x) - \frac{F(x) + F(-x)}{2} = -F_{odd} + \frac{F(x) - F(-x)}{2}.$$

The left hand side here is the sum of two even functions, so it is also even, and, similarly, the right-hand side is the sum of two odd functions, so it is odd. But then each side is simultaneously both even and odd, and has to be zero.

Thus, $F_{even}(x) = \frac{F(x)+F(-x)}{2}$ and $F_{odd}(x) = \frac{F(x)-F(-x)}{2}$, so the even-odd decomposition of a function is unique.

This decomposition might seem familiar from hyperbolic trig function formulas: The even part of e^x is $\frac{e^x+e^{-x}}{2} = \cosh x$, and the odd part of e^x is $\frac{e^x-e^{-x}}{2} = \sinh x$.