

Modified Input

Way back when we introduced the language of system, input and response we decided that the right hand side of our equations wasn't always the input. Sometimes it was a modified version of the input.

Example 1. Recall the heat diffusion equation

$$\dot{x} + kx = kT_e(t),$$

where $T_e(t)$ is the temperature of the environment. Consider $T_e(t)$ to be the input and find the system function.

Solution. Look at the equation for the unit impulse response

$$\dot{w} + kw = k\delta(t), \quad \text{rest IC.}$$

Notice that since $T_e(t)$ is the input, the unit impulse response comes by letting $T_e(t) = \delta(t)$. The Laplace transform now gives

$$(s + k)W(s) = k \quad \Rightarrow \quad W(s) = \frac{k}{s + k}.$$

Note well, that with modified input on the right hand side of the DE, the system function does not automatically have a 1 in the numerator.

You might have noticed that in the previous example we could have written $W(s) = \frac{1}{s/k + 1}$, which has our usual form. The next example shows that this is not always the case.

Example 2. Consider an LC circuit with input voltage $v(t)$. We'll assume L and C are set so the differential equation for the current $i(t)$ is

$$i''(t) + 4i = v'(t).$$

We consider the input to be $v(t)$ and the output to be $i(t)$. (We use primes instead of dots for the derivative because the i already has a dot.)

Finding the unit impulse response is tricky, because if we set $v(t) = \delta(t)$ then we will have $\delta'(t)$ on the right hand side of the DE. Let's avoid this by using the characterization of the transfer function as the ratio output/input. In this case, we'll have $W(s) = I(s)/V(s)$.

Laplace Directly:
 $W(s^2+4) = s$
 $\Rightarrow W(s) = \frac{s}{s^2+4}$

Assuming rest IC, we have $\mathcal{L}(v') = sV(s)$, where, as usual, we have let the uppercase letter be the Laplace transform of the lowercase one. Applying the Laplace transform gives

$$(s^2 + 4)I(s) = sV(s) \quad \Rightarrow \quad W(s) = \frac{I(s)}{V(s)} = \frac{s}{s^2 + 4}.$$

The s in the numerator guarantees this cannot be written in the form $1/p(s)$ for any polynomial $p(s)$.

As a concluding note, we'll say that we were too **pessimistic** about our ability to handle $\delta'(t)$. We might not know what it is, but we do know how to find its Laplace transform.

$$\mathcal{L}^{-1}(\delta'(t)) = s\mathcal{L}(\delta(t)) - \delta(0^-) = s.$$

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