

## Proof of Green's Formula

**Green's Formula:** For the equation

$$P(D)y = f(t), \quad y(t) = 0 \text{ for } t < 0 \quad (1)$$

the solution for  $t > 0$  is given by

$$y(t) = (f * w)(t) = \int_{0^-}^{t^+} f(\tau)w(t - \tau) d\tau, \quad (2)$$

where  $w(t)$  is the weight function (unit impulse response) for the system.

**Proof:** The proof of Green's formula is surprisingly direct. We will use the linear time invariance of the system combined with superposition and the definition of the integral as a limit of Riemann sums.

To avoid worrying about  $0^-$  and  $t^+$  we will assume that  $f(t)$  is continuous. With appropriate care, the proof will work for an  $f(t)$  that has jump discontinuities or contains delta functions.

As we saw in the session on Linear Operators in the last unit, linear time invariance means that

$$y(t) \text{ solves } P(D)y = f(t) \Rightarrow y(t - a) \text{ solves } P(D)y = f(t - a). \quad (3)$$

Or, in the language of input-response, if  $y(t)$  is the response to input  $f(t)$  then  $y(t - a)$  is the response to input  $f(t - a)$ .

First we will partition time into intervals of width  $\Delta t$ . So,  $t_0 = 0$ ,  $t_1 = \Delta t$ ,  $t_2 = 2\Delta t$ , etc.

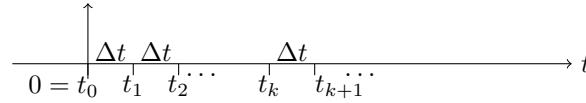
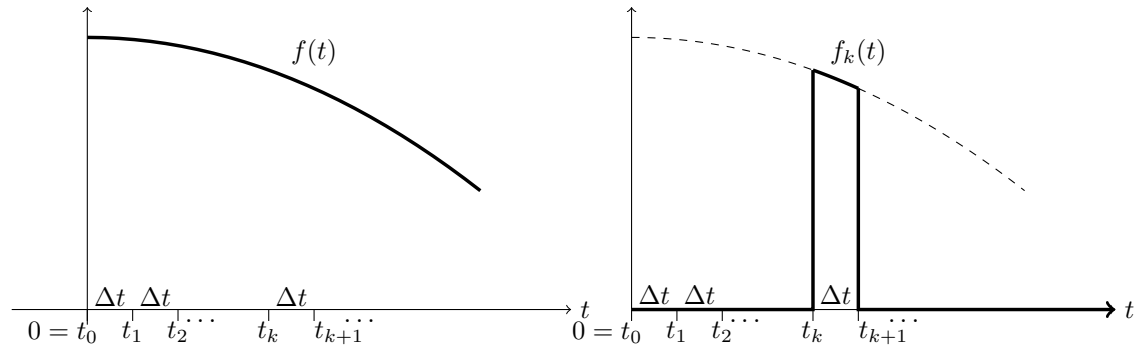


Figure 1: Division of the  $t$ -axis into small intervals.

Next we decompose the input signal  $f(t)$  into packets over each interval. The  $k$ th signal packet,  $f_k(t)$  coincides with  $f(t)$  between  $t_k$  and  $t_{k+1}$  and is 0 elsewhere

$$f_k(t) = \begin{cases} f(t) & \text{for } t_k < t < t_{k+1} \\ 0 & \text{elsewhere.} \end{cases}$$

Figure 2: The signal packet  $f_k(t)$ .

It is clear that for  $t > 0$  we have  $f(t)$  is the sum of the packets

$$f(t) = f_0(t) + f_1(t) + \dots + f_k(t) + \dots$$

As  $\Delta t \rightarrow 0$ ,  $f_k(t)$  becomes a impulse at  $t_k$ ,  $\delta(t - t_k)$ . The area under  $f_k(t)$  is  $\Delta t f(t_k)$ . then  $f_k(t) = [\Delta t f(t_k)] \delta(t - t_k)$  namely  $\int_{t_k}^{t_{k+1}} f_k(t) dt = \int_{t_k}^{t_{k+1}} \Delta t f(t_k) \delta(t - t_k) dt = \Delta t f(t_k) \int_{t_k}^{t_{k+1}} \delta(t - t_k) dt = \Delta t f(t_k)$

A single packet  $f_k(t)$  is concentrated entirely in a small neighborhood of  $t_k$  so it is approximately an impulse with the same size as the area under  $f_k(t)$ . [The area under  $f_k(t)$ ]  $\approx f(t_k) \Delta t$ . Hence,  $\int_{t_k}^{t_{k+1}} f_k(t) dt \approx f(t_k) \Delta t$   $f_k(t) \approx (f(t_k) \Delta t) \delta(t - t_k)$  ? How impulse function is introduced ?

The weight function  $w(t)$  is response to  $\delta(t)$ . So, by linear time invariance the response to  $f_k(t)$  is  $y_k(t) \approx (f(t_k) \Delta t) w(t - t_k)$ .

Why approximate as impulse? The value of impulse is Infinity. Not value but area matters. Impulse function is introduced to approximate area to make it compatible with unit impulse response to get response of  $f_k$  from  $w$ . response from slice of input.

By superposition  $\Rightarrow [f(t_k) \Delta t] w(t)$  is response to  $[f(t_k) \Delta t] \delta(t)$ . By replace  $t$  with  $t - t_k \Rightarrow [f(t_k) \Delta t] w(t - t_k)$  is response to  $[f(t_k) \Delta t] \delta(t - t_k)$ .

We want to find the response at a fixed time. Since  $t$  is already in use, we will let  $T$  be our fixed time and find  $y(T)$ .

Since  $f$  is the sum of  $f_k$ , superposition gives  $y$  is the sum of  $y_k$ . That is, at time  $T$

$$\begin{aligned} y(T) &= y_0(T) + y_1(T) + \dots \\ &\approx \left( f(t_0)w(T - t_0) + f(t_1)w(T - t_1) + \dots \right) \Delta t \end{aligned} \quad (4)$$

We can ignore all the terms where  $t_k > T$ . (Because then  $w(T - t_k) = 0$ , since  $T - t_k < 0$ .) If  $n$  is the last index where  $t_k < T$  we have

$$y(T) \approx \left( f(t_0)w(T - t_0) + f(t_1)w(T - t_1) + \dots + f(t_n)w(T - t_n) \right) \Delta t$$

This is a Riemann sum and as  $\Delta t \rightarrow 0$  it goes to an integral

$$y(T) = \int_0^T f(t)w(T-t) dt$$

Except for the change in notation this is Green's formula (2).

**Note on Causality:** Causality is the principle that the future does not affect the past. Green's theorem shows that the system (1) is causal. That is,  $y(t)$  only depends on the input up to time  $t$ . Real physical systems are causal.

There are non-causal systems. For example, an audio compressor that gathers information after time  $t$  before deciding how to compress the signal at time  $t$  is non-causal. Another example is the system with input  $f(t)$  and output  $y(t)$  where  $y$  is the solution to  $\dot{y} = f(t+1)$ .

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