Part II Problems and Solutions

Problem 1: *[Laplace transform]* **(a)** *Suppose that $F(s)$ is the Laplace transform of $f(t)$, and let $a > 0$. Find a formula for the Laplace transform of $g(t) = f(at)$ in terms of $F(s)$, by using the integral definition and making a change of variable. Verify your formula by using formulas and rules to compute both $\mathcal{L}(f(t))$ and $\mathcal{L}(f(at))$ with $f(t) = t^n$.*

(b) *Use your calculus skills: Show that if $h(t) = f(t) * g(t)$ then $H(s) = F(s)G(s)$. Do this by writing $F(s) = \int_0^\infty f(x)e^{-sx}\,dx$ and $G(s) = \int_0^\infty g(y)e^{-sy}\,dy$; expressing the product as a double integral; and changing coordinates using $x = t - \tau$, $y = \tau$.*

(c) *Use the integral definition to find the Laplace transform of the function $f(t)$ with $f(t) = 1$ for $0 < t < 1$ and $f(t) = 0$ for $t > 0$. What is the region of convergence of the integral?*

Solution: (a) $G(s) = \int_0^\infty f(at)e^{-st}\,dt$. To make this look more like $F(s) = \int_0^\infty f(t)e^{-st}\,dt$, make the substitution $u = at$. Then $du = a\,dt$ and

$$G(s) = \int_0^\infty f(u)e^{-su/a}\,\frac{du}{a} = \frac{1}{a}\int_0^\infty f(u)e^{-(s/a)u}\,du = \frac{1}{a}F\left(\frac{s}{a}\right).$$

For example, take $f(t) = t^n$, so $F(s) = \frac{n!}{s^{n+1}}$, $g(t) = (at)^n = a^n t^n$, $G(s) = \frac{a^n n!}{s^{n+1}}$. Now compute $\frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}\frac{n!}{(s/a)^{n+1}} = \frac{a^{n+1}}{a}\frac{n!}{s^{n+1}} = \frac{a^n n!}{s^{n+1}} = G(s)$.

(b) Compute $F(s)G(s) = \int_0^\infty\int_0^\infty f(x)e^{-sx}g(y)e^{-sy}\,dxdy = \iint_R f(x)g(y)e^{-s(x+y)}\,dxdy$, where R is the first quadrant. We can use the substitution is $x = t - \tau$, $y = \tau$. To convert to these coordinates, note that the Jacobian is $\det\begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial \tau} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \tau} \end{bmatrix} = \det\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1$ For fixed t, τ ranges over numbers between 0 and t, and t ranges over positive numbers. Since $x + y = t$, $F(s)G(s) = \int_0^\infty\int_0^t f(t-\tau)g(\tau)e^{-st}\,d\tau dt$

$$= \int_0^\infty \left(\int_0^t f(t-\tau)g(\tau)\,d\tau\right)e^{-st}\,dt = \int_0^\infty (f(t) * g(t))\,e^{-st}\,dt = \int_0^\infty h(t)e^{-st}\,dt = H(s).$$

(c) $F(s) = \int_0^\infty f(t)e^{-st}\,d\tau = \int_0^1 f(t)e^{-st}\,d\tau + \int_1^\infty 0e^{-st}\,d\tau$. The improper integral converges for any s; the region of convergence is the whole complex plane. Continuing, $F(s) = \left.\frac{1}{-s}e^{-st}\right|_0^1 = \frac{1-e^{-s}}{s}$.

[Why doesn't this blow up when $s \to 0$? The numerator goes to zero too, then, and the limit of the quotient (by l'Hopital for example) is 1.]