Part I Problems and Solutions

In each of the following three problems find a particular solution to the differential equation. Use complex exponentials where possible.

Problem 1: $y^{(3)} + y'' - y' + 2y = 2\cos x$

Solution: characteristic polynomial $p(s) = s^3 + s^2 - s + 2$; complex replacement: $z^{(3)} + z'' - 2z' + 2z = 2e^{ix}$. Complex solution from ERF:

$$z_p = \frac{2e^{ix}}{p(i)} = \frac{2e^{ix}}{i^3 + i^2 - i + 2} = \frac{2(1+2i)e^{ix}}{(1-2i)(1+2i)}$$

Thus

$$z_p = \frac{2+4i}{5}(\cos x + i \sin x)$$

, and

$$y_p = \mathrm{Re}(z_p) = \frac{2}{5}\cos x - \frac{4}{5}\sin x$$

Problem 2: $y'' - 2y' + 4y = e^x \cos x$

Solution: complex replacement: $z'' - 2z' + 4z = e^{(1+i)x}$. $p(s) = s^2 - 2s + 4$; $p(1+i) = (1+i)^2 - 2(1+i) + 4 = 2$, so $z_p = \frac{e^{(1+i)x}}{2}$, and thus

$$y_p = \mathrm{Re}(z_p) = \frac{1}{2}e^x \cos x$$

Problem 3: $y'' - 6y' + 9y = e^{3x}$

Solution: $p(s) = s^2 - 6s + 9 = (s-3)^2$ so $y_p = cx^2e^{3x}$. $p(3) = p'(3) = 0$, $p''(3) = 2 \neq 0$. This the generalized ERF gives

$$y_p = \frac{1}{2}x^2e^{3x}$$

Problem 4: *Find the real general solution to the DE*

$$\frac{d^3x}{dt^3} - x = e^{2t}$$

Solution: $p(r) = r^3 - 1$, $p(2) = 2^3 - 1 = 7 \neq 0$. Thus the ERF gives a particular solution to the inhomogenous DE $x^{(3)} - x = e^{2t}$ as $x_p = \frac{1}{p(2)}e^{2t} = \frac{1}{7}e^{2t}$.

x_h solution to the homogeneous eqn.$x^{(3)} - x = 0$, characteristic equation $p(r) = r^3 - 1 = 0 \rightarrow r^3 = 1 \rightarrow$ cube roots of 1.

Complex roots for $\sqrt[3]{1}$
n-th root.

$r = 1, e^{2\pi i/3}, e^{4\pi i/3} = 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - \frac{\sqrt{3}}{2}$

So $x_1 = e^t$, $x_2 = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$, $x_3 = e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$

Real solutions: $x_h = c_1x_1 + c_2x_2 + c_3x_3$ so the *general solution* $x = x_p + x_h$ is

$$x = \frac{1}{7}e^{2t} + c_1e^t + c_2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_3e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Problem 5: *Find a particular solution to the differential equation*

$$y'' - 4y = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$$

Solution: $y_p = y_{p,1} + y_{p,2}$ where $p(D)y_{p,1} = \frac{1}{2}e^{2x}$ and $p(D)y_{p,2} = \frac{1}{2}e^{-2x}$ by superposition principle. $p(r) = r^2 - 4$ so $p(2) = p(-2) = 0$. Using the generalized ERF $p'(r) = 2r$. Thus,

$$y_{p,1} = \frac{\frac{1}{2}}{p'(2)}xe^{2x} = \frac{1}{8}xe^{2x}$$

$$y_{p,2} = \frac{\frac{1}{2}}{p'(-2)}xe^{-2x} = -\frac{1}{8}xe^{-2x}$$

$$y_p = y_{p,1} + y_{p,2} = \frac{x}{8}\left(e^{2x} - e^{-2x}\right)$$