Complex Exponentials

Because of the importance of complex exponentials in differential equations, and in science and engineering generally, we go a little further with them. Euler's formula defines the exponential to a pure imaginary power. The definition of an exponential to an arbitrary complex power is:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b)). \tag{1}$$

We stress that the equation (1) is a definition, not a self-evident truth, since up to now no meaning has been assigned to the left-hand side. From (1) we see that

$$\mathrm{Re}(e^{a+ib}) = e^a\cos(b), \qquad \mathrm{Im}(e^{a+ib}) = e^a\sin(b). \tag{2}$$

The complex exponential obeys the usual law of exponents:

$$e^{z+z'} = e^z e^{z'}, \tag{3}$$

as is easily seen by combining (1) with the multiplication rule for complex numbers.

The complex exponential is expressed in terms of the sine and cosine functions by Euler's formula. Conversely, the sine and cosine functions can be expressed in terms of complex exponentials. There are two important ways of doing this, both of which you should learn:

$$\cos(x) = \mathrm{Re}(e^{ix}), \qquad\qquad \sin(x) = \mathrm{Im}(e^{ix}); \tag{4}$$

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}). \tag{5}$$

The equations in (5) follow easily from Euler's formula; their derivation is left as an exercise. Here are some examples of their use.

Example. Express $\cos^3(x)$ in terms of the functions $\cos(nx)$, for suitable n.

Solution. We use (5) and the binomial theorem, then (5) again:

$$\cos^3(x) = \frac{1}{8}(e^{ix} + e^{-ix})^3$$

$$= \frac{1}{8}(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix})$$

$$= \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x).$$

As a preliminary to the next example, we note that a function like

$$e^{ix} = \cos(x) + i\sin(x)$$

is a *complex-valued function of the real variable x*. Such a function may be written as

$$u(x) + iv(x) \qquad u, v \text{ real-valued}$$

and its derivative and integral with respect to x are <u>defined</u> to be

a) $D(u + iv) = Du + iDv$ b) $\displaystyle\int (u+iv)dx = \int u dx + i \int v dx.$ (6)

From this it follows easily that

$$D(e^{(a+ib)x}) = (a+ib)e^{(a+ib)x}, \text{ {\small Try to prove}}$$

and therefore

$$\int e^{(a+ib)x}\,dx = \frac{1}{a+ib}\,e^{(a+ib)x}. \tag{7}$$

Example. Calculate $\displaystyle\int e^x \cos 2x\,dx$ by using complex exponentials.

Solution. The usual method is a tricky use of two successive integration by parts. Using complex exponentials instead, the calculation is straightforward. We have

$$e^{(1+2i)x} = e^x \cos(2x) + ie^x \sin(2x), \qquad\qquad \text{by (1)}$$

therefore by (6b)

$$\int e^x \cos(2x)\,dx = \text{Re}\!\left(\int e^{(1+2i)x}dx\right).$$

Calculating the integral,

$$\int e^{(1+2i)x}dx = \frac{1}{1+2i}e^{(1+2i)x} \qquad\qquad \text{by (7)}$$
$$= \left(\frac{1}{5} - \frac{2}{5}i\right)(e^x \cos(2x) + ie^x \sin(2x)),$$

using (1) and complex division. According to the second line above, we want the real part of this last expression. Multiply and take the real part; you get the answer

$$\int e^x \cos(2x)\,dx = \frac{1}{5}e^x \cos(2x) + \frac{2}{5}e^x \sin(2x).$$

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In this differential equations course, we will make free use of complex exponentials in solving differential equations, and in doing formal calculations like the ones above. This is standard practice in science and engineering, and it's well worth getting used to.

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