General Linear ODE Systems and Independent Solutions

We have studied the homogeneous system of ODE's with constant coefficients,

$$\mathbf{x}' = A\,\mathbf{x}\,, \tag{1}$$

where A is an $n \times n$ matrix of constants ($n = 2, 3$). We described how to calculate the eigenvalues and corresponding eigenvectors for the matrix A, and how to use them to find n independent solutions to the system (1).

With this concrete experience in solving low-order systems with constant coefficients, what can be said when the coefficients are functions of the independent variable t? We can still write the linear system in the matrix form (1), but now the matrix entries will be functions of t:

$$\begin{aligned} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{aligned}\quad, \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2}$$

or in more abridged notation, valid for $n \times n$ linear homogeneous systems,

$$\mathbf{x}' = A(t)\,\mathbf{x}\,. \tag{3}$$

Note how the matrix becomes a function of t — we call it a *matrix-valued function* of t, since to each value of t the function rule assigns a matrix:

$$t_0 \to A(t_0) = \begin{pmatrix} a(t_0) & b(t_0) \\ c(t_0) & d(t_0) \end{pmatrix}$$

In the rest of this chapter we will often not write the variable t explicitly, but it is always understood that the matrix entries are functions of t.

We will sometimes use $n = 2$ or 3 in the statements and examples in order to simplify the exposition, but the definitions, results, and the arguments which prove them are essentially the same for higher values of n.

Definition 1 Solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ to (3) are called **linearly dependent** if there are constants c_i, not all of which are 0, such that

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0, \qquad \text{for all } t. \tag{4}$$

If there is no such relation, i.e., if

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0 \quad \text{for all } t \quad \Rightarrow \quad \text{all } c_i = 0, \tag{5}$$

the solutions are called **linearly independent**, or simply *independent*.

The phrase *for all* t is often in practice omitted, as being understood. This can lead to ambiguity. To avoid it, we will use the symbol $\equiv 0$ for **identically 0**, meaning *zero for all* t; the symbol $\not\equiv 0$ means *not identically 0*, i.e., there is some t-value for which it is not zero. For example, (4) would be written

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) \equiv 0\,.$$

Theorem 1 *If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is a linearly independent set of solutions to the $n \times n$ system $\boldsymbol{x}' = A(t)\boldsymbol{x}$, then the general solution to the system is*

$$\boldsymbol{x} = c_1\boldsymbol{x}_1 + \ldots + c_n\boldsymbol{x}_n. \tag{6}$$

Such a linearly independent set is called a **fundamental** *set of solutions.*

This theorem is the reason for expending so much effort to find two independent solutions, when $n = 2$ and A is a constant matrix. In this chapter, the matrix A is not constant; nevertheless, (6) is still true.

Proof. There are two things to prove:

(a) All vector functions of the form (6) really are solutions to $\mathbf{x}' = A\,\mathbf{x}$.

This is the *superposition principle* for solutions of the system; it's true because the system is *linear*. The matrix notation makes it really easy to prove. We have

$$\begin{aligned} (c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n)' &= c_1\mathbf{x}_1' + \ldots + c_n\mathbf{x}_n' \\ &= c_1A\,\mathbf{x}_1 + \ldots + c_nA\,\mathbf{x}_n, && \text{since } \mathbf{x}_i' = A\,\mathbf{x}_i\,; \\ &= A\,(c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n), && \text{by the distributive law.} \end{aligned}$$

(b) All solutions to the system are of the form (6).

This is harder to prove and will be the main result of the next note.