Proof of the Generalized Exponential Response Formula

Using the exponential shift rule, we can now give a proof of the general case of the ERF which we stated without proof in the session on Exponential Response. This is a slightly complicated proof and you can safely skip it if you are not interested.

Generalized Exponential Response Formula. *Let $p(D)$ be a polynomial operator with constant coefficients and $p^{(s)}$ its s-th derivative. Then*

$$p(D)x = e^{at}, \qquad \text{where } a \text{ is real or complex} \tag{1}$$

has the particular solution

$$x_p = \begin{cases} i) & \dfrac{e^{at}}{p(a)} & \text{if } p(a) \neq 0 \\ ii) & \dfrac{te^{at}}{p'(a)} & \text{if } p(a) = 0 \text{ and } p'(a) \neq 0 \\ iii) & \dfrac{t^2e^{at}}{p''(a)} & \text{if } p(a) = p'(a) = 0 \text{ and } p''(a) \neq 0 \\ \dots & & \\ iv) & \dfrac{t^s e^{at}}{p^{(s)}(a)} & \text{if } a \text{ is an } s\text{-fold zero} \end{cases}$$

Proof. That (i) is a particular solution to (1) follows immediately by using the linearity and substitution rules given earlier.

Prove equality:
$P(D)\, x_p = e^{at}$ when $x_p = \dfrac{e^{at}}{p(a)}$

$$p(D)x_p = p(D)\frac{e^{at}}{p(a)} = \frac{1}{p(a)}p(D)e^{at} = \frac{p(a)e^{at}}{p(a)} = e^{at}.$$

Since cases (ii) and (iii) are special cases of (iv) we skip right to that. For case (iv), we begin by noting that to say the polynomial $p(D)$ has the number a as an s-fold zero is the same as saying $p(D)$ has a factorization

$$p(D) = q(D)(D-a)^s, \qquad q(a) \neq 0. \tag{2}$$

We will first prove that (2) implies

$$p^{(s)}(a) = q(a)\, s!\,. \tag{3}$$

$\frac{d}{dD}p(D) = q'(D)(D-a)^s + q(D)\,s\,(D-a)^{s-1}$
$= (D-a)^{s-1}\left[q'(D)(D-a) + q(D)\,s\right]$

$q(D)$, $q(a)$, not the same q

$\underbrace{q(D),\ q(a)}_{\text{Irrelevant}}$

$\frac{d}{dD}(D-a)^k = k(D-a)^{k-1}$

$\frac{d^k}{dD^k}(D-a) = k!$

To prove this, let k be the degree of $q(D)$ and write it in powers of $(D-a)$:

$$\begin{aligned} q(D) &= q(a) + c_1(D-a) + \ldots + c_k(D-a)^k; \quad \text{then} \\ p(D) &= q(a)(D-a)^s + c_1(D-a)^{s+1} + \ldots + c_k(D-a)^{s+k}; \qquad (4) \\ p^{(s)}(D) &= q(a)\, s! + \text{positive powers of } D-a. \end{aligned}$$

Substituting a for D on both sides proves (3). □

Using (3), we can now prove (iv) easily using the exponential-shift rule. We have

$$\begin{aligned} p(D)\frac{e^{at}x^s}{p^{(s)}(a)} &= \frac{e^{at}}{p^{(s)}(a)}\, p(D+a)x^s, && \text{by linearity and ERF case (i);} \\ &= \frac{e^{at}}{p^{(s)}(a)}\, q(D+a)\, D^s x^s, && \text{by (2);} \\ &= \frac{e^{at}}{q(a)s!}\, q(D+a)\, s!, && \text{by (3);} \\ &= \frac{e^{at}}{q(a)s!}\, q(a)\, s! \;=\; e^{at}, \end{aligned}$$

$p(D) = q(D)(D-a)^s$

$D = D + a$

$\Rightarrow p(D+a) = q(D+a)D^s$

$\frac{d^s}{dx^s} x^s = s!$

$q(D) = q(a) + c_1(D-a) + \dots + c_k(D-a)^k$

$D = D + a$

$\Rightarrow q(D+a) = q(a) + c_1 D + \dots + c_k D^k$

$\Rightarrow q(D+a)s! = (q(a) + c_1 D + \dots + c_k D^k)\, s!$

$= q(a)s!$

where the last line follows from (4), since $s!$ is a constant:

$$q(D+a)s! = (q(a) + c_1 D + \ldots + c_k D^k)\, s! = q(a)s!\,.$$

Note: By linearity we could have stated the formula with a factor of B in the input and a corresponding factor of B to the output. That is, the DE

$$p(D)x = Be^{at}$$

has a particular solution

$$x_p = \frac{Be^{at}}{p(a)}, \quad \text{if } p(a) \neq 0 \text{ etc.}$$