

Definition of Laplace Transform

1. Definition of Laplace Transform

The Laplace transform of a function $f(t)$ of a real variable t is another function depending on a new variable s , which is in general complex. We will denote the Laplace transform of f by $\mathcal{L}f$. It is defined by the integral

$$(\mathcal{L}f)(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt, \quad (1)$$

for all values of s for which the integral converges.

There are a few things to note.

- $\mathcal{L}f$ is only defined for those values of s for which the improper integral on the right-hand side of (1) converges.
- We will allow s to be complex.
- As with convolution, the use of 0^- in the definition (1) is necessary to accommodate generalized functions containing $\delta(t)$. Many textbooks do not do this carefully, and hence their definition of the Laplace transform is not consistent with the properties they assert. In those cases where 0^- isn't needed we will use the less precise form

$$(\mathcal{L}f)(s) = \int_0^{\infty} f(t)e^{-st} dt. \quad (1')$$

- Also, as with convolution, the limits of integration mean that the Laplace transform is only concerned with functions on $(0^-, \infty)$.

2. Notation, $F(s)$

We will adopt the following conventions:

1. Writing $(\mathcal{L}f)(s)$ can be cumbersome so we will often use an uppercase letter to indicate the Laplace transform of the corresponding lowercase function:

$$(\mathcal{L}f)(s) = F(s), \quad (\mathcal{L}g)(s) = G(s), \text{ etc.}$$

For example, in the formula

$$\mathcal{L}(f') = sF(s) - f(0^-)$$

it is understood that we mean $F(s) = \mathcal{L}(f)$.

2. If our function doesn't have a name we will use the formula instead. For example, the Laplace transform of the function t^2 is written $\mathcal{L}(t^2)(s)$ or more simply $\mathcal{L}(t^2)$.
3. If in some context we need to modify $f(t)$, e.g. by applying a translation by a number a , we can write $\mathcal{L}(f(t-a))$ for the Laplace transform of this translation of f .
4. You've already seen several different ways to use parentheses. Sometimes we will even drop them altogether. So, if $f(t) = t^2$ then the following all mean the same thing

$$(\mathcal{L}f)(s) = F(s) = \mathcal{L}f(s) = \mathcal{L}(f(t))(s) = \mathcal{L}(t^2)(s); \quad \mathcal{L}f = F = \mathcal{L}(t^2).$$

3. First Examples

For the first few examples we will explicitly use a limit for the improper integral. Soon we will do this implicitly without comment.

Example 1. Let $f(t) = 1$, find $F(s) = \mathcal{L}f(s)$.

Solution. Using the definition (1') we have

$$\mathcal{L}(1) = F(s) = \int_0^\infty e^{-st} dt = \lim_{T \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_0^T = \lim_{T \rightarrow \infty} \frac{e^{-sT} - 1}{-s} \Big|_0^T.$$

The limit depends on whether s is positive or negative.

$$\lim_{T \rightarrow \infty} e^{-sT} = \begin{cases} 0 & \text{if } s > 0 \\ \infty & \text{if } s < 0. \end{cases}$$

Therefore,

$$\mathcal{L}(1) = F(s) = \begin{cases} \frac{1}{s} & \text{if } s > 0 \\ \text{diverges} & \text{if } s \leq 0. \end{cases}$$

(We didn't actually compute the case $s = 0$, but it is easy to see it diverges.)

Example 2. Compute $\mathcal{L}(e^{at})$.

Solution. Using the definition (1') we have

$$\begin{aligned} \mathcal{L}(e^{iat}) &= \int_0^\infty e^{iat} e^{-st} dt & \mathcal{L}(e^{at}) &= \int_0^\infty e^{at} e^{-st} dt = \lim_{T \rightarrow \infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^T = \lim_{T \rightarrow \infty} \frac{e^{(a-s)T} - 1}{a-s} \Big|_0^T \\ &= \int_0^\infty e^{(-s+ia)t} dt & & \\ &= \frac{e^{(-s+ia)t}}{-s+ia} \Big|_0^\infty & & \\ &= \frac{1}{-s+ia} \left(\lim_{t \rightarrow \infty} e^{-st} e^{iat} - 1 \right) & & \\ &= \frac{1}{-s+ia} (0 - 1) = \frac{1}{s-ia} \end{aligned}$$

The limit depends on whether $s > a$ or $s < a$.

$$\lim_{T \rightarrow \infty} e^{(a-s)T} = \begin{cases} 0 & \text{if } s > a \\ \infty & \text{if } s < a. \end{cases}$$

Therefore,

$$\mathcal{L}(e^{at}) = \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ \text{diverges} & \text{if } s \leq a. \end{cases}$$

(We didn't actually compute the case $s = a$, but it is easy to see it diverges.)

We have the first two entries in our table of Laplace transforms:

$$\begin{aligned} f(t) = 1 &\Rightarrow F(s) = 1/s, & s > 0 \\ f(t) = e^{at} &\Rightarrow F(s) = 1/(s-a), & s > a. \end{aligned}$$

4. Linearity

You will not be surprised to learn that the Laplace transform is linear. For functions f, g and constants c_1, c_2

$$\mathcal{L}(c_1f + c_2g) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g)$$

This is clear from the definition (1) of \mathcal{L} and the linearity of integration.

$$\int_0^\infty [c_1f(t) + c_2g(t)] e^{-st} dt = c_1 \int_0^\infty f(t) e^{-st} dt + c_2 \int_0^\infty g(t) e^{-st} dt$$

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