

18.03SC Practice Problems 10

Homogeneous second order linear equations with constant coefficients

Solution suggestions

1. Start with $\ddot{x} + \omega^2 x = 0$. What is the characteristic polynomial? What are its roots? What are the exponential solutions – the solutions of the form e^{at} ? These may be complex exponentials. What are their real and imaginary parts? Check that these are also solutions to the original equation. What is the general real solution?

The characteristic polynomial is $p(s) = s^2 + \omega^2$. Its roots are $\pm i\omega$. The exponential solutions are $e^{i\omega t}$ and $e^{-i\omega t}$. These are complex-valued functions, which are complex conjugates of each other. Up to sign, they have the same real and imaginary parts. Namely, $e^{\pm i\omega t} = \cos(\omega t) \pm i \sin(\omega t)$, so,

$$\operatorname{Re}(e^{i\omega t}) = \cos(\omega t), \quad \operatorname{Im}(e^{i\omega t}) = \sin(\omega t).$$

$$\operatorname{Re}(e^{-i\omega t}) = \cos(\omega t), \quad \operatorname{Im}(e^{-i\omega t}) = -\sin(\omega t).$$

We can check that these functions are also solutions by verifying that they satisfy the original equation:

$$\frac{d^2}{dt^2}(\cos(\omega t)) + \omega^2 \cos(\omega t) = -\omega^2 \cos(\omega t) + \omega^2 \cos(\omega t) = 0$$

and

$$\frac{d^2}{dt^2}(\sin(\omega t)) + \omega^2 \sin(\omega t) = -\omega^2 \sin(\omega t) + \omega^2 \sin(\omega t) = 0.$$

The same equation holds for $-\sin(\omega t)$, and, in fact for any linear combination of these two functions. The general real solution is

$$x(t) = a \cos(\omega t) + b \sin(\omega t).$$

In standard notation, this is the general sinusoid $A \cos(\omega t - \phi)$.

2. Suppose that $e^{-t/2} \cos(3t)$ is a solution of the equation $m\ddot{x} + b\dot{x} + kx = 0$.

(a) What can you say about m, b, k ?

We can write $e^{-t/2} \cos(3t) = \operatorname{Re} e^{(-1/2 \pm 3i)t}$, so $p(s) = ms^2 + bs + k$ has solutions $-\frac{1}{2} \pm 3i$. This means $p(s) = m(s + 1/2 - 3i)(s + 1/2 + 3i) = m(s^2 + s + \frac{37}{4})$. Then m can be any nonzero number, $b = m$, and $k = \frac{37}{4}m$.

We could have also solved this problem by the technique we had always used before – by substitution into the equation, but then the calculations would have been harder. In general, you should always interpret knowing some exponential

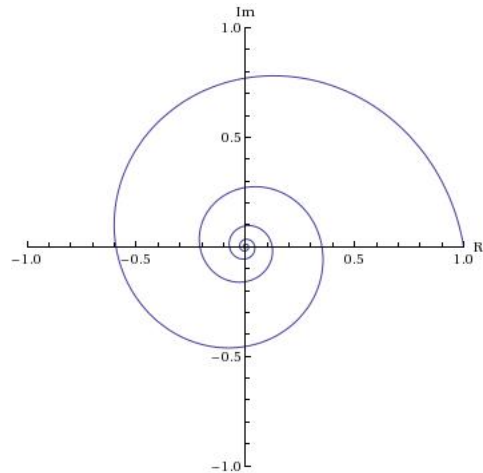
solutions of a homogeneous differential equation as knowing some of the roots of the characteristic polynomial.

(b) What are the exponential solutions (solutions of the form $e^{\alpha t}$) of this differential equation?

The exponential solutions are $e^{(-1/2+3i)t}$ and $e^{(-1/2-3i)t}$.

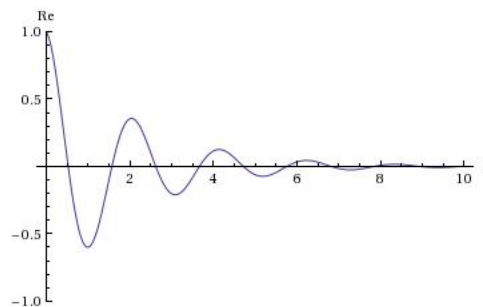
(c) Sketch the curve in the complex plane traced by one of the exponential solutions. Then sketch the graph of the real part, and explain the relationship.

Here is the trajectory traced by $e^{(-1/2+3i)t}$ in the complex plane.



The curve is a spiral. As t increases, it moves counterclockwise around the origin, and approaches zero.

Here is the graph of the real part as a function of t .



The graph of the real part describes a damped oscillation, i.e., it is roughly sinusoidal, with exponentially decreasing amplitude. It can be obtained from the graph of the complex-valued trajectory by envisioning moving along the trajectory in the complex plane as time t increases and projecting onto the real axis.

(d) What is the general solution?

All real-valued solutions are given by real linear combinations of the real part and the imaginary part of $e^{(-1/2+3i)t}$. So the general solution is $C_1 e^{-t/2} \cos(3t) +$

← Real part is solution

$C_2 e^{-t/2} \sin(3t)$ for any real numbers C_1 and C_2 , or equivalently, $A e^{-t/2} \cos(3t - \phi)$, where $A \geq 0$ and $\phi \in [0, 2\pi)$.

3. Let $\omega > 0$. A damped sinusoid $x(t) = A e^{-at} \cos(\omega t)$ has “pseudo-period” $2\pi/\omega$. The pseudo-period, and hence ω , can be measured from the graph: it is twice the distance between successive zeros of $x(t)$, which is always the same. Now what is the spacing between successive maxima of $x(t)$? Is it always the same, or does it differ from one successive pair of maxima to the next?

The extrema of $x(t) = A e^{-at} \cos(\omega t)$ occur when $\dot{x}(t) = 0$, i.e., $-a \cos(\omega t) = \omega \sin(\omega t)$. The extrema are achieved at t where $\tan(\omega t) = -a/\omega$. Since minima and maxima of $x(t)$ are alternating, the maxima occur at every other such t , and the spacing between successive maxima is twice the period of $\tan(\omega t)$, which is just the pseudo-period $2\pi/\omega$. This is always the same since it doesn't depend on t or x .

4. Suppose that successive maxima of $x(t) = A e^{-at} \cos(\omega t)$ occur at $t = t_0$ and $t = t_1$. What is the ratio $x(t_1)/x(t_0)$? (Hint: Compare $\cos(\omega t_0)$ and $\cos(\omega t_1)$.) Does this offer a means of determining the value of a from the graph?

As seen in Question 3, $t_1 - t_0 = 2\pi/\omega$. So $\cos(\omega t_1) = \cos(\omega t_0 + 2\pi) = \cos(\omega t_0)$ are the same, and the ratio is given by $x(t_1)/x(t_0) = e^{-a(t_1-t_0)}$. So a can be estimated graphically using the formula $a = \frac{\ln x(t_0) - \ln x(t_1)}{t_1 - t_0}$, where $(t_0, x(t_0))$ and $(t_1, x(t_1))$ are any two successive maxima.

5. For what value of b does $\ddot{x} + b\dot{x} + x = 0$ exhibit critical damping? For this value of b , what is the solution x_1 with $x_1(0) = 1$, $\dot{x}_1(0) = 0$? What is the solution x_2 with $x_2(0) = 0$, $\dot{x}_2(0) = 1$? (This is a “normalized pair” of solutions.) What is the solution such that $x(0) = 2$ and $\dot{x}(0) = 3$?

The characteristic polynomial is $p(s) = s^2 + bs + 1$. For the system to be critically damped, the characteristic polynomial must be a perfect square, i.e., $p(s)$ must equal $(s - k)^2$ for some k . Multiplying and comparing gives $b = -2k$ and $k^2 = 1$. Therefore, $b = \pm 2$. When $b = -2$, e^t is a solution, and it exhibits exponential growth instead of damping, so we reject that value of b . So the system is critically damped when $b = 2$.

For this value of b , the general solution is $x(t) = (C_0 + C_1 t)e^{-t}$. The corresponding initial conditions for each solution are $x(0) = C_0$ and $\dot{x}(0) = -C_0 + C_1$. By using the given initial conditions to solve for the constants, we see that the normalized pair of solutions is $x_1 = e^{-t} + te^{-t}$ and $x_2 = te^{-t}$.

We can use these normalized solutions to read off the solution that satisfies any given initial condition. In particular, the solution with $x(0) = 2$ and $\dot{x}(0) = 3$ is

$$x = x(0) \cdot x_1 + \dot{x}(0) \cdot x_2 = 2x_1 + 3x_2 = 2e^{-t} + 5te^{-t}.$$

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