

The Existence and Uniqueness Theorem for Linear Systems

For simplicity, we stick with $n = 2$, but the results here are true for all n . There are two questions about the following general linear system that we need to consider.

$$\begin{aligned} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{aligned} \quad ; \text{ in matrix form, } \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1)$$

The first is from the previous section: to show that all solutions are of the form

$$\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2,$$

where the \mathbf{x}_i form a fundamental set, that is, no \mathbf{x}_i is a constant multiple of the other). (The fact that we can write down *all* solutions to a linear system in this way is one of the main reasons why such systems are so important.)

An even more basic question for the system (1) is: how do we know that it *has* two linearly independent solutions? For systems with a constant coefficient matrix A , we showed in the previous chapters how to solve them explicitly to get two independent solutions. But the general non-constant linear system (1) does not have solutions given by explicit formulas or procedures.

The answers to these questions are based on following theorem.

Theorem 2 Existence and uniqueness theorem for linear systems.

If the entries of the square matrix $A(t)$ are continuous on an open interval I containing t_0 , then the initial value problem

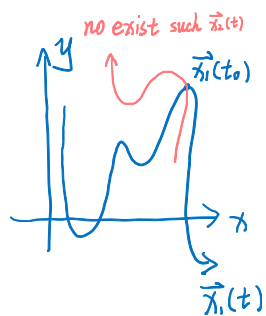
$$\mathbf{x}' = A(t) \mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (2)$$

has one and only one solution $\mathbf{x}(t)$ on the interval I .

The proof is difficult and we shall not attempt it. More important is to see how it is used. The following three theorems answer the questions posed for the 2×2 system (1). They are true for $n > 2$ as well, and the proofs are analogous.

In the following theorems, we assume the entries of $A(t)$ are continuous on an open interval I . Here the conclusions are valid on the interval I , for example, I could be the whole t -axis.

Theorem 2A Linear independence theorem.



Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two solutions to (1) on the interval I , such that at some point t_0 in I , the vectors $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(t_0)$ are linearly independent. Then

- a) the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent on I , and
- b) the vectors $\mathbf{x}_1(t_1)$ and $\mathbf{x}_2(t_1)$ are linearly independent at every point t_1 of I .

Proof. a) By contradiction. If they were dependent on I , one would be a constant multiple of the other, say $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$. Then $\mathbf{x}_2(t_0) = c_1\mathbf{x}_1(t_0)$, showing them dependent at t_0 . \square

b) By contradiction. If there were a point t_1 on I where they were dependent, say $\mathbf{x}_2(t_1) = c_1\mathbf{x}_1(t_1)$, then $\mathbf{x}_2(t)$ and $c_1\mathbf{x}_1(t)$ would be solutions to (1) which agreed at t_1 . Hence, by the uniqueness statement in Theorem 2, $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$ on all of I , showing them linearly dependent on I . \square

Given $\mathbf{x}_2(t_1)$, there is only one solution \mathbf{x}_2 . $\Rightarrow \mathbf{x}_2$ and $c_1\mathbf{x}_1$ are the same
 \hookrightarrow which contradict with 2A

Theorem 2B General solution theorem.

- a) The system (1) has two linearly independent solutions.
- b) If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are any two linearly independent solutions, then every solution \mathbf{x} can be written in the form (3), for some choice of c_1 and c_2 :

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2. \quad (3)$$

Proof. Choose a point $t = t_0$ in the interval I .

- a) According to Theorem 2, there are two solutions $\mathbf{x}_1, \mathbf{x}_2$ to (1), satisfying respectively the initial conditions

$$\mathbf{x}_1(t_0) = \mathbf{i}, \quad \mathbf{x}_2(t_0) = \mathbf{j}, \quad (4)$$

where \mathbf{i} and \mathbf{j} are the usual unit vectors in the xy -plane. Since the two solutions are linearly independent when $t = t_0$, they are linearly independent on I , by Theorem 2A.

- b) Let $\mathbf{u}(t)$ be a solution to (1) on I . Since \mathbf{x}_1 and \mathbf{x}_2 are independent at t_0 by Theorem 2, using the parallelogram law of addition we can find constants c'_1 and c'_2 such that

$$\mathbf{u}(t_0) = c'_1\mathbf{x}_1(t_0) + c'_2\mathbf{x}_2(t_0). \quad (5)$$

The vector equation (5) shows that the solutions $\mathbf{u}(t)$ and $c'_1\mathbf{x}_1(t) + c'_2\mathbf{x}_2(t)$ agree at t_0 . Therefore by the uniqueness statement in Theorem 2, they are equal on all of I ; that is,

$$\mathbf{u}(t) = c'_1\mathbf{x}_1(t) + c'_2\mathbf{x}_2(t) \quad \text{on } I.$$

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