Time Invariance

In the case of constant coefficient operators $p(D)$, there is an important and useful relationship between solutions to $p(D)x = q(t)$ for input signals $q(t)$ which start at different times t. The following result shows why these operators are called "Linear Time Invariant" (or LTI).

$p(D)x(t) = q(t)$
$t = t - c$
$\Rightarrow p(D)x(t-c) = q(t-c)$
Just variable substitution

Translation invariance. If $p(D)$ is a constant-coefficient differential operator and $p(D)x = q(t)$, then $p(D)y = q(t-c)$, where $y(t) = x(t-c)$.

This is the "time invariance" of $p(D)$. Here is an example of its use.

Example. Suppose that we know that $x_p(t) = \sqrt{2}\sin(t/2 - \pi/4)$ is a solution to the DE

$$2\ddot{x} + \dot{x} + x = \sin(t/2) \tag{1}$$

Find a solution y_p to

$\Downarrow$ $t/2 \to t/2 - \pi/3$

$$2\ddot{x} + \dot{x} + x = \sin(t/2 - \pi/3) \tag{2}$$

Solution. By translation-invariance, we have immediately that

$$y_p = \sqrt{2}\sin(t/2 - \pi/4 - \pi/3) = \sqrt{2}\sin(t/2 - 7\pi/12).$$