Table Entries: Repeated Quadratic Factors

1. Repeated Quadratic Factors

We will add three entries to our Laplace Table.

$$\mathcal{L}\left(\frac{1}{2\omega^3}(\sin(\omega t) - \omega t\cos(\omega t))\right) = \frac{1}{(s^2+\omega^2)^2} \tag{1}$$

$$\mathcal{L}\left(\frac{t}{2\omega}\sin(\omega t)\right) = \frac{s}{(s^2+\omega^2)^2} \tag{2}$$

$$\mathcal{L}\left(\frac{1}{2\omega}(\sin(\omega t) + \omega t\cos(\omega t))\right) = \frac{s^2}{(s^2+\omega^2)^2} \tag{3}$$

There are several ways to prove these formulas. We will give one using partial fractions by factoring the denominators on the frequency side into complex linear factors.

RH - side

Proof of 1. First some algebra:

$$\frac{1}{(s-a)^2(s+a)^2} = \frac{A}{(s-a)^2} + \frac{B}{s-a} + \frac{C}{(s+a)^2} + \frac{D}{s+a}$$

Cover-up gives us A and C. Undetermined coefficients then gives B and D:

$$A = \frac{1}{4a^2} = C, \quad D = \frac{1}{4a^3} = -B$$

This gives the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^2(s+a)^2}\right) = \frac{1}{4a^2}(te^{at} + te^{-at}) - \frac{1}{4a^3}(e^{at} - e^{-at}). \tag{4}$$

We will use this on the right hand side of (1), but first recall

$$e^{i\omega t} + e^{-i\omega t} = 2\cos(\omega t) \quad \text{and} \quad e^{i\omega t} - e^{-i\omega t} = 2i\sin(\omega t) \tag{5}$$

Let $a = i\omega$, then (4) and (5) combine to prove formula (1).

$$(s-i\omega)^2(s+i\omega)^2 = [(s-i\omega)(s+i\omega)]^2 = (s^2+\omega^2)^2$$

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) &= \mathcal{L}^{-1}\left(\frac{1}{(s-i\omega)^2(s+i\omega)^2}\right) \\
&= -\frac{1}{4\omega^2}(te^{i\omega t} + te^{-i\omega t}) + \frac{1}{4\omega^3 i}(e^{i\omega t} - e^{-i\omega t}). \\
&= -\frac{1}{2\omega^2}t\cos(\omega t) + \frac{1}{2\omega^3}\sin(\omega t).
\end{aligned}$$

The proofs of (2) and (3) are similar, and we will omit them.

2. Note on the Relation to Resonance:

Each of the formulas (1), (2), and (3) has a term with a factor of t. This is exactly what we saw with the response x in the resonance equation

$$\ddot{x} + \omega^2 x = \cos(\omega t),$$

which has solution $x(t) = t\sin(\omega t)/(2\omega)$.

Notice that $\mathcal{L}^{-1}(1/s^2) = t$ and the s-shift rule shows $\mathcal{L}^{-1}(1/(s-a)^2) = te^{at}$. So repeated factors on the frequency side always lead to multiplication by t on the time side. If the repeated factor has a higher power then we get multiplication by a higher power of t.