Integration and Differentiation

We can integrate a Fourier series term-by-term:

Example 1. Let

$$f(t) = 1 + \cos t + \frac{\cos 2t}{2} + \frac{\cos 3t}{3} + \ldots$$

then,

$$h(t) \;=\; \int_0^t f(u)\,du \;=\; t + \sin t + \frac{\sin 2t}{2^2} + \frac{\sin 3t}{3^2} + \ldots$$

Note: The integrated function $h(t)$ is not periodic (because of the t term), so the result is a series, but not a Fourier series.

We can also differentiate a Fourier series term-by-term to get the Fourier series of the derivative function.

Example 2. Let $f(t)$ be the period 2π triangle wave (continuous sawtooth) given on the interval $[-\pi, \pi)$ by $f(t) = |t|$. Its Fourier series is

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \ldots\right)$$

In the previous session we computed the Fourier series of a period 2 triangle wave. This series can then be obtained from that one by scaling by π in both time and the vertical dimension, using the methods we learned in the previous note.
The derivative of $f(t)$ is the square wave. (You should verify this). Differentiating the Fourier series of $f(t)$ term-by-term gives

$$f'(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots\right),$$

which is, indeed, the Fourier series of the period 2π square wave we found in the previous session.

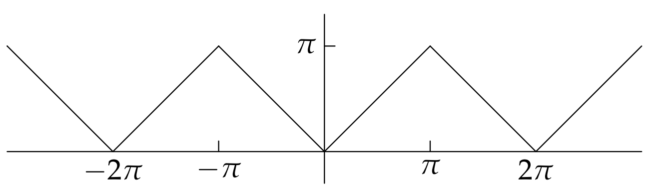

Figure 1: The period 2π triangle wave.

Example 3. What happens if you try to differentiate the square wave

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots\right)?$$

Solution. Differentiation term-by-term gives

$$\mathrm{sq}'(t) = \frac{4}{\pi}\left(\cos t + \cos 3t + \cos 5t + \ldots\right).$$

But, what is meant by $\mathrm{sq}'(t)$? Since $\mathrm{sq}(t)$ consists of horizontal segments its derivative at most places is 0. However we can't ignore the 'vertical' segments where the function has a *jump discontinuity*. For now, the best we can say is that the slope is infinite at these jumps and $\mathrm{sq}'(t)$ doesn't exist. Later in this unit we will learn about *delta functions* and *generalized derivatives*, which will allow us to make better sense of $\mathrm{sq}'(t)$.