

Second order Unit Impulse Response

1. Effect of a Unit Impulse on a Second order System

We consider a second order system

$$m\ddot{x} + b\dot{x} + kx = f(t). \quad (1)$$

Our first task is to derive the following. If the input $f(t)$ is an impulse $c\delta(t - a)$, then the system's response to $f(t)$ has the following properties.

1. The momentum $m\dot{x}(t)$ jumps by c units at $t = a$. That is,

$$m\dot{x}(a^+) - m\dot{x}(a^-) = c.$$

2. The position $x(t)$ is unchanged at $t = a$. That is,

$$x(a^+) = x(a^-).$$

Argument 1:

Recall the argument that we used before: If $x(t)$ had a jump at a then $\dot{x}(t)$ would contain a multiple of $\delta(t - a)$. So, $m\ddot{x}(t)$ would contain a multiple of the doublet $\delta'(t - a)$. This is impossible since the input $\delta(t - a)$ does not contain a doublet. This shows point (2) above.

To show point (1), we note that if $m\dot{x}(t)$ has a jump of c units at $t = a$ then $m\ddot{x}(t)$ contains the term $c\delta(t - a)$. This is needed to make the left-hand side of equation (1) match the right hand side when $f(t) = c\delta(t - a)$.

Argument 2:

Another way to show points (1) and (2) is a physical argument. A force acting on the mass over time changes its momentum. In fact, the best way to state Newton's second law is that

$$\frac{dp}{dt} = f(t),$$

where $p(t)$ is the momentum of a system and $f(t)$ is an external force acting on the system. If a force $f(t)$ acts over the time interval $[t_1, t_2]$ the total change of momentum due to the force is

$$\int_{t_1}^{t_2} f(t) dt.$$

Physicists call this the impulse of the force $f(t)$ over the interval $[t_1, t_2]$. If a very large force is applied over a very short time interval and has total impulse of 1 the result will be a sudden unit jump in the momentum of the system.

For a second order system the unit impulse function δ can be thought of as an idealization of this force. It is a force with total impulse 1 applied all at once.

Argument 3: A third argument that we will skip would be to solve equation (1) with a box function for input and take the limit as the box gets narrower and taller always with area 1.

2. Unit Impulse Response

We consider once again the damped harmonic oscillator equation

$$m\ddot{x} + b\dot{x} + kx = f(t).$$

The **unit impulse response** is the solution to this equation with input $f(t) = \delta(t)$ and rest initial conditions: $x(t) = 0$ for $t < 0$. That is, it is the solution to the initial value problem (IVP)

$$m\ddot{x} + b\dot{x} + kx = \delta(t), \quad x(0^-) = 0, \quad \dot{x}(0^-) = 0.$$

This could be a damped spring-mass system with mass m , damping constant b and spring constant k . The mass is at rest at equilibrium until time $t = 0$ when it is hit by a sudden very brief very intense force, rather like getting hit on the head by a hammer. The effect is to increase the momentum instantaneously, without changing the position of the mass.

Let $w(t)$ denote the solution we seek. The rest initial conditions tell us that $w(t) = 0$ for $t < 0$. We know from section 1 that the effect of the input is to cause a unit jump in the momentum at $t = 0$ and no change in position. We also know that, for $t > 0$, the input $\delta(t) = 0$. Putting this together, for $t > 0$ the $w(t)$ satisfies the equation

$$m\ddot{w} + b\dot{w} + kw = 0, \quad \dot{w}(0^+) = 1/m, \quad w(0^+) = 0.$$

This is a homogeneous constant coefficient linear differential equation which we have lots of practice in solving.

Example 1. Find the unit impulse response for the system

$$2\ddot{x} + 8\dot{x} + 26x = f(t). \tag{2}$$

Solution. We will use the standard notation $w(t)$ for the unit impulse response. We are looking for the response from rest to $f(t) = \delta(t)$. We know

$$\begin{aligned}
2\ddot{w}(t) &= \delta(t) + \dots \text{continuous things} \\
\int 2\ddot{w}(t) dt &= \int \delta(t) dt + \dots \\
&= \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} + \dots \\
\Rightarrow 2\dot{w}(0^+) &= 1
\end{aligned}$$

$w(t) = 0$ for $t < 0$. At $t = 0$ the input causes a unit jump in momentum, i.e., $2\dot{w}(0^+) = 1$. So, for $t > 0$ we have to solve

$$2\ddot{w} + 8\dot{w} + 26w = 0, \quad \dot{w}(0^+) = 1/2, w(0^+) = 0.$$

The roots of the characteristic polynomial are $-2 \pm 3i$. Which implies

$$w(t) = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t), \quad \text{for } t > 0.$$

The initial conditions give

$$\begin{aligned}
0 &= w(0^+) = c_1, \\
1/2 &= \dot{w}(0^+) = -2c_1 + 3c_2 \Rightarrow c_2 = 1/6.
\end{aligned}$$

Thus, the unit impulse response (in both cases and u -format) is

$$w(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{6} e^{-2t} \sin(3t) & \text{for } t > 0 \end{cases} = \frac{1}{6} e^{-2t} \sin(3t) u(t). \quad (3)$$

Figure 1 the graph of the unit impulse response. Notice that at $t = 0$ the graph has a corner. This corresponds to the slope \dot{w} jumping from 0 to $1/2$. For $t > 0$ the graph decays to 0 while oscillating.

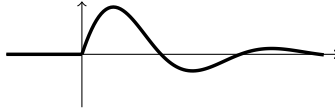


Figure 1. The unit impulse response for the system $2\ddot{x} + 8\dot{x} + 26x$.

3. Checking Example 1 by Substitution

With any differential equation you can verify a solution by plugging it into the equation. We will do that with example 1 to gain some more insight into why we get the solution.

First, we take the derivatives of the solution in equation (3) for $t \neq 0$

$$\begin{aligned}
\dot{w}(t) &= \begin{cases} \frac{1}{6} e^{-2t} (-2 \sin(3t) + 3 \cos(3t)) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} = \frac{1}{6} e^{-2t} (-2 \sin(3t) + 3 \cos(3t)) u(t) \\
\ddot{w}(t) &= \begin{cases} \frac{1}{6} e^{-2t} (-5 \sin(3t) - 12 \cos(3t)) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} = \frac{1}{6} e^{-2t} (-5 \sin(3t) - 12 \cos(3t)) u(t)
\end{aligned}$$

Next we look at the jumps at $t = 0$

$$\begin{aligned}
w(0^-) &= 0, & w(0^+) &= 0 \\
\dot{w}(0^-) &= 0, & \dot{w}(0^+) &= 1/2
\end{aligned}$$

Now we can compute the full generalized derivatives (we just give them in u -format)

$$\begin{aligned}\dot{w}(t) &= \frac{1}{6}e^{-2t}(-2\sin(3t) + 3\cos(3t)) u(t) \\ \ddot{w}(t) &= \frac{1}{2}\delta(t) + \frac{1}{6}e^{-2t}(-5\sin(3t) - 12\cos(3t)) u(t)\end{aligned}$$

Finally we substitute w for x in equation (2)

$$\begin{aligned}2\ddot{w}(t) &= \delta(t) - \frac{5}{3}e^{-2t}\sin(3t) - 4e^{-2t}\cos(3t) \\ 8\dot{w}(t) &= \frac{-8}{3}e^{-2t}\sin(3t) + 4e^{-2t}\cos(3t) \\ 26w(t) &= \frac{13}{3}e^{-2t}\sin(3t)\end{aligned}$$

$$2\ddot{w} + 8\dot{w} + 26w = \delta(t).$$

The Meaning of the Phrase ‘Unit Impulse Response’

As we’ve noted several times already, the response to a given input depends on what we consider to be the input. For example, if our system is

$$m\ddot{x} + b\dot{x} + kx = b\dot{y}$$

and we consider y to be the input, then the unit impulse response is the solution to

$$m\ddot{x} + b\dot{x} + kx = b\dot{\delta}(t) \quad \text{with rest IC.}$$

(Here, $\dot{\delta}$ is what we’ve called a *doublet*.) For $t > 0$ this is equivalent to

$$m\ddot{x} + b\dot{x} + kx = 0 \quad \text{with post IC} \quad x(0^+) = \frac{b}{m}, \quad \dot{x}(0^+) = -\frac{b^2}{m^2}$$

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18.03SC Differential Equations
Fall 2011

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