

18.03SC Practice Problems 29

Solving IVP's

Solution suggestions

1. Let $f(t) = e^{-t} \cos(3t)$.

(a) From the rules and tables, what is $F(s) = \mathcal{L}[f(t)]$?

From the formulas,

$$\mathcal{L}[\cos(3t)] = \frac{s}{s^2 + 9}.$$

So, by s -shift,

$$F(s) = \mathcal{L}[e^{-t} \cos(3t)] = \frac{s + 1}{(s + 1)^2 + 9}.$$

(b) Compute the derivative $f'(t)$ and its Laplace transform. Verify the t -derivative rule in this case.

Now we want to compute the derivative. Use the product rule to get that

$$f'(t) = e^{-t}(-\cos(3t) - 3\sin(3t)).$$

Now, from the tables, we have that $\mathcal{L}[\sin(3t)] = \frac{3}{s^2 + 9}$. Therefore, by linearity and s -shift,

$$\begin{aligned}\mathcal{L}[f'(t)] &= -\mathcal{L}[e^{-t} \cos(3t)] - 3\mathcal{L}[e^{-t} \sin(3t)] \\ &= -\frac{s + 1}{(s + 1)^2 + 9} - 3\frac{3}{(s + 1)^2 + 9} \\ &= \frac{-s - 10}{(s + 1)^2 + 9}.\end{aligned}$$

Also, $f(0^-) = 1$. So, using the expression we found in part (a) for $F(s)$, we get that

$$sF(s) - f(0^-) = \frac{s^2 + s}{(s + 1)^2 + 9} - 1 = \frac{-s - 10}{(s + 1)^2 + 9} = \mathcal{L}[f'(t)],$$

verifying the t -derivative rule in this case.

2. Use the Laplace transform to find the unit step and impulse response of the operator $D + 2I$.

The unit step response is a solution to $\dot{v} + 2v = u(t)$ which vanishes for $t < 0$.

Take the Laplace transform of both sides, denoting the transform of $v(t)$ by $V(s)$.

The transform of the left hand side, under rest preinitial conditions, is $\mathcal{L}[\dot{v}(t) + 2v(t)] = sV(s) + 2V(s)$. The transform of the right side is $\mathcal{L}[u(t)] = \frac{1}{s}$. So, in s -domain, the equation becomes

$$sV(s) + 2V(s) = \frac{1}{s},$$

and we get that $V(s) = \frac{1}{s(s+2)}$, which has inverse transform

$$\mathcal{L}^{-1} \left[\frac{1}{s(s+2)} \right] = \mathcal{L}^{-1} \left[\frac{1/2}{s} + \frac{-1/2}{s+2} \right] = \frac{1}{2} - \frac{1}{2} e^{-2t},$$

for $t > 0$. By definition, the unit step response is zero for $t < 0$, so this really means that the unit step response is

$$v(t) = \left(\frac{1}{2} - \frac{1}{2} e^{-2t} \right) u(t).$$

In turn, the unit impulse response is a solution to $\dot{w} + 2w = \delta(t)$ which vanishes for $t < 0$.

Again, take the Laplace transform of both sides under rest preinitial conditions, and solve for $\mathcal{L}[w(t)] = W(s)$.

The transform of the left hand side is the same as before, but now the right hand side is just $\mathcal{L}[\delta(t)] = 1$, so $W(s) = \frac{1}{s+2}$. Taking the inverse transform and multiplying by a step function to get a solution that vanishes for $t < 0$, we find that the unit impulse response is

$$w(t) = e^{-2t} u(t).$$

Note that, because $\delta(t)$ has such a simple transform, the unit impulse response $w(t)$ is usually easier to find using LT than other responses. Remark also that the derivative of $v(t)$ is $w(t)$. Why should this make sense?

3. Use the Laplace transform to find the solution to $\dot{x} + 2x = t^2$ with initial condition $x(0) = 1$.

Using the rules to take the transform of both sides, we get the equation $sX(s) - 1 + 2X(s) = \frac{2}{s^3}$, which has solution $X(s) = \frac{s^3+2}{s^3(s+2)}$. The partial fractions decomposition of this expression has the form

$$\frac{s^3+2}{s^3(s+2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+2}.$$

As before, we find the leading coefficients by Heaviside cover up: $A = 2/2 = 1$ and $D = (-8+2)/(-8) = 3/4$. We then set up two systems of equations for the remaining coefficients and deduce that $B = -1/2$ and $C = 1/4$. Then, from the tables, we can read off that the form of the inverse Laplace transform of this expression is

$$\mathcal{L}^{-1}[X(s)] = \frac{A}{2} t^2 - Bt + C + De^{-2t},$$

for $t > 0$, so the solution is

$$x(t) = \frac{1}{2} t^2 - \frac{1}{2} t + \frac{1}{4} + \frac{3}{4} e^{-2t},$$

for $t > 0$.

Solve each of the following by using the Laplace transform.

4. Find the unit impulse response of the operator $D + 3I$.

The unit impulse response of this operator is the solution to $\dot{w}(t) + 3w(t) = \delta(t)$ that vanishes for $t < 0$.

We use the Laplace transform. Taking the transform of both sides under rest preinitial conditions (i.e., using $w(0^-) = 0$), we get the equation

$$sW(s) + 3W(s) = 1,$$

or $W(s) = \frac{1}{s+3}$. This has inverse transform

$$\mathcal{L}^{-1}[W(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = e^{-3t},$$

for $t > 0$, so the unit impulse response is

$$w(t) = e^{-3t}u(t).$$

5. Find the solution to $\dot{x} + 3x = e^{-t}$ with rest initial conditions (so $x(0) = 0$).

Again, we use the Laplace transform to solve.

Taking the transform of both sides under rest initial conditions, we get that

$$sX(s) + 3X(s) = \frac{1}{s+1},$$

so, $X(s) = \frac{1}{(s+1)(s+3)}$. To take the inverse transpose, we use partial fractions:

$$X(s) = \frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3},$$

and, by the cover up method, $A = 1/(-1+3) = 1/2$ and $B = 1/(-3+1) = -1/2$. So, by linearity and the formula tables, we get that

$$x(t) = \mathcal{L}^{-1}[X(s)] = Ae^{-t} + Be^{-3t} = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t},$$

for $t > 0$.

6. Find the unit impulse response of $D^3 + D$.

The unit impulse response of this operator is the solution to $w^{(3)} + w' = \delta$ that vanishes for $t < 0$.

We proceed as before. Taking the transform of the equation, we get $s^3W(s) + sW(s) = 1$, so

$$W(s) = \frac{1}{s^3 + s} = \frac{1}{s(s^2 + 1)}.$$

By partial fractions,

$$W(s) = \frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1},$$

which has inverse transform $1 - \cos t$ for $t < 0$, so the unit impulse response is

$$w(t) = (1 - \cos t)u(t).$$

7. (a) Find the solution (for $t > 0$) to $\dot{x} + 3x = 1$ with $x(0) = 2$ by applying the Laplace transform to the equation.

The Laplace transform of this equation under the given initial condition is

$$sX(s) - 2 + 3X(s) = \frac{1}{s}$$

so $X(s) = \frac{(1/s)+2}{s+3} = \frac{1+2s}{s(s+3)}$. By partial fractions,

$$X(s) = \frac{1+2s}{s(s+3)} = \frac{(1/3)}{s} + \frac{(5/3)}{s+3},$$

and the solution is the inverse transform

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{3} + \frac{5}{3}e^{-3t},$$

for $t > 0$.

(b) Find the solution (for $t > 0$) to $\dot{x} + 3x = 1 + 2\delta(t)$ with rest initial conditions by applying the Laplace transform to the equation.

We take the Laplace transform of this new equation, now under rest conditions (i.e., using $x(0^-) = 0$), to get the equation

$$sX(s) + 3X(s) = \frac{1}{s} + 2,$$

and observe that this is satisfied by the same $X(s) = \frac{(1/s)+2}{s+3}$ as in (a). So the solution is again

$$x(t) = \frac{1}{3} + \frac{5}{3}e^{-3t},$$

for $t > 0$.

(c) Explain the relationship between these two problems.

The first problem asks to find the responses of an operator to an input under nonzero preinitial conditions. The second problem asks to find the response of the same operator to a slightly modified input under rest (zero) preinitial conditions. The input in the second problem has been modified in just the right way to get the same equation in s -domain. Effectively, the second problem is the same as the first problem, with the preinitial conditions “encoded on the input,” and both problems have the same response $x(t)$ for $t > 0$.

The intuition behind encoding initial conditions on the input in this way is that, in some sense, we are mimicking having nonzero preinitial conditions by starting out with a system at zero and then jolting it with the right impulse, in addition to the original input.

Note that sending in an appropriately weighted delta function worked here because we were dealing with a first-order operator. It is possible, but slightly more tricky, to figure out how to encode preinitial conditions on the input for higher-order operators as well. (Hint: you would need to know about derivatives of the delta function, which were not introduced in this class.)

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