Scaling and Shifting

There is a very useful class of shortcuts which allows us to use the known Fourier series of a function $f(t)$ to get the series for a function related to $f(t)$ by shifts and scale changes. We illustrate this technique with a collection of examples of related functions.

We let $\mathrm{sq}(t)$ be the standard odd, period 2π square wave.

$$\mathrm{sq}(t) = \begin{cases} -1 & \text{for } -\pi \le t < 0 \\ 1 & \text{for } 0 \le t < \pi \end{cases} \tag{1}$$

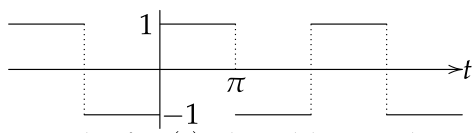

Figure 0: The graph of $\mathrm{sq}(t)$, the odd, period 2π square wave.

We already know the Fourier series for $\mathrm{sq}(t)$. It is

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n} \tag{2}$$

1. Shifting and Scaling in the Vertical Direction

Example 1. (Shifting) Find the Fourier series of the function $f_1(t)$ whose graph is shown.

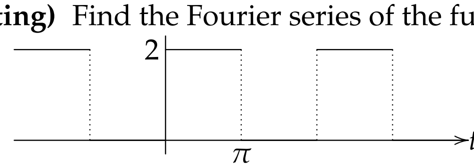

Figure 1: $f_1(t) = \mathrm{sq}(t)$ shifted up by 1 unit.

Solution. The graph in Figure 1 is simply the graph in Figure 0 shifted upwards one unit. That is, $f_1(t) = 1 + \mathrm{sq}(t)$. Therefore

$$f_1(t) = 1 + \frac{4}{\pi}\sum_{n \text{ odd}} \frac{sin(nt)}{n}.$$

Example 2. (Scaling) Let $f_2(t) = 2\,\mathrm{sq}(t)$. Sketch its graph and find its Fourier series.

Solution.

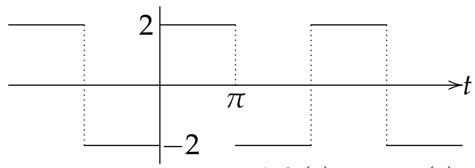

Figure 2: Graph of $f_2(t) = 2\,\mathrm{sq}(t)$.

The Fourier series of $f_2(t)$ comes from that of $\mathrm{sq}(t)$ by multiplying by 2.

$$f_2(t) = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{sin(nt)}{n}.$$

Example 3. We can combine shifting and scaling along the vertical axis. Let $f_3(t)$ be the function shown in Figure 3. Write it in terms of $\mathrm{sq}(t)$ and find its Fourier series.

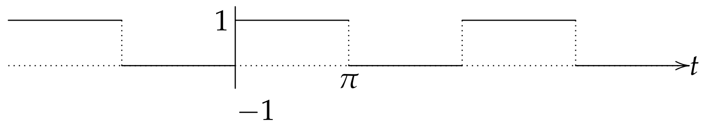

Figure 3: $f_3(t) = \mathrm{sq}(t)$ shifted by 1 and then scaled by 1/2.

Solution. $f_3(t) = \frac{1}{2}(1 + \mathrm{sq}(t)) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$.

2. Scaling and Shifting in t

Example 4. (Scaling in time) Find the Fourier series of the function $f_4(t)$ whose graph is shown.

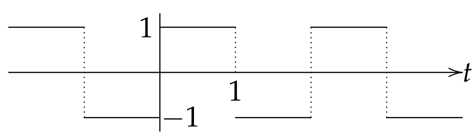

Figure 4: $\mathrm{sq}(t)$ scaled in time.

InFigure 4 the point marked 1 on the t-axis corresponds with the point marked π in Figure 0. This shows that $f_4(t) = \mathrm{sq}(\pi t)$ and therefore we replace t by πt in the Fourier series of $\mathrm{sq}(t)$.

$$f_4(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{sin(n\pi t)}{n}.$$

Example 5. **(Shifting in time)** Let $f_5(t) = \mathrm{sq}(t + \pi/2)$. Graph this function and find its Fourier series.

Solution. We have $f_5(t)$ is $\mathrm{sq}(t)$ shifted to the left by $\pi/2$. Therefore

$$f_5(t) = \frac{4}{\pi}\left(\sin(t+\pi/2) + \frac{\sin(3t+3\pi/2)}{3} + \ldots\right) = \frac{4}{\pi}\left(\cos t - \frac{\cos 3t}{3} + \ldots\right)$$

(To simplify the series we used the trig identities $\sin(\theta + \pi/2) = \cos(\theta)$ and $\sin(\theta + 3\pi/2) = -\cos(\theta)$ etc.)

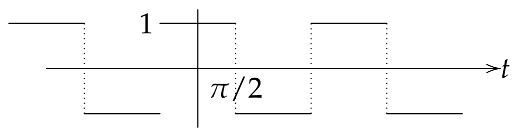

Figure 5: $\mathrm{sq}(t)$ shifted in time.

Notice that $f_5(t)$ is even, and so must have only cosine terms in its series, which is in fact confirmed by the simplified form above.