

Part II Problems and Solutions

Problem 1: [Second order ODEs via Laplace transform] Find the unit impulse response of the following operators by means of the Laplace transform.

(a) $3D^2 + 6D + 6I$.

(b) $D^4 - I$.

Solution: (a) $w(t)$ has Laplace transform $W(s) = \frac{1}{3s^2 + 6s + 6} = \frac{1}{3} \frac{1}{(s+1)^2 + 1} \cdot \mathcal{L}(\sin t) = \frac{1}{s^2+1}$, so by s -shift $w(t) = \frac{1}{3}u(t)e^{-t} \sin t$.

(b) $W(s) = \frac{1}{s^4 - 1}$. The roots of $s^4 - 1$ are ± 1 and $\pm i$, so we can write

$$\frac{1}{s^4 - 1} = \frac{a}{s-1} + \frac{b}{s+1} + \frac{c}{s-i} + \frac{d}{s+i}. \text{ Cover-up gives easily } a = b = \frac{1}{4},$$

$$c = \frac{i}{4}, d = -\frac{i}{4}. \text{ So}$$

$$w(t) = u(t) \frac{1}{4} (e^t + e^{-t} + ie^{it} - ie^{-it}) = u(t) \frac{1}{2} (\sinh(t) - \sin(t))$$

$$(\text{where } \sinh(t) = \frac{1}{2} (e^t + e^{-t}), \text{ the hyperbolic sine function})$$

$$e^{it} - e^{-it} = 2i \sin(t)$$

$$i(e^{it} - e^{-it}) = -2 \sin(t)$$

$$\frac{1}{s^4-1} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s-1}$$

$$s=-1 \quad C = \frac{1}{2(-2)} = -\frac{1}{4}$$

$$s=1 \quad D = \frac{1}{4}$$

$$\Rightarrow \frac{1}{s^4-1} = \frac{As+B}{s^2+1} - \frac{1}{4(s+1)} + \frac{1}{4(s-1)}$$

$$s=0 \Rightarrow -1 = B - \frac{1}{4} - \frac{1}{4}$$

$$\Rightarrow B = -\frac{1}{2}$$

$$s=2 \Rightarrow \frac{1}{15} = \frac{2A-\frac{1}{2}}{5} - \frac{1}{12} + \frac{1}{4}$$

$$\Rightarrow \frac{2}{30} = \frac{12A-3}{30} + \frac{5}{30}$$

$$\Rightarrow A = \frac{0}{12}$$

$$\Rightarrow \frac{1}{s^4-1} = \frac{-1}{2(s^2+1)} - \frac{1}{4(s+1)} + \frac{1}{4(s-1)}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^4-1}\right) = -\frac{1}{2} \sinh(t) - \frac{1}{4} e^{-t} + \frac{1}{4} e^t$$

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