

## The Transfer Function

### 1. Definition

We start with the definition (see equation (1)). In subsequent sections of this note we will learn other ways of describing the transfer function. (See equations (2) and (3).)

For any linear time invariant system the **transfer function** is

$$W(s) = \mathcal{L}(w(t)), \quad \text{where } w(t) \text{ is the unit impulse response.} \quad (1)$$

**Example 1.** Find the transfer function for the system  $\dot{x} + 3x = f(t)$ .

**Solution.** The unit impulse response is the solution to

$$\dot{w} + 3w = \delta(t), \quad \text{with rest IC.}$$

The Laplace transform method finds  $W(s)$  on the way to finding  $w(t)$ . Since we only want  $W(s)$  we can stop when we get there. Taking the Laplace transform of the DE we get

$$sW(s) - w(0^-) + 3W = 1 \quad \Rightarrow \quad W = \frac{1}{s+3}.$$

The 'annoying' term  $w(0^-) = 0$  because we have rest initial conditions. (Subsequent to this we will not bother writing the annoying terms when we have rest IC.)

### 2. Other Standard Terminology

The unit impulse response is also called the **weight function** and the transfer function is also called the **system function**. All of these terms are widely used and we will use them all to help you become familiar with them.

### 3. Formula for $W(s)$

**Example 2.** Find the transfer function for  $m\ddot{x} + b\dot{x} + kx = f(t)$ .

**Solution.** The unit impulse response is the solution to

$$m\ddot{w} + b\dot{w} + kw = \delta(t), \quad \text{with rest IC.}$$

By definition, the transfer function  $W(s) = \mathcal{L}(w)$ . So, we take the Laplace transform of the DE. There are no 'annoying terms' because with rest initial

conditions  $\mathcal{L}(\ddot{w}) = s^2 W(s)$  and  $\mathcal{L}(\dot{w}) = sW(s)$ . We get

$$(ms^2 + bs + k)W(s) = 1 \quad \Rightarrow \quad W(s) = \frac{1}{ms^2 + bs + k}.$$

In example 2, the differential operator is  $p(D) = mD^2 + bD + kI$ . That is, the characteristic polynomial is  $p(s) = ms^2 + bs + k$  and the transfer function is  $W(s) = 1/p(s)$ .

Exactly the same reasoning holds for operators of higher order.

**Formula:** For any polynomial operator  $p(D)$  the transfer function for the system

$$p(D)x = f(t)$$

is given by

$$W(s) = \frac{1}{p(s)}. \quad (2)$$

**Example 3.** Suppose  $W(s) = 1/(s^2 + 4)$  is the transfer function for a system  $p(D)x = f(t)$ . What is  $p(D)$ ?

**Solution.** Since  $W(s) = 1/p(s)$  we have  $p(s) = s^2 + 4$ , which implies  $p(D) = D^2 + 4I$ .

#### 4. Another Characterization of the Transfer Function

The best way to think of the transfer function is as a ratio of output to input. By this we mean the following.

Suppose we have an equation

$$p(D)x = f(t), \quad \text{with rest IC.}$$

Taking Laplace transform of both sides gives

$$p(s)X(s) = F(s) \quad \Rightarrow \quad X(s) = \frac{1}{p(s)}F(s) \stackrel{\text{assume } \frac{1}{p(s)} = W(s)}{=} W(s)F(s).$$

Solving for  $W(s)$  shows

$$W(s) = \frac{X(s)}{F(s)} = \frac{\text{output}}{\text{input}}. \quad (3)$$

#### 5. Conclusion

We have characterized the transfer functions in three different ways. Equations (1) and (3) are perfectly general and apply to any LTI system. Equation (2) is specific to constant coefficient linear differential equations.

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