Orthogonality Relations

We now explain the basic reason why the remarkable Fourier coefficent formulas work. We begin by repeating them from the last note:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\,dt, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\frac{\pi}{L}t)\,dt, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin(n\frac{\pi}{L}t)\,dt. \end{aligned} \tag{1}$$

The key fact is the following collection of integral formulas for sines and cosines, which go by the name of **orthogonality relations**:

$$\frac{1}{L}\int_{-L}^{L}\cos(n\tfrac{\pi}{L}t)\cos(m\tfrac{\pi}{L}t)\,dt = \begin{cases} 1 & n=m\neq 0 \\ 0 & n\neq m \\ 2 & n=m=0 \end{cases}$$

$$\frac{1}{L}\int_{-L}^{L}\cos(n\tfrac{\pi}{L}t)\sin(m\tfrac{\pi}{L}t)\,dt = 0$$

$$\frac{1}{L}\int_{-L}^{L}\sin(n\tfrac{\pi}{L}t)\sin(m\tfrac{\pi}{L}t)\,dt = \begin{cases} 1 & n=m\neq 0 \\ 0 & n\neq m \end{cases}$$

$$\cos nt \sin mt = \frac{e^{int}+e^{-int}}{2}\cdot\frac{e^{imt}-e^{-imt}}{2}$$

$$= \frac{1}{4}\left(e^{i(m+n)t}+e^{i(m-n)t}-e^{i(-m+n)t}-e^{i(-m-n)t}\right)$$

$$\int_{-\pi}^{\pi}\cos nt\sin mt\,dt = \frac{1}{4}$$

$$\frac{1}{m+n}e^{i(m+n)t}\Big|_{-\pi}^{\pi} - \frac{1}{-(m+n)}e^{-i(m+n)t}\Big|_{-\pi}^{\pi}$$

$$+\frac{1}{m-n}e^{i(m-n)t}\Big|_{-\pi}^{\pi} - \frac{1}{-(m-n)}e^{-i(m-n)t}\Big|_{-\pi}^{\pi}$$

$= 0$ (Symmetric cancel out)

Proof of the orthogonality relations: This is just a straightforward calculation using the periodicity of sine and cosine and either (or both) of these two methods:
Method 1: use $\cos at = \frac{e^{iat}+e^{-iat}}{2}$, and $\sin at = \frac{e^{iat}-e^{-iat}}{2i}$.
Method 2: use the trig identity $\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))$, and the similar trig identies for $\cos(\alpha)\sin(\beta)$ and $\sin(\alpha)\sin(\beta)$.

Using the orthogonality relations to prove the Fourier coefficient formula
Suppose we know that a periodic function $f(t)$ has a Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(n\frac{\pi}{L}t\right) + b_n\sin\left(n\frac{\pi}{L}t\right) \tag{2}$$

How can we find the values of the coefficients? Let's choose one coefficient, say a_2, and compute it; you will easily how to generalize this to any other coefficient. The claim is that the right-hand side of the Fourier coefficient formula (1), namely the integral

$$\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(2\frac{\pi}{L}t\right)\,dt.$$

is in fact the coefficent a_2 in the series (2). We can replace $f(t)$ in this integral by the series in (2) and multiply through by $\cos\left(2\frac{\pi}{L}t\right)$, to get

$$\frac{1}{L}\int_{-L}^{L}\frac{a_0}{2}\cos\left(2\frac{\pi}{L}t\right)+\sum_{n=1}^{\infty}\left(a_n\cos\left(n\frac{\pi}{L}t\right)\cos\left(2\frac{\pi}{L}t\right)+b_n\sin\left(n\frac{\pi}{L}t\right)\cos\left(2\frac{\pi}{L}t\right)\right)\,dt$$

Now the orthogonality relations tell us that almost every term in this sum will integrate to 0. In fact, the only non-zero term is the $n=2$ cosine term

$$\frac{1}{L}\int_{-L}^{L}a_2\cos\left(2\frac{\pi}{L}t\right)\cos\left(2\frac{\pi}{L}t\right)\,dt$$

and the orthogonality relations for the case $n=m=2$ show this integral is equal to a_2 as claimed.

Why the denominator of 2 in $\frac{a_0}{2}$?

Answer: it is in fact just a convention, but the one which allows us to have the same Fourier coefficent formula for a_n when $n=0$ and $n\geq 1$. (Notice that in the $n=m$ case for cosine, there is a factor of 2 only for $n=m=0$.)

Interpretation of the constant term $\frac{a_0}{2}$.

We can also interpret the constant term $\frac{a_0}{2}$ in the Fourier series of $f(t)$ as the average of the function $f(t)$ over one full period: $\frac{a_0}{2}=\frac{1}{2L}\int_{-L}^{L}f(t)\,dt$.