The Laplace Transform of the Delta Function

Since the Laplace transform is given by an integral, it should be easy to compute it for the delta function. The answer is

1. $\mathcal{L}(\delta(t)) = 1.$
2. $\mathcal{L}(\delta(t-a)) = e^{-as}$ for $a > 0.$

As expected, proving these formulas is straightforward as long as we use the precise form of the Laplace integral. For (1) we have:

$$\mathcal{L}(\delta(t)) = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = 1. \quad \text{converge for all } s$$

$$= e^{-s\cdot 0}\int_{0^-}^{\infty} \delta(t)\,dt$$

As we saw in a previous session, integrating e^{-st} against $\delta(t)$ amounts to evaluating e^{-st} at $t = 0$, and $e^0 = 1$. Similarly for the shifted version (2), integrating e^{-st} against $\delta(t-a)$ amounts to evaluating e^{-st} at $t = a$:

$$\mathcal{L}(\delta(t-a)) = \int_{0^-}^{\infty} \delta(t-a)e^{-st}\,dt = e^{-sa}. \quad \text{converge for all } s$$

$$= e^{-sa}\int_{0^-}^{\infty} \delta(t-a)\,dt$$

Notice that the two formulas are consistent: if we set $a = 0$ in formula (2) then we recover formula (1).