Proof of Green's Formula

Green's Formula: For the equation

$$P(D)y = f(t), \quad y(t) = 0 \text{ for } t < 0 \tag{1}$$

the solution for $t > 0$ is given by

$$y(t) = (f * w)(t) = \int_{0^-}^{t^+} f(\tau)w(t-\tau)\,d\tau, \tag{2}$$

where $w(t)$ is the weight function (unit impulse response) for the system.

Proof: The proof of Green's formula is surpisingly direct. We will use the linear time invariance of the system combined with superposition and the definition of the integral as a limit of Riemann sums.

To avoid worrying about 0^- and t^+ we will assume that $f(t)$ is continuous. With appropriate care, the proof will work for an $f(t)$ that has jump discontinuities or contains delta functions.

As we saw in the session on Linear Operators in the last unit, linear time invariance means that

$$y(t) \text{ solves } P(D)y = f(t) \;\Rightarrow\; y(t-a) \text{ solves } P(D)y = f(t-a). \tag{3}$$

Or, in the language of input-response, if $y(t)$ is the response to input $f(t)$ then $y(t-a)$ is the response to input $f(t-a)$.

First we will partition time into intervals of width Δt. So, $t_0 = 0,\ t_1 = \Delta t,\ t_2 = 2\Delta t,$ etc.

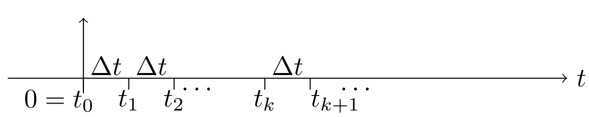

Figure 1: Division of the t-axis into small intervals.

Next we decompose the input signal $f(t)$ into packets over each interval. The kth signal packet, $f_k(t)$ coincides with $f(t)$ between t_k and t_{k+1} and is 0 elsewhere

$$f_k(t) = \begin{cases} f(t) & \text{for } t_k < t < t_{k+1} \\ 0 & \text{elsewhere.} \end{cases}$$

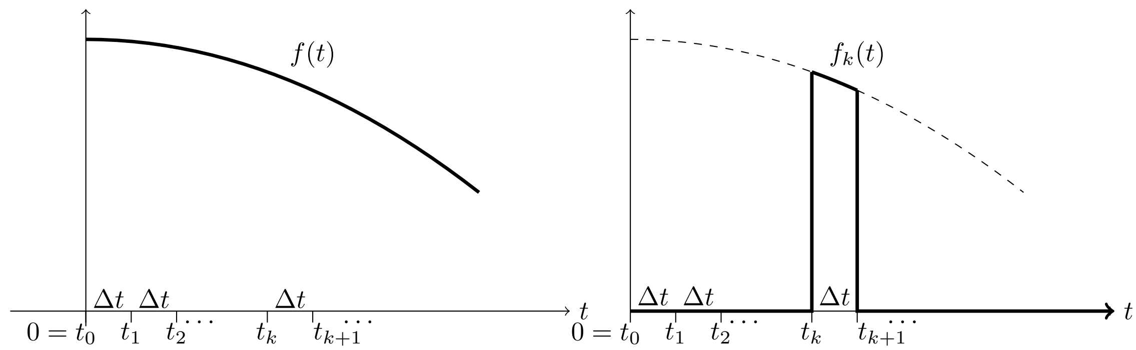

Figure 2: The signal packet $f_k(t)$.

It is clear that for $t > 0$ we have $f(t)$ is the sum of the packets

$$f(t) = f_0(t) + f_1(t) + \ldots + f_k(t) + \ldots$$

As $\Delta t \to 0$, $f_k(t)$ becomes a impulse at t_k, $\delta(t-t_k)$

The area under $f_k(t)$ is $\Delta t f(t_k)$

then $f_k(t) = [\Delta t f(t_k)]\,\delta(t-t_k)$

namely $\int_{t_k^-}^{t_k^+} f_k(t)\,dt = \int \Delta t f(t_k)\,\delta(t-t_k)\,dt = \Delta t f(t_k) \int_{t_k^-}^{t_k^+} \delta(t-t_k)\,dt = \Delta t f(t_k)$

A single packet $f_k(t)$ is concentrated entirely in a small neighborhood of t_k so it is approximately an impulse with the same size as the area under $f_k(t)$. The area under $f_k(t)$ $\approx f(t_k)\,\Delta t$. Hence,

Why approximate as impulse?

The value of impulse is Infinity

$\int_{t_k}^{t_{k+1}} f_k(t)\,dt \approx f(t_k)\,\Delta t = \int_{t_k}^{t_{k+1}} f(t_k)\Delta t\,\delta(t-t_k)\,dt$

$$f_k(t) \approx (f(t_k)\,\Delta t)\,\delta(t - t_k).$$

? How impulse function is introduced ?

The weight function $w(t)$ is response to $\delta(t)$. So, by linear time invariance the response to $f_k(t)$ is

and superposition

$$y_k(t) \approx (f(t_k)\,\Delta t)\,w(t - t_k).$$

$w(t)$ is response to $\delta(t)$

By superposition $\Rightarrow [f(t_k)\Delta t]\,w(t)$ is response to $[f(t_k)\Delta t]\,\delta(t)$

By replace t with $t-t_k$ $\Rightarrow [f(t_k)\Delta t]\,w(t-t_k)$ is response to $[f(t_k)\Delta t]\,\delta(t-t_k)$

We want to find the response at a fixed time. Since t is already in use, we will let T be our fixed time and find $y(T)$.

Since f is the sum of f_k, superposition gives y is the sum of y_k. That is, at time T

$$\begin{aligned} y(T) &= y_0(T) + y_1(T) + \ldots \\ &\approx \Big(f(t_0)w(T - t_0) + f(t_1)w(T - t_1) + \ldots \Big)\,\Delta t \end{aligned} \tag{4}$$

We can ignore all the terms where $t_k > T$. (Because then $w(T - t_k) = 0$, since $T - t_k < 0$.) If n is the last index where $t_k < T$ we have

$$y(T) \approx \Big(f(t_0)w(T - t_0) + f(t_1)w(T - t_1) + \ldots + f(t_n)w(T - t_n) \Big)\,\Delta t$$

This is a Riemann sum and as $\Delta t \to 0$ it goes to an integral

$$y(T) = \int_0^T f(t) w(T - t)\, dt$$

Except for the change in notation this is Green's formula (2).

Note on Causality: Causality is the principle that the future does not affect the past. Green's theorem shows that the system (1) is causal. That is, $y(t)$ only depends on the input up to time t. Real physical systems are causal.

There are non-causal systems. For example, an audio compressor that gathers information after time t before deciding how to compress the signal at time t is non-causal. Another example is the system with input $f(t)$ and output $y(t)$ where y is the solution to $\dot{y} = f(t + 1)$.