Part II Problems and Solutions

Problem 1: *[Second order ODEs via Laplace transform] Find the unit impulse response of the following operators by means of the Laplace transform.*

(a) $3D^2 + 6D + 6I$.

(b) $D^4 - I$.

Solution: (a) $w(t)$ has Laplace transform $W(s) = \dfrac{1}{3s^2+6s+6} = \dfrac{1}{3}\dfrac{1}{(s+1)^2+1}$. $\mathcal{L}(\sin t) = \frac{1}{s^2+1}$, so by s-shift $w(t) = \frac{1}{3}u(t)e^{-t}\sin t$.

(b) $W(s) = \dfrac{1}{s^4-1}$. The roots of s^4-1 are ± 1 and $\pm i$, so we can write $\dfrac{1}{s^4-1} = \dfrac{a}{s-1} + \dfrac{b}{s+1} + \dfrac{c}{s-i} + \dfrac{d}{s+i}$. Cover-up gives easily $a = b = \frac{1}{4}$, $c = \frac{i}{4}, d = -\frac{i}{4}$. So
$w(t) = u(t)\frac{1}{4}\left(e^t + e^{-t} + ie^{it} - ie^{-it}\right) = u(t)\frac{1}{2}\left(\sinh(t) - \sin(t)\right)$
(where $\sinh(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$, the hyperbolic sine function)

$e^{it} - e^{-it} = 2i\sin(t)$

$i\left(e^{it} - e^{-it}\right) = -2\sin(t)$

$\frac{1}{s^4-1} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s-1}$

$s=-1 \quad C = \frac{1}{2\cdot(-2)} = -\frac{1}{4}$

$s = 1 \quad D = \frac{1}{4}$

$\Rightarrow \frac{1}{s^4-1} = \frac{As+B}{s^2+1} - \frac{1}{4(s+1)} + \frac{1}{4(s-1)}$

$s=0 \Rightarrow -1 = B - \frac{1}{4} - \frac{1}{4}$

$\Rightarrow B = -\frac{1}{2}$

$s=2 \Rightarrow \frac{1}{15} = \frac{2A-\frac{1}{2}}{5} - \frac{1}{12} + \frac{1}{4}$

$\Rightarrow \frac{2}{30} = \frac{12A-3}{30} + \frac{5}{30}$

$\Rightarrow A = \frac{0}{12}$

$\Rightarrow \frac{1}{s^4-1} = \frac{-1}{2(s^2+1)} - \frac{1}{4(s+1)} + \frac{1}{4(s-1)}$

$\mathcal{L}^{-1}\left(\frac{1}{s^4-1}\right) = -\frac{1}{2}\sin(t) - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t}$