

Projections onto subspaces

Projections

If we have a vector \mathbf{b} and a line determined by a vector \mathbf{a} , how do we find the point on the line that is closest to \mathbf{b} ?

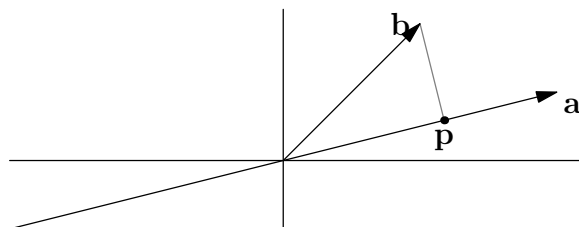


Figure 1: The point closest to \mathbf{b} on the line determined by \mathbf{a} .

We can see from Figure 1 that this closest point \mathbf{p} is at the intersection formed by a line through \mathbf{b} that is orthogonal to \mathbf{a} . If we think of \mathbf{p} as an approximation of \mathbf{b} , then the length of $\mathbf{e} = \mathbf{b} - \mathbf{p}$ is the error in that approximation.

We could try to find \mathbf{p} using trigonometry or calculus, but it's easier to use linear algebra. Since \mathbf{p} lies on the line through \mathbf{a} , we know $\mathbf{p} = x\mathbf{a}$ for some number x . We also know that \mathbf{a} is perpendicular to $\mathbf{e} = \mathbf{b} - x\mathbf{a}$:

$$\begin{aligned}\mathbf{a}^T(\mathbf{b} - x\mathbf{a}) &= 0 \\ x\mathbf{a}^T\mathbf{a} &= \mathbf{a}^T\mathbf{b} \\ x &= \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}},\end{aligned}$$

and $\mathbf{p} = x\mathbf{a} = \mathbf{a} \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}$. Doubling \mathbf{b} doubles \mathbf{p} . Doubling \mathbf{a} does not affect \mathbf{p} .

Projection matrix

We'd like to write this projection in terms of a *projection matrix* P : $\mathbf{p} = P\mathbf{b}$.

$$\mathbf{p} = x\mathbf{a} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}\mathbf{b},$$

so the matrix is:

$$P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}.$$

Note that $\mathbf{a}\mathbf{a}^T$ is a three by three matrix, not a number; matrix multiplication is not commutative.

The column space of P is spanned by \mathbf{a} because for any \mathbf{b} , $P\mathbf{b}$ lies on the line determined by \mathbf{a} . The rank of P is 1. P is symmetric. $P^2\mathbf{b} = P\mathbf{b}$ because

the projection of a vector already on the line through \mathbf{a} is just that vector. In general, projection matrices have the properties:

$$P^T = P \quad \text{and} \quad P^2 = P.$$

Why project?

As we know, the equation $A\mathbf{x} = \mathbf{b}$ may have no solution. The vector $A\mathbf{x}$ is always in the column space of A , and \mathbf{b} is unlikely to be in the column space. So, we project \mathbf{b} onto a vector \mathbf{p} in the column space of A and solve $A\hat{\mathbf{x}} = \mathbf{p}$.

Projection in higher dimensions

In \mathbb{R}^3 , how do we project a vector \mathbf{b} onto the closest point \mathbf{p} in a plane?

If \mathbf{a}_1 and \mathbf{a}_2 form a basis for the plane, then that plane is the column space of the matrix $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$.

We know that $\mathbf{p} = \hat{x}_1\mathbf{a}_1 + \hat{x}_2\mathbf{a}_2 = A\hat{\mathbf{x}}$. We want to find $\hat{\mathbf{x}}$. There are many ways to show that $\mathbf{e} = \mathbf{b} - \mathbf{p} = \mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to the plane we're projecting onto, after which we can use the fact that \mathbf{e} is perpendicular to \mathbf{a}_1 and \mathbf{a}_2 :

$$\mathbf{a}_1^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \quad \text{and} \quad \mathbf{a}_2^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0.$$

In matrix form, $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$. When we were projecting onto a line, A only had one column and so this equation looked like: $a^T(\mathbf{b} - x\mathbf{a}) = 0$.

Note that $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$ is in the nullspace of A^T and so is in the left nullspace of A . We know that everything in the left nullspace of A is perpendicular to the column space of A , so this is another confirmation that our calculations are correct.

We can rewrite the equation $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$ as:

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}.$$

When projecting onto a line, $A^T A$ was just a number; now it is a square matrix. So instead of dividing by $\mathbf{a}^T \mathbf{a}$ we now have to multiply by $(A^T A)^{-1}$

In n dimensions,

$$\begin{aligned} \hat{\mathbf{x}} &= (A^T A)^{-1} A^T \mathbf{b} \\ \mathbf{p} = A\hat{\mathbf{x}} &= A(A^T A)^{-1} A^T \mathbf{b} \\ P &= A(A^T A)^{-1} A^T. \end{aligned}$$

It's tempting to try to simplify these expressions, but if A isn't a square matrix we can't say that $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$. If A does happen to be a square, invertible matrix then its column space is the whole space and contains \mathbf{b} . In this case P is the identity, as we find when we simplify. It is still true that:

$$P^T = P \quad \text{and} \quad P^2 = P.$$

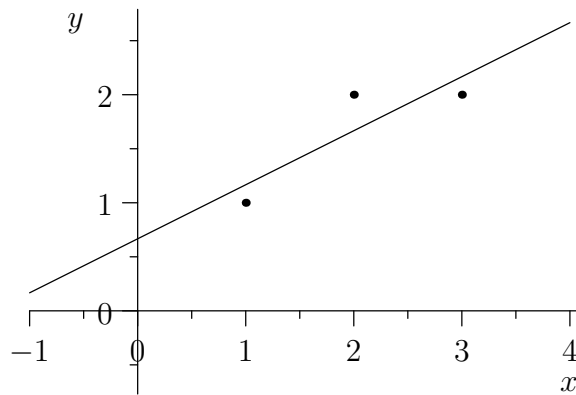


Figure 2: Three points and a line close to them.

Least Squares

Suppose we're given a collection of data points (t, b) :

$$\{(1, 1), (2, 2), (3, 2)\}$$

and we want to find the closest line $b = C + Dt$ to that collection. If the line went through all three points, we'd have:

$$\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2, \end{aligned}$$

which is equivalent to:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

A \mathbf{x} \mathbf{b}

In our example the line does not go through all three points, so this equation is not solvable. Instead we'll solve:

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}.$$

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