Cramer's rule, inverse matrix, and volume

We know a formula for and some properties of the determinant. Now we see how the determinant can be used.

Formula for A^{-1}

We know:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Can we get a formula for the inverse of a 3 by 3 or n by n matrix? We expect that $\frac{1}{\det A}$ will be involved, as it is in the 2 by 2 example, and by looking at the cofactor matrix $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ we might guess that cofactors will be involved.

In fact:

$$A^{-1} = \frac{1}{\det A}C^T$$

where C is the matrix of cofactors – please notice the transpose! Cofactors of row one of A go into column 1 of A^{-1}, and then we divide by the determinant.

The determinant of A involves products with n terms and the cofactor matrix involves products of $n-1$ terms. A and $\frac{1}{\det A}C^T$ might cancel each other. This is much easier to see from our formula for the determinant than when using Gauss-Jordan elimination.

To more formally verify the formula, we'll check that $AC^T = (\det A)I$.

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}.$$

The entry in the first row and first column of the product matrix is:

$$\sum_{j=1}^{n} a_{1j}C_{j1} = \det A.$$

(This is just the cofactor formula for the determinant.) This happens for every entry on the diagonal of AC^T.

To finish proving that $AC^T = (\det A)I$, we just need to check that the off-diagonal entries of AC^T are zero. In the two by two case, multiplying the entries in row 1 of A by the entries in column 2 of C^T gives $a(-b) + b(a) = 0$. This is the determinant of $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$. In higher dimensions, the product of the first row of A and the last column of C^T equals the determinant of a matrix whose first and last rows are identical. This happens with all the off diagonal matrices, which confirms that $A^{-1} = \frac{1}{\det A}C^T$.

$$\sum_{j=1}^{n} a_{2j}C_{j1} = \begin{vmatrix} a_{21} & \cdots & a_{2j} \\ a_{21} & \cdots & a_{2j} \\ a_{31} & \cdots & a_{3j} \\ \vdots & & \vdots \end{vmatrix} = 0$$ (1st and 2st rows are identical)

$$\sum_{j=1}^{n} a_{1j}C_{j2} = \begin{vmatrix} a_{11} & \cdots & a_{1j} \\ a_{11} & \cdots & a_{1j} \\ \vdots & & \vdots \end{vmatrix}$$

Replace the row of corresponding cofactor

This formula helps us answer questions about how the inverse changes when the matrix changes.

Cramer's Rule for $\mathbf{x} = A^{-1}\mathbf{b}$

We know that if $A\mathbf{x} = \mathbf{b}$ and A is nonsingular, then $\mathbf{x} = A^{-1}\mathbf{b}$. Applying the formula $A^{-1} = C^T/\det A$ gives us:

$$\mathbf{x} = \frac{1}{\det A} C^T \mathbf{b}.$$

Cramer's rule gives us another way of looking at this equation. To derive this rule we break $\mathbf{x}$ down into its components. Because the i'th component of $C^T\mathbf{b}$ is a sum of cofactors times some number, it is the determinant of some matrix B_j.

$$x_1 = \frac{1}{\det A}\left(C_{11}b_1 + C_{21}b_2 + C_{31}b_3 + \dots\right) = \frac{1}{\det A} \cdot \begin{vmatrix} b_1 & a_{21} & a_{22} \\ b_2 & a_{31} & a_{32} \\ \vdots & & \ddots \end{vmatrix}$$

$$x_j = \frac{\det B_j}{\det A},$$

where B_j is the matrix created by starting with A and then replacing column j with $\mathbf{b}$, so:

$$B_1 = \left[\begin{array}{cc} \mathbf{b} & \begin{array}{c}\text{last n-1} \\ \text{columns} \\ \text{of } A\end{array} \end{array}\right] \quad \text{and}$$

$$B_n = \left[\begin{array}{cc} \begin{array}{c}\text{first n-1} \\ \text{columns} \\ \text{of } A\end{array} & \mathbf{b} \end{array}\right].$$

This agrees with our formula $x_1 = \frac{\det B_1}{\det A}$. When taking the determinant of B_1 we get a sum whose first term is b_1 times the cofactor C_{11} of A.

Computing inverses using Cramer's rule is usually less efficient than using elimination.

$|\det A|$ = volume of box

Claim: $|\det A|$ is the volume of the box (*parallelepiped*) whose edges are the column vectors of A. (We could equally well use the row vectors, forming a different box with the same volume.)

If $A = I$, then the box is a unit cube and its volume is 1. Because this agrees with our claim, we can conclude that the volume obeys determinant property 1.

If $A = Q$ is an orthogonal matrix then the box is a unit cube in a different orientation with volume $1 = |\det Q|$. (Because Q is an orthogonal matrix, $Q^TQ = I$ and so $\det Q = \pm 1$.)

Swapping two columns of A does not change the volume of the box or (remembering that $\det A = \det A^T$) the absolute value of the determinant (property 2). If we show that the volume of the box also obeys property 3 we'll have proven $|\det A|$ equals the volume of the box.

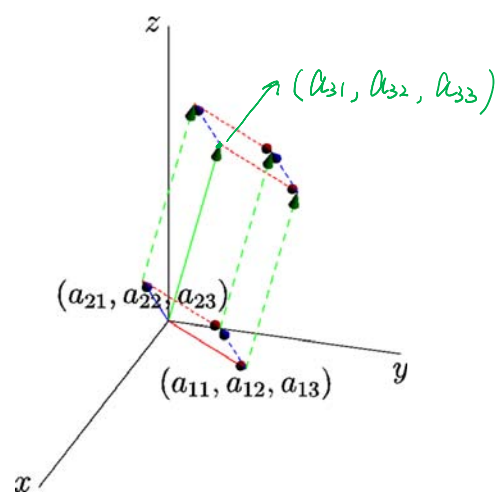

Figure 1: The box whose edges are the column vectors of A.

If we double the length of one column of A, we double the volume of the box formed by its columns. Volume satisfies property 3(a).

Property 3(b) says that the determinant is linear in the rows of the matrix:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Figure 2 illustrates why this should be true.

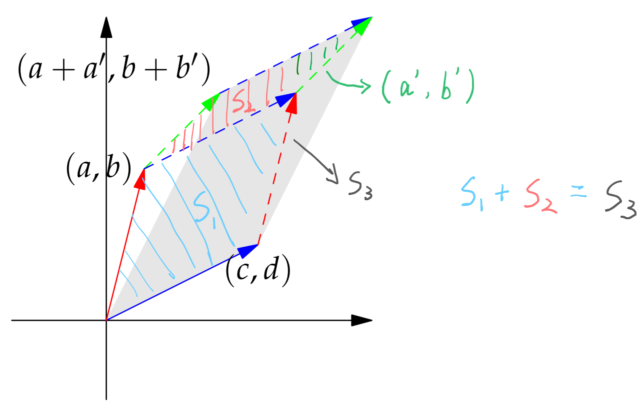

Figure 2: Volume obeys property 3(b).

Although it's not needed for our proof, we can also see that determinants obey property 4. If two edges of a box are equal, the box flattens out and has no volume.

Important note: If you know the coordinates for the corners of a box, then computing the volume of the box is as easy as calculating a determinant. In particular, the area of a parallelogram with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is $ad - bc$. The area of a triangle with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is half the area of that parallelogram, or $\frac{1}{2}(ad - bc)$. The area of a triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) is:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

$$\frac{1}{6}\underbrace{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}_{\text{Volumn of parallelepiped}} = \text{Volumn of tetrahedron}$$

with base $(x_1, y_1)(x_2, y_2)(x_3, y_3)$ and height 1