Exam 2 Review

Material covered by the exam

- Orthogonal matrices $Q = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix}$. $Q^T Q = I$.

  Projections – Least Squares "best fit" solution to $A\mathbf{x} = \mathbf{b}$.

  Gram-Schmidt process for getting an orthonormal basis from any basis.

- $\det A$

  Properties 1-3 that define the determinant.

  Big formula for the determinant with $n!$ terms, each with $+$ or $-$.

  Cofactors formula, leading to a formula for A^{-1}.

- Eigenvalues $A\mathbf{x} = \lambda\mathbf{x}$.

  $\det(A - \lambda I) = 0$.

  Diagonalization: If A has n independent eigenvectors, then $S^{-1}AS = \Lambda$ (this is $A\mathbf{x} = \lambda\mathbf{x}$ for all n eigenvectors at once).

  Powers of A: $A^k = (S\Lambda S^{-1})^k = S\Lambda^k S^{-1}$.

Sample questions

1. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

   a) Find the projection matrix P that projects onto $\mathbf{a}$.

      To answer this, we just use the formula for P. Ordinarily $P = A(A^TA)^{-1}A^T$, but here A is a column vector so:

      $$P = \frac{\mathbf{a}\,\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}.$$

   b) What is the rank of P?

      P has rank 1 because each of its columns is some multiple of its second column, or because it projects onto a one dimensional subspace.

   c) What is the column space of P?

      The line determined by $\mathbf{a}$.

   d) What are the eigenvalues of P?

      Since P has rank 1 we know it has a repeated eigenvalue of 0. We can use its trace or the fact that it's a projection matrix to find that P has the eigenvalue 1.

      The eigenvalues of P are 0, 0 and 1.

e) Find an eigenvector of P that has eigenvalue 1.

Eigenvector $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ has eigenvalue one. Because P is a projection matrix, any vector in the space it's projecting onto will be an eigenvector with eigenvalue 1.

f) Suppose $\mathbf{u}_{k+1} = P\mathbf{u}_k$ with initial condition $\mathbf{u}_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$. Find $\mathbf{u}_k$.

We're repeatedly projecting a vector onto a line:

$$\mathbf{u}_1 = P\mathbf{u}_0 = \mathbf{a}\frac{\mathbf{a}^T\mathbf{u}_0}{\mathbf{a}^T\mathbf{a}} = \mathbf{a}\frac{27}{9} = 3\mathbf{a} = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}.$$

$\mathbf{u}_2$ is the projection of $\mathbf{u}_1$ onto the line determined by $\mathbf{a}$. But $\mathbf{u}_1$ already lies on the line through $\mathbf{a}$. In fact, $\mathbf{u}_k = P^k\mathbf{u}_0 = P\mathbf{u}_0 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$.

g) The exam might have a difference equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$ in which A is not a projection matrix with $P^k = P$. In that case we would find its eigenvalues and eigenvectors to calculate $\mathbf{u}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$. Then $\mathbf{u}_k = c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + c_3\lambda_3^k\mathbf{x}_3$. (For the projection matrix P above, two eigenvalues are 0 and the third is 1, so two terms vanish and for the third $\lambda^k = 1$. Then $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3 = \cdots$.)

2. We're given the following data points:

t	y
1	4
2	5
3	8

.

a) Find the straight line through the origin that best fits these points.
The equation of this line will be $y = Dt$. There's only one unknown, D. We would like a solution to the 3 equations:

$$\begin{aligned} 1 \cdot D &= 4 \\ 2 \cdot D &= 5 \\ 3 \cdot D &= 8. \end{aligned}$$

or $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} D = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$, if we put this in the form $A\mathbf{x} = \mathbf{b}$. To find the best value for D we solve the equation:

$$A^TA\hat{D} = A^T\mathbf{b}$$
$$14\hat{D} = 38$$
$$\hat{D} = \frac{38}{14} = \frac{19}{7}.$$

We conclude that the best fit line through the origin is $y = \frac{19}{7}t$. We can roughly check our answer by noting that the line $y = 3t$ runs fairly close to the data points.

b) What vector did we just project onto what line?

There are two ways to think about least squares problems. The first is to think about the best fit line in the ty-plane. The other way is to think in terms of projections – we're projecting $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$ onto the column space of A (the line through $(1,2,3)$) to get as close as possible to a solution to $A\mathbf{x} = \mathbf{b}$.

3. The vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ determine a plane. Find two orthogonal vectors in the plane.

To answer this question we use the Gram-Shmidt process. We start with $\mathbf{a}_1$ and find a second vector $\mathbf{B}$ perpendicular to $\mathbf{a}_1$ by subtracting the component of $\mathbf{a}_2$ that lies in the $\mathbf{a}_1$ direction.

$$\begin{aligned} \mathbf{B} &= \mathbf{a}_2 - \frac{\mathbf{a}_1^T\mathbf{a}_2}{\mathbf{a}_1^T\mathbf{a}_1}\mathbf{a}_1 \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/7 \\ 6/7 \\ 9/7 \end{bmatrix} \\ &= \begin{bmatrix} 4/7 \\ 1/7 \\ -2/7 \end{bmatrix}. \end{aligned}$$

Because the dot product of $\mathbf{a}_1$ and $\mathbf{B}$ is zero, we know this answer is correct. These are our orthogonal vectors.

4. We're given a 4 by 4 matrix A with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

a) What conditions must the λ_i satisfy for the matrix to be invertible?

A is invertible if and only if none of the λ_i are 0.

If one of the λ_i is zero, then there is a non-zero vector in the nullspace of A and A is not invertible.

b) What is $\det A^{-1}$?

The eigenvalues of A^{-1} are the inverses of the eigenvalues of A, so $\det A^{-1} = \left(\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_4}\right)$.

$\det AA^{-1} = \det A \cdot \det A^{-1}$

$\det I = \det A \cdot \det A^{-1}$

$\det A^{-1} = \frac{\det I}{\det A} = \frac{1}{\prod_{i=1}^{n} \lambda_i}$

c) What is trace$(A+I)$?
We know the trace of A is the sum of the eigenvalues of A, so trace$(A+I) = (\lambda_1+1)+(\lambda_2+1)+(\lambda_3+1)+(\lambda_4+1) = \lambda_1+\lambda_2+\lambda_3+\lambda_4+4$.

5. Remember the family of *tridiagonal matrices*; for example:

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $D_n = \det A_n$.

a) Use cofactors to show that $D_n = aD_{n-1} + bD_{n-2}$ and find values for a and b.
Using the cofactor formula we find that the determinant of A_4 is:

$$\begin{aligned} D_4 &= 1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} + 0 - 0 \\ &= 1D_3 - 1\cdot 1D_2 \\ &= D_3 - D_2. \end{aligned}$$

In general, $D_n = D_{n-1} - D_{n-2}$. The answer is $a = 1$ and $b = -1$.

b) In part (a) you found a recurrence relation $D_n = aD_{n-1} + bD_{n-2}$. Find a way to predict the value of D_n for any n. Note: if your computations are all correct it should be true that $\lambda_1^6 = \lambda_2^6 = 1$.
We can quickly compute $D_1 = 1$ and $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$.
We set up the system for $D_n = D_{n-1} - D_{n-2}$:

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

to get an equation of the form $\mathbf{u}_k = A\mathbf{u}_{k-1}$.
To find the eigenvalues λ_i, solve $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0.$$

The quadratic formula gives us $\lambda = \dfrac{1 \pm \sqrt{1-4}}{2}$, so:

$$\boxed{\lambda_1 = \frac{1+\sqrt{3}i}{2} = e^{i\pi/3} \text{ and } \lambda_2 = \frac{1-\sqrt{3}i}{2} = e^{-i\pi/3}.}$$

The magnitude of these complex numbers is 1. This tells us that the system is stable. The fact that $\lambda_1^6 = \lambda_2^6$ tells us that $A^6 = I$ and so the sequence of vectors $\mathbf{u}_k = A^k \mathbf{u}_{k-1}$ will repeat every 6 steps.

To finish answering the problem, we can use the recurrence relation $D_n = D_{n-1} - D_{n-2}$ starting from $D_1 = 1$ and $D_2 = 0$ to find $D_3 = -1$, $D_4 = -1$, $D_5 = 0$ and $D_6 = 1$. The sequence will then repeat, with $D_7 = D_1 = 1$, $D_8 = D_2 = 0$ and so on. If $n = 6j + k$ for positive integers j and k, then $D_n = D_k$.

6. Consider the following family of symmetric matrices:

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}, \ldots$$

a) Find the projection matrix P onto the column space of A_3.

We know that A_3 is singular because column 3 is a multiple of column 1, so P is a projection matrix onto a plane. Columns 1 and 2 form a basis for the column space of A_3, so we could use the formula $P = A(A^TA)^{-1}A^T$ with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$ to find:

$$P = \begin{bmatrix} 1/5 & 0 & 2/5 \\ 0 & 1 & 0 \\ 2/5 & 0 & 4/5 \end{bmatrix}.$$

However, there may be a quicker way to solve this problem.

To check our work we multiply P by the column vectors of A to see that $PA = A$.

b) What are the eigenvalues and eigenvectors of A_3?

$$|A_3 - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = -\lambda^3 + 5\lambda.$$

Setting $|A_3 - \lambda I| = 0$ gives us $\lambda(-\lambda^2 + 5) = 0$, so $\lambda_1 = 0$, $\lambda_2 = \sqrt{5}$, $\lambda_3 = -\sqrt{5}$.

We check that the trace of A_3 equals the sum of its eigenvalues.

Next we solve $(A_3 - \lambda I)\mathbf{x} = \mathbf{0}$ to find our eigenvectors. A good strategy for doing this is to choose one component of $\mathbf{x}$ to set equal to 1, then determine what the other components of $\mathbf{x}$ must be for the product to equal the zero vector.

$(A_3 - 0I)\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

$(A_3 - \sqrt{5}I)\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x}_2 = \begin{bmatrix} 1 \\ \sqrt{5} \\ 2 \end{bmatrix}$.

$(A_3 + \sqrt{5}I)\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -\sqrt{5} \\ 2 \end{bmatrix}$.

If time permits, we can check this answer by multiplying each eigenvector by A_3.

c) (This is not difficult.) What is the projection matrix onto the column space of A_4?

How could this not be difficult? If A_4 is invertible, then its column space is $\mathbb{R}^4$ and the answer is $P = I$.

To confirm that A_4 is invertible, we can check that its determinant is non-zero. This is not difficult if we use cofactors:

$$\det A_4 = (-1)\left(1 \cdot \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}\right) = 9.$$

Because A_4 is invertible, the projection matrix onto its column space is I.

d) Bonus question: Prove or disprove that A_n is singular if n is odd and invertible if n is even.