# Final course review

Once more, we review questions from a previous exam to prepare ourselves for an upcoming exam.

1. Suppose we know that $A$ is an $m$ by $n$ matrix of rank $r$, $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no solution, and $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has exactly one solution.

   a) What can we say about $m$, $n$ and $r$?
   The product $A\mathbf{x}$ is a vector in three dimensions, so $m = 3$.

   The fact that $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no solution tells us that the column space is not all of $\mathbb{R}^3$. In addition, we know that the column space contains $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so $r$ is not zero: $1 \leq r < 3$.

   The fact that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has exactly one solution tells us that the nullspace of $A$ contains only the zero vector and so $n = r$. Hence $1 \leq n < 3$.

   b) Write down an example of a matrix $A$ that fits this description.

   The vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ must be in the column space, so we'll make it a column of $A$. The simplest way to answer this question is to stop here.

   $$A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

   In this solution, $n = r = 1$ and $m = 3$.

   To find a solution in which $n = r = 2$, add a second column. Make sure that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the column space:

   $$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   There are many other correct answers to this question.

c) Cross out all statements that are false about any matrix with the given properties (which are $1 \leq r = n$, $m = 3$).

   i. $\det A^T A = \det AA^T$
   ii. $A^T A$ is invertible
   iii. $AA^T$ is positive definite

   One good approach to this problem is to use our sample matrix to test each statement.

   i. If we leave this part to last, we can quickly answer it (**false**) using what we learn while answering the following two parts.
   ii. The matrix $A^T A$ is invertible if $r = n$; i.e. if the columns of $A$ are independent.
   The nullspace of our $A$ contains only the zero vector, so this statement is **true**.
   For each of our sample matrices, $A^T A$ equals the identity and so is invertible.
   Note that this means $\det A^T A \neq 0$.
   iii. We know that $m = 3$ and $r < 3$, so $AA^T$ will be a 3 by 3 matrix with rank less than 3; it **can't be positive definite**. (It is true that for any matrix $A$ with real valued entries, $AA^T$ is positive semidefinte.)
   For our test matrices, $AA^T$ has at least one row that's all zeros, so 0 is an eigenvalue (and is not positive).

   Note also that $\det AA^T = 0$ and so statement (i) must be false. (However, if $A$ and $B$ are square matrices then $\det BA = \det AB = \det A \det B$.)

d) Prove that $A^T\mathbf{y} = \mathbf{c}$ has at least one solution for every right hand side $\mathbf{c}$, and in fact has infinitely many solutions for every $\mathbf{c}$.

   We know $A^T$ is an $n$ by $m$ matrix with $m = 3$ and rank $r = n < m$. If $A^T$ has full row rank, the equation $A^T\mathbf{y} = \mathbf{c}$ is always solvable. We have $n$ rows and rank $r = n$, so $A^T$ has full row rank. Therefore $A^T\mathbf{y} = \mathbf{c}$ has a solution for every vector $\mathbf{c}$.

   The solvable system $A^T\mathbf{y} = \mathbf{c}$ will have infinitely many solutions if the nullspace of $A^T$ has positive dimension. We know $\dim(N(A^T)) = m - r > 0$, so $A^T\mathbf{y} = \mathbf{c}$ has infinitely many solutions for every $\mathbf{c}$.

2. Suppose the columns of $A$ are $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

   a) Solve $A\mathbf{x} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$.

   This is just the "column method" of multiplying matrices from the first lecture. Choose $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

b) True or false: if $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, then the solution to (2a) is not unique. Explain your answer.

**True**. Any scalar multiple of $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ will be a solution.

Another way of answering this is to note that $A^T$ has a nontrivial nullspace, and we can always add any vector in the nullspace to a solution $\mathbf{x}$ to get a different solution.

c) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are orthonormal (forget about (2b)). What combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is closest to $\mathbf{v}_3$?

If we imagine the right triangle out from the origin formed by $a\mathbf{v}_1 + b\mathbf{v}_2$ and $\mathbf{v}_3$, the Pythagorean theorem tells us that $0\mathbf{v}_1 + 0\mathbf{v}_2 = \mathbf{0}$ is the closest point to $\mathbf{v}_3$ in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

3. Suppose we have the Markov matrix

$$A = \begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}.$$

Note that the sum of the first two columns of $A$ equals twice the third column of $A$.

a) What are the eigenvalues of $A$?

Zero is an eigenvalue because the columns of $A$ are dependent. ($A$ is singular.)

One is an eigenvalue because $A$ is a Markov matrix.

The third eigenvalue is $-.2$ because the trace of $A$ is .8. So $\lambda = 0, 1, -.2$.

b) Let $\mathbf{u}_k = A^k\mathbf{u}(0)$. If $\mathbf{u}(0) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$, what is $\lim_{k\to\infty} \mathbf{u}_k$?

We'll start by computing $\mathbf{u}_k$ and then find the steady state. This means finding a general expression of the form:

$$\mathbf{u}_k = c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + c_3\lambda_3^k\mathbf{x}_3.$$

When we plug in the eigenvalues we found in part (3a), this becomes

$$\mathbf{u}_k = \mathbf{0} + c_2\mathbf{x}_2 + c_3(-.2)^k\mathbf{x}_3.$$

We see that as $\mathbf{k}$ approaches infinity, $c_2\mathbf{x}_2$ is the only term that does not go to zero.

The key eigenvector in any Markov process is the one with eigenvalue one.

To find $\mathbf{x}_2$, solve $(A - 1I)\mathbf{x}_2 = \mathbf{0}$:

$$\begin{bmatrix} -.8 & .4 & .3 \\ .4 & -.8 & .3 \\ .4 & .4 & -.6 \end{bmatrix} \mathbf{x}_2 = \mathbf{0}.$$

The best way to solve this might be by elimination. However, because the first two columns look like multiples of 4 and the third column looks like a multiple of 3, we might get lucky and guess $\mathbf{x}_2 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$.

This gives us $\mathbf{u}_\infty = c_2 \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$. We know that in a Markov process, the sum of the entries of $\mathbf{u}_k$ is the same for all $k$. The sum of the entries of $\mathbf{u}(0)$ is 10, so $c_2 = 1$ and $\mathbf{u}_\infty = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$.

4. Find a two by two matrix that:

a) projects onto the line spanned by $\mathbf{a} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

The formula for this matrix is $P = \dfrac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}$. This gives us

$$P = \begin{bmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{bmatrix}.$$

(To test this answer, we can quickly check that $\det P = 0$.)

b) has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 3$ and eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Here the formula we need is $A = S\Lambda S^{-1}$.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \\ A &= \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}. \end{aligned}$$

If time permits, we can check this by computing the products $A\mathbf{x}_i$.

c) has real entries and cannot be factored as $B^TB$ for any $B$.
We know that $B^TB$ will always be symmetric, so any asymmetric matrix has this property. For example, we could choose $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

d) is not symmetric, but has orthogonal eigenvectors.
We know that symmetric matrices have orthogonal eigenvectors, but so do other types of matrices (e.g. skew symmetric and orthogonal) when we allow complex eignevectors.
Two possible answers are:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{(skew symmetric)}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{(orthogonal).}$$

5. Applying the least squares method to the system

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \mathbf{b}$$

gives the best fit vector $\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 11/3 \\ -1 \end{bmatrix}$.

a) What is the projection $\mathbf{p}$ of $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ onto the column space of

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}?$$

We know that 11/3 times the first column minus 1 times the second column is the closest point $P$ in the column space to $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, so the answer is

$$A \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \frac{11}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 8/3 \\ 5/3 \end{bmatrix}.$$

b) Draw the straight line problem that corresponds to this system.
Plotting the entries of the second column of $A$ against the entries of $\mathbf{b}$ we get the three points shown in Figure 1. The best fit line is $\hat{c} + \hat{d}t$.

c) Find a different vector $\mathbf{b} \neq \mathbf{0} \in \mathbb{R}^3$ so that the least squares solution is $\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
We know that $\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix}$ is the projection of $\mathbf{b}$ onto the column space, so to get a zero projection we need to find a vector *orthogonal to the columns*.

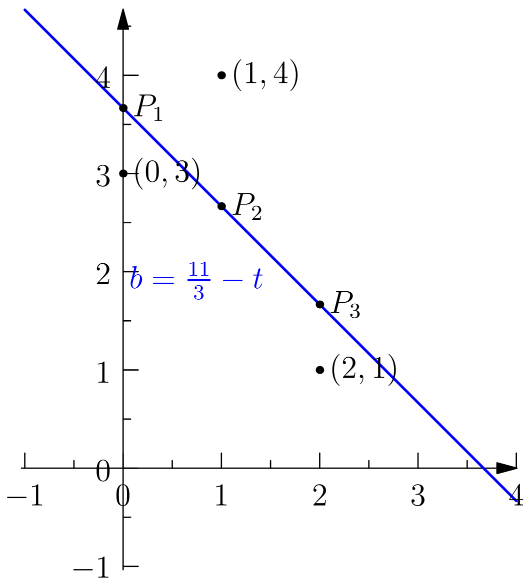


Figure 1: Three data points and their "best fit" line $\frac{11}{3} - t$.

We could get the answer $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ by inspection, or we could use the cross product of the columns to find a value for $\mathbf{b}$.

Thank you for taking this course!