LECTURE 15

Poisson process — II

- **Readings:** Finish Section 6.2.

- Review of Poisson process

- Merging and splitting

- Examples

- Random incidence

Review

- Defining characteristics
  - **Time homogeneity**: $P(k, \tau)$
  - **Independence**
  - **Small interval probabilities** (small δ):

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta, & \text{if } k=0, \\ \lambda\delta, & \text{if } k=1, \\ 0, & \text{if } k>1. \end{cases}$$

- N_τ is a Poisson r.v., with parameter $\lambda\tau$:

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

$\mathbf{E}[N_\tau] = \text{var}(N_\tau) = \lambda\tau$

- Interarrival times ($k = 1$): exponential:

$$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \qquad \mathbf{E}[T_1] = 1/\lambda$$

- Time Y_k to kth arrival: Erlang(k):

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$$

Poisson fishing

- Assume: Poisson, $\lambda = 0.6$/hour.
  - Fish for two hours.
  - if no catch, continue until first catch.

a) $\mathbf{P}$(fish for more than two hours)=

b) $\mathbf{P}$(fish for more than two and less than five hours)=

c) $\mathbf{P}$(catch at least two fish)=

$p(\#|\text{catch in 2 hours}) = p(k,2|\text{catch}\ldots) = \frac{p(k,2)}{1-p(0,2)}$, $k = 1, 2, \ldots$

$E[\#|\text{catch in 2 hours}]P(\ldots) + E(\#|\text{no catch in 2 hours})P(\ldots)$

d) $\mathbf{E}$[number of fish]=

$E[\#|\text{two hours}]\,P(2\text{ hours}) + E(\#|\text{more than } 2)\,P(\text{more } 2) = \overbrace{\left[\sum_{k=1}^{\infty} k\,p(k,2)\right]}^{= \sum_{k=0}^{\infty} k\,p(k,2) = \lambda\tau = 0.6\cdot 2}(1-p(0,2)) + (0.6\cdot 2+1)\,p(0,2)$ Wrong maybe

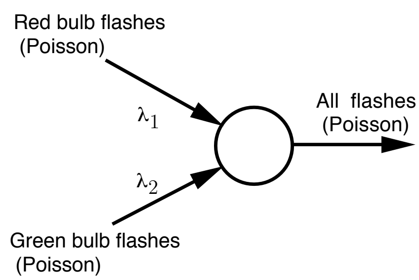
$= 0.6\cdot 2 + p(0,2)$

e) $\mathbf{E}$[future fishing time | fished for four hours]=

f) $\mathbf{E}$[total fishing time]=

Merging Poisson Processes (again)

- Merging of independent Poisson **processes** is Poisson



  - What is the probability that the next arrival comes from the first process?

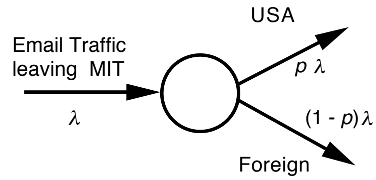
Light bulb example

- Each light bulb has independent, exponential(λ) lifetime
- Install three light bulbs.
  Find expected time until last light bulb dies out.

$$P(\max\{T_1, T_2, T_3\} \le t) = P(T_1 \le t \text{ and } T_2 \le t \text{ and } T_3 \le t)$$
$$= P(T_1 \le t)\, P(T_2 \le t)\, P(T_3 \le t)$$
$$= (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})(1 - e^{-\lambda_3 t})$$
$$f_T(t) = \frac{d}{dt} P(\max\{T_1, T_2, T_3\} \le t) = \frac{d}{dt}\left[-e^{-(\lambda_1+\lambda_2+\lambda_3)t} + e^{-(\lambda_1+\lambda_2)t} + \dots - e^{-\lambda_1 t} - \dots\right]$$
$$= \dots$$

Splitting of Poisson processes

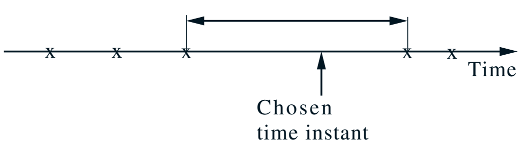
- Assume that email traffic through a server is a Poisson process.
  Destinations of different messages are independent.



- Each output stream is Poisson.

Random incidence for Poisson

- Poisson process that has been running forever
- Show up at some "random time"
  (really means "arbitrary time")



- What is the distribution of the length of the chosen interarrival interval?

Random incidence in "renewal processes"

- Series of successive arrivals
  - i.i.d. interarrival times
    (but not necessarily exponential)
- **Example:**
  Bus interarrival times are equally likely to be 5 or 10 minutes
- If you arrive at a "random time":
  - what is the probability that you selected a 5 minute interarrival interval?
  - what is the expected time to next arrival?