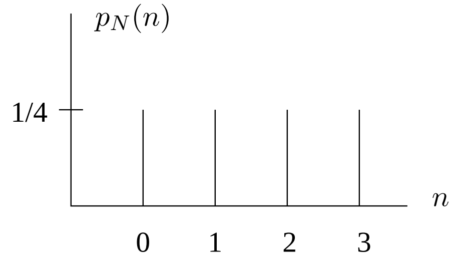
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Recitation 6 Solutions
September 28, 2010

1. (a) The first part can be completed without reference to anything other than the die roll:



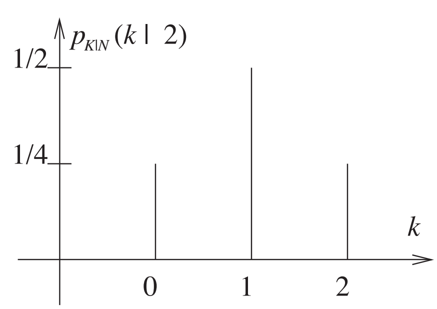
(b) When $N = 0$, the coin is not flipped at all, so $K = 0$. When $N = n$ for $n \in \{1, 2, 3\}$, the coin is flipped n times, resulting in K with a distribution that is conditionally binomial. The binomial probabilities are all multiplied by 1/4 because $p_N(n) = 1/4$ for $n \in \{0, 1, 2, 3\}$. The joint PMF $p_{N,K}(n, k)$ thus takes the following values and is zero otherwise:

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
$n = 0$	1/4	0	0	0
$n = 1$	1/8	1/8	0	0
$n = 2$	1/16	1/8	1/16	0
$n = 3$	1/32	3/32	3/32	1/32

(c) Conditional on $N = 2$, K is a binomial random variable. So we immediately see that

$$p_{K|N}(k|2) = \begin{cases} 1/4, & \text{if } k = 0, \\ 1/2, & \text{if } k = 1, \\ 1/4, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

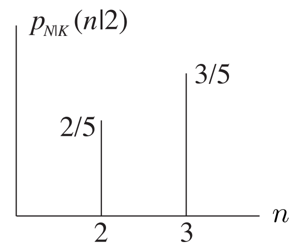
This is a normalized row of the table in the previous part.



(d) To get $K = 2$ heads, there must have been at least ~~3~~ 2 coin tosses, so only $N = \cancel{3}\ 2$ and $N = \cancel{4}\ 3$ have positive conditional probability given $K = 2$.

$$p_{N|K}(2 \mid 2) = \frac{\mathbf{P}(\{N = 2\} \cap \{K = 2\})}{\mathbf{P}(\{K = 2\})} = \frac{1/16}{1/16 + 1/32 + 1/32 + 1/32} = 2/5.$$

Similarly, $p_{N|K}(3 \mid 2) = 3/5$.



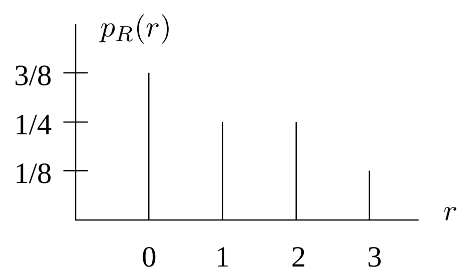
2. (a) $x = 0$ maximizes $\mathbf{E}[Y \mid X = x]$ since

$$\mathbf{E}[Y \mid X = x] = \begin{cases} 2, & \text{if } x = 0, \\ 3/2, & \text{if } x = 2, \\ 3/2, & \text{if } x = 4, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

(b) $y = 3$ maximizes $\text{var}(X \mid Y = y)$ since

$$\text{var}(X \mid Y = y) = \begin{cases} 0, & \text{if } y = 0, \\ 8/3, & \text{if } y = 1, \\ 1, & \text{if } y = 2, \\ 4, & \text{if } y = 3, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

(c)



(d) By traversing the points top to bottom and left to right, we obtain

$$\mathbf{E}[XY] = \frac{1}{8}(0 \cdot 3 + 4 \cdot 3 + 2 \cdot 2 + 4 \cdot 2 + 0 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 + 4 \cdot 0) = \frac{15}{4}.$$

Conditioning on A removes the point masses at $(0, 1)$ and $(0, 3)$. The conditional probability of each of the remaining point masses is thus $1/6$, and

$$\mathbf{E}[XY \mid A] = \frac{1}{6}(4 \cdot 3 + 2 \cdot 2 + 4 \cdot 2 + 2 \cdot 1 + 4 \cdot 1 + 4 \cdot 0) = 5.$$

3. See the textbook, Example 2.17, pages 105–106.