

Tutorial 6: Solutions

1. Let $Z = X + Y$. Using the 2 step CDF method,

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \leq z) \\ &= \mathbf{P}(X + Y \leq z) \end{aligned}$$

Using the Total Probability Theorem, we have

$$\begin{aligned} F_Z(z) &= \sum_x p_X(x) p(x + Y \leq z) = \int_y f_Y(y) \mathbf{P}(X + Y \leq z | y) dy \\ &= \sum_x p_X(x) p(Y \leq z - x) = \int_y f_Y(y) \mathbf{P}(X + y \leq z) dy \\ &= \sum_x p_X(x) F_Y(z - x) = \int_y f_Y(y) \underbrace{\mathbf{P}(X \leq z - y)}_{?} dy \\ &= \int_y f_Y(y) F_X(z - y) dy \end{aligned}$$

Differentiating both sides with respect to z , we obtain

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \sum_x p_X(x) f_Y(z - x) \end{aligned}$$

$f_Z(z) = \frac{d}{dz} \int_y f_Y(y) F_X(z - y) dy$
 $= \frac{d}{dz} \left(\int_y f_Y(y) F_X(z - y) dy \right) \frac{d}{dz} z$
 $= f_Y(y) F_X(z - y) \cdot 1$
 $= f_Y(y) F_X(z - y) ?$

2. We will condition on X and use the law of total variance

$$\text{var}(X + Y) = \mathbf{E}[\text{var}(X + Y | X)] + \text{var}(\mathbf{E}[X + Y | X]).$$

Given a value x of X , the random variable Y is uniformly distributed in the interval $[x, x + 1]$, and the random variable $X + Y$ is uniformly distributed in the interval $[2x, 2x + 1]$. Therefore, $\mathbf{E}[X + Y | X] = 0.5 + 2X$ and $\text{var}(X + Y | X) = 1/12$. Thus,

$$\text{var}(X + Y) = \text{var}(0.5 + 2X) + \mathbf{E}[1/12] = 4\text{var}(X) + \mathbf{E}[1/12] = \frac{5}{12}.$$

3. (a) Let X_i be independent Bernoulli random variables that are equal to 1 if the i th flip results in heads. Let N be the number of coin flips. We have $\mathbf{E}[X_i] = 1/2$, $\text{var}(X_i) = 1/4$, $\mathbf{E}[N] = 7/2$, and $\text{var}(N) = 35/12$. (The last equality is obtained from the formula for the variance of a discrete uniform random variable.) Therefore, the expected number of heads is

$$\mathbf{E}[X_i] \mathbf{E}[N] = \frac{7}{4},$$

and the variance is

$$\text{var}(X_i) \mathbf{E}[N] + (\mathbf{E}[X_i])^2 \text{var}(N) = \frac{1}{4} \cdot \frac{7}{2} + \frac{1}{4} \cdot \frac{35}{12} = \frac{77}{48}.$$

- (b) The experiment in part (b) can be viewed as consisting of two independent repetitions of the experiment in part (a). Thus, both the mean and the variance are doubled and become $7/2$ and $77/24$, respectively.

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