LECTURE 14

The Poisson process

- **Readings:** Start Section 6.2.

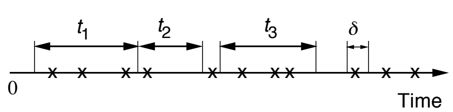
Lecture outline

- Review of Bernoulli process
- Definition of Poisson process
- Distribution of number of arrivals
- Distribution of interarrival times
- Other properties of the Poisson process

Bernoulli review

- Discrete time; success probability p
- Number of arrivals in n time slots: binomial pmf
- Interarrival times: geometric pmf
- Time to k arrivals: Pascal pmf
- Memorylessness

Definition of the Poisson process



- **Time homogeneity:** $P(k,\tau)$ = Prob. of k arrivals in interval of duration τ
- Numbers of arrivals in disjoint time intervals are **independent**
- **Small interval probabilities:** For VERY small δ:

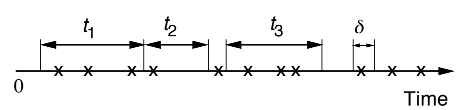
$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta, & \text{if } k=0; \\ \lambda\delta, & \text{if } k=1; \\ 0, & \text{if } k>1. \end{cases} \qquad \lim_{\delta\to 0}\frac{P(1,\delta)}{\delta} = \lambda$$

– λ: "arrival rate" = E[# of arrival] per unit time

E[# of arrival during interval $[t, t+\delta]$] = $\lambda\delta$

$\Rightarrow \lambda = \dfrac{E[\#\text{ of arrival in }[0,\delta]]}{\delta}$

PMF of Number of Arrivals N



- Finely discretize $[0,t]$: approximately Bernoulli
- N_t (of discrete approximation): binomial
- Taking $\delta \to 0$ (or $n \to \infty$) gives:

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0,1,\ldots$$

- $\mathbf{E}[N_t] = \lambda t, \qquad \text{var}(N_t) = \lambda t$

Example

- You get email according to a Poisson process at a rate of $\lambda = 5$ messages per hour. You check your email every thirty minutes.
- Prob(no new messages) $=$
- Prob(one new message) $=$

Interarrival Times

- Y_k time of kth arrival
- **Erlang** distribution:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$$

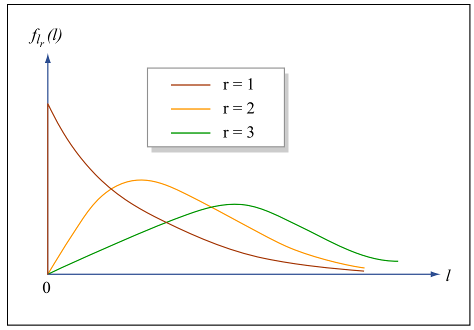
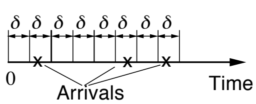


Image by MIT OpenCourseWare.

- Time of first arrival ($k = 1$): **exponential**: $f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0$
  - Memoryless property: The time to the next arrival is independent of the past

Bernoulli/Poisson Relation



$n = t/\delta$

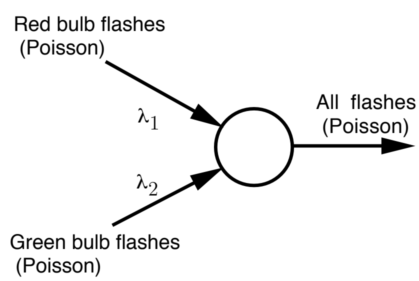
$p = \lambda\delta$

$np = \lambda t$

	POISSON	**BERNOULLI**
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

Merging Poisson Processes

- Sum of independent Poisson **random variables** is Poisson
- Merging of independent Poisson **processes** is Poisson



  - What is the probability that the next arrival comes from the first process?

6.041SC Probabilistic Systems Analysis and Applied Probability
Fall 2010