LECTURE 12

- **Readings:** Section 4.3;
  parts of Section 4.5
  (mean and variance only; no transforms)

Lecture outline

- Conditional expectation
  - Law of iterated expectations
  - Law of total variance
- Sum of a random number
  of independent r.v.'s
  - mean, variance

Conditional expectations

- Given the value y of a r.v. Y:

$$\mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

(integral in continuous case)

- Stick example: stick of length ℓ
  break at uniformly chosen point Y
  break again at uniformly chosen point X

- $\mathbf{E}[X \mid Y = y] = \dfrac{y}{2}$ (number)

$\mathbf{E}[X \mid Y] = \dfrac{Y}{2}$ (r.v.)

- **Law of iterated expectations:**

$$\mathbf{E}[\mathbf{E}[X \mid Y]] = \sum_y \mathbf{E}[X \mid Y = y] p_Y(y) = \mathbf{E}[X]$$

- In stick example:
  $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Y]] = \mathbf{E}[Y/2] = \ell/4$

$\mathrm{var}(X \mid Y)$ and its expectation

- $\mathrm{var}(X \mid Y = y) = \mathbf{E}\left[(X - \mathbf{E}[X \mid Y = y])^2 \mid Y = y\right]$
- $\mathrm{var}(X \mid Y)$: a r.v.
  with value $\mathrm{var}(X \mid Y = y)$ when $Y = y$
- **Law of total variance:**

$$\mathrm{var}(X) = \mathbf{E}[\mathrm{var}(X \mid Y)] + \mathrm{var}(\mathbf{E}[X \mid Y])$$

Proof:

(a) Recall: $\mathrm{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$

(b) $\mathrm{var}(X \mid Y) = \mathbf{E}[X^2 \mid Y] - (\mathbf{E}[X \mid Y])^2$

(c) $\mathbf{E}[\mathrm{var}(X \mid Y)] = \mathbf{E}[X^2] - \mathbf{E}[(\mathbf{E}[X \mid Y])^2]$

(d) $\mathrm{var}(\mathbf{E}[X \mid Y]) = \mathbf{E}[(\mathbf{E}[X \mid Y])^2] - (\mathbf{E}[X])^2$

Sum of right-hand sides of (c), (d):
$\mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \mathrm{var}(X)$

Section means and variances

Two sections:
$y = 1$ (10 students); $y = 2$ (20 students)

$$y = 1: \ \frac{1}{10}\sum_{i=1}^{10} x_i = 90 \qquad y = 2: \ \frac{1}{20}\sum_{i=11}^{30} x_i = 60$$

$$\mathbf{E}[X] = \frac{1}{30}\sum_{i=1}^{30} x_i = \frac{90 \cdot 10 + 60 \cdot 20}{30} = 70$$

$\mathbf{E}[X \mid Y = 1] = 90, \quad \mathbf{E}[X \mid Y = 2] = 60$

$Z = g(Y) = \begin{cases} 90, & Y=1 \\ 60, & Y=2 \end{cases} = E(X|Y)$

$$\mathbf{E}[X \mid Y] = \begin{cases} 90, & \text{w.p. } 1/3 \\ 60, & \text{w.p. } 2/3 \end{cases}$$

$g(1)=90$
$g(2)=60$

$$\mathbf{E}[\mathbf{E}[X \mid Y]] = \tfrac{1}{3} \cdot 90 + \tfrac{2}{3} \cdot 60 = 70 = \mathbf{E}[X]$$

$$\begin{aligned} \mathrm{var}(\mathbf{E}[X \mid Y]) &= \frac{1}{3}(90-70)^2 + \frac{2}{3}(60-70)^2 \\ &= \frac{600}{3} = 200 \end{aligned}$$

Section means and variances (ctd.)

$$\frac{1}{10}\sum_{i=1}^{10}(x_i - 90)^2 = 10 \qquad \frac{1}{20}\sum_{i=11}^{30}(x_i - 60)^2 = 20$$

$$\text{var}(X \mid Y = 1) = 10 \qquad \text{var}(X \mid Y = 2) = 20$$

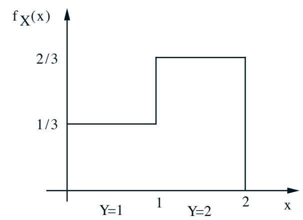
$$\text{var}(X \mid Y) = \begin{cases} 10, & \text{w.p. } 1/3 \\ 20, & \text{w.p. } 2/3 \end{cases}$$

$$\mathbf{E}[\text{var}(X \mid Y)] = \tfrac{1}{3}\cdot 10 + \tfrac{2}{3}\cdot 20 = \tfrac{50}{3}$$

$$\begin{aligned} \text{var}(X) &= \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y]) \\ &= \frac{50}{3} + 200 \\ &= (\text{average variability } \textbf{within} \text{ sections}) \\ &\quad + (\text{variability } \textbf{between} \text{ sections}) \end{aligned}$$

Example

$$\text{var}(X) = \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y])$$



$\mathbf{E}[X \mid Y = 1] =$ $\qquad$ $\mathbf{E}[X \mid Y = 2] =$

$\text{var}(X \mid Y = 1) =$ $\qquad$ $\text{var}(X \mid Y = 2) =$

$\mathbf{E}[X] =$

$\text{var}(\mathbf{E}[X \mid Y]) =$

Sum of a random number of independent r.v.'s

- N: number of stores visited
  (N is a nonnegative integer r.v.)
- X_i: money spent in store i
  - X_i assumed i.i.d.
  - independent of N
- Let $Y = X_1 + \cdots + X_N$

$$\begin{aligned} \mathbf{E}[Y \mid N = n] &= \mathbf{E}[X_1 + X_2 + \cdots + X_n \mid N = n] \\ &= \mathbf{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbf{E}[X_1] + \mathbf{E}[X_2] + \cdots + \mathbf{E}[X_n] \\ &= n\,\mathbf{E}[X] \end{aligned}$$

- $\mathbf{E}[Y \mid N] = N\,\mathbf{E}[X]$

$$\begin{aligned} \mathbf{E}[Y] &= \mathbf{E}[\mathbf{E}[Y \mid N]] \\ &= \mathbf{E}[N\,\mathbf{E}[X]] \\ &= \mathbf{E}[N]\,\mathbf{E}[X] \end{aligned}$$

Variance of sum of a random number of independent r.v.'s

- $\text{var}(Y) = \mathbf{E}[\text{var}(Y \mid N)] + \text{var}(\mathbf{E}[Y \mid N])$
- $\mathbf{E}[Y \mid N] = N\,\mathbf{E}[X]$
  $\text{var}(\mathbf{E}[Y \mid N]) = (\mathbf{E}[X])^2\,\text{var}(N)$
- $\text{var}(Y \mid N = n) = n\,\text{var}(X)$
  $\text{var}(Y \mid N) = N\,\text{var}(X)$
  $\mathbf{E}[\text{var}(Y \mid N)] = \mathbf{E}[N]\,\text{var}(X)$

$$\begin{aligned} \text{var}(Y) &= \mathbf{E}[\text{var}(Y \mid N)] + \text{var}(\mathbf{E}[Y \mid N]) \\ &= \mathbf{E}[N]\,\text{var}(X) + (\mathbf{E}[X])^2\,\text{var}(N) \end{aligned}$$