Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Tutorial 10 Solutions
November 18/19, 2010

1. Note that n is deterministic and H is a random variable.

   (a) Use X_1, X_2, ... to denote the (random) measured heights.

$$\begin{aligned} H &= \frac{X_1 + X_2 + \cdots + X_n}{n} \\ \mathbf{E}[H] &= \frac{\mathbf{E}[X_1 + X_2 + \cdots + X_n]}{n} = \frac{n\mathbf{E}[X]}{n} = h \\ \sigma_H &= \sqrt{\text{var}(H)} = \sqrt{\frac{n\,\text{var}(X)}{n^2}} \quad \text{(var of sum of } \textit{independent} \text{ r.v.s is sum of vars)} \\ &= \frac{1.5}{\sqrt{n}} \end{aligned}$$

   (b) We solve $\frac{1.5}{\sqrt{n}} < 0.01$ for n to obtain $n > 22500$.

   (c) Apply the Chebyshev inequality to H with $\mathbf{E}[H]$ and $\text{var}(H)$ from part (a):

$$\begin{aligned} \mathbf{P}(|H - h| \geq t) &\leq \left(\frac{\sigma_H}{t}\right)^2 \\ \mathbf{P}(|H - h| < t) &\geq 1 - \left(\frac{\sigma_H}{t}\right)^2 \end{aligned}$$

   To be "99% sure" we require the latter probability to be at least 0.99. Thus we solve

$$1 - \left(\frac{\sigma_H}{t}\right)^2 \geq 0.99$$

   with $t = 0.05$ and $\sigma_H = \frac{1.5}{\sqrt{n}}$ to obtain

$$n \geq \left(\frac{1.5}{0.05}\right)^2 \frac{1}{0.01} = 90000.$$

   (d) Intuitively, the variance of a random variable X that takes values in the range $[0, b]$ is maximum when X takes the value 0 with probability 0.5 and the value b with probability 0.5, in which case the variance of X is $b^2/4$ and its standard deviation is $b/2$.
   More formally, since $\mathbf{E}[(X - c)^2]$ is minimized when $c = \mathbf{E}[X]$, we have for any random variable X taking values in $[0, b]$,

$$\begin{aligned} E[(X-c)^2] &= E[X^2 + c^2 - 2cX] \\ &= E[X^2] + c^2 - 2cE(X) \\ &= c^2 - 2E(X)c + E(X^2) \end{aligned}$$

   minimize when $c = E(X)$

$$\Rightarrow E[(X-c)^2] \geq E[(X-E(X))^2] = Var(X)$$

$$\begin{aligned} \text{var}(X) &\leq \mathbf{E}[(X - \frac{b}{2})^2] \\ &= \mathbf{E}[X^2] - b\mathbf{E}[X] + \frac{b^2}{4} \\ &= \mathbf{E}[X(X - b)] + \frac{b^2}{4} \\ &\leq 0 + \frac{b^2}{4}, \end{aligned}$$

   since $0 \leq X \leq b \Rightarrow X(X - b) \leq 0$. Thus $\sigma_X \leq b/2$.
   In our example, we have $b = 3$, so $\sigma_X \leq 3/2$.

2. (a) Setting $s = 1$, we get $t_1 = 0$ and

$$\begin{aligned} t_2 &= 1 + \sum_{j=1}^{m} p_{ij} t_j \quad \forall i \neq s\,, \\ &= 1 + p_{22} t_2 \\ &\Rightarrow t_2 = 5/3\,. \end{aligned}$$

(b)

$$\begin{aligned} t_s^* &= 1 + \sum_{j=1}^{m} p_{sj} t_j \\ t_1^* &= 1 + p_{12} t_2 = 4/3\,. \end{aligned}$$

3. (a) $K = 2 + X_1 + X_2$, where X_1 and X_2 are independent exponential random variables with parameters 2/3 and 3/5.

$$\begin{aligned} E[K] &= 2 + 1/p_1 + 1/p_2 \\ &= 31/6\,. \\ \mathrm{var}(K) &= \frac{1 - p_1}{p_1^2} + \frac{1 - p_2}{p_2^2} \\ &= 67/36\,. \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{P}(A) &= \mathbf{P}(X_{999} \neq X_{1000} \neq X_{1001}) \\ &= \sum_{i=1}^{4} \mathbf{P}(A | X_{999} = i) \pi_i \\ &= 2/3\pi_1 + 2/3\pi_2 + 3/5\pi_3 + 3/5\pi_4 \\ &= 30/93 + 48/155 \approx 0.6323\,. \end{aligned}$$

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