# PMBM density

Multi-Object Tracking

Karl Granström

# A MOTIVATION FOR CHANGING THE BIRTH MODEL

In MBM filters, an MB birth model is used:

- In the prediction, add $N_k^{\mathrm{B}}$ birth components to each MB in the mixture
- If an object actually appeared and was detected, then in the update this detection can be associated to the birth component, and we can start to track the object.

But why add birth components before we know if there are any detections?

- Could the addition of new Bernoulli components, corresponding to the new potential objects, not be **measurement-driven**?
- Yes, it could, **if we use a PPP birth model** instead of a MB birth model.

# POISSON MULTI-BERNOULLI MIXTURE CONJUGATE PRIOR

## PMBM conjugate prior

With a Poisson birth, the Poisson Multi-Bernoulli Mixture (PMBM) density

$$\mathcal{PMBM}_{k|k}(\mathbf{x}_k)$$

is a multi-object conjugate prior to the standard point object transition density $p(\mathbf{x}_k \middle| \mathbf{x}_{k-1})$ and measurement model $p(\mathbf{z}_k \middle| \mathbf{x}_k)$,

$$\text{Prediction:} \quad \mathcal{PMBM}_{k|k-1}(\mathbf{x}_k) = \int p(\mathbf{x}_k | \mathbf{x}_{k-1}) \mathcal{PMBM}_{k-1|k-1}(\mathbf{x}_{k-1}) \, \delta \mathbf{x}_{k-1}$$

$$\text{Update:} \quad \mathcal{PMBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k | \mathbf{x}_k) \mathcal{PMBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k | \mathbf{x}'_k) \mathcal{PMBM}_{k|k-1}(\mathbf{x}'_k) \, \delta \mathbf{x}'_k}.$$

# THE PMBM MODEL

- Beyond multi-object conjugacy, why is the PMBM model useful for MOT?
- Some uncertainties in MOT:
  - Are there any objects? How many?
    - Detected objects: Bernoulli existence probabilities
    - Undetected objects: PPP intensity
  - If so, what are their states?
    - Detected objects: Bernoulli state densities
    - Undetected objects: PPP intensity
  - Data association? Captured by the MB mixture.
- The PMBM density nicely captures the relevant uncertainties.

# THE PBMB MODEL: DETECTED AND UNDETECTED OBJECTS

In the PMBM model, the set of objects $\mathbf{x}_k$ at time $k$

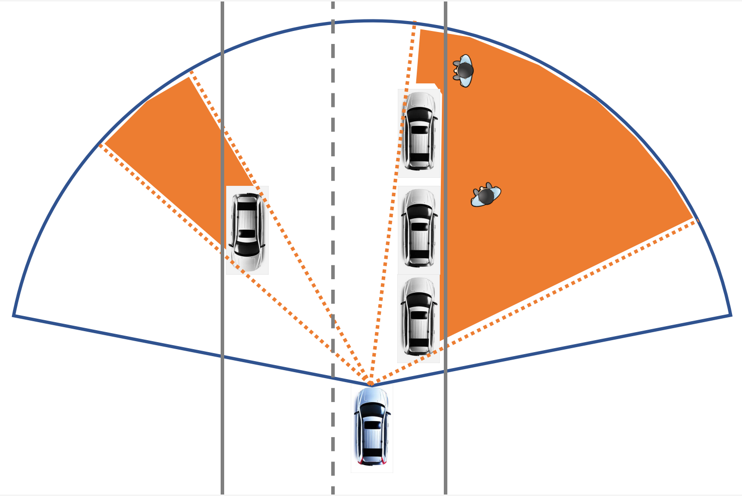
- Union of detected objects and undetected objects $\mathbf{x}_k = \mathbf{x}_k^d \uplus \mathbf{x}_k^{\mathrm{u}}$
  - Detected $\mathbf{x}_k^d$: objects the sensors have detected at least once
  - Undetected $\mathbf{x}_k^{\mathrm{u}}$: objects that have never detected

We are doing **tracking based on detections**, how could we **track undetected objects?!**

- Representation of their possible existence.
- Actually included in many tracking algorithms.
  - MBM filter: any Bernoulli to which a detection has never been associated
- Here it is made explicit.

# EXAMPLE: DETECTED AND UNDETECTED OBJECTS

## Autonomous car: Possibility of undetected objects in occluded areas



## PMBM DENSITY

- The Poisson Multi-Bernoulli Mixture density is defined as

$$\mathcal{PMBM}_{k|k}(\mathbf{x}_k) = \sum_{\mathbf{x}_k^{\mathrm{u}} \uplus \mathbf{x}_k^{d} = \mathbf{x}_k} \mathcal{P}_{k|k}^{\mathrm{u}}(\mathbf{x}_k^{\mathrm{u}}) \mathcal{MBM}_{k|k}^{d}(\mathbf{x}_k^{d})$$

- **PPP density** $\mathcal{P}_{k|k}^{\mathrm{u}}(\cdot)$ for undetected objects, typically with mixture intensity,

$$\lambda_{k|k}^{\mathrm{u}}(x_k) = \sum_{t=1}^{N_k^{\mathrm{u}}} w_{k|k}^{\mathrm{u},t} p_{k|k}^{\mathrm{u},t}(x_k), \qquad \left\{ \left( w_{k|k}^{\mathrm{u},t},\ p_{k|k}^{\mathrm{u},t}(\cdot) \right) \right\}_{t=1}^{N_k^{\mathrm{u}}}$$

- **Multi-Bernoulli Mixture density** $\mathcal{MBM}_{k|k}^{d}(\cdot)$ for detected objects, with parameters

$$\left\{ \left( \ell_{k|k}^{h_k},\ \left\{ \left( r_{k|k}^{i,h_k},\ p_{k|k}^{i,h_k}(\cdot) \right) \right\}_{i=1}^{N_k^{h_k}} \right) \right\}_{h_k=1}^{\mathcal{H}_k}$$

- PMBM density parameterised by the mixture intensity parameters, the MB log-weights, and the Bernoulli parameters of the MBs.

# PMBM DENSITY, GAUSSIAN DENSITIES

## Example: PPP mixture intensity and Bernoulli state pdfs are Gaussian

Gaussian mixture intensity,

$$\lambda^{\mathrm{u}}_{k|k}(x_k) = \sum_{t=1}^{N^{\mathrm{u}}_{k|k}} w^{\mathrm{u},t}_{k|k} \mathcal{N}\left(x^{\mathrm{u},t}_k \,;\, \mu^{\mathrm{u},t}_{k|k}, P^{i,h_k}_{k|k}\right)$$

Gaussian object densities,

$$p^{i,h_k}_{k|k}(x^{i,h_k}_k) = \mathcal{N}\left(x^{i,h_k}_k \,;\, \mu^{i,h_k}_{k|k}, P^{i,h_k}_{k|k}\right)$$

PMBM density parameters

$$\left\{\left(w^{\mathrm{u},t}_{k|k},\ \mu^{\mathrm{u},t}_{k|k},\ P^{\mathrm{u},t}_{k|k}\right)\right\}_{t=1}^{N^{\mathrm{u}}_k}, \quad \left\{\left(\ell^{h_k}_{k|k}, \left\{\left(r^{i,h_k}_{k|k},\ \mu^{i,h_k}_{k|k},\ P^{i,h_k}_{k|k}\right)\right\}_{i=1}^{N^{h_k}_k}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$$

# PMBM FILTER

If we design an MOT algorithm for the PMBM density, we get a PMBM filter

**PMBM filter: pseudo-code**

For $k = 1, 2, \ldots, K$

- **Prediction**
- **Update**
- **Reduction**
- **Estimation**

# PMBM prediction

## Multi-Object Tracking

Karl Granström

# POISSON MULTI-BERNOULLI MIXTURE PREDICTION

- **Posterior PMBM parameters**
- Prediction

$$\mathcal{PMBM}_{k+1|k}(\mathbf{x}_k) = \int p(\mathbf{x}_{k+1} \big| \mathbf{x}_k)\mathcal{PMBM}_{k|k}(\mathbf{x}_k)\,\delta\mathbf{x}_k$$

  with transition density $p(\mathbf{x}_{k+1} \big| \mathbf{x}_k)$ with
  - Probability of survival $P_k^{\mathrm{S}}(x_k)$
  - Transition density $\pi_{k+1}(x_{k+1}|x_k)$
  - PPP birth model: $\left\{w_{k+1}^{\mathrm{B},t},\ p_{k+1}^{\mathrm{B},i}(\cdot)\right\}_{i=1}^{N_{k+1}^{\mathrm{B}}}$
- **Predicted PMBM parameters**

# PMBM PREDICTION WITH PPP BIRTH, IN SUMMARY

## PMBM prediction with PPP birth

- Undetected and detected objects can be predicted independently
- Predicted undetected parameters consist of union of
  - predicted parameters from previous time step, and
  - birth parameters
- Each MB can be predicted independently of the other MBs
- Number of parameters increases (we add PPP birth params. to the undetected PPP)

# UNDETECTED PPP PREDICTION

Predicted PPP intensity for undetected objects,

$$\lambda^{\mathrm{u}}_{k+1|k}(x_{k+1}) = \int p(x_{k+1}|x_k)P^{\mathrm{S}}(x_k)\lambda^{\mathrm{u}}_{k|k}(x_k)\mathrm{d}x_k + \lambda^{\mathrm{B}}_{k+1}(x_{k+1})$$

The predicted intensity $\lambda^{\mathrm{u}}_{k+1|k}(x_{k+1})$ is the sum two intensities:

- Prediction of the surviving undetected objects $\int p(x_{k+1}|x_k)P^{\mathrm{S}}(x_k)\lambda^{\mathrm{u}}_{k|k}(x_k)\mathrm{d}x_k$
- Birth of new undetected objects $\lambda^{\mathrm{B}}_{k+1}(x_{k+1})$

## Mixture representations

$$\lambda^{\mathrm{u}}_{k+1|k}(x_{k+1}) = \sum_{t=1}^{N^{\mathrm{u}}_k} w^{\mathrm{u},t}_{k|k} \int p(x_{k+1}|x_k)P^{\mathrm{S}}(x_k)p^{\mathrm{u},t}_{k|k}(x_k)\mathrm{d}x_k + \sum_{i=1}^{N^{\mathrm{B}}_{k+1}} w^{\mathrm{B},i}_{k+1}p^{\mathrm{B},i}_{k+1}(x_{k+1})$$

# UNDETECTED PPP PREDICTION

## Undetected PPP prediction: pseudo-code

- **Posterior parameters:** $\left\{\left(w^{\mathrm{u},t}_{k|k},\ p^{\mathrm{u},t}_{k|k}(\cdot)\right)\right\}_{t=1}^{N^{\mathrm{u}}_k}$
- **Predicted parameters:**

$$\begin{aligned}
&\left\{\left(w^{\mathrm{u},t}_{k+1|k},\ p^{\mathrm{u},t}_{k+1|k}(\cdot)\right)\right\}_{t=1}^{N^{\mathrm{u}}_{k+1}} \\
&\quad = \left\{\mathrm{Predict}\left(w^{\mathrm{u},t'}_{k|k},\ p^{\mathrm{u},t'}_{k|k}(\cdot)\right)\right\}_{t'=1}^{N^{\mathrm{u}}_k} \bigcup \left\{\left(w^{\mathrm{B},t''}_{k+1},\ p^{\mathrm{B},t''}_{k+1}(\cdot)\right)\right\}_{t''=1}^{N^{\mathrm{B}}_{k+1}}
\end{aligned}$$

Increased number of mixture parameters $N^{\mathrm{u}}_{k+1} = N^{\mathrm{u}}_k + N^{\mathrm{B}}_{k+1}$

# PREDICTION OF A POSTERIOR MIXTURE COMPONENT

The predicted PPP weight and density,

$$\left(w^{\mathrm{u},t}_{k+1|k},\ p^{\mathrm{u},t}_{k+1|k}(\cdot)\right) = \mathrm{Predict}\left(w^{\mathrm{u},t}_{k|k},\ p^{\mathrm{u},t}_{k|k}(\cdot)\right)$$

are given by

$$w^{\mathrm{u},t}_{k+1|k} p^{\mathrm{u},t}_{k+1|k}(x_{k+1}) = w^{\mathrm{u},t}_{k|k} \int \pi_{k+1}(x_{k+1}|x_k) P^{\mathrm{S}}(x_k) p^{\mathrm{u},t}_{k|k}(x^i_k) \mathrm{d}x_k$$

and are

$$w^{\mathrm{u},t}_{k+1|k} = w^{\mathrm{u},t}_{k|k} P^{\mathrm{S}}_{\mathrm{u},t}$$

$$p^{\mathrm{u},t}_{k+1|k}(x_{k+1}) = \int \pi_{k+1}(x_{k+1}|x_k) \frac{P^{\mathrm{S}}(x_k) p^{\mathrm{u},t}_{k|k}(x^i_k)}{P^{\mathrm{S}}_{\mathrm{u},t}} \mathrm{d}x_k$$

where

$$P^{\mathrm{S}}_{\mathrm{u},t} = \int P^{\mathrm{S}}(x_k) p^{\mathrm{u},t}_{k|k}(x_k) \mathrm{d}x_k$$

# EXAMPLE: UNDETECTED PPP PREDICTION

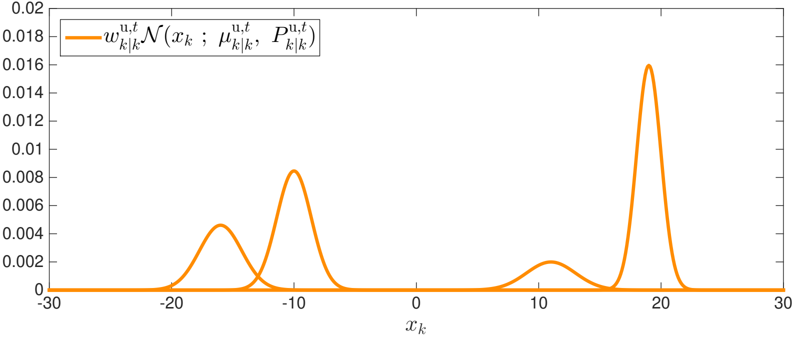
## Constant $P^{\mathrm{S}}$, linear Gaussian motion model

- $P^{\mathrm{S}}(x) = P^{\mathrm{S}}$ and $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}\left(x_{k+1}\,;\, F_{k+1}x_k, Q_{k+1}\right)$
- PPP birth with Gaussian mixture intensity $\sum_{i=1}^{N_{k+1}^{\mathrm{B}}} w_{k+1}^{\mathrm{B},i}\mathcal{N}\left(x_{k+1}\,;\, \mu_{k+1}^{\mathrm{B},i}, P_{k+1}^{\mathrm{B},i}\right)$
- Posterior intensity $\lambda_{k|k}^{\mathrm{u}}(x_k) = \sum_{t=1}^{N_k^{\mathrm{u}}} w_{k|k}^{\mathrm{u},t}\mathcal{N}\left(x_k\,;\, \mu_{k|k}^{\mathrm{u},t}, P_{k|k}^{\mathrm{u},t}\right)$
- Predicted intensity,

$$
\begin{aligned}
\lambda_{k+1|k}^{\mathrm{u}}(x_{k+1}) = &\sum_{t=1}^{N_k^{\mathrm{u}}} w_{k|k}^{\mathrm{u},t} P^{\mathrm{S}}\mathcal{N}\left(x_{k+1}\,;\, F_{k+1}\mu_{k|k}^{\mathrm{u},t}, F_{k+1}P_{k|k}^{\mathrm{u},t}F_{k+1}^{\mathrm{T}} + Q_{k+1}\right)\\
&+ \sum_{i=1}^{N_{k+1}^{\mathrm{B}}} w_{k+1}^{\mathrm{B},i}\mathcal{N}\left(x_{k+1}\,;\, \mu_{k+1}^{\mathrm{B},i}, P_{k+1}^{\mathrm{B},i}\right)
\end{aligned}
$$

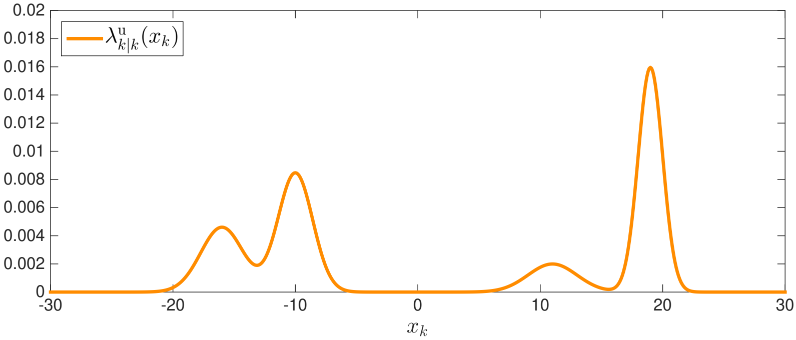
# EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
$0.02\mathcal{N}(x_{k+1}\,;\,-16, 3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10, 2) + 0.01\mathcal{N}(x_{k+1}\,;\,11, 4) + 0.04\mathcal{N}(x_{k+1}\,;\,19, 1)$



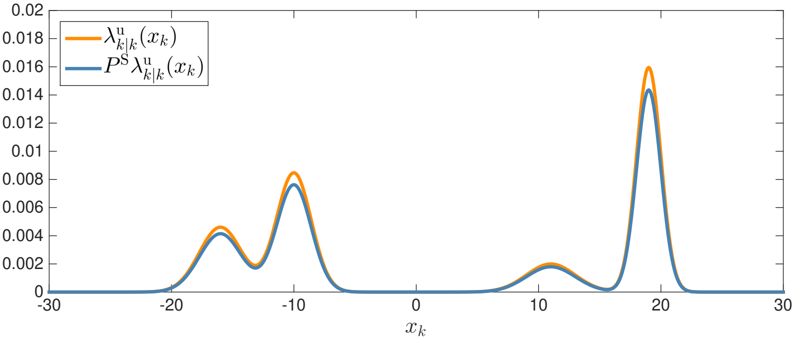
# EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
  $0.02\mathcal{N}(x_{k+1}\,;\,-16,3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10,2) + 0.01\mathcal{N}(x_{k+1}\,;\,11,4) + 0.04\mathcal{N}(x_{k+1}\,;\,19,1)$



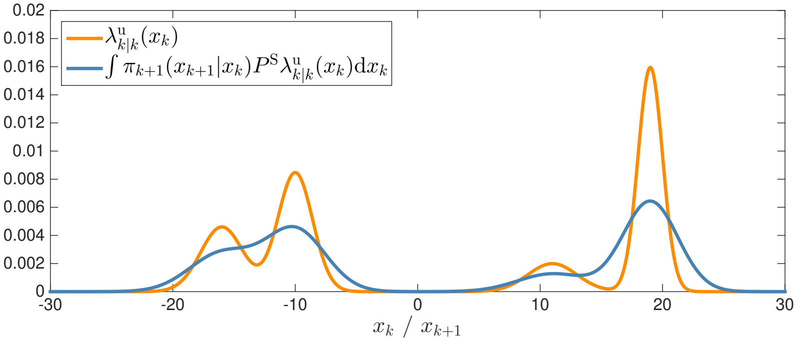
# EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
  $0.02\mathcal{N}(x_{k+1}\,;\,-16,3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10,2) + 0.01\mathcal{N}(x_{k+1}\,;\,11,4) + 0.04\mathcal{N}(x_{k+1}\,;\,19,1)$
- $P^{\mathrm{S}} = 0.9$ and random walk, $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}\,;\,x_k,4)$



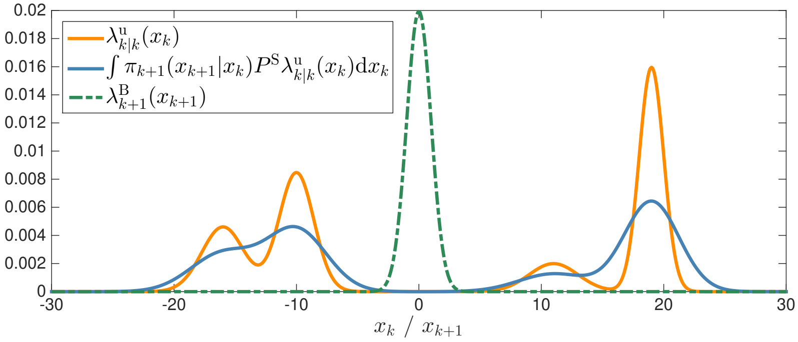
# EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
  $0.02\mathcal{N}(x_{k+1}\,;\,-16, 3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10, 2) + 0.01\mathcal{N}(x_{k+1}\,;\,11, 4) + 0.04\mathcal{N}(x_{k+1}\,;\,19, 1)$
- $P^{\mathrm{S}} = 0.9$ and random walk, $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}\,;\,x_k, 4)$



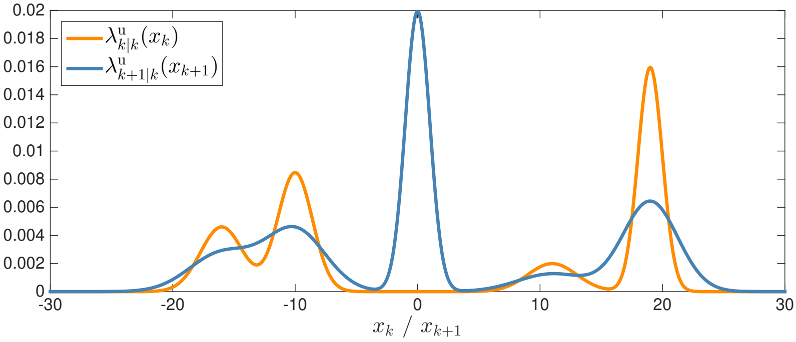
# EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
  $0.02\mathcal{N}(x_{k+1}\,;\,-16, 3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10, 2) + 0.01\mathcal{N}(x_{k+1}\,;\,11, 4) + 0.04\mathcal{N}(x_{k+1}\,;\,19, 1)$
- $P^{\mathrm{S}} = 0.9$ and random walk, $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}\,;\,x_k, 4)$
- $\lambda^{\mathrm{B}}_{k+1}(x_{k+1}) = 0.05\mathcal{N}(x_{k+1}\,;\,0, 1)$



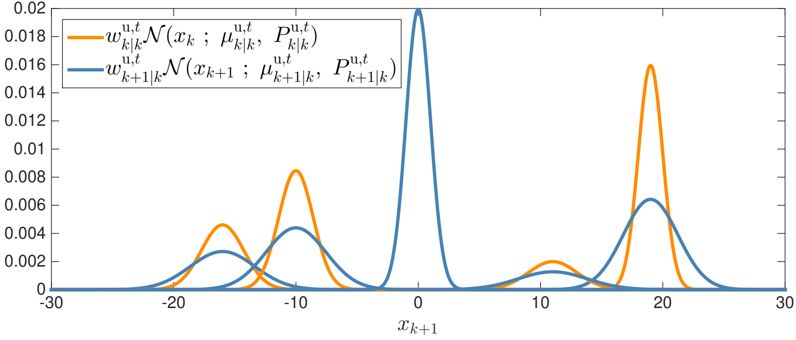
# EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
  $0.02\mathcal{N}(x_{k+1}\,;\,-16,3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10,2) + 0.01\mathcal{N}(x_{k+1}\,;\,11,4) + 0.04\mathcal{N}(x_{k+1}\,;\,19,1)$
- $P^{\mathrm{S}} = 0.9$ and random walk, $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}\,;\,x_k, 4)$
- $\lambda^{\mathrm{B}}_{k+1}(x_{k+1}) = 0.05\mathcal{N}(x_{k+1}\,;\,0,1)$



# EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
  $0.02\mathcal{N}(x_{k+1}\,;\,-16,3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10,2) + 0.01\mathcal{N}(x_{k+1}\,;\,11,4) + 0.04\mathcal{N}(x_{k+1}\,;\,19,1)$
- $P^{\mathrm{S}} = 0.9$ and random walk, $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}\,;\,x_k,4)$
- $\lambda^{\mathrm{B}}_{k+1}(x_{k+1}) = 0.05\mathcal{N}(x_{k+1}\,;\,0,1)$



# DETECTED MBM PREDICTION

## Detected MBM prediction: pseudo-code

- **Posterior parameters:** $\left\{\left(\ell_{k|k}^{h_k},\ \left\{\left(r_{k|k}^{i,h_k},\ p_{k|k}^{i,h_k}(\cdot)\right)\right\}_{i=1}^{N_k^{h_k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$
- **Predicted parameters:** $\left\{\left(\ell_{k|k}^{h_k},\ \left\{\left(r_{k+1|k}^{i,h_k},\ p_{k+1|k}^{i,h_k}(\cdot)\right)\right\}_{i=1}^{N_k^{h_k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$

  where, for each $h_k$ and each $i$,

  $$\left(r_{k+1|k}^{i,h_k},\ p_{k+1|k}^{i,h_k}(\cdot)\right)$$

  are computed the same way as in an MBM filter
- Same number of Bernoullis $N_k^{h_k}$

# BERNOULLI PREDICTION

The predicted Bernoulli parameters,

$$\left(r^{i,h_k}_{k+1|k},\ p^{i,h_k}_{k+1|k}(\cdot)\right)$$

are

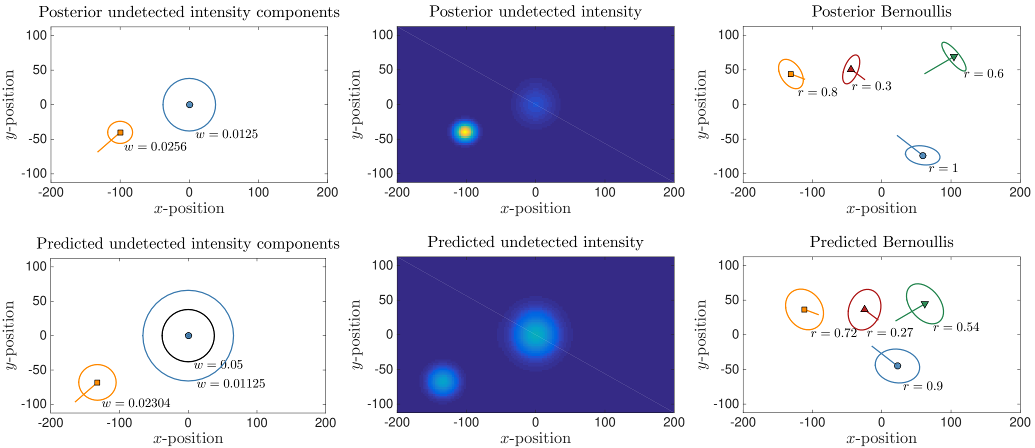
$$r^{i,h_k}_{k+1|k} = r^{i,h_k}_{k|k} P^{\mathrm{S}}_{i,h_k}$$

$$p^{i,h_k}_{k+1|k}(x^i_{k+1}) = \int \pi_{k+1}(x^i_{k+1}|x^i_k)\frac{P^{\mathrm{S}}(x^i_k)p^{i,h_k}_{k|k}(x^i_k)}{P^{\mathrm{S}}_{i,h_k}}\mathrm{d}x^i_k$$

where

$$P^{\mathrm{S}}_{i,h_k} = \int P^{\mathrm{S}}(x^i_k)p^{i,h_k}_{k|k}(x^i_k)\mathrm{d}x^i_k$$

# PMBM PREDICTION: 2D EXAMPLE, $P^S = 0.9$




- Constant velocity $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\,Fx_{k-1}, Q)$, and $P^S(x_{k-1}) = 0.9$
- Birth intensity with single mixture component, position in origin, zero velocity

# PMBM update: overview

Multi-Object Tracking

Karl Granström

# POISSON MULTI-BERNOULLI MIXTURE UPDATE

- **Prior PMBM parameters**
- Update

$$\mathcal{PMBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k \big| \mathbf{x}_k)\mathcal{PMBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k \big| \mathbf{x}'_k)\mathcal{PMBM}_{k|k-1}(\mathbf{x}'_k)\, \delta \mathbf{x}'_k}$$

with multi-object measurement model $p(\mathbf{z}_k \big| \mathbf{x}_k)$ with

- Probability of detection $P_k^{\mathrm{D}}(x_k)$
- Measurement model $g_k(z_k|x_k)$
- Poisson clutter intensity: $\lambda_c(z_k)$

- **Posterior PMBM parameters**

# POISSON MULTI-BERNOULLI MIXTURE UPDATE, IN SUMMARY

## PMBM update for the standard point object models

- For each prior MB, multiple data associations
- For each prior MB and each data association, we get an MB in the posterior MBM
- For each Bernoulli, two possibilities:
  - Either associated to one of the measurements,
  - or misdetected.
- Any measurement not associated to a prior Bernoulli is either
  - clutter
  - or from an object detected for the first time

  We get a **new Bernoulli**

# PMBM UPDATE: DATA ASSOCIATION

With an MB birth model,

- Initiation of potential new objects: add birth Bernoulli components in prediction
- When handling the data association, each detection is associated to
  - one of the prior Bernoullis, or
  - the clutter PPP

With a PPP birth model,

- Initiation of potential new objects: measurement driven.
- When handling the data association, each detection is associated to
  - one of the prior Bernoullis,
  - the undetected PPP or the clutter PPP

  **Important:** we treat the potential new objects and the clutter jointly!
- Convenient to reformulate assignment problem: assign the measurements

# DATA ASSOCIATION VARIABLE FOR $M_K$ DETECTIONS, 1

- Let there be $m_k$ measurements and $n_k$ objects.
- The association for measurement $z_k^j$ is denoted $\psi_k^j$
- An association for all $m_k$ measurement is denoted

$$\psi_k = \left[\psi_k^1,\ \psi_k^2, \dots,\ \psi_k^j, \dots,\ \psi_k^{m_k}\right]$$

- $\psi_k^j$ is defined similarly to how $\theta_k^i$ was defined,

$$\psi_k^j = \begin{cases} i & \text{if measurement } j \text{ is associated to object } i \\ 0 & \text{if measurement } j \text{ is associated either to a potential new object, or to clutter} \end{cases}$$

# DATA ASSOCIATION VARIABLE FOR $M_K$ DETECTIONS, 2

- $\Psi_k$ is the set of valid association events at time $k$.
- For $\psi_k \in \Psi_k$, the following must hold:

  1. Each measurement must be either from a previously detected object, or clutter/potential new object,

  $$\psi_k^j \in \{0, \ldots, n_k\}, \; \forall\, j \in \{1, \ldots, m_k\}$$

  2. **Point object assumption:** For any pair of two measurements, they cannot be associated to the same previously detected object,

  $$\forall\, j, j' \in \{1, \ldots, m_k\}, \; j \neq j', \text{ if } \psi_k^j \neq 0, \; \psi_k^{j'} \neq 0 \; \Rightarrow \; \psi_k^j \neq \psi_k^{j'}$$

# DATA ASSOCIATION VARIABLE FOR $M_K$ DETECTIONS, 3

- Note that 1. and 2. on the previous slide together implicitly ensures that we do not associate more than $n_k$ measurements to the $n_k$ objects.
- In what follows, unless otherwise stated, we consider associations $\psi_k \in \Psi_k$.
- Given a $\psi$, we can find the equivalent $\theta$, and vice versa.

$$\begin{aligned}
\psi^j = i \quad &\Leftrightarrow \quad \theta^i = j \\
\psi^j = 0 \quad &\Leftrightarrow \quad \nexists\, i \;:\; \theta^i = j \\
\nexists\, j \;:\; \psi^j = i \quad &\Leftrightarrow \quad \theta^i = 0
\end{aligned}$$

# HYPOTHESIS ORIENTED-PMBM UPDATE

## Hypothesis oriented-PMBM update: pseudo-code

**Input:** $\lambda^{\mathrm{u}}_{k|k-1}(x_k), \left\{ \left( \ell^{h_{k-1}} , \left\{ \left( r^{i,h_{k-1}}_{k|k-1}, \; p^{i,h_{k-1}}_{k|k-1}(\cdot) \right) \right\}_{i=1}^{N^{h_{k-1}}_{k-1}} \right) \right\}_{h_{k-1}=1}^{\mathcal{H}_{k-1}}$

Misdetection update: $\lambda^{\mathrm{u}}_{k|k}(x_k) = (1 - P^{\mathrm{D}}(x_k))\lambda^{\mathrm{u}}_{k|k-1}(x_k)$

Initialise: $h_k = 0$

For $h_{k-1} = 1, \ldots, \mathcal{H}_{k-1}$

Create cost matrix $L^{h_{k-1}}$, and compute $M_{h_{k-1}}$ associations $\psi_m$

For $m = 1, \ldots, M_{h_{k-1}}$

Increase: $h_k \leftarrow h_k + 1$

**Compute posterior MB parameters:** detected, misdetected & new Bernoulli, log-weight $\tilde{\ell}^{h_k}$

Set $\mathcal{H}_k = h_k$

Normalise log-weights $\ell^{h_k} \leftarrow \tilde{\ell}^{h_k}$

**Output:** $\lambda^{\mathrm{u}}_{k|k}(x_k), \left\{ \left( \ell^{h_k} , \left\{ \left( r^{i,h_k}_{k|k}, \; p^{i,h_k}_{k|k}(\cdot) \right) \right\}_{i=1}^{N^{h_k}_{k}} \right) \right\}_{h_k=1}^{\mathcal{H}_k}$

# PMBM update: details

Multi-Object Tracking

Karl Granström

# POISSON MULTI-BERNOULLI MIXTURE UPDATE, “BUILIDING BLOCKS”

- Undetected object: remain undetected, or detected for the first time
- Previously detected object: misdetected, or detected again

**Important “builiding blocks” of PMBM update:**

- update of PPP intensity for undetected objects that remain undetected
- update of potential new object detected for the first time $\Rightarrow$ **new Bernoulli**
- update of Bernoulli with associated measurement – similar to MBM-filter
- update of misdetected Bernoulli – similar to MBM-filter
- posterior log-weights

# UNDETECTED OBJECTS MISDETECTED AGAIN

- Posterior PPP intensity for objects that remain undetected,

$$\lambda^{\mathrm{u}}_{k|k}(x) = (1 - P^{\mathrm{D}}(x))\lambda^{\mathrm{u}}_{k|k-1}(x)$$

- Posterior intensity is **lower**/**higher** in areas where $(1 - P^{\mathrm{D}}(x))$ is **low**/**high**, because it is **unlikely**/**likely** that an object was not detected there
- Mixture representation of intensity

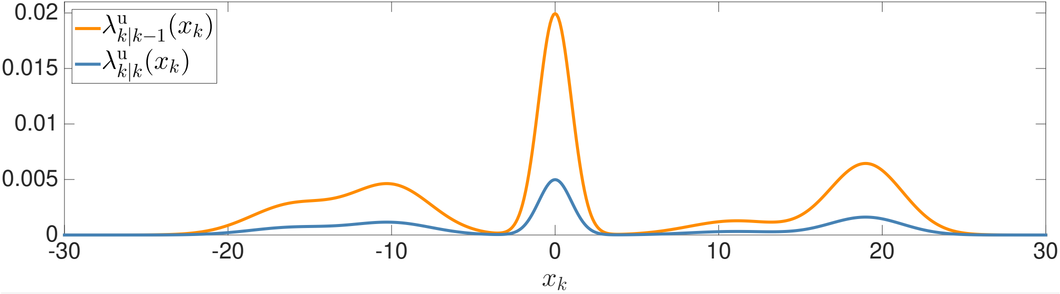
$$\begin{aligned}\lambda^{\mathrm{u}}_{k|k}(x_k) &= \sum_t w^{\mathrm{u},t}_{k|k-1}(1 - P^{\mathrm{D}}(x))p^{\mathrm{u},t}_{k|k-1}(x_k) \\ &= \sum_t w^{\mathrm{u},t}_{k|k-1}P^{\mathrm{MD}}_{\mathrm{u},t}\frac{(1 - P^{\mathrm{D}}(x))p^{\mathrm{u},t}_{k|k-1}(x_k)}{P^{\mathrm{MD}}_{\mathrm{u},t}} = \sum_t \underbrace{w^{\mathrm{u},t}_{k|k-1}P^{\mathrm{MD}}_{\mathrm{u},t}}_{w^{\mathrm{u},t}_{k|k}} \underbrace{\frac{(1 - P^{\mathrm{D}}(x))p^{\mathrm{u},t}_{k|k-1}(x_k)}{P^{\mathrm{MD}}_{\mathrm{u},t}}}_{p^{\mathrm{u},t}_{k|k}(x_k)}\end{aligned}$$

where $P^{\mathrm{MD}}_{\mathrm{u},t} = \int(1 - P^{\mathrm{D}}(x))p^{\mathrm{u},t}_{k|k-1}(x_k)\mathrm{d}x_k$

# POSTERIOR UNDETECTED INTENSITY EXAMPLE

## Constant $P^{\mathrm{D}}$, linear Gaussian models

- Prior intensity $\lambda^{\mathrm{u}}_{k|k-1}(x_k) = \sum_{t=1}^{5} w^{\mathrm{u},t}_{k|k-1}\mathcal{N}\left(x_k\,;\,\mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1}\right)$
- Probability of detection $P^{\mathrm{D}} = 0.75$
- Posterior intensity $\lambda^{\mathrm{u}}_{k|k}(x_k) = \sum_{t=1}^{5} 0.25 w^{\mathrm{u},t}_{k|k-1}\mathcal{N}\left(x_k\,;\,\mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1}\right)$



# OBJECT DETECTED FOR THE FIRST TIME: NEW BERNOULLI

If $\psi^j = 0$, then the measurement $z_k^j$ is either clutter or from a previously undetected object.

New Bernoulli component in the posterior MB, with parameters

$$r_{k|k}^j = \frac{\rho_{k|k-1}^{\mathrm{u}}(z_k^j)}{\lambda_c(z_k^j) + \rho_{k|k-1}^{\mathrm{u}}(z_k^j)}, \qquad \rho_{k|k-1}^{\mathrm{u}}(z_k^j) = \int P^{\mathrm{D}}(x_k) g_k(z_k^j|x_k) \lambda_{k|k-1}^{\mathrm{u}}(x_k) \mathrm{d}x_k$$

$$p_{k|k}^j(x_k) = \frac{P^{\mathrm{D}}(x_k) g_k(z_k^j|x_k) \lambda_{k|k-1}^{\mathrm{u}}(x_k)}{\rho_{k|k-1}^{\mathrm{u}}(z_k^j)}$$

**Posterior $r$ conditioned on $\psi$**

Relative intensity of: 1) detection from previously undetected object; and, 2) clutter

Predicted log-likelihood

$$\ell_k^{\mathrm{u},j} = \log\left(\lambda_c(z_k^j) + \rho_{k|k-1}^{\mathrm{u}}(z_k^j)\right)$$

# NEW BERNOULLI: PROBABILITY OF EXISTENCE $< 1$?

- Earlier we had that, conditioned on the data association, $r = 1$
- How can we now have a new Bernoulli for which $r < 1$?
- It represents two possibilities:
  - Detection was from clutter – no new object
    Likelihood $\lambda_c(z_k^j)$.
  - Detection was from a new object
    Likelihood $\rho^{\mathrm{u}}_{k|k-1}(z_k^j) = \int P^{\mathrm{D}}(x_k) g_k(z_k^j|x_k) \lambda^{\mathrm{u}}_{k|k-1}(x_k) \mathrm{d}x_k$.
- Represented compactly as a new Bernoulli with probability of existence

$$r^j_{k|k} = \frac{\rho^{\mathrm{u}}_{k|k-1}(z_k^j)}{\lambda_c(z_k^j) + \rho^{\mathrm{u}}_{k|k-1}(z_k^j)}$$

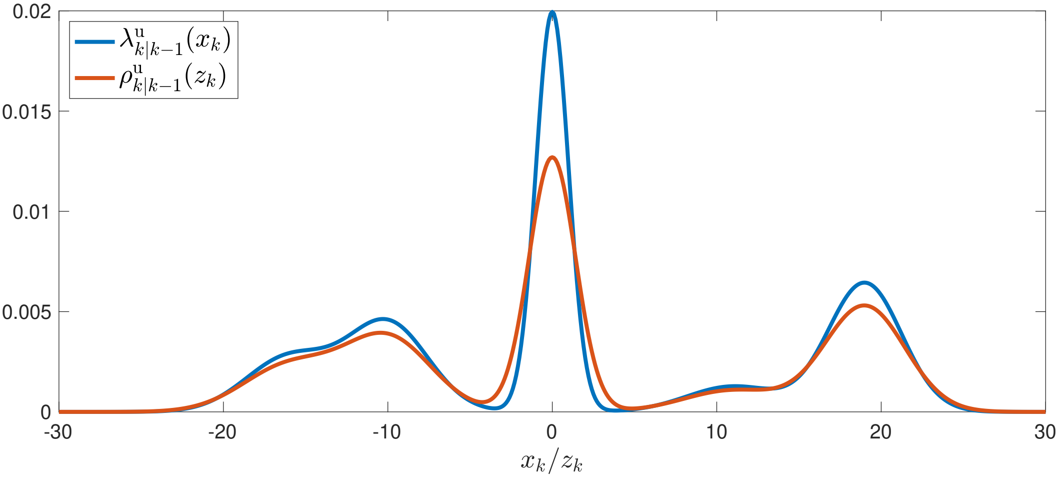
# NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE

## Constant $P^{\mathrm{D}}$, linear Gaussian models

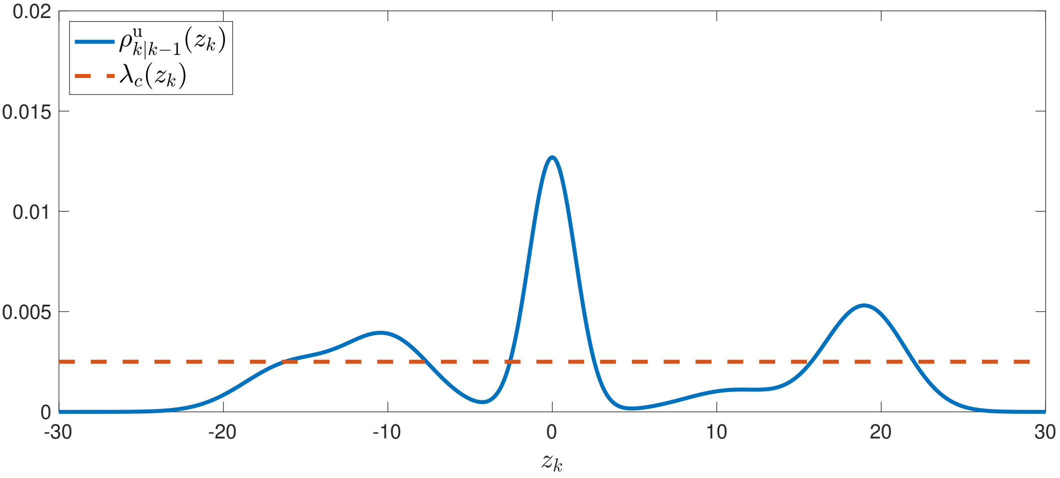
- Undetected intensity $\lambda^{\mathrm{u}}_{k|k-1}(x_k) = \sum_{t=1}^{5} w^{\mathrm{u},t}_{k|k-1} \mathcal{N}\left(x_k\,;\, \mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1}\right)$
- Probability of detection $P^{\mathrm{D}} = 0.9$, measurement model $g_k(z|x) = \mathcal{N}\left(z\,;\, x, R\right)$
- Likelihood $\rho^{\mathrm{u}}_{k|k-1}(z_k) = \sum_{t=1}^{5} P^{\mathrm{D}} w^{\mathrm{u},t}_{k|k-1} \mathcal{N}\left(z_k\,;\, \mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1} + R\right)$
- Clutter $\lambda_c(z_k) = \bar{\lambda}_c / V$
- Probability of existence of new Bernoulli

$$r_{k|k} = \frac{\sum_{t=1}^{5} P^{\mathrm{D}} w^{\mathrm{u},t}_{k|k-1} \mathcal{N}\left(z_k\,;\, \mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1} + R\right)}{\frac{\bar{\lambda}_c}{V} + \sum_{t=1}^{5} P^{\mathrm{D}} w^{\mathrm{u},t}_{k|k-1} \mathcal{N}\left(z_k\,;\, \mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1} + R\right)}$$

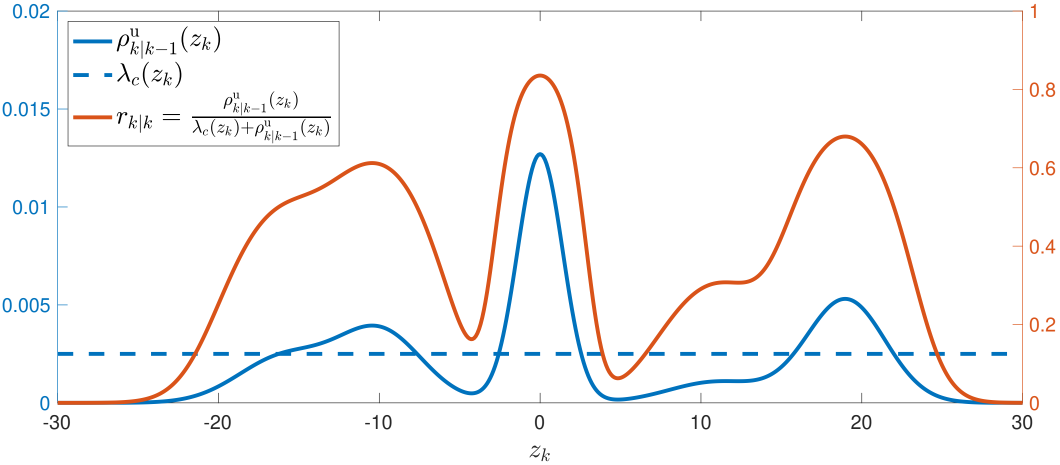
# NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE



# NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE



# NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE



# NEW BERNOULLI STATE DENSITY EXAMPLE

## Constant $P^{\mathrm{D}}$, linear Gaussian models

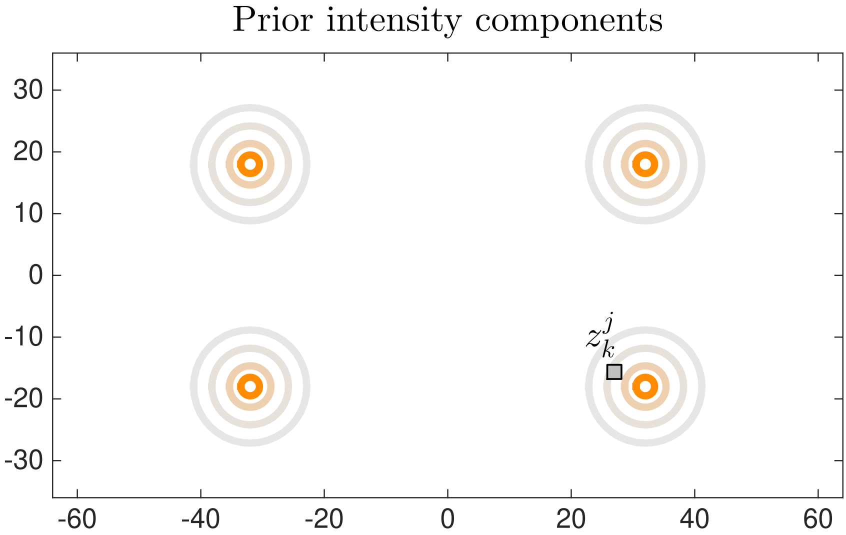
- Undetected intensity $\lambda^{\mathrm{u}}_{k|k-1}(x_k) = \sum_t w^{\mathrm{u},t}_{k|k-1}\mathcal{N}\left(x_k\,;\,\mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1}\right)$
- Probability of detection $P^{\mathrm{D}} = 0.9$, measurement model $g_k(z|x) = \mathcal{N}\left(z\,;\,Hx, R\right)$
- State density of new Bernoulli

$$p^j_{k|k}(x_k) = \sum_t w^{\mathrm{u},t,j}_k \mathcal{N}\left(x_k\,;\,\mu^{\mathrm{u},t}_{k|k-1} + K^{\mathrm{u},t}_k\left(z^j_k - \hat{z}^{\mathrm{u},t}_k\right), P^{\mathrm{u},t}_{k|k-1} - K^{\mathrm{u},t}_k H_k P^{\mathrm{u},t}_{k|k-1}\right)$$

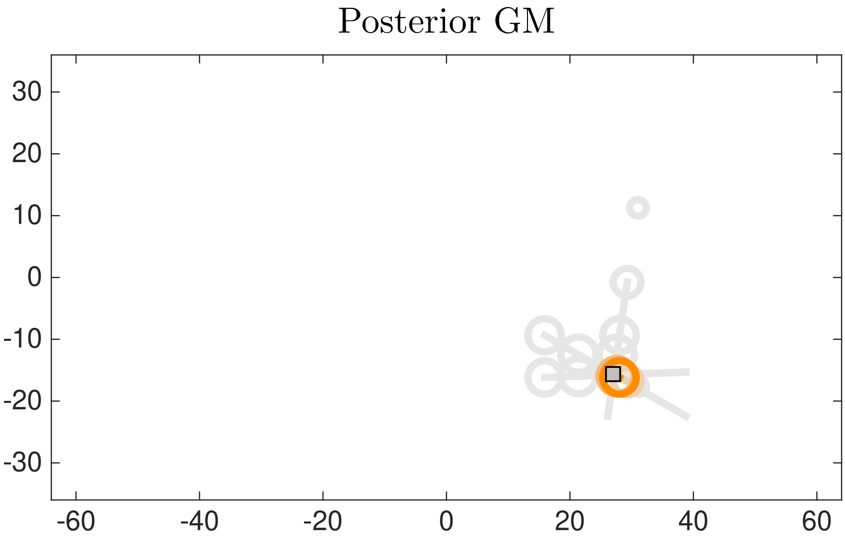
$$w^{\mathrm{u},t,j}_k = \frac{w^{\mathrm{u},t}_{k|k-1} P^{\mathrm{D}}\mathcal{N}\left(z^j_k\,;\,\hat{z}^{\mathrm{u},t}_k, S^{\mathrm{u},t}_k\right)}{\sum_{t'} w^{\mathrm{u},t'}_{k|k-1} P^{\mathrm{D}}\mathcal{N}\left(z^j_k\,;\,\hat{z}^{\mathrm{u},t'}_k, S^{\mathrm{u},t'}_k\right)}$$

- Pruning and merging used to reduce $p^j_{k|k}(x_k)$, often to a single Gaussian

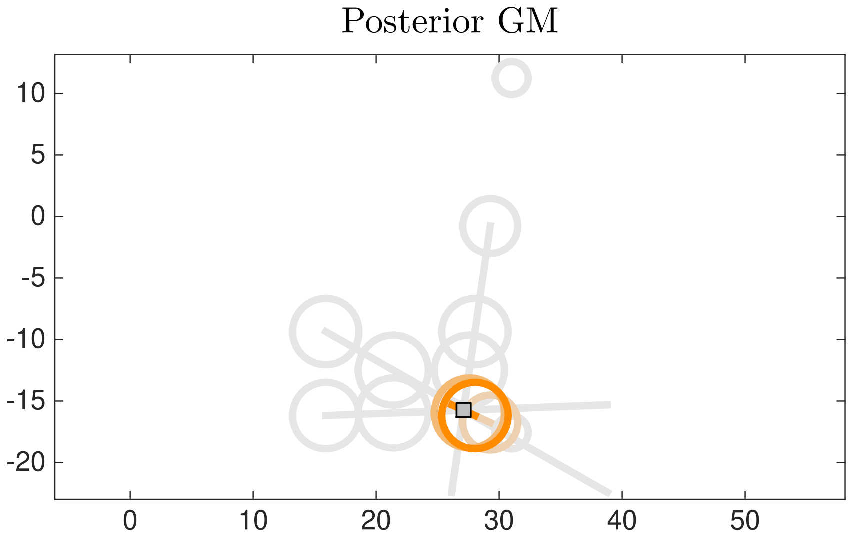
## NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



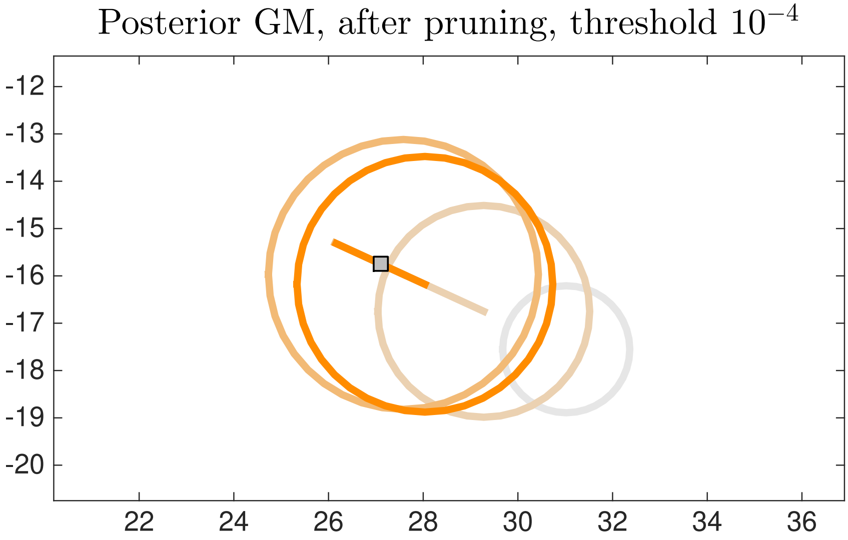
# NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



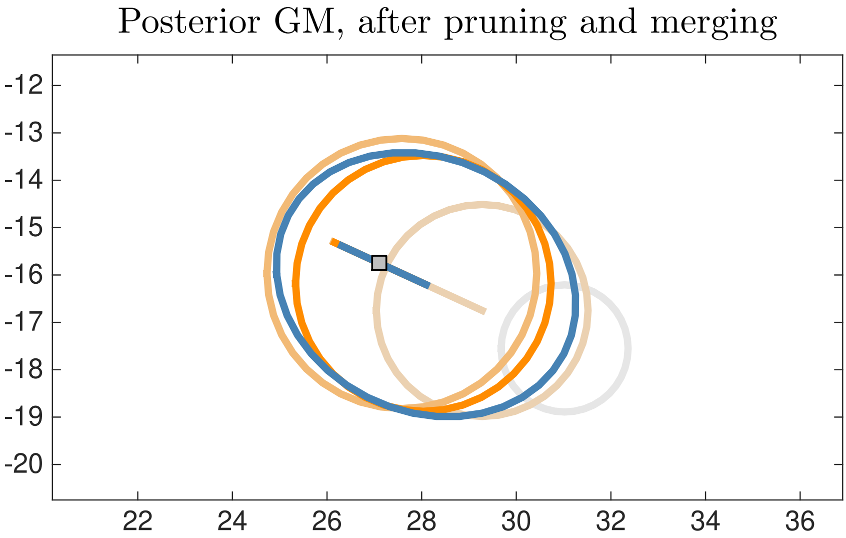
# NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



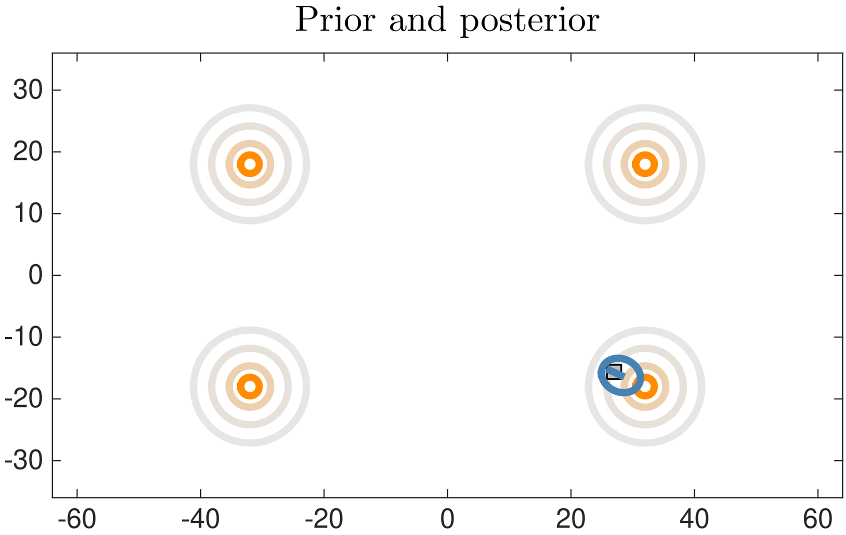
# NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



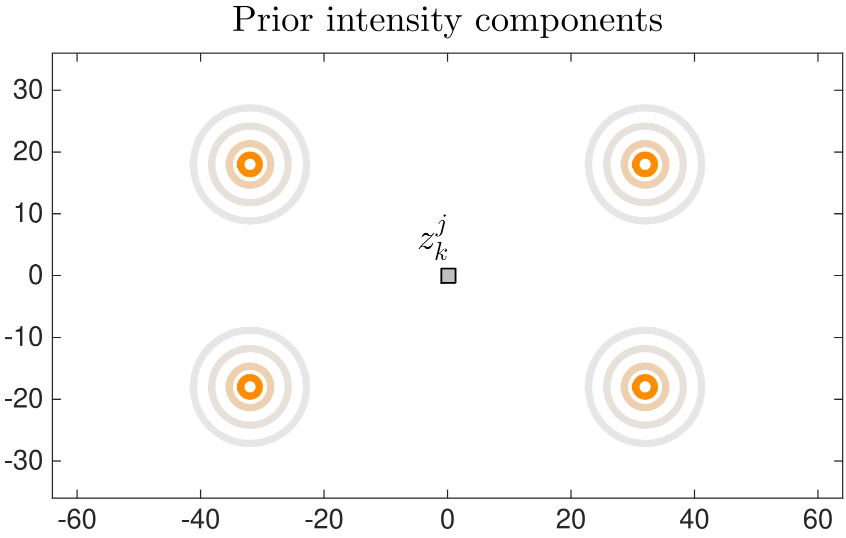
# NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



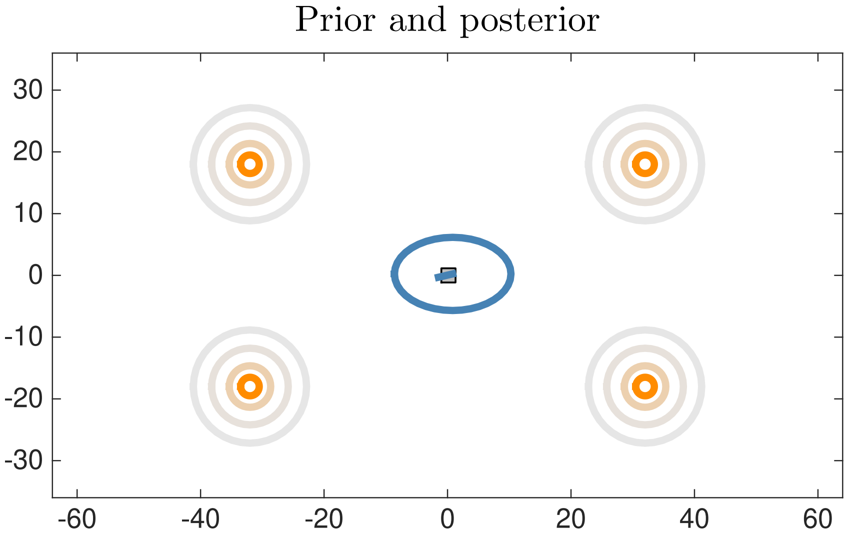
## NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



# NEW BERNOULLI STATE DENSITY VISUALIZATION, 2



# NEW BERNOULLI STATE DENSITY VISUALIZATION, 2



# BERNOULLI UPDATE: DETECTION AND MISDETECTION

For prior hypothesis $h$, if $\psi^j = i$,

$$r_{k|k}^{i,j,h} = 1$$

$$p_{k|k}^{i,j,h}(x_k^i) = \frac{P^{\mathrm{D}}(x_k^i) g_k(z_k^{\theta^i} | x_k^i) p_{k|k-1}^{i,h}(x_k^i)}{\int P^{\mathrm{D}}(x_k^i) g_k(z_k^{\theta^i} | x_k^i) p_{k|k-1}^{i,h}(x_k^i) \mathrm{d}x_k^i}$$

For prior hypothesis $h$, if $\nexists j : \psi^j = i$,

$$r_{k|k}^{i,0,h} = \frac{r_{k|k-1}^{i,h} P_{i,h}^{\mathrm{MD}}}{1 - r_{k|k-1}^{i,h} + r_{k|k-1}^{i,h} P_{i,h}^{\mathrm{MD}}}$$

$$p_{k|k}^{i,0,h}(x) = \frac{(1 - P^{\mathrm{D}}(x_k)) p_{k|k-1}^{i,h}(x)}{P_{i,h}^{\mathrm{MD}}}$$

where $P_{i,h}^{\mathrm{MD}} = \int \left(1 - P^{\mathrm{D}}\left(x_k^i\right)\right) p_{k|k-1}^{i,h}\left(x_k^i\right) \mathrm{d}x_k^i$

Predicted log-likelihoods

$$\ell_k^{i,j,h} = \log\left( r_{k|k-1}^{i,h} \int P^{\mathrm{D}}\left(x_k^i\right) g_k\left(z_k^j | x_k^i\right) p_{k|k-1}^{i,h}(x_k^i) \mathrm{d}x_k^i \right)$$

$$\ell_k^{i,0,h} = \log\left(1 - r_{k|k-1}^{i} + r_{k|k-1}^{i} P_{i,h}^{\mathrm{MD}}\right)$$

Note similarity to Bernoulli update in MBM filter

# NON-NORMALISED POSTERIOR LOG-WEIGHTS

For a prior MB $h$ and a data association $\psi_k$, the non-normalized posterior log-weight is

$$\begin{aligned}
\tilde{\ell}_{k|k}^{h_{k-1},\psi_k} &= \underbrace{\ell_{k|k-1}^{h_{k-1}}}_{\text{Prior}} + \underbrace{\sum_{i:\nexists j:\psi^j=i} \ell_k^{i,0,h}}_{\text{Misdetection}} + \underbrace{\sum_{j:\psi^j\neq 0} \ell_k^{\psi^j,j,h}}_{\text{Assoc. meas.}} + \underbrace{\sum_{j:\psi^j=0} \ell_k^{\mathrm{u},j}}_{\text{Clutter or potential new object}} \\
&= \ell_{k|k-1}^{h_{k-1}} + \sum_{i=1}^{N^h} \ell_k^{i,0,h} + \sum_{j:\psi^j\neq 0} \left[\ell_k^{\psi^j,j,h} - \ell_k^{\psi^j,0,h}\right] + \sum_{j:\psi^j=0} \ell_k^{\mathrm{u},j} \\
&= \ell_{k|k-1}^{h_{k-1}} + \sum_{j=1}^{m_k} \tilde{\ell}_k^{j,h} + \text{Constant independent of } \psi_k
\end{aligned}$$

where

$$\tilde{\ell}_k^{j,h} = \begin{cases} \ell_k^{\psi^j,j,h} - \ell_k^{\psi^j,0,h} & \text{if } \psi^j \neq 0 \\ \ell_k^{\mathrm{u},j} & \text{if } \psi^j = 0 \end{cases}$$

# EXAMPLE PMBM UPDATE VISUALIZATION

## Prior and model, 2D scenario

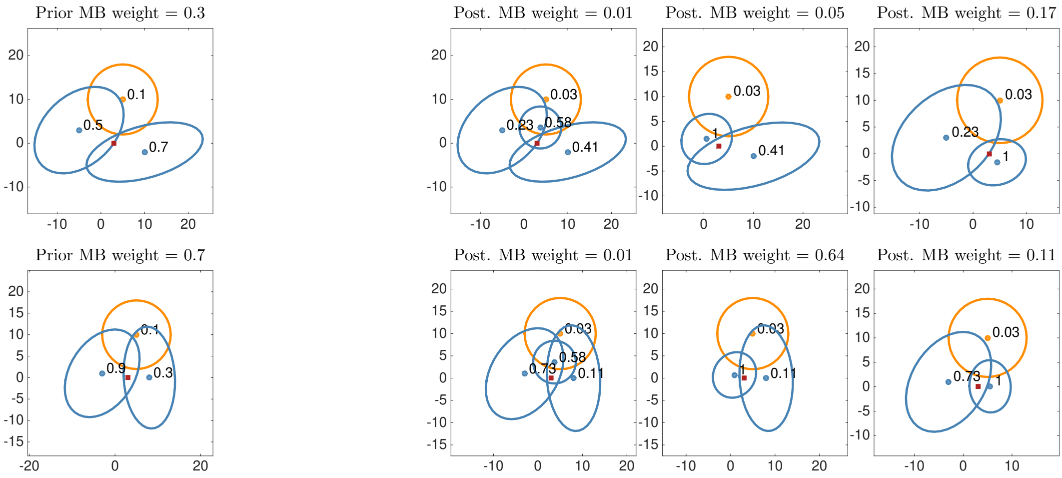
- PMBM with two MBs, each with two Bernoullis with Gaussian state densities, undetected PPP intensity with single Gaussian.
- Measurement model:

$$
\begin{aligned}
P^{\mathrm{D}} &= 0.75 \\
g_k(z|x) &= \mathcal{N}\left(z\,;\ x, 9\mathbf{I}_2\right) \\
\lambda_c(z) &= \begin{cases} 4\times 10^{-4} & z \in [-25,25]\times[-25,25] \\ 0 & z \notin [-25,25]\times[-25,25] \end{cases}
\end{aligned}
$$

- Single measurement $\Rightarrow$ 3 DAs for each prior MB $\Rightarrow$ Posterior MBM with 6 MBs

# EXAMPLE PMBM UPDATE VISUALIZATION

Undetected objects PPP (Orange), Detected objects MB (Blue), Measurement (Red)



# HANDLING THE DATA ASSOCIATIONS

The data association problem is handled analogously to tracking $n$ objects, and MBM filter:

- Use gating to remove very unlikely associations and group Bernoullis/measurements
- For each group, form cost matrix with negative log likelihoods
- Use some algorithm to find $M$ associations, e.g.,
  - Murty
  - Gibbs' sampling
- Truncate all other associations

# PMBM UPDATE: COST MATRIX

Let there be $m_k$ detections, and consider an MB $h$ with $N^h$ Bernoullis. The cost matrix is

$$L^h = \begin{bmatrix} -\ell^{1,1,h} & -\ell^{1,2,h} & \dots & -\ell^{1,N^h,h} & -\ell^{1,0} & \infty & \dots & \infty \\ -\ell^{2,1,h} & -\ell^{2,2,h} & \dots & -\ell^{2,N^h,h} & \infty & -\ell^{2,0} & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\ell^{m_k,1,h} & -\ell^{m_k,2,h} & \dots & -\ell^{m_k,N^h,h} & \infty & \infty & \dots & -\ell^{m_k,0} \end{bmatrix}$$

where

$$\ell^{j,0} = \log\left(\lambda_c(z_k^j) + \int P^{\mathrm{D}}(x_k) g_k(z_k^j|x_k)\lambda^{\mathrm{u}}_{k|k-1}(x_k)\mathrm{d}x_k\right)$$

$$\ell^{j,i,h} = \log\left(\frac{r^{i,h}_{k|k-1}\int P^{\mathrm{D}}\left(x_k^i\right) g_k(z_k^j|x_k^i) p^{i,h}_{k|k-1}(x_k^i)\mathrm{d}x_k^i}{1 - r^{i,h}_{k|k-1} + r^{i,h}_{k|k-1}\int\left(1 - P^{\mathrm{D}}\left(x_k^i\right)\right) p^{i,h}_{k|k-1}\left(x_k^i\right)\mathrm{d}x_k^i}\right)$$

# PMBM post processing

Multi-Object Tracking

Karl Granström

# PMBM POST PROCESSING

After prediction and update, we have a PMBM density $\mathcal{PMBM}_{k|k}(\mathbf{x}_k)$ with params.

$$\left\{\left(w_{k|k}^{\mathrm{u},t},\ p_{k|k}^{\mathrm{u},t}(\cdot)\right)\right\}_{t=1}^{N_k^{\mathrm{u}}},\quad \left\{\left(\ell_{k|k}^{h_k},\ \left\{\left(r_{k|k}^{i,h_k},\ p_{k|k}^{i,h_k}(\cdot)\right)\right\}_{i=1}^{N_k^{h_k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$$

- Reduction:
  - Reduce $N_k^{\mathrm{u}}$, $\mathcal{H}_k$ and $N_k^{h_k}$
  - Important for computational cost
  - Pruning, merging, capping, and **recycling**
- Estimation:
  - Extracting a set of estimated object states from the posterior density.

# PMBM REDUCTION

## PMBM reduction

- **MBM pruning:** prune MB $h_k$ if $\ell^{h_k}_{k|k} \leq \Gamma$
- **MBM capping:** if $\mathcal{H}_k > N_{\max}$, keep the $N_{\max}$ MBs with largest log-weights.

  After pruning and capping the MBM, remaining log-weights are re-normalized.
- **Bernoulli recycling:** in each MB $h_k$, recycle Bernoulli $i$ if $r^{i,h}_{k|k} < \Gamma^r$
- **PPP reduction:** pruning, merging and capping of the mixture intensity

Outside the scope of the course: MBM merging

# BERNOULLI RECYCLING

## Bernoulli recycling: basic idea

Instead of pruning Bernoullis with small $r$, approximate them as PPP, and add the intensity to undetected PPP intensity.

- KL-div minimising PPP approximation: intensity $=$ Bernoulli PHD,

$$\lambda_{k|k}^{h,i,\mathrm{REC}}(x_k) = r_{k|k}^{h,i} p_{k|k}^{h,i}(x_k)$$

- Undetected PPP intensity after recycling is

$$\lambda_{k|k}^{\mathrm{u},\mathrm{REC}}(x_k) = \lambda_{k|k}^{\mathrm{u}}(x_k) + \sum_{h} \sum_{i: r_{k|k}^{h,i} < \Gamma^r} \exp\left(\ell_{k|k}^{h}\right) r_{k|k}^{h,i} p_{k|k}^{h,i}(x_k)$$

Note that we must take the normalized hypothesis weight $\exp\left(\ell_{k|k}^{h}\right)$ into account

# WHY RECYCLING?

Following the recycling, we have to reduce the undetected PPP intensity.

**Why not just prune right away?**

- Recycling/pruning lowers the computational complexity, because there are fewer Bernoullis to consider in the data association.
- Pruning means that we lose all information contained in what is pruned.
- By recycling, the information is retained approximately as a PPP.
- The Bernoulli recycling threshold can therefore be considerably larger than a Bernoulli pruning threshold.
- Empirical studies show that Bernoulli **recycling leads to lower computational cost**, without sacrificing tracking performance.

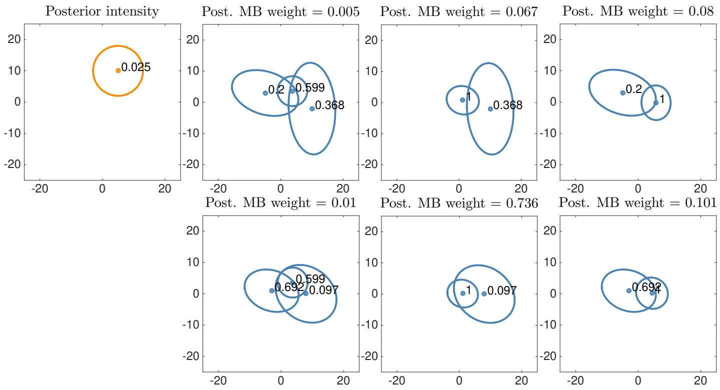
# REDUCING THE UNDETECTED INTENSITY

- Undetected intensity with mixture representation

$$\lambda^{\mathrm{u}}_{k|k}(x_k) = \sum_{t=1}^{N^{\mathrm{u}}_k} w^{\mathrm{u},t}_{k|k} p^{\mathrm{u},t}_{k|k}(x_k)$$

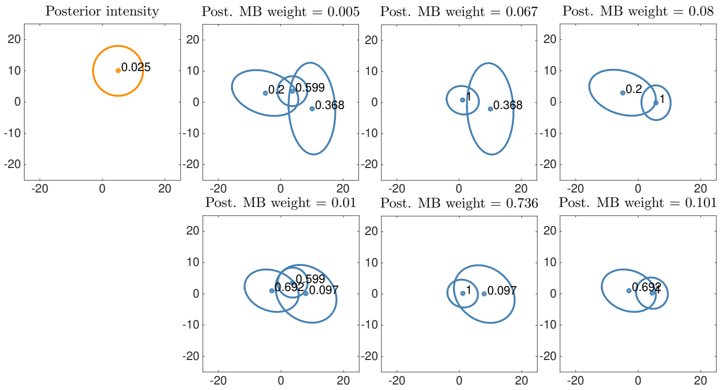
- The number of mixture components increases over time, due to:
  - the addition of birth in the prediction
  - the recycling following the update
- Reduced using pruning, merging and capping.

# EXAMPLE PMBM REDUCTION



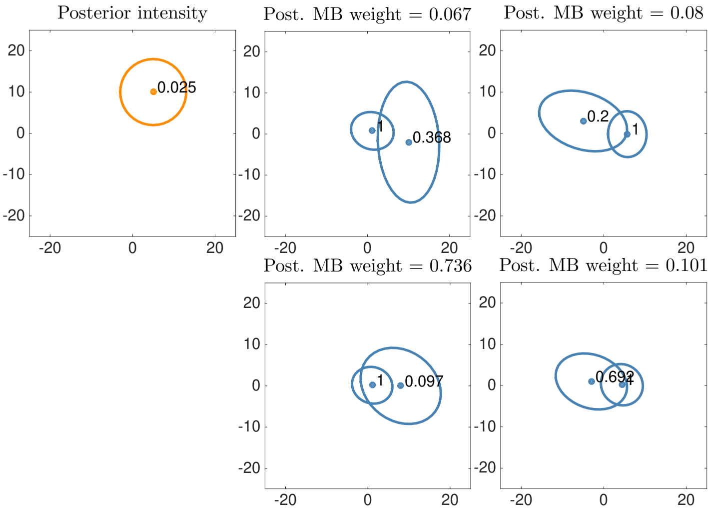
# EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$



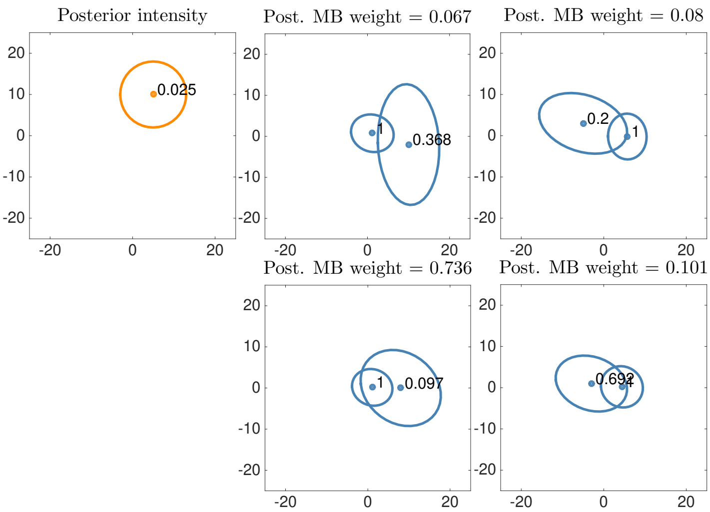
# EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$



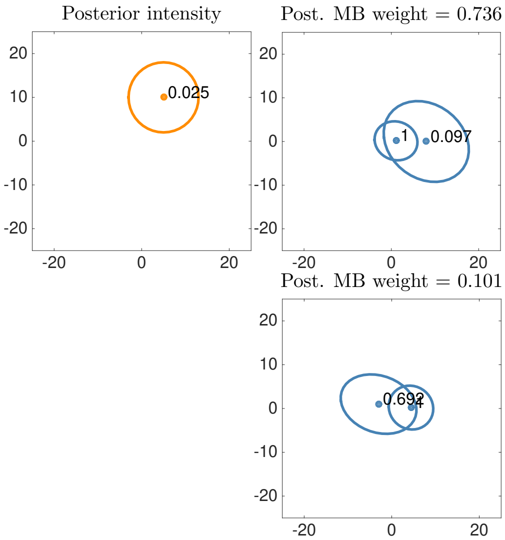
# EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$



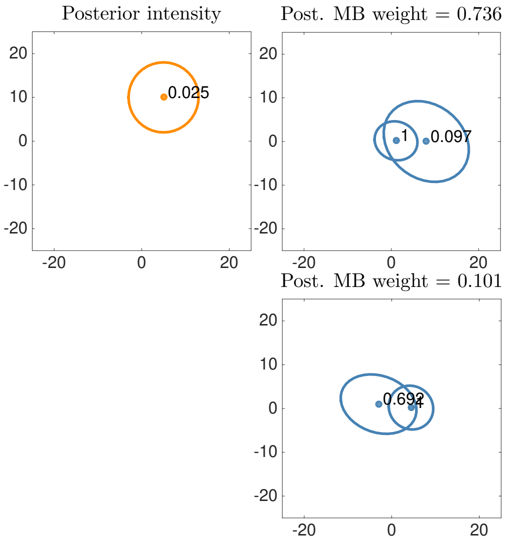
# EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$



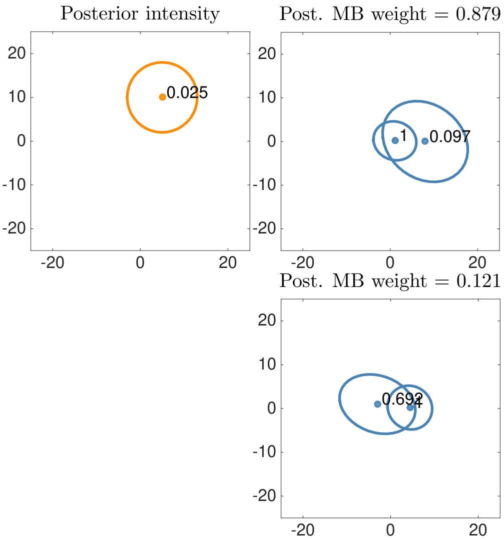
# EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\mathrm{max}} = 2$
- Re-normalize weights



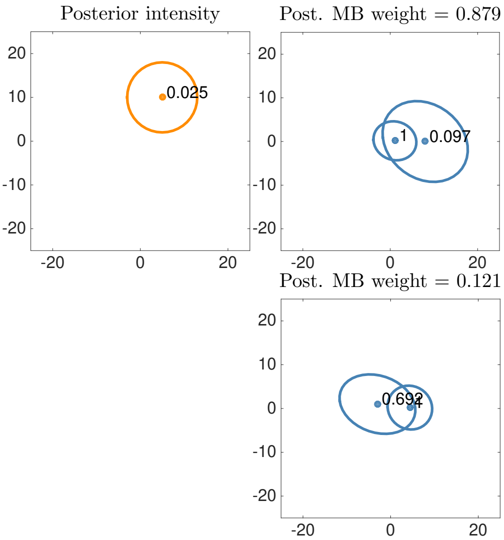
# EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\mathrm{max}} = 2$
- Re-normalize weights



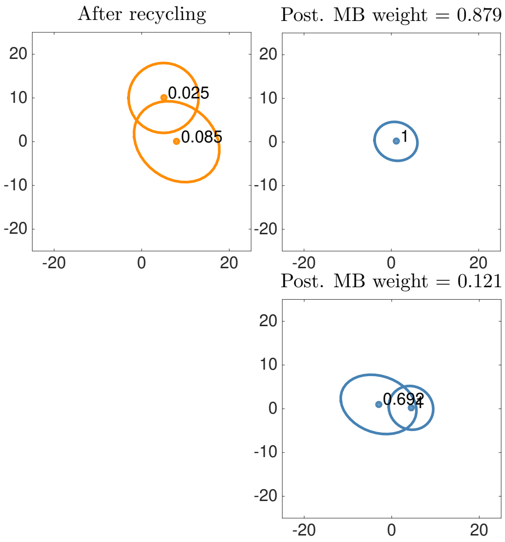
# EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\text{max}} = 2$
- Re-normalize weights
- Bernoulli recycling, $\Gamma^r = 0.1$



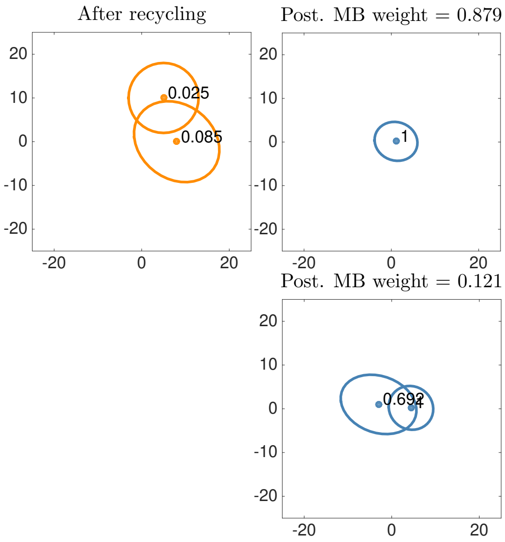
# EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$
- Re-normalize weights
- Bernoulli recycling, $\Gamma^r = 0.1$



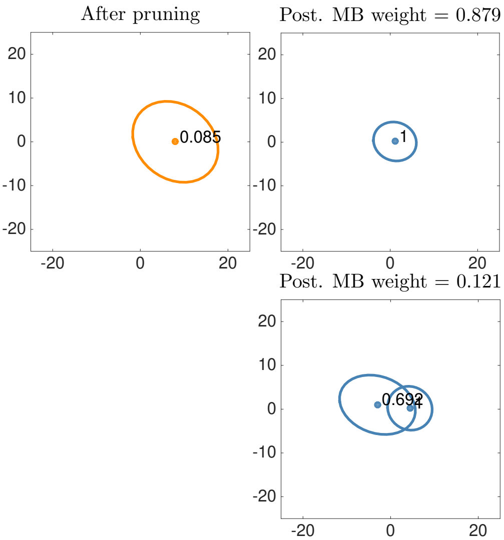
# EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\text{max}} = 2$
- Re-normalize weights
- Bernoulli recycling,
  $\Gamma^r = 0.1$
- PPP pruning
  $\Gamma^w = 0.05$



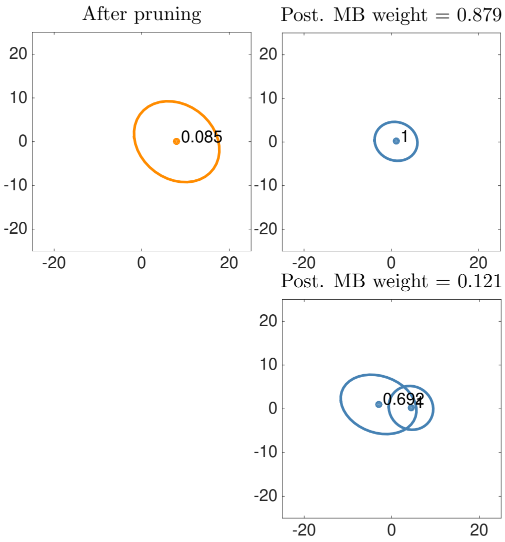
# EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\max} = 2$
- Re-normalize weights
- Bernoulli recycling,
  $\Gamma^r = 0.1$
- PPP pruning
  $\Gamma^w = 0.05$



# EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$
- Re-normalize weights
- Bernoulli recycling, $\Gamma^r = 0.1$
- PPP pruning $\Gamma^w = 0.05$
- **Note:** typically, $\Gamma$ and $\Gamma^w$ are smaller, and $N_{\max}$ is larger



# PMBM ESTIMATION

## Simple PMBM estimator

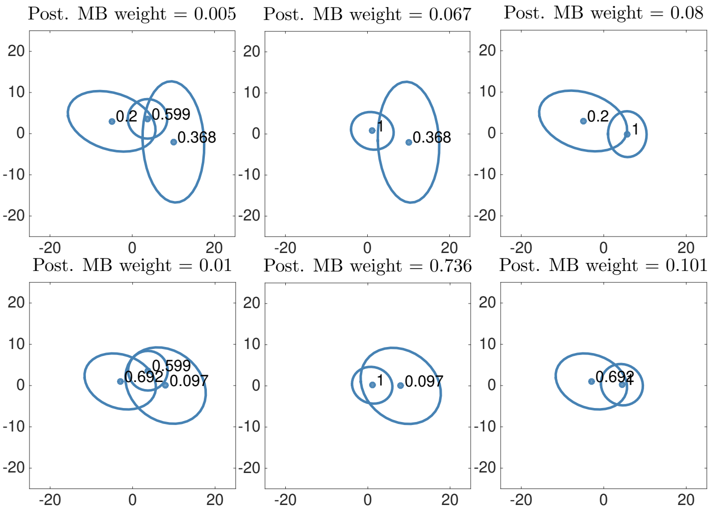
Generally we do not extract estimates from the undetected PPP.

- Initialise as empty set: $\hat{\mathbf{x}}_{k|k} = \emptyset$
- MB with largest weight: $h^\star = \max_{h_k} \ell_{k|k}^{h_k}$
- For $i = 1, \ldots, N_k^{h^\star}$, if $r_{k|k}^{h^\star,i} > \Gamma^e$: $\hat{\mathbf{x}}_{k|k} \leftarrow \hat{\mathbf{x}}_{k|k} \cap \hat{x}_{k|k}^{i,h^\star}$
- For example, expected value or MAP estimate,

$$\bar{x}_{k|k}^{i} = \int x_k p_{k|k}^{h^\star,i}(x_k)\mathrm{d}x_k, \qquad \hat{x}_{k|k}^{i,\mathrm{MAP}} = \arg\max_{x_k} p_{k|k}^{h^\star,i}(x_k)$$

# EXAMPLE PMBM ESTIMATOR

- Largest weight
- Extract: $\Gamma^e = 0.5$



# Implementation of Conjugate multi-object filters

## Multi-Object Tracking

Karl Granström

# HO-MHT AND TO-MHT

- Two MHT approaches to tracking $n$ objects:
  - Hypothesis oriented (HO): represent each global $n$ object hypothesis explicitly
  - Track oriented (TO): represent each object by local hypotheses. Global $n$ object hypotheses encoded by look-up table.
  - Track oriented is computationally more efficient.

- Similar alternatives for the MB mixture in PMBM and MBM filtering
  - Hypothesis oriented: represent each MB explicitly
  - Track oriented: represent each Bernoulli by local hypotheses. Each MB (global hypothesis) encoded by look-up table.
  - Again, track oriented is computationally more efficient.

# TRACK ORIENTED CONJUGATE MULTI-OBJECT FILTERS

## MBM filter

- Add new Bernoullis in prediction
- New local hypothesees in update
- Look-up table for MBs
- Reduction: remove local hypotheses and global hypotheses

## PMBM filter

- Initiate new Bernoullis in update
- New local hypothesees in update
- Look-up table for MBs
- Reduction: remove local hypotheses and global hypotheses

# IMPLEMENTATIONAL ASPECTS OF MBM FILTERS AND PMBM FILTERS

- Local and global hypotheses

- Bernoulli representation in the MBs
  - Uncertain existence, $r \in (0, 1)$
  - Certain existence, $r = 0$ or $r = 1$

# Local and Global Hypotheses in MBM filter

Multi-Object Tracking

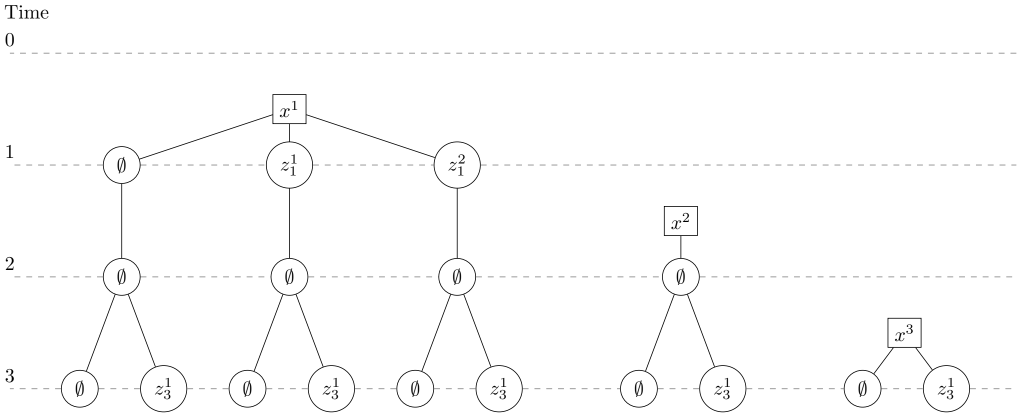
Karl Granström

# HYPOTHESES IN MBM FILTER

## Example: scenario setup

- MB birth with a single birth component in each time step
- Three time steps, measurement sets:
  - $\mathbf{z}_1 = \{z_1^1,\ z_1^2\}$
  - $\mathbf{z}_2 = \emptyset$
  - $\mathbf{z}_3 = \{z_3^1\}$
- At time $k = 0$, empty MBM. Not necessary, but most common.

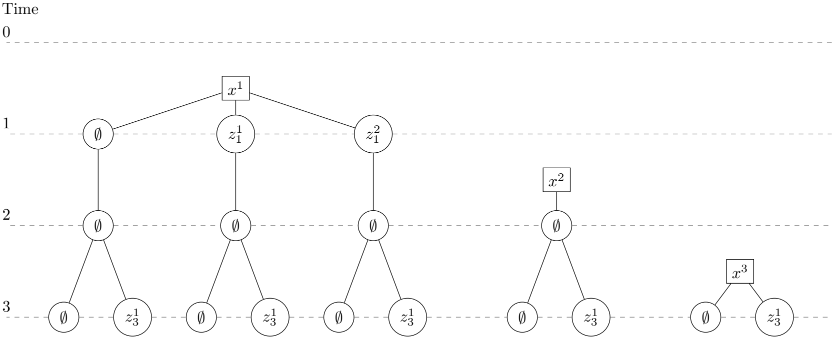
# LOCAL HYPOTHESIS TREES IN MBM FILTER




For each leaf node, we have $r$ and $p(x)$ conditioned on that association sequence

# MBM GLOBAL HYPOTHESES

Total number of MBs (global hypotheses): 12




For simplicity: local hypotheses indexed 1, 2, 3 . . . from left to right.

Look-up table:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \\ 5 & 1 & 1 \\ 5 & 1 & 2 \\ 5 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

# TRACK ORIENTED MBM FILTER

- Add new Bernoullis in the prediction
- For each Bernoulli, maintain local hypotheses
- Look-up table points out which local hypotheses are included in an MB
- MBM reduction affects both the local hypotheses and look-up table

# Local and Global Hypotheses in PMBM filter

## Multi-Object Tracking

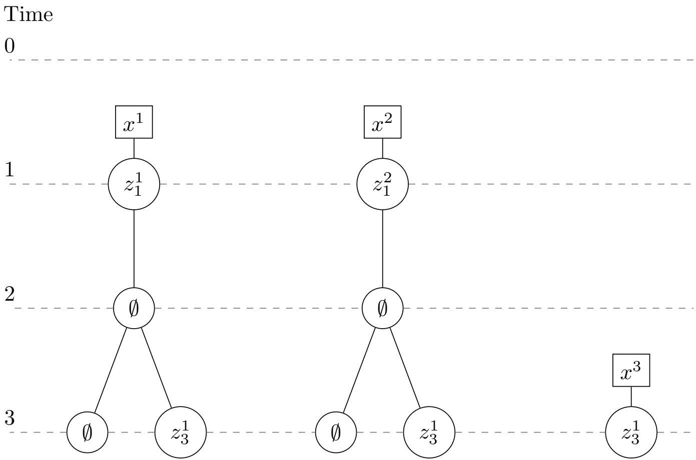
Karl Granström

# LOCAL HYPOTHESIS TREES IN PMBM FILTER

## Example: scenario setup

- PPP birth, i.e., initiation of new Bernoullis is measurement driven
- Three time steps, measurement sets:
  - $\mathbf{z}_1 = \{z_1^1,\ z_1^2\}$
  - $\mathbf{z}_2 = \emptyset$
  - $\mathbf{z}_3 = \{z_3^1\}$
- At time $k = 0$, empty MBM. Not necessary, but most common.

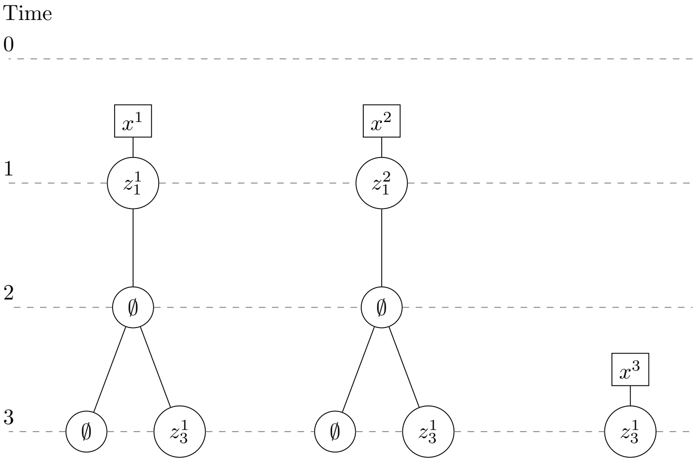
# LOCAL HYPOTHESIS TREES IN PMBM FILTER




For each leaf node, we have $r$ and $p(x)$ conditioned on that association sequence

# PMBM GLOBAL HYPOTHESES

Total number of MBs (global hypotheses): 3




Look-up table:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

For simplicity: local hypotheses indexed from left to right.

Note: each Bernoulli is not represented in each MB (global hypothesis)

# TRACK ORIENTED PMBM FILTER

- Add a new Bernoulli for each measurement
- For each Bernoulli, maintain local hypotheses
- Look-up table points out which local hypotheses are included in an MB
- MBM reduction affects both the local hypotheses and look-up table

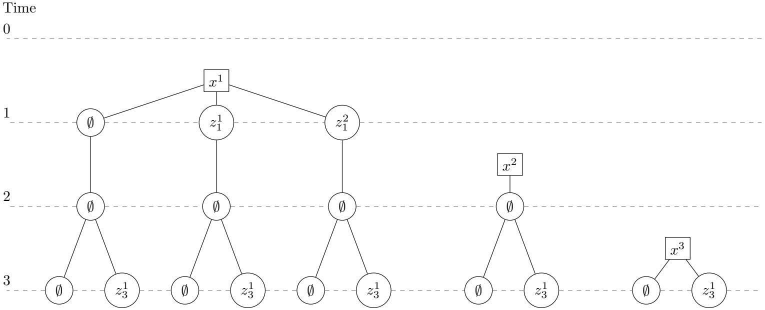
# Reduction of local and global hypotheses

Multi-Object Tracking

Karl Granström

# MULTI-BERNOULLI MIXTURE PRUNING/CAPPING

Global hypotheses are removed

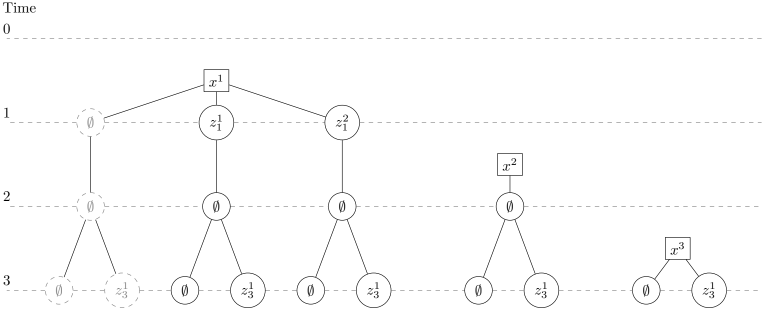



Before pruning/capping

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \\ 5 & 1 & 1 \\ 5 & 1 & 2 \\ 5 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

# MULTI-BERNOULLI MIXTURE PRUNING/CAPPING

Global hypotheses are removed



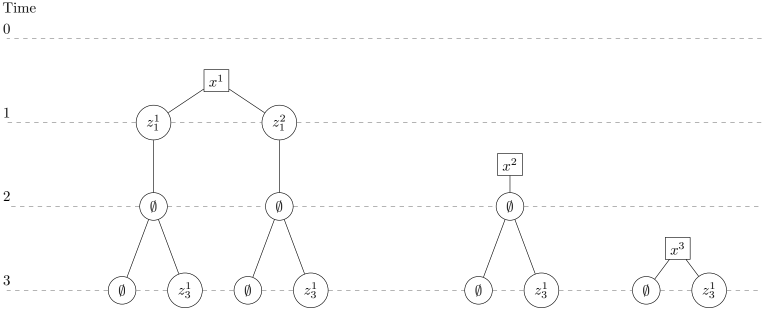

$$\begin{bmatrix} & & \\ & & \\ & & \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \\ 5 & 1 & 1 \\ 5 & 1 & 2 \\ 5 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

Four global hypotheses are pruned/capped

Some local hypotheses no longer included in an MB

# MULTI-BERNOULLI MIXTURE PRUNING/CAPPING

Global hypotheses are removed



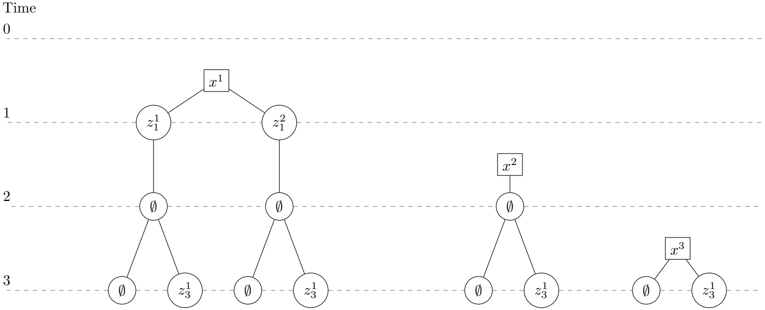

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

Prune un-used local hypotheses, adjust look-up table

If a Bernoulli has no local hypothesis included in any global hypothesis, naturally it can be pruned entirely.

# BERNOULLI PRUNING/RECYCLING

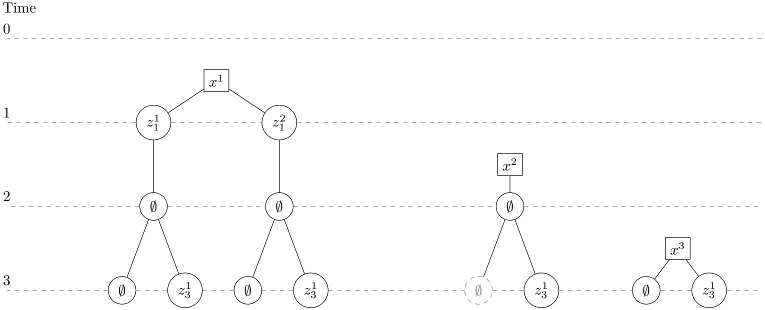
Local hypotheses are removed




$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

# BERNOULLI PRUNING/RECYCLING

Local hypotheses are removed

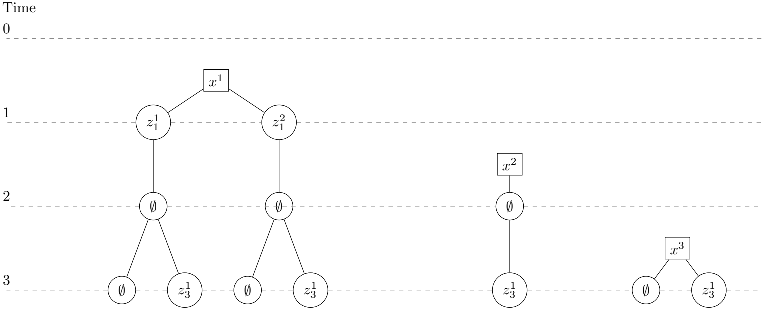



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

Example: prune local hypothesis corresponding to 2 misdetections

# BERNOULLI PRUNING/RECYCLING

Local hypotheses are removed




$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 0 & 2 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

Adjust look-up table

# GLOBAL HYPOTHESIS UNIQUENESS

Global hypotheses should be unique

Before reduction:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} \quad \begin{matrix} w^1 \\ w^2 \\ w^3 \\ w^4 \\ w^5 \\ w^6 \end{matrix}$$

After reduction:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 0 & 1 \\ 4 & 0 & 1 \end{bmatrix} \quad \begin{matrix} w^1 \\ w^2 \\ w^3 \\ w^4 \\ w^5 \\ w^6 \end{matrix}$$

Unique global hypotheses:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \quad \begin{matrix} w^1 + w^2 \\ w^3 \\ w^4 + w^5 \\ w^6 \end{matrix}$$

Important to adjust the weights accordingly.

# MBs with certain object existence

Multi-Object Tracking

Karl Granström

# OBJECT EXISTENCE IS BINARY

- In both MBM and PMBM there are Bernoullis with $0 < r < 1$.
- Expected cardinality is $r$
  - $r = 0.5 \Rightarrow$ we expect half an object
  - For example, how should an autonomous car react if there is half a car in front?
- In reality, an object is either there, or not.
- Compare to $n$ object tracking: an integer number of objects, no fractions of objects
- Can we have a multi-object density such that each hypothesis represents an integer number of objects?
- Yes, if we change to a so called $\text{MBM}_{01}$ representation.
- The $\delta$-GLMB filter can be interpreted as having a $\text{MBM}_{01}$ representation.

# EXPANDING A BERNOULLI TO CERTAIN EXISTENCE

- A Bernoulli $(r, p(\cdot))$ with $r \in (0,\ 1)$ represents two possibilities,
  - With probability $r$ we have exactly one object, with state pdf $p(\cdot)$.
  - With probability $1 - r$, we have exactly zero objects.
- We can represent this as a Bernoulli mixture density,

$$p(\mathbf{x}) = r\mathcal{B}^1(\mathbf{x}) + (1 - r)\mathcal{B}^2(\mathbf{x})$$

  where $\mathcal{B}^1(\mathbf{x})$ and $\mathcal{B}^2(\mathbf{x})$ are Bernoulli densities with parameters

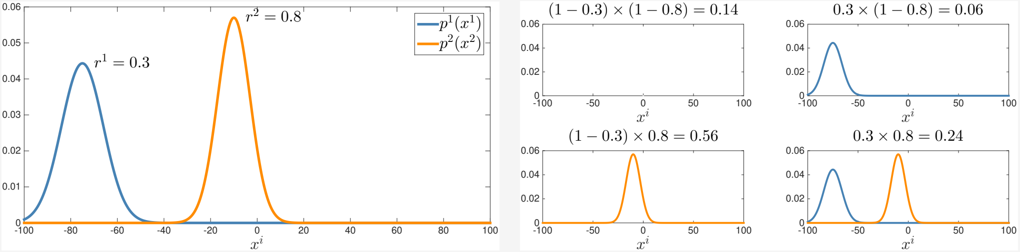
$$\left(r^1,\ p^1(\cdot)\right) = (1,\ p(\cdot)), \qquad \left(r^2,\ p^2(\cdot)\right) = (0,\ \text{any pdf})$$

- Instead of one Bernoulli with uncertain existence $r$, we have two hypotheses that each have certain existence: either one object ($r^1 = 1$) or zero objects ($r^2 = 0$).

# EXPANDING TWO BERNOULLIS TO CERTAIN EXISTENCE

An MB with two Bernoullis $(r^1, p^1(\cdot)($ and $(r^2, p^2(\cdot))$, with $r^1 \in (0,\ 1)$ and $r^2 \in (0,\ 1)$ corresponds to $2^2 = 4$ hypotheses with certain object existence:

## Example: MB with two Bernoullis



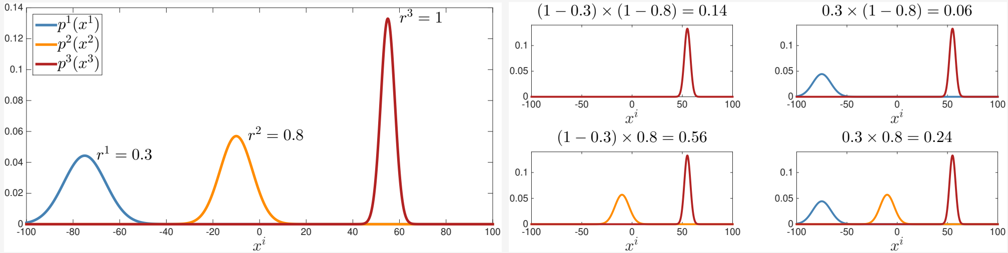

Each hypothesis is a special kind of MB:

1) Zero Bernoullis, 2) & 3) One Bernoulli, $r = 1$, 4) Two Bernoullis, $r = 1$

# EXPANDING AN MB TO CERTAIN EXISTENCE

- Consider an MB with $n$ Bernoullis, with parameters $\left(r^i, p^i(\cdot)\right)$.
- Let $n' \leq n$ of the Bernoullis have $r \in (0,\ 1)$, and let remaining $n - n'$ have $r = 1$.
- Can be expanded into a $\mathrm{MBM}_{01}$ with $2^{n'}$ MBs, where each Bernoulli has $r = 1$.

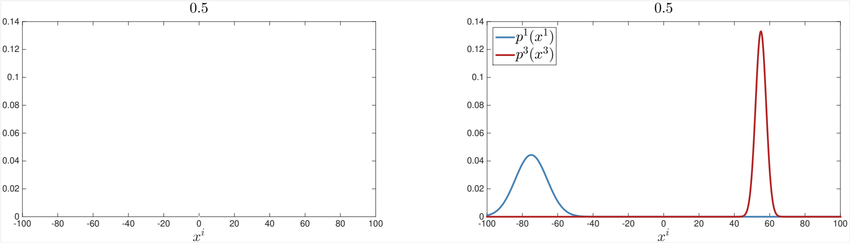
**Example: MB with $n = 3,\ n' = 2$ leads to $\mathrm{MBM}_{01}$ with $2^{n'} = 4$ MBs**



# MBM WITH CERTAIN EXISTENCE

- We denote this type of MBM as $\text{MBM}_{01}$
- Any MB(M) can be expanded into an $\text{MBM}_{01}$.
- Not all $\text{MBM}_{01}$ have a simpler MB equivalent.

## $\text{MBM}_{01}$ with two equally probable components




Cannot be simplified to MB with two Bernoullis

# WHEN IS THIS USEFUL?

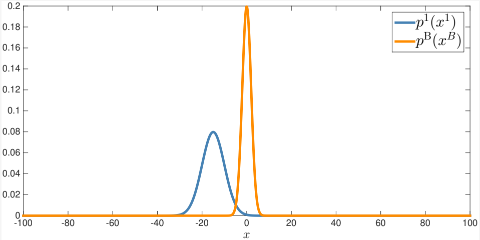
- $\text{MBM}_{01}$ representation can feel more intuitive, with an integer number of objects in each global hypothesis.
- Unusually specific birth model: If we know that new objects appear in the surveillance area together in groups, e.g., in pairs, the birth can be modeled as $\text{MBM}_{01}$.
- Facilitates other multi-object estimator:
  - Find MAP cardinality estimate
  - Find most probable global hypothesis with this cardinality
  - Extract estimates
- However, worse computational cost with $\text{MBM}_{01}$ representation.

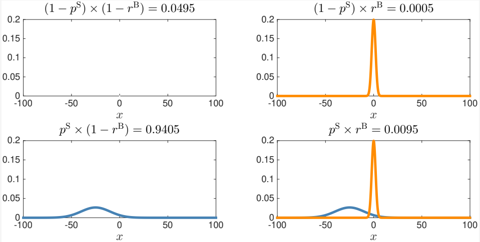
# PREDICTION OF AN MBM$_{01}$ DENSITY

**Objects may appear and disappear:**

MB$_{01}$ with $N$ Bernoullis, birth with $N^{\mathrm{B}}$ Bernoullis $\Rightarrow$ predicted MBM$_{01}$ with $2^{N+N^{\mathrm{B}}}$ hyps.

**One object (blue), $P^{\mathrm{S}} = 0.95$, and one birth (orange), $r^{\mathrm{B}} = 0.01$**



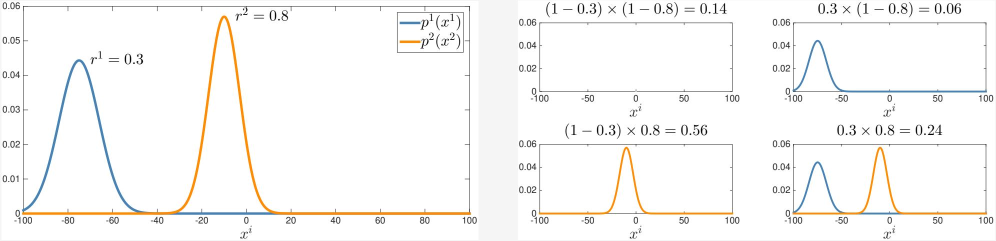




- Intractable in practice for large $N$ and $N^{\mathrm{B}}$, approximations are required.
- **Regular MB prediction does not require approximation.**

# UPDATE OF AN $\text{MBM}_{01}$ DENSITY

Posterior $\text{MBM}_{01}$ has many more components, compared to posterior MBM

## Update with $m_k$ measurements




**Number of data associations:**

$$\begin{aligned} \text{MB} \quad &: \quad N_A(m_k, 2) \\ \text{MBM}_{01} \quad &: \quad N_A(m_k, 0) + 2N_A(m_k, 1) + N_A(m_k, 2) \end{aligned}$$

# SUMMARY OF MB VS MBM$_{01}$

With an MBM$_{01}$ representation:

- Requires additional approximations in both the prediction and the update
- Generally requires a higher number of global hypotheses to achieve the same performance

Simulation studies have shown that the MBM$_{01}$ representation results in a higher computational cost to achieve the same tracking performance.