Nonlinear Control Theory

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Feedback Control

Stabilization via Linearization

Consider the nonlinear system

$$\dot{x} = f(x, u), \tag{1}$$

where $f(0,0) = 0$ and $f(x,u)$ is continuously differentiable in a domain $D_x \times D_u \subset R^n \times R^p$ that contains the origin.

Its linearization around $(x = 0, u = 0)$ can be calculated by

$$\dot{x} = Ax + Bu, \tag{2}$$

where

$$A = \frac{\partial f}{\partial x}(x,u)\bigg|_{x=0,u=0}, \quad B = \frac{\partial f}{\partial u}(x,u)\bigg|_{x=0,u=0} \tag{3}$$

Assume that (A, B) is controllable (or at least stabilizable).

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

The eigenvalues of $A - BK$ can be assigned to the desired locations in the open left-half complex plane.

Apply the state feedback control $u = -Kx$ to the nonlinear system:

$$\dot{x} = f(x, -Kx). \tag{4}$$

Its linearization around $x = 0$ is given by

$$\dot{x} = \left[\frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx)(-K) \right]_{x=0} x = (A - BK)x \tag{5}$$

where $A - BK$ is Hurwitz.

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- The origin is an asymptotically stable equilibrium point. (Theorem 4.7)
- The origin is an exponentially stable equilibrium point. (Theorem 4.13)
- Let Q be any positive-definite symmetric matrix. the Lyapunov equation

$$P(A - BK) + (A - BK)^T P = -Q \tag{6}$$

  has a unique positive definite solution P, and $V = x^T P x$ is a Lyapunov function for the closed-loop system in the neighborhood of the origin. (Theorem 4.6)
- $V(x)$ can be applied to estimate the region of attraction.

Example (12.2)

Consider the pendulum equation

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT \tag{7}$$

where $a = g/l > 0$, $b = k/m \geq 0$, $c = 1/ml^2 > 0$, θ is the angle subtended by the rod and the vertical axis, and T is the control torque applied to the pendulum. Suppose we want to stabilize the pendulum at an angle $\theta = \delta$.

The torque must have a steady-state component satisfying $0 = -a\sin\delta + cT_{ss}$.

Choose the state variables as $x_1 = \theta - \delta$, $x_2 = \dot{\theta}$ and the control variable as $u = T - T_{ss}$.

$$\dot{x}_1 = x_2, \tag{8}$$

$$\dot{x}_2 = -a[\sin(x_1 + \delta) - \sin\delta] - bx_2 + cu. \tag{9}$$

Linearization around the origin results in

$$A = \left[\begin{array}{cc} 0 & 1 \\ -a\cos(x_1+\delta) & -b \end{array} \right]_{x_1=0} = \left[\begin{array}{cc} 0 & 1 \\ -a\cos\delta & -b \end{array} \right]; \quad B = \left[\begin{array}{c} 0 \\ c \end{array} \right]. \tag{10}$$

The pair (A, B) is controllable. Taking $K = [k_1 \;\; k_2]$, and $A - BK$ is Hurwitz for

$$k_1 > -\frac{a\cos\delta}{c}, \quad k_2 > -\frac{b}{c}, \tag{11}$$

and the control torque can be given by

$$T = \frac{a\sin\delta}{c} - Kx = \frac{a\sin\delta}{c} - k_1(\theta - \delta) - k_2\dot{\theta}. \tag{12}$$

Output feedback stabilization

Consider the nonlinear system

$$
\begin{aligned}
\dot{x} =& f(x, u), && (13) \\
y =& h(x), && (14)
\end{aligned}
$$

where $h(0) = 0$, $h(x)$ is continuously differentiable in the domain $D_x \subset R^n$.

Only measurement of the output y is available for feedback!

Linearization around the origin results in the linear system

$$
\begin{aligned}
\dot{x} =& Ax + Bu, && (15) \\
y =& Cx, && (16)
\end{aligned}
$$

where $C = \frac{\partial h}{\partial x}(x)\big|_{x=0}$.

Assume that (A, B) is stabilizable, and (A, C) is detectable.

Design a linear dynamic output feedback controller

$$\dot{z} = Fz + Gy, \tag{17}$$
$$u = Lz + My, \tag{18}$$

such that the closed-loop matrix

$$\begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix} \tag{19}$$

is Hurwitz. An example is the observer-based controller

$$z = \hat{x}, \quad F = A - BK - HC, \quad G = H, \quad L = -K, \quad M = 0, \tag{20}$$

where $A - BK$ and $A - HC$ are Hurwitz.

Example

Reconsider the pendulum equation of Example 12.2, and suppose we measure the angle θ, but not the angular velocity $\dot{\theta}$.

An output variable can be taken as $y = x_1 = \theta - \delta$, and the state feedback controller of Example 12.2 can be implemented by using the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{x}_1), \tag{21}$$

where $H = [h_1 \ h_2]^T$. It can be verified that $A - HC$ is Hurwitz if

$$h_1 + b > 0, \quad h_1 b + h_2 + a\cos\delta > 0. \tag{22}$$

The control torque is given by $T = \frac{a \sin\delta}{c} - K\hat{x}$.

Motivating example

Consider the pendulum in Example 12.2:

$$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + cT \tag{23}$$

- Suppose that a and c are uncertain with nominal values a_0 and c_0.
- The goal: $\quad \theta \rightarrow \delta = 45°$.

The control is designed as before:

$$T = \frac{a_0 \sin\delta}{c_0} - k_1(\theta - \delta) - k_2\dot{\theta}, \tag{24}$$

where the uncertain parameters are replaced by their nominal values.

If $c = \frac{c_0}{2}$, $a = a_0$, $k_1 = \frac{3a_0}{c_0}$, then $\theta_{ss} = 36° \neq \delta$.

Integral control

For the nonlinear system:

$$\begin{cases} \dot{x} = f(x, u, w) \\ y = h(x, w) \end{cases}$$

where $w \in R^l$ is the constant disturbance.

- Assume: $\exists u_{ss},\ x_{ss},$ s.t. $0 = f(x_{ss}, u_{ss}, w),\ r = h(x_{ss}, w)$.
- Our goal: $y(t) \rightarrow r$ (constant reference value), as $t \rightarrow +\infty$.

Define $e \triangleq y - r$, and $\dot{\sigma} = e$. It follows that

$$\begin{cases} \dot{x} = f(x, u, w) \\ \dot{\sigma} = h(x, w) - r \end{cases}$$

Its equilibrium point is (x_{ss}, σ_{ss}).

Our goal is now: Design $u = \gamma(x, \sigma, e)$ such that $\sigma \to \sigma_{ss}$, thus $e = 0$.

But how to design $u = \gamma(x, \sigma, e)$?

1. Stabilization via linearization
2. Integral control
3. **Integral control via linearization**
4. Gain scheduling

Integral control via linearization

Consider linear feedback control:

$$u = -K_1 x - K_2 \sigma - K_3 e \tag{25}$$

The closed-loop system turns out:

$$\begin{cases} \dot{x} = f(x, -K_1 x - K_2 \sigma - K_3 \left(h(x, w) - r\right), w) \\ \dot{\sigma} = h(x, w) - r \end{cases} \tag{26}$$

Its equilibrium point (x_{ss}, σ_{ss}) satisfies

$$\begin{cases} 0 = f(x_{ss}, u_{ss}, w) \\ 0 = h(x_{ss}, w) - r \\ u_{ss} = -K_1 x_{ss} - K_2 \sigma_{ss} \end{cases}$$

Linearize (26) around its equilibrium point:

$$\dot{\xi}_\delta = (\mathcal{A} - \mathcal{B}\mathcal{K})\xi_\delta \tag{27}$$

where

$$\xi_\delta = \begin{bmatrix} x - x_{ss} \\ \sigma - \sigma_{ss} \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \mathcal{K} = \begin{bmatrix} K_1 + K_3 C & K_2 \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x}(x,u,w)\bigg|_{x=x_{ss},u=u_{ss}}, \ B = \frac{\partial f}{\partial u}(x,u,w)\bigg|_{x=x_{ss},u=u_{ss}}, \ C = \frac{\partial h}{\partial x}(x,w)\bigg|_{x=x_{ss}}$$

(A, B) controllable, & $\mathrm{Rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p$(full rank) $\Rightarrow$ $(\mathcal{A}, \mathcal{B})$ controllable

$\Rightarrow$ $\exists\, \mathcal{K} = \begin{bmatrix} K_1 + K_3 C & K_2 \end{bmatrix}$, such that $\mathcal{A} - \mathcal{B}\mathcal{K}$ is Hurwitz.

Usually $K_3 = 0$ can be assigned, and K_1 and K_2 can be obtained from $\mathcal{K}$.

The linear integral control can be obtained by

$$\begin{cases} \dot{\sigma} = e = y - r \\ u = -K_1 x - K_2 \sigma \end{cases} \tag{28}$$

Example – the pendulum system

- Pendulum model:

$$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + cT \tag{29}$$

- Goal:

$$\theta \quad \rightarrow \quad \delta \text{ (constant)} \tag{30}$$

Model transformation: $x_1 = \theta - \delta$, $x_2 = \dot{\theta}$, $u = T$

$$\left\{ \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1+\delta) - bx_2 + cu \end{aligned} \right. , \; y = x_1 \tag{31}$$

where $x_{ss} = [0,\ 0]^T,\ u_{ss} = \frac{a}{c}\sin\delta$

$$A = \begin{bmatrix} 0 & 1 \\ -a\cos\delta & -b \end{bmatrix},\ B = \begin{bmatrix} 0 \\ c \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{32}$$

It is obvious that(A, B) is controllable, $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is full rank.

Let $K_1 = [k_1,\ k_2]^T$ and $K_2 = k_3$, then

$$b + k_2 c > 0,\ (b + k_2 c)(a\cos\delta + k_1 c) - k_3 c > 0,\ k_3 c > 0 \quad \Rightarrow \quad \text{A.S}$$

If $\frac{a}{c} < \rho_1$ and $\frac{1}{c} < \rho_2$, then the closed-loop stability can be guaranteed by

$$k_2 > 0,\ k_3 > 0,\ k_1 > \rho_1 + \frac{k_3}{k_2}\rho_2 \tag{33}$$

and the linear integral control can be obtained by

$$\begin{cases} \dot{\sigma} = \theta - \delta \\ u = -k_1(\theta - \delta) - k_2\dot{\theta} - k_3\sigma \end{cases} \tag{34}$$

which is actually a PID control.

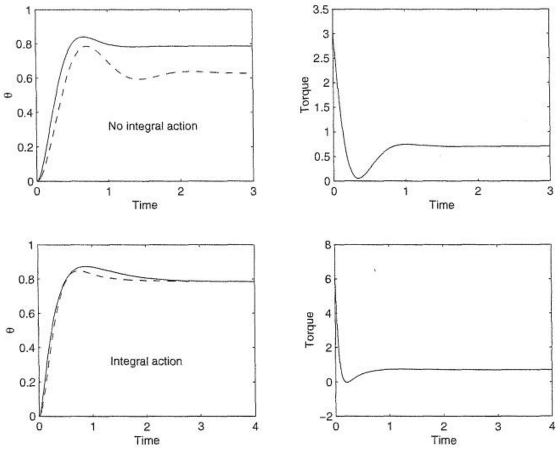

Figure: Simulation results for pendulum regulation under nominal (solid) and perturbed (dashed) parameters, with and without integral action

1. Stabilization via linearization
2. Integral control
3. Integral control via linearization
4. **Gain scheduling**

Gain scheduling

Operating points are parameterized by one or more variables (scheduling variables)

Example

Tank system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^h A(h)\mathrm{d}y\right) = w_i - k\sqrt{\rho g h} \tag{35}$$

where $h-$ hight of water, w_i- input flow rate, $A(h)-$ cross-sectional area.

Define the state $x = h$. The objective is to design w_i such that x tracks its reference r.

- Let $x = h$, and $u = w_i$, then

$$\dot{x} = \frac{1}{A(x)}(u - c\sqrt{x}) = f(x, u) \tag{36}$$

where $c = k\sqrt{\rho g}$ is uncertain.

- Let $y = x$ be the output, r be the scheduling variable.
- If $r = \alpha$ where α is constant, then integral control is applied here, such that

$$0 = u_{ss} - c\sqrt{x_{ss}}, \ \alpha = x_{ss} \tag{37}$$

- Integral control

$$\begin{cases} \dot{x} = f(x, u) \\ \dot{\sigma} = x - r \\ u = -k_1(\alpha)e - k_2(\alpha)\sigma \quad \text{(actually a PI control)} \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = f(x, -k_1(\alpha)e - k_2(\alpha)\sigma) \\ \dot{\sigma} = x - r \end{cases} \Rightarrow \sigma_{ss} = \frac{-u_{ss}}{k_2}$$

- Linearize around (x_{ss}, σ_{ss})

$$\dot{\xi}_\delta = \begin{bmatrix} a(\alpha) - b(\alpha)k_1(\alpha) & -b(\alpha)k_2(\alpha) \\ 1 & 0 \end{bmatrix} \xi_\delta + \begin{bmatrix} b(\alpha)k_1(\alpha) \\ -1 \end{bmatrix} r_\delta \tag{38}$$

where $\xi_\delta = [x_\delta, \sigma_\delta]^T$, $x_\delta = x - \alpha$, $\sigma_\delta = \sigma - \sigma_{ss}$, $r_\delta = r - \alpha$.

- Jacobian matrices

$$a(\alpha) = \frac{\partial f}{\partial x}\bigg|_{x_{ss},u_{ss}} = -\frac{c\sqrt{\alpha}}{2\alpha A(\alpha)}, \quad b(\alpha) = \frac{\partial f}{\partial u}\bigg|_{x_{ss},u_{ss}} = \frac{1}{A(\alpha)} \tag{39}$$

- Assume that the upper bound for c is known, and let

$$k_1(\alpha) = \frac{2\zeta\omega_n}{b(\alpha)}, \quad k_2(\alpha) = \frac{\omega_n^2}{b(\alpha)}, \quad 0 < \zeta < 1, \quad 2\zeta\omega_n >> |a(\alpha)| \tag{40}$$

such that

$$\dot{\xi}_\delta = \begin{bmatrix} a(\alpha) - 2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix} \xi_\delta + \begin{bmatrix} 2\zeta\omega_n \\ -1 \end{bmatrix} r_\delta, \quad y_\delta = [1 \ \ 0]\xi_\delta \tag{41}$$

and its transfer function

$$\Phi(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + [2\zeta\omega_n - a(\alpha)]s + \omega_n^2} \tag{42}$$

Time-varying r: $u = -k_1(r)e - k_2(r)\sigma$

- Closed-loop system:

$$\begin{cases} \dot{x} = f(x, -k_1(r)e - k_2(r)\sigma) \\ \dot{\sigma} = x - r \end{cases} \tag{43}$$

- In case of $r = \alpha$ where α is time-varying or switching, linearize around (x_{ss}, σ_{ss}):

$$\dot{\xi}_\delta = \begin{bmatrix} a(\alpha) - 2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix} \xi_\delta + \begin{bmatrix} 2\zeta\omega_n + \gamma(\alpha) \\ -1 \end{bmatrix} r_\delta, \quad y_\delta = [1 \;\; 0]\xi_\delta \tag{44}$$

where $\gamma(\alpha) = -b(\alpha)k_2'(\alpha)\sigma_{ss}(\alpha) = \frac{A'(\alpha)c\sqrt{\alpha}}{A^2(\alpha)}$.

- Now, the transfer function is $\Phi(s) = \frac{[2\zeta\omega_n + \gamma(\alpha)]s + \omega_n^2}{s^2 + [2\zeta\omega_n - a(\alpha)]s + \omega_n^2}$. Its zeros are different from those with constant α.

Is it possible to achieve the same performance as that with constant α?

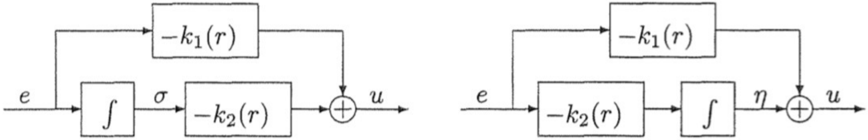

Figure: Modification of the gain-scheduled PI controller: original (left) and modified (right)

$$\begin{cases} \dot{x} = f(x, -k_1(r)e - k_2(r)\sigma) \\ \dot{\sigma} = x - r \end{cases} \rightarrow \begin{cases} \dot{x} = f(x, -k_1(r)e + \eta) \\ \dot{\eta} = -k_2(r)(x - r) \end{cases}$$

$$\begin{cases} \dot{x} = f(x, -k_1(r)e - k_2(r)\sigma) \\ \dot{\sigma} = x - r \end{cases} \rightarrow \begin{cases} \dot{x} = f(x, -k_1(r)e + \eta) \\ \dot{\eta} = -k_2(r)(x - r) \end{cases}$$

- Linearization around its working point:

$$\dot{z}_\delta = \begin{bmatrix} a(\alpha) - 2\zeta\omega_n & b(\alpha) \\ -\frac{\omega_n^2}{b(\alpha)} & 0 \end{bmatrix} z_\delta + \begin{bmatrix} 2\zeta\omega_n \\ \frac{\omega_n^2}{b(\alpha)} \end{bmatrix} r_\delta, \quad y_\delta = [1 \quad 0] z_\delta \tag{45}$$

where $z_\delta = [x_\delta, \eta_\delta]^T$, and $\xi_\delta = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-b(\alpha)}{\omega_n^2} \end{bmatrix} z_\delta$.

- The transfer function is now

$$\Phi(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + [2\zeta\omega_n - a(\alpha)]s + \omega_n^2} \tag{46}$$

which is the same as that with constant α.

Gain scheduling

General design process

1. Linearize the nonlinear model about a family of operating points, parameterized by the scheduling variables.
2. Design a parameterized family of linear controllers to achieve the specified performance at each operating point.
3. Construct a gain-scheduled controller such that
   - ⋆ the closed-loop system under the gain-scheduled controller has the same equilibrium point as the closed-loop system under the fixed-gain controller;
   - ⋆ the linearization under the gain-scheduled controller is equivalent to that under the fixed-gain controller.
4. Check the performance of the gain-scheduled controller by simulating the nonlinear closed-loop model.