

Lecture 2: Single object tracking in clutter

Version January 15, 2021

Multi-Object Tracking

Lennart Svensson

Section 1: **Introduction to SOT in clutter**

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An introduction to single object tracking in clutter

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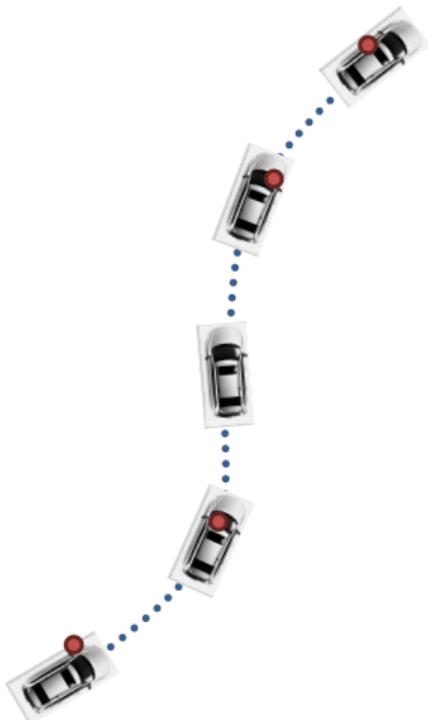
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DEFINITION OF SINGLE OBJECT TRACKING (SOT)

- SOT in clutter is a special case of multi-object tracking (MOT).

Key property

- In SOT, there is always **precisely one object present** at all times.
- Easier than MOT? Yes!
 - No need to infer the number of objects, or when objects appear/disappear.
 - Many fewer data association hypotheses.



SOT IN CLUTTER: MAIN CHALLENGES

Previous challenges still relevant:

- Time varying state variables.
- Noisy measurements.

New challenges

- missed detections
- clutter detections
- unknown data associations.

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WHY STUDY SOT IN CLUTTER?

Adding complexity bit by bit

- A simple(r) setting to learn about:
 - measurement models,
 - data association uncertainties,
 - tools to approximate posterior density.
- Important subproblem of MOT.
 - In some settings we only have one object to track (robot, athlete, vehicle).
 - Radar systems use SOT to control sensors and direct radar towards object.



Section 2: Motion and measurement models

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Single object motion and measurement models

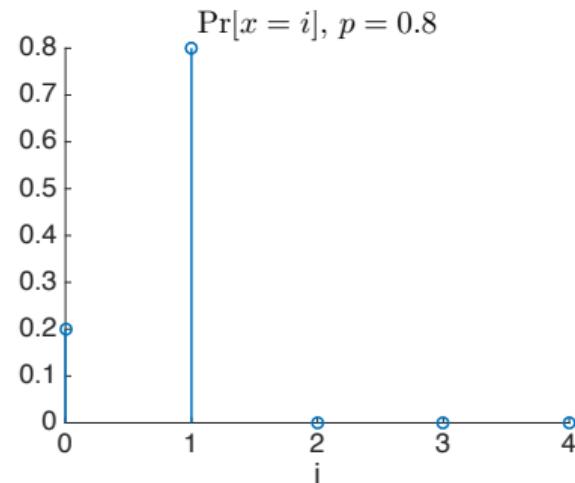
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PRELIMINARIES – BERNOUlli DISTRIBUTION

- The **Bernoulli distribution** is central to both SOT and MOT.
- If x is Bernoulli distributed with probability, $p \in [0, 1]$, it takes the values

$$x = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1 - p. \end{cases}$$



SINGLE OBJECT MOTION MODELS

Motion model

- The object state is a Markov chain that evolves as

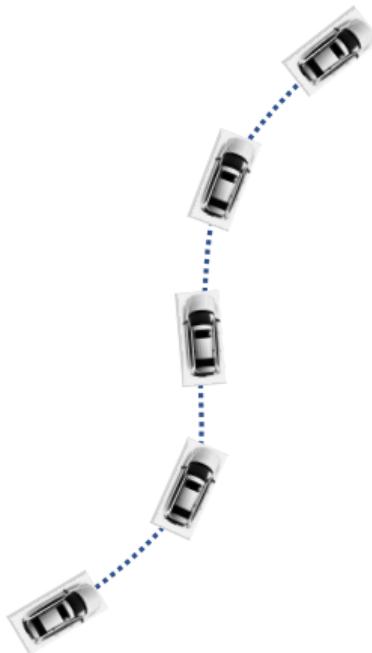
$$p(x_k | x_{k-1}) = \pi_k(x_k | x_{k-1}).$$

- For instance, we often assume that

$$x_k = f_{k-1}(x_{k-1}) + q_{k-1}, \quad q_{k-1} \sim \mathcal{N}(0, Q_{k-1}),$$

such that

$$\pi_k(x_k | x_{k-1}) = \mathcal{N}(x_k; f_{k-1}(x_{k-1}), Q_{k-1}).$$



SINGLE OBJECT MEASUREMENT MODELS (1)

Measurement model

- The object is detected with probability $P^D(x_k)$, and then generates a measurement from

$$p(o_k|x_k) = g_k(o_k|x_k).$$

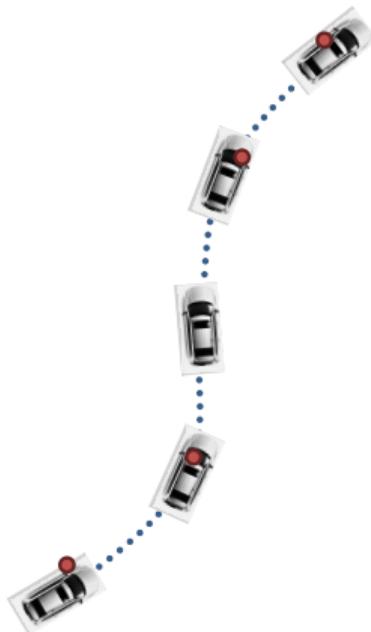
- For instance, we often assume that

$$o_k = h_k(x_k) + v_k, \quad v_k \sim \mathcal{N}(0, R_k),$$

such that

$$g_k(o_k|x_k) = N(o_k; h_k(x_k), R_k).$$

- Note:** object is **not always detected**.



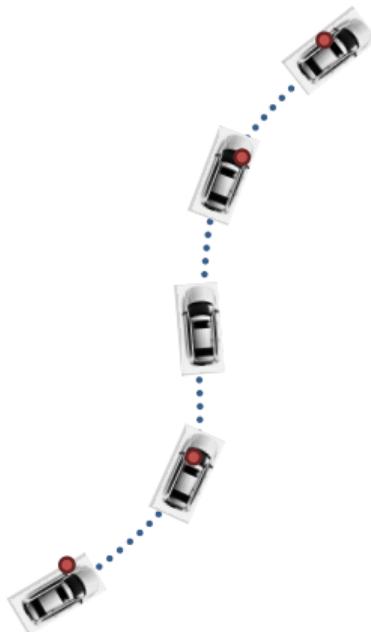
SINGLE OBJECT MEASUREMENT MODELS (2)

- We use a matrix (or a sequence) to represent the object detections,

$$O_k = \begin{cases} [] & \text{if object is undetected,} \\ o_k & \text{if object is detected.} \end{cases}$$

- We use $|O_k|$ to denote the number of column vectors in O_k .
- Given x_k , $|O_k|$ is Bernoulli distributed:

$$|O_k| = \begin{cases} 1 & \text{with probability } P^D(x_k), \\ 0 & \text{with probability } 1 - P^D(x_k). \end{cases}$$



SINGLE OBJECT MEASUREMENT MODEL (3)

- Using the matrix notation, we get

$$p(O_k | x_k) = \begin{cases} 1 - P^D(x_k) & \text{if } O_k = [] \\ P^D(x_k) g_k(o_k | x_k) & \text{if } O_k = o_k. \end{cases}$$

Note: this captures both the probability of detection and, if detected, the distribution of the detection.

- Given x_k , the set of vectors in O_k is a **Bernoulli random finite set**.

SINGLE OBJECT MEASUREMENT MODEL (4)

- Simple to generate object measurements O_k given x_k .

Algorithm Sampling O_k given x_k .

```
1: Initialize  $O_k = []$ 
2: if rand <  $P^D(x_k)$  then
3:    $o_k \sim g_k(\cdot | x_k)$ 
4:    $O_k = o_k$ 
5: end if
```

Example, samples of O_k

- Suppose $P^D(x_k) = 0.85$ and $g_k(o_k | x_k) = \mathcal{N}(o_k; [3, 2]^T, 0.3\mathbf{I})$.

SOT with known associations

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PROBLEM FORMULATION

- Suppose the data associations are known.
- Given a set of measurements, Z_k , we then know O_k .

Objective (this video)

- Recursively compute

$$p(x_k | O_{1:k}).$$

PREDICTION STEP

- Given a motion model $\pi_k(x_k|x_{k-1})$, how can we perform prediction?

Chapman-Kolmogorov equation

- We use the Chapman-Kolmogorov equation

$$p(x_k|O_{1:k-1}) = \int \pi_k(x_k|x_{k-1})p(x_{k-1}|O_{1:k-1}) dx_{k-1}.$$

Linear and Gaussian models

- If $p(x_{k-1}|O_{1:k-1}) = \mathcal{N}(x_{k-1}; \bar{x}_{k-1|k-1}, P_{k-1|k-1})$ and $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k; Fx_{k-1}, Q)$ then

$$p(x_k|O_{1:k-1}) = \mathcal{N}(x_k; F\bar{x}_{k-1|k-1}, FP_{k-1|k-1}F^T + Q).$$

UPDATE STEP

- Given a measurement model,

$$p(O_k|x_k) = \begin{cases} 1 - P^D(x_k) & \text{if } O_k = [] \\ P^D(x_k)g_k(o_k|x_k) & \text{if } O_k = o_k. \end{cases}$$

how can we perform the update?

- Recall:** posterior \propto prior \times likelihood.

Bayes' rule

- We get

$$\begin{aligned} p(x_k|O_{1:k}) &\propto p(x_k|O_{1:k-1})p(O_k|x_k) \\ &= \begin{cases} p(x_k|O_{1:k-1})(1 - P^D(x_k)) & \text{if } O_k = [] \\ p(x_k|O_{1:k-1})P^D(x_k)g_k(o_k|x_k) & \text{if } O_k = o_k. \end{cases} \end{aligned}$$

UPDATE STEP: INFORMATIVE $P^D(x_k)$

Example 1: informative $P^D(x_k)$

- Suppose we have a scalar state,

$$p(x_k | O_{1:k-1}) = \mathcal{N}(x_k; 0, 1).$$

- Consider a measurement model with $g_k(o_k | x_k) = p(o_k)$ and

$$P^D(x_k) = \begin{cases} 1 & \text{if } x_k \geq 0 \\ 0 & \text{if } x_k < 0. \end{cases}$$

- We get

$$p(x_k | O_{1:k}) \propto \begin{cases} p(x_k | O_{1:k-1})(1 - P^D(x_k)) & \text{if } O_k = [] \\ p(x_k | O_{1:k-1})P^D(x_k)g_k(o_k | x_k) & \text{if } O_k = o_k. \end{cases}$$

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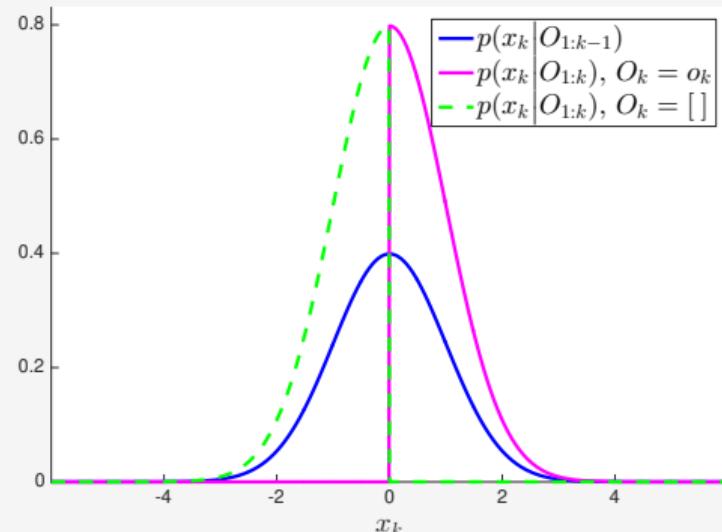
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UPDATE STEP: CONSTANT P^D

Example 2: constant $P^D(x_k)$

- Suppose $P^D(x_k)$ is constant.
- We have

$$p(x_k | O_{1:k}) \propto \begin{cases} p(x_k | O_{1:k-1})(1 - P^D(x_k)) & \text{if } O_k = [] \\ p(x_k | O_{1:k-1})P^D(x_k)g_k(o_k | x_k) & \text{if } O_k = o_k, \end{cases}$$

which simplifies to

$$p(x_k | O_{1:k}) \propto \begin{cases} p(x_k | O_{1:k-1}) & \text{if } O_k = [] \\ p(x_k | O_{1:k-1})g_k(o_k | x_k) & \text{if } O_k = o_k. \end{cases}$$

- In short, **standard update** using o_k but **only if object is detected**.

UPDATE STEP: CONSTANT P^D , LINEAR AND GAUSSIAN MODELS

Example 2: constant $P^D(x_k)$, continued

- Specifically, consider

$$p(x_k | O_{1:k-1}) = \mathcal{N}(x_k; \bar{x}_{k|k-1}, P_{k|k-1}), \quad g_k(o_k | x_k) = \mathcal{N}(o_k; H_k x_k, R_k).$$

- Then, $p(x_k | O_{1:k}) = \mathcal{N}(x_k; \bar{x}_{k|k}, P_{k|k})$ where

$$\bar{x}_{k|k} = \bar{x}_{k|k-1}, \quad P_{k|k} = P_{k|k-1}$$

when $O_k = []$, and, when $O_k = o_k$,

$$\bar{x}_{k|k} = \bar{x}_{k|k-1} + K_k(o_k - H_k \bar{x}_{k|k-1})$$

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1},$$

where $K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}$ is the Kalman gain.

- **Standard Kalman filter update**, but only **if object is detected**.

PREDICTION AND UPDATE: ILLUSTRATION

Constant P^D , linear and Gaussian models

- **Motion model** (constant velocity):

$$x_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_{k-1} + q_{k-1}$$

$$q_{k-1} \sim \mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0.5 \begin{bmatrix} 1/3 & 1/2 \\ 1/2 & 1 \end{bmatrix} \right)$$

$$p(x_0) = \mathcal{N} \left(x_0; \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0.3 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

- **Measurement model:**

$$P^D(x_k) = 0.85$$

$$o_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k, \quad v_k \sim \mathcal{N}(0, 1).$$

Standard clutter model: motivation

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PRELIMINARIES – BINOMIAL DISTRIBUTION

- This video uses two other scalar distributions, apart from the Bernoulli distribution.

Binomial distribution

- If x is binomially distribution with parameters $p \in [0, 1]$ and $j \in \mathbb{N}$,

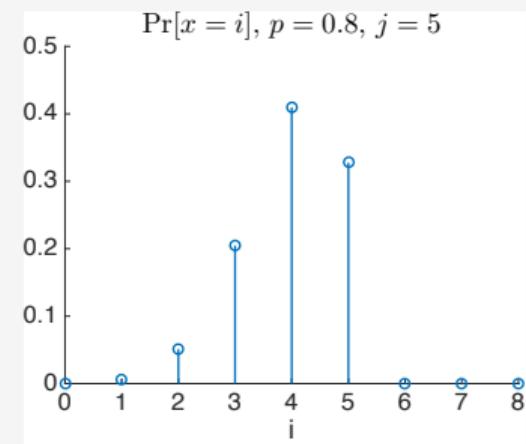
$$\Pr[x = i] = \binom{j}{i} p^i (1 - p)^{j-i}$$

where

$$\binom{j}{i} = \frac{j!}{i!(j-i)!}.$$

- It holds that $\mathbb{E}[x] = pj$.

Binomial example



PRELIMINARIES – POISSON DISTRIBUTION

Poisson distribution

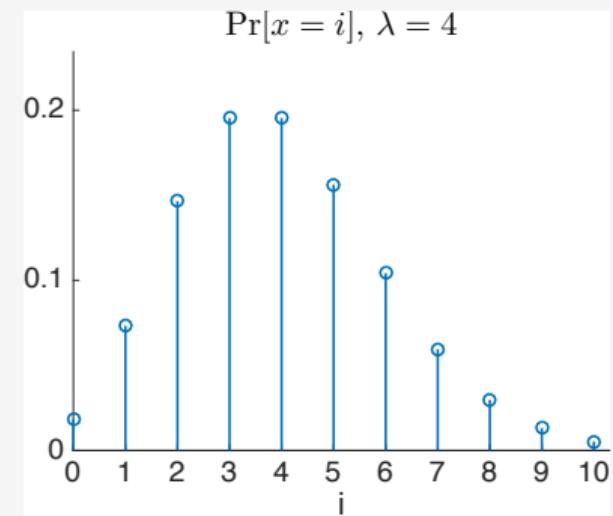
- If x is Poisson distributed with expected value $\lambda > 0$,

$$\Pr [x = i] = \text{Po}[i; \lambda] = \frac{\lambda^i \exp(-\lambda)}{i!}.$$

- It is useful to know that

$$\mathbb{E}[x] = \text{Var}(x) = \lambda.$$

Example of Poisson pmf



MODELLING CLUTTER

- Observed measurement matrix:

$$Z_k = \Pi(O_k, C_k),$$

where Π **randomly shuffles column vectors**, and C_k is **clutter**.

Example of Π

- If $Z = \Pi(o^1, c^1)$, then

$$\Pr [Z = [o^1, c^1]] =$$

$$\Pr [Z = [c^1, o^1]] = 0.5.$$

Parts of a clutter model

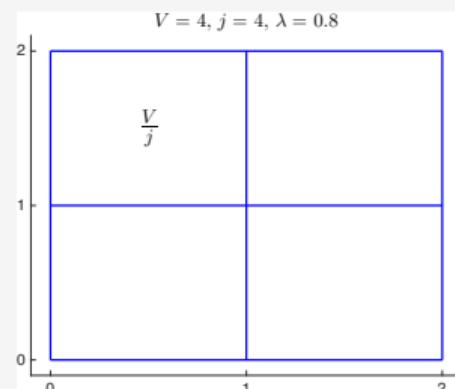
- We need a stochastic model for
 - number of detections, $|C_k|$,
 - vectors in C_k .
- Consider a field of view in \mathbb{R}^{n_z} of volume V .
- Let λ denote “expected number of clutter detections per unit volume”.

CLUTTER – LIMITED RESOLUTION

- Real sensors have limited resolution.
⇒ Nearby objects generate at most one detection.

A possible clutter model

- Split volume into j cells,
 $C_k = \Pi(C_k^{(1)}, \dots, C_k^{(j)})$, where $C_k^{(i)}$ denotes clutter in cell i .
- $C_k^{(1)}, \dots, C_k^{(j)}$ are assumed independent.
- $|C_k^{(i)}|$ is Bernoulli with probability $\lambda V/j$.
- Detections are uniformly distribution within their cells.
- According to model, $|C_k|$ is binomially distributed with parameters j and $\lambda V/j$.



CLUTTER – UNLIMITED RESOLUTION

- In this course we assume unlimited sensor resolution.

A new clutter model

- Let us increase j and make cells smaller!
Consequences:
 - Possible to obtain more detections.
 - Probability of detection in a single cell,
 $\lambda V/j$, decreases.
 - $\mathbb{E}[|C_k|] = V\lambda$ for all j .
- In the limit, as $j \rightarrow \infty$:
 - $|C_k|$ is Poisson distributed.
 - C_k is a **Poisson point process**.

Standard clutter model: the Poisson point process

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POISSON POINT PROCESSES

- The Poisson point process (PPP) is the **default model for clutter** $C_k = [c_k^1, \dots, c_k^{m_k^c}]$.

PPP introduced above

- The number of clutter is

$$m_k^c \sim \text{Po}(\lambda V).$$

- Given m_k^c , the vectors $c_k^1, \dots, c_k^{m_k^c}$ are i.i.d.

$$c_k^i \sim \text{unif}(\mathbf{V}),$$

where \mathbf{V} is our field of view.

Algorithm Sampling the PPP

```
1: Initialize  $C_k = []$ 
2: Generate  $m_k^c \sim \text{Po}(\lambda V)$ 
3: for  $i = 1$  to  $m_k^c$  do
4:   Generate  $c_k^i \sim \text{unif}(\mathbf{V})$ 
5:   Set  $C_k = [C_k, c_k^i]$ 
6: end for
```

GENERAL PARAMETRIZATIONS OF PPPS

- More generally, we parametrize PPPs using either
 - an **intensity function**, $\lambda_c(c) \geq 0$,
 - or
 - a combination of

$$\begin{cases} \bar{\lambda}_c = \int \lambda_c(c) dc & \text{rate} \\ f_c(c) = \frac{\lambda_c(c)}{\bar{\lambda}_c} & \text{spatial pdf.} \end{cases}$$

- Note:** the intensity can be computed from rate and spatial pdf:

$$\lambda_c(c) = \bar{\lambda}_c f_c(c).$$

In previous example

- Intensity function is

$$\lambda_c(c) = \begin{cases} \lambda & \text{if } c \in V \\ 0 & \text{otherwise.} \end{cases}$$

- Rate and spatial pdf are

$$\bar{\lambda}_c = \lambda V$$

$$f_c(c) = \begin{cases} \frac{1}{V} & \text{if } c \in V \\ 0 & \text{otherwise.} \end{cases}$$

GENERAL PPPs

Algorithm Sampling a general PPP

```
1: Initialize  $C_k = []$ 
2: Generate  $m_k^c \sim \text{Po}(\bar{\lambda}_c)$ 
3: for  $i = 1$  to  $m_k^c$  do
4:   Generate  $c_k^i \sim f_c(\cdot)$ 
5:   Set  $C_k = [C_k, c_k^i]$ 
6: end for
```

PPP distributions

- For $C_k = [c_k^1, \dots, c_k^{m_k^c}]$,

$$p(C_k) = p(C_k, m_k^c) = p(m_k^c)p(C_k | m_k^c)$$

$$= \text{Po}(m_k^c; \bar{\lambda}_c) \prod_{i=1}^{m_k^c} f_c(c_k^i)$$

$$= \frac{\exp(-\bar{\lambda}_c) \bar{\lambda}_c^{m_k^c}}{m_k^c!} \prod_{i=1}^{m_k^c} \frac{\lambda_c(c_k^i)}{\bar{\lambda}_c}$$

$$= \frac{\exp(-\bar{\lambda}_c)}{m_k^c!} \prod_{i=1}^{m_k^c} \lambda_c(c_k^i).$$

PROPERTIES AND SAMPLES OF PPPs

Algorithm Sampling a general PPP

-
- 1: Initialize $C_k = []$
 - 2: Generate $m_k^c \sim \text{Po}(\bar{\lambda}_c)$
 - 3: **for** $i = 1$ to m_k^c **do**
 - 4: Generate $c_k^i \sim f_c(\cdot)$
 - 5: Set $C_k = [C_k, c_k^i]$
 - 6: **end for**
-

Samples of original PPP

Complete measurement model – part 1

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MEASUREMENT MODEL

Objective

- We seek $p(Z_k | x_k)$, needed in the updated step.
- **Note:** $Z_k = \Pi(O_k, C_k)$.
- **Object detections:** the object is detected with probability $P^D(x_k)$, and, if detected, generates $o_k \sim g_k(o_k | x_k)$.
- **Clutter detections:** the number of clutter measurements is $m_k^c \sim \text{Po}(\bar{\lambda}_c)$, and the clutter measurements, $c_k^1, \dots, c_k^{m_k^c}$, are i.i.d., $c_k^i \sim f_c(c_k^i)$.

Main challenges

- 1) The width of Z_k is random.
- 2) We do not know which, if any, detection in Z_k that is an object detection.

NOTATION FOR DATA ASSOCIATION

- For brevity, we omit time indexes.

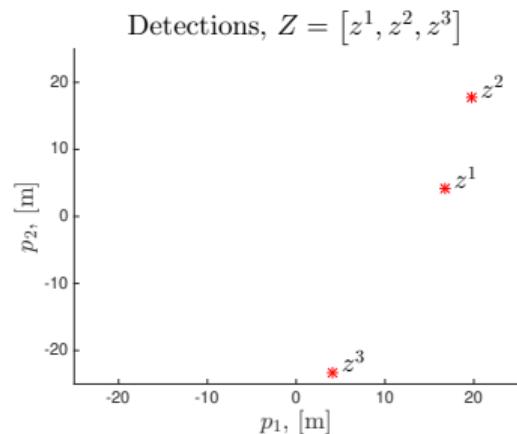
Data associations

- To describe the data association we use

$$\theta = \begin{cases} i > 0 & \text{if } z^i \text{ is an object detection,} \\ 0 & \text{if object is undetected.} \end{cases}$$

Data association example

- Suppose $Z = [z^1, z^2, z^3]$. If $\theta = 2$, then z^2 is an object detection whereas z^1 and z^3 are clutter. If $\theta = 0$, then the object is undetected and z^1 , z^2 and z^3 are all clutter detections.



DERIVING THE MEASUREMENT MODEL

- To find $p(Z|x)$, we introduce the variables m and θ

$$\begin{aligned} p(Z|x) &= p(Z, m|x) \\ &= \sum_{\theta=0}^m p(Z, m, \theta|x) \\ &= \sum_{\theta=0}^m p(Z|m, \theta, x)p(\theta, m|x). \end{aligned}$$

- As we will see, $p(Z|m, \theta, x)$ and $p(\theta, m|x)$ have simple expressions. This enables us to **express the measurement model!**
- Let us find an expression for $p(Z, m, \theta|x) = p(Z|m, \theta, x)p(\theta, m|x)$!

Complete measurement model – part 2

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PRIOR PROBABILITIES OF DATA ASSOCIATIONS

- **Objective:** find $p(Z, m, \theta|x) = p(Z|m, \theta, x)p(\theta, m|x)$.
- Expressions differ depending on:

$$\theta = 0, m : \begin{cases} \text{object is not detected,} \\ m \text{ clutter detections,} \end{cases}$$

$$\theta = i > 0, m > 0 : \begin{cases} \text{object is detected,} \\ m - 1 \text{ clutter detections,} \\ \text{object detection is given index } i. \end{cases}$$

- We now present $p(Z|m, \theta, x)$ and $p(\theta, m|x)$ for these two cases.

FINDING $p(Z, m, \theta | x)$, $\theta = 0$

- If $\theta = 0$,

$$p(Z | m, \theta, x) = \prod_{i=1}^m f_c(z^i)$$

$$p(\theta, m | x) = (1 - P^D(x)) \text{Po}(m; \bar{\lambda}_c)$$

which implies that

$$p(Z, m, \theta | x) = (1 - P^D(x)) \text{Po}(m; \bar{\lambda}_c) \prod_{i=1}^m f_c(z^i).$$

- Using $\text{Po}(m; \bar{\lambda}) = \frac{\exp(-\bar{\lambda}) \bar{\lambda}^m}{m!}$ and $f_c(z) = \lambda_c(z) / \bar{\lambda}_c$:

$$\begin{aligned} p(Z, m, \theta | x) &= (1 - P^D(x)) \frac{\exp(-\bar{\lambda}_c) \bar{\lambda}_c^m}{m!} \prod_{i=1}^m \frac{\lambda_c(z^i)}{\bar{\lambda}_c} \\ &= (1 - P^D(x)) \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i) \end{aligned}$$

FINDING $p(Z, m, \theta | x)$, $\theta = 1, \dots, m$

- For $\theta = 1, \dots, m$,

$$p(Z|m, \theta, x) = g_k(z^\theta|x) \prod_{\substack{i=1 \\ i \neq \theta}}^N f_c(z^i) = g_k(z^\theta|x) \frac{\prod_{i=1}^m f_c(z^i)}{f_c(z^\theta)}$$

$$p(\theta, m|x) = P^D(x) \text{Po}(m-1; \bar{\lambda}_c) \frac{1}{m}$$

which implies that

$$p(Z, m, \theta|x) = P^D(x) \text{Po}(m-1; \bar{\lambda}_c) \frac{1}{m} \frac{g_k(z^\theta|x)}{f_c(z^\theta)} \prod_{i=1}^m f_c(z^i).$$

- Using $\text{Po}(m-1; \bar{\lambda}) = \frac{\exp(-\bar{\lambda}_c) \bar{\lambda}_c^{m-1}}{(m-1)!}$ and $f_c(z) = \lambda_c(z)/\bar{\lambda}_c$:

$$\begin{aligned} p(Z, m, \theta|x) &= P^D(x) \frac{\exp(-\bar{\lambda}_c) \bar{\lambda}_c^{m-1}}{(m-1)!} \frac{1}{m} \frac{\bar{\lambda}_c g_k(z^\theta|x)}{\lambda_c(z^\theta)} \prod_{i=1}^m \frac{\lambda_c(z^i)}{\bar{\lambda}_c} \\ &= P^D(x) \frac{g_k(z^\theta|x)}{\lambda_c(z^\theta)} \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i) \end{aligned}$$

Example:

$$\begin{cases} Z = \{z^1, z^2\} \\ \theta = m = 2 \end{cases}$$

$$\begin{aligned} &\Rightarrow p(Z|m, \theta, x) \\ &= g_k(z^2|x) f_c(z^1) \\ &= g_k(z^2|x) \frac{f_c(z^1) f_c(z^2)}{f_c(z^2)} \\ &= g_k(z^2|x) \frac{\prod_{i=1}^m f_c(z^i)}{f_c(z^\theta)} \end{aligned}$$

THE COMPLETE MEASUREMENT MODEL

- Putting these equations together,

$$\begin{aligned} p(Z|x) &= \sum_{\theta=0}^m p(Z, m, \theta|x) \\ &= \overbrace{(1 - P^D(x)) \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i)}^{\theta=0} + \sum_{\theta=1}^m P^D(x) \frac{g_k(z^\theta|x)}{\lambda_c(z^\theta)} \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i) \\ &= \left[(1 - P^D(x)) + P^D(x) \sum_{\theta=1}^m \frac{g_k(z^\theta|x)}{\lambda_c(z^\theta)} \right] \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i). \end{aligned}$$

LIKELIHOOD VISUALIZATIONS

- We have found that

$$p(Z|x) = \sum_{\theta=0}^m p(Z, m, \theta|x) = \left[(1 - P^D(x)) + P^D(x) \sum_{\theta=1}^m \frac{g_k(z^\theta|x)}{\lambda_c(z^\theta)} \right] \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i).$$

Examples with different P^D

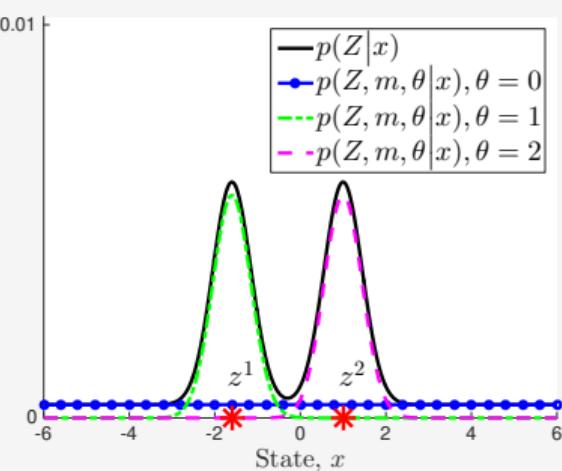
- The original example

$$P^D(x) = 0.85, \quad g_k(o|x) = \mathcal{N}(o; x, 0.2)$$

$$\lambda(c) = \begin{cases} 0.3 & \text{if } |c| \leq 5, \\ 0 & \text{otherwise,} \end{cases} \quad Z = [-1.6, 1].$$

- Likelihood is dominated by hypotheses

$\theta > 0$, for x “near” z^1 or z^2 .



LIKELIHOOD VISUALIZATIONS

- We have found that

$$p(Z|x) = \sum_{\theta=0}^m p(Z, m, \theta|x) = \left[(1 - P^D(x)) + P^D(x) \sum_{\theta=1}^m \frac{g_k(z^\theta|x)}{\lambda_c(z^\theta)} \right] \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i).$$

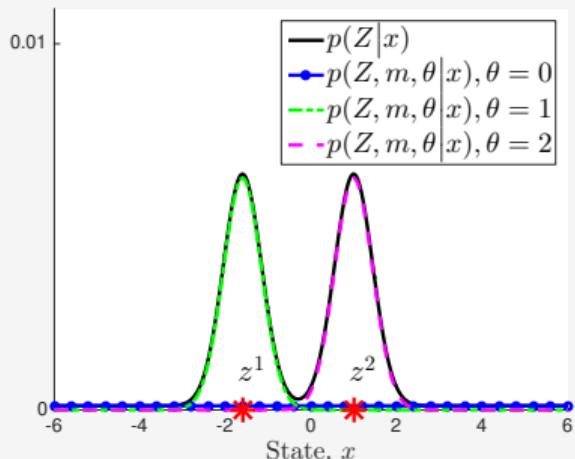
Examples with different P^D

- Now, with a larger P^D

$$P^D(x) = 0.95, \quad g_k(o|x) = \mathcal{N}(o; x, 0.2)$$

$$\lambda(c) = \begin{cases} 0.3 & \text{if } |c| \leq 5, \\ 0 & \text{otherwise,} \end{cases} \quad Z = [-1.6, 1].$$

- The hypothesis $\theta = 0$ contributes even less to the likelihood.



LIKELIHOOD VISUALIZATIONS

- We have found that

$$p(Z|x) = \sum_{\theta=0}^m p(Z, m, \theta|x) = \left[(1 - P^D(x)) + P^D(x) \sum_{\theta=1}^m \frac{g_k(z^\theta|x)}{\lambda_c(z^\theta)} \right] \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i).$$

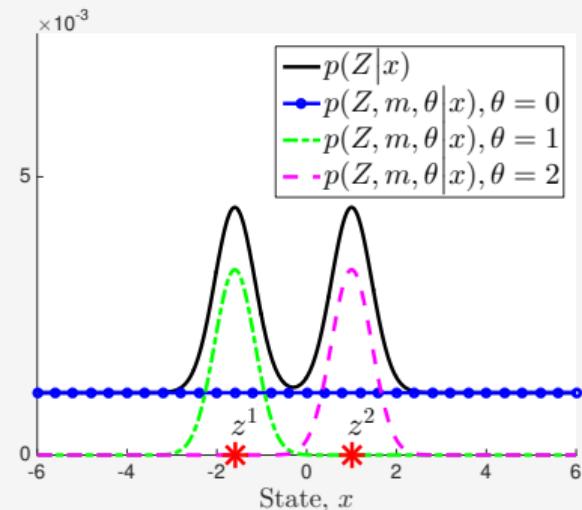
Examples with different P^D

- Finally, with a smaller P^D :

$$P^D(x) = 0.5, \quad g_k(o|x) = \mathcal{N}(o; x, 0.2)$$

$$\lambda(c) = \begin{cases} 0.3 & \text{if } |c| \leq 5, \\ 0 & \text{otherwise,} \end{cases} \quad Z = [-1.6, 1].$$

- Now, $\theta = 0$ is more likely and the likelihood is less informative.



Section 3: SOT, conceptual solution

Multi-Object Tracking

Lennart Svensson

Visualizing the SOT filtering recursions

Multi-Object Tracking

Lennart Svensson

POSTERIOR DENSITIES: BASIC STRUCTURE

- Let the sequences of measurements and data association hypotheses up to time k be denoted

$$Z_{1:k} = (Z_1, Z_2, \dots, Z_k), \quad \theta_{1:k} = (\theta_1, \dots, \theta_k).$$

Structure of posterior density

- In SOT, the filtering density can be written as

$$p(x_k | Z_{1:k}) = \sum_{\theta_{1:k}} p(x_k | Z_{1:k}, \theta_{1:k}) \Pr [\theta_{1:k} | Z_{1:k}],$$

- Proof using law of total probability

$$p(x_k | Z_{1:k}) = \sum_{\theta_{1:k}} p(x_k, \theta_{1:k} | Z_{1:k}).$$

POSTERIOR DENSITIES: BASIC STRUCTURE

- Let the sequences of measurements and data association hypotheses up to time k be denoted

$$Z_{1:k} = (Z_1, Z_2, \dots, Z_k), \quad \theta_{1:k} = (\theta_1, \dots, \theta_k).$$

Structure of posterior density

- In SOT, the filtering density can be written as

$$p(x_k | Z_{1:k}) = \sum_{\theta_{1:k}} p(x_k | Z_{1:k}, \theta_{1:k}) \Pr [\theta_{1:k} | Z_{1:k}],$$

whereas the predicted density is

$$p(x_{k+1} | Z_{1:k}) = \sum_{\theta_{1:k}} p(x_{k+1} | Z_{1:k}, \theta_{1:k}) \Pr [\theta_{1:k} | Z_{1:k}].$$

MODEL ASSUMPTIONS AND OBSERVATIONS

Prior density : $p(x_1) = \mathcal{N}(x_1; 0.5, 0.2)$

Motion model : $\pi_k(x_k | x_{k-1}) = \mathcal{N}(x_k; x_{k-1}, 0.35)$

Probability of detection: $P^D(x) = 0.9$

Object likelihood : $g_k(o_k | x_k) = \mathcal{N}(o_k; x_k, 0.2)$

Clutter intensity : $\lambda(c) = \begin{cases} 0.4 & \text{if } |c| \leq 4 \\ 0 & \text{otherwise} \end{cases}$

Observed detections : $Z_1 = [-1.3, 1.7], \quad Z_2 = [1.3],$
 $Z_3 = [-0.3, 2.3].$

- **Note:** in this example, $p(x_k | Z_{1:k}, \theta_{1:k})$ is computed using a Kalman filter (no update if undetected).

A VISUALIZATION OF THE UPDATE STEP, $k = 1$

- Circle corresponds to “object is undetected”.
- **Green curves** illustrate $p(x_1)$ and $p(x_1|Z_1)$, $Z_1 = [-1.3, 1.7]$.
- There are $m_1 + 1$ hypotheses.
- **Red dashed curve** is contribution from a single term to the posterior,

$$p(x_1|Z_1) = \sum_{\theta_1} p(x_1|\theta_1, Z_1) \Pr(\theta_1|Z_1).$$

A VISUALIZATION OF THE PREDICTION STEP, $k = 2$

- Green curves illustrate

$$p(x_2|Z_1) = \sum_{\theta_1} p(x_2|Z_1, \theta_1) \Pr[\theta_1|Z_1]$$

$$p(x_1|Z_1) = \sum_{\theta_1} p(x_1|Z_1, \theta_1) \Pr[\theta_1|Z_1],$$

whereas **red dashed curves** illustrate individual terms.

- There are $m_1 + 1$ hypotheses.

A VISUALIZATION OF THE UPDATE STEP, $k = 2$

- Green curves illustrate

$$p(x_2|Z_{1:2}) = \sum_{\theta_{1:2}} p(x_2|Z_{1:2}, \theta_{1:2}) \Pr[\theta_{1:2}|Z_{1:2}]$$

$$p(x_2|Z_1) = \sum_{\theta_1} p(x_2|Z_1, \theta_1) \Pr[\theta_1|Z_1],$$

whereas red dashed curves illustrate individual terms.

- There are $(m_1 + 1) \times (m_2 + 1)$ hypotheses.

A VISUALIZATION OF THE PREDICTION STEP, $k = 3$

- Green curves illustrate

$$p(x_2 | Z_{1:2}) = \sum_{\theta_{1:2}} p(x_2 | Z_{1:2}, \theta_{1:2}) \Pr[\theta_{1:2} | Z_{1:2}]$$

$$p(x_3 | Z_{1:2}) = \sum_{\theta_{1:2}} p(x_3 | Z_{1:2}, \theta_{1:2}) \Pr[\theta_{1:2} | Z_{1:2}],$$

whereas red dashed curves illustrate individual terms.

- There are $(m_1 + 1) \times (m_2 + 1)$ hypotheses.

A VISUALIZATION OF THE UPDATE STEP, $k = 3$

- Green curves illustrate

$$p(x_3 | Z_{1:2}) = \sum_{\theta_{1:2}} p(x_3 | Z_{1:2}, \theta_{1:2}) \Pr[\theta_{1:2} | Z_{1:2}]$$

$$p(x_3 | Z_{1:3}) = \sum_{\theta_{1:2}} p(x_3 | Z_{1:3}, \theta_{1:3}) \Pr[\theta_{1:3} | Z_{1:3}],$$

whereas red dashed curves illustrate individual terms.

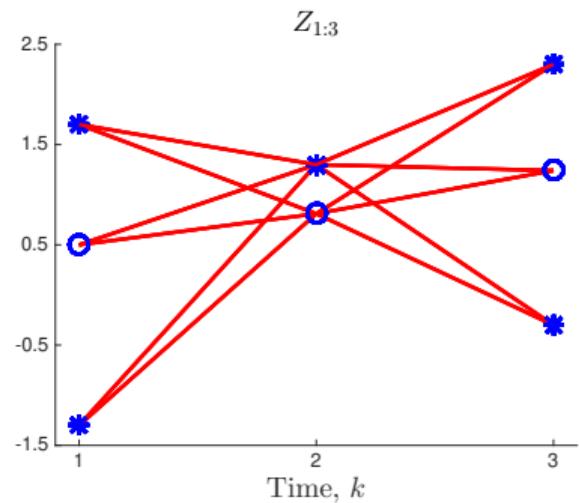
- There are $(m_1 + 1) \times (m_2 + 1) \times (m_3 + 1)$ hypotheses.

DATA ASSOCIATION HYPOTHESES

- We have $m_k + 1$ data association hypotheses at time k .
- The number of possible association sequences at time k is

$$\prod_{i=1}^k (m_i + 1) = (m_1 + 1) \times \cdots \times (m_k + 1),$$

which **grows quickly** with k .



Normalizing the posterior mixture of densities

Multi-Object Tracking

Lennart Svensson

MEASUREMENT UPDATE

- **Measurement likelihood:**

$$p(Z_k | x_k) = \sum_{\theta_k=0}^{m_k} p(Z_k, m_k, \theta_k | x_k).$$

- **Posterior density:**

$$\begin{aligned} p(x | Z) &\propto p(x)p(Z | x) \\ &= \sum_{\theta=0}^m p(x)p(Z, m, \theta | x). \end{aligned}$$

- **Desired form:**

$$p(x | Z) = \sum_{\theta=0}^m w_\theta p_\theta(x),$$

where

$$\begin{cases} w_\theta & \text{is a pmf } (\Pr[\theta | Z]) \\ p_\theta(x) & \text{is a pdf } (p(x | \theta, Z)). \end{cases}$$

PROBLEM FORMULATION

- Consider a probability density function

$$p(x) \propto g(x) = \sum_{\theta=0}^m g_{\theta}(x),$$

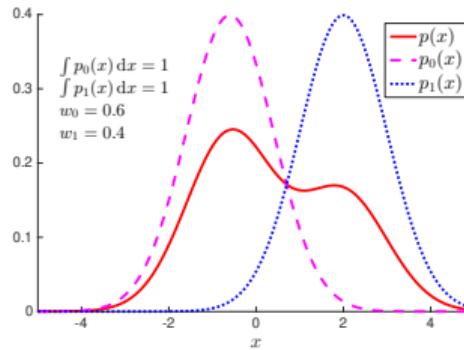
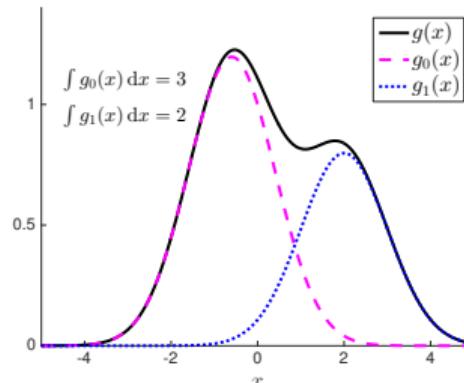
where $g_0(x), \dots, g_m(x)$ are non-negative functions with integrals

$$0 < \int g_i(x) dx < \infty.$$

- How can we express this pdf as a mixture of pdfs

$$p(x) = \sum_{\theta=0}^m w_{\theta} p_{\theta}(x)?$$

Example:



NORMALIZING A FUNCTION

- For $p(x) \propto g(x)$ there is a c :

$$p(x) = c g(x).$$

Example:

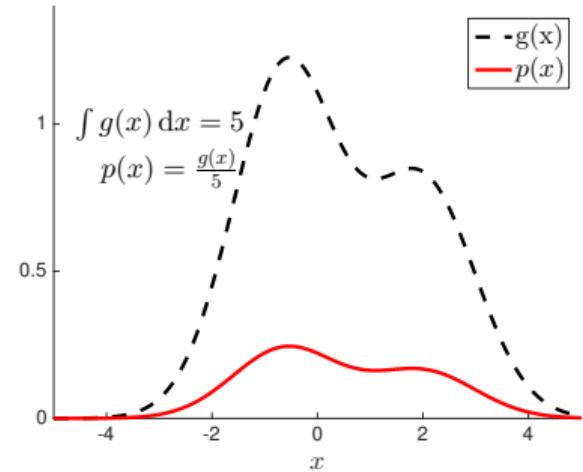
- Given that $p(x)$ is a pdf:

$$1 = \int p(x) dx = c \int g(x) dx$$
$$\Rightarrow c = \frac{1}{\int g(x) dx}.$$

Normalizing a density

- If $p(x) \propto g(x)$, then

$$p(x) = \frac{g(x)}{\int g(x') dx'}.$$



FACTORIZING $g_\theta(x)$

- Introducing

$$\begin{cases} \tilde{w}_\theta = \int g_\theta(x) dx \\ p_\theta(x) = \frac{g_\theta(x)}{\tilde{w}_\theta} \end{cases}$$

we get the factorization

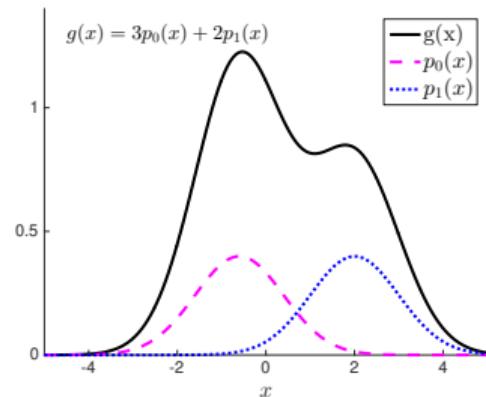
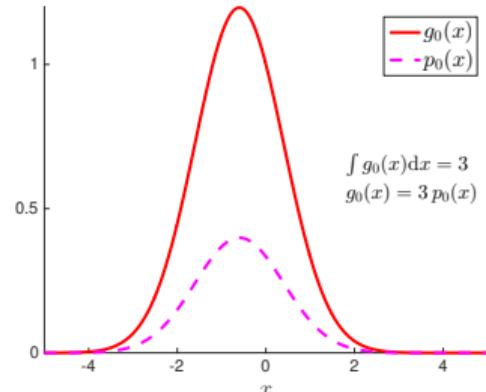
$$g_\theta(x) = \tilde{w}_\theta p_\theta(x)$$

where $p_\theta(x)$ is a pdf.

- It follows that

$$p(x) \propto g(x) = \sum_{\theta=0}^m \tilde{w}_\theta p_\theta(x).$$

Example:



NORMALIZING THE MIXTURE

- We know that

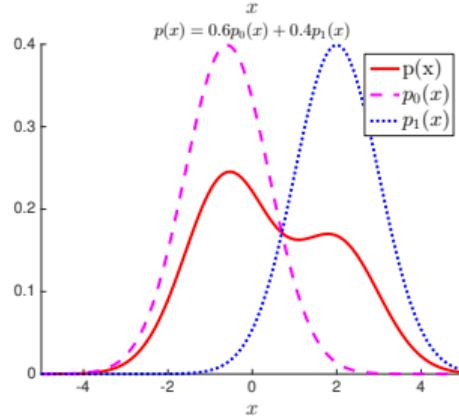
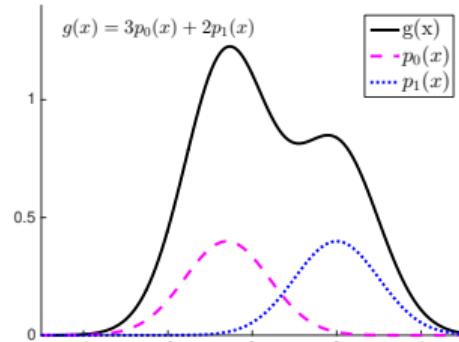
$$\begin{cases} p(x) \propto g(x) \Rightarrow p(x) = \frac{g(x)}{\int g(x') dx'} \\ g(x) = \sum_{\theta=0}^m g_{\theta}(x) = \sum_{\theta=0}^m \tilde{w}_{\theta} p_{\theta}(x) \end{cases}$$

where $\tilde{w}_{\theta} = \int g_{\theta}(x) dx$ and $p_{\theta}(x) = \frac{g_{\theta}(x)}{\tilde{w}_{\theta}}$.

- **Note:** $\int g(x) dx = \sum_{\theta=0}^m \int g_{\theta}(x) dx = \sum_{\theta=0}^m \tilde{w}_{\theta}$.
- Introducing **normalized weights**

$$w_{\theta} = \frac{\tilde{w}_{\theta}}{\sum_{i=0}^m \tilde{w}_i},$$
$$\Rightarrow p(x) = \frac{\sum_{\theta=0}^m \tilde{w}_{\theta} p_{\theta}(x)}{\sum_{i=0}^m \tilde{w}_i} = \sum_{\theta=0}^m w_{\theta} p_{\theta}(x).$$

Example:



MIXTURES OF DENSITIES: SUMMARY

Normalizing a mixture

- If

$$p(x) \propto \sum_{\theta=0}^m g_{\theta}(x),$$

it follows that

$$p(x) = \sum_{\theta=0}^m w_{\theta} p_{\theta}(x),$$

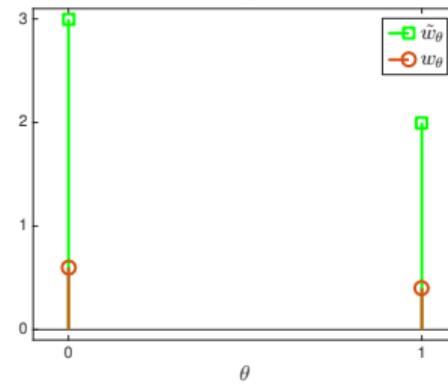
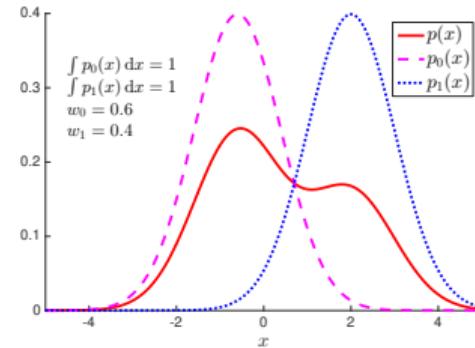
where $p_{\theta}(x) \propto g_{\theta}(x)$ and $w_{\theta} \propto \int g_{\theta}(x) dx$.

- **Note:** w_{θ} should be normalized to become a pmf.

- Specifically, we can set

$$\tilde{w}_{\theta} = \int g_{\theta}(x) dx, \quad p_{\theta}(x) = \frac{g_{\theta}(x)}{\tilde{w}_{\theta}}, \quad w_{\theta} = \frac{\tilde{w}_{\theta}}{\sum_i \tilde{w}_i}.$$

Example:



Interpretation of weights and densities

Multi-Object Tracking

Lennart Svensson

MEASUREMENT UPDATE

- **Posterior density:**

$$\begin{aligned} p(x|Z) &\propto p(x)p(Z|x) \\ &= \sum_{\theta=0}^m \underbrace{p(x)p(Z, m, \theta|x)}_{g_\theta(x)}. \end{aligned}$$

- **Final expression:**

$$p(x|Z) = \sum_{\theta=0}^m w_\theta p_\theta(x),$$

where

$$\begin{cases} w_\theta & \text{is a pmf, } \Pr[\theta|Z], \\ p_\theta(x) & \text{is a pdf, } p(x|\theta, Z). \end{cases}$$

INTERPRETATIONS: MAIN RESULTS

- One can show that $w_\theta p_\theta(x) = p(x, \theta | Z)$.

Three important consequences

- Weights are DA probabilities

$$w_\theta = \Pr(\theta | Z).$$

- Pdfs are conditional posterior pdfs

$$p_\theta(x) = \frac{p(x, \theta | Z)}{w_\theta} = \frac{p(x, \theta | Z)}{\Pr(\theta | Z)} = p(x | \theta, Z).$$

- An expression for posterior pdf

$$p(x | Z) = \sum_\theta w_\theta p_\theta(x).$$

A general update equation

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UPDATE STEP: AN ILLUSTRATION

- An illustrative example (revisited):

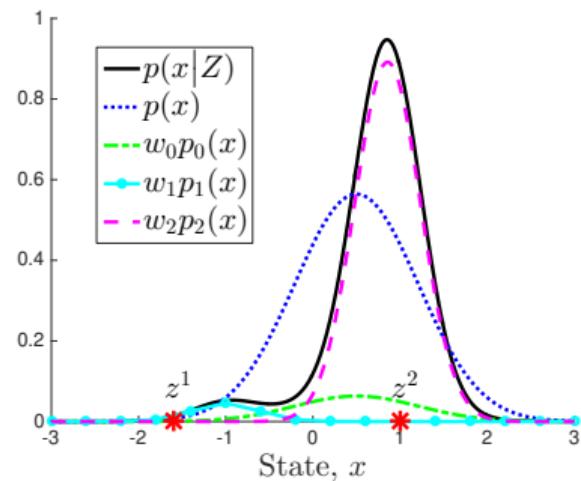
Prior density : $p(x) = \mathcal{N}(x; 0.5, 0.5)$,

Constant P^D : $P^D(x) = 0.85$,

Object likelihood : $g(o|x) = \mathcal{N}(o; x, 0.2)$

Clutter intensity : $\lambda_c(c) = \begin{cases} 0.3 & \text{if } |c| < 5 \\ 0 & \text{otherwise,} \end{cases}$

Observed detections : $Z = [-1.6, 1]$.



- We derive an expression

$$p(x|Z) = \sum_{\theta=0}^m w_\theta p_\theta(x)$$

for **general models** $p(x)$, $P^D(x)$, $\lambda_c(c)$ and $g(o|x)$.

UPDATE STEP (1)

- **Measurement model:**

$$p(Z|x) = \left[(1 - P^D(x)) + P^D(x) \sum_{\theta=1}^m \frac{g(z^\theta|x)}{\lambda_c(z^\theta)} \right] \frac{\exp(-\bar{\lambda}_c)}{m!} \prod_{i=1}^m \lambda_c(z^i).$$

- **Posterior density:**

$$\begin{aligned} p(x|Z) &\propto p(x)p(Z|x) \\ &\propto p(x) \left[(1 - P^D(x)) + P^D(x) \sum_{\theta=1}^m \frac{g(z^\theta|x)}{\lambda_c(z^\theta)} \right]. \end{aligned}$$

UPDATE STEP (2)

- The posterior density is,

$$p(x|Z) \propto p(x) \left[(1 - P^D(x)) + P^D(x) \sum_{\theta=1}^m \frac{g(z^\theta|x)}{\lambda_c(z^\theta)} \right].$$

Posterior probabilities and densities

- We get $p(x|Z) = \sum_{\theta=0}^m w_\theta p_\theta(x)$, where

$$\theta = 0$$

Object is undetected

$$\begin{cases} \tilde{w}_0 = \int p(x)(1 - P^D(x)) dx \\ p_0(x) = \frac{p(x)(1 - P^D(x))}{\int p(x')(1 - P^D(x')) dx'}, \end{cases}$$

$$\theta \in \{1, 2, \dots, m\}$$

z^θ is object detection

$$\begin{cases} \tilde{w}_\theta = \frac{1}{\lambda_c(z^\theta)} \int p(x) P^D(x) g(z^\theta|x) dx \\ p_\theta(x) = \frac{p(x) P^D(x) g(z^\theta|x)}{\int p(x') P^D(x') g(z^\theta|x') dx'}, \end{cases}$$

and $w_\theta \propto \tilde{w}_\theta$.

POSTERIOR DENSITY GIVEN θ

- Recall the object measurement model

$$p(O|x) = \begin{cases} 1 - P^D(x) & \text{if } O = [], \\ P^D(x)g(o|x) & \text{if } O = o. \end{cases}$$

- Given O , we get

$$p(x|O) \propto \begin{cases} p(x)(1 - P^D(x)) & \text{if } O = [], \\ p(x)P^D(x)g(o|x) & \text{if } O = o. \end{cases}$$

- By comparison,

$$p_\theta(x) \propto \begin{cases} p(x)(1 - P^D(x)) & \text{if } \theta = 0, \\ p(x)P^D(x)g(z^\theta|x) & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

Conclusion: $p_\theta(x)$ is identical to $p(x|O)$, with O defined by θ and Z .

Update equations for linear and Gaussian models

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Lennart Svensson

MODEL ASSUMPTIONS

- Suppose:

$$\text{Prior density : } p(x) = \mathcal{N}(x; \mu, P),$$

$$\text{Constant } P^D : \quad P^D(x) = P^D,$$

$$\text{Object measurement likelihood : } g(o|x) = \mathcal{N}(o; Hx, R),$$

$$\text{Clutter intensity : } \lambda_c(c) \geq 0.$$

- We express the posterior density on the form

$$p(x|Z) = \sum_{\theta=0}^m w_\theta p_\theta(x)$$

and study w_θ and $p_\theta(x)$.

POSTERIOR DENSITY GIVEN θ , (1)

- We found that

$$p_\theta(x) \propto \begin{cases} p(x)(1 - P^D(x)) & \text{if } \theta = 0, \\ p(x)P^D(x)g(z^\theta|x) & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

- When $P^D(x) = P^D$, this simplifies to

$$p_\theta(x) \propto \begin{cases} p(x) & \text{if } \theta = 0, \\ p(x)g(z^\theta|x) & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

- If $\theta > 0$, we update the prior using the likelihood $g(z^\theta|x)$.
No update if $\theta = 0$.

POSTERIOR DENSITY GIVEN θ , (2)

- When $\theta \in \{1, 2, \dots, m\}$, assuming $p(x) = \mathcal{N}(x; \mu, P)$ and $g(o|x) = \mathcal{N}(o; Hx, R)$,

$$p_\theta(x) \propto p(x)g(z^\theta|x) = \mathcal{N}(x; \mu, P)\mathcal{N}(z^\theta; Hx, R).$$

- With a Gaussian prior and a linear-Gaussian likelihood, we obtain a Gaussian posterior.
- We can use the **Kalman filter update** to compute the posterior density:

Predicted measurement covariance:

$$S = HPH^T + R$$

Kalman gain:

$$K = PH^T S^{-1}$$

Posterior mean:

$$\hat{x}_\theta = \mu + K(z^\theta - H\mu)$$

Posterior covariance:

$$P_+ = P - KHP$$

Posterior density:

$$p_\theta(x) = \mathcal{N}(x; \hat{x}_\theta, P_+).$$

VISUALIZING $p_\theta(x)$

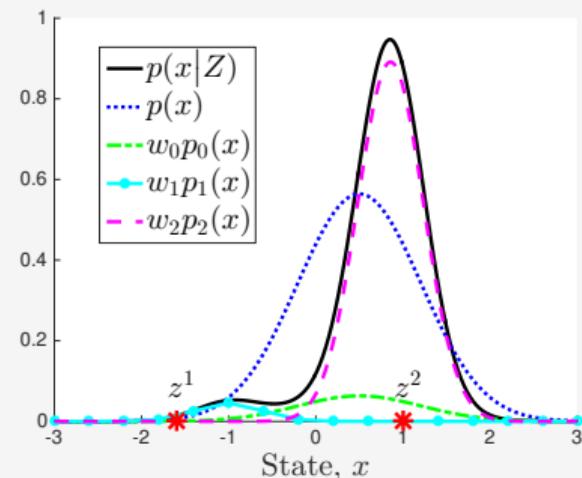
- When $\theta > 0$, $p_\theta(x)$ is obtained by a Kalman filter update of $p(x)$, assuming that z^θ is the object detection.

Example, revisited

- Suppose $p(x) = \mathcal{N}(x; 0.5, 0.5)$,
 $Z = [-1.6, 1]$, $P^D = 0.85$,
 $g(o|x) = \mathcal{N}(o; x, 0.2)$ and

$$\lambda_c(c) = \begin{cases} 0.3 & \text{if } |c| < 5 \\ 0 & \text{otherwise.} \end{cases}$$

- $p_1(x)$ and $p_2(x)$ are obtained from a Kalman filter update using z^1 and z^2 , respectively.



POSTERIOR PROBABILITIES OF θ , (1)

- We found that

$$\tilde{w}_\theta = \begin{cases} \int p(x)(1 - P^D(x)) dx & \text{if } \theta = 0, \\ \frac{1}{\lambda_c(z^\theta)} \int p(x)P^D(x)g(z^\theta|x) dx & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

- When $P^D(x) = P^D$, this simplifies to

$$\tilde{w}_\theta = \begin{cases} 1 - P^D & \text{if } \theta = 0, \\ \frac{P^D}{\lambda_c(z^\theta)} \int p(x)g(z^\theta|x) dx & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

- Here, $\int p(x)g(z^\theta|x) dx$ is the predicted density for the object measurement, evaluated at z^θ .

POSTERIOR PROBABILITIES OF θ , (2)

- When $\theta \in \{1, 2, \dots, m\}$, assuming $p(x) = \mathcal{N}(x; \mu, P)$ and $g(o|x) = \mathcal{N}(o; Hx, R)$,

$$\tilde{w}_\theta = \frac{P^D}{\lambda_c(z^\theta)} \int \mathcal{N}(x; \mu, P) \mathcal{N}(z^\theta; Hx, R) dx.$$

- Specifically, $\int \underbrace{\mathcal{N}(x; \mu, P)}_{p(x)} \underbrace{\mathcal{N}(z^\theta; Hx, R)}_{p(z^\theta|x, \theta)} dx = \mathcal{N}(z^\theta; H\mu, HPH^T + R),$

where we often use the Kalman filter notation: $\bar{z} = H\mu$ and $S = HPH^T + R$.

Note: this is the density of $z^\theta = Hx + v$ where $x \sim \mathcal{N}(\mu, P)$, and $v \sim \mathcal{N}(0, R)$.

- We conclude that

$$\tilde{w}_\theta = \frac{P^D \mathcal{N}(z^\theta; \bar{z}, S)}{\lambda_c(z^\theta)},$$

where $\mathcal{N}(z^\theta; \bar{z}, S)$ is called the **predicted likelihood**.

VISUALIZING w_θ

- We conclude that

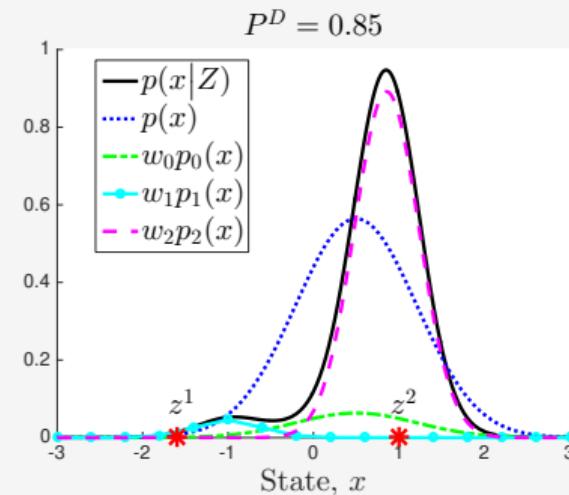
$$\tilde{w}_\theta = \begin{cases} 1 - P^D & \text{if } \theta = 0, \\ \frac{P^D \mathcal{N}(z^\theta; \bar{z}, S)}{\lambda_c(z^\theta)} & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

Examples, revisited, with different P^D

- We assume $p(x) = \mathcal{N}(x; 0.5, 0.5)$ and $g(o|x) = \mathcal{N}(o; x, 0.2)$

$$\Rightarrow \bar{z} = 0.5, \quad S = 0.5 + 0.2 = 0.7.$$

- $\tilde{w}_2 > \tilde{w}_1$ since z^2 is closer than z^1 to \bar{z} .



VISUALIZING w_θ

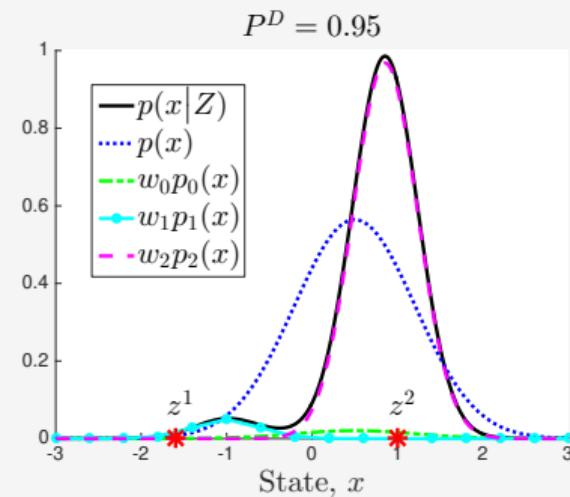
- We conclude that

$$\tilde{w}_\theta = \begin{cases} 1 - P^D & \text{if } \theta = 0, \\ \frac{P^D \mathcal{N}(z^\theta; \bar{z}, S)}{\lambda_c(z^\theta)} & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

Examples, revisited, with different P^D

- We assume $p(x) = \mathcal{N}(x; 0.5, 0.5)$ and $g(o|x) = \mathcal{N}(o; x, 0.2)$
 $\Rightarrow \bar{z} = 0.5, \quad S = 0.5 + 0.2 = 0.7.$

- $\tilde{w}_2 > \tilde{w}_1$ since z^2 is closer than z^1 to \bar{z} .
- w_0 decreases with P^D .



VISUALIZING w_θ

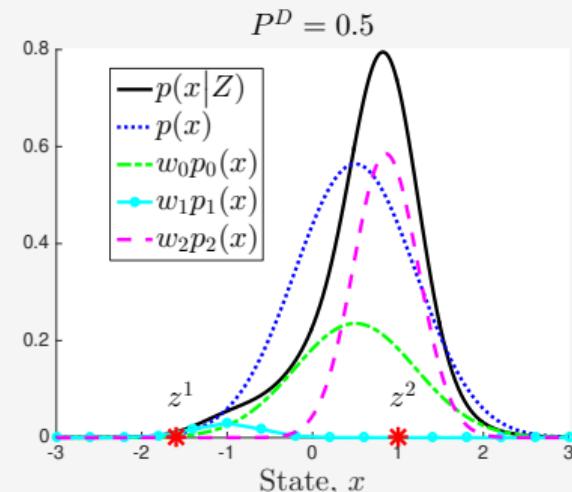
- We conclude that

$$\tilde{w}_\theta = \begin{cases} 1 - P^D & \text{if } \theta = 0, \\ \frac{P^D \mathcal{N}(z^\theta; \bar{z}, S)}{\lambda_c(z^\theta)} & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

Examples, revisited, with different P^D

- We assume $p(x) = \mathcal{N}(x; 0.5, 0.5)$ and $g(o|x) = \mathcal{N}(o; x, 0.2)$
 $\Rightarrow \bar{z} = 0.5, \quad S = 0.5 + 0.2 = 0.7.$

- $\tilde{w}_2 > \tilde{w}_1$ since z^2 is closer than z^1 to \bar{z} .
- w_0 decreases with P^D .



CONCLUSIONS

Closed form expressions

- If $p(x) = \mathcal{N}(x; \mu, P)$, $P^D(x) = P^D$ and $g(o|x) = \mathcal{N}(o; Hx, R)$:

$$p_\theta(x) = \begin{cases} p(x) & \text{if } \theta = 0, \\ \mathcal{N}(x; \hat{x}_\theta, P_+) & \text{if } \theta \in \{1, 2, \dots, m\}, \end{cases}$$
$$\tilde{w}_\theta = \begin{cases} 1 - P^D & \text{if } \theta = 0, \\ \frac{P^D \mathcal{N}(z^\theta; \bar{z}, S)}{\lambda_c(z^\theta)} & \text{if } \theta \in \{1, 2, \dots, m\}. \end{cases}$$

- **Remark:** Suppose

$$o = h(x) + v, \quad v \sim \mathcal{N}(0, R),$$

such that $g(o|x) = \mathcal{N}(o; h(x), R)$, where $h(x)$ is a nonlinear function.

- We can then **approximate** \hat{x}_θ , P_+ , \bar{z} and S **using**, e.g., **an extended Kalman filter**.

Prediction and update steps: conceptual solution, part 1

Multi-Object Tracking

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OVERVIEW OF RESULTS

Main result

- Suppose

$$p(x_{k-1} | \mathcal{Z}_{1:k-1}) = \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1})$$

where $w^{\theta_{1:k-1}} = \Pr[\theta_{1:k-1} | \mathcal{Z}_{1:k-1}]$ and $p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1}) = p(x_{k-1} | \theta_{1:k-1}, \mathcal{Z}_{1:k-1})$.

- We can then express the predicted and updated densities as

Predicted density
$$p(x_k | \mathcal{Z}_{1:k-1}) = \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k),$$

Updated density
$$p(x_k | \mathcal{Z}_{1:k}) = \sum_{\theta_{1:k}} w^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(x_k).$$

- 1) Here $\theta_{1:k} = [\theta_1, \dots, \theta_k]$ is a sequence of data association hypotheses.

OVERVIEW OF RESULTS

Main result

- Suppose

$$p(x_{k-1} | \mathcal{Z}_{1:k-1}) = \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1})$$

where $w^{\theta_{1:k-1}} = \Pr[\theta_{1:k-1} | \mathcal{Z}_{1:k-1}]$ and $p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1}) = p(x_{k-1} | \theta_{1:k-1}, \mathcal{Z}_{1:k-1})$.

- We can then express the predicted and updated densities as

$$p(x_k | \mathcal{Z}_{1:k-1}) = \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k), \quad p(x_k | \mathcal{Z}_{1:k}) = \sum_{\theta_{1:k}} w^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(x_k).$$

- 2) The posterior at time k is written on the same form, but contains more terms

$$\sum_{\theta_{1:k}} = \sum_{\theta_1=0}^{m_1} \sum_{\theta_2=0}^{m_2} \cdots \sum_{\theta_k=0}^{m_k}.$$

OVERVIEW OF RESULTS

Main result

- Suppose

$$p(x_{k-1} | \mathcal{Z}_{1:k-1}) = \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1})$$

where $w^{\theta_{1:k-1}} = \Pr[\theta_{1:k-1} | \mathcal{Z}_{1:k-1}]$ and $p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1}) = p(x_{k-1} | \theta_{1:k-1}, \mathcal{Z}_{1:k-1})$.

- We can then express the predicted and updated densities as

$$p(x_k | \mathcal{Z}_{1:k-1}) = \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k), \quad p(x_k | \mathcal{Z}_{1:k}) = \sum_{\theta_{1:k}} w^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(x_k).$$

- 3) We know how to compute $p(x_1 | \mathcal{Z}_1)$ on the above form.

We obtain a recursive algorithm to compute $p(x_k | \mathcal{Z}_{1:k})$ for any k .

PREDICTION STEP

- If

$$p(x_{k-1} | Z_{1:k-1}) = \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1})$$

then

$$\begin{aligned} p(x_k | Z_{1:k-1}) &= \int p(x_{k-1} | Z_{1:k-1}) p(x_k | x_{k-1}) dx_{k-1} \\ &= \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} \underbrace{\int p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1}) p(x_k | x_{k-1}) dx_{k-1}}_{\triangleq p_{k|k-1}^{\theta_{1:k-1}}(x_k)} \\ &= \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k). \end{aligned}$$

Prediction step

- Weights are unchanged, standard prediction of densities for each hypothesis.

PREDICTION STEP: LINEAR AND GAUSSIAN MOTION

- Suppose

$$x_k = F_{k-1}x_{k-1} + q_{k-1}, \quad q_{k-1} \sim \mathcal{N}(0, Q_{k-1}),$$

such that $\pi_k(x_k | x_{k-1}) = \mathcal{N}(x_k; F_{k-1}x_{k-1}, Q_{k-1})$.

- If $p_{k-1|k-1}^{\theta_{1:k-1}}(x_{k-1}) = \mathcal{N}(x_{k-1}, \hat{x}_{k-1|k-1}^{\theta_{1:k-1}}, P_{k-1|k-1}^{\theta_{1:k-1}})$, then

$$p_{k|k-1}^{\theta_{1:k-1}}(x_k) = \mathcal{N}(x_k; F_{k-1}\hat{x}_{k-1|k-1}^{\theta_{1:k-1}}, F_{k-1}P_{k-1|k-1}^{\theta_{1:k-1}}F_{k-1}^T + Q_{k-1}).$$

-
- Remark:** Suppose

$$x_k = f_{k-1}(x_{k-1}) + q_{k-1}, \quad q_{k-1} \sim \mathcal{N}(0, Q_{k-1}),$$

such that $\pi_k(x_k | x_{k-1}) = \mathcal{N}(x_k; f_{k-1}(x_{k-1}), R)$, where $f_{k-1}(x_{k-1})$ is a nonlinear function.

- We can then **approximate** $p_{k|k-1}^{\theta_{1:k-1}}(x_k)$ **using, e.g., an extended Kalman filter**.

MODEL ASSUMPTIONS FOR VISUALIZATION

Prior density : $p(x_1) = \mathcal{N}(x_1; 0.5, 0.2)$

Motion model : $\pi_k(x_k | x_{k-1}) = \mathcal{N}(x_k; x_{k-1}, 0.35)$

Probability of detection: $P^D(x) = 0.9$

Object likelihood : $g_k(o_k | x_k) = \mathcal{N}(o_k; x_k, 0.2)$

Clutter intensity : $\lambda(c) = \begin{cases} 0.4 & \text{if } |c| \leq 4 \\ 0 & \text{otherwise} \end{cases}$

Observed detections : $Z_1 = [-1.3, 1.7], \quad Z_2 = [1.3].$

A VISUALIZATION OF THE PREDICTION STEP, $k = 2$

- Green curves illustrate

$$p(x_2 | Z_1) = \sum_{\theta_1} w^{\theta_1} p_{2|1}^{\theta_1}(x_2)$$

$$p(x_1 | Z_1) = \sum_{\theta_1} w^{\theta_1} p_{1|1}^{\theta_1}(x_1),$$

whereas **red dashed curves** illustrate individual terms.

- There are $m_1 + 1$ hypotheses.

Prediction and update steps: conceptual solution, part 2

Multi-Object Tracking

Lennart Svensson

UPDATE STEP (1)

- **Measurement model**

$$p(Z_k | x_k) = \left[(1 - P^D(x_k)) + P^D(x_k) \sum_{\theta_k=1}^{m_k} \frac{g_k(z_k^{\theta_k} | x_k)}{\lambda_c(z_k^{\theta_k})} \right] \frac{\exp(-\bar{\lambda}_c)}{m_k!} \prod_{i=1}^m \lambda_c(z_k^i).$$

- For

$$p(x_k | Z_{1:k-1}) = \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k),$$

this implies that

$$\begin{aligned} p(x_k | Z_{1:k}) &\propto \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k) (1 - P^D(x_k)) \\ &\quad + \sum_{\theta_{1:k-1}} \sum_{\theta_k=1}^{m_k} \frac{1}{\lambda_c(z_k^{\theta_k})} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k) P^D(x_k) g_k(z_k^{\theta_k} | x_k). \end{aligned}$$

- **Note:** for every pair of hypotheses, $(\theta_{1:k-1}, \theta_k)$, we obtain a new hypothesis. We index this hypothesis using the vector $\theta_{1:k}$.

UPDATE STEP (2)

- The posterior density is,

$$p(x_k | Z_{1:k}) \propto \sum_{\theta_{1:k-1}} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k) (1 - P^D(x_k)) \\ + \sum_{\theta_{1:k-1}} \sum_{\theta_k=1}^{m_k} \frac{1}{\lambda_c(z_k^{\theta_k})} w^{\theta_{1:k-1}} p_{k|k-1}^{\theta_{1:k-1}}(x_k) P^D(x_k) g_k(z_k^{\theta_k} | x_k).$$

Posterior probabilities and densities

- We get $p(x_k | Z_{1:k}) = \sum_{\theta_{1:k}} w^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(x_k)$, where $w^{\theta_{1:k}} \propto \tilde{w}^{\theta_{1:k}}$ and

$$\theta_k = 0$$

Object is undetected

$$\begin{cases} \tilde{w}^{\theta_{1:k}} = w^{\theta_{1:k-1}} \int p_{k|k-1}^{\theta_{1:k-1}}(x_k) (1 - P^D(x_k)) dx_k \\ p_{k|k}^{\theta_{1:k}}(x_k) \propto p_{k|k-1}^{\theta_{1:k-1}}(x_k) (1 - P^D(x_k)), \end{cases}$$

$$\theta_k \in \{1, 2, \dots, m_k\}$$

z_k^{θ} is object detection

$$\begin{cases} \tilde{w}^{\theta_{1:k}} = \frac{w^{\theta_{1:k-1}}}{\lambda_c(z_k^{\theta_k})} \int p_{k|k-1}^{\theta_{1:k-1}}(x_k) P^D(x_k) g_k(z_k^{\theta_k} | x_k) dx_k \\ p_{k|k}^{\theta_{1:k}}(x_k) \propto p_{k|k-1}^{\theta_{1:k-1}}(x_k) P^D(x_k) g_k(z_k^{\theta_k} | x_k). \end{cases}$$

A LINEAR AND GAUSSIAN UPDATE STEP

Closed form expressions

- If $p_{k|k-1}^{\theta_{1:k-1}}(x_k) = \mathcal{N}(x_k; \mu_{k|k-1}^{\theta_{1:k-1}}, P_{k|k-1}^{\theta_{1:k-1}})$, $P^D(x_k) = P^D$ and $g_k(o_k|x_k) = \mathcal{N}(o_k; H_k x_k, R_k)$:

$$p_{k|k}^{\theta_{1:k}}(x_k) = \begin{cases} p_{k|k-1}^{\theta_{1:k-1}}(x_k) & \text{if } \theta_k = 0, \\ \mathcal{N}(x_k; \mu_{k|k}^{\theta_{1:k}}, P_{k|k}^{\theta_{1:k}}) & \text{if } \theta_k \in \{1, 2, \dots, m\}, \end{cases}$$
$$\tilde{w}^{\theta_{1:k}} = \begin{cases} w^{\theta_{1:k-1}}(1 - P^D) & \text{if } \theta_k = 0, \\ w^{\theta_{1:k-1}} \frac{P^D \mathcal{N}(z_k^\theta; \bar{z}_{k|k-1}^{\theta_{1:k-1}}, S_{k|k-1}^{\theta_{1:k-1}})}{\lambda_c(z_k^\theta)} & \text{if } \theta_k \in \{1, 2, \dots, m\}. \end{cases}$$

- Here $\mu_{k|k}^{\theta_{1:k}}$ and $P_{k|k}^{\theta_{1:k}}$ are the posterior mean and covariance given $Z_{1:k}$ and $\theta_{1:k}$.
- Similarly, $\bar{z}_{k|k-1}^{\theta_{1:k-1}}$ and $S_{k|k-1}^{\theta_{1:k-1}}$ are the predicted object measurement mean and covariance assuming the predicted density $p_{k|k-1}^{\theta_{1:k-1}}(x_k)$.

THE KALMAN FILTER UPDATE

Object measurement prediction:

$$\bar{z}_{k|k-1}^{\theta_{1:k-1}} = H_k \mu_{k|k-1}^{\theta_{1:k-1}}$$

Predicted measurement covariance:

$$S_{k|k-1}^{\theta_{1:k-1}} = H_k P_{k|k-1}^{\theta_{1:k-1}} H_k^T + R_k$$

Kalman gain:

$$K_k^{\theta_{1:k}} = P_{k|k-1}^{\theta_{1:k-1}} H_k^T (S_{k|k-1}^{\theta_{1:k-1}})^{-1}$$

Posterior mean:

$$\mu_{k|k}^{\theta_{1:k}} = \mu_{k|k-1}^{\theta_{1:k-1}} + K_k^{\theta_{1:k}} (z_k^{\theta_k} - \bar{z}_{k|k-1}^{\theta_{1:k-1}})$$

Posterior covariance:

$$P_{k|k}^{\theta_{1:k}} = P_{k|k-1}^{\theta_{1:k-1}} - K_k^{\theta_{1:k}} H_k P_{k|k-1}^{\theta_{1:k-1}}.$$

A VISUALIZATION OF THE UPDATE STEP, $k = 2$

- Green curves illustrate

$$p(x_2 | Z_{1:2}) = \sum_{\theta_{1:2}} w^{\theta_{1:2}} p_{2|2}^{\theta_{1:2}}(x_2)$$

$$p(x_2 | Z_1) = \sum_{\theta_1} w^{\theta_1} p_{2|1}^{\theta_1}(x_2),$$

whereas **red dashed curves** illustrate individual terms.

- There are $(m_1 + 1) \times (m_2 + 1)$ hypotheses.

CONCLUDING REMARKS

- We have presented a recursive algorithm to compute

$$p(x_k | Z_{1:k}) = \sum_{\theta_{1:k}} w^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(x_k).$$

- The posterior contains one term for **every possible sequence** of data associations.
- In linear and Gaussian settings (with constant P^D) the densities are computed **using a Kalman filter**.
- The number of hypotheses grows quickly

$$\prod_{i=1}^k (m_i + 1)$$

which means that we **need to introduce approximations**.

Section 4: SOT algorithms

Multi-Object Tracking

Lennart Svensson

An overview of different SOT algorithms

Multi-Object Tracking

Lennart Svensson

THE NEED FOR APPROXIMATIONS

- The number of hypotheses grows as

$$\prod_{i=1}^k (m_i + 1).$$

- It is therefore generally intractable to compute

$$p(x_k | Z_{1:k})$$

exactly, except for a small number of time steps.

- To obtain a feasible algorithm, we **need to introduce approximations**.
- We focus specifically on the **Gaussian mixture setting**, though principles apply more generally.

GAUSSIAN MIXTURE REDUCTION

- **Problem:** $p(x_k | Z_{1:k})$ is a Gaussian mixture with too many components.
- **Standard solution:** Find

$$\hat{p}(x_k | Z_{1:k}) \approx p(x_k | Z_{1:k})$$

where $\hat{p}(x_k | Z_{1:k})$ is a Gaussian mixture with fewer components.

- Once we have selected $\hat{p}(x_k | Z_{1:k})$, we start the next recursion assuming

$$p(x_k | Z_{1:k}) = \hat{p}(x_k | Z_{1:k}).$$

- By limiting the number of components in $\hat{p}(x_k | Z_{1:k})$ we obtain a feasible algorithm.

PRUNING AND MERGING

- The main techniques for mixture reduction are **pruning and merging**.

Pruning

- Remove hypotheses with small weights (and renormalize).

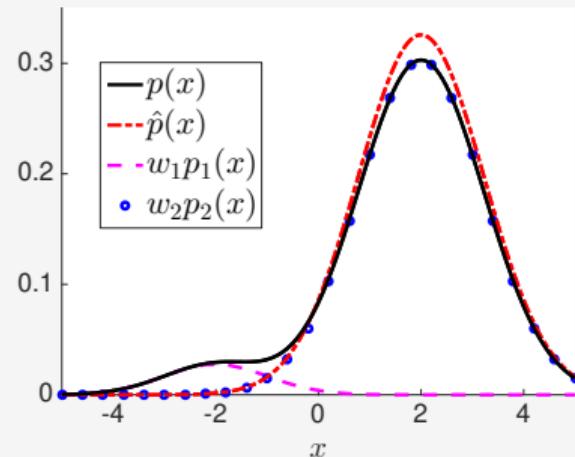
A pruning example

- Suppose $p(x)$ is given by

$$p(x) = w_1 p_1(x) + w_2 p_2(x) \text{ where}$$

$$\begin{cases} w_1 = 0.07, & p_1(x) = \mathcal{N}(x; -2, 1) \\ w_2 = 0.93, & p_2(x) = \mathcal{N}(x; 2, 1.5) \end{cases}$$

- Pruning first hypothesis gives $\hat{p}(x) = p_2(x)$.



PRUNING AND MERGING

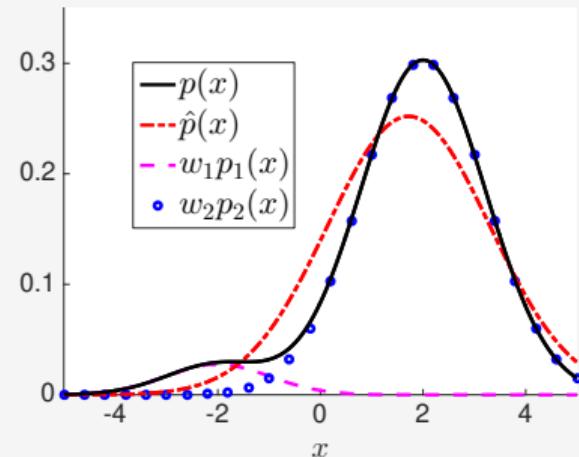
- The main techniques for mixture reduction are **pruning and merging**.

Merging

- Approximate a mixture of densities by a single density (often Gaussian).

A merging example

- Consider again $p(x) = w_1 p_1(x) + w_2 p_2(x)$ as above.
- We can select $\hat{p}(x)$ to match the first two moments of $p(x)$.
- Approximation also depends on w_1 and $p_1(x)$.



PRESENTED ALGORITHMS

- In the next videos, we present three algorithms for SOT in clutter:
 - Nearest neighbour (NN) filter, [pruning],
 - Probabilistic data association (PDA) filter [merging],
 - Gaussian sum filter (GSF). [pruning/merging]
- All of these are examples of **assumed density filters**
 - NN and PDA: Gaussian densities,
 - GSF: Gaussian mixture densities,that is, every recursion starts and ends with a density in that family.
- Apart from the above tracking algorithms, we also present **gating**, which is a technique to disregard unreasonable detections. [pruning]

Nearest neighbour filtering

Multi-Object Tracking

Lennart Svensson

LINEAR AND GAUSSIAN MODELS, PREDICTION STEP

- NN and PDA both assume **Gaussian posterior at time $k - 1$** :

$$p(x_{k-1} | Z_{1:k-1}) = \mathcal{N}(x_{k-1}; \bar{x}_{k-1|k-1}, P_{k-1|k-1}).$$

- We also assume a **linear and Gaussian motion model**:

$$x_k = F_{k-1}x_{k-1} + q_{k-1}, \quad q_{k-1} \sim \mathcal{N}(0, Q).$$

- **Predicted density** is therefore

$$p(x_k | Z_{1:k-1}) = \mathcal{N}(x_k; \bar{x}_{k|k-1}, P_{k|k-1})$$

where $\bar{x}_{k|k-1} = F_{k-1}\bar{x}_{k-1|k-1}$ and $P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^T + Q$.

- We sometimes use superscript NN, e.g., $p^{\text{NN}}(x_k | Z_{1:k-1})$, to clarify that it is an approximation obtained using the NN algorithm.

LINEAR AND GAUSSIAN MODELS, UPDATE STEP

Measurement model

We assume $P^D(x) = P^D$, $g_k(o|x) = \mathcal{N}(o; H_k x, R_k)$, general $\lambda_c(c)$.

Posterior density, given $p^{\text{NN}}(x_k | Z_{1:k-1}) = \mathcal{N}(x_k; \bar{x}_{k|k-1}^{\text{NN}}, P_{k|k-1}^{\text{NN}})$

Posterior is $\check{p}^{\text{NN}}(x_k | Z_{1:k}) = \sum_{\theta_k=0}^{m_k} \check{w}_k^{\theta_k} p_{k|k}^{\theta_k}(x_k)$ where $p_{k|k}^{\theta_k}(x_k) = \mathcal{N}(x_k; \hat{x}_k^{\theta_k}, P_k^{\theta_k})$
and

$$\theta_k = 0 : \begin{cases} \check{w}_k^{\theta_k} = 1 - P^D, & \hat{x}_k^{\theta_k} = \bar{x}_{k|k-1}^{\text{NN}}, \\ P_k^{\theta_k} = P_{k|k-1}^{\text{NN}}, \end{cases}$$

$$\theta_k \in \{1, \dots, m_k\} : \begin{cases} \check{w}_k^{\theta_k} = \frac{P^D \mathcal{N}(z_k^{\theta_k}; \bar{z}_{k|k-1}, S_k)}{\lambda_c(z_k^{\theta_k})}, & \hat{x}_k^{\theta_k} = \bar{x}_{k|k-1}^{\text{NN}} + K_k (z_k^{\theta_k} - \bar{z}_{k|k-1}), \\ P_k^{\theta_k} = P_{k|k-1}^{\text{NN}} - K_k H_k P_{k|k-1}^{\text{NN}}. \end{cases}$$

- How can we **approximate** $\check{p}^{\text{NN}}(x_k | Z_{1:k})$ as Gaussian?

NEAREST NEIGHBOUR FILTERING

Basic idea

- Prune all hypotheses except the most probable one.

Algorithm The NN filtering update.

- 1: Compute $\tilde{w}_k^{\theta_k}$, $\theta_k = 0, 1, \dots, m_k$.
- 2: Find

$$\theta_k^* = \arg \max_{\theta} \tilde{w}_k^{\theta}.$$

- 3: Compute $\hat{x}_k^{\theta_k^*}$ and $P_k^{\theta_k^*}$.
 - 4: Set $\bar{x}_{k|k}^{\text{NN}} = \hat{x}_k^{\theta_k^*}$ and $P_{k|k}^{\text{NN}} = P_k^{\theta_k^*}$.
-

- **Note:** we then assume that $p^{\text{NN}}(x_k | Z_{1:k}) = \mathcal{N}(x_k; \bar{x}_{k|k}^{\text{NN}}, P_{k|k}^{\text{NN}})$.

EXAMPLE FOR VISUALIZATION (SIMPLE)

Prior density : $p(x_1) = \mathcal{N}(x_1; 0.5, 0.2)$

Motion model : $\pi_k(x_k | x_{k-1}) = \mathcal{N}(x_k; x_{k-1}, 0.35)$

Probability of detection: $P^D(x) = 0.9$

Object likelihood : $g_k(o_k | x_k) = \mathcal{N}(o_k; x_k, 0.2)$

Clutter intensity : $\lambda(c) = \begin{cases} 0.4 & \text{if } |c| \leq 4 \\ 0 & \text{otherwise} \end{cases}$

Observed detections : $Z_1 = [-1.3, 1.7], \quad Z_2 = [1.3],$
 $Z_3 = [-0.3, 2.3], \quad Z_4 = [-2, 3]$
 $Z_5 = [2.6], \quad Z_6 = [-3.5, 2.8]$

A NEAREST NEIGHBOUR FILTERING EXAMPLE

Example: NN vs exact posterior

- We compare four densities:

$$p^{\text{NN}}(x_k | Z_{1:k-1}), \quad p(x_k | Z_{1:k})$$

$$\check{p}^{\text{NN}}(x_k | Z_{1:k}), \quad p^{\text{NN}}(x_k | Z_{1:k})$$

- We visualize the $m_k + 1$ hypotheses, and mark the **most probable in red**.
- The NN algorithm approximates the posterior fairly well.

Nearest neighbour filtering – additional remarks

Multi-Object Tracking

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WHY THE NAME “NEAREST NEIGHBOUR”?

- To find θ_k^* , we can do

$$\theta_k^+ = \arg \max_{\theta \in \{1, 2, \dots, m_k\}} \tilde{w}_k^{\theta},$$
$$\theta_k^* = \begin{cases} \theta_k^+ & \text{if } \tilde{w}_k^{\theta_k^+} \geq \tilde{w}_k^0 \\ 0 & \text{if } \tilde{w}_k^{\theta_k^+} < \tilde{w}_k^0. \end{cases}$$

- Roughly speaking, $z_k^{\theta_k^+}$ is the **“nearest” measurement** to $\bar{z}_{k|k-1}$.

NEAREST NEIGHBOUR?

$$\begin{aligned}\theta_k^+ &= \arg \max_{\theta \in \{1, 2, \dots, m_k\}} \tilde{w}_k^{\theta_k} \\ &= \arg \max_{\theta \in \{1, 2, \dots, m_k\}} \frac{P^D \mathcal{N}(z_k^\theta; \bar{z}_{k|k-1}, S_k)}{\lambda_c(z_k^\theta)} \\ &= \left\{ \text{If } \lambda_c(z_k^\theta) = \lambda_c, \forall \theta \in \{1, 2, \dots, m_k\} \right\} \\ &= \arg \max_{\theta \in \{1, 2, \dots, m_k\}} \frac{\exp \left(-\frac{1}{2} (z_k^\theta - \bar{z}_{k|k-1})^T S_k^{-1} (z_k^\theta - \bar{z}_{k|k-1}) \right)}{|2\pi S_k|^{1/2}} \\ &= \arg \min_{\theta \in \{1, 2, \dots, m_k\}} (z_k^\theta - \bar{z}_{k|k-1})^T S_k^{-1} (z_k^\theta - \bar{z}_{k|k-1})\end{aligned}$$

The nearest neighbour

- Under certain assumptions, $z_k^{\theta_k^+}$ is the “**nearest**” neighbour to $\bar{z}_{k|k-1}$, where S_k is used to define the distance.

EXAMPLE FOR VISUALIZATION (HARD)

Prior density : $p(x_1) = \mathcal{N}(x_1; 0.5, 0.2)$

Motion model : $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k; x_{k-1}, 0.35)$

Probability of detection: $P^D(x) = 0.9$

Object likelihood : $g_k(o_k|x_k) = \mathcal{N}(o_k; x_k, 0.2)$

Clutter intensity : $\lambda(c) = \begin{cases} 0.4 & \text{if } |c| \leq 4 \\ 0 & \text{otherwise} \end{cases}$

Observed detections : $Z_1 = [-1.3, 1.7], \quad Z_2 = [1.3],$
 $Z_3 = [-0.3, 2.3], \quad Z_4 = [-0.7, 3]$
 $Z_5 = [-1], \quad Z_6 = [-1.3]$

A SECOND NEAREST NEIGHBOUR FILTERING EXAMPLE

Example: NN vs exact posterior

- We compare four densities:

$$p^{\text{NN}}(x_k | Z_{1:k-1}), \quad p(x_k | Z_{1:k})$$

$$\check{p}^{\text{NN}}(x_k | Z_{1:k}), \quad p^{\text{NN}}(x_k | Z_{1:k})$$

- We visualize the $m_k + 1$ hypotheses, and mark the **most probable in red**.
- The NN may lose track of the object in complicated scenarios.

NEAREST NEIGHBOUR FILTERING: SUMMARY

Basic idea

- Prune all hypotheses except the most probable one.

Pros and cons

- ++ A fast algorithm which is simple to implement.
- ++ Works well in simple scenarios.
- Ignores uncertainties which increases the risk that we will lose track of the object.
- Performs poorly in complicated scenarios.

Probabilistic data association filtering

Multi-Object Tracking

Lennart Svensson

PROBLEM SETTING

- PDA approximates posterior and predicted densities as **Gaussian** :

$$p^{\text{PDA}}(x_{k-1} | Z_{1:k-1}) = \mathcal{N}(x_{k-1}; \bar{x}_{k-1|k-1}^{\text{PDA}}, P_{k-1|k-1}^{\text{PDA}}),$$
$$p^{\text{PDA}}(x_k | Z_{1:k-1}) = \mathcal{N}(x_k; \bar{x}_{k|k-1}^{\text{PDA}}, P_{k|k-1}^{\text{PDA}}).$$

Measurement model (same as for NN)

We assume $P^{\text{D}}(x) = P^{\text{D}}$, $g_k(o|x) = \mathcal{N}(o; H_k x, R_k)$, general $\lambda_c(c)$.

Posterior density, given $p^{\text{PDA}}(x_k | Z_{1:k-1}) = \mathcal{N}(x_k; \bar{x}_{k|k-1}^{\text{PDA}}, P_{k|k-1}^{\text{PDA}})$

Posterior is $\check{p}^{\text{PDA}}(x_k | Z_{1:k}) = \sum_{\theta_k=0}^{m_k} w_k^{\theta_k} p_{k|k}^{\theta_k}(x_k)$ where $p_{k|k}^{\theta_k}(x_k) = \mathcal{N}(x_k; \hat{x}_k^{\theta_k}, P_k^{\theta_k})$.

- How can we **approximate** $\check{p}^{\text{PDA}}(x_k | Z_{1:k})$ as Gaussian?

PROBABILISTIC DATA ASSOCIATION FILTERING

Basic idea

- Approximate posterior as a Gaussian density with the same mean and covariance as $\check{p}^{\text{PDA}}(x_k | Z_{1:k})$.
- We set

$$\bar{x}_{k|k}^{\text{PDA}} = \mathbb{E}_{\check{p}^{\text{PDA}}(x_k | Z_{1:k})} [x_k]$$

$$P_{k|k}^{\text{PDA}} = \text{Cov}_{\check{p}^{\text{PDA}}(x_k | Z_{1:k})} [x_k]$$

- **Note 1:** we then assume that $p^{\text{PDA}}(x_k | Z_{1:k}) = \mathcal{N}(x_k; \bar{x}_{k|k}^{\text{PDA}}, P_{k|k}^{\text{PDA}})$.

PROBABILISTIC DATA ASSOCIATION FILTERING

Basic idea

- Approximate posterior as a Gaussian density with the same mean and covariance as $\check{p}^{\text{PDA}}(x_k | Z_{1:k})$.

- We set

$$\bar{x}_{k|k}^{\text{PDA}} = \mathbb{E}_{\check{p}^{\text{PDA}}(x_k | Z_{1:k})} [x_k] = \sum_{\theta_k=0}^{m_k} w_k^{\theta_k} \hat{x}_k^{\theta_k}$$
$$P_{k|k}^{\text{PDA}} = \text{Cov}_{\check{p}^{\text{PDA}}(x_k | Z_{1:k})} [x_k] = \sum_{\theta_k=0}^{m_k} \underbrace{w_k^{\theta_k} P_k^{\theta_k}}_{\text{average cov.}} + \underbrace{w_k^{\theta_k} \left(\bar{x}_{k|k}^{\text{PDA}} - \hat{x}_k^{\theta_k} \right) \left(\bar{x}_{k|k}^{\text{PDA}} - \hat{x}_k^{\theta_k} \right)^T}_{\text{spread of mean}}.$$

- **Note 2:** this minimizes the Kullback-Leibler divergence

$$\int \check{p}^{\text{PDA}}(x_k | Z_{1:k}) \log \frac{\check{p}^{\text{PDA}}(x_k | Z_{1:k})}{\mathcal{N}(x_k; \bar{x}_{k|k}^{\text{PDA}}, P_{k|k}^{\text{PDA}})} dx_k.$$

MOMENTS OF A GAUSSIAN MIXTURE

Moments of a Gaussian mixture

- Suppose

$$p(x) = 0.5\mathcal{N}(x; -3, 2) + 0.5\mathcal{N}(x; 3, 2).$$

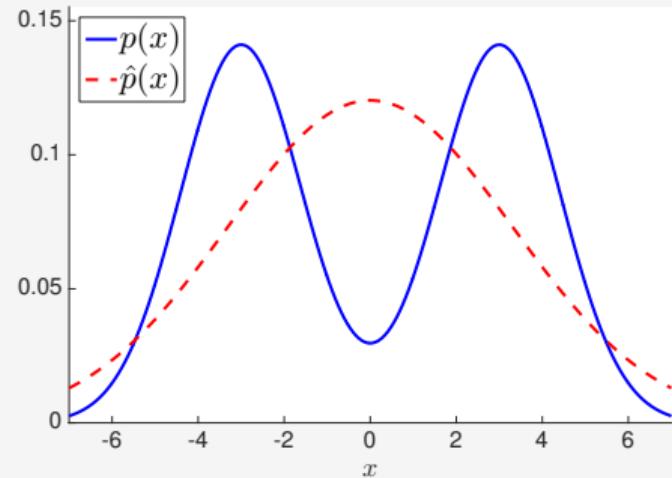
- It follows that

$$\mathbb{E}_{p(x)}[x] = 0.5 \times (-3) + 0.5 \times 3 = 0,$$

$$\begin{aligned}\text{Cov}_{p(x)}[x] &= \underbrace{0.5 \times 2 + 0.5 \times 2}_{=2} \\ &\quad + \underbrace{0.5 \times 3^2 + 0.5 \times (-3)^2}_{=9} = 11.\end{aligned}$$

- Similar to PDA, we can approximate $p(x)$ using

$$\hat{p}(x) = \mathcal{N}(x; 0, 11).$$



THE PDA FILTERING ALGORITHM

Basic idea

- Approximate posterior as a Gaussian density with the same mean and covariance as $\check{p}^{\text{PDA}}(x_k | Z_{1:k})$.

Algorithm The PDA filtering update.

1: Compute $w_k^{\theta_k}$, $\hat{x}_k^{\theta_k}$ and $P_k^{\theta_k}$, $\theta_k = 0, 1, \dots, m_k$.

2: Set

$$\bar{x}_{k|k}^{\text{PDA}} = \sum_{\theta_k=0}^{m_k} w_k^{\theta_k} \hat{x}_k^{\theta_k}.$$

3: Compute

$$P_{k|k}^{\text{PDA}} = \sum_{\theta_k=0}^{m_k} w_k^{\theta_k} P_k^{\theta_k} + w_k^{\theta_k} \left(\bar{x}_{k|k}^{\text{PDA}} - \hat{x}_k^{\theta_k} \right) \left(\bar{x}_{k|k}^{\text{PDA}} - \hat{x}_k^{\theta_k} \right)^T.$$

Probabilistic data association filtering – remarks and visualizations

Multi-Object Tracking

Lennart Svensson

PDA FILTERING: POSTERIOR MEAN

- For linear and Gaussian models, and $\bar{z}_{k|k-1} = H_k \bar{x}_{k|k-1}^{\text{PDA}}$,

$$\hat{x}_k^{\theta_k} = \begin{cases} \bar{x}_{k|k-1}^{\text{PDA}} & \text{if } \theta_k = 0 \\ \bar{x}_{k|k-1}^{\text{PDA}} + K_k(z_k^{\theta_k} - \bar{z}_{k|k-1}) & \text{if } \theta_k \in \{1, 2, \dots, m_k\}. \end{cases}$$

- Hence, the **posterior mean** is

$$\begin{aligned} \bar{x}_{k|k}^{\text{PDA}} &= \sum_{\theta_k=0}^{m_k} w_k^{\theta_k} \hat{x}_k^{\theta_k} \\ &= \sum_{\theta_k=0}^{m_k} w_k^{\theta_k} \bar{x}_{k|k-1}^{\text{PDA}} + \sum_{\theta_k=1}^{m_k} w_k^{\theta_k} K_k(z_k^{\theta_k} - \bar{z}_{k|k-1}) \\ &= \bar{x}_{k|k-1}^{\text{PDA}} + K_k \underbrace{\sum_{\theta_k=1}^{m_k} w_k^{\theta_k} (z_k^{\theta_k} - \bar{z}_{k|k-1})}_{\text{expected innovation}}. \end{aligned}$$

A FIRST PDA FILTERING EXAMPLE

Example: PDA vs NN and exact posterior

- We compare five densities:

$$\begin{aligned} p^{\text{PDA}}(x_k | Z_{1:k-1}), \quad p(x_k | Z_{1:k}), \\ \check{p}^{\text{PDA}}(x_k | Z_{1:k}), \quad p^{\text{PDA}}(x_k | Z_{1:k}), \\ p^{\text{NN}}(x_k | Z_{1:k}). \end{aligned}$$

- We visualize the $m_k + 1$ hypotheses.
- PDA yields larger posterior uncertainties than NN filtering.

A SECOND PDA FILTERING EXAMPLE

Example: PDA vs NN and exact posterior

- We compare five densities:

$$\begin{aligned} p^{\text{PDA}}(x_k | Z_{1:k-1}), \quad p(x_k | Z_{1:k}), \\ \check{p}^{\text{PDA}}(x_k | Z_{1:k}), \quad p^{\text{PDA}}(x_k | Z_{1:k}), \\ p^{\text{NN}}(x_k | Z_{1:k}). \end{aligned}$$

- We visualize the $m_k + 1$ hypotheses.
- In difference to NN, the PDA algorithm did not lose track of object.

PROBABILISTIC DATA ASSOCIATION: SUMMARY

Basic idea

- Approximate posterior as a Gaussian density with the same mean and covariance as $\check{p}^{\text{PDA}}(x_k | Z_{1:k})$.

Pros and cons

- ++ A fast algorithm which is simple to implement.
- ++ Works well in simple scenarios.
- ++ Acknowledges uncertainties slightly better than NN.
- Performs poorly in complicated scenarios.

Gaussian sum filtering

Multi-Object Tracking

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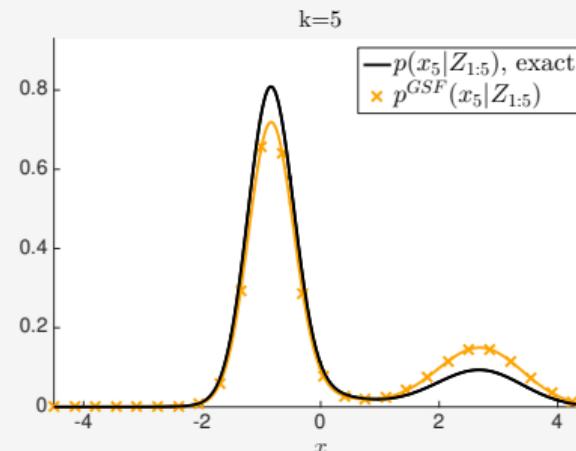
GAUSSIAN SUM FILTERING (1)

Gaussian sum filtering

- **Basic idea:** approximate the posterior as a Gaussian mixture with a few components.

Example

- In the figure to the right, the posterior contains 108 hypotheses.
- We prune all but 5 hypotheses (at all times).
- Approximation is significantly better than PDA and NN.



GAUSSIAN SUM FILTERING (2)

Gaussian sum filtering

- **Basic idea:** approximate the posterior as a Gaussian mixture with a few components.

Prediction and update of a Gaussian mixture

- Suppose $p_{k-1|k-1}^{h_{k-1}}(x_{k-1}) = \mathcal{N}(x_{k-1}; \hat{x}_{k-1|k-1}^{h_{k-1}}, P_{k-1|k-1}^{h_{k-1}})$ and

$$p^{GSF}(x_{k-1} | Z_{1:k-1}) = \sum_{h_{k-1}=1}^{\mathcal{H}_{k-1}} w_{k-1}^{h_{k-1}} p_{k-1|k-1}^{h_{k-1}}(x_{k-1}).$$

- Assuming “linear and Gaussian” models, posterior at time k is a Gaussian mixture

$$\check{p}^{GSF}(x_k | Z_{1:k}) = \sum_{h_k=1}^{\mathcal{H}_{k-1} \times (m_k+1)} \check{w}_k^{h_k} \check{p}_{k|k}^{h_k}(x_k).$$

- How can we approximate $\check{p}^{GSF}(x_k | Z_{1:k})$ as a Gaussian mixture with **fewer terms**?

PRUNING HYPOTHESES WITH SMALL WEIGHTS

Basic idea

- Prune all hypotheses whose weights are smaller than a threshold γ .

Example

- Suppose

$$p(x) = 0.7\mathcal{N}(x; \hat{x}^1, P^1) + 0.005\mathcal{N}(x; \hat{x}^2, P^2) + 0.295\mathcal{N}(x; \hat{x}^3, P^3)$$

and that $\gamma = 0.01$.

- Pruning then yields

$$\begin{aligned} p(x) \approx \hat{p}(x) &= \frac{0.7}{0.295 + 0.7} \mathcal{N}(x; \hat{x}^1, P^1) + \frac{0.295}{0.295 + 0.7} \mathcal{N}(x; \hat{x}^3, P^3) \\ &= \hat{w}_1 \mathcal{N}(x; \hat{x}^1, \hat{P}^1) + \hat{w}_2 \mathcal{N}(x; \hat{x}^2, \hat{P}^2). \end{aligned}$$

MERGING SIMILAR COMPONENTS

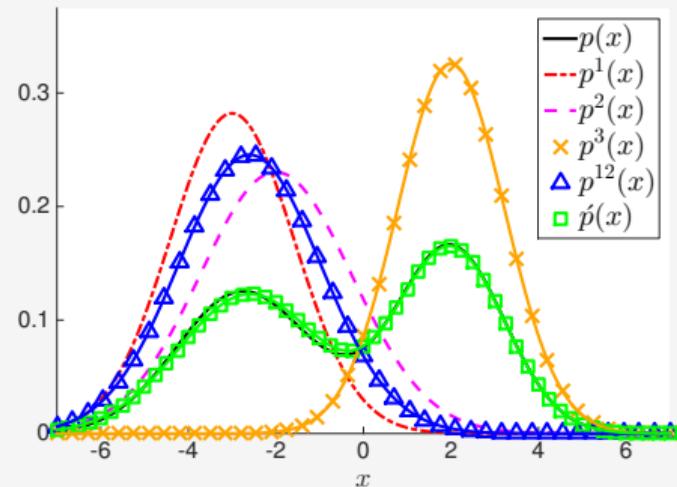
Merging two out of three components

- Suppose $p^1(x)$ and $p^2(x)$ are similar.
- Setting $w^{12} = w^1 + w^2$ we get

$$\begin{aligned} p(x) &= w^1 p^1(x) + w^2 p^2(x) + w^3 p^3(x) \\ &= w^{12} \left(\underbrace{\frac{w^1 p^1(x)}{w^{12}} + \frac{w^2 p^2(x)}{w^{12}}}_{\approx p^{12}(x), \text{ see PDA}} \right) + w^3 p^3(x) \\ &\approx w^{12} p^{12}(x) + w^3 p^3(x) = \hat{p}(x) \end{aligned}$$

- We select $p^{12}(x)$ to match moments of

$$\frac{w^1 p^1(x)}{w^{12}} + \frac{w^2 p^2(x)}{w^{12}}.$$



CAPPING THE NUMBER OF HYPOTHESES

Basic idea

- Prune hypotheses until we are left with at most N_{\max} hypotheses.

Algorithm Capping the number of hypotheses.

- 1: Input: N_{\max} , $w^i, \hat{x}^i, P^i, i = 1, \dots, \mathcal{H} > N_{\max}$.
 - 2: Output: $\acute{w}^i, \acute{x}^i, \acute{P}^i, i = 1, \dots, \acute{\mathcal{H}} = N_{\max}$
 - 3: $[out, ind] = \text{sort}([w^1, \dots, w^{\mathcal{H}}], \text{'descend'})$.
 - 4: % Gives a list 'ind' with indexes $w^{ind(1)} \geq w^{ind(2)} \geq \dots \geq w^{ind(\mathcal{H})}$.
 - 5: Compute $c = \sum_{i=1}^{N_{\max}} w_{k|k}^{ind(i)}$.
 - 6: **for** $i = 1$ **to** N_{\max} **do**
 - 7: Set $\acute{w}^i = w^{ind(i)}/c$, $\acute{x}^i = \hat{x}^{ind(i)}$ and $\acute{P}^i = P^{ind(i)}$.
 - 8: **end for**
-

SUMMARY OF MIXTURE REDUCTION STRATEGIES

- We have described three ways to reduce the number of hypotheses in

$$\check{p}^{GSF}(x_k | Z_{1:k}) = \sum_{h_k=1}^{\check{\mathcal{H}}_k} \check{w}_k^{h_k} \check{p}_{k|k}^{h_k}(x_k).$$

- These can be combined in different ways, e.g.,
 1. cap the number of hypotheses at N_{\max} , or,
 2. we can
 - i) remove hypotheses with weights $< \gamma$,
 - ii) merge similar components, and then,
 - iii) cap the number of hypotheses at N_{\max} .
 - The resulting Gaussian mixture is the GSF posterior

$$p^{GSF}(x_k | Z_{1:k}) = \sum_{h_k=1}^{\mathcal{H}_k} w_k^{h_k} p_{k|k}^{h_k}(x_k).$$

Gaussian sum filtering – prediction and update equations

Multi-Object Tracking

Lennart Svensson

PREDICTION AND UPDATE EQUATIONS

Prediction and update equations for Gaussian sum filters

- Suppose

$$p^{GSF}(x_{k-1} | Z_{1:k-1}) = \sum_{h_{k-1}=1}^{\mathcal{H}_{k-1}} w_{k-1}^{h_{k-1}} p_{k-1|k-1}^{h_{k-1}}(x_{k-1}).$$

- It then follows that the predicted and updated densities are

$$p^{GSF}(x_k | Z_{1:k-1}) = \sum_{h_{k-1}=1}^{\mathcal{H}_{k-1}} w_{k-1}^{h_{k-1}} p_{k|k-1}^{h_{k-1}}(x_k)$$

$$\check{p}^{GSF}(x_k | Z_{1:k}) = \sum_{h_k=1}^{\check{\mathcal{H}}_k} \check{w}_k^{h_k} \check{p}_{k|k}^{h_k}(x_k),$$

where $\check{\mathcal{H}}_k = \mathcal{H}_{k-1} \times (m_k + 1)$.

- In this video, we present equations for computing $p_{k|k-1}^{h_{k-1}}(x_k)$, $\check{w}_k^{h_k}$ and $\check{p}_{k|k}^{h_k}(x_k)$.

PREDICTION STEP

Chapman-Kolmogorov for every hypothesis

- For $h_{k-1} = 1, 2, \dots, \mathcal{H}_{k-1}$,

$$p_{k|k-1}^{h_{k-1}}(x_k) = \int \pi_k(x_k | x_{k-1}) p_{k-1|k-1}^{h_{k-1}}(x_{k-1}) dx_{k-1}.$$

Linear and Gaussian prediction

- If

$$\begin{cases} p_{k-1|k-1}^{h_{k-1}}(x_{k-1}) = \mathcal{N}(x_{k-1}; \hat{x}_{k-1|k-1}^{h_{k-1}}, P_{k-1|k-1}^{h_{k-1}}) dx_{k-1} \\ x_k = F_{k-1}x_{k-1} + q_{k-1}, \quad q_{k-1} \sim \mathcal{N}(0, Q_{k-1}), \end{cases}$$

then

$$p_{k|k-1}^{h_{k-1}}(x_k) = \mathcal{N}(x_k; \hat{x}_{k|k-1}^{h_{k-1}}, P_{k|k-1}^{h_{k-1}})$$

where

$$\hat{x}_{k|k-1}^{h_{k-1}} = F_{k-1}\hat{x}_{k-1|k-1}^{h_{k-1}}, \quad P_{k|k-1}^{h_{k-1}} = F_{k-1}P_{k-1|k-1}^{h_{k-1}}F_{k-1}^T + Q_{k-1}.$$

UPDATE STEP (1)

Updated weights and densities

- For every pair of hypotheses

$$h_{k-1} \in \{1, 2, \dots, \mathcal{H}_{k-1}\} \quad \text{and} \quad \theta_k \in \{0, 1, \dots, m_k\}$$

we obtain a new hypothesis h_k :

$$\check{w}_{k|k}^{h_k} \propto \begin{cases} w_{k-1}^{h_{k-1}} \int (1 - P^D(x_k)) p_{k|k-1}^{h_{k-1}}(x_k) dx_k & \text{if } \theta_k = 0 \\ \frac{w_{k-1}^{h_{k-1}}}{\lambda_c(z_k^{\theta_k})} \int p_{k|k-1}^{h_{k-1}}(x_k) P^D(x_k) g_k(z_k^{\theta_k} | x_k) dx_k & \text{if } \theta_k \in \{1, 2, \dots, m_k\}, \end{cases}$$

$$\check{p}_{k|k}^{h_k}(x_k) \propto \begin{cases} p_{k|k-1}^{h_{k-1}}(x_k)(1 - P^D(x_k)) & \text{if } \theta_k = 0 \\ p_{k|k-1}^{h_{k-1}}(x_k) P^D(x_k) g_k(z_k^{\theta_k} | x_k) & \text{if } \theta_k \in \{1, 2, \dots, m_k\}. \end{cases}$$

UPDATE STEP (2)

Updated weights and densities

- When P^D is constant and g_k is linear and Gaussian:

$$\check{w}_{k|k}^{h_k} \propto \begin{cases} w_{k-1}^{h_{k-1}} (1 - P^D(x_k)) & \text{if } \theta_k = 0 \\ \frac{w_{k-1}^{h_{k-1}} P^D \mathcal{N}(z_k^{\theta_k}; \bar{z}_{k|k-1}^{h_{k-1}}, S_{k,h_{k-1}})}{\lambda_c(z_k^{\theta_k})} & \text{if } \theta_k \in \{1, 2, \dots, m_k\}, \end{cases}$$

$$\check{p}_{k|k}^{h_k}(x_k) = \mathcal{N}(x_k; \check{x}_{k|k}^{h_k}, \check{P}_{k|k}^{h_k}) \text{ where}$$

$$\theta_k = 0 : \quad \begin{cases} \check{x}_{k|k}^{h_k} = \hat{x}_{k|k-1}^{h_{k-1}} \\ \check{P}_{k|k}^{h_k} = P_{k|k-1}^{h_{k-1}}, \end{cases}$$

$$\theta_k \in \{1, 2, \dots, m_k\} : \quad \begin{cases} \check{x}_{k|k}^{h_k} = \hat{x}_{k|k-1}^{h_{k-1}} + K_k^{h_{k-1}} (z_k^{\theta_k} - \bar{z}_{k|k-1}^{h_{k-1}}) \\ \check{P}_{k|k}^{h_k} = P_{k|k-1}^{h_{k-1}} - K_k^{h_{k-1}} H_k P_{k|k-1}^{h_{k-1}}. \end{cases}$$

UPDATE STEP (3)

- We obtain a hypothesis h_k for every pair of h_{k-1} and θ_k , but how can we index h_k ?
- Two possibilities:

$$1) \quad h_k = h_{k-1} + \mathcal{H}_{k-1}\theta_k$$

$$2) \quad h_k = 1 + \theta_k + (m_k + 1)(h_{k-1} - 1).$$

Both ensure that we have a one-to-one mapping between (h_{k-1}, θ_k) and h_k .

Indexing four new hypotheses

- If $\mathcal{H}_{k-1} = 2$, $m_k = 1$ and $h_k = h_{k-1} + \mathcal{H}_{k-1}\theta_k$:

$$h_{k-1} = 1, \theta_k = 0 \Leftrightarrow h_k = 1$$

$$h_{k-1} = 2, \theta_k = 0 \Leftrightarrow h_k = 2$$

$$h_{k-1} = 1, \theta_k = 1 \Leftrightarrow h_k = 3$$

$$h_{k-1} = 2, \theta_k = 1 \Leftrightarrow h_k = 4.$$

Gaussian sum filtering – estimation and visualizations

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STATE ESTIMATION

- If the posterior is a Gaussian mixture, how can we estimate x_k ?

Minimum mean square error (MMSE) estimation

- The posterior mean

$$\bar{x}_{k|k} = \mathbb{E} [x_k | \mathcal{Z}_{1:k}] = \sum_{h_k=1}^{\mathcal{H}_k} w_{k|k}^{h_k} \hat{x}_{k|k}^{h_k}$$

minimizes the MMSE, $\mathbb{E} [(x_k - \bar{x}_{k|k})^T (x_k - \bar{x}_{k|k}) | \mathcal{Z}_{1:k}]$.

Most probable hypothesis estimation

- For multi-modal densities, we sometimes prefer

$$h_k^* = \arg \max_h w_{k|k}^h$$

$$\hat{x}_{k|k} = \hat{x}_{k|k}^{h_k^*}.$$

A FIRST GSF FILTERING EXAMPLE

Example: GSF vs exact posterior

- We compare four densities:

$$p^{\text{GSF}}(x_k | Z_{1:k-1}), \quad p(x_k | Z_{1:k}),$$
$$\check{p}^{\text{GSF}}(x_k | Z_{1:k}), \quad p^{\text{GSF}}(x_k | Z_{1:k}).$$

- The number of hypotheses is capped at $N_{\max} = 5$.
- The GSF filter approximates the posterior well.

A SECOND GSF FILTERING EXAMPLE

Example: GSF vs exact posterior

- We compare four densities:

$$p^{\text{GSF}}(x_k | Z_{1:k-1}), \quad p(x_k | Z_{1:k}),$$
$$\check{p}^{\text{GSF}}(x_k | Z_{1:k}), \quad p^{\text{GSF}}(x_k | Z_{1:k}).$$

- The number of hypotheses is capped at $N_{\max} = 5$.
- The GSF approximates the posterior significantly better than NN and PDA.

GAUSSIAN SUM FILTERING – SUMMARY

Basic idea

- Approximate the posterior as a Gaussian mixture with a few components.

Pros and cons

- ++ Significantly more accurate than NN and PDA.
- ++ Complexity can be adjusted to computational resources.
- More complicated to implement than NN/PDA.
- More computationally demanding to run than NN/PDA.

- **Note:** even though GSFs looks much more accurate than NN and PDA, the difference is mostly noticeable in medium-difficult settings.

Gating to remove unlikely hypotheses

Multi-Object Tracking

Lennart Svensson

MOTIVATION

PDA with large m_k

- Suppose we have an amazing sensor:
 - large P^D ,
 - small λ_c ,
 - huge field of view.
- Excellent conditions, but m_k may be **very large**.
- Do we have to compute $w_k^{\theta_k}$, $\hat{x}_k^{\theta_k}$ and $P_k^{\theta_k}$ for hypotheses θ_k : $w_k^{\theta_k} \approx 0$?
- **Gating enables us to avoid this!** (Not only for PDA.)

- PDA:

$$\bar{x}_{k|k}^{\text{PDA}} = \sum_{\theta_k=0}^{m_k} w_k^{\theta_k} \hat{x}_k^{\theta_k}$$

$$P_{k|k}^{\text{PDA}} = \sum_{\theta_k=0}^{m_k} w_k^{\theta_k} P_k^{\theta_k}$$

$$+ w_k^{\theta_k} \left(\bar{x}_{k|k}^{\text{PDA}} - \hat{x}_k^{\theta_k} \right) \left(\bar{x}_{k|k}^{\text{PDA}} - \hat{x}_k^{\theta_k} \right)^T$$

BASIC IDEA

Idea

- Form a gate around the predicted measurement, and only consider detections within the gate.
- Gating leads to much fewer local hypotheses.
- A gate may be designed in many different ways, e.g., rectangular.
- Here, we study one which is natural for Gaussian distributions, namely the **ellipsoidal gate**.

ELLIPSOIDAL GATES: MOTIVATION AND DEFINITION

- We consider $\theta_k > 0$. Recall that

$$\tilde{w}_k^{h_k} = \frac{P^D(x_k) \mathcal{N}(z_k^{\theta_k}; \bar{z}_{k|k}^{h_{k-1}}, S_{k,h_{k-1}})}{\lambda_c(z_k^{\theta_k})}.$$

- We note that $\tilde{w}_k^{h_k}$ is “small” when the distance

$$d_{h_{k-1}, \theta_k}^2 = (z_k^{\theta_k} - \bar{z}_{k|k}^{h_{k-1}})^T S_{k,h_{k-1}}^{-1} (z_k^{\theta_k} - \bar{z}_{k|k}^{h_{k-1}})$$

is large (if $\lambda_c \approx \text{constant}$).

Ellipsoidal gate

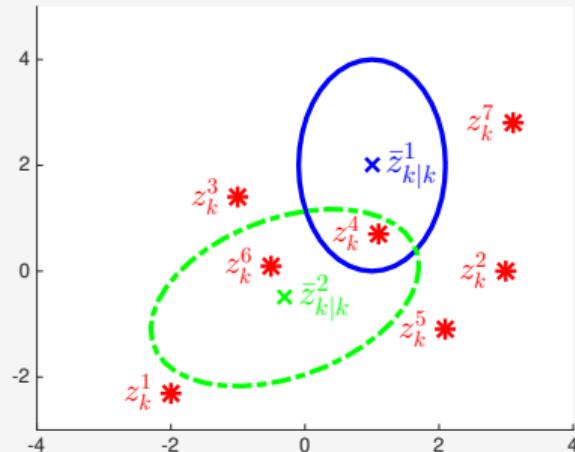
- Disregard $z_k^{\theta_k}$ as a clutter detection under hypothesis h_{k-1} , if

$$d_{h_{k-1}, \theta_k}^2 > G.$$

VISUALIZING GATING

Example: gating 2D measurements

- We have seven measurements and two predicted hypotheses, $h_{k-1} = 1$ and $h_{k-1} = 2$.
- The ellipsoids illustrate the two gates.
- For $h_{k-1} = 1$ only z_k^4 is inside the gate.
- For $h_{k-1} = 2$ all measurements except z_k^4 and z_k^6 are outside the gate.



SELECTING THE THRESHOLD G

- If G is small, we may have a “large” probability that the object detection is outside the gate.
- Given h_{k-1} and θ_k , where $\theta_k > 0$, the probability that the object measurement is inside the gate is

$$P_G = \Pr \left[d_{h_{k-1}, \theta_k}^2 < G \middle| h_{k-1}, \theta_k \right].$$

- One can show that

$$d_{h_{k-1}, \theta_k}^2 | h_{k-1}, \theta_k \sim \chi^2(n_z).$$

- A common strategy is to set a desired value for P_G , say, 99.5%, and then use the cumulative distribution of $\chi^2(n_z)$ to find G .

GATING – A SUMMARY

- Gating is a technique to disregard measurements as clutter (given h_{k-1}) without computing the weights.
- Gating can be combined with all tracking algorithms presented later.
- In Gaussian settings, the ellipsoidal gate is a natural choice which is simple to implement.
- It is important to find a reasonable value for the threshold G .