现代控制理论

——切换系统分析与综合

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本节基本内容

1. 马尔可夫跳变线性系统
2. 均方稳定性判据

马尔可夫链

- 离散状态、离散时间的马尔可夫过程
- 随机变量序列 $X = \{X_k : k > 0\}$ 满足

$$\mathrm{P}\left(X_{k+1} = x | X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k\right) = \mathrm{P}\left(X_{k+1} = x | X_k = x_k\right)$$

- 状态空间：X_i 的可能值构成的可数集 $\mathcal{N}$
- 齐次或者时齐：

$$\mathrm{P}\left(X_{k+1} = x | X_k = y\right) = \mathrm{P}\left(X_k = x | X_{k-1} = y\right) \ \forall k$$

- 不可约性、重现性、周期性、遍历性

马尔可夫过程

有限状态空间的齐次马尔可夫链

- 转移矩阵 $P = [p_{ij}], p_{ij} = P(X_{k+1} = j | X_k = i)$
- P^k：k 步转移后的转移矩阵
- $P(X_k = j) = \pi_{j,k} \Rightarrow$

$$\begin{bmatrix}\pi_{1,k+1} & \dots & \pi_{n,k+1}\end{bmatrix} = \begin{bmatrix}\pi_{1,k} & \dots & \pi_{n,k}\end{bmatrix} P = \begin{bmatrix}\pi_{1,1} & \dots & \pi_{n,1}\end{bmatrix} P^k$$

- 遍历性 $\Rightarrow$ 平稳分布：存在 π 使得 $\pi P = \pi$

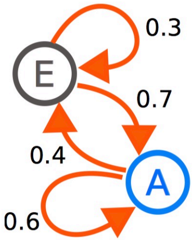
马尔可夫过程

有限状态空间的齐次马尔可夫链

- 转移矩阵 $P=[p_{ij}], p_{ij}=P(X_{k+1}=j|X_k=i)$
- P^k: k 步转移后的转移矩阵
- $P(X_k=j)=\pi_{j,k} \Rightarrow$

$$\begin{bmatrix}\pi_{1,k+1} & \dots & \pi_{n,k+1}\end{bmatrix}=\begin{bmatrix}\pi_{1,k} & \dots & \pi_{n,k}\end{bmatrix}P=\begin{bmatrix}\pi_{1,1} & \dots & \pi_{n,1}\end{bmatrix}P^k$$

- 遍历性 $\Rightarrow$ 平稳分布: 存在 π 使得 $\pi P=\pi$
- 例子:



The transition matrix is

$$P=\begin{bmatrix}0.3 & 0.7\\ 0.4 & 0.6\end{bmatrix}$$

with

$$\mathcal{N}=\{1,2\}$$

平稳分布 $\pi=\begin{bmatrix}4/11 & 7/11\end{bmatrix}$; $\lim\limits_{k\to\infty}P^k=\begin{bmatrix}4/11 & 7/11\\ 4/11 & 7/11\end{bmatrix}$

马尔可夫过程

有限状态空间的齐次马尔可夫链

- 转移矩阵 $P = [p_{ij}], p_{ij} = P(X_{k+1} = j | X_k = i)$
- P^k：k 步转移后的转移矩阵
- $P(X_k = j) = \pi_{j,k} \Rightarrow$

$$\begin{bmatrix}\pi_{1,k+1} & \dots & \pi_{n,k+1}\end{bmatrix} = \begin{bmatrix}\pi_{1,k} & \dots & \pi_{n,k}\end{bmatrix} P = \begin{bmatrix}\pi_{1,1} & \dots & \pi_{n,1}\end{bmatrix} P^k$$

- 遍历性 $\Rightarrow$ 平稳分布：存在 π 使得 $\pi P = \pi$

- 特例—**伯努利过程**：状态空间 $\{0, 1\}$, 独立同分布

  Q: 假设伯努利过程的状态空间为 $\{1, 2\}$ 且 $P(X_i = 1) = \bar{p}$，则伯努利过程的转移矩阵是什么样的?

马尔可夫跳变线性系统

系统描述

$$x(k+1) = A_{\gamma(k)}x(k),\ x(0) = x_0,\ \gamma(0) = \gamma_0 \tag{1}$$

- $x(k) \in \mathbb{R}^n$
- $A_{\gamma(k)} \in \{A_1, \ldots, A_m\}$; 当 $\gamma(k) = i$, $A_{\gamma(k)} = A_i$
- $\gamma(k) : \mathbb{R} \to \{1, 2, \ldots, m\}$ 马尔可夫链
  - 有限状态空间、齐次
  - 转移矩阵 $P = [p_{ij}]$:

$$p_{ij} = P(\gamma(k+1) = j|\gamma(k) = i)$$

  - $P(\gamma(k) = j) = \pi_{j,k}$; 初始分布 $\pi_{j,0}$ 给定

马尔可夫跳变线性系统

随机系统稳定性

- **平均稳定**：$\lim_{k\to\infty} E[x(k))] = 0$
- **均方稳定**：$\lim_{k\to\infty} E\left[x(k)x^T(k)\right] = 0$
- **依概率收敛**：$\mathrm{P}\left(\|x(k)\| \geq \epsilon\right) \to 0 \ \forall \epsilon > 0$
- **几乎处处收敛**：$\mathrm{P}\left(\lim_{k\to\infty} \|x(k)\| = 0\right) = 1$

$$\begin{array}{ccc} \text{均方稳定} & \Rightarrow & \text{平均稳定} \\ \Downarrow & & \Downarrow \\ \text{几乎处处收敛} & \Rightarrow & \text{依概率收敛} \end{array}$$

均方稳定性判据

$E\left[x(k)x(k)^T\right]$ **的动态特性:**

①

$$1_{\{\gamma(k)=i\}} = \begin{cases} 1 & \text{if } \gamma(k) = i \\ 0 & \text{otherwise} \end{cases}$$

$$C_k^i \triangleq E\left[x(k)x^T(k)1_{\{\gamma(k)=i\}}\right] = \pi_{i,k}E\left[x(k)x^T(k)|\gamma_k = i\right]$$

$$\begin{aligned}\Longrightarrow E\left[x(k)x^T(k)\right] &= E\left[x(k)x^T(k)\sum_{i=1}^{m}1_{\{\gamma(k)=i\}}\right] \\ &= \sum_{i=1}^{m}E\left[x(k)x^T(k)1_{\{\gamma(k)=i\}}\right] = \sum_{i=1}^{m}C_k^i\end{aligned}$$

$\Longrightarrow$ **建立** $C^i(k), i = 1, \ldots, m$ **的动态特性**

均方稳定性判据

② $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

$$
\begin{aligned}
\Longrightarrow C_k^i &= \pi_{i,k} E\left[x(k)x^T(k)|\gamma_k = i\right] \\
&= \pi_{i,k} E\left[x(k)x^T(k)\sum_{j=1}^{m} 1_{\{\gamma(k-1)=j\}}|\gamma_k = i\right] \\
&= \sum_{j=1}^{m} P(\gamma_{k-1} = j|\gamma_k = i)P(\gamma_k = i)E\left[x(k)x^T(k)|\gamma_k = i, \gamma_{k-1} = j\right] \\
&= \sum_{j=1}^{m} P(\gamma_{k-1} = j, \gamma_k = i)E\left[x(k)x^T(k)|\gamma_k = i, \gamma_{k-1} = j\right] \\
&= \sum_{j=1}^{m} P(\gamma_k = i|\gamma_{k-1} = j)P(\gamma_{k-1} = j)E\left[x(k)x^T(k)|\gamma_{k-1} = j\right] \\
&= \sum_{j=1}^{m} p_{ji}\pi_{j,k-1} E\left[x(k)x^T(k)|\gamma_{k-1} = j\right]
\end{aligned}
$$

均方稳定性判据

③

$$\gamma_{k-1} = j \Longrightarrow x(k) = A_j x(k-1)$$

$$\begin{aligned}
\Longrightarrow C_k^i &= \sum_{j=1}^{m} p_{ji}\pi_{j,k-1} E\left[x(k)x^T(k)|\gamma_{k-1}=j\right] \\
&= \sum_{j=1}^{m} p_{ji}\pi_{j,k-1} A_j E\left[x(k-1)x^T(k-1)|\gamma_{k-1}=j\right] A_j^T \\
&= \sum_{j=1}^{m} p_{ji} A_j C_{k-1}^j A_j^T \quad \text{动态特性}
\end{aligned}$$

均方稳定性判据

④ $\mathrm{vec}(ABC) = \left(C^T \otimes A\right) \mathrm{vec}(B)$

$$\mathrm{vec}(C_k^i) = \sum_{j=1}^{m} p_{ji}(A_j \otimes A_j)\mathrm{vec}(C_{k-1}^j)$$

$$\Downarrow$$

$$\begin{bmatrix} \mathrm{vec}(C_k^1) \\ \mathrm{vec}(C_k^2) \\ \vdots \\ \mathrm{vec}(C_k^m) \end{bmatrix} = \begin{bmatrix} p_{11}(A_1 \otimes A_1) & p_{21}(A_2 \otimes A_2) & \cdots & p_{m1}(A_m \otimes A_m) \\ p_{12}(A_1 \otimes A_1) & p_{22}(A_2 \otimes A_2) & \cdots & p_{m2}(A_m \otimes A_m) \\ \vdots & \vdots & & \vdots \\ p_{1m}(A_1 \otimes A_1) & p_{2m}(A_2 \otimes A_2) & \cdots & p_{mm}(A_m \otimes A_m) \end{bmatrix} \begin{bmatrix} \mathrm{vec}(C_{k-1}^1) \\ \mathrm{vec}(C_{k-1}^2) \\ \vdots \\ \mathrm{vec}(C_{k-1}^m) \end{bmatrix}$$

$$= \begin{bmatrix} p_{11}I & p_{21}I & \cdots & p_{m1}I \\ p_{12}I & p_{22}I & \cdots & p_{m2}I \\ \vdots & \vdots & & \vdots \\ p_{1m}I & p_{2m}I & \cdots & p_{mm}I \end{bmatrix} \begin{bmatrix} A_1 \otimes A_1 & & & \\ & A_2 \otimes A_2 & & \\ & & \ddots & \\ & & & A_m \otimes A_m \end{bmatrix} \begin{bmatrix} \mathrm{vec}(C_{k-1}^1) \\ \mathrm{vec}(C_{k-1}^2) \\ \vdots \\ \mathrm{vec}(C_{k-1}^m) \end{bmatrix}$$

$$= (P^T \otimes I)\mathrm{diag}(A_i \otimes A_i) \begin{bmatrix} \mathrm{vec}(C_{k-1}^1) \\ \mathrm{vec}(C_{k-1}^2) \\ \vdots \\ \mathrm{vec}(C_{k-1}^m) \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{vec}(C_k^1) \\ \mathrm{vec}(C_k^2) \\ \vdots \\ \mathrm{vec}(C_k^m) \end{bmatrix} = (P^T \otimes I)\mathrm{diag}(A_i \otimes A_i) \begin{bmatrix} \mathrm{vec}(C_{k-1}^1) \\ \mathrm{vec}(C_{k-1}^2) \\ \vdots \\ \mathrm{vec}(C_{k-1}^m) \end{bmatrix}$$

⑤ **均方稳定性:**

$$\lim_{k\to\infty} E\left[x(k)x^T(k)\right] = \lim_{k\to\infty}\sum_{i=1}^{m} C_k^i = 0 \Longleftrightarrow \lim_{k\to\infty}\begin{bmatrix} \mathrm{vec}(C_k^1) \\ \mathrm{vec}(C_k^2) \\ \vdots \\ \mathrm{vec}(C_k^m) \end{bmatrix} = 0$$

$$\Longleftrightarrow \rho((P^T \otimes I)\mathrm{diag}(A_i \otimes A_i)) < 1$$

均方稳定性判据

定理 1

马尔可夫跳变线性系统 *(1)* 是均方稳定的当且仅当

$$\rho\left((P^T \otimes I)diag(A_i \otimes A_i)\right) < 1$$

定理 2

当对任意 $i, j = 1, \ldots, m$, $p_{ij} = p_j$ 成立，则马尔可夫跳变线性系统 *(1)* 是均方稳定的当且仅当

$$\rho\left(E(A_i \otimes A_i)\right) \triangleq \rho\left(\sum_{i=1}^{m} p_i(A_i \otimes A_i)\right) < 1$$

- 伯努利过程：$m = 2$
- 当 $m = 1$：$\rho\left(A_1 \otimes A_1\right) < 1 \Leftrightarrow \rho\left(A_1\right) < 1$
  与经典线性系统结论一致

均方稳定性判据

定理 2 证明

- $p_{ij} = p_j \Longrightarrow \pi_{i,k} = p_i$
- $\gamma(k)$ 和 $x(k)$ 独立 $\Longrightarrow$

$$C_k^i = \pi_{i,k} E\left[x(k)x^T(k)|\gamma_k = i\right] = p_i E\left[x(k)x^T(k)\right]$$

-

$$\begin{aligned}
\mathrm{vec}(C_k^i) &= \sum_{j=1}^{m} p_{ji}(A_j \otimes A_j)\mathrm{vec}(C_{k-1}^j) \\
&\Longrightarrow p_i \mathrm{vec}(E\left[x(k)x^T(k)\right]) = \sum_{j=1}^{m} p_i(A_j \otimes A_j) p_j \mathrm{vec}(E\left[x(k-1)x^T(k-1)\right] \\
&\Longrightarrow \mathrm{vec}(E\left[x(k)x^T(k)\right]) = \sum_{j=1}^{m} p_j(A_j \otimes A_j)\mathrm{vec}(E\left[x(k-1)x^T(k-1)\right]) \\
&\Longrightarrow \mathrm{vec}(E\left[x(k)x^T(k)\right]) \to 0 \text{ 当且仅当 } \rho\left(\sum_{i=1}^{m} p_i(A_i \otimes A_i)\right) < 1
\end{aligned}$$

均方稳定性判据

例子 1:

- 两个子系统: $A_1 = \frac{4}{3}$(不稳定), $A_2 = \frac{1}{3}$(稳定)
  -
  $$\begin{aligned} P_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} &\Rightarrow (P_1^T \otimes I)\mathrm{diag}(A_i \otimes A_i) = \frac{1}{2}\begin{bmatrix} \frac{16}{9} & \frac{1}{9} \\ \frac{16}{9} & \frac{1}{9} \end{bmatrix} \\ &\Rightarrow \rho\left((P_1^T \otimes I)\mathrm{diag}(A_i \otimes A_i)\right) = \frac{17}{18} < 1 \\ &\Rightarrow \text{系统是均方稳定} \end{aligned}$$
  -
  $$\begin{aligned} P_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix} &\Rightarrow (P_2^T \otimes I)\mathrm{diag}(A_i \otimes A_i) = \begin{bmatrix} \frac{144}{90} & \frac{1}{10} \\ \frac{16}{9} & \frac{1}{9} \end{bmatrix} \\ &\Rightarrow \rho\left((P_2^T \otimes I)\mathrm{diag}(A_i \otimes A_i)\right) = 1.61 > 1 \\ &\Rightarrow \text{系统不是均方稳定} \end{aligned}$$

例子 2:

- 两个不稳定子系统 $A_1 = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$
  - $P = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}$

    Q: 请计算 $(P^T \otimes I)\text{diag}(A_i \otimes A_i)$
  - $\rho\left((P^T \otimes I)\text{diag}(A_i \otimes A_i)\right) = 0.4 < 1 \Rightarrow$ 均方稳定

均方稳定性判据

例子 3：

- 两个子系统 $A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0.5 \end{bmatrix}$，$A_2 = \begin{bmatrix} 0.5 & 0 \\ 2 & 0 \end{bmatrix}$
  - $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$
  - $\rho\left((P^T \otimes I)\mathrm{diag}(A_i \otimes A_i)\right) = 2.125 > 1 \Rightarrow$ 不是均方稳定

结论：子系统稳定不是保证马尔可夫跳变系统均方稳定的充分或者必要条件。均方稳定取决于转移矩阵和系统动态特性。