Nonlinear Control Theory

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Backstepping

1. **Integrator backstepping**
2. Backstepping for higher-order systems
3. Backstepping in case of unmatched uncertainty
4. Block backstepping

Integrator backstepping

Consider the nonlinear system

$$\begin{aligned}\dot{\eta} =& f(\eta) + g(\eta)\xi, \\ \dot{\xi} =& u,\end{aligned}$$

where

- $[\eta^T, \xi]^T \in R^{n+1}$ is the state and $u \in R$ is the control input.
- The functions $f : D \to R^n$ and $g : D \to R^n$ are smooth in a domain $D \subset R^n$ that contains $\eta = 0$ and $f(0) = 0$.
- Both f and g are known.

GOAL: Design state feedback control law u to stabilize the origin $\eta = 0$, $\xi = 0$.

Suppose that $\dot{\eta} = f(\eta) + g(\eta)\xi$ can be stabilized by a SMOOTH state feedback control $\xi = \phi(\eta)$ with $\phi(0) = 0$, and there exists a Lyapunov function $V(\eta)$ satisfying

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \quad \forall\, \eta \in D,$$

where $W(\eta)$ is positive definite.

Suppose that ξ is not exactly ϕ. We have

$$\begin{aligned}\dot{\eta} =& f(\eta) + g(\eta)\phi(\eta) + g(\eta)[\xi - \phi(\eta)],\\ \dot{\xi} =& u.\end{aligned}$$

The change of variables

$$z = \xi - \phi(\eta)$$

results in the system

$$\begin{aligned}\dot{\eta} =& f(\eta) + g(\eta)\phi(\eta) + g(\eta)z,\\ \dot{z} =& u - \dot{\phi},\end{aligned}$$

where $\dot{\phi} = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$. Let $v = u - \dot{\phi}$, and it holds that

$$\begin{aligned}\dot{\eta} =& f(\eta) + g(\eta)\phi(\eta) + g(\eta)z,\\ \dot{z} =& v.\end{aligned}$$

Select Lyapunov function $V_c(\eta,\xi) = V(\eta) + \frac{1}{2}z^2$, then,

$$
\begin{aligned}
\dot{V}_c =& \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta} g(\eta) z + zv \\
\leq & - W(\eta) + \frac{\partial V}{\partial \eta} g(\eta) z + zv.
\end{aligned}
$$

Choosing

$$
v = -\frac{\partial V}{\partial \eta} g(\eta) - kz, \quad k > 0
$$

yields

$$
\dot{V}_c \leq -W(\eta) - kz^2 < 0, \quad \Rightarrow \quad \text{Asymptotically stable}
$$

The control law is now

$$
\begin{aligned}
u =& v + \dot{\phi} \\
=& -\frac{\partial V}{\partial \eta} g(\eta) - k[\xi - \phi(\eta)] + \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi].
\end{aligned}
$$

If all assumptions hold globally, and $V(\eta)$ is radially unbounded, then the origin is globally asymptotically stable.

For more general system

$$\dot{\eta} = f(\eta) + g(\eta)\xi,$$
$$\dot{\xi} = f_a(\eta, \xi) + g_a(\eta, \xi)u,$$

where f_a and g_a are smooth, and $g_a(\eta, \xi) \neq 0$ in the domain of interest. The input transformation

$$u = \frac{1}{g_a(\eta, \xi)}[u_a - f_a(\eta, \xi)]$$

yields $\dot{\xi} = u_a$. Then, the control law to stabilize the system can be designed by

$$u = \frac{1}{g_a(\eta, \xi)}\left[-\frac{\partial V}{\partial \eta}g(\eta) - k[\xi - \phi(\eta)] + \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - f_a(\eta, \xi)\right].$$

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Backstepping for higher-order systems

Consider the strict-feedback systems of the form

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x) z_1, \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2, \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3, \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \cdots, z_{k-1}) + g_{k-1}(x, z_1, \cdots, z_{k-1}) z_k, \\
\dot{z}_k &= f_k(x, z_1, \cdots, z_k) + g_k(x, z_1, \cdots, z_k) u,
\end{aligned}
$$

where $x \in R^n$; z_1 to z_k are scalars; f_0 to f_k vanishes at the origin; $g_i(x, z_1, \cdots, z_i) \neq 0$ for $1 \leq i \leq k$ over the domain of interest.

The backstepping procedure starts with $\dot{x} = f_0(x) + g_0(x)z_1$, where z_1 is viewed as the control input.

Design $\phi_0(x)$ such that a Lyapunov function $V_0(x)$ exists, and

$$\frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)\phi_0(x)] < -W(x).$$

where $W(x)$ is a positive definite function.

It follows that

$$\dot{V}_0 = \frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)z_1] < -W(x) + \frac{\partial V_0}{\partial x}g_0(z_1 - \phi_0(x)).$$

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Let us then consider the dynamics of x and z_1 together:

$$\begin{aligned}\dot{x} =& f_0(x) + g_0(x)z_1, \\ \dot{z}_1 =& f_1(x, z_1) + g_1(x, z_1)z_2.\end{aligned}$$

where, if z_2 is considered as the control input, it is a special case of the integrator backstepping. The control $\phi_1(x, z_1)$ can be designed to stabilize x and z_1:

$$\phi_1(x, z_1) = \frac{1}{g_1(x, z_1)}\left[-\frac{\partial V_0}{\partial x}g_0 - k_1(z_1 - \phi_0) + \frac{\partial \phi_0}{\partial x}(f_0 + g_0 z_1) - f_1\right], \quad k_1 > 0.$$

and the Lyapunov function can be chosen by $V_1(x, z_1) = V_0(x) + \frac{1}{2}(z_1 - \phi_0(x))^2$:

$$\dot{V}_1 \leq -W(x) - k_1(z_1 - \phi_0)^2 + g_1(z_1 - \phi_0)(z_2 - \phi_1).$$

Let us then consider the dynamics of x, z_1 and z_2 together:

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x) z_1, \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2, \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3.
\end{aligned}
$$

where z_3 can be considered as the control input. The control $\phi_2(x, z_1, z_2)$ can be designed to stabilize x, z_1 and z_2:

$$
\phi_2(x, z_1, z_2) = \frac{1}{g_2}\left[-\frac{\partial V_1}{\partial z_1} g_1 - k_2(z_2 - \phi_1) + \frac{\partial \phi_1}{\partial x}(f_0 + g_0 z_1) + \frac{\partial \phi_1}{\partial z_1}(f_1 + g_1 z_2) - f_2\right].
$$

with $k_2 > 0$, and the Lyapunov function can be chosen by $V_2 = V_1 + \frac{1}{2}(z_2 - \phi_1)^2$:

$$
\dot{V}_2 \leq -W(x) - k_1(z_1 - \phi_0)^2 - k_2(z_2 - \phi_1)^2 + g_2(z_2 - \phi_1)(z_3 - \phi_2).
$$

Repeat the process for k times to obtain the overall stabilizing state feedback control:

$$
\begin{aligned}
u =& \phi_k(x, z_1, \cdots, z_k) \\
=& \frac{1}{g_k}\left[-\frac{\partial V_{k-1}}{\partial z_{k-1}} g_{k-1} - k_k(z_k - \phi_{k-1}) + \frac{\partial \phi_{k-1}}{\partial x}(f_0 + g_0 z_1) + \sum_{i=1}^{k-1} \frac{\partial \phi_{k-1}}{\partial z_i}(f_i + g_i z_{i+1}) - f_k\right].
\end{aligned}
$$

and the corresponding Lyapunov function $V_k(x, z_1, z_2, \cdots, z_k)$, such that

$$
\dot{V}_k \leq -W(x) - \sum_{i=1}^{k} k_i (z_i - \phi_{i-1})^2 < 0.
$$

Backstepping in case of unmatched uncertainty

Let us consider the single-input system

$$\begin{aligned}\dot{\eta} =& f(\eta) + g(\eta)\xi + \delta_\eta(\eta,\xi),\\ \dot{\xi} =& f_a(\eta,\xi) + g_a(\eta,\xi)u + \delta_\xi(\eta,\xi),\end{aligned}$$

defined on a domain $D \in R^{n+1}$ that contains the origin, where

- $\eta \in R^n$ and $\xi \in R$.
- $g_a(\eta,\xi) \neq 0$, and all functions are smooth for $(\eta,\xi) \in D$.
- f, g, f_a and g_a are known, and $f(0) = 0$, $f_a(0,0) = 0$.
- δ_η and δ_ξ are uncertain terms satisfying $\|\delta_\eta\|_2 \leq a_1\|\eta\|_2$ and $\|\delta_\xi\|_2 \leq a_2\|\eta\|_2 + a_3|\xi|_2$.

Start with $\dot{\eta} = f(\eta) + g(\eta)\xi + \delta_\eta$.

Suppose that we can find a state feedback control law $\phi(\eta)$ such that

$$\dot{\eta} = f + g\phi + \delta_\eta$$

is asymptotically stable.

There exists a Lyapunov function $V(\eta)$:

$$\frac{\partial V}{\partial \eta}\left[f + g\phi + \delta_\eta\right] \leq -b\|\eta\|_2^2, \quad b > 0.$$

Suppose further that

$$|\phi(\eta)| \leq a_4\|\eta\|_2, \quad \frac{\partial \phi}{\partial \eta} \leq a_5$$

over D.

Select the Lyapunov candidate $V_c(\eta,\xi) = V(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^2$, such that

$$\begin{aligned}\dot{V}_c =& \frac{\partial V}{\partial \eta}(f + g\phi + \delta_\eta) + \frac{\partial V}{\partial \eta} g(\xi - \phi) \\ &+ (\xi - \phi)\left[f_a + g_a u + \delta_\xi - \frac{\partial \phi}{\partial \eta}(f + g\xi + \delta_\eta)\right],\end{aligned}$$

and the control u can be designed by

$$u = \frac{1}{g_a}\left[\frac{\partial \phi}{\partial \eta}(f + g\xi) - \frac{\partial V}{\partial \eta} g - f_a - k(\xi - \phi)\right], \quad k > 0.$$

It follows that

$$\dot{V}_c \leq -b\|\eta\|_2^2 - k(\xi - \phi)^2 + (\xi - \phi)\left[\delta_\xi - \frac{\partial \phi}{\partial \eta}\delta_\eta\right].$$

It holds that

$$\begin{aligned}\dot{V}_c \leq & -b\|\eta\|_2^2 + 2a_6\|\eta\|_2|\xi-\phi| - (k-a_3)(\xi-\phi)^2 \\ = & \begin{bmatrix} \|\eta\|_2 \\ |\xi-\phi| \end{bmatrix}^T \begin{bmatrix} b & -a_6 \\ -a_6 & k-a_3 \end{bmatrix} \begin{bmatrix} \|\eta\|_2 \\ |\xi-\phi| \end{bmatrix}\end{aligned}$$

for some $a_6 \geq 0$. Choosing $k > a_3 + \frac{a_6^2}{b}$ yields

$$\dot{V}_c \leq -\sigma\left[\eta^2 + (\xi-\phi)^2\right]$$

for some $\sigma > 0$.

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Block backstepping

Consider a multi-input system

$$\dot{\eta} = f(\eta) + G(\eta)\xi,$$
$$\dot{\xi} = f_a(\eta, \xi) + G_a(\eta, \xi)u,$$

where

- $\eta \in R^n$, $\xi \in R^m$, and $u \in R^m$.
- f, f_a, G and G_a are smooth functions over the domain of interest.
- $f(0) = 0$, $f_a(0, 0) = 0$, and G_a is non-singular.

Suppose further that η can be stabilized by a state feedback control $\phi(\eta)$ with $\phi(0) = 0$, such that there exists a Lyapunov function $V(\eta)$ satisfying

$$\frac{\partial V}{\eta}\left[f(\eta) + G(\eta)\phi(\eta)\right] \leq -W(\eta)$$

for some positive definite function $W(\eta)$. Using

$$V_c = V(\eta) + \frac{1}{2}(\xi - \phi(\eta))^T(\xi - \phi(\eta)),$$

as a Lyapunov function candidate. It then holds that

$$\dot{V}_c = \frac{\partial V}{\partial \eta}\left[f + G\phi\right] + \frac{\partial V}{\partial \eta} G(\xi - \phi) + (\xi - \phi)^T\left[f_a + G_a u - \frac{\partial \phi}{\partial \eta}(f + G\xi)\right].$$

$$\dot{V}_c = \frac{\partial V}{\partial \eta}[f + G\phi] + \frac{\partial V}{\partial \eta} G(\xi - \phi) + (\xi - \phi)^T \left[f_a + G_a u - \frac{\partial \phi}{\partial \eta}(f + G\xi) \right].$$

Taking

$$u = G_a^{-1} \left[\frac{\partial \phi}{\partial \eta}(f + G\xi) - \left(\frac{\partial V}{\partial \eta} G \right)^T - f_a - k(\xi - \phi) \right], \quad k > 0$$

leads to

$$\dot{V}_c \leq -W(\eta) - k(\xi - \phi)^T (\xi - \phi),$$

showing that the origin ($\eta = 0,\ \xi = 0$) is asymptotically stable.