模型预测控制

Model Predictive Control

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- **Before proceeding to our main contents of MPC…**

…Let me talk something about our daily life.

7:00 am: Wake up 起床

7:05 am: Wash up 洗漱

7:15 am: Breakfast 早餐

7:30 am: Heading to class 出发上课

8:00 am: Have classes 上课

……

……

12:00 pm: Lunch 午餐

……

7:05 am: Wash up 洗漱

7:15 am: Breakfast 早餐

7:30 am: Heading to class 出发上课

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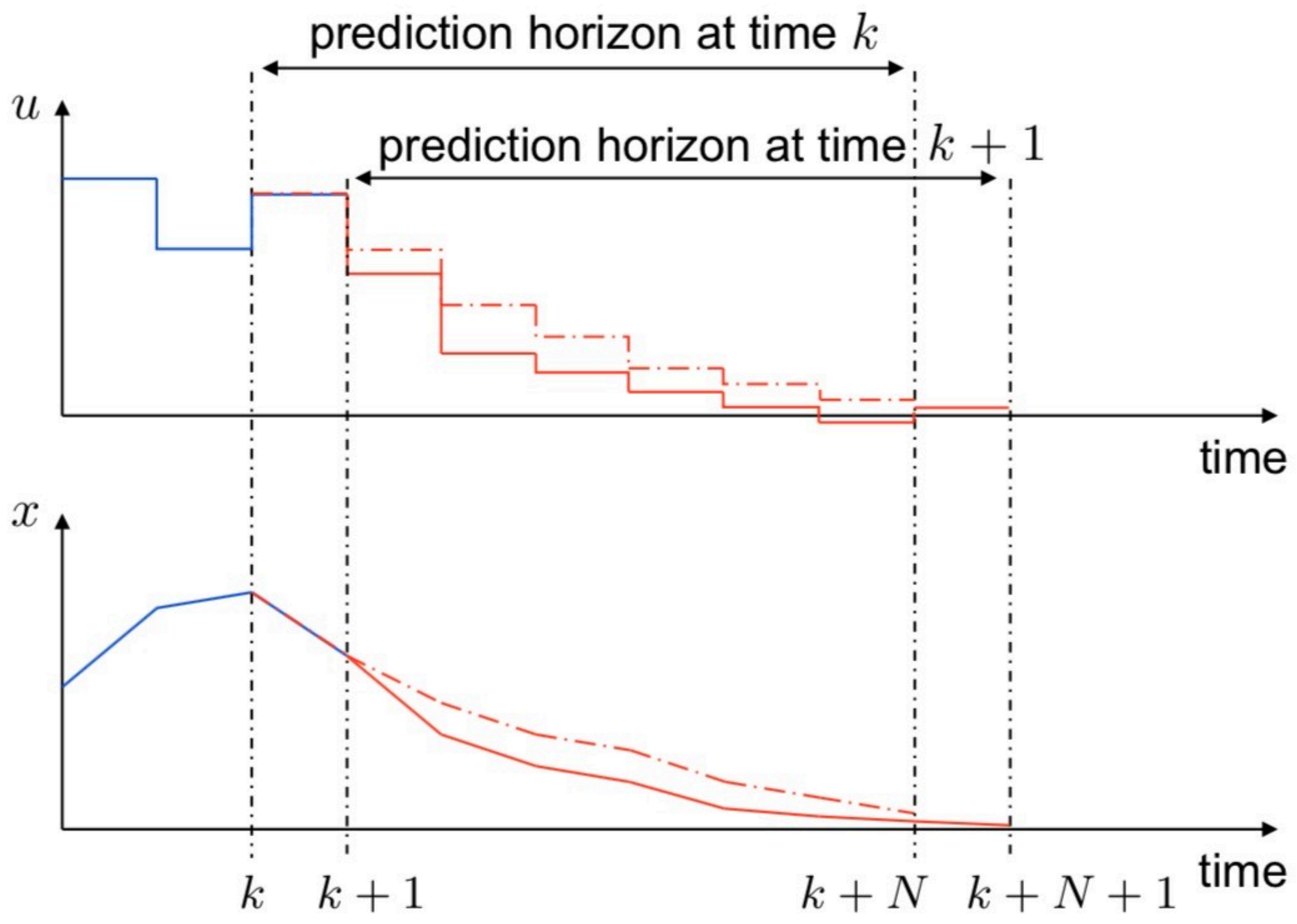
……

12:00 pm: Lunch 午餐

……

That is something similar to model predictive control…

- **Predict** from current information
- Calculate **optimization** based on prediction
- Implement the **first step optimal action**
- **Repeat** the above procedures at the next time step



In detail…

- Plant: $x(k+1) = f(x(k), u(k))$

The model can be linear, nonlinear, continuous, discrete, hybrid, deterministic, stochastic, or even data-based …

- Prediction:

Initial state: $x(k)$

Input series: $\{u(k), u(k+1), u(k+2), \cdots\}$

Determines $\Rightarrow$ **State series:** $x(k+1), x(k+2), \cdots$

- Optimization:

$l(x(i), u(i))$: Stage cost

$$U^*(k) = \arg \min_{u(k),u(k+1),\cdots} \sum_{i=k}^{\infty} l(x(i), u(i)) = \{u^*(k), u^*(k+1), \cdots\} \qquad \textbf{s.t.} \quad x \in \mathcal{X}, \, u \in \mathcal{U}$$

- Implement the first step optimal action: $u(k) = u^*(k)$

- Repeat the above procedures at time $k+1$

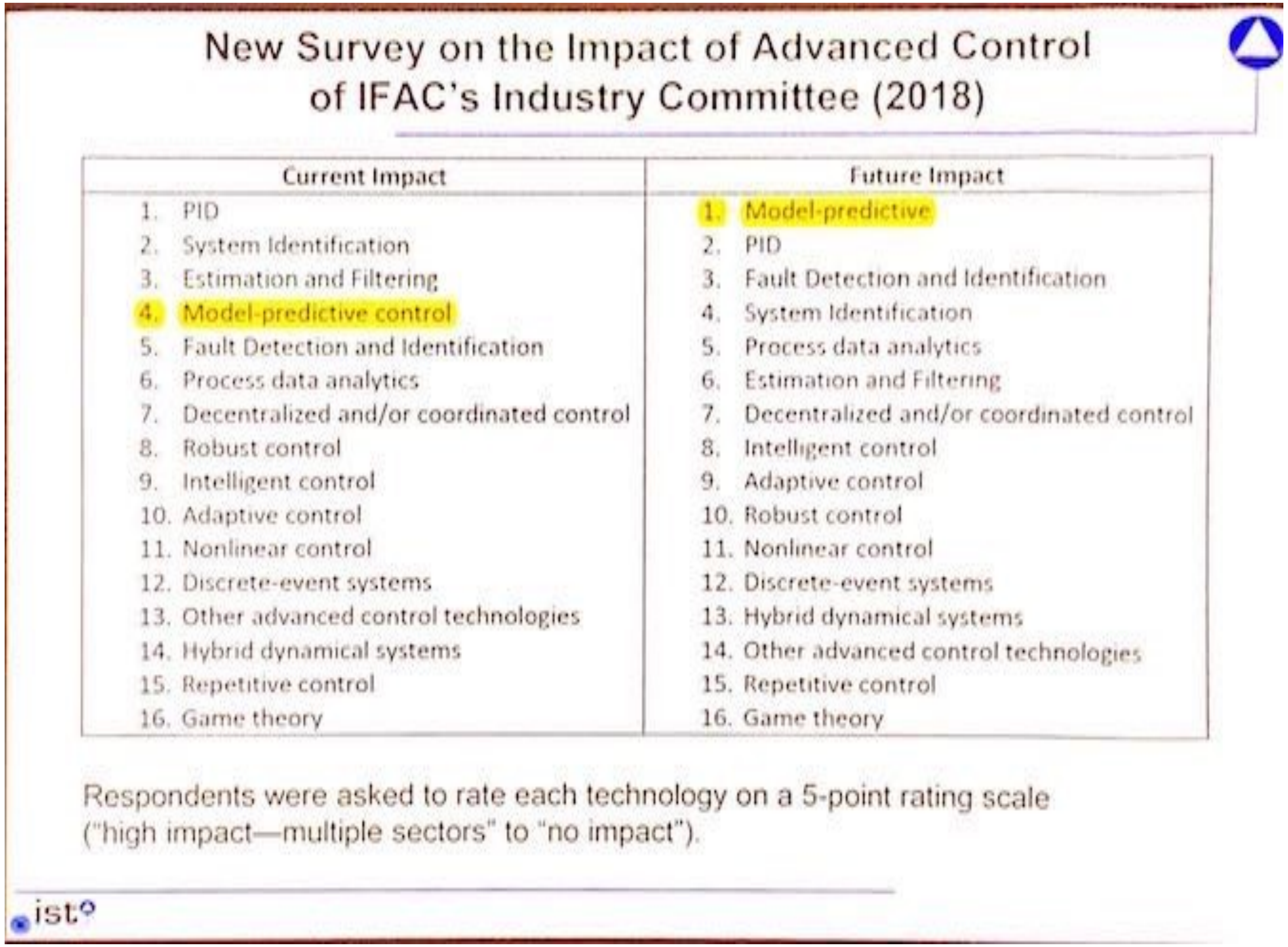
MPC provides feedback, since $u^*(k)$ is a function of $x(k)$.

- **Advantages**
    - **Handles constraints on control inputs and states**
        - e.g., actuator limits; safety, environmental, or economic constraints
    - **Approximately optimal control**
- **Disadvantages**
    - **Requires online optimization**
        - e.g., large computation for nonlinear and uncertain systems

TABLE 1 A list of the survey results in order of industry impact as perceived by the committee members.

Rank and Technology	High-Impact Ratings	Low- or No-Impact Ratings
PID control	100%	0%
Model predictive control	78%	9%
System identification	61%	9%
Process data analytics	61%	17%
Soft sensing	52%	22%
Fault detection and identification	50%	18%
Decentralized and/or coordinated control	48%	30%
Intelligent control	35%	30%
Discrete-event systems	23%	32%
Nonlinear control	22%	35%
Adaptive control	17%	43%
Robust control	13%	43%
Hybrid dynamical systems	13%	43%

T. Samad, "A Survey on Industry Impact and Challenges Thereof [Technical Activities]," in IEEE Control Systems Magazine, vol. 37, no. 1, pp. 17-18, Feb. 2017.



New Survey on the Impact of Advanced Control of IFAC's Industry Committee (2018)

Current Impact	Future Impact
1. PID	1. Model-predictive
2. System Identification	2. PID
3. Estimation and Filtering	3. Fault Detection and Identification
4. Model-predictive control	4. System Identification
5. Fault Detection and Identification	5. Process data analytics
6. Process data analytics	6. Estimation and Filtering
7. Decentralized and/or coordinated control	7. Decentralized and/or coordinated control
8. Robust control	8. Intelligent control
9. Intelligent control	9. Adaptive control
10. Adaptive control	10. Robust control
11. Nonlinear control	11. Nonlinear control
12. Discrete-event systems	12. Discrete-event systems
13. Other advanced control technologies	13. Hybrid dynamical systems
14. Hybrid dynamical systems	14. Other advanced control technologies
15. Repetitive control	15. Repetitive control
16. Game theory	16. Game theory

Respondents were asked to rate each technology on a 5-point rating scale ("high impact—multiple sectors" to "no impact").

ist

Frank Allgower, "From Stabilizing to Economic Model Predictive Control: A Paradigm Shift towards Increased Control Performance," Planary speech at 2019 Asian Control Conference.

Outline

- Unconstrained linear MPC
- Constrained linear MPC
- Output feedback MPC
- Robust MPC
- MPC toolbox in MATLAB

Unconstrained linear MPC

Lecture 1

- LTI system:

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$

Suppose (A, B) is stabilizable

Suppose there are no constraints on states or control inputs

- Define:

$x(i\,|\,k), u(i\,|\,k)$: Predition of i steps ahead from time k $\qquad x(0\,|\,k) = x(k), \quad u(0\,|\,k) = u(k)$

- Prediction:

$$x(1\,|\,k) = Ax(0\,|\,k) + Bu(0\,|\,k)$$

$$\begin{aligned} x(2\,|\,k) &= Ax(1\,|\,k) + Bu(1\,|\,k) = A[Ax(0\,|\,k) + Bu(0\,|\,k)] + B(1\,|\,k) \\ &= A^2x(0\,|\,k) + ABu(0\,|\,k) + Bu(1\,|\,k) \end{aligned}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\begin{aligned} x(i\,|\,k) &= Ax(i-1\,|\,k) + Bu(i-1\,|\,k) = \cdots \\ &= A^ix(0\,|\,k) + A^{i-1}Bu(0\,|\,k) + A^{i-2}Bu(1\,|\,k) + \cdots + Bu(i-1\,|\,k) \end{aligned}$$

- In compact form:

$$X(k) = Fx(k) + \Phi U(k)$$

$N:$ Control/predictive horizon

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$$X(k) \triangleq \begin{bmatrix} x(1|k) \\ x(2|k) \\ \vdots \\ x(N|k) \end{bmatrix} \qquad U(k) \triangleq \begin{bmatrix} u(0|k) \\ u(1|k) \\ \vdots \\ u(N-1|k) \end{bmatrix} \qquad F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \qquad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}$$

*** Recall in the previous page:**

$$x(1|k) = Ax(0|k) + Bu(0|k)$$

$$x(2|k) = A^2x(0|k) + ABu(0|k) + Bu(1|k)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x(i|k) = A^ix(0|k) + A^{i-1}Bu(0|k) + A^{i-2}Bu(1|k) + \cdots + Bu(i-1|k)$$

- Cost function (with finite control/predictive horizon)

$$J(k) = \sum_{i=1}^{N} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 = X^T(k)\mathcal{Q}X(k) + U(k)^T\mathcal{R}U(k)$$

$$\mathcal{Q} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix} \qquad \mathcal{R} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix} \qquad X(k) = Fx(k) + \Phi U(k)$$

$$\begin{aligned} J(k) &= (Fx(k) + \Phi U(k))^T\mathcal{Q}(Fx(k) + \Phi U(k)) + U^T(k)\mathcal{R}U(k) \\ &= x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k) \end{aligned}$$

- Minimizing the cost function by predictive control series

$$\nabla_U\Big|_{U=U*} = \frac{\partial J}{\partial U}\Big|_{U=U*} = 0$$

- Cost function

$$J(k) = x^T F^T \mathscr{Q} F x + 2x^T F^T \mathscr{Q} \Phi U + U^T (\Phi^T \mathscr{Q} \Phi + \mathscr{R}) U$$

- Minimizing the cost function by predictive control:

$$\nabla_U \Big|_{U=U^*} = \frac{\partial J}{\partial U}\Big|_{U=U^*} = 0$$

$$2x(k)^T F^T \mathscr{Q} \Phi + 2U(k)^T (\Phi^T \mathscr{Q} \Phi + \mathscr{R}) \Big|_{U(k)=U^*} = 0$$

$$U^*(k) = -(\Phi^T \mathscr{Q} \Phi + \mathscr{R})^{-1} \Phi^T \mathscr{Q} F x(k)$$

($R > 0,\ Q \geq 0;$ **or** $R \geq 0,\ Q > 0$**, and** Φ **is fully ranked)**

$$u^*(k) = -[I_{p\times p}\ 0\ \cdots\ 0](\Phi^T \mathscr{Q} \Phi + \mathscr{R})^{-1} \Phi^T \mathscr{Q} F x(k) = -K_{mpc} x(k)$$

*** The unconstrained linear MPC is actually linear feedback control !!**

- Example

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^2, \;\; u \in \mathbb{R}^1 \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

Set control horizon $N = 4$. Set weight matrices $Q = I_{2\times 2}$, $R = 0.1$. Then,

$$F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \\ 1.21 & 4.1 \\ 0 & 0.902 \\ 1.331 & 6.315 \\ 0 & 0.857 \\ 1.464 & 8.661 \\ 0 & 0.814 \end{bmatrix} \qquad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0790 & 0 & 0 & 0 \\ 0.1580 & 0 & 0 & 0 \\ 0.0750 & 0.0790 & 0 & 0 \\ 0.3239 & 0.1580 & 0 & 0 \\ 0.0713 & 0.0750 & 0.0790 & 0 \\ 0.4989 & 0.3239 & 0.1580 & 0 \\ 0.0677 & 0.0713 & 0.0750 & 0.0790 \end{bmatrix}$$

$$u^*(k) = -\left[I_{p\times p} \; 0 \; \cdots \; 0\right](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k) = -K_{mpc} x(k) = -\left[2.51 \quad 12.29\right] x(k)$$

- Closed-loop system

$$x(k+1) = (A - BK_{mpc})x(k)$$

Check wether $|\mathrm{eig}(A - BK_{mpc})| < 1$ to determine its stability

A necessary condition is that $(A,\ B)$ is stabilizable, such that K_{mpc} exists.

- Optimality does NOT necessarily indicate stability !!

✳ **Criterion for stability**

	Continuous-time systems	**Discrete-time systems**
Lyapunov indirect method:	$\mathrm{Re}\left(\mathrm{eig}(A - BK)\right) < 0$	$\lvert \mathrm{eig}(A - BK) \rvert < 1$
Lyapunov direct method:	$V(x(t)) > 0, \quad \dot{V}(x(t)) < 0$	$V(x(k)) > 0, \quad V(x(k+1)) - V(x(k)) < 0$

- Example:

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^2, \quad u \in \mathbb{R}^1 \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

Set weight matrices $Q = I_{2\times 2}$, $R = 0.1$.

	N=4	N=3	N=2	N=1
K	$[2.51 \quad 12.29]$	$[2.41 \quad 10.90]$	$[1.40 \quad 5.76]$	$[0 \quad 0.70]$
$\mathrm{eig}(A - BK)$	$0.5394 \pm 0.2872j$	$0.5944 \pm 0.3543j$	$0.7974 \pm 0.3601j$	$1.10, \quad 0.89$
	Stable	Stable	Stable	Unstable

Can we guarantee the closed-loop stability rather than check it by trial ??

Yes we can !!

- Try cost function with infinite control/predictive horizon: $N = \infty$

$$J(k) = \sum_{i=1}^{\infty} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2$$

- Suppose at time k, the optimal solution is $u^*(i\,|\,k)$ for $i = 0,1,\cdots$

  At time $k+1$, there exists at least one feasible solution

$$u(i\,|\,k+1) = u^*(i+1\,|\,k) \qquad \text{such that} \qquad x(i\,|\,k+1) = x^*(i+1\,|\,k) \qquad \text{for all} \quad i = 0,1,\cdots,$$

- Use the optimal cost function $J^*(k)$ as the Lyapunov candidate

$$J^*(k+1) - J^*(k) \le J(k+1) - J^*(k) = -\|x(1\,|\,k)\|_Q^2 - \|u(0\,|\,k)\|_R^2 < 0$$

Asymptotically stable

The remaining problem is… How to solve the infinite-horizon optimization ??

It seems like everything is almost the same

$$J(k) = \sum_{i=1}^{\infty} \|x(i \mid k)\|_Q^2 + \|u(i-1 \mid k)\|_R^2 = X^T(k)\mathcal{Q}X(k) + U(k)^T\mathcal{R}U(k)$$

$$= (Fx + \Phi U)^T\mathcal{Q}(Fx + \Phi U) + U^T\mathcal{R}U^T = x^TF^T\mathcal{Q}Fx + 2x^TF^T\mathcal{Q}\Phi U + U^T(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U$$

So,

$$u^*(k) = -\,[I_{p\times p}\ 0\ \cdots\ 0](\Phi^T\mathcal{Q}\Phi + \mathcal{R})^{-1}\Phi^T\mathcal{Q}Fx(k) = -\,K_{mpc}x(k)$$

Still linear state feedback, **except…** they are with **infinite** dimensions !!

$$\mathcal{Q} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \quad \mathcal{R} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \quad F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ \vdots \end{bmatrix} \quad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ \vdots & \cdots & \cdots & \cdots \end{bmatrix}$$

- For the infinite horizon optimization, the closed-loop system is asymptotically stable, so the infinite horizon cost function actually has a **finite** limit

$$J(k) = \sum_{i=1}^{\infty} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 = \|x(k)\|_P^2 - \|x(k)\|_Q^2$$

where $$P - (A-BK)^T P (A-BK) = Q + K^T R K$$ with $$|\mathrm{eig}(A-BK)| < 1$$

Discrete-time Lyapunov function

Proof:

$$x^T[P - (A-BK)^T P (A-BK)]x = x^T[Q + K^T R K]x$$

$$x^T P x - x^T (A-BK)^T P (A-BK) x = x^T Q x + x^T K^T R K x$$

$$x(k)^T P x(k) - x^T(k+1)^T P x(k+1) = x^T(k) Q x(k) + u^T(k) R u(k)$$

$$\|x(k)\|_P^2 = \sum_{i=k}^{\infty} \|x(i)\|_Q^2 + \|u(i)\|_R^2$$

- **What we need to do:**

A necessary condition is that (A, B) is stabilizable, such that the stabilizing K exists.

- find a linear feedback $u = -Kx$ that can asymtotically stabilize the system

- solve the discrete-time Lyapunov equation according to Q, R, K

$$P - (A - BK)^T P(A - BK) = Q + K^T RK$$

s.t. the equivalent finite horizon optimization is given by

$$\mathcal{Q} = \text{diag}[Q, Q, \cdots, Q, P]$$

$$\mathcal{R} = \text{diag}[R, R, \cdots, \cdots, R]$$

$$J(k) = \sum_{i=1}^{\infty} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2$$

$$= \|x(N\,|\,k)\|_P^2 + \|u(N-1\,|\,k)\|_R^2 + \sum_{i=1}^{N-1} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2$$

$$= X^T(k)\mathcal{Q}X(k) + U(k)^T\mathcal{R}U(k)$$

- calculate $\quad u^*(k) = -\,[I_{p\times p}\ 0\ \cdots\ 0](\Phi^T\mathcal{Q}\Phi + \mathcal{R})^{-1}\Phi^T\mathcal{Q}Fx(k) = -K_{mpc}x(k)$

Shall we calculate the optimal K ?? Not necessary !! Its existence matters

- Theoretically, however you choose K (s.t. $|\mathrm{eig}(A - BK)| < 1$), the following still holds:

$$J(k) = \sum_{i=1}^{\infty} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2$$

$$= \|x(N\,|\,k)\|_P^2 + \|u(N-1\,|\,k)\|_R^2 + \sum_{i=1}^{N-1} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2$$

$u(i\,|\,k+1) = u^*(i+1\,|\,k)$ for all $i = 0,1,\cdots,N-1$

$u(i\,|\,k+1) = -Kx(i\,|\,k+1)$ for all $i = N, N+1,\cdots$

such that $x(i\,|\,k+1) = x^*(i+1\,|\,k)$

"State shift"

$$J^*(k+1) - J^*(k) \leq J(k+1) - J^*(k) = -\|x(1\,|\,k)\|_Q^2 - \|u(0\,|\,k)\|_R^2 < 0$$

Asymptotically stable

- Example

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^2, \;\; u \in \mathbb{R}^1 \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

Set weight matrices $Q = I_{2\times2}$, $R = 0.1$. Set control horizon $N = 4$.

Take $K = [1.4 \;\; 5.76]$, such that $|\,\mathrm{eig}(A - BK)\,| < 1$

The solution of discrete Lyapunov equation

$$P - (A - BK)^T P(A - BK) = Q + K^T RK$$

can be calculated by $\qquad P = \begin{bmatrix} 5.2471 & 12.8188 \\ 12.8188 & 67.1313 \end{bmatrix}$

Use $\quad \mathcal{Q} = \mathrm{diag}[Q, Q, Q, P] \qquad \mathcal{R} = \mathrm{diag}[R, R, R, R]$

to calculate $\quad u^*(k) = -[1 \;\; 0 \;\; \cdots \;\; 0](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k) = -[2.6167 \;\; 12.9286] x(k)$

$$K_{mpc} = [2.6167 \;\; 12.9286]$$

Take $K = [1.4 \quad 5.76]$, such that $|\mathrm{eig}(A - BK)| < 1$

$$P - (A - BK)^T P(A - BK) = Q + K^T RK \qquad P = \begin{bmatrix} 5.2471 & 12.8188 \\ 12.8188 & 67.1313 \end{bmatrix}$$

	N=4	N=3	N=2	N=1
K	$[2.6167 \;\; 12.9286]$	$[2.7320 \;\; 13.2108]$	$[2.6607 \;\; 13.0972]$	$[2.1465 \;\; 13.6108]$
$\mathrm{eig}(A - BK)$	$0.5143 \pm 0.2654j$	$0.5032 \pm 0.2747j$	$0.5077 \pm 0.2637j$	$0.6776, \; 0.2972$
	Stable	Stable	Stable	Stable

Unconstrained MPC versus LQR ??

$$J(k) = \sum_{i=1}^{\infty} \|x(i \mid k)\|_Q^2 + \|u(i-1 \mid k)\|_R^2$$

- **<u>Unconstrained MPC</u>**:
  - Find the optimal control series $\{u(k), u(k+1), \cdots\}$ to optimize the cost function
  - It is proved to be a linear feedback control $u = -K_{mpc}x$

- **<u>Linear quadratic regulator (LQR)</u>**:
  - Find the optimal control gain K_{lqr} , such that $u = -K_{lqr}x$ optimizes the cost function

- If MPC is superior, or namely $J^*_{mpc} < J^*_{lqr}$

  then, for LQR, there exists at least one better feedback gain $K = K_{mpc}$!!

  This is a contradictory, since K_{lqr} is the optimal feedback gain.

- If LQR is superior, or namely $J^*_{lqr} < J^*_{mpc}$

  then, for MPC, there exists at least one better control series $\{K_{lqr}x(k), K_{lqr}x(k+1), \cdots\}$!!

  This is a contradictory, since $\{K_{mpc}x(k), K_{mpc}x(k+1), \cdots\}$ is the optimal control series.

- Consequently, it has to be $J^*_{mpc} = J^*_{lqr}$ and therefore, $K_{mpc} = K_{lqr}$

- Example

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^2, \;\; u \in \mathbb{R}^1 \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

Set weight matrices $Q = I_{2\times 2}$, $R = 0.1$. Set control horizon $N = 4$.

Take $K = K_{lqr} = [2.4950 \;\; 12.5106]$, and it is proved $|\mathrm{eig}(A - BK_{lqr})| < 1$

The solution of discrete Lyapunov equation

$$P - (A - BK)^T P (A - BK) = Q + K^T R K$$

can be calculated by $$P = \begin{bmatrix} 4.0402 & 8.5265 \\ 8.5265 & 31.5654 \end{bmatrix}$$

Use $\mathcal{Q} = \mathrm{diag}[Q, Q, Q, P]$ $\qquad \mathcal{R} = \mathrm{diag}[R, R, R, R]$ **Exactly the same with K_{lqr} !!**

to calculate $$u^*(k) = -[1 \;\; 0 \;\; \cdots \;\; 0](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k) = -[2.4950 \;\; 12.5106] x(k)$$

$$K_{mpc} = [2.4950 \;\; 12.5106]$$

It seems unconstrained MPC makes no sense, since it is the same with LQR ??

- **Yes, indeed.**
  - We can simply solve the Riccati equation, and simply get $K_{mpc} = K_{lqr}$
- **Except for**
  - Sometimes the finite-horizon performance is emphasized.
  - The closed-loop stability of constrained MPC is based on unconstrained MPC.

Summary

- **Predict** “future” performance using plant model
  - e.g., linear or nonlinear, discrete or continuous time
- **Optimize** “future” (open loop) control series
  - computationally much easier than optimizing over feedback laws
- Implement the **first** optimal control action
  - provides feedback to reduce effects of uncertainty
- **Repeat** the above process at the next sampling time