Nonlinear Control Theory

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Lyapunov Stability

Linear Systems

Linear systems:

$$\dot{x} = Ax.$$

- Equilibrium points
  - $*$ $\det A \neq 0$: one isolated equilibrium point
  - $*$ $\det A = 0$: equilibrium set $=$ the non-trivial null space of A, or $\{x \in R^n | Ax = 0\}$.
  - $*$ It is impossible that a linear system has multiple isolated equilibria.
- Solution: $x(t) = e^{At}x(0)$.
- Stability
  - $*$ All eigenvalues of A have negative real parts. $\Rightarrow$ Asymptotically stable
  - $*$ At least one eigenvalues of A have positive real parts. $\Rightarrow$ Unstable
  - $*$ What if some eigenvalues of A have zero real parts, and all others have negative real parts?

- Eigenvalues & Jordan blocks

$$P^{-1}AP = J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & J_r \end{bmatrix}, \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & & 1 \\ 0 & & & 0 & \lambda_i \end{bmatrix}_{m_i \times m_i}.$$

Note that $e^{At} = Pe^{Jt}P^{-1} = \sum_{i=1}^{r} \sum_{k=1}^{m_i} t^{k-1} e^{\lambda_i t} R_{ik}$, then,

$$\mathrm{Re}[\lambda(A)] \leq 0 \ \& \ m_i = 1 \text{ for all } \mathrm{Re}[\lambda_i(A)] = 0 \quad \Leftrightarrow \quad \textit{Stable}$$

(The algebraic multiplicity of λ_i with $\mathrm{Re}[\lambda_i] = 0$ is equal to its geometric multiplicity, or equivalently, $\mathrm{rank}(A - \lambda_i I) = n - q_i$ where q_i is the algebraic multiplicity.)

- Types of stability for linear systems
  - $*$ Stable $\Leftrightarrow$ Globally stable
  - $*$ Asymptotically stable $\Leftrightarrow$ Globally asymptotically stable
  - $*$ Asymptotically stable $\Leftrightarrow$ Exponentially stable $\Leftrightarrow$ Globally exponentially stable

  For autonomous linear systems (LTI systems), the above statements hold uniformly.

- A is **Hurwitz**: all its eigenvalues have negative real parts, or $\mathrm{Re}[\lambda(A)] < 0$.

- Lyapunov equation: $PA + A^T P = -Q$
  - $*$ A is Hurwitz, if and only if for any given $Q = Q^T > 0$, there exists $P = P^T > 0$ satisfying the Lyapunov equation.
  - $*$ A is Hurwitz. $\quad\Rightarrow\quad$ P is the unique solution of the Lyapunov equation.

Linearization

Nonlinear system:

$$\dot{x} = f(x),$$

where $f : D \to R^n$ is continuously differentiable, and $f(0) = 0$.

- By mean value theorem, $f(x) = f(0) + \frac{\partial f}{\partial x}\big|_{x=z} x$, where $z \in B_x = \{\|z\| \leq x\}$.
- $\dot{x} = Ax + g(x)$, where $A = \frac{\partial f}{\partial x}\big|_{x=0}$, and $g(x) = \frac{\partial f}{\partial x}\big|_{x=z} x - \frac{\partial f}{\partial x}\big|_{x=0} x$.
- For any $\gamma > 0$, there exists $r > 0$, such that the function $g(x)$ satisfies

$$\|g(x)\| \leq \left\| \frac{\partial f}{\partial x}\bigg|_{x=z} - \frac{\partial f}{\partial x}\bigg|_{x=0} \right\| \|x\| \leq \gamma \|x\|, \quad \forall x \in B_r \subset D,$$

  suggesting that we use $\dot{x} = Ax$ to approximate $\dot{x} = f(x)$ in a small neighborhood of the origin.

Theorem (4.7 Lyapunov's indirect method)

Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = f(x)$, where $f : D \to R^n$ is continuously differentiable, and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0},$$

then,

1. *The origin is asymptotically stable, if $\mathrm{Re}\lambda_i < 0$ for all eigenvalues of A.*
2. *The origin is unstable, if $\mathrm{Re}\lambda_i > 0$ for at least one eigenvalues of A.*

Proof:

- A is Hurwitz, then there exists $P = P^T > 0$ satisfying $PA + A^T P = -Q$ with $Q = Q^T > 0$.
- Use $V(x) = x^T P x$ as the Lyapunov candidate for $\dot{x} = f(x)$.

$$\begin{aligned}\dot{V}(x) =& x^T P\dot{x} + \dot{x}^T P x = x^T P(Ax + g(x)) + (Ax + g(x))^T P x \\ =& x^T (PA + A^T P)x + 2x^T P g(x) = -x^T Q x + 2x^T P g(x) \\ <& -x^T Q x + 2\gamma \|P\|_2 \|x\|_2^2 < -[\lambda_{min}(Q) - 2\gamma \|P\|_2] \|x\|_2^2.\end{aligned}$$

- Given any $\gamma < \frac{\lambda_{min}(Q)}{2\|P\|_2}$, there exists r, such that

$$\dot{V}(x) < 0, \quad \forall x \in B_r.$$

- Consequently, it is concluded that $x = 0$ is (locally) asymptotically stable. (Please prove the unstable part as an exercise.)

However, Theorem 4.7 fails when at least one eigenvalue of A has **zero real parts**, and others have negative real parts.

Example

Consider the scalar system $\dot{x} = ax^3$. Please use Theorem 4.7 and Theorem 4.2 to prove whether $x = 0$ is asymptotically stable. Can we obtain the same result?

Example

Consider again the pendulum equation with $b > 0$. Please use linearization to investigate the stability of the equilibrium points $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (\pi, 0)$.