



非线性函数的展开方法——泰勒展开的描述

$$f(x) = f(\bar{x}) + \frac{\partial f}{\partial x} \bigg|_{\bar{x}} \tilde{x} + \frac{1}{2!} \frac{\partial^2 f}{\partial x^2} \bigg|_{\bar{x}} \tilde{x}^2 + \frac{1}{3!} \frac{\partial^3 f}{\partial x^3} \bigg|_{\bar{x}} \tilde{x}^3 + \dots$$

$$f(x) = f(\bar{x}) + \left(\tilde{x}_1 \frac{\partial}{\partial x_1} + \tilde{x}_2 \frac{\partial}{\partial x_2} + \dots + \tilde{x}_n \frac{\partial}{\partial x_n} \right) f \bigg|_{\bar{x}} + \frac{1}{2!} \left(\tilde{x}_1 \frac{\partial}{\partial x_1} + \tilde{x}_2 \frac{\partial}{\partial x_2} + \dots + \tilde{x}_n \frac{\partial}{\partial x_n} \right)^2 f \bigg|_{\bar{x}} + \frac{1}{3!} \left(\tilde{x}_1 \frac{\partial}{\partial x_1} + \tilde{x}_2 \frac{\partial}{\partial x_2} + \dots + \tilde{x}_n \frac{\partial}{\partial x_n} \right)^3 f \bigg|_{\bar{x}} + \dots$$

$$\text{令 } D_{\tilde{x}}^k f = \left(\sum_{i=1}^n \tilde{x}_i \frac{\partial}{\partial x_i} \right)^k f(x) \bigg|_{\bar{x}}$$

$$f(x) = f(\bar{x}) + D_{\tilde{x}} f + \frac{1}{2!} D_{\tilde{x}}^2 f + \frac{1}{3!} D_{\tilde{x}}^3 f + \dots$$





非线性函数的展开——对称分布下的泰勒展开

$$y = h(x) = h(\bar{x}) + D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2h + \frac{1}{3!}D_{\tilde{x}}^3h + \dots$$

$$\bar{y} = E \left[h(\bar{x}) + D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2h + \frac{1}{3!}D_{\tilde{x}}^3h + \dots \right]$$

$$= h(\bar{x}) + E \left[D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2h + \frac{1}{3!}D_{\tilde{x}}^3h + \dots \right]$$

$$\left. \begin{aligned} E[x^i] &= \int_{-\infty}^{\infty} p(x)x^i dx \\ \text{\textcolor{red}{x}} \text{对称分布} \\ \text{\textcolor{red}{i}} \text{为奇数} \end{aligned} \right\} E[x^i] = 0$$

$$\begin{aligned} E[D_{\tilde{x}}h] &= E \left[\sum_{i=1}^n \tilde{x}_i \frac{\partial}{\partial x_i} h(x) \right]_{x=\bar{x}} \\ &= \sum_{i=1}^n E(\tilde{x}_i) \frac{\partial}{\partial x_i} h(x) \Big|_{x=\bar{x}} \\ &= 0 \end{aligned}$$



$$\begin{aligned} E[D_{\tilde{x}}^3h] &= E \left[\left(\sum_{i=1}^n \tilde{x}_i \frac{\partial}{\partial x_i} \right)^3 h(x) \right]_{x=\bar{x}} \\ &= E \left(\tilde{x}_1^3 + \tilde{x}_1^2 \tilde{x}_2 + \tilde{x}_1 \tilde{x}_2^2 + \dots \right) \frac{\partial^3}{\partial x_i^3} h(x) \Big|_{x=\bar{x}} \\ &= 0 \end{aligned}$$

$$\bar{y} = h(\bar{x}) + \frac{1}{2!}E[D_{\tilde{x}}^2h] + \frac{1}{4!}E[D_{\tilde{x}}^4h] + \dots$$



1) σ 点的定义

设 x 是 n 维向量，与 y 之间满足非线性关系 $y=h(x)$ ， $2n$ 个 σ 点的选择方法如下

$$\begin{cases} x^{(i)} = \bar{x} + \tilde{x}^{(i)} \\ \tilde{x}^{(i)} = \left(\sqrt{nP} \right)_i^T \\ \tilde{x}^{(n+i)} = -\left(\sqrt{nP} \right)_i^T \end{cases} \quad i = 1, \dots, n$$

矩阵平方根的第 i 行



P的作用

2) y 均值的计算

$$y^{(i)} = h(x^{(i)})$$

$$\bar{y}_u = \sum_{i=1}^{2n} W^{(i)} h(x^{(i)})$$

$$W^{(i)} = \frac{1}{2n} \quad i = 1, \dots, 2n$$

$$\bar{y}_u = \frac{1}{2n} \sum_{i=1}^{2n} h(x^{(i)})$$



\bar{y}_u 精度的分析 \bar{y}_u 代表UT变换均值

$$\begin{aligned} \bar{y}_u &= \frac{1}{2n} \sum_{i=1}^{2n} \left[h(\bar{x}) + D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots \right] \\ &= h(\bar{x}) + \frac{1}{2n} \sum_{i=1}^{2n} \left[D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots \right] \end{aligned}$$



6.5.2 Unscented Transformation

$$\bar{y}_u = h(\bar{x}) + \frac{1}{2n} \sum_{i=1}^{2n} \left[D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots \right] \rightarrow \bar{y}_u = h(\bar{x}) + \frac{1}{2n} \sum_{i=1}^{2n} \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \frac{1}{2n} \sum_{i=1}^{2n} \left[\frac{1}{4!} D_{\tilde{x}^{(i)}}^4 h + \frac{1}{6!} D_{\tilde{x}^{(i)}}^6 h + \dots \right]$$

$$\begin{aligned} \sum_{j=1}^{2n} D_{\tilde{x}^{(j)}}^{2k+1} h &= \sum_{j=1}^{2n} \left[\left(\sum_{i=1}^n \tilde{x}_i^{(j)} \frac{\partial}{\partial x_i} \right)^{2k+1} h(x) \right]_{x=\bar{x}} \\ &= \sum_{j=1}^{2n} \left[\sum_{i=1}^n (\tilde{x}_i^{(j)})^{2k+1} \frac{\partial^{2k+1}}{\partial x_i^{2k+1}} h(x) \right]_{x=\bar{x}} \\ &= \sum_{i=1}^n \left[\sum_{j=1}^{2n} (\tilde{x}_i^{(j)})^{2k+1} \frac{\partial^{2k+1}}{\partial x_i^{2k+1}} h(x) \right]_{x=\bar{x}} \\ &= \mathbf{0} \end{aligned}$$

$$\frac{1}{2n} \sum_{i=1}^{2n} \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h = \frac{1}{2n} \sum_{k=1}^{2n} \frac{1}{2!} \left(\sum_{i=1}^n \tilde{x}_i^{(k)} \frac{\partial}{\partial x_i} \right)^2 h(x) \Big|_{x=\bar{x}}$$

向量
 $\tilde{x}^{(k)} = (\sqrt{nP})^T_k$
↓
 $\tilde{x}_i^{(k)} = \tilde{x}^{(k)} \text{第} i \text{行}$
标量

$$= \frac{1}{4n} \sum_{k=1}^{2n} \sum_{i,j=1}^n \tilde{x}_i^{(k)} \tilde{x}_j^{(k)} \frac{\partial^2}{\partial x_i \partial x_j} h(x) \Big|_{x=\bar{x}}$$

$$= \frac{1}{4n} \sum_{i,j=1}^n \sum_{k=1}^{2n} \tilde{x}_i^{(k)} \tilde{x}_j^{(k)} \frac{\partial^2}{\partial x_i \partial x_j} h(x) \Big|_{x=\bar{x}}$$

$$= \frac{1}{2n} \sum_{i,j=1}^n \sum_{k=1}^n \tilde{x}_i^{(k)} \tilde{x}_j^{(k)} \frac{\partial^2}{\partial x_i \partial x_j} h(x) \Big|_{x=\bar{x}}$$

$$= \frac{1}{2n} \sum_{i,j=1}^n \sum_{k=1}^n (\sqrt{nP})_{ki} (\sqrt{nP})_{kj} \frac{\partial^2}{\partial x_i \partial x_j} h(x) \Big|_{x=\bar{x}}$$

$$= \frac{1}{2n} \sum_{i,j=1}^n nP_{ij} \frac{\partial^2}{\partial x_i \partial x_j} h(x) \Big|_{x=\bar{x}}$$

针对样本数

针对X维数

$$\bar{y}_u = h(\bar{x}) + \frac{1}{2} \sum_{i,j=1}^n P_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j} \Big|_{x=\bar{x}} + \frac{1}{2n} \sum_{i=1}^{2n} \left[\frac{1}{4!} D_{\tilde{x}^{(i)}}^4 h + \frac{1}{6!} D_{\tilde{x}^{(i)}}^6 h + \dots \right]$$



6.5.2 Unscented Transformation

$$\bar{y} = h(\bar{x}) + \frac{1}{2!} E[D_{\tilde{x}}^2 h] + \frac{1}{4!} E[D_{\tilde{x}}^4 h] + \dots$$

理想期望值

$$\begin{aligned} \frac{1}{2!} E[D_{\tilde{x}}^2 h] &= \frac{1}{2!} E \left[\left(\sum_{i=1}^n \tilde{x}_i \frac{\partial}{\partial x_i} \right)^2 h(x) \right]_{x=\bar{x}} \\ &= \frac{1}{2!} E \left[\left(\sum_{i,j=1}^n \tilde{x}_i \tilde{x}_j \frac{\partial^2 h}{\partial x_i \partial x_j} \right) \right]_{x=\bar{x}} \\ &= \frac{1}{2!} \sum_{i,j=1}^n E(\tilde{x}_i \tilde{x}_j) \frac{\partial^2 h}{\partial x_i \partial x_j} \bigg|_{x=\bar{x}} \\ &= \frac{1}{2} \sum_{i,j=1}^n P_{ij} \frac{\partial^2}{\partial x_i \partial x_j} h(x) \bigg|_{x=\bar{x}} \end{aligned}$$

$$\bar{y} = h(\bar{x}) + \frac{1}{2} \sum_{i,j=1}^n P_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j} \bigg|_{x=\bar{x}} + \frac{1}{4!} D_{\tilde{x}^{(i)}}^4 h + \frac{1}{6!} D_{\tilde{x}^{(i)}}^6 h + \dots$$

$$\bar{y}_u = h(\bar{x}) + \frac{1}{2} \sum_{i,j=1}^n P_{i,j} \frac{\partial^2 h}{\partial x_i \partial x_j} \bigg|_{x=\bar{x}} + \frac{1}{2n} \sum_{i=1}^{2n} \left[\frac{1}{4!} D_{\tilde{x}^{(i)}}^4 h + \frac{1}{6!} D_{\tilde{x}^{(i)}}^6 h + \dots \right]$$



因此 \bar{y}_u 精度可以达到**3**阶



3) y 方差的计算

$$P_u = \sum_{i=1}^{2n} W^{(i)} (y^{(i)} - \bar{y}_u)(y^{(i)} - \bar{y}_u)^T = \frac{1}{2n} \sum_{i=1}^{2n} (y^{(i)} - \bar{y}_u)(y^{(i)} - \bar{y}_u)^T$$



$$\bar{y}_u = h(\bar{x}) + \frac{1}{2n} \sum_{i=1}^{2n} \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \frac{1}{2n} \sum_{i=1}^{2n} \left[\frac{1}{4!} D_{\tilde{x}^{(i)}}^4 h + \frac{1}{6!} D_{\tilde{x}^{(i)}}^6 h + \dots \right]$$

$$y^{(i)} = h(x^{(i)}) = h(\bar{x}) + D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \frac{1}{3!} D_{\tilde{x}^{(i)}}^3 h + \dots$$

$$P_u = \frac{1}{2n} \sum_{i=1}^{2n} \left[\cancel{h(\bar{x})} + D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots - \cancel{h(\bar{x})} - \frac{1}{2n} \sum_{j=1}^{2n} \left(\frac{1}{2!} D_{\tilde{x}^{(j)}}^2 h + \frac{1}{4!} D_{\tilde{x}^{(j)}}^4 h + \dots \right) \right] \bullet$$



$$P_u = \frac{1}{2n} \sum_{i=1}^{2n} \left[\cancel{h(\bar{x})} + D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots - \cancel{h(\bar{x})} - \frac{1}{2n} \sum_{j=1}^{2n} \left(\frac{1}{2!} D_{\tilde{x}^{(j)}}^2 h + \frac{1}{4!} D_{\tilde{x}^{(j)}}^4 h + \dots \right) \right]^T$$

$$P_u = \frac{1}{2n} \sum_{i=1}^{2n} \left[D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots - \frac{1}{2n} \sum_{j=1}^{2n} \left(\frac{1}{2!} D_{\tilde{x}^{(j)}}^2 h + \frac{1}{4!} D_{\tilde{x}^{(j)}}^4 h + \dots \right) \right] \bullet$$

$$\left[D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots - \frac{1}{2n} \sum_{j=1}^{2n} \left(\frac{1}{2!} D_{\tilde{x}^{(j)}}^2 h + \frac{1}{4!} D_{\tilde{x}^{(j)}}^4 h + \dots \right) \right]^T$$



6.5.2 Unscented Transformation

$$P_u = \frac{1}{2n} \sum_{i=1}^{2n} \left[D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots - \frac{1}{2n} \sum_{j=1}^{2n} \left(\frac{1}{2!} D_{\tilde{x}^{(j)}}^2 h + \frac{1}{4!} D_{\tilde{x}^{(j)}}^4 h + \dots \right) \right] \cdot \left[D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \dots - \frac{1}{2n} \sum_{j=1}^{2n} \left(\frac{1}{2!} D_{\tilde{x}^{(j)}}^2 h + \frac{1}{4!} D_{\tilde{x}^{(j)}}^4 h + \dots \right) \right]^T$$

$$P_u = \frac{1}{2n} \sum_{i=1}^{2n} \left[D_{\tilde{x}^{(i)}} h (D_{\tilde{x}^{(i)}} h)^T + \frac{1}{2} D_{\tilde{x}^{(i)}} h (D_{\tilde{x}^{(i)}}^2 h)^T + \frac{1}{2} D_{\tilde{x}^{(i)}}^2 h (D_{\tilde{x}^{(i)}} h)^T + \alpha D_{\tilde{x}^{(i)}}^2 h (D_{\tilde{x}^{(j)}}^2 h)^T + \dots \right]$$

$$P_u = \frac{1}{2n} \sum_{i=1}^{2n} \left[D_{\tilde{x}^{(i)}} h (D_{\tilde{x}^{(i)}} h)^T \right] + \text{HOT}$$

4阶以上项

$$\left\{ \begin{aligned} P_u &\approx \frac{1}{2n} \sum_{i=1}^{2n} \sum_{j,k=1}^n \left(\tilde{x}_j^{(i)} \frac{\partial h}{\partial x_j} \bigg|_{\bar{x}} \right) \left(\tilde{x}_k^{(i)} \frac{\partial h}{\partial x_k} \bigg|_{\bar{x}} \right)^T \\ \tilde{x}_j^{(i)} &= -\tilde{x}_j^{(i+n)} = \left(\sqrt{nP} \right)_i^T \text{ 的第 } j \text{ 个元素} \end{aligned} \right.$$

$$P_u = \frac{1}{n} \sum_{i=1}^n \sum_{j,k=1}^n n P_{jk} \left(\frac{\partial h}{\partial x_j} \bigg|_{\bar{x}} \right) \left(\frac{\partial h}{\partial x_k} \bigg|_{\bar{x}} \right)^T = H P H^T$$

精确到3阶





6.5.3 无迹滤波(UKF)

Unscented Kalman Filter

1) EKF的不足

$$\begin{cases} X(k+1) = \varphi[X(k), k] + \Gamma[X(k), k]W(k) \\ Z(k+1) = h[X(k+1), k+1] + V(k+1) \end{cases}$$

$$\text{EKF} \begin{cases} K(k+1) = P(k+1|k)H^T(k+1)[H(k+1)P(k+1|k) \cdot H^T(k+1) + R_{k+1}]^{-1} \\ P(k+1|k) = \Phi(k+1, k)P(k|k)\Phi^T(k+1, k) + \Gamma[\hat{X}(k|k), k]Q_k\Gamma^T[\hat{X}(k|k), k] \\ P(k+1|k+1) = [I - K(k+1)H(k+1)]P(k+1|k) \end{cases}$$



$$\phi(k+1, k) = \left. \frac{\partial \varphi}{\partial X(k)} \right|_{X(k)=\hat{X}(k)} \quad H(k) = \left. \frac{\partial h}{\partial X(k)} \right|_{X(k)=\hat{X}(k|k-1)}$$

不准确的系统模型

$K(k), P(k+1|k), P(k+1|k+1)$ 不准确，滤波结果难以达到最优。



6.5.3 无迹滤波(UKF)

2) Kalman 滤波的分析

$$\hat{X} = m_x + COV(X, Z)(VarZ)^{-1}(Z - m_z)$$

$$\begin{cases} X(k+1) = \varphi[X(k), k] + \Gamma[X(k), k]W(k) \\ Z(k+1) = h[X(k+1), k+1] + V(k+1) \end{cases}$$

$$K(k+1) = P(k+1|k)H^T(k+1) \left[H(k+1)P(k+1|k) \cdot H^T(k+1) + R_{k+1} \right]^{-1}$$

$$P(k+1|k)H^T(k+1) = E[\tilde{X}(k+1|k)\tilde{X}^T(k+1|k)]H^T(k+1)$$

$$= E[\tilde{X}(k+1|k)\tilde{X}^T(k+1|k)H^T(k+1)]$$

$$= P_{xy}(k+1|k)$$

如以 **y** 表示观测

$$H(k+1)P(k+1|k) \cdot H^T(k+1) + R$$

$$= E[H(k+1)\tilde{X}(k+1|k)\tilde{X}^T(k+1|k)H^T(k+1)] + R$$

$$= E[\tilde{Z}(k+1|k)\tilde{Z}^T(k+1|k)]$$

$$= P_{yy}(k+1|k)$$

$$K(k+1) = P_{xy}(k+1|k)P_{yy}^{-1}(k+1|k)$$





6.5.3 无迹滤波(UKF)

Kalman滤波最优增益 K 的获得依赖于 $P(k/k)$ 、 $P(k/k-1)$ 。在非线形系统中 $P(k/k)$ 、 $P(k/k-1)$ 的解算存在较大误差，所以估计结果会有一定的偏差。

如果 $P_{xy}(k+1|k)$ 、 $P_{yy}(k+1|k)$ 能够通过其他途径较为精确地获得，估计结果将更准确。

3) 获得 $P_{xy}(k+1|k)$ 、 $P_{yy}(k+1|k)$ 的思想

$$P_{xy}(k) = E\{[X(k) - E(X(k))][Y(k) - E(Y(k))]^T\}$$

$$P_{yy}(k) = E\{[Y(k) - E(Y(k))][Y(k) - E(Y(k))]^T\}$$

对于每一步 k ，如果我们有足够的样本 $X(k)$ 以及相应的 $Y(k)$ ， $P_{xy}(k)$ 与 $P_{yy}(k)$ 就能够较为准确地获得。



6.5.3 无迹滤波(UKF)

4) 如何获得 $P_{xy}(k+1|k)$, $P_{yy}(k+1|k)$

关键问题:

如何进行 X 的有效采样

如何在 X 的基础上获得 Y 的统计信息。

方法

Unscented Transformation (UT) 可以获得非线性变量间的统计特性, UT 可以提供以下两种重要的内容:

X 的有效采样方法

对 $y=f(x)$ 的统计指标估计



6.5.3 无迹滤波(UKF)

5) Unscented Transform——sigma采样 采样算法

UT变换有多种采样算法，在此仅介绍对称采样。

如何采样

Sigma point

$$\begin{cases} X_0 = \bar{X} \\ X_i = \bar{X} + \left(\sqrt{(L + \lambda) P_x} \right)_i & i = 1, \dots, L \\ X_i = \bar{X} - \left(\sqrt{(L + \lambda) P_x} \right)_{i-L} & i = L + 1, \dots, 2L \end{cases}$$

矩阵平方根的
第*i*列

其中

L = 状态的维数

$$\lambda = \alpha^2 (L + k) - L$$

k usually set to 0 or 3 - L

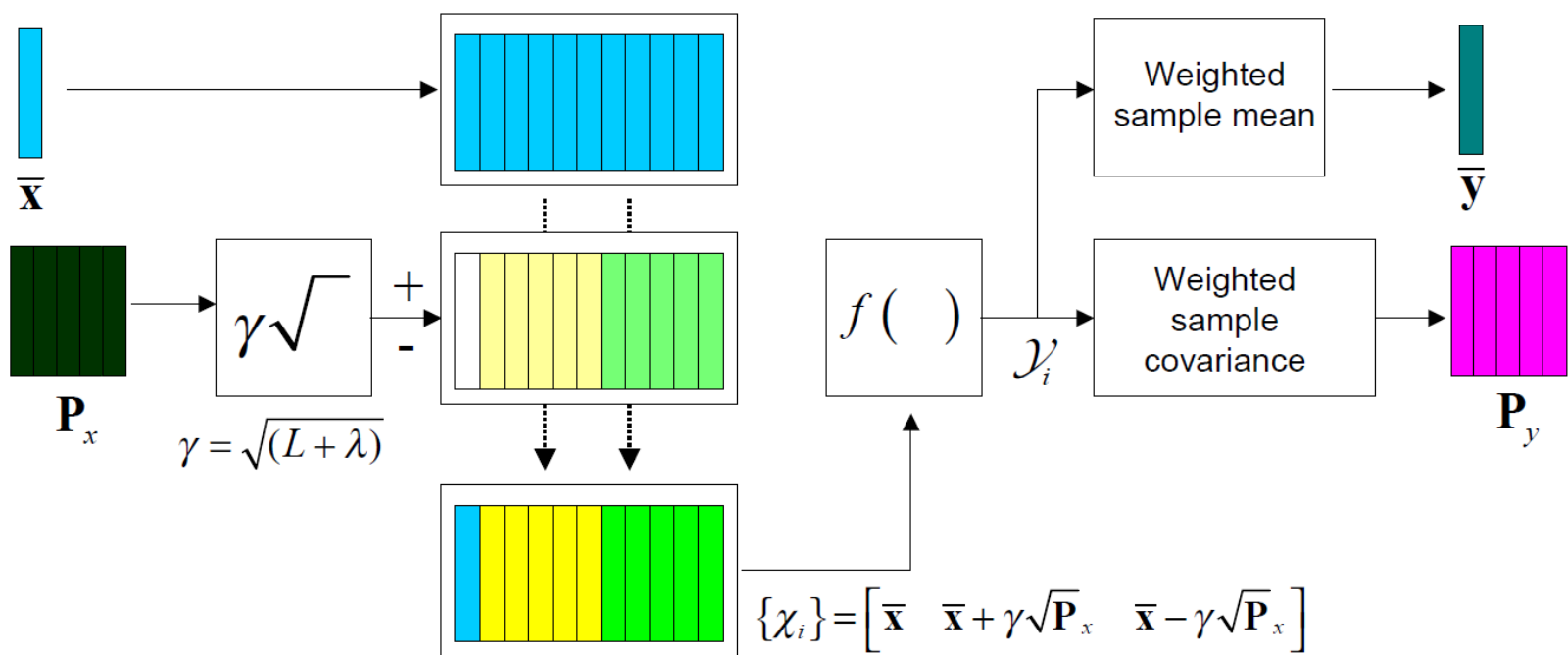
$$1e-4 \leq \alpha \leq 1$$

α 决定了sigma点在 \bar{X} 周围的分布, k 是相对次要的分布控制参数。





5) Unscented Transform





6.5.3 无迹滤波(UKF)

6) Unscented Transform——获得 **Y** 的统计特性

$$y_i = f(x_i) \quad i = 0, \dots, 2L.$$

$$\bar{y} = \sum_{i=0}^{2L} W_i^{(m)} y_i \quad i = 0, 1, \dots, L.$$

$$P_y \approx \sum_{i=0}^{2L} W_i^{(c)} (y_i - \bar{y})(y_i - \bar{y})^T$$

其中

$$W_0^{(m)} = \lambda / (L + \lambda)$$

$$W_0^{(c)} = \lambda / (L + \lambda) + (1 - \alpha^2 + \beta)$$

$$W_i^{(m)} = W_i^{(c)} = 1 / 2(L + \lambda) \quad i = 1, \dots, 2L$$

$$\lambda = \alpha^2 (L + K) - L$$

$$10^{-4} \leq \alpha \leq 1, \quad \kappa = 3 - L$$

β 与 x 的验前分布知识有关，正态分布最优值为2**。**

$$W_0^{(m)} = \lambda / (L + \lambda)$$

$$W_0^{(c)} = \lambda / (L + \lambda) + (1 - \alpha^2 + \beta)$$

$$W_i^{(m)} = W_i^{(c)} = 1 / 2(L + \lambda) \quad i = 1, \dots, 2L$$

$$\lambda = \alpha^2(L + K) - L$$

$$W_0^{(m)} = \lambda / (L + \lambda)$$

$$W_0^{(m)} = \frac{\lambda}{3\alpha^2} = \frac{3\alpha^2 - L}{3\alpha^2}$$

$$\lambda = \alpha^2(L + K) - L$$

$$(L + \lambda) = 3\alpha^2$$

$$K = 3 - L$$

$$W_0^{(m)} + 2LW_i^{(m)} = 1$$

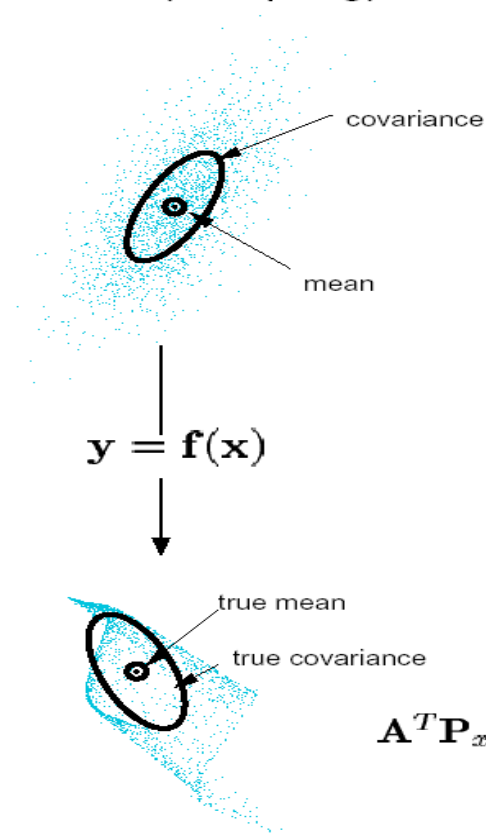
$$W_i^{(m)} = 1 / 2(L + \lambda)$$

$$W_i^{(m)} = \frac{1}{2} \cdot \frac{1}{3\alpha^2}$$

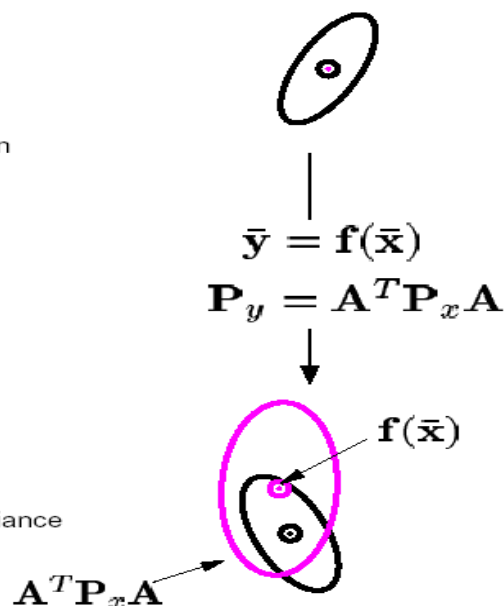


6.5.3 无迹滤波(UKF)

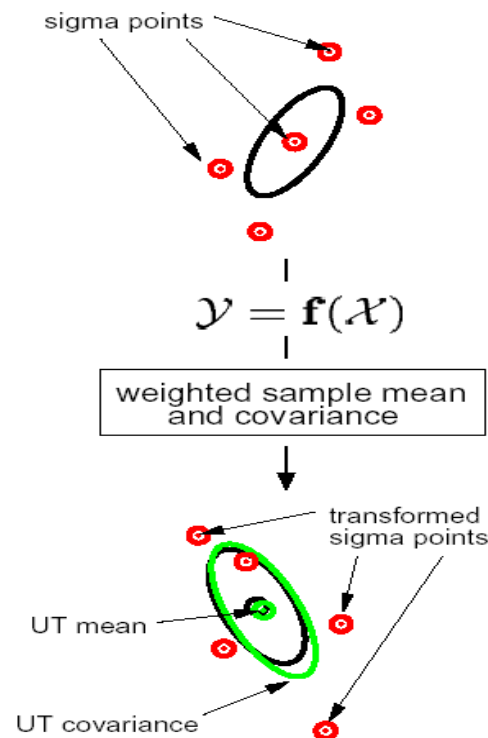
Actual (sampling)



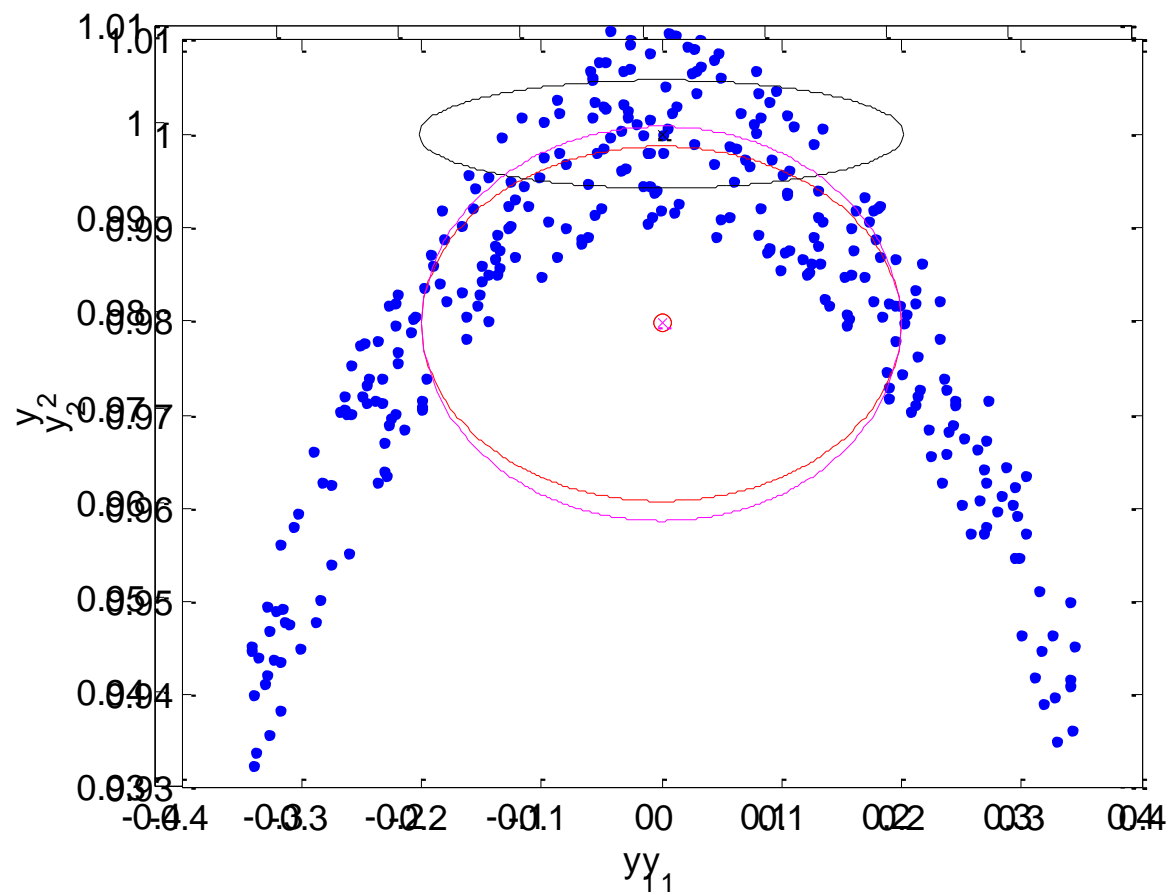
Linearized (EKF)



UT



The **EKF** only calculates the posterior mean and covariance accurately to the first order with all higher order moments truncated; however, **UT** calculates the mean and covariance to the second order.





6.5.3 无迹滤波(UKF)

7) 非加性噪声UKF方法

① 状态方程及初始条件

$$X_k^a = \begin{bmatrix} X_k^T & W_k^T & V_k^T \end{bmatrix}^T \quad P_k^a = \begin{bmatrix} P_k & 0 & 0 \\ 0 & Q_k & 0 \\ 0 & 0 & R_k \end{bmatrix}$$

对于加性噪声可以不采用扩维的方式。



扩维

$$\bar{X}_0 = E[X_0] \quad P_0 = E[(X_0 - \bar{X}_0)(X_0 - \bar{X}_0)^T]$$

$$\bar{X}_0^a = E[X^a] = [\bar{X}_0^T \quad 0 \quad 0]^T$$

$$P_0^a = E[(X_0^a - \bar{X}_0^a)(X_0^a - \bar{X}_0^a)^T] = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & Q_0 & 0 \\ 0 & 0 & R_0 \end{bmatrix}$$

② sigma point selecting

$$X_{k-1}^a = \begin{bmatrix} \bar{X}_{k-1}^a & \bar{X}_{k-1}^a \pm \sqrt{(L + \lambda) P_{k-1}^a} \end{bmatrix}$$



6.5.3 无迹滤波(UKF)

③ 时间更新

$$X_{k|k-1}^x = f\left(X_{k-1}^x, X_{k-1}^w\right)$$

$$\bar{X}_{k|k-1} = \sum_{i=0}^{2L} W_i^{(m)} X_{i,k|k-1}^x$$

$$P_{k|k-1} = \sum_{i=0}^{2L} W_i^{(c)} \left[X_{i,k|k-1}^x - \bar{X}_{k|k-1} \right] \left[X_{i,k|k-1}^x - \bar{X}_{k|k-1} \right]^T$$

$$Y_{k|k-1} = h\left(X_{k|k-1}^x, X_{k-1}^v\right)$$

$$\bar{y}_{k|k-1} = \sum_{i=0}^{2L} W_i^{(m)} Y_{i,k|k-1}$$





6.5.3 无迹滤波(UKF)

④ 量测更新

$$P_{yy} = \sum_{i=0}^{2L} W_i^{(c)} \left[Y_{i,k|k-1} - \bar{y}_{k|k-1} \right] \left[Y_{i,k|k-1} - \bar{y}_{k|k-1} \right]^T$$

$$P_{xy} = \sum_{i=0}^{2L} W_i^{(c)} \left[X_{i,k|k-1} - \bar{X}_{k|k-1} \right] \left[Y_{i,k|k-1} - \bar{y}_{k|k-1} \right]^T$$

$$K_k = P_{xy} P_{yy}^{-1}$$

$$\bar{X}_k = \bar{X}_{k|k-1} + K_k (y_k - \bar{y}_{k|k-1})$$

$$P_k = P_{k|k-1} - K_k P_{yy} K_k^T$$





6.5.3 无迹滤波(UKF)

8) 加性噪声UKF方法

$$X_{k+1} = f(X_k, U_k) + W_k$$

$$Y_k = h(X_k) + V_k$$

$$X_{k-1} = \begin{bmatrix} \hat{X}_{k-1} & \hat{X}_{k-1} \pm \sqrt{(L+\lambda)P_{k-1}} \end{bmatrix}$$

$$X_{k|k-1} = f(X_{k-1}, U_{k-1}) \longrightarrow \bar{X}_{k|k-1} = \sum_{i=0}^{2L} W_i^{(m)} X_{i,k|k-1}$$

$$P_{k|k-1} = \sum_{i=0}^{2L} W_i^{(c)} \left[X_{i,k|k-1} - \bar{X}_{k|k-1} \right] \left[X_{i,k|k-1} - \bar{X}_{k|k-1} \right]^T + Q_{k-1}$$

$$Y_{k|k-1} = h(X_{k|k-1}) \longrightarrow \bar{y}_{k|k-1} = \sum_{i=0}^{2L} W_i^{(m)} Y_{i,k|k-1}$$



8) 加性噪声UKF方法

$$P_{yy} = \sum_{i=0}^{2L} W_i^{(c)} \left[Y_{i,k|k-1} - \bar{y}_{k|k-1} \right] \left[Y_{i,k|k-1} - \bar{y}_{k|k-1} \right]^T + R_k$$

$$P_{xy} = \sum_{i=0}^{2L} W_i^{(c)} \left[X_{i,k|k-1} - \bar{X}_{k|k-1} \right] \left[Y_{i,k|k-1} - \bar{y}_{k|k-1} \right]^T$$

$$K_k = P_{xy} P_{yy}^{-1}$$

$$\bar{X}_k = \bar{X}_{k|k-1} + K_k (y_k - \bar{y}_{k|k-1})$$

$$P_k = P_{k|k-1} - K_k P_{yy} K_k^T$$



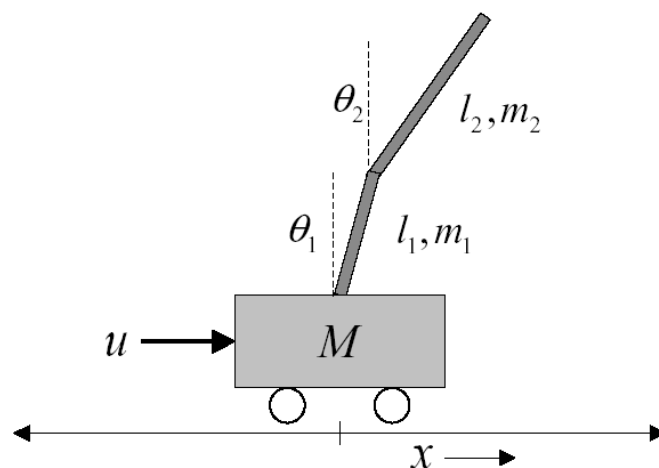


9) UKF的优点

UKF的优点

- 对非线性函数的概率密度分布进行近似，而不是对非线性函数进行近似，不需要知道非线性函数的显式表达式；
- 非线性函数统计量的精度至少达到**3**阶，对于采用特殊的采样策略，如高斯分布**4**阶采样和偏度采样等可达到更高阶精度；
- 计算量与**EKF**同阶；
- 不要求求导计算**Jacobian**矩阵,可以处理非可导的非线性函数

例子



$$\mathbf{\tilde{x}} = [x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2].$$

水平方向力平衡方程

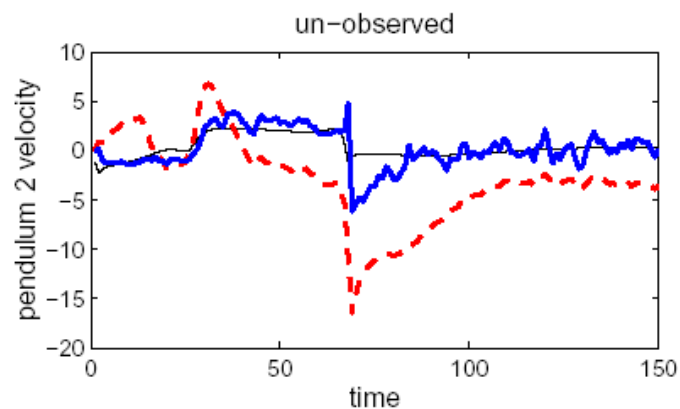
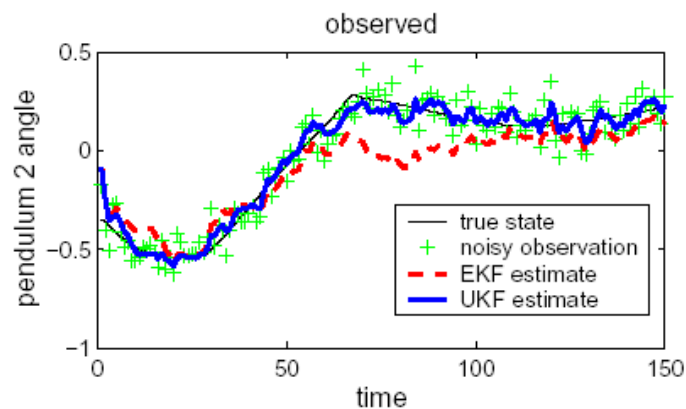
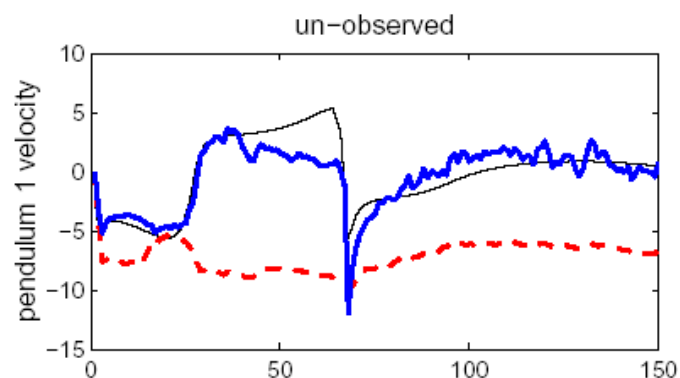
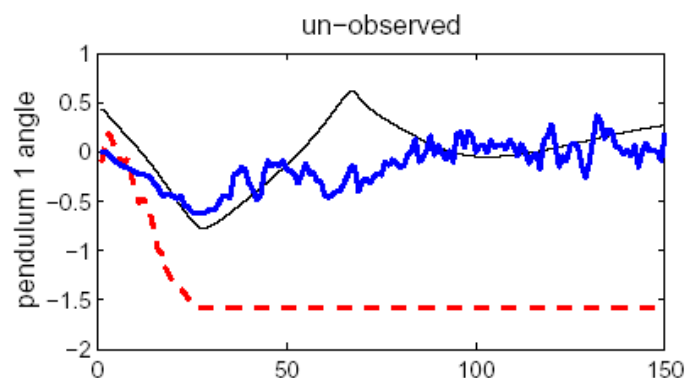
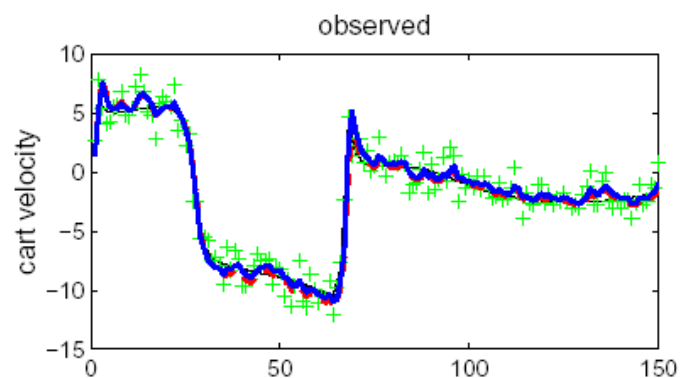
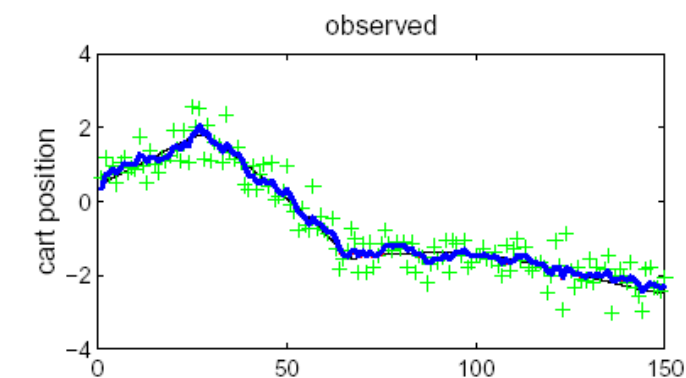
$$\begin{aligned} (M + m_1 + m_2)\ddot{x} - (m_1 + 2m_2)l_1\ddot{\theta}_1 \cos \theta_1 - m_2l_2\ddot{\theta}_2 \cos \theta_2 \\ = u + (m_1 + 2m_2)l_1\dot{\theta}_1^2 \sin \theta_1 + m_2l_2\dot{\theta}_2^2 \sin \theta_2 \end{aligned}$$

L1力矩平衡方程

$$\begin{aligned} -(m_1 + 2m_2)l_1\ddot{x} \cos \theta_1 + 4\left(\frac{m_1}{3} + m_2\right)l_1^2\ddot{\theta}_1 + 2m_2l_1l_2\ddot{\theta}_2 \cos(\theta_2 - \theta_1) \\ = (m_1 + 2m_2)gl_1 \sin \theta_1 + 2m_2l_1l_2\dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \end{aligned}$$

L2力矩平衡方程

$$\begin{aligned} -m_2\ddot{x}l_2 \cos \theta_2 + 2m_2l_1l_2\ddot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{4}{3}m_2l_2^2\ddot{\theta}_2 \\ = m_2gl_2 \sin \theta_2 - 2m_2l_1l_2\dot{\theta}_1^2 \sin(\theta_2 - \theta_1) \end{aligned}$$



相关内容请参阅

Optimal State Estimation

标称轨道及EKF 13.1~13.2

Unscented Kalman Filtering 14

Optimal Estimation of Dynamic Systems

EKF 3.6

Unscented Kalman Filtering 3.7



- [1] **S. Julier**, J. Uhlmann, and H. Durrant-Whyte, “A new approach for filtering nonlinear systems,” American Control Conference, pp. 1628-1632 (1995).
- [2] **S. Julier**, J. Uhlmann, and H. Durrant-Whyte, “A new method for the nonlin-lear transformation of means and covariances in filters and estimators,” IEEE Transactions on Automatic Control, 45(3), pp. 477-482 (March 2000).
- [3] **S. Julier** and J. Uhlmann, “Reduced sigma point filters for the propagation of means and covariances through nonlinear transformations,” American Control Conference, pp. 887-892, 2002.
- [4] **S. Julier**, “The spherical simplex unscented transformation,” American Control Conference, pp. 2430-2434, 2003.
- [5] **S. Julier** and J. Uhlmann, “Unscented filtering and nonlinear estimation,” Proceedings of the IEEE, 92(3), pp. 401-422 (March 2004).