第七章 次优滤波器的设计

7.1 Kalman滤波的计算特性

$$\hat{X}(k+1|k+1)=\hat{X}(k+1|k)+K(k+1)[Z(k+1)-H(k+1)\hat{X}(k+1|k)]$$

$$\hat{X}(k+1|k)]=\Phi(k+1,k)\hat{X}(k|k)$$

$$K(k+1)=P(k+1|k)H^T(k+1)[H(k+1)P(k+1|k)H^T(k+1)+R_{k+1}]^{-1}$$

$$P(k+1|k+1)=P(k+1|k)-$$

$$P(k+1|k)H^T(k+1)[H(k+1)P(k+1|k)H^T(k+1)+R_{k+1}]^{-1}\bullet$$

$$H(k+1)P(k+1|k)$$

$$P(k+1|k)=\Phi(k+1,k)P(k|k)\Phi^T(k+1,k)+\Gamma(k+1,k)Q_k\Gamma^T(k+1,k)$$

状态X为n维，观测Z为m维，Kalman滤波计算与存储量

加法运算： $4n^3+(1+4m)n^2+(2m^2+2m)n+m^3$

乘法运算： $4n^3+(4m-2)n^2-(2m+1)n+m^3$

存储空间： $4n^2+(2m+1)n+m^2+m$

7.2 简化增益阵的次优设计

$$S(t)=a+bt+ct^2$$

$$Z(t)=S(t)+V(t)$$

$$X_1(t)=S(t)$$
$$X_2(t)=\dot{S}(t)$$
$$X_3(t)=\ddot{S}(t)$$

$$\begin{bmatrix}\dot{X}_1\\\dot{X}_2\\\dot{X}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}W$$

$$Z(t)=\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}+V$$

此类系统为定常系统，且一致完全可观、可控，因此在噪声方差不变情况下，滤波稳定后，**P(k+1|k)**与**P(k+1|k+1)**都将趋于极限，根据最优增益**K**的计算公式，**K**也将趋于极限，因此在滤波过程中无需计算**K**阵。特别地，在滤波初始阶段也可以使用**K**的极限作为Kalman增益，滤波计算最终也将达到理想状态，这种情况下可视为次优滤波。

7.2.1 α-β滤波

1）连续系统的α-β滤波

$$S(t)=a+bt$$

$$Z(t)=S(t)+V(t)$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}W(t)$$

$$Z(t)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+V(t)$$

$$\boldsymbol{A}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad \boldsymbol{F}=\begin{bmatrix}0\\ 1\end{bmatrix},\quad \boldsymbol{H}=\begin{bmatrix}1 & 0\end{bmatrix},$$

$$\boldsymbol{E}\left[\boldsymbol{W}(\boldsymbol{t})\boldsymbol{W}^{T}(\boldsymbol{\tau})\right]=\boldsymbol{q}^{2}\delta(\boldsymbol{t}-\boldsymbol{\tau})$$

$$\boldsymbol{E}\left[\boldsymbol{V}(\boldsymbol{t})\boldsymbol{V}^{T}(\boldsymbol{\tau})\right]=\boldsymbol{r}^{2}\delta(\boldsymbol{t}-\boldsymbol{\tau})$$

$$\boldsymbol{E}[\boldsymbol{W}(\boldsymbol{t})]=0,\ \boldsymbol{E}[\boldsymbol{V}(\boldsymbol{t})]=0$$

$$\begin{bmatrix}F & AF\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \begin{bmatrix}H^{T} & A^{T}H^{T}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\dot{\hat{X}}(t|t)=A(t)\hat{X}(t|t)+K(t)[Z(t)-H(t)\hat{X}(t|t)]$$

$$K(t)=P(t|t)H^{T}(t)R^{-1}(t)$$

$$\dot{P}(t|t)=A(t)P(t|t)+P(t|t)A^{T}(t)-P(t|t)H^{T}(t)R^{-1}(t)H(t)P(t|t)+F(t)Q(t)F^{T}(t)$$

7.2.1 α-β滤波

1）连续系统的α-β滤波

$$\dot{P}(t|t)=A(t)P(t|t)+P(t|t)A^T(t)-P(t|t)H^T(t)R^{-1}(t)H(t)P(t|t)+F(t)Q(t)F^T(t)$$

$$\begin{bmatrix}\dot{p}_{11} & \dot{p}_{12}\\ \dot{p}_{21} & \dot{p}_{22}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}+\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}-\frac{1}{r^2}\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}+$$

$$q^2\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

滤波达到稳态时，**P**阵不变化

$$p_{11}=r\sqrt{2qr}$$

$$p_{12}=qr$$

$$p_{22}=q\sqrt{2qr}$$

$$K=\begin{bmatrix}k_1\\ k_2\end{bmatrix}=\frac{1}{r^2}\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}\sqrt{\dfrac{2q}{r}}\\ \dfrac{q}{r}\end{bmatrix}$$

$$\boxed{h=\frac{q}{r}\qquad K=\begin{bmatrix}k_1\\ k_2\end{bmatrix}=\begin{bmatrix}\sqrt{2h}\\ h\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}$$

7.2.1 α-β滤波

2）离散系统的α-β滤波

$$X(k+1)=\begin{bmatrix}1 & T\\0 & 1\end{bmatrix}X(k)+\begin{bmatrix}T^2/2\\T\end{bmatrix}W'(k)$$

$$X(k+1)=\begin{bmatrix}1 & T\\0 & 1\end{bmatrix}X(k)+W(k)$$

$$Z(t)=\begin{bmatrix}1 & 0\end{bmatrix}X(k)+V(t)$$

$$E[W'(k)]=E[V(k)]=0$$

$$E[W'(k)W'^T(j)]=\sigma^2\delta_{kj}$$

$$E[V(k)V^T(j)]=R\delta_{kj}$$

$$E[W(k)V^T(j)]=0$$

$$Q=\begin{bmatrix}T^2/2\\T\end{bmatrix}E[W'(k)W'^T(k)]\begin{bmatrix}T^2/2 & T\end{bmatrix}$$

$$=\begin{bmatrix}T^4/4 & T^3/2\\T^3/2 & T^2\end{bmatrix}\sigma^2$$

$$P(k+1|k)=P^-=\begin{bmatrix}P_{11}^- & P_{12}^-\\P_{21}^- & P_{22}^-\end{bmatrix}$$

$$P(k+1|k+1)=P^+=\begin{bmatrix}P_{11}^+ & P_{12}^+\\P_{21}^+ & P_{22}^+\end{bmatrix}$$

定常系统且Q、R恒定，则滤波稳定后P(k+1|k)与P(k+1|k+1)恒定不变。

7.2.1 α-β滤波

2）离散系统的α-β滤波

$$K(k+1)=P(k+1|k)H^T(k+1)[H(k+1)P(k+1|k)H^T(k+1)+R_{k+1}]^{-1}$$

$$\boldsymbol{P}(\boldsymbol{k}+1|\boldsymbol{k})=\begin{bmatrix}\boldsymbol{P}_{11}^- & \boldsymbol{P}_{12}^- \\ \boldsymbol{P}_{21}^- & \boldsymbol{P}_{22}^-\end{bmatrix}$$

$$\boldsymbol{K}=\begin{bmatrix}\boldsymbol{P}_{11}^- & \boldsymbol{P}_{12}^- \\ \boldsymbol{P}_{21}^- & \boldsymbol{P}_{22}^-\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{P}_{11}^- & \boldsymbol{P}_{12}^- \\ \boldsymbol{P}_{21}^- & \boldsymbol{P}_{22}^-\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}+\boldsymbol{R}\right)^{-1}=\frac{1}{\boldsymbol{P}_{11}^-+\boldsymbol{R}}\begin{bmatrix}\boldsymbol{P}_{11}^-\\ \boldsymbol{P}_{12}^-\end{bmatrix}=\begin{bmatrix}\boldsymbol{K}_1\\ \boldsymbol{K}_2\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta/\boldsymbol{T}\end{bmatrix}$$

$$P(k+1|k+1)=\left(I-KH\right)P^T(k+1|k)$$

$$\begin{bmatrix}\boldsymbol{P}_{11}^+ & \boldsymbol{P}_{12}^+ \\ \boldsymbol{P}_{21}^+ & \boldsymbol{P}_{22}^+\end{bmatrix}=\left(\boldsymbol{I}-\begin{bmatrix}\boldsymbol{K}_1\\ \boldsymbol{K}_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{P}_{11}^- & \boldsymbol{P}_{12}^- \\ \boldsymbol{P}_{21}^- & \boldsymbol{P}_{22}^-\end{bmatrix}=\begin{bmatrix}\left(\boldsymbol{I}-\boldsymbol{K}_1\right)\boldsymbol{P}_{11}^- & \left(\boldsymbol{I}-\boldsymbol{K}_1\right)\boldsymbol{P}_{12}^- \\ \left(\boldsymbol{I}-\boldsymbol{K}_1\right)\boldsymbol{P}_{12}^- & \boldsymbol{P}_{22}^- -\boldsymbol{K}_2\boldsymbol{P}_{12}^-\end{bmatrix}$$

7.2.1 α-β滤波

2）离散系统的α-β滤波

$$P(k+1|k)=\Phi(k+1,k)P(k|k)\Phi^T(k+1,k)+\Gamma(k+1,k)Q_k\Gamma^T(k+1,k)$$

$$\boldsymbol{P}^- = \Phi \boldsymbol{P}^+ \Phi^{\boldsymbol{T}} + \boldsymbol{Q}$$

$$\Phi = \begin{bmatrix} 1 & \boldsymbol{T} \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{P}^+ = \Phi^{-1}\left(\boldsymbol{P}^- - \boldsymbol{Q}\right)\Phi^{-\boldsymbol{T}}$$

$$\begin{bmatrix} \boldsymbol{P}_{11}^+ & \boldsymbol{P}_{12}^+ \\ \boldsymbol{P}_{21}^+ & \boldsymbol{P}_{22}^+ \end{bmatrix} = \begin{bmatrix} 1 & -\boldsymbol{T} \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \boldsymbol{P}_{11}^- & \boldsymbol{P}_{12}^- \\ \boldsymbol{P}_{21}^- & \boldsymbol{P}_{22}^- \end{bmatrix} - \begin{bmatrix} \boldsymbol{T}^4/4 & \boldsymbol{T}^3/2 \\ \boldsymbol{T}^3/2 & \boldsymbol{T}^2 \end{bmatrix} \boldsymbol{\sigma}^2 \right) \begin{bmatrix} 1 & 0 \\ -\boldsymbol{T} & 1 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{P}_{11}^+ & \boldsymbol{P}_{12}^+ \\ \boldsymbol{P}_{21}^+ & \boldsymbol{P}_{22}^+ \end{bmatrix} = \begin{bmatrix} \left(\boldsymbol{I}-\boldsymbol{K}_1\right)\boldsymbol{P}_{11}^- & \left(\boldsymbol{I}-\boldsymbol{K}_1\right)\boldsymbol{P}_{12}^- \\ \left(\boldsymbol{I}-\boldsymbol{K}_1\right)\boldsymbol{P}_{12}^- & \boldsymbol{P}_{22}^- - \boldsymbol{K}_2\boldsymbol{P}_{12}^- \end{bmatrix}$$

$$\boldsymbol{K}_1 = -\frac{1}{8}\left[\lambda^2+8\lambda-(\lambda-4)\sqrt{\lambda^2+8\lambda}\right]$$

$$\boldsymbol{K}_2 = \frac{1}{4\boldsymbol{T}}\left(\lambda^2+4\lambda-\lambda\sqrt{\lambda^2+8\lambda}\right)$$

$$\lambda = \frac{\boldsymbol{\sigma}^2\boldsymbol{T}^2}{\boldsymbol{R}}$$

$$\boldsymbol{P}_{11}^- = \frac{\boldsymbol{K}_1\boldsymbol{\sigma}^2}{1-\boldsymbol{K}_1} \qquad \boldsymbol{P}_{12}^- = \frac{\boldsymbol{K}_2\boldsymbol{\sigma}^2}{1-\boldsymbol{K}_1}$$

$$\boldsymbol{P}_{22}^- = \left(\frac{\boldsymbol{K}_1}{\boldsymbol{T}} + \frac{\boldsymbol{K}_2}{2}\right)\boldsymbol{P}_{12}^-$$

$$\boldsymbol{P}_{11}^+ = \boldsymbol{K}_1\boldsymbol{R} \qquad \boldsymbol{P}_{12}^+ = \boldsymbol{K}_2\boldsymbol{R}$$

$$\boldsymbol{P}_{22}^+ = \left(\frac{\boldsymbol{K}_1}{\boldsymbol{T}} - \frac{\boldsymbol{K}_2}{2}\right)\boldsymbol{P}_{12}^-$$

7.2.2 α-β-γ滤波

1）连续系统的α-β-γ滤波

$$S(t)=a+bt+ct^2$$

$$Z(t)=S(t)+V(t)$$

$$X_1(t)=S(t)$$
$$X_2(t)=\dot{S}(t)$$
$$X_3(t)=\ddot{S}(t)$$

$$\Rightarrow \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}W(t), \quad Z(t)=[1\quad 0\quad 0]\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}+V(t)$$

$$\begin{bmatrix}F & AF & A^2F\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \quad \begin{bmatrix}H^T & A^TH^T & (A^T)^2H^T\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

滤波达到稳态时，**P**阵不变化

$$\boxed{K=\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}2\sqrt[3]{h}\\2\sqrt[3]{h^2}\\h\end{bmatrix}=\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}, \qquad h=\frac{q}{r}}$$

7.2.2 α-β-γ滤波

2）离散系统的α-β-γ滤波

$$X(k+1)=\begin{bmatrix}1 & T & T^2/2\\ 0 & 1 & T\\ 0 & 0 & 1\end{bmatrix}X(k)+\begin{bmatrix}T^2/2\\ T\\ 1\end{bmatrix}W'(k)$$

$$X(k+1)=\begin{bmatrix}1 & T & T^2/2\\ 0 & 1 & T\\ 0 & 0 & 1\end{bmatrix}X(k)+W(k)$$

$$Z(t)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}X(k)+V(t)$$

$$Q=\begin{bmatrix}T^2/2\\ T\\ 1\end{bmatrix}E[W'(k)W'^T(k)]\begin{bmatrix}T^2/2 & T & 1\end{bmatrix}$$

$$=\begin{bmatrix}T^4/4 & T^3/2 & T^2/2\\ T^3/2 & T^2 & T\\ T^2/2 & T & 1\end{bmatrix}\sigma^2$$

$$K=\begin{bmatrix}K_1 & K_2 & K_3\end{bmatrix}^T=\begin{bmatrix}\alpha & \beta/T & \gamma/T^2\end{bmatrix}^T$$

$$\alpha=1-s^2 \qquad b=0.5\lambda-3$$

$$\beta=2\left(1-s^2\right) \qquad c=0.5\lambda+3$$

$$\gamma=2\lambda s \qquad p=c-\left(b^2/3\right)$$

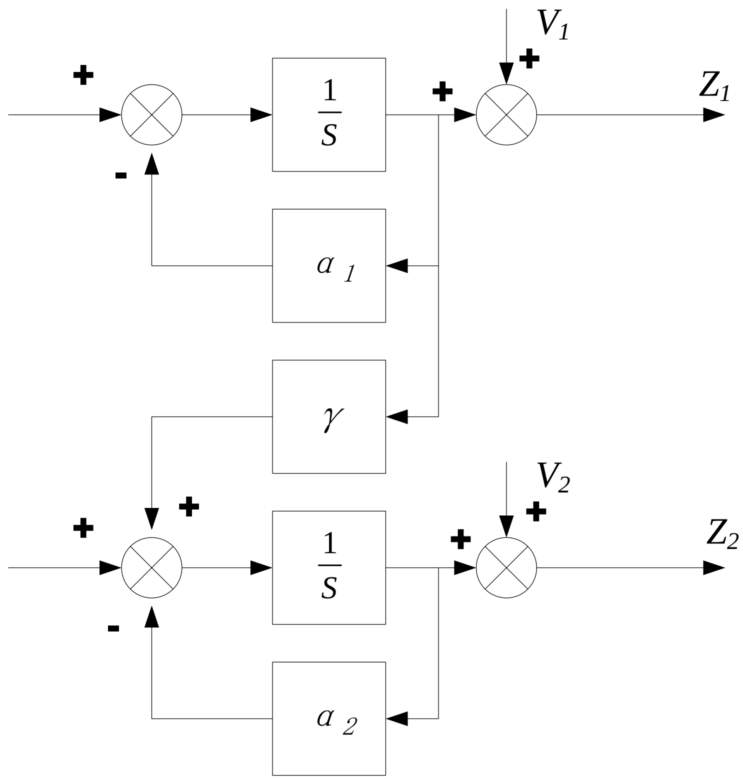
$$q=\frac{2b^3}{27}-\frac{bc}{3}-1 \qquad z=\left[\frac{-q+\sqrt{q^2+4p^3/27}}{2}\right]^{1/3}$$

$$s=z-\frac{p}{3z}-\frac{b}{3} \qquad \lambda=\frac{\sigma^2T^2}{R}$$

7.3简化模型的次优滤波器设计

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 0 \\ \gamma & -\alpha_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



若γ 很小，忽略其存在有

$$\begin{cases} \dot{x}_1(t) = -\alpha_1 x_1(t) + w_1 \\ z_1 = x_1(t) + v_1 \end{cases}$$

$$\begin{cases} \dot{x}_2(t) = -\alpha_2 x_2(t) + w_2 \\ z_2 = x_2(t) + v_2 \end{cases}$$

有关α-β及α-β-γ滤波内容请参见

Optimal State Estimation 7.3.1-7.3.2
Optimal Estimation of Dynamic Systems 7.4