Nonlinear Control Theory

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Introduction

1. **Nonlinear models and nonlinear phenomena**
2. Examples

Nonlinear models and nonlinear phenomena

Consider dynamical systems that are modeled by a finite number of coupled first-order ordinary differential equations:

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \cdots, x_n, u_1, \cdots, u_p), \\
\dot{x}_2 &= f_2(t, x_1, \cdots, x_n, u_1, \cdots, u_p), \\
&\vdots \\
\dot{x}_n &= f_n(t, x_1, \cdots, x_n, u_1, \cdots, u_p),
\end{aligned}
$$

where

- $x_1, \cdots, x_n$: state variables;
- $\dot{x}_1, \cdots, \dot{x}_n$: derivatives w.r.t. time t;
- $u_1, \cdots, u_p$: specified input variables.

Vector notations:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix}.$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

- One n-dimensional first-order vector differential equation (state equation):

$$\dot{x} = f(t,x,u).$$

- Output equation (for particular interest):

$$y = h(t,x,u).$$

- Unforced state equation:

$$\dot{x} = f(t, x).$$

  * Working with an unforced state equation does not necessarily mean that the input to the system is zero.
  * Possibly $u = \gamma(x)$ or $u = \gamma(t, x)$.

- Autonomous system (time-invariant system):

$$\dot{x} = f(x).$$

  * If the system is not autonomous, then it is called nonautonomous or time varying.

Definition (Equilibrium point)

A point $x = x^*$ in the state space is said to be an equilibrium point of $\dot{x} = f(t, x)$, if it has the property that whenever the state of the system starts at x^*, it will remain at x^* for all future time.

- For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real roots of the equation
$$f(x) = 0.$$
  * An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points.

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For linear systems, the state model take the special form:

$$\begin{aligned}\dot{x} &= A(t)x + B(t)u,\\ y &= C(t)x + D(t)u.\end{aligned}$$

- Linear systems: superposition principle.
- For nonlinear systems, the superposition principle does not hold any longer.
- Linearization around some operation points.
- However, linearization alone will not be sufficient.
  - Linearization predicts only “local” behavior; it cannot predict the “global” behavior throughout the state space.
  - There are “essentially nonlinear phenomena” that cannot be described by linear models

Essentially nonlinear phenomena:

- **Finite escape time:** A nonlinear system's state can go to infinity in finite time.
- **Multiple isolated equilibria:** A nonlinear system can have more than one isolated equilibrium point. The state may converge to one of several operating points, depending on the initial state.
- **Limit cycles:** There are nonlinear systems that can go into an oscillation of fixed amplitude and frequency, irrespective of the initial state.
- **Subharmonic, harmonic, or almost-periodic oscillations:** A nonlinear system under periodic excitation can oscillate with frequencies of submultiples or multiples of the input frequency.
- **Chaos:** The steady-state behavior of a nonlinear system is possibly not equilibrium, periodic or almost-periodic oscillation, but exhibits randomness, despite the deterministic nature of the system.
- **Multiple modes of behavior:** More than one limit cycle; harmonic, subharmonic, or more complicated steady-state behavior; discontinuous jump, etc.

1. Nonlinear models and nonlinear phenomena
2. **Examples**

Examples – Pendulum

Parameters and variables:

- l denotes the length of the rod, and m denotes the mass of the bob. Assume the rod is rigid and has zero mass.
- θ denotes the angle subtended by the rod and the vertical axis through the pivot point.
- g is the acceleration due to gravity.
- There is a frictional force resisting the motion, which is proportional to the speed of the bob with a coefficient of friction k

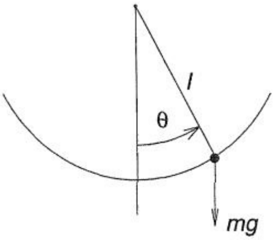


Figure: Pendulum

Using Newton's second law of motion:

$$ml\ddot{\theta} = -mg\sin\theta - k\dot{\theta}.$$

Take the state variables: $x_1 = \theta$ and $x_2 = \dot{\theta}$:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2.\end{aligned}$$

Set $\dot{x}_1 = \dot{x}_2 = 0$ to find the equilibrium points:

$$0 = x_2, \;\; 0 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \quad \Rightarrow \quad (x_1, x_2)_{eq} = (n\pi, 0), \;\; n = 0, \pm 1, \pm 2, \cdots$$

If the frictional resistance is neglected ($k = 0$), the resulting system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1\end{aligned}$$

is conservative.

* If the pendulum is given an initial push, it will keep oscillating forever with a nondissipative energy exchange between kinetic and potential energies.

If we can apply a torque T as the control input to the pendulum, then

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T.\end{aligned}$$

Example – Tunnel-Diode Circuit

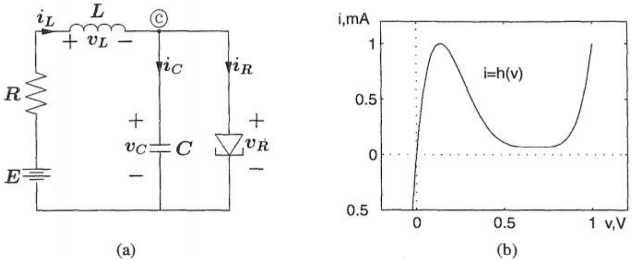


Figure: (a) Tunnel-diode circuit; (b) Tunnel-diode $v_R - i_R$ characteristic.

- $i_R = h(v_R), \quad i_C = C\frac{dv_C}{dt}, \quad v_L = L\frac{di_L}{dt}$
- States: $x_1 = v_C$ and $x_2 = i_L$; Input: $u = E$
- Kirchhoff's current law: $i_C + i_R - i_L = 0 \quad \Rightarrow \quad i_C = -h(x_1) + x_2$
- Kirchhoff's voltage law: $v_C - E + Ri_L + v_L = 0 \quad \Rightarrow \quad v_L = -x_1 - Rx_2 + u$
- State model:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2], \\ \dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u] \end{aligned}$$

The equilibrium points are determined by setting $\dot{x}_1 = \dot{x}_2 = 0$:

$$0 = \frac{1}{C}[-h(x_1) + x_2],$$
$$0 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

$$\Rightarrow \quad h(x_1) = \frac{E}{R} - \frac{1}{R}x_1.$$

Three isolated equilibrium points!!

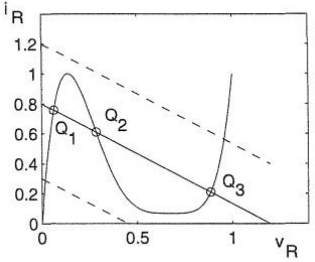


Figure: Equilibrium points of the tunnel diode circuit

Example – Mass-Spring System

- Newton's law of motion:

$$m\ddot{y} + F_f + F_{sp} = F$$

where F_f is a resistive force due to friction, and F_{sp} is the restoring force of the spring.

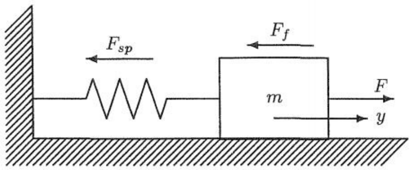


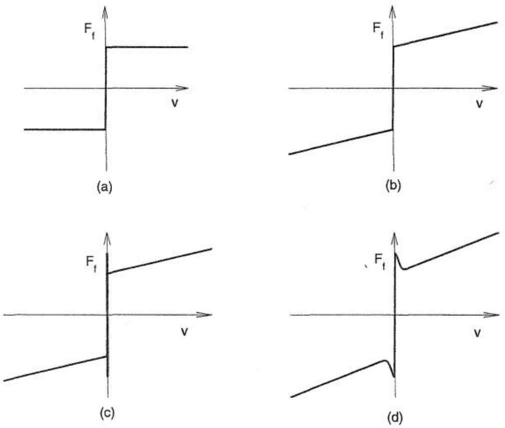
Figure: Mass-spring mechanical system

The restoring force of the spring:

- $F_{sp} = g(y)$, and $g(0) = 0$.
- For small displacement: $g(y) = ky$ with constant k.
- Large displacement, softening spring: $g(y) = k(1 - a^2y^2)y$ with $|ay| < 1$.
- Large displacement, hardening spring: $g(y) = k(1 + a^2y^2)y$.

The resistive force:

- Static friction $F_s = \pm\mu_s mg$ with $0 < \mu_s < 1$ to keep the mass at rest.
- Coulomb friction $F_c = -\mu_k mg$ for $v < 0$, and $F_c = \mu_k mg$ for $v > 0$.
- Viscous friction $F_v = h(v)$ with $h(0) = 0$ which is possibly nonlinear.



(a) Coulomb friction
(b) Coulomb plus linear viscous friction
(c) Static, Coulomb, and linear viscous friction
(d) Static, Coulomb, and linear viscous friction – Stribeck effect

- Hardening spring + linear viscous friction + a periodic external force $F = A\cos\omega t$

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t$$

Duffing's equation, which is a typical example of periodic excitation.

- Linear spring + static friction + Coulomb friction + linear viscous friction

$$m\ddot{y} + c\dot{y} + ky + \eta(y,\dot{y}) = 0$$

where

$$\eta(y,\dot{y}) = \begin{cases} \mu_k mg\,\mathrm{sign}(\dot{y}), & \text{for } |\dot{y}| > 0, \\ -ky, & \text{for } \dot{y} = 0 \text{ and } |y| \le \mu_s mg/k, \\ -\mu_s mg\,\mathrm{sign}(\dot{y}), & \text{for } \dot{y} = 0 \text{ and } |y| > \mu_s mg/k. \end{cases}$$

- Let $x_1 = y$ and $x_2 = \dot{y}$,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 - \frac{1}{m}\eta(x_1, x_2).\end{aligned}$$

* Equilibrium set

$$\dot{x}_1 = 0,\ \dot{x}_2 = 0 \quad \Rightarrow \quad x_2 = 0,\ -\mu_s mg/k \le x_1 \le \mu_s mg/k.$$

* Discontinuity

$$\text{When } x_2 > 0,\ \eta = \mu_k mg; \quad \text{when } x_2 < 0,\ \eta = -\mu_k mg.$$

This is an example of *piecewise linear analysis*, where a system is represented by linear models in various regions of the state space, certain coefficients changing from region to region.