模型预测控制

Model Predictive Control

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2022 Spring

Robust MPC

Lecture 4

- Robust MPC for bounded disturbances
- Robust MPC for model uncertainties

Problem Formulation

- In the real world, disturbances are inevitable !!
- Suppose that, the plant is given by

$$x(k+1) = Ax(k) + Bu(k) + Dw(k), \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^p,\ w \in \mathbb{R}^l$$

  where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ are state and control input, respectively; $w \in \mathbb{R}^l$ is the uncertain disturbance satisfying a known bound $w \in \mathcal{W}$.

- The matrices A, B, D are known with proper dimensions. (A, B) is controllable/stabilizable.
- The objective is to design robust MPC, such that x=0 is stabilized, and states and control satisfies their constraints

$$x \in \mathcal{X}, \quad u \in \mathcal{U}$$

Problem Formulation (cont'd)

- Since we are discussing constrained control, the bound for the disturbance should not be too large, and the control constraints should not be too small.

- <u>Assumption</u>: $\mathscr{X}$, $\mathscr{U}$, $\mathscr{W}$ are convex sets.

- <u>Definition (robust invariant set)</u>: A set S is a robust invariant set of the plant

$$x(k+1) = Ax(k) + Bu(k) + Dw(k),$$

  if there exists a feedback control $u = -Kx$, such that

$$x(k) \in S \qquad \Rightarrow \qquad x(k+1) \in S, \forall w \in \mathscr{W}.$$

- <u>Assumption</u>: For the plant with bounded disturbance, a robust invariant set S exists s.t.

$$S \in \mathscr{X}, \;\; KS \in \mathscr{U}.$$

Robust MPC with linear feedback control ("pre-stabilized" MPC)

- Since $w(k)$ is uncertain, we cannot use the real model

$$x(k+1) = Ax(k) + Bu(k) + Dw(k),$$

  to predict its future behavior.

- We have to use the nominal model for prediction

$$z(k+1) = Az(k) + Bv(k)$$

  where $z(0) = x(0)$.

- The structure of the robust MPC is "**nominal MPC** + **linear feedback control**".

MPC for the nominal plant

- The state sequence can be predicted by

$$Z(k) = Fz(k) + \Phi V(k)$$

where

$$F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$Z(k) = [z^T(1\,|\,k), \cdots, z^T(N\,|\,k)]^T, \quad V(k) = [v^T(0\,|\,k), \cdots, z^T(0\,|\,k)]^T$$

MPC for the nominal plant (cont'd)

- For the nominal system, the cost function is constructed by

$$
\begin{aligned}
J(k) &= \|z(N\,|\,k)\|_P^2 + \|v(N-1\,|\,k)\|_R^2 + \sum_{i=1}^{N-1} \|z(i\,|\,k)\|_Q^2 + \|v(i-1\,|\,k)\|_R^2 \\
&= Z^T(k)\mathcal{Q}Z(k) + V^T(k)\mathcal{R}V(k) \\
&= (Fz(k) + \Phi V(k))^T \mathcal{Q}(Fz(k) + \Phi V(k)) + V^T(k)\mathcal{R}V(k) \\
&= z^T(k)F^T\mathcal{Q}Fz(k) + 2z^T(k)F^T\mathcal{Q}\Phi V(k) + V^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})V(k),
\end{aligned}
$$

where

$$
\mathcal{Q} = \mathrm{diag}[Q, Q, \cdots, Q, P], \quad \mathcal{R} = \mathrm{diag}[R, R, \cdots, \cdots, R],
$$

$$
P - (A - BK)^T P(A - BK) = Q + K^T RK,
$$

and the feedback gain K satisfies $|\,\mathrm{eig}(A - BK)\,| < 1.$

MPC for the nominal plant (cont'd)

- The constrained optimization in the nominal MPC is formulated by

$$V^*(k) = \arg\min_{V(k)} \left[z^T(k)F^T\mathcal{Q}Fz(k) + 2z^T(k)F^T\mathcal{Q}\Phi V(k) + V^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})V(k) \right],$$

subject to

$$z(i\,|\,k) \in \mathcal{Z},\ v(i-1\,|\,k) \in \mathcal{V},\ \ \textbf{for}\ \ i = 1,2,\cdots,N,$$

$$z(N\,|\,k) \in \mathcal{Z}_f$$

where $\mathcal{Z}$ and $\mathcal{V}$ are "contractive" constraints, and they will be discussed later; $\mathcal{Z}_f$ is the terminal set (which can be chosen to be an invariant set).

MPC for the nominal plant (cont'd)

- The nominal MPC is implemented in receding horizon scheme:
  - At time k, implement the first optimal control action:

$$v(k) = v^*(0 \mid k) = [I_{p\times p}, 0, \cdots, 0]V^*(k)\,.$$

  - At time $k+1$, repeat the above procedures, and implement the first control action.

- It is clear that, the closed-loop nominal system satisfies recursive feasibility and closed-loop stability, i.e.,

$$z(k) \to 0 \quad \textbf{as} \quad k \to \infty,$$

and $z(k)$, $v(k)$ satisfy their constraints.

Robust MPC with linear feedback

- For the real plant

$$x(k+1) = Ax(k) + Bu(k) + Dw(k),$$

its control can be designed by

$$u(k) = v(k) - K\left(x(k) - z(k)\right),$$

where $v(k)$ is the nominal MPC, and the linear feedback gain K satisfies

$$|\,\mathrm{eig}(A - BK)\,| < 1.$$

- Define $e(k) = x(k) - z(k)$, such that

$$e(k+1) = x(k+1) - z(k+1) = Ae(k) - BKe(k) + Dw(k) = (A - BK)e(k) + Dw(k)\,.$$

Robust MPC with linear feedback (cont'd)

- Remember that we have set $z(0) = x(0)$, such that

$$
\begin{aligned}
e(0) &= x(0) - z(0) = 0, \\
e(1) &= A_K e(0) + Dw(0) = Dw(0), \\
e(2) &= A_K e(1) + Dw(1) = A_K Dw(0) + Dw(1), \\
e(3) &= A_K e(2) + Dw(2) = A_K^2 Dw(0) + A_K Dw(1) + Dw(2), \\
&\vdots \\
e(k) &= \cdots = \sum_{i=1}^{k} A_K^{i-1} Dw(k-i), \\
&\vdots
\end{aligned}
$$

where $A_K = A - BK$.

Robust MPC with linear feedback (cont'd)

- Since $w(k) \in \mathcal{W}$ and $|\mathrm{eig}(A - BK)| < 1$, it follows that $Dw(k) \in D\mathcal{W}$, and

$$e(k) = \sum_{i=1}^{k} A_K^{i-1} D w(k-i) \in \sum_{i=1}^{k} A_K^{i-1} D\mathcal{W} \subset \sum_{i=1}^{\infty} A_K^{i-1} D\mathcal{W} \triangleq \Gamma,$$

  where $\Gamma = \sum_{i=1}^{\infty} A_K^{i-1} D\mathcal{W}$ is finite, such that $e(k)$ is bounded.

- Definitions of set operations are given by
  - Multiplication with constant matrix: $D\mathcal{W} \triangleq \{Dw \in \mathbb{R}^n \mid \forall w \in \mathcal{W}\}$;
  - Addition: $\mathcal{W}_1 + \mathcal{W}_2 \triangleq \{w_1 + w_2 \mid \forall w_1 \in \mathcal{W}_1, \forall w_2 \in \mathcal{W}_2\}$.

- We have already see that $z = 0$ is exponentially stable. Consequently, $x(k) = z(k) + e(k)$ is ultimately bounded.

Feasibility (constraint satisfaction)

- Since $e(k) \in \Gamma$, it follows that $Ke(k) \in K\Gamma$.

- To guarantee $x \in \mathscr{X}$, $u \in \mathscr{U}$, the closed-loop nominal system should satisfy

$$z \in \mathscr{Z} = \mathscr{X} - \Gamma, \quad v \in \mathscr{V} = \mathscr{U} - K\Gamma,$$

Contractive constraints

and the terminal set $\mathscr{Z}_f$ should be an invariant set such that

$$z \in \mathscr{Z}_f \quad \Rightarrow \quad A_K z \in \mathscr{Z}_f \text{ and } u = -K_{mpc} z - Ke \in K_{mpc} \mathscr{Z}_f + K\Gamma \subset \mathscr{U}.$$

- Then it is clear that recursively stability is guaranteed.

How to calculate the contracted constraints ??

- Since $|\mathrm{eig}(A - BK)| = |\mathrm{eig}(A_K)| < 1$, there exists N_c such that

$$A_K^{N_c} D\mathcal{W} \subset \alpha D\mathcal{W},$$

where $0 \leq \alpha < 1$.

- It then follows that

$$\begin{aligned}\Gamma = \sum_{i=1}^{\infty} A_K^{i-1} D\mathcal{W} &= \Gamma_{N_c} + \sum_{i=N_c+1}^{\infty} A_K^{i-1} D\mathcal{W} \\ &= \Gamma_{N_c} + \sum_{i=1}^{\infty} A_K^{i-1} A_K^{N_c} D\mathcal{W} \subset \Gamma_{N_c} + \alpha\Gamma,\end{aligned}$$

such that $\Gamma \subset (1-\alpha)^{-1}\Gamma_{N_c}$.

How to calculate the contracted constraints ?? (cont'd)

- Consequently,
  - find N_c and the corresponding α, such that $A_K^{N_c} D\mathscr{W} \subset \alpha D\mathscr{W}$;
  - use the maximum $\|e\|$ from the set $\{e \in (1-\alpha)^{-1}\Gamma_{N_c}\}$ to calculate $\mathscr{Z}$ and $\mathscr{V}$.

- Or, simply estimate a conservatively large Γ to calculate $\mathscr{Z}$ and $\mathscr{V}$.

Example

- The plant with disturbance is given by

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix},\ D = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}$$

  the control constraint is given by $-4 \leq u \leq 4$;

  the bounded disturbance satisfies $-0.1 \leq w \leq 0.1$.

- In the nominal MPC design,
  - the control horizon is set to $N = 10$;
  - the feedback gain is set to $K = [2.5\ \ 12.5]$ to calculate the terminal weight P;
  - weighting matrices are set to $Q = I_{2\times 2},\ \ R = 0.1$;
  - the terminal set is assigned to $-0.1 \leq x_1 \leq 0.1,\ -0.1 \leq x_2 \leq 0.1$.

Example (cont'd)

- To calculate the contractive constraint:
  - Try $N_c = 10$, and it can be calculated that $\alpha \approx 0.2$;
  - Calculate

$$K\Gamma = (1-\alpha)^{-1}K\Gamma_{Nc} = 0.8^{-1}K\sum_{i=1}^{N_c} A_K^{i-1}D\mathscr{W} \approx [-1.375,\ 1.375];$$

  - The contractive constraint can be calculated by

$$-4 + 1.375 \leq v \leq 4 - 1.375.$$

- The robust MPC is implemented by

DEMO

$$u(k) = v(k) - Ke(k),$$

where $v(k)$ is the nominal MPC, and $K = [2.5\ \ 12.5]$.

Variation: update the nominal state by actual state

- At time k, can we update the nominal state by the actual state, i.e., $z(k) = x(k)$??

- Then a “switching” strategy can be designed to improve the overall performance:
  - Update $z(0 | k) = x(k)$, and solve the optimization to get $u^*(x(k))$;
  - Update $z(0 | k) = z(k)$, and solve the optimization to get $u^*(z(k))$;
  - Judge wether $J^*(x) \leq J^*(z)$: if yes, implement $u^*(x(k))$; otherwise, implement $u^*(z(k))$.

- The closed-loop stability is clear; more discussions should be given on feasibility.

Robust MPC for model uncertainty

- Suppose the plant is given by

$$x(k+1) = Ax(k) + B(u(k) + \delta_1(x(k), u(k))) + \delta_2(x(k)),$$

  where $\delta_1(x, u)$, $\delta_2(x)$ are model uncertainties.

- Assume that
  - the uncertainties vanishes with x and u, i.e., $\delta_1(0,0) = 0$, $\delta_2(0) = 0$;
  - $\|\delta_1(x, u)\| \leq \rho_1\|x\| + \kappa\|u\|$, where $\rho_1 > 0$ and $0 \leq \kappa < 1$ are known;
  - $\|\delta_2(x)\| \leq \rho_1\|x\|$, where $\rho_2 > 0$ is known.

Robust MPC for model uncertainty (cont'd)

- In this circumstance, we can consider

$$Dw = B\delta_1 + \delta_2, \qquad D\mathcal{W} = \{B\delta_1 + \delta_2 \mid \forall x \in \mathcal{X}, \forall u \in \mathcal{U}\}$$

  and the foregoing robust MPC still applies.

- In the previous section, it has been proved that, for bounded disturbance, the closed-loop system is **ultimately bounded**.
- In this section with model uncertainties, we would like to explore whether the final result could be **"asymptotically stable"**.

Robust MPC for model uncertainty (cont'd)

- For the nominal system, $z = 0$ is (locally) exponentially stable. (Same as previous)

- According to (discrete-time) converse Lyapunov theorem, $V_z(k) > 0$ exists, such that

$$V_z(z(k+1) - V_z(z(k)) \leq -\alpha_1 \|z\|^2,$$

with some positive number $\alpha_1 > 0$.

- For the error dynamics,

$$e(k+1) = A_K e(k) + Dw(k) = A_K e(k) + B\delta_1 + \delta_2,$$

select $V_e(e) = e^T P e > 0$ where P is the unique solution of

$$P - A_K^T P A_K = Q + K^T R K\,.$$

Robust MPC for model uncertainty (cont'd)

- For the overall system, take $V = V_e + \gamma V_z$ with $\gamma > 0$, such that $V(k) > 0$, and

$$V(k+1) - V(k) = -e^T(Q + K^TRK)e + w^TD^TPDw + 2w^TD^TPA_Ke - \alpha_1\|z\|^2,$$

  where $Dw = B\delta_1 + \delta_2$, such that

$$\|Dw\| \le \rho_3\|e\| + \rho_4\|z\|,$$

  for some positive numbers $\rho_1,\ \rho_2$.

- It then follows that $V(k+1) - V(k)$ is a quadratic function of $[\|e\|,\ \|z\|]^T$.

- If the uncertainty is sufficiently small, i.e., Dw is small such that $\rho_3,\ \rho_4$ are small, then $V(k+1) - V(k)$ is negative definite with sufficiently large γ, indicating that $[e, z]^T = 0$ is asymptotically stable, and $x = z + e = 0$ is **asymptotically stable**.

Example

- The plant with disturbance is given by

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix},$$

the control constraint is given by $-4 \le u \le 4$.

- Suppose we do not know (A, B) exactly, but we know their values are around

$$\bar{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},\ \bar{B} = \begin{bmatrix} 0 \\ 0.08 \end{bmatrix},$$

which can be used for prediction in the nominal MPC.

Example (cont'd)

- In the nominal MPC design,
  - the control horizon is set to $N = 10$;
  - the feedback gain is set to $K = [2.5 \;\; 12.5]$ to calculate the terminal weight P;
  - weighting matrices are set to $Q = I_{2\times 2}, \;\; R = 0.1$;
  - the terminal set is assigned to $-0.1 \leq x_1 \leq 0.1, \; -0.1 \leq x_2 \leq 0.1$.
- Since the uncertainty is small, the contractive constraint can be set with a conservatively large Γ, i.e., $-4 + 1 \leq v \leq 4 - 1$.
- The robust MPC is implemented by

DEMO

$$u(k) = v(k) - Ke(k),$$

where $v(k)$ is the nominal MPC, and $K = [2.5 \;\; 12.5]$.

Example

- The plant with disturbance is given by

$$x(k+1) = Ax(k) + Bu(k) + \delta(x(k), u(k))$$

where

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}, \quad \delta(x, u) = \begin{bmatrix} 0.4x_1^2 + 0.1x_2^3 \\ 0.5x_2^3 + 0.1u \end{bmatrix}.$$

the control constraint is given by $-4 \leq u \leq 4$.

✳ The uncertainty in this example can be seen as “linearization error”.

Example (cont'd)

- In the nominal MPC design,
  - the control horizon is set to $N = 10$;
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- Since the uncertainty is small, the contractive constraint can be set with a conservatively large Γ, i.e., $-4 + 1 \leq v \leq 4 - 1$.
- The robust MPC is implemented by

DEMO

$$u(k) = v(k) - Ke(k),$$

where $v(k)$ is the nominal MPC, and $K = [2.5 \;\; 12.5]$.

Summary

- Robust MPC: nominal MPC + linear feedback
    - For bounded disturbances
        - contractive constraints are calculated to guarantee the constraint satisfaction;
        - the final result is “ultimately bounded”.
    - Variation: update nominal states with actual states.
- Robust MPC in case of model uncertainty (vanishing disturbance)
    - The final result is “asymptotically stable”, if the uncertainty is small.