Nonlinear Control Theory

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Lyapunov Stability

The Invariance Principle

Consider again the pendulum with friction:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 - b x_2.$$

The derivative of the Lyapunov function $V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$ is calculated by

$$\dot{V} = a\dot{x}_1 \sin x_1 + x_2 \dot{x}_2 = -b x_2^2,$$

which is only negative semi-definite, and it only guarantees that $x = 0$ is stable (not asymptotically stable).

However, it should be noticed that, to maintain $\dot{V}(x) = 0$,

$$x_2 \equiv 0 \;\; \Rightarrow \;\; \dot{x}_2 \equiv 0 \;\; \Rightarrow \;\; \sin x_1 \equiv 0,$$

indicating that, over $-\pi < x_1 < \pi$, the situation $\dot{V} = 0$ only maintains at $x = 0$.

Definition

Let $x(t)$ be a solution of $\dot{x} = f(x)$.

- A point p is a positive limit point of $x(t)$, if there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$, such that $x(t_n) \to p$ as $n \to \infty$.
- The set of all positive limit points of $x(t)$ is called the positive limit set of $x(t)$, denoted by L^+.

- If $x(t)$ approaches an asymptotically stable equilibrium point $\bar{x}$, then $\bar{x}$ is the positive limit point of $x(t)$ and $L^+ = \bar{x}$.
- A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle.

Definition

- A set M is an invariant set with respect to $\dot{x} = f(x)$, if

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M, \ \forall t \in R.$$

  Examples: equilibrium points, limit cycles, etc.

- A set M is an positive invariant set with respect to $\dot{x} = f(x)$, if

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M, \ \forall t \geq 0.$$

  Examples: the set $\Omega = \{V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ in Ω_c.

Definition

- The distance from a point p to a set M is defined by

$$\mathrm{dist}(p, M) = \inf_{x \in M} \|p - x\|.$$

- $x(t)$ approaches a set M as t approaches infinity, if for each $\epsilon > 0$ there is $T > 0$ such that

$$\mathrm{dist}(x, M) < \epsilon, \quad \forall t > T.$$

Example

- Every solution $x(t)$ starting sufficiently near a stable limit cycle approaches the limit cycle as $t \to \infty$.
- However, $x(t)$ does not converge to any specific point on the limit cycle.

Lemma

If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a non-empty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$.

Theorem (4.4 LaSalle's theorem)

Let $f(x)$ be be a locally Lipschitz function defined over a domain $D \subset R^n$, and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Remark

In Theorem 4.4, $V(x)$ is ***not*** *required to be positive definite.*

Proof:

- $\dot{V}(x) \le 0, \ \forall x \in \Omega \quad \Rightarrow \quad V(x(t))$ is decreasing.
- $V(x)$ is continuous in Ω which is compact. $\Rightarrow$ Its limit exists, $\lim_{t\to\infty} V(x(t)) = a$.
- $x \in \Omega \quad \Rightarrow \quad x(t)$ is bounded. $\Rightarrow \quad L^+$ exists, and $L^+ \subset \Omega$.
- For any $p \in L^+$, there exists $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$, such that $\lim_{n\to\infty} x(t_n) = p$.
- $V(x)$ is continuous. $\Rightarrow \quad V(p) = \lim_{n\to\infty} V(x(t_n)) = a$.
- $V(x) = a, \ \forall x \in L^+ \quad \Rightarrow \quad \dot{V}(x) = 0, \ \forall x \in L^+$.
- $L^+ \subset M \subset E \subset D$.

Consequently, $x(t)$ approaches $L^+ \quad \Rightarrow \quad x(t)$ approaches M.

Theorem

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$, and $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D | \dot{V}(x) = 0\}$.

- *If no solution can stay identically in S other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable.*
- *Moreover, if $\Gamma \subset D$ is compact and positively invariant, then Γ is a subset of the region of attraction.*
- *Furthermore, if $D = R^n$, and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable.*

Remark

Here, $V(x)$ is required to be positive definite.

Example

Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - h_2(x_2),$$

where $h_i(0) = 0$, $yh_i(y) > 0$ for $0 < |y| < a$.

Consider the energy-like function $V(x) = \int_0^{x_1} h_1(y)\mathrm{d}y + \frac{1}{2}x_2^2$ and $D = \{-a < x_i < a\}$.

$$\begin{aligned} \dot{V}(x) =& h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \le 0, \\ \dot{V}(x) =& 0 \;\Rightarrow\; x_2 h_2(x_2) = 0 \;\Rightarrow\; x_2 = 0 \;\Rightarrow\; S = \{x \in D | x_2 = 0\}. \end{aligned}$$

$$x_2 \equiv 0 \;\Rightarrow\; \dot{x}_2 \equiv 0 \;\Rightarrow\; h_1(x_1) \equiv 0 \;\Rightarrow\; x_1 \equiv 0.$$

The only solution that can stay identically in S is $x(t) \equiv 0$. Thus, the origin is asymptotically stable.

Example (Cont'd)

Suppose that $a = \infty$, and

$$\int_0^y h_1(z)\mathrm{d}z \ \rightarrow \ \infty \text{ as } |y| \rightarrow \infty.$$

Then, $D = R^2$, and $V(x) = \int_0^{x_1} h_1(y)\mathrm{d}y + \frac{1}{2}x_2^2$ is radially unbounded.

$$S = \{x \in D | x_2 = 0\},$$

and the only solution that can stay identically in S is $x \equiv 0$.

Consequently, the origin is globally asymptotically stable.

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LaSalle's theorem can be used in cases where the system has an **equilibrium set**, rather than an isolated equilibrium point.

Example

Consider the case of a simple adaptive control:

$$\dot{y} = ay + u, \quad u = -k(t)y, \quad \dot{k} = \gamma y^2, \quad \gamma > 0.$$

Taking $x_1 = y$ and $x_2 = k$, the closed-loop system is given by

$$\dot{x}_1 = -(x_2 - a)x_1, \quad \dot{x}_2 = \gamma x_1^2,$$

where the line $\{x \in R^n | x_1 = 0\}$ is an equilibrium set.

Example (Cont'd)

Consider the Lyapunov candidate $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2$, where $b > a$. $V(x)$ is radially unbounded, and its derivative is calculated by

$$\dot{V}(x) = x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - b)\dot{x}_2 = -x_1^2(b - a) \leq 0.$$

Then, all conditions in LaSalle's theorem are satisfied, it is concluded that all solutions will converge to the invariant set $\{x \in R^n | x_1 = 0\}$.

That is, $x_1 \to 0$ as $t \to \infty$, and this result holds globally.

Consequently, $y = 0$ is globally asymptotically stable for the closed-loop system.