模型预测控制

Model Predictive Control

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2022 Spring

Constrained linear MPC

Lecture 1

Why constrained control ?? Because constraints are everywhere !!

Why constrained MPC ??   MPC = optimization + feedback control

- Unconstrained optimization

$$u* = \arg\min_u \; f^T u + u^T H u$$

$$u* = -\frac{1}{2} H^{-1} f$$

- Constrained optimization with equality constraints

$$u* = \arg\min_u \; f^T u + u^T H u$$

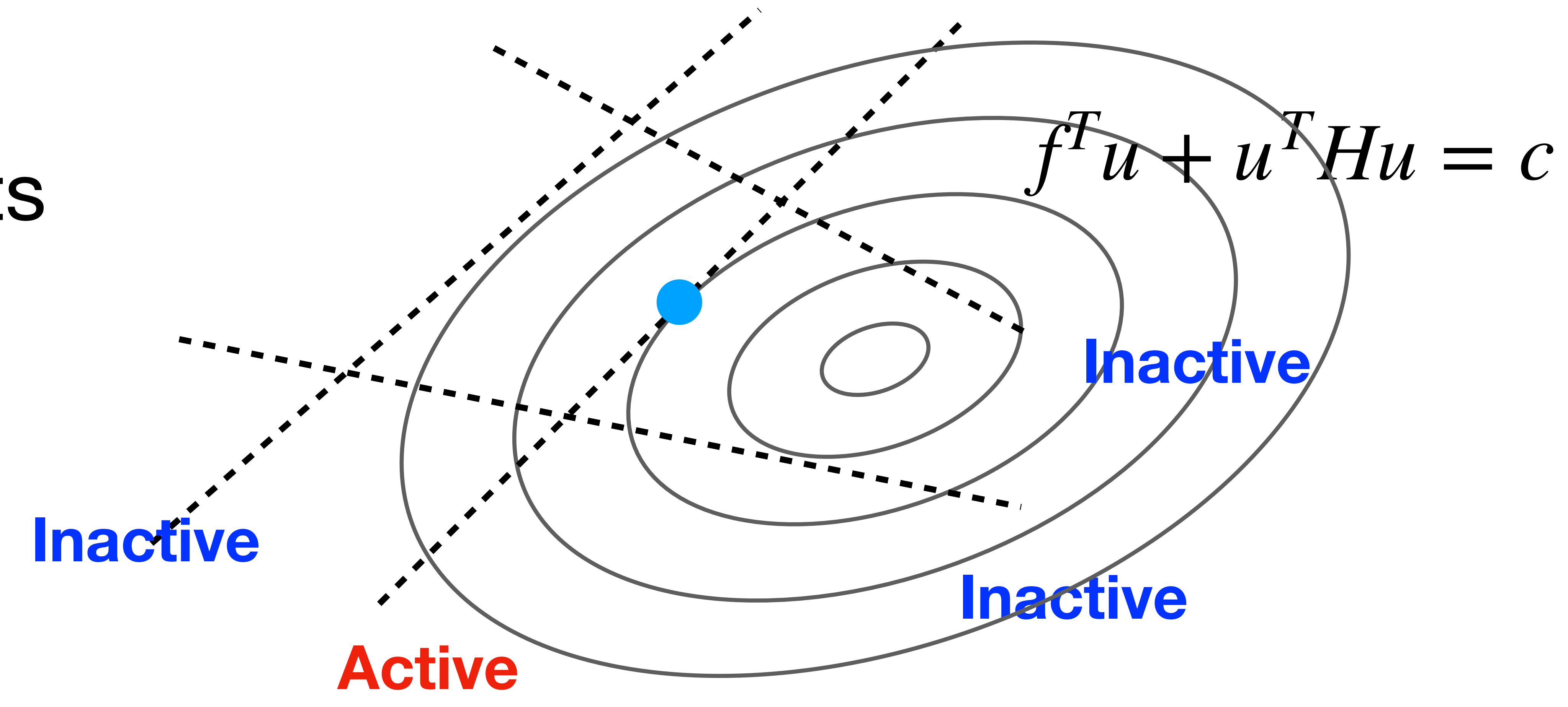
s.t. $A_{eq} u = b_{eq}$

$$[u*, \lambda*] = \arg\min_{u,\lambda} \; f^T u + u^T H u + \lambda(A_{eq} u - b_{eq})$$

- Constrained optimization with equality constraints

$$u* = \arg\min_u \; f^T u + u^T H u$$

s.t. $Au \leq b$



- Linear time invariant plant:

$$x(k+1) = Ax(k) + Bu(k)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$. Suppose that (A, B) is stabilizable.

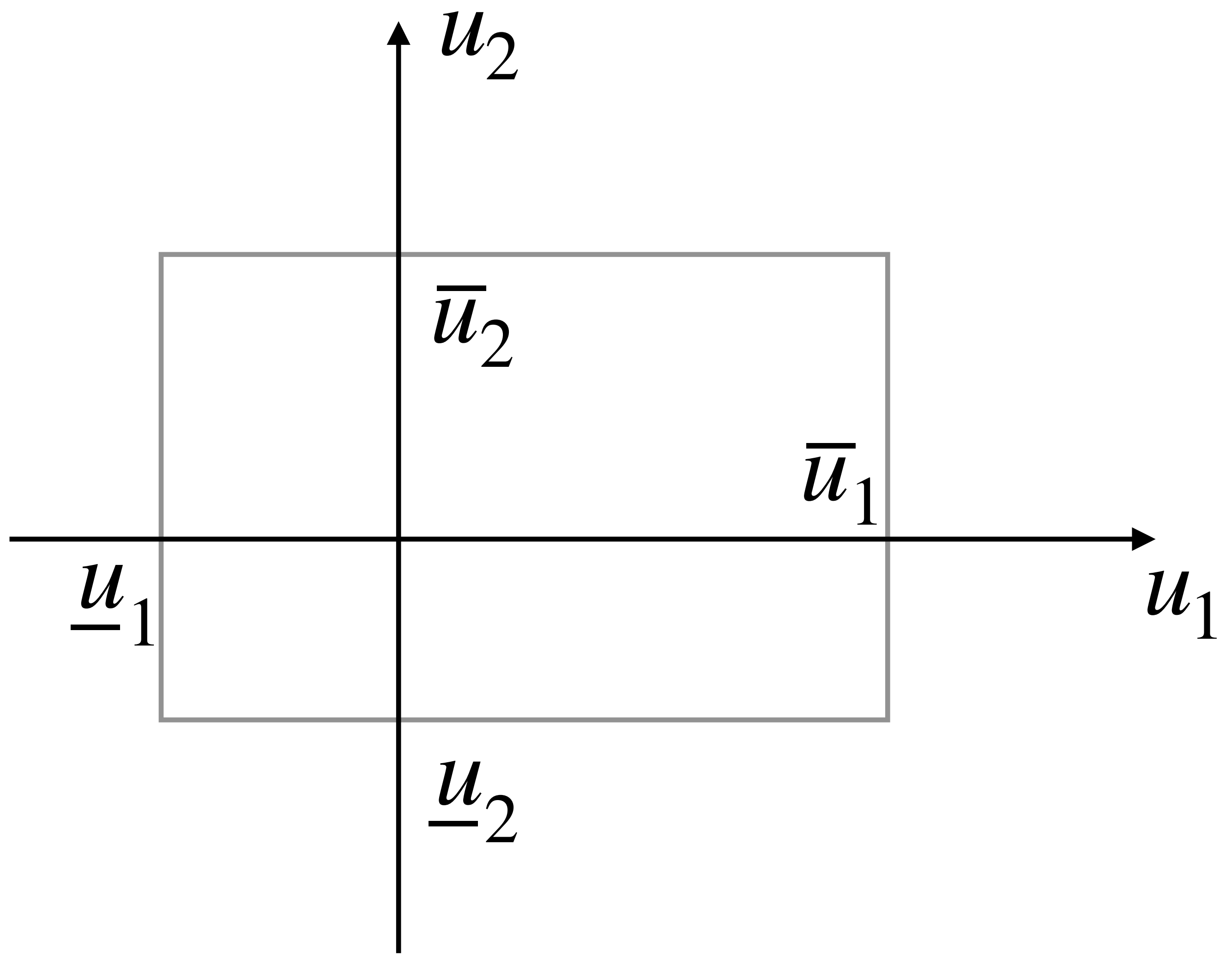
- The above LTI system is subject to state and input contraints:

$$x \in \mathcal{X},\ u \in \mathcal{U}\ .$$

Assume the above constraints are convex.

For example, constraints in rectangle form:

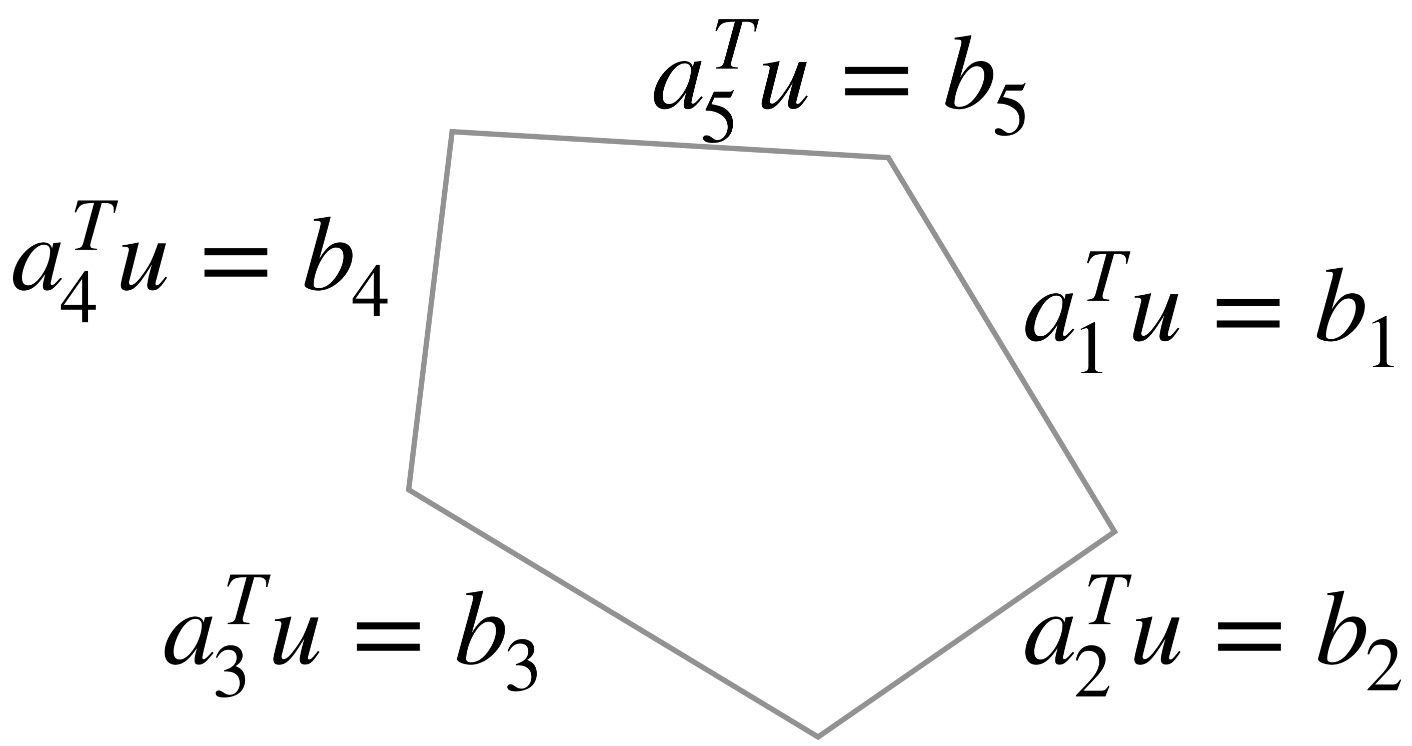
$$\underline{x}_i \leq x_i \leq \overline{x}_i,\ \underline{u}_i \leq u_i \leq \overline{u}_i\ .$$



where x_i and u_i are elements of x and u, respectively;

or, in polyhedra form:

$$A_x x \leq b_x,\ \ A_u u \leq b_u\ .$$



where A_x, b_x, A_u, b_u have proper dimensions.

- More generally, it is also possible that state and control constraints are coupled:

$$Gx + Hu \le \mathbf{1},$$

where G, H have proper dimensions.

- Cost/objective function

$$J(k) = \sum_{i=1}^{N} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 = X^T(k)\mathscr{Q}X(k) + U^T(k)\mathscr{R}U(k),$$

where N denotes the control/predictive horizon, and

$$\mathscr{Q} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix}, \qquad \mathscr{R} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}, \qquad X(k) = Fx(k) + \Phi U(k),$$

$$X(k) \triangleq [x^T(1\,|\,k), x^T(2\,|\,k), \cdots, x^T(N\,|\,k)]^T, \qquad U(k) \triangleq [u^T(0\,|\,k), u^T(1\,|\,k), \cdots, u^T(N-1\,|\,k)]^T.$$

Basic algorithm

- Model-based prediction:

$$X(k) = Fx(k) + \Phi U(k),$$

where

$$F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \qquad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}$$

- Cost function

$$\begin{aligned} J(k) &= X^T(k)\mathcal{Q}X(k) + U^T(k)\mathcal{R}U(k) \\ &= (Fx(k) + \Phi U(k))^T\mathcal{Q}(Fx(k) + \Phi U(k)) + U^T(k)\mathcal{R}U(k) \\ &= x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k) \,. \end{aligned}$$

Basic algorithm (cont'd)

- Constrained optimization

$$
\begin{aligned}
U^*(k) &= \arg\min_{U(k)} J(k) \\
&= \arg\min_{U(k)} \left[x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k)\right],
\end{aligned}
$$

s.t. $\quad Gx(i\,|\,k) + Hu(i\,|\,k) \leq 1, \qquad \forall i = 0,1,\cdots,N-1.$

- Implement the first action in the optimal control sequence, and discard others

$$u(k) = u^*(0\,|\,k) = [I_{p\times p}\ ,\ 0,\ 0,\ \cdots,\ 0]U^*(k)\,.$$

- At time k+1, repeat the above process. (Receding horizon control)

Example

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^2, \;\; u \in \mathbb{R}^1 \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

The linear system is subject to control constraints: $-2 \le u \le 2$.

Set control horizon $N = 4$. Set weight matrices $Q = I_{2\times2}$, $R = 0.1$.

At time k,

$$U^*(k) = \arg\min_{U(k)} \left[x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k) \right],$$

s.t. $\quad -2 \le u(i\,|\,k) \le 2, \qquad \forall i = 0,1,\cdots,N-1.$

$$u(k) = u^*(0\,|\,k) = [I_{p\times p}\ ,\ 0,\ 0,\ \cdots,\ 0]U^*(k)\,.$$

Demo

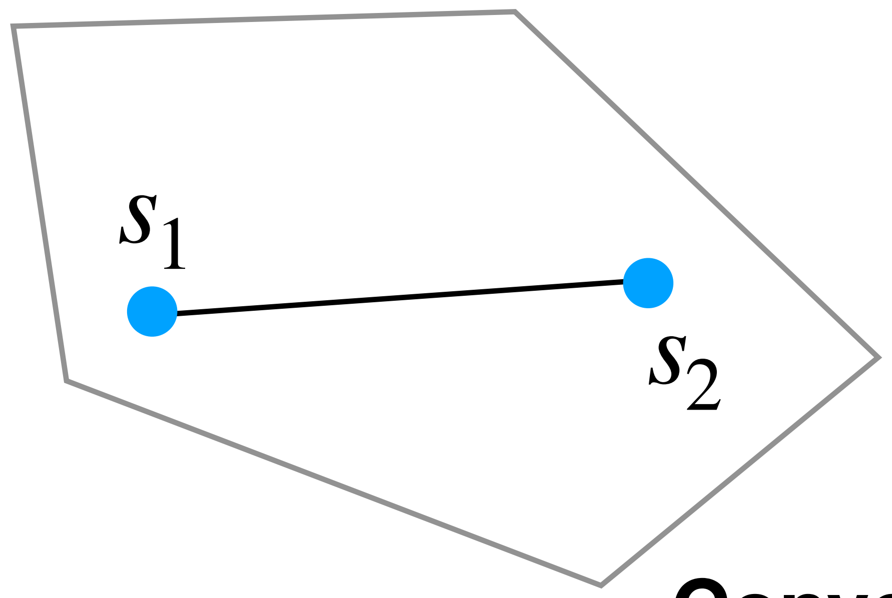
At time k+1, obtain $x(k+1)$ from the plant, and repeat the above process.

Feasibility of optimization

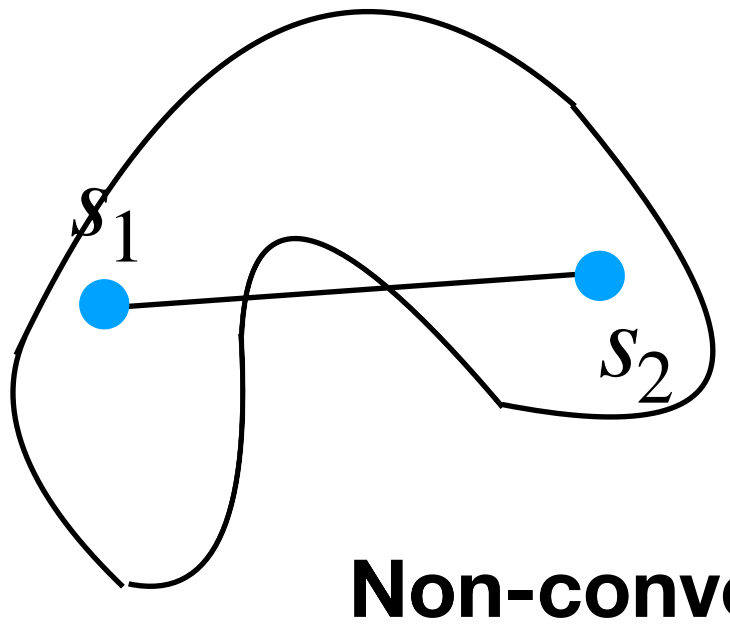
<u>Definition (Convex set)</u>: S is a convex set, if $\lambda s_1 + (1-\lambda)s_2 \in S$, $\forall s_1, s_2 \in S$ and $\forall \lambda \in [0,1]$.

<u>Definition (Convex function)</u>: The function $f : S \to \mathbb{R}$ is convex, if S is convex, and

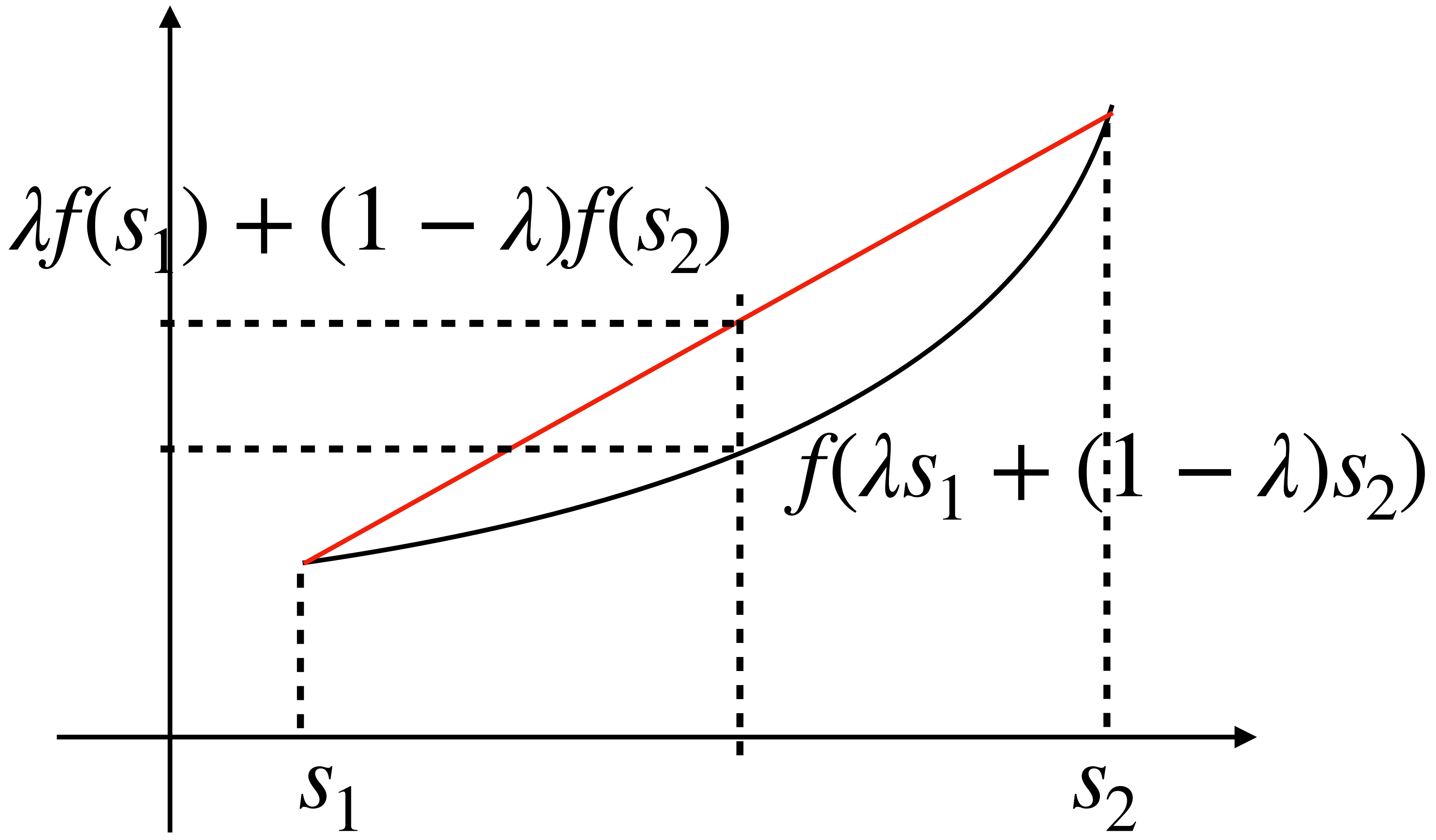
$f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda)f(s_2)$, $\forall s_1, s_2 \in S$ and $\forall \lambda \in [0,1]$.



Convex



Non-convex



Convex function

Example:

- The set given by polyhetron or polytope is a convex set.
- The quadratic cost function is a convex function.

Feasibility of optimization

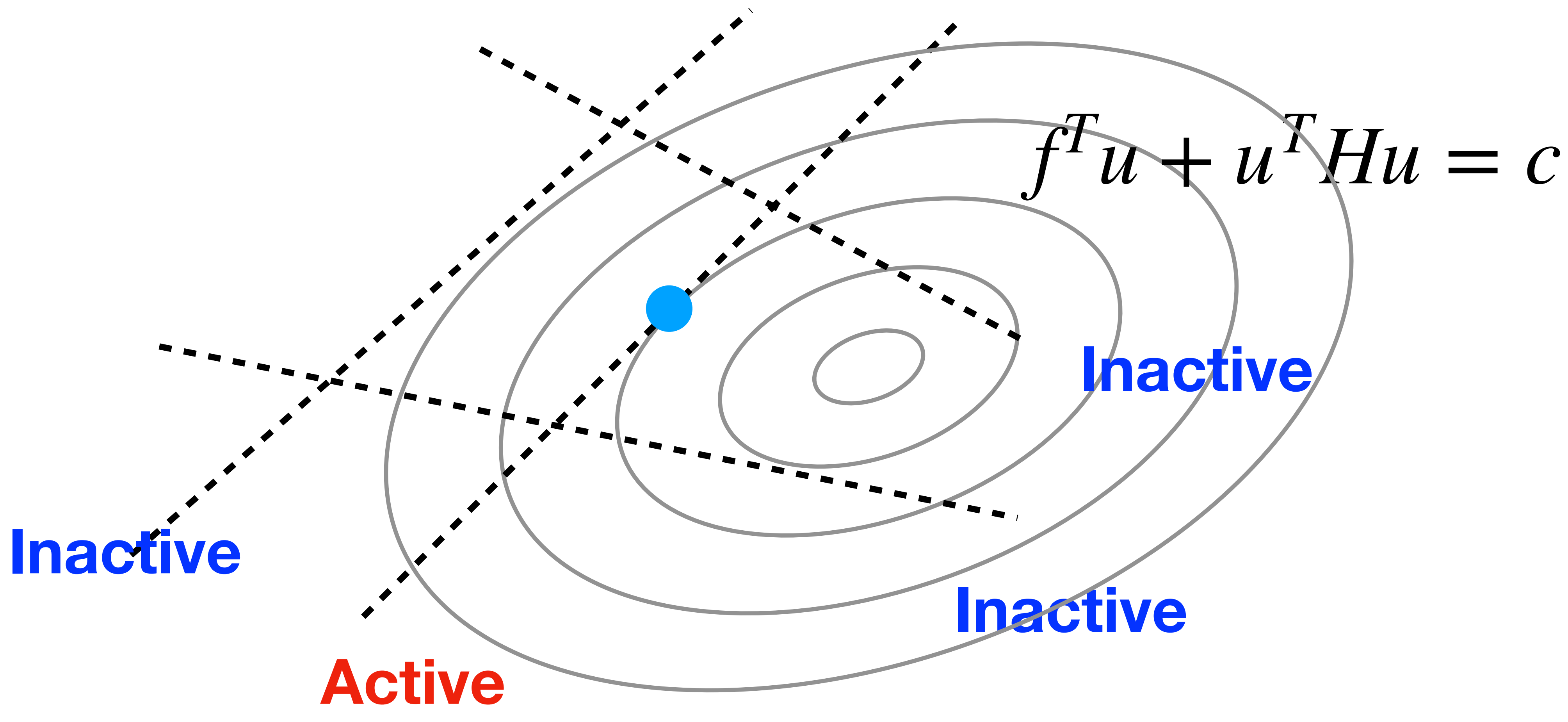
If the above cost function $J(k)$ is given by quadratic function with

(1) $R > 0,\ Q \geq 0$; or

(2) $R \geq 0,\ Q = CC^T$, fully controllable $(A,\ B)$, and fully observable $(A,\ C)$,

and the constraints form a non-empty convex set, then the optimization is “convex”.

- The “convex optimization” is feasible.
- There exists efficient algorithms to find the (global) optimal solution.
- For example, “active set” algorithm.



Stability of the closed-loop system

Optimality does not necessarily indicate stability !!

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^2, \;\; u \in \mathbb{R}^1 \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

The linear system is subject to control constraints: $-2 \leq u \leq 2$.

Set control horizon $N = 2$. Set weight matrices $Q = I_{2\times 2}$, $R = 0.1$.

At time k,

$$U^*(k) = \arg\min_{U(k)} \left[x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k) \right],$$

s.t. $\quad -2 \leq u(i\,|\,k) \leq 2, \qquad \forall i = 0,1,\cdots,N-1.$

$$u(k) = u^*(0\,|\,k) = [I_{p\times p}\;, \;0,\; 0,\;\; \cdots,\; 0]U^*(k)\,.$$

$\boxed{\textbf{Demo}}$

At time k+1, obtain $x(k+1)$ from the plant, and repeat the above process.

Stability of the closed-loop system

Several ways to guarantee the closed-loop stability of constrained MPC:

- Terminal equality constraint
- Infinite control horizon
- Terminal inequality constraints
- Terminal cost
- Constraint checking horizon
- ...

Stabilizing constraint: terminal equality constraint

- Add a terminal equality constraint at the end of the control horizon:

$$x(N|k) = 0.$$

Terminal equality constraint

- Suppose, at time k, there exist an optimal control sequence (satisfying all constraints)

$$U^*(k) = [u^{*T}(0|k), u^{*T}(1|k), \cdots, u^{*T}(N-1|k)]^T,$$

such that the corresponding state sequence is given by

$$X^*(k) = [x^{*T}(1|k), x^{*T}(2|k), \cdots, x^{*T}(N|k)]^T,$$

where $x^*(N|k) = 0$, as indicated by the terminal equality constraint.

- Suppose there exists neither uncertainties nor disturbances. Then, $u(k) = u^*(0|k)$ is implemented such that

$$x(k+1) = Ax(k) + Bu(k) = Ax^*(0|k) + Bu^*(0|k) = x^*(1|k)$$

Stabilizing constraint: terminal equality constraint (Cont'd)

- At time k+1, there exists at least one feasible control sequence

$$u(0 \mid k+1) = u^*(1 \mid k),\ u(1 \mid k+1) = u^*(2 \mid k),\ \cdots,\ u(N-2 \mid k+1) = u^*(N-1 \mid k),\ u(N-1 \mid k+1) = 0,$$

and correspondingly,

$$x(1 \mid k+1) = x^*(2 \mid k),\ x(2 \mid k+1) = x^*(3 \mid k),\ \cdots,\ x(N-1 \mid k+1) = x^*(N \mid k) = 0,\ x(N \mid k+1) = 0,$$

where the terminal state is calculated by

$$x(N \mid k+1) = Ax(N-1 \mid k+1) + Bu(N-1 \mid k+1) = 0.$$

- Feasibility at time k indicates feasibility at time k+1. **"Recursive feasibility"**

- Consequently, if the optimization is feasible initially (at time k=0), it will be feasible for all future time.

Stabilizing constraint: terminal equality constraint (Cont'd)

- Select $J^*(k)$ as the candidate Lyapunov function for the closed-loop system.

$$
\begin{aligned}
J^*(k+1) - J^*(k) &\le J(k+1) - J^*(k) \\
&= \sum_{i=1}^{N} \left(\|x(i\,|\,k+1)\|_Q^2 + \|u(i-1\,|\,k+1)\|_R^2 \right) - \sum_{i=1}^{N} \left(\|x^*(i\,|\,k)\|_Q^2 + \|u^*(i-1\,|\,k)\|_R^2 \right) \\
&= \sum_{i=1}^{N-1} \left(\|x^*(i+1\,|\,k)\|_Q^2 + \|u^*(i\,|\,k)\|_R^2 + 0 + 0 \right) - \sum_{i=1}^{N} \left(\|x^*(i\,|\,k)\|_Q^2 + \|u^*(i-1\,|\,k)\|_R^2 \right) \\
&= -\|x^*(1\,|\,k)\|_Q^2 - \|u^*(0\,|\,k)\|_R^2,
\end{aligned}
$$

implying that the closed-loop system is asymptotically stable, provided that the optimization is feasible initially.

"Initial and recursive feasibility indicates closed-loop stability"

MPC algorithm with terminal equality constraint

- Model-based prediction:

$$X(k) = Fx(k) + \Phi U(k), \qquad F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \qquad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}.$$

- Constrained optimization

$$U^*(k) = = \arg\min_{U(k)} \left[x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k)\right],$$

s.t. $\quad Gx(i\,|\,k) + Hu(i\,|\,k) \leq 1, \qquad \forall i = 0,1,\cdots,N-1,$

$$x(N\,|\,k) = 0.$$

- Implement the first action $u(k) = u^*(0\,|\,k)$ in the optimal control sequence.

- At time k+1, repeat the above process. (Receding horizon control)

Example

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^2, \;\; u \in \mathbb{R}^1 \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

The linear system is subject to control constraints: $-2 \leq u \leq 2$.

Set control horizon $N = 10$. Set weight matrices $Q = I_{2\times2}$, $R = 0.1$.

At time k,

$$U^*(k) = \arg\min_{U(k)} \left[x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k) \right],$$

s.t. $\qquad -2 \leq u(i\,|\,k) \leq 2, \qquad \forall i = 0,1,\cdots,N-1.$

$$x(N\,|\,k) = 0.$$

where $x(N\,|\,k) = [0,\ 0,\ \cdots,\ I_{n\times n}]X(k)$

$$= [0,\ 0,\ \cdots,\ I_{n\times n}]\left[Fx(k) + \Phi U(k)\right]$$

$$u(k) = u^*(0\,|\,k) = [I_{p\times p}\ ,\ 0,\ 0,\ \cdots,\ 0]U^*(k)\,.$$

At time k+1, obtain $x(k+1)$ from the plant, and repeat the above process.

Demo

Terminal equality constraint — Summary

The terminal equality constraint

- is structurally simple, and
- theoretically guarantees the closed-loop stability, provided that the optimization is feasible initially.

However, it

- may impede the feasibility of optimization.

Feasibility can be improved by increasing the control horizon N.

Infinite control horizon: $N = \infty$

- If the control horizon is infinite, i.e.,

$$J(k) = \sum_{i=1}^{\infty} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 = X^T(k)\mathcal{Q}X(k) + U^T(k)\mathcal{R}U(k),$$

and there exists one optimal/feasible solution initially, then,

$$\begin{aligned}
J^*(k+1) - J^*(k) &\leq J(k+1) - J^*(k) \\
&= \sum_{i=1}^{\infty} \left(\|x(i\,|\,k+1)\|_Q^2 + \|u(i-1\,|\,k+1)\|_R^2 \right) - \sum_{i=1}^{\infty} \left(\|x^*(i\,|\,k)\|_Q^2 + \|u^*(i-1\,|\,k)\|_R^2 \right) \\
&= \sum_{i=1}^{\infty} \left(\|x^*(i+1\,|\,k)\|_Q^2 + \|u^*(i\,|\,k)\|_R^2 \right) - \sum_{i=1}^{\infty} \left(\|x^*(i\,|\,k)\|_Q^2 + \|u^*(i-1\,|\,k)\|_R^2 \right) \\
&= -\|x^*(1\,|\,k)\|_Q^2 - \|u^*(0\,|\,k)\|_R^2,
\end{aligned}$$

which is negative definite, even without the terminal equality constraint.

How to deal with infinite number of decision variables?

$$J(k) = \sum_{i=1}^{\infty} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 = X^T(k)\mathcal{Q}X(k) + U^T(k)\mathcal{R}U(k),$$

$$U^*(k) = \left[u^{*T}(0\,|\,k), u^{*T}(1\,|\,k), \cdots, u^{*T}(i\,|\,k), \cdots\right]^T.$$

For a closed-loop linear system $x(k+1) = (A - BK)x(k)$ where $|\mathrm{eig}(A - BK)| < 1$, there always exists an invariant set $\Omega \subset \mathcal{X}$ around the origin, such that

$$x(k) \in \Omega \quad \Rightarrow \quad x(k+1) \in \Omega, \quad \text{and} \quad u(k) \in K\Omega \subset \mathcal{U}.$$

Whenever x is very close to the origin,

- the state x and the linear control $u = -Kx$ satisfy their constraints naturally, and
- the constrained MPC turns to be “unconstrained MPC”.

- In such a circumstances, the infinite horizon cost is actually "finite", i.e.

$$J(k) = \sum_{i=1}^{\infty} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2$$

$$= \|x(N+1\,|\,k)\|_P^2 + \|u(N\,|\,k)\|_R^2 + \sum_{i=1}^{N} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2$$

or, equivalently

$$J(k) = X^T(k)\mathcal{Q}X(k) + U^T(k)\mathcal{R}U(k),$$

where

$$\mathcal{Q} = \mathrm{diag}[Q, Q, \cdots, Q, P], \qquad \mathcal{R} = \mathrm{diag}[R, R, \cdots, \cdots, R],$$

and P is the unique solution of the Lyapunov equation

$$P - (A - BK)^T P (A - BK) = Q + K^T R K\,.$$

Terminal inequality constraint

- To ensure the "unconstrained MPC", a constraint should be added: $x(N\,|\,k) \in \mathcal{X}_f \subset \Omega\,.$

Recursively feasible ??

- Suppose, at time k, the optimization is feasible.
- Implement the first control action in the control sequence:

$$u(k) = u^*(0\,|\,k) \qquad \text{such that} \qquad x(k+1) = Ax(k) + Bu(k) = Ax^*(0\,|\,k) + Bu^*(0\,|\,k) = x^*(1\,|\,k)\,.$$

- At time k+1, at least one feasible control sequence exists:

$$u(i\,|\,k+1) = u^*(i+1\,|\,k), \qquad \text{for} \qquad i = 0,1,\cdots,N-1,$$

$$u(i\,|\,k+1) = -Kx(i\,|\,k+1) \qquad \text{for} \qquad i = N, N+1,\cdots,$$

such that

$$x(i\,|\,k+1) = x^*(i+1\,|\,k) \qquad \text{for} \qquad i = 0,1,\cdots.$$

It is clear that all states and control inputs satisfy their constraints.

MPC algorithm with terminal inequality constraint (Cont'd)

Given $A,\ B,\ N,\ P,\ Q$:

- Model-based prediction:

$$X(k) = Fx(k) + \Phi U(k), \qquad F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \qquad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}.$$

- Terminal invariant set:
    - find state feedback $u = -Kx$ such that $|\mathrm{eig}(A - BK)| < 1$.
    - fine an invariant set Ω such that: $x(k) \in \Omega \quad \Rightarrow \quad x(k+1) \in \Omega$, and $u(k) \in K\Omega \subset \mathcal{U}$.

- Solve Lyapunov equation to get P: $\qquad P - (A - BK)^T P(A - BK) = Q + K^T RK$.

MPC algorithm with terminal inequality constraint

- Constrained optimization

$$U^*(k) = \arg\min_{U(k)} \left[x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k)\right],$$

$$\text{s.t.} \quad Gx(i\,|\,k) + Hu(i\,|\,k) \leq 1, \qquad \forall i = 0,1,\cdots,N-1,$$

$$x(N\,|\,k) \in \mathcal{X}_f \subset \Omega\,.$$

where

$$\mathcal{Q} = \text{diag}[Q, Q, \cdots, Q, P], \qquad \mathcal{R} = \text{diag}[R, R, \cdots, \cdots, R],$$

- Implement the first action $u(k) = u^*(0\,|\,k)$ in the optimal control sequence.

- At time k+1, repeat the above process. (Receding horizon control)

Example

$$x(k+1) = Ax(k) + Bu(k) \qquad x \in \mathbb{R}^2, \ \ u \in \mathbb{R}^1 \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}$$

The linear system is subject to control constraints: $-2 \le u \le 2$.

Set control horizon $N = 4$. Set weight matrices $Q = I_{2\times 2}$, $R = 0.1$.

At time k,

$$U^*(k) = \arg\min_{U(k)} \left[x^T(k) F^T \mathcal{Q} F x(k) + 2x^T(k) F^T \mathcal{Q} \Phi U(k) + U^T(k)(\Phi^T \mathcal{Q} \Phi + \mathcal{R}) U(k) \right],$$

s.t. $\quad -2 \le u(i\,|\,k) \le 2, \qquad \forall i = 0,1,\cdots,N-1.$

$$x(N\,|\,k) \in \left[-0.2, +0.2\right] \times \left[-0.1, +0.1\right]$$

$$u(k) = u^*(0\,|\,k) = [I_{p\times p}\ , \ 0,\ 0,\ \cdots,\ 0] U^*(k)\,.$$

$$\boxed{\begin{gathered} \mathcal{Q} = \operatorname{diag}[Q, Q, \cdots, Q, P], \\ \mathcal{R} = \operatorname{diag}[R, R, \cdots, \cdots, R], \\ P - (A - BK)^T P (A - BK) = Q + K^T R K\,. \end{gathered}}$$

At time k+1, obtain $x(k+1)$ from the plant, and repeat the above process. **Demo**

An alternative algorithm - “dynamic equality constraints”

- To solve linear constrained MPC, we can also regard the plant as “dynamic constraints”

$$[X^*(k), U^*(k)] = \arg \min_{X(k),U(k)} \left[\|x(N+1\,|\,k)\|_P^2 + \|u(N\,|\,k)\|_R^2 + \sum_{i=1}^{N} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 \right]$$

$$\text{s.t.} \quad \begin{aligned} & Gx(i\,|\,k) + Hu(i\,|\,k) \le 1, && \forall i = 0,1,\cdots,N-1, \\ & x(0\,|\,k) = x(k), && \\ & x(i+1\,|\,k) = Ax(i\,|\,k) + Bu(i\,|\,k), && \forall i = 0,1,\cdots,N-1, \\ & x(N\,|\,k) \in \mathcal{X}_f \subset \Omega\,. && \end{aligned}$$

Linear time-varying MPC

$$[X^*(k), U^*(k)] = \arg \min_{X(k),U(k)} \left[\|x(N+1\,|\,k)\|_P^2 + \|u(N\,|\,k)\|_R^2 + \sum_{i=1}^{N} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 \right]$$

$$\begin{aligned}
\text{s.t.} \quad & Gx(i\,|\,k) + Hu(i\,|\,k) \leq 1, && \forall i = 0,1,\cdots,N-1, \\
& x(0\,|\,k) = x(k), && \\
& x(i+1\,|\,k) = A(i\,|\,k)x(i\,|\,k) + B(i\,|\,k)u(i\,|\,k), && \forall i = 0,1,\cdots,N-1, \\
& x(N\,|\,k) \in \mathcal{X}_f \subset \Omega\,. &&
\end{aligned}$$

Nonlinear MPC

$$[X^*(k), U^*(k)] = \arg \min_{X(k),U(k)} \left[\|x(N+1\,|\,k)\|_P^2 + \|u(N\,|\,k)\|_R^2 + \sum_{i=1}^{N} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 \right]$$

$$\begin{aligned}
\text{s.t.} \quad & Gx(i\,|\,k) + Hu(i\,|\,k) \leq 1, && \forall i = 0,1,\cdots,N-1, \\
& x(0\,|\,k) = x(k), && \\
& x(i+1\,|\,k) = f\big(x(i\,|\,k),\ u(i\,|\,k)\big), && \forall i = 0,1,\cdots,N-1, \\
& x(N\,|\,k) \in \mathcal{X}_f \subset \Omega\,. &&
\end{aligned}$$

Summary

- Constrained MPC
  - Stability and feasibility
    - Terminal equality constraint: “initial & recursive feasibility indicates closed-loop stability”
    - Infinite control horizon and terminal inequality constraints
  - Algorithms: “predictive cost function” or “dynamic equality constraints”
- Invariants:
  - Time-varying linear MPC and Nonlinear MPC: “dynamic equality constraints”