Nonlinear Control Theory

Bing Zhu

The Seventh Research Division
Beihang University, Beijing, P.R.China

2020 Spring

Feedback Linearization

1. **Motivation**
2. Input-Output Linearization
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Motivation

Let us start with the pendulum

$$\dot{x}_1 = x_2, \tag{1}$$
$$\dot{x}_2 = -a[\sin(x_1+\delta) - \sin\delta] - bx_2 + cu. \tag{2}$$

Its control u can be designed by

$$u = \frac{a}{c}[\sin(x_1+\delta) - \sin\delta] + \frac{v}{c}, \tag{3}$$

to cancel the nonlinear term $a[\sin(x_1+\delta) - \sin\delta]$. This cancelation results in

$$\dot{x}_1 = x_2, \tag{4}$$
$$\dot{x}_2 = -bx_2 + v. \tag{5}$$

The nonlinear stabilization has been reduced to linear stabilization.

Proceed to design a linear control

$$v = -k_1 x_1 - k_2 x_2, \tag{6}$$

to locate the eigenvalues of the closed-loop system

$$\dot{x}_1 = x_2, \tag{7}$$
$$\dot{x}_2 = -k_1 x_1 - (k_2 + b) x_2 \tag{8}$$

in the open left-half plane.

The overall state feedback control law is given by

$$u = \frac{a}{c}[\sin(x_1 + \delta) - \sin\delta] - \frac{1}{c}(k_1 x_1 + k_2 x_2). \tag{9}$$

Is there any structural property of the system that allows us to perform the cancelation?

The nonlinear system must satisfy

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)], \tag{10}$$

where

- $A \in R^{n\times n}$, $B \in R^{n\times p}$, (A, B) is controllable;
- the functions $\alpha: \ R^n \to R^p$ and $\gamma: \ R^n \to R^{p\times p}$ are defined in a domain $D \subset R^n$ that contains the origin,
- and the matrix $\gamma(x)$ is nonsingular for every $x \in D$.

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)], \qquad u = \alpha(x) + \beta(x)v, \quad \beta(x) = \gamma^{-1}(x).$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\dot{x} = Ax + Bv. \tag{11}$$

We can now design $v = -Kx$ such that $A - BK$ is Hurwitz.

The overall nonlinear control is given by

$$u = \alpha(x) - \beta(x)Kx. \tag{12}$$

What if the nonlinear system does not satisfy $\dot{x} = Ax + B\gamma(x)[u - \alpha(x)]$??

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= a \sin x_2, \\ \dot{x}_2 &= -x_1^2 + u,\end{aligned}$$

which does not satisfy (10). However, the transformation

$$\begin{aligned}z_1 &= x_1, \\ z_2 &= a \sin x_2 = \dot{x}_1,\end{aligned}$$

satisfies

$$\begin{aligned}\dot{z}_1 &= z_2, \\ \dot{z}_2 &= a \cos x_2(-x_1^2 + u).\end{aligned}$$

Definition

A continuously differentiable map with a continuously differentiable inverse is known as a **diffeomorphism**.

If the Jacobian matrix $[\partial T/\partial x]$ is nonsingular at a point $x_0 \in D$, then it follows from the inverse function theorem that there is a neighborhood N of x_0 such that T restricted to N is a diffeomorphism on N.

A map T is said to be a **global diffeomorphism**, if it is a diffeomorphism on R^n and $T(R^n) = R^n$.

Definition

A nonlinear system

$$\dot{x} = f(x) + G(x)u, \tag{13}$$

where $f : D \to R^n$ and $G : D \to R^{n\times p}$ are sufficiently smooth on a domain $D \subset R^n$, is said to be **feedback linearizable** (or input-state linearizable) if there exists a diffeomorphism $T : D \to R^n$ such that $D_z = T(D)$ contains the origin, and the change of variables $z = T(x)$ transforms the nonlinear plant into the form

$$\dot{z} = Az + B\gamma(x)[u - \alpha(x)], \tag{14}$$

with (A, B) controllable and $\gamma(x)$ non-singular for all $x \in D$.

Input-Output Linearization

Consider the SISO nonlinear system

$$\dot{x} = f(x) + g(x)u, \tag{15}$$

$$y = h(x), \tag{16}$$

where f, g, and h are sufficiently smooth in a domain $D \subset R^n$.

The derivative of y is given by

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \triangleq L_f h(x) + L_g h(x)u, \tag{17}$$

where $L_f h(x) = \frac{\partial h}{\partial x} f(x)$ is the Lie derivative of h along f.

If $L_g h(x) = 0$, then $\dot{y} = L_f h(x)$, independent of u.

Continue to calculate $\ddot{y}$:

$$\ddot{y} = \frac{\partial L_f h}{\partial x}[f(x) + g(x)u] \triangleq L_f^2 h(x) + L_g L_f h(x) u. \tag{18}$$

Once again, if $L_g L_f h(x) = 0$, then $\ddot{y} = L_f^2 h(x)$, independent of u.

We see that, if

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \cdots, \rho - 1; \qquad L_g L_f^{\rho-1} h(x) \neq 0, \tag{19}$$

then u does not appear in $y, \dot{y}, \cdots, y^{(\rho-1)}$, and appears in $y^{(\rho)}$:

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u. \tag{20}$$

The control

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left[-L_f^{\rho} h(x) + v \right] \tag{21}$$

reduces the input-output relationship to ρ-integrators:

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u \qquad \Rightarrow \qquad y^{(\rho)} = v. \tag{22}$$

where ρ is called the relative degree of the system.

Definition

The nonlinear system (15)–(16) is said to have relative degree ρ, $1 \leq \rho \leq n$, in a region $D_0 \subset D$, if

$$L_g L_f^{i-1} h(x) = 0, \;\; i = 1, 2, \cdots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0, \tag{23}$$

for all $x \in D_0$.

Example

Consider the controlled van der Pol equation

$$\dot{x}_1 = x_2, \tag{24}$$

$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u, \quad \epsilon > 0. \tag{25}$$

- With output $y = x_1$,

$$\dot{y} = \dot{x}_1 = x_2, \quad \ddot{y} = \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u. \tag{26}$$

The system has relative degree 2 in R^2.

- With output $y = x_2$,

$$\dot{y} = \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u. \tag{27}$$

The system has relative degree 1 in R^2.

- With output $y = x_1 + x_2^2$,

$$\dot{y} = x_2 + 2x_2[-x_1 + \epsilon(1 - x_1^2)x_2 + u]. \tag{28}$$

The system has relative degree 1 in $D_0 = \{x \in R^2 | \ x_2 \neq 0\}$.

Remark

For a linear system, the relative degree is $(n - m)$, where n and m are orders of the denominator and numerator polynomials, respectively.

For nonlinear system with relative degree ρ, apply the following state transformation:

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ ---- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} = \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix}, \tag{29}$$

where ϕ_1 to $\phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on $D_0 \subset D$, and satisfy $\frac{\partial \phi_i}{\partial x} g(x) = 0$ for $1 \leq i \leq n-\rho$, $\forall\ x \in D_0$.

Does this $T(x)$ always exist ??

Theorem

Consider the system (15)–(16), and suppose it has relative degree $\rho \leq n$ in D.

- *If $\rho = n$, then for every $x_0 \in D$, a neighborhood N of x_0 exists such that the map*

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \end{bmatrix} \tag{30}$$

  restricted to N, is a diffeomorphism on N.

- *If $\rho < n$, then for every $x_0 \in D$, a neighborhood N of x_0 and smooth functions $\phi_1(x), \cdots, \phi_{n-\rho}(x)$ exist such that $\frac{\partial \phi_i}{\partial x} g(x) = 0$ is satisfied for all $x \in N$ and the map $T(x)$ of (29), restricted to N, is a diffeomorphism on N.*

The condition $\frac{\partial \phi_i}{\partial x} g(x) = 0$ ensures that, when we calculate

$$\dot{\eta} = \frac{\partial \phi}{\partial x}[f(x) + g(x)u], \tag{31}$$

the u term cancels out:

$$\dot{\eta} = f_0(\eta, \xi), \tag{32}$$

$$\dot{\xi} = A_c \xi + B_c \gamma(x)[u - \alpha(x)], \tag{33}$$

$$y = C_c \xi, \tag{34}$$

where $\xi \in R^{\rho}$, $\eta \in R^{n-\rho}$, (A_c, B_c, C_c) is in the form of ρ integrators, and

$$f_0(\eta, \xi) = \frac{\partial \phi}{\partial x} f(x)\bigg|_{x=T^{-1}(z)}, \quad \gamma(x) = L_g L_f^{\rho-1} h(x), \quad \alpha(x) = -\frac{L_f^{\rho} h(x)}{L_g L_f^{\rho-1} h(x)}.$$

The ***normal form***

$$\begin{aligned}\dot{\eta} =& f_0(\eta, \xi),\\ \dot{\xi} =& A_c\xi + B_c\gamma(x)[u - \alpha(x)],\\ y =& C_c\xi,\end{aligned}$$

decomposes the original system into an external part ξ and an internal part η.

- The external part is linearized by $u = \alpha(x) + \beta(x)v$, where $\beta(x) = \gamma^{-1}(x)$.
- The internal part is un-observable by the same control.

Setting $\xi = 0$ results

$$\dot{\eta} = f_0(\eta, 0), \tag{35}$$

which is called the *zero dynamics*.

- The system is said to be *minimum phase*, if its zero dynamics has an asymptotically stable equilibrium point in the domain of interest.
- In the special case $\rho = n$, the variable η does not exist. The system has no zero dynamics, and by default, is said to be minimum phase.

Example

The controlled van der Pol equation:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 + u, \\
y &= x_2.
\end{aligned}
$$

- The system has relative degree one in R^2.
- Taking $\xi = y$ and $\eta = x_1$. The system is already in the normal form.
- The zero dynamics can be given by $\dot{x}_1 = 0$, which is not asymptotically stable.
- The system is NOT minimum phase.

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Example

The field controlled DC motor

$$
\begin{aligned}
\dot{x}_1 &= -ax_1 + u, \\
\dot{x}_2 &= -bx_2 + k - cx_1 x_3, \\
\dot{x}_3 &= \theta x_1 x_2, \\
y &= x_3.
\end{aligned}
$$

has relative degree two in the region $D_0 = \{x \in R^3 | x_2 \neq 0\}$.

- Set $\xi = [y,\ \dot{y}]^T$, and restrict $\xi = 0$, $\Rightarrow$ $x_3 = 0$ and $x_1 x_2 = 0$.
- The zero dynamics $\dot{x}_2 = -bx_2 + k$ has an asymptotically stable equilibrium at $x_2 = \frac{k}{b}$. The system is minimum phase.

To transform the system in the above example into normal form:

- find a function $\phi(x)$ satisfying $\frac{\partial \phi}{\partial x} g = 0$.
- $g = [1, 0, 0]^T \quad \Rightarrow \quad \frac{\partial \phi}{\partial x} g = \frac{\partial \phi}{\partial x_1} = 0$, and $\phi = x_2 - \frac{k}{b}$ satisfies this condition.
- $T(x) = [\phi(x),\ x_3,\ \theta x_1 x_2]^T$ is a diffeomorphism on $D_x = \{x \in R^3 \mid x_2 > 0\}$.
- $T(x)$ transforms the system into the normal form, and it also transforms the equilibrium point of the zero dynamics to the origin.

1. Motivation
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3. **Full-State Linearization**
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Full-State Linearization

If a sufficiently smooth function $h : D \to R$ exists such that the feedback linearizable SISO system

$$\begin{aligned}\dot{x} =& f(x) + g(x)u,\\ y =& h(x)\end{aligned}$$

has relative degree n in a region $D_0 \subset D$. Then the normal form reduces to

$$\begin{aligned}\dot{z} =& A_c z + B_c \gamma(x)[u - \alpha(x)],\\ y =& C_c z,\end{aligned}$$

where A_c, B_c and C_c are in their normal form, and there exists no zero dynamics.

Does this function h exist??

Definition (Lie bracket)

For two vector fields f and g on $D \subset R^n$, the **Lie bracket** $[f, g]$ is a third vector field defined by

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x), \tag{36}$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are Jacobian matrices.

The following notation is used to simplify this process:

$$\begin{aligned} \mathrm{ad}_f^0 g(x) &= g(x), \\ \mathrm{ad}_f g(x) &= [f, g](x), \\ \mathrm{ad}_f^k g(x) &= [f, \mathrm{ad}_f^{k-1} g](x), \quad k \geq 1. \end{aligned}$$

It is obvious that $[f, g] = -[g, f]$; and $[f, g] = 0$ for constant f and g.

Definition (Distribution)

For vector fields $f_1, \ f_2, \cdots, \ f_k$ on $D \subset R^n$, let

$$\Delta(x) = \mathrm{span}\{f_1(x), f_2(x), \cdots, f_k(x)\} \tag{37}$$

be the subspace of R^n spanned by the vectors $f_1(x), \ f_2(x), \cdots, \ f_k(x)$ at any fixed $x \in D$. The collection of all vector spaces $\Delta(x)$ for $x \in D$ is called a **distribution** and referred to by

$$\Delta = \mathrm{span}\{f_1, f_2, \cdots, f_k\}. \tag{38}$$

Definition

If $\Delta = \mathrm{span}\{f_1, f_2, \cdots, f_k\}$, where $f_1(x), \ f_2(x), \cdots, \ f_k(x)$ are linearly independent for all $x \in D$, then $\dim(\Delta(x)) = k$ for all $x \in D$. In this case, Δ is a nonsingular distribution on D, generated by $f_1, \ f_2, \cdots, \ f_k$.

Definition (Involutive distribution)

A distribution is **involutive**, if

$$g_1 \in \Delta \text{ and } g_2 \in \Delta \quad \Rightarrow \quad [g_1, g_2] \in \Delta. \tag{39}$$

If Δ is a nonsingular distribution on D, generated by $f_1, \ f_2, \cdots, \ f_k$, then it can be verified that Δ is involutive if and only if

$$[f_i, f_j] \in \Delta, \ \ \forall\, 1 \le i, j \le k. \tag{40}$$

Theorem (13.2)

The system $\dot{x} = f(x) + g(x)u$ is feedback linearizable if and only if there is a domain $D_0 \subset D$ such that

1. *the matrix $\mathcal{G}(x) = [g(x), \mathrm{ad}_f g(x), \cdots, \mathrm{ad}_f^{n-1} g(x)]$ has rank n for all $x \in D_0$;*
2. *the distribution $\mathcal{D} = \mathrm{span}\{g, \mathrm{ad}_f g, \cdots, \mathrm{ad}_f^{n-2} g\}$ is involutive in D_0.*

Example

Reconsider the example

$$\dot{x} = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \triangleq f(x) + g(x)u. \tag{41}$$

We have $\mathrm{ad}_f g = [f, g] = -\frac{\partial f}{\partial x} g = [-a \cos x_2, 0]^T$, and the matrix

$$\mathcal{G} = [g, \mathrm{ad}_f g] = \begin{bmatrix} 0 & -a \cos x_2 \\ 1 & 0 \end{bmatrix} \tag{42}$$

has rank 2 for all x provided that $\cos x_2 \neq 0$. The distribution $\mathcal{D} = \mathrm{span}\{g\}$ is involutive.

Consequently, the nonlinear system is feedback linearizable in $D_0 = \{x \in R^2 | \cos x_2 \neq 0\}$.

How can we find $h(x)$ in this example?

In this example, $h(x)$ should satisfy

$$\frac{\partial h}{\partial x} g = 0, \quad \frac{\partial (L_f h)}{\partial x} g \neq 0, \quad h(0) = 0. \tag{43}$$

- From $\frac{\partial h}{\partial x} g = 0$, we have $\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0$, indicating that h must be independent of x_2.
- Therefore, $L_f h(x) = \frac{\partial h}{\partial x_1} a \sin x_2$, and the condition

$$\frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos x_2 \neq 0 \tag{44}$$

  is satisfied in the domain D_0 by any choice of h independent of x_2.
- $h(x) = x_1$ or $h(x) = x_1 + x_1^3$ can be chosen.

Stabilization

Consider a partially feedback linearizable system

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi), \\ \dot{\xi} &= A\xi + B\gamma[u - \alpha(x)], \end{aligned}$$

where

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}.$$

Here, $T(x)$ is a diffeomorphism for $x \in D \subset R^n$; $D_z = T(D)$ contains the origin; (A, B) is controllable; $\gamma(x)$ is nonsingular for $x \in D$; $f_0(0, 0) = 0$, and $f_0(\eta, \xi)$, $\alpha(x)$ and $\gamma(x)$ are continuously differentiable.

GOAL: Design u to stabilize $z = 0$.

The state feedback control

$$u = \alpha(x) + \beta(x)v, \quad \beta(x) = \gamma^{-1}(x), \tag{45}$$

reduces the original system to a “triangular” form

$$\begin{aligned}\dot{\eta} =& f_0(\eta, \xi),\\ \dot{\xi} =& A\xi + Bv,\end{aligned}$$

where ξ can be easily stabilized by $v = -K\xi$ with K such that $(A - BK)$ is Hurwitz. Asymptotic stability of the origin of the full closed-loop system

$$\begin{aligned}\dot{\eta} =& f_0(\eta, \xi),\\ \dot{\xi} =& (A - BK)\xi,\end{aligned}$$

follows from asymptotic stability of the origin of $\dot{\eta} = f_0(\eta, 0)$.

Lemma (13.1)

The origin of the full closed-loop system is asymptotically stable, if the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable.

- It indicates that a minimum-phase input-output linearizable system can be stabilized by the state feedback control

$$u = \alpha(x) - \beta(x)KT_2(x), \tag{46}$$

which is independent of $T_1(x)$ (or independept of $\phi(x)$).

- Lemma 13.1 is valid only on bounded sets. Hence, it cannot be extended to show global asymptotic stability.

Lemma (13.2)

The origin of the full closed-loop system is globally asymptotically stable, if the origin of $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable.

- "Globally" minimum-phase does not guarantee global stabilization.
- The closed-loop system will be globally stable, if the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally exponentially stable, and $f_0(\eta, \xi)$ is globally Lipschitz in (η, ξ).
- Global Lipschitz conditions are sometimes referred to as **linear growth conditions.**

Example

Consider the nonlinear system

$$\begin{aligned}\dot{\eta} &= -\eta + \eta^2 \xi,\\ \dot{\xi} &= v.\end{aligned}$$

While the origin of $\dot{\eta} = -\eta$ is globally exponentially stable, the system $\dot{\eta} = -\eta + \eta^2 \xi$ is not input-to-state stable.

- The linear control $v = -k\xi$ with $k > 0$ stabilizes the origin of the full system LOCALLY.
- Global behavior can be evaluate by considering $\nu = \eta\xi$.

$$\dot{\nu} = \eta\dot{\xi} + \dot{\eta}\xi = -k\eta\xi - \eta\xi + \eta^2\xi^2 = -(1+k)\nu + \nu^2.$$

The set $\{\nu = \eta\xi < 1 + k\}$ is positively invariant.

- On the boundary $\eta\xi = 1 + k$:

$$\dot{\nu} = 0, \;\; \xi = e^{-kt}\xi(0), \;\; \eta = e^{kt}\eta(0) \quad \Rightarrow \quad \eta\xi \equiv 1 + k.$$

- Inside the set $\{\nu = \eta\xi < 1 + k\}$:

$$\nu\dot{\nu} \leq -(1 + k - \nu)\nu < 0 \quad \Rightarrow \quad \exists T > 0, \text{ s.t. } \nu(t) < \frac{1}{2}, \; \forall t > T$$

$$\dot{\eta} = -\eta + \eta\nu \quad \Rightarrow \quad \eta\dot{\eta} \leq -\frac{1}{2}\eta^2, \; \forall t > T \quad \Rightarrow \quad \eta \to 0.$$

- $\{\nu = \eta\xi < 1 + k\}$ is the exact region of attraction; thus, $v = -k\xi$ is locally stabilizing.

Robustness of feedback linearization

Suppose that $\alpha(x)$, $\beta(x) = \gamma^{-1}(x)$ and $T_2(x)$ are not known exactly. The feedback linearization control can now be implemented by

$$u = \hat{\alpha}(x) - \hat{\beta}(x) K \hat{T}_2(x), \tag{47}$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{T}_2$ are approximations of α, β and T_2.

The closed-loop system is now given by

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi), \\ \dot{\xi} &= A\xi + B\gamma(x)[\hat{\alpha}(x) - \hat{\beta}(x) K \hat{T}_2(x) - \alpha(x)]. \end{aligned}$$

By adding and subtracting the term $BK\xi$ to the right-hand side of the second equation,

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi),\\ \dot{\xi} &= (A - BK)\xi + B\delta(z),\end{aligned}$$

where

$$\delta(z) = \gamma(x)\left\{\hat{\alpha}(x) - \alpha(x) + [\beta(x) - \hat{\beta}(x)]KT_2(x) + \hat{\beta}(x)K[T_2(x) - \hat{T}_2(x)]\right\}\Big|_{x=T^{-1}(z)}.$$

The closed-loop system appears as a perturbation of the nominal system

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi),\\ \dot{\xi} &= (A - BK)\xi.\end{aligned}$$

Start from the full state feedback linearizable system ($\rho = n$) which can be reduced to

$$\dot{z} = (A - BK)z + B\delta(z). \tag{48}$$

Lemma (13.3)

Consider (48) *where* $(A - BK)$ *is Hurwitz. Let* $P - P^T > 0$ *be the solution of the Lyapunov equation*

$$P(A - BK) + (A - BK)^T P = -I, \tag{49}$$

and k *be a non-negative constant less than* $\frac{1}{2\|PB\|_2}$.

- *If* $\|\delta(z)\| \leq k\|z\|$ *for all* z*, the origin of* (48) *will be globally exponentially stable .*
- *If* $\|\delta(z)\| \leq k\|z\| + \epsilon$ *for all* z*, the state* z *will be globally ultimately bounded by* ϵc *for some* $c > 0$.

For the more general form ($\rho < n$)

$$\dot{\eta} = f_0(\eta, \xi), \tag{50}$$

$$\dot{\xi} = (A - BK)\xi + B\delta(z). \tag{51}$$

Lemma (13.3)

Consider (50)–(51) *where* $(A - BK)$ *is Hurwitz.*

- *If* $\|\delta(z)\| \leq \epsilon$ *for all* z *and* $\dot{\eta} = f_0(\eta, \xi)$ *is input-to-state stable, then the state* z *will be globally ultimately bounded by a class* $\mathcal{K}$ *function of* ϵ*.*
- *If* $\|\delta(z)\| \leq k\|z\|$ *in some neighbourhood of* $z = 0$ *with sufficiently small* k*, and the origin of* $\dot{\eta} = f_0(\eta, 0)$ *is exponentially stable, then* $z = 0$ *is an exponentially stable equilibrium point of* (50)–(51)*.*

Tracking

Consider an SISO input-output linearizable system (already in the normal form):

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi),\\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)],\\ y &= C_c\xi.\end{aligned}$$

where $f_0(0,0) = 0$ is assumed.

GOAL: Design state feedback control law such that y asymptotically tracks $r(t)$.

- $r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \geq 0$, and the ρ-th derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t;
- the signals $r, \cdots, r^{(\rho)}$ are available on-line.

Let

$$\mathcal{R} = \begin{bmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{bmatrix}, \quad e = \begin{bmatrix} \xi_1 - r \\ \vdots \\ \xi_\rho - r^{(\rho-1)} \end{bmatrix} = \xi - \mathcal{R}$$

The change of variable $e = \xi - \mathcal{R}$ yields

$$\begin{aligned} \dot{\eta} =& f_0(\eta, e + \mathcal{R}), \\ \dot{e} =& A_c e + B_c \left\{ \gamma(x)[u - \alpha(x)] - r^{(\rho)} \right\} \end{aligned}$$

The control law can then be designed by

$$u = \alpha(x) + \beta(x)[v + r^{(\rho)}]$$

where $\beta(x) = \gamma^{-1}(x)$.

The closed-loop system is now reduced to a cascaded form

$$\begin{aligned}\dot{\eta} =& f_0(\eta, e + \mathcal{R}),\\ \dot{e} =& A_c e + B_c v\end{aligned}$$

where the second equation can be stabilized by $v = -Ke$, if $A_c - B_c K$ is Hurwitz.

The complete state feedback control law can be designed by

$$u = \alpha(x) + \beta(x)\left\{-K[T_2(x) - \mathcal{R}] + r^{(\rho)}\right\},$$

and the closed-loop system is now:

$$\begin{aligned}\dot{\eta} =& f_0(\eta, e + \mathcal{R}),\\ \dot{e} =& (A_c - B_c K)e.\end{aligned}$$

A sufficient condition to ensure global tracking is ISS of $\dot{\eta} = f_0(\eta, \xi)$.