Nonlinear Control Theory

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Lyapunov Stability

- Stability of equilibrium points is usually characterized in the sense of **Lyapunov**, a Russian mathematician and engineer who laid the foundation of the theory, which now carries his name.
- An equilibrium point is **stable** if all solutions starting at nearby points stay nearby; otherwise, it is unstable.
- It is **asymptotically stable** if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.

Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x),$$

where $f: \ D \to R^n$ is a locally Lipschitz map.

- Suppose that $f(\bar{x}) = 0$, then $\bar{x}$ is an equilibrium point.
- For convenience, suppose that $\bar{x} = 0$, such that $f(0) = 0$.
  - If $\bar{x} \neq 0$, let $y = x - \bar{x}$, then $\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) = g(y)$, where $g(0) = 0$.
- The goal is to characterize and study the stability at the origin.

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Definition

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

- **stable** if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, such that

$$\|x(0)\| < \delta \;\Rightarrow\; \|x(t)\| < \epsilon, \;\; \forall\, t \geq 0;$$

- **unstable**, if it is not stable;
- **asymptotically stable** if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \;\Rightarrow\; \lim_{t \to +\infty} x(t) = 0.$$

Example

The pendulum equation

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a \sin x_1 - b x_2 \end{aligned}$$

has two equilibrium points, namely $(0, 0)$ and $(\pi, 0)$.

- Neglecting friction ($b = 0$), the phase portrait around $(0, 0)$ are closed-orbits, thus the origin is stable.
  - ∗ it is not asymptotically stable; trajectories starting off the equilibrium point do not tend to it.
- With $b > 0$, the equilibrium point at the origin becomes a stable focus, and it is asymptotically stable.
- The equilibrium point $(\pi, 0)$ is a saddle, thus it is un-stable.

How to determine stability?

Take the pendulum as example, and try its energy function

$$E(x) = \int_0^{x_1} a \sin y \mathrm{d}y + \frac{1}{2}x_2^2 = a(1 - \cos x_1) + \frac{1}{2}x_2^2.$$

- When $b = 0$ (no friction), the system is conservative.
  - $\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \quad \Rightarrow \quad E(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2 = c \quad \Rightarrow \quad \frac{a}{2}x_1^2 + \frac{1}{2}x_2^2 = c.$
- When $b > 0$, it can be calculated that $\frac{\mathrm{d}E}{\mathrm{d}t} \leq 0$, indicating that the energy keeps decreasing until it eventually reaches zero, showing that the trajectory tends to $x = 0$ as t tends to ∞.

In 1892, Lyapunov showed that certain other functions could be used instead of energy.

Theorem (4.1 Lyapunov theorem)

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$, and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function such that

$$V(0) = 0, \ and, \ V(x) > 0 \ in \ D - \{0\}, \quad \dot{V} \leq 0 \ in \ D,$$

then, $x = 0$ is stable. Moreover, if

$$\dot{V} < 0 \ in \ D - \{0\},$$

then $x = 0$ is asymptotically stable.

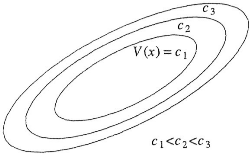

Figure: Level surfaces of a Lyapunov function

- The function $V(x)$ satisfying Theorem 4.1 is called **Lyapunov function**.
- The surface $V(x) = c$ for some $c > 0$ is called Lyapunov surface or level surface.
- $V(x)$ is **positive definite**: $V(0) = 0$ & $V(x) > 0$.
- $V(x)$ is **positive semi-definite**: $V(0) = 0$ & $V(x) \geq 0$.
- Negative definite, negative semi-definite...
- $V(x)$ is radially unbounded: $x \to \infty \ \Rightarrow \ V(x) \to \infty$.

Theorem (Lyapunov theorem (rephrase))

The origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.

If $V(x)$ is defined in quadratic form $V(x) = x^T Px$, it is easy to check sign definiteness:

- $V(x)$ is positive definite (or positive semi-definite), if and only if all eigenvalues of P are positive (or non-negative).
- $V(x) = x^T Px$ is positive definite (or positive semi-definite), we say that P is positive definite (or positive semi-definite), written by $P > 0$ (or $P \geq 0$).

Example

$$V(x) = [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x.$$

- $V(x)$ is positive definite if $a > \sqrt{5}$; $V(x)$ is positive semi-definite if $a \geq \sqrt{5}$.
- $V(x)$ is negative definite if $a < -\sqrt{5}$; $V(x)$ is negative semi-definite if $a \leq -\sqrt{5}$.
- $V(x)$ is in-definite if $-\sqrt{5} < a < \sqrt{5}$.

Example

Consider the first-order differential equation

$$\dot{x} = -g(x),$$

where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0, \ xg(x) > 0, \ \forall \ x \neq 0 \text{ and } x \in (-a, a).$$

Consider the function $V(x) = \int_0^x g(y)\mathrm{d}y, \ x \in (-a, a)$,

- $V(x)$ is continuously differentiable;
- $V(0) = 0$, and $V(x) > 0$ for $x \neq 0$;
- $\dot{V} = \frac{\partial V}{\partial x}\dot{x} = -g^2(x) < 0, \ \ \forall \ x \in (-a, a) - 0.$

According to Theorem 4.1, it is concluded that the origin is asymptotically stable.

Example

Pendulum equation with friction:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 - b x_2.$$

Choose $V(x) = a(1 - \cos x_1) + \frac{1}{2} x_2^2$, and it can be calculated that $\dot{V} = -b x_2^2 \leq 0$. Then, It is concluded that the origin is **stable**.
$\dot{V}$ is not negative definite because $\dot{V} = 0$ for all $x_2 = 0$ irrespective of x_1.

Remark

The conditions of Lyapunovs theorem are only sufficient.

Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate.

Try

$$V(x) = \frac{1}{2}x^T Px + a(1 - \cos x_1) = \frac{1}{2}[x_1 \ x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1),$$

where $p_{11} > 0$ and $p_{11}p_{22} - p_{12}^2 > 0$. It can be calculate that

$$\begin{aligned} \dot{V} =& (p_{11}x_1 + p_{12}x_2 + a \sin x_1)x_2 + (p_{12}x_1 + p_{22}x_2)(-a \sin x_1 - bx_2) \\ =& a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 + (p_{11} - p_{12}b)x_1x_2 + (p_{12} - p_{22}b)x_2. \end{aligned}$$

Select $p_{22} = 1$, $p_{11} = \frac{b^2}{2}$, and $p_{12} = \frac{b}{2}$, such that

$$\dot{V} = -abx_1 \sin x_1 - bx_2^2 < 0, \ \ \forall x_1 \in (-\pi, \pi), \ \forall x_2 \in R.$$

Then, it can be concluded that the origin is **asymptotically stable.**

Variable Gradient Method

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x), \quad \text{where } g(x) = \left(\frac{\partial V}{\partial x}\right)^T.$$

Goal: Choose $g(x)$ as the gradient of a positive definite function $V(x)$ that makes $\dot{V}(x)$ negative definite.

- $g(x)$ is the gradient of a scalar function if and only if

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \ \forall i, j = 1, 2, \cdots, n.$$

- Choose $g(x)$ such that $g^T(x) f(x)$ is negative definite, and then calculate $V(x)$.

- To obtain $V(x)$, calculate the integral:

$$V(x) = \int_0^x g(y)^T \mathrm{d}y = \int_0^x \sum_{i=1}^n g_i(y)\mathrm{d}y_i,$$

which is a line integral of gradient from 0 to x, and it is independent of the path.

- Calculate the integral along the axes:

$$V(x) = \int_0^{x_1} g_1(x_1, 0, \cdots, 0)\mathrm{d}y_1 + \int_0^{x_2} g(x_1, y_2, \cdots, 0)\mathrm{d}y_2 + \cdots$$
$$+ \int_0^{x_n} g_1(x_1, x_2, \cdots, y_n)\mathrm{d}y_n.$$

- Leave some parameters of $g(x)$ undetermined, and choose them to make $V(x)$ positive definite.

Example (4.5)

Consider the nonlinear system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h(x_1) - ax_2,$$

where $a > 0$, and $h(x)$ is locally Lipschitz, $h(0) = 0$, and $yh(y) > 0$ for all $y \neq 0$ and $y \in (-b, c)$ for some positive constants b and c.

We want to choose $g(x)$ satisfying

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \ \forall i, j = 1, 2, \cdots, n,$$
$$\dot{V} = g_1(x)x_2 + g_2(x)[-h(x_1) - ax_2] < 0, \ \forall x \neq 0,$$
$$V(x) = \int_0^x g(y)^T \mathrm{d}y > 0, \ \forall x \neq 0.$$

- Try

$$g(x) = [\phi_1(x_1) + \psi_1(x_2),\ \phi_2(x_1) + \psi_2(x_2)]^T.$$

- To satisfy $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$, choose

$$\psi_1(x_2) = \gamma x_2, \quad \phi_2(x_1) = \gamma x_1.$$

- Calculate

$$\dot{V}(x) = -\gamma x_1 h(x_1) - a x_2 \psi_2(x_2) + \gamma x_2^2 + x_2 \phi_1(x_1) - a\gamma x_1 x_2 - \psi_2(x_2) h(x_1).$$

- To cancel the cross-product terms, set

$$\psi_2(x_2) = \delta x_2, \quad \phi_1(x_1) = a\gamma x_1 + \delta h(x_1).$$

- $g = [a\gamma x_1 + \delta h(x_1) + \gamma x_2,\ \gamma x_1 + \delta x_2]^T$.
- $\dot{V}(x) = -\gamma x_1 h(x_1) - (a\delta - \gamma)x_2^2$.
- $V(x) = \int_0^{x_1}[a\gamma y_1 + \delta h(y_1)]\mathrm{d}y_1 + \int_0^{x_2}(\gamma x_1 + \delta y_2)\mathrm{d}y_2 = \frac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)\mathrm{d}y$.
- $P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$.
- Choose $\delta > 0$ and $0 < \gamma < a\delta$ guarantees asymptotic stability.

Theorem (4.2 Global asymptotic stability, Barbashin-Krasovskii theorem)

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : R^n \to R$ be a continuously differentiable function such that

- $V(0) = 0$ *and* $V(x) > 0, \forall x \neq 0$;
- $\|x\| \to \infty \quad \Rightarrow \quad V(x) \to \infty$ *(Radially unbounded);*
- $\dot{V}(x) < 0, \forall x \neq 0$;

then, $x = 0$ is ***globally*** *asymptotically stable.*

Example

Consider Example 4.5 again. Suppose that $yh(y) > 0$ holds for all $y \neq 0$. Then the Lyapunov function $V(x) = \frac{1}{2}x^T Px + \frac{1}{2}x_2^2$ is positive definite and radially unbounded. Its derivative $\dot{V}$ is negative definite. It is concluded that $x = 0$ is globally asymptotic stable.

Theorems 4.1 and 4.2 are all sufficient conditions. They cannot be used as criteria for ”unstability”.

Theorem (4.3 Chetaev’s theorem)

Let $x = 0$ be an equilibrium point for $\dot{x} = f(0)$. Let $V : D \to R$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Define a set $U = \{x \in B_r \mid V(x) > 0\}$, and suppose that $\dot{V}(x) > 0$ in U. Then, $x = 0$ is unstable.