Nonlinear Control Theory

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Lyapunov Redesign

1. **Stabilization**
2. Nonlinear damping

Stabilization

Consider the system

$$\dot{x} = f(t,x) + G(t,x)[u + \delta(t,x,u)], \tag{1}$$

where

- $x \in R^n$ is the state, and $u \in R^p$ is the control input.
- The functions f, G and δ are defined for $(t,x,u) \in [0,\infty) \times D \times R^p$.
- $D \subset R^n$ is a domain that contains the origin.
- f, G and δ are piecewise continuous in t and locally Lipschitz in x and u.
- f and G are known exactly, and δ is unknown due to model simplification and uncertainties.

Its nominal model:

$$\dot{x} = f(t,x) + G(t,x)u. \tag{2}$$

- Suppose that we have already design a stabilizing feedback control law $u = \psi(t, x)$, such that the origin of

$$\dot{x} = f(t, x) + G(t, x)\psi(t, x) \tag{3}$$

  is uniformly asymptotically stable.

- Suppose further that, with $u = \psi(t, x)$, there exists a continuously differentiable function $V(t, x)$, such that

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|), \tag{4}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(t, x) + G(t, x)\psi(t, x)] \le -\alpha_3(\|x\|), \tag{5}$$

  for all $(t, x) \in [0, \infty) \times D$, where α_i are class $\mathcal{K}$ functions.

Assumption

With $u = \psi(t,x) + v$, the uncertain term δ satisfies

$$\|\delta(t,x,\psi(t,x)+v)\| \leq \rho(t,x) + \kappa_0\|v\|, \quad 0 \leq \kappa_0 < 1, \tag{6}$$

where $\rho : [0,\infty) \times D \to R$ is a known non-negative continuous function.

- ρ is a measure of the size of the uncertainty, and it is not required to be small.
- With the knowledge of V, ρ and κ_0, it is possible to design $u = \psi(t,x) + v$ such that the closed-loop system is stabilized with uncertainties.
- The design of v is named "**Lyapunov redesign**".

Apply $u = \psi(t,x) + v$ to the system:

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x) + G(t,x)\left[v + \delta(t,x,\psi(t,x)+v)\right], \tag{7}$$

which is a perturbation of the nominal closed-loop system

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x). \tag{8}$$

Let us calculate $\dot{V}(t,x)$ along the trajectory of the perturbed system:

$$\begin{aligned}\dot{V} =& \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + G\psi) + \frac{\partial V}{\partial x}G(v+\delta) \le -\alpha_3(\|x\|) + \frac{\partial V}{\partial x}G(v+\delta)\\ \le& -\alpha_3(\|x\|) + w^T v + w^T\delta,\end{aligned}$$

where $w^T = \frac{\partial V}{\partial x}G$.

Two methods to design v such that $w^T v + w^T \delta \le 0$:

1. Suppose that $\|\delta(t, x, \psi(t, x) + v)\|_2 \le \rho(t, x) + \kappa_0 \|v\|_2$ for $0 \le \kappa_0 < 1$. It holds that

$$w^T v + w^T \delta \le w^T v + \|w\|_2 \|\delta\|_2 \le w^T v + \|w\|_2 [\rho(t, x) + \kappa_0 \|v\|_2].$$

Taking

$$v = -\eta(t, x) \frac{w}{\|w\|_2}, \quad \eta(t, x) \ge 0.$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$w^T v + w^T \delta \le -\eta \|w\|_2 + \rho \|w\|_2 + \kappa_0 \eta \|w\|_2 = -\eta(1 - \kappa_0) \|w\|_2 + \rho \|w\|_2.$$

Choose $\eta(t, x) \ge \frac{\rho(t,x)}{1-\kappa_0} \quad \Rightarrow \quad w^T v + w^T \delta \le 0.$

2. Suppose that $\|\delta(t, x, \psi(t, x) + v)\|_\infty \leq \rho(t, x) + \kappa_0 \|v\|_\infty$ for $0 \leq \kappa_0 < 1$. It holds that

$$w^T v + w^T \delta \leq w^T v + \|w\|_1 \|\delta\|_\infty \leq w^T v + \|w\|_1 [\rho(t, x) + \kappa_0 \|v\|_\infty].$$

Choose

$$v = -\eta(t, x)\mathrm{sgn}(w), \quad \eta(t, x) \geq \frac{\rho(t, x)}{1 - \kappa_0}.$$

Then,

$$\begin{aligned} w^T v + w^T \delta \leq & -\eta\|w\|_1 + \rho\|w\|_1 + \kappa_0 \eta \|w\|_1 \\ = & -\eta(1 - \kappa_0)\|w\|_1 + \rho\|w\|_1 \leq 0. \end{aligned}$$

Neither of the above two control laws are dis-continuous!

Consider the feedback control law:

$$v = \begin{cases} -\eta(t,x)\frac{w}{\|w\|_2}, & \text{if } \eta(t,x)\|w\|_2 \geq \epsilon, \\ -\eta^2(t,x)\frac{w}{\epsilon}, & \text{if } \eta(t,x)\|w\|_2 < \epsilon. \end{cases}$$

The derivative of V will be negative definite whenever $\eta(t,x)\|w\|_2 \geq \epsilon$.

Check $\dot{V}$ in case of $\eta(t,x)\|w\|_2 < \epsilon$:

$$\begin{aligned} \dot{V} &\leq -\alpha_3(\|x\|_2) + w^T\left[-\eta^2\frac{w}{\epsilon} + \delta\right] \\ &\leq -\alpha_3(\|x\|_2) - \frac{\eta^2}{\epsilon}\|w\|_2^2 + \rho\|w\|_2 + \kappa_0\|w\|_2\|v\|_2. \end{aligned}$$

$$
\begin{aligned}
\dot{V} \leq & -\alpha_3(\|x\|_2) - \frac{\eta^2}{\epsilon}\|w\|_2^2 + \rho\|w\|_2 + \frac{\kappa_0\eta^2}{\epsilon}\|w\|_2^2 \\
\leq & -\alpha_3(\|x\|_2) + (1-\kappa_0)\left(-\frac{\eta^2}{\epsilon}\|w\|_2^2 + \eta\|w\|_2\right),
\end{aligned}
$$

where the term $-\frac{\eta^2}{\epsilon}\|w\|_2^2 + \eta\|w\|_2$ attains a maximum value $\frac{\epsilon}{4}$ at $\eta\|w\|_2 = \frac{\epsilon}{2}$. Therefore,

$$
\dot{V} \leq -\alpha_3(\|x\|_2) + \frac{\epsilon(1-\kappa_0)}{4},
$$

whenever $\eta(t,x)\|w\|_2 < \epsilon$.

On the other hand, when $\eta(t,x)\|w\|_2 \geq \epsilon$, it holds that

$$\dot{V} \leq -\alpha_3(\|x\|_2) \leq -\alpha_3(\|x\|_2) + \frac{\epsilon(1-\kappa_0)}{4}.$$

Consequently, $\dot{V} \leq -\alpha_3(\|x\|_2) + \frac{\epsilon(1-\kappa_0)}{4}$ is satisfied irrespective of $\eta(t,x)\|w\|_2$.

Take $r > 0$ such that $B_r \subset D$, and choose $\epsilon < \frac{2\alpha_3\left(\alpha_2^{-1}(\alpha_1(r))\right)}{1-\kappa_0}$, and set $\mu = \alpha_3^{-1}\left(\frac{\epsilon(1-\kappa_0)}{2}\right) < \alpha_2^{-1}(\alpha_1(r))$. Then,

$$\dot{V} \leq -\alpha_3(\|x\|_2), \quad \forall \mu \leq \|x\|_2 < r.$$

and the trajectory of the closed-loop system will enter $\|x\|_2 \leq \mu$. $\Rightarrow$ $\boxed{U.U.B.}$

Under what condition does the continuous control guarantee asymptotic stability?

Suppose there is a ball $B_a = \{\|x\|_2 < a\},\ a \leq r$, such that

$$\alpha_3(\|x\|_2) \geq \phi^2(x),\ \eta(t,x) \geq \eta_0 > 0,\ \rho(t,x) \leq \rho_1\phi(x),$$

where $\phi : R^n \to R$ is a positive definite function of x, and $\phi(0) = 0$.

Choosing $\epsilon < \alpha_2^{-1}(\alpha_1(a))$ ensures that $x(t)$ be maintained in B_a after finite time.

When $\eta(t,x)\|w\|_2 < \epsilon$, it holds that

$$\begin{aligned}\dot{V} \leq & -\alpha_3(\|x\|_2) - \frac{\eta^2(1-\kappa_0)}{\epsilon}\|w\|_2^2 + \rho\|w\|_2 \\ \leq & -\frac{1}{2}\alpha_3(\|x\|_2) - \frac{1}{2}\phi^2(x) - \frac{\eta_0^2(1-\kappa_0)}{\epsilon}\|w\|_2^2 + \rho_1\phi(x)\|w\|_2.\end{aligned}$$

$$
\begin{aligned}
\dot{V} &\leq -\frac{1}{2}\alpha_3(\|x\|_2) - \frac{1}{2}\phi^2(x) - \frac{\eta_0^2(1-\kappa_0)}{\epsilon}\|w\|_2^2 + \rho_1\phi(x)\|w\|_2 \\
&= -\frac{1}{2}\alpha_3(\|x\|_2) - \frac{1}{2}\begin{bmatrix} \phi(x) \\ \|w\|_2 \end{bmatrix}^T \begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & \frac{2\eta_0^2(1-\kappa_0)}{\epsilon} \end{bmatrix} \begin{bmatrix} \phi(x) \\ \|w\|_2 \end{bmatrix}.
\end{aligned}
$$

Consequently, choose $\epsilon < \frac{2\eta_0^2(1-\kappa_0)}{\rho_1^2}$, we have

$$
\dot{V} \leq -\frac{1}{2}\alpha_3(\|x\|_2).
$$

And, in another aspect, $\dot{V} \leq -\alpha_3(\|x\|_2) \leq -\frac{1}{2}\alpha_3(\|x\|_2)$ in case of $\eta(t,x)\|w\|_2 \geq \epsilon$.

$\Rightarrow$ $\boxed{\text{U.A.S.}}$

Nonlinear Damping

Reconsider the nonlinear system with $\delta(t, x, u) = \Gamma(t, x)\delta_0(t, x, u)$:

$$\dot{x} = f(t, x) + G(t, x)\left[u + \Gamma(t, x)\delta_0(t, x, u)\right],$$

where

- f, G and Γ are known.
- δ_0 is bounded but unknown. We do NOT even known its bounds.
- There exists a nominal control law $u = \psi(t, x)$ and a Lypuanov function $V(x)$ for the nominal system, such that

$$\alpha_1(\|x\|_2) \le V(x) \le \alpha_2(\|x\|_2), \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + g\psi) \le -\alpha_3(\|x\|_2).$$

GOAL: **Design $u = \psi(t,x) + v$ to robustly stabilize the closed-loop system.**

- The derivative of V along the trajectory of the closed-loop system:

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + g\psi) + \frac{\partial V}{\partial x}G(v + \Gamma\delta_0) \leq -\alpha_3(\|x\|_2) + w^T(v + \Gamma\delta_0).$$

- Taking

$$v = -kw\|\Gamma(t,x)\|_2^2, \quad k > 0,$$

yields that

$$\dot{V} \leq -\alpha_3(\|x\|_2) - k\|w\|_2^2\|\Gamma\|_2^2 + k_0\|w\|_2\|\Gamma\|_2.$$

where $k_0 > 0$ is an (unknown) upper bound on $\|\delta_0\|$.

- The term $-k\|w\|_2^2\|\Gamma\|_2^2 + k_0\|w\|_2\|\Gamma\|_2$ attains its maximum value $\frac{k_0^2}{4k}$ at $\|w\|_2\|\Gamma\|_2 = \frac{k_0}{2k}$.
- Consequently, it holds that

$$\dot{V} \leq -\alpha_3(\|x\|_2) + \frac{k_0^2}{4k},$$

indicating that the solution of the closed-loop system is Uniformly bounded.

- The Lyapunov redesign $v = -kw\|\Gamma(t,x)\|_2^2$ is called **nonlinear damping**.