Nonlinear Control Theory

Bing Zhu

The Seventh Research Division
Beihang University, Beijing, P.R.China

2020 Spring

Fundamental Properties

Existence & Uniqueness

Existence & Uniqueness: A deterministic system can be repeated exactly.

Consider the nonlinear system and its initial condition:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0.$$

Assume:

- f(t,x) is continuous in x;
- f(t,x) is piecewise continuous in t.

Remark

The assumption that $f(t, x)$ be piecewise continuous in t allows us to include the case when $f(t, x)$ depends on a time-varying input experiencing step changes with time.

Example

The scalar equation

$$\dot{x} = x^{\frac{1}{3}}, \quad x(0) = 0$$

has a solution $x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$. However, this solution is NOT UNIQUE, since $x(t) \equiv 0$ is another solution.

Theorem (3.1 Local existence and uniqueness)

Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition:

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|,$$

$\forall x, y \in B = \{x \in R^n \,|\, \|x - x_0\| \leq r\}, \forall t \in [t_0, t_1]$. *Then, there exists some $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.*

- A function $f(x)$ satisfying $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$ is said to be Lipschitz in x, and $L > 0$ is called a Lipschitz constant.
- A function $f(x)$ is *locally Lipschitz* on a domain (open and connected set) $D \subset R^n$ if each point of D has a neighborhood D_0 such that f satisfies the Lipschitz condition for all points in D_0 with some Lipschitz constant L_0.
- f is Lipschitz on a set W, if it satisfies the Lipschitz condition for all points in W, with the same Lipschitz constant L.
- A locally Lipschitz function on a domain D is NOT necessarily Lipschitz on D, since the Lipschitz condition may not hold uniformly (with the same constant L) for all points in D.
- A locally Lipschitz function on a domain D is Lipschitz on every compact (closed and bounded) subset of D (Exercise 3.19).

- A function $f(x)$ is globally Lipschitz, if it is Lipschitz on R^n.
- "Globally Lipschitz" can be extended to a function $f(t,x)$, provided the Lipschitz condition holds uniformly in t for all t in a given interval of time.

Example

- $f(t,x)$ is locally Lipschitz in x on $[a,b] \times D \subset R \times R^n$, if each point $x \in D$ has a neighborhood D_0 such that f satisfies Lipschitz condition on $[a,b] \times D_0$ with some Lipschitz constant L_0.
- $f(t,x)$ is locally Lipschitz in x on $[t_0, \infty) \times D$, if it is locally Lipschitz in x on $[a,b] \times D$ for every compact interval $[a,b] \subset [t_0, \infty)$.
- $f(t,x)$ is Lipschitz in x on $[a,b] \times W$, if it satisfies Lipschitz condition for all $t \in [a,b]$ and all points in W, with the same Lipschitz constant L.

- When $f : R \to R$, the Lipschitz condition can be rewritten by

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L,$$

implying that on a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L.

- A function that has infinite slope at some point is not locally Lipschitz at that point.

Example

- Any discontinuous function is not locally Lipschitz at the point of discontinuity.
- The function $f(x) = x^{\frac{1}{3}}$ is not locally Lipschitz at $x = 0$, since $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \to \infty$ as $x \to 0$.

Example

if $|f'(x)|$ is bounded by a constant k over the interval of interest, then $f(x)$ is Lipschitz on the same interval with Lipschitz constant $L = k$.

Lemma (Extension to vector-valued functions)

Let $f : [a, b] \times D \to R^m$ be continuous for some domain $D \subset R^n$. Suppose that $\frac{\partial f}{\partial x}$ exists and is continuous on $[a, b] \times D$. If, for a convex subset $W \subset D$, there is a constant $L \geq 0$ such that

$$\left\| \frac{\partial f}{\partial x}(t, x) \right\| \leq L$$

on $[a, b] \times W$, then

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $t \in [a, b]$, $x \in W$, and $y \in W$.

- The Lipschitz property of a function is stronger than continuity.
- If $f(x)$ is Lipschitz on W, then it is uniformly continuous on W (Exercise 3.20).
- The converse is not true, as seen from the function $f(x) = x^{\frac{1}{3}}$, which is continuous, but not locally Lipschitz at $x = 0$
- The Lipschitz property is weaker than continuous differentiability.

Lemma

- *If $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ are continuous on $[a,b] \times D$, for some domain $D \subset R^n$, then f is locally Lipschitz in x on $[a,b] \times D$.*
- *If $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ are continuous on $[a,b] \times R^n$, then f is globally Lipschitz in x on $[a,b] \times R^n$ if and only if $\frac{\partial f}{\partial x}(t,x)$ is uniformly bounded on $[a,b] \times R^n$.*

Theorem 3.1 is a local theorem since it guarantees existence and uniqueness only over an interval $[t_0, t_0 + \delta]$, where δ may be very small.

Example

Consider the scalar system

$$\dot{x} = -x^2, \quad x(0) = -1$$

. The function $f(x) = -x^2$ is locally Lipschitz for all $x \in R$. Hence, it is Lipschitz on any compact subset of R. The unique solution

$$x(t) = \frac{1}{t-1}$$

exists over $[0, 1)$. As $t \to 1$, $x(t)$ leaves any compact set (its trajectory has a *finite escaping time* at $t = 1$).

Theorem (3.2)

Suppose that $f(t,x)$ is piece-wise continuous in t and satisfies

$$\|f(t,x)-f(t,y)\| \leq L\|x-y\|, \quad \forall x,y \in R^n, \quad \forall t \in [t_0,t_1].$$

Then, the state equation $\dot{x}=f(t,x)$, with $x(t_0)=x_0$, has a unique solution over $[t_0,t_1]$.

Theorem (3.3)

Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, $x_0 \in W$, and suppose it is known that every solution of

$$\dot{x}=f(t,x), \quad x(0)=x_0$$

lies entirely in W, Then, there is a unique solution that is defined for all $t \geq t_0$.

Example

Consider the scalar system

$$\dot{x} = -x^3 = f(x).$$

- The function $f(x)$ does not satisfy a global Lipschitz condition since the Jacobian $\frac{\partial f}{\partial x} = -3x^2$ is not globally bounded.
- Nevertheless, for any initial state $x(t_0) = x_0$, the equation has the unique solution

$$x(t) = \mathrm{sign}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}$$

  which is well-defined for all $t \geq t_0$.
- At any time instant, $x(t)\dot{x}(t)$ is negative. Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $W = \{x \in R \mid |x| \leq |a|\}$. Thus, the equation has a unique solution for all $t \geq t_0$.

1. Existence & Uniqueness
2. **Continuous Dependence on Initial Conditions and Parameters**
3. Differentiability of Solutions and Sensitivity Equations
4. Comparison Principle

Continuous Dependence on Initial Conditions and Parameters

Continuous dependence on initial conditions

- Continuous dependence on the initial time t_0 is obvious from the integral expression

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\mathrm{d}s.$$

- The solution depends continuously on $y(t_0) = y_0$ on the compact time interval $[t_0, t_1]$, if solutions starting at nearby points are defined on the same time interval and remain close to each other in that interval.
  - ⋆ Given $\epsilon > 0$, there is $\delta > 0$ such that for all z_0 in the ball $\{x \in R^n | \, \|x - y_0\| < \delta\}$, the equation $\dot{x} = f(t, x)$ has a unique solution $z(t)$ defined on $[t_0, t_1]$, with $z(t_0) = z_0$, and satisfies $\|z(t) - y(t)\| < \epsilon$ for all $t \in [t_0, t_1]$.

Continuous dependence on parameters

- Assume that f depends continuously on a set of constant parameters: $f = f(t, x, \lambda)$, where $\lambda \in R^p$.
  - ⋆ The constant parameters could represent physical parameters of the system, and the study of perturbation of these parameters accounts for modeling errors or changes in the parameter values due to aging.
- Let $x(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ defined on $[t_0, t_1]$, with $x(t_0, \lambda_0) = x_0$.
- The solution is said to depend continuously on λ, if for any $\epsilon > 0$, there is $\delta > 0$ such that for all λ in the ball $\{\lambda \in R^p | \, \|\lambda - \lambda_0\| < \delta\}$, the equation $\dot{x} = f(t, x, \lambda)$ has a unique solution $x(t, \lambda)$ defined on $[t_0, t_1]$, with $x(t_0, \lambda) = x_0$, and satisfies $\|x(t, \lambda) - x(t, \lambda_0)\| < \epsilon$ for all $t \in [t_0, t_1]$.

Theorem

Let $f(t,x)$ be piece-wise continuous in t and Lipschitz in x on $[t_0, t_1] \times W$ with a Lipschitz constant L, where $W \subset R^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t,y), \quad y(t_0) = y_0,$$

and

$$\dot{z} = f(t,z) + g(t,z), \quad z(t_0) = z_0,$$

such that $y(t),\ z(t) \in W,\ \forall\ t \in [t_0, t_1]$. Suppose that

$$\|g(t,x)\| \leq \mu, \quad \forall\ (t,x) \in [t_0, t_1] \times W$$

for some $\mu > 0$. Then,

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| \exp[L(t - t_0)] + \frac{\mu}{L}\{\exp[L(t - t_0)] - 1\}, \ \ \forall t \in [t_0, t_1].$$

Theorem

Let $f(t,x,\lambda)$ be continuous in (t,x,λ) and locally Lipschitz in x (uniformly in t and λ) on $[t_0,t_1] \times D \times \{\|\lambda-\lambda_0\| \le c\}$, where $D \subset R^n$ is an open connected set. Let $y(t,\lambda_0)$ be a solution of $\dot{x} = f(t,x,\lambda_0)$ with $y(t_0,\lambda_0) = y_0 \in D$. Suppose $y(t,\lambda_0)$ is defined and belongs to D for all $t \in [t_0,t_1]$. Then, given $\epsilon > 0$, there is $\delta > 0$ such that if

$$\|z_0 - y_0\| < \delta, \quad \|\lambda - \lambda_0\| < \delta,$$

then there is a unique solution $z(t,\lambda)$ of $\dot{x} = f(t,x,\lambda)$ defined on $[t_0,t_1]$, with $z(t_0,\lambda) = z_0$, and $z(t,\lambda)$ satisfies

$$\|z(t,\lambda) - y(t,\lambda_0)\| < \epsilon, \quad \forall\, t \in [t_0,t_1].$$

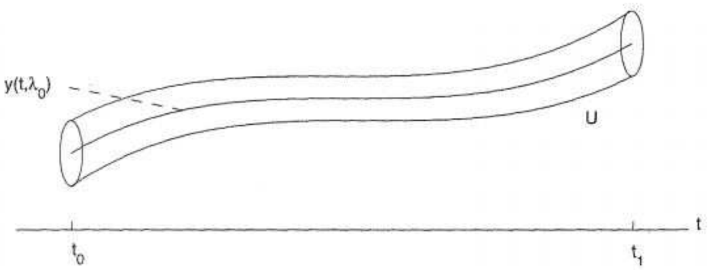

Figure: A tube constructed around the solution $y(t, \lambda_0)$

Differentiability of Solutions and Sensitivity Equations

- Suppose that $f(t, x, \lambda)$ is continuous in (t, x, λ) and has continuous first partial derivatives with respect to x and λ for all $(t, x, \lambda) \in [t_0, t_1] \times R^n \times R^P$.
- Let λ_0 be a nominal value of λ, and suppose that the nominal state equation

$$\dot{x} = f(t, x, \lambda), \quad x(t_0) = x_0$$

  has a unique solution $x(t, \lambda)$ over $[t_0, t_1]$ that is close to $x(t, \lambda_0)$.
- The continuous differentiability of f with respect to x and λ implies that the solution $x(t, \lambda)$ is differentiable with respect to λ near λ_0.

Mathematically,

$$x(t,\lambda) = x_0 + \int_{t_0}^{t} f(s, x(s,\lambda), \lambda)\mathrm{d}s.$$

Taking partial derivatives with respect to λ yields

$$x_\lambda(t,\lambda) \triangleq \frac{\partial x(t,\lambda)}{\partial \lambda} = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s,\lambda), \lambda) x_\lambda(s,\lambda) + \frac{\partial f}{\partial \lambda}(s, x(s,\lambda), \lambda)\right] \mathrm{d}s,$$

where $\frac{\partial x_0}{\partial \lambda} = 0$, since x_0 is independent of λ. Differentiating with respect to t,

$$\frac{\partial x_\lambda(t,\lambda)}{\partial t} = A(t,\lambda)x_\lambda(t,\lambda) + B(t,\lambda), \quad x_\lambda(t_0,\lambda) = 0,$$

where

$$A(t,\lambda) = \left.\frac{\partial f(t,x,\lambda)}{\partial x}\right|_{x=x(t,\lambda)}, \quad B(t,\lambda) = \left.\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right|_{x=x(t,\lambda)}.$$

Let $S(t) = x_\lambda(t, \lambda_0)$; then $S(t)$ is the unique solution of the equation

$$\dot{S}(t) = A(t, \lambda_0)S(t) + B(t, \lambda_0), \quad S(t_0) = 0.$$

The function $S(t)$ is called the *sensitivity function*, and the above equation is called the *sensitivity equation*.

- Provide first-order estimates of the effect of parameter variations on solutions.
- Approximate the solution when λ is sufficiently close to its nominal value λ_0.

$$\begin{aligned} x(t, \lambda) =& x(t, \lambda_0) + S(t)(\lambda - \lambda_0) + \text{higher-order terms}, \\ x(t, \lambda) \approx& x(t, \lambda_0) + S(t)(\lambda - \lambda_0). \end{aligned}$$

The procedure for calculating the sensitivity function:

1. Solve the nominal state equation for the nominal solution $x(t, \lambda_0)$.
2. Evaluate the Jacobian matrices

$$A(t,\lambda_0) = \left.\frac{\partial f(t,x,\lambda)}{\partial x}\right|_{x=x(t,\lambda_0),\ \lambda=\lambda_0}, \quad B(t,\lambda_0) = \left.\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right|_{x=x(t,\lambda_0),\ \lambda=\lambda_0}.$$

3. Solve the sensitivity equation for $S(t)$

Remark

The nonlinear nominal state equation and the linear time-varying sensitivity equation have to be solved numerically.

An alternative solution for calculating $S(t)$:

$$\begin{aligned} \dot{x} &= f(t, x, \lambda_0), & x(t_0) &= x_0 \\ \dot{S} &= \left.\frac{\partial f(t, x, \lambda)}{\partial x}\right|_{\lambda=\lambda_0} S + \left.\frac{\partial f(t, x, \lambda)}{\partial \lambda}\right|_{\lambda=\lambda_0}, & S(t_0) &= 0, \end{aligned}$$

which is an $(n + np)$ augmented equation and can be solved numerically.

Remark

If the original state equation is autonomous, that is, $f(t, x, \lambda) = f(x, \lambda)$*, then the augmented equation will be autonomous as well.*

Example

Consider the phase-locked-loop model

$$\begin{aligned}\dot{x}_1 &= x_2 && = f_1(x_1, x_2),\\ \dot{x}_2 &= -c\sin x_1 - (a + b\cos x_1)x_2 && = f_2(x_1, x_2).\end{aligned}$$

and suppose the parameters a, b, and c have the nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$. The nominal system is given by

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\sin x_1 - x_2.\end{aligned}$$

The Jacobian matrices are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -c\cos x_1 + bx_2 \sin x_1 & -(a + b\cos x_1) \end{bmatrix},$$

$$\frac{\partial f}{\partial \lambda} = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} & \frac{\partial f}{\partial c} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2\cos x_1 & -\sin x_1 \end{bmatrix}.$$

Evaluate the Jacobian matrices at nominal parameters:

$$\left.\frac{\partial f}{\partial x}\right|_{a=1,b=0,c=1} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix},$$

$$\left.\frac{\partial f}{\partial \lambda}\right|_{a=1,b=0,c=1} = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2\cos x_1 & -\sin x_1 \end{bmatrix}.$$

Let

$$S = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} \end{bmatrix}_{a=1,b=0,c=1}$$

Then,

$$\begin{aligned}
\dot{x}_1 &= x_2, & x_1(0) &= x_{10}, \\
\dot{x}_2 &= -\sin x_1 - x_2, & x_2(0) &= x_{20}, \\
\dot{x}_3 &= x_4, & x_3(0) &= 0, \\
\dot{x}_4 &= -x_3 \cos x_1 - x_4 - x_2, & x_4(0) &= 0, \\
\dot{x}_5 &= x_6, & x_5(0) &= 0, \\
\dot{x}_6 &= -x_5 \cos x_1 - x_6 - x_2 \cos x_1, & x_6(0) &= 0, \\
\dot{x}_7 &= x_8, & x_7(0) &= 0, \\
\dot{x}_8 &= -x_7 \cos x_1 - x_8 - \sin x_1, & x_8(0) &= 0.
\end{aligned}$$

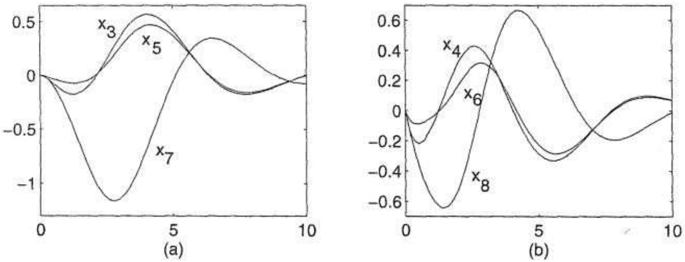

Figure: Sensitivity function with initial conditions $x_{10} = x_{20} = 1$.

The solution is more sensitive to variations in the parameter c than to variations in the parameters a and b.

Comparison Principle

Definition

If the derivative of a scalar differentiable function $v(t)$ satisfies inequality of the form $\dot{v}(t) \leq f(t, v(t))$ for all t in a certain time interval, then the inequality is called a *differential inequality*, and the function $v(t)$ satisfying the inequality is called a solution of the differential inequality.

- Suppose that $v(t)$ is not necessarily differentiable, but has an upper right-hand derivative $D^+v(t)$.
- If $v(t)$ is differentiable at t, then $D^+v(t) = \dot{v}(t)$.
- If $\frac{1}{h}|v(t+h) - v(t)| \leq g(t,h), \ \forall h \in (0, b]$ and $\lim_{h\to 0^+} g(t,h) = g_0(t)$, then $D^+v(t) \leq g_0(t)$.

Lemma (3.4 Comparison Lemma)

Consider the scalar differential equation

$$\dot{u} = f(t, u), \quad u(t_0) = u_0,$$

where $f(t, u)$ *is continuous in* t *and locally Lipschitz in* u *for all* $t \geq 0$ *and all* $u \in J \subset R$*. Let* $[t_0, T)$ *(*T *could be infinity) be the maximal interval of existence of the solution* $u(t)$*, and suppose* $u(t) \in J$ *for all* $t \in [t_0, T)$*. Let* $v(t)$ *be a continuous function whose upper right-hand derivative* $D^+v(t)$ *satisfies the differential inequality*

$$D^+v(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in J$ *for all* $t \in [t_0, T)$*. Then,* $v(t) \leq u(t)$ *for all* $t \in [t_0, T)$*.*

Example

The scalar differential equation

$$\dot{x} = f(x) = -(1 + x^2)x, \quad x(0) = a$$

has a unique solution on $[0, t_1)$ for some $t_1 > 0$, because $f(x)$ is locally Lipschitz.

Let $v(t) = x^2(t)$. The function $v(t)$ is differentiable and its derivative is given by

$$\dot{v}(t) = 2x\dot{x} = -2x^2 - 2x^4 \leq -2x^2.$$

Hence, $v(t)$ satisfies the differential inequality

$$\dot{v}(t) \leq -2v(t), \;\; v(0) = a^2$$

Let $u(t)$ be the solution of the differential equation

$$\dot{u} = -2u, \;\; u(0) = a^2 \quad \Rightarrow \quad u(t) = a^2 e^{-2t},$$

Then, $x(t)$ is defined for all $t \geq 0$ and satisfies $|x(t)| = \sqrt{v(t)} \leq e^{-t}|a|, \; \forall t \geq 0.$