Nonlinear Control Theory

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Advanced Stability Analysis

1. **The Center Manifold Theorem**
2. Region of Attraction
3. Invariance-like Theorem

The Center Manifold Theorem

Consider the autonomous system:

$$\dot{x} = f(x) \tag{1}$$

where $f : D \to R^n$ is continuously differentiable, and $D \subset R^n$ is a domain that contains the origin.

Suppose that the origin is an equilibrium point. Theorem 4.7 indicates that:

- the origin is **locally asymptotically stable**, if all eigenvalues of its Jacobian matrix have negative real parts;
- the origin is **unstable**, if at least one eigenvalue of its Jacobian matrix has positive real part.

What if some eigenvalues of the Jacobian matrix have **zero real parts**, and others have negative real parts??

Definition

A **k-dimensional manifold** is the solution of the equation

$$\eta(x) = 0, \tag{2}$$

where $\eta : R^n \to R^{n-k}$ is sufficiently smooth.

Examples:

- The unit circle $\{x \in R^2 | x_1^2 + x_2^2 = 1\}$ is a **one-dimensional manifold** in R^2;
- The unit sphere $\{x \in R^n | \sum_{i=1}^n x_i^2 = 1\}$ is an $(n-1)$**-dimensional manifold** in R^n.

Definition

A manifold $\{\eta(x) = 0\}$ is said to be an **invariant manifold** if

$$\eta(x(0)) = 0 \quad \Rightarrow \quad \eta(x(t)) \equiv 0, \ \ \forall t \in [0, t_1) \subset R, \tag{3}$$

where $[0, t_1)$ is any time interval over which the solution $x(t)$ is defined.

Suppose that the Jacobian matrix

$$A = \frac{\partial f}{\partial x}(0), \tag{4}$$

has k eigenvalues with zero real parts and $m = n - k$ eigenvalues with negative real parts.

There always exists a similarity transformation T, such that

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \tag{5}$$

where all eigenvalues of A_1 have zero real parts, and all eigenvalues of A_2 have negative real parts.

The original system:

$$\dot{x} = f(x) = Ax + \tilde{f}(x), \quad \tilde{f}(x) = f(x) - \frac{\partial f}{\partial x}(0)x. \tag{6}$$

where $\tilde{f}(x)$ is at least twice differentiable.

The transformation

$$\begin{bmatrix} y \\ z \end{bmatrix} = Tx; \quad y \in R^k, \; z \in R^m, \tag{7}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\dot{y} = A_1 y + g_1(y, z), \tag{8}$$
$$\dot{z} = A_2 z + g_2(y, z), \tag{9}$$

where

$$g_i(0,0) = 0, \quad \frac{\partial g_i}{\partial y}(0,0) = 0, \quad \frac{\partial g_i}{\partial z}(0,0) = 0. \tag{10}$$

Definition

If $z = h(y)$ is an invariant manifold for (8)–(9) and h is smooth, then it is called a **center manifold** if

$$h(0) = 0, \quad \frac{\partial h}{\partial y}(0) = 0. \tag{11}$$

Theorem (8.1)

If

- *g_1 and g_2 are twice continuously differentiable and satisfy (10), and*
- *all eigenvalues of A_1 have zero real parts, and all eigenvalues of A_2 have negative real parts,*

then there exist a constant $\delta > 0$ and a continuously differentiable function $h(y)$, defined for all $\|y\| < \delta$, such that $z = h(y)$ is a center manifold for (8)–(9).

If the initial state lies in the center manifold ($z(0) = h(y(0))$), then $z(t) \equiv h(y(t))$.

In this case, the motion in the manifold can be described by

$$\dot{y} = A_1 y + g_1(y, h(y)), \tag{12}$$

which we refer to as the *reduced system*.

If $z(0) \neq h(y(0))$, then the deviation from the center manifold can be described by

$$w = z - h(y), \tag{13}$$

and

$$\begin{aligned} \dot{y} &= A_1 y + g_1(y, w + h(y)), && (14) \\ \dot{w} &= A_2 (w + h(y)) + g_2 (y, w + h(y)) - \frac{\partial h}{\partial y}(y) \left(A_1 y + g_1(y, w + h(y))\right). && (15) \end{aligned}$$

The motion in the manifold: $w(t) \equiv 0 \quad \Rightarrow \quad \dot{w} \equiv 0.$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\boxed{0 = A_2 h(y) + g_2\left(y, h(y)\right) - \frac{\partial h}{\partial y}(y)\left(A_1 y + g_1(y, h(y))\right).} \tag{16}$$

The function $h(y)$ must satisfy the partial differential equation (16).

Subtracting (16) from (15) yields

$$\dot{y} = A_1 y + g_1(y, h(y)) + N_1(y, w), \tag{17}$$

$$\dot{w} = A_2 w + N_2(y, w), \tag{18}$$

where

$$N_1(y, w) = g_1(y, w + h(y)) - g_1(y, h(y)), \tag{19}$$

$$N_2(y, w) = g_2(y, w + h(y)) - g_2(y, h(y)) - \frac{\partial h}{\partial y}(y) N_1(y, w), \tag{20}$$

and

$$N_i(y, 0) = 0, \quad \frac{\partial N_i}{\partial w}(0, 0) = 0, \quad i = 1, 2. \tag{21}$$

Consequently, in the domain $\left\| \begin{matrix} y \\ z \end{matrix} \right\|_2 < \rho$, N_1 and N_2 satisfy

$$\|N_i(y, w)\|_2 \leq k_i \|w\|, \quad i = 1, 2, \tag{22}$$

where the positive constants k_1 and k_2 can be made arbitrarily small by choosing ρ small enough, such that (18) is asymptotically stable.

Theorem (8.2)

Under the assumptions of Theorem 8.1, if the origin $y = 0$ of the reduced system (12) *is asymptotically stable (respectively, unstable), then the origin of the full system* (8) *and* (9) *is also asymptotically stable (respectively, unstable).*

Corollary (8.1)

Under the assumptions of Theorem 8.1, if the origin $y = 0$ of the reduced system (12) is stable, and there is a continuously differentiable Lyapunov function $V(y)$ such that

$$\frac{\partial V}{\partial y}(A_1 y + g_1(y, h(y))) \leq 0, \tag{23}$$

in some neighborhood of $y = 0$, then the origin of the full system (8)–(9) is stable.

Corollary (8.2)

Under the assumptions of Theorem 8.1, the origin $y = 0$ of the reduced system (12) is asymptotically stable if and only if the origin of the full system (8)–(9) is asymptotically stable.

As aforementioned, the center manifold can be obtained by solving the partial differential equation

$$\mathcal{N}(h(y)) \triangleq \frac{\partial h}{\partial y}(y)\left(A_1 y + g_1(y, h(y))\right) - A_2 h(y) - g_2\left(y, h(y)\right) = 0 \tag{24}$$

with boundary conditions

$$h(0) = 0, \;\; \frac{\partial h}{\partial y}(0) = 0. \tag{25}$$

Unfortunately, we do not know how to solve PARTIAL differential equations rigorously!

An efficient way is to approximate its solution via Taylor Series.

Theorem (8.3)

If a continuously differentiable function $\phi(y)$ *with* $\phi(0) = 0$ *and* $\frac{\partial \phi}{\partial y}(0) = 0$ *can be found such that* $\mathcal{N}(\phi(y)) = \mathcal{O}\left(\|y\|^p\right)$ *for some* $p > 1$*, then for sufficiently small* $\|y\|$*,*

$$h(y) - \phi(y) = \mathcal{O}\left(\|y\|^p\right), \tag{26}$$

and the reduced system can be represented as

$$\dot{y} = A_1 y + g_1(y, \phi(y)) + \mathcal{O}\left(\|y\|^{p+1}\right), \tag{27}$$

where $\mathcal{O}(\cdot)$ *denotes the order of magnitude.*

Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2, & (28) \\ \dot{x}_2 &= -x_2 + a x_1^2 + b x_1 x_2, & (29) \end{aligned}$$

where $a \neq 0$. Its linearization around the origin gives

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \tag{30}$$

which has eigenvalues at 0 and -1.

Let $y = x_1 + x_2$ and $z = -x_2$. The system can be transformed to

$$\dot{y} = a(y+z)^2 - b(yz + z^2), \tag{31}$$
$$\dot{z} = -z - a(y+z)^2 + b(yz + z^2). \tag{32}$$

Its center manifold satisfies

$$\begin{aligned}\mathcal{N}(h(y)) = & h'(y)\left[a(y+h(y))^2 - b(yh(y) + h^2(y))\right] + h(y) \\ & + a(y+h(y))^2 - b(yh(y) + h^2(y)) = 0, \qquad h(0) = h'(0) = 0.\end{aligned}$$

Set $h(y) = h_2 y^2 + h_3 y^3 + \cdots$. Start with the simplest one $h(y) = 0 + \mathcal{O}(|y|^2)$, and the reduced system can be calculated by

$$\dot{y} = ay^2 + \mathcal{O}(|y|^3) \tag{33}$$

where ay^2 is the dominant term, and the reduced system is unstable. Consequently, by Theorem 8.2, the origin of the full system is unstable.

Example

Consider the system

$$\dot{y} = yz, \tag{34}$$

$$\dot{z} = -z + ay^2. \tag{35}$$

which is already in (y, z) representation.

Its center manifold equation with the boundary condition is

$$h'(y)[yh(y)] + h(y) - ay^2 = 0, \quad h(0) = h'(0) = 0. \tag{36}$$

Start by trying $\phi(y) = 0$. The reduced system is

$$\dot{y} = \mathcal{O}(|y|^3), \tag{37}$$

and we cannot reach any conclusion on whether it is stable or not.

Then, we try $h(y) = h_2 y^2 + \mathcal{O}(|y|^3)$.
Substitute it into the center manifold equation yields $h_2 = a$, and the reduced system

$$\dot{y} = ay^3 + \mathcal{O}(|y|^4). \tag{38}$$

is asymptotically stable if $a < 0$, and unstable if $a > 0$.

- Consequently, by Theorem 8.2, it can be claimed that the full system is asymptotically stable if $a < 0$, and unstable if $a > 0$.
- If $a = 0$, the center manifold equation with boundary condition reduces to

  $$h'(y)[yh(y)] + h(y) = 0, \quad h(0) = h'(0) = 0, \tag{39}$$

  which has the exact solution $h(y) = 0$. The reduced system $\dot{y} = 0$ is stable with $V(y) = y^2$ as a Lyapunov function. Consequently, by Corollary 8.1, we conclude that the full system is stable if $a = 0$.

1. The Center Manifold Theorem
2. **Region of Attraction**
3. Invariance-like Theorem

Region of Attraction

- Let $x = 0$ be an asymptotically stable equilibrium point for

$$\dot{x} = f(x) \tag{40}$$

  where $f : D \to R^n$ is locally Lipschitz, and $D \subset R^n$ contains the origin.
- Let $\phi(t; x)$ be its solution that starts at initial state x at time $t = 0$.

Definition

The **region of attraction** of the origin, denoted by R_A, is defined by

$$R_A = \{x \in D \mid \phi(t; x) \text{ is defined } \forall t > 0 \text{ and } \phi(t; x) \to 0 \text{ as } t \to \infty\}. \tag{41}$$

Lemma

If $x = 0$ is an asymptotically stable equilibrium point for (40), then its region of attraction R_A is an open, connected, invariant set. Moreover, the boundary of R_A is formed by trajectories.

Example

Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \frac{1}{3}x_1^3 - x_2, \end{aligned}$$

which has three isolated equilibrium points at $(0, 0)$, $(\sqrt{3}, 0)$, and $(-\sqrt{3}, 0)$.

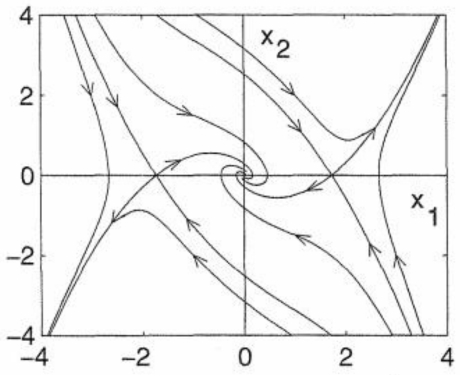

Figure: Phase portrait

- The origin is a stable focus, and the other two equilibrium points are saddle points.
- The origin is locally asymptotically stable, and the other two equilibrium points are unstable.
- The stable trajectories of the saddle points form the two boundaries of the region of attraction.

Can we estimate the region of attraction via Lyapunov's method?

If D is a domain that contains the origin, and if

- there exists a Lyapunov function $V(x)$ that is positive definite in D,
- $\dot{V}(x)$ is negative definite in D, or negative semi-definite, but no solution can stay identically in $\{\dot{V} = 0\}$ except for $x = 0$,

then its solution is locally asymptotically stable.

Is D an estimation of the region of attraction?

Example

Consider (again) the second-order system

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \frac{1}{3}x_1^3 - x_2,\end{aligned}$$

which has three isolated equilibrium points at $(0,0)$, $(\sqrt{3},0)$, and $(-\sqrt{3},0)$.

A Lyapunov function can be given by

$$V(x) = \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2 > 0, \tag{42}$$

and

$$\dot{V}(x) = -\frac{1}{2}x_1^2\left(1 - \frac{1}{3}x_1^2\right) - \frac{1}{2}x_2^2. \tag{43}$$

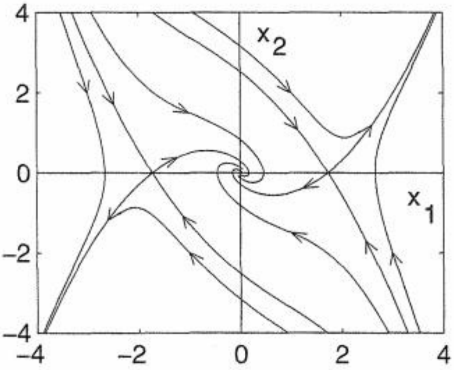

Figure: Phase portrait

- Define the domain

$$D = \left\{ x \in R^2 \mid -\sqrt{3} < x_1 < \sqrt{3} \right\}.$$

It can be seen that $V(x) > 0$ and $\dot{V} < 0$ in $D - \{0\}$.

- However, D is not a subset of R_A.

The set D is not an invariant set!

A simplest estimation is the set

$$\Omega_c = \{x \in R^n \mid V(x) < c\}, \tag{44}$$

when Ω_c is bounded and contained in D.

- For $V(x) = x^T P x$ and $D = \{\|x\|_2 < r\}$, the invariance of $\Omega_c \subset D$ can be ensured by

$$c < \min_{\|x\|_2 = r} x^T P x = \lambda_{min}(P) r^2. \tag{45}$$

- For $V(x) = x^T P x$ and $D = \{|b_i^T x| < r_i,\ i = 1, \cdots, p\}$ where $b_i \in R^n$, the invariance of $\Omega_c \subset D$ can be ensured by

$$c < \min_{1 \le i \le p} \frac{r_i^2}{b_i^T P^{-1} b_i}. \tag{46}$$

Example

Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -4(x_1+x_2)-h(x_1+x_2),\end{aligned}$$

where $h: \ R \to \ R$ is a locally Lipschitz function satisfying

$$h(0)=0, uh(u)\geq 0, \ \forall \ |u|\leq 1.$$

Select the quadratic function

$$V(x) = 2x_1^2 + 2x_1x_2 + x_2^2 > 0. \tag{47}$$

Its derivative is given by

$$\dot{V} \leq -2x_1^2 - 6(x_1+x_2)^2 < 0, \quad \forall \, |x_1+x_2| \leq 1. \tag{48}$$

- Therefore, $\dot{V}(x)$ is negative semi-definite in the set

$$D = \{x \in R^2 \mid |x_1 + x_2| = |b^T x| \leq 1\},$$

where $b = [1, 1]^T$, and the origin is locally asymptotically stable.

- The region of attraction can be estimated in the form $\Omega_c = \{V(x) \leq c\}$, where the largest c can be given by

$$c = \min_{|x_1+x_2|=1} x^T P x = \frac{1}{b^T P^{-1} b} = 1. \tag{49}$$

- The set Ω_c with $c = 1$ is an estimate of the region of attraction.

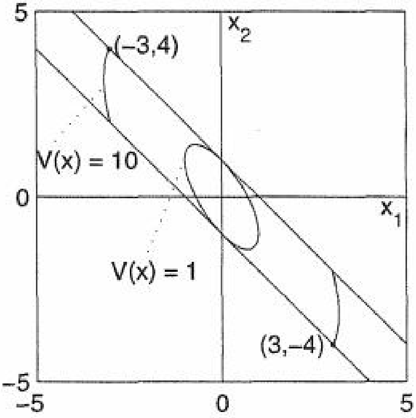

Figure: Phase portrait

Is there any better estimate of R_A?

- Examining the vector field at the boundary $|\sigma| = |x_1 + x_2| = 1$:

$$
\begin{aligned}
\frac{d}{dt}\sigma^2 =& 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \\
=& 2|x_2| - 8 \leq 0, \quad \forall\ |x_2| \leq 4.
\end{aligned}
$$

- It indicates that the trajectory cannot leave the set D through the segment of the boundary $|\sigma| = 1$ with $|x_2| \leq 4$.

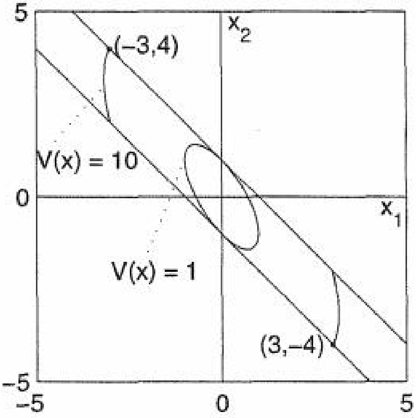

Figure: Phase portrait

- Let $V(x) = c_1$ and $V(x) = c_2$ be the Lyapunov surfaces intersecting the boundary $|x_1 + x_2| = 1$ at $x_2 = 4$ and $x_2 = -4$, respectively.

$$\begin{aligned} c_1 &= V(x)|_{x_1=-3, x_2=4} = 10, \\ c_2 &= V(x)|_{x_1=3, x_2=-4} = 10. \end{aligned}$$

- The region of attraction can be estimated by

$$\Omega = \{x \in R^2 \mid V(x) \leq 10, \ |x_1 + x_2| \leq 1\}.$$

1. The Center Manifold Theorem
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3. **Invariance-like Theorem**

Invariance-like Theorems

Lemma (8.2, Barbalat Lemma)

Let $\phi: R \to R$ be a uniformly continuous function on $[0, \infty)$, Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)d\tau$ exists and is finite, then $\phi(t) \to 0$ as $t \to \infty$.

Proof (by contradiction)

If it is not true, then there is a positive constant k_1 such that for every $T > 0$, we can find $T_1 > T$ with $|\phi(T_1)| \geq k_1$. Since $\phi(t)$ is uniformly continuous, there is a positive constant k_2 such that $|\phi(t+\tau) - \phi(t)| < \frac{k_1}{2}$ for all $t \geq 0$ and all $0 \leq \tau \leq k_2$. Hence,

$$\begin{aligned} |\phi(t)| &= |\phi(t) - \phi(T_1) + \phi(T_1)| \geq |\phi(T_1)| - |\phi(t) - \phi(T_1)| \\ &\geq k_1 - \frac{1}{2}k_1 = \frac{1}{2}k_1, \qquad \forall\, t \in [T_1, T_1 + k_2] \end{aligned}$$

Proof (Continued)

$$\left|\int_{T_1}^{T_1+k_2} \phi(t)dt\right| = \int_{T_1}^{T_1+k_2} |\phi(t)|\, dt \geq \frac{k_2 k_1}{2},$$

where the equality holds, since $\phi(t)$ retains the same sign for $T_1 \leq t \leq T_1 + k_2$. Thus, $\int_0^t \phi(\tau)d\tau$ cannot converge to a finite limit as $t \to \infty$, a contradiction.

Theorem (8.4)

Let $D \subset R^n$ be the domain that contains $x = 0$, and suppose that $f(t,x)$ is piecewise continuous in t and locally Lipschitz in x, uniformly in t, on $[0,\infty) \times D$. Furthermore, suppose $f(t,0)$ is uniformly bounded for all $t \geq 0$. Let $V : [0,\infty) \times D \to R$ be a continuously differentiable function such that

$$W_1(x) \leq V(t,x) \leq W_2(x), \quad \dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W(x), \ \forall t \in [0,\infty), \ x \in D,$$

where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions, and $W(x)$ is a continuous positive semi-definite function on D. Choose $r > 0$ such that $B_r \subset D$, and let $\rho < \min_{|x|=r} W_1(x)$. Then, all solutions of $\dot{x} = f(t,x)$ with $x(t_0) \in \{x \in B_r | \ W_2(x) \leq \rho\}$ are bounded and satisfy

$$W(x(t)) \to 0, \ \text{as } t \to \infty. \tag{50}$$

Moreover, if all the assumptions hold globally and $W_1(x)$ is radially unbounded, the statement is true for all $x(t_0) \in R^n$.

More discussions about Theorem 8.4:

- The limit $W(x(t)) \to 0$ implies that $x(t)$ approaches E as $t \to \infty$, where

$$E = \{x \in D|\ W(x) = 0\}. \tag{51}$$

  Therefore, the positive limit set of $x(t)$ is a subset of E.
- $x(t) \to E$ is much weaker than the invariance principle for autonomous systems, which states that $x(t)$ approaches the largest invariant set in E.
- For autonomous systems, the positive limit set is an invariant set (Lemma 4.1). This is also true for periodic non-autonomous systems.
- For general non-autonomous systems, Lemma 4.1 is not applicable.

Example (Model reference adaptive control, MRAC)

A model reference adaptive control system, with plant $\dot{y}_p = a_p y_p + k_p u$, and reference model $\dot{y}_m = a_m y_m + k_m r$, is given by

$$\begin{aligned}\dot{e}_o =& a_m e_o + k_p \phi_1 r(t) + k_p \phi_2 [e_o + y_m(t)],\\ \dot{\phi}_1 =& -\gamma e_o r(t),\\ \dot{\phi}_2 =& -\gamma e_o [e_o + y_m(t)],\end{aligned}$$

where $\gamma > 0$ is the adaptation gain, $e_o = y_p - y_m$ is the output error, and ϕ_1 and ϕ_2 are the parameter errors. It is assumed that

- $k_p > 0$, and the reference model must have $a_m < 0$.
- The input of the reference model $r(t)$ is piecewise continuous and bounded.

- Using $V = \frac{1}{2}\left[\frac{1}{k_p}e_o^2 + \frac{1}{\gamma}(\phi_1^2 + \phi_2^2)\right]$ as a Lyapunov candidate, we obtain

$$\dot{V} = \frac{a_m}{k_p}e_o^2 + e_o(\phi_1 r - \phi_2 e_o + \phi_2 y_m) - \phi_1 e_o r - \phi_2 e_o(e_o + y_m) = \frac{a_m}{k_p}e_o^2 \leq 0. \quad (52)$$

  By applying Theorem 8.4, we conclude that, for any $c > 0$ and for all initial states in the set $\{V \leq c\}$, all states are bounded for all $t > t_0$ and $\lim_{t\to\infty} e_o(t) = 0$.

- How about the behaviors of ϕ_1 and ϕ_2? Are they also converging?

- For example, if r and y_m are nonzero constant signals, the closed-loop system will have an equilibrium subspace $\{e_o = 0,\ \phi_2 = \frac{a_m}{k_m}\phi_1\}$, which shows that, in general, ϕ_1 and ϕ_2 do not converge to zero.