# Advanced Topics in Control: Distributed Systems and Control

## Lecture 4: Discrete-Time Averaging

Dr. Mohammad H. (Vahid) Mamduhi, March 14th, 2022

ETH zürich
AUTOMATIC CONTROL LABORATORY

## Brief review

1. elements of graphs, types of graphs
2. neighbours, subgraphs, paths, cycles
3. globally reachable node, directed spanning tree
4. strongly connected, aperiodic, condensation
5. adjacency matrix (degree, path, connectivity)
6. equivalent definition for irreducibility
7. equivalent definition for aperiodicity

## Clarification

1. "If ∃ directed path $i \to j$, then ∃ simple directed path $i \to j$ with length $\leq n-1$." only holds for the case when $i \neq j$.
2. "$(A^k)_{ij}$>0 ⇒ ∃ directed path $i \to j$ with length $k$".

   Here the path is not necessarily simple and can pass through $j$ for multiple times.

**Equivalent definition for aperiodicity**

Consider a directed graph $G$.

**Definition**: $\exists$ directed path $i \to i$ with length $k \Rightarrow k$ is *a recurrence time* of node $i$.

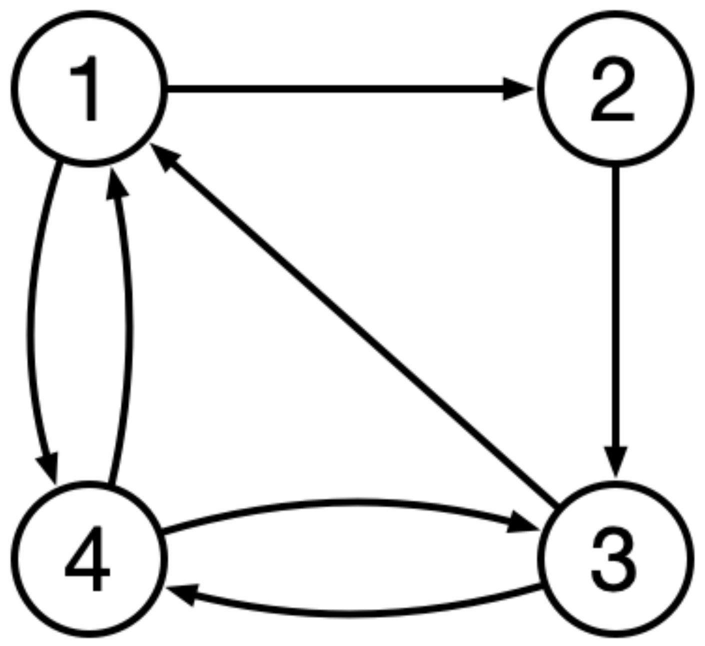
*period* of node $i$: the greatest common divisor of all its recurrence times.

**Theorem 1**: (1) All nodes belonging to the same strongly-connected component of $G$ have the same period;

(2) If $G$ is strongly connected, then

$G$ is aperiodic $\Leftrightarrow$ The period of each node is 1.

E.g.




Node 1: recurrence times 2 (1-4-1), 3 (1-2-3-1), 4 (1-2-3-4-1), 5 (1-2-3-1-4-1), … $\Rightarrow$ period = 1

Node 2: recurrence times 3 (2-3-1-2), 4 (2-3-4-1-2), 5 (2-3-1-4-1-2), 6 (2-3-1-2-3-1-2) … $\Rightarrow$ period = 1

Node 3: recurrence times 2 (3-4-3), 3 (3-1-2-3), 4 (3-4-1-4-3), 5 (3-1-2-3-4-3), … $\Rightarrow$ period = 1

Node 4: recurrence times 2 (4-1-4), 3 (4-3-1-4), 5 (4-1-4-3-4), 5 (4-1-2-3-1-4), … $\Rightarrow$ period = 1

## Equivalent definition for aperiodicity

Consider a directed graph $G$.

**Definition**: $\exists$ directed path $i \to i$ with length $k \Rightarrow k$ is *a recurrence time* of node $i$.

*period* of node $i$: the greatest common divisor of all its recurrence times.

**Theorem 1**: (1) All nodes belonging to the same strongly-connected component of $G$ have the same period;

(2) If $G$ is strongly connected, then

$G$ is aperiodic $\Leftrightarrow$ The period of each node is 1.

Application:

**Corollary 2:** Consider a strongly connected digraph $G$ with non-negative link weights. If $\exists k \in \mathbb{N}$ s.t. the $i$-th column of $A^k$ is entry-wise positive. then $A$ is primitive.

**Corollary 2:** Consider a strongly connected digraph $G$ with non-negative link weights. If $\exists k \in \mathbb{N}$ s.t. the $i$-th column of $A^k$ is entry-wise positive. then $A$ is aperiodic.

## Algebraic graph theory II: network structure → behavior of discrete-time averaging

"Behavior": 1) converge? 2) consensus? 3) consensus value

**Discrete-time averaging algorithm**

$$x(t+1) = Ax(t), \text{ where } A \text{ is row-stochastic.}$$

**In this lecture, we always assume $A$ to be row-stochastic.**

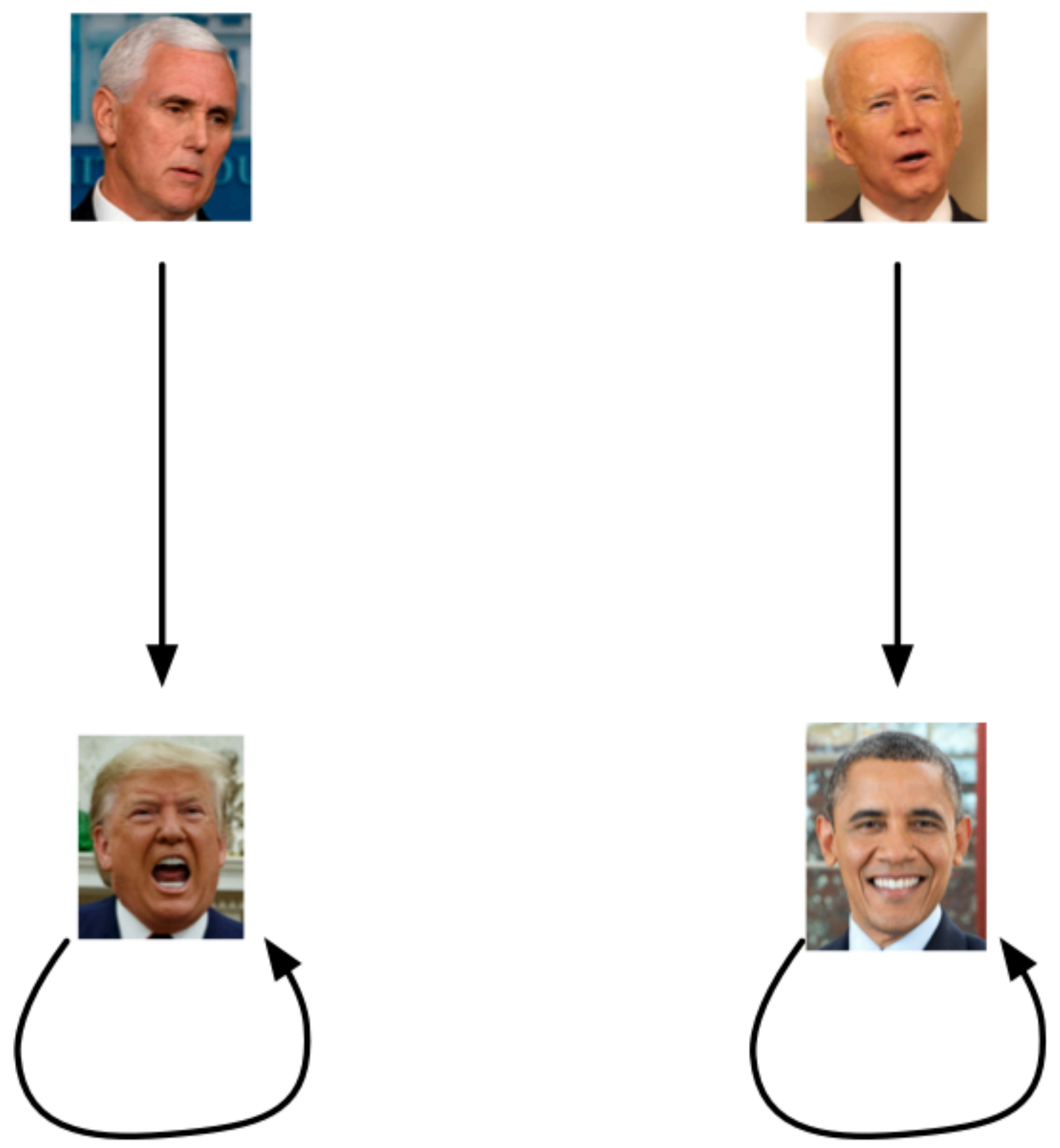
**Example 1: converge? consensus?**

Pence's opinion (1) = Trump's opinion (0)

Trump's opinion (1) = Trump's opinion (0)

Biden's opinion (1) = Obama's opinion (0)

Obama's opinion (1) = Obama's opinion (0)



It converges. Consensus ⇔ Obama's opinion (0) = Trump's opinion (0).

# Algebraic graph theory II: network structure → behavior of discrete-time averaging

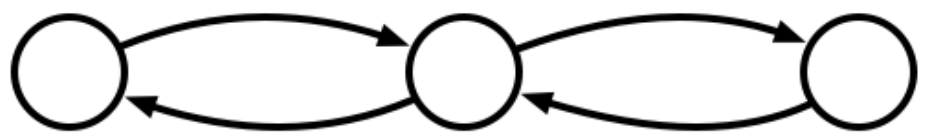
"Behaviour": 1) converge? 2) consensus? 3) consensus value

**Discrete-time averaging algorithm**

$$x(t+1) = Ax(t), \text{ where } A \text{ is row-stochastic.}$$

**In this lecture, we always assume $A$ to be row-stochastic.**

**Example 2: converge? consensus?**



$$\Rightarrow [A] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ e.g. } A = \begin{bmatrix} 0 & 1 & 0 \\ 0.3 & 0 & 0.7 \\ 0 & 1 & 0 \end{bmatrix}$$

## Algebraic graph theory II: network structure → behavior of discrete-time averaging

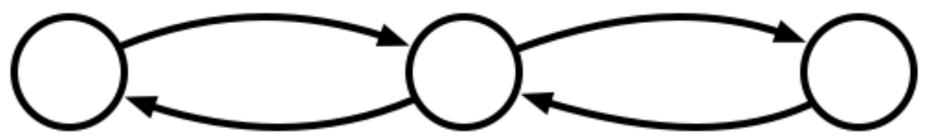
"Behavior": 1) converge? 2) consensus? 3) consensus value

### Discrete-time averaging algorithm

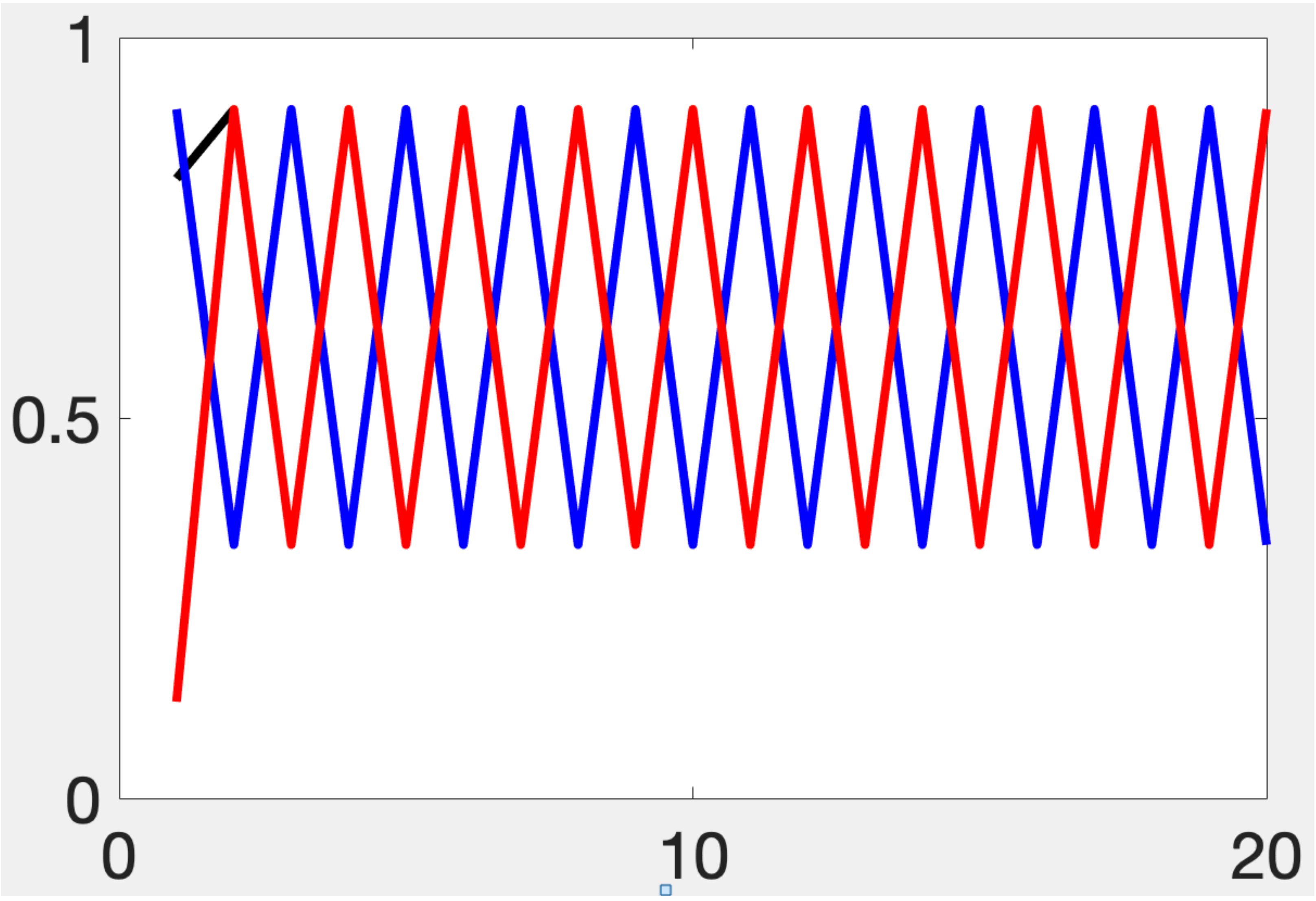
$$x(t+1) = Ax(t), \text{ where } A \text{ is row-stochastic.}$$

**In this lecture, we always assume $A$ to be row-stochastic.**

### Example 2: converge? consensus?



$$\Rightarrow [A] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ e.g. } A = \begin{bmatrix} 0 & 1 & 0 \\ 0.3 & 0 & 0.7 \\ 0 & 1 & 0 \end{bmatrix}$$



# Algebraic graph theory II: network structure → behavior of discrete-time averaging

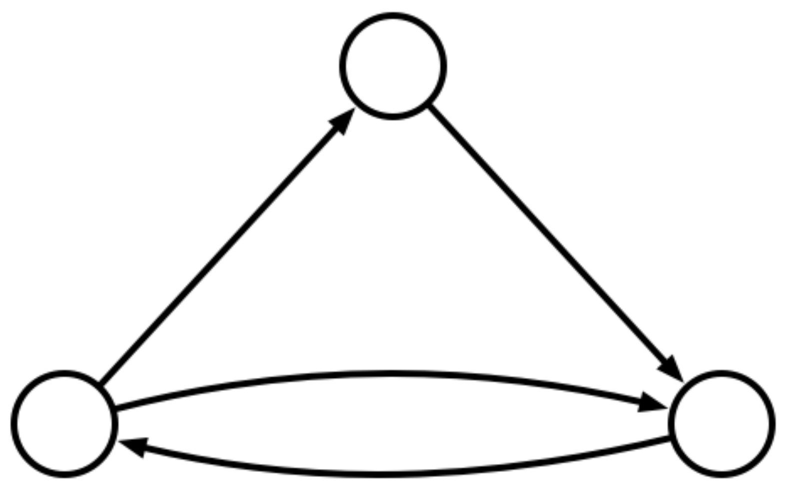
"Behavior": 1) converge? 2) consensus? 3) consensus value

## Discrete-time averaging algorithm

$$x(t+1) = Ax(t), \text{ where } A \text{ is row-stochastic.}$$

**In this lecture, we always assume $A$ to be row-stochastic.**

## Example 3: converge? consensus?



$$\Rightarrow [A] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{e.g. } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.3 & 0.7 & 0 \end{bmatrix}$$

## Algebraic graph theory II: network structure → behavior of discrete-time averaging

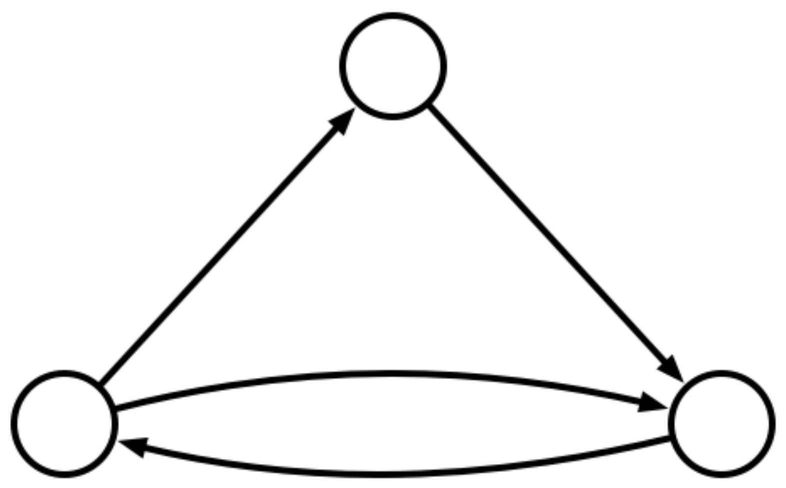
"Behavior": 1) converge? 2) consensus? 3) consensus value

### Discrete-time averaging algorithm

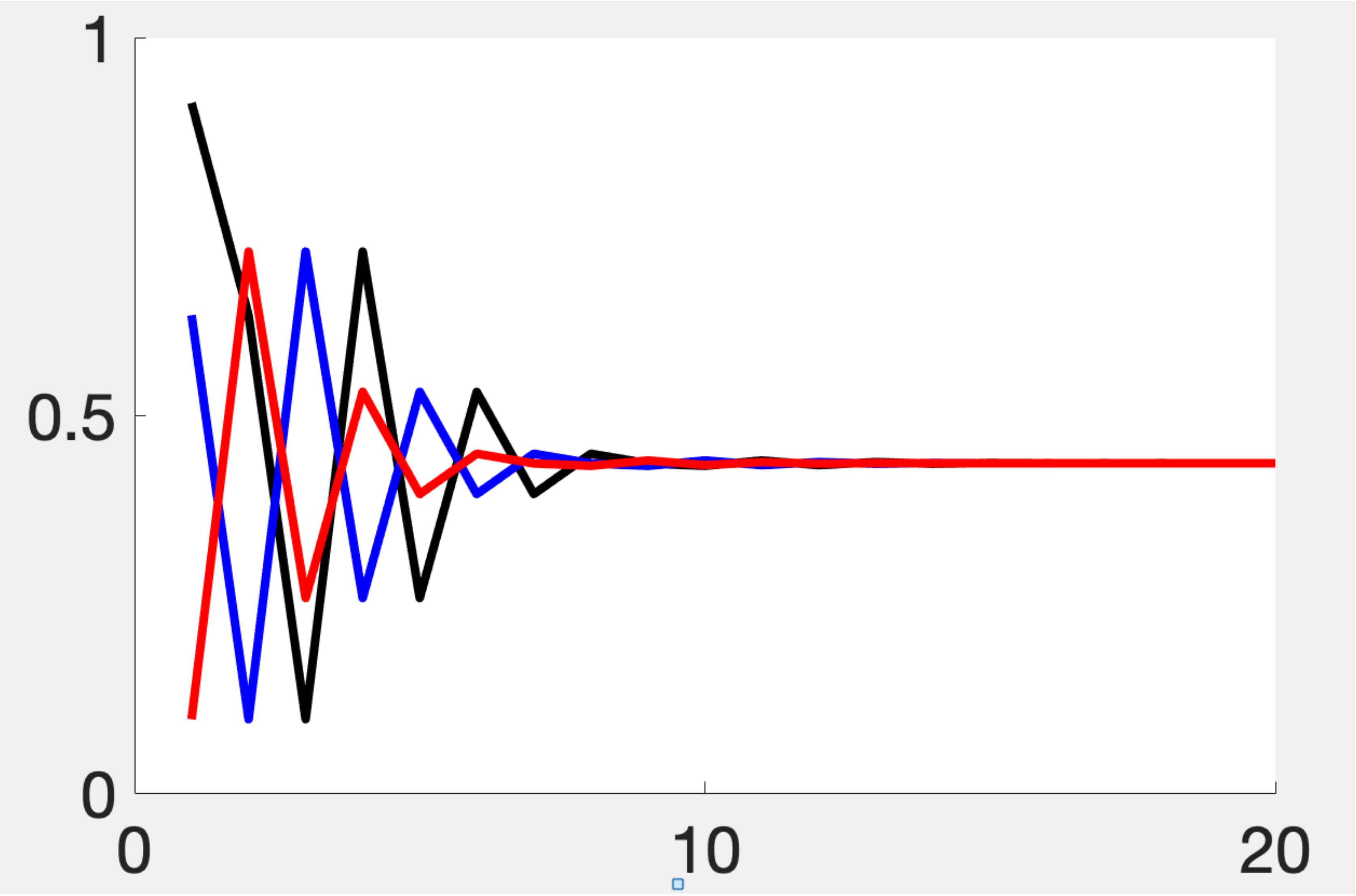
$$x(t+1) = Ax(t), \text{ where } A \text{ is row-stochastic.}$$

In this lecture, we always assume $A$ to be row-stochastic.

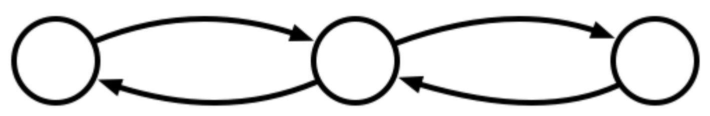
### Example 3: converge? consensus?



$$\Rightarrow [A] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{e.g. } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.3 & 0.7 & 0 \end{bmatrix}$$



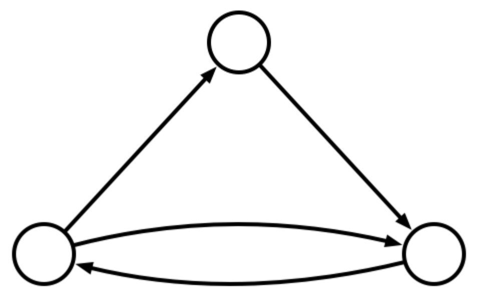
Example 2



not converge

strongly connected, periodic

Example 3



converge to consensus

strongly connected, aperiodic

**Theorem 2.13 in the textbook**

$A$ is non-negative

$\rho(A)$ is a simple eigenvalue

$|\mu| < \rho(A)$, $\forall$ other $\mu \in$ spec($A$)

$\Rightarrow$

$A/\rho(A)$ is semi-convergent &

$\lim_{t\to\infty} A^t/\rho(A)^t = vw^\top$

where

$Av = \rho(A)v$

$w^\top A = \rho(A)w^\top$

$v \geq 0,\ w \geq 0,\ w^\top v = 1$

$\Uparrow$

$\rho(A)$ is a simple eigenvalue

$|\mu| < \rho(A)$, $\forall$ other $\mu \in$ spec($A$)

$v = 1_n,\ w > 0$ (unique up to rescaling)

$\Leftarrow$

**Perron-Frobenius theorem**

$A$ is primitive

$A$ is row-stochastic

$\Leftarrow$

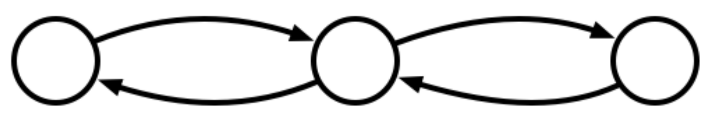
**Algebraic graph theory**

$G$ is strongly-connected & aperiodic

Suppose $G$ is strongly connected.

The discrete-time averaging algorithm converges to consensus. $\Leftrightarrow$ The digraph $G$ is aperiodic.

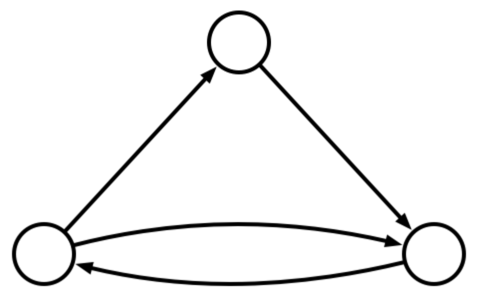
Example 2



not converge

strongly connected, periodic

Example 3



converge to consensus

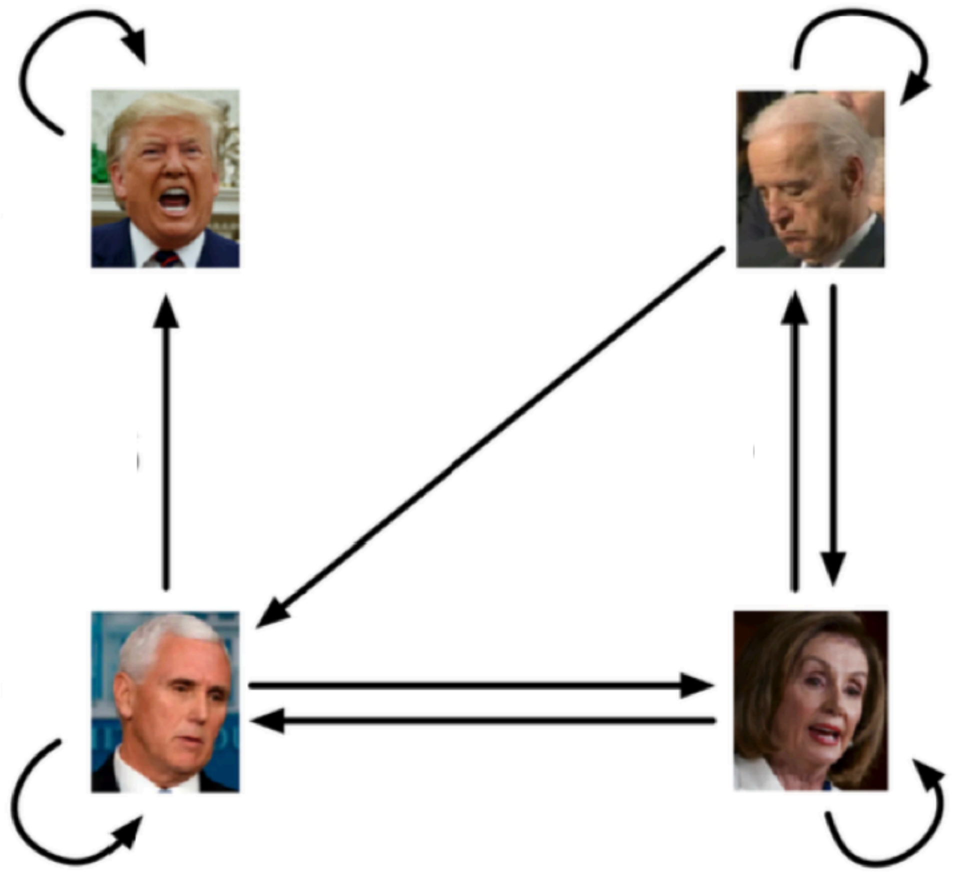
strongly connected, aperiodic

$$A \text{ is primitive } \Rightarrow x(\infty) = (w^\top x(0))1_n = \left( \sum_{i=1}^{n} w_i x_i(0) \right) 1_n$$

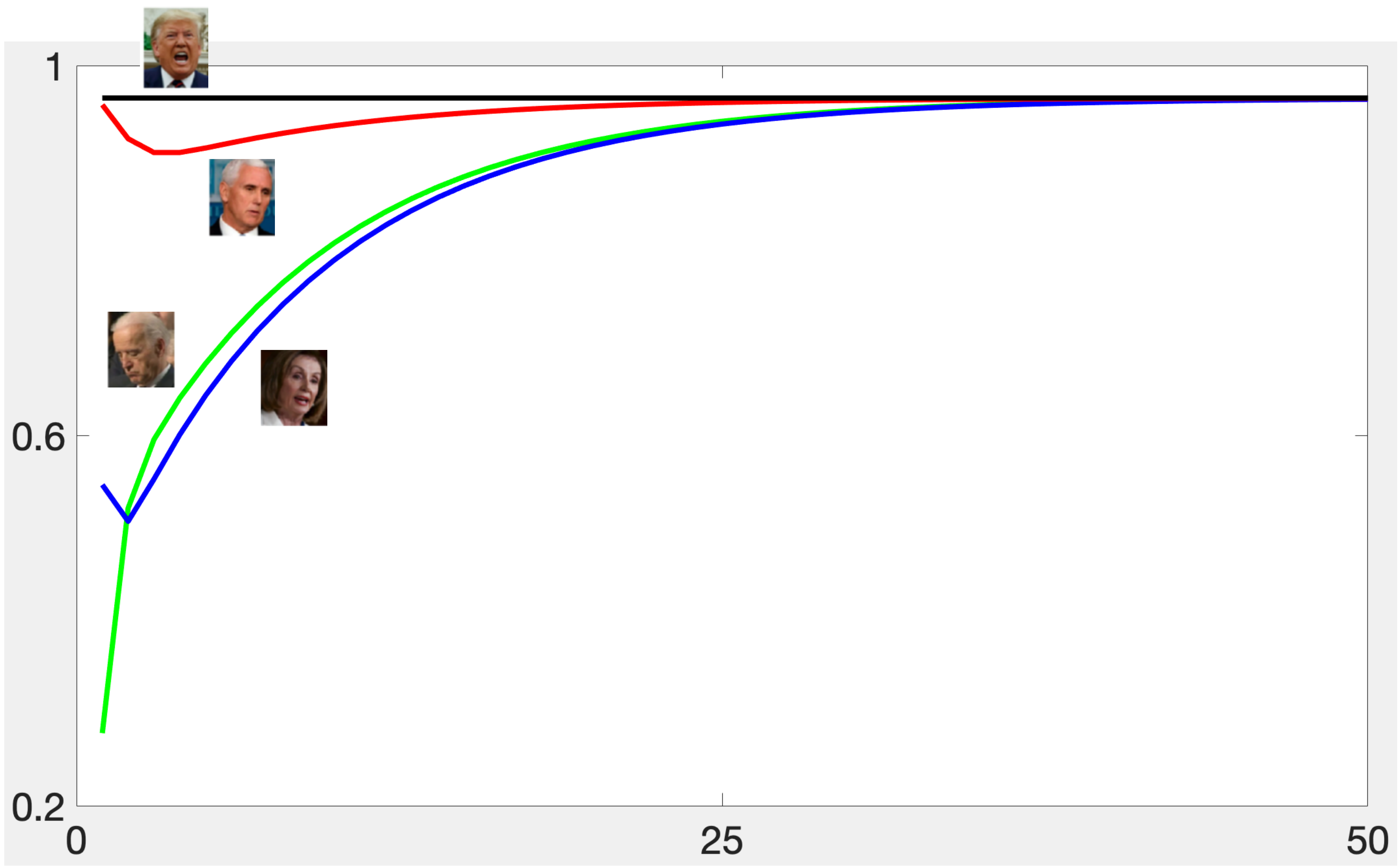
$\Rightarrow$

**DeGroot opinion dynamics model: social power ~ eigenvector centrality.**

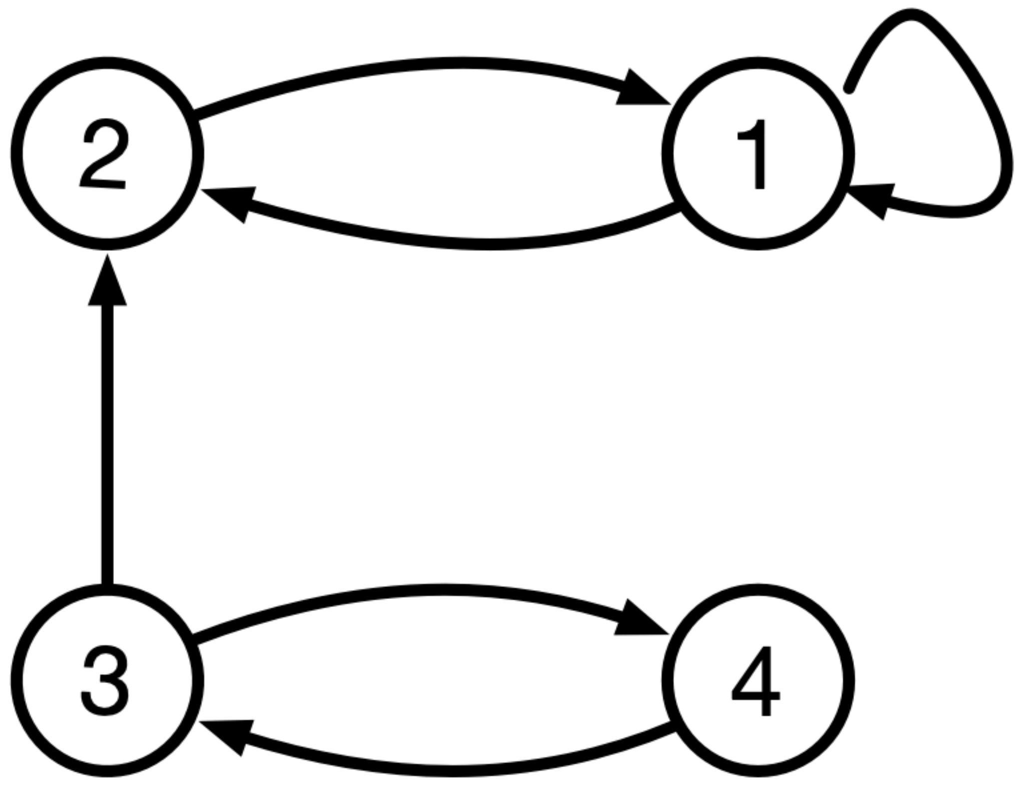
$A$ is primitive $\Leftrightarrow$ $x(t)$ converges to consensus? "$\Leftarrow$" does not hold.

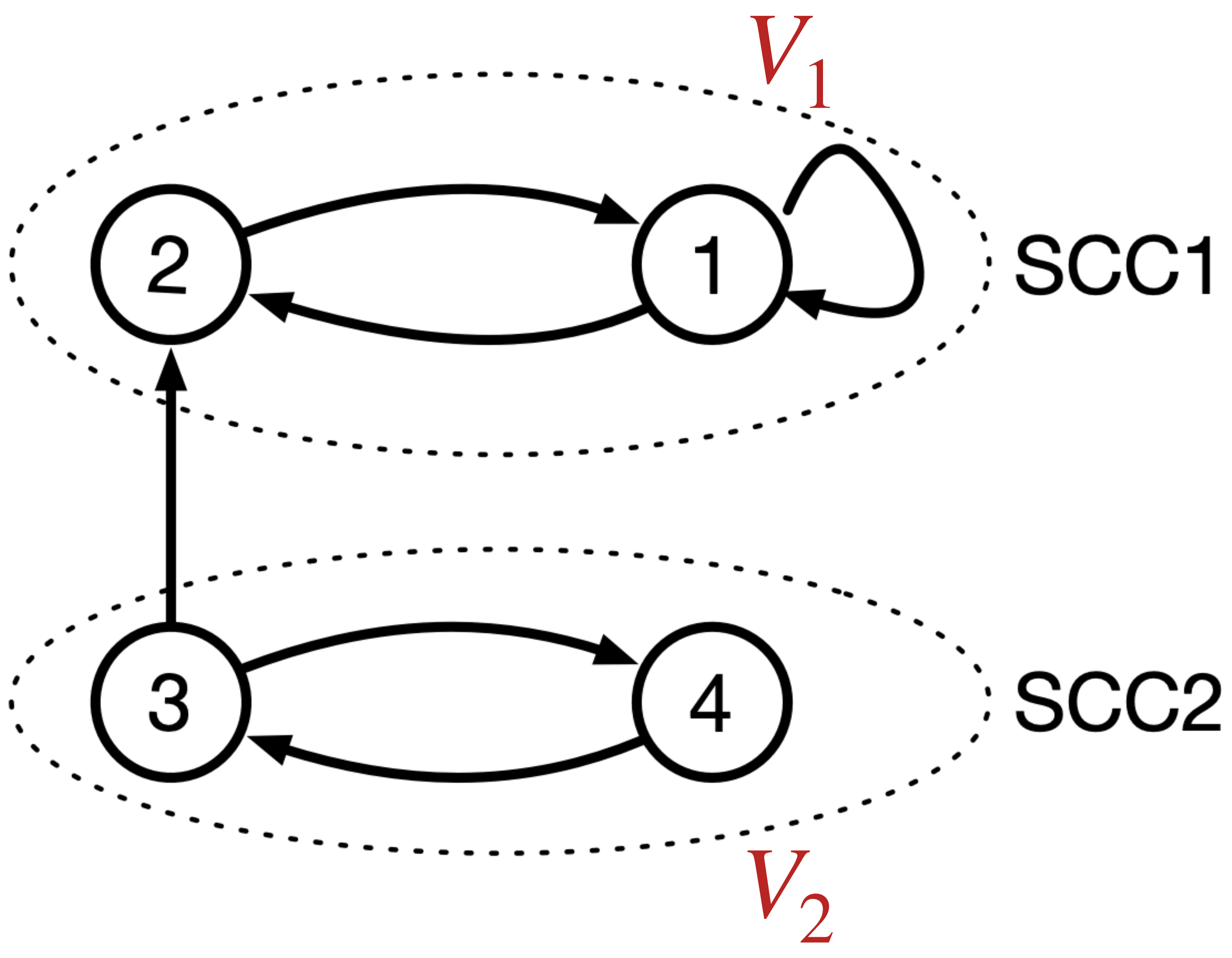


$$A = \begin{bmatrix} 0.4 & 0.4 & 0.2 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0.1 & 0.3 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



**Example 4: converge? consensus?**

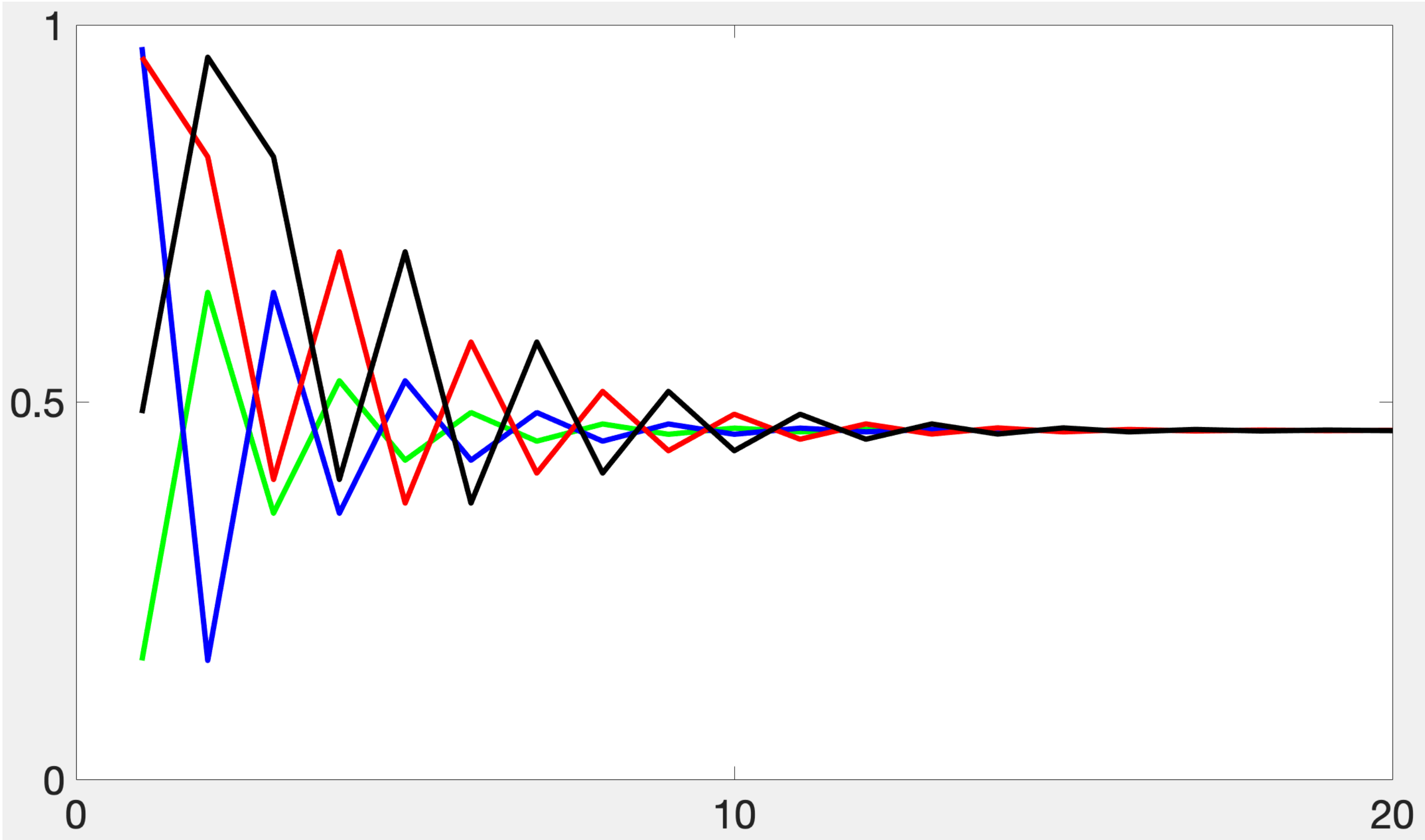



$$
\begin{array}{ccc}
 & A_{11} & A_{12} = 0 \\
[A] = & \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] & \\
 & A_{21} \neq 0 & A_{22}
\end{array}
$$

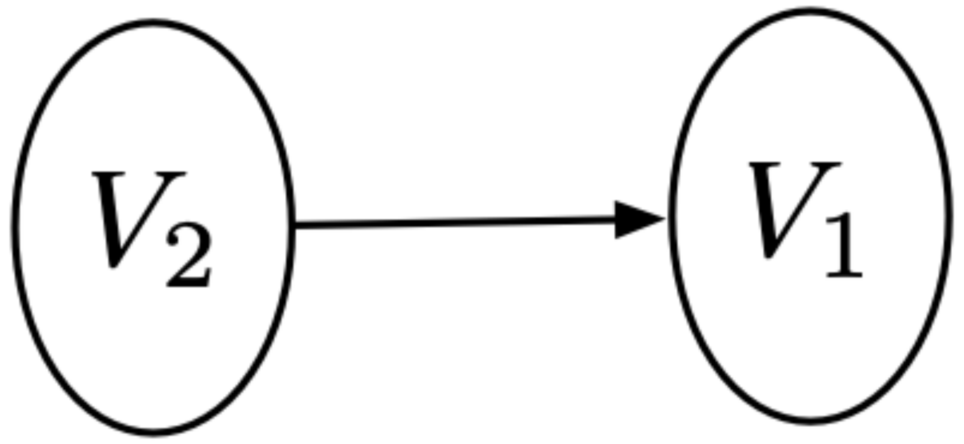
**e.g.**

$$
A = \begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$




1. ∃ globally reachable node
2. The SCC containing the globally reachable node is aperiodic.

## Example 4 in general




globally reachable
strongly connected
aperiodic

1. $\exists$ globally reachable node
2. The SCC containing the globally reachable node is aperiodic.

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

$$x(t+1) = Ax(t) \quad \Leftrightarrow \quad \begin{bmatrix} x^{(1)}(t+1) \\ x^{(2)}(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^{(1)}(t) \\ x^{(2)}(t) \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} x^{(1)}(t+1) & = A_{11}x^{(1)}(t) \\ x^{(2)}(t+1) & = A_{21}x^{(1)}(t) + A_{22}x^{(2)}(t) \end{cases}$$

(1) $x^{(1)}(t+1) = A_{11}x^{(1)}(t)$, where $A_{11}$ is row-stochastic and primitive $\Rightarrow x^{(1)}(t) \rightarrow \left( {w^{(1)}}^{\top} x^{(1)}(0) \right) 1_{n_1}$

(2) What about $x^{(2)}(t+1) = A_{21}x^{(1)}(t) + A_{22}x^{(2)}(t)$?

(2) What about $x^{(2)}(t+1) = A_{21}x^{(1)}(t) + A_{22}x^{(2)}(t)$?

1) **Suppose $A$ is semi-convergent.** $\Rightarrow \lim_{t\to\infty} x(t) = x^*$ exists. $\Rightarrow x^{*(2)} = A_{21}x^{*(1)} + A_{22}x^{*(2)}$

2) **If $\rho(A_{22}) < 1$, then** $(I - A_{22})\,x^{*(2)} = A_{21}x^{*(1)} \;\Rightarrow\; x^{*(2)} = (I - A_{22})^{-1} A_{21}\, x^{*(1)} = \left(w^{(1)\top} x^{(1)}(0)\right)(I - A_{22})^{-1} A_{21}\, 1_{n_1}$

3) Since $A$ is row-stochastic, $A_{21}1_{n_1} + A_{22}1_{n_2} = 1_{n_2} \Rightarrow (I - A_{22})^{-1}A_{21}1_{n_1} = 1_{n_2} \Rightarrow x^{*(2)} = \left(w^{(1)\top} x^{(1)}(0)\right) 1_{n_2}$

$$\Rightarrow \qquad x^* = [\; w^{(1)\top} \quad 0 \;] \begin{bmatrix} x^{(1)}(0) \\ x^{(2)}(0) \end{bmatrix} 1_n$$

Moreover, $\rho(A_{22}) < 1 \;\Rightarrow\; A$ is semi-convergent. (Why?)

**Row-substochastic matrix:** A non-negative matrix $A \in \mathbb{R}^{n\times n}$ is said to be row-substochastic if its row-sums are at most one, with at least one row-sum being strictly less than one.

- Any row-substochastic matrix $A$ that has at least one row-sum equal to one, satisfies:

$$\min_i (A1_n)_i < \max_i (A1_n)_i$$

$\rho(A_{22}) < 1$ ?

**Lemma 3 (The. 4.11 in textbook):** Consider a non-negative matrix $A$ associated with a directed weighted graph $G$.

(i) $\min_i (A\mathbb{1}_n)_i \leq \rho(A) \leq \max_i (A\mathbb{1}_n)_i$

(ii) If $\min_i (A\mathbb{1}_n)_i < \max_i (A\mathbb{1}_n)_i$, i.e, $A$ is row-substochastic then the following two statements are equivalent.

(a) $\forall i$ such that $(A\mathbb{1}_n)_i = \max_k (A\mathbb{1}_n)_k$, $\exists$ a directed path from $i$ to some $j$ such that $(A\mathbb{1}_n)_j < \max_k (A\mathbb{1}_n)_k$;

(b) $\rho(A) < \max_k (A\mathbb{1}_n)_k$.

$$\left.\begin{array}{l} \text{1) } \exists j \text{ s.t. } (A_{22}\mathbb{1}_{n_2})_j < 1\text{, i.e., } A_{22} \text{ is row-substochastic!} \\ \text{2) } \max_k (A_{22}\mathbb{1}_{n_2})_k \leq 1 \\ \text{3) } \forall i \text{ such that } (A_{22}\mathbb{1}_{n_2})_i = 1\text{, } \exists \text{ directed path from } i \text{ to at least one such } j \text{ in 1).} \end{array}\right\} \overset{\textbf{Lemma 3}}{\Longrightarrow} \quad \rho(A_{22}) < 1$$

## Theorem 4 (Thm. 5.1 in textbook)

**Theorem 5.1 (Consensus for row-stochastic matrices with a globally-reachable aperiodic strongly-connected component).** *Let $A$ be a row-stochastic matrix and let $G$ be its associated digraph. The following statements are equivalent:*

(A1) *the eigenvalue $1$ is simple and all other eigenvalues $\mu$ satisfy $|\mu| < 1$;*

(A2) *$A$ is semi-convergent and $\lim_{k\to\infty} A^k = \mathbb{1}_n w^\mathsf{T}$, where $w \in \mathbb{R}^n$ satisfies $w \geq 0$, $\mathbb{1}_n^\mathsf{T} w = 1$, and $w^\mathsf{T} A = w^\mathsf{T}$; and*

(A3) *$G$ contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic.*

*If any, and therefore all, of the previous conditions are satisfied, then the matrix $A$ is said to be* indecomposable *and*

(i) *$w \geq 0$ is the left dominant eigenvector of $A$ and $w_i > 0$ if and only if node $i$ is globally reachable;*

(ii) *the solution to the averaging model (5.1) $x(k+1) = Ax(k)$ satisfies*

$$\lim_{k\to\infty} x(k) = \big(w^\mathsf{T} x(0)\big)\mathbb{1}_n;$$

(iii) *if additionally $A$ is doubly-stochastic, then $w = \frac{1}{n}\mathbb{1}_n$ (since $A^\mathsf{T}\mathbb{1}_n = \mathbb{1}_n$ and $\frac{1}{n}\mathbb{1}_n^\mathsf{T}\mathbb{1}_n = 1$) so that*

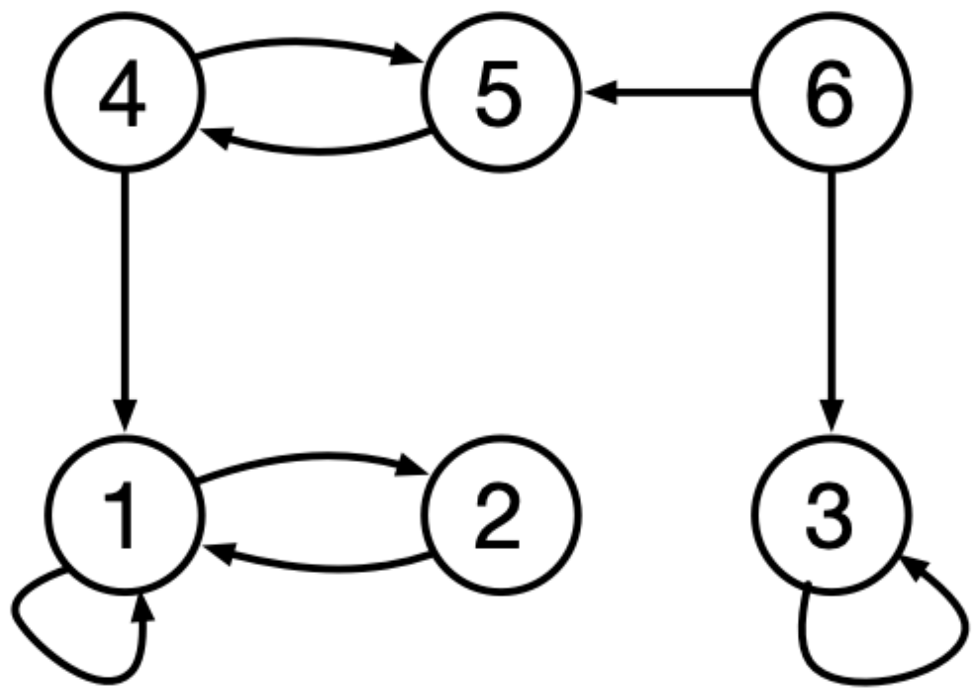
$$\lim_{k\to\infty} x(k) = \frac{\mathbb{1}_n^\mathsf{T} x(0)}{n}\mathbb{1}_n = \mathrm{average}\big(x(0)\big)\mathbb{1}_n.$$

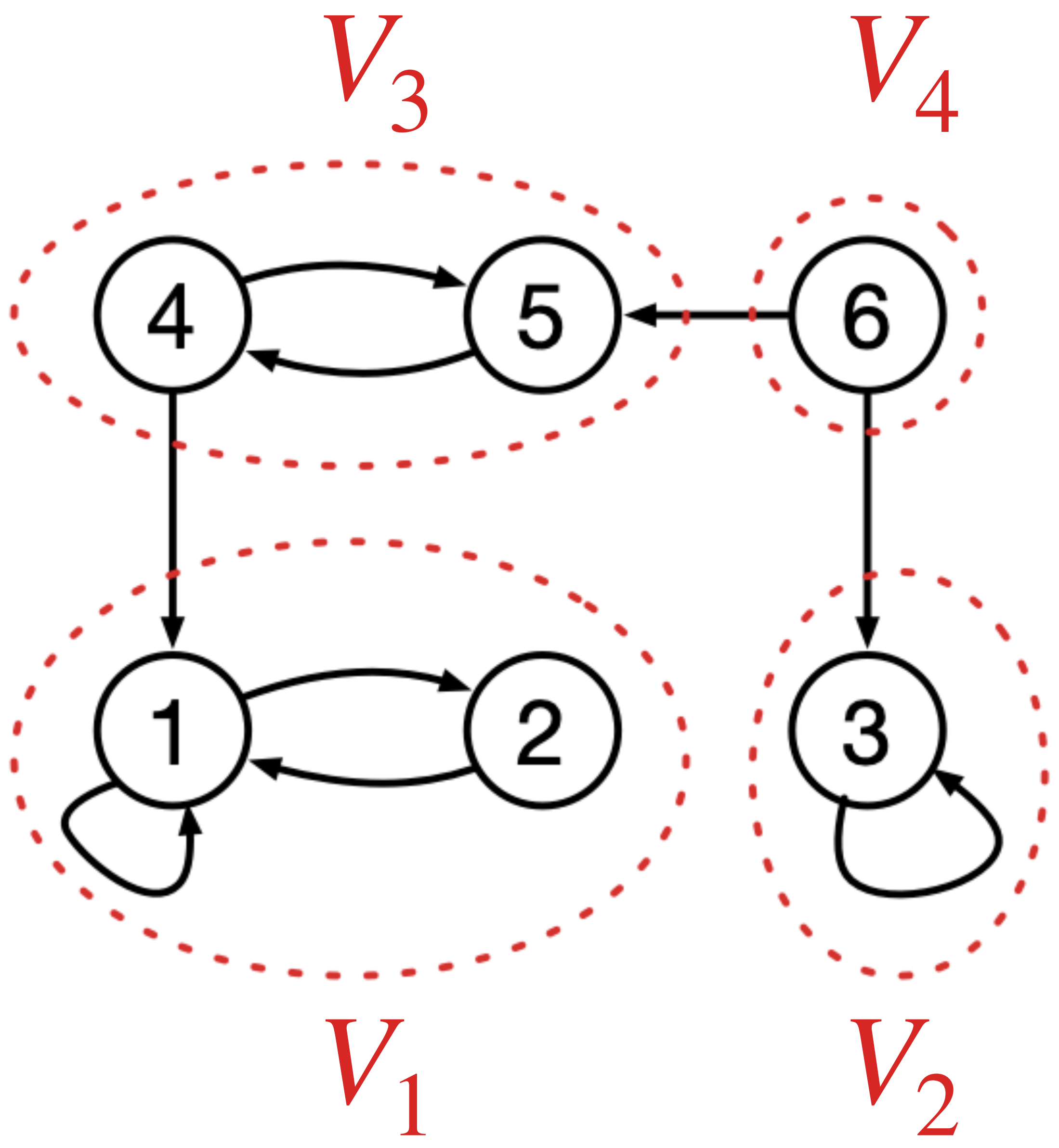
(A3) $\Rightarrow$ (A1): We have already proved it. (Why?)

(A1) $\Rightarrow$ (A2): By applying Theorem 2.13 in textbook.

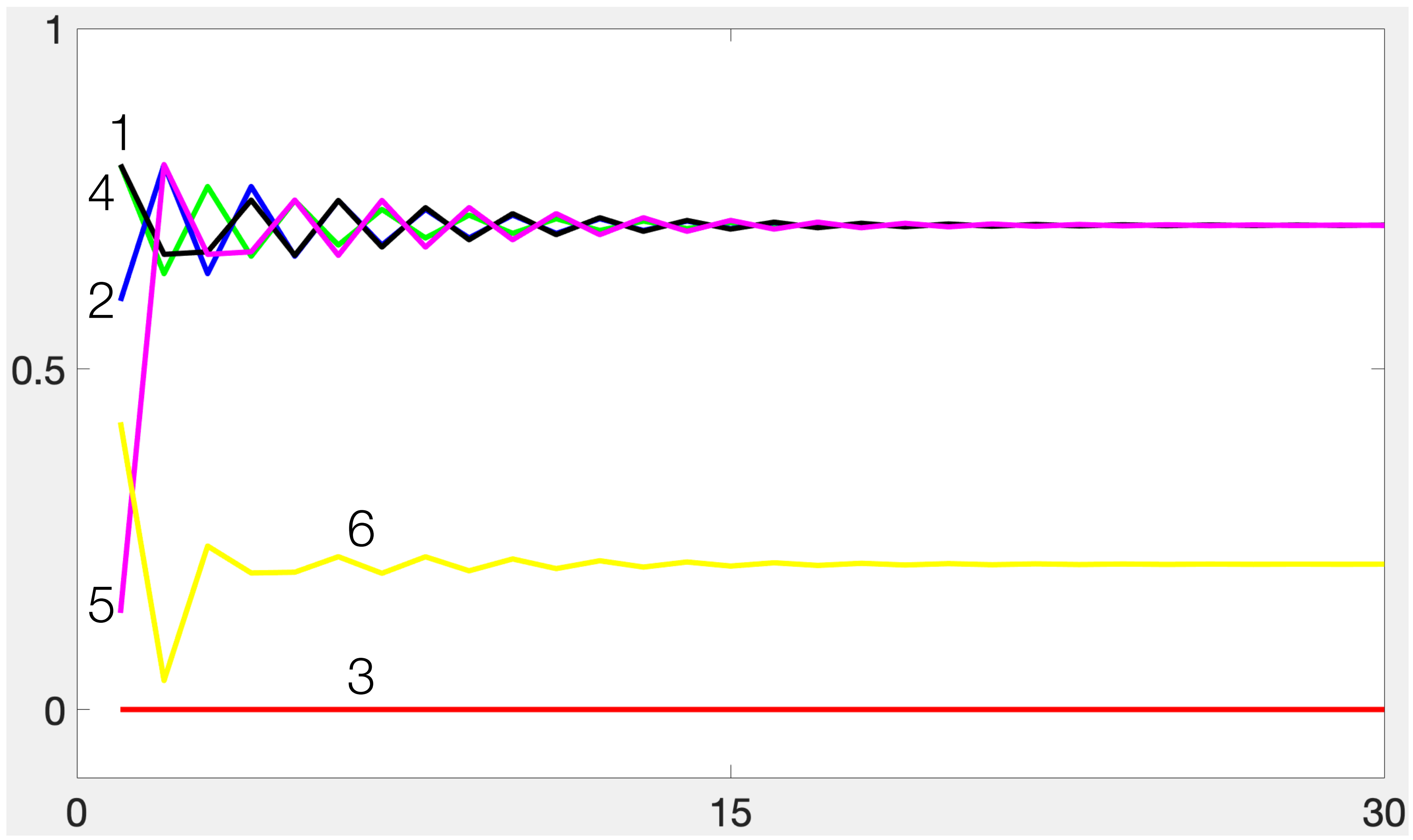
(A2) $\Rightarrow$ (A3):

**Example 5: What if $\nexists$ globally reachable node?**




$$[A] = \left[\begin{array}{cc|c|cc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$




1) $C(G)$ has multiple sinks;

2) Each sink in $C(G)$ is an aperiodic SCC in $G$;

## Theorem 5 (Thm. 5.2 in textbook)

**Theorem 5.2 (Convergence for row-stochastic matrices with multiple aperiodic sinks).** *Let $A$ be a row-stochastic matrix, $G$ be its associated digraph, and $n_s \geq 2$ be the number of sinks in the condensation digraph $C(G)$. The following statements are equivalent:*

(A1) *the eigenvalue 1 is semi-simple with multiplicity $n_s$ and all other eigenvalues $\mu$ satisfy $|\mu| < 1$,*

(A2) *$A$ is semi-convergent, and*

(A3) *each sink of $C(G)$, regarded as a subgraph of $G$, is aperiodic.*

*If any, and therefore all, of the previous conditions are satisfied, then*

(i) *the left eigenvectors $w^p \in \mathbb{R}^n$, $p \in \{1, \dots, n_s\}$, of $A$ corresponding to the eigenvalue 1 can be selected to satisfy: $w^p \geq 0$, $\mathbb{1}_n^\top w^p = 1$, and $w_i^p > 0$ if and only if node $i$ belongs to sink $p$,*

(ii) *the solution to the averaging model $x(k+1) = Ax(k)$ with initial condition $x(0)$ satisfies*

$$\lim_{k\to\infty} x_i(k) = \begin{cases} (w^p)^\top x(0), & \text{if node } i \text{ belongs to sink } p, \\ \displaystyle\sum_{p=1}^{n_s} z_{i,p}\big((w^p)^\top x(0)\big), & \text{otherwise,} \end{cases}$$

*where $z_{i,p}$, $p \in \{1, \dots, n_s\}$, are convex combination coefficients and $z_{i,p} > 0$ if and only if there exists a directed path from node $i$ to the sink $p$.*

**(A3) $\Rightarrow$ (A1): Lemma 3 $\Rightarrow \rho(\tilde{A}) < 1$**

**(A1) $\Rightarrow$ (A2): By Jordan decomposition**

**(A2) $\Rightarrow$ (A3):**

$$\left.\begin{array}{l} \text{1) } A \text{ is semi-convergent} \Rightarrow A_{ii} \text{ is semi-convergent } \Rightarrow |\mu| < \rho(A_{ii}), \forall \text{ other } \mu \in \text{spec}(A) \\ \text{2) } A_{ii} \text{ is row-stochastic \& irreducible} \Rightarrow v^{(i)} = 1_{n_i}, w^{(i)} > 0, \rho(A_{ii}) = 1 \end{array}\right\} \xRightarrow{\text{Thm. 2.13}} A_{ii}^t \to 1_{n_i} w^{(i)\top}$$

$\Rightarrow$ Every node has period 1 $\Rightarrow$ The digraph associated with $A_{ii}$ is aperiodic.

**Theorem 5 (ii)**

## Centrality Measures/Scores

It is of great interest for network science applications to determine the relative importance of a node in a graph!

**Various measures for centrality based on adjacency matrix:** Degree centrality, Eigenvector centrality, Katz centrality, PageRank centrality (we review the first two, for more please check Ch. 5 of Prof. Bullo's lecture notes)

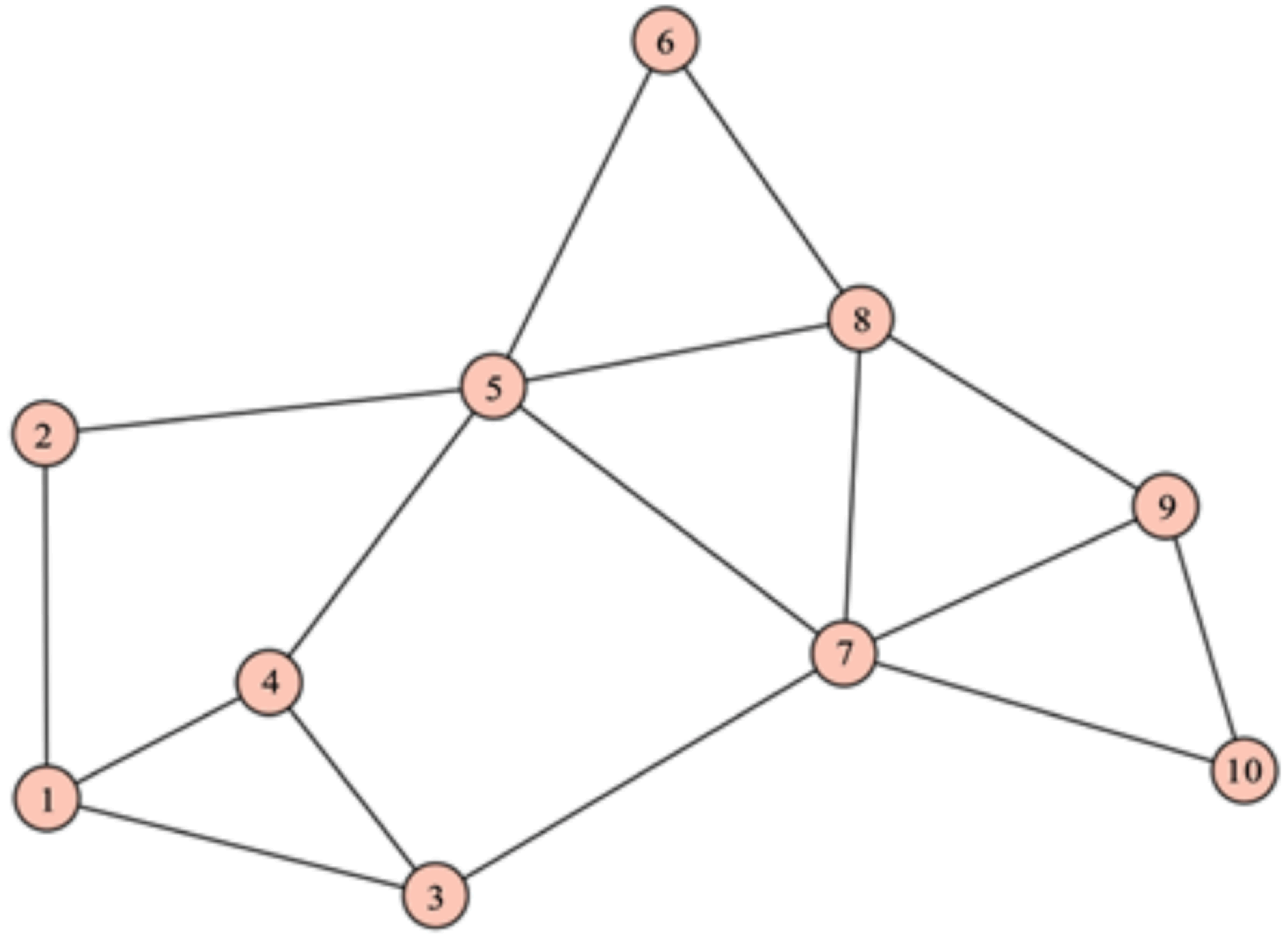
1. **Degree centrality:** For a weighted digraph $G$, the degree centrality of node $i$, $c_{\text{degree}}(i)$, is the in-degree of node $i$:

$$c_{\text{degree}}(i) = \sum_{j=1}^{n} a_{ji}$$

2. **Eigenvector centrality:** For a weighted digraph $G$ with globally reachable nodes, the eigenvector centrality vector $c_{\text{ev}}$ is defined as the left dominant eigenvector (associated with the dominant eigenvalue) of the adjacency matrix. Eigenvector centrality satisfies:

$$A^\top c_{\text{ev}} = \rho(A) c_{\text{ev}} \quad \Leftrightarrow \quad c_{\text{ev}}(i) = \frac{1}{\rho(A)} \sum_{j=1}^{n} a_{ji} c_{\text{ev}}(j)$$

## Centrality Measures/Scores: An example




1. **Degree centrality:** the most important node is the one with the largest in-degree, i.e., nodes {5,7}.

2. **Eigenvector centrality:**
Eigenvalues of the adjacency matrix:

$$\{-2.46, -1.93, -1.54, -0.66, -0.47, -0.33, 0.37, 1.39, 2.11, 3.53\}$$

The greatest eigenvalue = 3.53

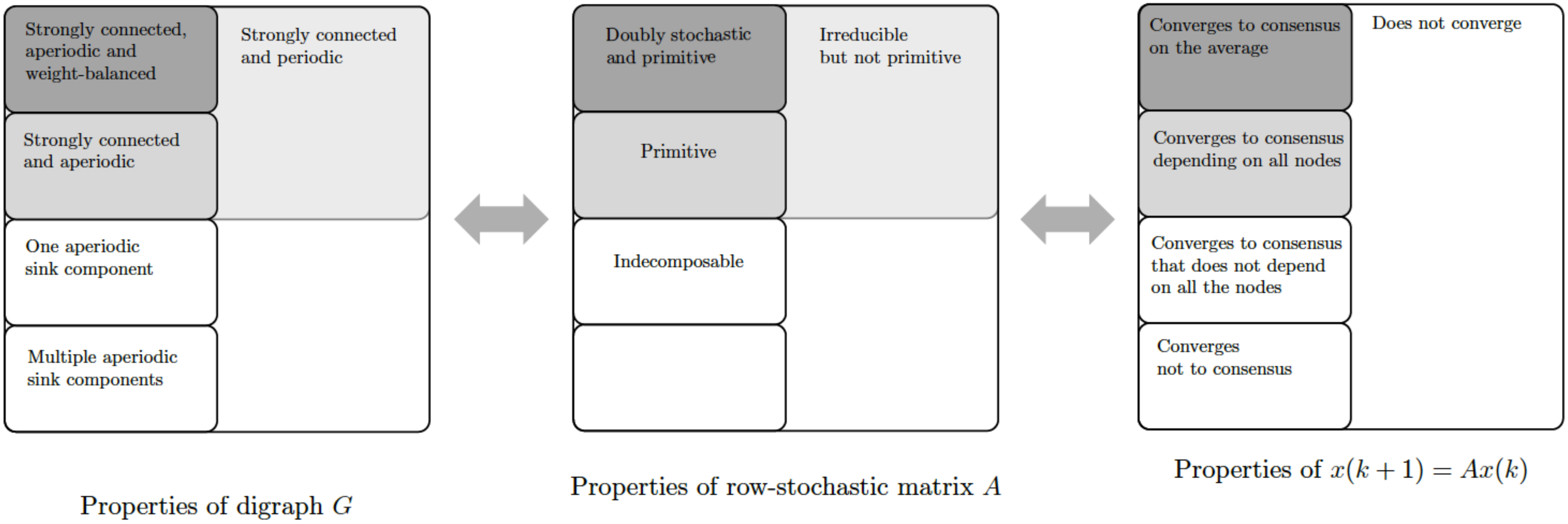
Eigenvector associated with the greatest eigenvalue:

$$v = \{0.19, 0.18, 0.26, 0.25, 0.44, 0.24, 0.46, 0.41, 0.31, 0.22\} = c_{\text{ev}}$$

The most important node is node 7.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

## Summary



Strongly connected, aperiodic and weight-balanced
Strongly connected and periodic
Strongly connected and aperiodic
One aperiodic sink component
Multiple aperiodic sink components
Properties of digraph $G$
Doubly stochastic and primitive
Irreducible but not primitive
Primitive
Indecomposable
Properties of row-stochastic matrix $A$
Converges to consensus on the average
Does not converge
Converges to consensus depending on all nodes
Converges to consensus that does not depend on all the nodes
Converges not to consensus
Properties of $x(k+1) = Ax(k)$