Nonlinear Control Theory

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Lyapunov Stability

Comparison Functions

Definition

- A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$, belongs to class $\mathcal{K}$, if it is strictly increasing and $\alpha(0) = 0$.
- It belongs to class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Definition

A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, +\infty)$, belongs to class $\mathcal{KL}$ if

- for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r, and
- for each fixed r, the mapping is decreasing with respect to s, and $\beta(r, s) \to 0$ as $s \to \infty$.

Example

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = \frac{1}{1+r^2} > 0$. It belongs to class $\mathcal{K}$. Does it belong to class $\mathcal{K}_\infty$?
- $\alpha(r) = r^c$ where $c > 0$. Does it belong to class $\mathcal{K}$? or $\mathcal{K}_\infty$?
- $\alpha(r) = \min[r, r^2]$ belongs to class $\mathcal{K}_\infty$. It is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class $\mathcal{K}$ or $\mathcal{K}_\infty$ functions.
- $\beta(r, s) = \frac{r}{ksr+1}$ with $k > 0$ is strictly increasing in r (why?) and strictly decreasing in s (why?). Moreover, $\beta(r, s) \to 0$ as $s \to \infty$. Therefore, it belongs to class $\mathcal{KL}$.
- $\beta(r, s) = r^c e^{-s}$ with $c > 0$. Does it belongs to class $\mathcal{KL}$?

Some useful properties of class $\mathcal{K}$ and class $\mathcal{KL}$ functions:

Lemma

Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$, α_3 and α_4 be class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}. Then,

- α_1^{-1} *is defined on* $[0, \alpha(a))$ *and belongs to class* $\mathcal{K}$.
- α_3^{-1} *is defined on* $[0, \infty)$ *and belongs to class* $\mathcal{K}_\infty$.
- $\alpha_1 \circ \alpha_2$ *belongs to class* $\mathcal{K}$.
- $\alpha_3 \circ \alpha_4$ *belongs to class* $\mathcal{K}_\infty$.
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ *belongs to class* $\mathcal{KL}$.

Please prove this Lemma for an exercise.

Lemma

- *Let $V : D \to R$ be a continuously positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r)$, such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in B_r.$$

- *If $D = R^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$, and the foregoing inequality will hold for all $x \in R^n$.*
- *Moreover, if $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$.*

Example

For a quadratic positive definite function $V(x) = x^T Px$, the following inequality holds:

$$\lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2.$$

Nonautonomous Systems

Consider the non-autonomous system

$$\dot{x} = f(t, x),$$

where $f : [0, \infty) \times D \to R^n$ is piece-wise continuous in t, and locally Lipschitz in x; D is a domain that contains $x = 0$.

- The origin is an equilibrium point at $t = 0$ if $f(t, 0) = 0, \ \forall t \geq 0$.
- An equilibrium point at the origin could be a translation of a non-zero equilibrium point (or a non-zero solution) of the system.
  - * Suppose that $\bar{y}(\tau)$ is a solution of $\frac{\mathrm{d}y}{\mathrm{d}\tau} = g(\tau, y), \ \forall \tau \geq a$. The change of variables $x = y - \bar{y}(\tau)$ and $t = \tau - a$ transforms the system into

$$\dot{x} = g(\tau, y) - \dot{\bar{y}} = g(t + a, x + \bar{y}(t + a)) - \dot{\bar{y}}(t + a) \triangleq f(t, x).$$

Since $\dot{\bar{y}}(t + a) = g(t + a, \bar{y}(t + a)), \ \forall t \geq 0$, the origin $x = 0$ is an equilibrium point at $t = 0$.

Difference between autonomous systems and non-autonomous systems:

- the solution of an autonomous system depends only on $t - t_0$;
- the solution of a non-autonomous system may depend on both t and t_0.

Definition

The origin is a stable equilibrium point for $\dot{x} = f(t, x)$, if for each $\epsilon > 0$, and any $t_0 \geq 0$, there exists $\delta = \delta(\epsilon, t_0) \geq 0$, such that

$$\|x(t_0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \; \forall t \geq t_0.$$

Remark

The existence of δ for every t_0 does not necessarily guarantee that there is one constant δ, dependent only on ϵ, that would work for all t_0.

Example

The linear first-order system

$$\dot{x} = (6t\sin t - 2t)x$$

has the solution:

$$x(t) = x(t_0)e^{6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2}.$$

For any t_0, the term t_0 dominates, indicating that $x(t)$ is bounded for any $t \geq t_0$ by a constant $c(t_0)$ dependent on t_0:

$$|x(t)| < |x(t_0)|c(t_0), \quad \forall t \geq t_0.$$

For any $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{c(t_0)}$ (dependent on t_0) showing that $x = 0$ is stable, but not uniformly in t_0.

Definition

The equilibrium point of $\dot{x} = f(t, x)$ is

- **uniformly stable**, if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, independent of t_0, such that such that

$$\|x(t_0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \ \forall t \geq t_0 \geq 0.$$

- **uniformly asymptotically stable**, if it is uniformly stable, and there exists a positive constant c independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0.
- **globally uniformly asymptotically stable**, if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon \to \infty} \delta(\epsilon) \to \infty$, and for each pair of $\eta > 0$ and $c > 0$, there exits $T = T(\eta, c) > 0$ such that

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta, c), \ \forall \|x(t_0)\| < c.$$

Lemma

The equilibrium $x = 0$ of $\dot{x} = f(t, x)$ is

- *uniformly stable if and only if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \le \alpha(\|x(t_0)\|), \quad \forall t \ge t_0 \ge 0, \ \ \forall \|x(t_0)\| < c.$$

- *uniformly asymptotically stable if and only if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall t \ge t_0 \ge 0, \ \ \forall \|x(t_0)\| < c.$$

- *globally uniformly asymptotically stable if and only if the foregoing inequality is satisfied for any initial state $x(t_0)$.*

Lemma (Cont'd)

- ***exponentially stable*** *if there exist positive constants c, k and λ, such that*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \|x(t_0)\| < c.$$

- ***globally exponentially stable*** *if the foregoing inequality is satisfied for any initial state $x(t_0)$.*

Remark

The foregoing statements could also be seen as definitions of exponential stability and global exponential stability.

Theorem

Let $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$, and $D \subset R^n$ be a domain containing the origin. Let $V : [0, \infty) \times D \to R$ be a continuously differentiable function such that

$$W_1(x) \leq V(x) \leq W_2(x), \qquad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0, \quad \forall t \geq 0, \ \forall x \in D,$$

where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly stable.

Proof:

- Choose $r > 0$ and $c > 0$ such that $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$. Then, $\{x \in B_r | W_1(x) \leq c\}$ is in the interior of B_r.
- Define a time-dependent set $\Omega_{t,c} = \{x \in B_r | V(t, x) \leq c\}$. It contains $\{x \in B_r | W_2(x) \leq c\}$, since $W_2(x) \leq c \ \Rightarrow \ V(t, x) \leq c$. (To be continued)

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Proof: (Cont'd)

- The set $\Omega_{t,c}$ is a subset of $\{x \in B_r | W_1(x) < c\}$, since $V(t,x) \leq c \Rightarrow W_1(x) \leq c$.
- Thus, $\{x \in B_r | W_2(x) \leq c\} \subset \Omega_{t,c} \subset \{x \in B_r | W_1(x) < c\} \subset B_r \subset D$ for all $t \geq 0$.

- The surface $V(t,x) = c$ is time-dependent, but it is surrounded by $W_1(x) = c$ and $W_2(x) = c$.
- $\dot{V}(x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$ indicates that, for any $t_0 \leq 0$ and $x_0 \in \Omega_{t_0,c}$, the solution starting at (x_0, t_0) stays in $\Omega_{t_0,c}$ for all $t \geq t_0$.
- Therefore, any solution starting in $\{x \in B_r | W_2(x) \leq c\}$ stays in $\Omega_{t,c}$, and consequently in $\{x \in B_r | W_1(x) \leq c\}$.

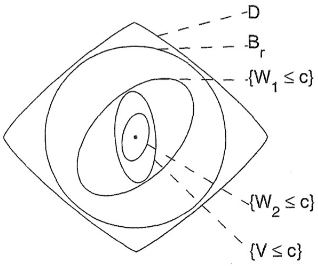

Figure: Geometric representation of sets

Proof: (Cont'd)

- Hence, the solution is bounded and defined for all $t \geq t_0$.
- Moreover, $\dot{V}(t,x) \leq 0 \quad \Rightarrow \quad V(t,x) \leq V(t_0, x_0), \ \forall t \geq t_0$.
- By foregoing Lemma, there exist class $\mathcal{K}$ functions α_1 and α_2 defined on $[0, r]$, such that

$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|).$$

- Combining the preceding two inequalities,

$$\|x(t)\| \leq \alpha_1^{-1}(V(t,x(t))) \leq \alpha_1^{-1}(V(t_0,x_0)) \leq \alpha_1^{-1}(\alpha_2(\|x_0\|)).$$

- Since $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function, the inequality $\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x_0\|))$ shows that the origin is uniformly stable. (End of proof)

Theorem

- *Suppose the assumptions of foregoing theorem are strengthened to*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \forall t \ge 0, \ \forall x \in D,$$

where $W_3(x)$ is a continuous positive definite function on D. Then, $x = 0$ is uniformly asymptotically stable.

- *Moreover, if r and c are chosen such that $B_r = \{\|x\| < r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r | W_2(x) \le c\}$ satisfies*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall t \ge t_0 \ge 0$$

for some $\mathcal{KL}$ function β.

- *Finally, if $D \subset R^n$ and $W_1(x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.*

Terminology: A function is said to be

- positive semi-definite if $V(t,x) \geq 0$;
- positive definite if $V(t,x) \geq W_1(x)$ for some positive definite function $W_1(x)$;
- radially unbounded if $V(t,x) \geq W_1(x)$ and $W_1(x)$ is radially unbounded;
- decrescent if $V(t,x) < W_2(x)$ for some positive definite function $W_2(x)$;
- negative definite (semi-definite) if $-V(t,x)$ is positive definite (semi-definite).

The foregoing theorems say that the origin is

- **uniformly stable** if there is a continuously differentiable, positive definite, decrescent function $V(t,x)$, whose derivative along the trajectories of the system is negative semidefinite.
- **uniformly asymptotically stable** if the derivative is negative definite.
- **globally uniformly asymptotically stable** if the conditions for uniform asymptotic stability hold globally with a radially unbounded $V(t,x)$.

Theorem

Suppose the assumptions of the previous theorems are satisfied with

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a, \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a, \;\; \forall t \geq 0, \;\; \forall x \in D,$$

where k_1, k_2, k_3 and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable.

Example

The scalar system $\dot{x} = -[1+g(t)]x^3$ with $g(t) \geq 0, \forall t \geq 0$.

Choose $V(x) = \frac{1}{2}x^2$, and its derivative with time can be calculated by

$$\dot{V}(x) = -[1+g(t)]x^4 \leq -x^4, \quad \forall x \in R, \;\; \forall t > 0.$$

The origin is globally uniformly asymptotically stable. (Not exponentially, why?)

Proof:

- Trajectories starting in $\{k_2\|x\|^a < c\}$, for sufficiently c, remains bounded $\forall t \geq t_0$.
- Inequalities in the theorem indicates that $\dot{V} \leq -\frac{k_3}{k_2}V$, and by comparison Lemma,

$$V(t, x(t)) \leq V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}.$$

- Hence,

$$\|x(t)\| \leq \left[\frac{V(t, x(t))}{k_1}\right]^{\frac{1}{a}} \leq \left[\frac{V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right]^{\frac{1}{a}} \leq \left[\frac{k_2\|x(t_0)\|^a e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right]^{\frac{1}{a}}.$$

- Thus, the origin is exponentially stable.
- If all the assumptions hold globally, the foregoing inequality holds $\forall x \in R^n$, and the origin is globally exponentially stable.

Example

Consider the non-autonomous system

$$\dot{x}_1 = -x_1 - g(t)x_2, \quad \dot{x}_2 = x_1 - x_2,$$

with $0 \leq g(t) \leq k$ and $\dot{g}(t) \leq g(t)$, $\forall t \geq 0$.

- Choose $V(t,x) = x_1^2 + [1 + g(t)]x_2^2$, and it satisfies

$$x_1^2 + x_2^2 \leq V(t,x) \leq x_1^2 + (1+k)x_2^2.$$

- Its derivative with time satisfies

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2 \leq -x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x.$$

- The origin is globally exponentially stable.