Nonlinear Control Theory

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Lyapunov Stability

Converse Theorems

- How can we search for Lyapunov functions to satisfy the foregoing theorems?

  Unfortunately, we do not have a systematic way for all systems.

- Can we at least prove the existence of Lyapunov functions?

  Yes, we can!

Theorem (4.14 Converse theorem for exponential stability)

Let $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$, where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{\|x\| < r\}$ and the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded on D, uniformly in t. Let k, λ and r_0 be positive constants with $r_0 < \frac{r}{k}$. Let $D_0 = \{\|x\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|x(t)\| \le k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall x(t_0) \in D_0, \quad \forall t \ge t_0 \ge 0.$$

Then, there exists a function $V : [0, \infty) \times D_0 \to R$ satisfying the inequalities

$$c_1\|x\|^2 \le V(t, x) \le c_2\|x\|^2, \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|$$

for some positive contants c_1, c_2, c_3 and c_4. Moreover, if $r = \infty$ and the origin is globally exponentially stable, then $V(t, x)$ is defined and satisfies the aforementioned inequalities on R^n. Furthermore, if the system is autonomous, V can be chosen independent of t.

The foregoing theorem can be used to prove exponential stability of the linearization is a necessary and sufficient condition for exponential stability of the corresponding nonlinear system.

Theorem

Let $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$, where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{\|x\| < r\}$ and the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded on D, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}(t, x)\bigg|_{x=0}.$$

Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system, if and only if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x.$$

Proof: (Sufficiency was proved in the previous section. Here is the proof of necessity.)

- The linear system can be written by

$$\dot{x} = f(t,x) - [f(t,x) - A(t)x] = f(t,x) - g(t,x).$$

  where $\|g(t,x)\|_2 \leq L\|x\|_2^2, \forall x \in D, \forall t \geq 0$.

- Choose $r_0 = \min\{c, \frac{r}{k}\}$. Then all conditions in the foregoing theorem are satisfied, and there exists $V(t,x)$ satisfying the foregoing inequalities.

- Use $V(t,x)$ as the Lyapunov candidate for the linear system,

$$\begin{aligned}\dot{V} =& \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) - \frac{\partial V}{\partial x} g(t,x) \leq -c_3\|x\|_2^2 + c_4 L\|x\|_2^3 \\ <& -(c_3 - c_4 L\rho)\|x\|_2^2 \qquad \forall \|x\|_2 < \rho.\end{aligned}$$

  The choice $\rho < \min\{r_0, \frac{c_3}{c_4 L}\}$ ensures that $V(t,x)$ is negative definite in $\|x\|_2 < \rho$. It is then concluded that $x = 0$ is exponentially stable for the linear system.

Corollary

Let $x = 0$ be be an equilibrium point of $\dot{x} = f(x)$, where $f(x)$ is continuously differentiable in some neighborhood of $x = 0$. Let $A = \frac{\partial f}{\partial x}\big|_{x=0}$. Then $x = 0$ is an exponentially stable equilibrium point for $\dot{x} = f(x)$ if and only if A is Hurwitz.

Example

Consider the system $\dot{x} = -x^3$. It is globally asymptotically stable. However, it is not exponentially stable, as can be seen from its linearization $\dot{x} = 0$ which is not exponentially stable.

Theorem (4.16 Converse theorem for uniform asymptotic stability)

Suppose the conditions in Theorem 4.14 are all satisfied, except that

$$\|x\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall x(t_0) \in D_0, \ \forall t \geq t_0 \geq 0,$$

where $\beta(\cdot,\cdot)$ belongs to class $\mathcal{KL}$; r_0 is a positive constant such that $\beta(r_0, 0) < r$; and $D_0 = \{\|x\| < r_0\}$. Then, there exists a continuously differentiable function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x\|), \quad \left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|)$$

where α_1, α_2, α_3 and α_4 are class $\mathcal{K}$ functions defined on $[0, r_0)$. If the system is autonomous, V can be chosen independent of t.

Theorem

Let $x = 0$ be an asymptotically stable equilibrium point for $\dot{x} = f(x)$, where f is locally Lipschitz on a domain $D \subset R^n$ that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \to \infty \ \text{ as } \ x \to \partial R_A, \quad \frac{\partial V}{\partial x} f \leq -W(x), \ \ \forall x \in R_A,$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of R_A. When $R_A = R^n$, $V(x)$ is radially unbounded.