模型预测控制

Model Predictive Control

诸 兵，Bing Zhu
第七研究室，北京航空航天大学
The Seventh Research Division, Beihang University

2022 Spring

Output feedback MPC

Lecture 3

- Unconstrained output feedback MPC
- Constrained output feedback MPC

Problem Formulation

- Consider the LTI system

$$x(k+1) = Ax(k) + Bu(k), \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^p$$

- Suppose that full state information cannot be directly measured, i.e., only partial information can be measured online:

$$y = Cx, \qquad\qquad y \in \mathbb{R}^m,\ m \leq n$$

- Assume that (A, B) is controllable/stabilizable, and (A, C) is observable/detectable.

- The objective is to design unconstrained/constrained MPC, using only the measured information $y(k)$, such that the origin of the LTI system is asymptotically stable.

Recall the full-state feedback Unconstrained MPC

- LTI plant

$$x(k+1) = Ax(k) + Bu(k),$$

- Model-based prediction

$$X(k) = Fx(k) + \Phi U(k)$$

where

$$F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \qquad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}.$$

$$X(k) \triangleq [x^T(1\,|\,k), x^T(2\,|\,k), \cdots, x^T(N\,|\,k)]^T, \qquad U(k) \triangleq [u^T(0\,|\,k), u^T(1\,|\,k), \cdots, u^T(N-1\,|\,k)]^T.$$

Recall the full-state feedback Unconstrained MPC (cont'd)

- Cost function

$$
\begin{aligned}
J(k) &= \|x(N+1\,|\,k)\|_P^2 + \|u(N\,|\,k)\|_R^2 + \sum_{i=1}^{N} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 \\
&= X^T(k)\mathcal{Q}X(k) + U^T(k)\mathcal{R}U(k) \\
&= (Fx(k) + \Phi U(k))^T\mathcal{Q}(Fx(k) + \Phi U(k)) + U^T(k)\mathcal{R}U(k) \\
&= x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k)\,.
\end{aligned}
$$

where

$$
\mathcal{Q} = \text{diag}[Q, Q, \cdots, Q, P], \qquad \mathcal{R} = \text{diag}[R, R, \cdots, \cdots, R],
$$

and the matrix P is the unique solution of the discrete-time Lyapunov equation:

$$
P - (A - BK)^T P(A - BK) = Q - K^T RK
$$

where K is chosen such that $|\text{eig}(A - BK)| < 1$.

Recall the full-state feedback Unconstrained MPC (cont'd)

- For unconstrained optimization, analytic solution exists such that

$$\nabla_U J \Big|_{U=U^*} = \frac{\partial J}{\partial U}\Bigg|_{U=U^*} = 0$$

or, explicitly

$$U^*(k) = -(\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k)$$

- At each time instant k, only the first control action is implemented, i.e.

$$u^*(k) = -[I_{p\times p}\ 0\ \cdots\ 0](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F x(k) = -K_{mpc} x(k)$$

which is actually a linear feedback control.

What if the state information is not fully available ??

Linear observer to estimate the state information

- If state information is not fully available, and only partial information $y = Cx$ is available online, then the separation principle applies here.

- For LTI system, the linear observer can be designed by

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + H(y - C\hat{x})$$

where $\hat{x}$ is the estimated state, and H is the observer gain.

- Define the estimation error $\tilde{x} \triangleq x - \hat{x}$. Its dynamics can be calculated by

$$\tilde{x}(k+1) = A\tilde{x}(k) - H(Cx - C\hat{x}) = (A - HC)\tilde{x}(k)$$

If the observer gain is designed such that $|\mathrm{eig}(A - HC)| < 1$, then the estimation error is asymptotically stable.

Output feedback unconstrained MPC

- The output feedback unconstrained MPC can be designed by simply replace the real state by estimated state:

$$u(k) = -K_{mpc}\hat{x}(k),$$

where $\hat{x}$ is estimated by the linear observer:

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + H(y - C\hat{x})$$

- According to the principle of separation, the feedback gain and the observer gain can be designed independently, and closed-loop stability is guaranteed, provided that

$$|\mathrm{eig}(A - BK_{mpc})| < 1 \quad \text{and} \quad |\mathrm{eig}(A - HC)| < 1$$

Proof of closed-loop stability in the framework of separation principle

- With the output feedback unconstrained MPC, the closed-loop system is now given by

$$x(k+1) = Ax(k) - BK_{mpc}\hat{x}(k),$$

$$\hat{x}(k+1) = A\hat{x}(k) - BK_{mpc}\hat{x}(k) + H(Cx - C\hat{x}),$$

  or in compact form:

$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} A & -BK_{mpc} \\ HC & A - BK_{mpc} - HC \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$$

  or, equivalently,

$$\begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} = \begin{bmatrix} A - BK_{mpc} & BK_{mpc} \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}$$

- Consequently, the closed-loop system is asymptotically stable, provided that

$$|\,\mathrm{eig}(A - BK_{mpc})\,| < 1 \quad \text{and} \quad |\,\mathrm{eig}(A - HC)\,| < 1$$

Example

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k), \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Set control horizon $N = 4$. Set weight matrices $Q = I_{2\times 2}$, $R = 0.1$.

Take $K = [1.4 \quad 5.76]$, such that $|\mathrm{eig}(A - BK)| < 1$.

The solution of discrete Lyapunov equation

$$P - (A - BK)^T P (A - BK) = Q + K^T R K$$

can be calculated by

$$P = \begin{bmatrix} 5.2471 & 12.8188 \\ 12.8188 & 67.1313 \end{bmatrix}$$

Set $\mathcal{Q} = \mathrm{diag}[Q, Q, Q, P]$, $\quad \mathcal{R} = \mathrm{diag}[R, R, R, R]$.

Calculate $K_{mpc} = [1 \ 0 \ \cdots \ 0](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F = [2.6167 \quad 12.9286]$.

Example (cont'd)

Output feedback: $u(k) = -K_{mpc}\hat{x}(k) = -[2.6167 \quad 12.9286]\hat{x}(k)$,

The estimated state is updated by linear observer:

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + H(y - C\hat{x})$$

where

$$H = \begin{bmatrix} 1.05 \\ 0.226 \end{bmatrix}, \quad \text{such that} \quad \mathrm{eig}(A - HC) = 0.5 \pm j0.5$$

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Observer in "unconstrained MPC framework"

- The observer gain can be calculated within unconstrained MPC framework

- Consider the dual system (A^T, C^T, B^T) of the plant (A, B, C). Its linear feedback gain can be used as the observer gain of the original plant.

- Prediction

$$Z(k) = F_d z(k) + \Phi_d V(k)$$

where

$$F_d = \begin{bmatrix} A^T \\ (A^T)^2 \\ \vdots \\ (A^T)^N \end{bmatrix}, \qquad \Phi_d = \begin{bmatrix} C^T & 0 & & \\ A^T C^T & C^T & 0 & \\ \vdots & & & \\ (A^T)^{N-1} C^T & (A^T)^{N-2} C^T & \cdots & C^T \end{bmatrix}.$$

Observer in "unconstrained MPC framework" (cont'd)

- Cost function

$$
\begin{aligned}
\hat{J}(k) &= \|z(N+1\,|\,k)\|_S^2 + \|v(N\,|\,k)\|_R^2 + \sum_{i=1}^{N} \|z(i\,|\,k)\|_Q^2 + \|v(i-1\,|\,k)\|_R^2 \\
&= Z^T(k)\mathcal{Q}Z(k) + V^T(k)\mathcal{R}V(k) \\
&= z^T(k)F_d^T\mathcal{Q}F_d z(k) + 2z^T(k)F_d^T\mathcal{Q}\Phi_d V(k) + V^T(k)(\Phi_d^T\mathcal{Q}\Phi_d + \mathcal{R})V(k)\,.
\end{aligned}
$$

where

$$
\mathcal{Q} = \mathrm{diag}[Q, Q, \cdots, Q, S], \qquad \mathcal{R} = \mathrm{diag}[R, R, \cdots, R],
$$

and the matrix P is the unique solution of the discrete-time Lyapunov equation

$$
S - (A^T - C^T H^T)^T S (A^T - C^T H^T) = Q + HRH^T,
$$

where $|\,\mathrm{eig}(A - HC)\,| < 1$.

Observer in “unconstrained MPC framework” (cont’d)

- The optimization can be solved analytically by

$$\nabla_V \hat{J}\Big|_{V=V^*} = \frac{\partial \hat{J}}{\partial V}\Bigg|_{V=V^*} = 0.$$

- The MPC for the dual system can be calculated by

$$v^*(k) = -\left[I_{p\times p}\ 0\ \cdots\ 0\right](\Phi_d^T \mathcal{Q} \Phi_d + \mathcal{R})^{-1} \Phi_d^T \mathcal{Q} F_d z(k) = -H_{mpc}^T z(k)\,.$$

- Consequently, according to the dual principle, the observer gain can be designed by

$$H_{mpc} = \left\{\left[I_{p\times p}\ 0\ \cdots\ 0\right](\Phi_d^T \mathcal{Q} \Phi_d + \mathcal{R})^{-1} \Phi_d^T \mathcal{Q} F_d\right\}^T.$$

Example

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k), \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Calculate $$K_{mpc} = [1\ 0\ \cdots\ 0](\Phi^T \mathcal{Q} \Phi + \mathcal{R})^{-1} \Phi^T \mathcal{Q} F = [2.6167 \quad 12.9286].$$

Use the dual system to calculate the observer gain:

$$A^T = \begin{bmatrix} 1.1 & 0 \\ 2 & 0.95 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 & 0.079 \end{bmatrix}.$$

Select $H = [1.05 \quad 0.226]^T$ such that $|\mathrm{eig}(A^T - C^T H^T)| < 1$.

Solve the Lyapunov equation to get S

$$S - (A^T - C^T H^T)^T S (A^T - C^T H^T) = Q + HRH^T, \qquad S = \begin{bmatrix} 11.2788 & 3.1419 \\ 3.1419 & 2.378 \end{bmatrix}.$$

Example (cont'd)

Use S to calculate

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$$\mathcal{Q} = \mathrm{diag}[Q, Q, \cdots, Q, S],$$

such that

$$H_{mpc} = \left\{ [I_{p\times p} \; 0 \; \cdots \; 0](\Phi_d^T \mathcal{Q} \Phi_d + \mathcal{R})^{-1} \Phi_d^T \mathcal{Q} F_d \right\}^T = \begin{bmatrix} 1.8342 \\ 0.3571 \end{bmatrix}.$$

The output feedback unconstrained MPC is designed by

$$u(k) = -K_{mpc}\hat{x}(k) = -[2.6167 \quad 12.9286]\hat{x}(k),$$

where the estimated state is updated by

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + H_{mpc}(y - C\hat{x}) = A\hat{x}(k) + Bu(k) + \begin{bmatrix} 1.8342 \\ 0.3571 \end{bmatrix}(y - C\hat{x}).$$

Output feedback unconstrained MPC — Summary

- Simply replace the real states by estimated states.
- States are estimated by linear observer.
- Principle of separation applies to analyze the closed-loop stability.
- The observer gain can also be designed in the framework of unconstrained MPC.

Recall full-state feedback constrained MPC

- LTI plant

$$x(k+1) = Ax(k) + Bu(k), \qquad \textbf{s.t.} \quad Gx(k) + Hu(k) \leq \mathbf{1}$$

- Model-based prediction

$$X(k) = Fx(k) + \Phi U(k)$$

where

$$F = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \qquad \Phi = \begin{bmatrix} B & 0 & & \\ AB & B & 0 & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}.$$

$$X(k) \triangleq [x^T(1\,|\,k), x^T(2\,|\,k), \cdots, x^T(N\,|\,k)]^T, \qquad U(k) \triangleq [u^T(0\,|\,k), u^T(1\,|\,k), \cdots, u^T(N-1\,|\,k)]^T.$$

Recall full-state feedback constrained MPC (cont'd)

- Cost function with terminal cost (infinite horizon)

$$
\begin{aligned}
J(k) &= \|x(N+1\,|\,k)\|_P^2 + \|u(N\,|\,k)\|_R^2 + \sum_{i=1}^{N} \|x(i\,|\,k)\|_Q^2 + \|u(i-1\,|\,k)\|_R^2 \\
&= X^T(k)\mathcal{Q}X(k) + U^T(k)\mathcal{R}U(k) \\
&= (Fx(k) + \Phi U(k))^T\mathcal{Q}(Fx(k) + \Phi U(k)) + U^T(k)\mathcal{R}U(k) \\
&= x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k)\,.
\end{aligned}
$$

where

$$
\mathcal{Q} = \mathrm{diag}[Q, Q, \cdots, Q, P], \qquad \mathcal{R} = \mathrm{diag}[R, R, \cdots, \cdots, R],
$$

and the matrix P is the unique solution of the discrete-time Lyapunov equation:

$$
P - (A - BK)^T P(A - BK) = Q - K^T RK
$$

where K is chosen such that $|\mathrm{eig}(A - BK)| < 1$.

Recall full-state feedback constrained MPC (cont'd)

- Constrained optimization

$$U^*(k) = \arg\min_{U(k)} \left[x^T(k)F^T\mathcal{Q}Fx(k) + 2x^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k)\right],$$

$$\text{s.t.} \quad Gx(i\,|\,k) + Hu(i\,|\,k) \leq 1, \qquad \forall i = 0,1,\cdots,N-1,$$

$$x(N\,|\,k) \in \mathcal{X}_f \subset \Omega,$$

where Ω is an invariant set for $x(k+1) = (A - BK)x(k)$, and $K\Omega \subset \mathcal{U}$.

- Receding horizon implementation
  - At time k, implement the first control action.
  - At time k+1, repeat prediction, optimization and the first control action.

Output feedback constrained MPC

- Constrained optimization

$$U^*(k) = \arg\min_{U(k)} \left[\hat{x}^T(k)F^T\mathcal{Q}F\hat{x}(k) + 2\hat{x}^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k)\right],$$

$$\text{s.t.} \quad Gx(i|k) + Hu(i|k) \leq 1, \qquad \forall i = 0,1,\cdots,N-1,$$

$$x(N|k) \in \mathcal{X}_f \subset \Omega,$$

where Ω is an invariant set for $x(k+1) = (A - BK)x(k)$, and $K\Omega \subset \mathcal{U}$.

- For implementation purpose, the above constraints are guaranteed conservatively by

$$Gx(i|k) + Hu(i|k) \leq \mathbf{1} \quad \Leftarrow \quad G\hat{x}(i|k) + Gu(i|k) \leq \mathbf{1} - \max[G\tilde{x}(i|k)]$$

$$x(N|k) \in \mathcal{X}_f \subset \Omega \quad \Leftarrow \quad \hat{x}(N|k) \in \mathcal{X}_f \ominus (A - HC)^N\mathcal{X}_0$$

where it is assumed that $\tilde{x}(k) \in \mathcal{X}_0$.

Output feedback constrained MPC — estimation and prediction

- The state is estimated by linear observer

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + H(y - C\hat{x})$$

where $|\mathrm{eig}(A - HC)| < 1$, such that

$$\tilde{x}(k+1) = (A - HC)\tilde{x}(k)$$

is globally exponentially stable.

- State sequence is predicted based on "current estimated state" (as indicated by the foregoing cost function)

$$\hat{x}(i+1 \mid k) = A\hat{x}(i \mid k) + Bu(i \mid k), \quad \text{for} \quad i = 0,1,\cdots,N-1.$$

leading to some "predictive error" (which is the function of $\tilde{x}(k)$).

Output feedback constrained MPC — closed-loop stability

- It has been proved that, with full-state feedback MPC, the closed-loop system is (at least locally) exponentially stable.
- Estimation error of the observer is globally exponentially stable.
- It then follows that, the principle of separation applies,
- The constrained MPC and the observer can be designed independently, such that the closed-loop system is (locally) asymptotically stable.
- Moreover, if the full-state feedback MPC achieves global exponential stability, then the closed-loop system is input-to-state stable with respect to the estimated error, and the closed-loop system is globally asymptotically stable.

Example

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k), \qquad A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

subject to control constraint: $-4 \leq u \leq 4$

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We have calculate that, the state can be estimated by the linear observer

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + H_{mpc}(y - C\hat{x}) = A\hat{x}(k) + Bu(k) + \begin{bmatrix} 1.8342 \\ 0.3571 \end{bmatrix}(y - C\hat{x}).$$

The constrained optimization can be constructed by

$$U^*(k) = \arg\min_{U(k)} \left[\hat{x}^T(k)F^T\mathcal{Q}F\hat{x}(k) + 2\hat{x}^T(k)F^T\mathcal{Q}\Phi U(k) + U^T(k)(\Phi^T\mathcal{Q}\Phi + \mathcal{R})U(k)\right],$$

subject to $-4 \leq u(i\,|\,k) \leq 4, \quad$ for $\; i = 0,1,\cdots,N-1$

$$\|\hat{x}(N\,|\,k)\| \leq 0.1$$

We do not have to find the contracted terminal set explicitly.
A sufficiently small terminal set is fine

Output feedback constrained MPC with predictive error feedback

- Constrained optimization

$$U^*(k) = \arg\min_{U(k)} \left[z^T(k) F^T \mathcal{Q} F z(k) + 2 z^T(k) F^T \mathcal{Q} \Phi V(k) + V^T(k)(\Phi^T \mathcal{Q} \Phi + \mathcal{R}) V(k) \right],$$

$$\text{s.t.} \quad Gx(i\,|\,k) + Hu(i\,|\,k) \le 1, \qquad \forall i = 0,1,\cdots,N-1,$$

$$x(N\,|\,k) \in \mathcal{X}_f \subset \Omega,$$

where Ω is an invariant set for $x(k+1) = (A - BK)x(k)$, and $K\Omega \subset \mathcal{U}$.

- State sequence is predicted based on "current estimated state" (as indicated by the foregoing cost function)

$$z(i+1\,|\,k) = Az(i\,|\,k) + Bv(i\,|\,k), \quad \text{for} \quad i = 0,1,\cdots,N-1.$$

$$z(0\,|\,k) = \hat{x}(k)$$

leading to some "predictive error" (which is the function of $\tilde{x}(k)$).

Output feedback constrained MPC with predictive error feedback (cont'd)

- The state is estimated by linear observer

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + H(y - C\hat{x}) = A\hat{x}(k) + Bu(k) + HC\tilde{x}$$

  where $|\mathrm{eig}(A - HC)| < 1$, such that

$$\tilde{x}(k+1) = (A - HC)\tilde{x}(k)$$

  is globally exponentially stable.

- Define $e = \hat{x} - z$, and $u(k)$ is designed by

$$u(k) = v(k) - K_e e$$

  where K_e is designed such that $|\mathrm{eig}(A - BK_e)| < 1$.

- Contracted constraints have to be used to guarantee the constraint satisfaction.

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Summary

- Output feedback unconstrained MPC
  - Principle of separation
    - The MPC linear feedback gain and the observer can be design independently.
    - Use the estimated state to calculate the linear feedback control.
  - The observer gain can also be calculated within the MPC framework (with dual principle)
- Output feedback constrained MPC
  - Principle of separation
  - Contracted constraints to conservatively guarantee the original constraints.