Nonlinear Control Theory

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Lyapunov Stability

Linear Time-varying Systems

Linear time-varying systems:

$$\dot{x} = A(t)x.$$

Its solution:

$$x(t) = \Phi(t, t_0)x(t_0),$$

where $\Phi(t, t_0)$ is the state transition matrix.

Theorem

The equilibrium point $x = 0$ of $\dot{x} = A(t)x$ is (globally) uniformly asymptotically stable, if and only if the state transition matrix satisfies the inequality

$$\|\Phi(t, t_0)\| \leq ke^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 \geq 0,$$

for some positive constants k and λ.

- For linear systems, uniform asymptotic stability $\Leftrightarrow$ exponential stability
- Eigenvalue criterion for linear time invariant systems is not applicable for linear time-varying systems

Example

Consider the LTV system with

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix}.$$

For each t, $\lambda[A(t)] = -0.25 \pm 0.25\sqrt{7}j$. The eigenvalues are independent of t, and have **negative real parts**. However, the origin is **unstable**, since its transition matrix is calculated by

$$\Phi(t, 0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix}.$$

Theorem

Let $x = 0$ be an exponential stable equilibrium point of $\dot{x} = A(t)x$. Suppose $A(t)$ is continuous and bounded. Let $Q(t)$ be a continuous, bounded, positive definite, symmetric matrix. Then, there is a continuously differentiable, bounded, positive definite, symmetric matrix $P(t)$ satisfying

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t).$$

Hence, $V(t, x) = x^T P(t)x$ is a Lyapunov function satisfying the theorem of exponential stability.

Linearization

Non-autonomous system:

$$\dot{x} = f(t, x),$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, and $D = \{\|x\|_2 < r\}$.

Suppose that

- $x = 0$ is an equilibrium point at $t = 0$, that is $f(t, 0) = 0$, $\forall t \geq 0$.
- The Jacobian matrix is bounded and Lipschitz on D, uniformly in t, thus,

$$\left\| \frac{\partial f_i}{\partial x}(t, x_1) - \frac{\partial f_i}{\partial x}(t, x_2) \right\|_2 \leq L_1 \|x_1 - x_2\|_2, \quad \forall x_1, x_2 \in D, \ \forall t \geq 0, \ \forall 1 \leq i \leq n.$$

- By mean value theorem, $f_i(t,x) = f_i(t,0) + \frac{\partial f_i}{\partial x}\Big|_{x=z} x$, where $z \in B_x = \{\|z\| \leq x\}$.
- $\dot{x} = A(t)x + g(t,x)$, where $A = \frac{\partial f}{\partial x}\big|_{x=0}$, and $g_i(t,x) = \frac{\partial f_i}{\partial x}\Big|_{x=z} x - \frac{\partial f_i}{\partial x}\Big|_{x=0} x$.
- The function $g(t,x)$ satisfies

$$\|g(t,x)\|_2 \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial f_i}{\partial x}\Big|_{x=z} - \frac{\partial f_i}{\partial x}\Big|_{x=0} \right\|_2^2 \right)^{\frac{1}{2}} \|x\|_2 \leq L\|x\|_2^2,$$

  where $L = \sqrt{n}L_1$.

Theorefore, in a small neighborhood of the origin, we may approximate $\dot{x} = f(t,x)$ by its linearization about the origin.

Theorem (4.13 Lyapunov's indirect method for non-autonomous systems)

Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = f(t, x)$, where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, and $D = \{\|x\|_2 < r\}$, and the Jacobian matrix $\frac{\partial f}{\partial x}(t, x)$ is bounded and Lipschitz on D, uniformly in t. Let

$$A = \frac{\partial f}{\partial x}(t, x)\bigg|_{x=0}.$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x.$$

Proof:

- If the linear time-varying system is exponentially stable, and $A(t)$ is continuous and bounded, then there exists a continuously differentiable, bounded, positive definite symmetric matrix $P(t)$ satisfying

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t),$$

  where $Q(t)$ is continuous, positive definite, and symmetric.

- Use $V(t,x) = x^T P(t)x$ as Lyapunov candidate,

$$\begin{aligned}\dot{V}(t,x) =& x^T P(t)f(t,x) + f^T(t,x)P(t)x + x^T\dot{P}(t)x \\ =& x^T\left[P(t)A(t) + A^T(t)P(t) + \dot{P}(t)\right]x + 2x^T P(t)g(t,x) \\ =& -x^T Q(t)x + 2x^T P(t)g(t,x).\end{aligned}$$

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\begin{aligned}
\dot{V}(t,x) =& x^T P(t) f(t,x) + f^T(t,x) P(t) x + x^T \dot{P}(t) x \\
=& x^T \left[P(t)A(t) + A^T(t)P(t) + \dot{P}(t) \right] x + 2x^T P(t) g(t,x) \\
=& -x^T Q(t) x + 2x^T P(t) g(t,x) \\
\leq& -c_3 \|x\|_2^2 + 2c_2 L \|x\|_2^3 \\
\leq& -(c_3 - 2c_2 L\rho)\|x\|_2^2, \qquad \forall \|x\|_2 < \rho.
\end{aligned}
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Choosing $\rho < \min\{r, \frac{c_3}{2c_2L}\}$ ensures that $V(t,x)$ is negative definite in $\|x\|_2 < \rho$. Therefore, it is conclude that the origin is exponentially stable.