# 现代控制理论

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# 2.2 状态反馈自适应控制

## 2.2.1问题描述

考虑一类可转化为如下参数严反馈形式的系统：

$$
\begin{aligned}
\dot{x}_i &= x_{i+1} + \theta^T \varphi_i(\bar{x}_i), i = 1, \cdots, \rho - 1 \\
\dot{x}_\rho &= \varphi_0(x, \xi) + \theta^T \varphi_\rho(x, \xi) + b\eta(x, \xi)u \\
\dot{\xi} &= \Psi(x, \xi) + \theta^T \Phi(x, \xi) \\
y &= x_1
\end{aligned}
\tag{2.2.1}
$$

其中 $x = \left[x_1, \cdots, x_\rho\right]^T \in \mathbb{R}^\rho$ 和 $\xi \in \mathbb{R}^{n-\rho}$ 为可测状态， $\bar{x}_i = [x_1, \cdots, x_i]^T, u \in \mathbb{R}$和$y \in \mathbb{R}$分别为输入和输出， $\varphi_0(x, \xi) \in \mathbb{R}$、$\eta(x, \xi) \in \mathbb{R}$、 $\varphi_i(\bar{x}_i) \in \mathbb{R}^r$ 和 $\varphi_\rho(x, \xi) \in \mathbb{R}^r$ 为已知光滑函数，$b \in \mathbb{R}$和$\theta \in \mathbb{R}^r$为未知常数。

# 2.2 状态反馈自适应控制

## □ 控制目的

在全状态可测的条件下，设计控制信号$u$使得

- 闭环系统内所有信号有界；
- 被控对象输出$y(t)$跟踪给定的期望轨迹$y_d(t)$。

## □ 假设

- 假设1： $b\eta(x,\xi) \neq 0$，且$b$的符号已知。
- 假设2：子系统$\dot{\xi} = \Psi(x,\xi) + \theta^T\Phi(x,\xi)$输入到状态稳定，其中$x$视为输入。
- 假设3： $y_d(t)$及其前$\rho$阶导数已知且有界。

## 2.2.2 控制器设计

第1步：根据式(2.2.1)，跟踪误差$z_1 = y - y_d$的导数可表示为

$$\dot{z}_1 = x_2 + \theta^T \omega_1 - \dot{y}_d \tag{2.2.2}$$

其中$\omega_1 = \varphi_1(x_1)$。定义第一个准Lyapunov函数：

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \tag{2.2.3}$$

其中正定对称矩阵$\Gamma \in \mathbb{R}^{r \times r}$为设计参数，$\tilde{\theta} := \hat{\theta} - \theta$， $\hat{\theta}$为$\theta$的估计。微分式(2.2.3)有

$$\dot{V}_1 = z_1(x_2 + \theta^T \omega_1 - \dot{y}_d) + \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \tag{2.2.4}$$

定义

$$z_2 = x_2 - \alpha_1 \tag{2.2.5}$$

其中$\alpha_1$为第1个待设计的**镇定函数**。

将式(2.2.5)代入式(2.2.4)得

$$\dot{V}_1 = z_1(z_2 + \alpha_1 + \theta^T \omega_1 - \dot{y}_d) + \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \tag{2.2.6}$$

选取

$$\alpha_1 = -c_1 z_1 - \hat{\theta}^T \omega_1 + \dot{y}_d \tag{2.2.7}$$

其中$c_1 > 0$为设计参数。于是有

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 + \tilde{\theta}^T \Gamma^{-1} \left( \dot{\hat{\theta}} - \Gamma \omega_1 z_1 \right) \tag{2.2.8}$$

针对$\hat{\theta}$，定义第1个**调节函数**(记为$\tau_1$)如下：

$$\tau_1 = \Gamma \omega_1 z_1 \tag{2.2.9}$$

然后可得

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 + \tilde{\theta}^T \Gamma^{-1} \left( \dot{\hat{\theta}} - \tau_1 \right) \tag{2.2.10}$$

第2步：注意到$\alpha_1$是$\left(x_1, y_d, \dot{y}_d, \hat{\theta}\right)$的光滑函数, $z_2 = x_2 - \alpha_1$的导数可表示为

$$\dot{z}_2 = x_3 + \theta^T \omega_2 + \beta_2 - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \tag{2.2.11}$$

其 中 $\omega_2 = \varphi_2(\bar{x}_2) - \frac{\partial \alpha_1}{\partial x_1} \varphi_1(x_1)$ ， $\beta_2 = -\frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial y_d} \dot{y}_d - \frac{\partial \alpha_1}{\partial \dot{y}_d} \ddot{y}_d$。选取第2个准Lyapunov函数

$$V_2 = V_1 + \frac{1}{2} z_2^2 \tag{2.2.12}$$

# 2.2 状态反馈自适应控制

根据式(2.2.10)-(2.2.12)可以证明

$$\dot{V}_2 = -c_1 z_1^2 + z_2\left(z_1 + x_3 + \theta^T\omega_2 + \beta_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) + \tilde{\theta}^T\Gamma^{-1}\left(\dot{\hat{\theta}} - \tau_1\right) \tag{2.2.13}$$

定义

$$z_3 = x_3 - \alpha_2 \tag{2.2.14}$$

其中$\alpha_2$为第2个待设计的镇定函数。

# 2.2 状态反馈自适应控制

将式(2.2.14)代入式(2.2.13)并用$\hat{\theta}-\tilde{\theta}$替代$\theta$，可得

$$\dot{V}_2 = -c_1 z_1^2 + z_2\left(z_1 + z_3 + \alpha_2 + \hat{\theta}^T\omega_2 + \beta_2 - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}\right) + \tilde{\theta}^T \Gamma^{-1}\left(\dot{\hat{\theta}} - \tau_1 - \Gamma\omega_2 z_2\right) \tag{2.2.15}$$

选取**镇定函数**$\alpha_2$和**调节函数**$\tau_2$如下：

$$\alpha_2 = -c_2 z_2 - z_1 - \hat{\theta}^T\omega_2 - \beta_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}}\tau_2 \tag{2.2.16}$$

$$\tau_2 = \tau_1 + \Gamma\omega_2 z_2 \tag{2.2.17}$$

其中$c_2>0$为设计参数。则有

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}\left(\tau_2 - \dot{\hat{\theta}}\right) + \tilde{\theta}^T \Gamma^{-1}\left(\dot{\hat{\theta}} - \tau_2\right) \tag{2.2.18}$$

## 2.2 状态反馈自适应控制

第$i$步$(3 \le i \le \rho - 1)$：注意到$\alpha_{i-1}$是$\left(\bar{x}_{i-1}, y_d, \dot{y}_d, \cdots, y_d^{(i-1)}, \hat{\theta}\right)$的光滑函数，$z_i = x_i - \alpha_{i-1}$的导数可表示为

$$\dot{z}_i = x_{i+1} + \theta^T \omega_i + \beta_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \tag{2.2.19}$$

其中 $\omega_i = \varphi_i(\bar{x}_i) - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k(\bar{x}_k)$， $\beta_i = -\sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} - \sum_{k=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(k)}} y_d^{(k+1)}$。选取第$i$个准Lyapunov函数

$$V_i = V_{i-1} + \frac{1}{2} z_i^2 \tag{2.2.20}$$

其中$V_{i-1}$ 的导数满足

$$\dot{V}_{i-1} = -\sum_{k=1}^{i-1} c_k z_k^2 + z_{i-1} z_i + \sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \left(\tau_{i-1} - \dot{\hat{\theta}}\right) + \tilde{\theta}^T \Gamma^{-1} \left(\dot{\hat{\theta}} - \tau_{i-1}\right) \tag{2.2.21}$$

## 2.2 状态反馈自适应控制

根据式(2.2.19)-(2.2.21)可以证明

$$\dot{V}_i = -\sum_{k=1}^{i-1} c_k z_k^2 + z_i\left(z_{i-1} + x_{i+1} + \theta^T\omega_i + \beta_i - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) + \sum_{k=2}^{i-1} z_k \frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\left(\tau_{i-1} - \dot{\hat{\theta}}\right) + \tilde{\theta}^T\Gamma^{-1}\left(\dot{\hat{\theta}} - \tau_{i-1}\right) \quad (2.2.22)$$

定义

$$z_{i+1} = x_{i+1} - \alpha_i \quad (2.2.23)$$

其中$\alpha_i$为第$i$个待设计的镇定函数。

## 2.2 状态反馈自适应控制

将式(2.2.23)代入式(2.2.22)并用$\hat{\theta}-\tilde{\theta}$替代$\theta$，有

$$\dot{V}_i=-\sum_{k=1}^{i-1}c_kz_k^2+z_i\left(z_{i-1}+z_{i+1}+\alpha_i+\hat{\theta}^T\omega_i+\beta_i-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right)$$
$$+\sum_{k=2}^{i-1}z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\left(\tau_{i-1}-\dot{\hat{\theta}}\right)+\tilde{\theta}^T\Gamma^{-1}\left(\dot{\hat{\theta}}-\tau_{i-1}-\Gamma\omega_iz_i\right)\quad(2.2.24)$$

令

$$\alpha_i=-c_iz_i-z_{i-1}-\hat{\theta}^T\omega_i-\beta_i+\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\tau_i+\sum_{k=2}^{i-1}z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\Gamma\omega_i\quad(2.2.25)$$

$$\tau_i=\tau_{i-1}+\Gamma\omega_iz_i\quad(2.2.26)$$

其中$c_i>0$为设计参数。则式(2.2.24)变为

$$\dot{V}_i=-\sum_{k=1}^{i}c_kz_k^2+z_iz_{i+1}+\sum_{k=2}^{i}z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\left(\tau_i-\dot{\hat{\theta}}\right)$$
$$+\tilde{\theta}^T\Gamma^{-1}\left(\dot{\hat{\theta}}-\tau_i\right)\quad(2.2.27)$$

## 2.2 状态反馈自适应控制

第$\rho$步：注意到$\alpha_{\rho-1}$是$\left(\bar{x}_{\rho-1}, y_d, \dot{y}_d, \cdots, y_d^{(\rho-1)}, \hat{\theta}\right)$的光滑函数，$z_\rho = x_\rho - \alpha_{\rho-1}$的导数可表示为

$$\dot{z}_\rho = b\eta(x,\xi)u + \theta^T\omega_\rho + \beta_\rho - \frac{\partial\alpha_{\rho-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{2.2.28}$$

其中 $\omega_\rho = \varphi_\rho(x,\xi) - \sum_{k=1}^{\rho-1}\frac{\partial\alpha_{\rho-1}}{\partial x_k}\varphi_k(\bar{x}_k)$，$\beta_\rho = \varphi_0(x,\xi) - \sum_{k=1}^{\rho-1}\frac{\partial\alpha_{\rho-1}}{\partial x_k}x_{k+1} - \sum_{k=0}^{\rho-1}\frac{\partial\alpha_{\rho-1}}{\partial y_d^{(k)}}y_d^{(k+1)}$。选取

$$V_\rho = V_{\rho-1} + \frac{1}{2}z_\rho^2 + \frac{|b|}{2\gamma}\tilde{p}^2 \tag{2.2.29}$$

其中$\tilde{p} := \hat{p} - p$，$\hat{p}$为$p = \frac{1}{b}$的估计，$\gamma > 0$为设计参数。

# 2.2 状态反馈自适应控制

根据式(2.2.29)和(2.2.28)并在式(2.2.27)中令$i = \rho - 1$，我们有

$$\dot{V}_\rho = -\sum_{k=1}^{\rho-1} c_k z_k^2 + z_\rho \left( z_{\rho-1} + b\eta(x,\xi)u + \theta^T \omega_\rho + \beta_\rho - \frac{\partial \alpha_{\rho-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right)$$
$$+ \sum_{k=2}^{\rho-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \left( \tau_{\rho-1} - \dot{\hat{\theta}} \right) + \tilde{\theta}^T \Gamma^{-1} \left( \dot{\hat{\theta}} - \tau_{\rho-1} \right) + \frac{|b|}{\gamma} \tilde{p} \dot{\hat{p}}$$
$$= -\sum_{k=1}^{\rho} c_k z_k^2 + z_\rho b\eta(x,\xi)u - z_\rho \bar{u} + \sum_{k=2}^{\rho} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \left( \tau_\rho - \dot{\hat{\theta}} \right)$$
$$+ \tilde{\theta}^T \Gamma^{-1} \left( \dot{\hat{\theta}} - \tau_\rho \right) + \frac{|b|}{\gamma} \tilde{p} \dot{\hat{p}} \quad (2.2.30)$$

其中

## 2.2 状态反馈自适应控制

$$\bar{u} = -c_\rho z_\rho - z_{\rho-1} - \hat{\theta}^T \omega_\rho - \beta_\rho + \frac{\partial \alpha_{\rho-1}}{\partial \hat{\theta}} \tau_\rho + \sum_{k=2}^{\rho-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \omega_\rho \quad (2.2.31)$$

$$\tau_\rho = \tau_{\rho-1} + \Gamma \omega_\rho z_\rho \quad (2.2.32)$$

$c_\rho > 0$为设计参数。根据式(2.2.30)，令

$$\dot{\hat{\theta}} = \tau_\rho \quad (2.2.33)$$

控制信号选取为

$$u = \frac{1}{\eta(x,\xi)} \hat{p} \bar{u} \quad (2.2.34)$$

其中$\hat{p}$对应的自适应律为

$$\dot{\hat{p}} = -\mathrm{sign}(b) \gamma z_\rho \bar{u} \quad (2.2.35)$$

将式(2.2.33)-(2.2.35)及关系式$b\hat{p}-b\tilde{p}=bp=1$代入式(2.2.30)可得

$$\dot{V}_\rho=-\sum_{k=1}^{\rho}c_k z_k^2 \tag{2.2.36}$$

## 2.2.3 稳定性分析

定理2.2：考虑由被控对象(2.2.1)、控制律(2.2.34)和自适应律(2.2.33)、(2.2.35)组成的闭环系统。假定假设1-3成立，则闭环系统内所有信号全局一致有界且$\lim_{t\to+\infty}[y(t)-y_d(t)]=0$。

证明：同时积分式(2.2.36)两端可得

$$V_\rho(t)-V_\rho(0)=-\sum_{k=1}^{\rho}\int_0^t c_k z_k^2(\tau)d\tau\,,\forall t\geq 0 \tag{2.2.37}$$

# 2.2 状态反馈自适应控制

由式(2.2.37)可知，$V_\rho$、$z_k$、$\hat{\theta}$和$\hat{p}$有界。利用式(2.2.7)、(2.2.14)、(2.2.16)、(2.2.23)和(2.2.25)可以证明，$\alpha_{k-1}(k=2,\cdots,\rho)$和$x$有界。结合$x$的有界性和假设2可得ξ有界,而根据式(2.2.31)和(2.2.34)可知，$\bar{u}$和控制信号$u$有界。由此可以推出闭环系统内所有信号全局一致有界。此外，由式(2.2.37)有$\int_0^{+\infty} z_k^2(\tau)d\tau \leq V_\rho(0)/c_k$，即$z_k \in L_2$；由式(2.2.2)、(2.2.11)、(2.2.19)和(2.2.28)有$\dot{z}_k \in L_\infty$。因此，根据Barbalat引理，$\lim\limits_{t\to+\infty} z_k(t)=0$，这意味着$\lim\limits_{t\to+\infty}[y(t)-y_d(t)]=0$，证毕。

**Barbalat引理**：对于信号 $h(t)$，若：1) $h(t)$ 有界；2) $\dot{h}(t)$ 有界；3) $\int_0^{+\infty} \| h(t) \|^2 dt$ 存在且有界，则有 $\lim\limits_{t\to+\infty} h(t)=0$ 。