Synchronization

Nonlinear Oscillators and Mobile Robots

Mathias Hudoba de Badyn

Advanced Topics in Control
May 9, 2022

Announcements

1. Homework 3 is due Friday (May 13th at 23:59)

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2. Project ideas should start being finalized

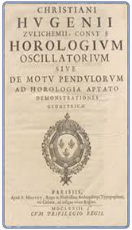
A brief history of synchronization

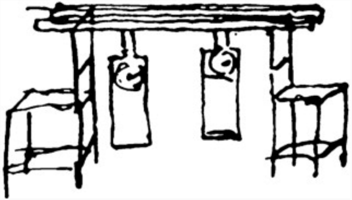
Christiaan Huygens (1629 – 1695)

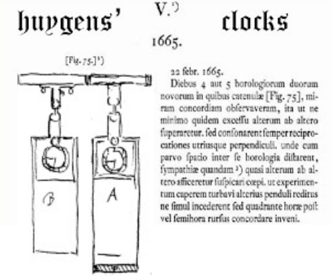
- physicist & mathematician
- engineer & horologist

observed “*an odd kind of sympathy*” between coupled & heterogeneous clocks

[Letter to Royal Society of London, 1665]







huygens' V.[1] clocks

1665.

[Fig. 75.][1]

22 febr. 1665.

Diebus 4 aut 5 horologiorum duorum novorum in quibus catenulæ [Fig. 75], miram concordiam obſervaveram, ita ut ne minimo quidem exceſſu alterum ab altero ſuperaretur. ſed conſonarent ſemper reciprocationes utriusque perpendiculi. unde cum parvo ſpatio inter ſe horologia diſtarent, ſympathiæ quandam [2]) quasi alterum ab altero afficeretur ſuſpicari cœpi. ut experimentum caperem turbavi alterius penduli reditus ne ſimul incederent ſed quadrante horæ poſt vel ſemihora rurſus concordare inveni.

Recent reviews, experiments, & analysis

[M. Bennet et al. ’02, M. Kapitaniak et al. ’12]

A brief history of synchronization

Sync of 32 metronomes at Ikeguchi Laboratory, Saitama University, 2012

`https://www.youtube.com/watch?v=5v5eBf2KwF8`

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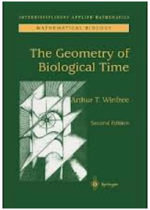

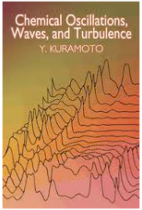

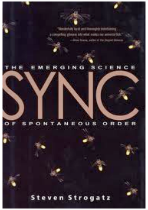

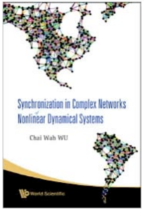

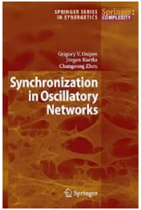

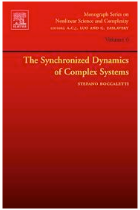

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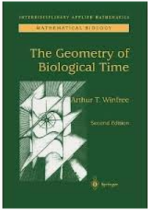

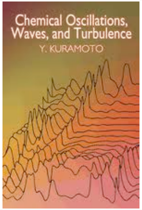

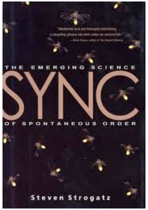

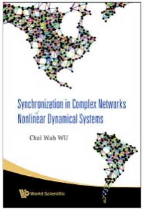

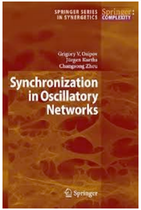

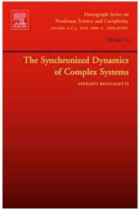

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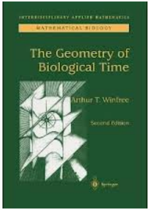

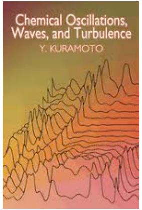

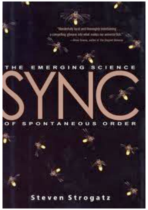

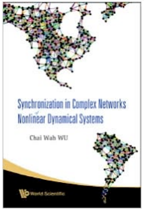

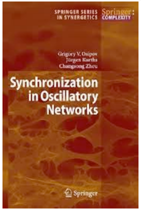

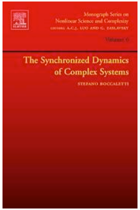

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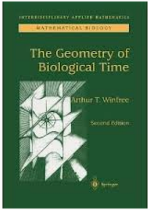

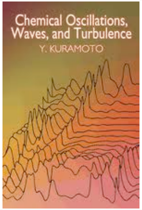

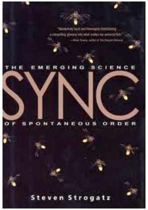

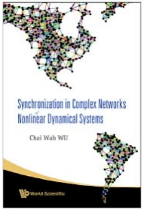

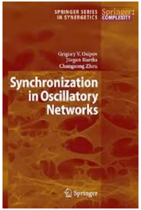

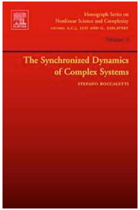

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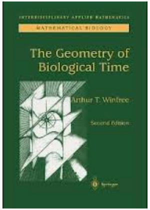
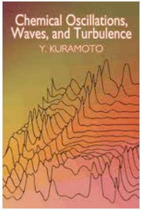
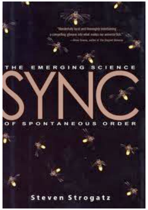
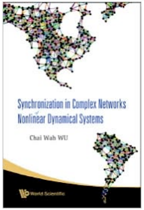
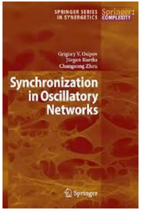
- ...and countless technological applications (reviewed later)

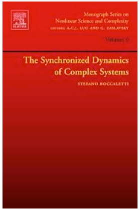






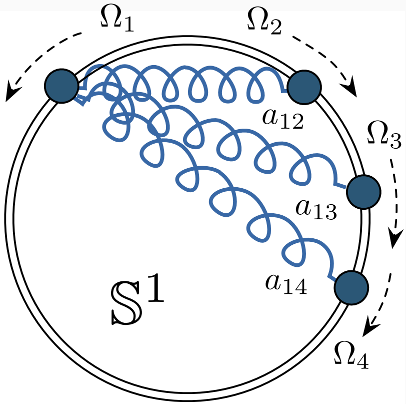






Example 1: mechanical spring network

- phase angles $\theta_i \in \mathbb{S}^1$



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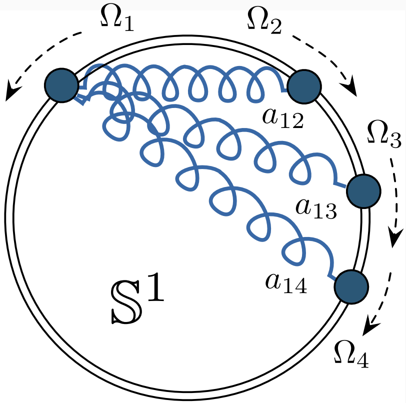
- phase angles $\theta_i \in \mathbb{S}^1$
- inertia constants $M_i > 0$



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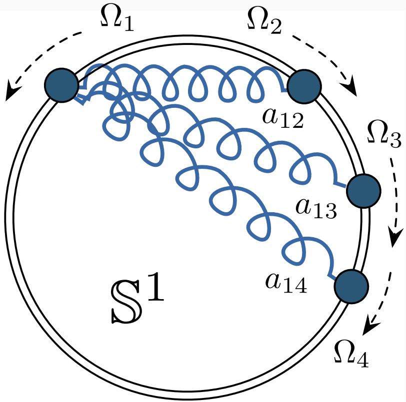

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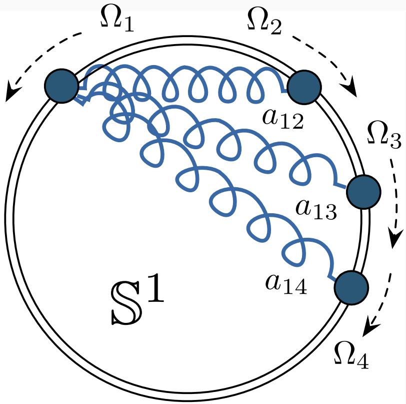

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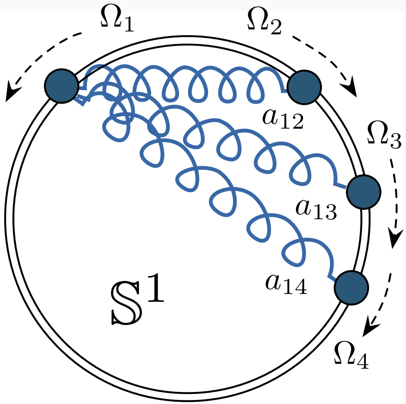

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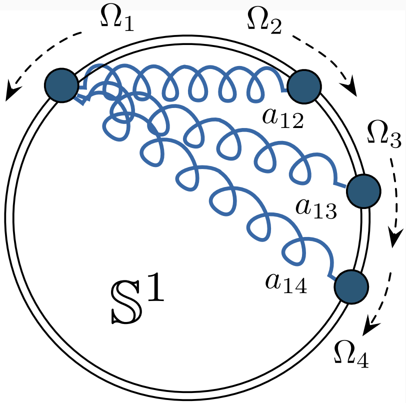
- spring network potential energy
  $U(\theta) = \sum_{\{i,j\}\in\mathcal{E}} a_{ij}\big(1 - \cos(\theta_i - \theta_j)\big)$



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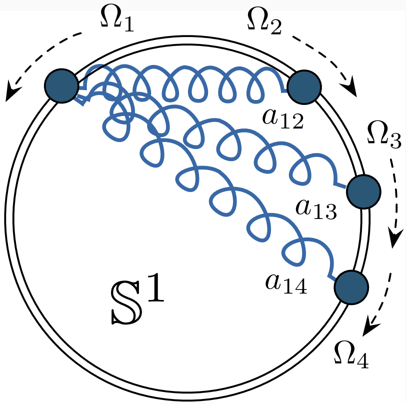
$\Rightarrow$ elastic torque $T_i^{\text{int}} = -\frac{\partial}{\partial \theta_i} U(\theta)$



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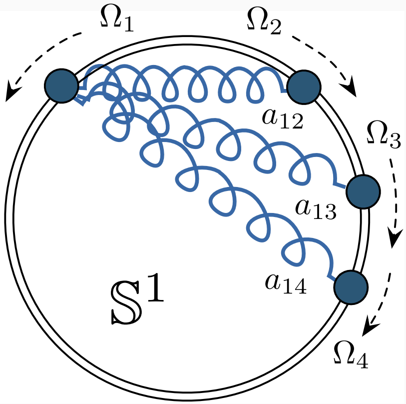

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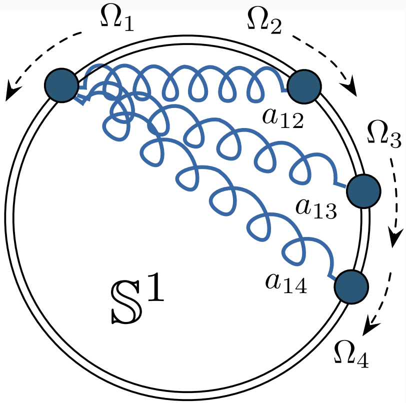


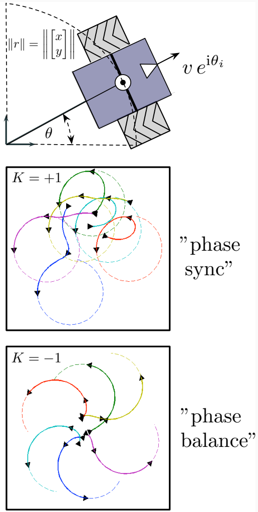
Particle dynamics:

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = \boldsymbol{\Omega}_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Example 2: flocking, schooling, & vehicle coordination

- Network of *Dubins' vehicles* with speed v and steering control $u_i(r, \theta)$

$$\dot{r}_i = v\, e^{\mathrm{i}\theta_i}$$
$$\dot{\theta}_i = u_i(r, \theta)$$



[R. Sepulchre et al. '07, D. Klein et al. '09, L Consolini et al '10]

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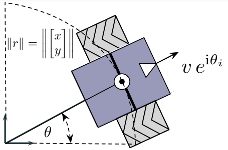

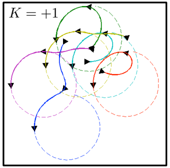

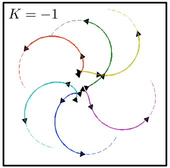

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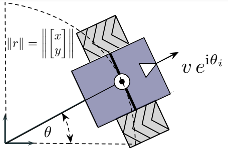
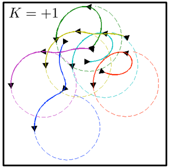
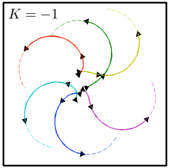
- *relative sensing control* $u_i = f_i(\theta_i - \theta_j)$ for neighbors $\{i, j\} \in \mathcal{E}$ yields closed-loop





"phase sync"



"phase balance"

[R. Sepulchre et al. '07, D. Klein et al. '09, L Consolini et al '10]

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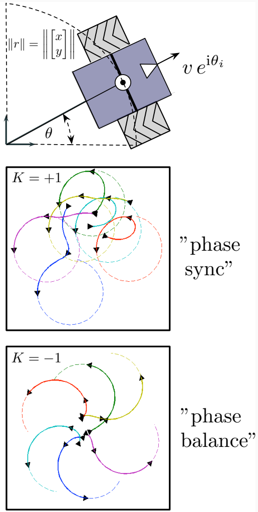
$$\dot{\theta}_i \;=\; \omega_0(t) - K\cdot\sum_j a_{ij}\sin(\theta_i - \theta_j)$$



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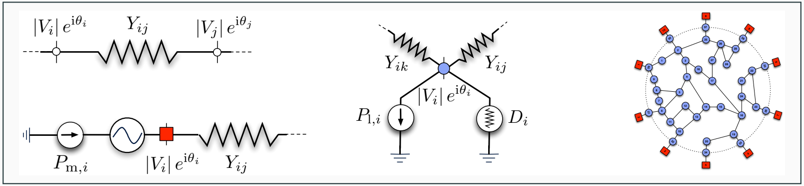
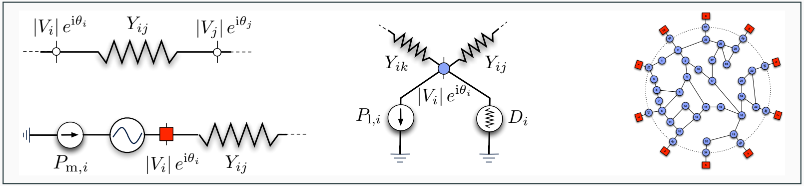


- power transfer on line $i \rightsquigarrow j$: $\underbrace{|V_i||V_j||Y_{ij}|}_{a_{ij} = \text{max power transfer}} \cdot \sin(\theta_i - \theta_j)$

■: swing eq with $P_{m,i} > 0$ $\qquad M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_{m,i} - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

●: $P_{l,i} < 0$ and $D_i \geq 0$ $\qquad D_i \dot{\theta}_i = P_{l,i} - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

Example 3: AC power transmission network



- power transfer on line $i \rightsquigarrow j$: $\underbrace{|V_i||V_j||Y_{ij}|}_{a_{ij}=\text{max power transfer}} \cdot \sin(\theta_i - \theta_j)$
- power balance at node i: $\underbrace{P_i}_{\text{power injection}} = \sum_j a_{ij} \sin(\theta_i - \theta_j)$

■: swing eq with $P_{m,i} > 0$ $\qquad M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_{m,i} - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

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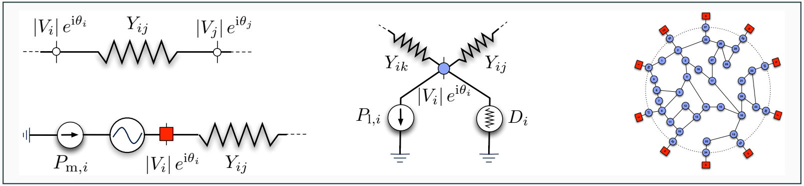

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- structure-preserving model [A. Bergen & D. Hill '81]:

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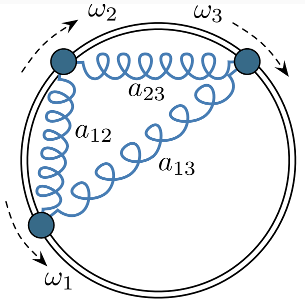
coupled phase oscillators

Canonical coupled phase oscillator model

Coupled oscillator model:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



- n **oscillators** with phase $\theta_i \in \mathbb{S}^1$

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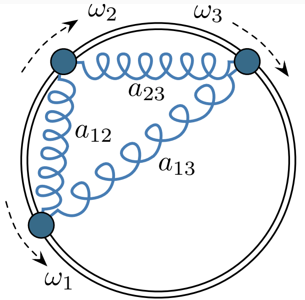

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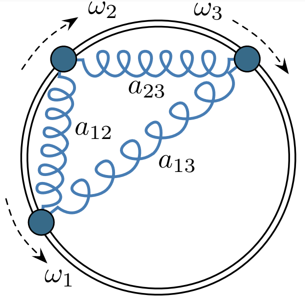

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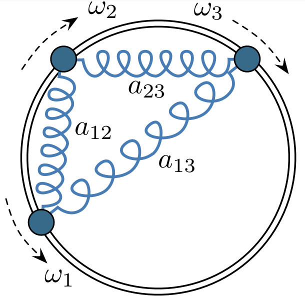

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- undirected & connected **graph** $G = (\mathcal{V}, \mathcal{E}, A)$

Applications of coupled phase oscillators in sciences

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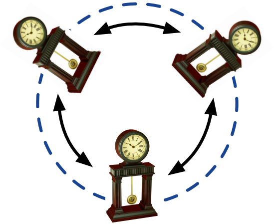
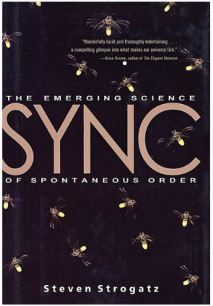


- Sync in a population of fireflies
  [G.B. Ermentrout '90, Y. Zhou et al. '06, ...]



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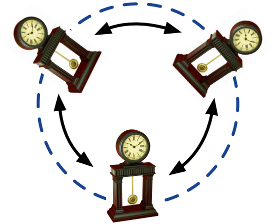

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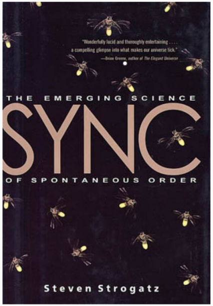

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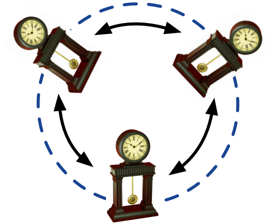

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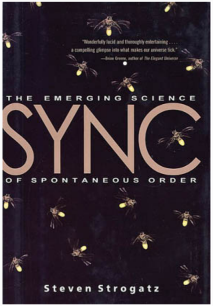

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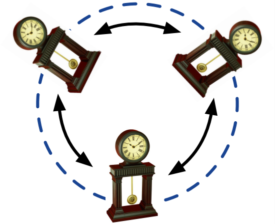

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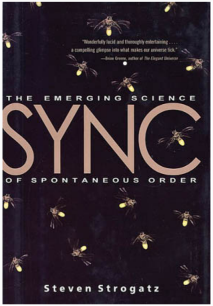
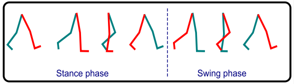
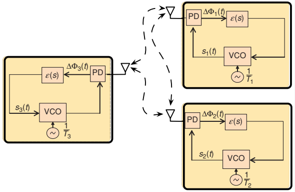
- Countless other sync phenomena in physics, biology, chemistry, social networks etc.
  [A. Winfree ’67, S.H. Strogatz ’00, J. Acebrón ’01, ...]

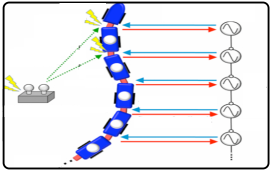


Technological applications of coupled phase oscillators

- Particle filtering to estimate limit cycles [A. Tilton & P. Mehta et al. ’12]







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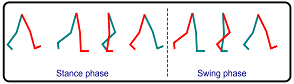

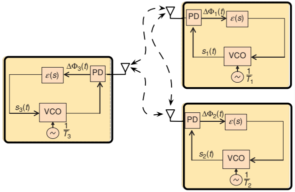

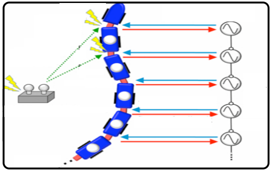

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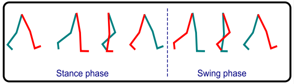

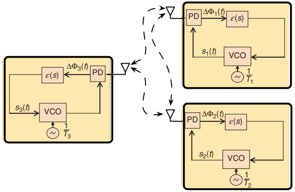

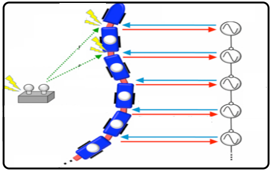

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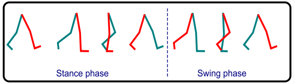

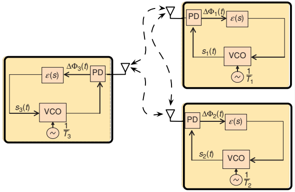

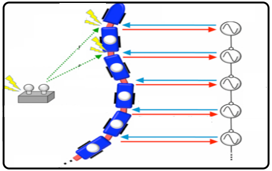

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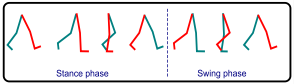
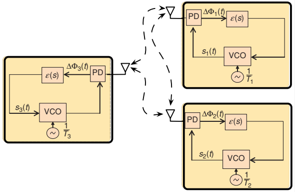
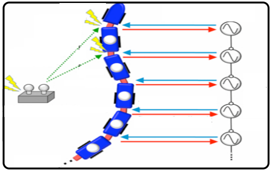
- Carrier sync without phase-locked loops [M. Rahman et al. '11]







Nanoelectromechanical Oscillators

Science

Exotic states in a simple network of nanoelectromechanical oscillators

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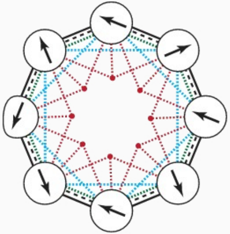

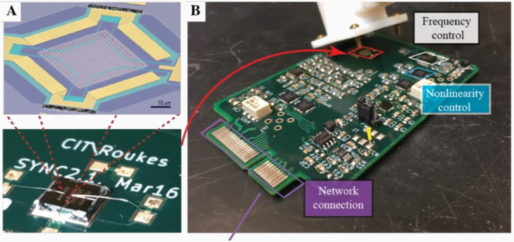

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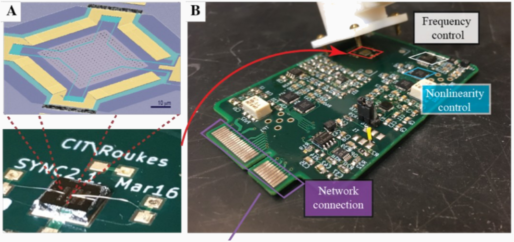
Topology influences the dynamics

- Symmetries describe stable states



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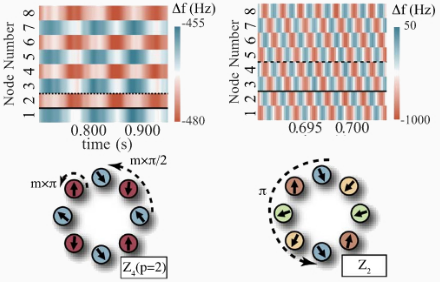

Subgroup	Subspace dimension	Generators	Phase pattern
D_8	1	σ, κ	$\{a, a, a, a, a, a, a, a\}$
$D_8(+, -)$	1	$(\kappa, 1), (\kappa\sigma, -1)$	$\{a, -a, a, -a, a, -a, a, -a\}$
$Z_8(p), p \in 1, 2, 3$	1	$\sigma\omega^p$	$\{a, \omega^p a, \omega^{2p} a, \omega^{3p} a, \omega^{4p} a, \omega^{5p} a, \omega^{6p} a, \omega^{7p} a\}$
$D_4(+, -)$	1	$(\sigma\kappa, 1), (\kappa\sigma, -1)$	$\{a, a, -a, -a, a, a, -a, -a\}$
$D_4(\kappa)$	2	$\sigma^2\kappa, \kappa$	$\{a, b, a, b, a, b, a, b\}$
$Z_4(p), p \in 1, 2$	2	$\sigma^2\omega^{2p}$	$\{a, b, i^p a, i^p b, i^{2p} a, i^{2p} b, i^{3p} a, i^{3p} b\}$
$D_2(\kappa)$	3	$\sigma^4\kappa, \kappa$	$\{a, b, c, b, a, b, c, b\}$
$D_1(\kappa)$	5	κ	$\{a, b, c, d, e, d, c, b\}$
$D_2(\kappa\sigma)$	2	$\sigma^3\kappa, \kappa\sigma$	$\{a, b, b, a, a, b, b, a\}$
$D_1(\kappa\sigma)$	4	$\sigma^7\kappa, \kappa\sigma$	$\{a, b, c, d, d, c, b, a\}$
$D_2(-, -)$	2	$(\sigma^3\kappa, -1), (\kappa\sigma, -1)$	$\{a, b, -b, -a, a, b, -b, -a\}$
$D_1(-, -)$	4	$(\sigma^7\kappa\ -1), (\kappa\sigma, -1)$	$\{a, b, c, d, -d, -c, -b, -a\}$
$D_2(+, -)$	2	$(\sigma^3\kappa, 1), (\kappa\sigma, -1)$	$\{a, b, b, a, -a, -b, -b, -a\}$
Z_2	4	σ^4	$\{a, b, c, d, a, b, c, d\}$
$Z_2(p = 1)$	4	$\sigma^4\omega^4$	$\{a, b, c, d, -a, -b, -c, -d\}$

preliminaries

Geometry of the circle $\mathbb{S}^1$ and the n-torus $\mathbb{T}^n$

- **Parametrization:** we parametrize the *unit circle* $\mathbb{S}^1$ by assuming
  (i) angles are measured counterclockwise; and
  (ii) angles take value in $[-\pi, \pi)$, and we associate $+\pi$ and $-\pi$.

  The *n-torus* is the cartesian product of n circles: $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

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- **Rigid rotation:** given the angle $\alpha \in [-\pi, \pi)$, the *rotation* of the n-tuple $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$ by α, denoted by $\mathrm{rot}_\alpha(\theta)$, is the counterclockwise rotation of each entry $(\theta_1, \ldots, \theta_n)$ by α.

Order parameter (for homogenous coupling $a_{ij} = K/n$)

Define the **order parameter** (centroid) by $\boxed{re^{\mathrm{i}\psi} = \frac{1}{n}\sum_{j=1}^{n} e^{\mathrm{i}\theta_i}}$

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Intuition: synchronization = entrainment by mean field $re^{i\psi}$

K small &
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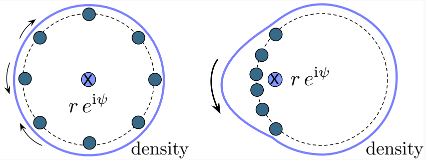

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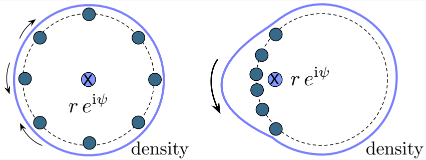

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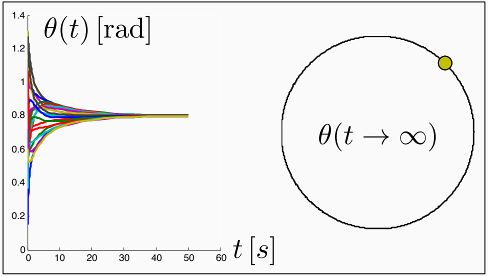
potential formulation in terms of order-parameter: $$U(\theta) = \frac{Kn}{2}\left(1 - r^2\right)$$

Phase synchronization ($r = 1$)



Theorem

Consider the coupled oscillator model with identical natural frequencies and a connected, undirected, and weighted graph. Then

1. **global convergence:** the oscillators converge to the set of equilibria; &

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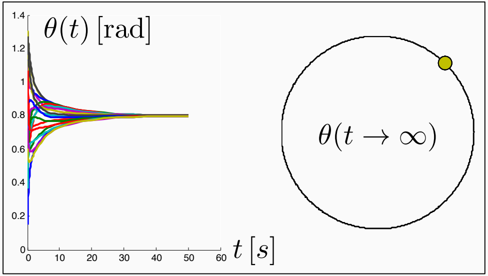

Theorem

Consider the coupled oscillator model with identical natural frequencies and a connected, undirected, and weighted graph. Then

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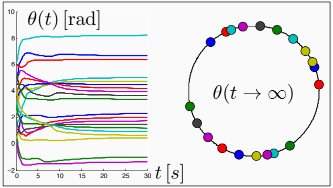
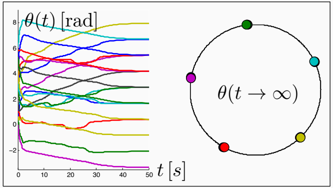
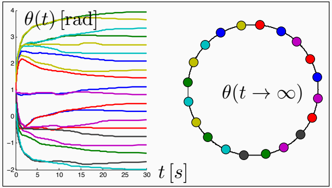
2. **local phase sync:** phase synchronization is a locally stable equilibrium.

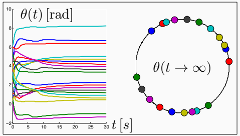
Phase balancing ($r = 0$)

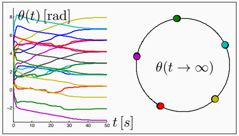


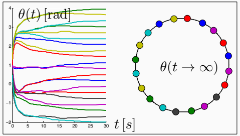




Phase balancing







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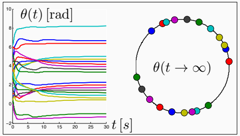

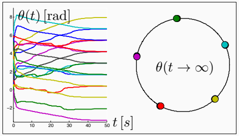

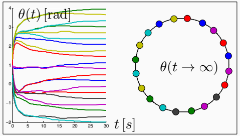

Theorem:

Consider the coupled oscillator model with identical natural frequencies and a connected, undirected, and weighted graph. Then

1. **global convergence:** the oscillators converge to the set of equilibria; &
2. **local phase balancing:** for a complete graph with homogeneous weights $a_{ij} = K/n$, phase balancing is locally asymptotically stable.

formation control

part I: specifying formations

Mobile robotics: Outline

1. Today: Specifying formations & rigidity

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    1.1 Lots of definitions and theorems

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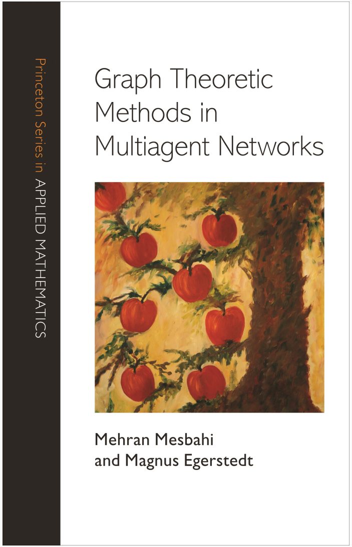
3. Next week: Control!

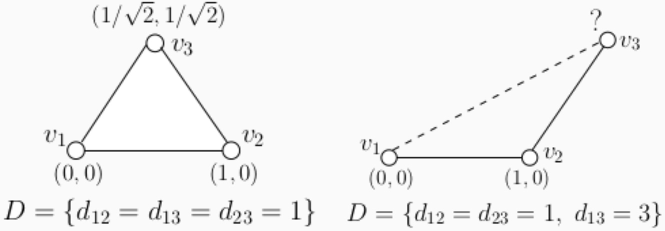

Princeton Series in APPLIED MATHEMATICS
Graph Theoretic Methods in Multiagent Networks
Mehran Mesbahi
and Magnus Egerstedt

Formation Specification: Shapes

Desired Interagent distances

$$D = \{d_{ij} \in \mathbb{R} \ : \ d_{ij} > 0; i, j = 1, \ldots, n, i \neq j\}$$

D must be **feasible:** there exist points $\Xi := \{\xi_1, \ldots, \xi_n \mid \xi_i \in \mathbb{R}^p\}$ such that $\|\xi_i - \xi_j\| = d_{ij}$,

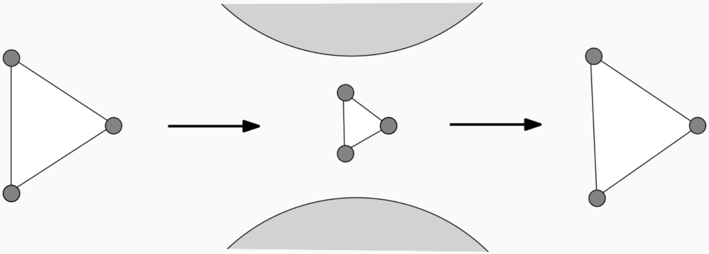


(a) Feasible formation (b) Infeasible formation

Formation Specification: Shapes

We refer to the **scale invariant** formation D as any set of distances D' such that $D' = \alpha D$ for any $\alpha \in \mathbb{R}_+$.

Used in 'cluttered' environments where strict interagent distances are not required.



Formation Specification: Shapes

Interagent distances

$$D = \{d_{ij} = d_{ji} \geq 0,\ i,j = 1,\ldots,n,\ i \neq j\}$$

formation	specification	interpretation
scale invariant	D	$\|x_i - x_j\| = \alpha d_{ij}$ for some $\alpha > 0$
rigid	D	$\|x_i - x_j\| = d_{ij}$
translational invariant	Ξ	$x_i = \xi_i + \tau$ for some $\tau \in \mathbf{R}^p$

formation control

part II: rigidity

Rigidity

Define a **formation graph** as $\mathcal{G}_f = (\mathcal{V}, \mathcal{E}, w)$, where

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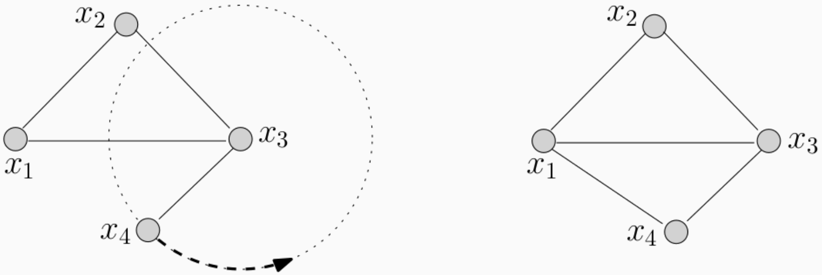
- **rigid**, if $\|x_i(t) - x_j(t)\|$ is constant for all $i \neq j$.

Rigidity

Definition

A framework $(\Xi, \mathcal{G}_f)$ is called **rigid** if and only if all edge-consistent trajectories of the framework are rigid trajectories.

Otherwise, the framework is called flexible.



Left: Flexible framework with edge-consistent trajectories.

Right: Rigid framework

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- Denote the velocities $\dot{x}_i =: u_i$ satisfying (1) as an **infinitesimal motion**

$$(\dot{x}_i(t) - \dot{x}_j(t))^T (x_i(t) - x_j(t)) = 0 \text{ for all } ij \in \mathcal{E} \qquad (1)$$

Let

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

be an infinitesimal motion applied at the points $\Xi := \{\xi_1, \ldots, \xi_n\}$.

We can write (1) in matrix form as,

$$R(\mathcal{G}(\Xi))u = 0,$$

where $R(\mathcal{G}(\Xi))$ is known as the **rigidity matrix**

Rigidity – Some final definitions

Definition

A framework $\mathcal{G}(\Xi)$ is **infinitesimally rigid** if $R(\mathcal{G}(\Xi))u = 0$

Some properties of $R(\mathcal{G}(\Xi))$:

- Has $|\mathcal{E}|$ rows
- Has pn columns (p is the state dimension)
- Is an algebraic representation of a graph plus the points Ξ

We want something that just depends on the graph

Theorem – Planar Case

A framework with $n \geq 2$ points in $\mathbb{R}^2$ is infinitesimally rigid if and only if $\mathrm{rank}[R(\mathcal{G}(\Xi))] = 2n - 3$

Theorem

Infinitesimal rigidity implies rigidity

We now have an algebraic characterization of rigidity

Rigid Graphs

Definition

A graph is **rigid** if there exists Ξ such that $\mathcal{G}(\Xi)$ is infinitesimally rigid

Theorem

If $\mathcal{G}$ is rigid, then the set of all Ξ such that $\mathcal{G}(\Xi)$ is infinitesimally rigid is a dense, open subset of $\mathbb{R}^{pn}$.

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Theorem

If $\mathcal{G}$ is rigid, then the set of all Ξ such that $\mathcal{G}(\Xi)$ is infinitesimally rigid is a dense, open subset of $\mathbb{R}^{pn}$.

1. Rigid graphs have a lot of feasible configurations!
2. Feasible configurations can be approximated arbitrarily well (remember what dense + open means)

Announcements

1. Synchronization and applications

Announcements

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2. Rigidity of graphs

Announcements

1. Synchronization and applications
2. Rigidity of graphs
3. Next time: Formation specification via relative states & control