Nonlinear Control Theory

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Lyapunov Stability

Boundedness and Ultimate Boundedness

Example

Consider the scalar non-autonomous system

$$\dot{x} = -x + \delta \sin t, \quad x(t_0) = a, \quad a > \delta > 0.$$

It can be regarded as a linear control system $\dot{x} = -x + u$ with $u = \delta \sin t$. Its solution can be given by

$$x(t) = e^{-t+t_0} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau \mathrm{d}\tau,$$

which is bounded by

$$\|x(t)\| \leq e^{-t+t_0} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \mathrm{d}\tau = e^{-t+t_0} a + \delta \left[1 - e^{-t+t_0}\right] \leq a.$$

Example (Cont'd)

- The solution is bounded for all $t \geq t_0$, uniformly in t_0.
- The bound $\|x(t)\| \leq a$ is conservative, since the exponential decaying term is not considered.
- For any number b satisfying $\delta < b < a$, it holds that

$$\|x(t)\| \leq b, \quad \forall t \geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right).$$

- The bound b, independent of t_0, is a better estimation of the bound than a after the transient process.
- The solution is uniformly bounded by a, and is uniformly ultimately bounded by b.

Example (Cont'd)

Boundedness and ultimate boundedness can be analysed via **Lyapunov function.**

- Select $V(x) = \frac{1}{2}x^2$ which is positive definite and decrescent, its derivative along the solution is calculated by

$$\dot{V} = x\dot{x} = -x^2 + \delta x \sin t \leq -x^2 + \delta|x| < 0, \quad \forall |x| > \delta.$$

- With $c > \frac{\delta^2}{2}$, solution starting from $\{V(x) \leq c\}$ will remain therein for all future time, indicating that it is uniformly bounded.
- Pick any number ϵ such that $\frac{\delta^2}{2} < \epsilon < c$. Then $\dot{V} < 0$ in $\{\epsilon \leq V \leq c\}$, implying that the solution will ultimately enter $\{V = \frac{1}{2}x^2 \leq \epsilon\}$.
- It is concluded that the solution is ultimately bounded by $|x| \leq \sqrt{2\epsilon}$.

Consider the nonlinear system $\dot{x} = f(t, x)$, where $f : [0, \infty) \times D \to R^n$ is piece-wise continuous in t and Lipschitz in x, and $D \subset R^n$ contains the origin.

Definition

The solutions of $\dot{x} = f(t, x)$ are

- **uniformly bounded**, if there exists $c > 0$, independent of t_0, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, but independent of t_0, such that

$$\|x(t_0)\| \leq a \quad \Rightarrow \quad \|x(t)\| \leq \beta, \ \ \forall t \geq t_0.$$

- **globally uniformly bounded**, if the foregoing inequalities hold for arbitrarily large a.

Definition (Cont'd)

The solutions of $\dot{x} = f(t, x)$ are

- **uniformly ultimately bounded** with ultimate bound b, if there exists $c > 0$, independent of t_0, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0, such that
$$\|x(t_0)\| \leq a \quad \Rightarrow \quad \|x(t)\| \leq b, \;\; \forall t \geq t_0 + T.$$
- **globally uniformly ultimately bounded** if the foregoing inequalities hold for arbitrarily large a.

Remark

Drop "uniformly" if $\dot{x} = f(x)$.

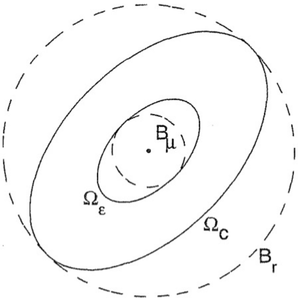
Lyapunov analysis on ultimate boundedness:

- Let $V(x)$ be a cont. diff. positive definite function, and suppose the sets

$$\begin{aligned}\Omega_c =& \{V(x) \le c\},\ \Omega_\epsilon = \{V(x) \le \epsilon\},\\ \Lambda =& \{\epsilon \le V(x) \le c\}\end{aligned}$$

  are compact for some $c > \epsilon > 0$.

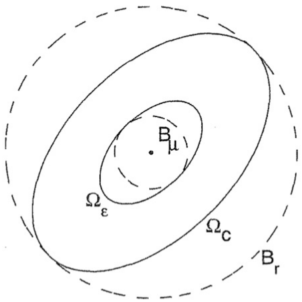
- Suppose $\dot{V}(t,x) \le -W_3(x)$ for all $x \in \Lambda$ and all $t \ge t_0$, where $W(x)$ is continuous and positive definite.



- The sets Ω_c and Ω_ϵ are positively invariant, since $\dot{V}$ is negative definite on boundaries $\partial\Omega_c$ and $\partial\Omega_\epsilon$.
- Set $k = \min_{x\in\Lambda} W_3(x) > 0$, then

$$\dot{V}(t,x) \le -k, \quad \forall x \in \Lambda,\ \forall t \ge t_0 \ge 0,$$
$$V(t,x) \le V(t_0, x(t_0)) - k(t-t_0) \le c - k(t-t_0),$$

implying that $x(t)$ enters Ω_ϵ within finite time interval $[t_0, t_0 + \frac{c-\epsilon}{k}]$.

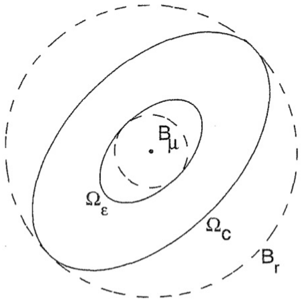


- Suppose that $\dot{V}(t,x) \le -W_3(x)$ holds for all $\mu \le \|x\| \le r$ and $t \ge t_0 \ge 0$, where r is arbitrarily larger than μ.
- Let α_1 and α_2 be class $\mathcal{K}$ functions such that

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|).$$

- Choose $c = \alpha_1(r)$ to guarantee $\Omega_c \subset B_r$.
  Choose $\epsilon = \alpha_2(\mu)$ to guarantee $B_\mu \subset \Omega_\epsilon$.
  Choose $\mu < \alpha_2^{-1}(\alpha_1(r))$ to guarantee $\epsilon < c$.
- The ultimate bound:

$$V \le \epsilon \ \Rightarrow \ \alpha_1(x) \le \epsilon \ \Leftrightarrow \|x\| \le \alpha^{-1}(\epsilon)$$
$$\Rightarrow \ \|x\| \le \alpha_1^{-1}(\alpha_2(\mu)).$$



Theorem

Suppose that $B_\mu \subset D \subset R^n$. Let $V : [0,\infty) \times D \to R$ be continuously differentiable and

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \;\; \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \;\; \forall \|x\| \geq \mu > 0, \; \forall t > 0,$$

where α_1 and α_2 are class $\mathcal{K}$ functions, and $W_3(x)$ is a continuous positive definite function. Take $\mu < \alpha_2^{-1}(\alpha_1(r))$ where $r > 0$. Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0) \leq \alpha_2^{-1}(\alpha_1(r))$, there is $T = T(x(t_0), \mu) \geq 0$, such that

$$\|x(t)\| \leq \beta\left(\|x(t_0)\|, t - t_0\right), \;\; \forall t_0 \leq t \leq t_0 + T,$$
$$\|x(t)\| \leq \alpha_1^{-1}\left(\alpha_2(\mu)\right), \;\; \forall t \geq t_0 + T.$$

Moreover, if $D = R^n$, and α_1 belongs to class $\mathcal{K}_\infty$, then the foregoing inequalities hold for any $x(t_0) \in R^n$ with no restrictions on how large μ is.

Example

A mass-spring system with a hardening spring, linear viscous damping, and a periodic external force can be represented by the Duffing's equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t.$$

Taking $x_1 = y$ and $x_2 = \dot{y}$ yields

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 + M\cos\omega t,$$

where $M \geq 0$ is proportional to the amplitude of the periodic external force.

Select Lyapunov candidate

$$V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \frac{1}{2}x_1^4 = x^TPx + \frac{1}{2}x_1^4,$$

which is positive definite and radially unbounded.

Example (Cont'd)

Its derivative along the solution of the system can be calculated by

$$\begin{aligned}\dot{V}(x) &= -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega t \leq -\|x\|_2^2 - x_1^4 + M\sqrt{5}\|x\|_2 \\ &\leq -(1-\theta)\|x\|_2^2 - x_1^4 - \theta\|x\|_2^2 + M\sqrt{5}\|x\|_2 \\ &\leq -(1-\theta)\|x\|_2^2 - x_1^4, \quad \forall \|x\|_2 \geq \frac{M\sqrt{5}}{\theta},\end{aligned}$$

showing that the foregoing theorem is satisfied with $\mu = \frac{M\sqrt{5}}{\theta}$. It is concluded that the solutions are globally uniformly ultimately bounded.

Example (Cont'd)

To find the ultimate bound, we have to find α_1 and α_2:

$$\lambda_{min}(P)\|x\|_2^2 \leq V(x) \leq \lambda_{max}(P)\|x\|_2^2 + \frac{1}{2}\|x\|_2^4,$$

indicating that

$$\alpha_1(r) = \lambda_{min}(P)r^2, \quad \alpha_2(r) = \lambda_{max}(P)r^2 + \frac{1}{2}r^4,$$

and the ultimate bound can be given by

$$b = \alpha_1^{-1}\left(\alpha_2(\mu)\right) = \sqrt{\frac{\lambda_{max}(P)\mu^2 + \frac{1}{2}\mu^4}{\lambda_{min}(P)}}.$$