Section 3:
Tracking *n* objects in clutter
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Multi-Object Tracking

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Introduction

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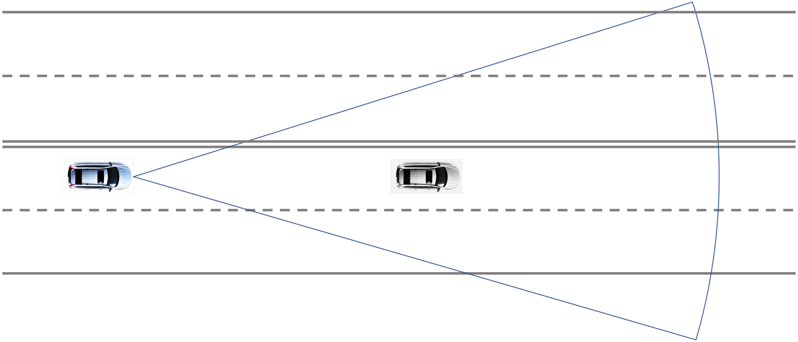
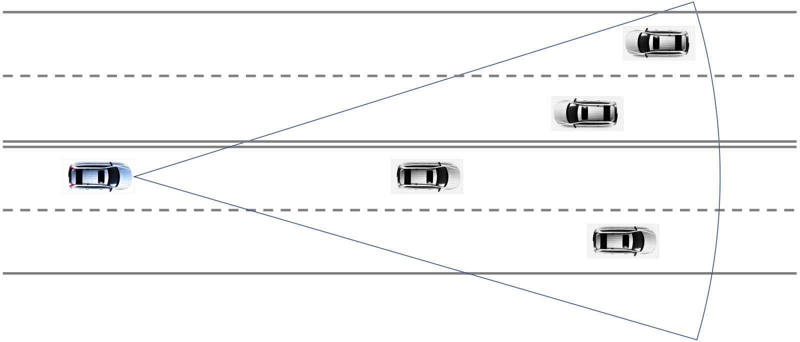
TRACKING N OBJECTS

- Previously: single object tracking
  - Exactly one object
  - Object state x_k
  - Missed detections
  - Clutter



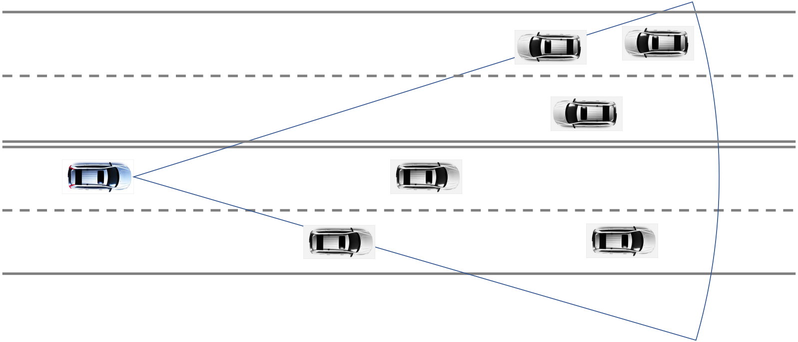
TRACKING N OBJECTS

- Now: generalisation to tracking n objects
  - n is assumed to be both known and constant
  - Object states $X_k = [x_k^1,\ x_k^2, \ldots,\ x_k^i, \ldots,\ x_k^n]$
  - Missed detections
  - Clutter



TRACKING N OBJECTS

- Later: relax the assumption that n is known and constant.
- **Multiple Object Tracking (MOT)**
  - An unknown, and time-varying, number of objects
  - Missed detections
  - Clutter



WHAT IS THE DIFFERENCE FROM TRACKING ONE OBJECT?

- Measurement sources:
  - Previously: two sources, the single object, and clutter.
  - Now: $n + 1$ sources, the n objects, and clutter.
- Data association:
  - Many more possible data associations
  - Considerably more difficult

Main challenge

The handling of the many data association hypotheses.

WHAT WILL WE LEARN?

We need the following:

- A model for the measurements from all n objects and the clutter.
- A model the motion of all n objects.
- A prior for the states of the n objects.
- Methods for handling the data association

Algorithms for tracking n objects:

- Global Nearest Neighbor (GNN) filter
- Joint Probabilistic Data Association (JPDA) filter
- Multi Hypothesis Tracker (MHT)

Initial prior density

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INTIAL PRIOR DENSITY FOR N OBJECTS

Prior density for n objects:

- We need a prior density for the n states,

$$p(X_0) = p\left(x_0^1,\ x_0^2, \ldots,\ x_0^i, \ldots,\ x_0^n\right)$$

Independent random variables:

- n random variables x^i, $i \in \{1, \ldots, n\}$ are independent if the joint density factorises,

$$p(x^1, \ldots, x^n) = \prod_{i=1}^{n} p^i(x^i)$$

  i.e., can be described as a product of densities

INTIAL PRIOR DENSITY FOR N OBJECTS

Assumption in n object tracking:

- Initially, the object states are independent,

$$p(X_0) = \prod_{i=1}^{n} p^i\left(x_0^i\right)$$

Assumed density filtering:

- For example, Gaussian densities

$$p(X_0) = \prod_{i=1}^{n} \mathcal{N}\left(x_0^i\,;\ \mu_0^i,\ P_0^i\right)$$

MIXTURE INTIAL PRIOR DENSITY FOR N OBJECTS

Mixture density:

- Although less common, the initial prior can also be mixture density,

$$p\left(X_0\right) = \sum_h w_0^h \prod_{i=1}^n p^{i,h}\left(x_0^i\right)$$

Note that in each mixture component the objects are independent.

Assumed density filtering:

- For example, Gaussian densities

$$p\left(X_0\right) = \sum_h w_0^h \prod_{i=1}^n \mathcal{N}\left(x_0^i\,;\ \mu_0^{i,h},\ P_0^{i,h}\right)$$

Modelling the measurements

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Introduction to modelling the measurements

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MODELLING CLUTTERED MEASUREMENTS FROM N OBJECTS

- Measurements Z_k, where m_k is the number of measurements
- Clutter C_k and object detections O_k, $Z_k = \Pi(O_k, C_k)$
- The clutter C_k is Poisson with intensity $\lambda_c(c) = \bar{\lambda}_c f_c(c)$
- Object detections $O_k = [O_k^1, \ldots, O_k^i, \ldots, O_k^n]$
- For O_k^i, same measurement model as in single object tracking
  - $O_k^i = [\,]$ with probability $1 - P^{\mathrm{D}}(x_k^i)$
  - $O_k^i = o_k^i$ with probability $P^{\mathrm{D}}(x_k^i)$, likelihood $g_k(o_k^i | x_k^i)$
- What we need: an n-object measurement likelihood

$$p(Z_k | X_k)$$

Includes clutter, detections and missed detections from the n objects.

MEASUREMENT LIKELIHOOD FOR N OBJECTS

We express this as a sum over data associations,

$$p_{Z|X}(Z|X) = \sum_{\theta \in \Theta} p_{Z,\theta|X}(Z,\ \theta|X) = \sum_{\theta \in \Theta} p_{Z|X,\theta,m}(Z|X,\ \theta,\ m) p_{\theta,m|X}(\theta,\ m|X)$$

where we have dropped the time indexing for brevity.

- Association variable $\theta \in \Theta$ for n objects
- Set of all valid associations Θ
- Joint likelihood $p_{Z,\theta|X}(Z,\ \theta|X)$
  - Association conditioned measurement model $p_{Z|X,\theta,m}(Z|X,\ \theta,\ m)$
  - Association prior $p_{\theta,m|X}(\theta,\ m|X)$

Data association variable for n objects

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DATA ASSOCIATION VARIABLE FOR N OBJECTS, 1

- $Z_k = \left[z_k^1,\ z_k^2, \dots,\ z_k^j, \dots,\ z_k^{m_k} \right]$
- θ_k^i is the association for the object with state x_k^i:

$$\theta_k^i = \begin{cases} j & \text{if object } i \text{ is associated to measurement } j \\ 0 & \text{if object } i \text{ is undetected} \end{cases}$$

- Association for all n:

$$\theta_k = \left[\theta_k^1,\ \theta_k^2, \dots,\ \theta_k^i, \dots,\ \theta_k^n \right]$$

EXAMPLE DATA ASSOCIATIONS

$n = 2$, $m = 2$

- Two objects $X = \left[x^1,\ x^2\right]$
- Two measurements $Z = \left[z^1,\ z^2\right]$
- Example association 1: $\theta = \left[\theta^1,\ \theta^2\right] = [1,\ 0]$
    - Object x^1 associated to detection z^1.
    - Object x^2 associated to misdetection.
- Example association 2: $\theta = \left[\theta^1,\ \theta^2\right] = [2,\ 1]$
    - Object x^1 associated to detection z^2.
    - Object x^2 associated to detection z^1.

DATA ASSOCIATION VARIABLE FOR N OBJECTS, 2

- Set of valid associations Θ_k
- For $\theta_k \in \Theta_k$, the following must hold:
  1. Each object must be either detected or misdetected,
  $$\theta_k^i \in \{0, \ldots, m_k\}, \ \forall\, i \in \{1, \ldots, n\}$$
  2. Any pair of detected objects cannot be associated to the same measurement,
  $$\forall\, i, i' \in \{1, \ldots, n\}, \ i \neq i', \ \text{if } \theta_k^i \neq 0, \ \theta_k^{i'} \neq 0 \ \Rightarrow \ \theta_k^i \neq \theta_k^{i'}$$
  Part of **the point object assumption**
- 1. and 2. together ensures that at most n measurements are associated
- In what follows, we consider $\theta_k \in \Theta_k$ unless otherwise stated

EXAMPLE VALID AND INVALID ASSOCIATIONS

Valid associations, $n = 2$, $m = 2$

1. $\theta = [0,\ 0]$
2. $\theta = [1,\ 0]$
3. $\theta = [2,\ 0]$
4. $\theta = [0,\ 1]$
5. $\theta = [0,\ 2]$
6. $\theta = [1,\ 2]$
7. $\theta = [2,\ 1]$

Invalid associations, $n = 2$, $m = 2$

$\theta^i \notin \{0, 1, 2, \ldots, m\}$:

- $\theta = [0,\ 3]$
- $\theta = [-1,\ 1]$

$\theta^i = \theta^{i'}$

- $\theta = [1,\ 1]$
- $\theta = [2,\ 2]$

DATA ASSOCIATION VARIABLE FOR N OBJECTS, 3

Given Z_k and θ_k, we know

- The association to object i,

$$O_k^i = \begin{cases} z_k^{\theta_k^i} & \text{if } \theta_k^i \neq 0 \\ [\,] & \text{if } \theta_k^i = 0 \end{cases}$$

- Which z_k^j are included in C_k

$$j \in \{1, \ldots, m_k\} \; : \; \nexists i \in \{1, \ldots, n\},\ \theta_k^i = j, \qquad \text{Abbreviated: } j \; : \; \nexists \theta_k^i = j$$

- Number of object detections, and number of clutter detections,

$$m_k^o = \sum_{i \in \{1,\ldots,n\} : \theta_k^i \neq 0} 1 = \sum_{i : \theta_k^i \neq 0} 1, \qquad \text{and} \qquad m_k^c = m_k - m_k^o$$

EXAMPLE OBJECT ASSOCIATIONS AND CLUTTER

$n = 2$, $m = 2$

Two measurements $Z = \left[z^1,\ z^2\right]$, and two objects $X = \left[x^1,\ x^2\right]$

θ	O^1	O^2	C	m^o	m^c
$[0,\ 0]$	$[\,]$	$[\,]$	$[z^1, z^2]$	0	2
$[1,\ 0]$	z^1	$[\,]$	z^2	1	1
$[2,\ 0]$	z^2	$[\,]$	z^1	1	1
$[0,\ 1]$	$[\,]$	z^1	z^2	1	1
$[0,\ 2]$	$[\,]$	z^2	z^1	1	1
$[1,\ 2]$	z^1	z^2	$[\,]$	2	0
$[2,\ 1]$	z^2	z^1	$[\,]$	2	0

Association prior and association conditioned likelihood

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DATA ASSOCIATION PRIOR

Dropping time indexing for brevity, the association prior is

$$p_{\theta,m|X}(\theta,\ m|X) = \underbrace{\prod_{i:\theta_k^i=0}\left(1-P^{\mathrm{D}}(x_k^i)\right)\prod_{i:\theta_k^i\neq 0}P^{\mathrm{D}}(x_k^i)}_{(1)}\underbrace{\mathrm{Po}(m^c|\bar{\lambda}_c)}_{(2)}\underbrace{\frac{1}{\binom{m}{m^o}m^o!}}_{(3)}$$

$$=\frac{m^c!}{m!}\mathrm{Po}(m^c|\bar{\lambda}_c)\prod_{i:\theta_k^i=0}\left(1-P^{\mathrm{D}}(x_k^i)\right)\prod_{i:\theta_k^i\neq 0}P^{\mathrm{D}}(x_k^i)$$

where

1. probability of detecting a **specific** set of m^o objects,
2. probability of $m^c = m - m^o$ clutter detection,
3. probability of the specific association, i.e., 1 over the number of ways to select m^o detections and associating them to the specific objects.

ASSOCIATION CONDITIONED LIKELIHOOD

Simplifying assumption:

Given θ and m, the measurements are independent

- Under this assumption, we get

$$p_{Z|X,\theta,m}(Z|X,\theta,\ m) = \prod_{j:\nexists\theta^i=j} f_c(z^j) \prod_{i:\theta^i\neq 0} g(z^{\theta^i}|x^i)$$

MEASUREMENT LIKELIHOOD FOR N OBJECTS

The n object measurement likelihood is

$$
\begin{aligned}
p_{Z|X}(Z|X) &= \sum_{\theta\in\Theta} p_{Z|X,\theta,m}(Z|X,\theta,m)\,p_{\theta,m|X}(\theta,m|X) \\
&= \sum_{\theta\in\Theta} \left[\prod_{j:\nexists\theta^i=j} f_c(z^j) \prod_{i:\theta^i\neq 0} g(z^{\theta^i}|x^i)\right] \left[\frac{m^c!}{m!}\mathrm{Po}(m^c|\bar{\lambda}_c) \prod_{i:\theta^i=0}\left(1 - P^{\mathrm{D}}(x^i)\right) \prod_{i:\theta^i\neq 0} P^{\mathrm{D}}(x^i)\right] \\
&= \sum_{\theta\in\Theta} \frac{e^{-\bar{\lambda}_c}}{m!} \prod_{j:\nexists\theta^i=j} \lambda_c(z^j) \prod_{i:\theta^i=0}\left(1 - P^{\mathrm{D}}(x^i)\right) \prod_{i:\theta^i\neq 0} P^{\mathrm{D}}(x^i) g(z^{\theta^i}|x^i)
\end{aligned}
$$

- Accounts for all n objects,
- and for all m measurements

SIMPLIFIED MEASUREMENT LIKELIHOOD

$$p_{Z|X}(Z|X) = \sum_{\theta \in \Theta} \frac{e^{-\bar{\lambda}_c}}{m!} \prod_{j : \nexists \theta^i = j} \lambda_c(z^j) \prod_{i:\theta^i = 0} \left(1 - P^{\mathrm{D}}(x^i)\right) \prod_{i:\theta^i \neq 0} P^{\mathrm{D}}(x^i) g(z^{\theta^i} | x^i)$$

Multiplication by $1 = \frac{\prod_{j:\exists \theta^i = j} \lambda_c(z^j)}{\prod_{j:\exists \theta^i = j} \lambda_c(z^j)} = \frac{\prod_{i:\theta^i \neq 0} \lambda_c(z^{\theta^i})}{\prod_{i:\theta^i \neq 0} \lambda_c(z^{\theta^i})}$

$$p_{Z|X}(Z|X) = \sum_{\theta \in \Theta} \frac{e^{-\bar{\lambda}_c}}{m!} \prod_{j=1}^{m} \lambda_c(z^j) \prod_{i:\theta^i = 0} \left(1 - P^{\mathrm{D}}(x^i)\right) \prod_{i:\theta^i \neq 0} \frac{P^{\mathrm{D}}(x^i) g(z^{\theta^i} | x^i)}{\lambda_c(z^{\theta^i})}$$

$$\propto \sum_{\theta \in \Theta} \underbrace{\prod_{i:\theta^i = 0} \left(1 - P^{\mathrm{D}}(x^i)\right)}_{\text{Missed detections}} \underbrace{\prod_{i:\theta^i \neq 0} \frac{P^{\mathrm{D}}(x^i) g(z^{\theta^i} | x^i)}{\lambda_c(z^{\theta^i})}}_{\text{Detections}}$$

Measurement likelihood examples

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MEASUREMENT LIKELIHOOD, LINEAR GAUSSIAN EXAMPLE

Linear Gaussian model

- Linear Gaussian measurement model: $g(z|x) = \mathcal{N}(z\,;\, Hx, R)$
- Constant probability of detection: $P^{\mathrm{D}}(x) = P^{D}$
- Uniform clutter: $\lambda(c) = \bar{\lambda}_c / V$

We then get the following *n* object measurement likelihood,

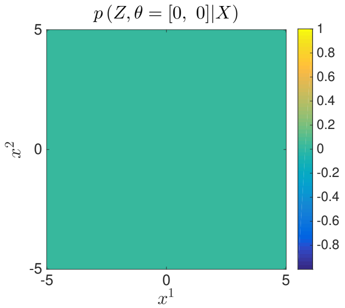
$$p_{Z|X}(Z|X) \propto \sum_{\theta \in \Theta} \left(1 - P^{\mathrm{D}}\right)^{n - m^o} \prod_{i:\theta^i \neq 0} \frac{P^{\mathrm{D}} \mathcal{N}\left(z^{\theta^i}\,;\, Hx^i, R\right)}{\bar{\lambda}_c / V}$$

LIKELIHOOD VISUALIZATIONS

Original example

Scalar object states

- $X = [x^1,\ x^2]$
- $Z = [z^1,\ z^2] = [-1.6,\ 1]$
- $P^{\mathrm{D}}(x) = 0.85$
- $\lambda_c(c) = 0.3,\ c \in [-5,\ 5]$
- $g(z|x) = \mathcal{N}(z\,;\ x, 0.2)$



1): Both objects are misdetected

$$p(Z, \theta = [0, 0]|X) \approx 5 \cdot 10^{-5}$$

LIKELIHOOD VISUALIZATIONS

Original example

Scalar object states

- $X = [x^1,\ x^2]$
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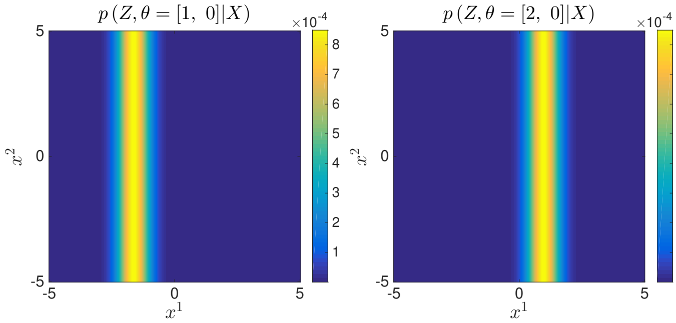

2) & 3): x^1 detected, x^2 misdetected

$$p(Z, \theta = [1, 0]|X) \approx 9.5 \cdot 10^{-4} \cdot \mathcal{N}\left(-1.6\,;\ x^1, 0.2\right)$$

$$p(Z, \theta = [2, 0]|X) \approx 9.5 \cdot 10^{-4} \cdot \mathcal{N}\left(1\,;\ x^1, 0.2\right)$$

LIKELIHOOD VISUALIZATIONS

Original example

Scalar object states

- $X = [x^1,\ x^2]$
- $Z = [z^1,\ z^2] = [-1.6,\ 1]$
- $P^{\mathrm{D}}(x) = 0.85$
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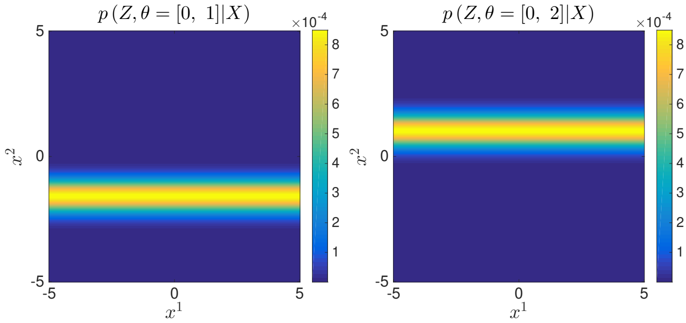

4) & 5): x^1 misdetected, x^2 detected

$$p(Z, \theta = [0, 1]|X) \approx 9.5 \cdot 10^{-4} \cdot \mathcal{N}(-1.6\,;\ x^2, 0.2)$$
$$p(Z, \theta = [0, 2]|X) \approx 9.5 \cdot 10^{-4} \cdot \mathcal{N}(1\,;\ x^2, 0.2)$$

LIKELIHOOD VISUALIZATIONS

Original example

Scalar object states

- $X = [x^1,\ x^2]$
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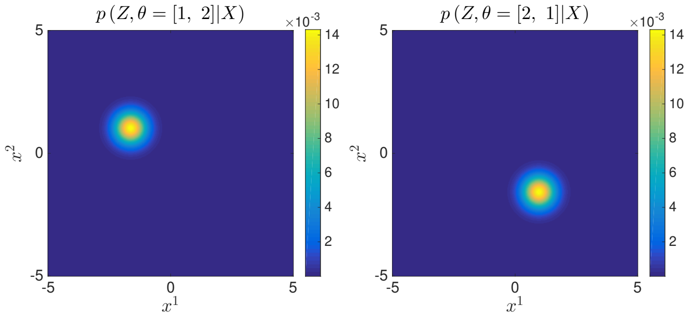
- $\lambda_c(c) = 0.3,\ c \in [-5,\ 5]$
- $g(z|x) = \mathcal{N}(z\,;\ x, 0.2)$



6) & 7): Both objects are detected

$$p(Z,\theta = [1,2]|X) \approx 0.02 \cdot \mathcal{N}(-1.6\,;\ x^1, 0.2)\,\mathcal{N}(1\,;\ x^2, 0.2)$$
$$p(Z,\theta = [2,1]|X) \approx 0.02 \cdot \mathcal{N}(1\,;\ x^1, 0.2)\,\mathcal{N}(-1.6\,;\ x^2, 0.2)$$

LIKELIHOOD VISUALIZATIONS

Original example

Scalar object states

- $X = [x^1,\ x^2]$
- $Z = [z^1,\ z^2] = [-1.6,\ 1]$
- $P^{\mathrm{D}}(x) = 0.85$
- $\lambda_c(c) = 0.3,\ c \in [-5,\ 5]$
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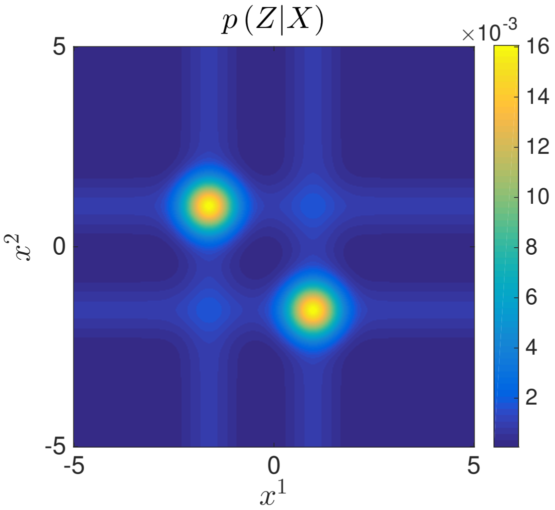
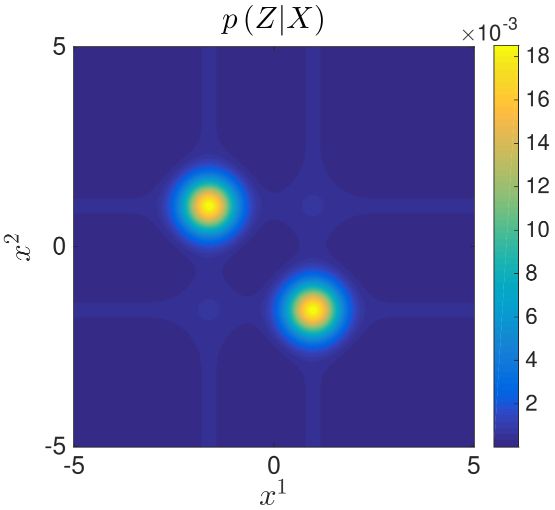


LIKELIHOOD VISUALIZATIONS

Larger P^{D}

Scalar object states

- $X = [x^1,\ x^2]$
- $Z = [z^1,\ z^2] = [-1.6,\ 1]$
- $P^{\mathrm{D}}(x) = 0.95$
- $\lambda(c) = 0.3,\ c \in [-5,\ 5]$
- $g(z|x) = \mathcal{N}(z\,;\ x, 0.2)$

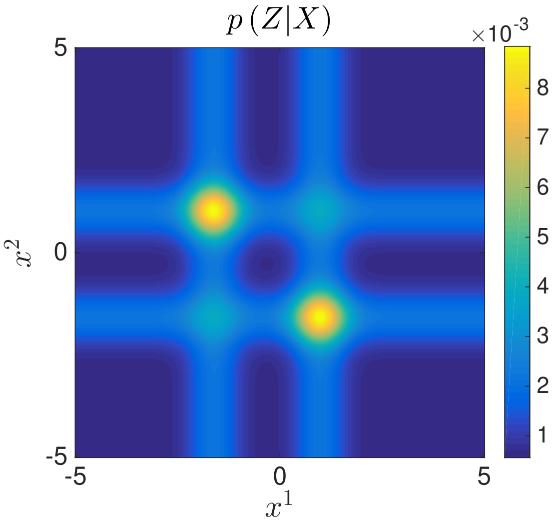


LIKELIHOOD VISUALIZATIONS

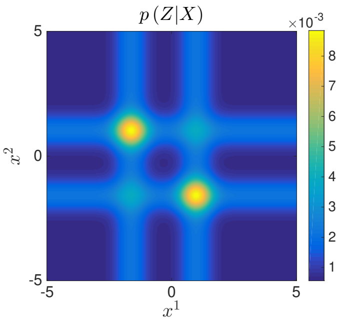
Smaller P^{D}

Scalar object states

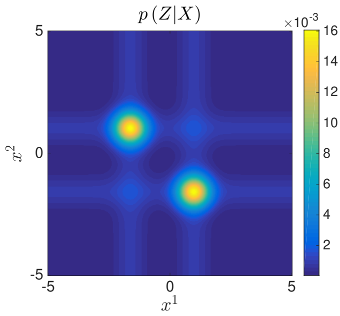
- $X = [x^1,\ x^2]$
- $Z = [z^1,\ z^2] = [-1.6,\ 1]$
- $P^{\mathrm{D}}(x) = 0.5$
- $\lambda(c) = 0.3,\ c \in [-5,\ 5]$
- $g(z|x) = \mathcal{N}(z\,;\ x, 0.2)$



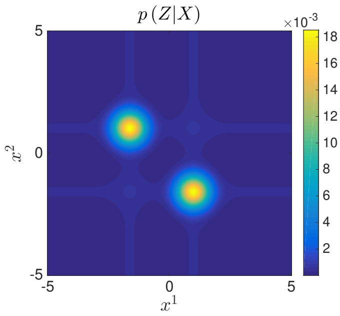
SYMMETRIC MEASUREMENT LIKELIHOOD



$P^{\mathrm{D}} = 0.5$



$P^{\mathrm{D}} = 0.85$



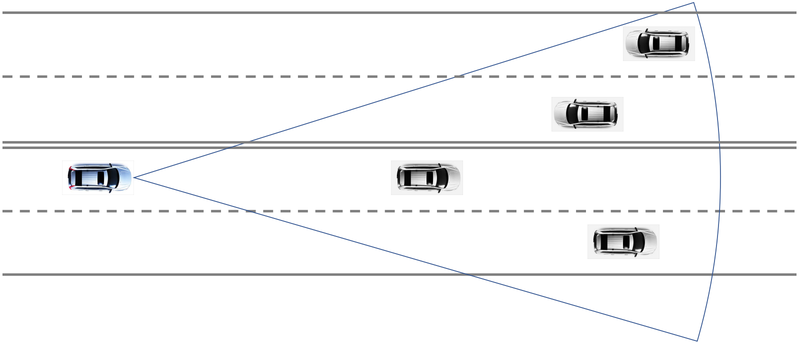
$P^{\mathrm{D}} = 0.95$

$p(Z|X)$ is symmetric with respect to the line $x^1 = x^2$. Due to

- the unknown data association
- the indexing x^1, x^2 is arbitrary

INDEXING IS ARBITRARY

We wish to track the states of the four vehicles: position, speed, heading, etc.



How we index (order) the four vehicles will not affect their states, hence, we say that the indexing is arbitrary.

SUMMARY

- Standard point object measurement model
  - Poisson clutter $\lambda_c(c)$
  - Probability of object detection $P^{\mathrm{D}}(x)$
  - Object measurement likelihood $g_k(z|x)$
- n object data association θ
  - Each object is either detected or misdetected.
  - Each measurement is associated to one source.
- n object measurement likelihood

$$p_{Z|X}(Z|X) = \sum_{\theta \in \Theta} p_{Z,\theta|X}(Z,\ \theta|X)$$

- Likelihood is symmetric

Number of associations

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NUMBER OF WAYS TO ASSOCIATE M MEAS. TO N OBJ.?

- The posterior density for n objects is a mixture density, and the number of mixture components increases due to the unknown data association.

- In single object tracking in clutter, the number of hypotheses increses with a factor $m_k + 1$ in each time step.

- What is the corresponding number for tracking n objects?
- How fast does the number of mixture components in the posterior density increase?

NUMBER OF WAYS TO ASSOCIATE M MEAS. TO N OBJ.?

m measurements, n objects

- $m^o \in \{0, 1, \dots, \min(m, n)\}$ are from objects
- $\binom{n}{m^o}$ — ways to select m^o objects from n objects.
- $\binom{m}{m^o}$ — ways to select m^o detections from m detections.
- $m^o!$ — ways to assign the selected detections and the selected objects.

$$
\begin{aligned}
N_A(m, n) &= \sum_{m^o=0}^{\min(m,n)} \binom{n}{m^o} \binom{m}{m^o} m^o! \\
&= \sum_{m^o=0}^{\min(m,n)} \frac{m!\,n!}{m^o!(m - m^o)!(n - m^o)!}
\end{aligned}
$$

NUMBER OF ASSOCIATIONS FOR A SINGLE OBJECT

We can verify that for $n = 1$, we get the right number of associations:

$$\begin{aligned} N_A(m,1) &= \sum_{m^o=0}^{\min(m,1)} \binom{1}{m^o}\binom{m}{m^o} m^o! \\ &= \binom{1}{0}\binom{m}{0}0! + \binom{1}{1}\binom{m}{1}1! \\ &= 1 \cdot 1 \cdot 1 + 1 \cdot m \cdot 1 \\ &= 1 + m \end{aligned}$$

One misdetection associations, plus m different measurement associations.

NUMBER OF ASSOCIATIONS

$N_A(m, n)$

$m\backslash n$	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2	1	3	7	13	21	31	43	57	73
3	1	4	13	34	73	136	229	358	529
4	1	5	21	73	209	501	1045	1961	3393
5	1	6	31	136	501	1546	4051	9276	19081
6	1	7	43	229	1045	4051	13327	37633	93289
7	1	8	57	358	1961	9276	37633	130922	394353
8	1	9	73	529	3393	19081	93289	394353	1441729

NUMBER OF MIXTURE COMPONENTS

- The number of mixture components in the posterior density at time k is

$$N_0 \prod_{t=1}^{k} N_A(m_t, n)$$

  where N_0 is the number of mixture components in the intial prior density. Often, $N_0 = 1$.

- Compare to the expression for tracking a single object:

$$N_0 \prod_{t=1}^{k} (1 + m_t) = N_0 \prod_{t=1}^{k} N_A(m_t, 1)$$

NUMBER OF MIXTURE COMPONENTS

$N_0 = 1$, and $m_1 = 3$, $m_2 = 5$, $m_3 = 4$

- $n = 1$

$$(1 + 3) \cdot (1 + 5) \cdot (1 + 4) = 120$$

- $n = 2$

$$N_A(3, 2) \cdot N_A(5, 2) \cdot N_A(4, 2) = 13 \cdot 31 \cdot 21 = 8463$$

- $n = 3$

$$N_A(3, 3) \cdot N_A(5, 3) \cdot N_A(4, 3) = 34 \cdot 136 \cdot 73 = 337552$$

SUMMARY

- Number of associations

$$N_A(m, n) = \sum_{m^o=0}^{\min(m,n)} \binom{n}{m^o} \binom{m}{m^o} m^o!$$

- Number of posterior mixture components

$$N_0 \prod_{t=1}^{k} N_A(m_t, n)$$

- The number of mixture components increases very fast.
- Computationally tractable tracking algorithms require approximations.

Posterior density

Multi-Object Tracking

Karl Granström

Introduction

Multi-Object Tracking

Karl Granström

N OBJECT POSTERIOR DENSITY

n objects

- Prior: $p(X_k|Z_{1:k-1})$
- Likelihood: $p(Z_k|X_k)$
- Posterior:

$$p(X_k|Z_{1:k}) \propto p(Z_k|X_k)p(X_k|Z_{1:k-1})$$

- Structure

$$p(X_k|Z_{1:k}) = \sum_{\theta_k} p(X_k|Z_{1:k}, \theta_k) \Pr[\theta_k|Z_{1:k-1}]$$

Single object

- Prior: $p(x_k|Z_{1:k-1})$
- Likelihood: $p(Z_k|x_k)$
- Posterior:

$$p(x_k|Z_{1:k}) \propto p(Z_k|x_k)p(x_k|Z_{1:k-1})$$

- Structure

$$p(x_k|Z_{1:k}) = \sum_{\theta_k} p(x_k|Z_{1:k}, \theta_k) \Pr[\theta_k|Z_{1:k-1}]$$

WHAT WILL WE LEARN?

- Posterior density with uni-modal prior

$$p(X|Z) \propto p(Z|X)p(X)$$

- Posterior with mixture prior

$$p(X|Z) \propto \sum_{h} \sum_{\theta \in \Theta} \tilde{w}^{h,\theta} p^{h,\theta}(X|Z)$$

- General expression for the posterior density

$$p(X_k|Z_{1:k}) \propto \sum_{\theta_{1:k}} \tilde{w}^{\theta_{1:k}} p^{\theta_{1:k}}(X_k|Z_{1:k})$$

Posterior density with uni-modal prior

Multi-Object Tracking

Karl Granström

PRIOR AND LIKELIHOOD

- Uni-modal independent prior,

$$p(X) = \prod_{i=1}^{n} p^i(x^i)$$

- Measurement likelihood

$$p(Z|X) = \sum_{\theta\in\Theta} \frac{e^{-\bar{\lambda}_c}}{m!} \prod_{j=1}^{m} \lambda_c(z^j) \prod_{i':\theta^{i'}=0} \left(1 - P^{\mathrm{D}}(x^{i'})\right) \prod_{i:\theta^i\neq 0} \frac{P^{\mathrm{D}}(x^i) g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})}$$

$$\propto \sum_{\theta\in\Theta} \underbrace{\prod_{i':\theta^{i'}=0} \left(1 - P^{\mathrm{D}}(x^{i'})\right)}_{\text{Missed detections}} \underbrace{\prod_{i:\theta^i\neq 0} \frac{P^{\mathrm{D}}(x^i) g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})}}_{\text{Detections}}$$

POSTERIOR DENSITY

The posterior density is given by the Bayes update,

$$
\begin{aligned}
p(X|Z) &\propto p(Z|X)p(X) \\
&\propto \sum_{\theta\in\Theta} \prod_{i':\theta^{i'}=0} \left(1 - P^{\mathrm{D}}(x^{i'})\right) \prod_{i:\theta^i\neq 0} \frac{P^{\mathrm{D}}(x^i)g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})} \prod_{i''=1}^{n} p^{i''}(x^{i''}) \\
&= \sum_{\theta\in\Theta} \underbrace{\prod_{i':\theta^{i'}=0} \left(1 - P^{\mathrm{D}}(x^{i'})\right) p^{i'}(x^{i'})}_{\text{Misdetected objects}} \underbrace{\prod_{i:\theta^i\neq 0} \frac{P^{\mathrm{D}}(x^i)g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})} p^i(x^i)}_{\text{Detected objects}}
\end{aligned}
$$

In each of the products, we have un-normalized posterior object densities.

NORMALIZING THE POSTERIOR OBJECT DENSITIES

Normalizing a density

If $p(x) \propto g(x)$, then $p(x) = \frac{g(x)}{\int g(x')\,\mathrm{d}x'}$

Posterior density, misdetected object, $\theta^i = 0$

$$g(x^i) = (1 - P^{\mathrm{D}}(x^i))p(x^i) \qquad \Rightarrow \qquad p^{i,\theta^i}(x^i) = \frac{g(x^i)}{\tilde{w}^{\theta^i}}, \text{ and } \tilde{w}^{\theta^i} = \int g(x^i)\mathrm{d}x^i$$

Posterior density, detected object, $\theta^i \neq 0$

$$g(x^i) = \frac{P^{\mathrm{D}}(x^i)g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})}p^i(x^i) \qquad \Rightarrow \qquad p^{i,\theta^i}(x^i) = \frac{g(x^i)}{\tilde{w}^{\theta^i}}, \text{ and } \tilde{w}^{\theta^i} = \int g(x^i)\mathrm{d}x^i$$

POSTERIOR DENSITY

Normalizing the individual object densities gives the following posterior n object density,

$$p(X|Z) \propto \sum_{\theta \in \Theta} \prod_{i':\theta^{i'}=0} \left(1 - P^{\mathrm{D}}(x^{i'})\right) p^{i'}(x^{i'}) \prod_{i:\theta^i \neq 0} \frac{P^{\mathrm{D}}(x^i) g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})} p^i(x^i)$$

$$= \sum_{\theta \in \Theta} \prod_{i=1}^{n} \tilde{w}^{\theta^i} p^{i,\theta^i}(x^i)$$

where

$$\tilde{w}^{\theta^i} = \begin{cases} \int \left(1 - P^{\mathrm{D}}(x^i)\right) p^i(x^i) \mathrm{d}x^i & \text{if } \theta^i = 0 \\ \int \frac{P^{\mathrm{D}}(x_k^i) g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})} p^i(x^i) \mathrm{d}x^i & \text{if } \theta^i \neq 0 \end{cases}$$

$$p^{i,\theta^i}(x^i) = \begin{cases} \frac{1}{\tilde{w}^{\theta^i}} \left(1 - P^{\mathrm{D}}(x^i)\right) p^i(x^i) & \text{if } \theta^i = 0 \\ \frac{1}{\tilde{w}^{\theta^i}} \frac{P^{\mathrm{D}}(x_k^i) g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})} p^i(x^i) & \text{if } \theta^i \neq 0 \end{cases}$$

Note that $p(X|Z)$ is not a normalized n object mixture density.

NORMALIZED POSTERIOR DENSITY

We can write this as a mixture of n object densities,

$$p(X|Z) \propto \sum_{\theta\in\Theta} \prod_{i=1}^{n} \tilde{w}^{\theta^i} p^{i,\theta^i}(x^i) = \sum_{\theta\in\Theta} \prod_{i'=1}^{n} \tilde{w}^{\theta^{i'}} \prod_{i=1}^{n} p^{i,\theta^i}(x^i)$$

$$= \sum_{\theta\in\Theta} \tilde{w}^{\theta} \prod_{i=1}^{n} p^{i,\theta^i}(x^i) = \sum_{\theta\in\Theta} \tilde{w}^{\theta} p^{\theta}(X)$$

and we can then obtain a normalised posterior density by normalizing the weights,

$$p(X|Z) = \sum_{\theta\in\Theta} w^{\theta} p^{\theta}(X)$$

$$w^{\theta} = \frac{\tilde{w}^{\theta}}{\sum_{\theta'} \tilde{w}^{\theta'}} = \frac{\prod_i \tilde{w}^{\theta^i}}{\sum_{\theta'} \prod_{i'} \tilde{w}^{\theta^{i'}}}$$

NORMALIZING THE POSTERIOR DIRECTLY 1/2

We can normalized the posterior n object density directly. Bayes update is,

$$p(X|Z) = \frac{p(Z|X)p(X)}{p(Z)} = \frac{p(Z|X)p(X)}{\int p(Z|X')p(X')\mathrm{d}X'}$$

where

$$\int g(X)\mathrm{d}X = \iint \cdots \int g\left([x^1,\ x^2,\ \ldots\ x^n]\right) \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n$$

The normalizing factor

$$p(Z) = \int p(Z|X)p(X)\mathrm{d}X = \int \left(\sum_{\theta \in \Theta} p(Z,\theta|X) \right) p(X)\mathrm{d}X$$

$$= \sum_{\theta \in \Theta} \int p(Z,\theta|X)p(X)\mathrm{d}X = \sum_{\theta \in \Theta} p(Z,\theta)$$

NORMALIZING THE POSTERIOR DIRECTLY 2/2

The normalizing factor

$$
\begin{aligned}
p(Z) &= \sum_{\theta \in \Theta} p(Z, \theta) \\
&= \sum_{\theta \in \Theta} \underbrace{\prod_{i' : \theta^{i'} = 0} \int \left(1 - P^{\mathrm{D}}(x^{i'})\right) p^{i'}(x^{i'}) \mathrm{d}x^{i'}}_{\text{Misdetected objects}} \underbrace{\prod_{i : \theta^i \neq 0} \int \frac{P^{\mathrm{D}}(x^i) g(z^{\theta^i} | x^i)}{\lambda_c(z^{\theta^i})} p^i(x^i) \mathrm{d}x^i}_{\text{Detected objects}} \\
&= \sum_{\theta \in \Theta} \prod_{i=1}^{n} \tilde{w}^{\theta^i}
\end{aligned}
$$

Indeed, $p(Z) = \sum_\theta p(Z, \theta) = \sum_\theta \prod_i \tilde{w}^{\theta^i}$ is in the denominator of w^θ.

GAUSSIAN PRIOR AND LINEAR GAUSSIAN LIKELIHOOD (1/2)

Gaussian prior and likelihood

$$p(X) = \prod_{i=1}^{n} \mathcal{N}\left(x^i\,;\,\mu^i,\,P^i\right)$$

$$p(Z|X) \propto \sum_{\theta\in\Theta} \prod_{i':\theta^{i'}=0} \left(1 - P^{\mathrm{D}}\right) \prod_{i:\theta^i\neq 0} \frac{P^{\mathrm{D}}\mathcal{N}\left(z^{\theta^i}\,;\,Hx^i, R\right)}{\bar{\lambda}/V}$$

Posterior with Gaussian object densities,

$$p(X|Z)$$

$$\propto \sum_{\theta\in\Theta} \prod_{i':\theta^{i'}=0} \left(1 - P^{\mathrm{D}}\right) \mathcal{N}\left(x^{i'}\,;\,\mu^{i'},\,P^{i'}\right) \prod_{i:\theta^i\neq 0} \frac{P^{\mathrm{D}}\mathcal{N}\left(z^{\theta^i}\,;\,Hx^i, R\right)}{\bar{\lambda}/V} \mathcal{N}\left(x^i\,;\,\mu^i,\,P^i\right)$$

GAUSSIAN PRIOR AND LINEAR GAUSSIAN LIKELIHOOD (2/2)

Gaussian prior and likelihood

$$p(X|Z) \propto \sum_{\theta \in \Theta} \tilde{w}^{\theta} \prod_{i} \mathcal{N}\left(x^{i}\,;\; \mu^{i,\theta^{i}},\; P^{i,\theta^{i}}\right)$$

$$\tilde{w}^{\theta} = \left(1 - P^{\mathrm{D}}\right)^{n-m^{o}} \left(\frac{P^{\mathrm{D}}}{\bar{\lambda}/V}\right)^{m^{o}} \prod_{i:\theta^{i} \neq 0} \mathcal{N}\left(z^{\theta^{i}}; \hat{z}^{i}, S^{i}\right)$$

where

$$\mu^{i,\theta^{i}} = \begin{cases} \mu^{i} + K^{i}\left(z^{\theta^{i}} - \hat{z}^{i}\right) & \text{if } \theta^{i} \neq 0 \\ \mu^{i} & \text{if } \theta^{i} = 0 \end{cases}$$

$$P^{i,\theta^{i}} = \begin{cases} P^{i} - K^{i} H P^{i} & \text{if } \theta^{i} \neq 0 \\ P^{i} & \text{if } \theta^{i} = 0 \end{cases}$$

$$\hat{z}^{i} = H\mu^{i}$$

$$S^{i} = HP^{i}H^{\mathrm{T}} + R$$

$$K^{i} = P^{i}H^{\mathrm{T}}\left(S^{i}\right)^{-1}$$

POSTERIOR VISUALIZATIONS

Original example

Scalar object states, measurements

- $X = [x^1,\ x^2]$
- $Z = [z^1,\ z^2] = [-1.6,\ 1]$
- $N_A(2,2) = 7$

Measurement model

- $P^{\mathrm{D}}(x) = 0.85$
- $\lambda_c(c) = 0.3,\ c \in [-5,\ 5]$
- $g(z|x) = \mathcal{N}\left(z\,;\ x, 0.2\right)$

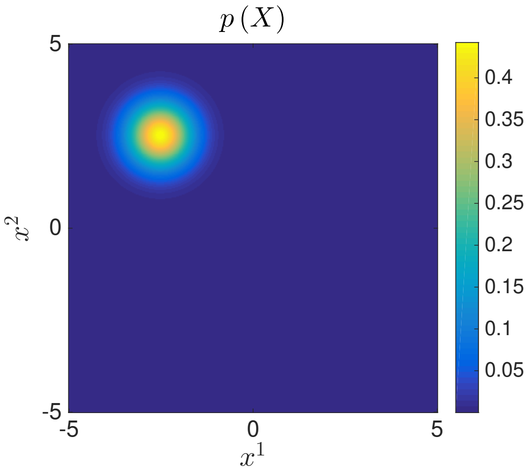
Prior $p(X) = p^1(x^1)p^2(x^2)$

- $p^1(x^1) = \mathcal{N}\left(x^1\,;\ -2.5, 0.36\right)$
- $p^2(x^2) = \mathcal{N}\left(x^2\,;\ 2.5, 0.36\right)$

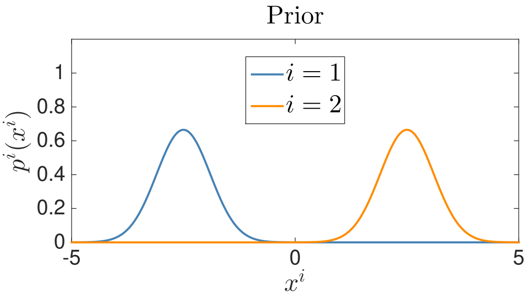
Visualizations

- n object: $p^{\theta}(X) = p(X|Z,\theta)$
- Marginal: $p^{i,\theta^i}(x^i) = p^i(x^i|Z,\theta)$
- n object: $p(X|Z)$
- Marginal: $p^i(x^i|Z)$

POSTERIOR VISUALIZATIONS: PRIOR DENSITY

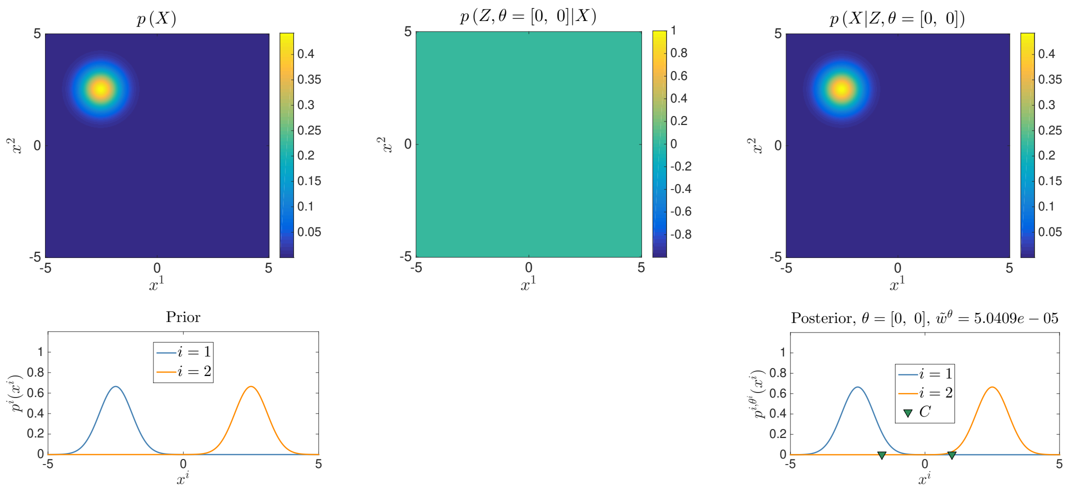


Joint *n* object density

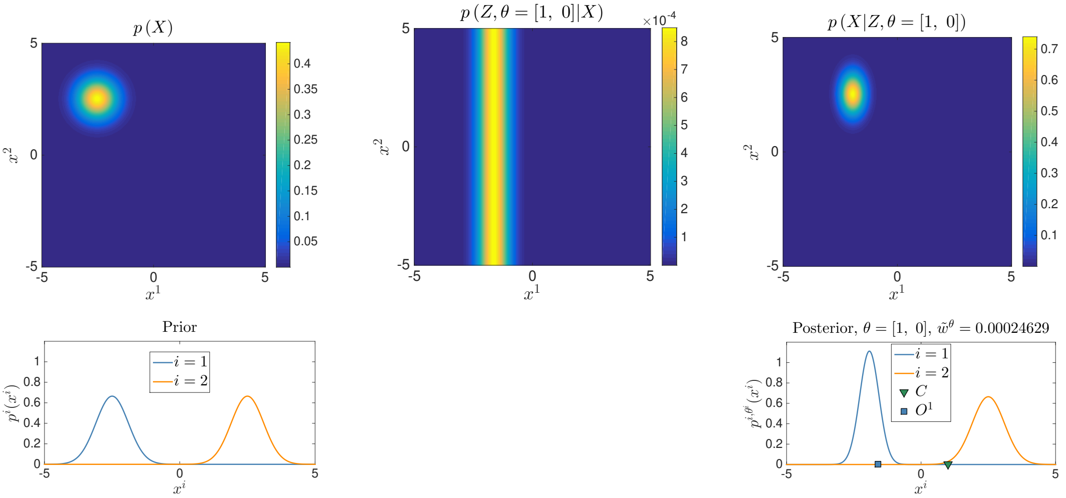


Corresponding object densities

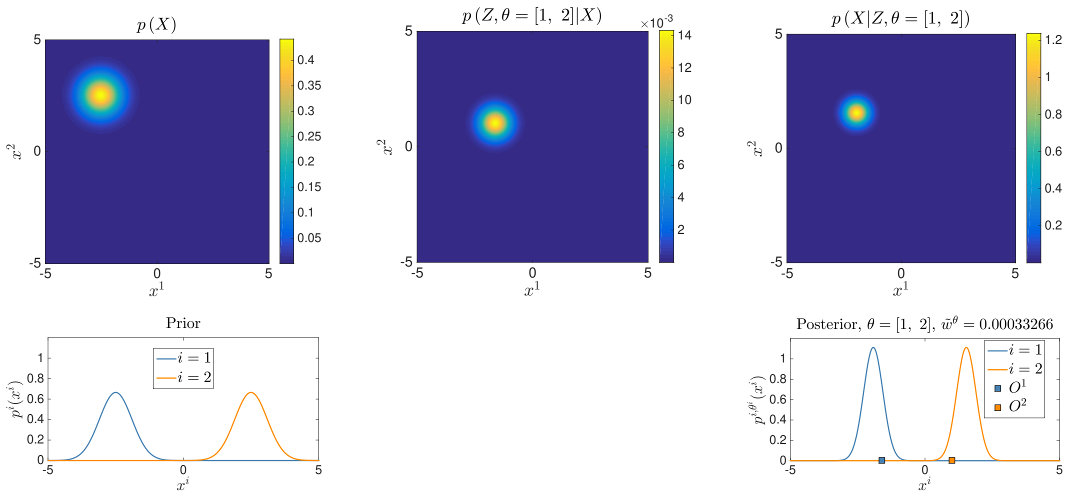
POSTERIOR VISUALIZATIONS: $\theta = [0, 0]$, $Z = [Z^1,\ Z^2] = [-1.6,\ 1]$



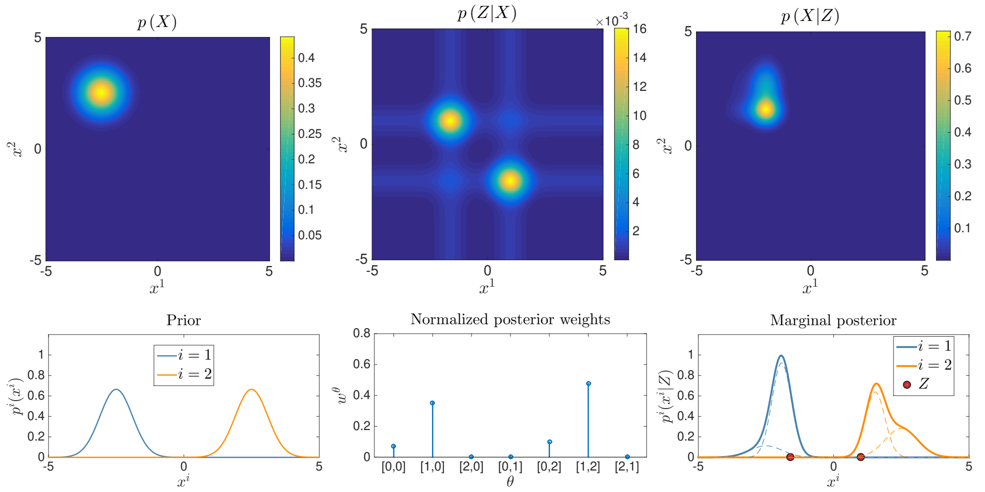
POSTERIOR VISUALIZATIONS: $\theta = [1, 0]$, $Z = [Z^1,\ Z^2] = [-1.6,\ 1]$



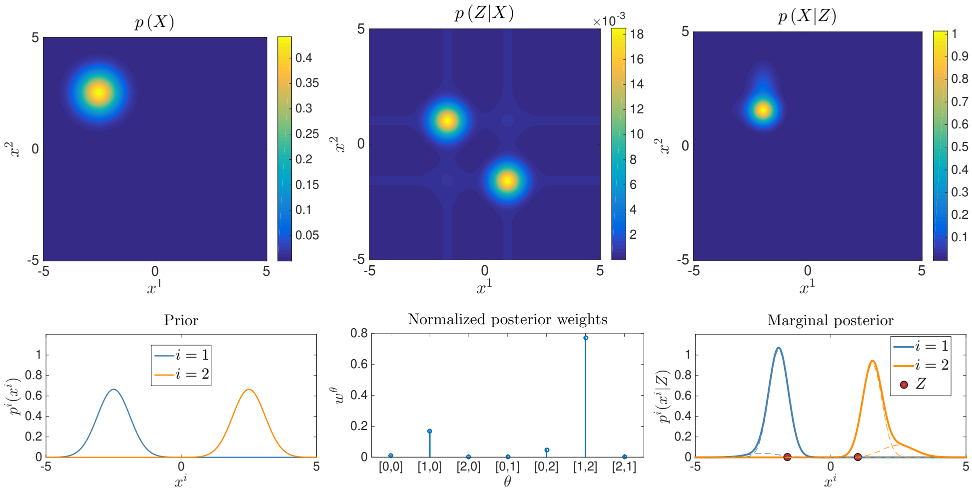
POSTERIOR VISUALIZATIONS: $\theta = [1, 2]$, $Z = [Z^1,\ Z^2] = [-1.6,\ 1]$



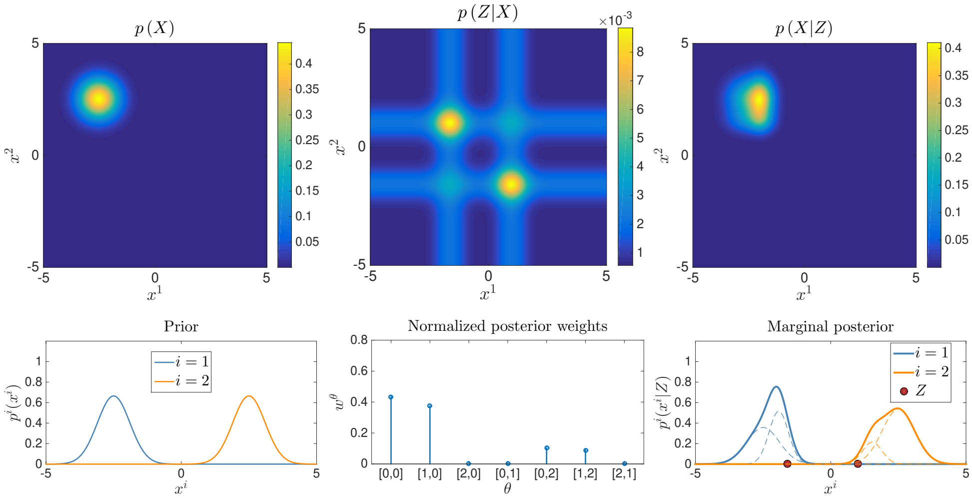
POSTERIOR VISUALIZATION: FULL POSTERIOR, $Z = [Z^1,\ Z^2] = [-1.6,\ 1]$



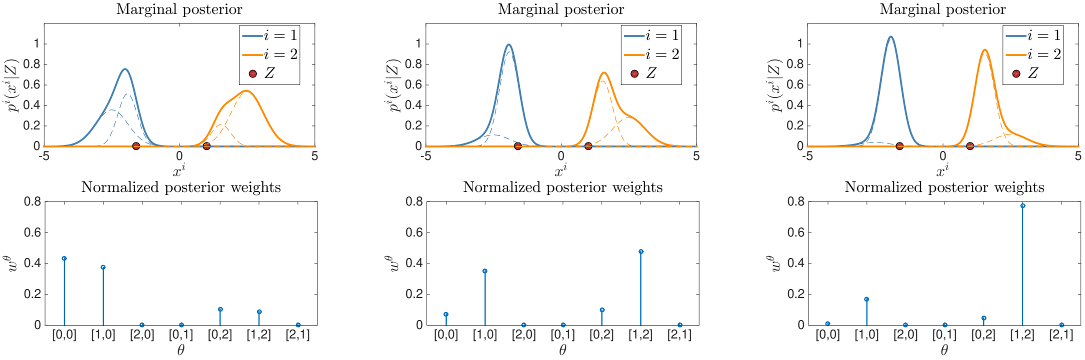
POSTERIOR VISUALIZATION: HIGHER P^{D}, $Z = [Z^1,\ Z^2] = [-1.6,\ 1]$



POSTERIOR VISUALIZATION: LOWER P^{D}, $Z = [Z^1,\ Z^2] = [-1.6,\ 1]$



POSTERIOR VISUALIZATION: COMPARISON, $Z = [Z^1,\ Z^2] = [-1.6,\ 1]$



$P^{\mathrm{D}} = 0.5$

$P^{\mathrm{D}} = 0.85$

$P^{\mathrm{D}} = 0.95$

General expression for the posterior density

Multi-Object Tracking

Karl Granström

CONCEPTUAL SOLUTION

- Single object tracking,

$$p_{x_k|Z_{1:k}}(x_k|Z_{1:k}) = \sum_{\theta_{1:k}} w^{\theta_{1:k}} p^{\theta_{1:k}}(x_k)$$

$$\sum_{\theta_{1:k}} = \sum_{\theta_1=0}^{m_1} \sum_{\theta_2=0}^{m_2} \cdots \sum_{\theta_k=0}^{m_k}$$

- *n* object tracking,

$$p_{X_k|Z_{1:k}}(X_k|Z_{1:k}) = \sum_{\theta_{1:k}\in\Theta_{1:k}} w^{\theta_{1:k}} p^{\theta_{1:k}}(X_k)$$

$$\sum_{\theta_{1:k}\in\Theta_{1:k}} = \sum_{\theta_1\in\Theta_1} \sum_{\theta_2\in\Theta_2} \cdots \sum_{\theta_k\in\Theta_k}$$

POSTERIOR DENSITY WITH MIXTURE PRIOR

A mixture prior,

$$p(X) = \sum_h w^h p^h(X) = \sum_h \Pr[h] p(X|h)$$

gives a mixture posterior, with a component for each combination of h and θ,

$$\begin{aligned}
p(X|Z) \propto p(Z|X)p(X) &= \left[\sum_{\theta\in\Theta} p(Z,\theta|X)\right]\left[\sum_h w^h p^h(X)\right] \\
&= \sum_h \sum_{\theta\in\Theta} w^h p(Z,\theta|X)p^h(X) = \sum_h \sum_{\theta\in\Theta} w^h \tilde{w}^{\theta|h} \frac{p(Z,\theta|X)p^h(X)}{\tilde{w}^{\theta|h}} \\
&= \sum_h \sum_{\theta\in\Theta} \tilde{w}^{h,\theta} p^{h,\theta}(X)
\end{aligned}$$

where $\tilde{w}^{\theta|h} = \int p(Z,\theta|X)p^h(X)\mathrm{d}X = p(Z,\theta|h)$

NORMALIZED POSTERIOR DENSITY WITH MIXTURE PRIOR

- Mixture prior,

$$p(X) = \sum_h w^h p^h(X) = \sum_h \Pr[h] p(X|h)$$

- Mixture posterior,

$$p(X|Z) \propto \sum_h \sum_{\theta \in \Theta} \tilde{w}^{h,\theta} p^{h,\theta}(X)$$

- Normalised mixture posterior,

$$p(X|Z) = \sum_h \sum_{\theta \in \Theta} w^{h,\theta} p^{h,\theta}(X) = \sum_h \sum_{\theta \in \Theta} \Pr[h, \theta] p(X|h, \theta)$$

$$w^{h,\theta} = \frac{\tilde{w}^{\theta|h} w^h}{\sum_h \sum_{\theta \in \Theta} \tilde{w}^{\theta|h} w^h}$$

GENERAL EXPRESSION FOR THE POSTERIOR DENSITY

- It can be shown that the n object posterior density at time k is

$$p_{k|k}(X_k) = p_{X_k|Z_{1:k}}(X_k|Z_{1:k}) \propto \sum_{\theta_{1:k}} \tilde{w}_{k|k}^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(X_k)$$

$$\tilde{w}_{k|k}^{\theta_{1:k}} = \prod_{i=1}^{n} \prod_{t=1}^{k} \tilde{w}^{\theta_t^i|\theta_{1:t-1}}$$

$$p_{k|k}^{\theta_{1:k}}(X_k) = \prod_{i=1}^{n} p_{k|k}^{i,\theta_{1:k}^i}(x_k^i)$$

  where $\tilde{w}^{\theta_1^i|\theta_0} = \tilde{w}^{\theta_1^i}$.

- The un-normalised weight $\tilde{w}_{k|k}^{\theta_{1:k}}$ of a sequence of data associations $\theta_{1:k}$ is given by the product of the un-normalised weights $\tilde{w}^{\theta_t^i|\theta_{1:t-1}^i}$ for each association for each object and each time step.

GENERAL EXPRESSION FOR THE POSTERIOR DENSITY

- The normalised posterior density can be expressed as

$$p_{k|k}(X_k) = \sum_{\theta_{1:k}} w_{k|k}^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(X_k) = \sum_{\theta_1 \in \Theta_1} \dots \sum_{\theta_k \in \Theta_k} \Pr[\theta_{1:k}|Z_{1:k}] p(X|\theta_{1:k}, Z_{1:k})$$

  with normalized weights

$$w_{k|k}^{\theta_{1:k}} = \frac{\tilde{w}_{k|k}^{\theta_{1:k}}}{\sum_{\theta'_{1:k}} \tilde{w}_{k|k}^{\theta'_{1:k}}} = \frac{\prod_{i=1}^{n} \prod_{t=1}^{k} \tilde{w}^{\theta_t^i|\theta_{1:t-1}^i}}{\sum_{\theta'_1} \dots \sum_{\theta'_k} \prod_{i=1}^{n} \prod_{t=1}^{k} \tilde{w}^{\theta_t^{i'}|\theta_{1:t-1}^{i'}}}$$

- Due to the rapidly increasing number of mixture components, the exact posterior is intractable, and approximations are necessary.
- Different tracking algorithms correspond to different approximations.

Predicting the n object density

Multi-Object Tracking

Karl Granström

N OBJECT PREDCTION

- Posterior

$$p_{k-1|k-1}(X_{k-1})$$

- Transition density

$$p_k(X_k|X_{k-1})$$

- Chapman-Kolmogorov prediction for n objects,

$$p_{k|k-1}(X_k) = \int p_k(X_k|X_{k-1})p_{k-1|k-1}(X_{k-1})\mathrm{d}X_{k-1}$$

MODELLING THE MOTION OF N OBJECTS

n object motion:

- We need to describe how the n states evolve from x_{k-1}^i to x_k^i, i.e., we need the transition density

$$p_k(X_k|X_{k-1}) = p_k\left(x_k^1,\ x_k^2, \ldots,\ x_k^i, \ldots,\ x_k^n \middle|\ x_{k-1}^1,\ x_{k-1}^2, \ldots,\ x_{k-1}^i, \ldots,\ x_{k-1}^n\right)$$

Simplifying assumption:

- The time evolution of the objects is independent

$$p_k(X_k|X_{k-1}) = \prod_{i=1}^{n} \pi_k\left(x_k^i \middle|\ x_{k-1}^i\right)$$

Typically, the same transition density is used for all objects

GAUSSIAN N OBJECT TRANSITION DENSITY

Gaussian transition density

- Gaussian state transition density for single object,

$$\pi_k\left(x_k \middle|\, x_{k-1}\right) = \mathcal{N}\left(x_k\,;\, f_{k-1}(x_{k-1}), Q_{k-1}\right)$$

- The transition density for the n objects is then

$$p_k\left(X_k \middle|\, X_{k-1}\right) = \prod_{i=1}^{n} \mathcal{N}\left(x_k^i\,;\, f_{k-1}(x_{k-1}^i), Q_{k-1}\right)$$

- **Note:** using the same motion model $(f_{k-1}(\cdot), Q_{k-1})$ for all objects does not mean that the estimated motion parameters (speed, heading, acceleration, turn-rate, etc) are the same.

PREDICTED DENSITY

- Independent posterior,

$$p_{k-1|k-1}(X_{k-1}) = \prod_{i=1}^{n} p^{i}_{k-1|k-1}(x^{i}_{k-1})$$

- Predicted density, assuming independent motion,

$$\begin{aligned}
p_{k|k-1}(X_k) &= \int p_k(X_k|X_{k-1})p_X(X_{k-1})\mathrm{d}X \\
&= \iint \cdots \int \left[\prod_{i=1}^{n} \pi_k\left(x^i_k \middle| \, x^i_{k-1}\right)\right]\left[\prod_{i'=1}^{n} p^{i'}_{k-1|k-1}(x^{i'}_{k-1})\right]\mathrm{d}x^1_{k-1}\mathrm{d}x^2_{k-1}\cdots\mathrm{d}x^n_{k-1} \\
&= \prod_{i}^{n}\int \pi_k(x^i_k|x^i_{k-1})p^i_{k-1|k-1}(x^i_{k-1})\mathrm{d}x^i_{k-1} = \prod_{i}^{n} p^i_{k|k-1}(x^i_k)
\end{aligned}$$

- Each object predicted independently of other objects.

PREDICTED MIXTURE DENSITY

- Mixture posterior,

$$p_{k-1|k-1}(X_{k-1}) = \sum_h w^h_{k-1|k-1} p^h_{k-1|k-1}(X_{k-1})$$

- Predicted mixture density,

$$\begin{aligned}
p_{k|k-1}(X_k) &= \int p_k(X_k|X_{k-1}) p_X(X_{k-1}) \mathrm{d}X \\
&= \int p_k(X_k|X_{k-1}) \left[\sum_h w^h p^h_{k-1|k-1}(X_{k-1}) \right] \mathrm{d}X \\
&= \sum_h w^h_{k-1|k-1} \int p_k(X_k|X_{k-1}) p^h_{k-1|k-1}(X_{k-1}) \mathrm{d}X = \sum_h w^h_{k|k-1} p^h_{k|k-1}(X_k)
\end{aligned}$$

- Each hypothesis can be predicted independently of other hypotheses, and the hypothesis probabilities (weights) remain the same, $w^h_{k|k-1} = w^h_{k-1|k-1}$.

LINEAR GAUSSIAN TRANSITION DENSITY, GAUSSIAN POSTERIOR

Gaussian mixture posterior and linear Gaussian transition density

$$p_{k-1|k-1}(X_{k-1}) = \sum_h w^h_{k-1|k-1} \prod_{i=1}^{n} \mathcal{N}\left(x^i_{k-1}\,;\, m^{i,h}_{k-1|k-1}, P^{i,h}_{k-1|k-1}\right)$$

$$p_k\left(X_k \middle|\, X_{k-1}\right) = \prod_{i=1}^{n} \mathcal{N}\left(x^i_k\,;\, F_{k-1}x^i_{k-1}, Q_{k-1}\right)$$

Predicted density

$$p_{k|k-1}(X_k) = \sum_h w^h_{k|k-1} \prod_{i}^{n} \mathcal{N}\left(x^i_k\,;\, m^{i,h}_{k|k-1}, P^{i,h}_{k|k-1}\right)$$

$$\begin{aligned} w^h_{k|k-1} &= w^h_{k-1|k-1} \\ m^{i,h}_{k|k-1} &= F_{k-1} m^{i,h}_{k-1|k-1} \\ P^{i,h}_{k|k-1} &= F_{k-1} P^{i,h}_{k-1|k-1} F^{\mathrm{T}}_{k-1} + Q_{k-1} \end{aligned}$$

PREDICTION VISUALIZATIONS: 1D, RANDOM WALK

Linear Gaussian example

Scalar object states

- $X = [x^1,\ x^2]$

Motion model: random walk

- $\pi_k(x_k|x_{k-1}) = \mathcal{N}\left(x_k\ ;\ x_{k-1}, 0.25\right)$

Posterior

- $p_{k-1|k-1}(X) = \sum_{h=1}^{4} w_{k-1|k-1}^{h} \prod_{i=1}^{2} \mathcal{N}\left(x^i\ ;\ \mu_{k-1|k-1}^{i,h}, P_{k-1|k-1}^{i,h}\right)$

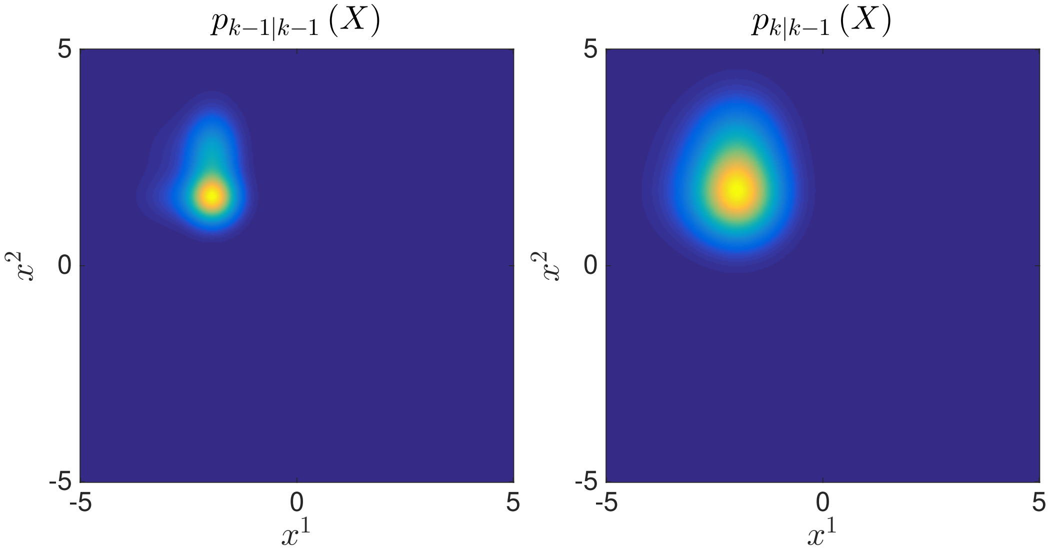
Prediction

- $p_{k|k-1}(X) = \sum_{h=1}^{4} w_{k-1|k-1}^{h} \prod_{i=1}^{2} \mathcal{N}\left(x^i\ ;\ \mu_{k-1|k-1}^{i,h}, P_{k-1|k-1}^{i,h} + 0.25\right)$

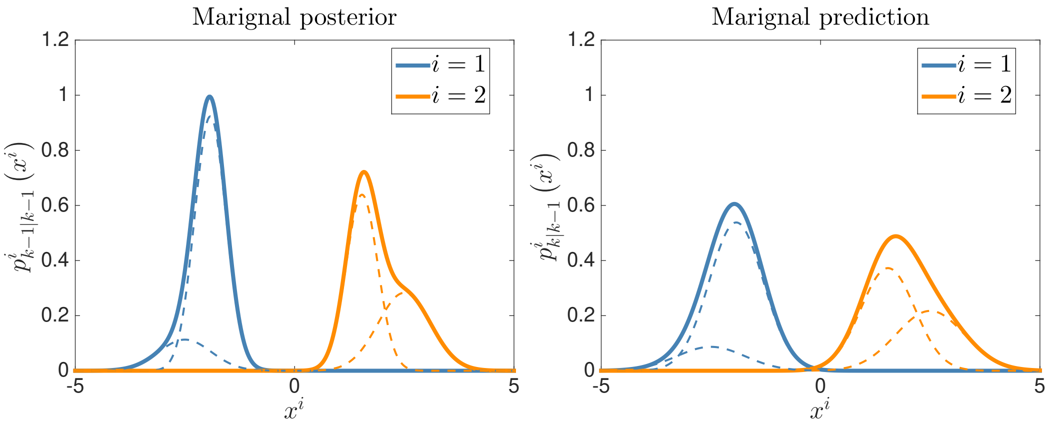
Visualizations

- n object: $p_{k|k-1}(X_k)$
- Marginal: $p_{k|k-1}^{i}(x_k^i)$

PREDICTION VISUALIZATIONS: 1D, RANDOM WALK



PREDICTION VISUALIZATIONS: 1D, RANDOM WALK



PREDICTION VISUALIZATIONS: 2D, CONSTANT VELOCITY

Linear Gaussian example: state with 2D object position, 2D velocity

Four objects

- $X_k = [x_k^1,\ x_k^2,\ x_k^3,\ x_k^4]$

Motion model: constant velocity

- $\pi_k(x_k|x_{k-1}) = \mathcal{N}\left(x_k\,;\, Fx_{k-1}, Q\right)$

Posterior

- $p_{k-1|k-1}(X) = \prod_{i=1}^{4} \mathcal{N}\left(x^i\,;\, \mu^i_{k-1|k-1}, P^i_{k-1|k-1}\right)$

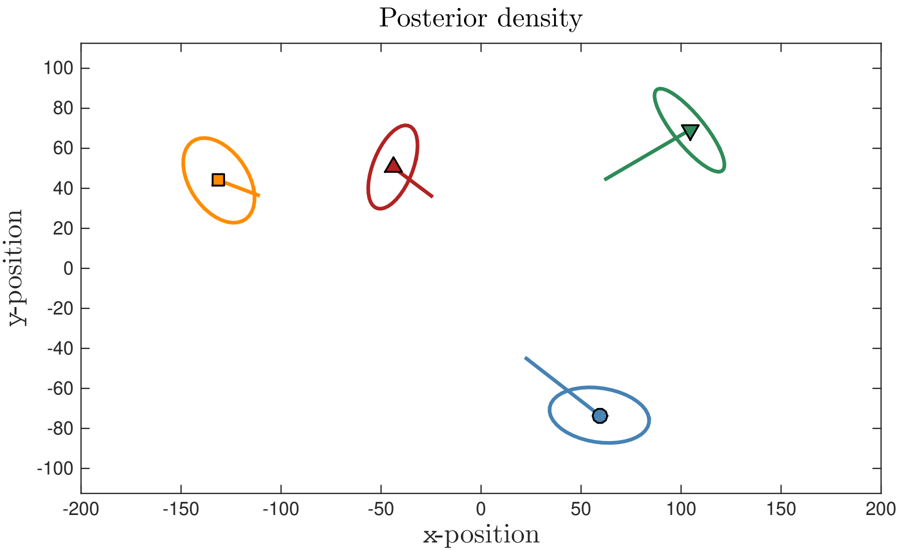
Prediction

- $p_{k|k-1}(X) = \prod_{i=1}^{4} \mathcal{N}\left(x^i\,;\, F\mu^i_{k-1|k-1}, FP^i_{k-1|k-1}F^{\mathsf{T}} + Q\right)$

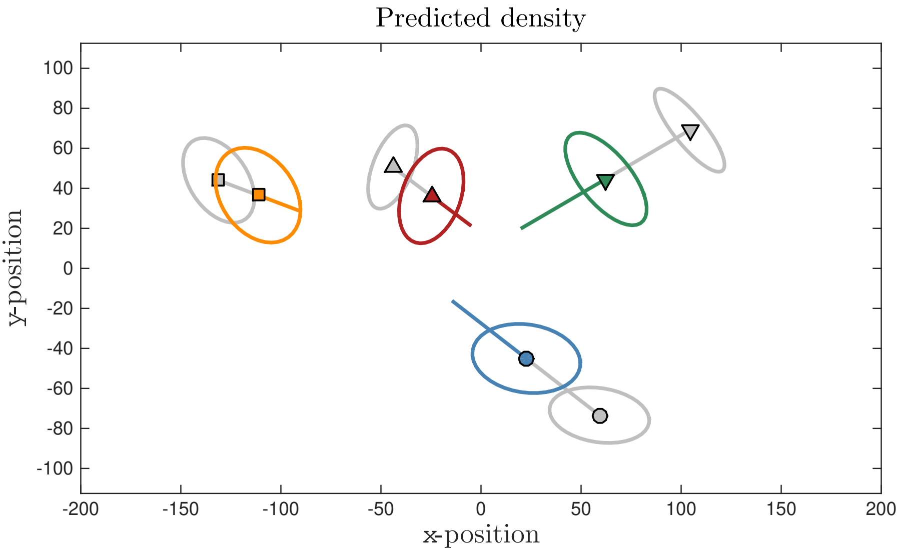
Visualizations

- Marginal: $p^i_{k|k-1}(x^i_k)$

PREDICTION VISUALIZATIONS: 2D, CONSTANT VELOCITY



PREDICTION VISUALIZATIONS: 2D, CONSTANT VELOCITY



SUMMARY

- Objects move independent of one another
- n object hypotheses can be predicted independently
- Hypothesis weights unaffected by the prediction.

Independent objects?

Multi-Object Tracking

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ARE THE OBJECTS INDEPENDENT?

Several independence assumptions in the modelling:

- Initial prior density
- Measurements
- Object motion

Do all these assumptions mean that the object states

$$X = \left[x^1,\ x^2,\ \dots\ x^n\right]$$

are independent?

No, actually not! At least not in the general case.

ARE THE OBJECTS INDEPENDENT?

- Mixture densities,

$$p(X) = \sum_h w^h p^h(X) = \sum_h w^h \prod_{i=1}^n p^{i,h}(x^i)$$
$$= \sum_h \Pr[h] p(X|h) = \sum_h \Pr[h] \prod_{i=1}^n p^i(x^i|h)$$

  where each hypothesis h correspond to sequence of n object data associations.

- Conditioned on a specific hypothesis h the objects are independent,

$$p(X|h) = \prod_{i=1}^n p^i(x^i|h)$$

ARE THE OBJECTS INDEPENDENT?

- If the objects were independent, we should be able to factorize the n object density,

$$p(X) = \sum_h w^h \prod_{i=1}^n p^{i,h}(x^i) \quad \stackrel{?}{=} \quad \prod_{i=1}^n \left(\sum_{h^i} w^{h^i} p^{i,h^i}(x^i) \right)$$

- **In the general case**, the mixture densities cannot be factorized,

$$p(X) = \sum_h w^h \prod_{i=1}^n p^{i,h}(x^i) \quad \stackrel{!}{\neq} \quad \prod_{i=1}^n \left(\sum_{h^i} w^{h^i} p^{i,h^i}(x^i) \right)$$

- However, there are tracking algorithms that are exceptions to this, e.g.,
  - Global Nearest Neighbor (GNN) filter
  - Joint Probabilistic Data Association (JPDA) filter

  Approximate the posterior n object densities, such that the objects are independent.

Data association as an optimisation problem

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Introduction

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AN INCREASING NUMBER OF HYPOTHESES

Consider one Bayes recursion:

- Mixture posterior,

$$p_{k-1|k-1}(X_{k-1}) = \sum_h w^h_{k-1|k-1} p^h_{k-1|k-1}(X_{k-1})$$

- Predicted mixture with same number of hypotheses

$$p_{k|k-1}(X_k) = \sum_h w^h_{k|k-1} p^h_{k|k-1}(X_k)$$

- Updated mixture posterior

$$p_{k|k}(X_k) \propto \sum_h \sum_{\theta_k \in \Theta_k} \tilde{w}^{\theta_k|h} w^h_{k|k-1} p^{h,\theta_k}_{k|k}(X_k)$$

For every predicted hypothesis, we get $N_A(m_k, n)$ new hypotheses.

HANDLING THE INCREASING NUMBER OF HYPOTHESES

- The number of hypotheses in the posterior density increases very rapidly
- We have to approximate to achieve computational tractability.

- **Idea:** Find a subset of data associations $\tilde{\Theta}_k \subset \Theta_k$, such that $|\tilde{\Theta}_k| \ll |\Theta_k|$, and for which $\theta_k \in \tilde{\Theta}_k$ have large weights $\tilde{w}^{\theta_k|h}$.

- **Challenge:** Avoid computing all $\theta_k \in \Theta_k$ and comparing their weights.

- **A solution:** Pose as optimisation problem, specifically as an **assignment problem**.

ASSIGNMENT PROBLEM

- Three workers: w^1, w^2, w^3. Three tasks: t^1, t^2, t^3
- The costs for each worker and task are

Worker\Task	t^1	t^2	t^3
w^1	5	8	7
w^2	8	12	7
w^3	4	8	5

- Properties:
  - Each worker w^i can only solve one task.
  - Each task t^j can only be solved by one worker.
- **Assignment problem:** assign workers to tasks, such that each worker has something to do, each task is performed, and the total cost is as low as possible.
- Equivalent to assign tasks to workers. Same assignment is optimal.

EXAMPLE ASSIGNMENTS

Example assignments

- $w^1 \rightarrow t^1$
- $w^2 \rightarrow t^2$
- $w^3 \rightarrow t^3$

W\T	t^1	t^2	t^3
w^1	**5**	8	7
w^2	8	**12**	7
w^3	4	8	**5**

- Cost $= 5 + 12 + 5 = 22$

- $w^1 \rightarrow t^3$
- $w^2 \rightarrow t^1$
- $w^3 \rightarrow t^2$

W\T	t^1	t^2	t^3
w^1	5	8	**7**
w^2	**8**	12	7
w^3	4	**8**	5

- Cost $= 8 + 8 + 7 = 23$

- $w^1 \rightarrow t^2$
- $w^2 \rightarrow t^3$
- $w^3 \rightarrow t^1$

W\T	t^1	t^2	t^3
w^1	5	**8**	7
w^2	8	12	**7**
w^3	**4**	8	5

- Cost $= 4 + 8 + 7 = 19$

OPTIMAL EXAMPLE ASSIGNMENT

Example assignments

- $w^1 \to t^2$
- $w^2 \to t^3$
- $w^3 \to t^1$

W\T	t^1	t^2	t^3
w^1	5	**8**	7
w^2	8	12	**7**
w^3	**4**	8	5

- Cost $= 4 + 8 + 7 = 19$

- For this example, $w^1 \to t^2$, $w^2 \to t^3$, $w^3 \to t^1$ is optimal, i.e., lowest cost.

COST MATRIX AND ASSIGNMENT MATRIX

- Cost matrix L with elements $L^{i,j}$, row i and column j
  - $L^{i,j}$ is the cost of assigning w^i to t^j.
- Assignment matrix A with elements $A^{i,j}$
  - $A^{i,j} = 1$ if w^i is assigned to t^j.
  - $A^{i,j} = 0$ otherwise
- The cost of an assignment can be expressed as

$$\text{Cost} = \sum_i \sum_j A^{i,j} L^{i,j} = \text{tr}\left(A^{\mathrm{T}} L\right)$$

3 workers, 3 tasks

Cost matrix

$$L = \begin{bmatrix} 5 & 8 & 7 \\ 8 & 12 & 7 \\ 4 & 8 & 5 \end{bmatrix}$$

Assignment matrix

$$w^1 \to t^2$$

$$w^2 \to t^3$$

$$w^3 \to t^1$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

EXAMPLE ASSIGNMENTS

Example assignments

- $w^1 \to t^1$
- $w^2 \to t^2$
- $w^3 \to t^3$

$$\text{Cost} = \operatorname{tr}\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}5 & 8 & 7\\8 & 12 & 7\\4 & 8 & 5\end{bmatrix}\right) = 5 + 12 + 5 = 22$$

- $w^1 \to t^3$
- $w^2 \to t^1$
- $w^3 \to t^2$

$$\text{Cost} = \operatorname{tr}\left(\begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}5 & 8 & 7\\8 & 12 & 7\\4 & 8 & 5\end{bmatrix}\right) = 8 + 8 + 7 = 23$$

- $w^1 \to t^2$
- $w^2 \to t^3$
- $w^3 \to t^1$

$$\text{Cost} = \operatorname{tr}\left(\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}5 & 8 & 7\\8 & 12 & 7\\4 & 8 & 5\end{bmatrix}\right) = 4 + 8 + 7 = 19$$

OPTIMAL ASSIGNMENT PROBLEM

- Given L, we seek the solution $A^\star$ to the following constrained minimization problem,

$$
\begin{array}{lll}
\text{minimize} & \operatorname{tr}\left(A^{\mathsf{T}} L\right) & \\
\text{subject to} & A^{i,j} \in \{0,1\}, \quad \forall\, i,j & \text{(Either assigned, or not)} \\
 & \sum_j A^{i,j} = 1, \quad \forall i & \text{(Each worker is assigned to exactly one task)} \\
 & \sum_i A^{i,j} = 1, \quad \forall j & \text{(Each task is assigned to exactly one worker)}
\end{array}
$$

- Data association as an assignment problem
  - Objects instead of workers
  - Detections instead of tasks
  - Assign objects to detections (or detections to objects)
- We will learn
  - Object tracking assignment matrix A, with constraints
  - Object tracking cost matrix L
  - Standard solvers

DA as an optimisation problem

Multi-Object Tracking

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DATA ASSOCIATION AS AN OPTIMISATION PROBLEM

- Posterior n object mixture density (time indexing omitted for brevity)

$$p(X) \propto \sum_{h} \sum_{\theta \in \Theta} \tilde{w}^{\theta|h} w^{h} p^{h,\theta}(X)$$

- For a given hypothesis h, the optimal data association $\theta^{\star} \in \Theta$ has maximum weight,

$$\tilde{w}^{\theta^{\star}|h} \geq \tilde{w}^{\theta|h}, \ \forall\, \theta \in \Theta$$

- We wish to solve the optimisation problem

$$\theta^{\star} = \arg\max_{\theta \in \Theta} \tilde{w}^{\theta|h} = \arg\max_{\theta \in \Theta} \prod_{i=1}^{n} \tilde{w}^{\theta^{i}|h}$$

- Equivalently, we can solve,

$$\theta^{\star} = \arg\max_{\theta \in \Theta} \log\left(\prod_{i=1}^{n} \tilde{w}^{\theta^{i}|h}\right) = \arg\max_{\theta \in \Theta} \sum_{i=1}^{n} \log\left(\tilde{w}^{\theta^{i}|h}\right)$$

A REFORMULATION OF THE OPTIMISATION PROBLEM

- Equivalently, minimise the sum of negative log-weights,

$$\theta^\star = \arg\max_{\theta \in \Theta} \sum_{i=1}^{n} \log\left(\tilde{w}^{\theta^i|h}\right) = \arg\min_{\theta \in \Theta} - \sum_{i=1}^{n} \log\left(\tilde{w}^{\theta^i|h}\right)$$
$$= \arg\min_{\theta \in \Theta} \sum_{i=1}^{n} - \log\left(\tilde{w}^{\theta^i|h}\right)$$

- Express as an **assignment problem** with cost matrix L and assignment matrix A,

$$A^\star = \arg\min_{A} \operatorname{tr}\left(A^\top L\right)$$

- **We can apply standard combinatorial optimisation algorithms to solve for $A^\star$!**

Assignment matrix in n object tracking

Multi-Object Tracking

Karl Granström

ASSIGNMENT MATRIX IN *N* OBJECT TRACKING

- A should have a structure such that it corresponds to a unique $\theta \in \Theta$.
- The assignment matrix A is an $n \times (m + n)$ matrix
  - n objects
  - m detections and n misdetections
- $A^{i,j} \in \{0, 1\}$, either an object and detection/misdetection is assigned, or not
- Relationship between the association θ and the assignment matrix A:

$$\begin{array}{rlcl} \text{Detection:} & \theta^i = j & \Rightarrow & A^{i,j} = 1 \\ \text{Misdetection:} & \theta^i = 0 & \Rightarrow & A^{i,m+i} = 1 \end{array}$$

and remaining elements are all zero.

EXAMPLE ASSIGNMENT MATRIX

$n = 2$, $m = 1$, $N_A(1, 2) = 3$

- Two misdetections,

$$\theta^1 = 0 \;\Rightarrow\; A^{1,2} = 1$$
$$\theta^2 = 0 \;\Rightarrow\; A^{2,3} = 1$$

$$\theta = \begin{bmatrix} 0 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- x^1 associated to z^1,

$$\theta^1 = 1 \;\Rightarrow\; A^{1,1} = 1$$
$$\theta^2 = 0 \;\Rightarrow\; A^{2,3} = 1$$

$$\theta = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- x^2 associated to z^1,

$$\theta^1 = 0 \;\Rightarrow\; A^{1,2} = 1$$
$$\theta^2 = 1 \;\Rightarrow\; A^{2,1} = 1$$

$$\theta = \begin{bmatrix} 0 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Relation between θ and A:

$$\theta^i = j \;\Rightarrow\; A^{i,j} = 1$$
$$\theta^i = 0 \;\Rightarrow\; A^{i,m+i} = 1$$

and remaining elements are all zero.

EXAMPLE ASSIGNMENT MATRIX

$n = 2$, $m = 1$, $N_A(1,2) = 3$

- Two misdetections,

$$\theta = \begin{bmatrix} 0 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- x^1 associated to z^1,

$$\theta = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- x^2 associated to z^1,

$$\theta = \begin{bmatrix} 0 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Note the structure of A:

- $A^{1,3} = A^{2,2} = 0$ for all θ
- $\sum_j A^{i,j} = 1 \; \forall i$
- $\sum_i A^{i,j} \in \{0, 1\} \; \forall j$

STRUCTURE OF ASSIGNMENT MATRIX

- A structure such that A to correspond to a unique $\theta \in \Theta$ is

$$A = \left[\begin{array}{cccc|cccc} A^{1,1} & A^{1,2} & \dots & A^{1,m} & A^{1,m+1} & 0 & \dots & 0 \\ A^{2,1} & A^{2,2} & \dots & A^{2,m} & 0 & A^{2,m+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{n,1} & A^{n,2} & \dots & A^{n,m} & 0 & 0 & \dots & A^{n,m+n} \end{array}\right]$$

- Left $n \times m$ sub-matrix: detections
- Diagonal of right $n \times n$ sub-matrix: misdetections
- $\sum_j A^{i,j} = 1 \ \forall i$
- $\sum_i A^{i,j} \in \{0, 1\} \ \forall j$

$n = 2$, $m = 1$

$$A = \left[\begin{array}{c|cc} A^{1,1} & A^{1,2} & 0 \\ A^{2,1} & 0 & A^{2,3} \end{array}\right]$$

WHY ELEMENTS THAT ALWAYS ARE ZERO?

- If right sub-matrix replaced by vector: at most one misdetected object, i.e., not all data associations can be represented by A.

$n = 2$, $m = 1$

$$A = \left[\begin{array}{c|c} A^{1,1} & A^{1,2} \\ A^{2,1} & A^{2,2} \end{array}\right] \quad \Rightarrow \quad \text{Only two unique assignment matrices: } \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right]$$

- If right sub-matrix replaced by full matrix: multiple different assignment matrices correspond to the same data association

$n = 2$, $m = 1$

$$A = \left[\begin{array}{c|cc} A^{1,1} & A^{1,2} & A^{1,3} \\ A^{2,1} & A^{2,2} & A^{2,3} \end{array}\right] \quad \Rightarrow \quad \text{Correspond to same } \theta\text{: } A = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], A' = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

Cost matrix in n object tracking

Multi-Object Tracking

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LOG-LIKELIHOODS

- Minimise the sum of negative log-weights,

$$\theta^\star = \arg\min_{\theta\in\Theta} \sum_{i=1}^{n} -\log\left(\tilde{w}^{\theta^i|h}\right)$$

- Given hypothesis h, the un-normalised association weights are

$$\tilde{w}^{\theta^i|h} = \begin{cases} \int \left(1 - P^{\mathrm{D}}(x^i)\right) p^{i,h}(x^i)\mathrm{d}x^i & \text{if } \theta^i = 0 \\ \int \frac{P^{\mathrm{D}}(x^i) g_k(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})} p^{i,h}(x^i)\mathrm{d}x^i & \text{if } \theta^i \neq 0 \end{cases}$$

- Given hypothesis h

$$\ell^{i,0,h} = \log\left(\int \left(1 - P^{\mathrm{D}}(x^i)\right) p^{i,h}(x^i)\mathrm{d}x^i\right) \text{ is log-likelihood of misdetecting } x^i$$

$$\ell^{i,j,h} = \log\left(\int \frac{P^{\mathrm{D}}(x^i) g_k(z^j|x^i)}{\lambda_c(z^j)} p^{i,h}(x^i)\mathrm{d}x^i\right) \text{ is log-likelihood of associating } x^i \text{ to } z^j$$

EXAMPLE

Linear Gaussian models

- $P^{\mathrm{D}}(x) = P^{\mathrm{D}}$
- $\lambda_c(c) = \bar{\lambda}_c / V$
- $g_k(z|x) = \mathcal{N}(z\,;\, Hx, R)$
- $p^{i,h}(x^i) = \mathcal{N}\left(x^i\,;\, \mu^{i,h}, P^{i,h}\right)$

We get the following log-likelihoods,

$$
\begin{aligned}
\ell^{i,0,h} &= \log\left(1 - P^{\mathrm{D}}\right) \\
\ell^{i,j,h} &= \log\left(\frac{P^{\mathrm{D}} V}{\bar{\lambda}_c}\right) - \frac{1}{2}\log\left(\det\left(2\pi S^{i,h}\right)\right) - \frac{1}{2}\left(z^j - \hat{z}^{i,h}\right)^{\mathrm{T}}\left(S^{i,h}\right)^{-1}\left(z^j - \hat{z}^{i,h}\right)
\end{aligned}
$$

where $\hat{z}^{i,h} = H\mu^{i,h}$ and $S^{i,h} = HP^{i,h}H^{\mathrm{T}} + R$

COST MATRIX FOR AN ASSIGNMENT PROBLEM

- Minimise the sum of negative log-weights,

$$\theta^\star = \arg\min_{\theta \in \Theta} \sum_{i=1}^{n} -\log\left(\tilde{w}^{\theta^i|h}\right)$$

- The association cost can be re-written as

$$\sum_{i=1}^{n} -\log\left(\tilde{w}^{\theta^i|h}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} A^{i,j}\left(-\ell^{i,j,h}\right) + \sum_{i'=1}^{n} A^{i',m+i'}\left(-\ell^{i',0,h}\right)$$

provided that A is the assignment matrix that corresponds to the association θ

- We need a cost matrix L_h that allows the cost to be written as

$$\sum_{i=1}^{n} -\log\left(\tilde{w}^{\theta^i|h}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m+n} A^{i,j} L^{i,j,h} = \mathrm{tr}(A^{\mathrm{T}} L^h),$$

while L^h is also facilitating finding assignments A that correspond to unique $\theta \in \Theta$.

COST MATRIX

The cost matrix L is an $n \times (m+n)$ matrix (h omitted for brevity)

$$L = \left[\begin{array}{cccc|cccc} -\ell^{1,1} & -\ell^{1,2} & \dots & -\ell^{1,m} & -\ell^{1,0} & \infty & \dots & \infty \\ -\ell^{2,1} & -\ell^{2,2} & \dots & -\ell^{2,m} & \infty & -\ell^{2,0} & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\ell^{n,1} & -\ell^{n,2} & \dots & -\ell^{n,m} & \infty & \infty & \dots & -\ell^{n,0} \end{array}\right]$$

- Left $n \times m$ sub-matrix: detections
- Diagonal of right $n \times n$ sub-matrix: misdetections
- Off-diagonal of right $n \times n$ sub-matrix:
  - $-\log(0) = \infty$
  - Avoids that optimisation returns multiple A that correspond to same $\theta \in \Theta$.
  - Note: for valid associations, the corresponding elements in A are always zero!

ASSIGNMENT COST

- With an assignment matrix A, and a cost matrix L as on the previous slide, the assignment cost can be expressed in the desired way,

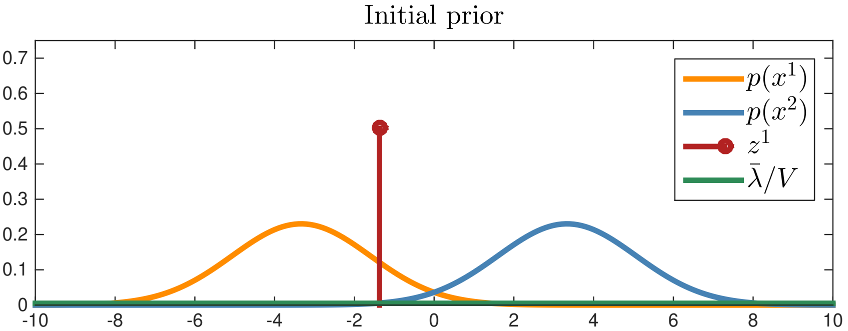
  $$\mathrm{tr}(A^{\mathsf{T}}L) = \sum_{i=1}^{n}\sum_{j=1}^{m+n} A^{i,j}L^{i,j}$$

  and non-unique assignments will have cost $= \infty$.
- For a hypothesis h, the association weight relates to A and L^h as follows

  $$\tilde{w}^{\theta|h} = \exp\left(-\mathrm{tr}\left(A^{\mathsf{T}}L^h\right)\right)$$

EXAMPLE

1D scenario, $n = 2$, $m = 1$, $N_A(1,2) = 3$



$$L = \begin{bmatrix} -\ell^{1,1} & -\ell^{1,0} & \infty \\ -\ell^{2,1} & \infty & -\ell^{2,0} \end{bmatrix} = \begin{bmatrix} -1.4745 & 0.2877 & \infty \\ -0.9210 & \infty & 0.2877 \end{bmatrix}$$

EXAMPLE

1D scenario for $n = 2$, $m = 1$, $N_A(1,2) = 3$

$$L = \begin{bmatrix} -\ell^{1,1} & -\ell^{1,0} & \infty \\ -\ell^{2,1} & \infty & -\ell^{2,0} \end{bmatrix} = \begin{bmatrix} -1.4745 & 0.2877 & \infty \\ -0.9210 & \infty & 0.2877 \end{bmatrix}$$

$$\text{Two misdetections: } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \operatorname{tr}(A^\intercal L) = -\left(\ell^{1,0} + \ell^{2,0}\right) = 0.5754$$

$$x^2 \text{ associated to } z^1\text{: } A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{tr}(A^\intercal L) = -\left(\ell^{1,0} + \ell^{2,1}\right) = -0.6333$$

$$x^1 \text{ associated to } z^1\text{: } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \operatorname{tr}(A^\intercal L) = -\left(\ell^{1,1} + \ell^{2,0}\right) = -1.1868$$

EXAMPLE

1D scenario for $n = 2$, $m = 1$, $N_A(1,2) = 3$

$$L = \begin{bmatrix} -1.4745 & 0.2877 & \infty \\ -0.9210 & \infty & 0.2877 \end{bmatrix}$$

$$\theta = \begin{bmatrix} 0 & 0 \end{bmatrix} \Rightarrow \mathrm{tr}(A^\top L) = 0.5754 \qquad w^\theta = 0.10$$

$$\theta = \begin{bmatrix} 0 & 1 \end{bmatrix} \Rightarrow \mathrm{tr}(A^\top L) = -0.6333 \qquad w^\theta = 0.33$$

$$\theta = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow \mathrm{tr}(A^\top L) = -1.1868 \qquad w^\theta = 0.57$$

Note: $w^\theta \propto \tilde{w}^\theta = \exp(-\mathrm{tr}(A^\top L))$

Optimisation algorithms for data association

Multi-Object Tracking

Karl Granström

OPTIMAL ASSIGNMENT PROBLEM

$$
\begin{array}{lll}
\text{minimize} & \operatorname{tr}\left(A^{\mathsf{T}} L\right) & \\
\text{subject to} & A^{i,j} \in \{0,1\}, & i,j \in \{1,\ldots,n\} \times \{1,\ldots,n+m\} \\
& \sum_{j=1}^{n+m} A^{i,j} = 1, & i \in \{1,\ldots,n\} \\
& \sum_{i=1}^{n} A^{i,j} \in \{0,1\}, & j \in \{1,\ldots,n+m\}
\end{array}
$$

Some solvers:

- Hungarian, Auction, Jonker-Volgenant-Castanon (JVC): find best solution $A^{\star}$
- Murty's: finds M best assignments, ranked in order of increasing assignment cost
- Gibbs sampling: sub-optimal, but computationally efficient, alternative to finding M best assignments.

FINDING THE BEST ASSIGNMENT(S)

- Hungarian, Auction, and JVC are algorithms that compute the optimal $A^\star$, i.e,.

$$\tilde{w}^{\theta^\star|h} \geq \tilde{w}^{\theta|h}, \ \forall\, \theta \in \Theta$$

  where $\theta^\star$ is the corresponding data association.

- Murty's algorithm is an example of an algorithm that computes M ranked assingments $A^{\star_1}, \ldots, A^{\star_M}$, such that

$$\tilde{w}^{\theta^{\star 1}|h} \geq \tilde{w}^{\theta^{\star 2}|h} \geq \ldots \geq \tilde{w}^{\theta^{\star M}|h}$$
$$\tilde{w}^{\theta^{\star M}|h} \geq \tilde{w}^{\theta|h}, \ \forall\, \theta \in \Theta \backslash \{\theta^{\star m}\}_{m=1}^{M}$$

  where $\theta^{\star_1}, \ldots, \theta^{\star_M}$ are the corresponding data associations.

FINDING GOOD ASSIGNMENTS

- Gibbs sampling is a Markov-Chain Monte-Carlo method.
- It can be used to find M assignments $A^{\star 1}, \ldots, A^{\star M}$ with high likelihoods,

$$\tilde{w}^{\theta^{\star 1}|h} \geq \tilde{w}^{\theta^{\star 2}|h} \geq \ldots \geq \tilde{w}^{\theta^{\star M}|h}$$

  where $\theta^{\star 1}, \ldots, \theta^{\star M}$ are the corresponding associations.
- No guarantees, it is possible that

$$\tilde{w}^{\theta^{\star M}|h} < \tilde{w}^{\theta|h}, \text{ for some } \theta \in \Theta \backslash \{\theta^{\star m}\}_{m=1}^{M}$$

- Computationally efficient; empirical evidence shows that it can yield good tracking performance.

SUMMARY

- Find a subset of data associations $\tilde{\Theta}_k \subset \Theta_k$, such that $|\tilde{\Theta}_k| \ll |\Theta_k|$, and for which $\theta_k \in \tilde{\Theta}_k$ have large weights $\tilde{w}^{\theta_k|h}$.
- Cast this task as an optimal assignment problem
  - Input: Cost matrix L with negative log-likelihoods
  - Output: Assignment(s) A that correspond to association(s) $\theta \in \Theta$
- Algorithms
  - $|\tilde{\Theta}_k| = 1$ best association: Hungarian, Auction, JVC
  - $|\tilde{\Theta}_k| = M$ best associations: Murty
  - $|\tilde{\Theta}_k| = M$ good associations: Gibbs sampling

Gating for n object tracking

Multi-Object Tracking

Karl Granström

MOTIVATION

- High number of possible data associations

- Just like in SOT, we would like to a simple way to “reject” the very unlikely ones.

- By using gating, we can
  - drastically reduce the number of data associations
  - lower the computational burden
  - partition the data association problem into sub-problems, each of which has fewer objects and fewer detections

BASIC IDEA

Idea

Basically the same as in SOT:

- Form a gate around the predicted measurements
- Consider only detections within the gates

- Gating often leads to a drastic reduction in the number of data associations

- Just like in SOT, for Gaussian object densities, we consider ellipsoidal gating

ELLIPSOIDAL GATING FOR GAUSSIAN DENSITIES

- Ellipsoidal gating distance for detection j and object i under hypothesis h

$$d^2_{i,j,h} = \left(z^j_k - \hat{z}^{i,h}_{k|k-1}\right)^{\mathrm{T}} \left(S^{i,h}_k\right)^{-1} \left(z^j_k - \hat{z}^{i,h}_{k|k-1}\right)$$

  where $\hat{z}^{i,h}_{k|k-1}$ is the predicted object detection and $S^{i,h}_k$ is the innovation covariance.

- Gating threshold G with probability P_G. If

$$d^2_{i,j,h} \leq G$$

  then z^j_k is considered as possible detection from object i,
  and otherwise that association is ruled out under hypothesis h.

- **Comparison:** the log-likelihood for data association is

$$\ell^{i,j,h} = \log\left(\frac{P^{\mathrm{D}} V}{\bar{\lambda}_c}\right) - \frac{1}{2}\log\left(\det\left(2\pi S^{i,h}_k\right)\right) - \frac{1}{2}\left(z^j_k - \hat{z}^{i,h}_{k|k-1}\right)^{\mathrm{T}} \left(S^{i,h}_k\right)^{-1} \left(z^j_k - \hat{z}^{i,h}_{k|k-1}\right)$$

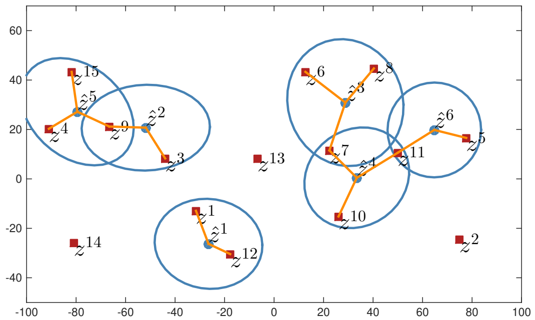
  Note: the larger $d^2_{i,j,h}$ is, the smaller the log-likelihood is.

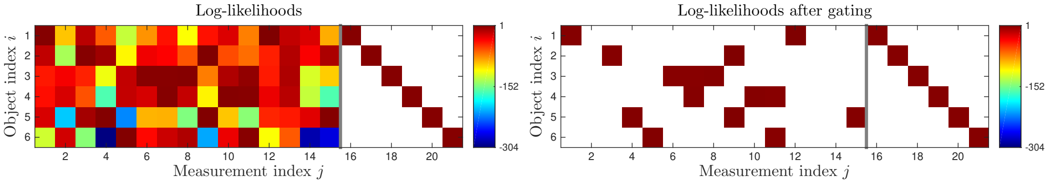
- If z^j_k falls outside the gate of x^i_k, we set $\ell^{i,j,h} = -\infty$

GATING EXAMPLE

2D position measurements

- $m = 15$ detections
- $n = 6$ objects
- $N_A(15, 6) = 6'315'001$ DAs
- Ellipsoidal gating

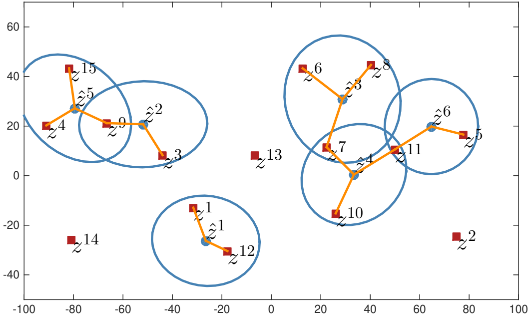




GROUPING BY GATING

2D position measurements

- We have three separate groups of detections and predicted detections.



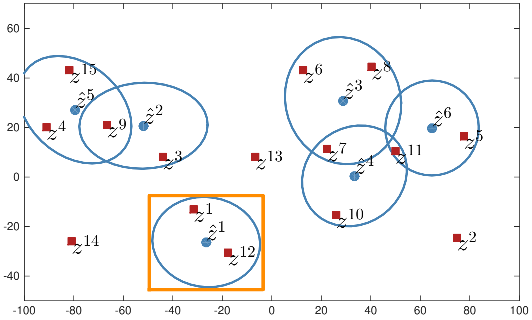
Grouping by gating

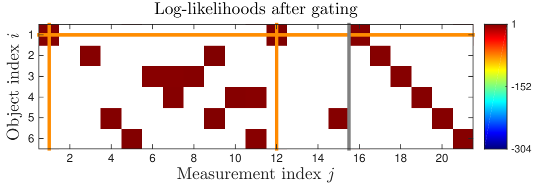
- **Idea:** use the gating to group the detections and objects into smaller groups
- **Motivation:** handling the DA for each group is computationally cheaper

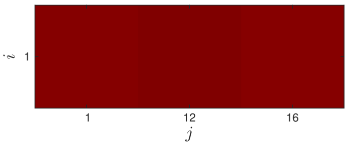
GROUPING BY GATING EXAMPLE

Tot.: $n = 6$, $m = 15$ $N_A(15, 6) > 6 \cdot 10^6$ **Gr. 1:** $n = 1$, $m = 2$, $N_A(2, 1) = 3$

Gr. 2: $n = 2$, $m = 4$ $N_A(4, 2) = 21$ **Gr. 3:** $n = 3$, $m = 6$, $N_A(6, 3) = 229$



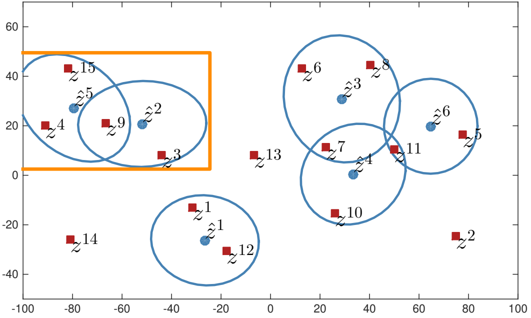


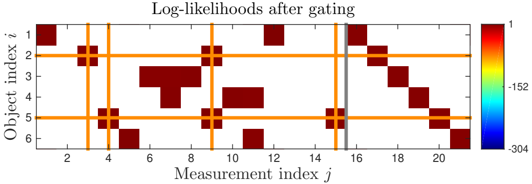


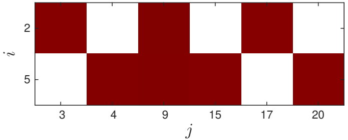
GROUPING BY GATING EXAMPLE

Tot.: $n = 6, \quad m = 15 \quad N_A(15, 6) > 6 \cdot 10^6$ **Gr. 1:** $n = 1, \quad m = 2, \quad N_A(2, 1) = 3$

Gr. 2: $n = 2, \quad m = 4 \quad N_A(4, 2) = 21$ **Gr. 3:** $n = 3, \quad m = 6, \quad N_A(6, 3) = 229$







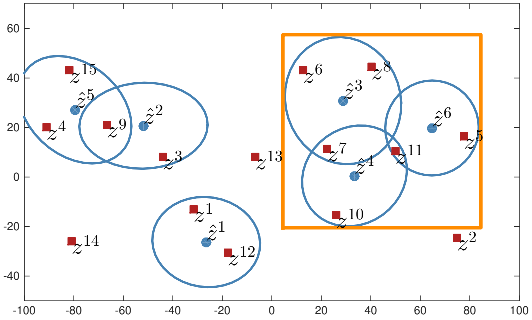
GROUPING BY GATING EXAMPLE

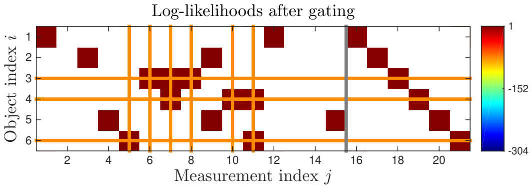
Tot.: $n = 6,\quad m = 15\quad N_A(15,6) > 6 \cdot 10^6$

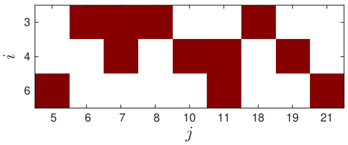
Gr. 1: $n = 1,\quad m = 2,\quad N_A(2,1) = 3$

Gr. 2: $n = 2,\quad m = 4\quad N_A(4,2) = 21$

Gr. 3: $n = 3,\quad m = 6,\quad N_A(6,3) = 229$



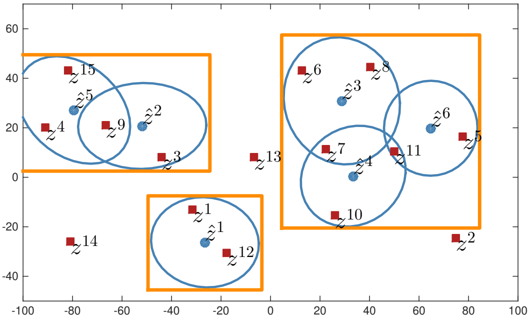


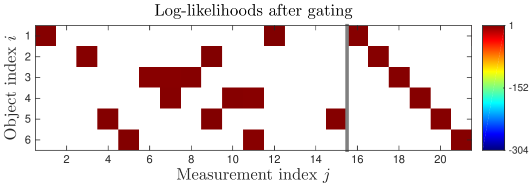


GROUPING BY GATING EXAMPLE

$$\textbf{Tot.:}\quad n=6,\quad m=15\quad N_A(15,6)>6\cdot 10^6 \qquad \textbf{Gr. 1:}\quad n=1,\quad m=2,\quad N_A(2,1)=3$$

$$\textbf{Gr. 2:}\quad n=2,\quad m=4\quad N_A(4,2)=21 \qquad \textbf{Gr. 3:}\quad n=3,\quad m=6,\quad N_A(6,3)=229$$





After gating and grouping:

$$3\times 21\times 229 = 14‘427 \ll 6‘315‘001$$

Considering the gating in the groups:

$$3\times 11\times 41 = 1‘353 < 14‘427$$

SUMMARY

- Gating disregards unlikely data associations
- Reduces the number of hypotheses in the posterior *n* object density
- When the object densities are Gaussian, ellipsoidal gates are a natural choice
- The threshold G, or, equivalently, P_G, must be chosen carefully
  - Too small G: actually probable θ rejected
  - Too large G: too many θ, not enough computational savings

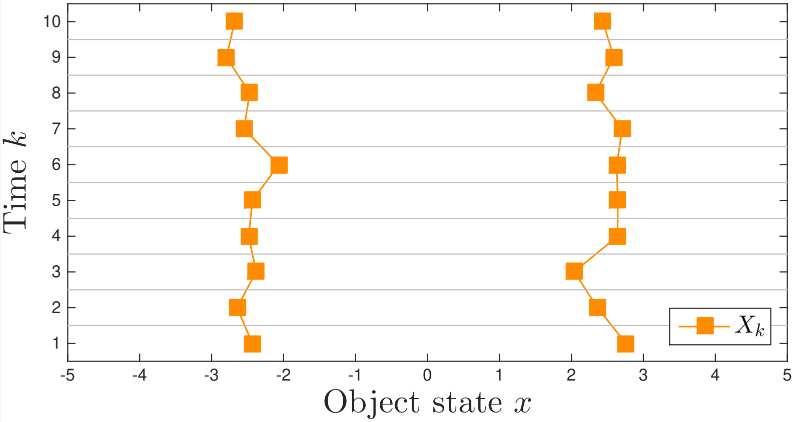
n Object Tracking Algorithms

Multi-Object Tracking

Karl Granström

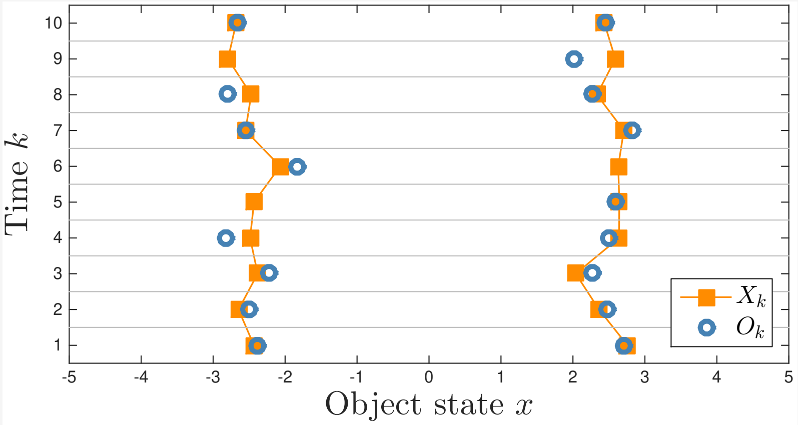
TRACKING N OBJECTS

$n = 2$, 1D states



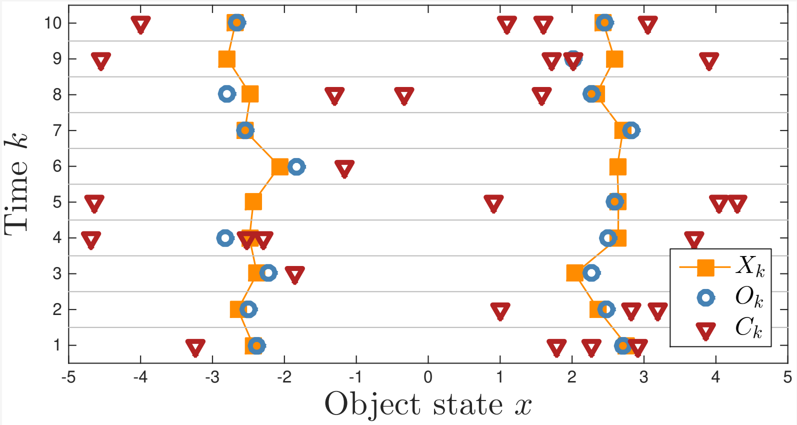
TRACKING N OBJECTS

$n = 2$, 1D states: detections, missed detections



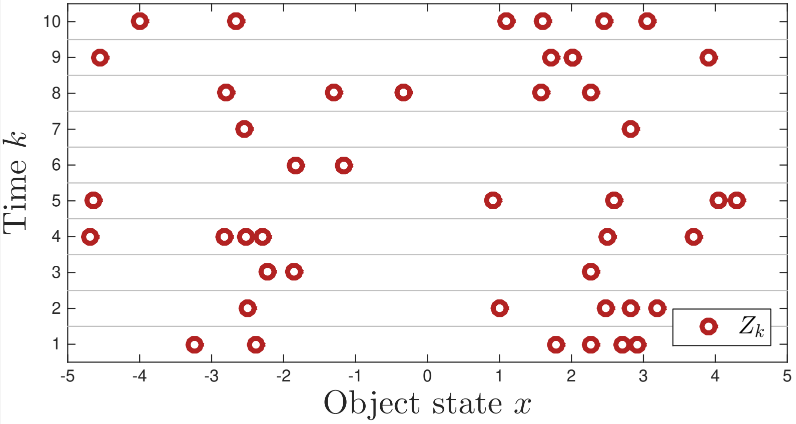
TRACKING N OBJECTS

$n = 2$, 1D states: detections, missed detections, clutter



TRACKING N OBJECTS

$n = 2$, 1D states: measurements



POSTERIOR DENSITY

- We want to process the measurements Z_k and estimate the posterior density,

$$p_{k|k}(X_k) = \sum_{\theta_{1:k} \in \Theta_{1:k}} w_{k|k}^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(X_k) = \sum_{\theta_{1:k} \in \Theta_{1:k}} \Pr[\theta_{1:k}|Z_{1:k}] p(X_k|\theta_{1:k}, Z_{1:k})$$

- The exact posterior is intractable, and approximations are necessary.
- Different tracking algorithms correspond to different approximations
- Two ways of dealing with the hypotheses:
    - Pruning
    - Merging

TWO WAYS OF DEALING WITH THE HYPOTHESES

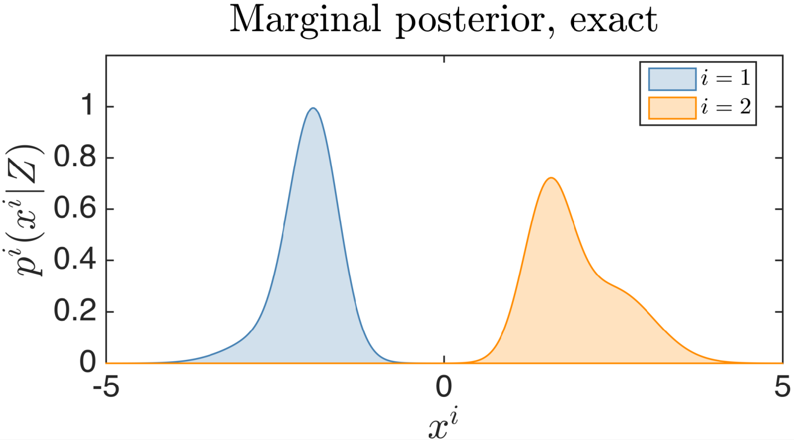
- Pruning:
  - Truncate hypotheses with small weight
  - Can be understood as approximating truncated weights as zero
  - Remaining weight re-normalized
- Merging:
  - Merge multiple hypotheses into a single hypothesis.
  - Mixture density approximated by single density
  - Merge, e.g., such that Kullback-Leibler (KL) divergence is minimized
- Pruning and merging can be combined in several different ways, e.g.,
  - Prune all except a fixed number
  - Prune only those with small weights
  - Merge hypotheses that are “similar”, e.g., small pairwise KL-divergence
  - Merge all hypotheses into a single hypothesis

THREE *N* OBJECT TRACKING ALGORITHMS

- Greedy, take the best association and prune all others
  - Compute optimal assignment
  - Posterior density approximated by a density with a single hypothesis
  - Global nearest neighbor (GNN) filter
- Merge all hypotheses into single hypothesis
  - Compute marginal association probabilities
  - Posterior density approximated by a density with a single hypothesis
  - Joint Probabilistic Data Association (JPDA) filter
- Maintain multiple hypotheses with highest weights, prune the rest
  - Compute M best assignments
  - Posterior density approximated by a density with multiple hypotheses
  - Multi-Hypothesis Tracker (MHT)
  - GNN can be seen as a special case, $M = 1$

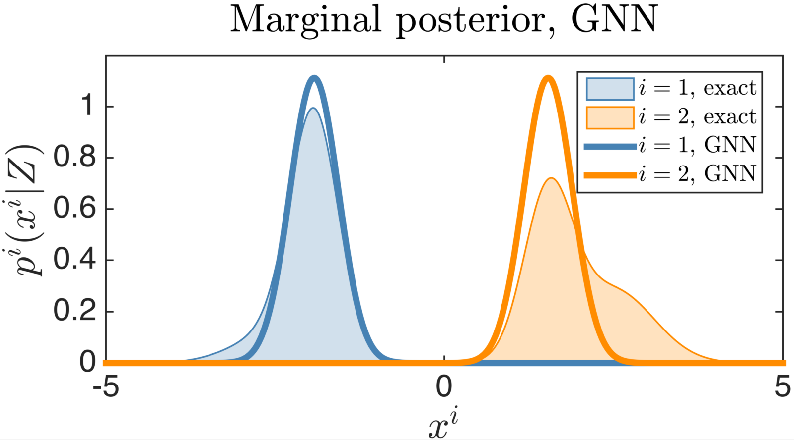
TWO OBJECTS, 1D STATES

Theorectially exact posterior



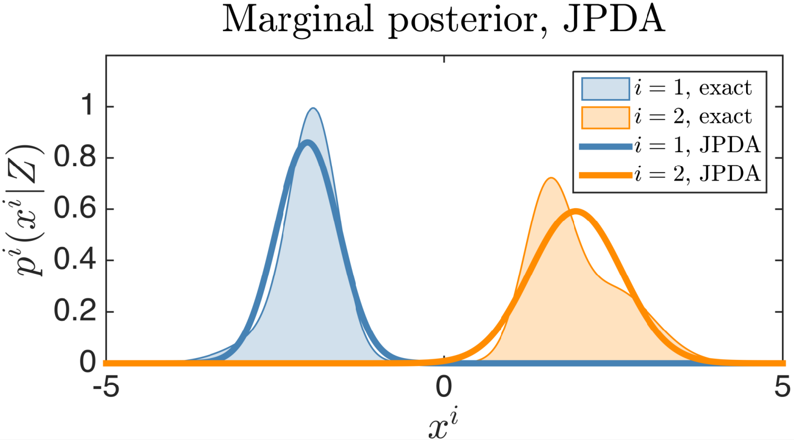
GREEDY, TAKE THE BEST ASSOCIATION

Global nearest neighbour (GNN) filter



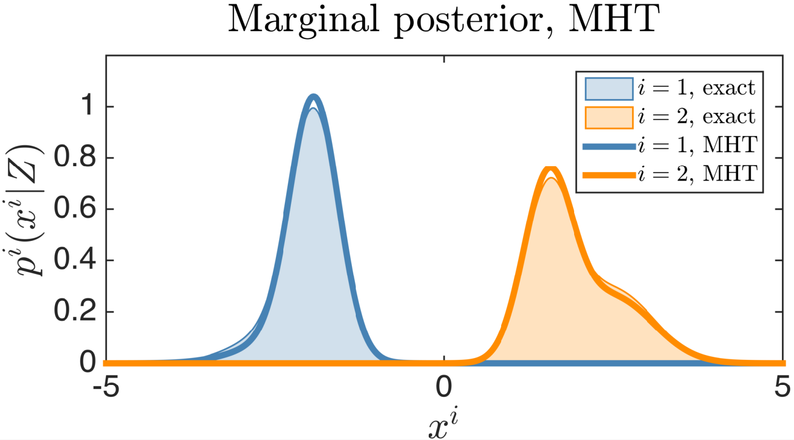
MERGE ALL HYPOTHESES INTO SINGLE HYPOTHESIS

Joint Probabilistic Data Association (JPDA) filter



MAINTAIN MULTIPLE HYPOTHESES

Multi-Hypothesis Tracker (MHT)



Global nearest neighbor

Multi-Object Tracking

Karl Granström

Basic idea, prediction and update

Multi-Object Tracking

Karl Granström

GLOBAL NEAREST NEIGHBOR (GNN) TRACKING

- **Basic idea:**
  - Be greedy: In each update, find $\theta_k^\star$, and prune all other $\theta_k \in \Theta_k$.
- Exact posterior density

$$p_{k|k}(X_k) = \sum_{\theta_{1:k} \in \Theta_{1:k}} w_{k|k}^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(X_k) \qquad \text{is approximated by} \qquad p_{k|k}^{\mathrm{GNN}}(X_k) = p_{k|k}^{\theta_{1:k}^\star}(X_k)$$

where $\theta_{1:k}^\star$ is the sequence of optimal data associations,

$$\theta_1^\star, \qquad \theta_2^\star|\theta_1^\star, \qquad \dots \qquad \theta_k^\star|\theta_{1:k-1}^\star$$

- GNN density parameterized by the object densities

$$p_{k|k}^{\theta_{1:k}^{\star,i}}(x_k^i), \qquad \text{for } i = 1, 2, \dots, n$$

BASIC GNN RECURSION

GNN: Prediction and update pseudo-code

For $k = 1, 2, \ldots, K$

Prediction:

For $i = 1, \ldots, n$: Chapman-Kolmogorov prediction,

$$p^i_{k|k-1}(x^i_k) = \int \pi(x^i_k | x^i_{k-1}) p^i_{k-1|k-1}(x^i_{k-1}) \mathrm{d}x^i_{k-1}$$

Update:

Create cost matrix L_k

Compute optimal association $\theta^\star_k$

For $i = 1, \ldots, n$:

$$p^i_{k|k}(x^i_k) \propto \begin{cases} P^{\mathrm{D}}(x^i_k) g_k(z_k^{\theta^{\star,i}} | x^i_k) p^i_{k|k-1}(x^i_k) & \theta^{\star,i} \neq 0 \text{ Bayes update} \\ (1 - P^{\mathrm{D}}(x^i_k)) p^i_{k|k-1}(x^i_k) & \theta^{\star,i} = 0 \end{cases}$$

GNN ESTIMATOR

- Estimation means to compute object estimates using the posterior density

- Common object estimator: expected value

$$\bar{x}^i_{k|k} = \int x^i_k p^{\mathrm{GNN},i}_{k|k}(x^i_k)\mathrm{d}x^i_k$$

  where $p^{\mathrm{GNN},i}_{k|k}(x^i_k)$ is the marginal density.

- For Gaussian densities, the expected value is one of the density parameters

LINEAR GAUSSIAN GNN: MODELS

- $P^{\mathrm{D}}(x) = P^{\mathrm{D}}$
- $\lambda(c) = \bar{\lambda}/V$
- $g_k(z|x) = \mathcal{N}(z\,;\, Hx, R)$
- $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\, Fx_{k-1}, Q)$
- $p_0(X_0) = \prod_{i=1}^{n} \mathcal{N}\left(x_0^i\,;\, \mu_0^i,\, P_0^i\right)$
- Posterior density parameterized by $\left\{\mu_{k|k}^i, P_{k|k}^i\right\}_{i=1}^{n}$.

LINEAR GAUSSIAN GNN: PREDICTION AND UPDATE

- Predicted density parameters given by Kalman prediction

$$\mu^i_{k|k-1} = F\mu^i_{k-1|k-1}$$
$$P^i_{k|k-1} = FP^i_{k-1|k-1}F^{\mathrm{T}} + Q$$

- Posterior density parameters given by the Kalman update

$$\mu^i_{k|k} = \begin{cases} \mu^i_{k|k-1} + K^i\left(z^{\theta^{\star,i}} - H\mu^i_{k|k-1}\right) & \text{if } \theta^{\star,i}_k \neq 0 \\ \mu^i_{k|k-1} & \text{if } \theta^{\star,i}_k = 0 \end{cases}$$

$$P^i_{k|k} = \begin{cases} P^i_{k|k-1} - K^i HP^i_{k|k-1} & \text{if } \theta^{\star,i}_k \neq 0 \\ P^i_{k|k-1} & \text{if } \theta^{\star,i}_k = 0 \end{cases}$$

$$K^i = P^i_{k|k-1}H^{\mathrm{T}}\left(HP^i_{k|k-1}H^{\mathrm{T}} + R\right)^{-1}$$

GNN examples

Multi-Object Tracking

Karl Granström

VISUALIZATIONS

Original example

Two objects, scalar states

- $X = [x^1,\ x^2]$

Measurement model

- $P^{\mathrm{D}}(x) = 0.85$
- $\lambda_c(c) = 0.3,\ c \in [-5,\ 5]$
- $g(z|x) = \mathcal{N}(z\,;\,x, 0.2)$

Motion model: random walk

- $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\,x_{k-1}, 0.25)$

Initial prior $p_0(X_0) = p_0^1(x_0^1)p_0^2(x_0^2)$

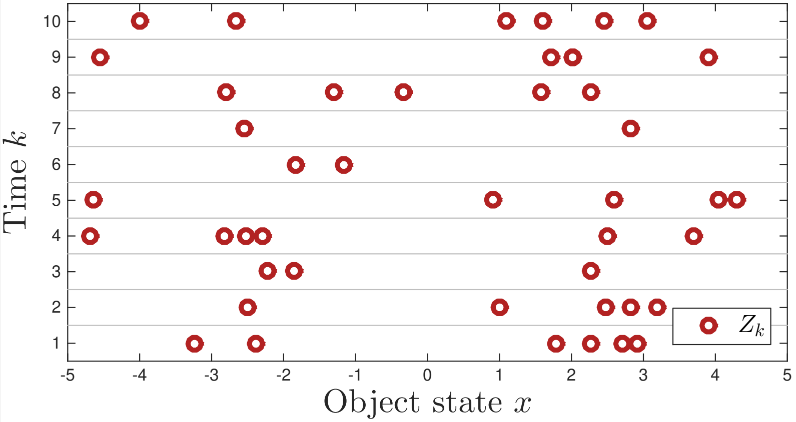
- $p_0^1(x_0^1) = \mathcal{N}(x_0^1\,;\,-2.5, 0.36)$
- $p_0^2(x_0^2) = \mathcal{N}(x_0^2\,;\,2.5, 0.36)$

Visualizations

- Marginal: $p^i_{k|k}(x^i_k)$
- Estimates: $\bar{x}^i_{k|k}$

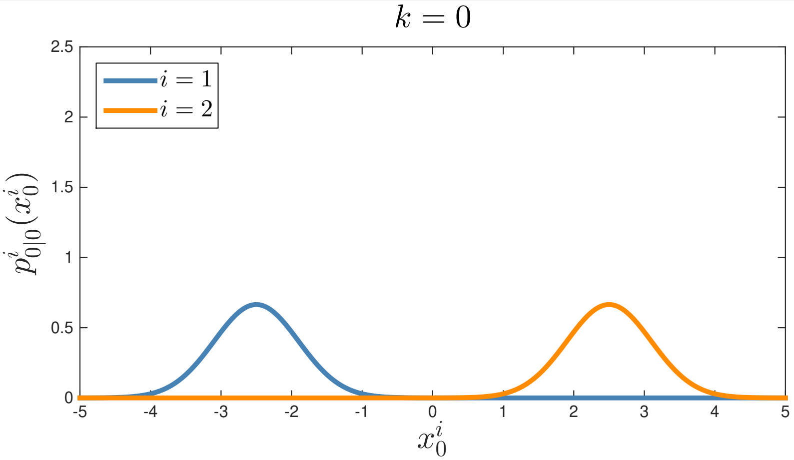
TRACKING N OBJECTS

$n = 2$, 1D states: measurements, $P^{\mathrm{D}} = 0.85$



TRACKING *N* OBJECTS USING GNN FILTER

Global Nearest Neighbor recursion



TRACKING *N* OBJECTS USING GNN FILTER

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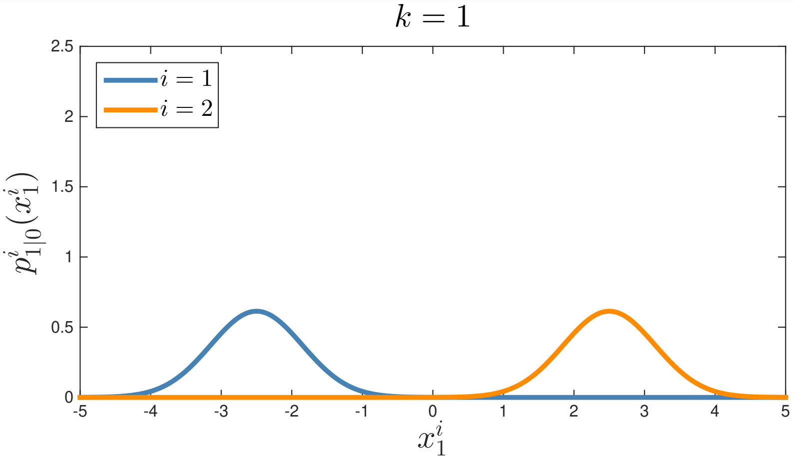

TRACKING N OBJECTS USING GNN FILTER

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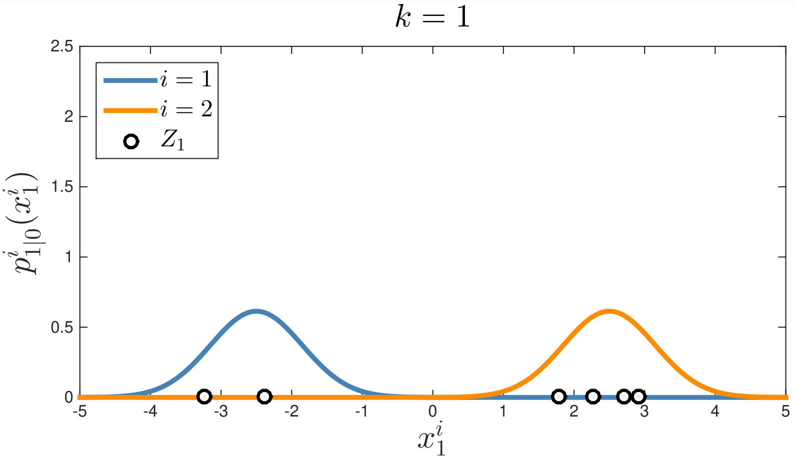

TRACKING *N* OBJECTS USING GNN FILTER

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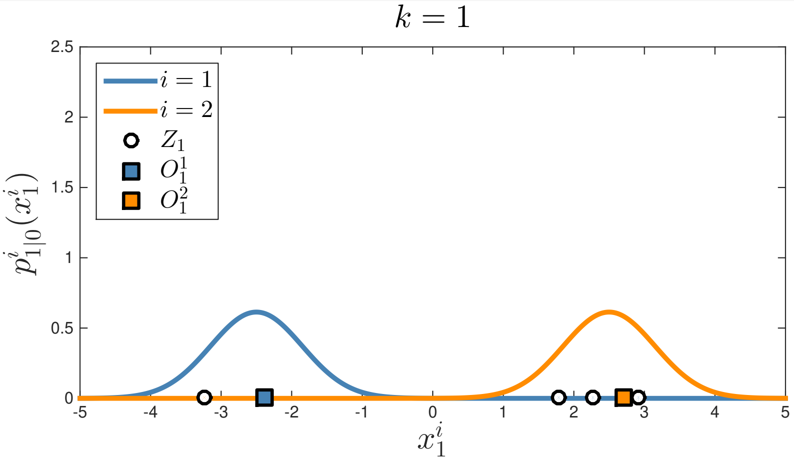
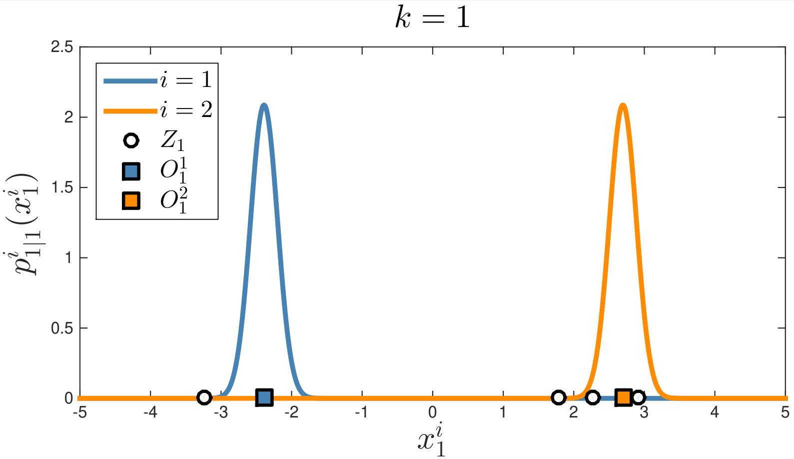


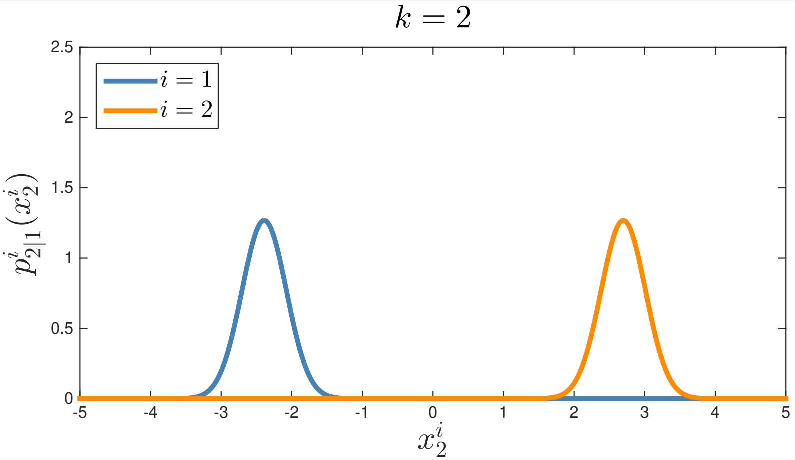
TRACKING *N* OBJECTS USING GNN FILTER

Global Nearest Neighbor recursion



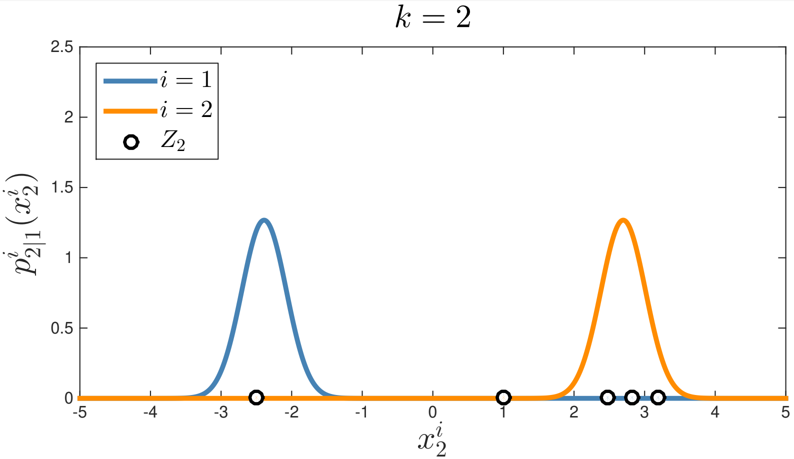
TRACKING N OBJECTS USING GNN FILTER

Global Nearest Neighbor recursion



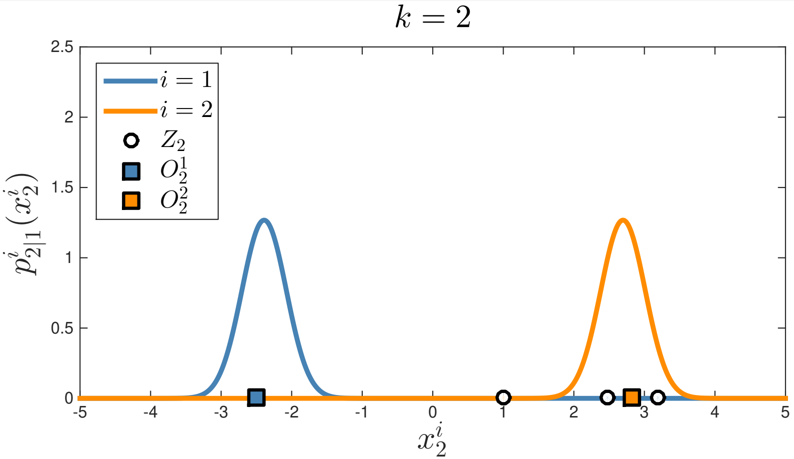
TRACKING *N* OBJECTS USING GNN FILTER

Global Nearest Neighbor recursion



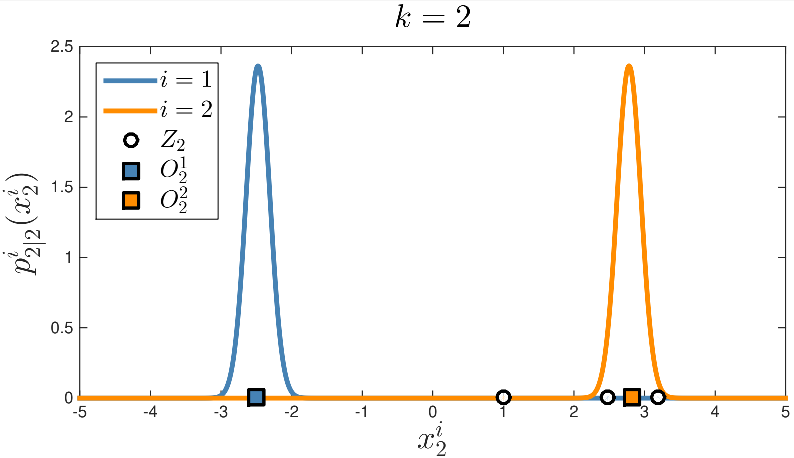
TRACKING *N* OBJECTS USING GNN FILTER

Global Nearest Neighbor recursion



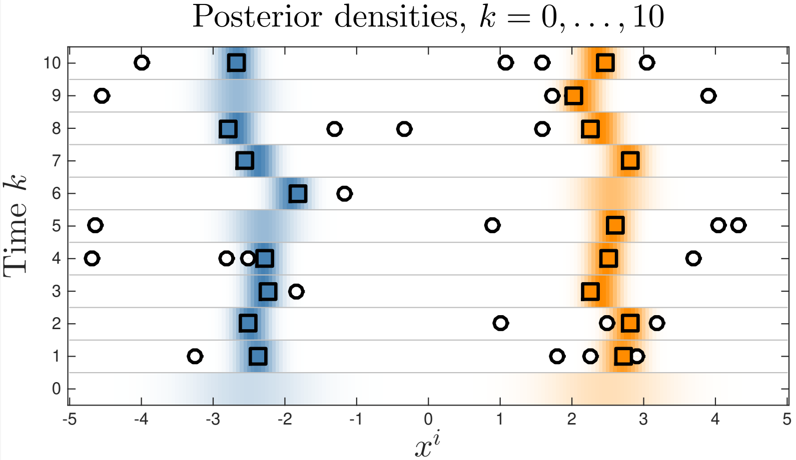
TRACKING *N* OBJECTS USING GNN FILTER

Global Nearest Neighbor recursion

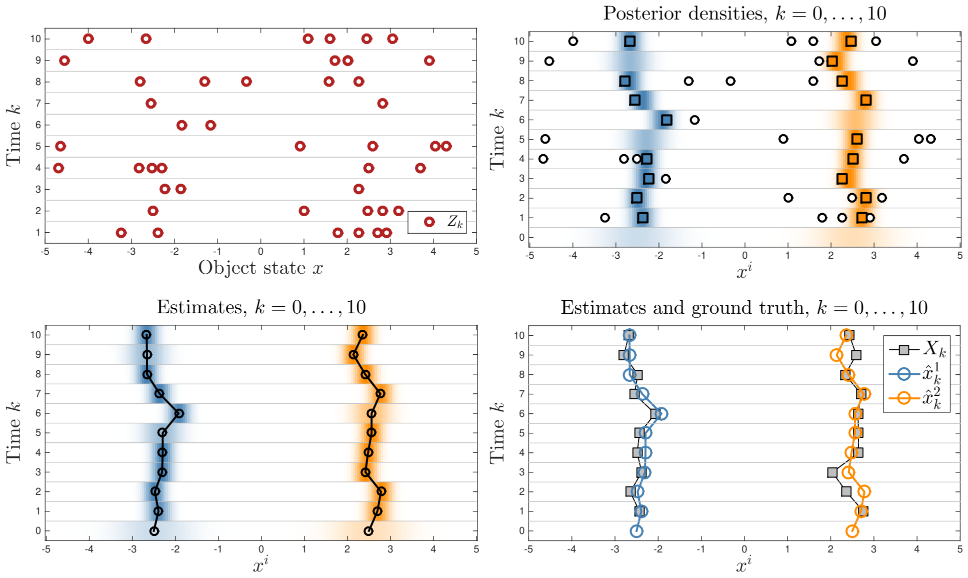


TRACKING *N* OBJECTS USING GNN FILTER

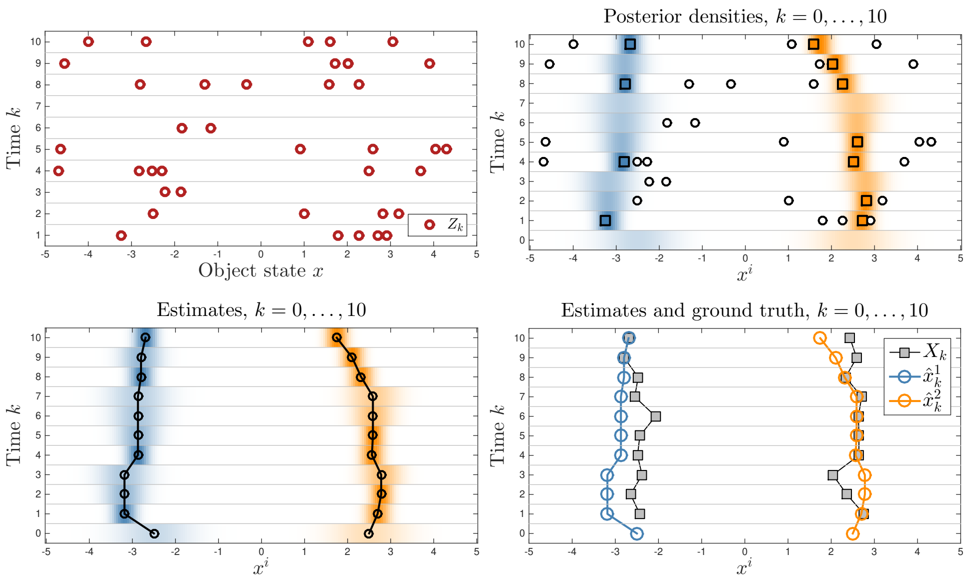
GNN: marginal posteriors visualized as heatmaps



TRACKING N OBJECTS



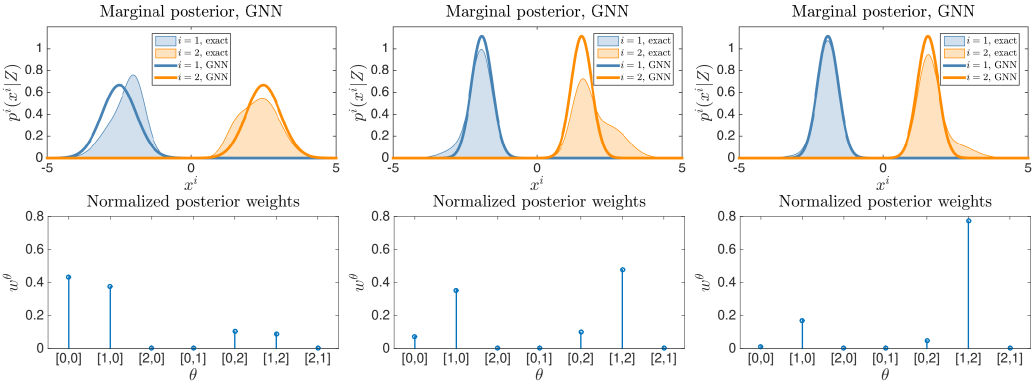
TRACKING N OBJECTS, LOWER $P^{\mathrm{D}} = 0.50$



DENSITY APPROXIMATION

Global nearest neighbor (GNN) filter

$$X = [x^1,\ x^2], \quad Z = [-1.6,\ 1], \quad p(X) = \mathcal{N}\left(x^1;\ -2.5, 0.36\right)\mathcal{N}\left(x^2;\ 2.5, 0.36\right)$$



$P^{\mathrm{D}} = 0.5$ $\quad$ $P^{\mathrm{D}} = 0.85$ $\quad$ $P^{\mathrm{D}} = 0.95$

GNN SUMMARY

Pros:

- Computationally cheap.
- Relatively simple to implement.
- Works fairly well in simple scenarios with high SNR, i.e., high P^{D}, low $\bar{\lambda}$, small R.

Cons:

- Actually not guaranteed that

$$w^{\theta^{\star}_{1:k}} \geq w^{\theta_{1:k}} \ \forall \theta_{1:k} \in \Theta_{1:k}$$

- A single *n*-object hypothesis not always sufficient to represent the uncertainty.
- Can give poor tracking performance in moderate to low SNR,
  or in high SNR when the objects are close to each other.

Joint Probabilistic Data Association

Multi-Object Tracking

Karl Granström

JPDA basic idea, marginal association probabilities

Multi-Object Tracking

Karl Granström

JOINT PROBABILISTIC DATA ASSOCIATION (JPDA) TRACKING

- **Basic idea:**
  - Merge the marginal posterior densities.
  - Compute marginal association probabilities jointly, i.e., for all objects.
- Exact posterior density

  $$p_{k|k}(X_k) = \sum_{\theta_{1:k} \in \Theta_{1:k}} w_{k|k}^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(X_k) \qquad \text{is approximated by} \qquad p_{k|k}^{\text{JPDA}}(X_k) = p_{k|k}^{\beta_{1:k}}(X_k)$$

  where $\beta_{1:k}$ is the sequence of marginal association probabilities,

  $$\beta_1, \qquad \beta_2|\beta_1, \qquad \dots \qquad \beta_k|\beta_{1:k-1}$$

- JPDA density parameterized by the object densities

  $$p_{k|k}^{i,\beta_{1:k}}(x_k^i), \qquad \text{for } i = 1, 2, \dots, n$$

MARGINAL ASSOCIATION PROBABILITY

- Marginal probability that object i is associated to measurement j at time k

$$\beta_k^{i,j} = \Pr\left(\theta_k^i = j \middle| Z_{1:k-1}\right) = \sum_{\theta_k \in \Theta_k : \theta_k^i = j} w^{\theta_k} \propto \sum_{\theta_k \in \Theta_k : \theta_k^i = j} \tilde{w}^{\theta_k}$$

- Marginal probability that object i was not detected at time k

$$\begin{aligned} \beta_k^{i,0} = \Pr\left(\theta_k^i = 0 \middle| Z_{1:k-1}\right) &= 1 - \sum_j \beta_k^{i,j} \\ &= \sum_{\theta_k \in \Theta_k : \theta_k^i = 0} w^{\theta_k} \propto \sum_{\theta_k \in \Theta_k : \theta_k^i = 0} \tilde{w}^{\theta_k} \end{aligned}$$

- Note: this requires a summation over the set of all valid data association Θ_k.
- Note: important to compute $\beta^{i,j}$ jointly!

VISUALIZING THE MARGINAL ASSOCIATION PROBABILITIES

A single detection, between 1 and 4 objects

Objects, measurements

- $1 - 4$ objects
- Scalar object state(s)
- Single measurement z

Measurement model

- $P^{\mathrm{D}}(x) = 0.85$
- $\lambda_c(c) = 0.3, \; c \in [-5, \; 5]$
- $g(z|x) = \mathcal{N}\left(z\,;\; x, 0.2\right)$

Prior

- $p^1(x^1) = \mathcal{N}\left(x^1\,;\; -2.5, 0.36\right)$
- $p^2(x^2) = \mathcal{N}\left(x^2\,;\; 2.8, 0.09\right)$
- $p^3(x^3) = \mathcal{N}\left(x^3\,;\; -2.0, 0.01\right)$
- $p^4(x^4) = \mathcal{N}\left(x^4\,;\; 3.2, 0.04\right)$

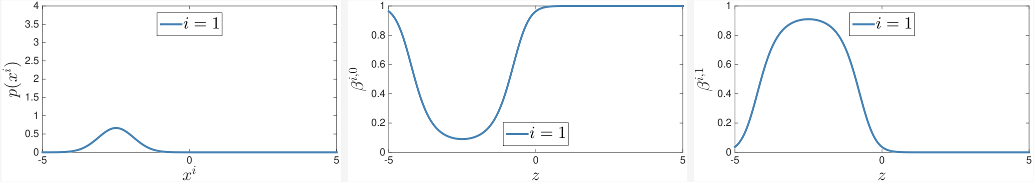
Visualizations

- Probability of no detection $\beta^{i,0}$
- Probability of detection $\beta^{i,1}$

VISUALIZING THE MARGINAL ASSOCIATION PROBABILITIES

A single detection, between 1 and 4 objects

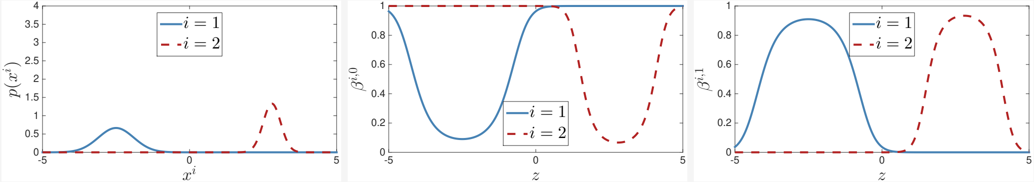
$n = 1$ object, and $m = 1$ measurement



VISUALIZING THE MARGINAL ASSOCIATION PROBABILITIES

A single detection, between 1 and 4 objects

$n = 2$ objects, and $m = 1$ measurement

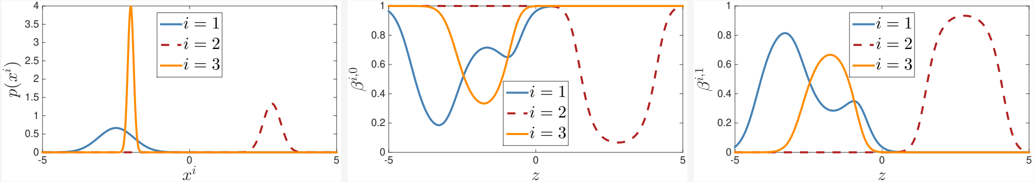


Means further apart, w.r.t. the covariances, have smaller effect on $\beta^{i,j}$

VISUALIZING THE MARGINAL ASSOCIATION PROBABILITIES

A single detection, between 1 and 4 objects

$n = 3$ objects, and $m = 1$ measurement



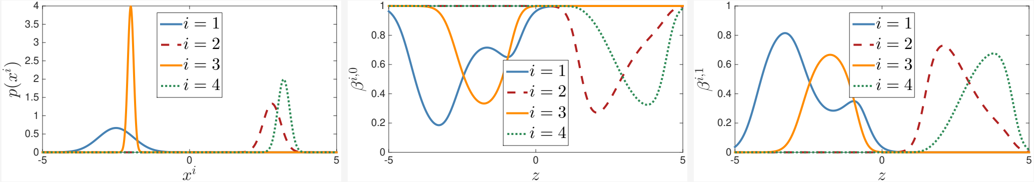
Means further apart, w.r.t. the covariances, have smaller effect on $\beta^{i,j}$

Means closer together, w.r.t. the covariances, have significant effect on $\beta^{i,j}$

VISUALIZING THE MARGINAL ASSOCIATION PROBABILITIES

A single detection, between 1 and 4 objects

$n = 4$ objects, and $m = 1$ measurement



Means further apart, w.r.t. the covariances, have smaller effect on $\beta^{i,j}$

Means closer together, w.r.t. the covariances, have significant effect on $\beta^{i,j}$

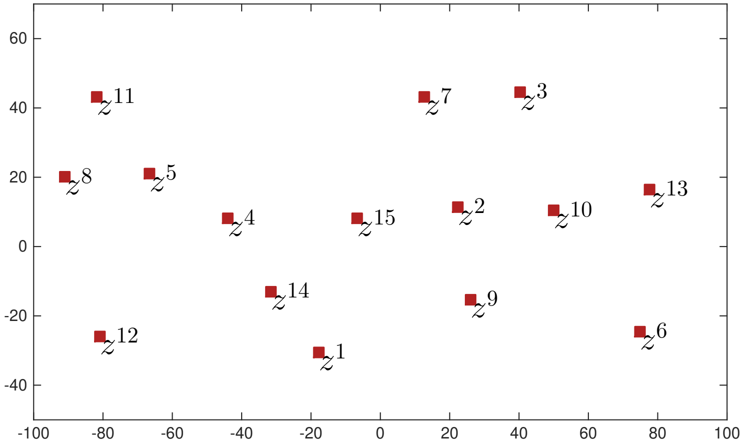
MARGINAL ASSOCIATION PROBABILITIES EXAMPLE

2D position meas.

- $m = 15$ detections
- $n = 6$ objects
- Ellipsoidal gating

Marginal association probabilities

$$\beta_k^{i,j} = \Pr\left(\theta_k^i = j \middle| Z_{1:k-1}\right) = \sum_{\theta_k \in \Theta_k : \theta_k^i = j} w^{\theta_k}$$



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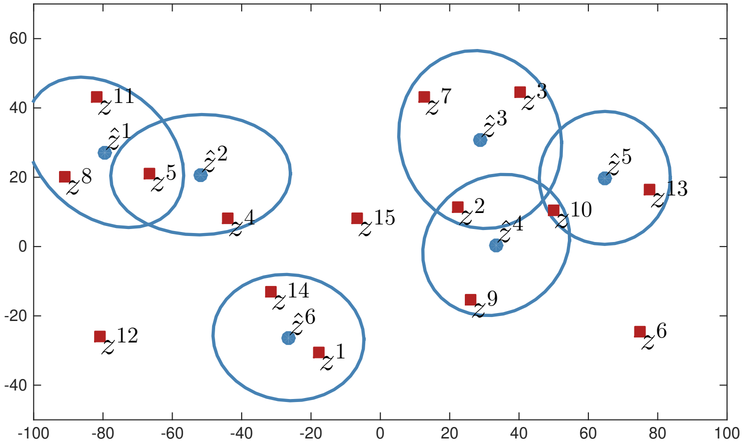

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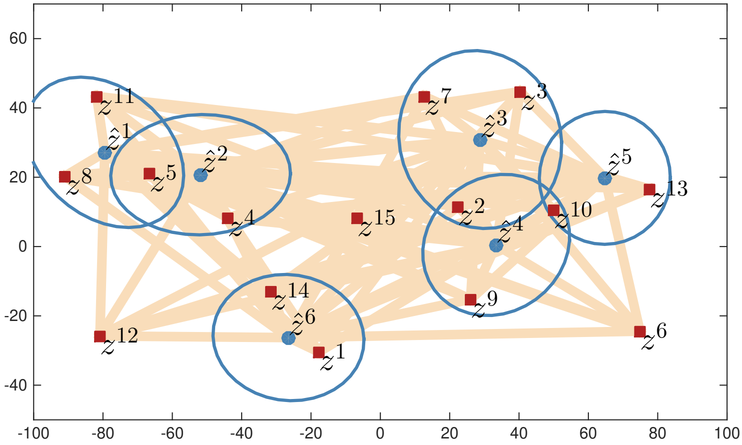

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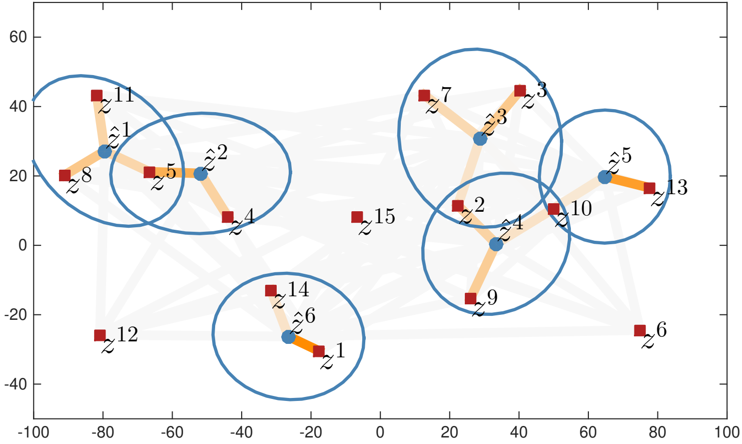

Color intensity proportional to $\beta_k^{i,j}$

MARGINAL ASSOCIATION PROBABILITIES EXAMPLE

2D position meas.

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- Ellipsoidal gating

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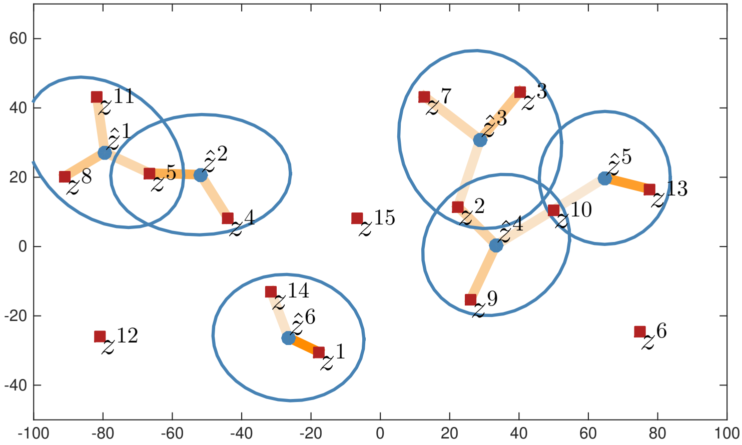


Color intensity proportional to $\beta_k^{i,j}$

If gate is large enough, the gating leads to negligible errors

JPDA prediction and update

Multi-Object Tracking

Karl Granström

JPDA UPDATE: CONCEPTUAL IDEA (1/2)

- Prior density

$$p_{k|k-1}(X_k) = \prod_{i=1}^{n} p^i_{k|k-1}(x^i)$$

- Marginal posterior for object i

$$p^i_{k|k}(x^i_k) = \beta^{i,0}_k p^i_{k|k-1}(x^i) + \sum_{j=1}^{m_k} \beta^{i,j}_k p^{i,j}_{k|k-1}(x^i)$$

where $p^{i,j}_{k|k-1}(x^i)$ is the posterior density that results from updating the prior $p^i_{k|k-1}(x^i)$ with measurement z^j_k.

JPDA UPDATE: CONCEPTUAL IDEA (2/2)

- Merged marginal posterior for object i

$$p^{i,\beta}_{k|k}(x^i_k) = \mathrm{MERGE}\left(\beta^{i,0}_k p^i_{k|k-1}(x^i) + \sum_{j=1}^{m_k} \beta^{i,j}_k p^{i,j}_{k|k-1}(x^i)\right)$$

  where $\mathrm{MERGE}(\cdot)$ is a function that merges the densities into a single density.

  - Typically: moment matching that minimizes the KL-divergence.
  - For Gaussian densities: match the mean and the covariance.

- Posterior density for n objects

$$p^{\beta}_{k|k}(X_k) = \prod_{i=1}^{n} p^{i,\beta}_{k|k}(x^i_k)$$

BASIC JPDA RECURSION

JPDA: Prediction and update pseudo-code

For $k = 1, 2, \ldots, K$

Prediction:

For $i = 1, \ldots, n$: Chapman-Kolmogorov prediction,

$$p_{k|k-1}^{i,\beta_{1:k-1}}(x_k^i) = \int \pi(x_k^i | x_{k-1}^i) p_{k-1|k-1}^{i,\beta_{1:k-1}}(x_{k-1}^i) \mathrm{d}x_{k-1}^i$$

Update:

Compute marginal association probabilities $\beta_k^{i,j}$

For $i = 1, \ldots, n$: JPDA update,

$$p_{k|k}^{i,\beta_{1:k}}(x_k^i) = \mathrm{MERGE}\left(\beta_k^{i,0} p_{k|k-1}^{i,\beta_{1:k-1}}(x^i) + \sum_{j=1}^{m_k} \beta_k^{i,j} p_{k|k-1}^{i,\beta_{1:k},j}(x^i)\right)$$

JPDA ESTIMATOR

- Common object estimator: expected value

$$\bar{x}^i_{k|k} = \int x^i_k p^{i,\beta_{1:k}}_{k|k}(x^i_k)\mathrm{d}x^i_k$$

  where $p^{i,\beta_{1:k}}_{k|k}(x^i_k)$ is the marginal density.

- For Gaussian densities, the expected value is one of the density parameters

LINEAR GAUSSIAN JPDA: MODELS

- $P^{\mathrm{D}}(x) = P^{\mathrm{D}}$
- $\lambda(c) = \bar{\lambda}/V$
- $g_k(z|x) = \mathcal{N}(z\,;\, Hx, R)$
- $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\, Fx_{k-1}, Q)$

- $p_0(X_0) = \prod_{i=1}^{n} \mathcal{N}\left(x_0^i\,;\, \mu_0^i,\, P_0^i\right)$
- Posterior density parameterized by $\left\{\mu_{k|k}^i, P_{k|k}^i\right\}_{i=1}^{n}$.

- For the prediction, the Kalman filter prediction is used.

JPDA UPDATED MEAN AND COVARIANCE

Predicted parameters $\left\{\mu^i_{k|k-1}, P^i_{k|k-1}\right\}_{i=1}^{n}$

Updated mean

$$\varepsilon_k^{i,j} = z_k^j - H\mu^i_{k|k-1}$$

$$\varepsilon_k = \sum_{j=1}^{m_k} \beta_k^{i,j}\varepsilon_k^{i,j}$$

$$\mu^i_{k|k} = \mu^i_{k|k-1} + K_k^i \varepsilon_k$$

Updated covariance

$$\bar{P}_k^i = P^i_{k|k-1} - K_k^i\left(HP^i_{k|k-1}H^{\mathrm{T}} + R\right)\left(K_k^i\right)^{\mathrm{T}}$$

$$\tilde{P}_k^i = K_k^i\left(\left[\sum_{j=1}^{m_k}\beta_k^{i,j}\varepsilon_k^{i,j}\left(\varepsilon_k^{i,j}\right)^{\mathrm{T}}\right] - \varepsilon_k\varepsilon_k^{\mathrm{T}}\right)\left(K_k^i\right)^{\mathrm{T}}$$

$$P^i_{k|k} = \beta_k^{i,0}P^i_{k|k-1} + \left(1 - \beta_k^{i,0}\right)\bar{P}_k^i + \tilde{P}_k^i$$

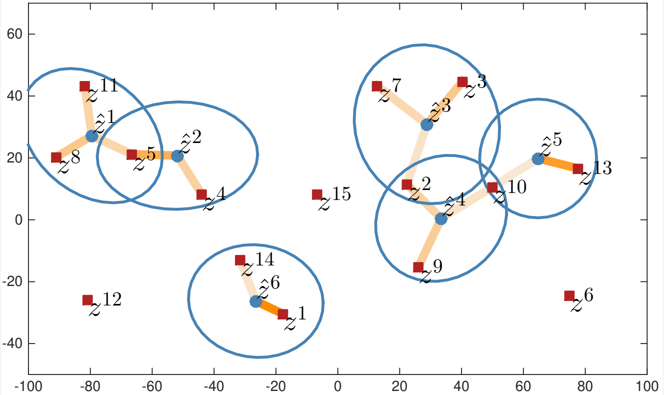
where

$$K_k^i = P^i_{k|k-1}H^{\mathrm{T}}\left(HP^i_{k|k-1}H^{\mathrm{T}} + R\right)^{-1}$$

JPDA UPDATE EXAMPLE

JPDA update

- $n = 6$ objects
- $m = 15$ measurements
- Focus on object x^6

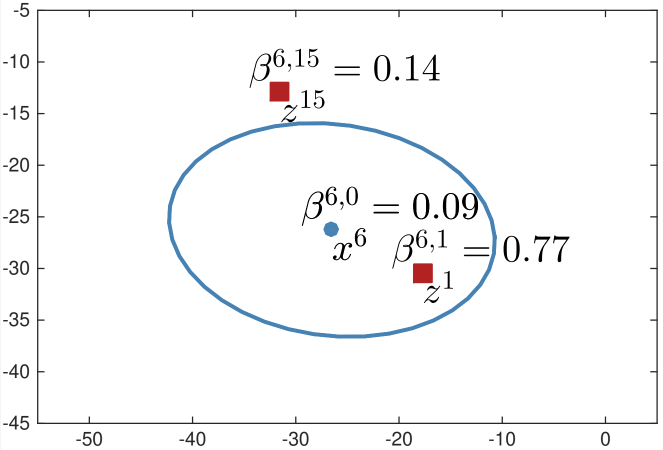


JPDA UPDATE EXAMPLE

JPDA update

Marginal association probabilities

- Missed detection
- Association to z^1
- Association to z^{15}

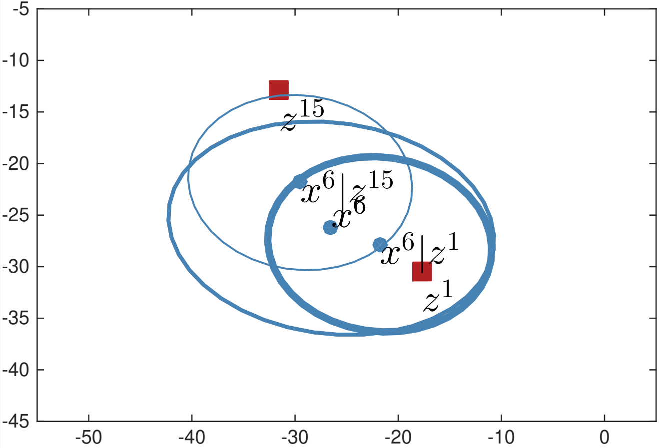


JPDA UPDATE EXAMPLE

JPDA update

Gaussians resulting from

- Missed detection
- Association to z^1
- Association to z^{15}

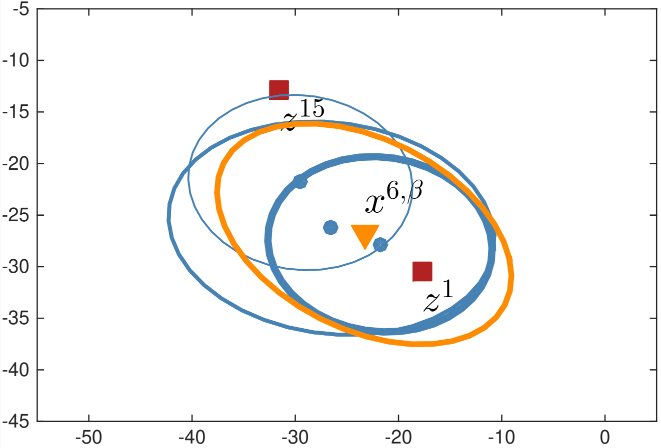


JPDA UPDATE EXAMPLE

JPDA update

Merged Gaussian

Merged mean closest to component resulting from z^1, because that association probability was highest.



APPROXIMATING THE MARGINAL ASSOCIATION PROBABILITIES

- Sum over all valid associations,

$$\beta_k^{i,j} = \sum_{\theta_k \in \Theta_k : \theta_k^i = j} w^{\theta_k}$$

  Can have high computational cost, or be intractable.

- Approximate computation of the marginal association probabilities:
  - Cheap JPDA
  - Suboptimal JPDA
  - Fast JPDA

- Find the best M associations $\theta_k^{\star 1:M}$,

$$\beta_k^{i,j} \approx \frac{\sum_{\theta_k \in \theta_k^{\star 1:M} : \theta_k^i = j} \tilde{w}^{\theta_k}}{\sum_{\theta_k \in \theta_k^{\star 1:M}} \tilde{w}^{\theta_k}}$$

JPDA examples

Multi-Object Tracking

Karl Granström

VISUALIZATIONS

Original example

Two objects, scalar states

- $X = [x^1,\ x^2]$

Measurement model

- $P^{\mathrm{D}}(x) = 0.85$
- $\lambda_c(c) = 0.3,\ c \in [-5,\ 5]$
- $g(z|x) = \mathcal{N}\left(z\,;\ x, 0.2\right)$

Motion model: random walk

- $\pi_k(x_k|x_{k-1}) = \mathcal{N}\left(x_k\,;\ x_{k-1}, 0.25\right)$

Initial prior $p_0(X_0) = p_0^1(x_0^1)p_0^2(x_0^2)$

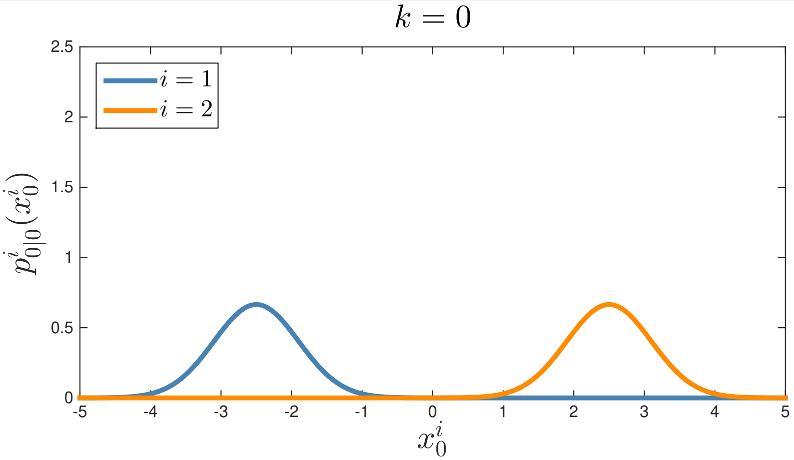
- $p_0^1(x_0^1) = \mathcal{N}\left(x_0^1\,;\ -2.5, 0.36\right)$
- $p_0^2(x_0^2) = \mathcal{N}\left(x_0^2\,;\ 2.5, 0.36\right)$

Visualizations

- Marginal: $p^i_{k|k}(x^i_k)$
- Estimates: $\bar{x}^i_{k|k}$

TRACKING *N* OBJECTS USING JPDA FILTER

JPDA recursion



TRACKING N OBJECTS USING JPDA FILTER

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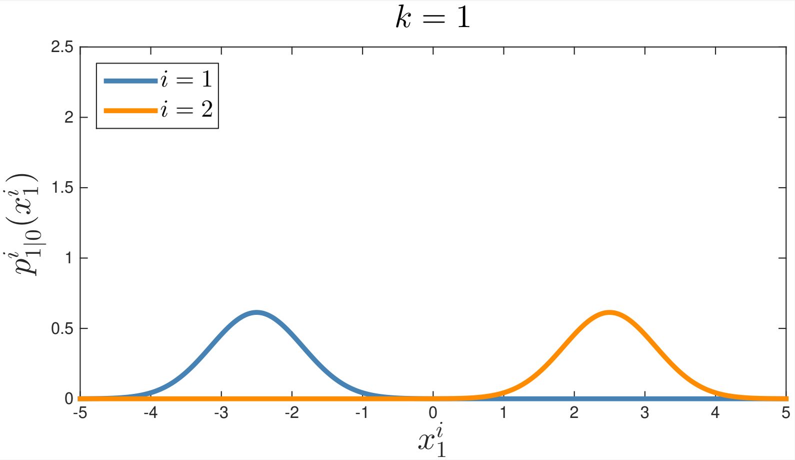
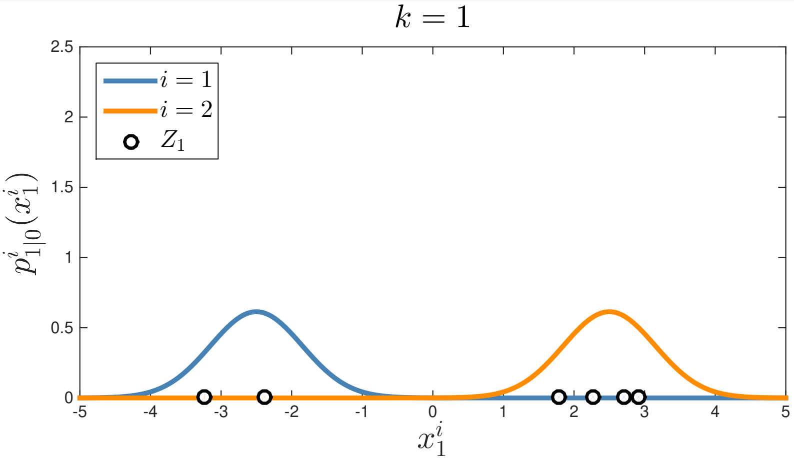


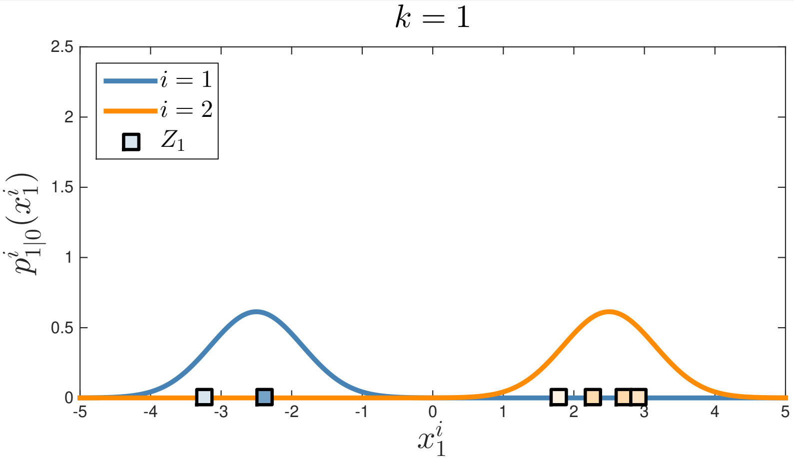
TRACKING N OBJECTS USING JPDA FILTER

JPDA recursion



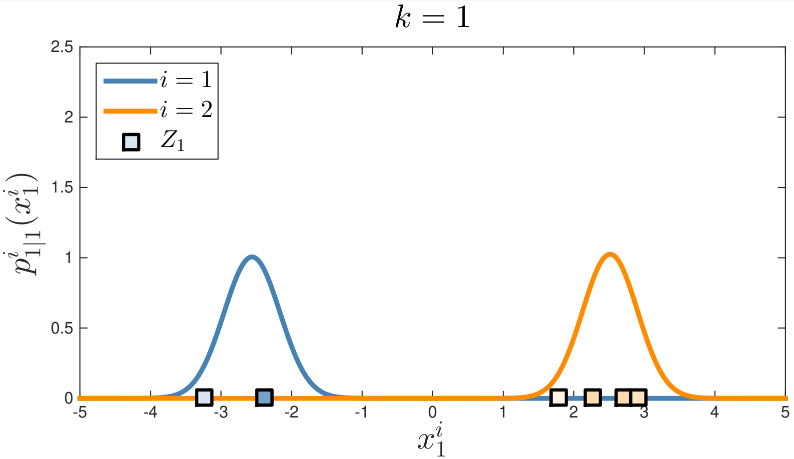
TRACKING N OBJECTS USING JPDA FILTER

JPDA recursion



TRACKING N OBJECTS USING JPDA FILTER

JPDA recursion



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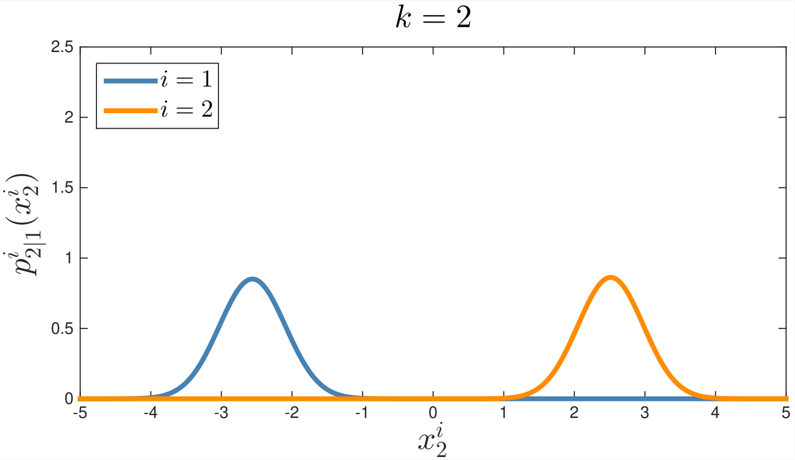

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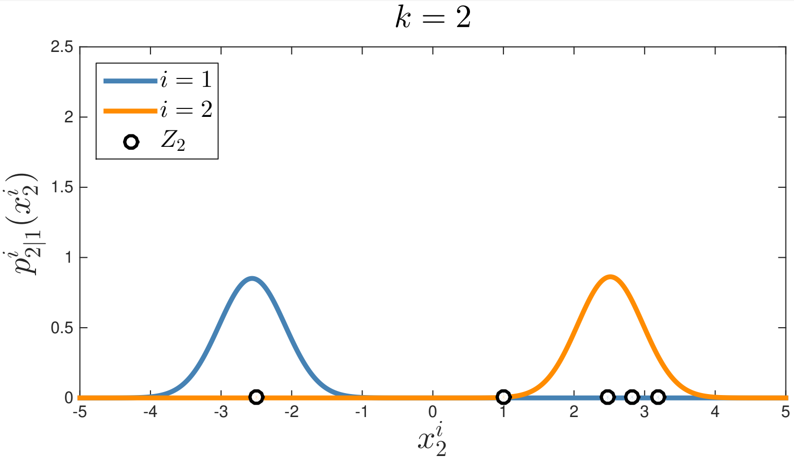

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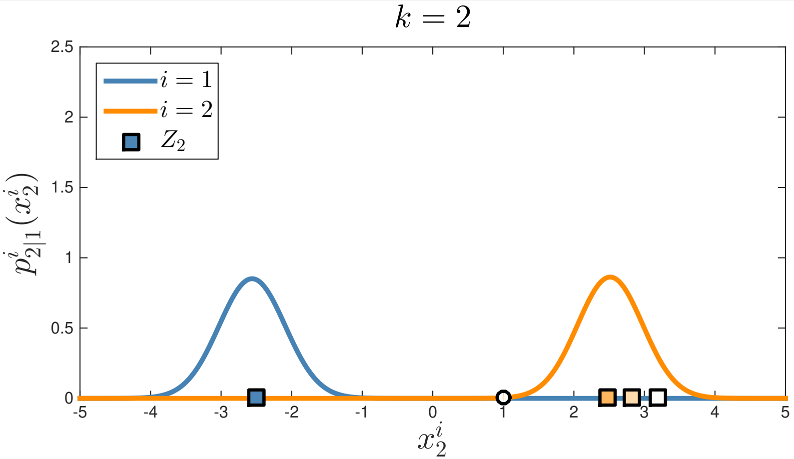
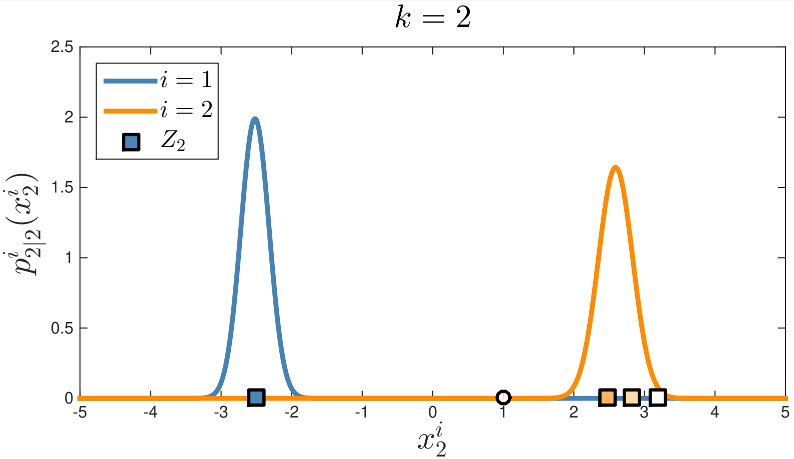
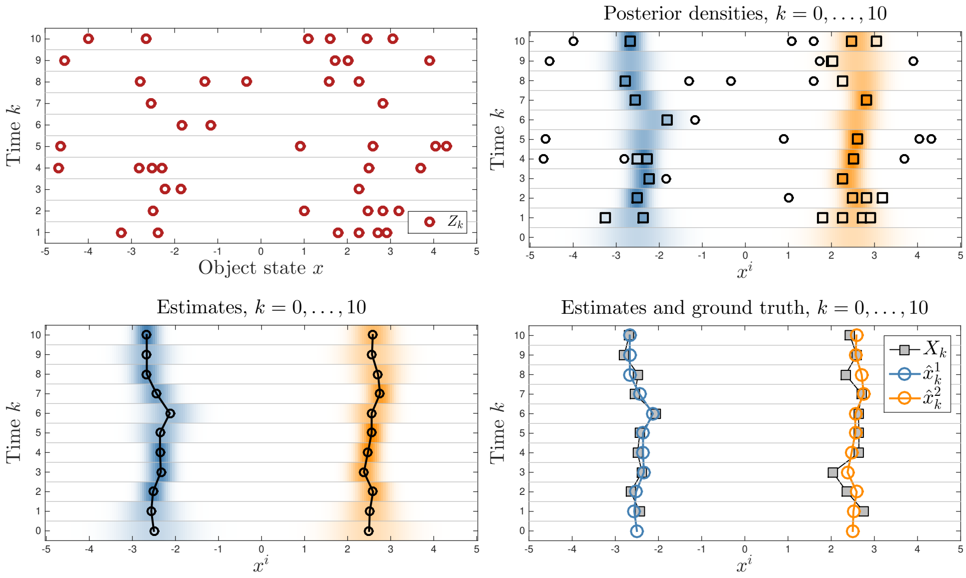


TRACKING *N* OBJECTS USING JPDA FILTER

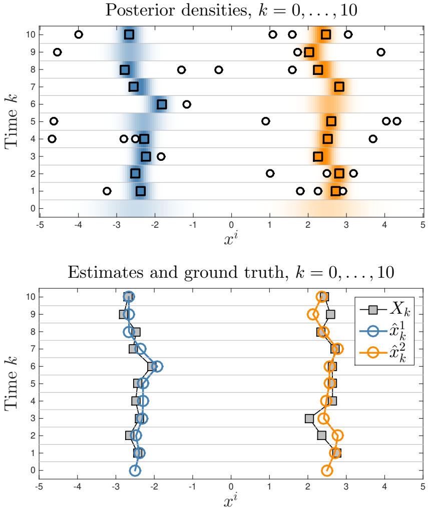
JPDA recursion



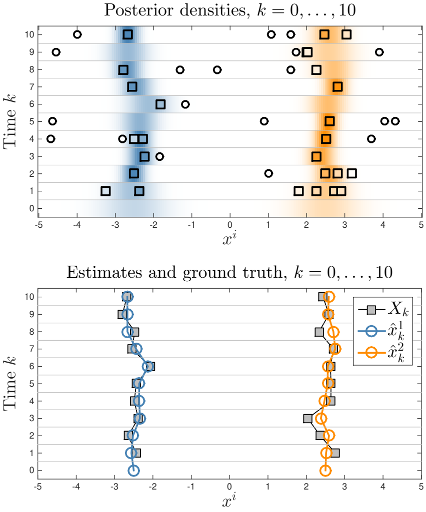
TRACKING *N* OBJECTS, $P^{\mathrm{D}} = 0.85$



GNN VS JPDA, $P^{\mathrm{D}} = 0.85$

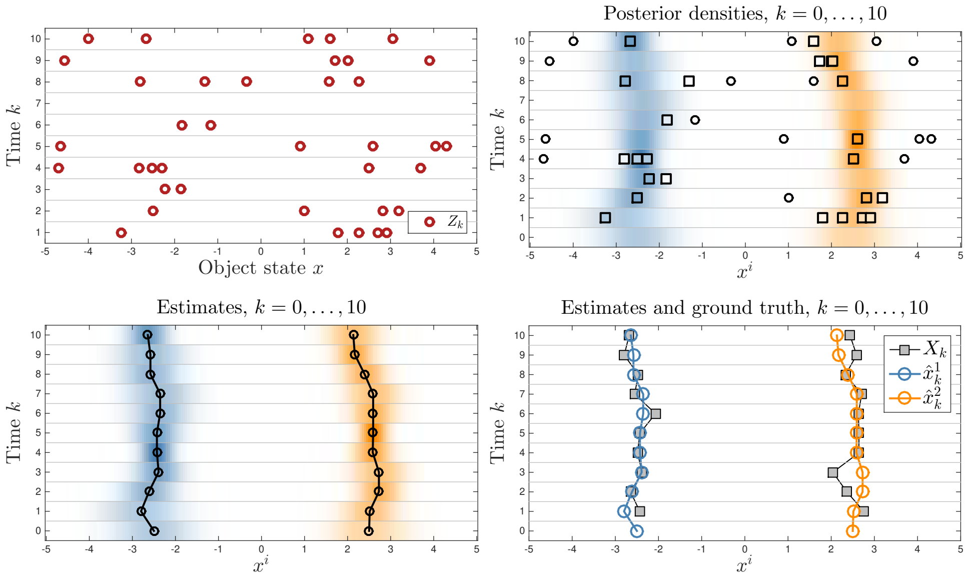


GNN filter

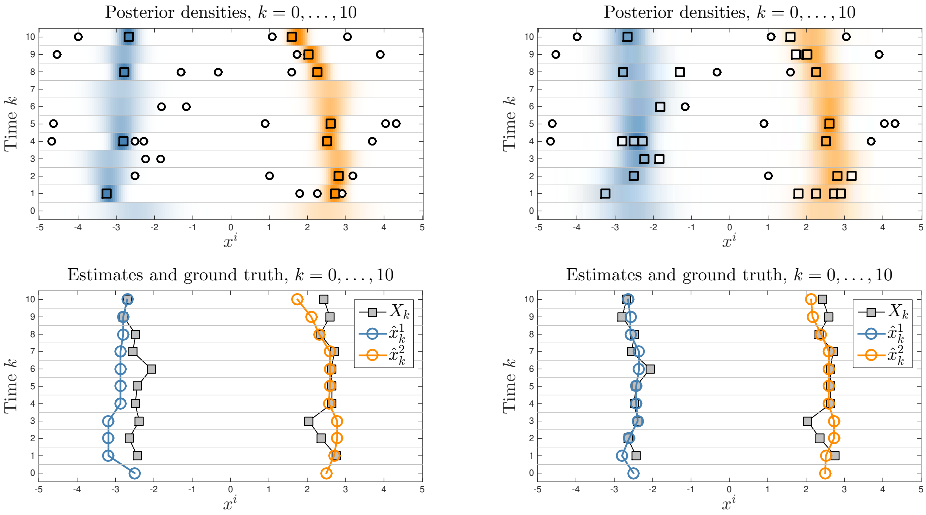


JPDA filter

TRACKING N OBJECTS, LOWER $P^{\mathrm{D}} = 0.5$



GNN VS JPDA, LOWER $P^{\mathrm{D}} = 0.5$



GNN filter

JPDA filter

DENSITY APPROXIMATION

Joint Probabilistic Data Association (JPDA) filter

$$X = [x^1,\ x^2], \quad Z = [-1.6,\ 1], \quad p(X) = \mathcal{N}\left(x^1\,;\,-2.5, 0.36\right)\mathcal{N}\left(x^2\,;\,2.5, 0.36\right)$$

$P^{\mathrm{D}} = 0.5$ $\qquad P^{\mathrm{D}} = 0.85$ $\qquad P^{\mathrm{D}} = 0.95$

DENSITY APPROXIMATION

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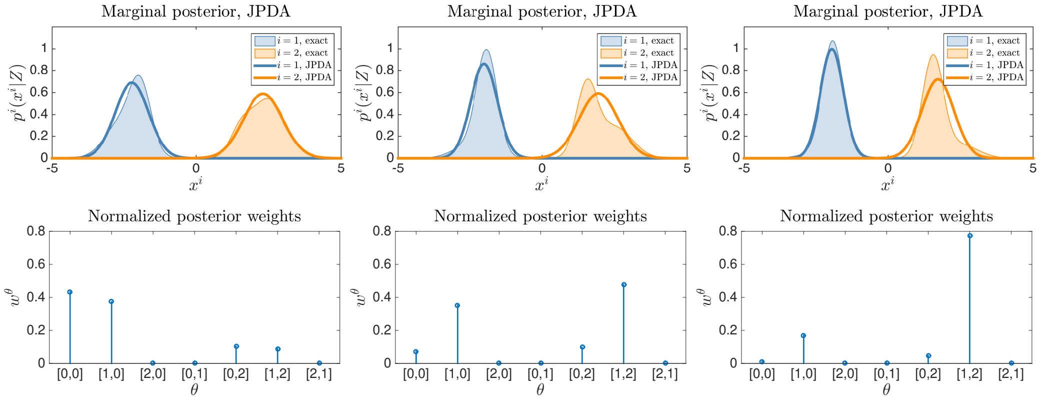

$P^{\mathrm{D}} = 0.5$ $\quad$ $P^{\mathrm{D}} = 0.85$ $\quad$ $P^{\mathrm{D}} = 0.95$

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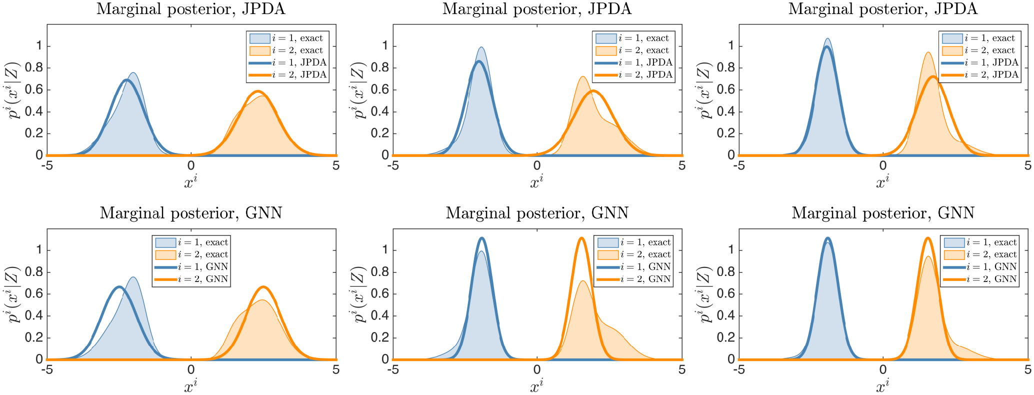


$P^{\mathrm{D}} = 0.5$ $\qquad$ $P^{\mathrm{D}} = 0.85$ $\qquad$ $P^{\mathrm{D}} = 0.95$

JPDA SUMMARY

Pros:

- Works better than GNN in scenarios with lower SNR.
- Computationally cheap, although not as cheap as GNN.
- Relatively simple to implement, although more involved than GNN.

Cons:

- Can give poor tracking performance in complicated scenarios, e.g., when the objects are close to each other.

Multi Hypothesis Tracker

Multi-Object Tracking

Karl Granström

Basic idea, HO prediction and update

Multi-Hypothesis Tracker

Karl Granström

MULTI HYPOTHESIS TRACKER (MHT)

- **Basic idea:**
  - In each update, find the best M associations θ_k^m, and prune all other $\theta_k \in \Theta_k$.
  - Reduce such that at most $N_{\max}$ hypotheses are included in the posterior density.
- Exact posterior density

$$p_{k|k}(X_k) = \sum_{\theta_{1:k} \in \Theta_{1:k}} w_{k|k}^{\theta_{1:k}} p_{k|k}^{\theta_{1:k}}(X_k) \qquad \text{is approx. by} \qquad p_{k|k}^{\mathrm{MHT}}(X_k) = \sum_{h_k=1}^{\mathcal{H}_k} w_{k|k}^{h_k} p_{k|k}^{h_k}(X_k)$$

  where the hypotheses h_k correspond to different data association sequences $\theta_{1:k}$, and $\mathcal{H}_k \leq N_{\max}$
- Two variants: Hypothesis Oriented (HO) MHT and Track Oriented (TO) MHT

BASIC HO-MHT RECURSION

HO-MHT density parameterized by log-weights $\ell^{h_k} = \log(w_{k|k}^{h_k})$ and densities,

$$\ell^{h_k}, \left\{p_{k|k}^{i,h_k}(x_k^i)\right\}_{i=1}^{n}, \qquad \text{for } h_k = 1, 2, \ldots, \mathcal{H}_k$$

HO-MHT: pseudo-code

For $k = 1, 2, \ldots, K$

Prediction: For each hypothesis and each object: Chapman-Kolmogorov prediction

Update: compute multiple associations, construct posterior hypotheses

Reduction: Pruning and capping

HO-MHT PREDICTION

- Predict each hypothesis independent of other hypotheses
- Predict each object independent of other objects
- Hypothesis log-weights not affected by prediction

HO-MHT prediction: pseudo-code

- **Input:** $\left\{ \ell^{h_{k-1}} \, , \, \left\{ p^{i,h_{k-1}}_{k-1|k-1}(x^i_{k-1}) \right\}_{i=1}^{n} \right\}_{h=1}^{\mathcal{H}_{k-1}}$
- **Output:** $\left\{ \ell^{h_{k-1}} \, , \, \left\{ p^{i,h_{k-1}}_{k|k-1}(x^i_{k}) \right\}_{i=1}^{n} \right\}_{h=1}^{\mathcal{H}_{k-1}}$

  where

$$p^{i,h_{k-1}}_{k|k-1}(x^i_k) = \int \pi(x^i_k|x^i_{k-1}) p^{i,h_{k-1}}_{k-1|k-1}(x^i_{k-1}) \mathrm{d}x^i_{k-1}$$

HO-MHT UPDATE

HO-MHT update: pseudo-code

Input: $\left\{\ell^{h_{k-1}}\ ,\ \left\{p^{i,h_{k-1}}_{k|k-1}(x^i_{k-1})\right\}_{i=1}^{n}\right\}_{h_{k-1}=1}^{\mathcal{H}_{k-1}}$

Initialise $h_k = 0$

For $h_{k-1} = 1, \ldots, \mathcal{H}_{k-1}$

Create cost matrix $L^{h_{k-1}}$

Compute $M_{h_{k-1}}$ associations θ_m

For $m = 1, \ldots, M_{h_{k-1}}$

Increase: $h_k \leftarrow h_k + 1$

Compute posterior parameters

Set $\mathcal{H}_k = h_k$

Normalise log-weights $\ell^{h_k} \leftarrow \tilde{\ell}^{h_k}$

Output: $\left\{\ell^{h_k}\ ,\ \left\{p^{i,h_k}_{k|k}(x^i_k)\right\}_{i=1}^{n}\right\}_{h_k=1}^{\mathcal{H}_k}$

Compute posterior parameters: $\ell^{h_k}\ ,\ \left\{p^{i,h_k}_{k|k}(x^i_k)\right\}_{i=1}^{n}$

Initialise log-weight: $\tilde{\ell}^{h_k} = \ell^{h_{k-1}}$

For $i = 1, \ldots, n$

$$p^{i,h_k}_{k|k}(x^i_k) \propto$$

$$\begin{cases} P^{\mathrm{D}}(x^i_k) g_k(z_k^{\theta^{\star,i}} | x^i_k) p^{i,h_{k-1}}_{k|k-1}(x^i_k) & \theta^{\star,i} \neq 0 \\ (1 - P^{\mathrm{D}}(x^i_k)) p^{i,h_{k-1}}_{k|k-1}(x^i_k) & \theta^{\star,i} = 0 \end{cases}$$

log-weight: $\tilde{\ell}^{h_k} \leftarrow \tilde{\ell}^{h_k} + \ell^{i,\theta^i_m,h_{k-1}}$

where $\ell^{i,\theta^i_m,h_{k-1}}$ is the log-likelihood of θ^i_m

SETTING THE NUMBER OF MHT ASSOCIATIONS

How do we choose $M_{h_{k-1}}$?

- A constant, e.g., 10, 100, or 1000.
- $M_{h_{k-1}} = \max\left(1,\ \lfloor N_{\max} \exp(\ell^{h_{k-1}}) \rceil\right)$, where $\lfloor \cdot \rceil$ rounds to the nearest integer.

  Predictable number of hypotheses in the posterior, because

$$\mathcal{H}_k = \sum_{h_{k-1}} M_{h_{k-1}} = \sum_{h_{k-1}} \lfloor N_{\max} \exp(\ell^{h_{k-1}}) \rceil \approx \sum_{h_{k-1}} N_{\max} \exp(\ell^{h_{k-1}}) = N_{\max}$$

- Practical computational requirements often point towards an answer.

NORMALISING THE MHT LOG-WEIGHTS

- $\mathcal{H}_k$ hypotheses with un-normalised log-weights $\tilde{\ell}^{h_k}$.
- Normalized log-weights ℓ^{h_k} can be computed as

$$\ell^{h_k} = \tilde{\ell}^{h_k} - \tilde{\ell}^{h_k^{\max}} - \log\left(1 + \sum_{h_k \neq h_k^{\max}} e^{\left(\tilde{\ell}^{h_k} - \tilde{\ell}^{h_k^{\max}}\right)}\right)$$

  where $\tilde{\ell}^{h_k^{\max}}$ is the largest un-normalised log-weight.
- **Note:** avoid computing the un-normalised weights $\tilde{w}^{h_k} = \exp(\tilde{\ell}^{h_k})$ and normalising them. Work with log-weights as this avoids some potential numerical problems.

REDUCING THE MHT POSTERIOR DENSITY

MHT reduction

- **Pruning:** prune an hypothesis if

$$\ell_{k|k}^{h_k} \leq \Gamma$$

  Typically, $\Gamma \leq \log(0.01)$, often smaller.

- **Capping:** after the pruning, if

$$\mathcal{H}_k > N_{\max}$$

  then keep the $N_{\max}$ hypotheses with largest log-weights.

After pruning and capping, remaining log-weights are re-normalised.

Possible to do merging in MHT, but this is outside scope of course.

HO-MHT: examples

Multi-Hypothesis Tracker

Karl Granström

MHT ESTIMATOR

- Possible to use marginal densities $p_{k|k}^{\mathrm{MHT},i}(x_k^i)$ to compute estimates for each object, e.g., the expected value or the Maximum A Posteriori (MAP) estimate,

$$\text{Expected value:} \qquad \bar{x}_{k|k}^i = \int x_k^i p_{k|k}^{\mathrm{MHT},i}(x_k^i)\mathrm{d}x_k^i = \sum_{h_k=1}^{\mathcal{H}_k} w_{k|k}^{h_k} \int x_k^i p_{k|k}^{i,h_k}(x_k^i)\mathrm{d}x_k^i$$

$$\text{MAP estimate:} \qquad \hat{x}_{k|k}^{i,\mathrm{MAP}} = \max_{x_k^i} p_{k|k}^{\mathrm{MHT},i}(x_k^i) = \max_{x_k^i} \sum_{h_k=1}^{\mathcal{H}_k} w_{k|k}^{h_k} x_k^i p_{k|k}^{i,h_k}(x_k^i)$$

- Using marginal densities can be complicated on account of the high number of hypotheses, and the fact that the marginal densities often are multi-modal.
- Common object estimator in MHT: expected value from the most probable hypothesis

$$h^\star = \max_{h_k} \ell_{k|k}^{h_k} \qquad\qquad \bar{x}_{k|k}^i = \int x_k^i p_{k|k}^{i,h^\star}(x_k^i)\mathrm{d}x_k^i$$

VISUALIZATIONS

Original example

Two objects, scalar states

- $X = [x^1,\ x^2]$

Measurement model

- $P^{\mathrm{D}}(x) = 0.85$
- $\lambda_c(c) = 0.3,\ c \in [-5,\ 5]$
- $g(z|x) = \mathcal{N}(z\,;\ x, 0.2)$

Motion model: random walk

- $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\ x_{k-1}, 0.25)$

Initial prior $p_0(X_0) = p_0^1(x_0^1)p_0^2(x_0^2)$

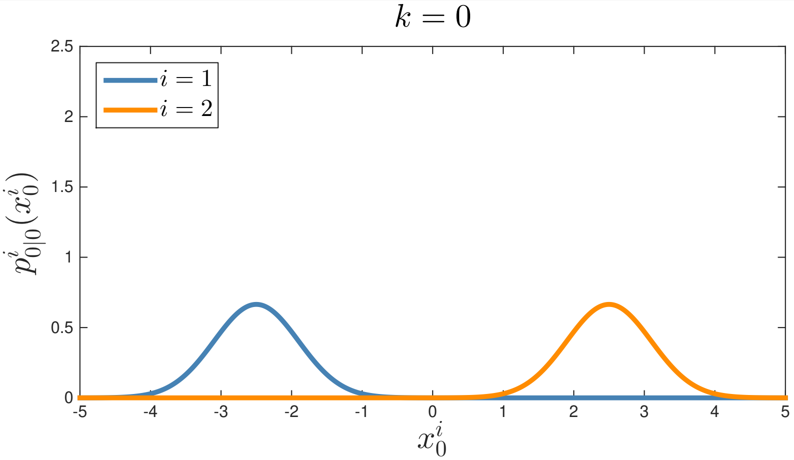
- $p_0^1(x_0^1) = \mathcal{N}(x_0^1\,;\ -2.5, 0.36)$
- $p_0^2(x_0^2) = \mathcal{N}(x_0^2\,;\ 2.5, 0.36)$

Visualizations

- Marginal: $p_{k|k}^i(x_k^i)$
- Estimates: $\bar{x}_{k|k}^i$

TRACKING *N* OBJECTS USING MHT

Multi Hypothesis Tracker recursion



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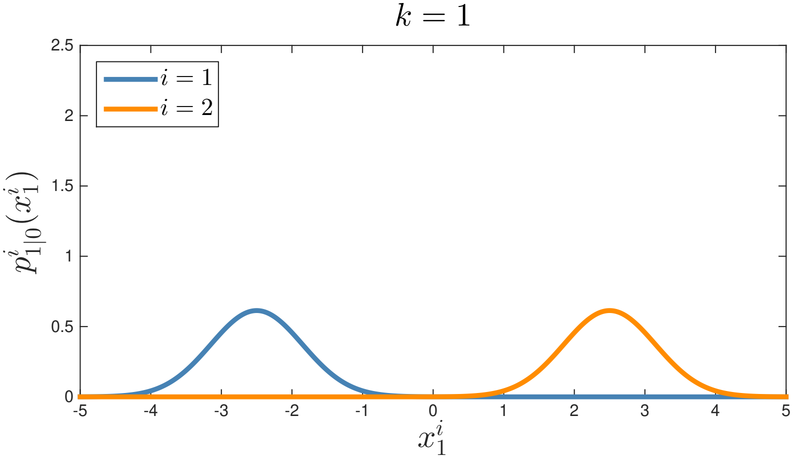

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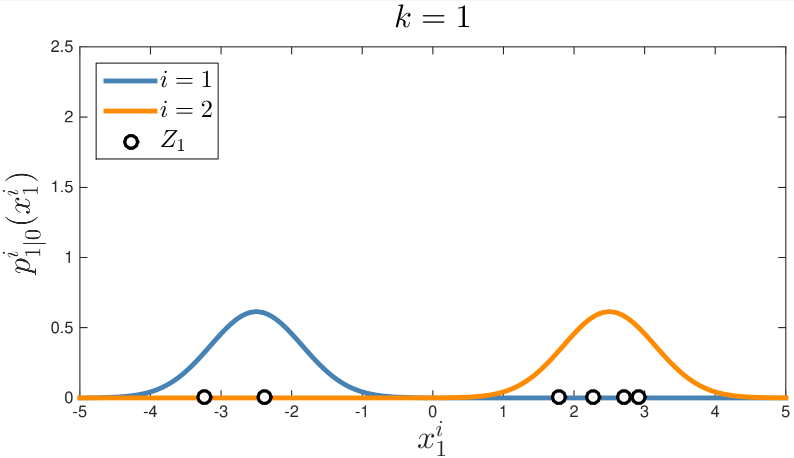
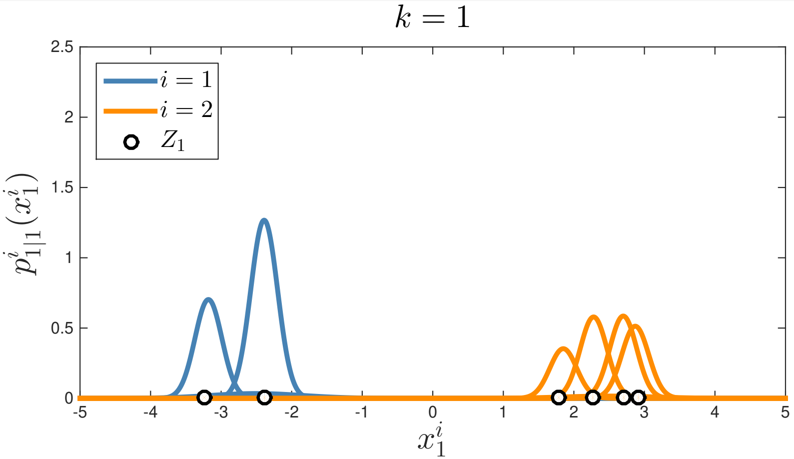
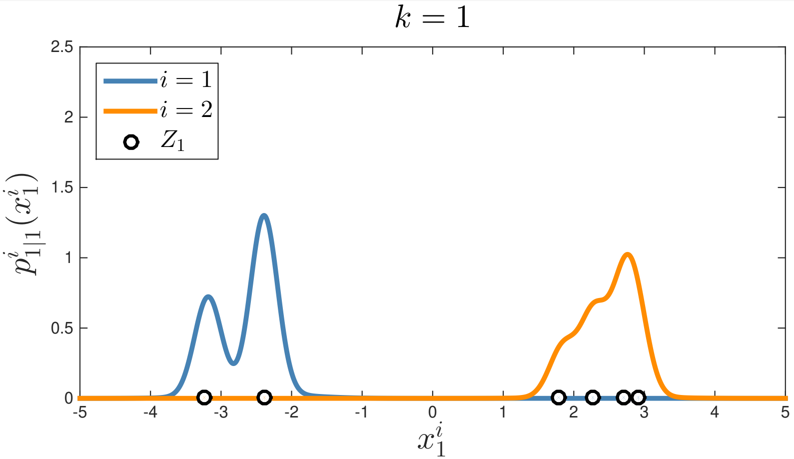


TRACKING N OBJECTS USING MHT

Multi Hypothesis Tracker recursion



Multi Hypothesis Tracker recursion



TRACKING N OBJECTS USING MHT

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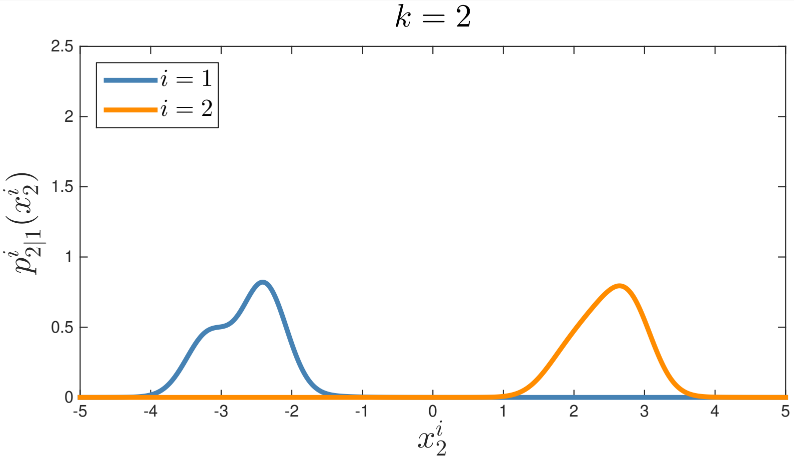

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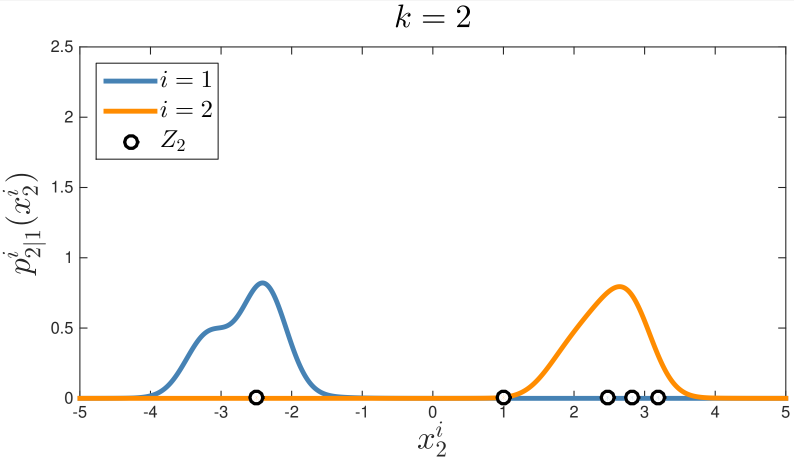

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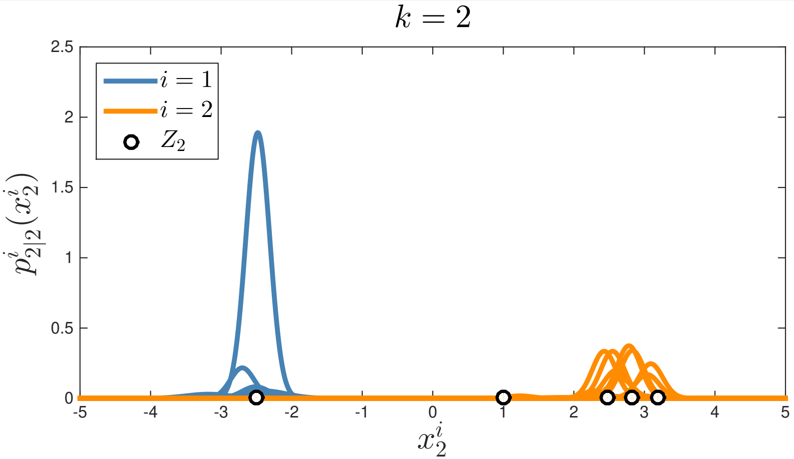

TRACKING N OBJECTS USING MHT

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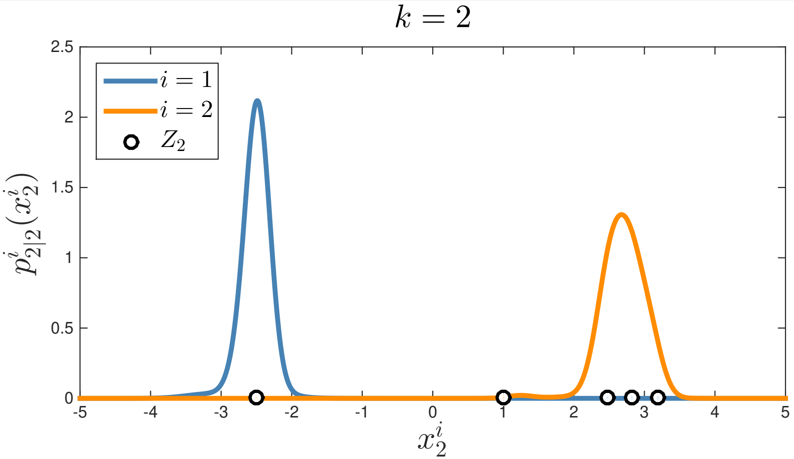
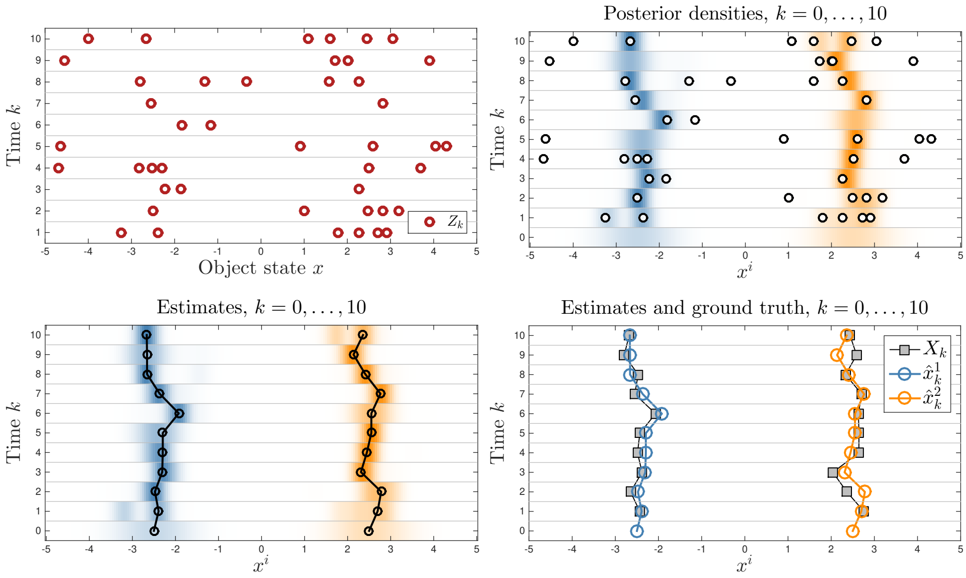
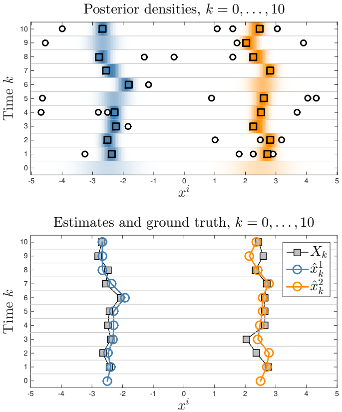


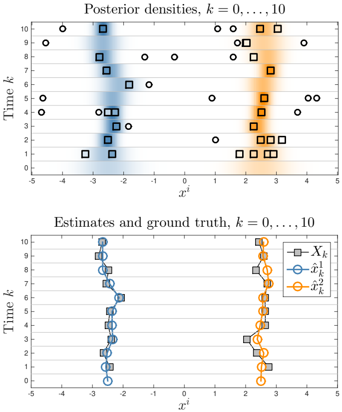
TRACKING *N* OBJECTS



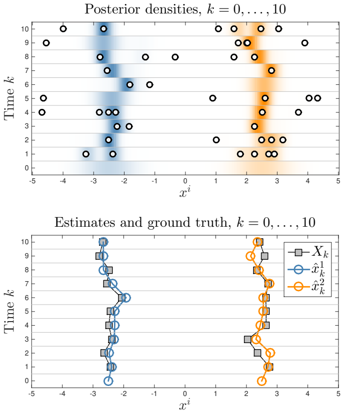
GNN, JPDA, MHT, $P^{\mathrm{D}} = 0.85$



GNN filter

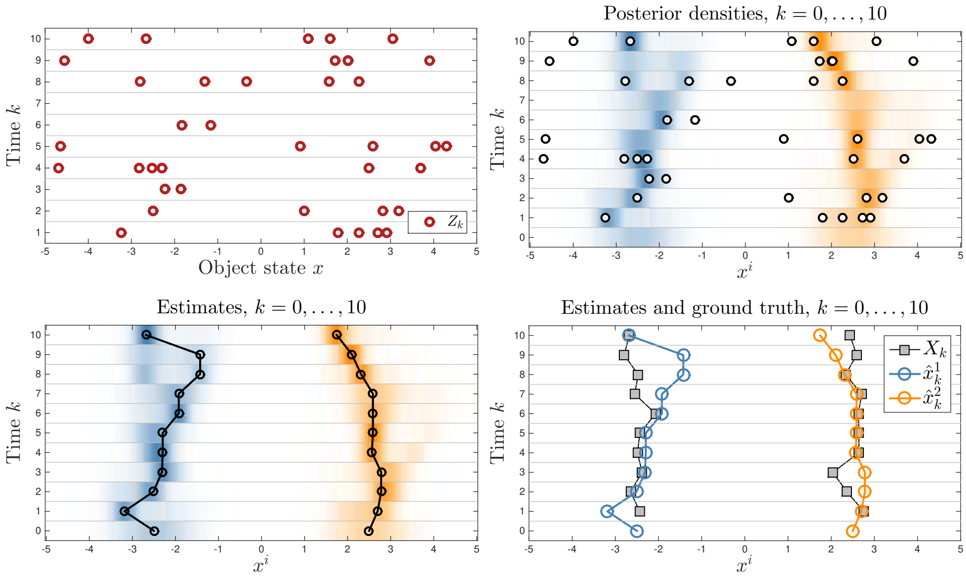


JPDA filter

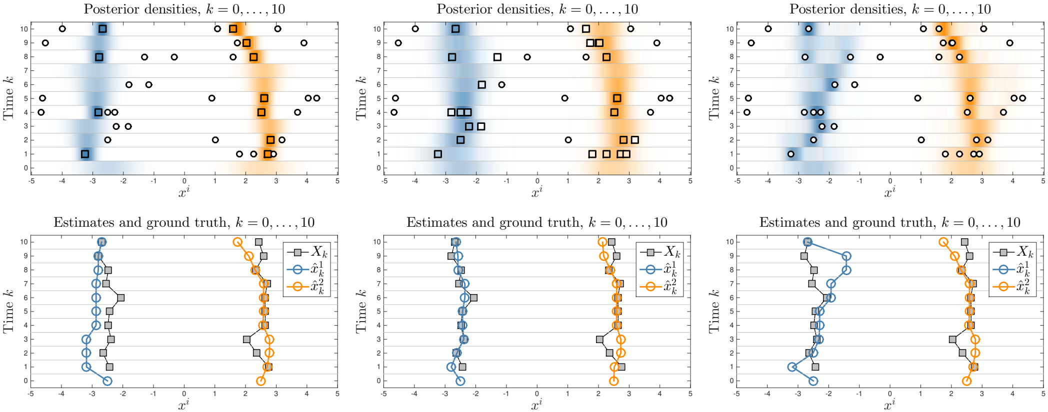


MHT

TRACKING N OBJECTS, LOWER $P^{\mathrm{D}} = 0.50$



GNN, JPDA, MHT, LOWER $P^{\mathrm{D}} = 0.5$



GNN filter

JPDA filter

MHT

DENSITY APPROXIMATION

Multi-Hypothesis Tracker (MHT)

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$P^{\mathrm{D}} = 0.5$

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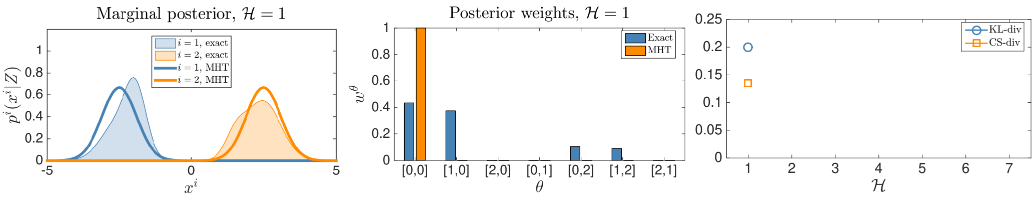

Kullback-Leibler-divergence (KL-div) and Cauchy-Schwarz-divergence (CS-div)

When $N_{\max} = 1$, MHT becomes equal to GNN

DENSITY APPROXIMATION

Multi-Hypothesis Tracker (MHT)

$X = [x^1,\ x^2],\quad Z = [-1.6,\ 1],\quad p(X) = \mathcal{N}\left(x^1;\ -2.5, 0.36\right)\mathcal{N}\left(x^2;\ 2.5, 0.36\right)$

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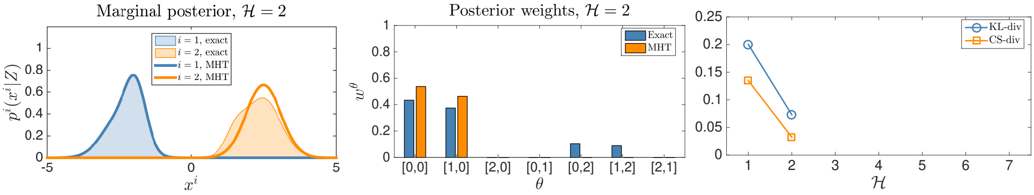

Kullback-Leibler-divergence (KL-div) and Cauchy-Schwarz-divergence (CS-div)

DENSITY APPROXIMATION

Multi-Hypothesis Tracker (MHT)

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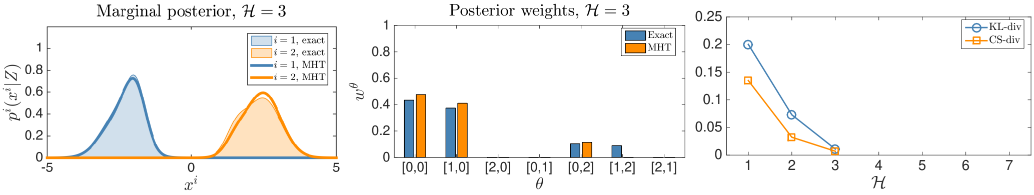

Kullback-Leibler-divergence (KL-div) and Cauchy-Schwarz-divergence (CS-div)

DENSITY APPROXIMATION

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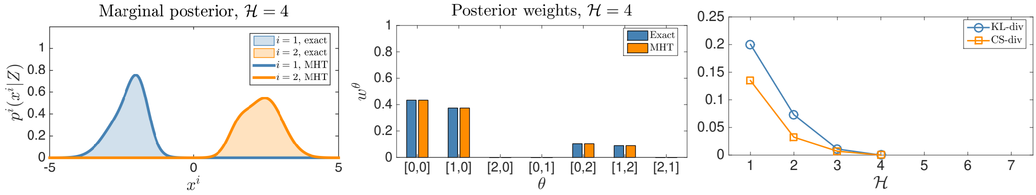

Kullback-Leibler-divergence (KL-div) and Cauchy-Schwarz-divergence (CS-div)

DENSITY APPROXIMATION

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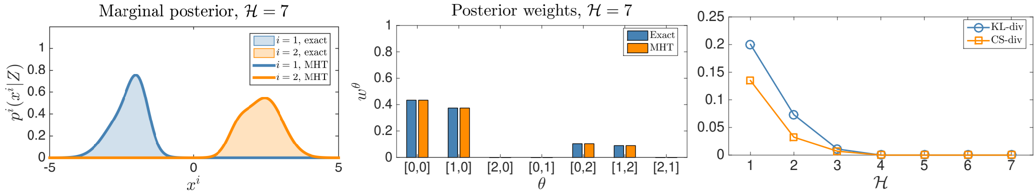
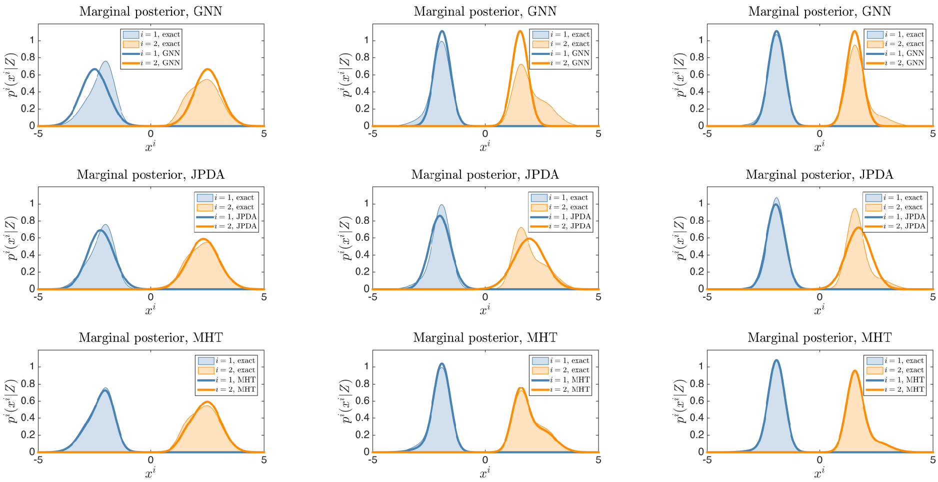


Kullback-Leibler-divergence (KL-div) and Cauchy-Schwarz-divergence (CS-div)

DENSITY APPROXIMATION: GNN, JPDA, MHT



$P^{\mathrm{D}} = 0.5$ $P^{\mathrm{D}} = 0.85$ $P^{\mathrm{D}} = 0.95$

HO-MHT SUMMARY

Pros:

- For large enough $N_{\max}$, typically sufficient to represent the uncertainty of the scenario.
- Works well in simple scenarios with low SNR, i.e., low P^D, high $\bar{\lambda}$, large R.

Cons:

- Computationally more expensive, compared to GNN and JPDA.
- More complicated to implement.
- Not guaranteed that most probable association sequence is represented in mixture, but if $N_{\max}$ is large enough, that is typically the case.

Representation of the hypotheses in MHT

Multi-Object Tracking

Karl Granström

GLOBAL HYPOTHESES AND LOCAL HYPOTHESES

- **Global hypothesis:** for all n objects, corresponds to association sequence

  $$\theta_{1:k}$$

  Assuming a single initial global hypothesis,

  $$\mathcal{H}_k = \prod_{t=1}^{k} N_A(m_t, n)$$

- **Local hypothesis:** for a single object, corresponds to association sequence

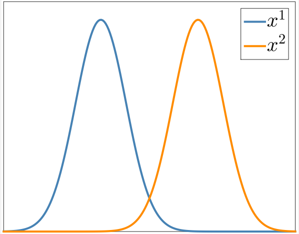
  $$\theta^i_{1:k}$$

  Assuming a single initial local hypothesis,

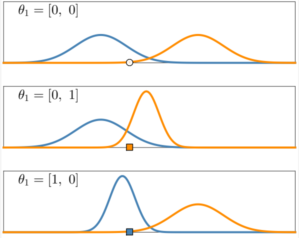
  $$\mathcal{H}^i_k = \prod_{t=1}^{k} (m_t + 1),$$

EXAMPLE

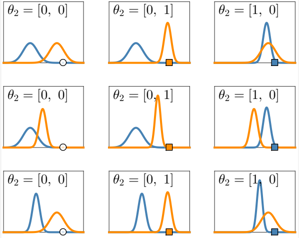
Number of global hypotheses and local hypotheses



Initial prior



Posterior, $k = 1$



Posterior, $k = 2$

- Number of global hypotheses: $\mathcal{H}_0 = 1$, $\mathcal{H}_1 = 3$, $\mathcal{H}_2 = 9$
- Number of local hypotheses: $\mathcal{H}_0^i = 1$, $\mathcal{H}_1^i = 2$, $\mathcal{H}_2^i = 4$
- **Note:** some local hypotheses occur in multiple global hypotheses!

IMPLICATIONS FOR MHT

- What does this mean for HO-MHT?
  - Each global hypothesis represented independently of the other hypotheses.
  - Local hypotheses are then represented multiple times.
  - Inefficient in terms of memory and computations.
- More efficient to have
  - a single copy of each local hypothesis
  - a look-up-table for the global hypotheses:
    which local hypotheses are included in the global hypothesis?
- This is called **Track Oriented (TO) MHT**

TO-MHT

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Hypothesis trees and look-up tables

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HYPOTHESIS TREES AND GLOBAL LOOK-UP TABLES

- Number of global hypotheses and number of local hypotheses

$$\mathcal{H}_k = \prod_{t=1}^{k} N_A(m_t, n), \qquad \mathcal{H}_k^i = \prod_{t=1}^{k} (m_t + 1),$$

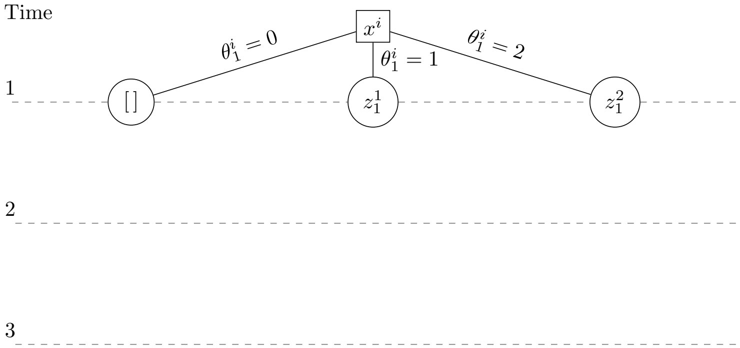
- For each object, we get a **hypothesis tree**
  - Leaf = local hypothesis, i.e., association sequence $\theta_{1:k}^i$
  - For each leaf: $p_{k|k}^i(x_k^i | Z_{1:k}, \theta_{1:k}^i)$
  - At time k, $\mathcal{H}_k^i$ leafs in tree
- Global hypotheses stored implicitly in **look-up-table** $\mathbb{H}_k$.
  - $\mathbb{H}_k(h_k, i)$: local hypothesis from object i is included in hypothesis h_k.
  - At time k, $\mathcal{H}_k$ global hypotheses in look-up table

OBJECT HYPOTHESIS TREE

Hypothesis tree

- $\mathcal{H}_k^i = \prod_{t=1}^{k}(m_t + 1)$
- Three time steps:

$$
\begin{aligned}
m_1 &= 2 & \mathcal{H}_1^i &= 3 \\
m_2 &= 0 & \mathcal{H}_2^i &= 3 \\
m_3 &= 1 & \mathcal{H}_3^i &= 6
\end{aligned}
$$

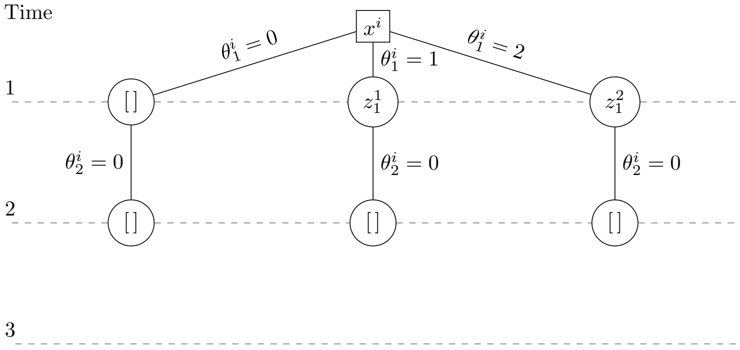


OBJECT HYPOTHESIS TREE

Hypothesis tree

- $\mathcal{H}_k^i = \prod_{t=1}^{k} (m_t + 1)$
- Three time steps:

$$m_1 = 2 \quad \mathcal{H}_1^i = 3$$
$$m_2 = 0 \quad \mathcal{H}_2^i = 3$$
$$m_3 = 1 \quad \mathcal{H}_3^i = 6$$



OBJECT HYPOTHESIS TREE

Hypothesis tree

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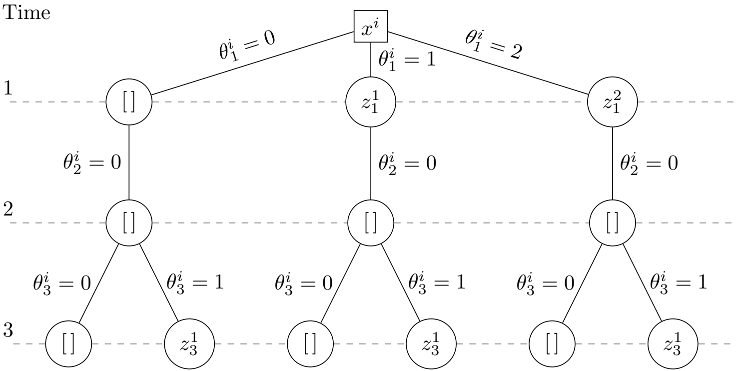
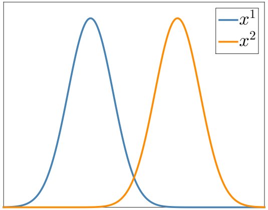


HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Look-up table

Initial prior



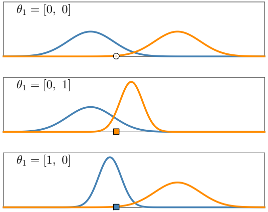
Hypothesis trees

For simplicity, local hypotheses indexed from left to right

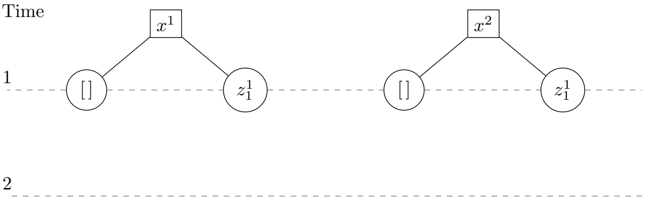
HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Posterior, $k = 1$



Hypothesis trees



Look-up table

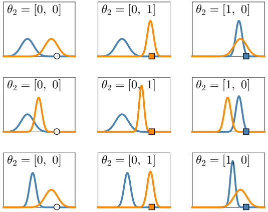
$$\mathbb{H}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

For simplicity, local hypotheses indexed from left to right

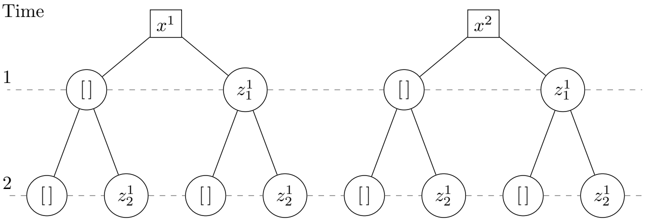
HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Posterior, $k = 2$



Hypothesis trees



For simplicity, local hypotheses indexed from left to right

Look-up table

$$\mathbb{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 3 \\ 1 & 4 \\ 2 & 3 \\ 3 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix}$$

TO-MHT prediction and update

Multi-Object Tracking

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BASIC TO-MHT RECURSION

TO-MHT density parameterized by a global look-up table $\mathbb{H}_k$, log-weights ℓ^{h_k} and local hypotheses for each object,

$$\mathbb{H}_k, \qquad \left\{\ell^{h_k}\right\}_{h_k=1}^{\mathcal{H}_k}, \qquad \left\{\left\{p_{k|k}^{i,h_k^i}(x_k^i)\right\}_{h_k^i=1}^{\mathcal{H}_k^i}\right\}_{i=1}^{n}$$

TO-MHT: pseudo-code

For $k = 1, 2, \ldots, K$

Prediction: For each local hypothesis: Chapman-Kolmogorov prediction

Local update: for each object, compute updated local hypotheses

Global update: compute $\mathbb{H}_k$ and ℓ^{h_k} of the posterior global hypotheses

Reduction: Pruning and capping, same as HO-MHT

TO-MHT PREDICTION

- Predict each local hypothesis independent of other local hypotheses

TO-MHT prediction: pseudo-code

- **Input:** $\mathbb{H}_{k-1}$, $\left\{\ell^{h_{k-1}}\right\}_{h_{k-1}=1}^{\mathcal{H}_{k-1}}$, $\left\{\left\{p_{k-1|k-1}^{i,h_{k-1}^i}(x_{k-1}^i)\right\}_{h_{k-1}^i=1}^{\mathcal{H}_{k-1}^i}\right\}_{i=1}^n$
- **Output:** $\mathbb{H}_{k-1}$, $\left\{\ell^{h_{k-1}}\right\}_{h_{k-1}=1}^{\mathcal{H}_{k-1}}$, $\left\{\left\{p_{k|k-1}^{i,h_{k-1}^i}(x_k^i)\right\}_{h_{k-1}^i=1}^{\mathcal{H}_{k-1}^i}\right\}_{i=1}^n$

  where

$$p_{k|k-1}^{i,h_{k-1}^i}(x_k^i) = \int \pi(x_k^i|x_{k-1}^i)p_{k-1|k-1}^{i,h_{k-1}^i}(x_{k-1}^i)\mathrm{d}x_{k-1}^i$$

TO-MHT: UPDATE LOCAL HYPOTHESES

TO-MHT local hypothesis update: pseudo-code

For $i = 1, \ldots, n$: **Input:** $\left\{ p_{k|k-1}^{i,h_{k-1}^i}(x_k^i) \right\}_{h_{k-1}^i=1}^{\mathcal{H}_{k-1}^i}$

For $h_{k-1}^i = 1, \ldots, \mathcal{H}_{k-1}^i$

For $j = 0, 1, \ldots, m_k$

Index: $h_k^i = (h_{k-1}^i - 1)(m_k + 1) + 1 + j$

Density: $p_{k|k}^{i,h_k^i}(x_k^i) \propto \begin{cases} P^{\mathrm{D}}(x_k^i) g_k(z_k^j | x_k^i) p_{k|k-1}^{i,h_{k-1}^i}(x_k^i) & j \neq 0 \\ (1 - P^{\mathrm{D}}(x_k^i)) p_{k|k-1}^{i,h_{k-1}^i}(x_k^i) & j = 0 \end{cases}$

Association log-likelihood ℓ^{i,j,h_{k-1}^i}

Number of local hypotheses: $\mathcal{H}_k^i = \mathcal{H}_{k-1}^i(m_k + 1)$

Output: $\left\{ p_{k|k}^{i,h_k^i}(x_k^i) \right\}_{h_k^i=1}^{\mathcal{H}_k^i}$, association log-likelihoods ℓ^{i,j,h_{k-1}^i} for $j = 0, \ldots, m_k$ and $h_{k-1}^i = 1, \ldots, \mathcal{H}_{k-1}^i$

TO-MHT UPDATE: LOOK-UP-TABLE

TO-MHT look-up table update: pseudo-code

Input: $\mathbb{H}_{k-1}$, $\left\{\ell^{h_{k-1}}\right\}_{h_{k-1}=1}^{\mathcal{H}_{k-1}}$, association log-likelihoods $\ell^{i,j,h^i_{k-1}}$

Set $h_k = 0$

For $h_{k-1} = 1, \ldots, \mathcal{H}_{k-1}$

Create cost matrix $L^{h_{k-1}}$ using $\ell^{i,j,h^i_{k-1}}$

Compute $M^{h_{k-1}}$ assignments θ_m

For $m = 1, \ldots, M^{h_{k-1}}$

Increase: $h_k \leftarrow h_k + 1$

Initialise log-weight: $\tilde{\ell}^{h_k} = \ell^{h_{k-1}}$

Update look-up table

Normalise log-weights $\ell^{h_k} \leftarrow \tilde{\ell}^{h_k}$

Output: $\mathbb{H}_k$, $\left\{\ell^{h_k}\right\}_{h_k=1}^{\mathcal{H}_k}$

Update look-up table

For $i = 1, \ldots, n$

$\mathbb{H}_k(h_k, i) =$
$(\mathbb{H}_{k-1}(h_{k-1}, i) - 1)(m_k + 1) + 1 + \theta^i_m$

Increase log-weight:
$\tilde{\ell}^{h_k} \leftarrow \tilde{\ell}^{h_k} + \ell^{i,\theta^i_m,h_{k-1}}$

TO-MHT VS HO-MHT

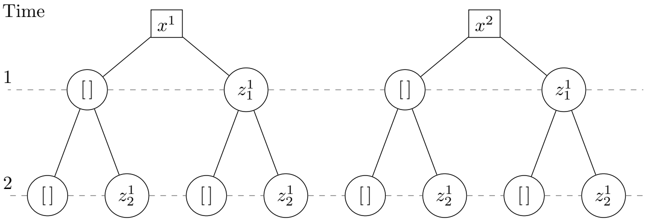
- Same as HO-MHT
  - Deciding the number of data associations
  - Normalizing the log-weights

- TO-MHT: pruning and capping just like HO-MHT, with one important difference:
  - When global hypotheses are pruned, one or more local hypotheses may no longer be included in any global hypothesis.
  - In that case, those local hypotheses are also pruned.

PRUNING OF HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Hypothesis trees



Look-up table, weights (%)

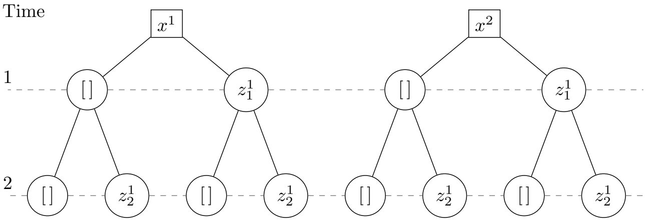
$$\mathbb{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 3 \\ 1 & 4 \\ 2 & 3 \\ 3 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix}, \quad w^\theta = \begin{bmatrix} 1.8 \\ 10.7 \\ 1.6 \\ 6 \\ 1.6 \\ 5.2 \\ 10.4 \\ 62.5 \\ 0.1 \end{bmatrix}$$

PRUNING OF HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Look-up table, weights (%)

Hypothesis trees



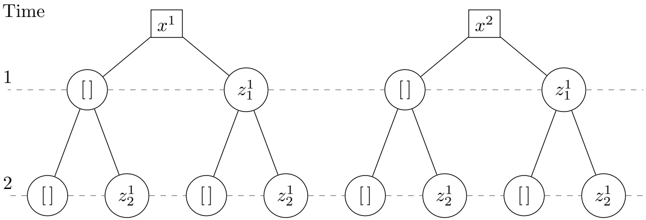
$$\mathbb{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 3 \\ 1 & 4 \\ 2 & 3 \\ 3 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix}, \quad \tilde{w}^{\theta} = \begin{bmatrix} 0 \\ 10.7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10.4 \\ 62.5 \\ 0 \end{bmatrix}$$

Prune global hypotheses with weights lower than 10%.

PRUNING OF HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Hypothesis trees



Look-up table, weights (%)

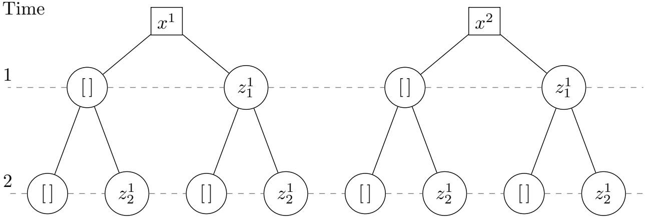
$$\mathbb{H}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}, \quad w^\theta = \begin{bmatrix} 12.8 \\ 12.4 \\ 74.8 \end{bmatrix}$$

Remove from $\mathbb{H}_2$ and re-normalize weights

PRUNING OF HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Hypothesis trees



Look-up table, weights (%)

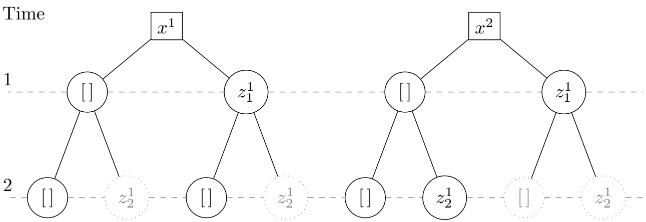
$$\mathbb{H}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}, \quad w^\theta = \begin{bmatrix} 12.8 \\ 12.4 \\ 74.8 \end{bmatrix}$$

Local hypotheses no longer used:
2 and 4 for x^1, and 3 and 4 for x^2.

PRUNING OF HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Hypothesis trees



Look-up table, weights (%)

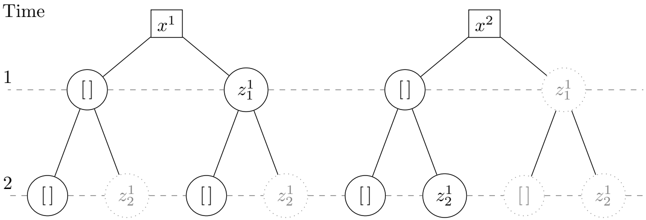
$$\mathbb{H}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}, \quad w^\theta = \begin{bmatrix} 12.8 \\ 12.4 \\ 74.8 \end{bmatrix}$$

Note implication for association $x^2 \leftrightarrow z_1^1$

PRUNING OF HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Hypothesis trees



Look-up table, weights (%)

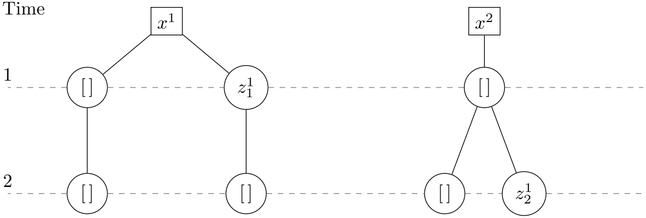
$$\mathbb{H}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}, \quad w^\theta = \begin{bmatrix} 12.8 \\ 12.4 \\ 74.8 \end{bmatrix}$$

Remove branches without leaf nodes at current time

PRUNING OF HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Hypothesis trees



Look-up table, weights (%)

$$\mathbb{H}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}, \quad w^\theta = \begin{bmatrix} 12.8 \\ 12.4 \\ 74.8 \end{bmatrix}$$

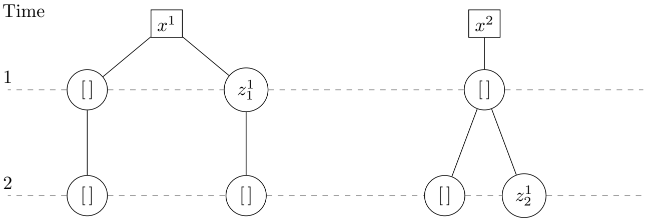
Pruned local hypothesis trees.

Look-up table must be adjusted accordingly.

PRUNING OF HYPOTHESIS TREES AND LOOK-UP TABLE

Example: $n = 2$, $m_1 = 1$, $m_2 = 1$

Hypothesis trees



Look-up table, weights (%)

$$\mathbb{H}_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}, \quad w^{\theta} = \begin{bmatrix} 12.8 \\ 12.4 \\ 74.8 \end{bmatrix}$$

Pruned local hypothesis trees and global look-up table, re-normalized weights.

TO-MHT: PROS AND CONS

Pros:

- Same tracking performance as HO-MHT, generally better than GNN and JPDA
- Computationally cheaper than HO-MHT
- More memory efficient than HO-MHT

Cons:

- Computationally more expensive than GNN and JPDA.
- A bit more complicated to implement than HO-MHT
- Not guaranteed that most probable association sequence is represented in mixture, but if $N_{\max}$ is large enough, that is typically the case.

Note: TO-MHT is the *de facto* standard in some industries.

Outlook on *n* object tracking

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EXTENSIONS OF MHT

- Use gating to group the targets into independent sub-sets
  - Reduces n object MHT into several MHTs for $< n$ objects.
    - Generally cheaper to run MHT with lower n and lower m_k.
    - Can be executed in parallel.
  - Complicated when previously “independent objects” begin to “interact”, i.e., when they can no longer be gated into separate groups. Also complicated as groups separate.

- N-scan MHT
  - Formulate data association for N latest time steps as constrained optimal assignment problem
  - No explicit look-up-table. With local hypothesis trees, possible to implicitly represent an enormous amount of global hypotheses.

TRACKING AN UNKNOWN NUMBER OF OBJECTS?

Can GNN, JPDA and MHT be used for tracking when n is unknown and time-varying, i.e., for **Multiple Object Tracking (MOT)**?

- Yes!
- Requires tracking initiation and track deletion
  to handle that the objects appear and disappear.
  - Score based
  - M/N logic
  - Measurement driven initiation empirically known to be good

- We will use Random Finite Sets (RFS) to model MOT.
- MOT algorithms will be derived based on the RFS models.