粒子滤波部分主要参考材料

Optimal State Estimation **15**

Optimal Estimation of Dynamic Systems **4.10**

Theory and Implementation of Particle Filters

University of Ottawa

非线性滤波理论与目标跟踪应用 **7.1~7.4**

占荣辉、张军等 国防工业出版社

A Tutorial on Particle Filters for Online Nonlinear/Non-Gaussian Bayesian Tracking，IEEE TRANSACTIONS ON SIGNAL PROCESSING

1. 贝叶斯估计

1）贝叶斯定理及其应用

$$p(\boldsymbol{A}|B)=\frac{p(B|A)\,p(A)}{p(B)}$$

贝叶斯定理阐述了条件概率密度的计算思想，而条件概率密度是我们依据给定信息进行推理、判断、估计的重要的方法。贝叶斯定理应用非常广泛，例如贝叶斯估计、贝叶斯学习、贝叶斯网络、贝叶斯分类、贝叶斯推断等。

2）贝叶斯定理与估计相关的应用

状态估计问题是利用观测**Z**及噪声**W**、**V**的统计特性，在状态方程的基础上对状态**X**估计，实质上也是一种条件概率密度的应用问题。

极大似然、极大验后（最大后验）估计就是建立在条件概率基础上的估计准则，如果条件概率密度的计算使用贝叶斯公式，就可称之为贝叶斯估计。通常情况下贝叶斯估计是针对验后而言的。

3）贝叶斯估计 Bayesian Estimation

贝叶斯估计有不同的形式，Optimal Estimation of Dynamic Systems 给出了贝叶斯状态估计、最小风险估计两种应用。

① 贝叶斯状态估计（贝叶斯滤波）

$$X_{k+1} = f\left(X_k, \mathrm{U}_k, \mathrm{W}_k\right)$$

$$\mathrm{Z}_k = h\left(X_k\right) + \mathrm{V}_k$$

贝叶斯状态估计的目的是实现对系统状态的准确估计，其最典型的应用就是**极大验后（后验）估计**的贝叶斯计算实现。

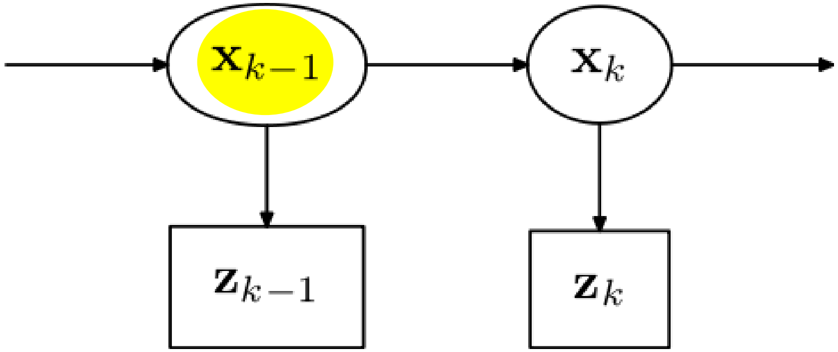
贝叶斯状态估计能有效对待噪声的作用吗？

包含有系统噪声W的作用

$$p\left(\boldsymbol{X} \mid Z\right) = \frac{p\left(Z \mid X\right) p\left(X\right)}{p\left(Z\right)}$$

包含有观测噪声V的作用

1. 贝叶斯估计



$$p(z_k \mid x_k, x_{k-1}, \cdots x_0) = p(z_k \mid x_k)$$

$$p(x_k \mid x_{k-1}, \cdots x_0) = p(x_k \mid x_{k-1})$$

$p(z_i, x_i \mid x_{i-1})$

$$p(x_0, \cdots x_k, z_0, \cdots, z_k) = p(x_0)\prod_{i=1}^{k} p(z_i \mid x_i) p(x_i \mid x_{i-1})$$

$$\boxed{p(x_k \mid z_{1:k-1}) = \int p(x_k \mid x_{k-1}) p(x_{k-1} \mid z_{1:k-1}) dx_{k-1}}$$

Chapman-Kolmogorov equation

$$p[x_k \mid z_{1:k}] = \frac{p[x_k, z_{1:k}]}{p(z_{1:k})}$$

$$= \frac{p(x_k, z_k \mid z_{1:k-1}) p(z_{1:k-1})}{p(z_k \mid z_{1:k-1}) p(z_{1:k-1})}$$

不是贝叶斯公式

$$= \frac{p(z_k \mid x_k, z_{1:k-1}) p(x_k \mid z_{1:k-1})}{p(z_k \mid z_{1:k-1})}$$

$$= \frac{p(z_k \mid x_k) p(x_k \mid z_{1:k-1})}{p(z_k \mid z_{1:k-1})}$$

利用贝叶斯公式可以得到相同的结果

$\mathbf{Z_k}$未知下预测**PDF**

$$p[x(k) \mid z_{1:k}] = \frac{p(z_k \mid x_k) p(x_k \mid z_{1:k-1})}{p(z_k \mid z_{1:k-1})} = \alpha p(z_k \mid x_k) p(x_k \mid z_{1:k-1})$$

$\mathbf{Z_k}$已知修正

$$p(z_k \mid z_{k-1}) = \int p(z_k \mid x_k) p(x_k \mid z_{k-1}) dx_k$$

与x_k取值无关

1. 贝叶斯估计

3）贝叶斯估计 Bayesian Estimation

② 最小风险估计

目标是评价代价函数$c(X^*|X)$，即估计值为X^*，而真实值为X的代价，由于X未知，确切的代价函数无法评估，通常假设X服从某验后概率的分布，其目标函数为

$$J_{MR}\left(X^*\right)=\int_{-\infty}^{+\infty} c\left(X^* \mid X\right) p\left(\boldsymbol{X} \mid Z\right) dX$$

利用贝叶斯公式，风险函数可以写为

$$J_{MR}\left(X^*\right)=\int_{-\infty}^{+\infty} c\left(X^* \mid X\right) p\left(\boldsymbol{X} \mid Z\right) dX=\int_{-\infty}^{+\infty} c\left(X^* \mid X\right) \frac{p\left(Z \mid X\right) p\left(X\right)}{p\left(Z\right)} dX$$

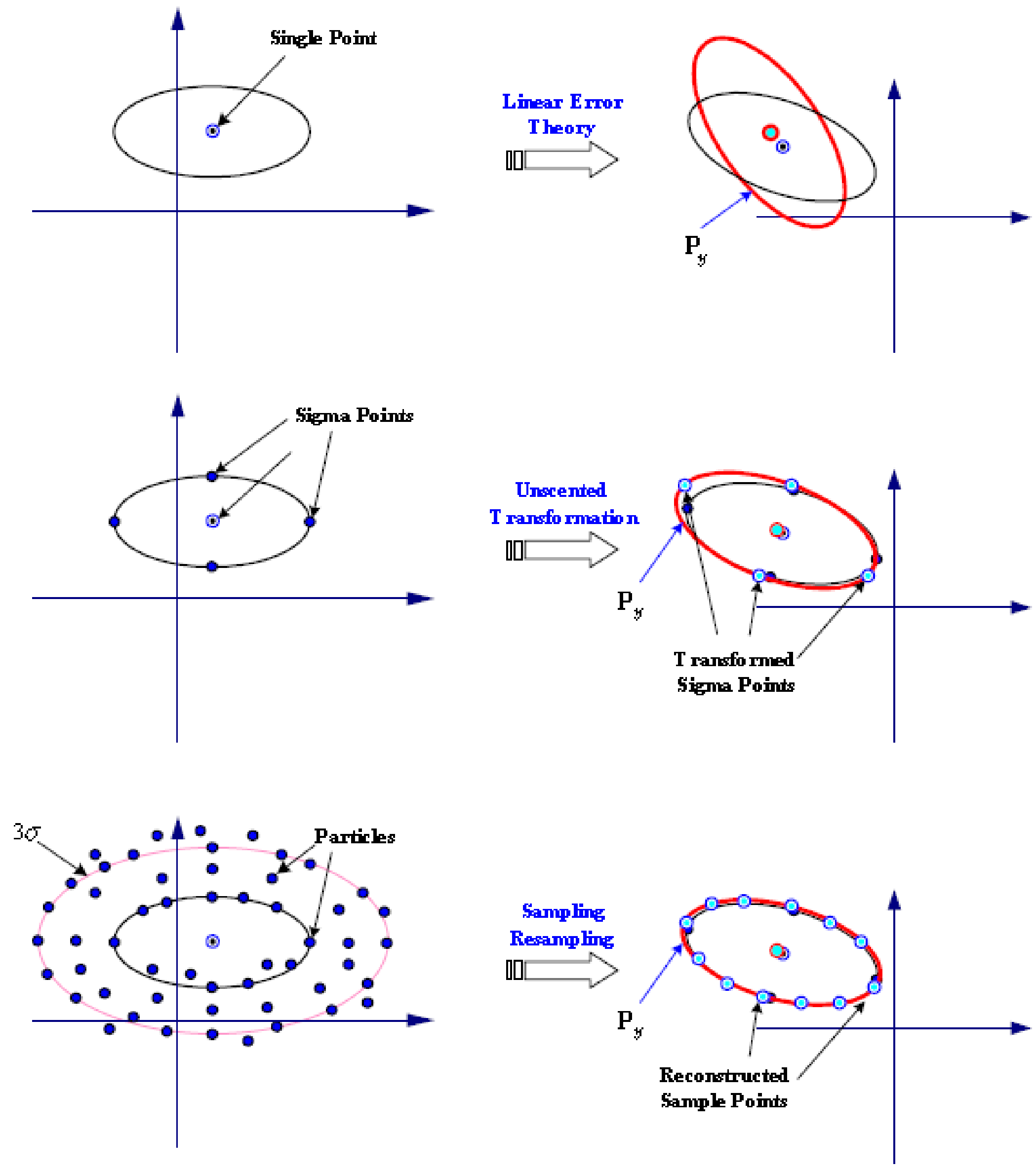
4）其他相关概念

贝叶斯推断 Bayesian Inference，重点在于获得验后概率密度，而不是对状态值的估计。

2. 粒子滤波的问题提出

1) 粒子滤波的思想

The particle filter is a statistical, **brute-force** approach to estimation that often works well for problems that are difficult for the conventional Kalman filter (i.e., **systems that are highly nonlinear**).



粒子滤波的其他名称

sequential importance sampling
bootstrap filtering
the condensation algorithm
interacting particle approximations
Monte Carlo filtering
sequential Monte Carlo (SMC) filtering

注意粒子滤波与**UKF**中采样的区别

2. 粒子滤波问题的引出

2) 粒子滤波的数学描述

以离散非线性系统为例

$$X_{k+1} = f\left(X_k, \mathrm{U}_k, \mathrm{W}_k\right)$$

$$\mathrm{Z}_k = h\left(X_k\right) + \mathrm{V}_k$$

极大验后估计是将概率密度最大情况下对应的状态作为最优估计解，如果有足够的样本，可以通过模拟数学期望的方式获得状态最优估计。

假设在k时刻有N个状态样本，则可以对k+1时刻状态进行一步预测

$$\hat{\mathrm{X}}_{k+1|k}^{(i)} = f\left(X_k^{(i)}, \mathrm{U}_k, \mathrm{W}_k\right)$$

k+1时刻状态最优估计解的数学期望为

$$E\left[X_{k+1|k+1} \mid \mathrm{Z}_{1:k+1}\right] = \int_{-\infty}^{+\infty} X_{k+1|k}\, p(X_{k+1} \mid \mathrm{Z}_{1:k+1})\, dX_{k+1} \approx \frac{1}{N}\sum_{i=1}^{N} \hat{\boldsymbol{X}}_{k+1|k}^{(i)}$$

如果积分公式可以准确地利用离散样本计算，则可以实现粒子滤波，其实现的方法是蒙特卡洛法。

3. 蒙特卡洛法的基本原理

1）蒙特卡洛法解决的典型问题

自变量 $x_1, x_2, \cdots, x_n$的分布特性已知，求y的分布特性。

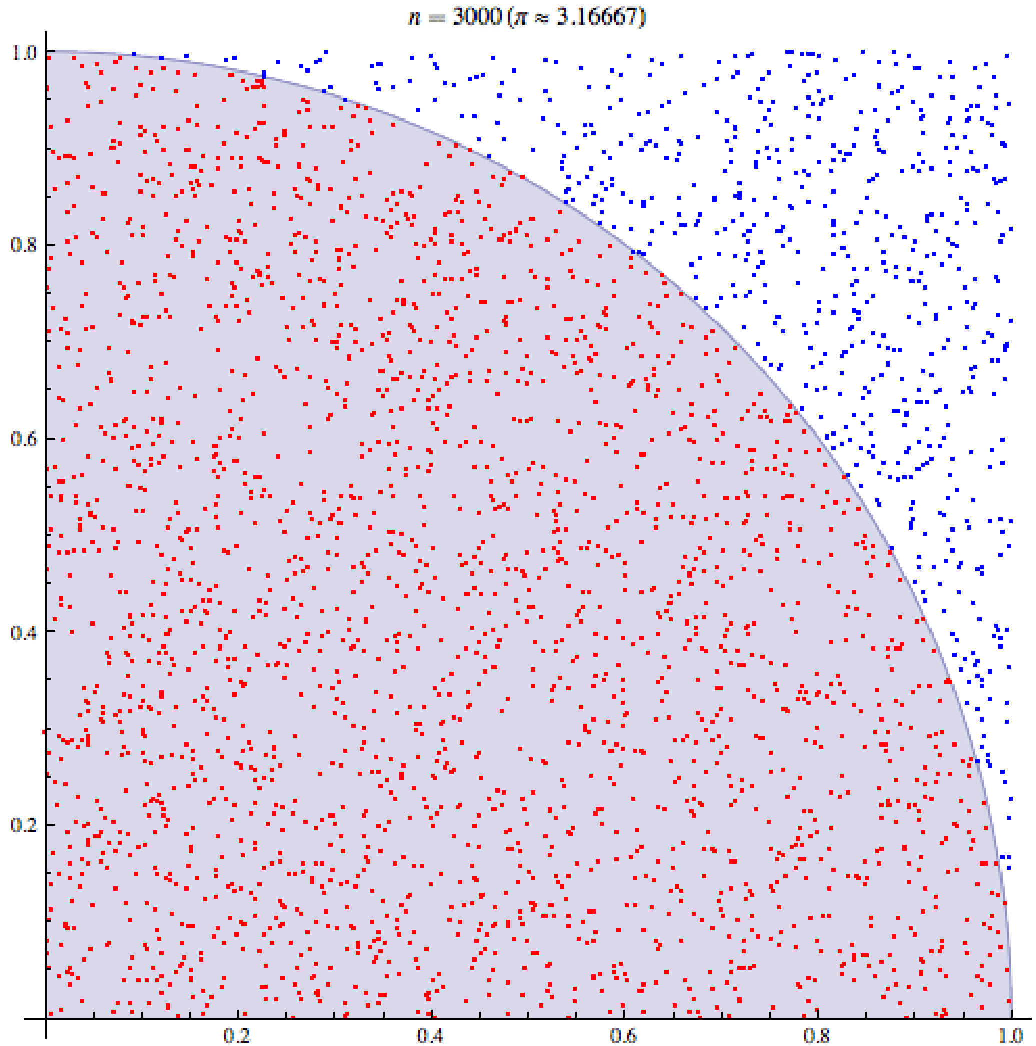
$$y = f(x_1, x_2, \cdots x_n)$$

Monte Carlo methods有多种形式，但关键实现方式如下：

- ●确定输入的可能值域范围
- ●在输入变量值域范围内按照其概率密度进行随机抽样
- ●按照给定的函数进行输出计算
- ●对输出结果进行统计

3. 蒙特卡洛法的基本原理

1）蒙特卡洛法解决的典型问题



3. 蒙特卡洛法的基本原理

2）蒙特卡洛法解决的基本原理

$$y_k = g(x_k) + V_k$$

$$E[g(x_k)] = \int g(x_k)p(x_k)dx_k$$

完备采样

如果使用**N**个**x**的样本，是否能够获得**g(x_k)**的近似数学期望？

$$E[g(x_k)] \approx \bar{y}_k = \frac{1}{N}\sum_{i=1}^{N} g(x_k^{(i)})$$

什么情况下会合理？

如果按照***p(x_k)***对***$x^{(i)}_k$***进行采样，那么***g($x^{(i)}_k$)***的出现已经包含了***$x^{(i)}_k$***的***p(x_k)***的概率信息，因此无需再对***g($x^{(i)}_k$)***进行概率加权。

同理

$$Var(y_k) \approx \frac{\sum_{i=1}^{N}\left[g(x_k^{(i)}) - \bar{y}_k\right]\left[g(x_k^{(i)}) - \bar{y}_k\right]^T}{N}$$

3. 蒙特卡洛法的基本原理

2）蒙特卡洛法解决的基本原理

$$y = f(x_1, x_2, \cdots x_n)$$

已知各自变量的分布信息为

$$\begin{aligned} &x_1\text{------}p_1 \\ &x_2\text{------}p_2 \\ &\vdots \\ &x_n\text{------}p_n \end{aligned}$$

按照各分量分布随机产生样本

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}, \cdots \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_n^{(i)} \end{bmatrix} \cdots \begin{bmatrix} x_1^{(m)} \\ x_2^{(m)} \\ \vdots \\ x_n^{(m)} \end{bmatrix}$$

$$E[y] \approx \bar{y} = \frac{\sum_{i=1}^{m} f(x_1^{(i)}, x_2^{(i)}, \cdots x_n^{(i)})}{m}$$

$$Var(y) \approx \frac{\sum_{i=1}^{m} \left[f(x_1^{(i)}, \cdots x_n^{(i)}) - \bar{y} \right] \left[f(x_1^{(i)}, \cdots x_n^{(i)}) - \bar{y} \right]^T}{m}$$

3. 蒙特卡洛法的基本原理

3）$p(x_k|y_{1:k})$未知情况下的实现方法(重要性采样原理)

$$E[g(\mathbf{x}_k)] = \int g(\mathbf{x}_k) p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) d\mathbf{x}_k$$

$p(\mathbf{x}_k \mid \mathbf{y}_{1:k})$ 未知无法采样

$$E[g(\mathbf{x}_k)] = \int g(\mathbf{x}_k) \frac{p(\mathbf{x}_k \mid \mathbf{y}_{1:k})}{q(\mathbf{x}_k \mid \mathbf{y}_{1:k})} q(\mathbf{x}_k \mid \mathbf{y}_{1:k}) d\mathbf{x}_k$$

$$= \int g(\mathbf{x}_k) w_k q(\mathbf{x}_k \mid \mathbf{y}_{1:k}) d\mathbf{x}_k$$

按照$q(\mathbf{x}_k \mid \mathbf{y}_{1:k})$进行采样

$$\approx \frac{1}{N} \sum_{i=1}^{N} g(\mathbf{x}_k^{(i)}) w_k^{(i)}$$

定义

$\mathbf{y}_{1:k}$ —测量序列$y_1 \sim y_k$

Importance Sampling，**IS**

$q(\mathbf{x}_k \mid \mathbf{y}_{1:k})$

Importance density function，**IDF**

W中依然包含$p(x_k|y_{1:k})$，需要进一步处理以便实施。

3. 蒙特卡洛法的基本原理

3）$p(x_k|y_{1:k})$未知情况下的实现方法(重要性采样原理)

$$E[g(x_k)] \approx \frac{1}{N}\sum_{i=1}^{N} g(x_k^{(i)})$$

依照$p(x_k|y_k)$对$x^{(i)}{}_k$进行采样，真实的分布特性。

完备采样

$$E[g(x_k)] \approx \frac{1}{N}\sum_{i=1}^{N} g(x_k^{(i)})w_k^{(i)}$$

依照$q(x_k|y_k)$对$x^{(i)}{}_k$进行采样，非真实分布特性。

重要性采样

不同E的理解

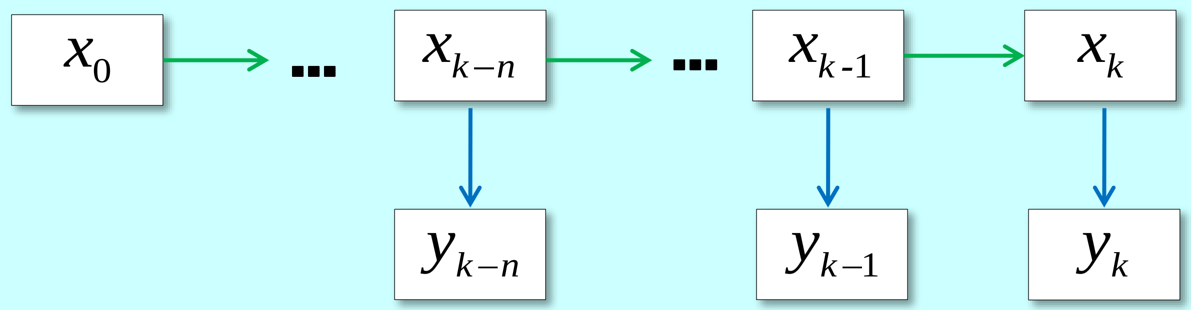
$$\mathbf{y}_k = g\left(\mathbf{x}_k, \mathbf{v}_k\right)$$

$$E\left[g(x_k)\right] = \int g(x_k)\,p(x_k)dx_k$$

$$\mathbf{x}_{k+1} = f\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right)$$

$$\mathbf{y}_k = g\left(\mathbf{x}_k, \mathbf{v}_k\right)$$

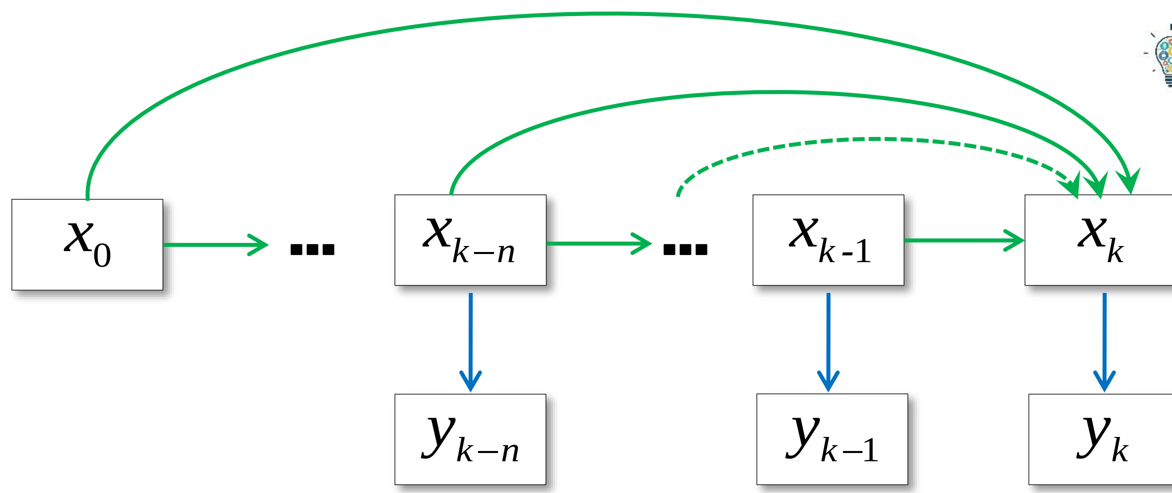
$$E\left[g(\mathbf{x}_k)\right] = \int g(\mathbf{x}_k)\,p(\mathbf{x}_k \mid \mathbf{y}_{1:k})d\mathbf{x}_k$$



$$\mathbf{x}_{k+1} = f\left(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots \mathbf{x}_0, \mathbf{u}_k, \mathbf{w}_k\right)$$

$$\mathbf{y}_k = g\left(\mathbf{x}_k, \mathbf{v}_k\right)$$

$$E\left[g(\mathbf{x}_{0:k})\right] = \int g(\mathbf{x}_{0:k})\,p(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k})d\mathbf{x}_{0:k}$$



4. 重要性采样

1）重要性采样的方式

按概率密度 $p(\mathbf{x}_k \mid \mathbf{y}_{1:k})$采样是特殊的蒙特卡洛采样方式
更一般的形式是从完整的 $p(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k})$ 中进行采样。

$$E\left[g(\mathbf{x}_{0:k})\right] = \int g(\mathbf{x}_{0:k}) p(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k}) d\mathbf{x}_{0:k}$$

$q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k})$容易获得

重要性概率密度函数

$$E\left[g(\mathbf{x}_{0:k})\right] = \int g(\mathbf{x}_{0:k}) \frac{p(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k})}{q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k})} q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k}) d\mathbf{x}_{0:k}$$

$$= \int g(\mathbf{x}_{0:k}) \frac{p(\mathbf{y}_{1:k} \mid \mathbf{x}_{0:k}) p(\mathbf{x}_{0:k})}{p(\mathbf{y}_{1:k}) q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k})} q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k}) d\mathbf{x}_{0:k}$$

$$w_k = \frac{p(\mathbf{y}_{1:k} \mid \mathbf{x}_{0:k}) p(\mathbf{x}_{0:k})}{q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k})} = \frac{p(\mathbf{x}_{0:k}, \mathbf{y}_{1:k})}{q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k})}$$

重要性权重

$$= \int g(\mathbf{x}_{0:k}) \frac{w_k}{p(\mathbf{y}_{1:k})} q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k}) d\mathbf{x}_{0:k} = \frac{\int g(\mathbf{x}_{0:k}) w_k q(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k}) d\mathbf{x}_{0:k}}{p(\mathbf{y}_{1:k})}$$

提出$p(y_{1:k})$

2）重要性权重的归一化

$$E\left[g(\mathbf{x}_{0:k})\right]=\frac{\int g(\mathbf{x}_{0:k})w_k q(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k})d\mathbf{x}_{0:k}}{p\left(\mathbf{y}_{1:k}\right)} \qquad w_k=\frac{p\left(\mathbf{y}_{1:k}\mid\mathbf{x}_{0:k}\right)p\left(\mathbf{x}_{0:k}\right)}{q(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k})}$$

$$p\left(\mathbf{y}_{1:k}\right)=\int p(\mathbf{y}_{1:k},\mathbf{x}_{0:k})d\mathbf{x}_{0:k}$$

$$=\int\frac{p(\mathbf{y}_{1:k}\mid\mathbf{x}_{0:k})p(\mathbf{x}_{0:k})}{q(\mathbf{y}_{1:k}\mid\mathbf{x}_{0:k})}q(\mathbf{y}_{1:k}\mid\mathbf{x}_{0:k})d\mathbf{x}_{0:k}$$

$$=\int w_k q(\mathbf{y}_{1:k}\mid\mathbf{x}_{0:k})d\mathbf{x}_{0:k}$$

$$E\left[g(\mathbf{x}_{0:k})\right]=\frac{\int g(\mathbf{x}_{0:k})w_k q(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k})d\mathbf{x}_{0:k}}{\int w_k q(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k})d\mathbf{x}_{0:k}}$$

按照$q(x_{0:k}|y_{1:k})$选择x_k的样本

$$\bar{w}_k^{(i)}=\frac{w_k^{(i)}}{\sum_{i=1}^{N}w_k^{(i)}}$$

$$w_k^{(i)}=\frac{p\left(\mathbf{x}_{0:k}^{(\mathrm{i})},\mathbf{y}_{1:k}\right)}{q(\mathbf{x}_{0:k}^{(\mathrm{i})}\mid\mathbf{y}_{1:k})}$$

$$E\left[g(\mathbf{x}_{0:k})\right]=\sum_{i=1}^{N}g(\mathbf{x}_k^{(i)})\frac{w_k^{(i)}}{\sum_{i=1}^{N}w_k^{(i)}}=\sum_{i=1}^{N}g(\mathbf{x}_k^{(i)})\bar{w}_k^{(i)}$$

$$E\left[g(\mathbf{x}_{0:k})\right]\approx\frac{\frac{1}{N}\sum_{i=1}^{N}g(\mathbf{x}_{0:k}^{(i)})w_k^{(i)}}{\frac{1}{N}\sum_{i=1}^{N}w_k^{(i)}}$$

归一化处理w

5. 序贯重要性采样(SIS)

1）序贯重要性采样的处理思想

$$E\left[g(\mathbf{x}_{0:k})\right]=\sum_{i=1}^{N}g(\mathbf{x}_{0:k}^{(i)})\bar{w}_k^{(i)}$$

$$\bar{w}_k^{(i)}=\frac{w_k^{(i)}}{\sum_{i=1}^{N}w_k^{(i)}}\qquad w_k^{(i)}=\frac{p\left(\mathbf{y}_{1:k}\mid\mathbf{x}_{0:k}^{(i)}\right)p\left(\mathbf{x}_{0:k}^{(i)}\right)}{q(\mathbf{x}_{0:k}^{(i)}\mid\mathbf{y}_{1:k})}=\frac{p\left(\mathbf{x}_{0:k}^{(i)},\mathbf{y}_{1:k}\right)}{q(\mathbf{x}_{0:k}^{(i)}\mid\mathbf{y}_{1:k})}$$

如果$\boldsymbol{w}_k^{(i)}$能够实现递推计算，可有效简化处理过程。

当重要性概率密度函数具有递推关系时

Sequential Importance Sampling
SIS

$$q\left(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k}\right)=q\left(\mathbf{x}_k\mid\mathbf{x}_{0:k-1},\mathbf{y}_{1:k}\right)q\left(\mathbf{x}_{0:k-1}\mid\mathbf{y}_{1:k-1}\right)$$

$\boldsymbol{w}_k^{(i)}$可以获得递推解算形式，因此称为**序贯重要性采样**。

5. 序贯重要性采样(SIS)

2）重要性权重的计算方法

$$w_k=\frac{p(\mathbf{y}_{1:k}\mid\mathbf{x}_{0:k})\,p(\mathbf{x}_{0:k})}{q(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k})}=\frac{p(\mathbf{x}_{0:k},\mathbf{y}_{1:k})}{q(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k})}$$

$$\begin{aligned}p(\mathbf{x}_{0:k},\mathbf{y}_{1:k})&=p(\mathbf{x}_k,\mathbf{x}_{0:k-1},\mathbf{y}_k,\mathbf{y}_{1:k-1})\\&=p(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{0:k-1},\mathbf{y}_{1:k-1})\,p(\mathbf{x}_{0:k-1},\mathbf{y}_{1:k-1})\\&=p(\mathbf{y}_k|\mathbf{x}_k,\mathbf{x}_{0:k-1},\mathbf{y}_{1:k-1})\,p(\mathbf{x}_k|\mathbf{x}_{0:k-1},\mathbf{y}_{1:k-1})\,p(\mathbf{x}_{0:k-1},\mathbf{y}_{1:k-1})\\&=p(\mathbf{y}_k|\mathbf{x}_k)\,p(\mathbf{x}_k|\mathbf{x}_{k-1})\,p(\mathbf{x}_{0:k-1},\mathbf{y}_{1:k-1})\end{aligned}$$

$$\mathbf{x}_{k+1}=f(\mathbf{x}_k,\mathbf{u}_k,\mathbf{w}_k)$$
$$\mathbf{y}_k=h(\mathbf{x}_k)+\mathbf{v}_k$$

$$\begin{aligned}q(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k})&=q(\mathbf{x}_k\mid\mathbf{x}_{0:k-1},\mathbf{y}_{1:k})q(\mathbf{x}_{0:k-1}\mid\mathbf{y}_{1:k})\\&=q(\mathbf{x}_k\mid\mathbf{x}_{0:k-1},\mathbf{y}_{1:k})q(\mathbf{x}_{0:k-1}\mid\mathbf{y}_{1:k-1})\end{aligned}$$

$$w_k=\frac{p(\mathbf{x}_{0:k},\mathbf{y}_{1:k})}{q(\mathbf{x}_{0:k}\mid\mathbf{y}_{1:k})}=\frac{p(\mathbf{y}_k|\mathbf{x}_k)\,p(\mathbf{x}_k|\mathbf{x}_{k-1})\,p(\mathbf{x}_{0:k-1},\mathbf{y}_{1:k-1})}{q(\mathbf{x}_k\mid\mathbf{x}_{0:k-1},\mathbf{y}_{1:k})q(\mathbf{x}_{0:k-1}\mid\mathbf{y}_{1:k-1})}=\frac{p(\mathbf{y}_k|\mathbf{x}_k)\,p(\mathbf{x}_k|\mathbf{x}_{k-1})}{q(\mathbf{x}_k\mid\mathbf{x}_{0:k-1},\mathbf{y}_{1:k})}w_{k-1}$$

5. 序贯重要性采样(SIS)

3）重要性密度函数的分析

$$w_k = \frac{p\left(\mathbf{y}_k|\mathbf{x}_k\right)p\left(\mathbf{x}_k|\mathbf{x}_{k\text{-}1}\right)}{q(\mathbf{x}_k \mid \mathbf{x}_{0:k-1},\mathbf{y}_{1:k})}w_{k-1}$$

马尔科夫过程 无后效性或无记忆性

A discrete-time Markov chain is a sequence of random variables X_1, X_2, X_3, ... with the Markov property, namely that the probability of moving to the next state depends only on the present state and not on the previous states

$$\Pr(X_k = x \mid X_{k\text{-}1} = x_{k\text{-}1}, X_{k\text{-}2} = x_{k-2},\cdots, X_0 = x_0) = \Pr(X_k = x \mid X_{k-1} = x_{k\text{-}1})$$

如果 $q(\mathbf{x}_k \mid \mathbf{x}_{0:k-1},\mathbf{y}_{1:k})$=$q(\mathbf{x}_k \mid \mathbf{x}_{k-1},\mathbf{y}_k)$，可以简化计算，通常粒子滤波都做此假设。

$$w_k = \frac{p\left(\mathbf{y}_k|\mathbf{x}_k\right)p\left(\mathbf{x}_k|\mathbf{x}_{k\text{-}1}\right)}{q(\mathbf{x}_k \mid \mathbf{x}_{k-1},\mathbf{y}_k)}w_{k-1} \xrightarrow{\text{每一个粒子i}} w_k^{(i)} = \frac{p\left(\mathbf{y}_k|\mathbf{x}_k^{(i)}\right)p\left(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)}\right)}{q(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)},\mathbf{y}_k)}w_{k-1}^{(i)}$$

5. 序贯重要性采样(SIS)

4）最优重要性密度函数的选择

$$w_k^{(i)} = \frac{p\left(\mathbf{y}_k|\mathbf{x}_k^{(i)}\right) p\left(\mathbf{x}_k^{(i)}\middle|\mathbf{x}_{k-1}^{(i)}\right)}{q(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)} w_{k-1}^{(i)}$$

重要性密度函数选择依据

重要性密度函数最优值选择的依据是能够**<u>最小化重要性权值方差，</u>**抑制估计结果过早收敛到非最优解上。

$$q(\mathbf{x}_k \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k) = p(\mathbf{x}_k \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)$$

常规密度函数选择（Bootstrap Filter，BF）

$$w_k^{(i)} = \frac{p\left(\mathbf{y}_k|\mathbf{x}_k^{(i)}\right) p\left(\mathbf{x}_k^{(i)}\middle|\mathbf{x}_{k-1}^{(i)}\right)}{q(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)} w_{k-1}^{(i)} \quad \longrightarrow \quad \boxed{w_k^{(i)} \propto p\left(\mathbf{y}_k|\mathbf{x}_k^{(i)}\right) w_{k-1}^{(i)}}$$

$$\boxed{q(\mathbf{x}_k \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k) = p(\mathbf{x}_k \mid \mathbf{x}_{k-1}^{(i)})}$$

5. 序贯重要性采样(SIS)

$$w_k^{(i)} = \frac{p\left(\mathbf{y}_k \mid \mathbf{x}_k^{(i)}\right) p\left(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)}\right)}{q(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)} w_{k-1}^{(i)}$$

$$q(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{y}_k) = p(\mathbf{x}_k \mid \mathbf{x}_{k-1})$$

$$\boxed{w_k^{(i)} = p\left(\mathbf{y}_k \mid \mathbf{x}_k^{(i)}\right) w_{k-1}^{(i)}}$$

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k)$$

$$\mathbf{y}_k = h(\mathbf{x}_k, k) + \mathbf{v}_k$$

$$\mathbf{y}_k \sim \left(h(\hat{x}_{k|k-1}, k), R_k\right)$$

$$p\left(\mathbf{y}_k \mid \mathbf{x}_k^{(i)}\right)$$

$\mathbf{y}_k^{(i)} = h\left(\mathbf{x}_{k|k-1}^{(i)}, k\right)$ 预测观测

$\mathbf{y}_k$ 实际观测

归一化处理权重

$$\overline{w}_k^{(i)} = \frac{w_k^{(i)}}{\sum_{i=1}^{N} w_k^{(i)}}$$

依据蒙特卡洛法思想，得到滤波解

$$\hat{\mathbf{x}}_{k|k} = E(\mathbf{x}_k \mid \mathbf{y}_k) \approx \sum_{i=1}^{N} \overline{w}_k^{(i)} \mathbf{x}_{k|k-1}^{(i)}$$

$$P_{k|k} \approx \frac{1}{N} \sum_{i=1}^{N} \overline{w}_k^{(i)} (\mathbf{x}_{k|k-1}^{(i)} - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_{k|k-1}^{(i)} - \hat{\mathbf{x}}_{k|k})^T$$

6. 重采样及粗糙化

1）重采样的思想 Resampling

序贯重要采样不能避免**粒子退化(Degeneracy)**问题，经若干次递推后，除少数粒子外，大部分粒子权重小到可以忽略。

重采样的思想：消除权值较小的粒子，对权值较大的粒子进行多份复制。**重采样后每个粒子的权重相等**。

$$\hat{N}_{eff} = \frac{1}{\sum_{i=1}^{N}\left(\overline{w}_k^{(i)}\right)^2}$$

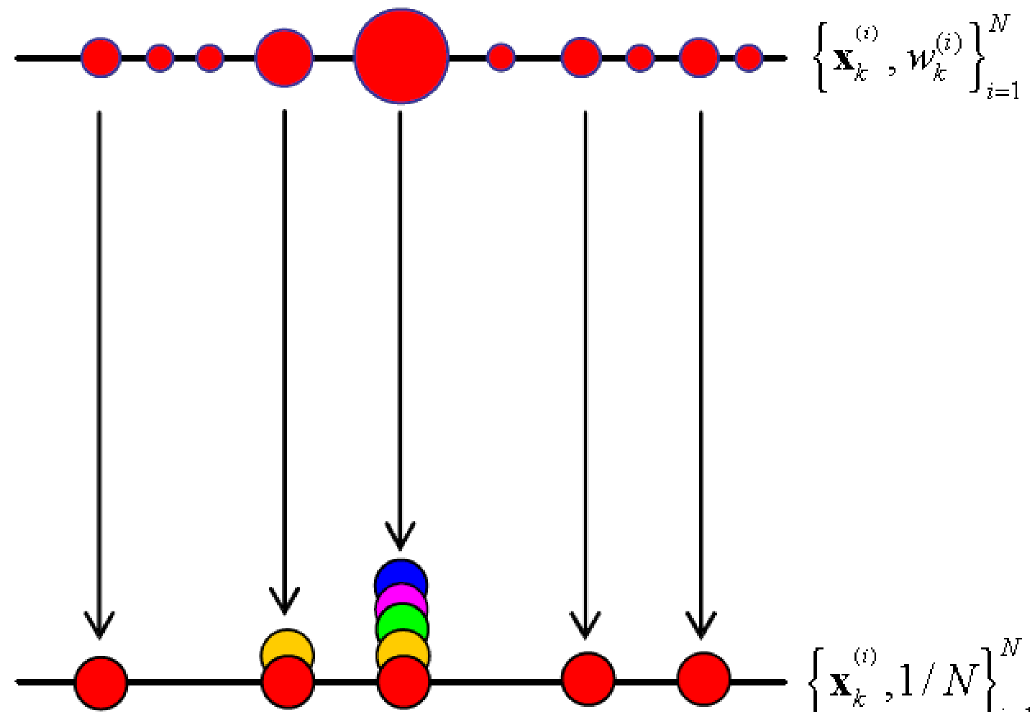
有效采样数量

$$\hat{N}_{eff}\begin{cases}\approx 1 & \text{严重退化} \\ \approx N & \text{权重相同}\end{cases}$$

$$\hat{N}_{eff} \le N_{threshold}$$

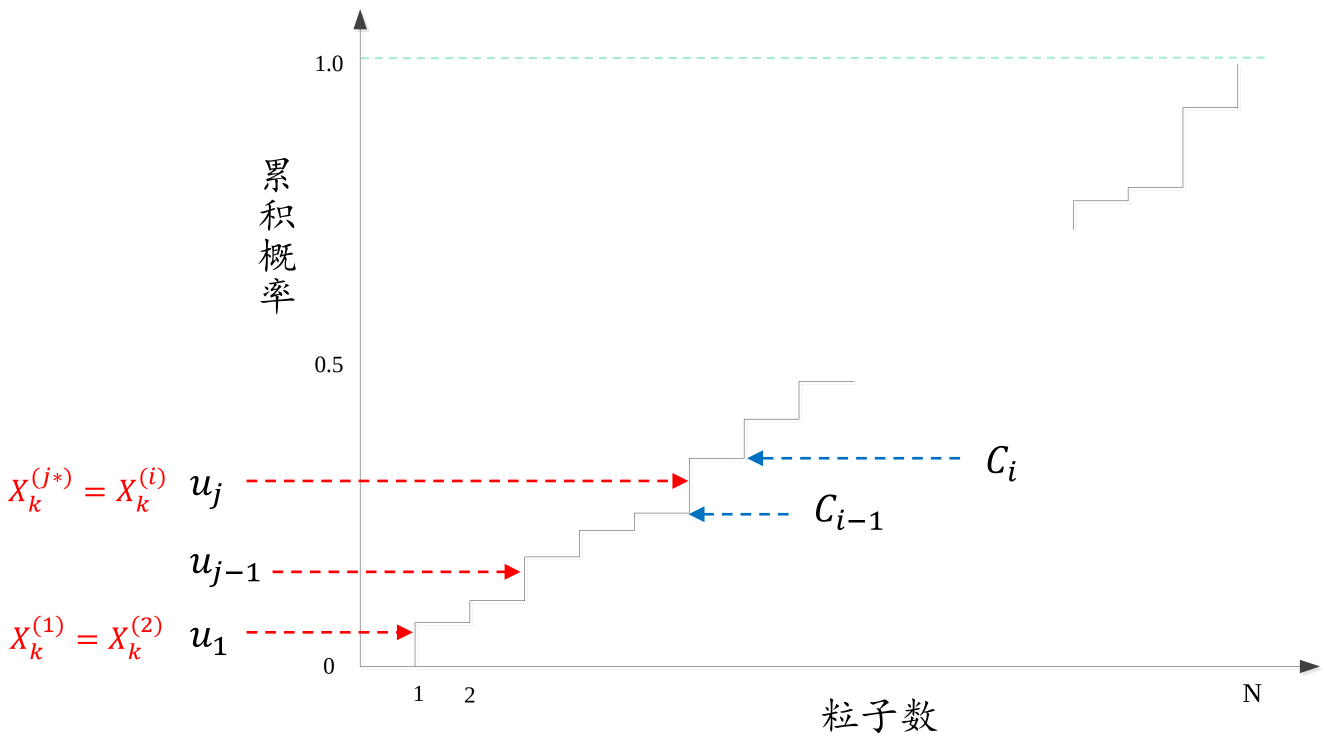
进行重采样

阈值可取为2N/3



2）重采样的实现

$C_1=0$
For i=2:N
    $C_i=C_{i-1}+w_k^{(i)}$
End For
Start at the bottom of the CDF: i=1
Draw a starting point: $u_1\sim[0,1/N]$
For j=1:N
    $u_j = u_1 + (j-1)/N$
    While $u_j > C_i$
        i=i+1
    End While
    $X_k^{(j*)} = X_k^{(i)}$
    $w_k^{(j)} = 1/N$
End For

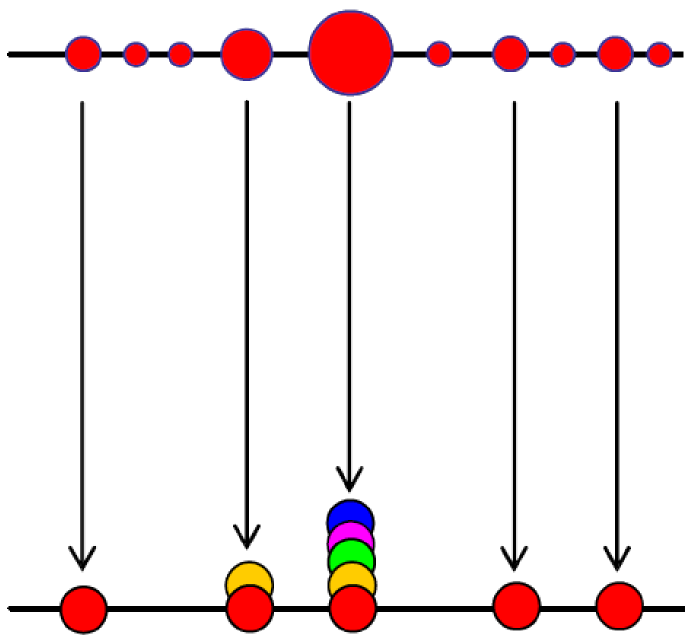


这里给出的仅是重采样方案中的一种

6. 重采样及粗糙化

3）粗糙化 （Roughening）

为避免粒子退化，重采样后对每一个粒子增加随机噪声。



$$\left\{\mathbf{x}_k^{(i)}, \mathrm{w}_k^{(i)}\right\}_{i=1}^{N}$$

$$\left\{\mathbf{x}_k^{(i)}, 1/\mathrm{N}\right\}_{i=1}^{N}$$

$$\mathbf{x}_k^{(i)} = \mathbf{x}_k^{(i)} + \Delta\mathbf{x}_k^{(i)}$$

$$\Delta\mathbf{x}_k^{(i)} = \begin{bmatrix} \Delta\mathbf{x}_k^{(i)}(1) \\ \vdots \\ \Delta\mathbf{x}_k^{(i)}(\mathrm{m}) \\ \vdots \\ \Delta\mathbf{x}_k^{(i)}(\mathrm{n}) \end{bmatrix}$$

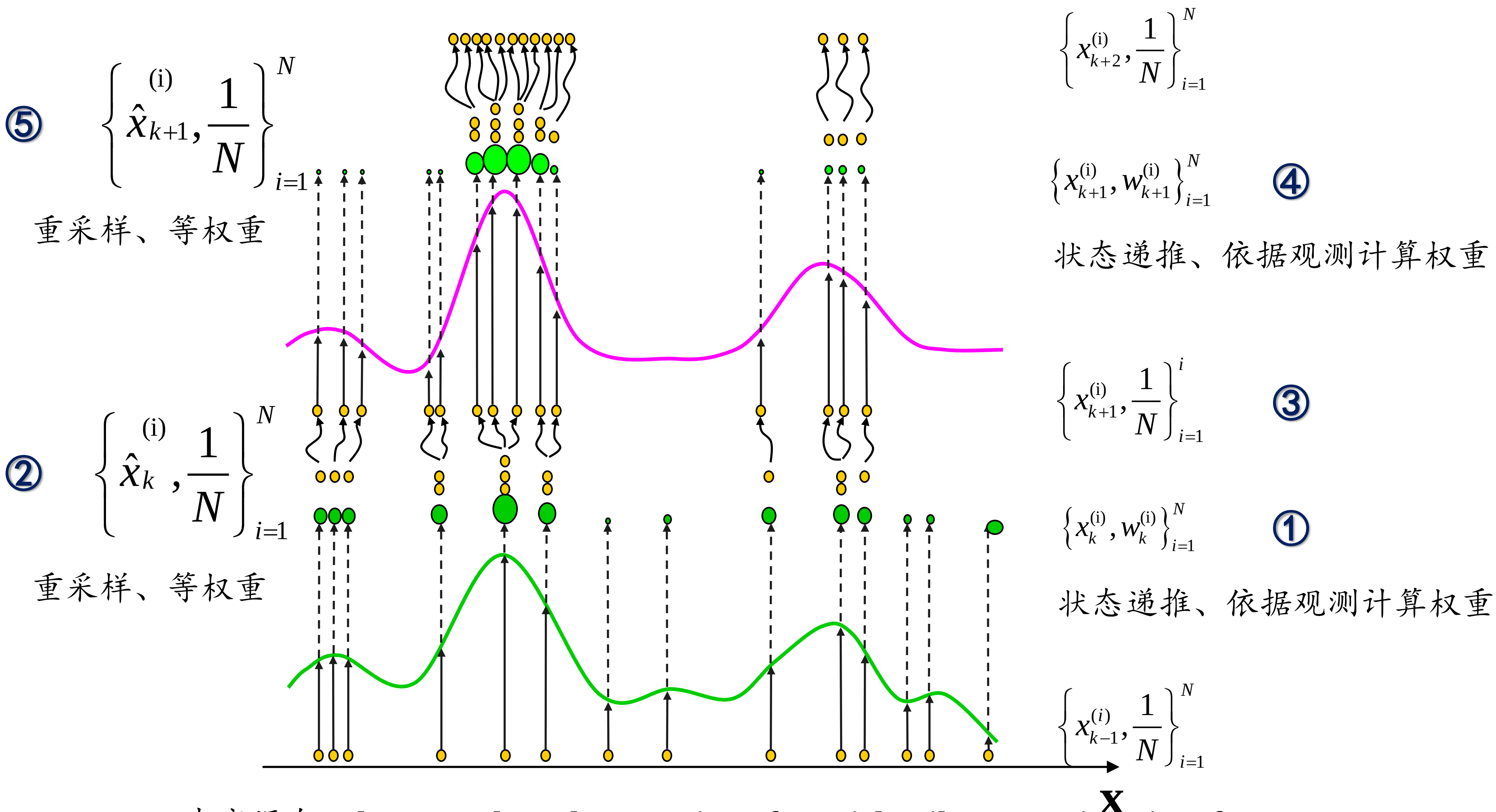
$$\Delta\mathbf{x}_k^{(i)}(\mathrm{m}) \sim \left(0,\ \underbrace{K}_{\text{调节常数 e.g. 0.2}} M(\mathrm{m}) \underbrace{\mathrm{N}}_{\text{样本数}}^{-1/\overbrace{n}^{\text{总维数}}}\right)$$

$$M(m) = \max \underbrace{\left|\mathbf{x}_k^{(i)}(m) - \mathbf{x}_k^{(j)}(m)\right|}_{\text{样本第m维的最大差值}} \quad (m = 1, \cdots, \mathrm{n})$$

6. 重采样及粗糙化

4）重采样实现的过程

$$w_k^{(i)} = p\left(\mathbf{y}_k \mid \mathbf{x}_k^{(i)}\right) w_{k-1}^{(i)}$$



内容源自 **Theory and Implementation of Particle Filters, University of Ottawa**

北京航空航天大学 BEIHANG UNIVERSITY

7. 粒子滤波的基本过程

非重采样方式

预测

$$x_{k|k-1}^{(i)} = f(x_{k-1}^{(i)})$$

$$y_{k|k-1}^{(i)} = g(x_{k|k-1}^{(i)})$$

似然PDF

$$p(y_k \mid x_{k|k-1}^{(i)}), \quad [y_{k|k-1}^{(i)} - y_k] \sim N(0, R)$$

权重计算

$$w_k^{(i)} = p\left(y_k \mid x_k^{(i)}\right) w_{k-1}^{(i)}$$

权重归一

$$\bar{w}_k^{(i)} = \frac{w_k^{(i)}}{\sum_{i=1}^{N} w_k^{(i)}}$$

滤波输出

$$\hat{x}_{k|k} = E(x_k \mid y_k) \approx \sum_{i=1}^{N} \bar{w}_k^{(i)} x_{k|k-1}^{(i)}$$

$$P_{k|k} \approx \frac{1}{N} \sum_{i=1}^{N} \bar{w}_k^{(i)} (x_{k|k-1}^{(i)} - \hat{x}_{k|k})(x_{k|k-1}^{(i)} - \hat{x}_{k|k})^T$$

粒子更新

重采样方式 BF

$$x_{k|k-1}^{(i)} = f(x_{k-1}^{(i)})$$

$$y_{k|k-1}^{(i)} = g(x_{k|k-1}^{(i)})$$

$$p(y_k \mid x_{k|k-1}^{(i)}), \quad [y_{k|k-1}^{(i)} - y_k] \sim N(0, R)$$

$$w_k^{(i)} = p\left(y_k \mid x_k^{(i)}\right) w_{k\text{-}1}^{(i)}$$

$$\bar{w}_k^{(i)} = \frac{w_k^{(i)}}{\sum_{i=1}^{N} w_k^{(i)}}$$

$$\hat{x}_{k|k} = E(x_k \mid y_k) \approx \sum_{i=1}^{N} w_k^{(i)} x_{k|k-1}^{(i)}$$

$$P_{k|k} \approx \frac{1}{N} \sum_{i=1}^{N} w_k^{(i)} (x_{k|k-1}^{(i)} - \hat{x}_{k|k})(x_{k|k-1}^{(i)} - \hat{x}_{k|k})^T$$

$x_k^{(i)}$重采样,粗糙化， $w_k^{(i)} = 1/N$

7. 粒子滤波的基本过程

MODEL

- ***States***:

  $\mathbf{x}_k=[x_k, V_{xk}, y_k, V_{yk}]^T$

- ***Observations***: z_k

- ***Noise***

  $\mathbf{u}_k \sim N(0,\sigma_u^2), v_k \sim N(0,\sigma_v^2)$

- ***State equation***:

  $\mathbf{x}_k=\mathbf{F}\mathbf{x}_{k-1}+\mathbf{G}\mathbf{u}_k$

- ***Observation equation***:

  $z_k=\text{atan}(y_k/x_k)+v_k$

ALGORITHM

- **Particle generation**
  - Generate M random numbers $\mathbf{u}_k^{(m)} \sim N(0,\sigma_u^2)$
  - Particle computation

$$\hat{\mathbf{x}}_k^{(m)} = F\hat{\mathbf{x}}_{k-1}^{(m)} + G\mathbf{u}_k^{(m)}$$

- **Weight computation**

$$w_k^{*(m)} = N\left(z_k - \text{atan}\frac{y_k^{(m)}}{x_k^{(m)}}, \sigma_v^2\right)$$

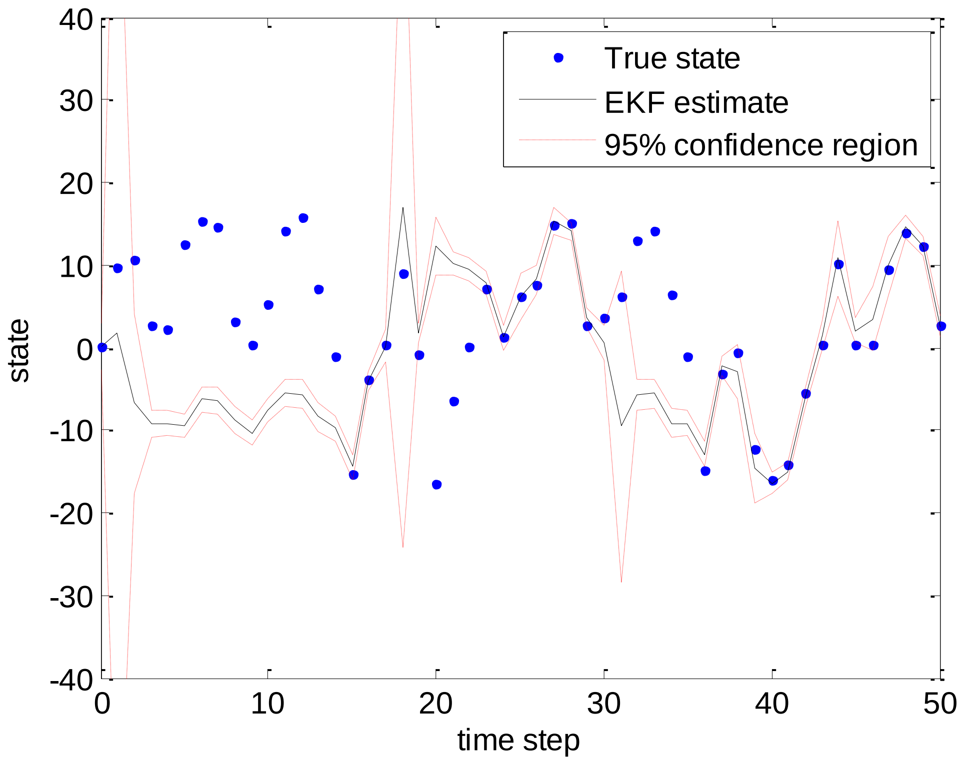
- **Weight normalization**
- **Computation of the estimates**
- **Resampling & Roughening** $\left\{\hat{\mathbf{x}}_k^{(m)}, \frac{1}{M}\right\}_{m=1}^{M} \sim \left\{\mathbf{x}_k^{(m)}, w_k^{(m)}\right\}_{m=1}^{M}$

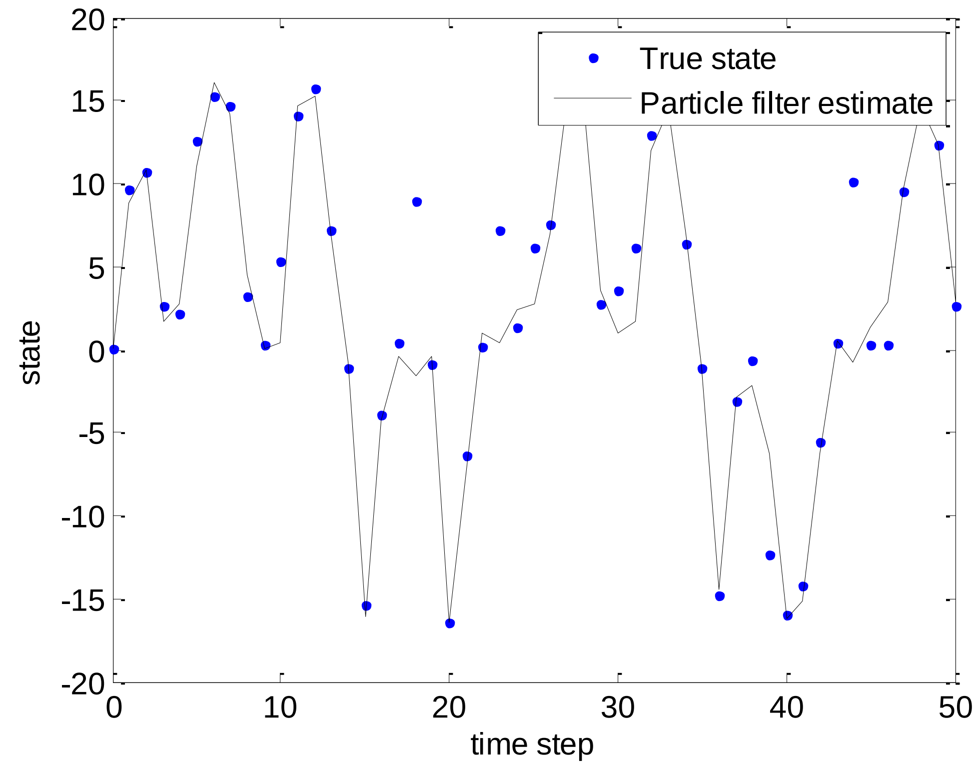
内容源自 **Theory and Implementation of Particle Filters, University of Ottawa**

7. 粒子滤波的基本过程

$$x_k = \frac{1}{2}x_{k-1} + \frac{25x_{k-1}}{1+x_{k-1}^2} + 8\cos\left[1.2(\mathrm{k}-1)\right] + w_k$$

$$y_k = \frac{1}{20}x_{k-1}^2 + v_k$$





插图源自 **Optimal State Estimation**