Nonlinear Control Theory

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Sliding Mode Control

1. **A motivating example**
2. Stabilization
3. Tracking

Motivating example

Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= h(x) + g(x)u,\end{aligned}$$

where h and g are *unknown* nonlinear functions, $h(0) = 0$ and $g(x) \geq g_0 > 0$ for all x.

If we can design u that constraints the motion of the system to the manifold $s = a_1 x_1 + x_2 = 0$, then the motion is governed by $\dot{x}_1 = -a_1 x_1$ which is asymptotically stable with $a_1 > 0$.

- The rate of convergence can be controlled by a_1.
- The manifold is independent of h and g.

Let us look into the dynamics of s:

$$\dot{s} = a_1 \dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x)u. \tag{1}$$

Suppose that h and g satisfy

$$\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \leq \varrho(x), \tag{2}$$

for some *known* function $\varrho(x)$.

With $V = \frac{1}{2}s^2$, we have

$$\dot{V} = s\dot{s} = s[a_1 x_2 + h(x)] + g(x)su \leq g(x)|s|\varrho(x) + g(x)su. \tag{3}$$

Design the control:

$$u = -\beta(x)\mathrm{sgn}(s) = \begin{cases} -\beta(x), & s > 0, \\ 0, & s = 0, \\ \beta(x), & s < 0, \end{cases} \tag{4}$$

where $\beta(x) \geq \varrho(x) + \beta_0$ with $\beta_0 > 0$, such that

$$\dot{V} = s\dot{s} \leq g(x)|s|\varrho(x) - g(x)[\varrho(x) + \beta_0]s\mathrm{sgn}(s) = -g(x)\beta_0|s| \leq -g_0\beta_0|s|. \tag{5}$$

Take $W(s) = \sqrt{2V} = |s|$. We have

$$D^+W = \frac{1}{|s|}\dot{V} \leq -g_0\beta_0, \quad \text{and} \quad W(s(t)) \leq W(s(0)) - g_0\beta_0 t. \tag{6}$$

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In summary,

- **Reaching phase:** trajectories starting off $s = 0$ move towards it and reach it in finite time since $\dot{V} \leq -g_0\beta_0|s|$.
- **Sliding phase:** the motion is confined to the manifold $s = 0$, and governed by the reduced order system $\dot{x}_1 = -a_1x_1$.
- The manifold $s = 0$ is called the sliding manifold, and the control $u = -\beta(x)\mathrm{sgn}(s)$ is called sliding mode control.
- The control is independent of h and g. The only necessary information is the upper bound $\varrho(x)$.

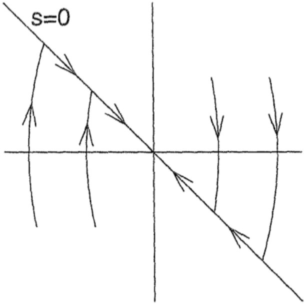

Figure: A typical phase portrait under sliding mode control

Extension:

If, in some domain of interest, h and g satisfy

$$\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \leq k_1, \tag{7}$$

where k_1 is a non-negative *constant*.

The sliding mode control can be simplified to

$$u = -k\,\mathrm{sgn}(s), \quad k > k_1. \tag{8}$$

This form of sliding mode control usually leads to a finite region of attraction.

Is (7) a possible region of attraction? Unfortunately NO!

From $\dot{x}_1 = x_2 = -a_1 x_1 + s$ and $V_1 = \frac{1}{2}x_1^2$, in the region $\{|s| < c\}$, we have

$$\dot{V}_1 = x_1 \dot{x}_1 = -a_1 x_1^2 + x_1 s \le -a_1 x_1^2 + |x_1| c \le 0, \quad \forall\, |x_1| \ge \frac{c}{a_1}. \tag{9}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$|x_1(0)| \le \frac{c}{a_1} \quad \Rightarrow \quad |x_1(t)| \le \frac{c}{a_1},\ \forall\, t \ge 0, \tag{10}$$

and the set

$$\Omega = \left\{ |x_1(t)| \le \frac{c}{a_1},\ |s| \le c \right\} \tag{11}$$

is positively invariant.

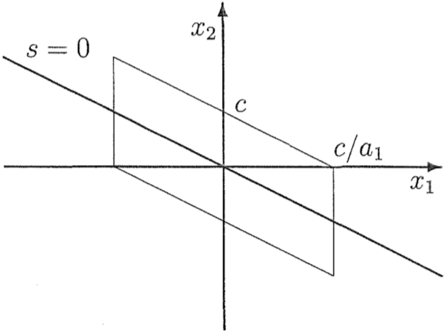

Figure: Estimation of region of attraction

- If c is large enough, any compact set can be contained in Ω.
- If k can be chosen arbitrarily large, the foregoing sliding mode control can achieve semi-global stabilization.

Chattering phenomenon

- In reality, there usually exists delay when the control switches.
- Repetition of this process creates the oscillation, which is called *chattering*.
- Low control accuracy, high heat losses, high wear of mechanical parts.
- Possibly excites un-modeled high-frequency dynamics, possibly leads to instability.

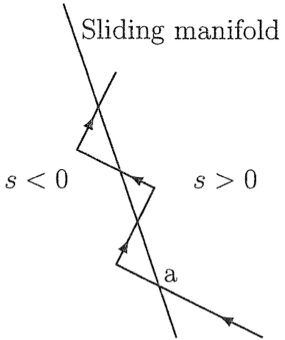

Figure: Chattering due to delay in control switching

Consider the pendulum system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\frac{g_0}{l}\sin(x_1+\delta_1)-\frac{k_0}{m}x_2+\frac{1}{ml^2}u.\end{aligned}$$

where $x_1 = \theta - \delta_1$ and $x_2 = \dot{\theta}$.

To stabilize the pendulum at $\delta_1 = \frac{\pi}{2}$, the sliding mode control is designed by

$$u = -k\,\mathrm{sgn}(s) = -k\,\mathrm{sgn}(a_1x_1 + x_2). \tag{12}$$

If the sliding mode control is implemented ideally, then

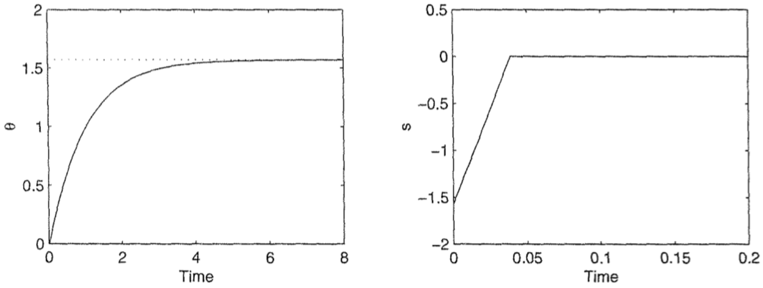

Figure: The pendulum can be stabilized by ideal SMC

However, If the sliding mode control is delayed by un-modeled actuator dynamics $\frac{1}{(0.01s+1)^2}$, then

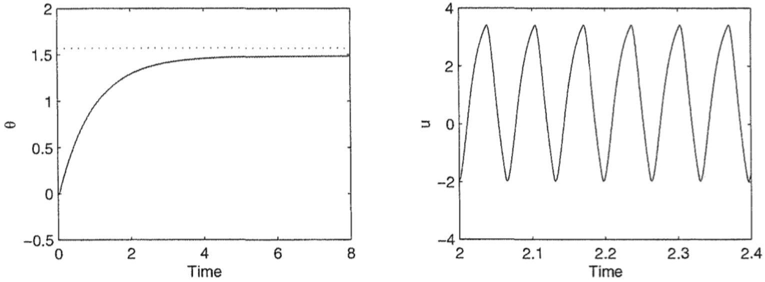

Figure: Chattering appears in case of delayed switching

Two possible ways to reduce or eliminate chattering:

1. Divide the control into continuous and switching components to reduce the amplitude of the switching one.

$$u = -\frac{a_1 x_2 + \hat{h}(x)}{\hat{g}(x)} + v, \quad \text{with } v = -\beta(x)\mathrm{sgn}(s), \tag{13}$$

where $\beta(x) = \varrho(x) + \beta_0$, $\left|\frac{\delta(x)}{g(x)}\right| \le \varrho(x)$, $\delta(x) = a_1\left[1 - \frac{g(x)}{\hat{g}(x)}\right] x_2 + h(x) - \frac{g(x)}{\hat{g}(x)}\hat{h}(x)$.

2. Replace the signum function by a high-slope saturation function.

$$u = -\beta(x)\mathrm{sat}(\frac{s}{\epsilon}). \tag{14}$$

If steady state error exists, it can be eliminated by integral control.

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Problem:

Design a state feedback control law to stabilize the origin of the following system

$$\dot{x} = f(x) + B(x)[G(x)E(x)u + \delta(t, x, u)], \tag{15}$$

where

- $x \in R^n$ and $u \in R^p$ are states and the control input;
- f, B, G and E are sufficiently smooth functions in a domain $D \subset R^n$ that contains the origin, and $f(0) = 0$;
- the function δ is piecewise continuous in t and sufficiently smooth in (x, u) for $(t, x, u) \in [0, \infty) \times D \times R^P$;
- f, B, and E are known, while G and δ could be uncertain;
- $E(x)$ is a nonsingular matrix; $B(x)$ is column full-rank; and $G(x)$ is a diagonal matrix whose diagonal elements $g_i(x) \geq g_0 > 0, \ \forall \ x \in D$.

Solution by sliding model control:

- Let $T: D \to R^n$ be a diffeomorphism such that $\frac{\partial T}{\partial x}B(x) = [0, I_{p\times p}]^T$. Then,

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x), \quad \Rightarrow \quad \begin{cases} \dot{\eta} = f_a(\eta, \xi), \\ \dot{\xi} = f_b(\eta, \xi) + G(x)E(x)u + \delta(t, x, u), \end{cases} \tag{16}$$

  where $\eta \in R^{n-p}$, $\xi \in R^p$.

- Assume that we could find $\xi = \phi(\eta)$, such that $\dot{\eta} = f_a(\eta, \phi(\eta))$ can be stabilized.
- The sliding manifold can be designed by $s = \xi - \phi(\eta)$, such that

$$\dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(x)E(x)u + \delta(t, x, u). \tag{17}$$

- The control u can be designed by

$$u = E^{-1}(x)\left\{-L(x)\left[f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta}f_a(\eta,\xi)\right] + v\right\}, \tag{18}$$

where $L(x) = \hat{G}^{-1}(x)$.

- Substitute (18) into the dynamics of the sliding manifold yields

$$\dot{s}_i = g_i(x)v_i + \Delta_i(t,x,v), \quad 1 \le i \le p, \tag{19}$$

where Δ_i is the ith component of

$$\begin{aligned}\Delta(t,x,u) =& \delta\left(t, x, -E^{-1}(x)L(x)\left(f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta}f_a(\eta,\xi)\right) + E^{-1}(x)v\right) \\ &+ [I - G(x)L(x)]\left[f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta}f_a(\eta,\xi)\right],\end{aligned}$$

and g_i is the ith diagonal element of G.

- It is assumed that the ratio $\frac{\Delta_i}{g_i}$ satisfies

$$\left|\frac{\Delta_i(t,x,v)}{g_i(x)}\right| \le \varrho(x) + \kappa_0\|v\|_\infty, \quad \forall\,(t,x,v) \in [0,\infty)\times D\times R^p, \tag{20}$$

where $\varrho(x) \ge 0$ and $\kappa_0 \in [0,1)$ are known.

- Select $V_i = \frac{1}{2}s_i^2$ as a Lyapunov function candidate.

$$\dot{V}_i = s_i\dot{s}_i = s_i g_i(x)v_i + s_i\Delta_i(t,x,v) \le g_i(x)\left(s_i v_i + |s_i|\left(\varrho(x)+\kappa_0\|v\|_\infty\right)\right). \tag{21}$$

- The control v_i can be designed by

$$v_i = -\beta(x)\mathrm{sgn}(s_i), \;\; 1\le i\le p, \quad \beta(x) \ge \frac{\varrho(x)}{1-\kappa_0} + \beta_0, \;\; \forall\, x\in D, \tag{22}$$

where $\beta_0 \ge 0$.

- It then follows that

$$\begin{aligned}\dot{V}_i \leq & g_i(x)\left[-\beta(x)+\varrho(x)+\kappa_0\beta(x)\right]|s_i| \\ = & g_i(x)\left[-(1-\kappa_0)\beta(x)+\varrho(x)\right]|s_i| \\ \leq & g_i(x)\left[-\varrho(x)-(1-\kappa_0)\beta_0+\varrho(x)\right]|s_i| \\ \leq & -g_0\beta_0(1-\kappa_0)|s_i|.\end{aligned}$$

- The negativeness of $\dot{V}_i$ ensures that all trajectories starting off the manifold $s=0$ reach it in finite time, and those on the manifold cannot leave it.

The procedures to design a sliding mode controller for stabilizing:

1. Design the sliding manifold $\xi = \phi(\eta)$ to stabilize the reduced order system.
2. Take the control as $u = E^{-1}(x)\left\{-L(x)\left[f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta} f_a(\eta,\xi)\right] + v\right\}$, or $u = E^{-1}v$.
3. Estimate $\varrho(x)$ and κ_0, where the uncertain remaining term Δ depends on the choice made in the previous step.
4. Choose $\beta(x) \geq \frac{\varrho(x)}{1-\kappa_0} + \beta_0$ and $v_i = -\beta(x)\mathrm{sgn}(s_i)$.

Remark

The key feature of SMC is its robustness to matched uncertainties. Provided that β is large enough to handle the uncertainties, the reaching phase can be guaranteed. The sliding phase is independent of the matched uncertainties.

To eliminate chattering:

$$v_i = \beta(x)\mathrm{sat}\left(\frac{s_i}{\epsilon}\right), \quad 1 \le i \le p. \tag{23}$$

The derivative of $V_i = \frac{1}{2}s_i^2$ in the region of $|s_i| \ge \epsilon$ can be calculated by

$$\begin{aligned}
\dot{V}_i \le & g_i(x)\left[-\beta(x)s_i\mathrm{sat}\left(\frac{s_i}{\epsilon}\right) + \varrho(x)|s_i| + \kappa_0\beta(x)|s_i|\right] \\
\le & g_i(x)\left[-(1-\kappa_0)\beta(x) + \varrho(x)\right]|s_i| \\
\le & g_0\beta_0(1-\kappa_0)|s_i|,
\end{aligned}$$

indicating that whenever $|s_i(0)| \ge \epsilon$, $|s_i(t)|$ will decrease until it reaches the boundary layer $\{|s_i| \le \epsilon\}$ in finite time and remains inside thereafter.

Suppose that, with the sliding manifold $\xi = \phi(\eta)$, there exists a (continuously differentiable) Lyapunov function $V(\eta)$, such that

$$\alpha_1(\|\eta\|) \le V(\eta) \le \alpha_2(\|\eta\|), \tag{24}$$

$$\frac{\partial V}{\partial \eta} f_a(\eta, \phi(\eta) + s) \le -\alpha_3(\eta), \quad \forall \; \|\eta\| \ge \gamma \|s\|, \tag{25}$$

for all $(\eta, \xi) \in T(D)$.

$$|s_i| \le c,\ 1 \le i \le p \;\Rightarrow\; \|s\| \le k_1 c \;\Rightarrow\; \dot{V} \le -\alpha_3(\eta),\ \|\eta\| \ge \gamma(k_1 c).$$

A class-$\mathcal{K}$ function $\alpha(r) = \alpha_2(\gamma(k_1 r))$ can be defined, such that

$$\begin{aligned} V(\eta) \ge \alpha(c) \;&\Rightarrow\; V(\eta) \ge \alpha_2(\gamma(k_1 c)) \;\Rightarrow\; \alpha_2(\|\eta\|) \ge \alpha_2(\gamma(k_1 c)) \\ &\Rightarrow\; \|\eta\| \ge \gamma(k_1 c) \;\Rightarrow\; \dot{V} \le -\alpha_3(\eta) \le -\alpha_3(\gamma(k_1 c)). \end{aligned}$$

It follows that $\{V(\eta) \leq c_0\}$ with $c_0 \geq \alpha(c)$ is positively invariant.

$$\Omega = \{V(\eta) \leq c_0\} \times \{|s_i| \leq c,\ 1 \leq i \leq p\} \tag{26}$$

The set Ω can be served as our "region of attraction".

After some finite time, we will have $|s_i(t)| \leq \epsilon$, and the trajectory will enter the positively invariant set

$$\Omega_\epsilon = \{V(\eta) \leq \alpha(\epsilon)\} \times \{|s_i| \leq \epsilon,\ 1 \leq i \leq p\}, \tag{27}$$

which can be made arbitrarily small by selecting ϵ small enough.

Theorem (14.1)

Consider the system (16). *Suppose that there exist* $\phi(\eta)$, $V(\eta)$, $\varrho(x)$, *and* κ_0 *satisfying* (20), (24), *and* (25). *Let u and v be given by* (18) *and* (22), *respectively. Suppose* ϵ, $c > \epsilon$ *and* $c_0 \leq \alpha(c)$ *are chosen such that the set* Ω *is contained in* $T(D)$. *Then, for all* $(\eta(0),\ \xi(0)) \in \Omega$, *the trajectory* $(\eta(t),\ \xi(t))$ *is bounded for all* $t \geq 0$, *and reaches the positively invariant set* Ω_ϵ, *defined in* (27), *in finite time. Moreover, if the assumptions hold globally and* $V(\eta)$ *is radially unbounded, the foregoing conclusion hold for any initial state.*

- For non-vanishing uncertainties, ultimate boundedness is the best to be expected.
- If the the uncertainties vanishes at the origin, then asymptotic stability is possible.

Theorem (14.2)

Suppose all assumptions of Theorem 14.1 are satisfied with $\varrho(0) = 0$ and $\kappa_0 = 0$. Suppose further that the origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is exponentially stable. Then, there exists $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$, the origin of the closed-loop system is exponentially stable, and Ω is a subset of its region of attraction. Moreover, if the assumptions hold globally, the origin will be globally uniformly asymptotically stable.

How about unmatched uncertainties?

Suppose the plant is given by

$$\dot{x} = f(x) + B(x)[G(x)E(x)u + \delta(t,x,u)] + \delta_1(x), \tag{28}$$

where $\delta_1(x)$ is the unmatched uncertainty.

It follows from the state transformation that

$$\dot{\eta} = f_a(\eta,\xi) + \delta_a(\eta,\xi), \tag{29}$$

$$\dot{\xi} = f_b(\eta,\xi) + G(x)E(x)u + \delta(t,x,u) + \delta_b(\eta,\xi), \tag{30}$$

where δ_a and δ_b are unmatched, and $\begin{bmatrix} \delta_a \\ \delta_b \end{bmatrix} = \frac{\partial T}{\partial x}\delta_1$.

- The sliding mode control **guarantees robustness for any matched uncertainty**, provided an upper bound is known and the needed control effort can be provided.
- There is **no such guarantee for unmatched uncertainties.** We may have to restrict its size to robustly stabilize the system on the sliding manifold.

Example

Consider the second-order system

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \tag{31}$$

$$\dot{x}_2 = \theta_2 x_2^2 + x_1 + u, \tag{32}$$

where θ_1 and θ_2 are unknown parameters satisfying $|\theta_1| < a$ and $|\theta_2| < b$ with known bounds a and b.

- Uncertainty due to θ_2 is matched, while uncertainty due to θ_1 is un-matched.
- Start by designing $x_2 = -kx_1, \ (k > a)$ to stabilize (31), since

$$x_1 \dot{x}_1 = -kx_1^2 + \theta_1 x_1^2 \sin(-kx_1) \leq -(k-a)x_1^2. \tag{33}$$

- The sliding manifold is $s = kx_1 + x_2 = 0$, and

$$\dot{s} = \theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_2). \tag{34}$$

- Design

$$u = -x_1 - kx_2 + v, \tag{35}$$

  to cancel the known terms on the right-hand side, such that $\dot{s} = v + \Delta(x)$, where $\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$, and therefore $|\Delta(x)| \le ak|x_1| + bx_2^2$.

- Take $\beta(x) = ak|x_1| + bx_2^2 + \beta_0$, and

$$v = -\beta(x)\mathrm{sgn}(s) \;\Rightarrow\; u = -x_1 - kx_2 - \beta(x)\mathrm{sgn}(s).$$

  This controller, or its continuous approximation with sufficiently small ϵ, globally stabilizes the origin.

1. A motivating example
2. Stabilization
3. **Tracking**

Consider the SISO system:

$$\begin{aligned}\dot{x} =& f(x) + \delta_1(x) + g(x)[u + \delta(t, x, u)],\\ y =& h(x),\end{aligned}$$

where x, u and y are the state, control input and controlled output, respectively.

Assume that

- f, g, h and δ_1 are sufficiently smooth in a domain $D \subset R^n$;
- δ is piecewise continuous in t and sufficiently smooth in (x, u) for $(t, x, u) \in [0, \infty) \times D \times R$;
- f and h are known, while g, δ and δ_1 could be uncertain;
- the nominal system has relative degree ρ in D, that is

$$L_g h(x) = \cdots = L_g L_f^{\rho-2} h(x) = 0, \quad L_g L_f^{\rho-1} h(x) \geq a > 0, \quad \forall\ x \in D.$$

The goal is to design a state feedback control such that **the output y asymptotically tracks a reference signal** $r(t)$, where

- $r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \geq 0$, and the ρth derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t;
- the signals $r, \cdots, r^{(\rho)}$ are available on line.

With input-output linearization (Section 13.2 in the textbook), the system can be transformed into the normal form by

$$\begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} = \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho} \\ ---- \\ h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \end{bmatrix} = T(x), \tag{36}$$

where ϕ_1 to $\phi_{n-\rho}$ satisfy $\frac{\partial \phi_i}{\partial x} g(x) = 0$, for $1 \leq i \leq n - \rho$, $\forall\ x \in D$.

Assume that the change of variables preserves the normal form structure; or equivalently, δ_1 belongs to the null space of $\left[\frac{\partial L_f^i h}{\partial x}\right]$:

$$\left[\frac{\partial h}{\partial x}\right]\delta_1(x) = 0, \ \left[\frac{\partial (L_f h)}{\partial x}\right]\delta_1(x) = 0, \cdots, \left[\frac{\partial (L_f^{\rho-2} h)}{\partial x}\right]\delta_1(x) = 0,$$

such that

$$\dot{\xi}_1 = \frac{\partial h}{\partial x}[f + \delta_1 + g(u + \delta)] = L_f h(x) = \xi_2, \tag{37}$$

$$\dot{\xi}_2 = \frac{\partial L_f h}{\partial x}[f + \delta_1 + g(u + \delta)] = L_f^2 h(x) = \xi_3, \tag{38}$$

$$\vdots \tag{39}$$

$$\dot{\xi}_{\rho-1} = \cdots = \xi_\rho. \tag{40}$$

$$
\begin{aligned}
\dot{\eta} =& f_0(\eta, \xi), \\
\dot{\xi}_1 =& \xi_2, \\
\dot{\xi}_2 =& \xi_3, \\
& \vdots \\
\dot{\xi}_{\rho-1} =& \xi_\rho, \\
\dot{\xi}_\rho =& L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t, x, u)], \\
\\
y =& \xi_1 .
\end{aligned}
$$

Define

$$\mathcal{R} = \begin{bmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{bmatrix}, \quad e = \begin{bmatrix} \xi_1 - r \\ \vdots \\ \xi_\rho - r^{(\rho-1)} \end{bmatrix} = \xi - \mathcal{R}. \tag{41}$$

It follows that

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi), \\
\dot{e}_1 &= e_2, \\
&\vdots \\
\dot{e}_{\rho-1} &= e_\rho, \\
\dot{e}_\rho &= L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t, x, u)] - r^{(\rho)}.
\end{aligned}$$

For simplification, it is assumed that $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable.

The reduced system

$$\begin{aligned}\dot{e}_1 &= e_2,\\ &\vdots\\ \dot{e}_{\rho-1} &= e_\rho,\end{aligned}$$

can be stabilized by $e_\rho = -(k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1})$.

The sliding manifold can be designed by

$$s = (k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1}) + e_\rho = 0. \tag{42}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$\dot{s} = k_1 e_2 + \cdots + k_{\rho-1} e_\rho + L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t,x,u)] - r^{(\rho)}.$$

The sliding mode control can be designed by

$$u = -\frac{1}{L_{\hat{g}} L_f^{\rho-1} h(x)} \left[k_1 e_2 + \cdots + k_{\rho-1} e_\rho + L_f^\rho h(x) - r^{(\rho)} \right] + v, \tag{43}$$

to cancel the known terms, where $\hat{g}$ is a nominal model of $g(x)$.

$$\dot{s} = k_1 e_2 + \cdots + k_{\rho-1} e_\rho + L_f^\rho h(x) + L_{\delta_1} L_f^{\rho-1} h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t, x, u)] - r^{(\rho)}.$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$\dot{s} = L_g L_f^{\rho-1} h(x) v + \Delta(t, x, v). \tag{44}$$

Suppose that

$$\left|\frac{\Delta(t,x,v)}{L_g L_f^{\rho-1} h(x)}\right| \leq \varrho(x) + \kappa_0 |v|, \quad 0 \leq \kappa_0 < 1, \tag{45}$$

for all $(t, x, v) \in [0, \infty) \times D \times R$, where $\varrho(x)$ and κ_0 are known.

The dis-continuous term can be designed by

$$v = -\beta(x)\mathrm{sgn}(s), \tag{46}$$

where $\beta(x) \geq \frac{\varrho(x)}{1-\kappa_0} + \beta_0$ with $\beta_0 > 0$, and its continuous approximation is obtained by replacing $\mathrm{sgn}(s)$ by $\mathrm{sat}\left(\frac{s}{\epsilon}\right)$.

- With the "continuous" sliding mode controller, there exists a finite time T_1 (possibly dependent on ϵ and initial states), and a positive constant k (independent of ϵ and initial states), such that $|y(t) - r(t)| \leq k\epsilon$, $\forall\ t \geq T_1$. (Exercise 14.3)