第六章 非线性滤波

6.1 非线性系统

通用离散 非线性系统 可以表示为

$$\begin{cases} X(k+1)=\varphi[X(k),W(k),k] \\ Z(k+1)=h[X(k+1),k+1]+V(k+1) \end{cases}$$

在本章有关传统kalman非线性滤波方法的介绍中，仅以下简化的情况

$$\begin{cases} X(k+1)=\varphi[X(k),k]+\Gamma[X(k),k]W(k) \\ Z(k+1)=h[X(k+1),k+1]+V(k+1) \end{cases}$$

6.2 围绕标称轨道线性化滤波

Nominal orbit filtering

1）系统模型

$$\begin{cases} X(k+1)=\varphi[X(k),k]+\Gamma[X(k),k]W(k) \\ Z(k+1)=h[X(k+1),k+1]+V(k+1) \end{cases}$$

$$\begin{cases} E[W(k)]=E[V(k)]=0 \\ E[W(k)W^T(j)]=Q_k\delta_{kj} \\ E[V(k)V^T(j)]=R_k\delta_{kj} \\ E[W(k)V^T(j)]=0 \end{cases}$$

2) 标称轨道的定义

标称轨道是不考虑系统噪声情况下，系统状态方程的解

Nominal orbit	$X^*(k+1)=\varphi[X^*(k),k]$ $X_0^*=E[X_0]=m_0$
State error	$\delta X(k)=X(k)-X^*(k)$

6.2 围绕标称轨道线性化滤波

3）状态方程线性化

$$X(k+1)=\varphi[X(k),k]+\Gamma[X(k),k]W(k)$$

Expand *φ[X(k),k]* in Taylor series about *X*(t)*, dropping all but the first term of the power series for *φ[X(k),k]*

$$X(k+1)\approx\underline{\varphi[X^*(k),k]}+\frac{\partial\varphi}{\partial X_k^*}[X(k)-X^*(k)]+\Gamma[X(k),k]W(k)$$

$$\delta X(k+1)=\frac{\partial\varphi}{\partial X^*(k)}\delta X(k)+\Gamma[X^*(k),k]W(k)$$

Jacobian matrix

雅可比矩阵

$$\frac{\partial\varphi}{\partial X^*(k)}=\left.\frac{\partial\varphi}{\partial X(k)}\right|_{X(k)=X^*(k)}=\left.\begin{bmatrix}\frac{\partial\varphi^{(1)}}{\partial X^{(1)}(k)} & \cdots & \frac{\partial\varphi^{(1)}}{\partial X^{(n)}(k)}\\ \vdots & \cdots & \vdots\\ \frac{\partial\varphi^{(n)}}{\partial X^{(1)}(k)} & \cdots & \frac{\partial\varphi^{(n)}}{\partial X^{(n)}(k)}\end{bmatrix}\right|_{X(k)=X^*(k)}$$

6.2 围绕标称轨道线性化滤波

4）观测方程线性化

$$Z(k+1)=h[X(k+1),k+1]+V(k+1)$$

nominal orbit

$$Z^*(k+1)=h[X^*(k+1),k+1]$$

Expanded about nominal orbit

$$Z(k+1)\approx h[X^*(k+1),k+1]+\frac{\partial h}{\partial X^*(k+1)}[X(k+1)-X^*(k+1)]+V(k+1)$$

$$\delta Z(k+1)=\frac{\partial h}{\partial X^*(k+1)}\delta X(k+1)+V(k+1)$$

Jacobian matrix

雅可比矩阵

$$\frac{\partial h}{\partial X^*(k+1)}=\left.\begin{bmatrix} \frac{\partial h^{(1)}}{\partial X^{(1)}(k+1)} & \cdots & \frac{\partial h^{(1)}}{\partial X^{(n)}(k+1)} \\ \vdots & \cdots & \vdots \\ \frac{\partial h^{(n)}}{\partial X^{(1)}(k+1)} & \cdots & \frac{\partial h^{(n)}}{\partial X^{(n)}(k+1)} \end{bmatrix}\right|_{X(k+1)=X^*(k+1)}$$

6.2 围绕标称轨道线性化滤波

5）围绕标称轨道线性化的滤波方程组

$$\delta\hat{X}(k+1|k+1)=\delta\hat{X}(k+1|k)+K(k+1)\left[\delta Z(k+1)-\frac{\partial h}{\partial X^*(k+1)}\delta\hat{X}(k+1|k)\right]$$

$$\delta\hat{X}(k+1|k)=\frac{\partial\varphi}{\partial X^*(k)}\delta\hat{X}(k|k)$$

$$K(k+1)=P(k+1|k)\left[\frac{\partial h}{\partial X^*(k+1)}\right]^T\left\{\frac{\partial h}{\partial X^*(k+1)}P(k+1|k)\left[\frac{\partial h}{\partial X^*(k+1)}\right]^T+R_{k+1}\right\}^{-1}$$

$$P(k+1|k+1)=\left[I-K(k+1)\frac{\partial h}{\partial X^*(k+1)}\right]P(k+1|k)$$

计算特性

$$P(k+1|k)=\frac{\partial\varphi}{\partial X^*(k+1)}P(k|k)\left[\frac{\partial\varphi}{\partial X^*(k+1)}\right]^T+\Gamma[X^*(k),k)]Q_k\Gamma^T[X^*(k),k)]$$

6.3 扩展Kalman滤波(EKF)

Extended Kalman Filtering

1）与围绕标称轨道的线性化滤波的区别

围绕标称轨道的线性化滤波是围绕标称轨道进行泰勒展开，*EKF*是围绕状态滤波估计值展开。

2) 状态方程的线性化

$$X(k+1)=\varphi[X(k),k]+\Gamma[X(k),k]W(k)$$

Expanded about $\hat{X}(k|k)$

$$X(k+1)\approx\varphi[\hat{X}(k|k),k]+\left.\frac{\partial\varphi}{\partial X}\right|_{X(k)=\hat{X}(k|k)}[X(k)-\hat{X}(k|k)]+\Gamma[\hat{X}(k|k),k]W(k)$$

assuming $\varphi[\hat{X}(k|k),k]-\left.\frac{\partial\varphi}{\partial X}\right|_{X(k)=\hat{X}(k|k)}\hat{X}(k|k)=f(k)$

$$X(k+1)=\underline{\Phi(k+1,k)}X(k)+\Gamma[\hat{X}(k|k),k]W(k)+\underline{f(k)}$$

作为输入项处理

6.3 扩展Kalman滤波(EKF)

3）观测方程线性化

$$Z(k+1)=h[X(k+1),k+1]+V(k+1)$$

Expanded about $\hat{X}(k|k)$

$$Z(k+1)\approx h[\hat{X}(k+1|k),k+1]+\underline{\left.\frac{\partial h}{\partial X}\right|_{\hat{X}(k+1|k)}}[X(k+1)-\hat{X}(k+1|k)]+V(k+1)$$

Ordering $-\left.\frac{\partial h}{\partial X}\right|_{\hat{X}(k+1|k)}\hat{X}(k+1|k)+h[\hat{X}(k+1|k),k+1]=Y(k+1)$

$$Z(k+1)=\underline{H(k+1)}X(k+1)+Y(k+1)+V(k+1)$$

4) 滤波方程组

显然线性化处理后的系统是标准的带有输入项系统，可以采用第二章中的结论进行直接处理。

6.3 扩展Kalman滤波(EKF)

$$\hat{X}(k+1|k+1) = \hat{X}(k+1|k) + K(k+1)\left\{Z(k+1) - h[\hat{X}(k+1|k), k+1]\right\}$$

$$\hat{X}(k+1,k) = \Phi(k+1,k)\hat{X}(k,k) + f(k) = \varphi[\hat{X}(k,k),k]$$

$$\Phi(k+1,k) = \left.\frac{\partial\varphi}{\partial X}\right|_{X(k)=\hat{X}(k|k)} \qquad f(k) = \varphi[\hat{X}(k|k),k] - \left.\frac{\partial\varphi}{\partial X}\right|_{X(k)=\hat{X}(k|k)} \hat{X}(k|k)$$

$$K(k+1) = P(k+1|k)H^T(k+1)\left[H(k+1)P(k+1|k)\cdot H^T(k+1) + R_{k+1}\right]^{-1}$$

$$P(k+1|k) = \Phi(k+1,k)P(k|k)\Phi^T(k+1,k) + \Gamma[\hat{X}(k|k),k]Q_k\Gamma^T[\hat{X}(k|k),k]$$

$$P(k+1|k+1) = [I - K(k+1)H(k+1)]P(k+1|k)$$

$$\hat{X}_0 = E[X_0] = m_0, P_0 = VarX_0$$

$$H(k+1) = \left.\frac{\partial h}{\partial X}\right|_{\hat{X}(k+1|k)}$$

6.3 扩展Kalman滤波(EKF)

5）连续系统的 EKF

$$\begin{cases} \dot{X}(t) = f[X(t)] + W(t) \\ Z(t) = h[X(t)] + V(t) \end{cases}$$

离散化

$$X(t+\Delta t) = X(t) + \dot{X}(t)\Delta t + \frac{1}{2!}\ddot{X}(t)(\Delta t)^2 + \cdots = X(t) + f(X)\Delta t + \frac{\partial f}{\partial X} f(X)\frac{(\Delta t)^2}{2!} + \cdots$$

$$\ddot{X}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \vdots \\ \ddot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial f_1}{\partial X}\right)^T \frac{dX}{dt} \\ \left(\frac{\partial f_2}{\partial X}\right)^T \frac{dX}{dt} \\ \vdots \\ \left(\frac{\partial f_n}{\partial X}\right)^T \frac{dX}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \frac{\partial f}{\partial X} f(X)$$

$$\left.\frac{\partial f}{\partial X}\right|_{X=X(k)} = A[X(k)]$$

6.3 扩展Kalman滤波(EKF)

$$\begin{cases} X(k+1) = X(k) + f[X(k)]\Delta t + A[X(k)]f[X(k)]\dfrac{(\Delta t)^2}{2} + W(k) \\ Z(k) = h[X(k)] + V(k) \end{cases}$$

$$f[X(k)]\Delta t + A[X(k)]f[X(k)]\dfrac{(\Delta t)^2}{2} \dashrightarrow \mathrm{U(k)}$$

利用离散滤波方程

$$\hat{X}(k+1|k+1) = \hat{X}(k+1|k) + K(k+1)\{Z(k+1) - h[\hat{X}(k+1|k)]\}$$

$$\hat{X}(k+1|k) = \hat{X}(k|k) + f[\hat{X}(k|k)]\Delta t + A[\hat{X}(k|k)]f[\hat{X}(k|k)]\frac{(\Delta t)^2}{2}$$

$$K(k+1) = P(k+1|k)H^T(k+1)\left[H(k+1)P(k+1|k)H^T(k+1) + R_{k+1}\right]^{-1}$$

$$H(k) = \left.\frac{\partial h}{\partial X}\right|_{X=\hat{X}(k|k-1)}$$

$$P(k+1|k) = \Phi(k)P(k|k)\Phi^T(k) + Q_k$$

$$\Phi(k) = I$$

$$P(k+1|k+1) = [I - K(k+1)H(k+1)]P(k+1|k)$$

$$\left.\frac{\partial f}{\partial X}\right|_{X=X(k)} = A[X(k)]$$

6.3 扩展Kalman滤波(EKF)

$$X(t+\Delta t)=X(t)+\dot{X}(t)\Delta t+\frac{1}{2!}\ddot{X}(t)(\Delta t)^2+\frac{1}{3!}\dddot{X}(t)(\Delta t)^3\cdots$$

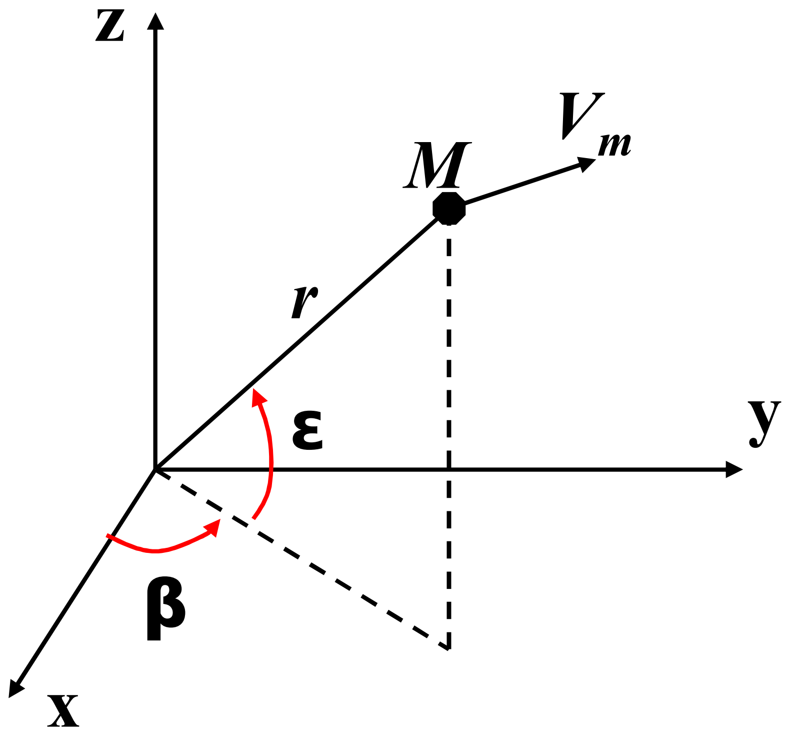
EKF在实现上，可以按照泰勒展开精度进行分类，常用有一阶、二阶、三阶不同形式。本课程给出的是二阶方法。

是不是采用高阶导就一定能够提高估计精度呢?

答案是否定的，采用高阶导的条件是噪声幅度小，否则求导会引入更大估计误差。

EKF Example

雷达跟踪一匀速飞行目标，利用Kalman滤波提高目标运动参数估计精度。



$$x = r\cos\varepsilon\sin\beta$$

$$y = r\cos\varepsilon\cos\beta$$

$$z = r\sin\varepsilon$$

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \ddot{z} = 0$$

$$\ddot{r} = r\dot{\varepsilon}^2 + r\cos^2\varepsilon \bullet \dot{\beta}^2$$

$$\ddot{\beta} = \frac{1}{r\cos\varepsilon}\left(2r\dot{\beta}\dot{\varepsilon}\sin\varepsilon - 2\dot{r}\dot{\beta}\cos\varepsilon\right)$$

$$\ddot{\varepsilon} = -\left(\sin\varepsilon\cos\varepsilon \bullet \dot{\beta}^2 + \frac{2}{r}\dot{r}\dot{\varepsilon}\right)$$

EKF Example

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} r \\ \beta \\ \varepsilon \\ \dot{r} \\ \dot{\beta} \\ \dot{\varepsilon} \end{bmatrix} \qquad \dot{X} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_1 x^2{}_6 + x_1 \cos^2 x_3 \bullet x^2{}_5 \\ 2x_5 x_6 tg x_3 - 2x_4 x_5 / x_1 \\ -\sin 2x_3 \bullet x_5 / 2 - 2x_4 x_6 / x_1 \end{bmatrix} = f(X)$$

$$\ddot{r} = r\dot{\varepsilon}^2 + r\cos^2\varepsilon \bullet \dot{\beta}^2$$

$$\ddot{\beta} = \frac{1}{r\cos\varepsilon}\left(2r\dot{\beta}\dot{\varepsilon}\sin\varepsilon - 2\dot{r}\dot{\beta}\cos\varepsilon\right)$$

$$\ddot{\varepsilon} = -\left(\sin\varepsilon\cos\varepsilon \bullet \dot{\beta}^2 + \frac{2}{r}\dot{r}\dot{\varepsilon}\right)$$

$$A(X) = \frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

EKF Example

$$\dot{X} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_1 x^2{}_6 + x_1 \cos^2 x_3 \bullet x^2{}_5 \\ 2x_5 x_6 tg x_3 - 2x_4 x_5 / x_1 \\ -\sin 2x_3 \bullet x_5 / 2 - 2x_4 x_6 / x_1 \end{bmatrix} = f(X) \qquad A(X) = \frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$A(X) f(X) = \begin{bmatrix} x_1 x_6^2 + x_1 x_5^2 \cos^2 x_3 \\ 2x_5 x_6 tg x_3 - 2x_4 x_5 / x_1 \\ -x_5^2 \sin 2x_3 / 2 - 2x_4 x_6 / x_1 \\ -3x_4 x_6^2 - 3x_4 x_5^2 \cos^2 x_3 \\ -2x_5^2 - 12 x_4 x_5 x_6 tg x_3 / x_1 + 6x_4^2 x_5 / x_1^2 + 6x_5 x_6^2 tg x_3 \\ 6x_4^2 x_6 / x_1^2 + 3x_4 x_5^2 \sin 2x_3 / x_1 - 2x_6^3 - 3x_5^2 x_6 \end{bmatrix}$$

EKF Example

$$A(X)f(X)=\begin{bmatrix} x_1x_6^2+x_1x_5^2\cos^2 x_3 \\ 2x_5x_6tgx_3-2x_4x_5/x_1 \\ -x_5^2\sin 2x_3/2-2x_4x_6/x_1 \\ \text{-}3x_4x_6^2-3x_4x_5^2\cos^2 x_3 \\ -2x_5^2-12x_4x_5x_6tgx_3/x_1+6x_4^2x_5/x_1^2+6x_5x_6^2tgx_3 \\ 6x_4^2x_6/x_1^2+3x_4x_5^2\sin 2x_3/x_1-2x_6^3-3x_5^2x_6 \end{bmatrix}$$

状态方程

$$X(k+1)=X(k)+f[X(k)]\Delta t+A[X(k)]f[X(k)]\frac{(\Delta t)^2}{2}+W(k)$$

$$X(k+1)=X(k)+f[\hat{X}(k)]\Delta t+A[\hat{X}(k)]f[\hat{X}(k)]\frac{(\Delta t)^2}{2}+W(k)$$

$$U(k)=f[\hat{X}(k)]\Delta t+A[\hat{X}(k)]f[\hat{X}(k)]\frac{(\Delta t)^2}{2}$$

$$X(k+1)=X(k)+U(k)+W(k)$$

EKF Example

观测方程

$$Z=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} r+v_1 \\ \beta+v_2 \\ \varepsilon+v_3 \end{bmatrix}$$

$$Z(k)=H(k)X(k)+V(k)$$

$$E\left[V(k)V^T(k)\right]=R_k\delta_{k,j}$$

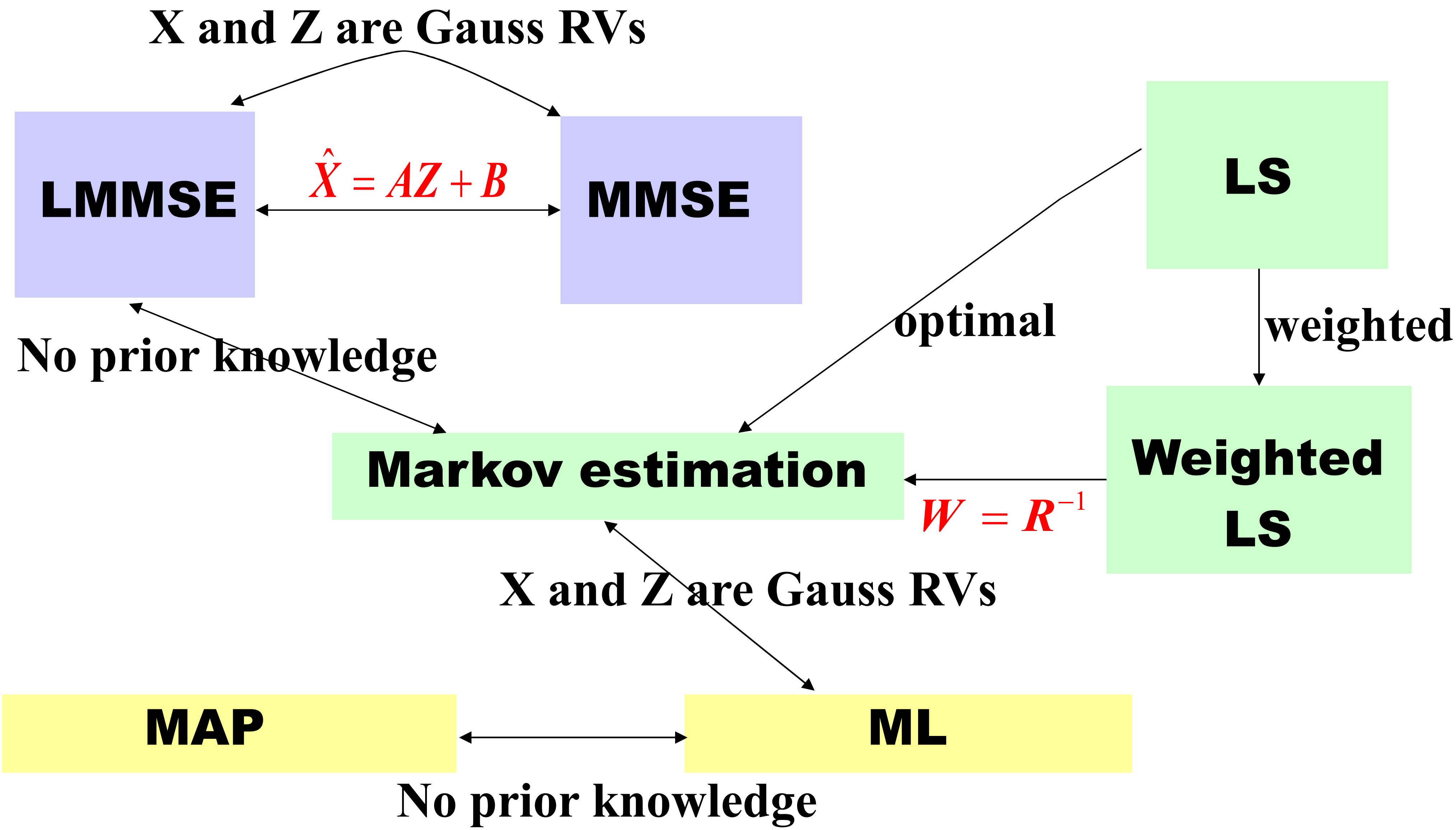
$$H(k)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

求解计算中，需要注意单位的选择，如果距离单位为米，角度单位为弧度，一定会造成大条件数，滤波计算将发散。

可以将距离单位设置为公里、10公里等其他值，将角度单位改为度，会大幅度提高计算稳定性。

实际应用中一定要注意状态中各维的数值均衡。

各估计准则间的关系


X and Z are Gauss RVs
LMMSE
$\hat{X} = AZ + B$
MMSE
LS
optimal
weighted
No prior knowledge
Markov estimation
Weighted LS
$W = R^{-1}$
X and Z are Gauss RVs
MAP
ML
No prior knowledge

6.4 近似条件均值滤波

1) MMSE

可以处理非线性系统

估计解的形式

$$\hat{X} = \mathrm{E}\left[\mathrm{X} \mid \mathrm{Z}\right]$$

如果***X***, ***Z***服从正态分布，

$$\hat{X} = m_x + COV(X,Z)(VarZ)^{-1}(Z - m_Z)$$

$$J = VarX - COV(X,Z)(VarZ)^{-1}COV(X,Z)$$

$$\hat{X} = \mathrm{E}\left[\mathrm{X} \mid \mathrm{Z}\right] = m_x + COV(X,Z)(VarZ)^{-1}(Z - m_Z)$$

$$P = Var(X \mid Z) = VarX - COV(X,Z)(VarZ)^{-1}COV(X,Z)$$

通常条件概率密度*$P[X(k+1)|Z_1^{k+1}]$*不完全服从正态分布，但接近正态分布。

6.4 近似条件均值滤波

2) 解的形式

$$\hat{X} = m_x + COV(X,Z)(VarZ)^{-1}(Z - m_Z)$$
$$P = VarX - COV(X,Z)(VarZ)^{-1}COV(X,Z)$$

$$E[X(k+1)|Z_1^{k+1}] = E\left[X(k+1)|Z(k+1),Z_1^k\right]$$
$$= E[X(k+1)|Z_1^k] + COV\{[X(k+1),Z(k+1)]\Big|Z_1^k\}\bullet \{Var[Z(k+1)\Big|Z_1^k]\}^{-1}\bullet\{Z(k+1)-E[Z(k+1)\Big|Z_1^k]\}$$

$$\hat{X}(k+1\big|k+1) = \hat{X}(k+1\big|k) + \underline{COV\{[X(k+1),Z(k+1)]\Big|Z_1^k\}}\bullet \underline{\{Var[Z(k+1)\Big|Z_1^k]\}^{-1}}\bullet\{Z(k+1)-\underline{E[Z(k+1)\Big|Z_1^k]}\}$$

$$Var\left[X(k+1)|Z_1^{k+1}\right] = Var\left[X(k+1)|Z(k+1),Z_1^k\right]$$
$$= Var[X(k+1)|Z_1^k] - COV\{[X(k+1),Z(k+1)]\Big|Z_1^k\}\bullet \{Var[Z(k+1)\Big|Z_1^k]\}^{-1}\bullet COV^T\{X(k+1),Z(k+1)\Big|Z_1^k]\}$$

$$P(k+1\big|k+1) = P(k+1\big|k) - \underline{COV\{[X(k+1),Z(k+1)]\Big|Z_1^k\}}\bullet \underline{\{Var[Z(k+1)\Big|Z_1^k]\}^{-1}}\bullet \underline{COV^T\{X(k+1),Z(k+1)\Big|Z_1^k]\}}$$

对应形式

6.4 近似条件均值滤波(一阶)

$$\hat{X}(k+1|k+1)=\hat{X}(k+1|k)+COV\left\{[X(k+1),Z(k+1)]\middle|Z_1^k\right\}\bullet$$
$$Var[Z(k+1)|Z_1^k]^{-1}\bullet\left\{Z(k+1)-E[Z(k+1)|Z_1^k]\right\}$$

$$\varphi[X(k),k]\approx\varphi[\hat{X}(k|k),k]+\left.\frac{\partial\varphi}{\partial X}\right|_{X(k)=\hat{X}(k|k)}[X(k)-\hat{X}(k|k)]$$

$$h[X(k+1),k+1]\approx h[\hat{X}(k+1|k),k+1]+\frac{\partial h}{\partial\hat{X}(k+1|k)}[X(k+1)-\hat{X}(k+1)]$$

$$E[Z(k+1)|Z_1^k]=E\left\{h\left[X(k+1),k+1\right]+V(k+1)\middle|Z_1^k\right\}$$
$$\approx E\left\{h\left[\hat{X}(k+1|k),k+1\right]+\frac{\partial h_{k+1}}{\partial\hat{X}(k+1|k)}\left[X(k+1)-\hat{X}(k+1|k)\right]\middle|Z_1^k\right\}$$
$$=h\left[\hat{X}(k+1|k),k+1\right]$$

$$h_{k+1}=h\left[\hat{X}(k+1|k),k+1\right]$$

6.4 近似条件均值滤波(一阶)

$$\hat{X}(k+1|k+1) = \hat{X}(k+1|k) + COV\left\{[X(k+1), Z(k+1)]\middle|Z_1^k\right\} \bullet$$

$$Var[Z(k+1)|Z_1^k]^{-1} \bullet \left\{Z(k+1) - \underline{E[Z(k+1)|Z_1^k]}\right\}$$

$$Var[Z(k+1)|Z_1^k] = E\left\{\left\{Z(k+1) - E[Z(k+1)\,|\,Z_1^k]\right\} \bullet \left\{Z(k+1) - E[Z(k+1)\,|\,Z_1^k]\right\}^T\right\}$$

$E[Z(k+1)\,|\,Z_1^k]$
$= h\left[\hat{X}(k+1|k), k+1\right]$

$$\approx E\left\{\begin{array}{l}\left\{\dfrac{\partial h_{k+1}}{\partial \hat{X}(k+1\,|\,k)}\left[X(k+1) - \hat{X}(k+1\,|\,k)\right] + V(k+1)\right\} \bullet \\ \left\{\dfrac{\partial h_{k+1}}{\partial \hat{X}(k+1\,|\,k)}\left[X(k+1) - \hat{X}(k+1\,|\,k)\right] + V(k+1)\right\}^T \middle| Z_1^k\end{array}\right\}$$

$$= \frac{\partial h_{k+1}}{\partial \hat{X}(k+1\,|\,k)} P(k+1\,|\,k)\left[\frac{\partial h_{k+1}}{\partial \hat{X}(k+1\,|\,k)}\right]^T + R_{k+1}$$

$$= H(k+1)P(k+1\,|\,k)H(k+1)^T + R_{k+1}$$

6.4 近似条件均值滤波(一阶)

$$\hat{X}(k+1|k+1)=\hat{X}(k+1|k)+COV\left\{[X(k+1),Z(k+1)]\middle|Z_1^k\right\}\bullet Var[Z(k+1)|Z_1^k]^{-1}\bullet\left\{Z(k+1)-E[Z(k+1)|Z_1^k]\right\}$$

$$COV\left\{[X(k+1),Z(k+1)]\middle|Z_1^k\right\}$$

$$=E\left\{\left[X(k+1)-\hat{X}(k+1|k)\right]\bullet\left[Z(k+1)-Z(k+1|k)\right]^T\middle|Z_1^k\right\}$$

$$\approx E\left\{\left[X(k+1)-\hat{X}(k+1|k)\right]\bullet\left\{\frac{\partial h_{k+1}}{\partial\hat{X}(k+1|k)}\left[X(k+1)-\hat{X}(k+1|k)\right]+V(k+1)\right\}^T\middle|Z_1^k\right\}$$

$$=P(k+1|k)H^T(k+1)$$

6.4 近似条件均值滤波(一阶)

$$\hat{X}(k+1|k+1)=\hat{X}(k+1|k)+COV\{[X(k+1),Z(k+1)]|Z_1^k\}\bullet Var[Z(k+1)|Z_1^k]^{-1}\bullet\{Z(k+1)-E[Z(k+1)|Z_1^k]\}$$

$$\hat{X}(k+1|k)=E\{[\varphi[X(k),k]+\Gamma[X(k),k]W(k)]|Z_1^k\}$$

$$\approx E\left\{\left[\varphi[\hat{X}(k|k),k]+\frac{\partial\varphi_k}{\partial\hat{X}(k|k)}[X(k)-\hat{X}(k|k)]+\Gamma[\hat{X}(k|k),k]W(k)\right]\middle|Z_1^k\right\}$$

$$=\varphi[\hat{X}(k|k),k]$$

6.4 近似条件均值滤波(一阶)

$$P(k+1|k+1) = P(k+1|k) - COV\{[X(k+1), Z(k+1)]|Z_1^k\} \bullet \{Var[Z(k+1)|Z_1^k]\}^{-1} \bullet COV^T\{X(k+1), Z(k+1)|Z_1^k]\}$$

$$P(k+1|k) = E\left\{\left[X(k+1) - \hat{X}(k+1|k)\right]\left[X(k+1) - \hat{X}(k+1|k)\right]^T \Big| Z_1^k\right\}$$

$$\approx E\left\{\left\{\frac{\partial\varphi_k}{\partial\hat{X}(k|k)}\left[X(k) - \hat{X}(k|k)\right] + \Gamma\left[\hat{X}(k|k), k\right]W(k)\right\} \bullet \left\{\frac{\partial\varphi_k}{\partial\hat{X}(k|k)}\left[X(k) - \hat{X}(k|k)\right] + \Gamma\left[\hat{X}(k|k), k\right]W(k)\right\}^T \Big| Z_1^k\right\}$$

$$\frac{\partial\varphi_k}{\partial\hat{X}(k|k)} = \phi(k+1, k)$$

$$= \phi(k+1, k)P(k|k)\phi^T(k+1, k) + \Gamma\left[\hat{X}(k|k), k\right]Q_k\Gamma^T\left[\hat{X}(k|k), k\right]$$

6.4 近似条件均值滤波(一阶)

与EKF求解公式完全相同

$$\hat{X}(k+1|k+1)=\hat{X}(k+1|k)+K(k+1)\left\{Z(k+1)-h[\hat{X}(k+1|k),k+1]\right\}$$

$$\hat{X}(k+1,k)=\Phi(k+1,k)\hat{X}(k,k)+f(k)=\varphi[\hat{X}(k,k),k]$$

$$\Phi(k+1,k)=\frac{\partial\varphi}{\partial X}\bigg|_{X(k)=\hat{X}(k|k)}$$

$$K(k+1)=P(k+1|k)H^T(k+1)\left[H(k+1)P(k+1|k)\cdot H^T(k+1)+R_{k+1}\right]^{-1}$$

$$P(k+1|k)=\Phi(k+1,k)P(k|k)\Phi^T(k+1,k)+\Gamma[\hat{X}(k|k),k]Q_k\Gamma^T[\hat{X}(k|k),k]$$

$$P(k+1|k+1)=[I-K(k+1)H(k+1)]P(k+1|k)$$

$$\hat{X}_0=E[X_0]=m_0,P_0=VarX_0$$

$$H(k+1)=\frac{\partial h}{\partial X}\bigg|_{\hat{X}(k+1|k)}\qquad f(k)=\varphi[\hat{X}(k|k),k]-\frac{\partial\varphi}{\partial X}\bigg|_{X(k)=\hat{X}(k|k)}\hat{X}(k|k)$$

6.5 Unscented Kalman Filtering

EKF算法的分析

$$\hat{X}(k+1|k+1)=\hat{X}(k+1|k)+K(k+1)\left\{Z(k+1)-h[\hat{X}(k+1|k),k+1]\right\}$$

$$\hat{X}(k+1,k)=\Phi(k+1,k)\hat{X}(k,k)+f(k)=\varphi[\hat{X}(k,k),k]$$

$$\Phi(k+1,k)=\left.\frac{\partial\varphi}{\partial X}\right|_{X(k)=\hat{X}(k|k)}$$

$$K(k+1)=P(k+1|k)H^T(k+1)\left[H(k+1)P(k+1|k)\cdot H^T(k+1)+R_{k+1}\right]^{-1}$$

$$P(k+1|k)=\Phi(k+1,k)P(k|k)\Phi^T(k+1,k)+\Gamma[\hat{X}(k|k),k]Q_k\Gamma^T[\hat{X}(k|k),k]$$

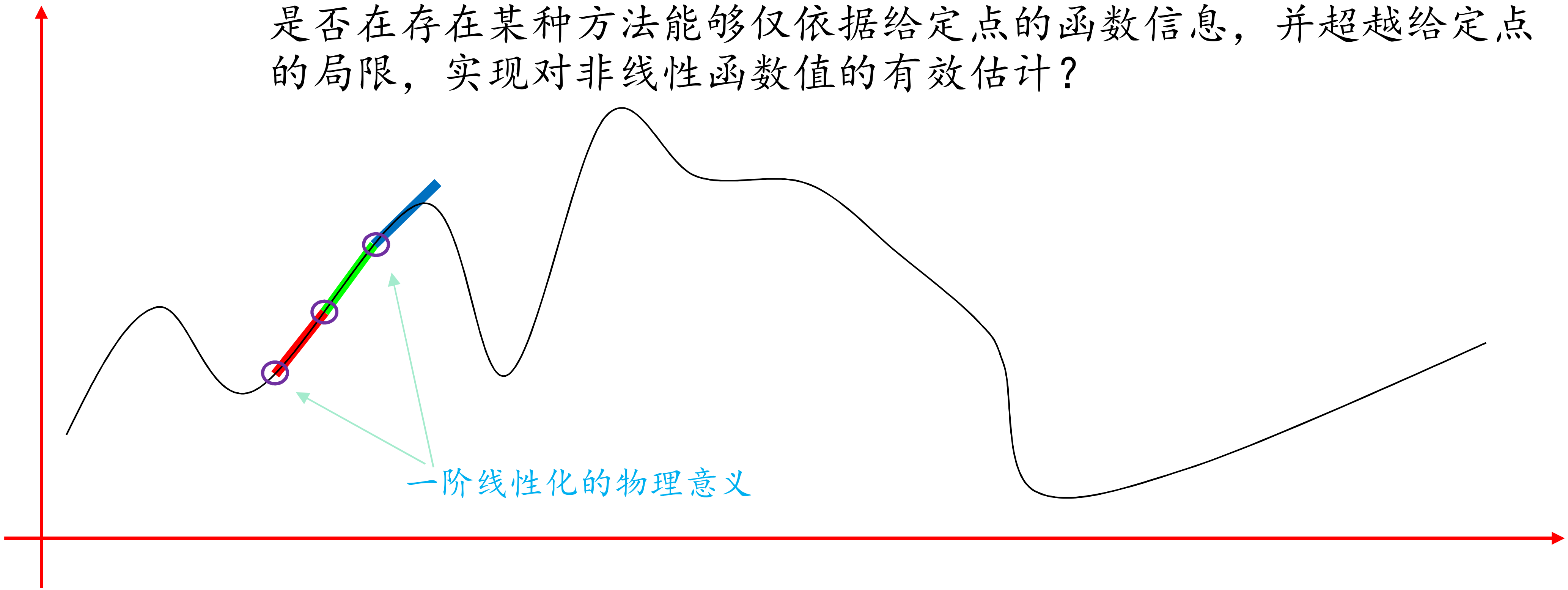
$$P(k+1|k+1)=[I-K(k+1)H(k+1)]P(k+1|k)$$

$$\hat{X}_0=E[X_0]=m_0,P_0=VarX_0$$

$$H(k+1)=\left.\frac{\partial h}{\partial X}\right|_{\hat{X}(k+1|k)} \qquad f(k)=\varphi[\hat{X}(k|k),k]-\left.\frac{\partial\varphi}{\partial X}\right|_{X(k)=\hat{X}(k|k)}\hat{X}(k|k)$$

EKF算法的分析

是否在存在某种方法能够仅依据给定点的函数信息，并超越给定点的局限，实现对非线性函数值的有效估计？

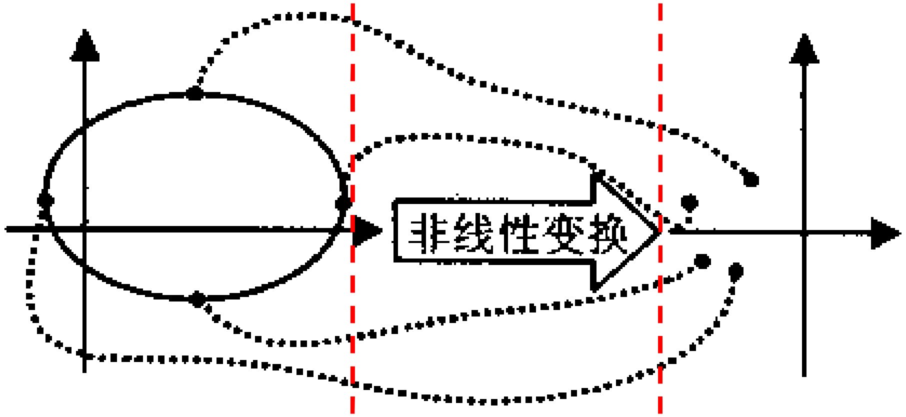


仅仅提高泰勒展开阶数，不能达到提高滤波精度的目的

如果计算步长足够的小，**EKF**仍然是有效的处理方法

UT变换的思想

Simon Julier 1995年提出UT变换



X的分布 | Y=F(X) | Y的分布

概率密度函数与**Kalman**滤波计算关系并不直接，但在滤波计算中，如果协方差阵、数学期望能不采用线性化的方式被高精度有效计算，那么以**MMSE**的角度出发为例，是可以获得高精度估计解的。**UT**变换具有这个功能。

UKF的优点

- 对非线性函数的概率密度分布进行近似，而不是对非线性函数进行近似，不需要知道非线性函数的显式表达式；
- 非线性函数统计量的精度至少达到3阶，对于采用特殊的采样策略，如高斯分布4阶采样和偏度采样等可达到更高阶精度；
- 计算量与EKF同阶；
- 不需要求导计算Jacobian矩阵，可以处理非可导的非线性函数。

6.5.1 非线性变换下均值与方差

1) 一阶近似下均值

极坐标到直角坐标转换问题

$$\begin{cases} y_1 = r\cos\theta \\ y_2 = r\sin\theta \end{cases} \Rightarrow y = h(x)$$

设 $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ \theta \end{bmatrix}$

假设x_1数学期望为1，方差为σ_r；x_2数学期望为$\pi/2$，方差为σ_θ。假设r、θ相互独立，并且他们的概率密度是围绕数学期望对称分布的。

$$\begin{aligned} \bar{y} &= E\left[h(x)\right] \\ &\approx E\left[h(\bar{x}) + \left.\frac{\partial h}{\partial x}\right|_{\bar{x}} (x-\bar{x}) \right] \\ &= h(\bar{x}) + \left.\frac{\partial h}{\partial x}\right|_{\bar{x}} E\left(x-\bar{x}\right) \end{aligned}$$

$$= h(\bar{x})$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

6.5.1 非线性变换下均值与方差

2) 实际数学期望

$$\begin{cases} r = \overline{r} + \tilde{r} \\ \theta = \overline{\theta} + \tilde{\theta} \end{cases}$$

$$\begin{aligned}\overline{y}_1 &= E\left(r\cos\theta\right) \\ &= E\left[\left(\overline{r}+\tilde{r}\right)\cos\left(\overline{\theta}+\tilde{\theta}\right)\right] \\ &= E\left[\left(\overline{r}+\tilde{r}\right)\left(\cos\overline{\theta}\cos\tilde{\theta}-\sin\overline{\theta}\sin\tilde{\theta}\right)\right]\end{aligned}$$

$\tilde{r}$、$\tilde{\theta}$**相互独立，对称分布**

$$\overline{y}_1 = \overline{r}\underbrace{\cos\overline{\theta}E\left[\cos\tilde{\theta}\right]-\overline{r}\sin\overline{\theta}E}_{}\left[\sin\tilde{\theta}\right]$$

$$\overline{y}_1 = \overline{r}\underline{\cos\overline{\theta}}_{\,\mathbf{0}}E\left[\cos\tilde{\theta}\right]-\overline{r}\sin\overline{\theta}\underline{E\left[\sin\tilde{\theta}\right]}_{\,\mathbf{0}}$$

$$\overline{y}_1 = 0$$

$$\begin{aligned}\overline{y}_2 &= E\left(r\sin\theta\right) \\ &= E\left[\left(\overline{r}+\tilde{r}\right)\sin\left(\overline{\theta}+\tilde{\theta}\right)\right] \\ &= E\left[\left(\overline{r}+\tilde{r}\right)\left(\sin\overline{\theta}\cos\tilde{\theta}+\cos\overline{\theta}\sin\tilde{\theta}\right)\right]\end{aligned}$$

$\tilde{r}$、$\tilde{\theta}$**相互独立，对称分布**

$$\overline{y}_2 = \overline{r}\sin\overline{\theta}E\left(\cos\tilde{\theta}\right)$$

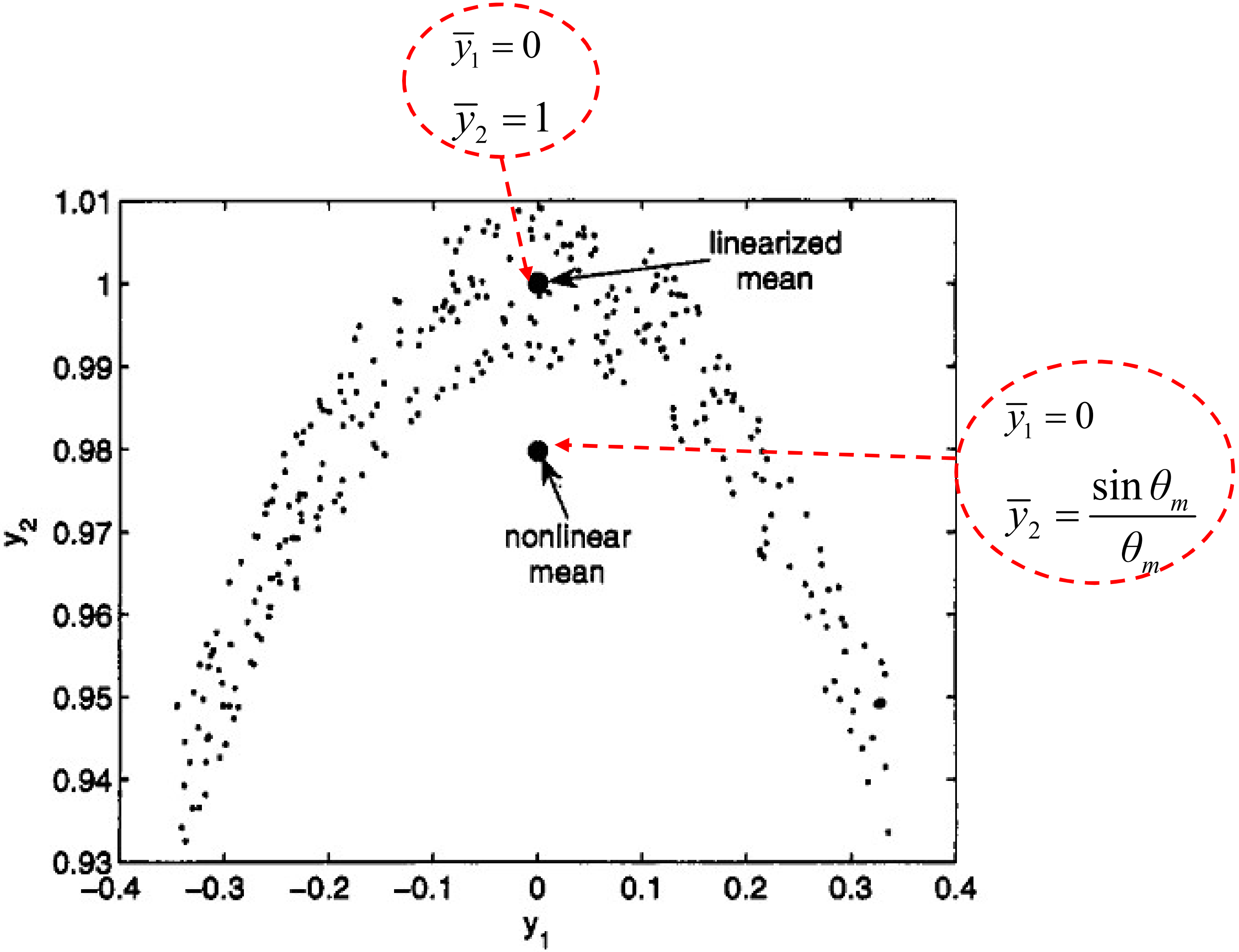
$$\overline{y}_2 = E\left(\cos\tilde{\theta}\right)$$

如果假设$\tilde{\theta}$**在**$\pm\theta_m$**间服从均匀分布**

$$\overline{y}_2 = \frac{\sin\theta_m}{\theta_m}$$

6.5.1 非线性变换下均值与方差

3) 一阶近似与实际均值的比较



6.5.1 非线性变换下均值与方差

4) 非线性函数的展开方法——泰勒展开的描述

$$f(x)=f(\bar{x})+\left.\frac{\partial f}{\partial x}\right|_{\bar{x}}\tilde{x}+\frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{\bar{x}}\tilde{x}^2+\frac{1}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{\bar{x}}\tilde{x}^3+\cdots$$

$$f(x)=f(\bar{x})+\left.\left(\tilde{x}_1\frac{\partial}{\partial x_1}+\tilde{x}_2\frac{\partial}{\partial x_2}+\cdots+\tilde{x}_n\frac{\partial}{\partial x_n}\right)f\right|_{\bar{x}}+\frac{1}{2!}\left.\left(\tilde{x}_1\frac{\partial}{\partial x_1}+\tilde{x}_2\frac{\partial}{\partial x_2}+\cdots+\tilde{x}_n\frac{\partial}{\partial x_n}\right)^2 f\right|_{\bar{x}}+$$
$$\frac{1}{3!}\left.\left(\tilde{x}_1\frac{\partial}{\partial x_1}+\tilde{x}_2\frac{\partial}{\partial x_2}+\cdots+\tilde{x}_n\frac{\partial}{\partial x_n}\right)^3 f\right|_{\bar{x}}+\cdots$$

令 $$D_{\tilde{x}}^k f=\left.\left(\sum_{i=1}^{n}\tilde{x}_i\frac{\partial}{\partial x_i}\right)^k f(x)\right|_{\bar{x}}$$

$$f(x)=f(\bar{x})+D_{\tilde{x}}f+\frac{1}{2!}D_{\tilde{x}}^2 f+\frac{1}{3!}D_{\tilde{x}}^3 f+\cdots$$

6.5.1 非线性变换下均值与方差

5) 非线性函数的展开——对称分布下的泰勒展开

$$y = h(x) = h(\bar{x}) + D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2 h + \frac{1}{3!}D_{\tilde{x}}^3 h + \cdots$$

$$\bar{y} = \mathrm{E}\left[h(\bar{x}) + D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2 h + \frac{1}{3!}D_{\tilde{x}}^3 h + \cdots\right]$$

$$= h(\bar{x}) + \mathrm{E}\left[D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2 h + \frac{1}{3!}D_{\tilde{x}}^3 h + \cdots\right]$$

$$\left.\begin{array}{l} E\left[x^i\right] = \int_{-\infty}^{\infty} p(x) x^i dx \\ x\text{对称分布} \\ i\text{为奇数} \end{array}\right\} E\left[x^i\right] = 0$$

$$E\left[D_{\tilde{x}}h\right] = E\left[\sum_{i=1}^{n} \tilde{x}_i \frac{\partial}{\partial x_i} h(x)\Big|_{x=\bar{x}}\right]$$

$$= \sum_{i=1}^{n} E(\tilde{x}_i) \frac{\partial}{\partial x_i} h(x)\Big|_{x=\bar{x}}$$

$$= 0$$

$$\mathrm{E}\left[D_{\tilde{x}}^3 \mathrm{h}\right] = \mathrm{E}\left[\left(\sum_{i=1}^{n} \tilde{x}_i \frac{\partial}{\partial x_i}\right)^3 \mathrm{h}(\mathrm{x})\Big|_{\mathrm{x}=\bar{\mathrm{x}}}\right]$$

$$= \mathrm{E}\left(\tilde{x}_1^{\ 3} + \tilde{x}_1^{\ 2}\tilde{x}_2 + \tilde{x}_1\tilde{x}_2^2 + \ldots\right) \frac{\partial^3}{\partial x_i} \mathrm{h}(\mathrm{x})\Big|_{\mathrm{x}=\bar{\mathrm{x}}}$$

$$= 0$$

$$\boxed{\bar{\mathrm{y}} = h(\bar{x}) + \frac{1}{2!}\mathrm{E}\left[D_{\tilde{x}}^2 h\right] + \frac{1}{4!}\mathrm{E}\left[D_{\tilde{x}}^4 h\right] + \cdots}$$

6.5.1 非线性变换下均值与方差

$$\bar{y} = h(\bar{x}) + \frac{1}{2!}\mathrm{E}\left[D_{\tilde{x}}^2 h\right] + \frac{1}{4!}\mathrm{E}\left[D_{\tilde{x}}^4 h\right] + \cdots$$

如果近似到2阶，则变换后的均值为

$$h(x) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ \theta \end{bmatrix}$$

$$\bar{y} \approx h(\bar{x}) + \frac{1}{2!}\mathrm{E}\left[D_{\tilde{x}}^2 h\right]$$

$$= h(\bar{x}) + \frac{1}{2}E\left[\left(\sum_{i=1}^{2}\tilde{x}_i \frac{\partial}{\partial x_i}\right)^2 h(x)\Big|_{x=\bar{x}}\right]$$

$$= h(\bar{x}) + \frac{1}{2}\left[E(\tilde{x}_1^2)\frac{\partial^2 h(x)}{\partial x_1^2}\Big|_{x=\bar{x}} + 2E(\tilde{x}_1\tilde{x}_2)\frac{\partial^2 h(x)}{\partial x_1 \partial x_2}\Big|_{x=\bar{x}} + E(\tilde{x}_2^2)\frac{\partial^2 h(x)}{\partial x_2^2}\Big|_{x=\bar{x}}\right]$$

$$= h(\bar{x}) + \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \sigma_\theta^2 \begin{bmatrix} -r\cos\theta \\ -r\sin\theta \end{bmatrix}\Big|_{x=\bar{x}}\right)$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2}\sigma_\theta^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\boxed{\begin{aligned} \bar{y}_1 &\approx 0 \\ \bar{y}_2 &\approx 1 - \frac{\sigma_\theta^2}{2} \end{aligned}}$$

6.5.1 非线性变换下均值与方差

6) 非线性变换后方差的求解

$$P_y = E\left[(y-\bar{y})(y-\bar{y})^T\right]$$

设$\tilde{x}$服从均值为***0***的对称分布。

$$\bar{y} = h(\bar{x}) + \frac{1}{2!}E\left[D_{\tilde{x}}^2 h\right] + \frac{1}{4!}E\left[D_{\tilde{x}}^4 h\right] + \cdots$$

$$y-\bar{y} = \left[\cancel{h(\bar{x})} + \underline{D_{\tilde{x}} h + \frac{1}{2!}D_{\tilde{x}}^2 h + \frac{1}{3!}D_{\tilde{x}}^3 h + \cdots}\right] - \left[\cancel{h(\bar{x})} + \underline{\frac{1}{2!}\mathrm{E}\left[D_{\tilde{x}}^2 h\right] + \frac{1}{4!}\mathrm{E}\left[D_{\tilde{x}}^4 h\right] + \cdots}\right]$$

$$P_y = \underline{E\left[D_{\tilde{x}} h\left(D_{\tilde{x}} h\right)^T\right] + E\left[\frac{1}{3!}D_{\tilde{x}} h\left(D_{\tilde{x}}^3 h\right)^T + \frac{1}{2!2!}D_{\tilde{x}}^2 h\left(D_{\tilde{x}}^2 h\right)^T + \frac{1}{3!}D_{\tilde{x}}^3 h\left(D_{\tilde{x}} h\right)^T\right]} + \underline{E\left(\frac{1}{2!}D_{\tilde{x}}^2 h\right)E\left(\frac{1}{2!}D_{\tilde{x}}^2 h\right)^T} + \cdots$$

$$\mathrm{E}\left[D_{\tilde{x}}\mathrm{h}\left(D_{\tilde{x}}\mathrm{h}\right)^{\mathrm{T}}\right] = \mathrm{E}\left[\left(\sum_{i=1}^{n}\tilde{x}_i \left.\frac{\partial \mathrm{h}}{\partial x_i}\right|_{\mathrm{x}=\bar{\mathrm{x}}}\right)\left(\sum_{i=1}^{n}\tilde{x}_i \left.\frac{\partial \mathrm{h}}{\partial x_i}\right|_{\mathrm{x}=\bar{\mathrm{x}}}\right)^{\mathrm{T}}\right]$$

6.5.1 非线性变换下均值与方差

$$E\left[D_{\tilde{x}}h\left(D_{\tilde{x}}h\right)^T\right]=E\left[\left(\sum_{i=1}^{n}\tilde{x}_i\frac{\partial h}{\partial x_i}\bigg|_{x=\bar{x}}\right)\left(\sum_{i=1}^{n}\tilde{x}_i\frac{\partial h}{\partial x_i}\bigg|_{x=\bar{x}}\right)^T\right]$$

$$=E\left[\sum_{i,j}\tilde{x}_i\frac{\partial h}{\partial x_i}\bigg|_{x=\bar{x}}\frac{\partial h^T}{\partial x_j}\bigg|_{x=\bar{x}}\tilde{x}_j\right]$$

$$=\sum_{i,j}H_iE\left(\tilde{x}_i\tilde{x}_j\right)H^T{}_j$$

$$=\sum_{i,j}H_iP_{ij}H^T{}_j$$

$$E\left[D_{\tilde{x}}h\left(D_{\tilde{x}}h\right)^T\right]=\frac{\partial h}{\partial x}\bigg|_{x=\bar{x}}P\frac{\partial h^T}{\partial x}\bigg|_{x=\bar{x}}=HPH^T$$

$$\boxed{\begin{aligned}P_y=HPH^T+E\left[\frac{1}{3!}D_{\tilde{x}}h\left(D_{\tilde{x}}^3h\right)^T+\frac{1}{2!2!}D_{\tilde{x}}^2h\left(D_{\tilde{x}}^2h\right)^T+\frac{1}{3!}D_{\tilde{x}}^3h\left(D_{\tilde{x}}^3h\right)^T\right]+\\E\left(\frac{1}{2!}D_{\tilde{x}}^2h\right)E\left(\frac{1}{2!}D_{\tilde{x}}^2h\right)^T+\cdots\end{aligned}}$$

$$D_{\tilde{x}}^{k} f = \left(\sum_{i=1}^{n} \tilde{x}_i \frac{\partial}{\partial x_i} \right)^{k} f\left(x\right)\Big|_{\bar{x}}$$

$$h = \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix}$$

$$D_{\tilde{x}} h = \begin{bmatrix} \frac{\partial h_1}{\partial x_1}\tilde{x}_1 + \frac{\partial h_1}{\partial x_2}\tilde{x}_2 + \cdots + \frac{\partial h_1}{\partial x_n}\tilde{x}_n \\ \vdots \\ \frac{\partial h_m}{\partial x_1}\tilde{x}_1 + \frac{\partial h_m}{\partial x_2}\tilde{x}_2 + \cdots + \frac{\partial h_m}{\partial x_n}\tilde{x}_n \end{bmatrix}\Bigg|_{x=\bar{x}} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}\Bigg|_{x=\bar{x}} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}$$

$$E\left[D_{\tilde{x}} h \left(D_{\tilde{x}} h \right)^{T} \right] = E\left\{ \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}\Bigg|_{x=\bar{x}} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}^{T} \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}^{T} \Bigg|_{x=\bar{x}} \right\}$$

$$= HPH^{T}$$

6.5.1 非线性变换下均值与方差

$$P_y = HPH^T + E\left[\frac{1}{3!}D_{\tilde{x}}h\left(D_{\tilde{x}}^3h\right)^T + \frac{1}{2!2!}D_{\tilde{x}}^2h\left(D_{\tilde{x}}^2h\right)^T + \frac{1}{3!}D_{\tilde{x}}^3h\left(D_{\tilde{x}}^3h\right)^T\right] + E\left(\frac{1}{2!}D_{\tilde{x}}^2h\right)E\left(\frac{1}{2!}D_{\tilde{x}}^2h\right)^T + \cdots$$

如果只使用1阶近似，对于坐标转换问题有

$$\left.\begin{array}{l}\begin{cases} y_1 = r\cos\theta \\ y_2 = r\sin\theta \end{cases} \\ \mathrm{H} = \left.\dfrac{\partial h}{\partial x}\right|_{x=\bar{x}} = \left.\begin{bmatrix} \cos\theta & -\mathrm{r}\sin\theta \\ \sin\theta & \mathrm{r}\cos\theta \end{bmatrix}\right|_{x=\bar{x}} \end{array}\right\} \Rightarrow \mathrm{H} = \left.\begin{bmatrix} \cos\theta & -\mathrm{r}\sin\theta \\ \sin\theta & \mathrm{r}\cos\theta \end{bmatrix}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

且 $P_{\mathrm{x}} = \mathrm{E}\left(\begin{bmatrix} \mathrm{r}-\bar{\mathrm{r}} \\ \theta-\bar{\theta} \end{bmatrix}\begin{bmatrix} \mathrm{r}-\bar{\mathrm{r}} \\ \theta-\bar{\theta} \end{bmatrix}^{\mathrm{T}}\right) = \begin{bmatrix} \sigma_{\mathrm{r}}^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}$

$$P_y \approx HP_xH^T$$

$$P_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

6.5.1 非线性变换下均值与方差

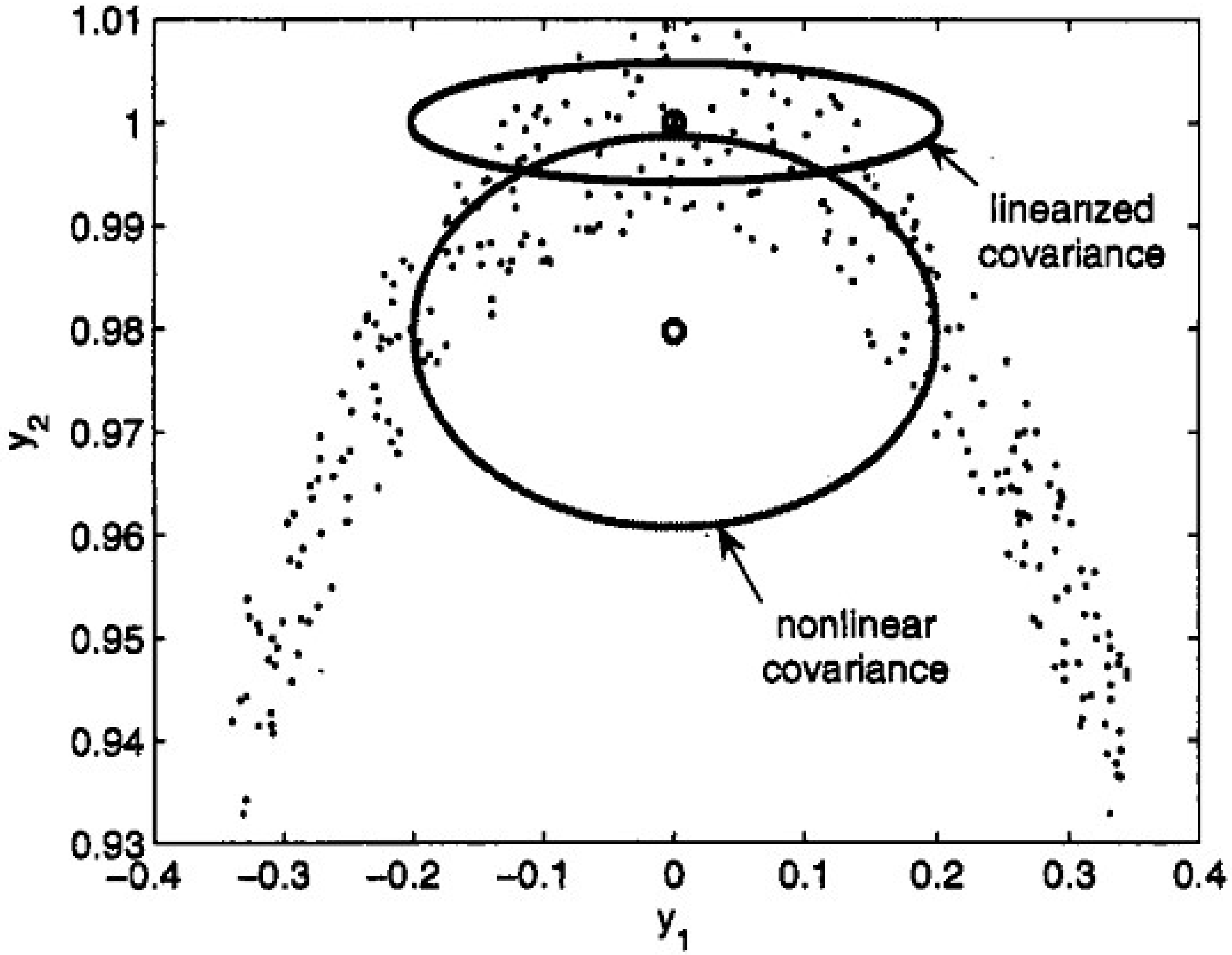
如果使用解析解，对于坐标转换问题有

$$\begin{cases} y_1 = r\cos\theta \\ y_2 = r\sin\theta \end{cases} \qquad \bar{y}_1 = 0 \qquad \bar{y}_2 = \frac{\sin\theta_m}{\theta_m}$$

$$P_y = E\left[(y-\bar{y})(y-\bar{y})^T \right]$$

$$P_y = \begin{bmatrix} \dfrac{(1+\sigma_r^2)(1-\sin 2\theta_m/2\theta_m)}{2} & 0 \\ 0 & \dfrac{(1+\sigma_r^2)(1+\sin 2\theta_m/2\theta_m)-\sin^2\theta_m/\theta_m^2}{2} \end{bmatrix}$$

6.5.1 非线性变换下均值与方差



相关内容请参阅

Optimal State Estimation

标称轨道及EKF	**13.1~13.2**
Unscented Kalman Filtering	**14**

Optimal Estimation of Dynamic Systems

EKF	**3.6**
Unscented Kalman Filtering	**3.7**