Nonlinear Control Theory

Bing Zhu

The Seventh Research Division
Beihang University, Beijing, P.R.China

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Control based on Lyapunov Function

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Consider the single input nonlinear system

$$\dot{x} = f(x) + g(x)u,$$

where $x \in R^n$, $u \in R$; $f(x)$ and $g(x)$ are locally Lipschitz with $f(0) = 0$.

Suppose there exists a locally Lipschitz state feedback control $u = \chi(x)$, such that the origin of

$$\dot{x} = f(x) + g(x)\chi(x)$$

is asymptotically stable. According to converse Lyapunov theorem, there exists a smooth Lyapunov function $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \quad \forall x \neq 0.$$

Moreover, if the origin of $\dot{x} = f(x) + g(x)\chi(x)$ is globally asymptotically stable, then $V(x)$ is radially unbounded, and the inequality

$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0$$

holds globally.

Definition

A continuously differentiable positive definite function $V(x)$ is a Control Lyapunov Function (CLF) for $\dot{x} = f(x) + g(x)u$, if

$$\frac{\partial V}{\partial x} g(x) = 0 \text{ for } x \in D, \ x \neq 0 \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x) < 0. \tag{1}$$

It is a Global CLF if it is radially unbounded and (1) holds with $D = R^n$.

$\dot{x} = f(x) + g(x)u$ is asymptotically stabilizable only if it has a CLF.

Is it also sufficient? YES!

- Sontag's Formula:

$$u = \phi(x) = \begin{cases} -\frac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\frac{\partial V}{\partial x} g}, & \text{if } \frac{\partial V}{\partial x} g \neq 0, \\ 0, & \text{if } \frac{\partial V}{\partial x} g = 0. \end{cases}$$

- If $\frac{\partial V}{\partial x} g = 0$, then $\dot{V} = \frac{\partial V}{\partial x} f < 0$.
- If $\frac{\partial V}{\partial x} g \neq 0$, then

$$\dot{V} = \frac{\partial V}{\partial x} f - \left[\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} \right] = -\sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} < 0.$$

It indicates that the origin is A.S.. If $V(x)$ is a global CLF, then the origin is G.A.S..

To guarantee that $\phi(x)$ is an effective controller, it must be sufficiently smooth.

Lemma (9.6)

If $f(x)$, $g(x)$ and $V(x)$ are smooth, then $\phi(x)$ will be smooth for $x \neq 0$. If they are of class $\mathcal{C}^{l+1}$ for $l \geq 1$, then $\phi(x)$ will be of class $\mathcal{C}^{l}$. Continuity at $x = 0$:

- *$\phi(x)$ is continuous at $x = 0$, if $V(x)$ has the small control property; namely, given any $\epsilon > 0$, there exists $\delta > 0$ such that if $x \neq 0$ and $\|x\| < \delta$, then there is u with $\|u\| < \epsilon$ such that*
$$\frac{\partial V}{\partial x}[f(x) + g(x)u] < 0.$$
- *$\phi(x)$ is locally Lipschitz at $x = 0$, if there is a locally Lipschitz function $\chi(x)$, with $\chi(0) = 0$, such that*
$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \text{ for } x \neq 0.$$

How can we find a CLF?

If we know of any stabilizing control with a corresponding Lyapunov function V, then V is a CLF.

- Feedback Linearization

$$z = T(x), \; u = -Kz \quad \Rightarrow \quad \dot{z} = (A - BK)z,$$

$$P(A - BK) + (A - BK)^T P = -Q, \;\; Q = Q^T > 0,$$

  Then $V = z^T P z = T^T(x) P T(x)$ is a CLF.

- Backstepping

Example

Consider the system

$$\dot{x} = x - x^3 + u.$$

The feedback linearization control

$$u = \chi(x) = -x + x^3 - \alpha x$$

with $\alpha > 0$ is capable of asymptotically stabilizing the system. The closed-loop system is now given by

$$\dot{x} = -\alpha x,$$

and the corresponding Lyapunov function $V(x) = \frac{1}{2}x^2$ is a CLF with

$$\frac{\partial V}{\partial x} g = x, \quad \frac{\partial V}{\partial x} f = x(x - x^3).$$

By Sontag's Formula,

$$u = \phi(x) = -\frac{\frac{\partial V}{\partial x} f + \sqrt{(\frac{\partial V}{\partial x} f)^2 + (\frac{\partial V}{\partial x} g)^4}}{\frac{\partial V}{\partial x} g}$$
$$= -x + x^3 - x\sqrt{(1 - x^2)^2 + 1}.$$

Compare with feedback linearization control

$$u = \chi(x) = -x + x^3 - \alpha x$$

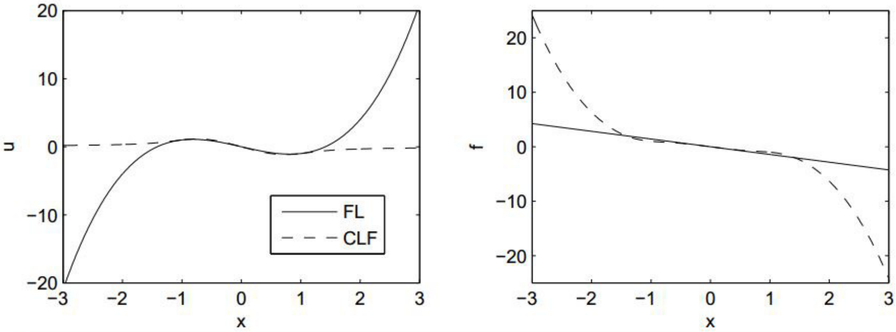

Figure: Comparison of CLF and feedback linearization in u and $\dot{x} = f$ with $\alpha = \sqrt{2}$

Robustness Property

Lemma (9.7)

Suppose

- f, g, *and* V *satisfy the conditions of Lemma 9.6;*
- ϕ *is given by Sontag's formula.*

Then,

- *the origin of* $\dot{x} = f(x) + g(x)k\phi(x)$ *is asymptotically stable for all* $k \geq \frac{1}{2}$.
- *If* V *is a global CLF, then the origin is globally asymptotically stable.*

Proof: Let

$$q(x) = \frac{1}{2}\left[-\frac{\partial V}{\partial x}f + \sqrt{\left(\frac{\partial V}{\partial x}f\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4}\right]$$

Since $V(x)$ is positive definite and smooth,

$$\frac{\partial V}{\partial x}(0) = 0 \quad \Rightarrow \quad q(0) = 0.$$

For $x \neq 0$,

$$\frac{\partial V}{\partial x}g \neq 0 \quad \Rightarrow \quad q > 0;$$

$$\frac{\partial V}{\partial x}g = 0 \quad \Rightarrow \quad q = -\frac{\partial V}{\partial x}f > 0.$$

$q(x)$ is positive definite.

$$u = k\phi(x) \quad \Rightarrow \quad \dot{x} = f(x) + g(x)k\phi(x),$$

$$\Rightarrow \quad \dot{V} = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} g k \phi.$$

For $x \neq 0$,

- $\frac{\partial V}{\partial x} g = 0 \quad \Rightarrow \quad \dot{V} = \frac{\partial V}{\partial x} f < 0.$
- $\frac{\partial V}{\partial x} g \neq 0$,

$$\begin{aligned}
\dot{V} &= -q + q + \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} g k \phi \\
&= -q - \left(k - \frac{1}{2}\right) \left[\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} \right] < 0.
\end{aligned}$$

The origin is A.S., if $k \geq \frac{1}{2}$. If $V(x)$ is a global CLF, then the origin is G.A.S..

Example

Reconsider $\dot{x} = x - x^3 + u$. Compare $u = \chi(x)$ with $u = \phi(x)$.

- By Lemma 9.7 the origin of $\dot{x} = x - x^3 + k\phi(x)$ is G.A.S. for all $k \geq \frac{1}{2}$.
- The origin of

$$\dot{x} = x - x^3 + k\chi(x) = -[k(1+\alpha) - 1]x + (k-1)x^3$$

is not G.A.S. for any $k > 1$. It is locally E.S. for $k > \frac{1}{1+\alpha}$ with region of attraction

$$\left\{|x| < \sqrt{1 + \frac{k\alpha}{k-1}}\right\} \quad \rightarrow \quad \left\{|x| < \sqrt{1+\alpha}\right\} \text{ as } k \rightarrow \infty.$$