非线性函数的展开方法——泰勒展开的描述

$$f(x)=f(\bar{x})+\left.\frac{\partial f}{\partial x}\right|_{\bar{x}}\tilde{x}+\frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{\bar{x}}\tilde{x}^2+\frac{1}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{\bar{x}}\tilde{x}^3+\cdots$$

$$f(x)=f(\bar{x})+\left.\left(\tilde{x}_1\frac{\partial}{\partial x_1}+\tilde{x}_2\frac{\partial}{\partial x_2}+\cdots+\tilde{x}_n\frac{\partial}{\partial x_n}\right)f\right|_{\bar{x}}+\frac{1}{2!}\left.\left(\tilde{x}_1\frac{\partial}{\partial x_1}+\tilde{x}_2\frac{\partial}{\partial x_2}+\cdots+\tilde{x}_n\frac{\partial}{\partial x_n}\right)^2 f\right|_{\bar{x}}+$$

$$\frac{1}{3!}\left.\left(\tilde{x}_1\frac{\partial}{\partial x_1}+\tilde{x}_2\frac{\partial}{\partial x_2}+\cdots+\tilde{x}_n\frac{\partial}{\partial x_n}\right)^3 f\right|_{\bar{x}}+\cdots$$

令 $D_{\tilde{x}}^k f=\left.\left(\sum_{i=1}^{n}\tilde{x}_i\frac{\partial}{\partial x_i}\right)^k f(x)\right|_{\bar{x}}$

$$f(x)=f(\bar{x})+D_{\tilde{x}}f+\frac{1}{2!}D_{\tilde{x}}^2 f+\frac{1}{3!}D_{\tilde{x}}^3 f+\cdots$$

非线性函数的展开——对称分布下的泰勒展开

$$y = h(x) = h(\bar{x}) + D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2 h + \frac{1}{3!}D_{\tilde{x}}^3 h + \cdots$$

$$\bar{y} = \mathrm{E}\left[h(\bar{x}) + D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2 h + \frac{1}{3!}D_{\tilde{x}}^3 h + \cdots\right]$$

$$= h(\bar{x}) + \mathrm{E}\left[D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2 h + \frac{1}{3!}D_{\tilde{x}}^3 h + \cdots\right]$$

$$\left.\begin{array}{l} E\left[x^i\right] = \int_{-\infty}^{\infty} p(x)x^i dx \\ x\text{对称分布} \\ i\text{为奇数} \end{array}\right\} E\left[x^i\right] = 0$$

$$E\left[D_{\tilde{x}}h\right] = E\left[\left.\sum_{i=1}^{n} \tilde{x}_i \frac{\partial}{\partial x_i} h(x)\right|_{x=\bar{x}}\right]$$

$$= \left.\sum_{i=1}^{n} E(\tilde{x}_i) \frac{\partial}{\partial x_i} h(x)\right|_{x=\bar{x}}$$

$$= 0$$

$$\mathrm{E}\left[D_{\tilde{x}}^3 \mathrm{h}\right] = \mathrm{E}\left[\left.\left(\sum_{i=1}^{n} \tilde{x}_i \frac{\partial}{\partial x_i}\right)^3 \mathrm{h}(\mathrm{x})\right|_{\mathrm{x}=\bar{\mathrm{x}}}\right]$$

$$= \mathrm{E}\left(\tilde{x}_1^{\ 3} + \tilde{x}_1^{\ 2}\tilde{x}_2 + \tilde{x}_1\tilde{x}_2^2 + \ldots\right)\left.\frac{\partial^3}{\partial x_i}\mathrm{h}(\mathrm{x})\right|_{\mathrm{x}=\bar{\mathrm{x}}}$$

$$= 0$$

$$\boxed{\bar{\mathrm{y}} = h(\bar{x}) + \frac{1}{2!}\mathrm{E}\left[D_{\tilde{x}}^2 h\right] + \frac{1}{4!}\mathrm{E}\left[D_{\tilde{x}}^4 h\right] + \cdots}$$

3）Potter平方根滤波——————P阵分解

对称非负定阵可以分解为

$$P = SS^T = \begin{bmatrix} S_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} \begin{bmatrix} S_{11} & \cdots & S_{n1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_{nn} \end{bmatrix}$$

$$S_{ii} = \sqrt{P_{ii} - \sum_{j=1}^{i-1} S_{ij}^2} \qquad (i < j)$$

$$S_{ij} = \begin{cases} o \\ \dfrac{1}{S_{jj}}(P_{ij} - \sum_{k=1}^{j-1} S_{ik} S_{jk}) \quad (i > j) \end{cases}$$

$$\begin{cases} P(k|k) = S(k|k)S^T(k|k) \\ P(k|k-1) = S(k|k-1)S^T(k|k-1) \end{cases}$$

$$P(k+1|k) = \Phi(k+1,k)S(k|k)S^T(k|k)\Phi^T(k+1,k) + \Gamma(k+1,k)Q_k\Gamma^T(k+1,k)$$

样本点的生成，核心思想：依据P阵生成多维向量样本

$$X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_n \end{bmatrix} \qquad \tilde{X}=\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \\ \tilde{x}_n \end{bmatrix} \qquad E\left[\tilde{X}\tilde{X}^T\right]=P$$

$$P=SS^T=\begin{bmatrix} S_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}\begin{bmatrix} S_{11} & \cdots & S_{n1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_{nn} \end{bmatrix}=\sqrt{P}\sqrt{P}^T$$

$$\tilde{X}^{(i)}=\begin{bmatrix} ns_{i1} \\ \vdots \\ ns_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

转置、放大

$$\begin{bmatrix} S_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}$$

取出第i行

6.5.2 Unscented Transformation

1) σ点的定义

设x是n维向量，与y之间满足非线性关系$y=h(x)$，$2n$个σ点的选择方法如下

$$\begin{cases} x^{(i)} = \bar{x} + \tilde{x}^{(i)} \\ \tilde{x}^{(i)} = \left(\sqrt{nP}\right)_i^T \qquad i = 1, \cdots n \\ \tilde{x}^{(n+i)} = -\left(\sqrt{nP}\right)_i^T \qquad i = 1, \cdots n \end{cases}$$

矩阵平方根的第i行

P的作用

2) y均值的计算

$$y^{(i)} = h(x^{(i)})$$

$$\bar{y}_u = \sum_{i=1}^{2n} W^{(i)} h(x^{(i)})$$

$$W^{(i)} = \frac{1}{2n} \qquad i = 1, \cdots, 2n$$

$$\boxed{\bar{y}_u = \frac{1}{2\text{n}} \sum_{i=1}^{2n} h(x^{(i)})}$$

$\bar{y}_u$ 精度的分析 $\bar{y}_u$代表UT变换均值

$$\bar{y}_u = \frac{1}{2n} \sum_{i=1}^{2n} \left[h(\bar{x}) + D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \cdots \right]$$

$$= h(\bar{x}) + \frac{1}{2n} \sum_{i=1}^{2n} \left[D_{\tilde{x}^{(i)}} h + \frac{1}{2!} D_{\tilde{x}^{(i)}}^2 h + \cdots \right]$$

6.5.2 Unscented Transformation

$$\bar{y}_u = h(\bar{x}) + \frac{1}{2n}\sum_{i=1}^{2n}\left[D_{\tilde{x}^{(i)}}h + \frac{1}{2!}D^2_{\tilde{x}^{(i)}}h + \cdots\right] \Rightarrow \bar{y}_u = h(\bar{x}) + \underline{\frac{1}{2n}\sum_{i=1}^{2n}\frac{1}{2!}D^2_{\tilde{x}^{(i)}}h} + \frac{1}{2n}\sum_{i=1}^{2n}\left[\frac{1}{4!}D^4_{\tilde{x}^{(i)}}h + \frac{1}{6!}D^6_{\tilde{x}^{(i)}}h + \cdots\right]$$

$$\sum_{j=1}^{2n}D^{2k+1}_{\tilde{x}^{(j)}}h = \sum_{j=1}^{2n}\left[\left(\sum_{i=1}^{n}\tilde{x}_i^{(j)}\frac{\partial}{\partial x_i}\right)^{2k+1}h(x)\Big|_{x=\bar{x}}\right]$$

$$= \sum_{j=1}^{2n}\left[\sum_{i=1}^{n}\left(\tilde{x}_i^{(j)}\right)^{2k+1}\frac{\partial^{2k+1}}{\partial x_i^{2k+1}}h(x)\Big|_{x=\bar{x}}\right]$$

$$= \sum_{i=1}^{n}\left[\sum_{j=1}^{2n}\left(\tilde{x}_i^{(j)}\right)^{2k+1}\frac{\partial^{2k+1}}{\partial x_i^{2k+1}}h(x)\Big|_{x=\bar{x}}\right]$$

=0

针对样本数 针对X维数

向量

$$\tilde{x}^{(k)} = \left(\sqrt{nP}\right)_k^T$$

$$\tilde{x}_i^{(k)} = \tilde{x}^{(k)}\text{第}i\text{行}$$

标量

$$\underline{\frac{1}{2n}\sum_{i=1}^{2n}\frac{1}{2!}D^2_{\tilde{x}^{(i)}}h} = \frac{1}{2n}\sum_{k=1}^{2n}\frac{1}{2!}\left(\sum_{i=1}^{n}\tilde{x}_i^{(k)}\frac{\partial}{\partial x_i}\right)^2 h(x)\Big|_{x=\bar{x}}$$

$$= \frac{1}{4n}\sum_{k=1}^{2n}\sum_{i,j=1}^{n}\tilde{x}_i^{(k)}\tilde{x}_j^{(k)}\frac{\partial^2}{\partial x_i\partial x_j}h(x)\Big|_{x=\bar{x}}$$

$$= \frac{1}{4n}\sum_{i,j=1}^{n}\sum_{k=1}^{2n}\tilde{x}_i^{(k)}\tilde{x}_j^{(k)}\frac{\partial^2}{\partial x_i\partial x_j}h(x)\Big|_{x=\bar{x}}$$

$$= \frac{1}{2n}\sum_{i,j=1}^{n}\sum_{k=1}^{n}\tilde{x}_i^{(k)}\tilde{x}_j^{(k)}\frac{\partial^2}{\partial x_i\partial x_j}h(x)\Big|_{x=\bar{x}}$$

$$= \frac{1}{2n}\sum_{i,j=1}^{n}\sum_{k=1}^{n}\underbrace{\left(\sqrt{nP}\right)_{ki}}_{\text{标量}}\left(\sqrt{nP}\right)_{kj}\frac{\partial^2}{\partial x_i\partial x_j}h(x)\Big|_{x=\bar{x}}$$

$$= \frac{1}{2n}\sum_{i,j=1}^{n}nP_{ij}\frac{\partial^2}{\partial x_i\partial x_j}h(x)\Big|_{x=\bar{x}}$$

$$\boxed{\bar{y}_u = h(\bar{x}) + \frac{1}{2}\sum_{i,j=1}^{n}P_{ij}\frac{\partial^2 h}{\partial x_i\partial x_j}\Big|_{x=\bar{x}} + \frac{1}{2n}\sum_{i=1}^{2n}\left[\frac{1}{4!}D^4_{\tilde{x}^{(i)}}h + \frac{1}{6!}D^6_{\tilde{x}^{(i)}}h + \cdots\right]}$$

6.5.2 Unscented Transformation

$$\bar{y}=h(\bar{x})+\frac{1}{2!}E\left[D_{\tilde{x}}^{2}h\right]+\frac{1}{4!}E\left[D_{\tilde{x}}^{4}h\right]+\cdots$$

理想期望值

$$\frac{1}{2!}E\left[D_{\tilde{x}}^{2}h\right]=\frac{1}{2!}E\left[\left.\left(\sum_{i=1}^{n}\tilde{x}_i\frac{\partial}{\partial x_i}\right)^{2}h(x)\right|_{x=\bar{x}}\right]$$

$$=\frac{1}{2!}E\left[\left.\left(\sum_{i,j=1}^{n}\tilde{x}_i\tilde{x}_j\frac{\partial^2 h}{\partial x_i\partial x_j}\right)\right|_{x=\bar{x}}\right]$$

$$=\frac{1}{2!}\sum_{i,j=1}^{n}E\left(\tilde{x}_i\tilde{x}_j\right)\left.\frac{\partial^2 h}{\partial x_i\partial x_j}\right|_{x=\bar{x}}$$

$$=\frac{1}{2}\sum_{i,j=1}^{n}P_{ij}\left.\frac{\partial^2}{\partial x_i\partial x_j}h(x)\right|_{x=\bar{x}}$$

无穷对称样本

$$\bar{y}=h(\bar{x})+\frac{1}{2}\sum_{i,j=1}^{n}P_{ij}\left.\frac{\partial^2 h}{\partial x_i\partial x_j}\right|_{x=\bar{x}}+\frac{1}{4!}D_{\tilde{x}^{(i)}}^{4}h+\frac{1}{6!}D_{\tilde{x}^{(i)}}^{6}h+\cdots$$

$$\bar{y}_u=h(\bar{x})+\frac{1}{2}\sum_{i,j=1}^{n}P_{i,j}\left.\frac{\partial^2 h}{\partial x_i\partial x_j}\right|_{x=\bar{x}}+\frac{1}{2n}\sum_{i=1}^{2n}\left[\frac{1}{4!}D_{\tilde{x}^{(i)}}^{4}h+\frac{1}{6!}D_{\tilde{x}^{(i)}}^{6}h+\cdots\right]$$

因此 $\bar{y}_u$ 精度可以达到3阶

重要样本的算术平均与二阶项的数学期望相同

3) y方差的计算

$$y^{(i)} = h(x^{(i)})$$

$$\boxed{P_u = \sum_{i=1}^{2n} W^{(i)}(y^{(i)} - \bar{y}_u)(y^{(i)} - \bar{y}_u)^T = \frac{1}{2\mathrm{n}}\sum_{i=1}^{2n}(y^{(i)} - \bar{y}_u)(y^{(i)} - \bar{y}_u)^T}$$

$$\bar{y}_u = h(\bar{x}) + \frac{1}{2n}\sum_{i=1}^{2n}\frac{1}{2!}D^2_{\tilde{x}^{(i)}}h + \frac{1}{2n}\sum_{i=1}^{2n}\left[\frac{1}{4!}D^4_{\tilde{x}^{(i)}}h + \frac{1}{6!}D^6_{\tilde{x}^{(i)}}h + \cdots\right]$$

$$y^{(i)} = h\left(x^{(i)}\right) = h\left(\bar{x}\right) + D_{\tilde{x}^{(i)}}h + \frac{1}{2!}D^2_{\tilde{x}^{(i)}}h + \frac{1}{3!}D^3_{\tilde{x}^{(i)}}h + \cdots$$

$$P_u = \frac{1}{2n}\sum_{i=1}^{2n}\left[\cancel{h(\bar{x})} + D_{\tilde{x}^{(i)}}h + \frac{1}{2!}D^2_{\tilde{x}^{(i)}}h + \cdots - \cancel{h(\bar{x})} - \frac{1}{2n}\sum_{j=1}^{2n}\left(\frac{1}{2!}D^2_{\tilde{x}^{(j)}}h + \frac{1}{4!}D^4_{\tilde{x}^{(j)}}h + \cdots\right)\right]\bullet$$

$$\left[\cancel{h(\bar{x})} + D_{\tilde{x}^{(i)}}h + \frac{1}{2!}D^2_{\tilde{x}^{(i)}}h + \cdots - \cancel{h(\bar{x})} - \frac{1}{2n}\sum_{j=1}^{2n}\left(\frac{1}{2!}D^2_{\tilde{x}^{(j)}}h + \frac{1}{4!}D^4_{\tilde{x}^{(j)}}h + \cdots\right)\right]^T$$

$$P_u = \frac{1}{2n}\sum_{i=1}^{2n}\left[D_{\tilde{x}^{(i)}}h + \frac{1}{2!}D^2_{\tilde{x}^{(i)}}h + \cdots - \frac{1}{2n}\sum_{j=1}^{2n}\left(\frac{1}{2!}D^2_{\tilde{x}^{(j)}}h + \frac{1}{4!}D^4_{\tilde{x}^{(j)}}h + \cdots\right)\right]\bullet$$

$$\left[D_{\tilde{x}^{(i)}}h + \frac{1}{2!}D^2_{\tilde{x}^{(i)}}h + \cdots - \frac{1}{2n}\sum_{j=1}^{2n}\left(\frac{1}{2!}D^2_{\tilde{x}^{(j)}}h + \frac{1}{4!}D^4_{\tilde{x}^{(j)}}h + \cdots\right)\right]^T$$

6.5.2 Unscented Transformation

$$P_u=\frac{1}{2n}\sum_{i=1}^{2n}\left[D_{\tilde{x}^{(i)}}h+\frac{1}{2!}D_{\tilde{x}^{(i)}}^2h+\cdots-\frac{1}{2n}\sum_{j=1}^{2n}\left(\frac{1}{2!}D_{\tilde{x}^{(j)}}^2h+\frac{1}{4!}D_{\tilde{x}^{(j)}}^4h+\cdots\right)\right]\bullet$$

$$\left[D_{\tilde{x}^{(i)}}h+\frac{1}{2!}D_{\tilde{x}^{(i)}}^2h+\cdots-\frac{1}{2n}\sum_{j=1}^{2n}\left(\frac{1}{2!}D_{\tilde{x}^{(j)}}^2h+\frac{1}{4!}D_{\tilde{x}^{(j)}}^4h+\cdots\right)\right]^T$$

$$P_u=\frac{1}{2n}\sum_{i=1}^{2n}\left[D_{\tilde{x}^{(i)}}h\left(D_{\tilde{x}^{(i)}}h\right)^{\mathrm{T}}+\underbrace{\frac{1}{2}D_{\tilde{x}^{(i)}}h\left(D_{\tilde{x}^{(i)}}^2h\right)^{\mathrm{T}}}_{0}+\underbrace{\frac{1}{2}D_{\tilde{x}^{(i)}}^2h\left(D_{\tilde{x}^{(i)}}h\right)^{\mathrm{T}}}_{0}+\alpha D_{\tilde{x}^{(i)}}^2h\left(D_{\tilde{x}^{(j)}}^2h\right)^{\mathrm{T}}+\cdots\right]$$

$$P_u=\frac{1}{2n}\sum_{i=1}^{2n}\left[D_{\tilde{x}^{(i)}}h\left(D_{\tilde{x}^{(i)}}h\right)^T\right]+\underline{HOT}$$

4阶以上项

$$\begin{cases}P_u\approx\dfrac{1}{2n}\displaystyle\sum_{i=1}^{2n}\sum_{j,k=1}^{n}\left(\tilde{x}_j^{(i)}\dfrac{\partial h}{\partial x_j}\bigg|_{\bar{x}}\right)\left(\tilde{x}_k^{(i)}\dfrac{\partial h}{\partial x_j}\bigg|_{\bar{x}}\right)^T\\ \tilde{x}_j^{(i)}=-\tilde{x}_j^{(i+n)}=\left(\sqrt{nP}\right)_i^T \text{的第j个元素}\end{cases}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned}P_u&=\frac{1}{n}\sum_{i=1}^{n}\sum_{j,k=1}^{n}nP_{jk}\left(\frac{\partial h}{\partial x_j}\bigg|_{\bar{x}}\right)\left(\frac{\partial h}{\partial x_j}\bigg|_{\bar{x}}\right)^T\\&=HPH^T\end{aligned}}$$

精确到3阶

6.5.3 无迹滤波(UKF)

Unscented Kalman Filter

1) EKF的不足

$$\begin{cases} X(k+1)=\varphi[X(k),k]+\Gamma[X(k),k]W(k) \\ Z(k+1)=h[X(k+1),k+1]+V(k+1) \end{cases}$$

EKF

$$\begin{cases} K(k+1)=P(k+1|k)H^T(k+1)\left[H(k+1)P(k+1|k)\cdot H^T(k+1)+R_{k+1}\right]^{-1} \\ P(k+1|k)=\Phi(k+1,k)P(k|k)\Phi^T(k+1,k)+\Gamma[\hat{X}(k|k),k]Q_k\Gamma^T[\hat{X}(k|k),k] \\ P(k+1|k+1)=[I-K(k+1)H(k+1)]P(k+1|k) \end{cases}$$

$$\phi(k+1,k)=\frac{\partial\varphi}{\partial X(k)}\bigg|_{X(k)=\hat{X}(k)} \qquad H(k)=\frac{\partial h}{\partial X(k)}\bigg|_{X(k)=\hat{X}(k|k-1)}$$

不准确的系统模型

$K(k),P(k+1|k),P(k+1|k+1)$ 不准确，滤波结果难以达到最优。

6.5.3 无迹滤波(UKF)

2) Kalman 滤波的分析

$$\hat{X} = m_x + COV(X,Z)(VarZ)^{-1}(Z - m_Z)$$

$$\begin{cases} X(k+1) = \varphi[X(k),k] + \Gamma[X(k),k]W(k) \\ Z(k+1) = h[X(k+1),k+1] + V(k+1) \end{cases}$$

$$K(k+1) = P(k+1|k)H^T(k+1)\left[H(k+1)P(k+1|k)\cdot H^T(k+1) + R_{k+1}\right]^{-1}$$

$$\begin{aligned} P(k+1|k)H^T(k+1) &= E[\tilde{X}(k+1|k)\tilde{X}^T(k+1|k)]H^T(k+1) \\ &= E[\tilde{X}(k+1|k)\tilde{X}^T(k+1|k)H^T(k+1)] \\ &= P_{xy}(k+1|k) \end{aligned}$$

如以**y**表示观测

$$\begin{aligned} &H(k+1)P(k+1|k)\cdot H^T(k+1) + R \\ &= E[H(k+1)\tilde{X}(k+1|k)\tilde{X}^T(k+1|k)H^T(k+1)] + R \\ &= E[\tilde{Z}(k+1|k)\tilde{Z}^T(k+1|k)] \\ &= P_{yy}(k+1|k) \end{aligned}$$

$$\boxed{K(k+1) = P_{xy}(k+1|k)P^{-1}{}_{yy}(k+1|k)}$$

6.5.3 无迹滤波(UKF)

Kalman滤波最优增益K的获得依赖于$P(k|k)$、$P(k|k-1)$。在非线性系统中$P(k|k)$、$P(k|k-1)$ 的解算存在较大误差，所以估计结果会有一定的偏差。

如果 $P_{xy}(k+1|k)$，$P_{yy}(k+1|k)$能够通过其他途径较为精确地获得，估计结果将更准确.

3) 获得 $P_{xy}(k+1|k)$，$P_{yy}(k+1|k)$ 的思想

$$P_{xy}(k)=E\left\{[X(k)-E(X(k))][Y(k)-E(Y(k))]^T\right\}$$

$$P_{yy}(k)=E\left\{[Y(k)-E(Y(k))][Y(k)-E(Y(k))]^T\right\}$$

对于每一步k, 如果我们有足够的样本$X(k)$以及相应的$Y(k)$, $P_{xy}(k)$与$P_{yy}(k)$ 就能够较为准确地获得。

6.5.3 无迹滤波(UKF)

4) 如何获得 $P_{xy}(k+1|k)$，$P_{yy}(k+1|k)$

关键问题:

如何进行 X 的有效采样

如何在X的基础上获得Y的统计信息。

方法

Unscented Transformation (*UT*) 可以获得非线性变量间的统计特性，*UT* 可以提供以下两种重要的内容:

X的有效采样方法

对$y=f(x)$的统计指标估计

6.5.3 无迹滤波(UKF)

5) Unscented Transform——sigma采样

采样算法

UT变换有多种采样算法，在此仅介绍优化后的对称采样。

如何采样

矩阵平方根的第i列

Sigma point

$$
\begin{cases}
X_0 = \bar{X} \\
X_i = \bar{X} + \left(\sqrt{(L+\lambda)P_x}\right)_i & i = 1, \cdots L \\
X_i = \bar{X} - \left(\sqrt{(L+\lambda)P_x}\right)_{i-L} & i = L+1, \cdots, 2L
\end{cases}
$$

L为状态X的维数

其中

L=状态的维数

$\lambda = \alpha^2 (L+\kappa) - L$

κ usually set to 0 or $3-L$

$1\text{e}-4 \le \alpha \le 1$

α决定了sigma点在$\bar{X}$周围的分布，*k*是相对次要的分布控制参数。

北京航空航天大學 BEIHANG UNIVERSITY

6.5.3 无迹滤波(UKF)

6) Unscented Transform——获得 *Y* 的统计特性

$$y_i = f(x_i) \qquad i = 0, \cdots, 2L.$$

$$\bar{y} = \sum_{i=0}^{2L} \mathrm{W}_{\mathrm{i}}^{(\mathrm{m})} y_{\mathrm{i}} \qquad i = 0, 1, \cdots, L.$$

$$P_y \approx \sum_{i=0}^{2L} W_i^{(c)} (y_i - \bar{y})(y_i - \bar{y})^T$$

其中

$$W_0^{(m)} = \lambda / (L + \lambda)$$

$$W_0^{(c)} = \lambda / (L + \lambda) + (1 - \alpha^2 + \beta)$$

$$W_i^{(m)} = W_i^{(c)} = 1 / 2(L + \lambda) \qquad i = 1, \cdots 2L$$

$$\lambda = \alpha^2 (L + \kappa) - L$$

$10^{-4} \le \alpha \le 1$， κ=3-L

β与x的验前分布知识有关，正态分布最优值为2。

7） 优化采样方案的理解

$$W_0^{(m)} = \lambda / (L+\lambda)$$

$$W_0^{(c)} = \lambda / (L+\lambda) + (1-\alpha^2+\beta)$$

$$W_i^{(m)} = W_i^{(c)} = 1/2(L+\lambda) \qquad i = 1,\cdots 2L$$

$$\lambda = \alpha^2(L+K) - L$$

$$1e-4 \le \alpha \le 1$$

$$W_0^{(m)} = \lambda / (L+\lambda)$$

$$\lambda = \alpha^2(L+\kappa) - L$$

$$\kappa = 3 - L$$

$$W_i^{(m)} = 1/2(L+\lambda)$$

$$W_0^{(m)} = \frac{\lambda}{3\alpha^2} = \frac{3\alpha^2 - L}{3\alpha^2}$$

$$(L+\lambda) = 3\alpha^2$$

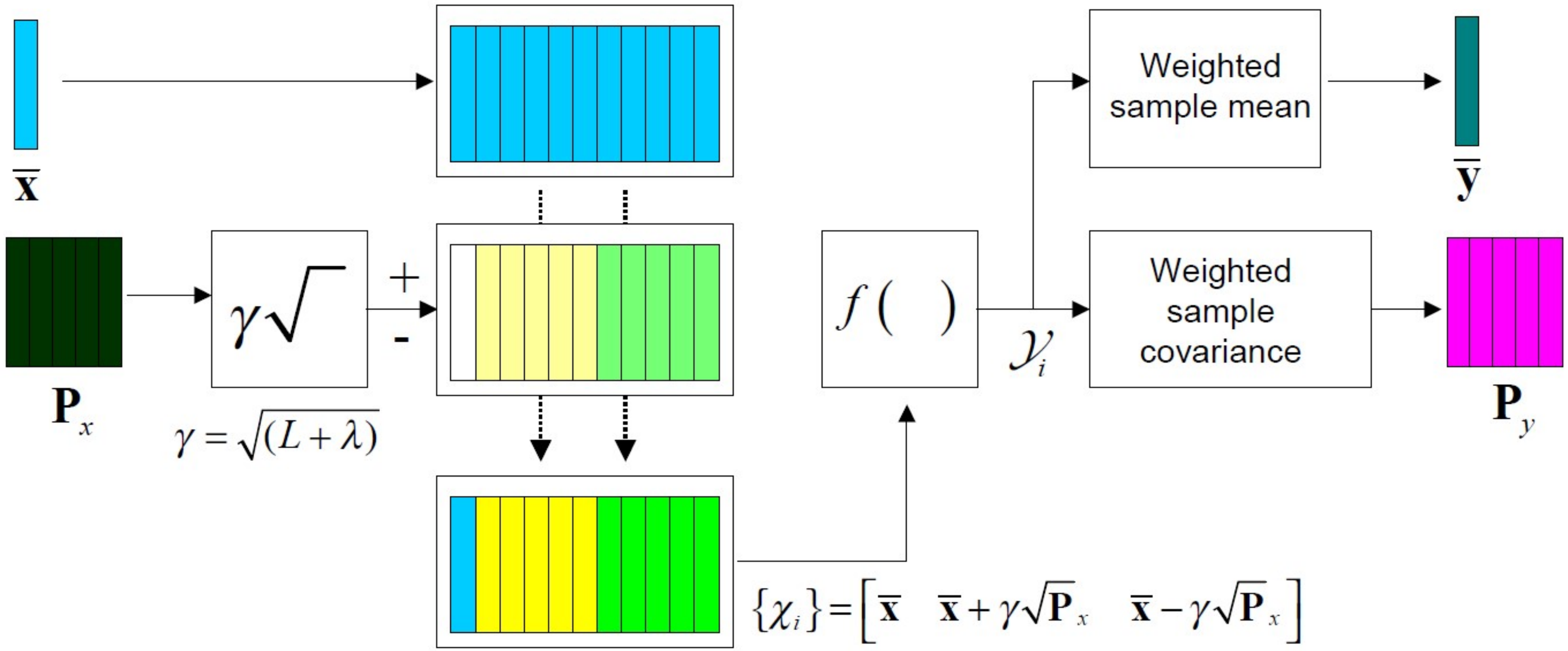
$$W_i^{(m)} = \frac{1}{2} \bullet \frac{1}{3\alpha^2}$$

$$\lambda = 3\alpha^2 - L$$

$$W_0^{(m)} + 2LW_i^{(m)} = 1$$

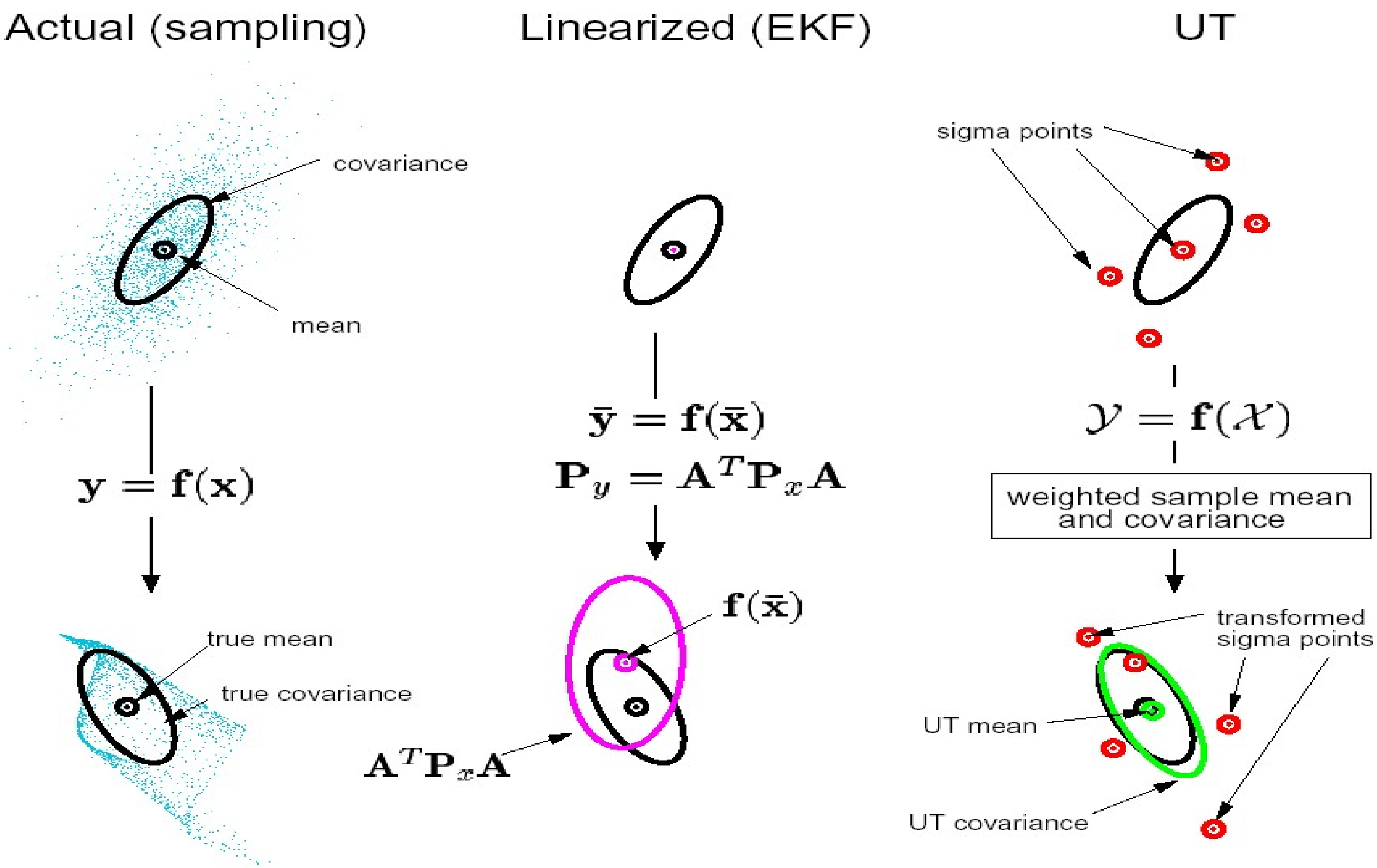
6.5.3 无迹滤波(UKF)

5) Unscented Transform



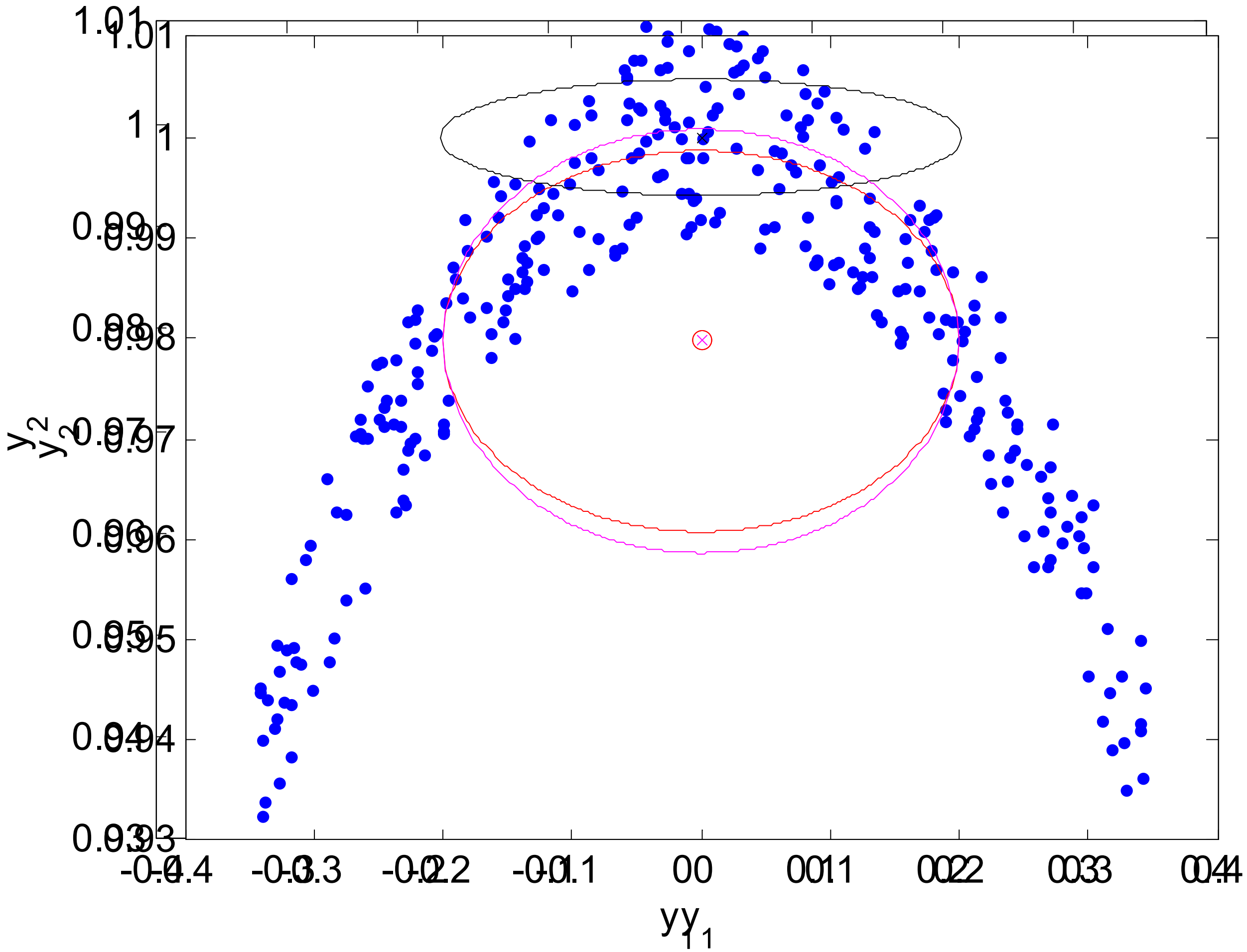
该图片引自Eric A Wan and Rudolph wan der Merwe The Unscented Kalman Filter

6.5.3 无迹滤波(UKF)



The **EKF** only calculates the posterior mean and covariance accurately to the first order with all higher order moments truncated; however, **UT** calculates the mean and covariance to the second order.

该图片引自Eric A Wan and Rudolph wan der Merwe The Unscented Kalman Filter


1.01
1
0.99
0.98
0.97
0.96
0.95
0.94
0.93
y2
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
y1

6.5.3 无迹滤波(UKF)

7）非加性噪声UKF方法

① 状态方程及初始条件

对于加性噪声可以不采用扩维的方式。

$$X_k^a = \begin{bmatrix} X_k^T & W_k^T & V_k^T \end{bmatrix}^T \quad P_k^a = \begin{bmatrix} P_k & 0 & 0 \\ 0 & Q_k & 0 \\ 0 & 0 & R_k \end{bmatrix}$$

扩维

$$\bar{X}_0 = E\left[X_0\right] \quad P_0 = E\left[\left(X_0 - \bar{X}_0\right)\left(X_0 - \bar{X}_0\right)^T\right]$$

$$\bar{X}_0^a = E\left[X^a\right] = \begin{bmatrix} \bar{X}_0^T & 0 & 0 \end{bmatrix}^T$$

$$P_0^a = E\left[\left(X_0^a - \bar{X}_0^a\right)\left(X_0^a - \bar{X}_0^a\right)^T\right] = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & Q_0 & 0 \\ 0 & 0 & R_0 \end{bmatrix}$$

② sigma point selecting

$$X_{k-1}^a = \begin{bmatrix} \bar{X}_{k-1}^a & \bar{X}_{k-1}^a \pm \sqrt{(L+\lambda)P_{k-1}^a} \end{bmatrix}$$

6.5.3 无迹滤波(UKF)

③ 时间更新

$$X_{k|k-1}^{x}=f\left(X_{k-1}^{x},X_{k-1}^{w}\right)$$

$$\bar{X}_{k|k-1}=\sum_{i=0}^{2L}W_i^{(m)}X_{i,k|k-1}^{x}$$

$$P_{k|k-1}=\sum_{i=0}^{2L}W_i^{(c)}\left[X_{i,k|k-1}^{x}-\bar{X}_{k|k-1}\right]\left[X_{i,k|k-1}^{x}-\bar{X}_{k|k-1}\right]^{T}$$

$$Y_{k|k-1}=h\left(X_{k|k-1}^{x},X_{k-1}^{v}\right)$$

$$\bar{y}_{k|k-1}=\sum_{i=0}^{2L}W_i^{(m)}Y_{i,k|k-1}$$

6.5.3 无迹滤波(UKF)

④ 量测更新

$$P_{yy}=\sum_{i=0}^{2L}W_i^{(c)}\left[Y_{i,k|k-1}-\bar{y}_{k|k-1}\right]\left[Y_{i,k|k-1}-\bar{y}_{k|k-1}\right]^T$$

$$P_{xy}=\sum_{i=0}^{2L}W_i^{(c)}\left[X_{i,k|k-1}-\bar{X}_{k|k-1}\right]\left[Y_{i,k|k-1}-\bar{y}_{k|k-1}\right]^T$$

$$K_k=P_{xy}P_{yy}^{-1}$$

$$\bar{X}_k=\bar{X}_{k|k-1}+K_k(y_k-\bar{y}_{k|k-1})$$

$$P_k=P_{k|k-1}-K_kP_{yy}K_k^T$$

无R

6.5.3 无迹滤波(UKF)

8）加性噪声UKF方法

$$X_{k+1} = f\left(X_k, \mathrm{U}_k\right) + \mathrm{W}_k$$

$$Y_k = h\left(X_k\right) + \mathrm{V}_k$$

$$X_{k-1} = \left[\hat{X}_{k-1} \quad \hat{X}_{k-1} \pm \sqrt{(L+\lambda) P_{k-1}}\right]$$

$$X_{k|k-1} = f\left(X_{k-1}, \mathrm{U}_{k-1}\right) \longrightarrow \bar{X}_{k|k-1} = \sum_{i=0}^{2L} W_i^{(m)} X_{i,k|k-1}$$

$$P_{k|k-1} = \sum_{i=0}^{2L} W_i^{(c)} \left[X_{i,k|k-1} - \bar{X}_{k|k-1}\right]\left[X_{i,k|k-1} - \bar{X}_{k|k-1}\right]^T + Q_{k\text{-}1}$$

有Q

$$Y_{k|k-1} = h\left(X_{k|k-1}\right) \longrightarrow \bar{y}_{k|k-1} = \sum_{i=0}^{2L} W_i^{(m)} Y_{i,k|k-1}$$

8）加性噪声UKF方法

$$P_{yy}=\sum_{i=0}^{2L}W_i^{(c)}\left[Y_{i,k|k-1}-\bar{y}_{k|k-1}\right]\left[Y_{i,k|k-1}-\bar{y}_{k|k-1}\right]^T+R_k$$

$$P_{xy}=\sum_{i=0}^{2L}W_i^{(c)}\left[X_{i,k|k-1}-\bar{X}_{k|k-1}\right]\left[Y_{i,k|k-1}-\bar{y}_{k|k-1}\right]^T$$

$$K_k=P_{xy}P_{yy}^{-1}$$

$$\bar{X}_k=\bar{X}_{k|k-1}+K_k(y_k-\bar{y}_{k|k-1})$$

$$P_k=P_{k|k-1}-K_kP_{yy}K_k^T$$

有R

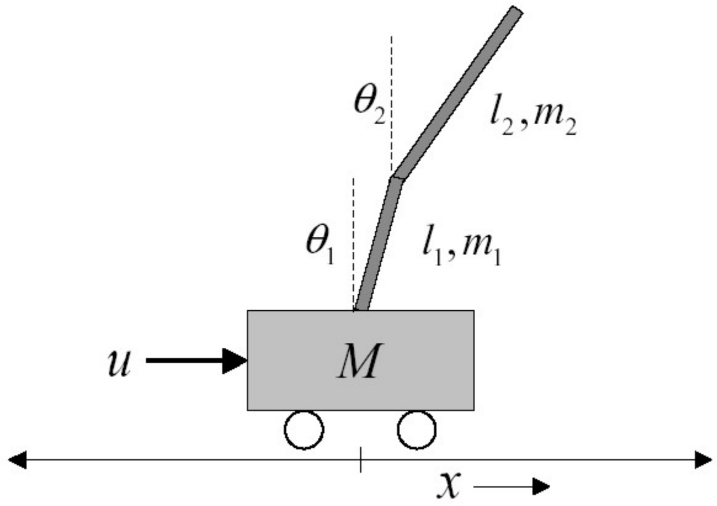
6.5.3 无迹滤波(UKF)

9） UKF的优点

UKF的优点

- 对非线性函数的概率密度分布进行近似，而不是对非线性函数进行近似，不需要知道非线性函数的显式表达式;
- 非线性函数统计量的精度至少达到*3*阶，对于采用特殊的采样策略，如高斯分布*4*阶采样和偏度采样等可达到更高阶精度;
- 计算量与*EKF*同阶;
- 不需要求导计算*Jacobian*矩阵,可以处理非可导的非线性函数

例子



$$\mathbf{x} \doteq [x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2].$$

水平方向力平衡方程

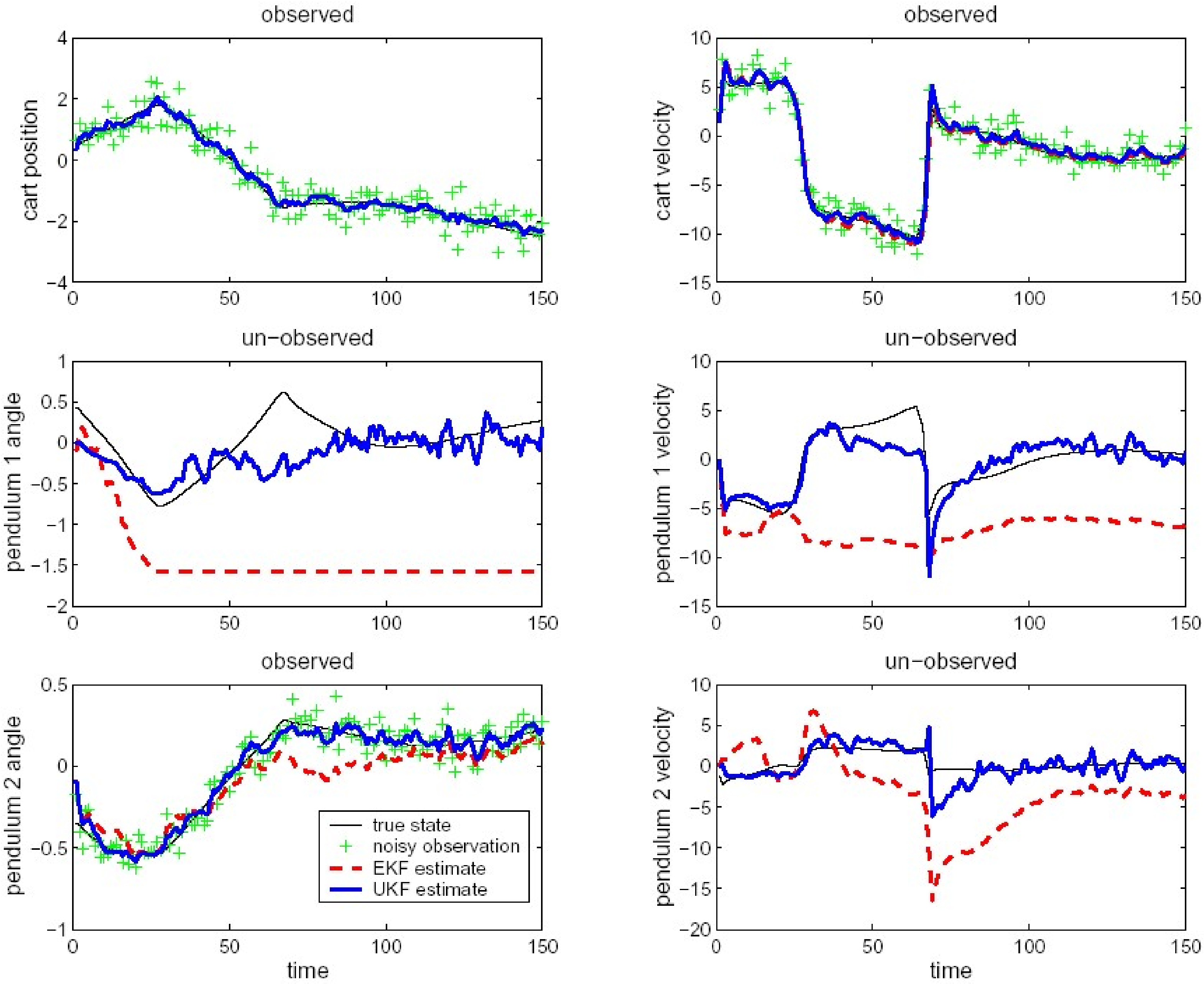
$$(M+m_1+m_2)\ddot{x} - (m_1+2m_2)l_1\ddot{\theta}_1\cos\theta_1 - m_2l_2\ddot{\theta}_2\cos\theta_2 = u + (m_1+2m_2)l_1\dot{\theta}_1^{\,2}\sin\theta_1 + m_2l_2\dot{\theta}_2^{\,2}\sin\theta_2$$

L1力矩平衡方程

$$-(m_1+2m_2)l_1\ddot{x}\cos\theta_1 + 4\left(\frac{m_1}{3}+m_2\right)l_1^{\,2}\ddot{\theta}_1 + 2m_2l_1l_2\ddot{\theta}_2\cos(\theta_2-\theta_1) = (m1+2m_2)gl_1\sin\theta_1 + 2m_2l_1l_2\dot{\theta}_2^{\,2}\sin(\theta_2-\theta_1)$$

L2力矩平衡方程

$$-m_2\ddot{x}l_2\cos\theta_2 + 2m_2l_1l_2\ddot{\theta}_1\cos(\theta_2-\theta_1) + \frac{4}{3}m_2l_2^{\,2}\ddot{\theta}_2 = m_2gl_2\sin\theta_2 - 2m_2l_1l_2\dot{\theta}_1^{\,2}\sin(\theta_2-\theta_1)$$


observed
cart position
observed
cart velocity
un−observed
pendulum 1 angle
un−observed
pendulum 1 velocity
observed
pendulum 2 angle
un−observed
pendulum 2 velocity
time
time
true state
noisy observation
EKF estimate
UKF estimate

相关内容请参阅

Optimal State Estimation

标称轨道及EKF	**13.1~13.2**
Unscented Kalman Filtering	**14**

Optimal Estimation of Dynamic Systems

EKF	**3.6**
Unscented Kalman Filtering	**3.7**

[1] **S. Julier**, J. Uhlmann, and H. Durrant-Whyte, “A new approach for filtering nonlinear systems,” American Control Conference, pp. 1628-1632 (1995).

[2] **S. Julier**, J. Uhlmann, and H. Durrant-Whyte, “A new method for the nonlin-lear transformation of means and covariances in filters and estimators,” IEEE Transactions on Automatic Control, 45(3), pp. 477-482 (March 2000).

[3] **S. Julier** and J. Uhlmann, “Reduced sigma point filters for the propagation of means and covariances through nonlinear transformations,” American Control Conference, pp. 887-892, 2002.

[4] **S. Julier**, “The spherical simplex unscented transformation,” American Control Conference, pp. 2430-2434, 2003.

[5] **S. Julier** and J. Uhlmann, “Unscented filtering and nonlinear estimation,” Proceedings of the IEEE, 92(3), pp. 401-422 (March 2004).

[6] T. Lefebvre, H. Bruyninckx, and J. De Schuller. Comment on ‘A new method for the nonlinear transformation of means and covariances in filters and estimators’. IEEE Trans. Autom. Control, 2002, 47(8): 1406-1409.