现代控制理论

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2.1 反步设计方法

一阶系统控制

$$\dot{x} = u + \varphi(x)$$
$$y = x$$

目的：设计u，使得$y \to y_d$

定义$z = y - y_d$，则

$$\dot{z} = \dot{y} - \dot{y}_d = u + \varphi(x) - \dot{y}_d$$

构造准Lyapunov函数：$V = \frac{1}{2}z^2$

$$\dot{V} = z\dot{z} = z[u + \varphi(x) - \dot{y}_d]$$

设计：$u = -cz - \varphi(x) + \dot{y}_d$，则 $\dot{V} = -cz^2 = -2cV$

可得：V以指数形式收敛到0，$z = y - y_d$以指数形式收敛到0。

二阶系统控制

$$\dot{x}_1 = x_2 + \varphi_1(x_1)$$
$$\dot{x}_2 = u + \varphi_2(x_1, x_2)$$
$$y = x_1$$

第1步

定义：$z_1 = y - y_d$

$\dot{z}_1 = \dot{y} - \dot{y}_d = x_2 + \varphi_1(x_1) - \dot{y}_d$

构造第1个准Lyapunov函数：$V_1 = \frac{1}{2} z_1^2$

$\dot{V}_1 = z_1[x_2 + \varphi_1(x_1) - \dot{y}_d] = z_1[x_2 - \alpha_1 + \alpha_1 + \varphi_1(x_1) - \dot{y}_d]$

α_1：镇定函数

2.1 反步设计方法

设计：$\alpha_1 = -c_1 z_1 - \varphi_1(x_1) + \dot{y}_d = \alpha_1(x_1, y_d, \dot{y}_d)$

定义：$z_2 = x_2 - \alpha_1$

$\dot{V}_1 = z_1 z_2 - c_1 z_1^2$

第2步

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = u + \varphi_2(x_1, x_2) - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_1}{\partial \dot{y}_d}\ddot{y}_d$$

构造第2个准Lyapunov函数： $V_2 = V_1 + \frac{1}{2} z_2^2$

$$\dot{V}_2 = \dot{V}_1 + z_2 \dot{z}_2$$

$$= z_1 z_2 - c_1 z_1^2 + z_2\left[u + \varphi_2(x_1, x_2) - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_1}{\partial \dot{y}_d}\ddot{y}_d\right]$$

2.1 反步设计方法

设计：$u=-c_2z_2-\varphi_2(x_1,x_2)+\frac{\partial\alpha_1}{\partial x_1}\dot{x}_1+\frac{\partial\alpha_1}{\partial y_d}\dot{y}_d+\frac{\partial\alpha_1}{\partial\dot{y}_d}\ddot{y}_d-z_1$

$\dot{V}_2=-c_1z_1^2-c_2z_2^2$

令$c=\min\{c_1,c_2\}$

$\dot{V}_2\le-2cV_2$

V_2将指数衰减到0，z_1、z_2将指数衰减到0。

三阶系统控制

$$\dot{x}_1 = x_2 + \varphi_1(x_1)$$
$$\dot{x}_2 = x_3 + \varphi_2(x_1, x_2)$$
$$\dot{x}_3 = u + \varphi_3(x_1, x_2, x_3)$$
$$y = x_1$$

第1步

$$z_1 = y - y_d$$
$$\dot{z}_1 = \dot{x}_1 - \dot{y}_d = x_2 + \varphi_1(x_1) - \dot{y}_d$$
$$V_1 = \frac{1}{2} z_1^2$$

2.1 反步设计方法

$$\dot{V}_1 = z_1(x_2 - \alpha_1 + \alpha_1 + \varphi_1(x_1) - \dot{y}_d)$$

设计第1个镇定函数：$\alpha_1 = -c_1 z_1 - \varphi_1(x_1) + \dot{y}_d$。则

$$\dot{V}_1 = z_1(x_2 - \alpha_1) - c_1 z_1^2$$

第2步

$$z_2 = x_2 - \alpha_1 = x_2 - \alpha_1(x_1, y_d, \dot{y}_d)$$

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = \dot{x}_2 - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_1}{\partial \dot{y}_d}\ddot{y}_d$$

2.1 反步设计方法

$$\dot{z}_2 = x_3 + \varphi_2(x_1, x_2) - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_1}{\partial \dot{y}_d}\ddot{y}_d$$

构造第2个准Lyapunov函数： $V_2 = V_1 + \frac{1}{2}z_2^2$

$$\dot{V}_2 = z_1 z_2 - c_1 z_1^2 + z_2[(x_3 - \alpha_2) + \alpha_2 + \varphi_2(x_1, x_2)$$

$$- \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_1}{\partial \dot{y}_d}\ddot{y}_d]$$

设计第2个镇定函数:

$$\alpha_2 = -c_2 z_2 - \varphi_2(x_1, x_2) + \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 + \frac{\partial \alpha_1}{\partial y_d}\dot{y}_d + \frac{\partial \alpha_1}{\partial \dot{y}_d}\ddot{y}_d - z_1$$

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2(x_3 - \alpha_2)$$

第3步

$$z_3 = x_3 - \alpha_2 = x_3 - \alpha_2(x_1, x_2, y_d, \dot{y}_d, \ddot{y}_d)$$

$$\dot{z}_3 = \dot{x}_3 - \dot{\alpha}_2$$

$$= \dot{x}_3 - \frac{\partial \alpha_2}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_2}{\partial x_2}\dot{x}_2 - \frac{\partial \alpha_2}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_2}{\partial \dot{y}_d}\ddot{y}_d - \frac{\partial \alpha_2}{\partial \ddot{y}_d}\dddot{y}_d$$

$$= u + \varphi_3(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_2}{\partial x_2}\dot{x}_2 - \frac{\partial \alpha_2}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_2}{\partial \dot{y}_d}\ddot{y}_d$$

$$- \frac{\partial \alpha_2}{\partial \ddot{y}_d}\dddot{y}_d$$

2.1 反步设计方法

构造第3个准Lyapunov函数： $V_3 = V_2 + \frac{1}{2} z_3^2$

$$\dot{V}_3 = z_3 z_2 - c_1 z_1^2 - c_2 z_2^2 + z_3 [u + \varphi_3(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_2}{\partial x_2}\dot{x}_2 - \frac{\partial \alpha_2}{\partial y_d}\dot{y}_d - \frac{\partial \alpha_2}{\partial \dot{y}_d}\ddot{y}_d - \frac{\partial \alpha_2}{\partial \ddot{y}_d}\dddot{y}_d]$$

设计：

$$u = -c_3 z_3 - \varphi_3(x_1, x_2, x_3) + \frac{\partial \alpha_2}{\partial x_1}\dot{x}_1 + \frac{\partial \alpha_2}{\partial x_2}\dot{x}_2 + \frac{\partial \alpha_2}{\partial y_d}\dot{y}_d + \frac{\partial \alpha_2}{\partial \dot{y}_d}\ddot{y}_d + \frac{\partial \alpha_2}{\partial \ddot{y}_d}\dddot{y}_d - z_2$$

2.1 反步设计方法

则：$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2$

取$c = \min\{c_1, c_2, c_3\}$

则： $\dot{V}_3 \leq -2cV_3$

可得： V_3将指数衰减到0，z_1、z_2、z_3将指数衰减到0。

简化的反步控制—动态面控制

三阶系统控制

$$\dot{x}_1 = x_2 + \varphi_1(x_1)$$
$$\dot{x}_2 = x_3 + \varphi_2(x_1, x_2)$$
$$\dot{x}_3 = u + \varphi_3(x_1, x_2, x_3)$$

第1步

定义：$z_1 = y - y_d$，其导数

$$\dot{z}_1 = x_2 + \varphi_1(x_1) - \dot{y}_d$$

定义：$V_1 = \frac{1}{2} z_1^2$。微分V_1有

$$\dot{V}_1 = z_1(x_2 + \varphi_1 - \dot{y}_d)$$

选取

$$\alpha_1 = -c_1 z_1 - \varphi_1 + \dot{y}_d$$

2.1 反步设计方法

然后有

$$\dot{V}_1 = -c_1 z_1^2 + z_1(x_2 - \alpha_1)$$

令α_1通过时间常数为τ_1的一阶低通滤波器：

$$\tau_1 \dot{\varepsilon}_1 + \varepsilon_1 = \alpha_1$$

第2步

定义：$z_2 = x_2 - \varepsilon_1$，其导数满足

$$\dot{z}_2 = x_3 + \varphi_2 - \dot{\varepsilon}_1$$

定义：$V_2 = \frac{1}{2} z_2^2$。微分V_2有

$$\dot{V}_2 = z_2(x_3 + \varphi_2 - \dot{\varepsilon}_1)$$

选取

$$\alpha_2 = -c_2 z_2 - \varphi_2 + \dot{\varepsilon}_1$$

2.1 反步设计方法

然后有

$$\dot{V}_2 = -c_2 z_2^2 + z_2(x_3 - \alpha_2)$$

令α_2通过时间常数为τ_2的一阶低通滤波器：

$$\tau_2 \dot{\varepsilon}_2 + \varepsilon_2 = \alpha_2$$

第3步

定义：$z_3 = x_3 - \varepsilon_2$，其导数满足

$$\dot{z}_3 = u + \varphi_3 - \dot{\varepsilon}_2$$

定义：$V_3 = \frac{1}{2} z_3^2$。微分V_3有

$$\dot{V}_3 = z_3(u + \varphi_3 - \dot{\varepsilon}_2)$$

选取

$$u = -c_3 z_3 - \varphi_3 + \dot{\varepsilon}_2$$

代入得

$$\dot{V}_3 = -c_3 z_3^2$$

2.1 反步设计方法

定义滤波器误差： $Y_1=\varepsilon_1-\alpha_1$， $Y_2=\varepsilon_2-\alpha_2$

容易验证

$$\dot{\varepsilon}_1=-\frac{1}{\tau_1}Y_1,\dot{\varepsilon}_2=-\frac{1}{\tau_2}Y_2$$

$$x_2=(x_2-\varepsilon_1)+(\varepsilon_1-\alpha_1)+\alpha_1=z_2+Y_1+\alpha_1$$

$$x_3=(x_3-\varepsilon_2)+(\varepsilon_2-\alpha_2)+\alpha_2=z_3+Y_2+\alpha_2$$

微分Y_1得

$$\dot{Y}_1=\dot{\varepsilon}_1-\dot{\alpha}_1=-\frac{1}{\tau_1}Y_1-\dot{\alpha}_1$$

由于α_1是$(z_1,y_d,\dot{y}_d)$的光滑函数，故而

$$\dot{\alpha}_1=\frac{\partial\alpha_1}{\partial z_1}\dot{z}_1+\frac{\partial\alpha_1}{\partial y_d}\dot{y}_d+\frac{\partial\alpha_1}{\partial\dot{y}_d}\ddot{y}_d$$

2.1 反步设计方法

根据上式可以验证$\dot{\alpha}_1 = \eta_1(z_1, z_2, Y_1, y_d, \dot{y}_d, \ddot{y}_d)$，其中$\eta_1$是光滑函数，于是

$$\dot{Y}_1 = -\frac{1}{\tau_1} Y_1 - \eta_1$$

微分Y_2得

$$\dot{Y}_2 = -\frac{1}{\tau_2} Y_2 - \dot{\alpha}_2$$

由于α_2是$(z_1, z_2, Y_1, y_d, \dot{y}_d)$的光滑函数，可以验证

$\dot{\alpha}_2 = \eta_2(z_1, z_2, z_3, Y_1, Y_2, y_d, \dot{y}_d, \ddot{y}_d)$，其中$\eta_2$是光滑函数。

定义如下准Lyapunov函数：

$$V = \sum_{i=1}^{3} V_i + \frac{1}{2} \sum_{i=1}^{2} Y_i^2$$

2.1 反步设计方法

定理2.1：对于任意的常量b_1和b_2，若

$$y_d^2+\dot{y}_d^2+\ddot{y}_d^2\le b_1$$
$$V(0)\le b_2$$

则存在设计参数使得闭环系统内所有信号有界，且z_1可收敛至一任意小的残集内。

证明：微分V可得

$$\dot{V}=-\sum_{i=1}^{3}c_i z_i^2+z_1(x_2-\alpha_1)+z_2(x_3-\alpha_2)+\sum_{i=1}^{2}Y_i\,\dot{Y}_i$$
$$=-\sum_{i=1}^{3}c_i z_i^2+z_1(z_2+Y_1)+z_2(z_3+Y_2)$$
$$+\sum_{i=1}^{2}Y_i\left(-\frac{1}{\tau_i}\,Y_i\,-\eta_i\right)$$

2.1 反步设计方法

利用以下关系式：

$$z_1(z_2+Y_1)\le \tfrac{1}{2}z_1^2+z_2^2+Y_1^2$$
$$z_2(z_3+Y_2)\le \tfrac{1}{2}z_2^2+z_3^2+Y_2^2$$

可得

$$\dot V\le -\sum_{i=1}^{3}(c_i-\tfrac{3}{2})z_i^2-\sum_{i=1}^{2}\left(\frac{1}{\tau_i}-1\right)Y_i^2+\sum_{i=1}^{2}|Y_i||\eta_i|$$

定义如下紧集：

$$\Xi_1=\left\{y_d^2+\dot y_d^2+\ddot y_d^2\le b_1\right\}$$
$$\Xi_2=\left\{\textstyle\sum_{i=1}^{3}z_i^2+\sum_{i=1}^{2}Y_i^2\le 2b_2\right\}$$

2.1 反步设计方法

连续函数$|\eta_1|$和$|\eta_2|$在$\Xi_1 \times \Xi_2$上有最大值，分别记为M_1和M_2，则对于任意的常数λ_1和λ_2，有

$$|Y_i||\eta_i| \leq \frac{M_i^2}{4\lambda_i}Y_i^2 + \lambda_i, i = 1,2$$

进而可得

$$\dot{V} \leq -\sum_{i=1}^{3}\left(c_i - \frac{3}{2}\right)z_i^2 - \sum_{i=1}^{2}\left(\frac{1}{\tau_i} - 1 - \frac{M_i^2}{4\lambda_i}\right)Y_i^2 + D$$

其中$D = \lambda_1 + \lambda_2$。

选取设计参数使得

$$c_i \geq \frac{3}{2} + k, i = 1,2,3$$

$$\frac{1}{\tau_i} \geq 1 + \frac{M_i^2}{4\lambda_i} + k, i = 1,2$$

2.1 反步设计方法

于是

$$\dot{V} \le -2kV + D$$

令$k > \frac{D}{2b_2}$，则当$V = b_2$时，$\dot{V} \le 0$。因此$V \le b_2$是不变集，即若$V(0) \le b_2$，则$V(t) \le b_2$，$\forall t \ge 0$。由此可知闭环系统全半局稳定。此外

$$0 \le V(t) \le \frac{D}{2k} + \left[V(0) - \frac{D}{2k}\right] e^{-2kt}$$

进而

$$\lim_{t \to +\infty} |z_1(t)| \le \lim_{t \to +\infty} \sqrt{2V(t)} \le \sqrt{\frac{D}{k}}$$

因此，增大k可使得跟踪误差收敛至一任意小的残集内。