Advanced Topics in Control: Distributed Systems and Control

Lecture 8: Markov Chains and Stochastic Stability (Continue) — Event-Triggered Control

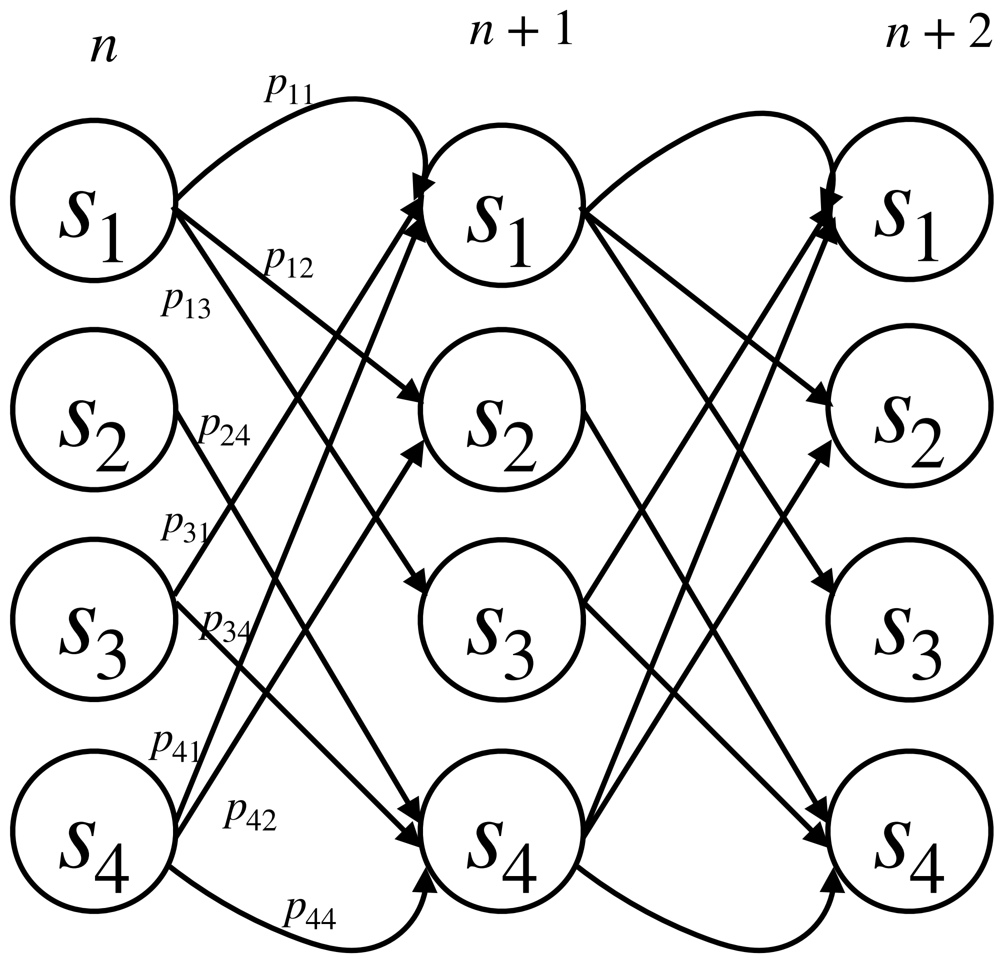
Dr. Mohammad H. (Vahid) Mamduhi, April 11th, 2022

ETH zürich
AUTOMATIC CONTROL LABORATORY

Summary of Lecture 7 on Markov chains

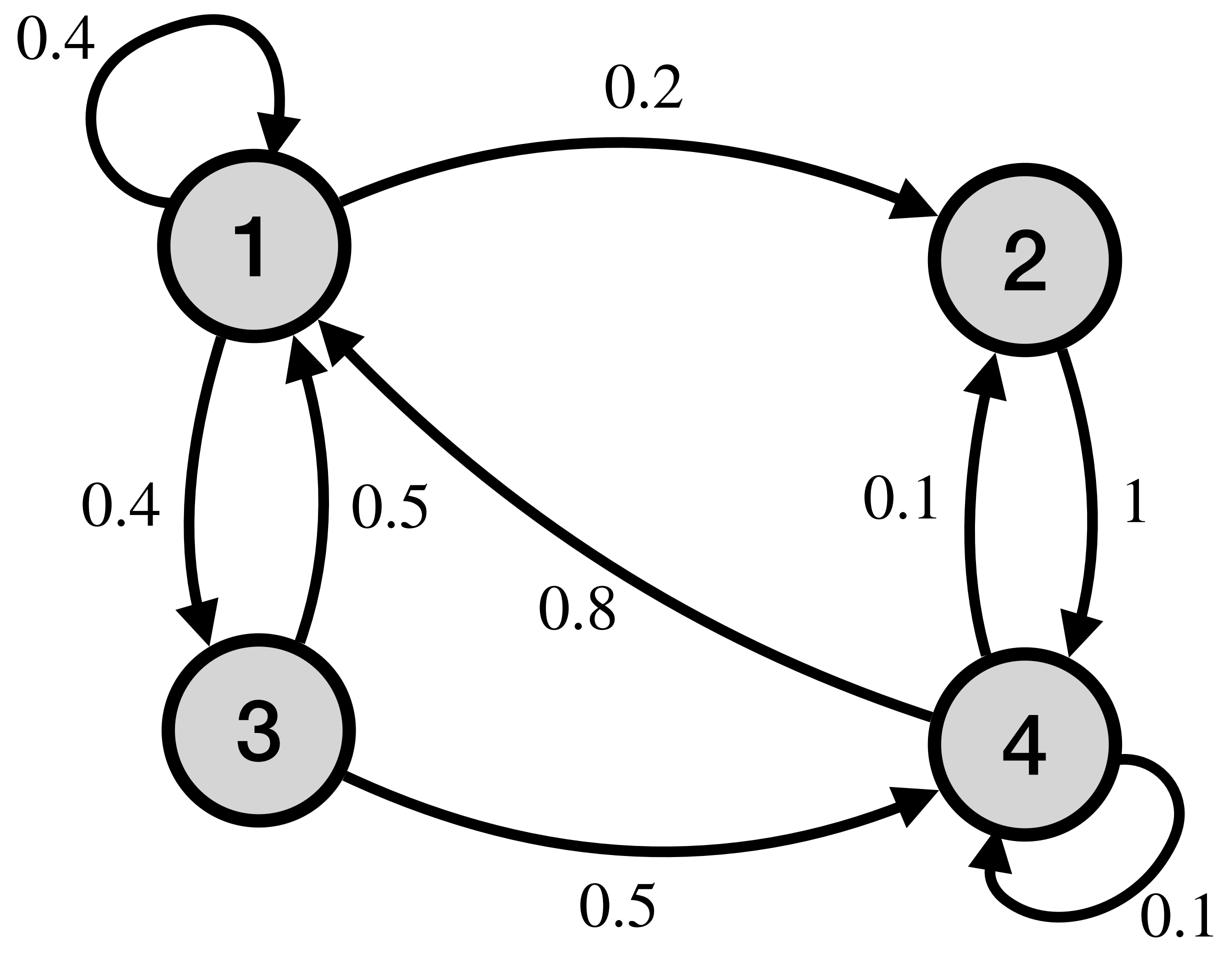
Time-homogeneous Markov chain on countable space $(\Omega, \mathscr{A}, \mathsf{P})$:

- $\mathsf{P}(\phi_{n+m} \in \mathscr{B} \,|\, \phi_n = x_n, \ldots, \phi_0 = x_0) = \mathsf{P}(\phi_{n+m} = \mathscr{B} \,|\, \phi_n = x_n) = p^m(x_n, \mathscr{B})$
- Irreducibility $\Leftrightarrow$ representative graph is strongly connected
- Occupation time η_A/first return time τ_A: the number of times/first time after $n = 0$, that the chain $\{\phi_1, \phi_2, \ldots\}$ visits A
- Markov chain is defined on a discrete-time finite space $\Rightarrow$ irreducibility implies recurrence



Transition probability matrix P_Φ

$\Leftrightarrow$



Adjacency matrix A

Discrete time averaging algorithm:

$$x(t+1) = Ax(t) \quad \Leftrightarrow \quad x(t+1) = P_\Phi x(t)$$

- [x] **A is primitive → converge to consensus!**
- [ ] **What about P_Φ?**

Summary of Lecture 7 on Markov chains

For a Markov chain $\Phi = \{\phi_0, \phi_1, \phi_2, \ldots\}$ on a countable space $(\Omega, \mathscr{A}, \mathrm{P})$ and its representative graph $G(\mathscr{V}, \mathscr{E}, \{a_e\}_{e\in\mathscr{E}})$:

- Φ is irreducible $\Leftrightarrow$ $\exists$ a directed path from any node in $\mathscr{V} = \mathscr{A}$ to any other node $\Leftrightarrow$ G strongly connected
- Φ is strongly aperiodic $\Leftrightarrow$ G is aperiodic and has at least one self-cycle $\Leftrightarrow$ period of each state is one and $\exists \omega \in \mathscr{A}$ s.t. $p(\omega, \omega) > 0$
- Φ is aperiodic $\Leftrightarrow$ G is aperiodic $\Leftrightarrow$ period of each state is one
- $\sum_{x_l \in \mathscr{A}} P(x_i, x_l) = 1$ $\Leftrightarrow$ Out-degree of every node is one $\Leftrightarrow$ adjacency matrix row-stochastic
- Φ irreducible and aperiodic $\Leftrightarrow$ G strongly connected and aperiodic $\Leftrightarrow$ adjacency matrix primitive
- A state is (Harris) recurrent $\Leftrightarrow$ the corresponding node in G is globally reachable

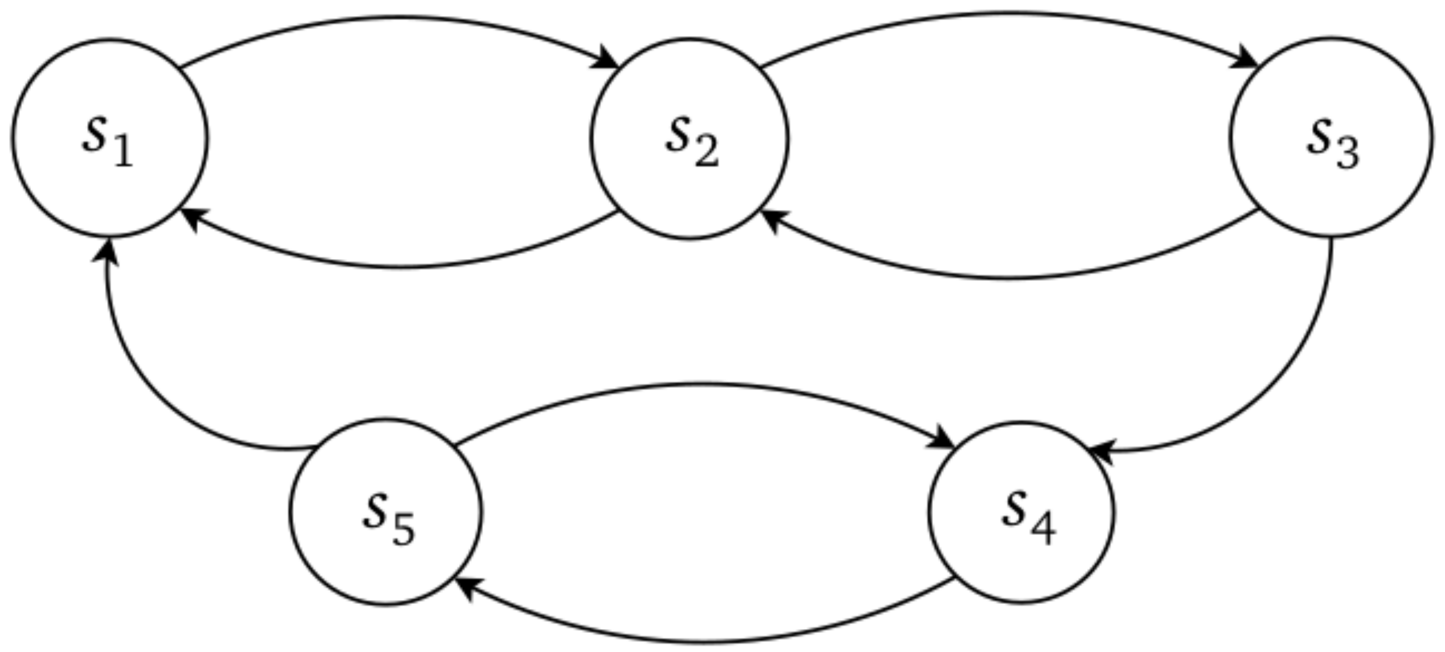
Stochastic Stability of Markov Processes: (continue…) Recurrence

Theorem: If a Markov chain is defined on a discrete-time finite space, then irreducibility implies recurrence (proof as exercise!)

Irreducibility $\Leftrightarrow L(A) = 1, \forall\, x \in A \in \mathscr{A} \Leftrightarrow Q(A) = 1,\, A \in \mathscr{A}$

Equivalently:

$\exists$ a path from any node to any other node $\Leftrightarrow$ all states reachable from other states with finite transitions $\Leftrightarrow$ all nodes globally reachable in finite time



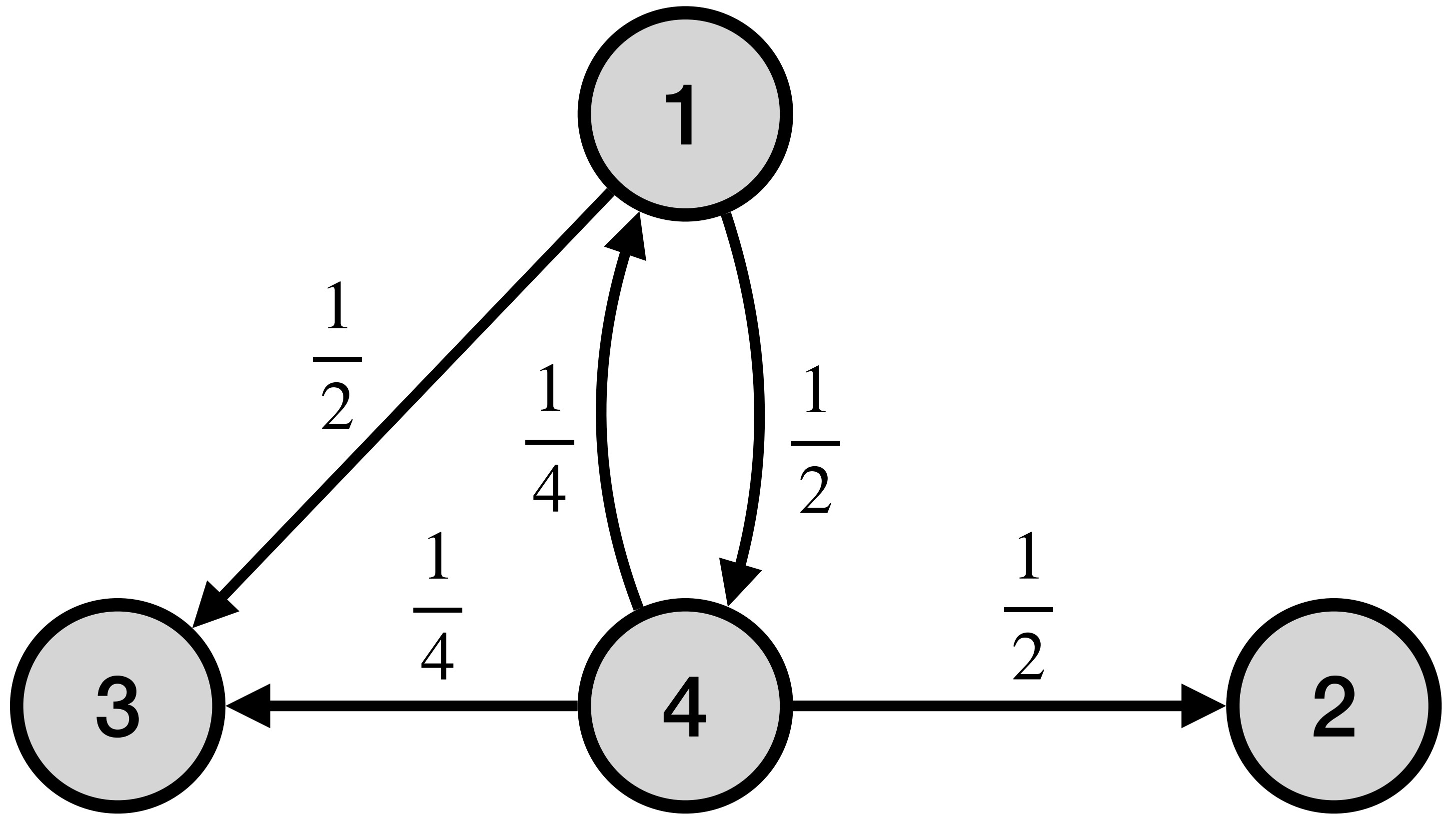
A recurrent/irreducible Markov chain on a discrete state space

Example: MC is reducible and no global reachable node (not Harris recurrent!). What are the recurrence probabilities?

Answer: Recurrent states are 2 and 3:

- $R_2(s_i)$ = P(recurrence to 2 from state i): $R_2(s_2) = 1,\, R_2(s_3) = 0$
- $R_3(s_i)$ = P(recurrence to 3 from state i): $R_3(s_2) = 0,\, R_3(s_3) = 1$

**Calculations on the board!*



Graph representation of a homogeneous Markov chain with two recurrent classes

Solution:

Concepts of Stochastic Stability for Markov Chains: Ergodicity

Definition: A homogeneous Markov chain on a countable space is said to be ***ergodic*** if it is irreducible (positive Harris recurrent) and aperiodic

1. If a chain is ergodic: a unique stationary probability distribution Π exists, independent of the initial distributions, such that the Markov chain evolves according to Π when $n \to \infty$, i.e., considering

$$\Pi(A) = \lim_{n\to\infty} \text{Pr}[\phi_n \in A \,|\, \phi_m]$$

If $\Pi(A)$ is finite and independent of $\phi_m, m < \infty$, the chain is ergodic.

☑ If a Markov chain is **periodic** or has **more than one recurrent class**, then it is ***not ergodic***!

2. If a chain is ergodic, the unique probability distribution Π is ***invariant*** $\quad \to \quad \Pi = P_\Phi^n \, \Pi$

Equivalence: Remember consensus algorithm $x(t+1) = Ax(t)$ where A is primitive? $x(\infty) = (w^\top x(0))1_n$

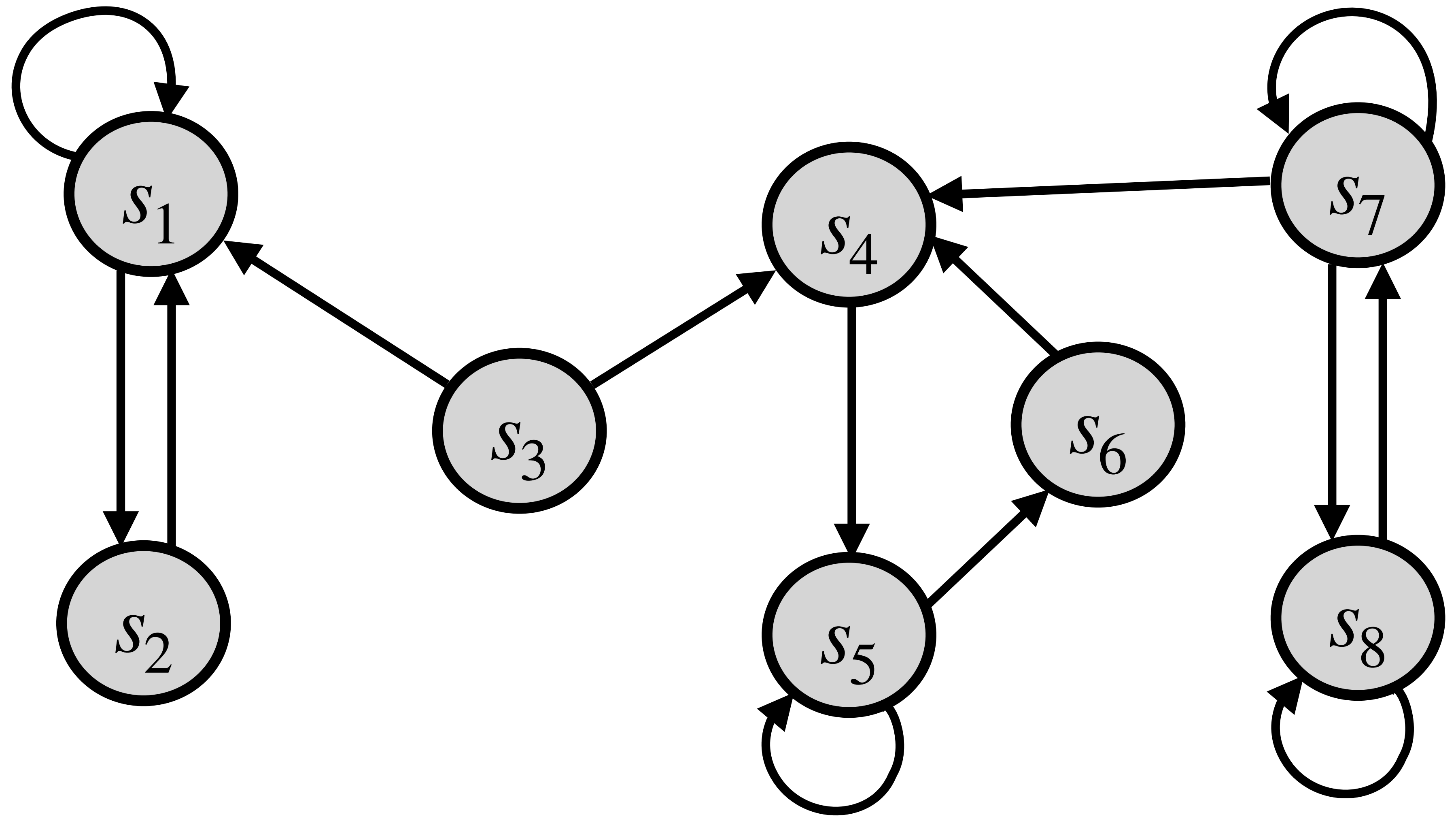
$$\text{Primitive} \overset{?}{\Leftrightarrow} \text{ergodic} \quad \& \quad \Pi \overset{?}{\Leftrightarrow} w^\top$$

Concepts of Stochastic Stability for Markov Chains: Ergodicity

Example 1: For the given finite state Markov chain we have:

- transient states: $\{s_3, s_7, s_8\}$
- recurrent class 1: $\{s_1, s_2\}$, reachable only from s_3
- recurrent class 2: $\{s_4, s_5, s_6\}$, reachable only from s_3 and s_7

☑ The chain is **not ergodic**! (no globally reachable node)



Intuition: stationary probability of being in a certain state depends on the initial state, e.g., chain starts from s_2, never gets out of the loop $\{s_1, s_2\}$!

☐ Revisit example 5 in Lecture 4 for a graph with no globally reachable node, and with aperiodic SCC sinks (recurrent classes?) What can be concluded?

Example 2: Derive the invariant probability distribution for the given transition probability matrix P_Φ for a 4-state MC

$$P_\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

** Calculations on the board!*

Stochastic Stability for Markov Chains: Drift criteria

Define the real-valued non-negative measurable function $V(\Phi) : \Omega \mapsto \mathbb{R}^+$. For a discrete state Markov chain $\Phi = \{\phi_0, \phi_1, \ldots, \phi_n, \phi_{n+1}, \ldots\}$ on a countable space, the drift function is defined as:

$$\Delta V(\phi_n) = \mathrm{E}[V(\phi_{n+1}) \,|\, \phi_n] - V(\phi_n), \;\; \phi_n \in \Omega$$

Lyapunov-based stochastic stability: Let the irreducible Markov chain $\Phi = \{\phi_n\}$ be defined on the space Ω with sigma-algebra $\mathscr{A}$. Then Φ is ergodic if a finite set $A \in \mathscr{A}$ exists s.t. for $V(\Phi) : \Omega \mapsto \mathbb{R}^+$, $V(\phi) < \infty, \;\; \phi \in A$:

$$\Delta V(\phi) \leq -1, \;\; \phi \in \Omega \backslash A$$

** Recall the consensus algorithm $x(t+1) = Ax(t)$. If A is primitive $\rightarrow x(t \rightarrow \infty) = \lim_{t \to \infty} A^t x(0) = (w^\top x(0)) 1_n$!

Corollary: Let Φ be irreducible and strongly aperiodic. Then Φ is ergodic and $\exists A \in \mathscr{A}$ s.t. for any initial state $x \in \Omega$:

$$\lim_{n \to \infty} P^n(x, A) - \Pi = 0$$

☑ How about this? $\lim_{n \to \infty} P^n(x, A) \;\; \Leftrightarrow \;\; \lim_{t \to \infty} A^t \;\; \rightarrow \;\; \Pi \;\; \Leftrightarrow \;\; 1_n w^\top$

Stability for General Dynamical Systems

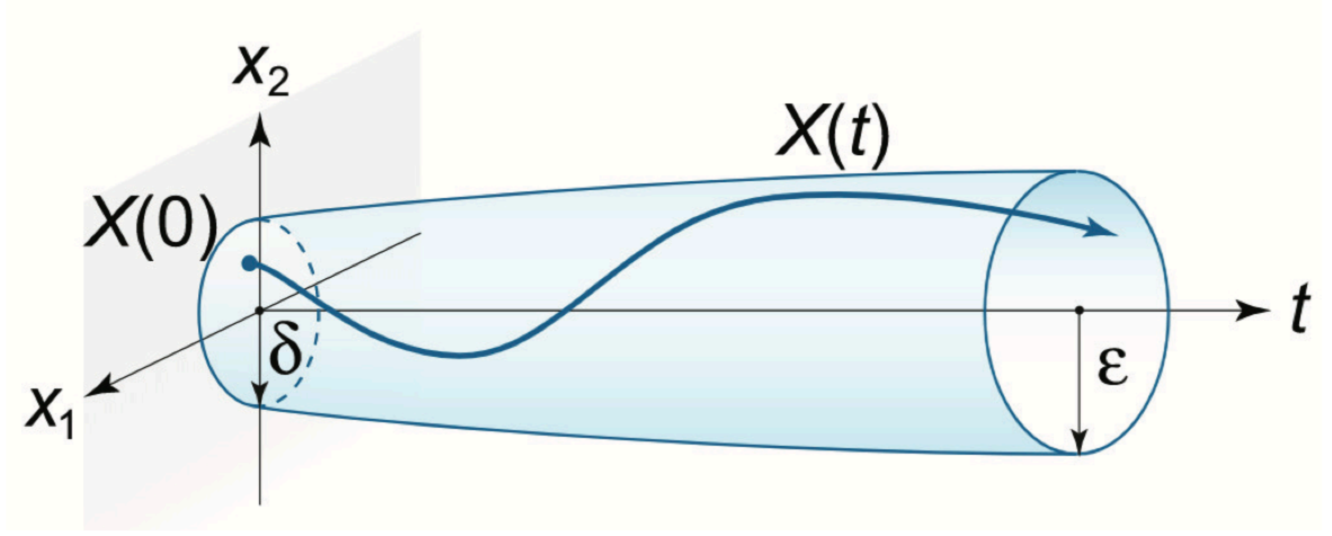
Consider the discrete-time stochastic dynamical system

$$x(t+1) = f(x(t), w(t)), \quad x(t_0) = x(0)$$

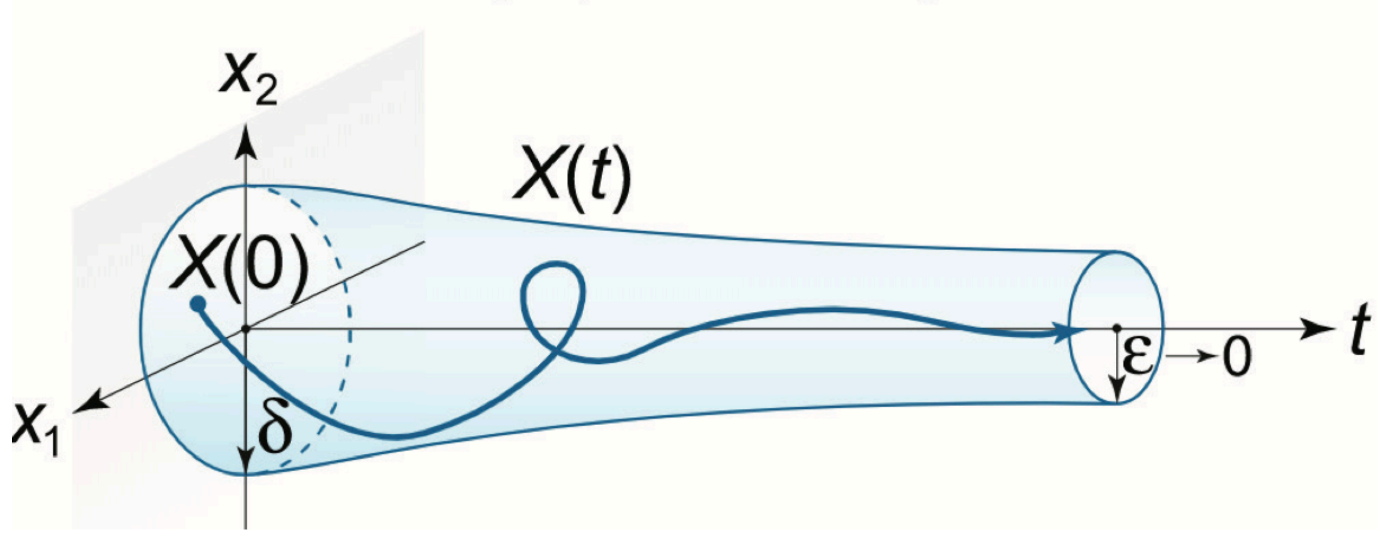
f is a (Lipschitz) continuous function and $w(t), t \in \mathbb{N}_0$ is a sequence of Markov random variables. We are interested in stability of the equilibrium $x(t, x(0)) = 0$

Recall stability notions for deterministic system $x(t+1) = f(x(t))$. $\forall \varepsilon > 0, \ \exists \delta > 0$:

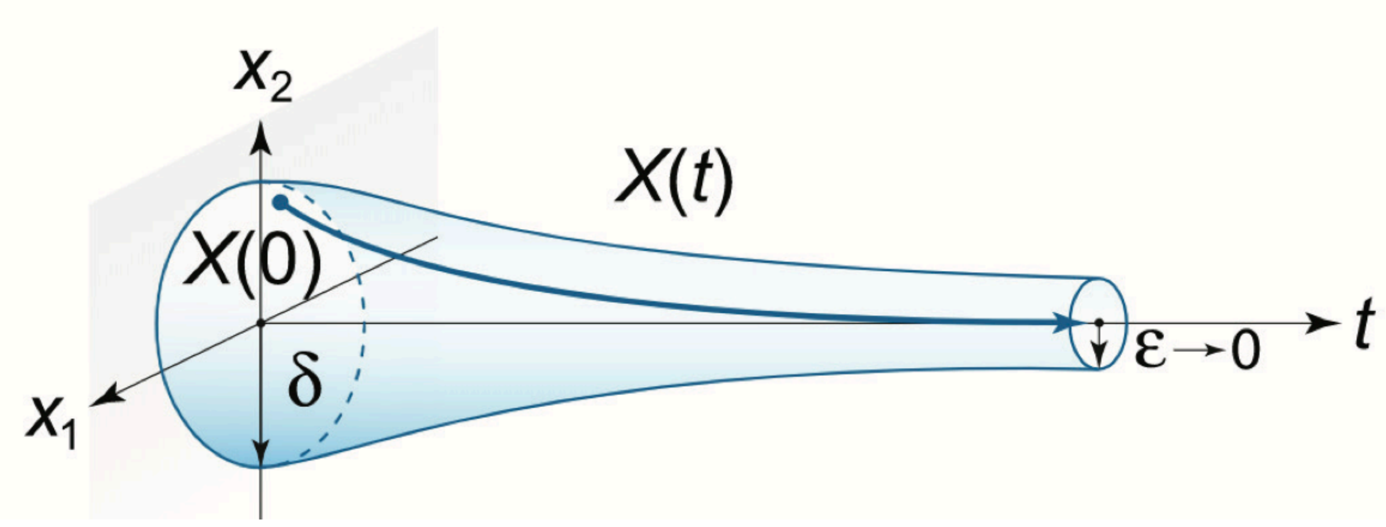
1. **Lyapunov stability:** trajectory starts in δ-vicinity of the equilibrium, always remains in its ε-vicinity
2. **Asymptotic stability:** trajectory starts in δ-vicinity of equilibrium, it stays in its ε-vicinity and eventually converges to the equilibrium
3. **Exponential stability:** trajectory starts in δ-vicinity of equilibrium, it remains in its ε-vicinity and smoothly converges to equilibrium with a particular rate



Lyapunov stability



Asymptotic stability



Exponential stability

Stability for General Dynamical Systems: LTI Discrete Time Example

Consider the discrete-time deterministic LTI system $x(t+1) = Ax(t)$. Under what conditions the system is Lyapunov stable/exponential stable?

Solution: Define $V(x(t)) = x(t)^\top Q x(t)$, with Q a positive definite symmetric matrix. So, $V(x(t)) > 0, \forall x(t) \neq 0$.

$$\Delta V = V(x(t+1)) - V(x(t)) = x(t)^\top A^\top Q A x(t) - x(t)^\top Q x(t) = x(t)^\top (A^\top Q A - Q) x(t)$$

If $A^\top Q A - Q < 0$, we have $\Delta V < 0, \forall x(t) \neq 0$.

$\Rightarrow$ If all eigenvalues of A are less than or equal 1, with at least one strictly less than 1, system is **Lyapunov stable**!

$\Rightarrow$ If all eigenvalues of A are strictly less than 1, system is **exponentially stable**! Also exists a positive def. symmetric matrix P such that $A^\top Q A - Q = -P$. We know $x(t)^\top P x(t) \geq \lambda_{min}(p) x(t)^\top x(t)$, and $x(t)^\top Q x(t) \leq \lambda_{max}(Q) x(t)^\top x(t)$:

$$\frac{\Delta V}{V} = \frac{-x(t)^\top P x(t)}{x(t)^\top Q x(t)} < -\frac{\lambda_{min}(P)}{\lambda_{max}(Q)} = -\alpha, \forall x(t) \neq 0 \quad \rightarrow \quad \Delta V < -\alpha V \ \ (\alpha > 0 \text{ convergence rate})$$

Stability for General Dynamical Systems: LTV Discrete Time Example

Consider the discrete-time deterministic LTV system $x(t+1) = A(t)x(t), \;\; x(t_0) = x_0$. Under what conditions the system is asymptotic stable/exponential stable for the equilibrium at the origin?

Solution: We generally write the LTV system as $x(t) = \Phi(t, t_0)x_0$, where $\Phi(t, t_0) = A(t-1)A(t-2)\ldots A(t_0)$ is state transition matrix. To converge to the origin, we know $x(t) \to 0$ when $t \to \infty$. Since $\Phi(t, t') = \Phi(t', t)^{-1}$, we can write

$$x(t) = \Phi(t,0)\;\Phi(0,t_0)\;x_0 \;\;\to\;\; \|x(t)\| \leq \|\Phi(t,0)\|\|\Phi(0,t_0)\|\|x_0\|, \;\; det(A(t)) \neq 0, \forall t$$

We know $\Phi(0,t_0)$ is bounded for any finite $t_0 \geq 0$ (linear system!) and also $x(t_0) = x_0$ is finite. Hence, to achieve asymptotic stability, i.e., $\lim_{t\to\infty} \|x(t)\| = 0$ it is enough to have $\|\Phi(t,0)\| \to 0$ as $t \to \infty$.

Theorem: For LTI and LTV systems $x(t+1) = Ax(t)$ and $x(t+1) = A(t)x(t)$:

asymptotic stability $\Leftrightarrow$ exponential stability!

**Proof as exercise!*

Stability for Stochastic Dynamical Systems

☑ Exponential stability $\Rightarrow$ Asymptotic stability $\Rightarrow$ Lyapunov stability

Generally for stochastic systems, stability is discussed in the sense of expectation or probability, because trajectories vary from one sample to another due to random dynamics, i.e., $w(t)$

$$x(t+1) = f(x(t), w(t)), \quad x(t_0) = x(0)$$

Lyapunov-based stochastic stability:

1. Lyapunov stability in probability: For any $\varepsilon_1 > 0$, $\varepsilon_2 \in [0,1]$, there exists $\delta(\varepsilon_1, \varepsilon_2, t_0)$, s.t. $\|x(0)\| < \delta$ implies

$$\mathsf{P}\{\sup_{t>t_0} \|x(t, x_0)\| > \varepsilon_1\} < \varepsilon_2$$

2. Lyapunov stability in the m^{th} mean: For any $\varepsilon > 0$, there exists $\delta(\varepsilon, t_0)$, s.t. $\|x(0)\|_m < \delta$ implies

$$\mathsf{E}\{\sup_{t>t_0} \|x(t, x(0))\|_m^m\} \leq \varepsilon$$

Stability for Stochastic Dynamical Systems

Asymptotic stochastic stability:

1. Asymptotic stability in probability: if Lyapunov stable in probability and $\exists \delta > 0$ s.t. $\|x(0)\| < \delta$ implies

$$\lim_{\tau \to \infty} \mathsf{P}\{\sup_{t \geq \tau} \|x(t, x(0))\| > \varepsilon\} = 0$$

2. Asymptotic stability in the m^{th} mean: if Lyapunov stable in m^{th} mean and $\exists \delta > 0$ s.t. $\|x(0)\| < \delta$ implies

$$\lim_{\tau \to \infty} \mathsf{E}\{\sup_{t \geq \tau} \|x(t, x(0))\|_m^m\} = 0$$

Exponential stochastic stability:

Exponential stability of the m^{th} mean: if $\exists \delta > 0$ and constants α, β s.t. $\|x(0)\| < \delta$ implies

$$\mathsf{E}\{\|x(t, x_0)\|_m^m\} \leq \beta \|x_0\|_m^m \, e^{-\alpha(t-t_0)}$$

☑ Exponential stability of m^{th} mean $\nLeftarrow$ Asymptotic stability in m^{th} mean $\Rightarrow$ Lyapunov stability in m^{th} mean

☑ Stability in the mean is stronger than stability in probability!

Exercise: A Linear Time-Invariant (LTI) system

Consider a discrete-time LTI stochastic system with states $x(t)$, control input $u(t)$, and measurements $y(t)$ at a time t:

$$x(t+1) = Ax(t) + Bu(t) + w(t), \quad t \in \mathbb{N}_0$$
$$y(t) = x(t) + v(t)$$

Where A, B are constant, and $v_k \sim \mathcal{N}(0,\sigma_v^2), w_k \sim \mathcal{N}(0,\sigma_w^2)$ are i.i.d. Gaussian random noise processes. Assume the control inputs are computed by the following linear state feedback law with gain $L(t)$:

$$u(t) = -L(t)\,\hat{x}(t)$$

Where, the state estimates are computed by a Kalman filter, as follows:

$$\hat{x}(t) = \hat{x}(t)^- + K(t)(y(t) - \hat{x}(t)^-), \quad \hat{x}(t)^- = A\hat{x}(t-1) + Bu(t-1)$$
$$K(t) = P(t)^-(P(t)^- + \sigma_v^2)^{-1}$$
$$P(t)^- = AP(t-1)A^\top + \sigma_w^2, \quad P(t) = P(t)^- - K(t)(P(t)^- + \sigma_v^2)K(t)^\top$$

Define the estimation error as $e(t) = y(t) - \hat{x}(t)$. Is $e(t)$ a Markov chain? What can be said about second moment stability of the closed-loop system $x(t+1) = f(x(t), e(t), w(t), v(t))$ with states $[x(t), e(t)]^\top$?

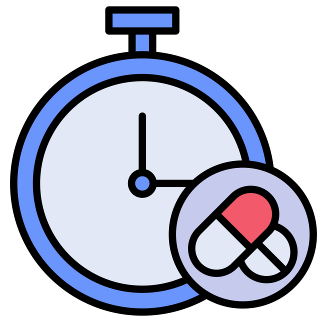
Event-Triggered Scheme

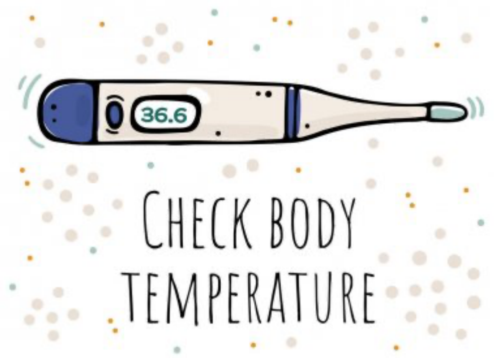
When execution of an action (control, sampling, actuation, communication, etc.) is performed based on the occurrence of a **defined "event"**, we call the action scheme ***event-triggered*** or ***event-based***.

Compared with **Time-Triggered** scheme, where an action is executed at pre-defined and equidistant time instances!

We do things in event-based fashion in our daily lives as well, e.g., when sick:

- Take medication every 8 hours → Time-triggered
- Take medication If body temperature above 38 → Event-triggered





Intuition: Actions often entail costs (implementation/resources), so why not perform an action only when necessary?

Major benefit: reduction of actions and consequently reduction of consuming costly resources

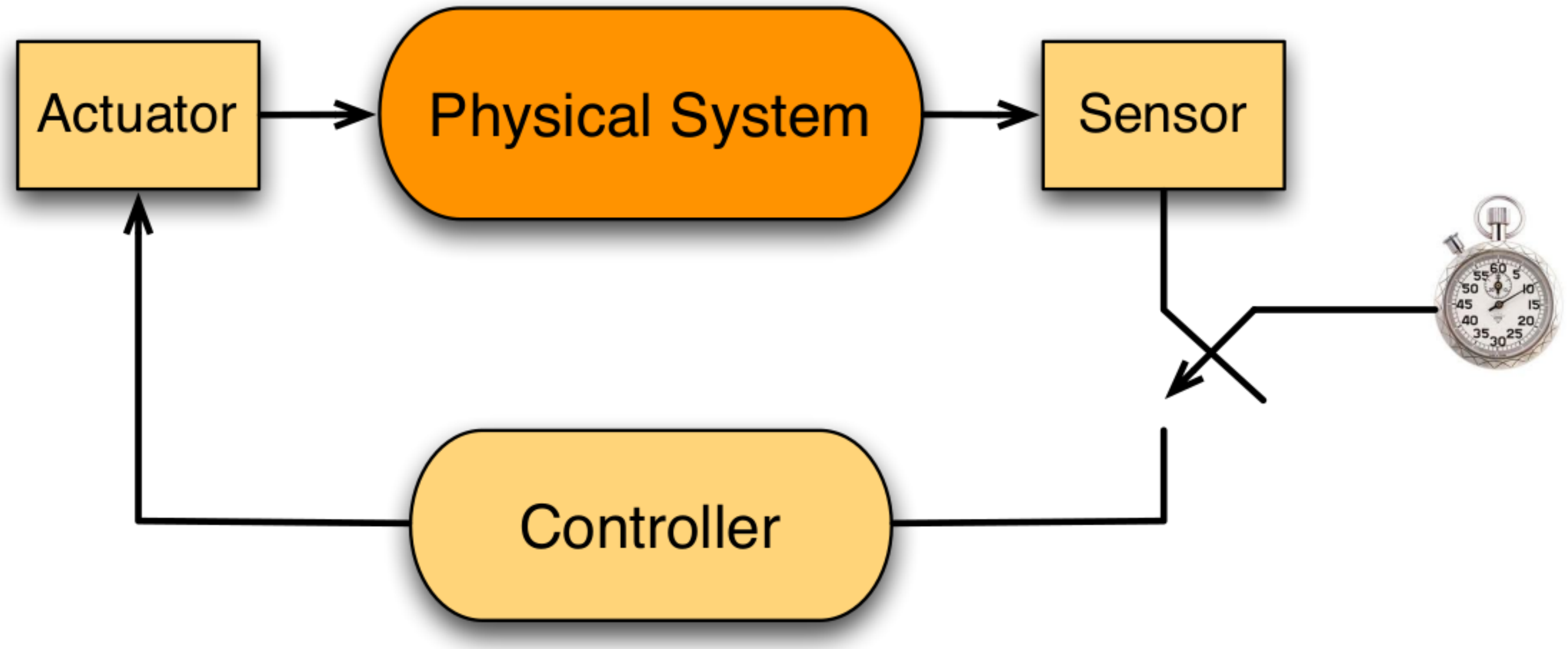
Event-Triggered vs. Time-Triggered Sampling

Time-triggered sampling is an open-loop scheme where sampling is performed using a clock

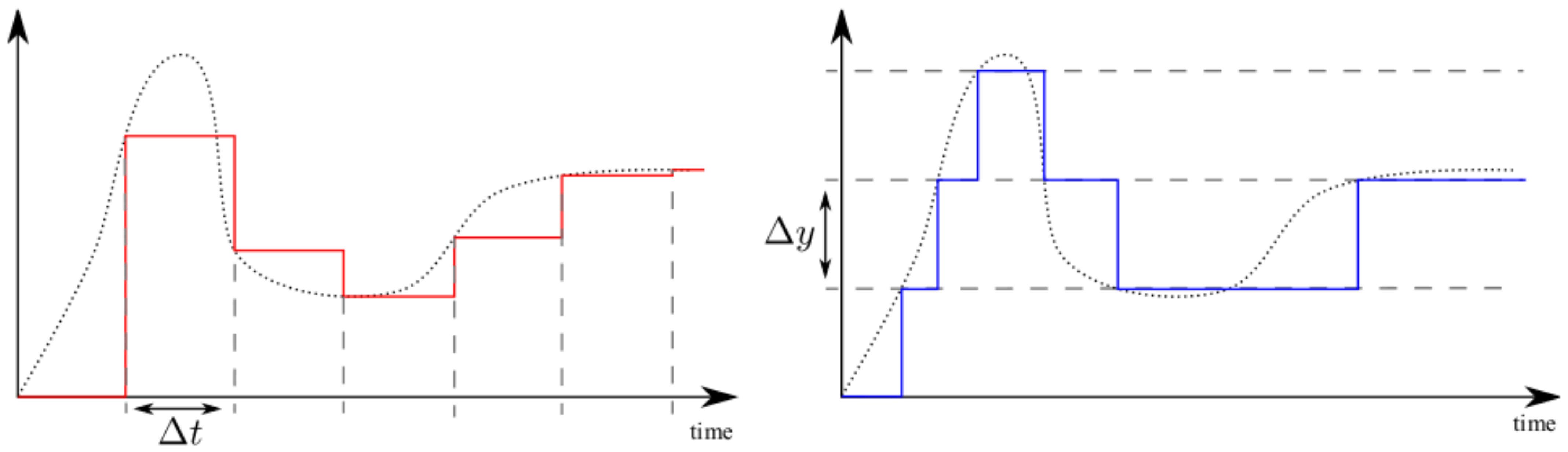
- Pre-specified sampling instances
- Deterministic and synchronous sampling
- Not resource efficient and not flexible

Event-triggered sampling uses feedback in the sampling process

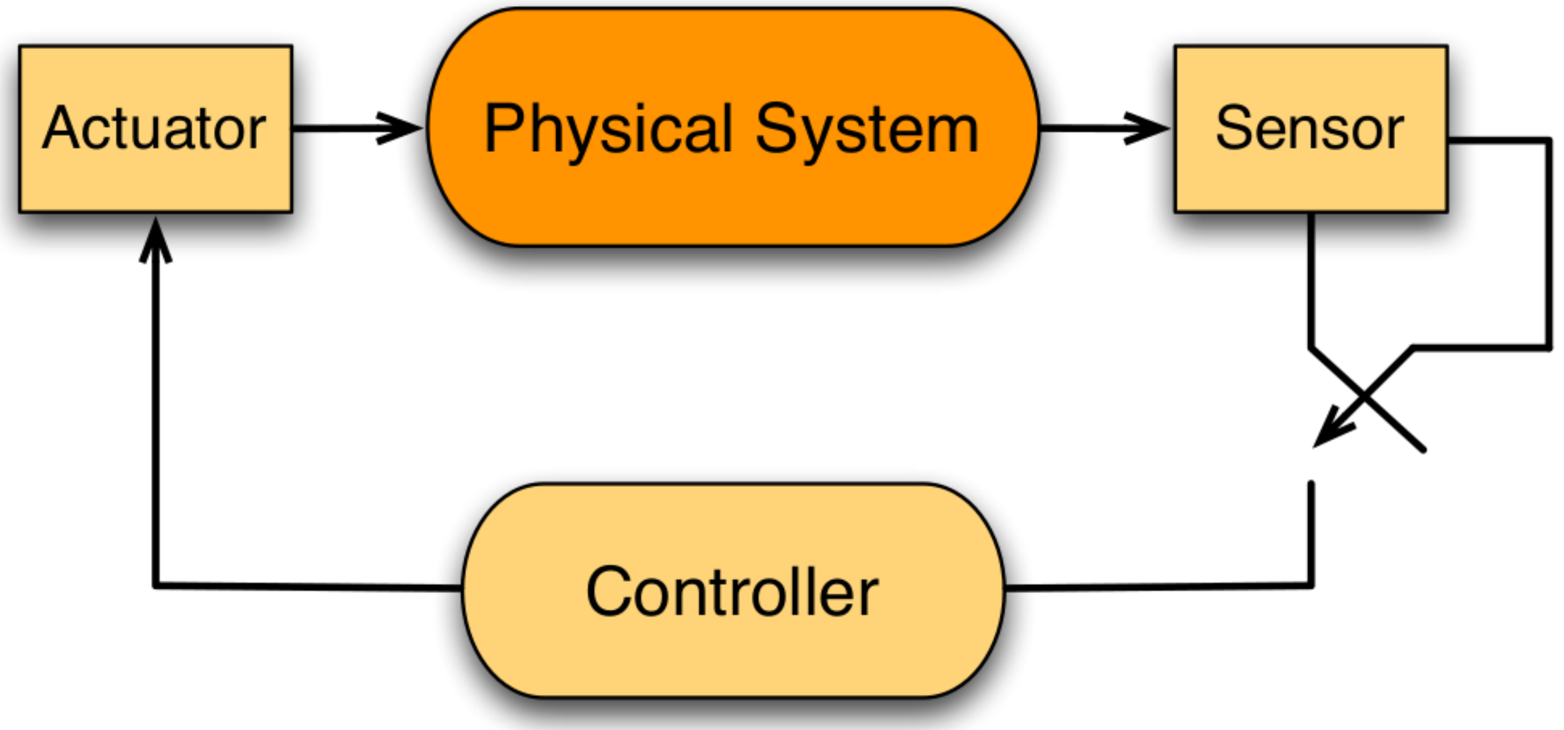
- Real-time sampling instances
- Asynchronous sampling and difficult to predict
- Resource efficient (sample only when performance not satisfactory)



Time-triggered sampling



Time-triggered (left) vs. event-triggered (right) sampling of a signal



Event-triggered sampling

Event-Triggered vs. Time-Triggered Sampling: A more complex scenario

Event-triggered actions: events can be designed for various functionalities across a control system

A single control system

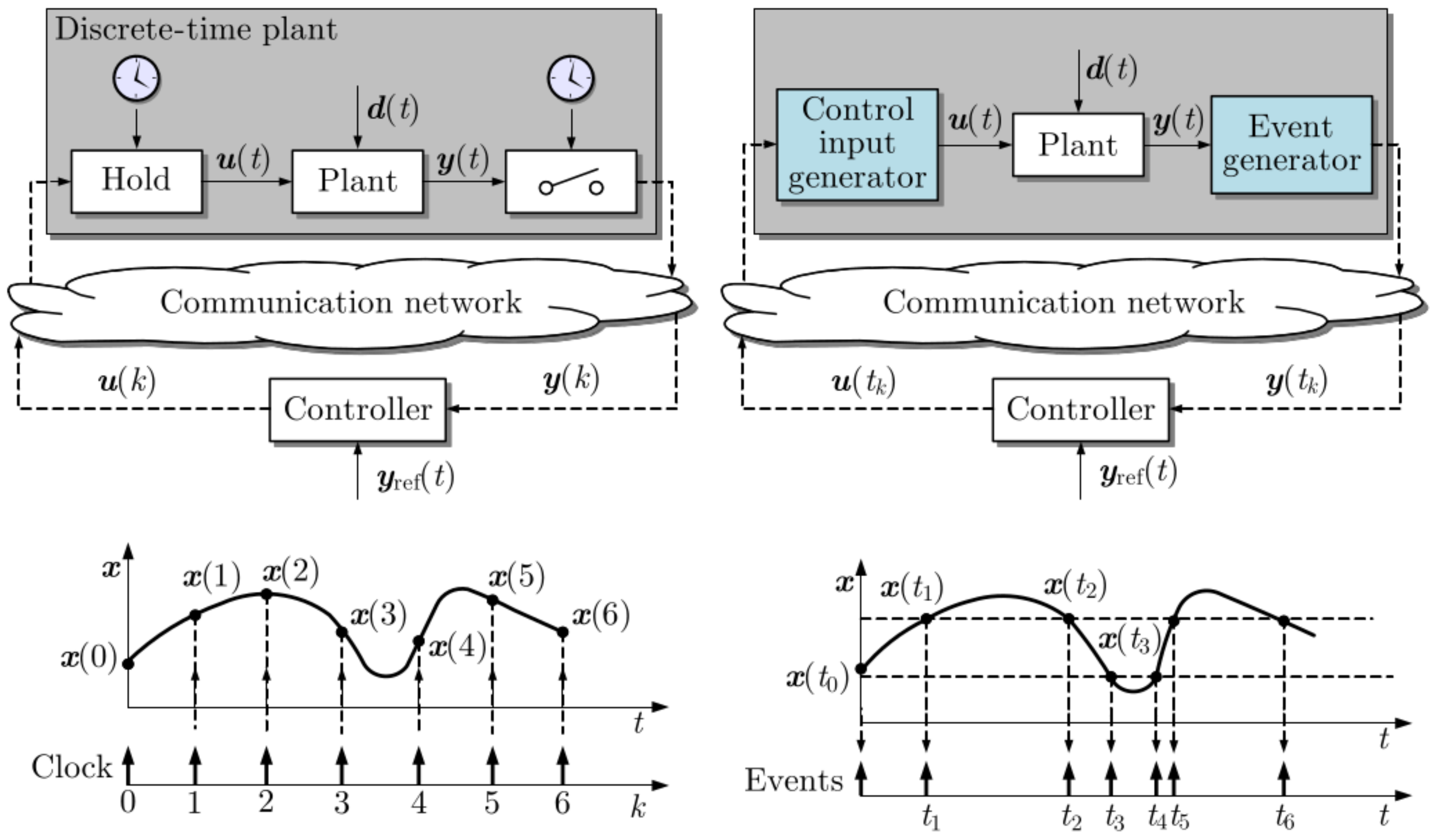
- When sample the system states
- When update the control input/actuate

A networked/multi-agent system

- When to communicate and with whom
- Which loop to be closed, if resources limited

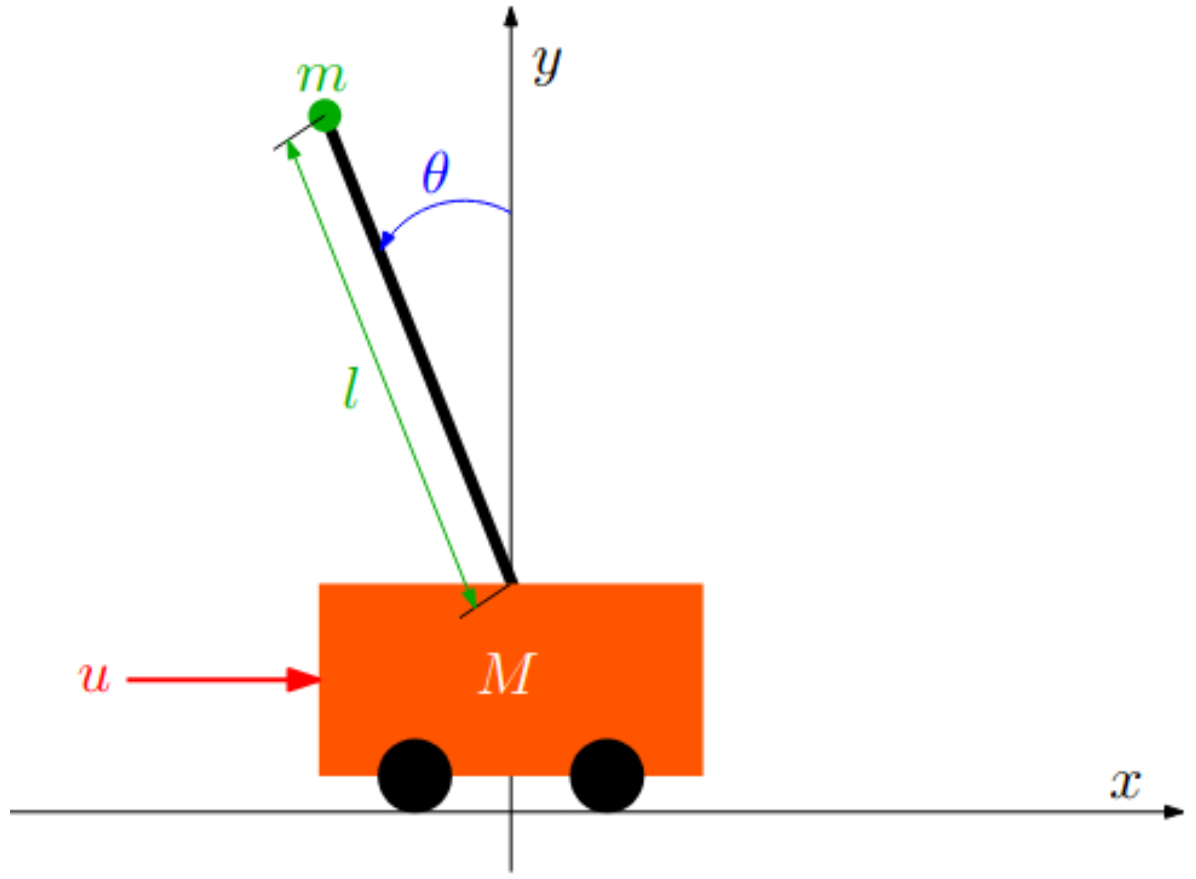
Challenges of event-triggered schemes:

- Complex to design and implement
- Optimal events often difficult to be found
- Requires constant monitoring of the events
- Triggering instances often not predictable



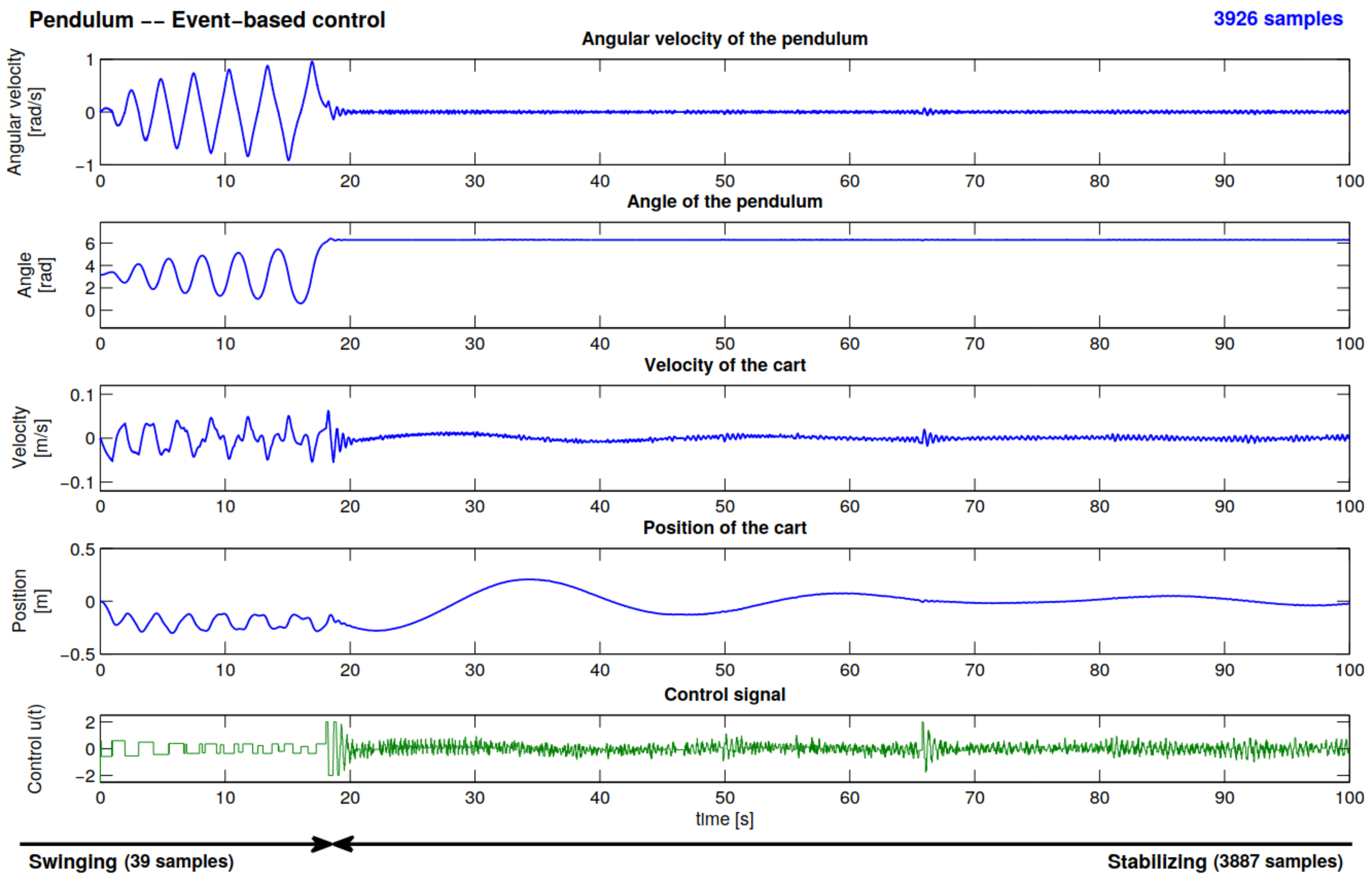
Time-triggered vs. Event-triggered schemes in a networked control system

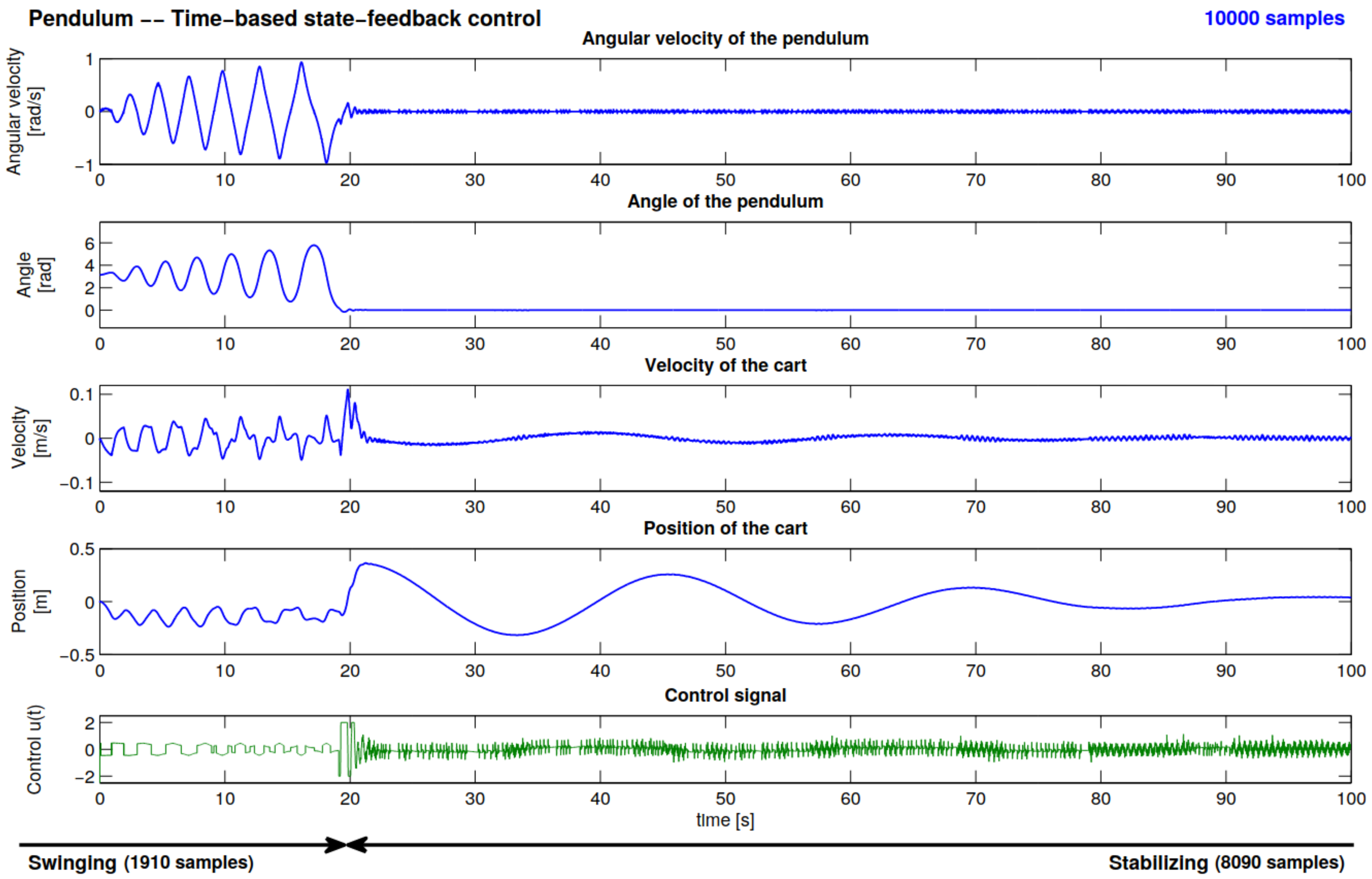
Event-Triggered vs. Time-Triggered Scheme: Inverted Pendulum



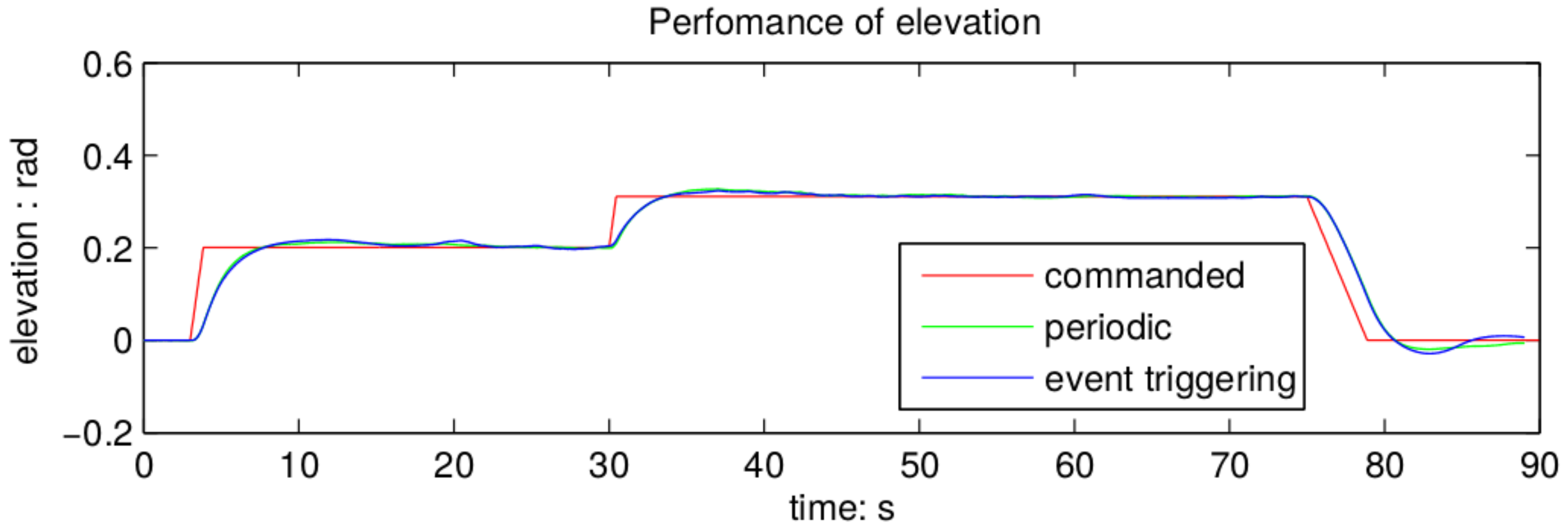
Compare the sampling instances:

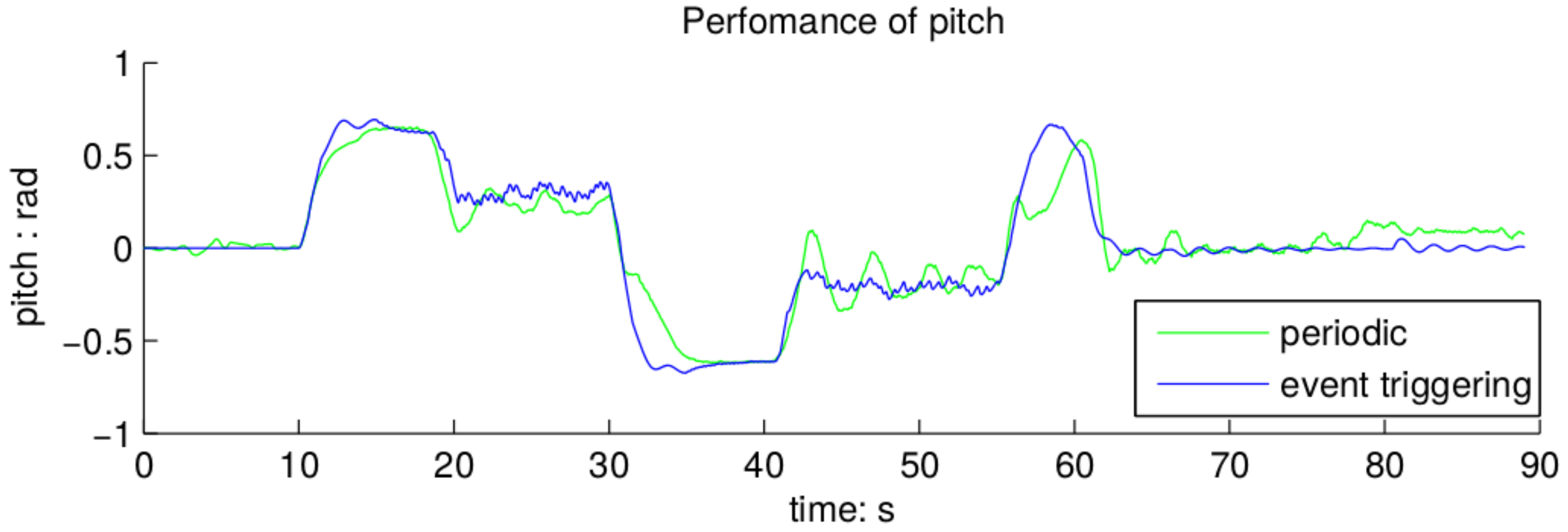
- Time-triggered: 10000 samples
- Event-triggered: 3926 samples
- ~60% reduction in sampling for similar performance

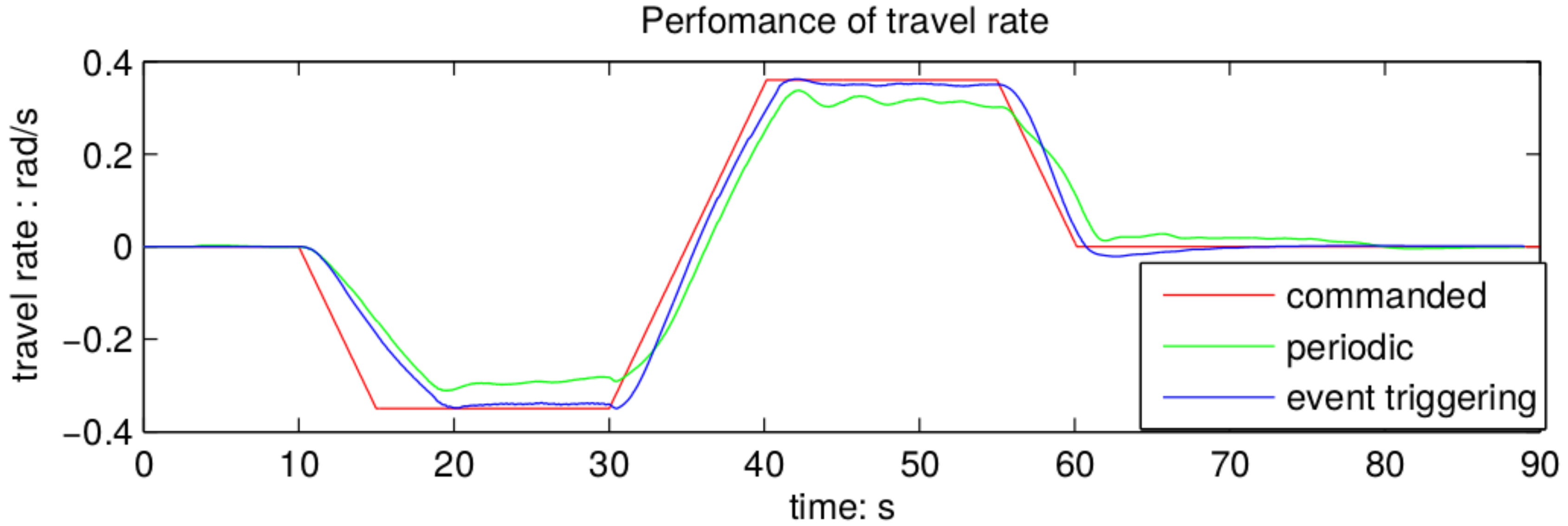


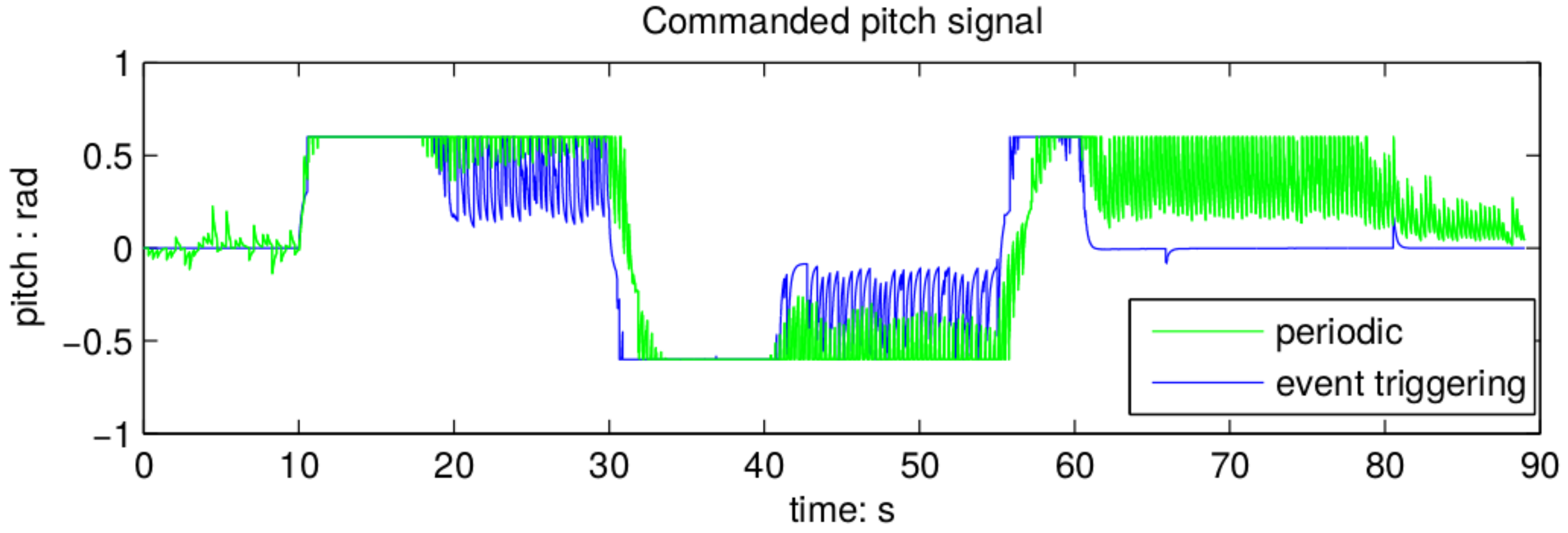


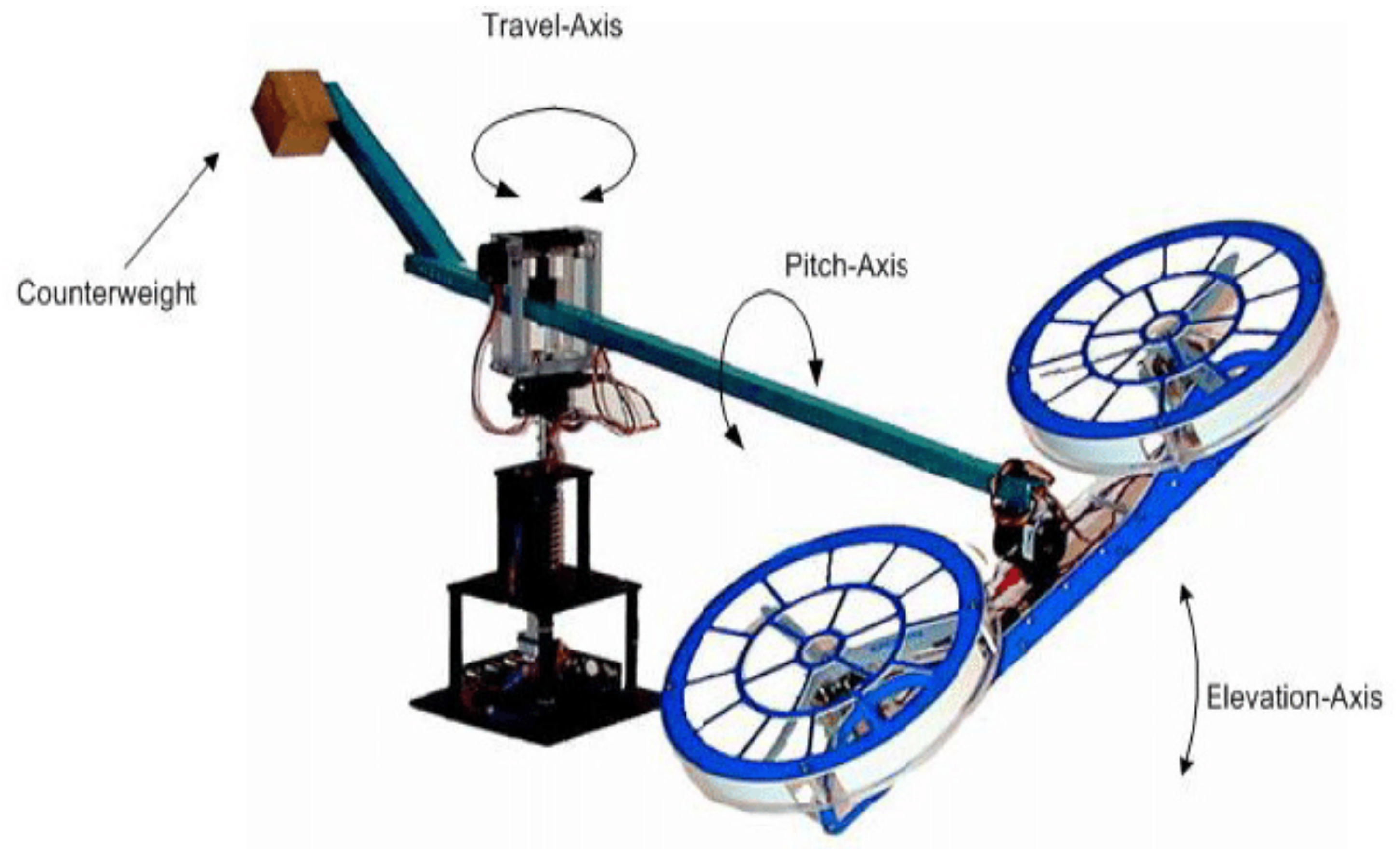
Event-Triggered vs. Time-Triggered Scheme: 3DOF Helicopter

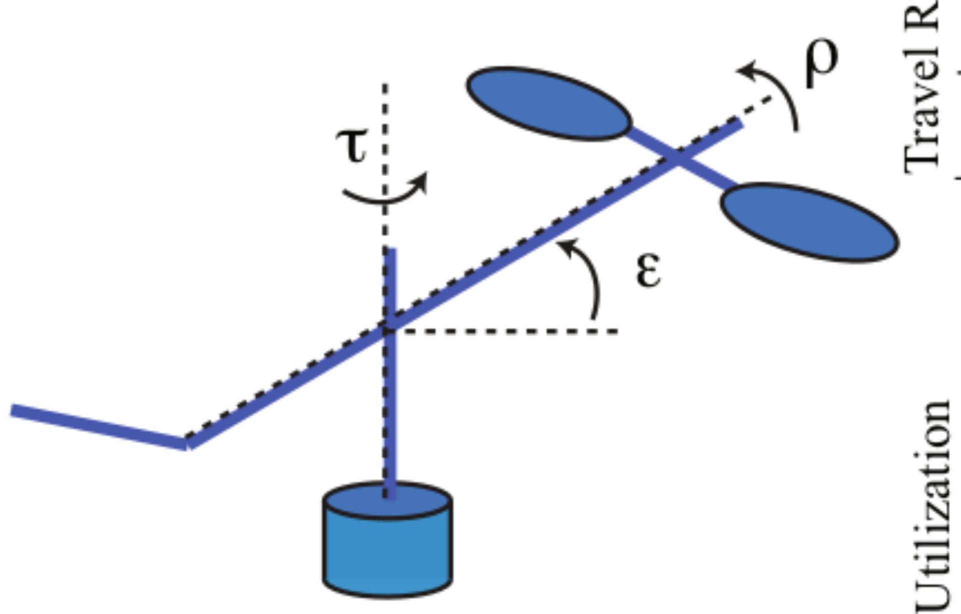


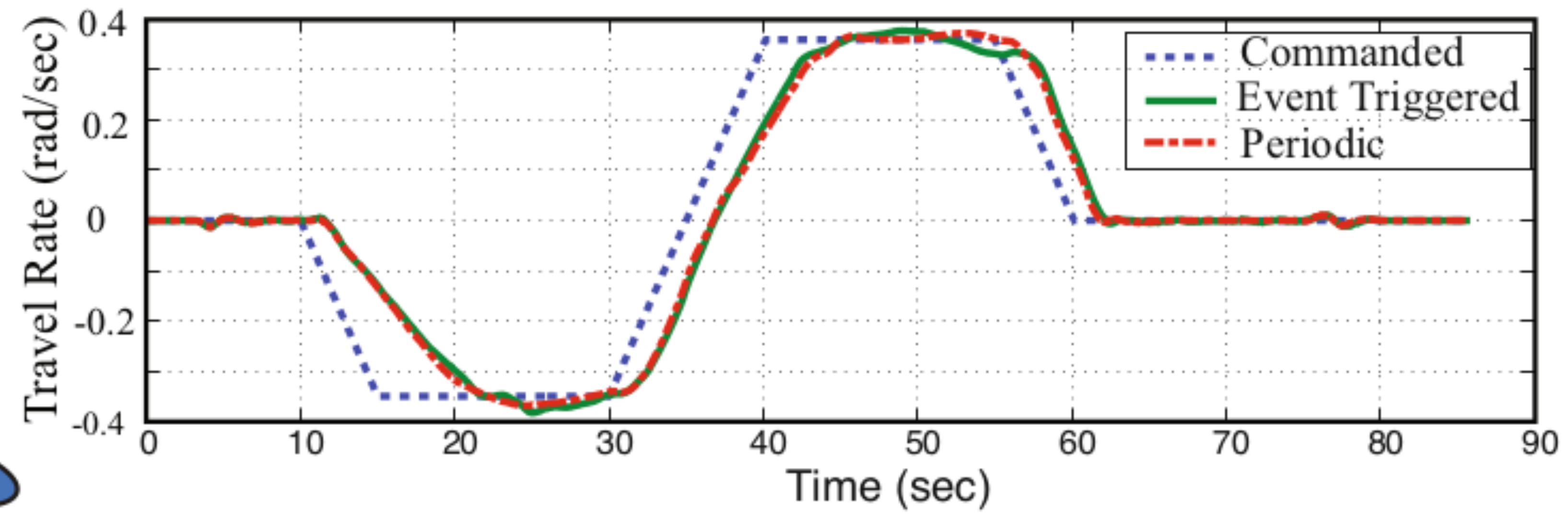


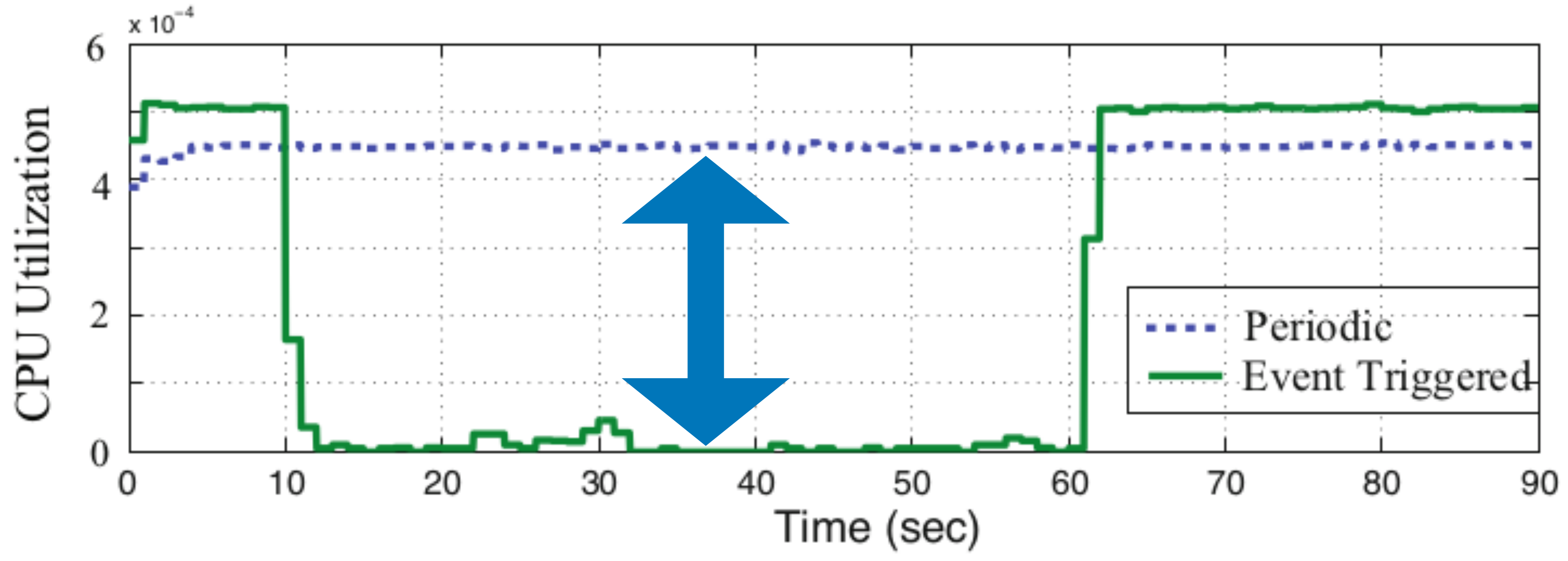








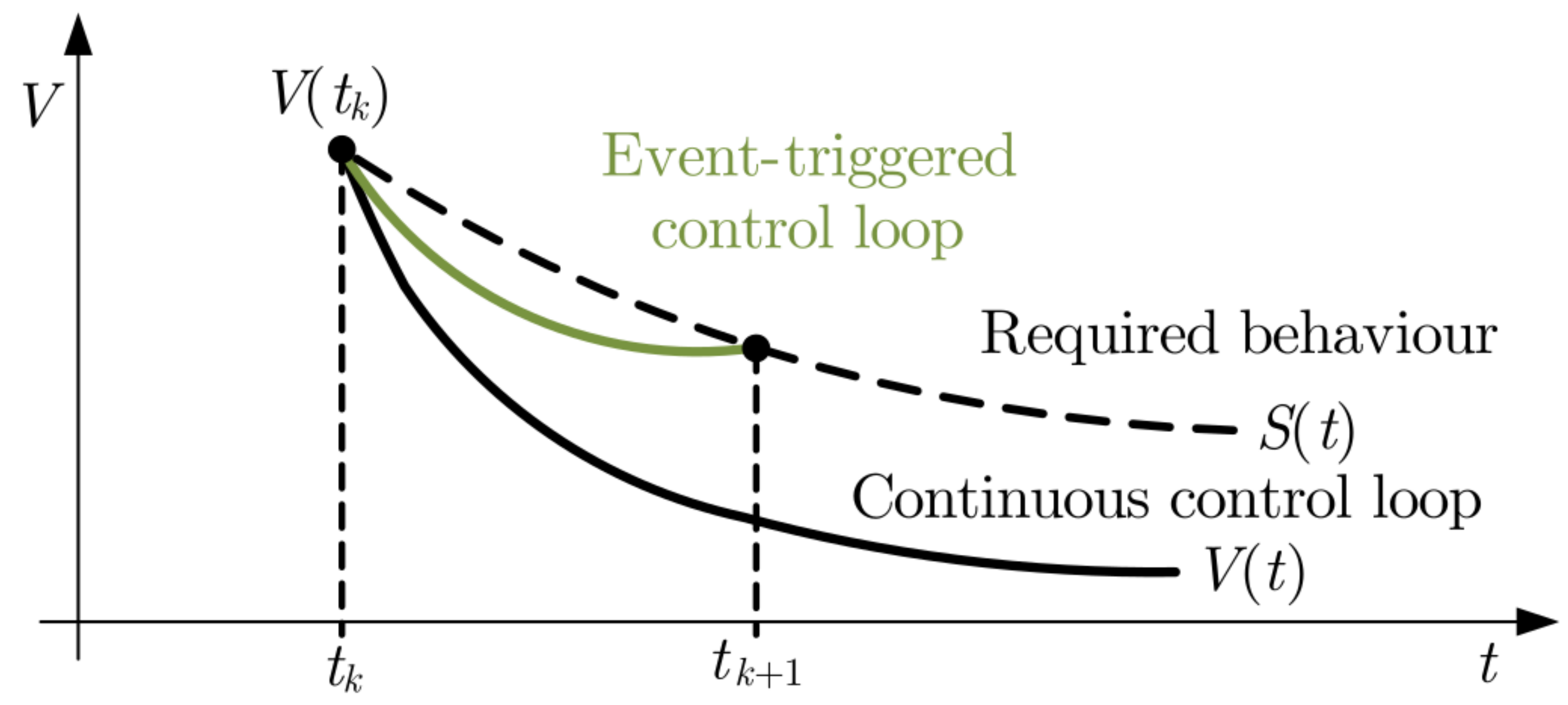




Asynchronous Triggering Schemes

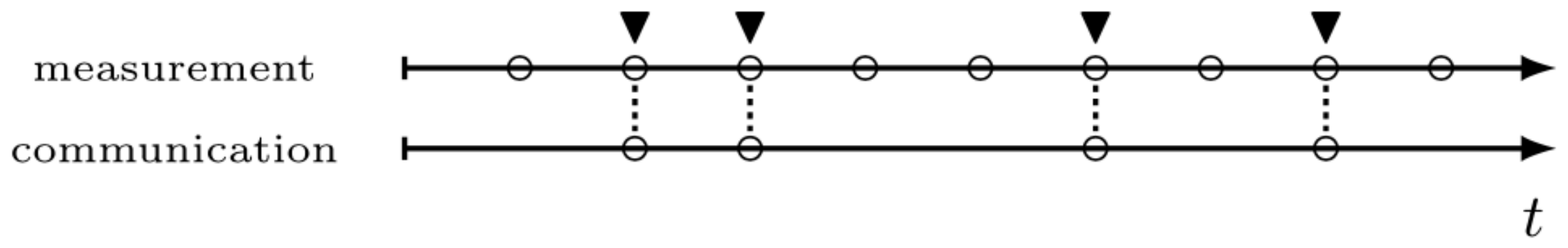
Event-triggered:

- Deterministic triggering functions: events are often deterministically defined with thresholds on parameters, e.g., the states of the system, estimation error, cost value, etc.
- Stochastic triggering functions: if events have stochastic nature, the triggering times might be defined in probabilisitic fashion
- Mixed triggering functions: events are triggered partly deterministically and partly stochastically



Periodic event-triggered:

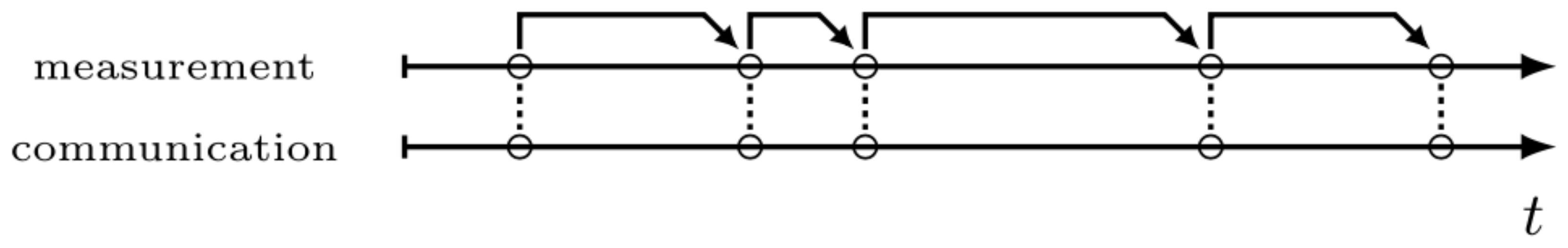
- Events are not continuously monitored but periodically, and if triggering condition is met, the action will be performed



(a) Event-triggered control.

Self-triggered:

- Triggering times are predicted ahead of time but still based on the current situation of the system



(b) Self-triggered control.

*** More on the design and analysis of event-triggered schemes in the next lecture!*

Next Lecture…

- Design gf event-triggered schemes (deterministic triggering rules, stochastic triggering rules, mixed triggering rules)
- Event-triggered sampling, event-triggered control/actuation, and event-triggered communication
- Implementation of event-triggered functions for single-loop and multi-loop (networked) control systems
- Stability and performance of closed-loop systems under event-triggered functions

References

[1]. Lichun Li, "Event-triggered state estimation and output feedback in Cyber Physical Systems", PhD Dissertation, 2011.

[2]. Alberto Bemporad, Maurice Heemels, Mikael Johansson, "Networked Control Systems", Springer, 2010.

[3]. S. Durand, J. F. Guerrero Castellanos, N. Marchand, W. F. Guerrero Sanchez, "Event-based Control of the Inverted Pendulum: Swing up and Stabilisation", Control Engineering and Applied Informatics, 2013.