Nonlinear Control Theory

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Second-Order Systems

Consider the 2-order system:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) = f_1(x),\\ \dot{x}_2 &= f_2(x_1, x_2) = f_2(x).\end{aligned}$$

Definition

Let $x(t) = (x1(t), x2(t))$ be a solution that starts at initial state $x_0 = (x_{10}, x_{20})$. The locus in the $x_1 - x_2$ plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a trajectory or orbit.

Definition

The $x_1 - x_2$ plane is called the state plane or phase plane. The family of all trajectories is called the phase portrait.

Definition

The vector field $f(x) = [f_1(x), f_2(x)]^T$ is tangent to the trajectory at point x because

$$\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}.$$

Example

Consider the function $f(x) = [2x_1^2, x_2]^T$. Represent $f(x)$ as a vector based at x; that is, assign to x the directed line segment from x to $x + f(x)$.

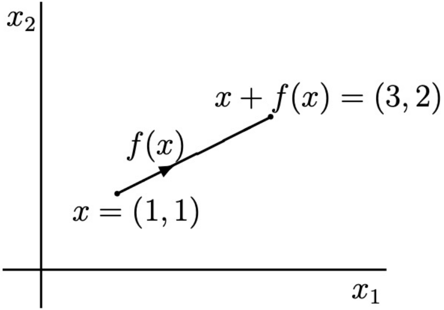

Figure: The vector $f = [2x_1^2, x_2]^T$ at $x = [1, 1]^T$

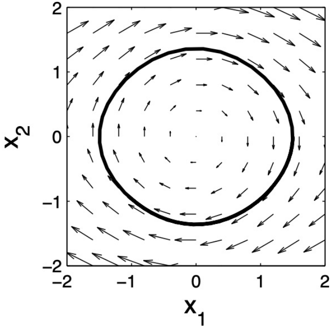

Figure: The phase portrait of $\dot{x}_1 = x_2,\ \dot{x}_2 = -\sin x_1$

Numerical construction of the phase portrait:

1. Select a bounding box in the state plane
2. Select an initial point x_0 and calculate the trajectory through it by solving

$$\dot{x} = f(x), \quad x(0) = x_0,$$

   in forward time (with positive t) and in reverse time (with negative t)

$$\dot{x} = -f(x), \quad x(0) = x_0$$

3. Repeat the process interactively
4. Use simulink

1. **Qualitative behavior of linear systems**
2. Qualitative behavior near equilibrium points
3. Multiple equilibria
4. Limit cycles

Qualitative behavior of linear systems

Start with linear system $\dot{x} = Ax$ where A is a 2×2 real matrix.

- For a given initial condition x_0, its solution:

$$x(t) = M \exp(J_r t) M^{-1} x_0.$$

- Suppose that A has distinct eigenvalues, then either

$$J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \text{or} \quad J_r = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

depending on whether the eigenvalues are real or complex.

Case 1. Both eigenvalues are real:

$$M = [v_1,\ v_2],$$

where $v_1,\ v_2$ are eigenvectors associated with $\lambda_1,\ \lambda_2$.

- The transformation $z = M^{-1}x$ results in

$$\dot{z}_1 = \lambda_1 z_1,\ \ \dot{z}_2 = \lambda_2 z_2.$$

- For given initial conditions z_{10} and z_{20},

$$z_1(t) = z_{10}e^{\lambda_1 t},\ \ z_2(t) = z_{20}e^{\lambda_2 t} \quad \Rightarrow \quad z_2 = c z_1^{\frac{\lambda_2}{\lambda_1}},\ \ c = \frac{z_{20}}{(z_{10})^{\frac{\lambda_2}{\lambda_1}}}$$

The shape of the phase portrait dependes on the signs of λ_1 and λ_2.

- Both eigenvalues are real and negative:

$$\lambda_2 < \lambda_1 < 0$$

  - Both $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero as $t \to \infty$.
  - $e^{\lambda_2 t}$ tends to zero faster than $e^{\lambda_1 t}$.
  - Call λ_2 the fast eigenvalue (v_2 the fast eigenvector) and λ_1 the slow eigenvalue (v_1 the slow eigenvector).
  - The trajectory tends to the origin along the curve

$$z_2 = c z_1^{\frac{\lambda_2}{\lambda_1}}, \quad \frac{\lambda_2}{\lambda_1} > 1, \quad \frac{dz_2}{dz_1} = c\frac{\lambda_2}{\lambda_1} z_1^{[\lambda_2/\lambda_1 - 1]}$$

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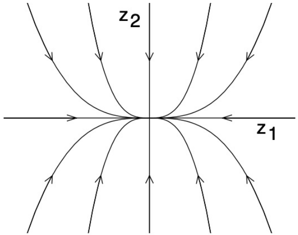


Figure: Stable node: $\lambda_2 < \lambda_1 < 0$
(If $\lambda_2 > \lambda_1 > 0$, reverse arrowheads $\Rightarrow$ Unstable node)

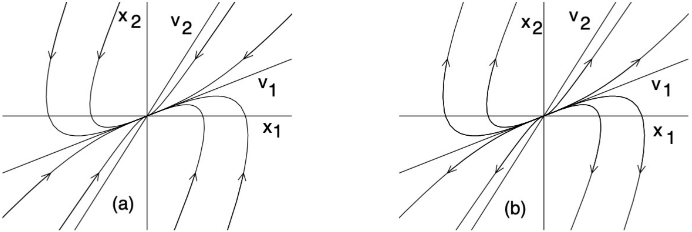


Figure: Stable node and unstable node in $x_1 - x_2$ plane

- The signs of eigenvalues are different:

$$\lambda_2 < 0 < \lambda_1$$

  * $e^{\lambda_1 t} \to \infty$ while $e^{\lambda_2 t} \to 0$ as $t \to \infty$
  * Call λ_2 the stable eigenvalue (v_2 the stable eigenvector) and λ_1 the unstable eigenvalue (v_1 the unstable eigenvector)
  * $z_2 = c z_1^{\lambda_2/\lambda_1}$, where $\lambda_2/\lambda_1 < 0$
  * Saddle

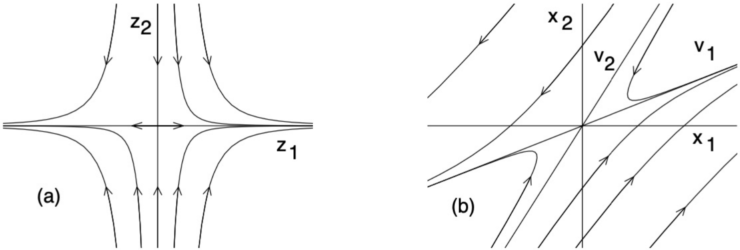


Figure: Phase portrait of a saddle point

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$

$$\Rightarrow \quad \dot{z}_1 = \alpha z_1 - \beta z_2, \quad \dot{z}_2 = \beta z_1 + \alpha z_2$$

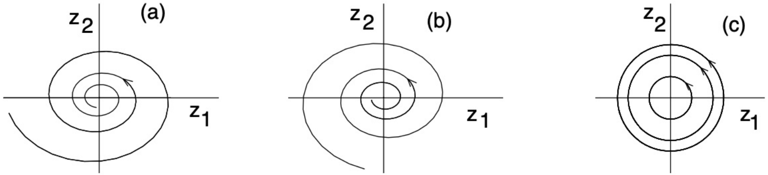
Let $r = \sqrt{z_1^2 + z_2^2}$ and $\theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$, then it holds that

$$r(t) = r_0 e^{\alpha t}, \quad \theta(t) = \theta_0 + \beta t$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$\alpha < 0 \quad \Rightarrow \quad r(t) \to 0$ as $t \to \infty$, or $\alpha > 0 \quad \Rightarrow \quad r(t) \to \infty$ as $t \to \infty$

or $\alpha = 0 \quad \Rightarrow \quad r(t) \equiv r_0 \ \forall\, t$



$\alpha < 0$ | $\alpha > 0$ | $\alpha = 0$

Stable Focus | Unstable Focus | Center

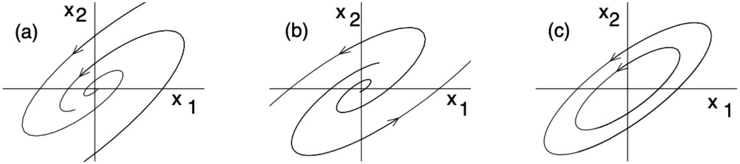


Figure: Focus and center

Effects of perturbation

$$A \quad \to \quad A + \delta A \quad (\delta A \text{ arbitrarily small})$$

- The eigenvalues of a matrix depend continuously on its parameters.
- A node (with distinct eigenvalues), a saddle or a focus is structurally stable because the qualitative behavior remains the same under arbitrarily small perturbations in A.
- A center is not structurally stable

$$\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \quad \text{eigenvalues} = \mu \pm j.$$

$$\mu < 0 \quad \Rightarrow \quad \text{stable focus;} \quad \mu > 0 \quad \Rightarrow \quad \text{unstable focus}$$

Qualitative behavior near equilibrium points

Can we determine the type of the equilibrium point of a nonlinear system by linearization?

Let $p = [p_1, p_2]^T$ be an equilibrium point of the system

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2),$$

where f_1 and f_2 are continuously differentiable.

Expand f_1 and f_2 in Taylor series about $[p_1, p_2]^T$,

$$\begin{aligned}\dot{x}_1 &= f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.}\\ \dot{x}_2 &= f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.},\end{aligned}$$

where $a_{11} = \left.\frac{\partial f_1(x_1,x_2)}{\partial x_1}\right|_{x=p}$, $a_{12} = \left.\frac{\partial f_1(x_1,x_2)}{\partial x_2}\right|_{x=p}$, $a_{21} = \left.\frac{\partial f_2(x_1,x_2)}{\partial x_1}\right|_{x=p}$, $a_{22} = \left.\frac{\partial f_2(x_1,x_2)}{\partial x_2}\right|_{x=p}$

Let $y_1 = x_1 - p_1$ and $y_2 = x_2 - p_2$. Since $f_1(p_1, p_2) = 0$, $f_2(p_1, p_2) = 0$, it holds that

$$\begin{aligned} \dot{y}_1 &= \dot{x}_1 = a_{11} y_1 + a_{12} y_2 + \text{H.O.T.} \\ \dot{y}_2 &= \dot{x}_2 = a_{21} y_1 + a_{22} y_2 + \text{H.O.T.} \end{aligned}$$

indicating that

$$\dot{y} \approx Ay$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=p} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

Eigenvalues of A and the type of equilibrium point of the nonlinear system:

1. $\lambda_2 < \lambda_1 < 0 \quad \Rightarrow$ Stable node
2. $\lambda_2 > \lambda_1 > 0 \quad \Rightarrow$ Unstable node
3. $\lambda_2 < 0 < \lambda_1 \quad \Rightarrow$ Saddle
4. $\alpha \pm j\beta, \ \alpha < 0 \quad \Rightarrow$ Stable focus
5. $\alpha \pm j\beta, \ \alpha > 0 \quad \Rightarrow$ Unstable focus
6. $\pm j\beta \quad \Rightarrow$ Linearization fails

Example

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 =& -x_2 - \mu x_1(x_1^2 + x_2^2),\\ \dot{x}_2 =& x_1 - \mu x_2(x_1^2 + x_2^2).\end{aligned}$$

where $x = 0$ is an equilibrium point. Linearization around $x = 0$ yields

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Eigenvalues have zero real parts. Let $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. We have

$$\dot{r} = -\mu r^3, \quad \dot{\theta} = 1,$$

indicating that the origin is a stable focus when $\mu > 0$ and unstable focus when $\mu < 0$.

Multiple equilibria

Example

Tunnel-diode circuit

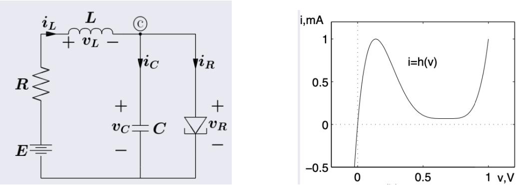


Figure: Tunnel-diode circuit

$$x_1 = v_C, \qquad x_2 = i_L$$

$$\begin{aligned}\dot{x}_1 =& 0.5[-h(x_1) + x_2],\\ \dot{x}_2 =& 0.2(x_1 - 1.5x_2 + 1.2),\end{aligned}$$

where $h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$. There exist multiple equilibria:

$$\begin{aligned}Q_1 =& [0.063,\ 0.758]^T\\ Q_2 =& [0.285,\ 0.61]^T\\ Q_3 =& [0.884,\ 0.21]^T\end{aligned}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h\prime(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix} \Rightarrow \lambda(A_1) = -3.57, \ -0.33$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix} \Rightarrow \lambda(A_2) = 1.77, \ -0.25$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix} \Rightarrow \lambda(A_3) = -1.33, \ -0.4$$

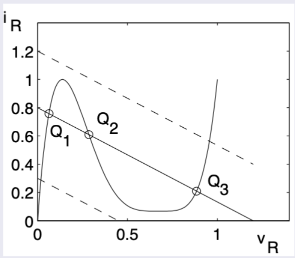


Figure: Multiple equilibria of the tunnel-diode circuit

Q_1 and Q_3 are stable nodes; Q_2 is a saddle.

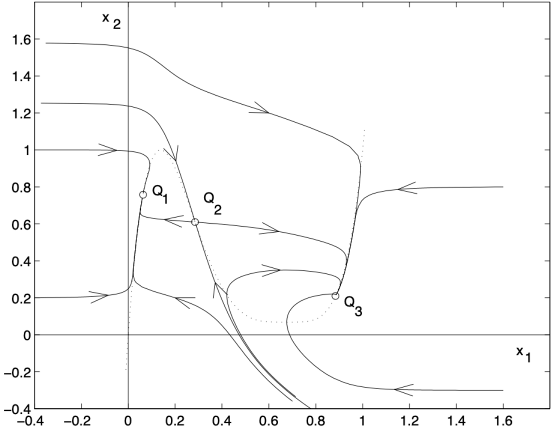


Figure: The phase portrait of multiple equilibria of the tunnel-diode circuit

Limit cycles

A system oscillates if it has a non-trivial periodic solution:

$$x(t+T) = x(t), \quad \forall\, t > 0.$$

Linear (harmonic) oscillator:

$$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z.$$

Its solution can be calculated by

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \quad z_2(t) = r_0 \sin(\beta t + \theta_0),$$

where $r_0 = \sqrt{z_1^2(0) + z_2^2(0)}$, and $\theta_0 = \tan^{-1}\left[\frac{z_2(0)}{z_1(0)}\right]$.

The linear oscillation is not practical

- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation).
- The amplitude of oscillation depends on the initial conditions (The same problems exist with oscillation of nonlinear systems due to a center equilibrium point, e.g., pendulum without friction).

Example

Van der Pol Oscillator:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2.\end{aligned}$$

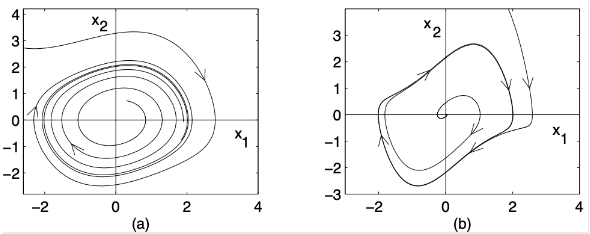


Figure: Phase portraint of Van der Pol Oscillator: (a) $\epsilon = 0.2$, (b) $\epsilon = 1$

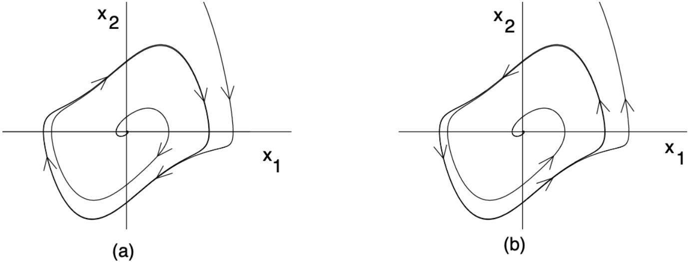


Figure: Phase portraint of (a) stable limit cycle and (b) unstable limit cycle

How can we determine whether there exist limit cycles (or periodic solutions)?

Lemma (Poincaré-Bendixon Criterion)

Consider the 2-order system $\dot{x} = f(x)$, and let M be a closed bounded subset of the plane such that

- *M contains no equilibrium points, or contains only one equilibrium point such that the Jacobian matrix $\frac{\partial f}{\partial x}$ at this point has eigenvalues with positive real parts. (Hence, the equilibrium point is unstable focus or unstable node.)*
- *Every trajectory starting in M stays in M for all future time.*

Then, M contains a periodic orbit.

Example

The nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2),\\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

has a unique equilibrium point at the origin, and the Jacobian matrix

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 1 - 3x_1^2 - x_2^2 & 1 - 2x_1 x_2 \\ -2 - 2x_1 x_2 & 1 - x_1^2 - 3x_2^2 \end{bmatrix}_{x=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

has eigenvalues $1 \pm j\sqrt{2}$.

- Let $M = \{V(x) \le c\}$, where $V(x) = x_1^2 + x_2^2$ and $c > 0$. It is clear that M is closed, bounded, and contains only one equilibrium point at which the Jacobian matrix has eigenvalues with positive real parts.

- On the surface $V(x) = c$, we have

$$\begin{aligned}\frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 =& 2x_1[x_1 + x_2 - x_1(x_1^2 + x_2^2)] + 2x_2[-2x_1 + x_2 - x_2(x_1^2 + x_2^2)] \\ =& 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 - 2x_1 x_2 \\ \le& 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2) \\ =& 3c - 2c^2.\end{aligned}$$

By choosing $c \ge 1.5$, we can ensure that all trajectories are trapped inside M. Hence, by the Poincaré-Bendixson criterion, we conclude that there is a periodic orbit in M.