Section 5: Multi-object conjugate priors
Version May 3, 2019

Multi-Object Tracking

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Introduction to multi-object conjugate priors

Multi-Object Tracking

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Conjugacy

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GAUSSIAN CONJUGATE PRIOR

For linear Gaussian measurement models

$$g_k(z|x) = \mathcal{N}(z\,;\,Hx, R)\,,$$

the Gaussian object state density,

$$p(x) = \mathcal{N}(x\,;\,m, P)\,,$$

is **Conjugate Prior**, meaning that the Bayes posterior object state density,

$$p(x|z) = \frac{g_k(z|x)p(x)}{p(z)} = \mathcal{N}(x\,;\,m', P')$$

will also be Gaussian, with mean m' and covariance P' given by the Kalman update.

DEFINITION OF CONJUGATE PRIOR

Definition: Conjugate prior

If $\mathscr{L}$ is a class of measurement models $g_k(z|x)$, and $\mathscr{F}$ is a class of prior distributions for x, then the class $\mathscr{F}$ is **conjugate** for $\mathscr{L}$ if

$$p(x|z) \in \mathscr{F} \text{ for all } g_k(z|x) \in \mathscr{L} \text{ and } p(x) \in \mathscr{F}.$$

Gaussian measurement models

If $\mathscr{L}$ is the class of Gaussian measurement models $g_k(z|x) = \mathcal{N}(z\,;\, Hx, R)$, then the class of conjugate prior distributions $\mathscr{F}$ is Gaussian, $p(x) = \mathcal{N}(x\,;\, m, P)$.

A list of conjugate priors can be found on, e.g., Wikipedia

“CONJUGATE PREDICTION”

- Prediction is also important in MOT.
- Consider linear Gaussian transition density, and a Gaussian posterior,

$$\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\,Fx_{k-1}, Q)$$
$$p(x_{k-1}) = \mathcal{N}(x_{k-1}\,;\,m', P')$$

- Then, the Chapman-Kolmogorov predicted density

$$p(x_k) = \int \pi_k(x_k|x_{k-1})p(x_{k-1})\mathrm{d}x_{k-1} = \mathcal{N}(x\,;\,m, P)$$

  is also Gaussian, with mean m and covariance P given by the Kalman prediction.
- We can think of $\mathcal{N}(x_{k-1}\,;\,m', P')$ as being “*conjugate*” to $\mathcal{N}(x_k\,;\,Fx_{k-1}, Q)$.

SINGLE-OBJECT CONJUGATE PRIOR

Definition: single-object conjugate prior

A single-object density is conjugate if, given an initial prior of this form, then all subsequent predicted and posterior distributions have the same form as the initial prior.

$$\text{Prediction:} \qquad p(x_k | z_{1:k-1}) = \int \pi_k(x_k | x_{k-1}) p(x_{k-1} | z_{1:k-1}) \, \mathrm{d}x_{k-1}$$

$$\text{Update:} \qquad p(x_k | z_{1:k}) = \frac{g_k(z_k | x_k) p(x_k | z_{1:k-1})}{\int g_k(z_k | x'_k) p(x'_k | z_{1:k-1}) \, \mathrm{d}x'_k}.$$

Note that this generalizes conjugacy

Here we have densities for exactly one object with state x_k.

MULTI-OBJECT CONJUGATE PRIOR

Definition: Multi-object conjugate prior

A multi-object density is conjugate if, given an initial prior of this form, then all subsequent predicted and posterior distributions have the same form as the initial prior.

$$\text{Prediction:} \qquad p(\mathbf{x}_k | \mathbf{z}_{1:k-1}) = \int p(\mathbf{x}_k | \mathbf{x}_{k-1}) p(\mathbf{x}_{k-1} | \mathbf{z}_{1:k-1}) \, \delta \mathbf{x}_{k-1}$$

$$\text{Update:} \qquad p(\mathbf{x}_k | \mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_k | \mathbf{x}_k) p(\mathbf{x}_k | \mathbf{z}_{1:k-1})}{\int p(\mathbf{z}_k | \mathbf{x}'_k) p(\mathbf{x}'_k | \mathbf{z}_{1:k-1}) \, \delta \mathbf{x}'_k}.$$

This generalizes conjugacy further

Here we have multi-object densities for a multi-object state $\mathbf{x}_k$.

MULTI-OBJECT CONJUGATE PRIOR

For the standard point object models, two multi-object conjugate priors are:

Multi-Bernoulli Mixture (both without, and with labels)

Prediction: $$\mathcal{MBM}_{k|k-1}(\mathbf{x}_k) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})\mathcal{MBM}_{k-1|k-1}(\mathbf{x}_{k-1})\,\delta\mathbf{x}_{k-1}$$

Update: $$\mathcal{MBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)\mathcal{MBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k|\mathbf{x}'_k)\mathcal{MBM}_{k|k-1}(\mathbf{x}'_k)\,\delta\mathbf{x}'_k}.$$

Poisson Multi-Bernoulli Mixture

Prediction: $$\mathcal{PMBM}_{k|k-1}(\mathbf{x}_k) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})\mathcal{PMBM}_{k-1|k-1}(\mathbf{x}_{k-1})\,\delta\mathbf{x}_{k-1}$$

Update: $$\mathcal{PMBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)\mathcal{PMBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k|\mathbf{x}'_k)\mathcal{PMBM}_{k|k-1}(\mathbf{x}'_k)\,\delta\mathbf{x}'_k}.$$

WHAT IS THE SIGNIFICANCE OF CONJUGACY?

- Thanks to the conjugacy, we can
  - given the assumed models, know what the theoretically exact density is,
  - find computationally tractable approximations,
    e.g., using tools such as minimisation of the Kullback-Leibler divergence
  - analyse the approximation error
- We know which parameters we need to fully represent the multi-object density.
  - Compare to a Kalman filter, where we propagate mean and covariance,

$$\ldots \to \begin{matrix} m_{k-1|k-1} \\ P_{k-1|k-1} \end{matrix} \to \begin{matrix} m_{k-1|k} \\ P_{k-1|k} \end{matrix} \to \begin{matrix} m_{k|k} \\ P_{k|k} \end{matrix} \to \begin{matrix} m_{k+1|k} \\ P_{k+1|k} \end{matrix} \to \ldots$$

  - With multi-object conjugate prior densities, similarly, we propagate the set of parameters that define the multi-object density.

CONJUGATE MULTI-OBJECT ALGORITHM

Different tracking filters can be derived from multi-object conjugate priors.

Conjugate multiple object tracking algorithm: pseudo-code

For $k = 1, 2, \ldots, K$

Chapman-Kolmogorov prediction

Bayes update

Reduction: Pruning and capping. Merging outside scope of course.

Estimation: Extract estimated objects

The tracking algorithms are similar in structure to MHT.

Object birth and object death

Multi-Object Tracking

Karl Granström

EXAMPLE: AN AUTONOMOUS VEHICLE WITH MULTIPLE SENSORS

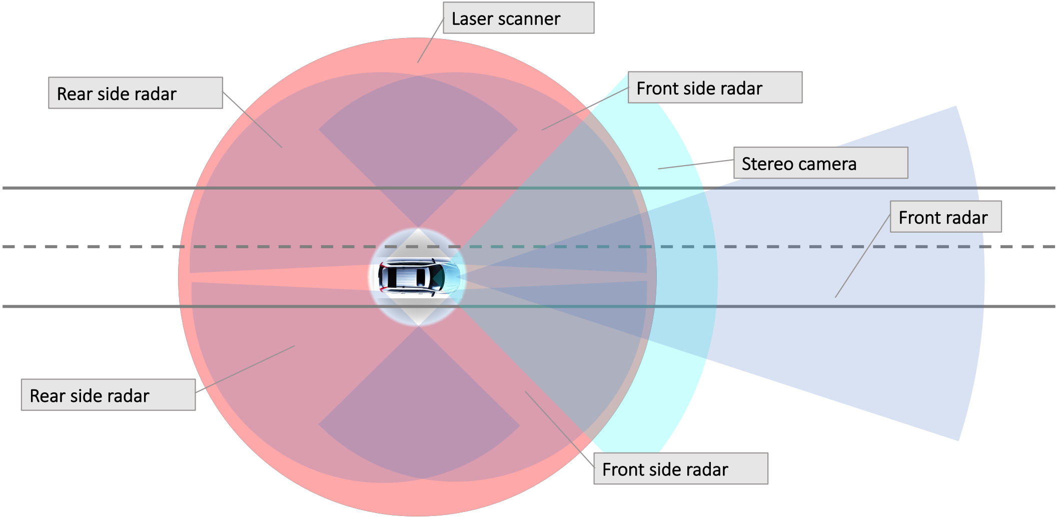


Image not to scale

TRACKING IN AN AREA OF INTEREST

- In MOT, we are interested in tracking objects in some area that is of interest to us.

Self-driving vehicle

The area around the vehicle that is in the union of the sensors' fields of view.

- In object tracking literature this area of interest is sometimes called
  - "surveillance area"
  - "tracking volume"
- Surveillance area not limited sensors' fields of view

Camera mounted to UAV

UAV flys over an area, gimbal mounted camera scans underneath UAV.

OBJECT BIRTH AND OBJECT DEATH

- We track objects in the surveillance area.
- A **true object** is one that is in the surveillance area at least one time step.
  - Can be simplified to “objects that are in the surveillance area for several scans”

Object birth

When the object first **enters the surveillance area**, i.e., when the object first appears.

Object death

When the object **leaves the surveillance area**, i.e., when the object disappears.

TIME EVOLUTION OF THE SET OF OBJECTS

- Time evolution of the set of objects

$$\mathbf{x}_k \quad \rightarrow \quad \mathbf{x}_{k+1}$$

- Set of objects at time $k+1$ is the union of surviving objects, and new-born objects

$$\mathbf{x}_{k+1} = \mathbf{x}^{\mathrm{S}}_{k+1} \cup \mathbf{x}^{\mathrm{B}}_{k+1}$$

- Object birth: $\mathbf{x}^{\mathrm{B}}_{k+1}$

- Object death: the objects at time k that do not survive to $k+1$, and are therefore not in $\mathbf{x}^{\mathrm{S}}_{k+1}$

EXAMPLE: 2D LIDAR

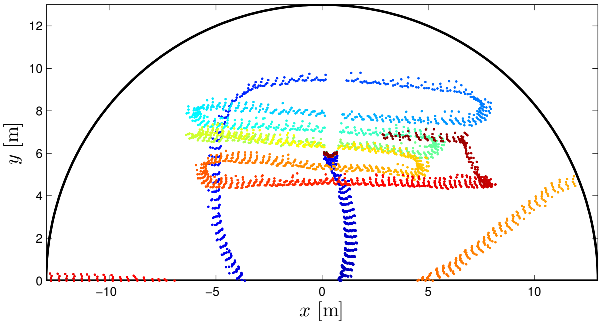
2D Lidar, 400 scans

Birth at time steps:

- 22
- 38
- 283
- 345

Death at time steps:

- 310
- 362



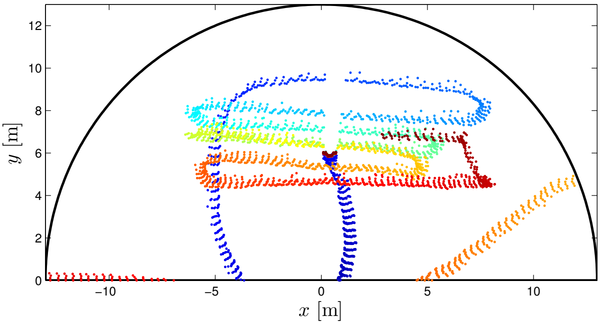
Color illustrates time step, from blue to red

EXAMPLE: 2D LIDAR BIRTH

2D Lidar, 400 scans

At birth,

- the object position is somewhere on the edge of the FOV.
- motion is more uncertain, but can reasonably be assumed to point into the FOV.

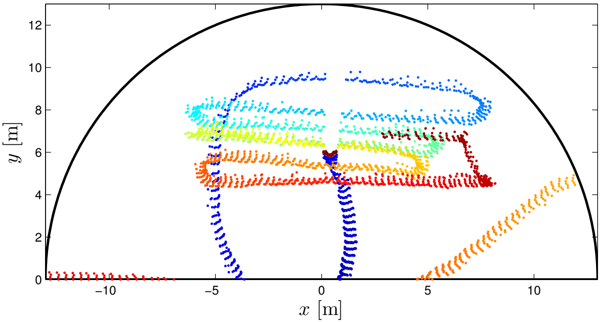


Color illustrates time step, from blue to red

EXAMPLE: 2D LIDAR DEATH

2D Lidar, 400 scans

- We cannot track outside FOV of a stationary sensor.
- Reasonable to assume that it is unlikely that a pedestrian just disappears in the middle of the FOV.



Color illustrates time step, from blue to red

STANDARD RFS BIRTH AND DEATH MODELS

To model object birth, we use models for

- the number of objects we expect to be born, i.e., a birth cardinality pmf,
- the states of the newborn objects, i.e., a birth state density.

To model object death, we use a model for

- the object survival probability

Note that by modelling survival, we get a model for death.

Bernoulli birth model

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BERNOULLI BIRTH

Bernoulli birth model

At time k an object is born with probability r_k^{B}, and if an object is born, the object state has density $p_k^{\mathrm{B}}(x)$

- The probability of birth r_k^{B} gives us the birth cardinality pmf, which is Bernoulli

$$\Pr[n_k^{\mathrm{B}} = j] = \begin{cases} 1 - r_k^{\mathrm{B}} & j = 0 \\ r_k^{\mathrm{B}} & j = 1 \\ 0 & j = 2, 3, \ldots \end{cases}$$

  where n_k^{B} denotes the number of births at time k.

- The state density $p_k^{\mathrm{B}}(x)$ tells us about the object's initial state.

MULTI BERNOULLI BIRTH

Multi Bernoulli (MB) birth model

Union of independent Bernoulli births. Parameters $\left\{\left(r_k^{\mathrm{B},i},\ p_k^{\mathrm{B},i}(\cdot)\right)\right\}_{i=1}^{N_k^{\mathrm{B}}}$.

- Expected number of births $\mathbb{E}\left[n_k^{\mathrm{B}}\right] = \sum_i r_k^{\mathrm{B},i}$
- MB birth cardinality distribution given by convolution of the Bernoulli birth pmfs
- If $r_k^{\mathrm{B},i} = r$ for all i, the birth cardinality pmf is Binomial,

$$\Pr\left[n_k^{\mathrm{B}} = j\right] = \binom{n_k^{\mathrm{B}}}{j} r^j (1-r)^{n_k^{\mathrm{B}}-j}$$

- Densities for the objects' initial states $p_k^{\mathrm{B},i}(x)$

GAUSSIAN MIXTURE BIRTH DENSITIES

Example: Gaussian mixture birth densities

- Linear Gaussian motion and measurement models

$$\pi_k(x_k|x_{k-1}) = \mathcal{N}\left(x_k\,;\,Fx_{k-1}, Q\right)$$
$$g_k(z_k|x_k) = \mathcal{N}\left(z_k\,;\,Hx_k, R\right)$$

- Gaussian object densities if intial prior is Gaussian,

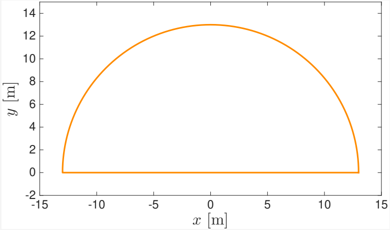
$$p_k^{\mathrm{B},i}(x) = \mathcal{N}\left(x_k\,;\,\mu_k^{\mathrm{B},i}, P_k^{\mathrm{B},i}\right)$$

- Parameters: $\left\{\left(r_k^{\mathrm{B},i}, \mu_k^{\mathrm{B},i}, P_k^{\mathrm{B},i}\right)\right\}_{i=1}^{N_k^{\mathrm{B}}}$

MODELLING THE BIRTH FOR A 2D LIDAR SENSOR

Example: MB birth for a 2D lidar sensor

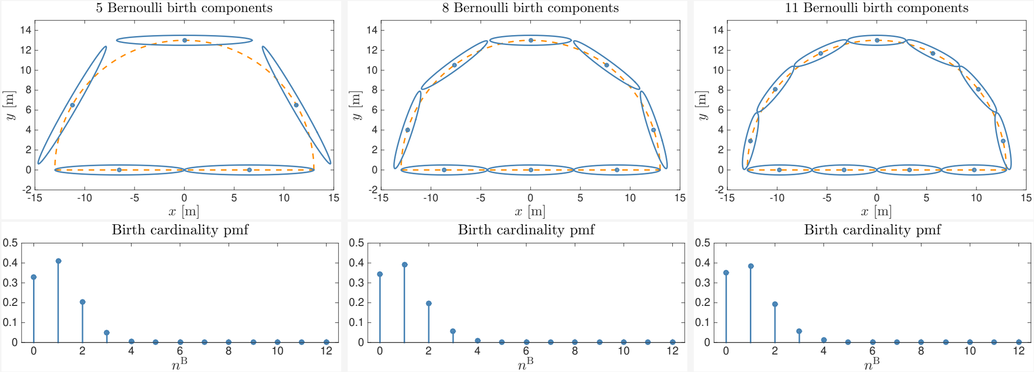
- Object state: 2D position and velocity
- Birth state densities $p_k^{\mathrm{B},i}(x)$: Gaussians
- Positions located around edge of FOV
- Velocity vectors could, e.g., be pointing into FOV, or be all-zero-vectors.



- Set Gaussian position covariances to follow edge of FOV.
- Set $r_k^{\mathrm{B},i} = \frac{\mathbb{E}[n_k^{\mathrm{B}}]}{N_k^{\mathrm{B}}}$, where $\mathbb{E}\left[n_k^{\mathrm{B}}\right]$ is some number adjusted to the scenario.

MODELLING THE BIRTH FOR A 2D LIDAR SENSOR

Example: MB birth for a 2D lidar sensor

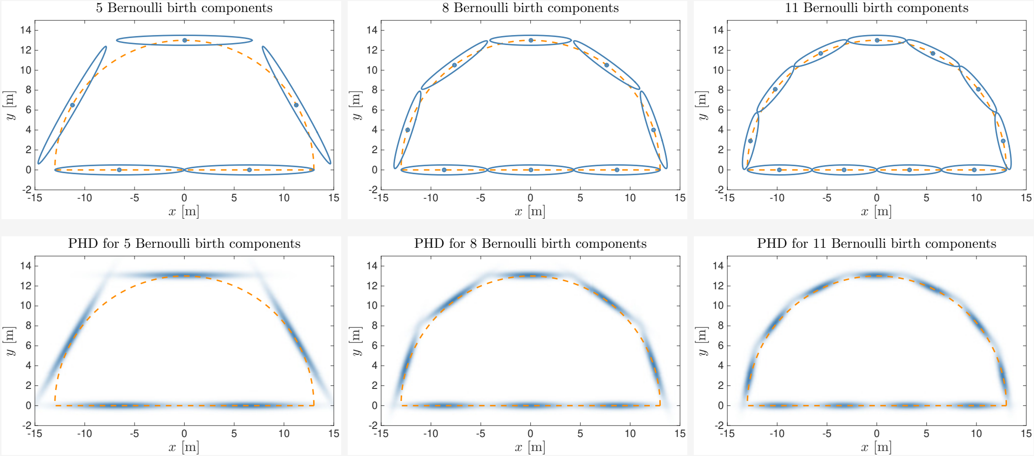


Here, the $r_k^{\mathrm{B},i}$ were set so that $\mathbb{E}\left[n_k^{\mathrm{B}}\right] = 1$

The more Bernoullis we use, the better the MB can follow the edge of the FOV.

MODELLING THE BIRTH FOR A 2D LIDAR SENSOR, BIRTH PHD

Example: 2D lidar sensor, MB births and their PHDs

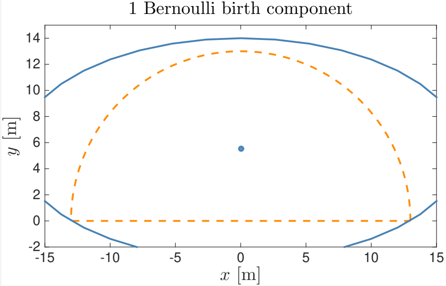


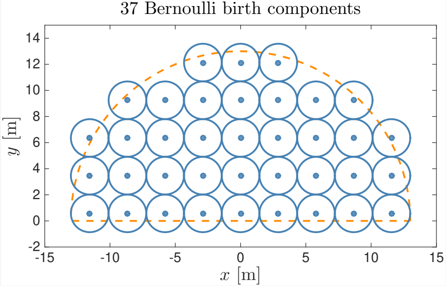
MODELLING THE BIRTH FOR A 2D LIDAR SENSOR

Example: objects that appear inside the FOV

We are not restricted to birth on the edge of the FOV:

- Single Bernoulli with Gaussian density, and covariance that covers FOV
- Several Bernoullis that cover the FOV





Generally, in practice the birth parameters are adapted to the tracking scenario!

MULTI-BERNOULLI MIXTURE BIRTH

Multi Bernoulli Mixture (MBM) birth model

Weighted mixture of birth MBs $\left\{\left(w_k^{\mathrm{B},j},\ \left\{\left(r_k^{\mathrm{B},i,j},\ p_k^{\mathrm{B},i,j}(\cdot)\right)\right\}_{i=1}^{N_k^{\mathrm{B},j}}\right)\right\}_j$.

- Not very common in practice. Can be useful for unusually specific birth models.

Example: Zero or three objects are born

- Probability p, three objects are born: $w_k^{\mathrm{B},1} = p$, $N_k^{\mathrm{B},j} = 3$, $r_k^{\mathrm{B},i,1} = 1$ for $i = 1, 2, 3$, and some appropriate $p_k^{\mathrm{B},i,1}(\cdot)$ for $i = 1, 2, 3$.
- Probability $1 - p$, no objects are born: $w_k^{\mathrm{B},2} = 1 - p$, $N_k^{\mathrm{B},2} = 0$

SUMMARY

- Object birth can be modelled by a Multi Bernoulli (Mixture)

- The parameters
  - Number of Bernoullis N_k^{B}
  - Probabilities of birth $r_k^{\mathrm{B},i}$
  - Birth state densities $p_k^{\mathrm{B},i}(\cdot)$

  should be adapted to the specific tracking scenario.

- Generally: the more Bernoullis, the more specific the MB birth model can be, but the price is increased computational cost, due to more possible data associations.

Poisson birth model

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POISSON BIRTH

Poisson birth model

At time k a possibly empty set of new-born objects $\mathbf{x}_k^{\mathrm{B}}$ appears, distributed according to a Poisson Point Process with intensity $\lambda_k^{\mathrm{B}}(x_k)$.

- Birth cardinality pmf is a Poisson pmf, with rate $\bar{\lambda}_k^{\mathrm{B}}$,

$$\Pr\left[n_k^{\mathrm{B}} = j\right] = e^{-\bar{\lambda}_k^{\mathrm{B}}} \frac{\left(\bar{\lambda}_k^{\mathrm{B}}\right)^j}{j!}$$

  where $\bar{\lambda}_k^{\mathrm{B}} = \int \lambda_k^{\mathrm{B}}(x_k)\mathrm{d}x_k$ and $\mathbb{E}\left[n_k^{\mathrm{B}}\right] = \bar{\lambda}_k^{\mathrm{B}}$.
- State density for each new-born object is

$$\frac{\lambda_k^{\mathrm{B}}(x_k)}{\bar{\lambda}_k^{\mathrm{B}}}$$

MIXTURE REPRESENTATION FOR INTENSITY

- Typically the Poisson intensity has a mixture representation,

$$\lambda_k^{\mathrm{B}}(x_k) = \sum_{i=1}^{N_k^{\mathrm{B}}} w_k^{\mathrm{B},i} p_k^{\mathrm{B},i}(x_k)$$

- Birth parameters: $\left\{ \left(w_k^{\mathrm{B},i}, p_k^{\mathrm{B},i}(\cdot) \right) \right\}_{i=1}^{N_k^{\mathrm{B}}}$.

- Expected number of births: $\mathbb{E}\left[n_k^{\mathrm{B}}\right] = \sum_{i=1}^{N_k^{\mathrm{B}}} w_k^{\mathrm{B},i}$.

GAUSSIAN MIXTURE INTENSITY

Example: Gaussian mixture intensity

- Linear Gaussian motion and measurement models

$$\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\,Fx_{k-1}, Q)$$
$$g_k(z_k|x_k) = \mathcal{N}(z_k\,;\,Hx_k, R)$$

- Gaussian object densities if intial prior is Gaussian,

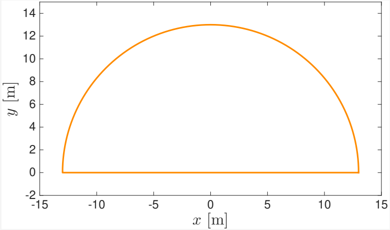
$$\lambda_k^{\mathrm{B}}(x_k) = \sum_{i=1}^{N_k^{\mathrm{B}}} w_k^{\mathrm{B},i} \mathcal{N}\left(x_k\,;\,\mu_k^{\mathrm{B},i}, P_k^{\mathrm{B},i}\right)$$

- Parameters: $\left\{\left(w_k^{\mathrm{B},i}, \mu_k^{\mathrm{B},i}, P_k^{\mathrm{B},i}\right)\right\}_{i=1}^{N_k^{\mathrm{B}}}$

MODELLING THE BIRTH FOR A 2D LIDAR SENSOR

Example: PPP birth for a 2D lidar sensor

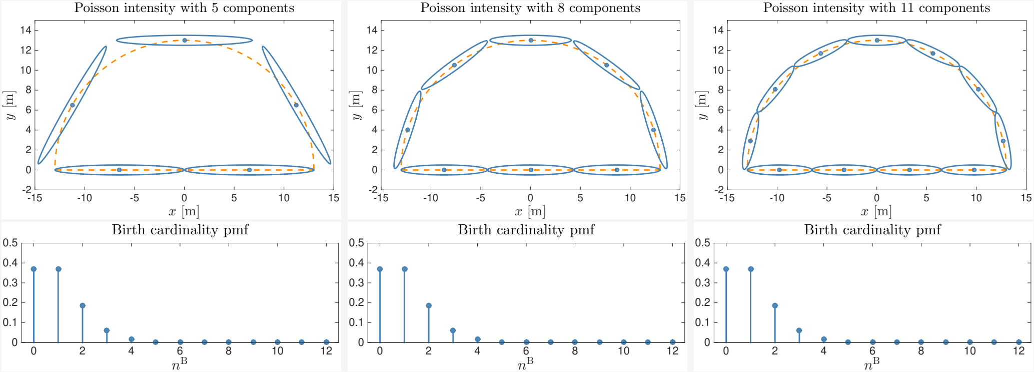
- Object state: 2D position and velocity
- Birth intensity: Gaussian mixture
- Positions located around edge of FOV
- Velocity vectors could, e.g., be pointing into FOV, or be all-zero-vectors.



- Set Gaussian position covariances to follow edge of FOV.
- Set $w_k^{\mathrm{B},i} = \frac{\mathbb{E}[n_k^{\mathrm{B}}]}{N_k^{\mathrm{B}}}$, where $\mathbb{E}\left[n_k^{\mathrm{B}}\right]$ is some number adjusted to the scenario.

MODELLING THE BIRTH FOR A 2D LIDAR SENSOR

Example: PPP birth for a 2D lidar sensor

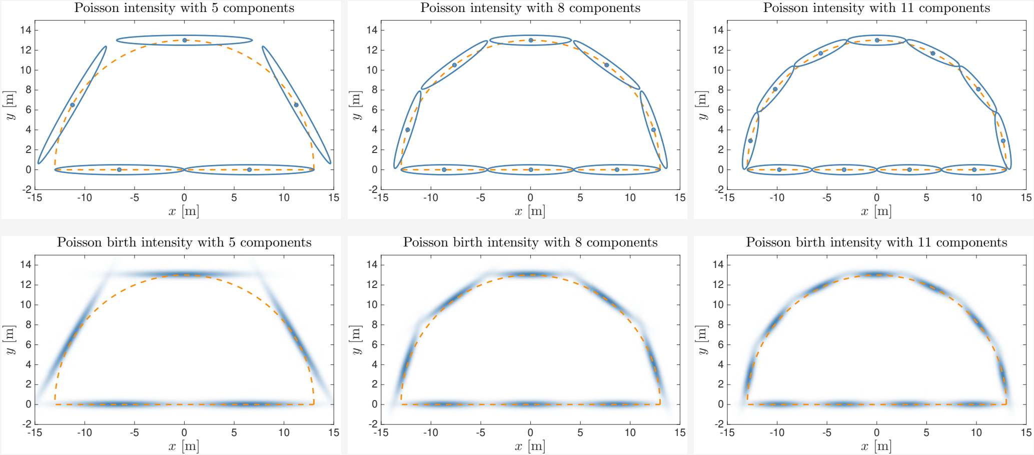


Here, the $w_k^{\mathrm{B},i}$ were set so that $\mathbb{E}\left[n_k^{\mathrm{B}}\right] = 1$

The more mixture components we use, the better $\lambda_k^{\mathrm{B}}(\cdot)$ can follow the edge of the FOV.

MODELLING THE BIRTH FOR A 2D LIDAR SENSOR

Example: 2D lidar sensor, PPP birth components and intensity

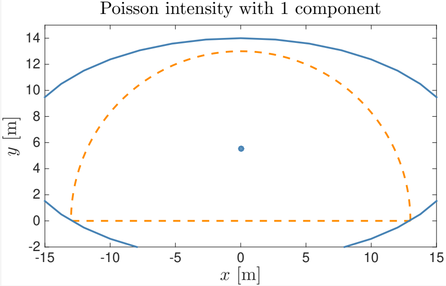


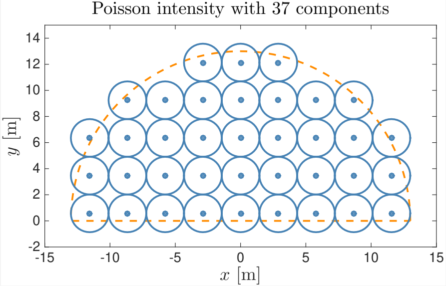
MODELLING THE BIRTH FOR A 2D LIDAR SENSOR

Example: objects that appear inside the FOV

We are not restricted to birth on the edge of the FOV:

- Intensity with a single Gaussian whose covariance covers the FOV
- Intensity with multiple Gaussians that together cover the FOV





Generally, in practice the birth parameters are adapted to the tracking scenario!

PPP OR MB BIRTH?

- What are the differences between the birth models?
  - Poisson Point Process
  - Multi-Bernoulli

- The MOT algorithms have different properties. More about this later.

SUMMARY

- Object birth can be modelled by a Poisson Point Process
- The Poisson intensity typically has a mixture representation. The parameters
  - Number of mixture components N_k^{B}
  - Component weights $w_k^{\mathrm{B},i}$
  - Component densities $p_k^{\mathrm{B},i}(\cdot)$

  should be adapted to the specific tracking scenario.
- Generally: the more mixture components in the Poisson intensity, the more specific the Poisson Point Process birth model can be, but the price is an increased computational cost.

The standard object death model

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OBJECT DEATH

Object death

When the object **leaves the surveillance area**, i.e., when the object disappears.

- We model object death implicitly, by modelling object survival.
- **The probability of survival**,

$$P^{S}(x_k)$$

  is the probability that an object with state x_k survives to the next time step $k+1$.
- The probability that the object remains in the surveillance area.

MODELLING THE PROBABILITY OF SURVIVAL

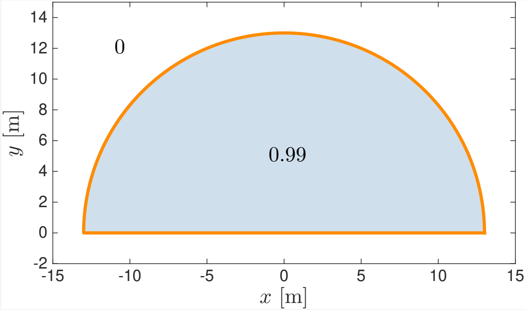
- The probability of survival is often large in the surveillance area and zero outside.

2D lidar sensor

Probability of survival

$$P^S(x_k) = \begin{cases} 0.99 & x_k \in SA \\ 0 & \text{else} \end{cases}$$

where *SA* is short for surveillance area



MBM density

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MULTI-BERNOULLI MIXTURE CONJUGATE PRIOR

MBM conjugate prior

With an MB(M) birth, the Multi-Bernoulli Mixture (MBM) density

$$\mathcal{MBM}_{k|k}(\mathbf{x}_k)$$

is a multi-object conjugate prior to the standard point object transition density $p(\mathbf{x}_k | \mathbf{x}_{k-1})$ and measurement model $p(\mathbf{z}_k | \mathbf{x}_k)$,

$$\text{Prediction:} \quad \mathcal{MBM}_{k|k-1}(\mathbf{x}_k) = \int p(\mathbf{x}_k | \mathbf{x}_{k-1}) \mathcal{MBM}_{k-1|k-1}(\mathbf{x}_{k-1}) \, \delta\mathbf{x}_{k-1}$$

$$\text{Update:} \quad \mathcal{MBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k | \mathbf{x}_k) \mathcal{MBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k | \mathbf{x}'_k) \mathcal{MBM}_{k|k-1}(\mathbf{x}'_k) \, \delta\mathbf{x}'_k}.$$

THE MBM MODEL

- Beyond multi-object conjugacy, why is the MBM model useful for MOT?

- Some uncertainties in MOT:
  - Are there any objects? How many? The Bernoulli existence probabilities.
  - If so, what are their states? The Bernoulli state densities.
  - Data association? Captured by the MB mixture.

- The MBM density nicely captures the relevant uncertainties.

MBM DENSITY

- The MBM density is defined as

$$\mathcal{MBM}_{k|k}(\mathbf{x}_k) = \sum_{h_k=1}^{\mathcal{H}_k} w_{k|k}^{h_k} \mathcal{MB}_{k|k}^{h_k}(\mathbf{x}_k) = \sum_{h_k=1}^{\mathcal{H}_k} w_{k|k}^{h_k} \sum_{\uplus \mathbf{x}_k^i = \mathbf{x}_k} \prod_{i=1}^{N_k^{h_k}} \mathcal{B}_{k|k}^{i,h_k}\left(\mathbf{x}_k^i\right)$$

  where $\mathcal{MB}(\mathbf{x})$ and $\mathcal{B}(\mathbf{x})$ denote Multi-Bernoulli density and Bernoulli density.

- Each hypothesis h_k corresponds to a sequence of data associations.
- Fully parameterised by the log-weights $\ell_{k|k}^{h_k} = \log(w_{k|k}^{h_k})$, and the Bernoulli parameters,

$$\left\{ \left(\ell_{k|k}^{h_k}, \left\{ \left(r_{k|k}^{i,h_k}, p_{k|k}^{i,h_k}(\cdot) \right) \right\}_{i=1}^{N_k^{h_k}} \right) \right\}_{h_k=1}^{\mathcal{H}_k}$$

MBM DENSITY, GAUSSIAN OBJECT DENSITIES

Example: the Bernoulli state pdfs are Gaussian

Gaussian object densities,

$$p_{k|k}^{i,h_k}(x_k^{i,h_k}) = \mathcal{N}\left(x_k^{i,h_k}\,;\,\mu_{k|k}^{i,h_k}, P_{k|k}^{i,h_k}\right)$$

MBM density parameters

$$\left\{\left(\ell_{k|k}^{h_k},\ \left\{\left(r_{k|k}^{i,h_k},\ \mu_{k|k}^{i,h_k},\ P_{k|k}^{i,h_k}\right)\right\}_{i=1}^{N_k^{h_k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$$

MBM FILTER

If we design an MOT algorithm for the MBM density, we get an MBM filter

MBM filter: pseudo-code

For $k = 1, 2, \ldots, K$

Prediction

Update

Reduction

Estimation

MBM prediction

Multi-Object Tracking

Karl Granström

MULTI-BERNOULLI MIXTURE PREDICTION

- **Posterior MBM parameters**
- Prediction

$$\mathcal{MBM}_{k+1|k}(\mathbf{x}_k) = \int p(\mathbf{x}_{k+1} \big| \mathbf{x}_k) \mathcal{MBM}_{k|k}(\mathbf{x}_k) \, \delta \mathbf{x}_k$$

  with transition density $p(\mathbf{x}_{k+1} \big| \mathbf{x}_k)$ with
  - Probability of survival $P_k^{\mathrm{S}}(x_k)$
  - Transition density $\pi_{k+1}(x_{k+1}|x_k)$
  - MB birth model: $\left\{ r_{k+1}^{\mathrm{B},i}, \; p_{k+1}^{\mathrm{B},i}(\cdot) \right\}_{i=1}^{N_{k+1}^{\mathrm{B}}}$
- **Predicted MBM parameters**

MBM PREDICTION WITH MB BIRTH, IN SUMMARY

MBM prediction with MB birth

- Each MB can be predicted independently of the other MBs
- Predicted MB parameters consist of union of
  - predicted Bernoulli parameters, and
  - Bernoulli birth parameters
- Number of parameters increases (we add Bernoulli birth parameters to each MB)

MBM PREDICTION WITH MB BIRTH

MBM prediction with MB birth: pseudo-code

- **Posterior parameters:** $\left\{ \left(\ell_{k|k}^{h_k}, \left\{ \left(r_{k|k}^{i,h_k}, p_{k|k}^{i,h_k}(\cdot) \right) \right\}_{i=1}^{N_k^{h_k}} \right) \right\}_{h_k=1}^{\mathcal{H}_k}$
- **Predicted parameters:** $\left\{ \left(\ell_{k|k}^{h_k}, \left\{ \left(r_{k+1|k}^{i,h_k}, p_{k+1|k}^{i,h_k}(\cdot) \right) \right\}_{i=1}^{N_{k+1}^{h_k}} \right) \right\}_{h_k=1}^{\mathcal{H}_k}$

  where, for each h_k,

$$
\begin{aligned}
&\left\{ \left(r_{k+1|k}^{i,h_k}, p_{k+1|k}^{i,h_k}(\cdot) \right) \right\}_{i=1}^{N_{k+1}^{h_k}} \\
&= \left\{ \mathrm{Predict} \left(r_{k|k}^{i',h_k}, p_{k|k}^{i',h_k}(\cdot) \right) \right\}_{i'=1}^{N_k^{h_k}} \bigcup \left\{ r_{k+1}^{\mathrm{B},i''}, p_{k+1}^{\mathrm{B},i''}(\cdot) \right\}_{i''=1}^{N_{k+1}^{\mathrm{B}}}
\end{aligned}
$$

  Increased number of Bernoullis $N_{k+1}^{h_k} = N_k^{h_k} + N_{k+1}^{\mathrm{B}}$

BERNOULLI PREDICTION

The predicted Bernoulli parameters,

$$\left(r^{i,h_k}_{k+1|k},\ p^{i,h_k}_{k+1|k}(\cdot)\right) = \mathrm{Predict}\left(r^{i,h_k}_{k|k},\ p^{i,h_k}_{k|k}(\cdot)\right)$$

are

$$\begin{aligned}
r^{i,h_k}_{k+1|k} &= r^{i,h_k}_{k|k} P^{\mathrm{S}}_{i,h_k} \\
p^{i,h_k}_{k+1|k}(x^i_{k+1}) &= \frac{\int \pi_{k+1}(x^i_{k+1}|x^i_k) P^{\mathrm{S}}(x^i_k) p^{i,h_k}_{k|k}(x^i_k) \mathrm{d}x^i_k}{P^{\mathrm{S}}_{i,h_k}} \\
&= \int \pi_{k+1}(x^i_{k+1}|x^i_k) \frac{P^{\mathrm{S}}(x^i_k) p^{i,h_k}_{k|k}(x^i_k)}{P^{\mathrm{S}}_{i,h_k}} \mathrm{d}x^i_k
\end{aligned}$$

where

$$P^{\mathrm{S}}_{i,h_k} = \int P^{\mathrm{S}}(x^i_k) p^{i,h_k}_{k|k}(x^i_k) \mathrm{d}x^i_k$$

BERNOULLI PREDICTION: EXAMPLE

Example: Constant probability of survival, linear Gaussian model

- $P^{\mathrm{S}}(x) = P^{\mathrm{S}}$
- $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}\left(x_{k+1}\,;\,F_{k+1}x_k, Q_{k+1}\right)$
- Posterior Bernoulli parameters: $r_{k|k}^{i,h_k},\ \mu_{k|k}^{i,h_k},\ P_{k|k}^{i,h_k}$
- Predicted parameters: probability of existence, mean, covariance

$$
\begin{aligned}
r_{k+1|k}^{i} &= r_{k|k}^{i} P^{\mathrm{S}} \\
\mu_{k+1|k}^{i} &= F_{k+1}\mu_{k|k}^{i} \\
P_{k+1|k}^{i} &= F_{k+1}P_{k|k}^{i}F_{k+1}^{\mathrm{T}} + Q_{k+1}
\end{aligned}
$$

2D EXAMPLE, $P^S = 0.9$

Linear Gaussian example: state with 2D object position, 2D velocity

Four Bernoullis, Gaussian densities

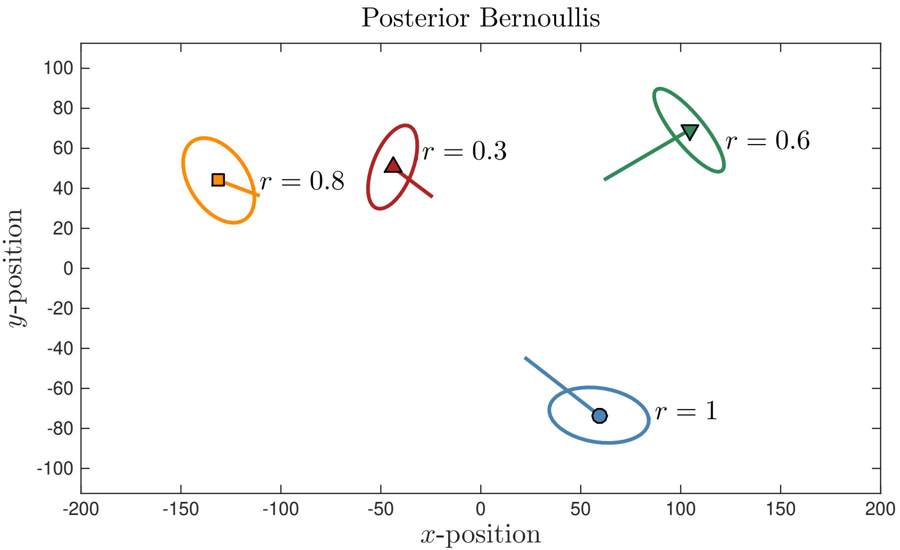
- $r^i_{k|k},\ m^i_{k|k},\ P^i_{k|k}$ for $i = 1, 2, 3, 4$

Motion model: constant velocity, constant probability of survival

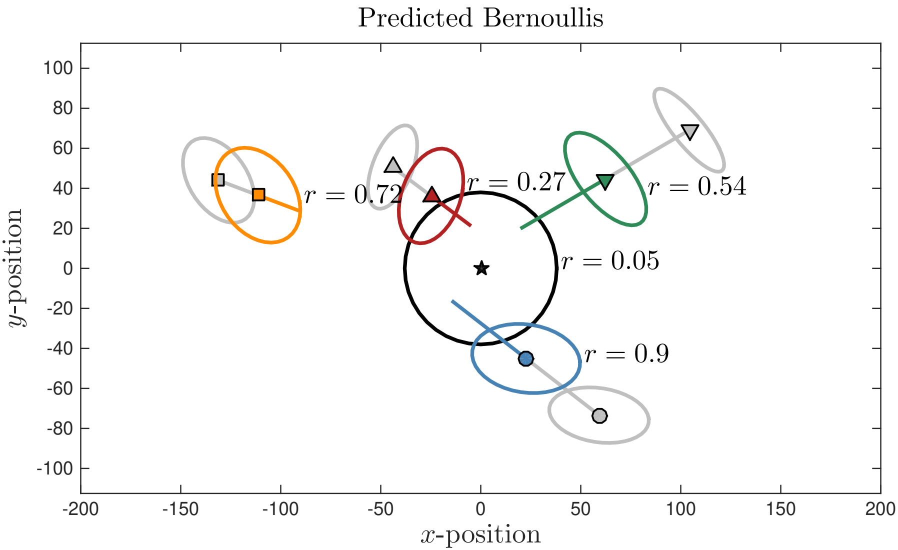
- $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\ Fx_{k-1}, Q)$
- $P^S(x_{k-1}) = 0.9$
- Single birth Bernoulli, position in origin, zero velocity

Visualizations: Bernoulli parameters: $r^i_{k+1|k},\ \mu^i_{k+1|k},\ P^i_{k+1|k}$ for $i = 1, 2, 3, 4, 5$

2D EXAMPLE, $P^{S} = 0.9$



2D EXAMPLE, $P^{S} = 0.9$



1D EXAMPLE, $P^{\mathrm{S}} = 0.99$

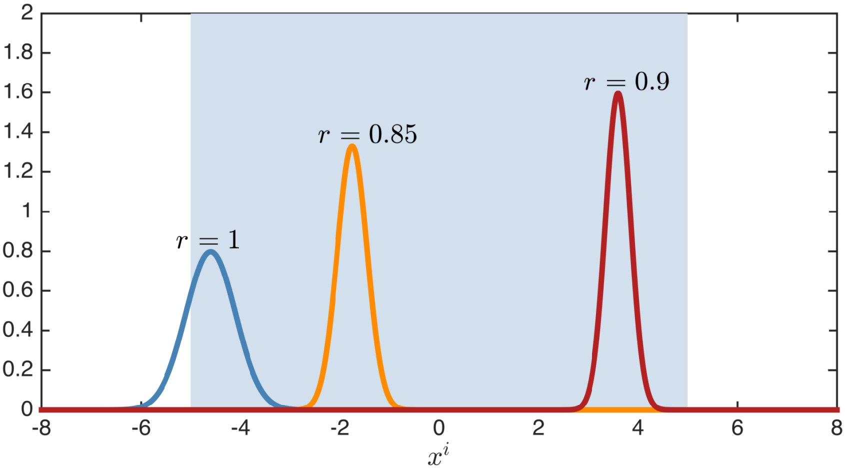
Linear Gaussian example

- Three Bernoullis, scalar states and Gaussian densities
- Random walk motion model: $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\,x_{k-1}, 0.25)$
- Probability of survival: $P^{\mathrm{S}}(x) = \begin{cases} 0.99 & x \in [-5\,,\,5] \\ 0 & \text{otherwise} \end{cases}$
- Two birth Bernoullis: positions -2.5 and 2.5, probability of birth 0.01

Visualization: predicted density, and probability of existence

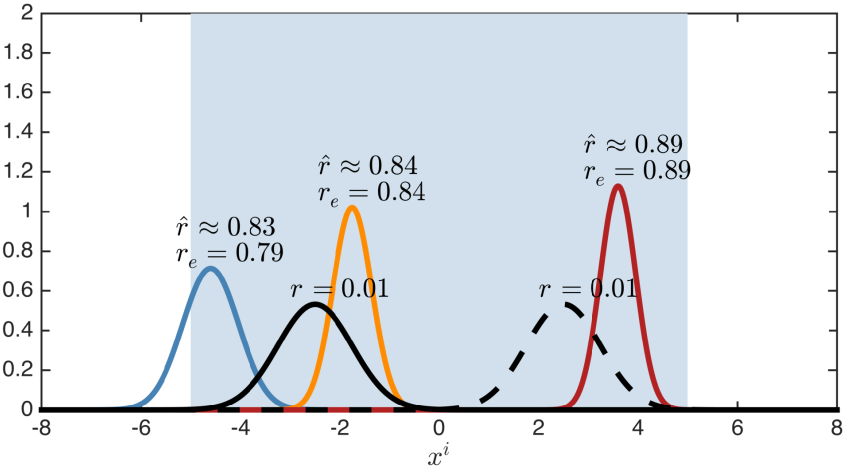
- Exact: $r^{i,h_k}_{k+1|k} = r^{i,h_k}_{k|k} \int P^{\mathrm{S}}(x^i_k) p^{i,h_k}_{k|k}(x^i_k) \mathrm{d}x^i_k$
- Sigma point approximation of integral, $\int P^{\mathrm{S}}(x^i_k) p^{i,h_k}_{k|k}(x^i_k) \mathrm{d}x^i_k \approx \sum_j w_j P^{\mathrm{S}}(x_j)$

1D EXAMPLE, $P^S = 0.99$



Posterior Bernoulli parameters

1D EXAMPLE, $P^S = 0.99$



Predicted Bernoulli parameters: r_e exact probability of existence; $\hat{r}$ approximation

MBM update

Multi-Object Tracking

Karl Granström

MULTI-BERNOULLI MIXTURE UPDATE

- **Prior MBM parameters**
- Update

$$\mathcal{MBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)\mathcal{MBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k|\mathbf{x}'_k)\mathcal{MBM}_{k|k-1}(\mathbf{x}'_k)\,\delta\mathbf{x}'_k}$$

  with multi-object measurement model $p(\mathbf{z}_k|\mathbf{x}_k)$ with
  - Probability of detection $P_k^{\mathrm{D}}(x_k)$
  - Measurement model $g_k(z_k|x_k)$
  - Poisson clutter intensity: $\lambda_c(z_k)$
- **Posterior MBM parameters**

MULTI-BERNOULLI MIXTURE UPDATE, IN SUMMARY

MBM update for the standard point object models

- For each prior MB, multiple data associations
- For each prior MB and each data association, we get an MB in the posterior MBM
- For each Bernoulli, two possibilities:
  - Either associated to one of the measurements,
  - or misdetected.

Important “builiding blocks” of MBM update:

- update of Bernoulli with associated measurement
- update of misdetected Bernoulli
- posterior log-weights

HYPOTHESIS ORIENTED-MBM UPDATE

Hypothesis oriented-MBM update: pseudo-code

Input: $\left\{ \left(\ell^{h_{k-1}} \, , \, \left\{ \left(r_{k|k-1}^{i,h_{k-1}}, \, p_{k|k-1}^{i,h_{k-1}}(\cdot) \right) \right\}_{i=1}^{N_{k-1}^{h_{k-1}}} \right) \right\}_{h_{k-1}=1}^{\mathcal{H}_{k-1}}$

Initialise $h_k = 0$

For $h_{k-1} = 1, \ldots, \mathcal{H}_{k-1}$

Create cost matrix $L^{h_{k-1}}$, and compute $M_{h_{k-1}}$ associations θ_m

For $m = 1, \ldots, M_{h_{k-1}}$

Increase: $h_k \leftarrow h_k + 1$

Compute posterior MB parameters: detected & misdetected Bernoulli, log-weight

Set $\mathcal{H}_k = h_k$

Normalise log-weights $\ell^{h_k} \leftarrow \tilde{\ell}^{h_k}$

Output: $\left\{ \left(\ell^{h_k} \, , \, \left\{ \left(r_{k|k}^{i,h_k}, \, p_{k|k}^{i,h_k}(\cdot) \right) \right\}_{i=1}^{N_k^{h_k}} \right) \right\}_{h_k=1}^{\mathcal{H}_k}$

BERNOULLI UPDATE: ASSOCIATED MEASUREMENT

For prior hypothesis h, if $\theta^i = j$, the updated parameters for the ith Bernoulli are

$$r_{k|k}^{i,\theta^i,h} = 1$$

$$p_{k|k}^{i,\theta^i,h}(x_k^i) = \frac{P^{\mathrm{D}}(x_k^i) g_k(z_k^{\theta^i}|x_k^i) p_{k|k-1}^{i,h}(x_k^i)}{\int P^{\mathrm{D}}(x_k^i) g_k(z_k^{\theta^i}|x_k^i) p_{k|k-1}^{i,h}(x_k^i) \mathrm{d}x_k^i}$$

Posterior r conditioned on θ

Must equal 1, because we cannot associate to an object that does not exist at time k.

Predicted log-likelihood

$$\ell_k^{i,\theta^i,h} = \log\left(r_{k|k-1}^{i,h} \int P^D\left(x_k^i\right) g_k\left(z_k^{\theta^i}|x_k^i\right) p_{k|k-1}^{i,h}(x_k^i) \mathrm{d}x_k^i\right)$$

BERNOULLI UPDATE: ASSOCIATED MEASUREMENT

Example: Constant P^{D}, linear Gaussian $g_k(z|x)$

- Prior parameters: $r_{k|k-1}^{i,h}$, $\mu_{k|k-1}^{i,h}$, $P_{k|k-1}^{i,h}$
- Posterior parameters:

$$
\begin{aligned}
r_{k|k}^{i,\theta^i,h} &= 1, \\
\mu_{k|k}^{i,\theta^i,h} &= \mu_{k|k}^{i,h} + P_{k|k-1}^{i,h} H_k^{\mathrm{T}} \left(S_k^{i,h} \right)^{-1} \left(z_k^j - H_k \mu_{k|k-1}^{i,h} \right), \\
P_{k|k}^{i,\theta^i,h} &= P_{k|k-1}^{i,h} - P_{k|k-1}^{i,h} H_k^{\mathrm{T}} \left(S_k^{i,h} \right)^{-1} H_k P_{k|k-1}^{i,h}
\end{aligned}
$$

  where $S_k^{i,h} = H_k P_{k|k-1}^{i,h} H_k^{\mathrm{T}} + R_k$ is the innovation covariance.

- Predicted log-likelihood

$$
\ell_k^{i,\theta^i,h} = \log \left(r_{k|k-1}^{i,h} P^D \mathcal{N} \left(z_k^{\theta^i} ; H_k \mu_{k|k-1}^{i,h}, S_k^{i,h} \right) \right)
$$

BERNOULLI UPDATE: NO MEASUREMENT ASSOCIATED

For prior hypothesis h, if $\theta^i = 0$, the updated parameters for the ith Bernoulli are

$$r_{k|k}^{i,0,h} = \frac{r_{k|k-1}^{i,h} P_{i,h}^{\mathrm{MD}}}{1 - r_{k|k-1}^{i,h} + r_{k|k-1}^{i,h} P_{i,h}^{\mathrm{MD}}} \qquad p_{k|k}^{i,0,h}(x) = \frac{(1 - P^{\mathrm{D}}(x_k)) p_{k|k-1}^{i,h}(x)}{P_{i,h}^{\mathrm{MD}}}$$

where $P_{i,h}^{\mathrm{MD}} = \int \left(1 - P^D\left(x_k^i\right)\right) p_{k|k-1}^{i,h}\left(x_k^i\right) \mathrm{d}x_k^i$

Posterior r conditioned on θ

Relative probability of:

1) existence $r_{k|k-1}^{i,h}$ and misdetection $P_{i,h}^{\mathrm{MD}}$, 2) non-existence $1 - r_{k|k-1}^{i,h}$.

Predicted log-likelihood

$$\ell_k^{i,0,h} = \log\left(1 - r_{k|k-1}^{i} + r_{k|k-1}^{i} P_{i,h}^{\mathrm{MD}}\right)$$

BERNOULLI UPDATE: NO MEASUREMENT ASSOCIATED

Example: Constant P^{D}, linear Gaussian $g_k(z|x)$

- Prior parameters: $r^{i,h}_{k|k-1}$, $\mu^{i,h}_{k|k-1}$, $P^{i,h}_{k|k-1}$
- Posterior parameters:

$$
\begin{aligned}
r^{i,0,h}_{k|k} &= \frac{r^{i,h}_{k|k-1}(1-P^{\mathrm{D}})}{1-r^{i,h}_{k|k-1}+r^{i,h}_{k|k-1}(1-P^{\mathrm{D}})} \\
\mu^{i,0,h}_{k|k} &= \mu^{i,h}_{k|k-1} \\
P^{i,0,h}_{k|k} &= P^{i,h}_{k|k-1}
\end{aligned}
$$

- Predicted log-likelihood

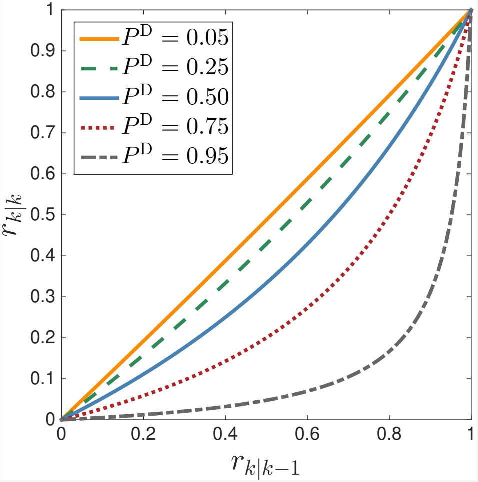
$$
\ell^{i,0,h}_{k} = \log\left(1-r^{i,h}_{k|k-1}+r^{i,h}_{k|k-1}(1-P^{\mathrm{D}})\right)
$$

BERNOULLI UPDATE: NO MEASUREMENT ASSOCIATED

Example: Constant P^{D}, linear Gaussian models, $\theta^i = 0$

Posterior probability of existence, given $\theta^i = 0$,

$$r_{k|k}^{i,0,h} = \frac{r_{k|k-1}^{i,h}(1 - P^{\mathrm{D}})}{1 - r_{k|k-1}^{i,h} + r_{k|k-1}^{i,h}(1 - P^{\mathrm{D}})}$$



NON-NORMALISED POSTERIOR LOG-WEIGHTS

For a prior MB h_{k-1} and a data association θ_k, the non-normalized posterior log-weight is

$$\begin{aligned}
\tilde{\ell}_{k|k}^{h_{k-1},\theta_k} &= \underbrace{\ell_{k|k-1}^{h_{k-1}}}_{\text{Prior}} + \underbrace{\sum_{i:\theta^i=0} \ell_k^{i,0,h}}_{\text{Misdetection}} + \underbrace{\sum_{i:\theta^i\neq 0} \ell_k^{i,\theta^i,h}}_{\text{Assoc. meas.}} + \underbrace{\sum_{j:\nexists\theta^i=j} \log\left(\lambda_c\left(z_k^j\right)\right)}_{\text{Clutter}} \\
&= \ell_{k|k-1}^{h_{k-1}} + \sum_{i:\theta^i=0} \ell_k^{i,0,h} + \sum_{i:\theta^i\neq 0} \left[\ell_k^{i,\theta^i,h} - \log\left(\lambda_c\left(z_k^{\theta^i}\right)\right)\right] + \sum_{j=1}^{m_k} \log\left(\lambda_c\left(z_k^j\right)\right) \\
&= \ell_{k|k-1}^{h_{k-1}} + \sum_{i=1}^{N_{k-1}^{h_{k-1}}} \tilde{\ell}_k^{i,\theta^i,h} + \text{Constant independent of } h_{k-1} \text{ and } \theta_k
\end{aligned}$$

where

$$\tilde{\ell}_k^{i,\theta^i,h} = \begin{cases} \ell_k^{i,\theta^i,h} - \log\left(\lambda_c\left(z_k^{\theta^i}\right)\right) & \text{if } \theta^i \neq 0 \\ \ell_k^{i,0,h} & \text{if } \theta^i = 0 \end{cases}$$

Compare to the non-normalized posterior weight in n-object tracking

EXAMPLE MBM UPDATE VISUALIZATION

Prior and model, 2D scenario

- MBM with two MBs, each with two Bernoullis with Gaussian state densities
- Measurement model:

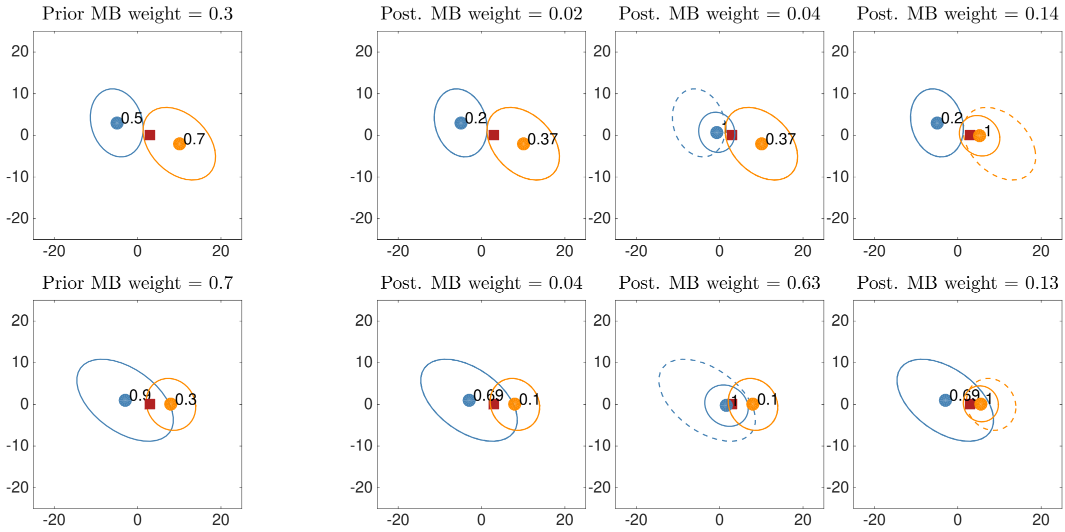
$$P^{\mathrm{D}} = 0.75$$

$$g_k(z|x) = \mathcal{N}(z\,;\, x, 9\mathbf{I}_2)$$

$$\lambda_c(z) = \begin{cases} 4 \times 10^{-4} & z \in [-25, 25] \times [-25, 25] \\ 0 & z \notin [-25, 25] \times [-25, 25] \end{cases}$$

- Single measurement $\Rightarrow$ 3 DAs for each prior MB $\Rightarrow$ Posterior MBM with 6 MBs

EXAMPLE MBM UPDATE



HANDLING THE DATA ASSOCIATIONS

The data association problem is handled analogously to tracking n objects:

- Use gating to remove very unlikely associations and group Bernoullis/measurements
- For each group, form cost matrix with negative log likelihoods
- Use some algorithm to find M associations, e.g.,
  - Murty
  - Gibbs' sampling
- Truncate all other associations

COST MATRIX

Let there be m detections, and consider a prior MB h with N^h Bernoullis. The cost matrix is

$$L^h = \begin{bmatrix} -\ell^{1,1,h} & -\ell^{1,2,h} & \dots & -\ell^{1,m,h} & -\ell^{1,0,h} & \infty & \dots & \infty \\ -\ell^{2,1,h} & -\ell^{2,2,h} & \dots & -\ell^{2,m,h} & \infty & -\ell^{2,0,h} & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\ell^{N^h,1,h} & -\ell^{N^h,2,h} & \dots & -\ell^{N^h,m,h} & \infty & \infty & \dots & -\ell^{N^h,0,h} \end{bmatrix}$$

where

$$\ell^{i,j,h} = \log\left(\frac{r^{i,h}_{k|k-1} \int P^D\left(x^i_k\right) g_k\left(z^j_k | x^i_k\right) p^{i,h}_{k|k-1}(x^i_k)\mathrm{d}x^i_k}{\lambda_c\left(z^j_k\right)}\right)$$

$$\ell^{i,0,h} = \log\left(1 - r^{i,h}_{k|k-1} + r^{i,h}_{k|k-1} \int \left(1 - P^D\left(x^i_k\right)\right) p^{i,h}_{k|k-1}\left(x^i_k\right)\mathrm{d}x^i_k\right)$$

MBM post processing

Multi-Object Tracking

Karl Granström

MBM POST PROCESSING

After the MBM prediction and the MBM update we have an MBM density $\mathcal{MBM}_{k|k}(\mathbf{x}_k)$ with parameters

$$\left\{\left(\ell_{k|k}^{h_k},\ \left\{\left(r_{k|k}^{i,h_k},\ p_{k|k}^{i,h_k}(\cdot)\right)\right\}_{i=1}^{N_k^{h_k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$$

- Reduction:
  - Reduce $\mathcal{H}_k$ and $N_k^{h_k}$
  - Important for computational cost
  - Pruning, and capping. Merging is outside the scope of the course
- Estimation:
  - Extracting a set of estimated object states from the posterior density.

MBM REDUCTION

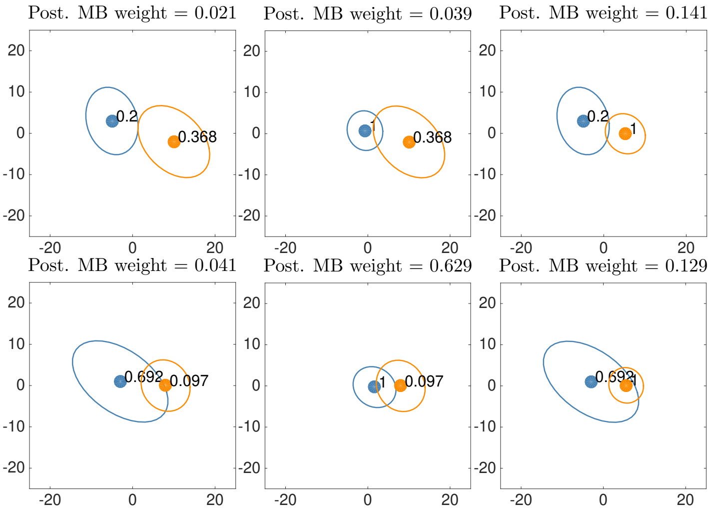
MBM reduction

- **MBM pruning:** prune MB h_k if $\ell_{k|k}^{h_k} \leq \Gamma$
- **MBM capping:** if $\mathcal{H}_k > N_{\max}$, keep the $N_{\max}$ MBs with largest log-weights.
- **Bernoulli pruning:** in each MB h_k, prune Bernoulli i if $r_{k|k}^{i,h} < \Gamma^r$

After pruning and capping, remaining log-weights are re-normalized.

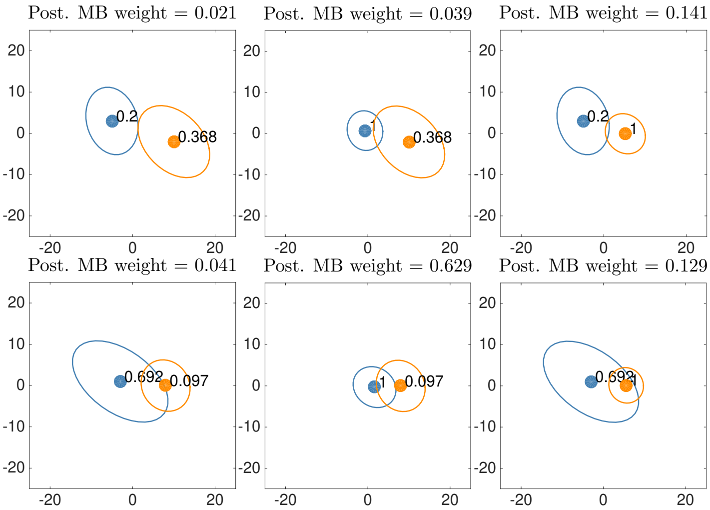
Outside the scope of the course: MBM merging

EXAMPLE MBM REDUCTION



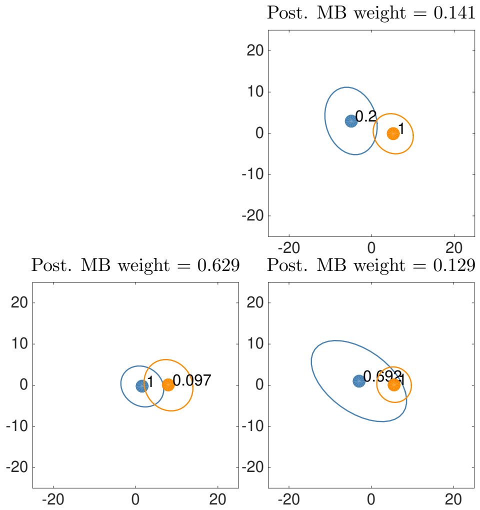
EXAMPLE MBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$



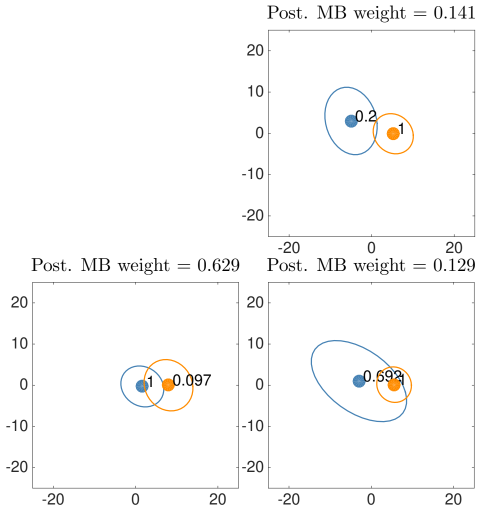
EXAMPLE MBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$



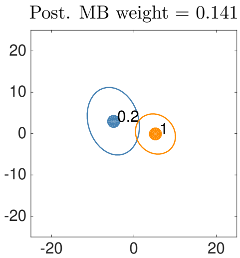
EXAMPLE MBM REDUCTION

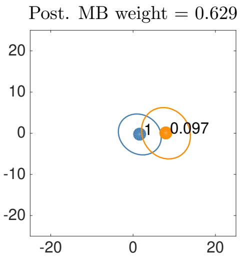
- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$



EXAMPLE MBM REDUCTION

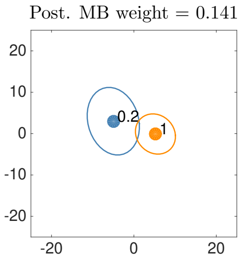
- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\max} = 2$

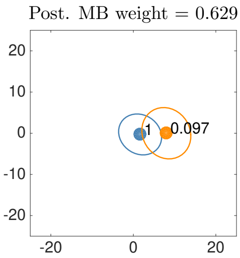




EXAMPLE MBM REDUCTION

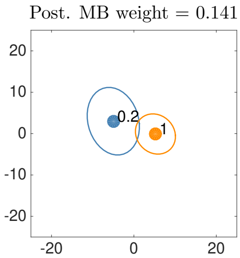
- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$
- Bernoulli pruning, $\Gamma^r = 0.1$

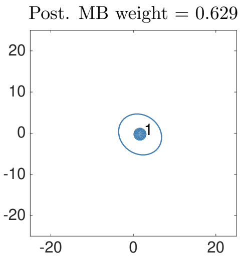




EXAMPLE MBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\text{max}} = 2$
- Bernoulli pruning,
  $\Gamma^r = 0.1$





EXAMPLE MBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\max} = 2$
- Bernoulli pruning,
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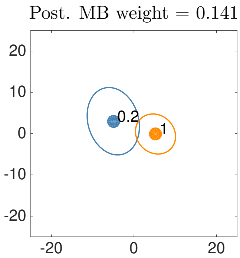
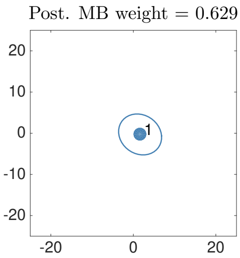
- Re-normlize weights





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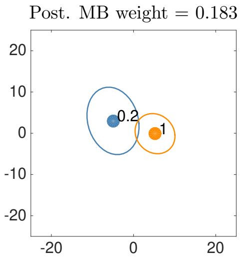
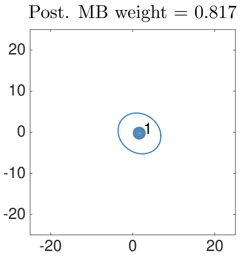
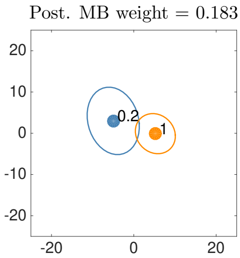
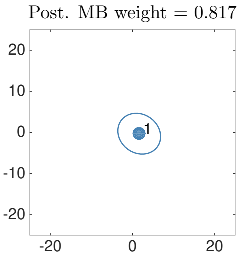




EXAMPLE MBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$
- Bernoulli pruning, $\Gamma^r = 0.1$
- Re-normlize weights
- **Note:** typically, Γ and Γ^r are much smaller, and $N_{\max}$ is much larger





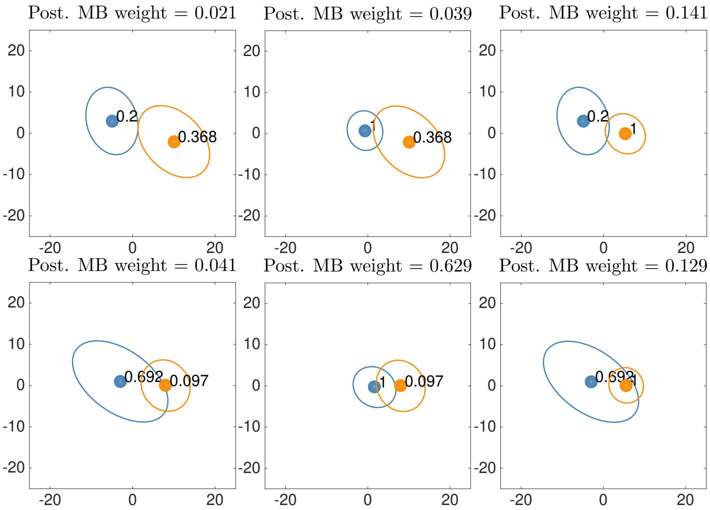
MBM ESTIMATION

Simple MBM estimator

- Initialise as empty set: $\hat{\mathbf{x}}_{k|k} = \emptyset$
- MB with largest weight: $h^\star = \max_{h_k} \ell_{k|k}^{h_k}$
- For $i = 1, \ldots, N_k^{h^\star}$, if $r_{k|k}^{h^\star,i} > \Gamma^e$: $\hat{\mathbf{x}}_{k|k} \leftarrow \hat{\mathbf{x}}_{k|k} \cap \hat{x}_{k|k}^{i,h^\star}$
- For example, expected value or MAP estimate,

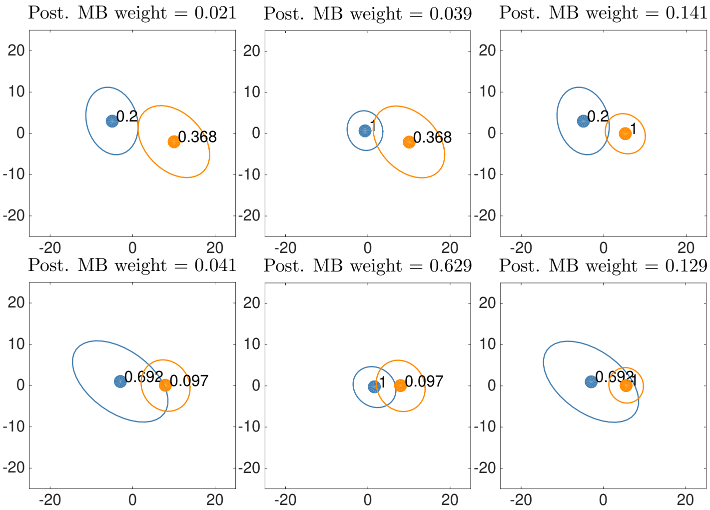
$$\bar{x}_{k|k}^{i} = \int x_k p_{k|k}^{h^\star,i}(x_k)\mathrm{d}x_k, \qquad \hat{x}_{k|k}^{i,\mathrm{MAP}} = \arg\max_{x_k} p_{k|k}^{h^\star,i}(x_k)$$

EXAMPLE MBM ESTIMATOR



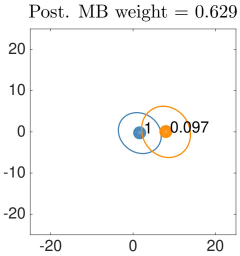
EXAMPLE MBM ESTIMATOR

- Largest weight



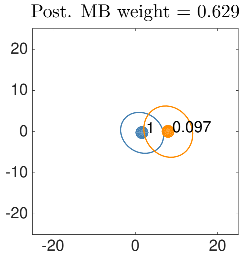
EXAMPLE MBM ESTIMATOR

- Largest weight



EXAMPLE MBM ESTIMATOR

- Largest weight
- Extract: $\Gamma^e = 0.5$



EXAMPLE MBM ESTIMATOR

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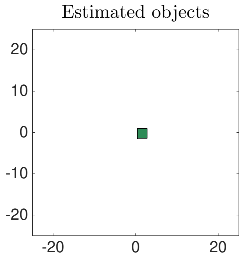


PMBM density

Multi-Object Tracking

Karl Granström

A MOTIVATION FOR CHANGING THE BIRTH MODEL

In MBM filters, an MB birth model is used:

- In the prediction, add $N_k^{\mathbb{B}}$ birth components to each MB in the mixture
- If an object actually appeared and was detected, then in the update this detection can be associated to the birth component, and we can start to track the object.

But why add birth components before we know if there are any detections?

- Could the addition of new Bernoulli components, corresponding to the new potential objects, not be **measurement-driven**?
- Yes, it could, **if we use a PPP birth model** instead of a MB birth model.

POISSON MULTI-BERNOULLI MIXTURE CONJUGATE PRIOR

PMBM conjugate prior

With a Poisson birth, the Poisson Multi-Bernoulli Mixture (PMBM) density

$$\mathcal{PMBM}_{k|k}(\mathbf{x}_k)$$

is a multi-object conjugate prior to the standard point object transition density $p(\mathbf{x}_k \middle| \mathbf{x}_{k-1})$ and measurement model $p(\mathbf{z}_k \middle| \mathbf{x}_k)$,

$$\begin{aligned}
\text{Prediction:} \quad & \mathcal{PMBM}_{k|k-1}(\mathbf{x}_k) = \int p(\mathbf{x}_k \middle| \mathbf{x}_{k-1}) \mathcal{PMBM}_{k-1|k-1}(\mathbf{x}_{k-1}) \, \delta\mathbf{x}_{k-1} \\
\text{Update:} \quad & \mathcal{PMBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k \middle| \mathbf{x}_k) \mathcal{PMBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k \middle| \mathbf{x}'_k) \mathcal{PMBM}_{k|k-1}(\mathbf{x}'_k) \, \delta\mathbf{x}'_k}.
\end{aligned}$$

THE PMBM MODEL

- Beyond multi-object conjugacy, why is the PMBM model useful for MOT?
- Some uncertainties in MOT:
  - Are there any objects? How many?
    - Detected objects: Bernoulli existence probabilities
    - Undetected objects: PPP intensity
  - If so, what are their states?
    - Detected objects: Bernoulli state densities
    - Undetected objects: PPP intensity
  - Data association? Captured by the MB mixture.
- The PMBM density nicely captures the relevant uncertainties.

THE PBMB MODEL: DETECTED AND UNDETECTED OBJECTS

In the PMBM model, the set of objects $\mathbf{x}_k$ at time k

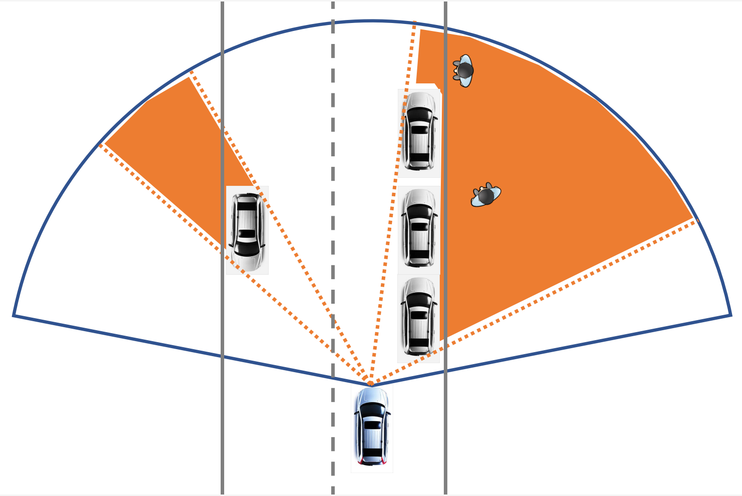
- Union of detected objects and undetected objects $\mathbf{x}_k = \mathbf{x}_k^d \uplus \mathbf{x}_k^{\mathrm{u}}$
  - Detected $\mathbf{x}_k^d$: objects the sensors have detected at least once
  - Undetected $\mathbf{x}_k^{\mathrm{u}}$: objects that have never detected

We are doing **tracking based on detections**, how could we **track undetected objects?!**

- Representation of their possible existence.
- Actually included in many tracking algorithms.
  - MBM filter: any Bernoulli to which a detection has never been associated
- Here it is made explicit.

EXAMPLE: DETECTED AND UNDETECTED OBJECTS

Autonomous car: Possibility of undetected objects in occluded areas



PMBM DENSITY

- The Poisson Multi-Bernoulli Mixture density is defined as

$$\mathcal{PMBM}_{k|k}(\mathbf{x}_k) = \sum_{\mathbf{x}_k^{\mathrm{u}} \uplus \mathbf{x}_k^{d} = \mathbf{x}_k} \mathcal{P}_{k|k}^{\mathrm{u}}(\mathbf{x}_k^{\mathrm{u}}) \mathcal{MBM}_{k|k}^{d}(\mathbf{x}_k^{d})$$

- **PPP density** $\mathcal{P}_{k|k}^{\mathrm{u}}(\cdot)$ for undetected objects, typically with mixture intensity,

$$\lambda_{k|k}^{\mathrm{u}}(x_k) = \sum_{t=1}^{N_k^{\mathrm{u}}} w_{k|k}^{\mathrm{u},t} p_{k|k}^{\mathrm{u},t}(x_k), \qquad \left\{ \left(w_{k|k}^{\mathrm{u},t},\ p_{k|k}^{\mathrm{u},t}(\cdot) \right) \right\}_{t=1}^{N_k^{\mathrm{u}}}$$

- **Multi-Bernoulli Mixture density** $\mathcal{MBM}_{k|k}^{d}(\cdot)$ for detected objects, with parameters

$$\left\{ \left(\ell_{k|k}^{h_k},\ \left\{ \left(r_{k|k}^{i,h_k},\ p_{k|k}^{i,h_k}(\cdot) \right) \right\}_{i=1}^{N_k^{h_k}} \right) \right\}_{h_k=1}^{\mathcal{H}_k}$$

- PMBM density parameterised by the mixture intensity parameters, the MB log-weights, and the Bernoulli parameters of the MBs.

PMBM DENSITY, GAUSSIAN DENSITIES

Example: PPP mixture intensity and Bernoulli state pdfs are Gaussian

Gaussian mixture intensity,

$$\lambda^{\mathrm{u}}_{k|k}(x_k) = \sum_{t=1}^{N^{\mathrm{u}}_{k|k}} w^{\mathrm{u},t}_{k|k} \mathcal{N}\left(x^{\mathrm{u},t}_k \, ; \, \mu^{\mathrm{u},t}_{k|k}, P^{i,h_k}_{k|k}\right)$$

Gaussian object densities,

$$p^{i,h_k}_{k|k}(x^{i,h_k}_k) = \mathcal{N}\left(x^{i,h_k}_k \, ; \, \mu^{i,h_k}_{k|k}, P^{i,h_k}_{k|k}\right)$$

PMBM density parameters

$$\left\{\left(w^{\mathrm{u},t}_{k|k}, \; \mu^{\mathrm{u},t}_{k|k}, \; P^{\mathrm{u},t}_{k|k}\right)\right\}_{t=1}^{N^{\mathrm{u}}_k}, \quad \left\{\left(\ell^{h_k}_{k|k}, \; \left\{\left(r^{i,h_k}_{k|k}, \; \mu^{i,h_k}_{k|k}, \; P^{i,h_k}_{k|k}\right)\right\}_{i=1}^{N^{h_k}_k}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$$

PMBM FILTER

If we design an MOT algorithm for the PMBM density, we get a PMBM filter

PMBM filter: pseudo-code

For $k = 1, 2, \ldots, K$

- **Prediction**
- **Update**
- **Reduction**
- **Estimation**

PMBM prediction

Multi-Object Tracking

Karl Granström

POISSON MULTI-BERNOULLI MIXTURE PREDICTION

- **Posterior PMBM parameters**
- Prediction

$$\mathcal{PMBM}_{k+1|k}(\mathbf{x}_k) = \int p(\mathbf{x}_{k+1} \big| \mathbf{x}_k) \mathcal{PMBM}_{k|k}(\mathbf{x}_k) \, \delta \mathbf{x}_k$$

  with transition density $p(\mathbf{x}_{k+1} \big| \mathbf{x}_k)$ with
  - Probability of survival $P_k^{\mathrm{S}}(x_k)$
  - Transition density $\pi_{k+1}(x_{k+1}|x_k)$
  - PPP birth model: $\left\{ w_{k+1}^{\mathrm{B},t}, \; p_{k+1}^{\mathrm{B},i}(\cdot) \right\}_{i=1}^{N_{k+1}^{\mathrm{B}}}$
- **Predicted PMBM parameters**

PMBM PREDICTION WITH PPP BIRTH, IN SUMMARY

PMBM prediction with PPP birth

- Undetected and detected objects can be predicted independently
- Predicted undetected parameters consist of union of
  - predicted parameters from previous time step, and
  - birth parameters
- Each MB can be predicted independently of the other MBs
- Number of parameters increases (we add PPP birth params. to the undetected PPP)

UNDETECTED PPP PREDICTION

Predicted PPP intensity for undetected objects,

$$\lambda^{\mathrm{u}}_{k+1|k}(x_{k+1}) = \int p(x_{k+1}|x_k)P^{\mathrm{S}}(x_k)\lambda^{\mathrm{u}}_{k|k}(x_k)\mathrm{d}x_k + \lambda^{\mathrm{B}}_{k+1}(x_{k+1})$$

The predicted intensity $\lambda^{\mathrm{u}}_{k+1|k}(x_{k+1})$ is the sum two intensities:

- Prediction of the surviving undetected objects $\int p(x_{k+1}|x_k)P^{\mathrm{S}}(x_k)\lambda^{\mathrm{u}}_{k|k}(x_k)\mathrm{d}x_k$
- Birth of new undetected objects $\lambda^{\mathrm{B}}_{k+1}(x_{k+1})$

Mixture representations

$$\lambda^{\mathrm{u}}_{k+1|k}(x_{k+1}) = \sum_{t=1}^{N^{\mathrm{u}}_k} w^{\mathrm{u},t}_{k|k} \int p(x_{k+1}|x_k)P^{\mathrm{S}}(x_k)p^{\mathrm{u},t}_{k|k}(x_k)\mathrm{d}x_k + \sum_{i=1}^{N^{\mathrm{B}}_{k+1}} w^{\mathrm{B},i}_{k+1}p^{\mathrm{B},i}_{k+1}(x_{k+1})$$

UNDETECTED PPP PREDICTION

Undetected PPP prediction: pseudo-code

- **Posterior parameters:** $\left\{\left(w^{\mathrm{u},t}_{k|k},\ p^{\mathrm{u},t}_{k|k}(\cdot)\right)\right\}_{t=1}^{N^{\mathrm{u}}_k}$
- **Predicted parameters:**

$$\begin{aligned}&\left\{\left(w^{\mathrm{u},t}_{k+1|k},\ p^{\mathrm{u},t}_{k+1|k}(\cdot)\right)\right\}_{t=1}^{N^{\mathrm{u}}_{k+1}}\\&\quad=\left\{\mathrm{Predict}\left(w^{\mathrm{u},t'}_{k|k},\ p^{\mathrm{u},t'}_{k|k}(\cdot)\right)\right\}_{t'=1}^{N^{\mathrm{u}}_k}\bigcup\left\{\left(w^{\mathrm{B},t''}_{k+1},\ p^{\mathrm{B},t''}_{k+1}(\cdot)\right)\right\}_{t''=1}^{N^{\mathrm{B}}_{k+1}}\end{aligned}$$

Increased number of mixture parameters $N^{\mathrm{u}}_{k+1}=N^{\mathrm{u}}_k+N^{\mathrm{B}}_{k+1}$

PREDICTION OF A POSTERIOR MIXTURE COMPONENT

The predicted PPP weight and density,

$$\left(w^{\mathrm{u},t}_{k+1|k},\ p^{\mathrm{u},t}_{k+1|k}(\cdot)\right) = \mathrm{Predict}\left(w^{\mathrm{u},t}_{k|k},\ p^{\mathrm{u},t}_{k|k}(\cdot)\right)$$

are given by

$$w^{\mathrm{u},t}_{k+1|k} p^{\mathrm{u},t}_{k+1|k}(x_{k+1}) = w^{\mathrm{u},t}_{k|k} \int \pi_{k+1}(x_{k+1}|x_k) P^{\mathrm{S}}(x_k) p^{\mathrm{u},t}_{k|k}(x^i_k) \mathrm{d}x_k$$

and are

$$w^{\mathrm{u},t}_{k+1|k} = w^{\mathrm{u},t}_{k|k} P^{\mathrm{S}}_{\mathrm{u},t}$$

$$p^{\mathrm{u},t}_{k+1|k}(x_{k+1}) = \int \pi_{k+1}(x_{k+1}|x_k) \frac{P^{\mathrm{S}}(x_k) p^{\mathrm{u},t}_{k|k}(x^i_k)}{P^{\mathrm{S}}_{\mathrm{u},t}} \mathrm{d}x_k$$

where

$$P^{\mathrm{S}}_{\mathrm{u},t} = \int P^{\mathrm{S}}(x_k) p^{\mathrm{u},t}_{k|k}(x_k) \mathrm{d}x_k$$

EXAMPLE: UNDETECTED PPP PREDICTION

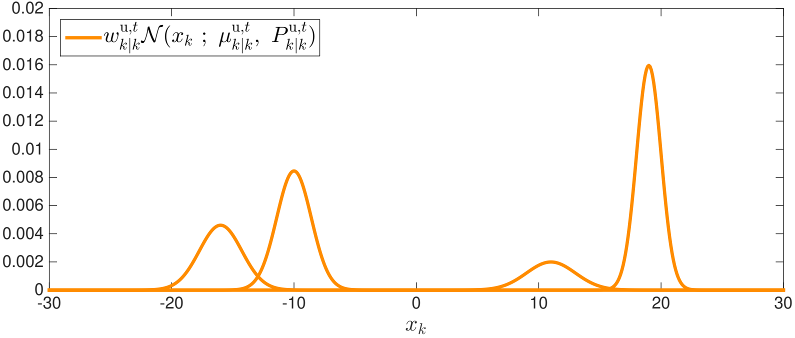
Constant P^{S}, linear Gaussian motion model

- $P^{\mathrm{S}}(x) = P^{\mathrm{S}}$ and $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}\left(x_{k+1}\,;\,F_{k+1}x_k, Q_{k+1}\right)$
- PPP birth with Gaussian mixture intensity $\sum_{i=1}^{N_{k+1}^{\mathrm{B}}} w_{k+1}^{\mathrm{B},i}\mathcal{N}\left(x_{k+1}\,;\,\mu_{k+1}^{\mathrm{B},i}, P_{k+1}^{\mathrm{B},i}\right)$
- Posterior intensity $\lambda_{k|k}^{\mathrm{u}}(x_k) = \sum_{t=1}^{N_k^{\mathrm{u}}} w_{k|k}^{\mathrm{u},t}\mathcal{N}\left(x_k\,;\,\mu_{k|k}^{\mathrm{u},t}, P_{k|k}^{\mathrm{u},t}\right)$
- Predicted intensity,

$$
\begin{aligned}
\lambda_{k+1|k}^{\mathrm{u}}(x_{k+1}) = & \sum_{t=1}^{N_k^{\mathrm{u}}} w_{k|k}^{\mathrm{u},t} P^{\mathrm{S}}\mathcal{N}\left(x_{k+1}\,;\,F_{k+1}\mu_{k|k}^{\mathrm{u},t}, F_{k+1}P_{k|k}^{\mathrm{u},t}F_{k+1}^{\mathrm{T}} + Q_{k+1}\right) \\
& + \sum_{i=1}^{N_{k+1}^{\mathrm{B}}} w_{k+1}^{\mathrm{B},i}\mathcal{N}\left(x_{k+1}\,;\,\mu_{k+1}^{\mathrm{B},i}, P_{k+1}^{\mathrm{B},i}\right)
\end{aligned}
$$

EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
  $0.02\mathcal{N}(x_{k+1}\,;\,-16,3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10,2) + 0.01\mathcal{N}(x_{k+1}\,;\,11,4) + 0.04\mathcal{N}(x_{k+1}\,;\,19,1)$



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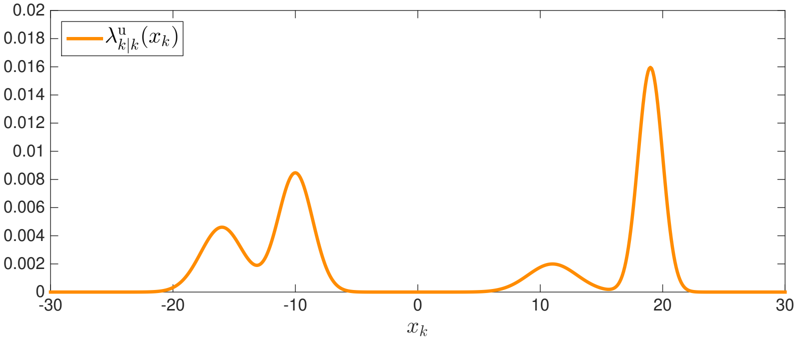

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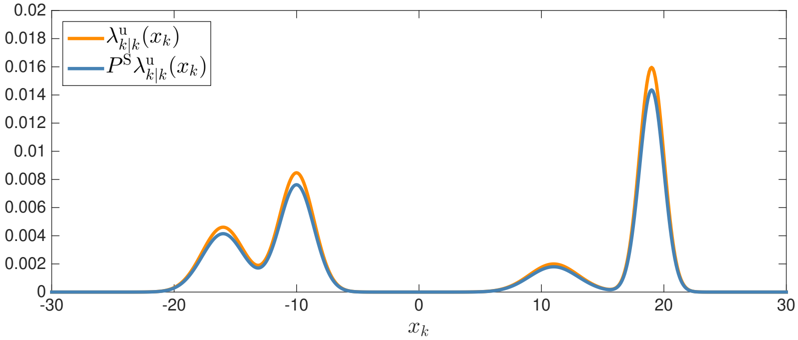
- $P^{\mathrm{S}} = 0.9$ and random walk, $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}\,;\,x_k,4)$



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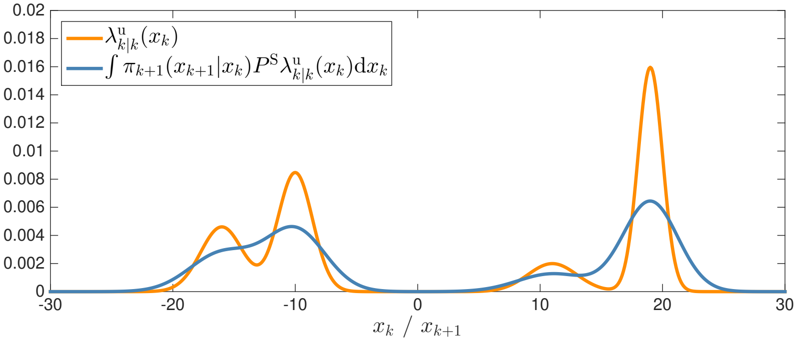

EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

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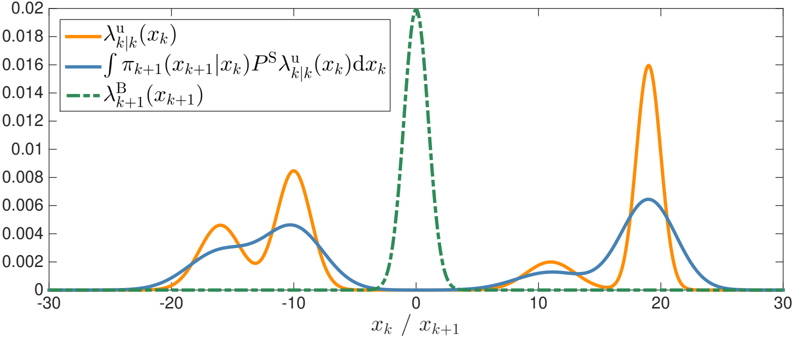
- $P^{\mathrm{S}} = 0.9$ and random walk, $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}\,;\,x_k,4)$
- $\lambda^{\mathrm{B}}_{k+1}(x_{k+1}) = 0.05\mathcal{N}(x_{k+1}\,;\,0,1)$



EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^{\mathrm{u}}_{k|k}(x_k) =$
  $0.02\mathcal{N}(x_{k+1}\,;\,-16, 3) + 0.03\mathcal{N}(x_{k+1}\,;\,-10, 2) + 0.01\mathcal{N}(x_{k+1}\,;\,11, 4) + 0.04\mathcal{N}(x_{k+1}\,;\,19, 1)$
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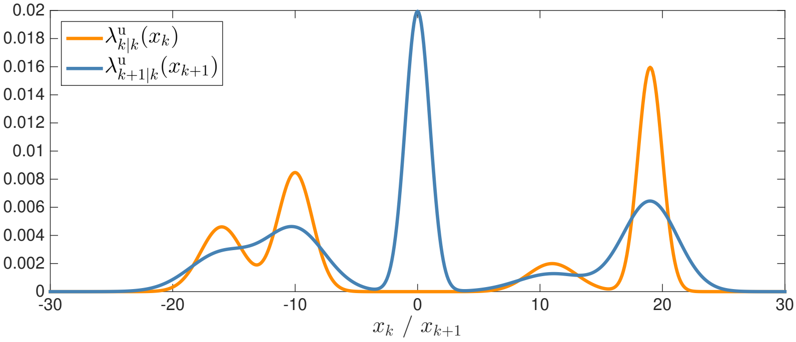

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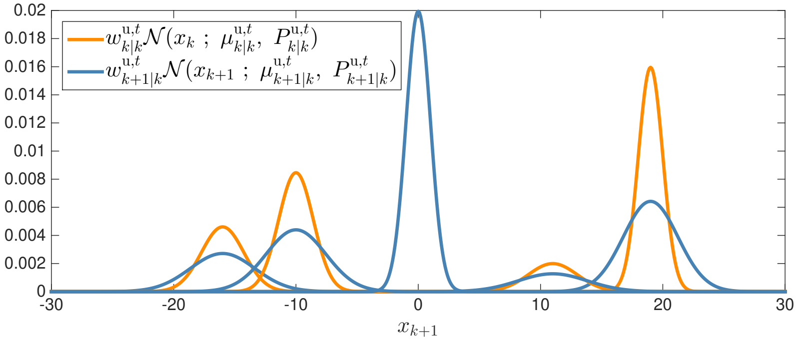
- $\lambda^{\mathrm{B}}_{k+1}(x_{k+1}) = 0.05\mathcal{N}(x_{k+1}\,;\,0, 1)$



DETECTED MBM PREDICTION

Detected MBM prediction: pseudo-code

- **Posterior parameters:** $\left\{\left(\ell_{k|k}^{h_k},\ \left\{\left(r_{k|k}^{i,h_k},\ p_{k|k}^{i,h_k}(\cdot)\right)\right\}_{i=1}^{N_k^{h_k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$
- **Predicted parameters:** $\left\{\left(\ell_{k|k}^{h_k},\ \left\{\left(r_{k+1|k}^{i,h_k},\ p_{k+1|k}^{i,h_k}(\cdot)\right)\right\}_{i=1}^{N_k^{h_k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$

  where, for each h_k and each i,

  $$\left(r_{k+1|k}^{i,h_k},\ p_{k+1|k}^{i,h_k}(\cdot)\right)$$

  are computed the same way as in an MBM filter
- Same number of Bernoullis $N_k^{h_k}$

BERNOULLI PREDICTION

The predicted Bernoulli parameters,

$$\left(r^{i,h_k}_{k+1|k},\ p^{i,h_k}_{k+1|k}(\cdot)\right)$$

are

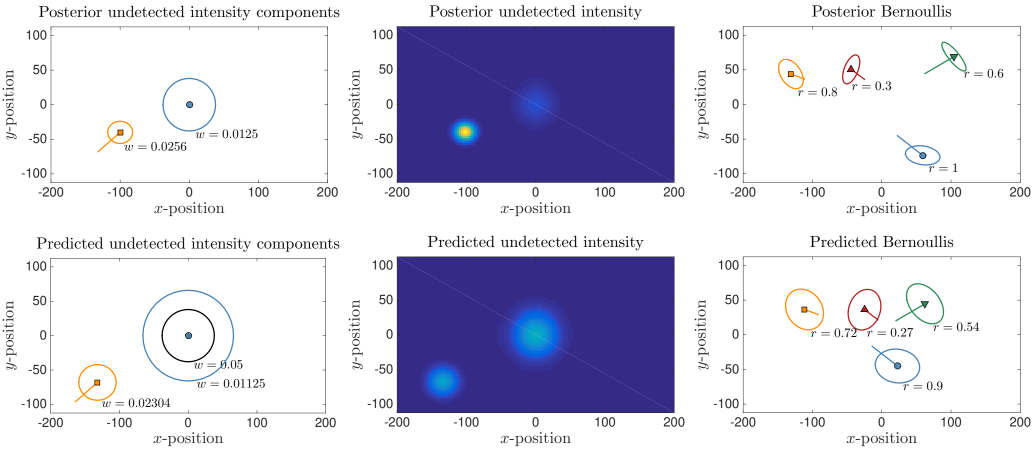
$$r^{i,h_k}_{k+1|k} = r^{i,h_k}_{k|k} P^{\mathrm{S}}_{i,h_k}$$

$$p^{i,h_k}_{k+1|k}(x^i_{k+1}) = \int \pi_{k+1}(x^i_{k+1}|x^i_k)\frac{P^{\mathrm{S}}(x^i_k)p^{i,h_k}_{k|k}(x^i_k)}{P^{\mathrm{S}}_{i,h_k}}\mathrm{d}x^i_k$$

where

$$P^{\mathrm{S}}_{i,h_k} = \int P^{\mathrm{S}}(x^i_k)p^{i,h_k}_{k|k}(x^i_k)\mathrm{d}x^i_k$$

PMBM PREDICTION: 2D EXAMPLE, $P^S = 0.9$



- Constant velocity $\pi_k(x_k|x_{k-1}) = \mathcal{N}(x_k\,;\,Fx_{k-1}, Q)$, and $P^S(x_{k-1}) = 0.9$
- Birth intensity with single mixture component, position in origin, zero velocity

PMBM update: overview

Multi-Object Tracking

Karl Granström

POISSON MULTI-BERNOULLI MIXTURE UPDATE

- **Prior PMBM parameters**
- Update

$$\mathcal{PMBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k \big| \mathbf{x}_k) \mathcal{PMBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k \big| \mathbf{x}'_k) \mathcal{PMBM}_{k|k-1}(\mathbf{x}'_k) \, \delta \mathbf{x}'_k}$$

  with multi-object measurement model $p(\mathbf{z}_k \big| \mathbf{x}_k)$ with
  - Probability of detection $P_k^{\mathrm{D}}(x_k)$
  - Measurement model $g_k(z_k | x_k)$
  - Poisson clutter intensity: $\lambda_c(z_k)$
- **Posterior PMBM parameters**

POISSON MULTI-BERNOULLI MIXTURE UPDATE, IN SUMMARY

PMBM update for the standard point object models

- For each prior MB, multiple data associations
- For each prior MB and each data association, we get an MB in the posterior MBM
- For each Bernoulli, two possibilities:
  - Either associated to one of the measurements,
  - or misdetected.
- Any measurement not associated to a prior Bernoulli is either
  - clutter
  - or from an object detected for the first time

  We get a **new Bernoulli**

PMBM UPDATE: DATA ASSOCIATION

With an MB birth model,

- Initiation of potential new objects: add birth Bernoulli components in prediction
- When handling the data association, each detection is associated to
  - one of the prior Bernoullis, or
  - the clutter PPP

With a PPP birth model,

- Initiation of potential new objects: measurement driven.
- When handling the data association, each detection is associated to
  - one of the prior Bernoullis,
  - the undetected PPP or the clutter PPP

  Important: we treat the potential new objects and the clutter jointly!
- Convenient to reformulate assignment problem: assign the measurements

DATA ASSOCIATION VARIABLE FOR M_K DETECTIONS, 1

- Let there be m_k measurements and n_k objects.
- The association for measurement z_k^j is denoted ψ_k^j
- An association for all m_k measurement is denoted

$$\psi_k = \left[\psi_k^1,\ \psi_k^2, \ldots,\ \psi_k^j, \ldots,\ \psi_k^{m_k}\right]$$

- ψ_k^j is defined similarly to how θ_k^i was defined,

$$\psi_k^j = \begin{cases} i & \text{if measurement } j \text{ is associated to object } i \\ 0 & \text{if measurement } j \text{ is associated either to a potential new object, or to clutter} \end{cases}$$

DATA ASSOCIATION VARIABLE FOR M_K DETECTIONS, 2

- Ψ_k is the set of valid association events at time k.
- For $\psi_k \in \Psi_k$, the following must hold:

1. Each measurement must be either from a previously detected object, or clutter/potential new object,

$$\psi_k^j \in \{0, \ldots, n_k\}, \ \forall\, j \in \{1, \ldots, m_k\}$$

2. **Point object assumption:** For any pair of two measurements, they cannot be associated to the same previously detected object,

$$\forall\, j, j' \in \{1, \ldots, m_k\},\ j \neq j',\ \text{if } \psi_k^j \neq 0,\ \psi_k^{j'} \neq 0 \ \Rightarrow\ \psi_k^j \neq \psi_k^{j'}$$

DATA ASSOCIATION VARIABLE FOR M_K DETECTIONS, 3

- Note that 1. and 2. on the previous slide together implicitly ensures that we do not associate more than n_k measurements to the n_k objects.
- In what follows, unless otherwise stated, we consider associations $\psi_k \in \Psi_k$.
- Given a ψ, we can find the equivalent θ, and vice versa.

$$
\begin{aligned}
\psi^j = i \quad &\Leftrightarrow \quad \theta^i = j \\
\psi^j = 0 \quad &\Leftrightarrow \quad \nexists\, i \ : \ \theta^i = j \\
\nexists\, j \ : \ \psi^j = i \quad &\Leftrightarrow \quad \theta^i = 0
\end{aligned}
$$

HYPOTHESIS ORIENTED-PMBM UPDATE

Hypothesis oriented-PMBM update: pseudo-code

Input: $\lambda^{\mathrm{u}}_{k|k-1}(x_k)$, $\left\{\left(\ell^{h_{k-1}}, \left\{\left(r^{i,h_{k-1}}_{k|k-1}, p^{i,h_{k-1}}_{k|k-1}(\cdot)\right)\right\}_{i=1}^{N^{h_{k-1}}_{k-1}}\right)\right\}_{h_{k-1}=1}^{\mathcal{H}_{k-1}}$

Misdetection update: $\lambda^{\mathrm{u}}_{k|k}(x_k) = (1 - P^{\mathrm{D}}(x_k))\lambda^{\mathrm{u}}_{k|k-1}(x_k)$

Initialise: $h_k = 0$

For $h_{k-1} = 1, \dots, \mathcal{H}_{k-1}$

Create cost matrix $L^{h_{k-1}}$, and compute $M_{h_{k-1}}$ associations ψ_m

For $m = 1, \dots, M_{h_{k-1}}$

Increase: $h_k \leftarrow h_k + 1$

Compute posterior MB parameters: detected, misdetected & new Bernoulli, log-weight $\tilde{\ell}^{h_k}$

Set $\mathcal{H}_k = h_k$

Normalise log-weights $\ell^{h_k} \leftarrow \tilde{\ell}^{h_k}$

Output: $\lambda^{\mathrm{u}}_{k|k}(x_k)$, $\left\{\left(\ell^{h_k}, \left\{\left(r^{i,h_k}_{k|k}, p^{i,h_k}_{k|k}(\cdot)\right)\right\}_{i=1}^{N^{h_k}_{k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$

PMBM update: details

Multi-Object Tracking

Karl Granström

POISSON MULTI-BERNOULLI MIXTURE UPDATE, “BUILIDING BLOCKS”

- Undetected object: remain undetected, or detected for the first time
- Previously detected object: misdetected, or detected again

Important “builiding blocks” of PMBM update:

- update of PPP intensity for undetected objects that remain undetected
- update of potential new object detected for the first time $\Rightarrow$ **new Bernoulli**
- update of Bernoulli with associated measurement – similar to MBM-filter
- update of misdetected Bernoulli – similar to MBM-filter
- posterior log-weights

UNDETECTED OBJECTS MISDETECTED AGAIN

- Posterior PPP intensity for objects that remain undetected,

$$\lambda^{\mathrm{u}}_{k|k}(x) = (1 - P^{\mathrm{D}}(x))\lambda^{\mathrm{u}}_{k|k-1}(x)$$

- Posterior intensity is **lower**/**higher** in areas where $(1 - P^{\mathrm{D}}(x))$ is **low**/**high**, because it is **unlikely**/**likely** that an object was not detected there
- Mixture representation of intensity

$$\lambda^{\mathrm{u}}_{k|k}(x_k) = \sum_t w^{\mathrm{u},t}_{k|k-1}(1 - P^{\mathrm{D}}(x))p^{\mathrm{u},t}_{k|k-1}(x_k)$$

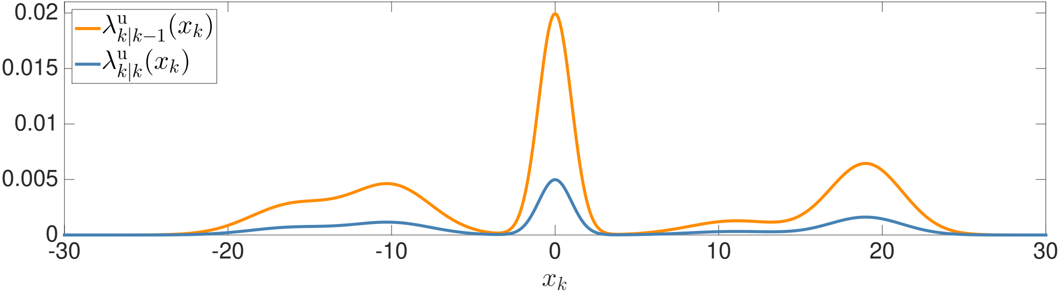
$$= \sum_t w^{\mathrm{u},t}_{k|k-1} P^{\mathrm{MD}}_{\mathrm{u},t} \frac{(1 - P^{\mathrm{D}}(x))p^{\mathrm{u},t}_{k|k-1}(x_k)}{P^{\mathrm{MD}}_{\mathrm{u},t}} = \sum_t \underbrace{w^{\mathrm{u},t}_{k|k-1} P^{\mathrm{MD}}_{\mathrm{u},t}}_{w^{\mathrm{u},t}_{k|k}} \underbrace{\frac{(1 - P^{\mathrm{D}}(x))p^{\mathrm{u},t}_{k|k-1}(x_k)}{P^{\mathrm{MD}}_{\mathrm{u},t}}}_{p^{\mathrm{u},t}_{k|k}(x_k)}$$

where $P^{\mathrm{MD}}_{\mathrm{u},t} = \int (1 - P^{\mathrm{D}}(x))p^{\mathrm{u},t}_{k|k-1}(x_k)\mathrm{d}x_k$

POSTERIOR UNDETECTED INTENSITY EXAMPLE

Constant P^{D}, linear Gaussian models

- Prior intensity $\lambda^{\mathrm{u}}_{k|k-1}(x_k) = \sum_{t=1}^{5} w^{\mathrm{u},t}_{k|k-1}\mathcal{N}\left(x_k\,;\,\mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1}\right)$
- Probability of detection $P^{\mathrm{D}} = 0.75$
- Posterior intensity $\lambda^{\mathrm{u}}_{k|k}(x_k) = \sum_{t=1}^{5} 0.25 w^{\mathrm{u},t}_{k|k-1}\mathcal{N}\left(x_k\,;\,\mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1}\right)$



OBJECT DETECTED FOR THE FIRST TIME: NEW BERNOULLI

If $\psi^j = 0$, then the measurement z_k^j is either clutter or from a previously undetected object.

New Bernoulli component in the posterior MB, with parameters

$$r_{k|k}^j = \frac{\rho_{k|k-1}^{\mathrm{u}}(z_k^j)}{\lambda_c(z_k^j) + \rho_{k|k-1}^{\mathrm{u}}(z_k^j)}, \qquad \rho_{k|k-1}^{\mathrm{u}}(z_k^j) = \int P^{\mathrm{D}}(x_k) g_k(z_k^j|x_k) \lambda_{k|k-1}^{\mathrm{u}}(x_k) \mathrm{d}x_k$$

$$p_{k|k}^j(x_k) = \frac{P^{\mathrm{D}}(x_k) g_k(z_k^j|x_k) \lambda_{k|k-1}^{\mathrm{u}}(x_k)}{\rho_{k|k-1}^{\mathrm{u}}(z_k^j)}$$

Posterior r conditioned on ψ

Relative intensity of: 1) detection from previously undetected object; and, 2) clutter

Predicted log-likelihood

$$\ell_k^{\mathrm{u},j} = \log\left(\lambda_c(z_k^j) + \rho_{k|k-1}^{\mathrm{u}}(z_k^j)\right)$$

NEW BERNOULLI: PROBABILITY OF EXISTENCE < 1?

- Earlier we had that, conditioned on the data association, $r = 1$
- How can we now have a new Bernoulli for which $r < 1$?
- It represents two possibilities:
  - Detection was from clutter – no new object
    Likelihood $\lambda_c(z_k^j)$.
  - Detection was from a new object
    Likelihood $\rho^{\mathrm{u}}_{k|k-1}(z_k^j) = \int P^{\mathrm{D}}(x_k) g_k(z_k^j|x_k) \lambda^{\mathrm{u}}_{k|k-1}(x_k) \mathrm{d}x_k$.
- Represented compactly as a new Bernoulli with probability of existence

$$r^j_{k|k} = \frac{\rho^{\mathrm{u}}_{k|k-1}(z_k^j)}{\lambda_c(z_k^j) + \rho^{\mathrm{u}}_{k|k-1}(z_k^j)}$$

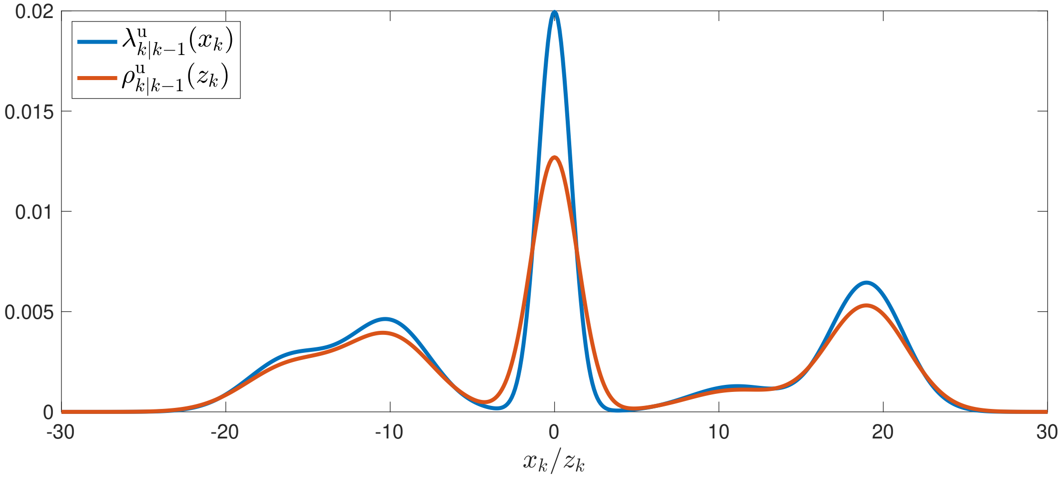
NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE

Constant P^{D}, linear Gaussian models

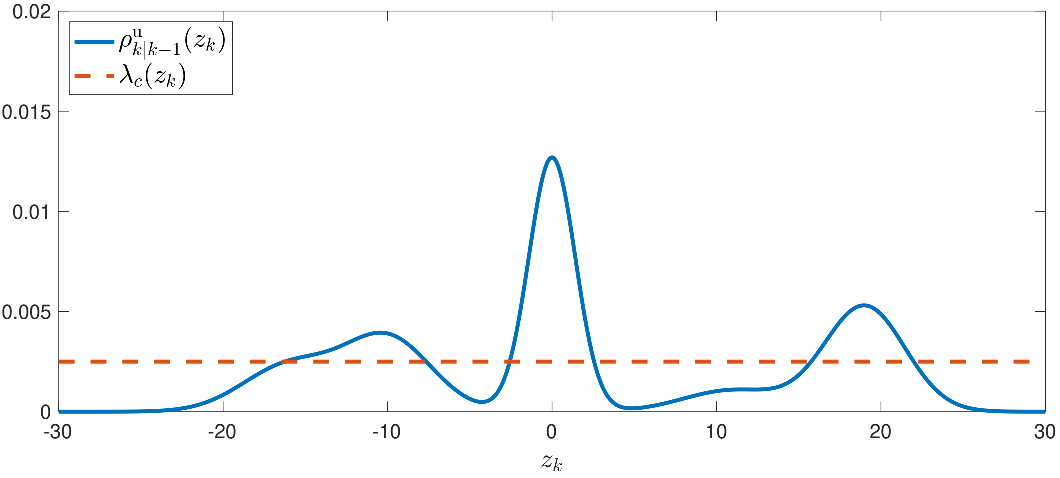
- Undetected intensity $\lambda^{\mathrm{u}}_{k|k-1}(x_k) = \sum_{t=1}^{5} w^{\mathrm{u},t}_{k|k-1} \mathcal{N}\left(x_k \,;\, \mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1}\right)$
- Probability of detection $P^{\mathrm{D}} = 0.9$, measurement model $g_k(z|x) = \mathcal{N}\left(z \,;\, x, R\right)$
- Likelihood $\rho^{\mathrm{u}}_{k|k-1}(z_k) = \sum_{t=1}^{5} P^{\mathrm{D}} w^{\mathrm{u},t}_{k|k-1} \mathcal{N}\left(z_k \,;\, \mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1} + R\right)$
- Clutter $\lambda_c(z_k) = \bar{\lambda}_c / V$
- Probability of existence of new Bernoulli

$$r_{k|k} = \frac{\sum_{t=1}^{5} P^{\mathrm{D}} w^{\mathrm{u},t}_{k|k-1} \mathcal{N}\left(z_k \,;\, \mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1} + R\right)}{\frac{\bar{\lambda}_c}{V} + \sum_{t=1}^{5} P^{\mathrm{D}} w^{\mathrm{u},t}_{k|k-1} \mathcal{N}\left(z_k \,;\, \mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1} + R\right)}$$

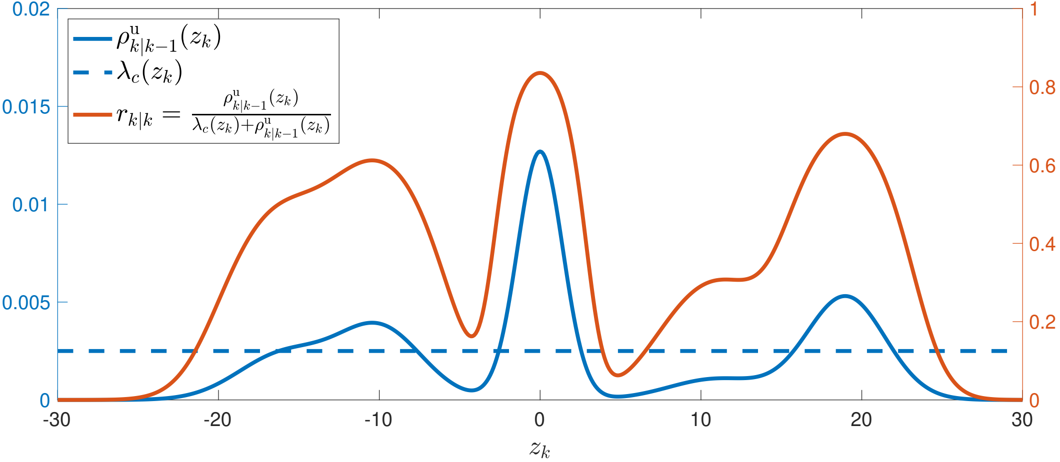
NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE



NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE



NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE



NEW BERNOULLI STATE DENSITY EXAMPLE

Constant P^{D}, linear Gaussian models

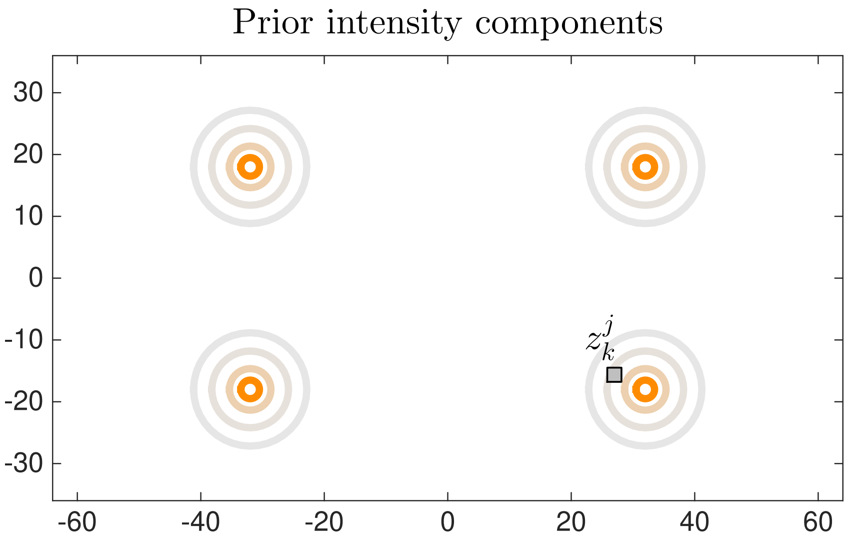
- Undetected intensity $\lambda^{\mathrm{u}}_{k|k-1}(x_k) = \sum_t w^{\mathrm{u},t}_{k|k-1}\mathcal{N}\left(x_k\,;\,\mu^{\mathrm{u},t}_{k|k-1}, P^{\mathrm{u},t}_{k|k-1}\right)$
- Probability of detection $P^{\mathrm{D}} = 0.9$, measurement model $g_k(z|x) = \mathcal{N}(z\,;\,Hx, R)$
- State density of new Bernoulli

$$p^j_{k|k}(x_k) = \sum_t w^{\mathrm{u},t,j}_k \mathcal{N}\left(x_k\,;\,\mu^{\mathrm{u},t}_{k|k-1} + K^{\mathrm{u},t}_k\left(z^j_k - \hat{z}^{\mathrm{u},t}_k\right), P^{\mathrm{u},t}_{k|k-1} - K^{\mathrm{u},t}_k H_k P^{\mathrm{u},t}_{k|k-1}\right)$$

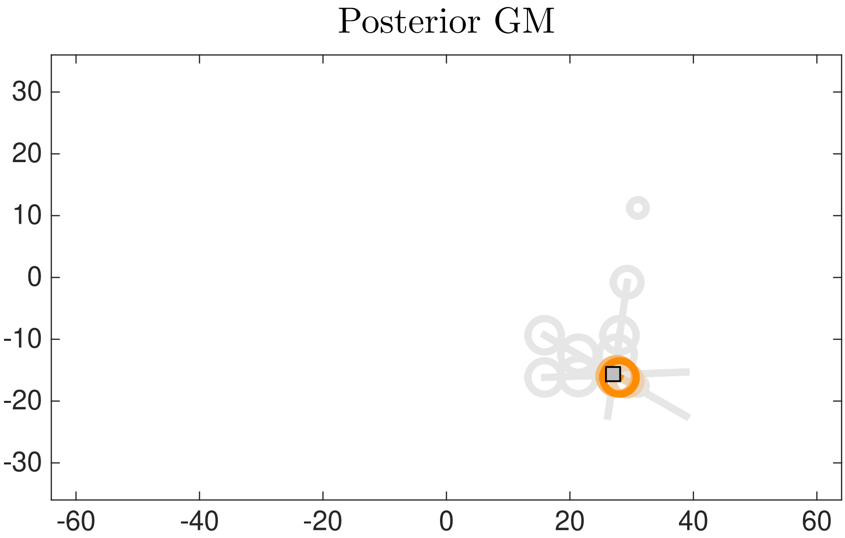
$$w^{\mathrm{u},t,j}_k = \frac{w^{\mathrm{u},t}_{k|k-1} P^{\mathrm{D}} \mathcal{N}\left(z^j_k\,;\,\hat{z}^{\mathrm{u},t}_k, S^{\mathrm{u},t}_k\right)}{\sum_{t'} w^{\mathrm{u},t'}_{k|k-1} P^{\mathrm{D}} \mathcal{N}\left(z^j_k\,;\,\hat{z}^{\mathrm{u},t'}_k, S^{\mathrm{u},t'}_k\right)}$$

- Pruning and merging used to reduce $p^j_{k|k}(x_k)$, often to a single Gaussian

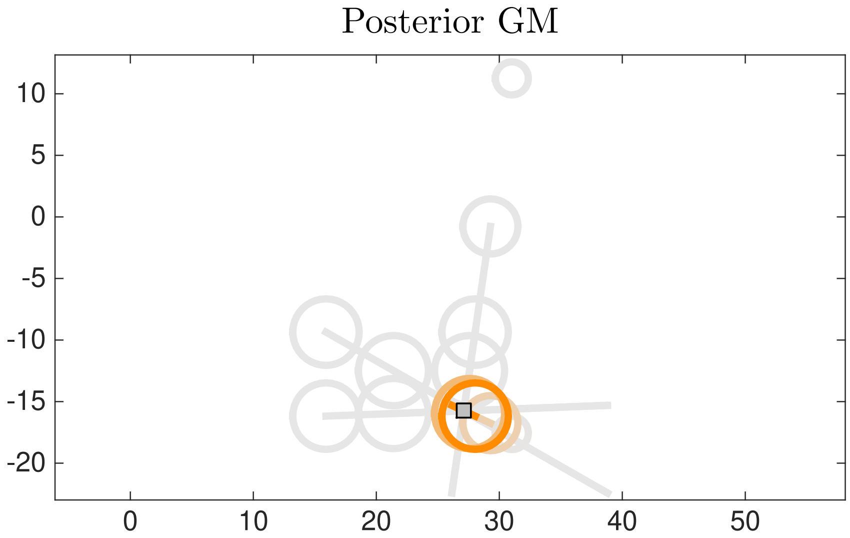
NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



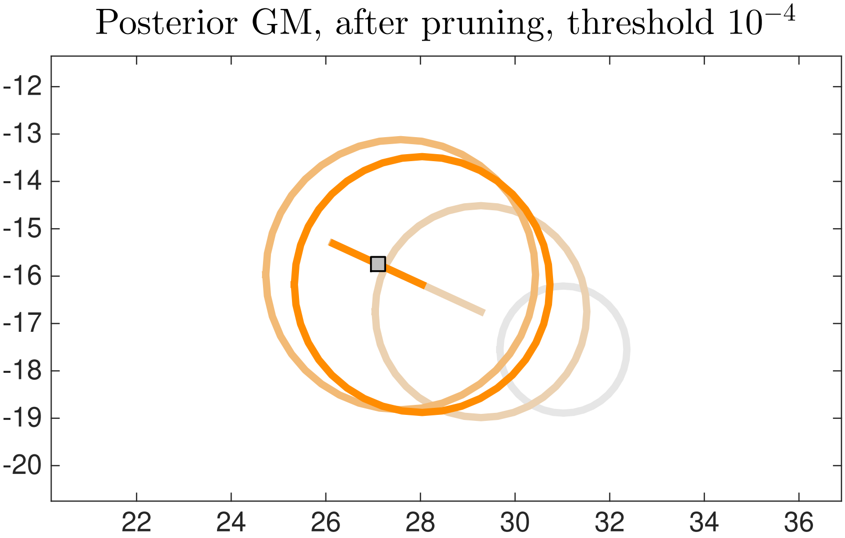
NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



NEW BERNOULLI STATE DENSITY VISUALIZATION, 1

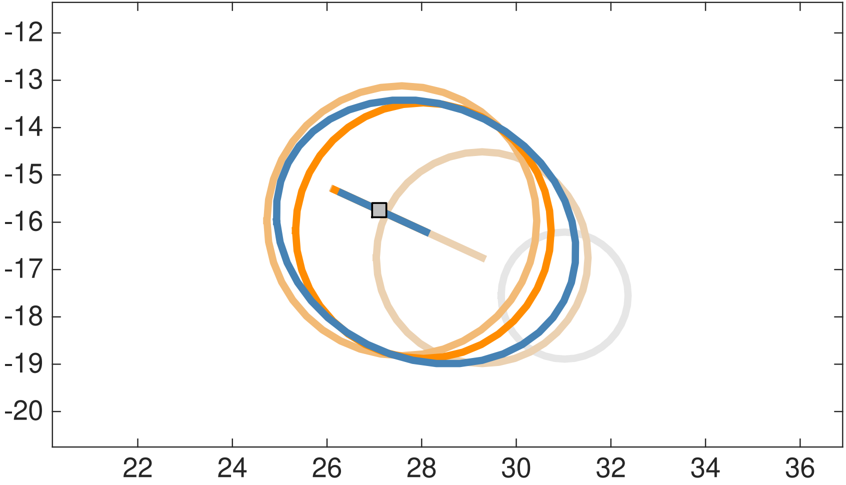


NEW BERNOULLI STATE DENSITY VISUALIZATION, 1

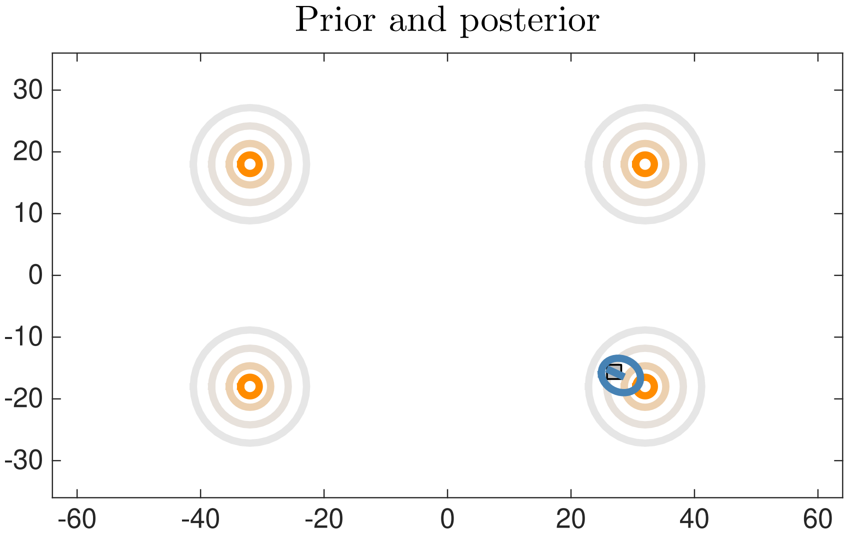


NEW BERNOULLI STATE DENSITY VISUALIZATION, 1

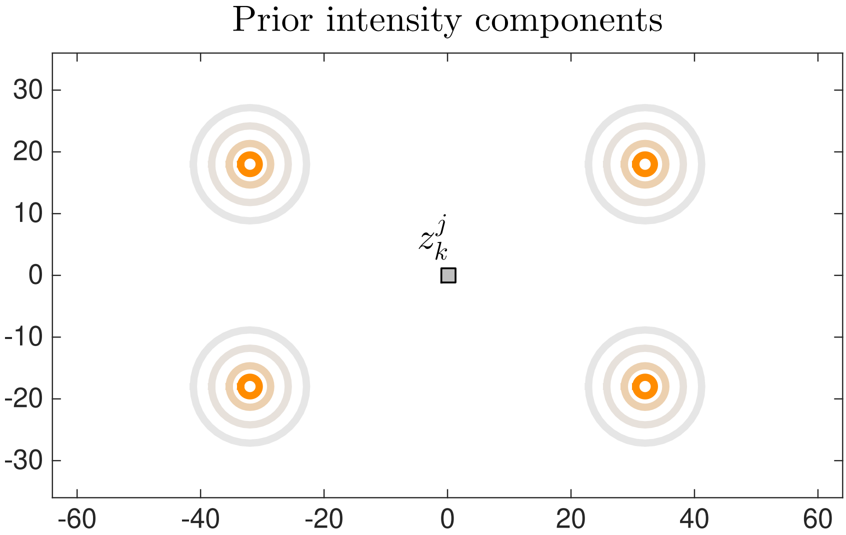
Posterior GM, after pruning and merging



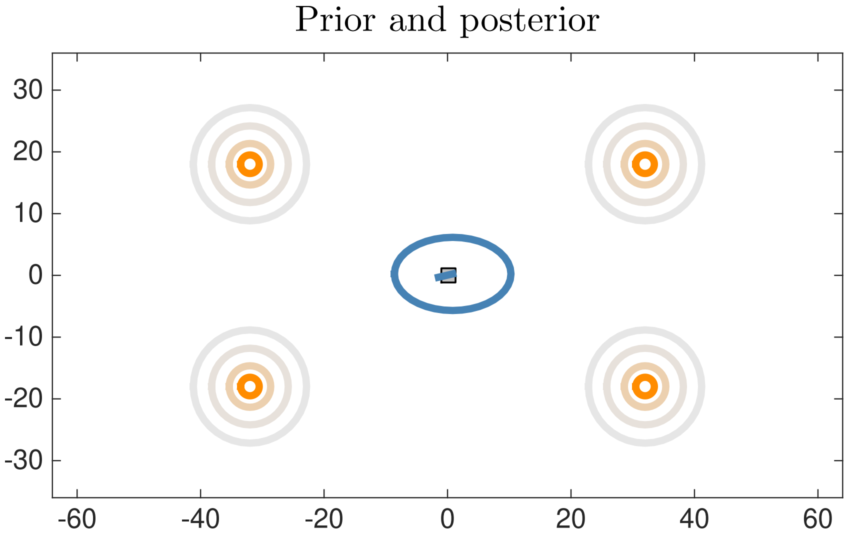
NEW BERNOULLI STATE DENSITY VISUALIZATION, 1



NEW BERNOULLI STATE DENSITY VISUALIZATION, 2



NEW BERNOULLI STATE DENSITY VISUALIZATION, 2



BERNOULLI UPDATE: DETECTION AND MISDETECTION

For prior hypothesis h, if $\psi^j = i$,

$$r_{k|k}^{i,j,h} = 1$$

$$p_{k|k}^{i,j,h}(x_k^i) = \frac{P^{\mathrm{D}}(x_k^i) g_k(z_k^{\theta^i} | x_k^i) p_{k|k-1}^{i,h}(x_k^i)}{\int P^{\mathrm{D}}(x_k^i) g_k(z_k^{\theta^i} | x_k^i) p_{k|k-1}^{i,h}(x_k^i) \mathrm{d}x_k^i}$$

For prior hypothesis h, if $\nexists j : \psi^j = i$,

$$r_{k|k}^{i,0,h} = \frac{r_{k|k-1}^{i,h} P_{i,h}^{\mathrm{MD}}}{1 - r_{k|k-1}^{i,h} + r_{k|k-1}^{i,h} P_{i,h}^{\mathrm{MD}}}$$

$$p_{k|k}^{i,0,h}(x) = \frac{(1 - P^{\mathrm{D}}(x_k)) p_{k|k-1}^{i,h}(x)}{P_{i,h}^{\mathrm{MD}}}$$

where $P_{i,h}^{\mathrm{MD}} = \int \left(1 - P^{\mathrm{D}}\left(x_k^i\right)\right) p_{k|k-1}^{i,h}\left(x_k^i\right) \mathrm{d}x_k^i$

Predicted log-likelihoods

$$\ell_k^{i,j,h} = \log \left(r_{k|k-1}^{i,h} \int P^{\mathrm{D}}\left(x_k^i\right) g_k\left(z_k^j | x_k^i\right) p_{k|k-1}^{i,h}(x_k^i) \mathrm{d}x_k^i \right)$$

$$\ell_k^{i,0,h} = \log \left(1 - r_{k|k-1}^i + r_{k|k-1}^i P_{i,h}^{\mathrm{MD}} \right)$$

Note similarity to Bernoulli update in MBM filter

NON-NORMALISED POSTERIOR LOG-WEIGHTS

For a prior MB h and a data association ψ_k, the non-normalized posterior log-weight is

$$
\begin{aligned}
\tilde{\ell}_{k|k}^{h_{k-1},\psi_k} &= \underbrace{\ell_{k|k-1}^{h_{k-1}}}_{\text{Prior}} + \underbrace{\sum_{i:\nexists j:\psi^j=i} \ell_k^{i,0,h}}_{\text{Misdetection}} + \underbrace{\sum_{j:\psi^j\neq 0} \ell_k^{\psi^j,j,h}}_{\text{Assoc. meas.}} + \underbrace{\sum_{j:\psi^j=0} \ell_k^{\mathrm{u},j}}_{\text{Clutter or potential new object}} \\
&= \ell_{k|k-1}^{h_{k-1}} + \sum_{i=1}^{N^h} \ell_k^{i,0,h} + \sum_{j:\psi^j\neq 0} \left[\ell_k^{\psi^j,j,h} - \ell_k^{\psi^j,0,h}\right] + \sum_{j:\psi^j=0} \ell_k^{\mathrm{u},j} \\
&= \ell_{k|k-1}^{h_{k-1}} + \sum_{j=1}^{m_k} \tilde{\ell}_k^{j,h} + \text{Constant independent of } \psi_k
\end{aligned}
$$

where

$$
\tilde{\ell}_k^{j,h} = \begin{cases} \ell_k^{\psi^j,j,h} - \ell_k^{\psi^j,0,h} & \text{if } \psi^j \neq 0 \\ \ell_k^{\mathrm{u},j} & \text{if } \psi^j = 0 \end{cases}
$$

EXAMPLE PMBM UPDATE VISUALIZATION

Prior and model, 2D scenario

- PMBM with two MBs, each with two Bernoullis with Gaussian state densities, undetected PPP intensity with single Gaussian.
- Measurement model:

$$P^{\mathrm{D}} = 0.75$$

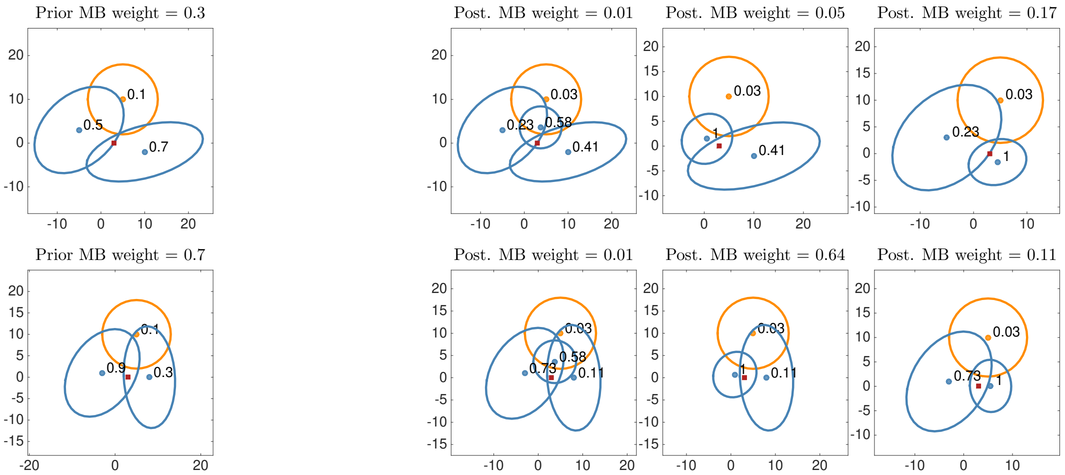
$$g_k(z|x) = \mathcal{N}(z\,;\,x, 9\mathbf{I}_2)$$

$$\lambda_c(z) = \begin{cases} 4 \times 10^{-4} & z \in [-25, 25] \times [-25, 25] \\ 0 & z \notin [-25, 25] \times [-25, 25] \end{cases}$$

- Single measurement $\Rightarrow$ 3 DAs for each prior MB $\Rightarrow$ Posterior MBM with 6 MBs

EXAMPLE PMBM UPDATE VISUALIZATION

Undetected objects PPP (Orange), Detected objects MB (Blue), Measurement (Red)



HANDLING THE DATA ASSOCIATIONS

The data association problem is handled analogously to tracking n objects, and MBM filter:

- Use gating to remove very unlikely associations and group Bernoullis/measurements
- For each group, form cost matrix with negative log likelihoods
- Use some algorithm to find M associations, e.g.,
  - Murty
  - Gibbs' sampling
- Truncate all other associations

PMBM UPDATE: COST MATRIX

Let there be m_k detections, and consider an MB h with N^h Bernoullis. The cost matrix is

$$L^h = \begin{bmatrix} -\ell^{1,1,h} & -\ell^{1,2,h} & \dots & -\ell^{1,N^h,h} & -\ell^{1,0} & \infty & \dots & \infty \\ -\ell^{2,1,h} & -\ell^{2,2,h} & \dots & -\ell^{2,N^h,h} & \infty & -\ell^{2,0} & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\ell^{m_k,1,h} & -\ell^{m_k,2,h} & \dots & -\ell^{m_k,N^h,h} & \infty & \infty & \dots & -\ell^{m_k,0} \end{bmatrix}$$

where

$$\ell^{j,0} = \log\left(\lambda_c(z_k^j) + \int P^{\mathrm{D}}(x_k) g_k(z_k^j|x_k)\lambda^{\mathrm{u}}_{k|k-1}(x_k)\mathrm{d}x_k\right)$$

$$\ell^{j,i,h} = \log\left(\frac{r^{i,h}_{k|k-1}\int P^{\mathrm{D}}\left(x_k^i\right) g_k(z_k^j|x_k^i) p^{i,h}_{k|k-1}(x_k^i)\mathrm{d}x_k^i}{1 - r^{i,h}_{k|k-1} + r^{i,h}_{k|k-1}\int\left(1 - P^{\mathrm{D}}\left(x_k^i\right)\right) p^{i,h}_{k|k-1}\left(x_k^i\right)\mathrm{d}x_k^i}\right)$$

PMBM post processing

Multi-Object Tracking

Karl Granström

PMBM POST PROCESSING

After prediction and update, we have a PMBM density $\mathcal{PMBM}_{k|k}(\mathbf{x}_k)$ with params.

$$\left\{\left(w_{k|k}^{\mathrm{u},t},\ p_{k|k}^{\mathrm{u},t}(\cdot)\right)\right\}_{t=1}^{N_k^{\mathrm{u}}}, \quad \left\{\left(\ell_{k|k}^{h_k},\ \left\{\left(r_{k|k}^{i,h_k},\ p_{k|k}^{i,h_k}(\cdot)\right)\right\}_{i=1}^{N_k^{h_k}}\right)\right\}_{h_k=1}^{\mathcal{H}_k}$$

- Reduction:
  - Reduce N_k^{u}, $\mathcal{H}_k$ and $N_k^{h_k}$
  - Important for computational cost
  - Pruning, merging, capping, and **recycling**
- Estimation:
  - Extracting a set of estimated object states from the posterior density.

PMBM REDUCTION

PMBM reduction

- **MBM pruning:** prune MB h_k if $\ell_{k|k}^{h_k} \leq \Gamma$
- **MBM capping:** if $\mathcal{H}_k > N_{\max}$, keep the $N_{\max}$ MBs with largest log-weights.

  After pruning and capping the MBM, remaining log-weights are re-normalized.
- **Bernoulli recycling:** in each MB h_k, recycle Bernoulli i if $r_{k|k}^{i,h} < \Gamma^r$
- **PPP reduction:** pruning, merging and capping of the mixture intensity

Outside the scope of the course: MBM merging

BERNOULLI RECYCLING

Bernoulli recycling: basic idea

Instead of pruning Bernoullis with small r, approximate them as PPP, and add the intensity to undetected PPP intensity.

- KL-div minimising PPP approximation: intensity $=$ Bernoulli PHD,

$$\lambda^{h,i,\mathrm{REC}}_{k|k}(x_k) = r^{h,i}_{k|k} p^{h,i}_{k|k}(x_k)$$

- Undetected PPP intensity after recycling is

$$\lambda^{\mathrm{u,REC}}_{k|k}(x_k) = \lambda^{\mathrm{u}}_{k|k}(x_k) + \sum_{h} \sum_{i : r^{h,i}_{k|k} < \Gamma^r} \exp\left(\ell^h_{k|k}\right) r^{h,i}_{k|k} p^{h,i}_{k|k}(x_k)$$

Note that we must take the normalized hypothesis weight $\exp\left(\ell^h_{k|k}\right)$ into account

WHY RECYCLING?

Following the recycling, we have to reduce the undetected PPP intensity.

Why not just prune right away?

- Recycling/pruning lowers the computational complexity, because there are fewer Bernoullis to consider in the data association.
- Pruning means that we lose all information contained in what is pruned.
- By recycling, the information is retained approximately as a PPP.
- The Bernoulli recycling threshold can therefore be considerably larger than a Bernoulli pruning threshold.
- Empirical studies show that Bernoulli **recycling leads to lower computational cost**, without sacrificing tracking performance.

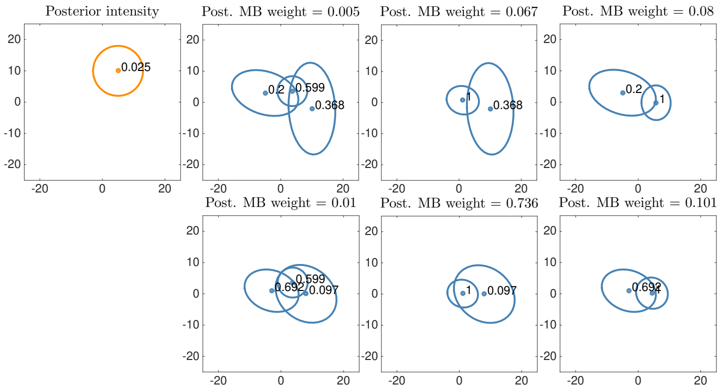
REDUCING THE UNDETECTED INTENSITY

- Undetected intensity with mixture representation

$$\lambda^{\mathrm{u}}_{k|k}(x_k) = \sum_{t=1}^{N^{\mathrm{u}}_k} w^{\mathrm{u},t}_{k|k} p^{\mathrm{u},t}_{k|k}(x_k)$$

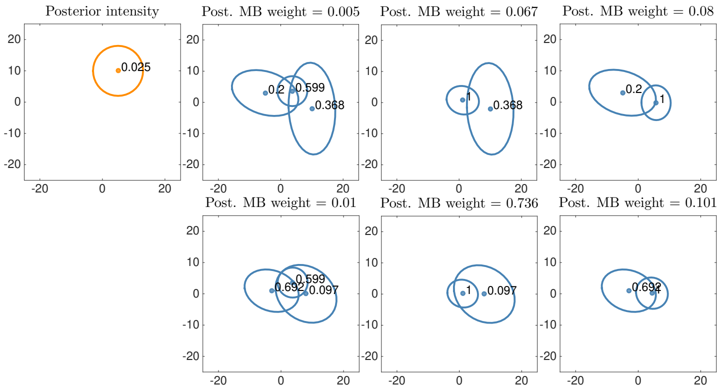
- The number of mixture components increases over time, due to:
  - the addition of birth in the prediction
  - the recycling following the update
- Reduced using pruning, merging and capping.

EXAMPLE PMBM REDUCTION



EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$



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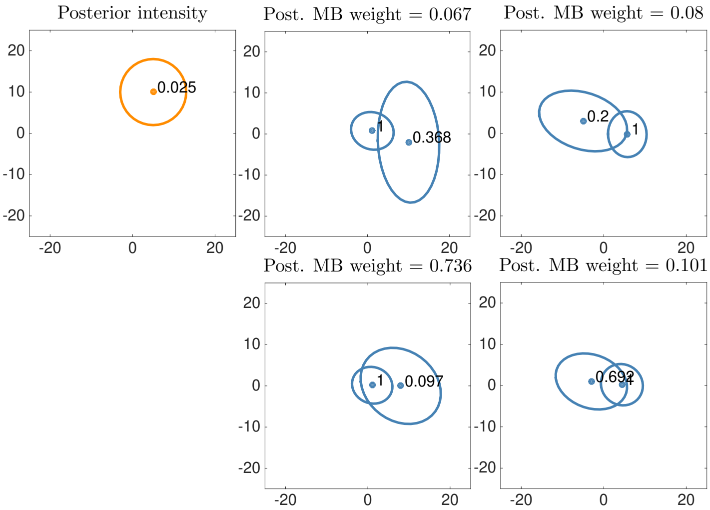
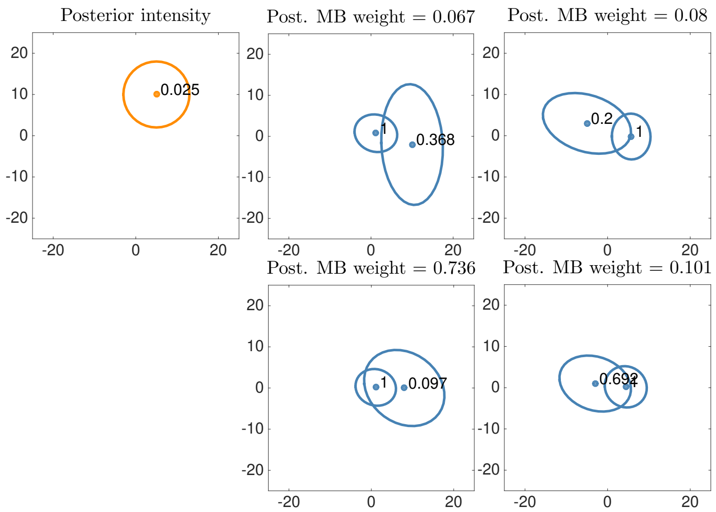


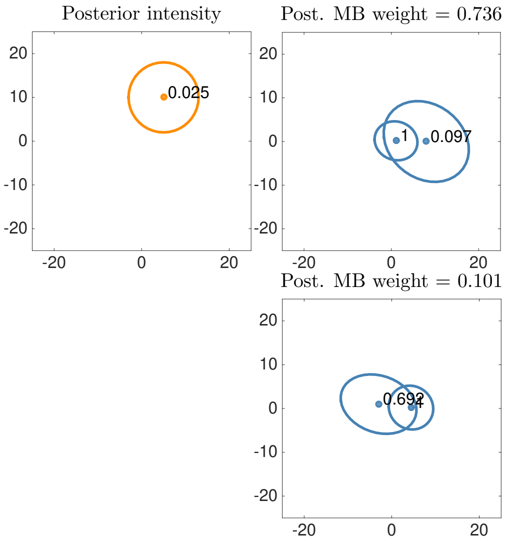
EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\max} = 2$



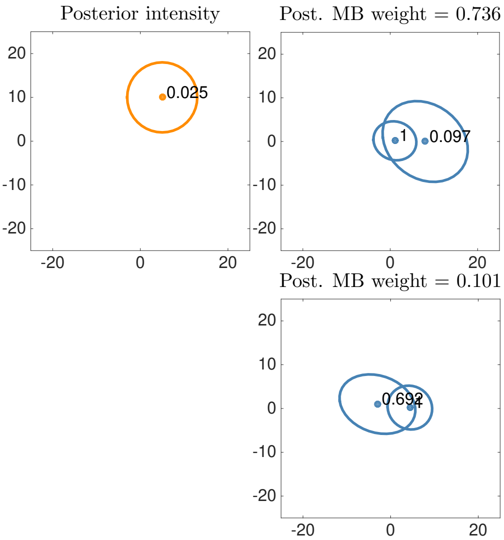
EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\max} = 2$



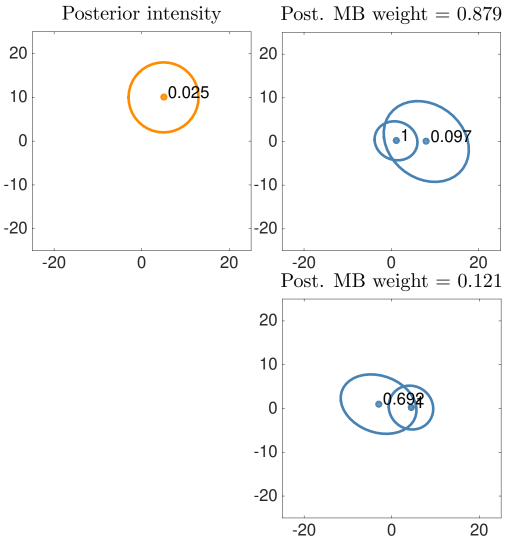
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$
- Re-normalize weights



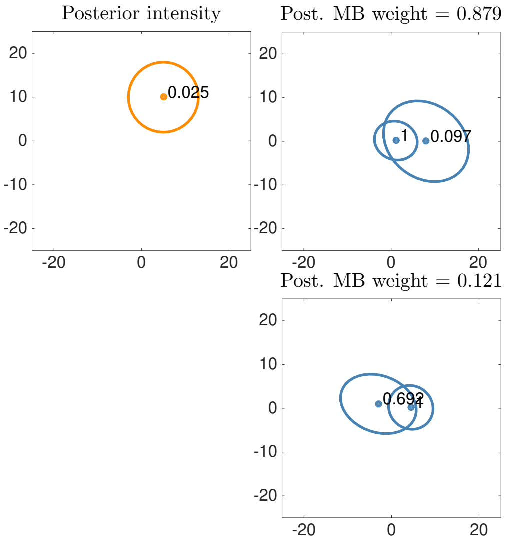
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\mathrm{max}} = 2$
- Re-normalize weights



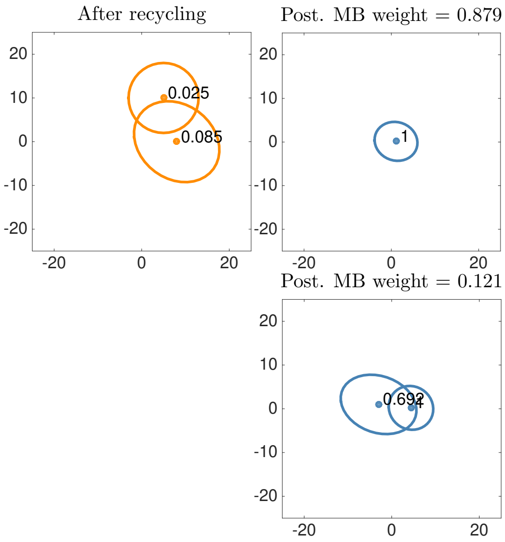
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$
- Re-normalize weights
- Bernoulli recycling, $\Gamma^r = 0.1$



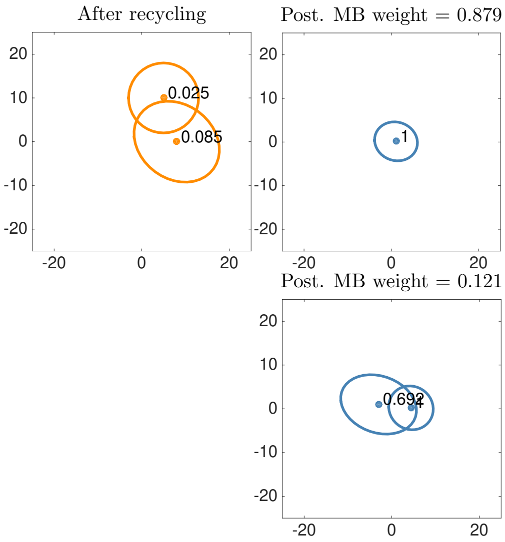
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\text{max}} = 2$
- Re-normalize weights
- Bernoulli recycling, $\Gamma^r = 0.1$



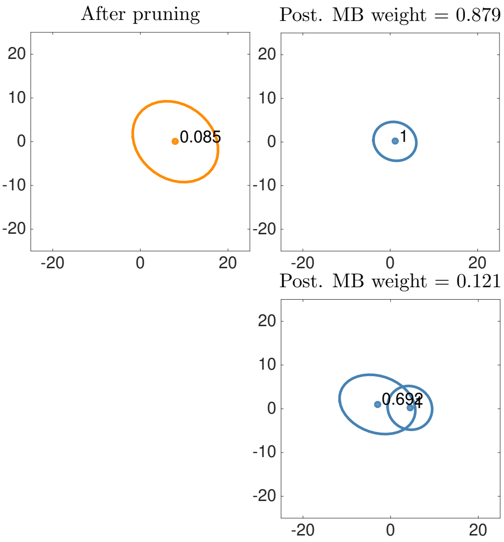
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$
- Re-normalize weights
- Bernoulli recycling, $\Gamma^r = 0.1$
- PPP pruning $\Gamma^w = 0.05$



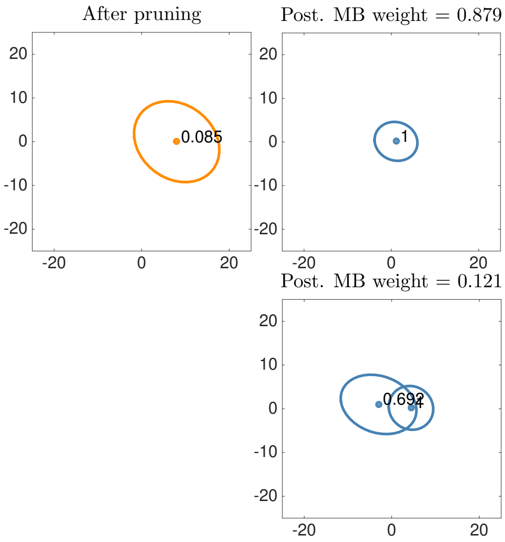
EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\max} = 2$
- Re-normalize weights
- Bernoulli recycling,
  $\Gamma^r = 0.1$
- PPP pruning
  $\Gamma^w = 0.05$



EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\max} = 2$
- Re-normalize weights
- Bernoulli recycling, $\Gamma^r = 0.1$
- PPP pruning $\Gamma^w = 0.05$
- **Note:** typically, Γ and Γ^w are smaller, and $N_{\max}$ is larger



PMBM ESTIMATION

Simple PMBM estimator

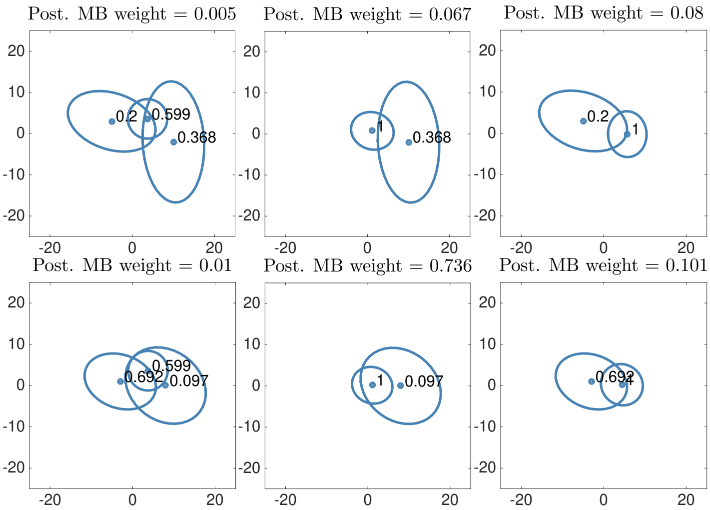
Generally we do not extract estimates from the undetected PPP.

- Initialise as empty set: $\hat{\mathbf{x}}_{k|k} = \emptyset$
- MB with largest weight: $h^\star = \max_{h_k} \ell_{k|k}^{h_k}$
- For $i = 1, \ldots, N_k^{h^\star}$, if $r_{k|k}^{h^\star,i} > \Gamma^e$: $\hat{\mathbf{x}}_{k|k} \leftarrow \hat{\mathbf{x}}_{k|k} \cap \hat{x}_{k|k}^{i,h^\star}$
- For example, expected value or MAP estimate,

$$\bar{x}_{k|k}^{i} = \int x_k p_{k|k}^{h^\star,i}(x_k)\mathrm{d}x_k, \qquad \hat{x}_{k|k}^{i,\mathrm{MAP}} = \arg\max_{x_k} p_{k|k}^{h^\star,i}(x_k)$$

EXAMPLE PMBM ESTIMATOR

- Largest weight
- Extract: $\Gamma^e = 0.5$



Implementation of Conjugate multi-object filters

Multi-Object Tracking

Karl Granström

HO-MHT AND TO-MHT

- Two MHT approaches to tracking n objects:
  - Hypothesis oriented (HO): represent each global n object hypothesis explicitly
  - Track oriented (TO): represent each object by local hypotheses. Global n object hypotheses encoded by look-up table.
  - Track oriented is computationally more efficient.

- Similar alternatives for the MB mixture in PMBM and MBM filtering
  - Hypothesis oriented: represent each MB explicitly
  - Track oriented: represent each Bernoulli by local hypotheses. Each MB (global hypothesis) encoded by look-up table.
  - Again, track oriented is computationally more efficient.

TRACK ORIENTED CONJUGATE MULTI-OBJECT FILTERS

MBM filter

- Add new Bernoullis in prediction
- New local hypothesees in update
- Look-up table for MBs
- Reduction: remove local hypotheses and global hypotheses

PMBM filter

- Initiate new Bernoullis in update
- New local hypothesees in update
- Look-up table for MBs
- Reduction: remove local hypotheses and global hypotheses

IMPLEMENTATIONAL ASPECTS OF MBM FILTERS AND PMBM FILTERS

- Local and global hypotheses

- Bernoulli representation in the MBs
  - Uncertain existence, $r \in (0, 1)$
  - Certain existence, $r = 0$ or $r = 1$

Local and Global Hypotheses in MBM filter

Multi-Object Tracking

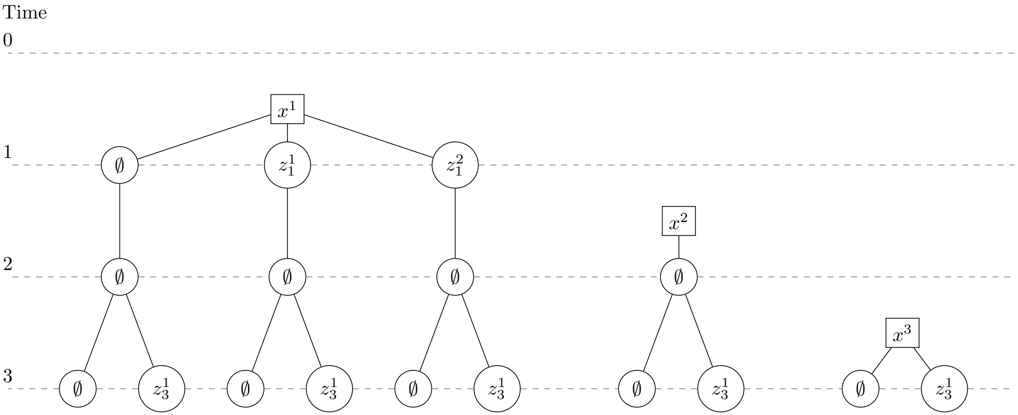
Karl Granström

HYPOTHESES IN MBM FILTER

Example: scenario setup

- MB birth with a single birth component in each time step
- Three time steps, measurement sets:
  - $\mathbf{z}_1 = \{z_1^1,\ z_1^2\}$
  - $\mathbf{z}_2 = \emptyset$
  - $\mathbf{z}_3 = \{z_3^1\}$
- At time $k = 0$, empty MBM. Not necessary, but most common.

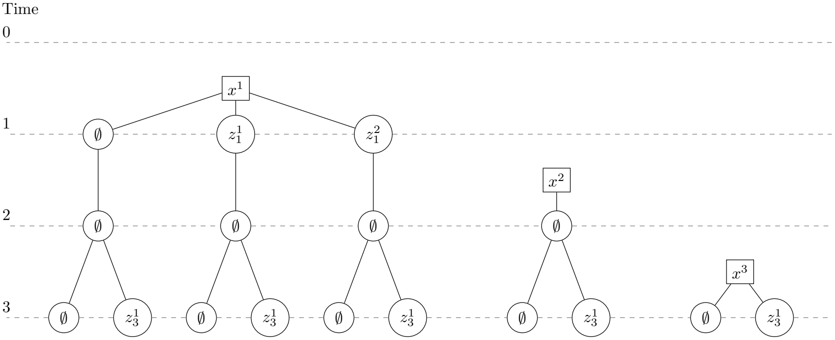
LOCAL HYPOTHESIS TREES IN MBM FILTER



For each leaf node, we have r and $p(x)$ conditioned on that association sequence

MBM GLOBAL HYPOTHESES

Total number of MBs (global hypotheses): 12



For simplicity: local hypotheses indexed 1, 2, 3 . . . from left to right.

Look-up table:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \\ 5 & 1 & 1 \\ 5 & 1 & 2 \\ 5 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

TRACK ORIENTED MBM FILTER

- Add new Bernoullis in the prediction
- For each Bernoulli, maintain local hypotheses
- Look-up table points out which local hypotheses are included in an MB
- MBM reduction affects both the local hypotheses and look-up table

Local and Global Hypotheses in PMBM filter

Multi-Object Tracking

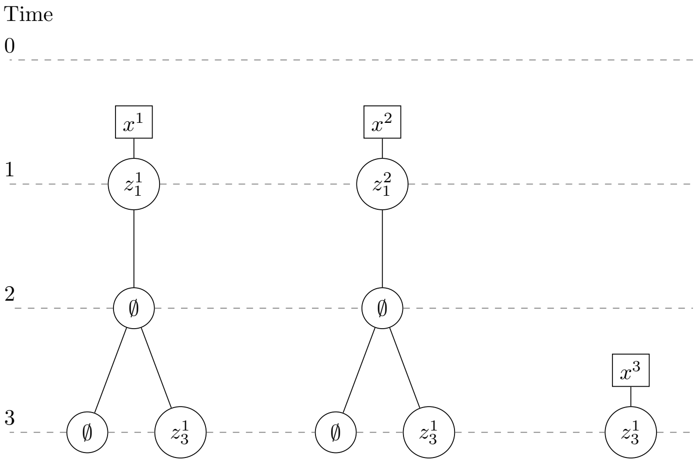
Karl Granström

LOCAL HYPOTHESIS TREES IN PMBM FILTER

Example: scenario setup

- PPP birth, i.e., initiation of new Bernoullis is measurement driven
- Three time steps, measurement sets:
  - $\mathbf{z}_1 = \{z_1^1,\ z_1^2\}$
  - $\mathbf{z}_2 = \emptyset$
  - $\mathbf{z}_3 = \{z_3^1\}$
- At time $k = 0$, empty MBM. Not necessary, but most common.

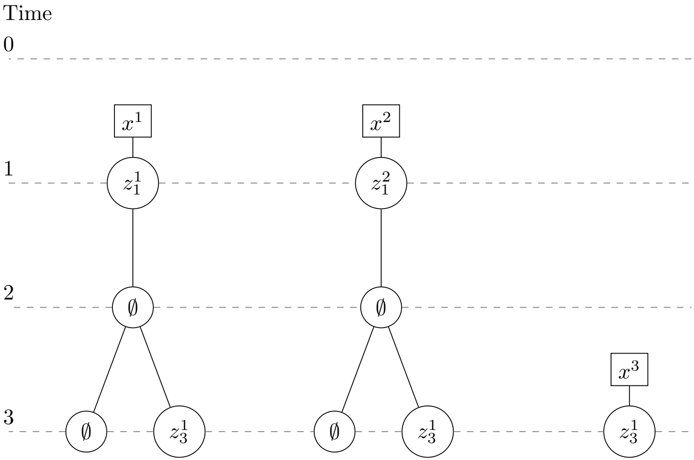
LOCAL HYPOTHESIS TREES IN PMBM FILTER



For each leaf node, we have r and $p(x)$ conditioned on that association sequence

PMBM GLOBAL HYPOTHESES

Total number of MBs (global hypotheses): 3



Look-up table:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

For simplicity: local hypotheses indexed from left to right.

Note: each Bernoulli is not represented in each MB (global hypothesis)

TRACK ORIENTED PMBM FILTER

- Add a new Bernoulli for each measurement
- For each Bernoulli, maintain local hypotheses
- Look-up table points out which local hypotheses are included in an MB
- MBM reduction affects both the local hypotheses and look-up table

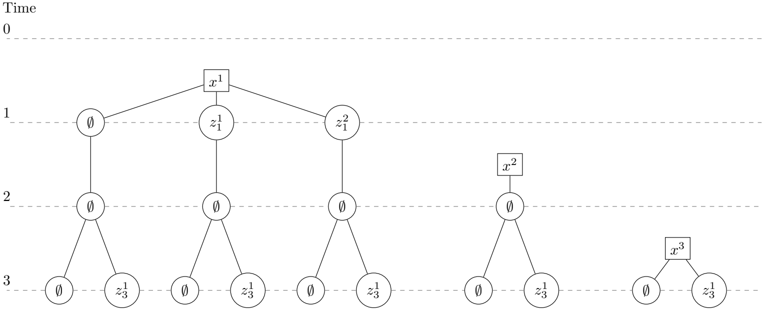
Reduction of local and global hypotheses

Multi-Object Tracking

Karl Granström

MULTI-BERNOULLI MIXTURE PRUNING/CAPPING

Global hypotheses are removed

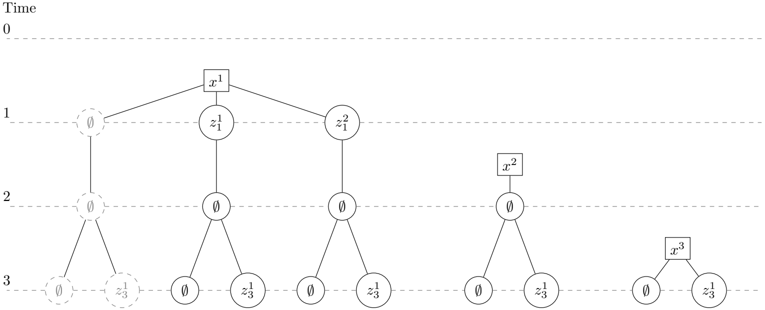


Before pruning/capping

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \\ 5 & 1 & 1 \\ 5 & 1 & 2 \\ 5 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

MULTI-BERNOULLI MIXTURE PRUNING/CAPPING

Global hypotheses are removed



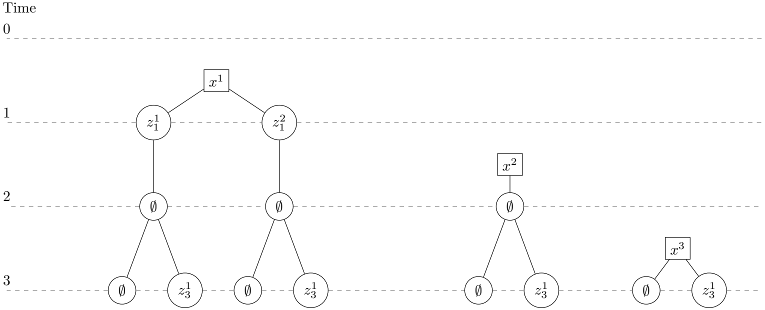
Four global hypotheses are pruned/capped

Some local hypotheses no longer included in an MB

$$\begin{bmatrix} 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \\ 5 & 1 & 1 \\ 5 & 1 & 2 \\ 5 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

MULTI-BERNOULLI MIXTURE PRUNING/CAPPING

Global hypotheses are removed

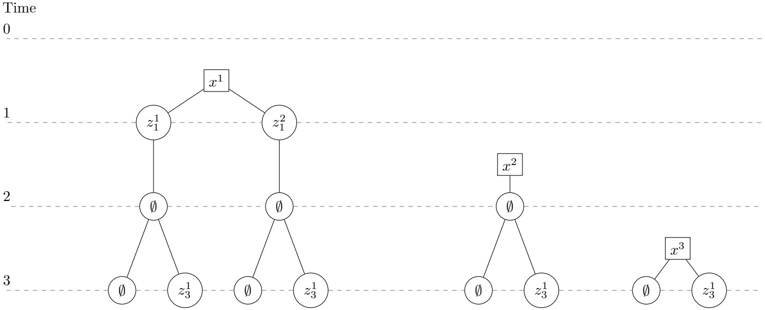


$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

Prune un-used local hypotheses, adjust look-up table
If a Bernoulli has no local hypothesis included in any global hypothesis, naturally it can be pruned entirely.

BERNOULLI PRUNING/RECYCLING

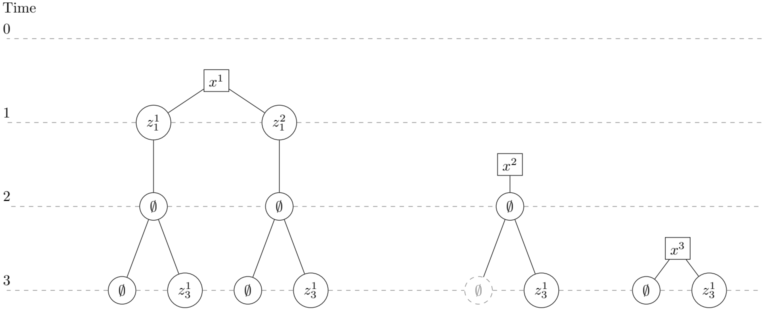
Local hypotheses are removed



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

BERNOULLI PRUNING/RECYCLING

Local hypotheses are removed

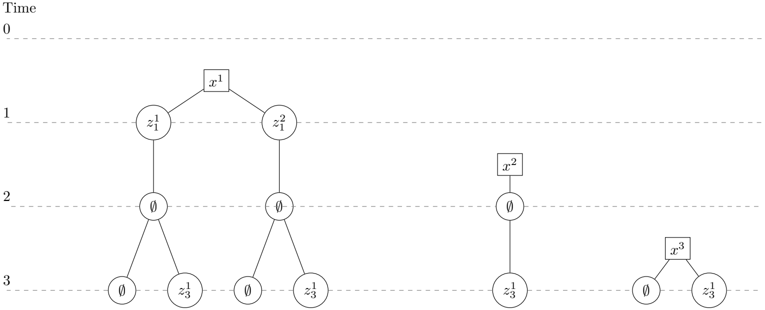


$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

Example: prune local hypothesis corresponding to 2 misdetections

BERNOULLI PRUNING/RECYCLING

Local hypotheses are removed



$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 0 & 2 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

Adjust look-up table

GLOBAL HYPOTHESIS UNIQUENESS

Global hypotheses should be unique

Before reduction:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} \quad \begin{matrix} w^1 \\ w^2 \\ w^3 \\ w^4 \\ w^5 \\ w^6 \end{matrix}$$

After reduction:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 0 & 1 \\ 4 & 0 & 1 \end{bmatrix} \quad \begin{matrix} w^1 \\ w^2 \\ w^3 \\ w^4 \\ w^5 \\ w^6 \end{matrix}$$

Unique global hypotheses:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \quad \begin{matrix} w^1 + w^2 \\ w^3 \\ w^4 + w^5 \\ w^6 \end{matrix}$$

Important to adjust the weights accordingly.

MBs with certain object existence

Multi-Object Tracking

Karl Granström

OBJECT EXISTENCE IS BINARY

- In both MBM and PMBM there are Bernoullis with $0 < r < 1$.
- Expected cardinality is r
  - $r = 0.5 \Rightarrow$ we expect half an object
  - For example, how should an autonomous car react if there is half a car in front?
- In reality, an object is either there, or not.
- Compare to n object tracking: an integer number of objects, no fractions of objects
- Can we have a multi-object density such that each hypothesis represents an integer number of objects?
- Yes, if we change to a so called MBM_{01} representation.
- The δ-GLMB filter can be interpreted as having a MBM_{01} representation.

EXPANDING A BERNOULLI TO CERTAIN EXISTENCE

- A Bernoulli $(r, p(\cdot))$ with $r \in (0,\ 1)$ represents two possibilities,
  - With probability r we have exactly one object, with state pdf $p(\cdot)$.
  - With probability $1 - r$, we have exactly zero objects.
- We can represent this as a Bernoulli mixture density,

$$p(\mathbf{x}) = r\mathcal{B}^1(\mathbf{x}) + (1 - r)\mathcal{B}^2(\mathbf{x})$$

  where $\mathcal{B}^1(\mathbf{x})$ and $\mathcal{B}^2(\mathbf{x})$ are Bernoulli densities with parameters

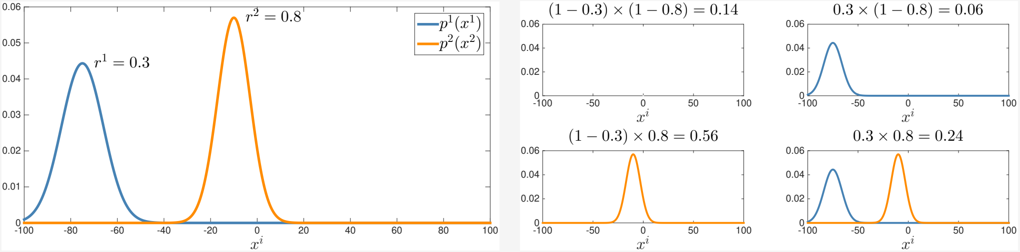
$$\left(r^1,\ p^1(\cdot)\right) = (1,\ p(\cdot))\,, \qquad \left(r^2,\ p^2(\cdot)\right) = (0,\ \text{any pdf})$$

- Instead of one Bernoulli with uncertain existence r, we have two hypotheses that each have certain existence: either one object ($r^1 = 1$) or zero objects ($r^2 = 0$).

EXPANDING TWO BERNOULLIS TO CERTAIN EXISTENCE

An MB with two Bernoullis $(r^1, p^1(\cdot)($ and $(r^2, p^2(\cdot))$, with $r^1 \in (0,\ 1)$ and $r^2 \in (0,\ 1)$ corresponds to $2^2 = 4$ hypotheses with certain object existence:

Example: MB with two Bernoullis



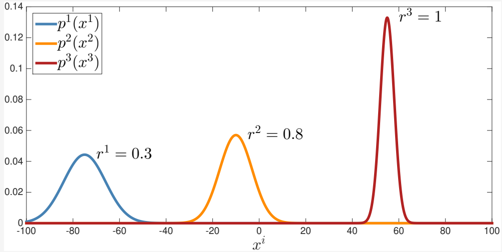
Each hypothesis is a special kind of MB:

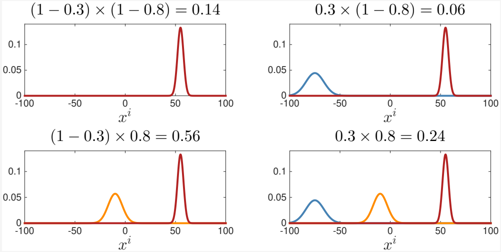
1) Zero Bernoullis, 2) & 3) One Bernoulli, $r = 1$, 4) Two Bernoullis, $r = 1$

EXPANDING AN MB TO CERTAIN EXISTENCE

- Consider an MB with n Bernoullis, with parameters $\left(r^i, p^i(\cdot)\right)$.
- Let $n' \leq n$ of the Bernoullis have $r \in (0,\ 1)$, and let remaining $n - n'$ have $r = 1$.
- Can be expanded into a MBM_{01} with $2^{n'}$ MBs, where each Bernoulli has $r = 1$.

Example: MB with $n = 3,\ n' = 2$ leads to MBM_{01} with $2^{n'} = 4$ MBs

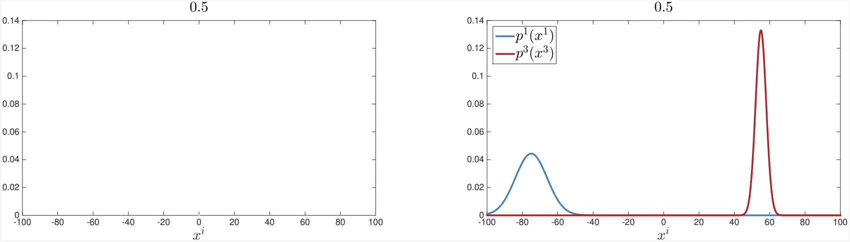




MBM WITH CERTAIN EXISTENCE

- We denote this type of MBM as MBM_{01}
- Any MB(M) can be expanded into an MBM_{01}.
- Not all MBM_{01} have a simpler MB equivalent.

MBM_{01} with two equally probable components



Cannot be simplified to MB with two Bernoullis

WHEN IS THIS USEFUL?

- MBM_{01} representation can feel more intuitive, with an integer number of objects in each global hypothesis.

- Unusually specific birth model: If we know that new objects appear in the surveillance area together in groups, e.g., in pairs, the birth can be modeled as MBM_{01}.

- Facilitates other multi-object estimator:
  - Find MAP cardinality estimate
  - Find most probable global hypothesis with this cardinality
  - Extract estimates

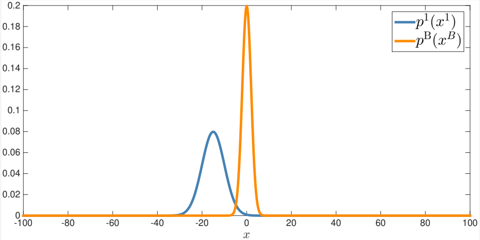
- However, worse computational cost with MBM_{01} representation.

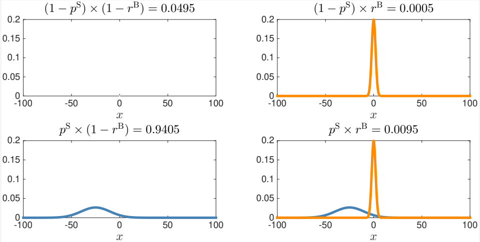
PREDICTION OF AN MBM$_{01}$ DENSITY

Objects may appear and disappear:

MB$_{01}$ with N Bernoullis, birth with N^{B} Bernoullis $\Rightarrow$ predicted MBM$_{01}$ with $2^{N+N^{\mathrm{B}}}$ hyps.

One object (blue), $P^{\mathrm{S}} = 0.95$, and one birth (orange), $r^{\mathrm{B}} = 0.01$



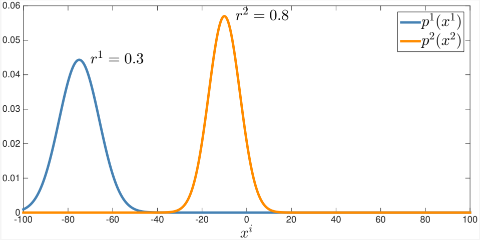


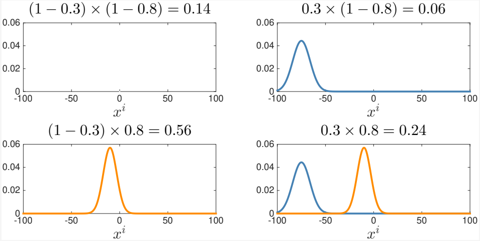
- Intractable in practice for large N and N^{B}, approximations are required.
- **Regular MB prediction does not require approximation.**

UPDATE OF AN MBM$_{01}$ DENSITY

Posterior MBM$_{01}$ has many more components, compared to posterior MBM

Update with m_k measurements





Number of data associations:

$$\begin{aligned} \text{MB} &: \quad N_A(m_k, 2) \\ \text{MBM}_{01} &: \quad N_A(m_k, 0) + 2N_A(m_k, 1) + N_A(m_k, 2) \end{aligned}$$

SUMMARY OF MB VS MBM$_{01}$

With an MBM$_{01}$ representation:

- Requires additional approximations in both the prediction and the update
- Generally requires a higher number of global hypotheses to achieve the same performance

Simulation studies have shown that the MBM$_{01}$ representation results in a higher computational cost to achieve the same tracking performance.

Forming trajectories using labels

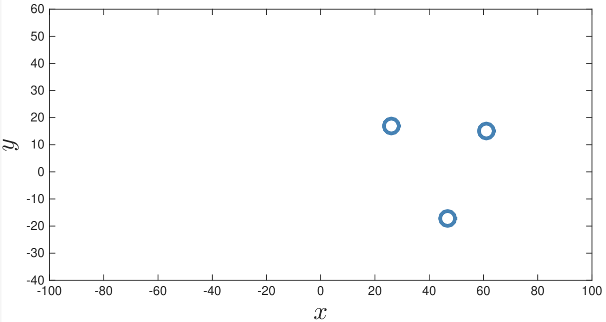
Multi-Object Tracking

Karl Granström

HOW CAN WE GET TRAJECTORIES?

- PMBM/MBM filter outputs $\hat{\mathbf{x}}_{k|k}$
- Estimate at time k of
  - Number of objects
  - States of the objects
- Trajectory: sequence of states, from initial time to end time.
- Two approaches
  - Sets of labelled objects
  - Sets of trajectories

Set of three state estimates (positions)



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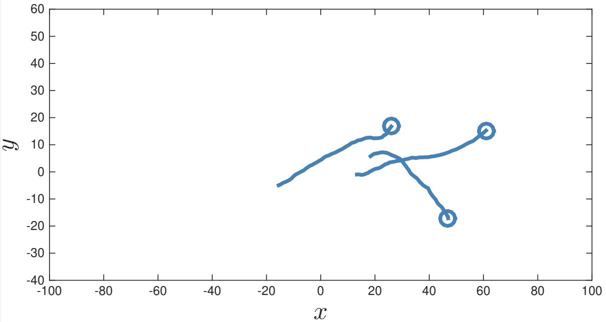
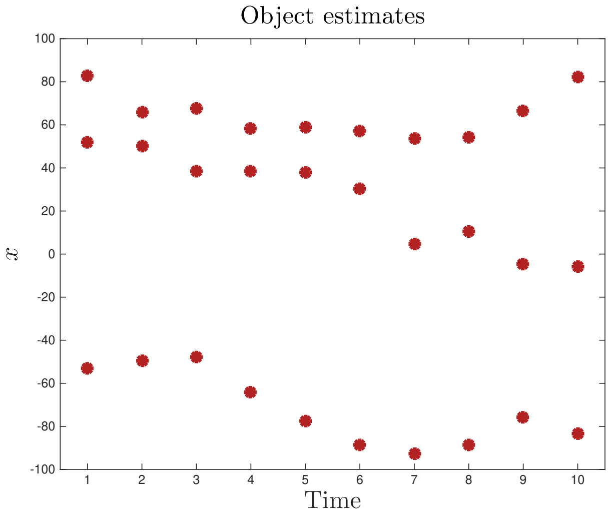
Set of three trajectories



SETS OF LABELLED OBJECTS: BASIC IDEA

- To obtain trajectories would require post-processing. Not directly available from $\hat{\mathbf{x}}_{1|1}, \dots, \hat{\mathbf{x}}_{10|10}$.
- **Simple idea:** assign to each estimate a unique identifier, **a label**
- Connect estimates with same label into trajectories.
- New RFS: Labelled Set.



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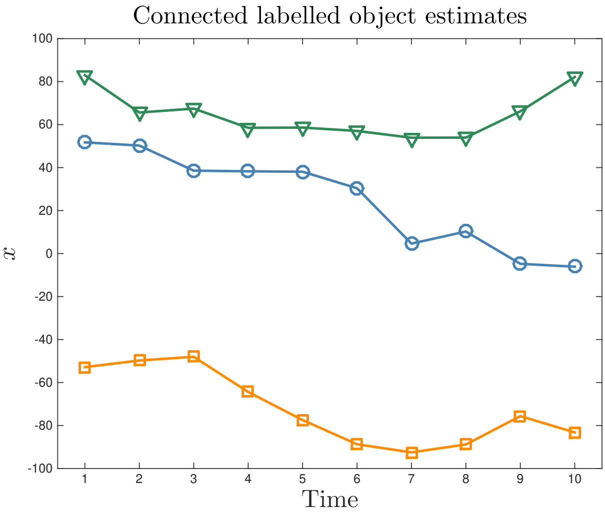


Labelled multi-Bernoulli

Multi-Object Tracking

Karl Granström

LABELLED BERNOULLIS

- A labelled object state is denoted (x, L)
- A labelled Bernoulli RFS is, just like the name suggests, parameterised by
  - Probability of existence r
  - Object state density $p(x)$
  - Label L
- A labelled MB (LMB) RFS is the union of labelled Bernoulli RFSs,

$$\left\{\left(r^i,\ p^i(\cdot),\ L^i\right)\right\}_i$$

  where the **labels are unique**.
- An LMB whose labels are not unique defeats its own purpose.
- The definition of the LMB multi-object density ensures the uniqueness.

LABELLED MBM FILTER

- Earlier we saw that with MB birth, we get an MBM filter.
- If the birth model is labelled MB, we get a labelled MBM (LMBM) filter
- Label uniqueness over time is ensured by labelling the birth appropriately, e.g.,
  - Labelled multi-Bernoulli birth with parameters

$$\left\{\left(r_k^{\mathrm{B},i},\ p_k^{\mathrm{B},i}(\cdot),\ L_k^{\mathrm{B},i}\right)\right\}_i$$

  - The labels are tuples

$$L_k^{\mathrm{B},i} = (k, \alpha_k(i))$$

    where the identifiers $\alpha_k(i)$ are such that $\alpha_k(i) \neq \alpha_k(i')$ if $i \neq i'$.
  - A simple and natural choice is $\alpha_k(i) = i$.

LABELLED MBM MULTI-OBJECT CONJUGATE PRIOR

Labelled MBM (LMBM) multi-object conjugate prior

Prediction: $$\mathcal{LMBM}_{k|k-1}(\mathbf{x}_k) = \int p(\mathbf{x}_k | \mathbf{x}_{k-1}) \mathcal{LMBM}_{k-1|k-1}(\mathbf{x}_{k-1}) \, \delta \mathbf{x}_{k-1}$$

Update: $$\mathcal{LMBM}_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{z}_k | \mathbf{x}_k) \mathcal{LMBM}_{k|k-1}(\mathbf{x}_k)}{\int p(\mathbf{z}_k | \mathbf{x}'_k) \mathcal{LMBM}_{k|k-1}(\mathbf{x}'_k) \, \delta \mathbf{x}'_k}.$$

One of the first multi-object conjugate priors for point object tracking to be presented in MOT literature is called the **δ-Generalised Labelled Multi-Bernoulli (δ-GLMB) filter**.

- Labelled Multi-Bernoulli birth
- MBM_{01} density representation

LABELLED MBM FILTER: PREDICTION AND UPDATE

- An object label neither changes over time, nor changes due to the detection process
- Therefore, the labels are unaffected by the prediction and the update
- The prediction and update of the labelled Bernoulli parameters r and $p(\cdot)$ in an LMBM filter is the same as the prediction and update of Bernoulli parameters r and $p(\cdot)$ in the MBM filter.
- Some challenges when using labels to form trajectories: gaps and switching

Labelled PMBM

Multi-Object Tracking

Karl Granström

LABELLED PMBM FILTER

- We obtained a labelled MBM filter by using labelled MB birth.
- Obvious idea: A labelled PMBM filter by using a labelled PPP birth.
- Is this theoretically sound? No
- But can we do it in practice anyway, using some heuristics? Yes

THE PROBLEM WITH LABELLED PPP

- A labelled Bernoulli $(r, p(\cdot), L)$ represents either no object, or one object
  - If we sample and obtain an object, it must have label L
  - This generalises to LMB and LMBM — we know what the unique labels are

- A labelled PPP with intensity $\lambda(x)$ and some label distribution.
  Represents any finite number of labelled objects,
  $$(x^1, L^1),\ (x^2, L^2),\ \ldots\ (x^i, L^i),\ \ldots$$
  The labelled PPP objects are iid. If we sample,
  - The states x^i are iid, with PPP intensity $\lambda(x)$
  - The labels are iid, with some label distribution over the labels $L^1,\ L^2,\ \ldots\ L^i,\ \ldots$
  - It is possible (non-zero probability) that many objects have the same label
  - **Label uniqueness cannot be guaranteed with labelled PPP**

HEURISTICALLY LABELLED PMBM FILTER

- Idea: heuristically modify the PMBM algorithm
  - When a new Bernoulli is initiated by a measurement z_k^j, it is given a unique identifier, e.g., (k, j), sometimes called meta-data
  - Lacks a certain theoretical elegance, but does work in practice

- **Heuristically "labelled" PMBM filter:**
  - No labels for the undetected PPP
  - Meta-data used to heuristically label Bernoullis when they are first initiated; treatment of labelled Bernoulli in subsequent time steps is same as in LMBM.
  - Lacks a sound theoretical basis, not a true labelled filter
  - In comparison, labelled MBM has a sound theoretical basis

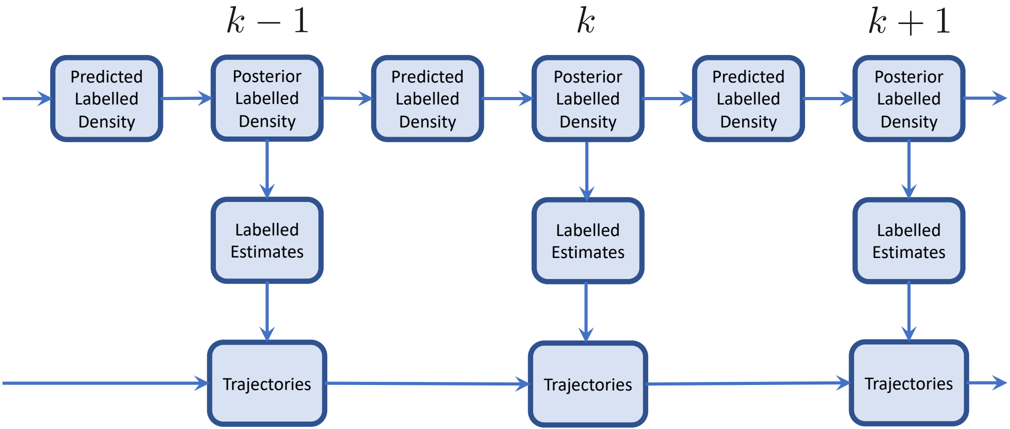
- Importantly: "labelled" PMBM does not solve the inherent challenges involved with using labels to form trajectories, e.g., gaps and switching.

Challenges when using labels to form trajectories

Multi-Object Tracking

Karl Granström

USING LABELS TO FORM TRAJECTORIES



PROS AND CONS OF USING LABELS TO FORM TRAJECTORIES

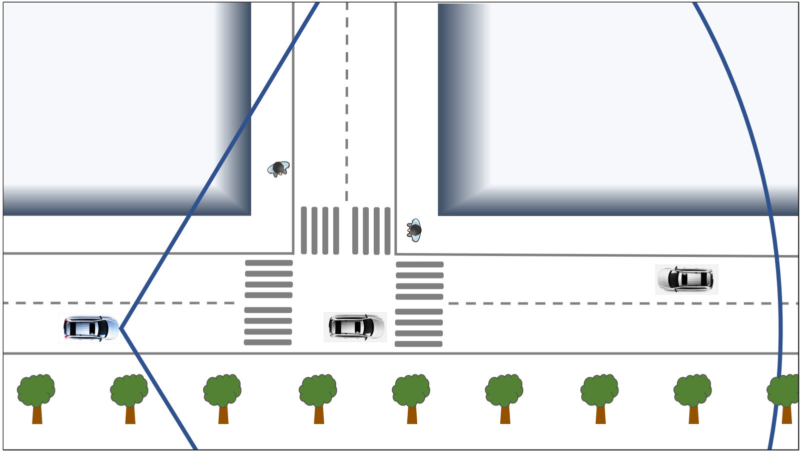
Pros

- Simple and intuitive idea
- Unique labels are a way to try to capture the object identity, which is unique
- Requires little modification to add labels to an unlabelled tracking algorithm

Cons

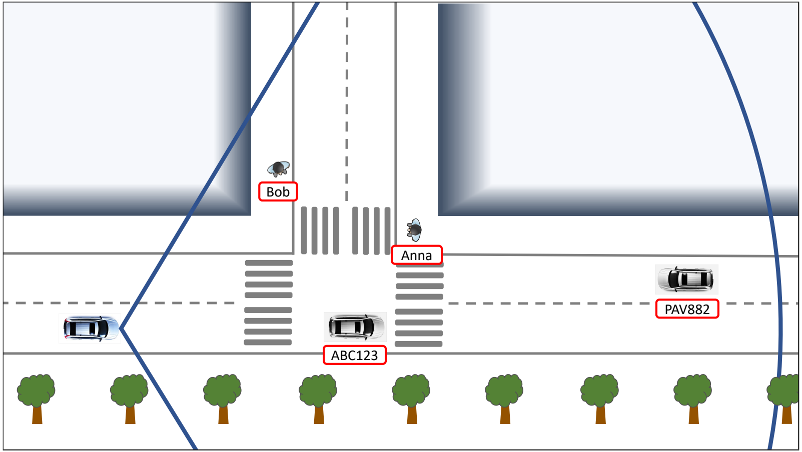
- Typically, the labels are implicit. The true object identity (the explicit label) is not measured by the sensor(s), it is unobservable.
- Missed detections can lead to gaps in the trajectory formed by labelled estimates
- Physically unrealistic switching when multi Bernoulli birth is i.i.d., and in challenging scenarios when objects get in close proximity and then separate
- Difficulty fusing labelled estimates from different sensors

IMPLICIT AND EXPLICIT LABELS



Autonomous vehicle in urban area

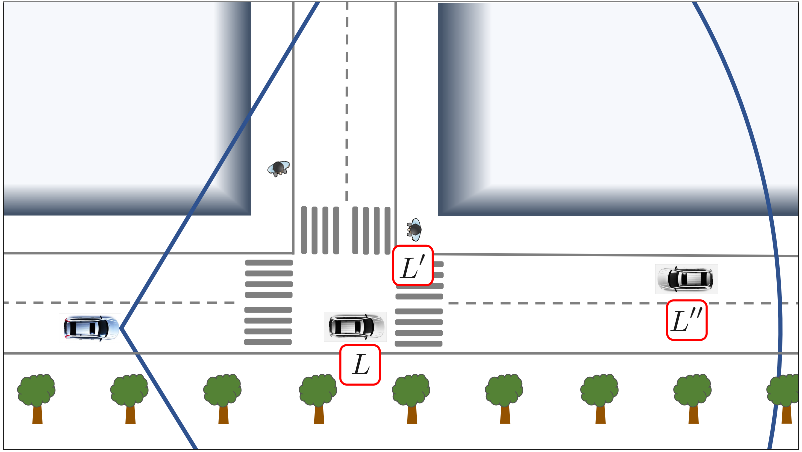
IMPLICIT AND EXPLICIT LABELS



Explicit label: corresponds to ID, e.g., person's name, vehicle's registration number

Explicit labels are most often unobservable, i.e., cannot be known

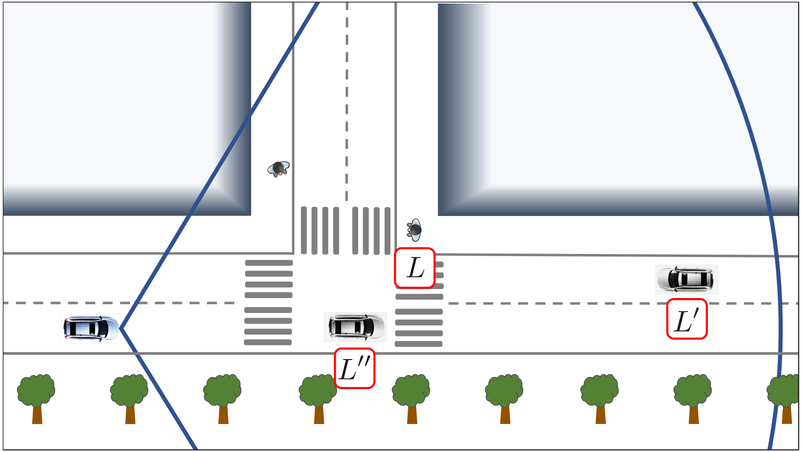
IMPLICIT AND EXPLICIT LABELS



Implicit label: static and unique label assigned when object is born

Implicit labels lack a clear physical meaning

IMPLICIT AND EXPLICIT LABELS



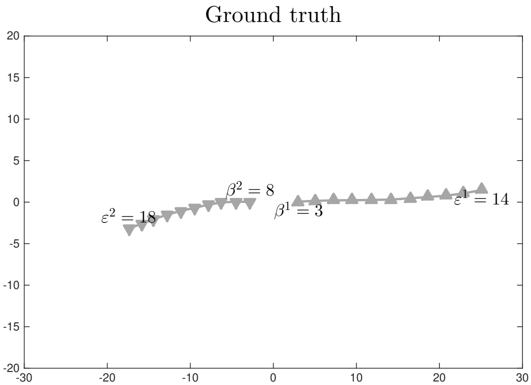
Assignment of unique implicit labels is arbitrary

A different unique assignment can form the same trajectories

GAPS IN THE TRAJECTORY: SETUP

Two objects

- Twenty time steps
- Two objects, appear at time step β^i, disappear at time step ε^i
- Left object misdetected at times $k = 11, 12, 13$



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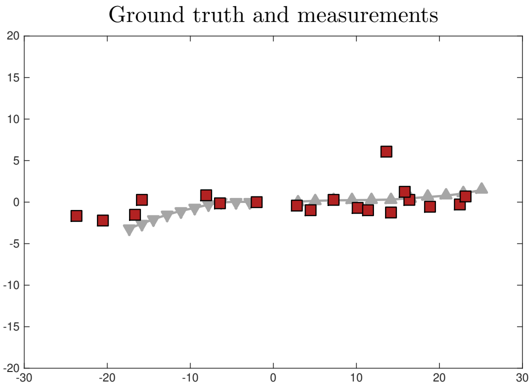

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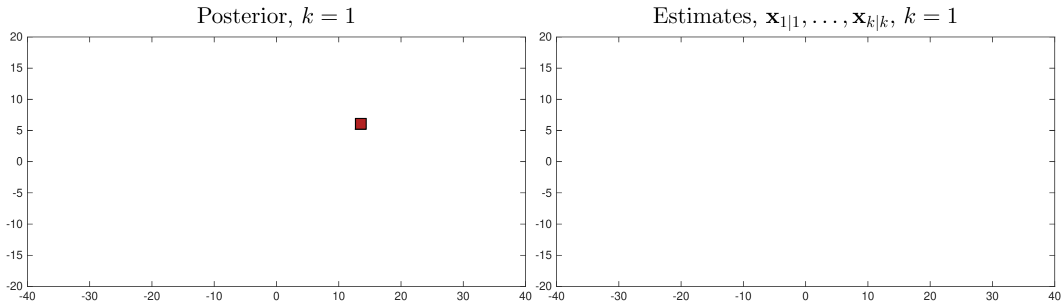

- Measurements - red squares
- Labelled Bernoulli
    - Gaussian state densities
    - Bernoulli labels illustrated by different colors
    - Probability of existence written in %

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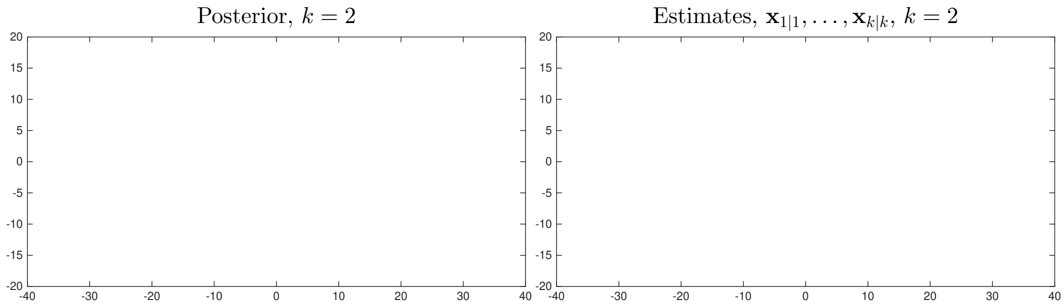

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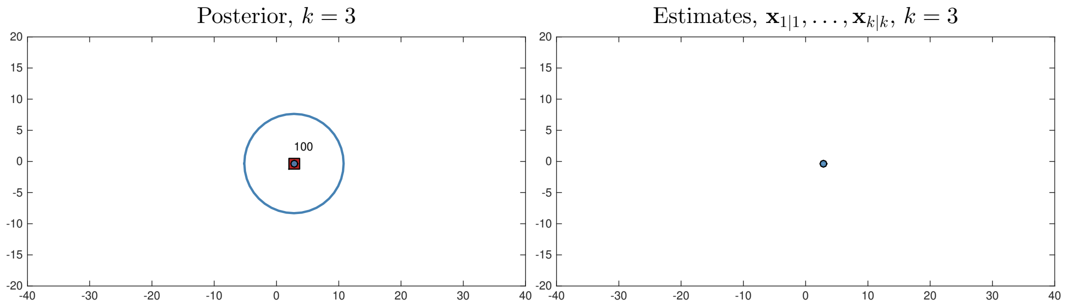

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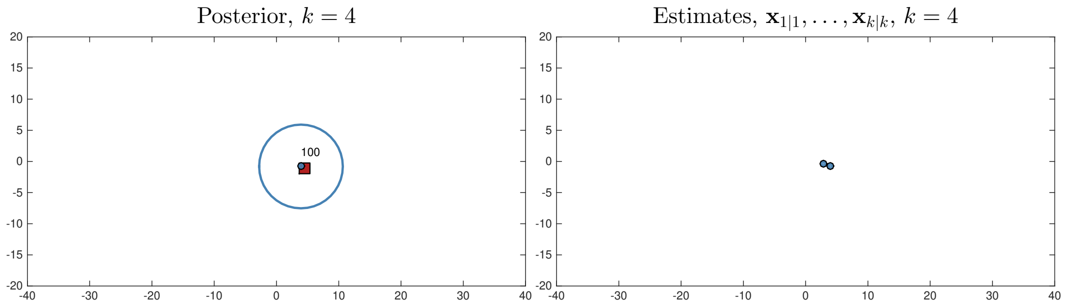

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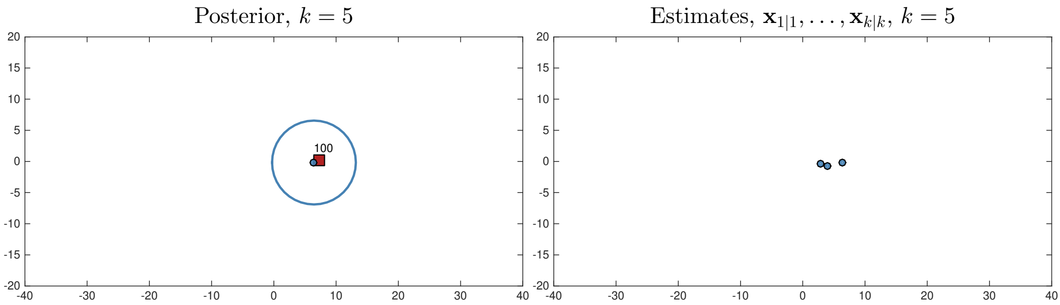

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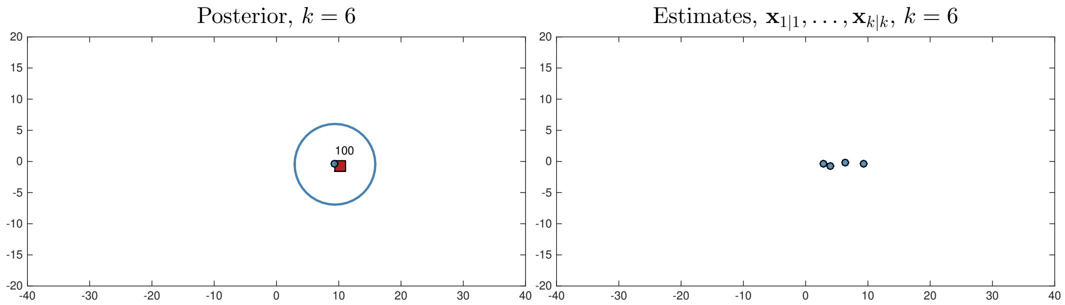

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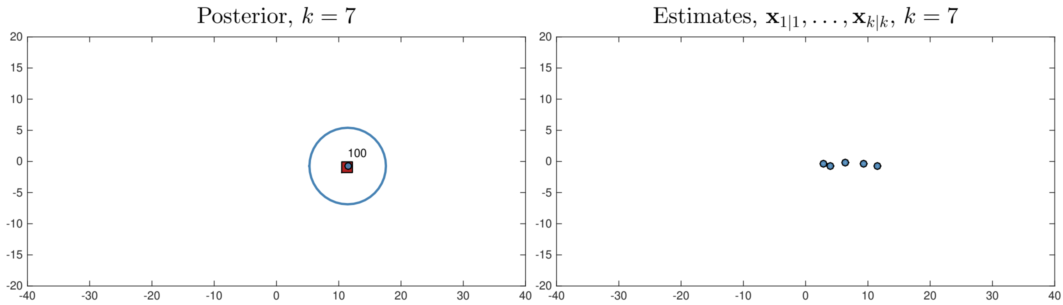

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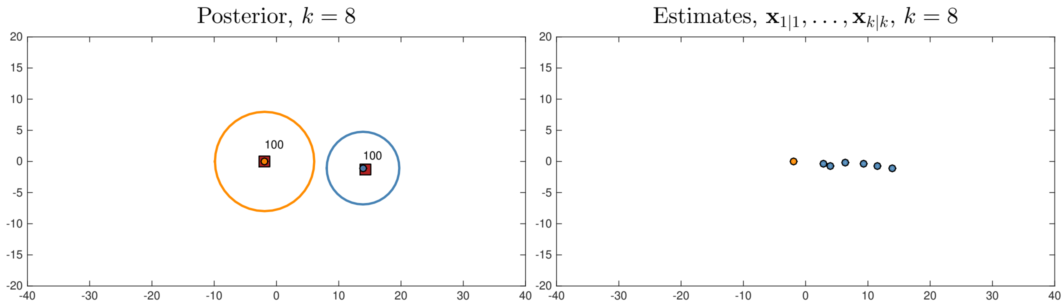

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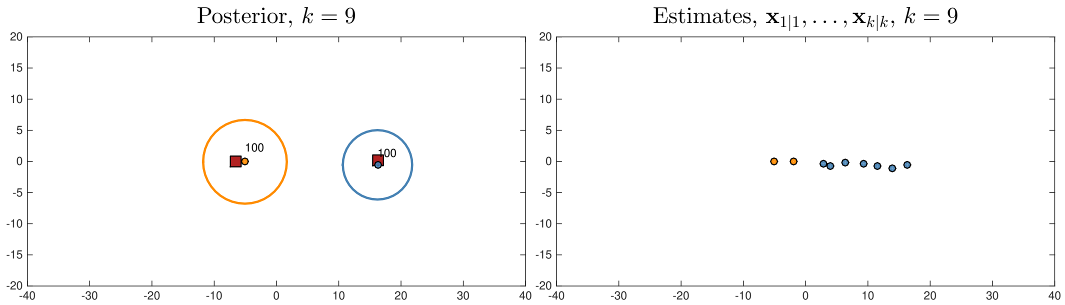

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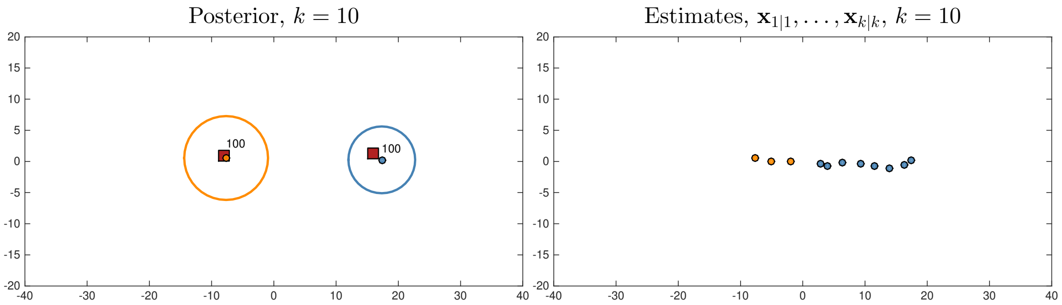

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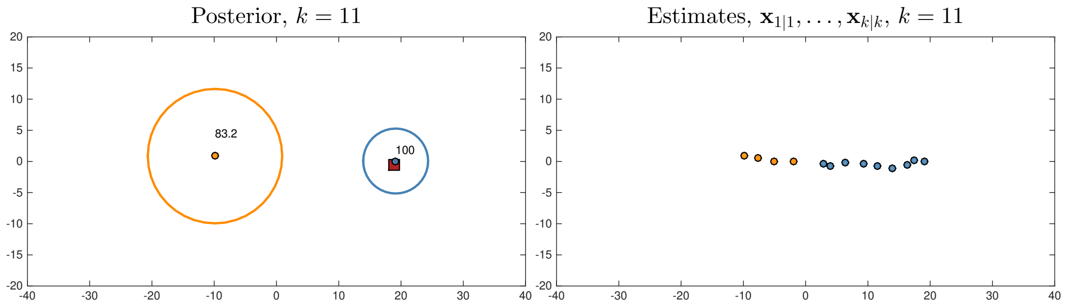

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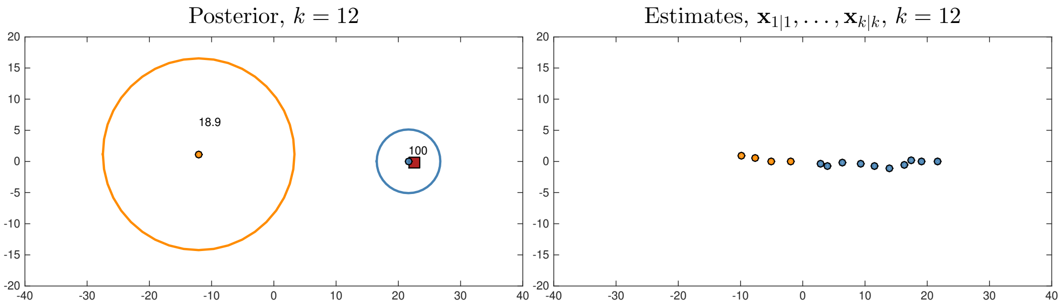

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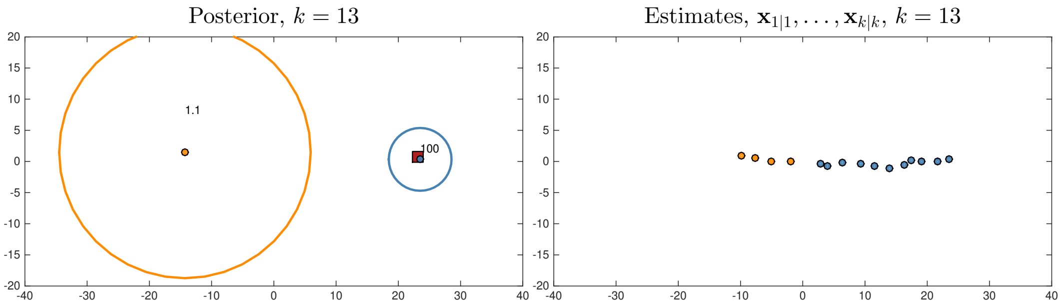

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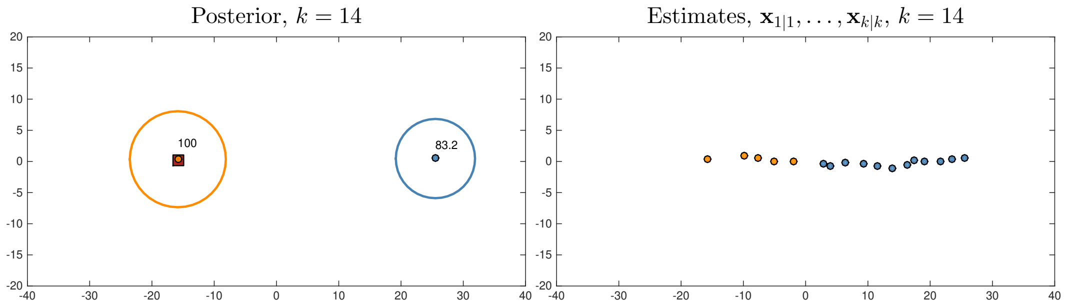

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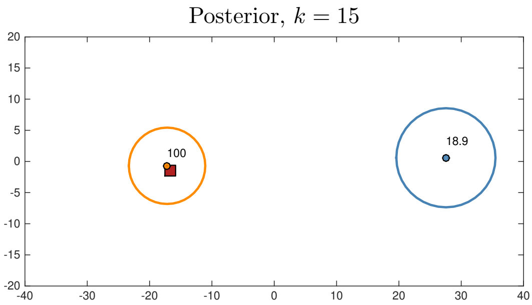

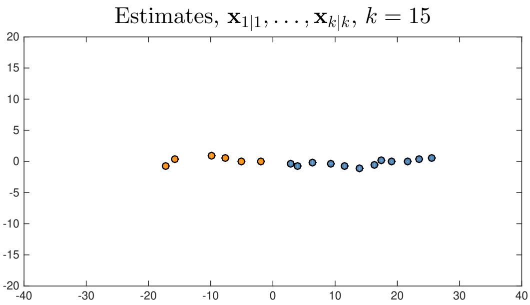

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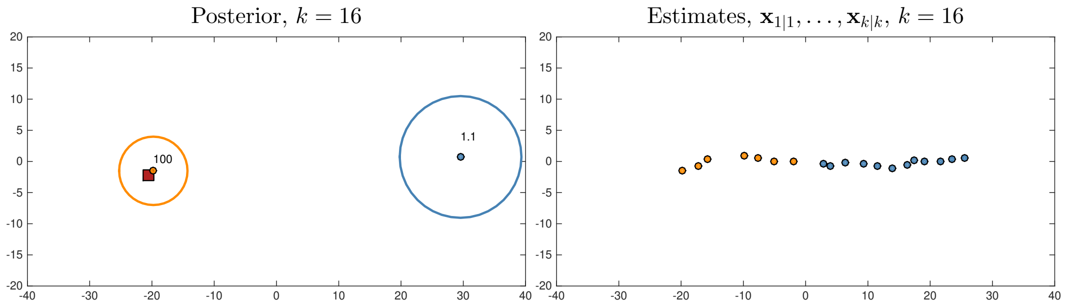

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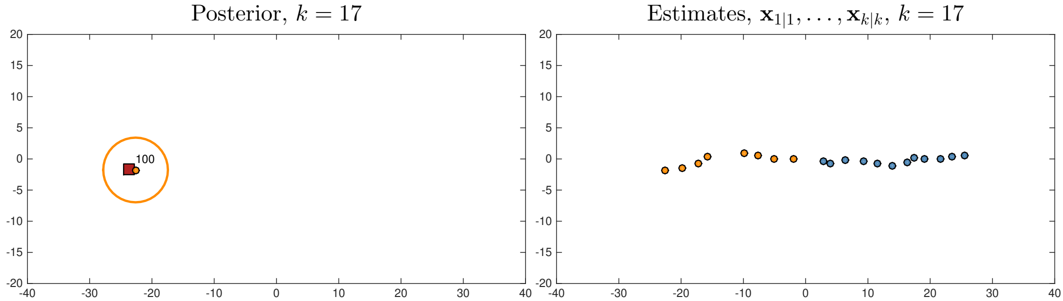

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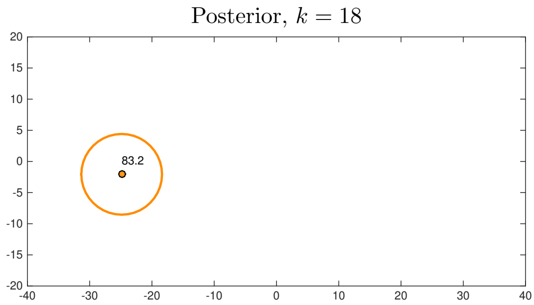

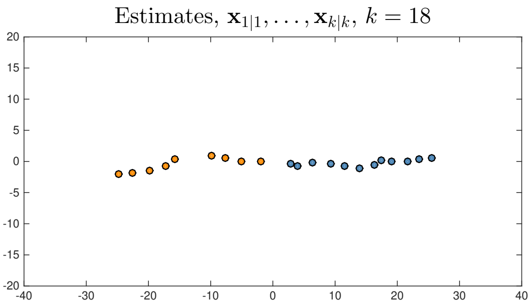

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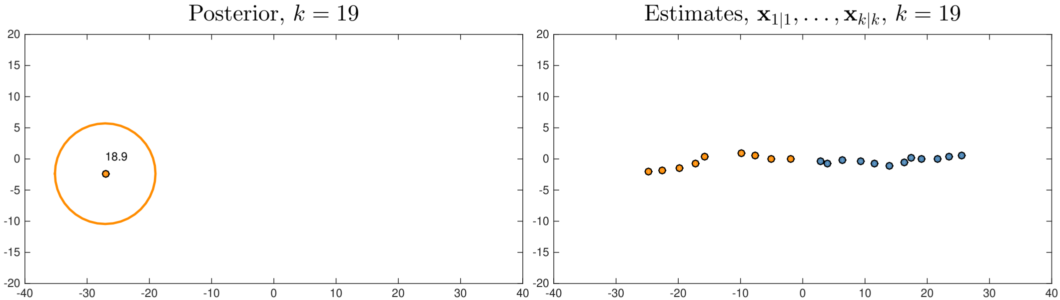

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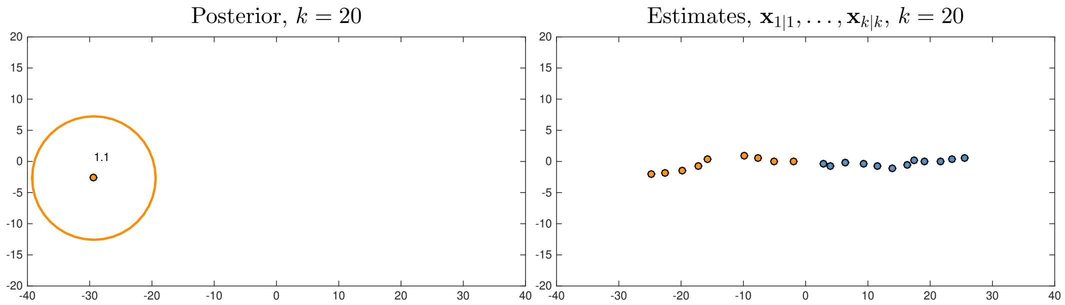

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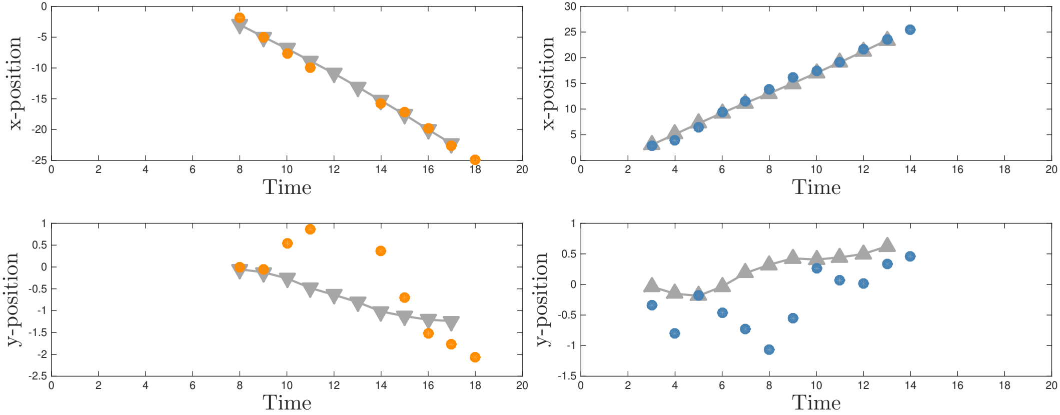
GAPS IN THE TRAJECTORY: TRACKING RESULTS



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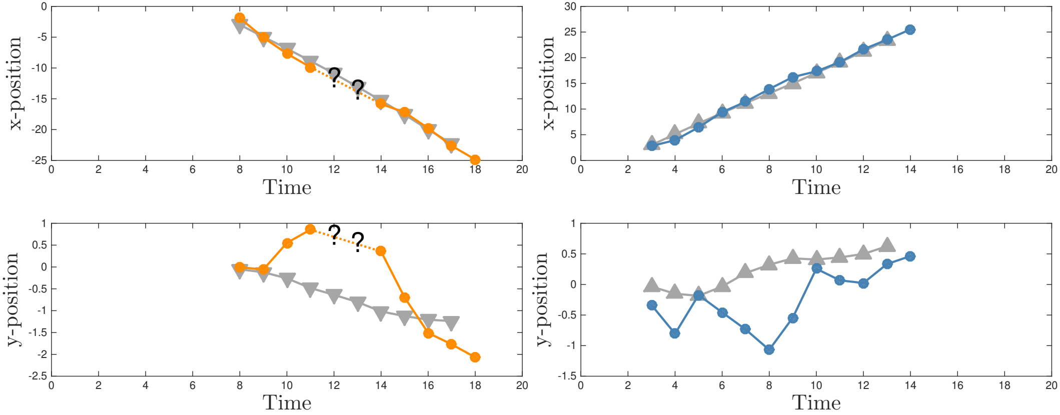
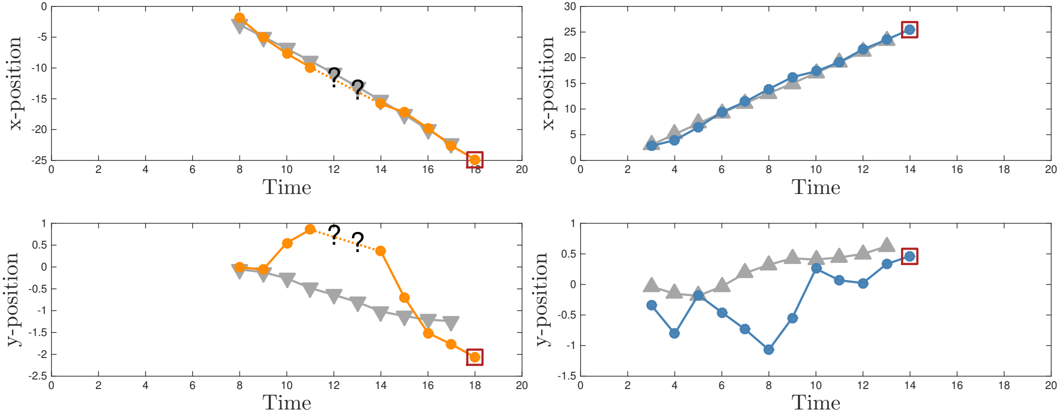


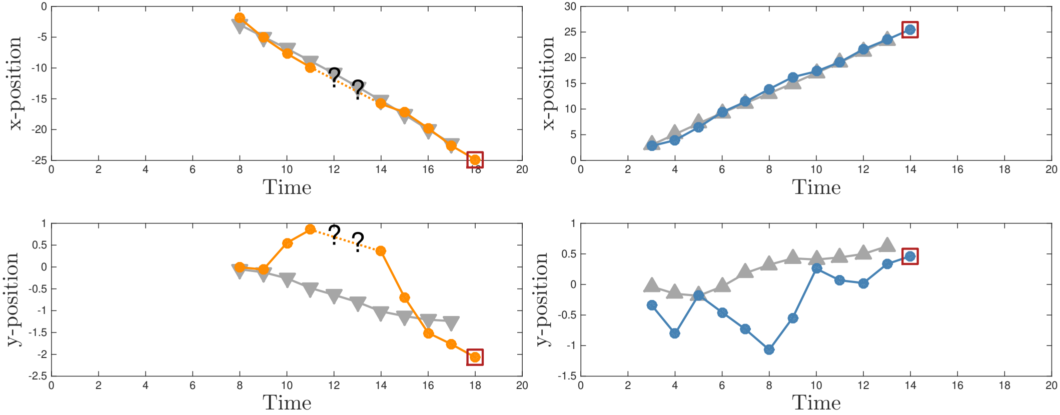
- Two gaps in the orange trajectory: missed objects at times $k = 12$ and 13

GAPS IN THE TRAJECTORY: TRACKING RESULTS



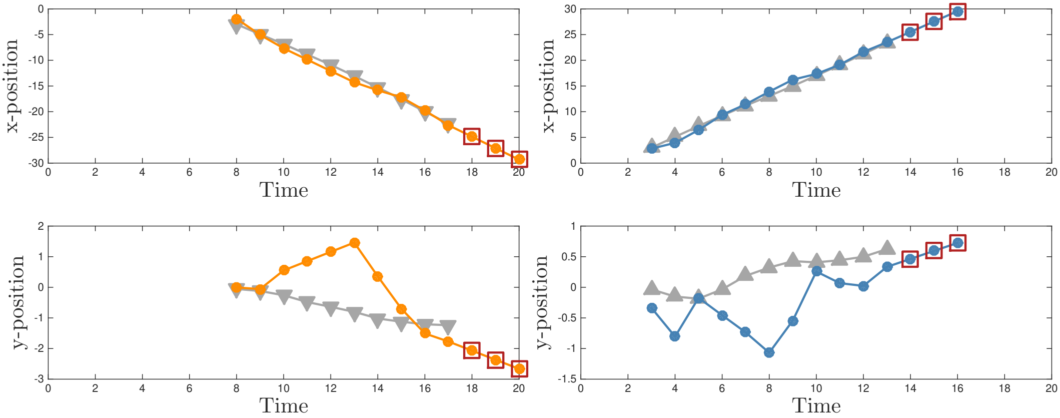
- Two gaps in the orange trajectory: missed objects at times $k = 12$ and 13
- False objects extractions at times $k = 14$ and 18, due to $r > \Gamma^e$
- Total missed/false objects: 4

GAPS IN THE TRAJECTORY: THE SIMPLE FIX



- “Confirmed object”: $r > \Gamma^e$ for some number of time steps
- Estimation: extract from Bernoullis that is either confirmed or has $r > \Gamma^e$

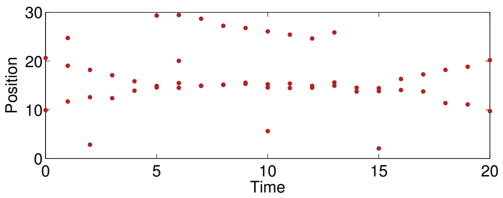
GAPS IN THE TRAJECTORY: THE SIMPLE FIX IS NOT A GOOD SOLUTION

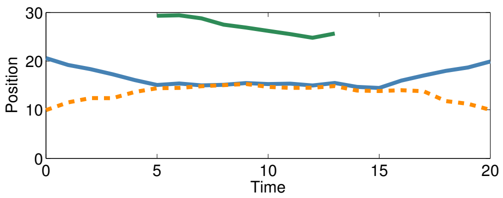


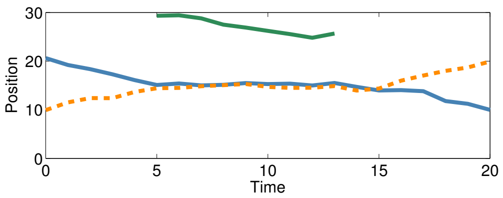
- Gaps are filled, no missed objects
- Additional false object extractions at times $k = 15$, 16, 19, and 20
- Total missed/false objects: 6. Not clear if results are improved overall.

SCENARIO WITH (APPROX) EQUI-PROBABLE GLOBAL HYPOTHESES

Ambiguous scenario with two (approx) equally probable hypotheses





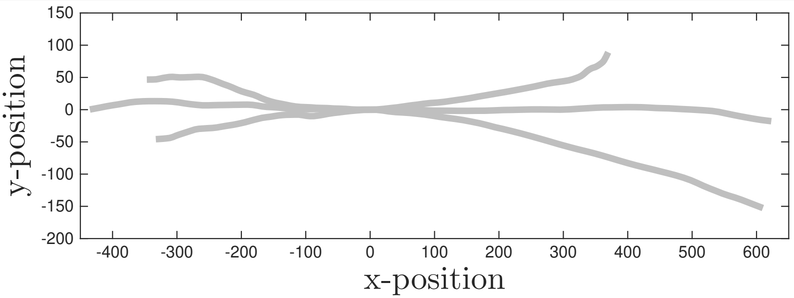


UNREALISTIC SWITCHING: SETUP

Three objects that become very close, and then separate

Objects move left to right, are initially well-separated

100 time steps. After time 50, ambiguous which object goes where



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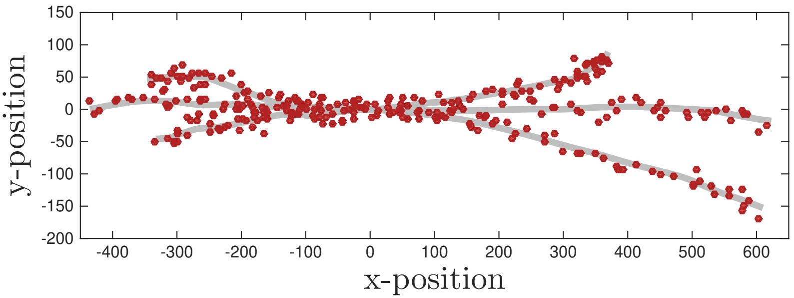
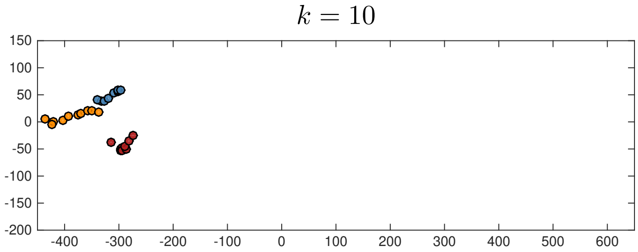
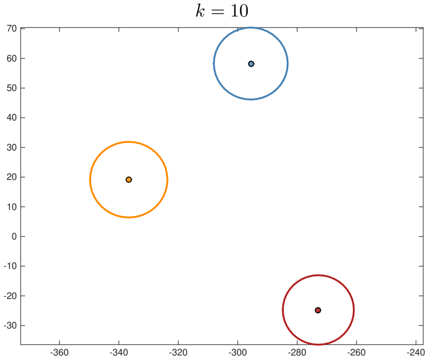


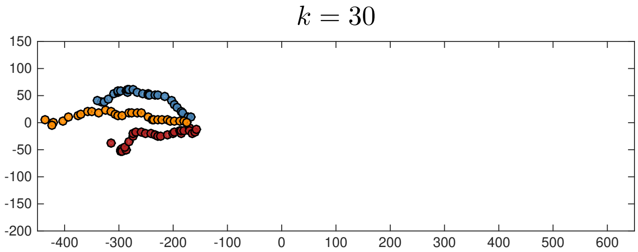
UNREALISTIC SWITCHING: TRACKING RESULTS

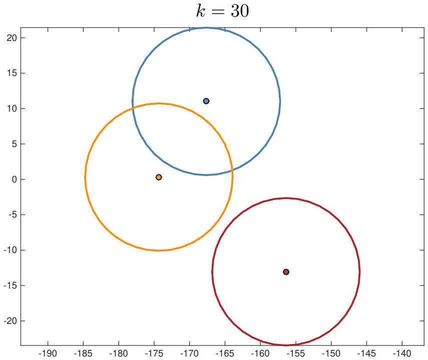




Extracted labeled objects (left) and hypothesis from which they were extracted (right)

UNREALISTIC SWITCHING: TRACKING RESULTS





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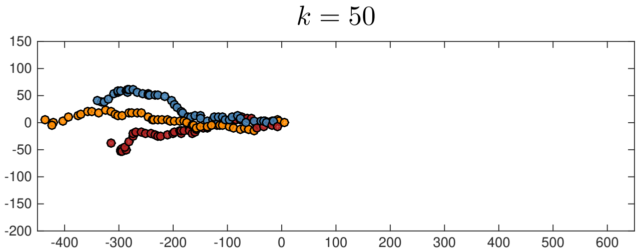

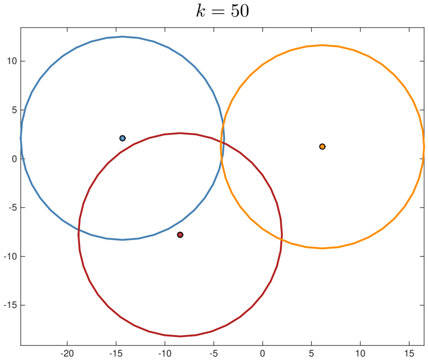

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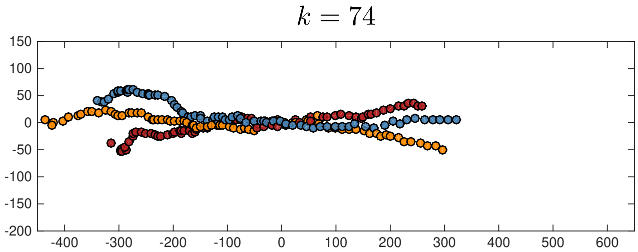

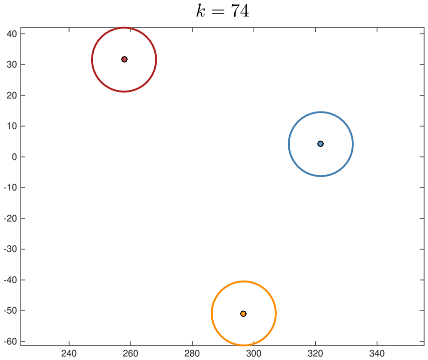

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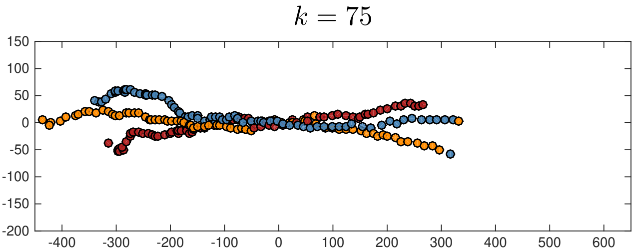

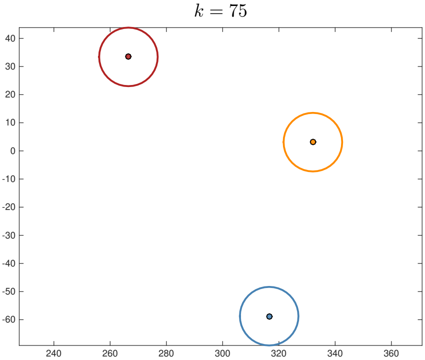

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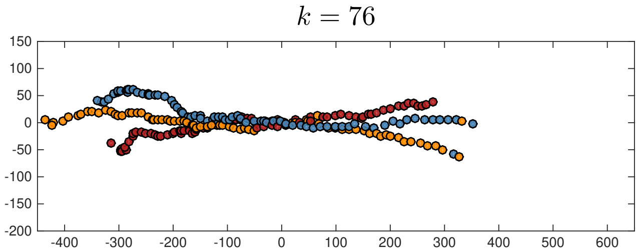

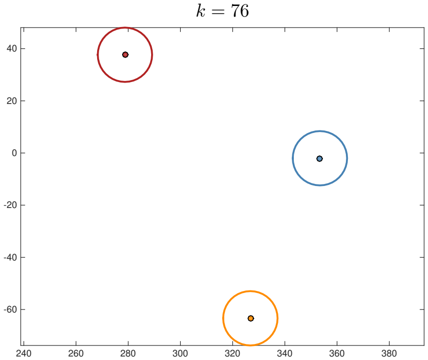

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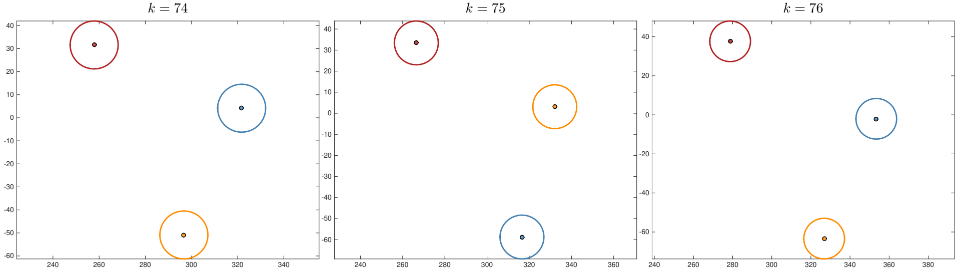
UNREALISTIC SWITCHING: TRACKING RESULTS

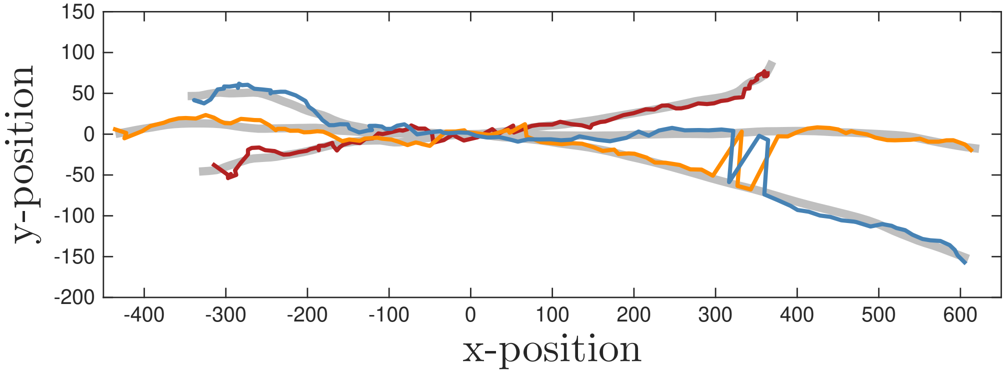


Hypotheses from which estimates were extracted

Physically unrealistic switching problem:

The hypothesis that the estimates are extracted from at time 76 is not a direct "descendant" of the hypothesis at time 75, which in turn was not a "descendant" of the hypothesis at time 74.

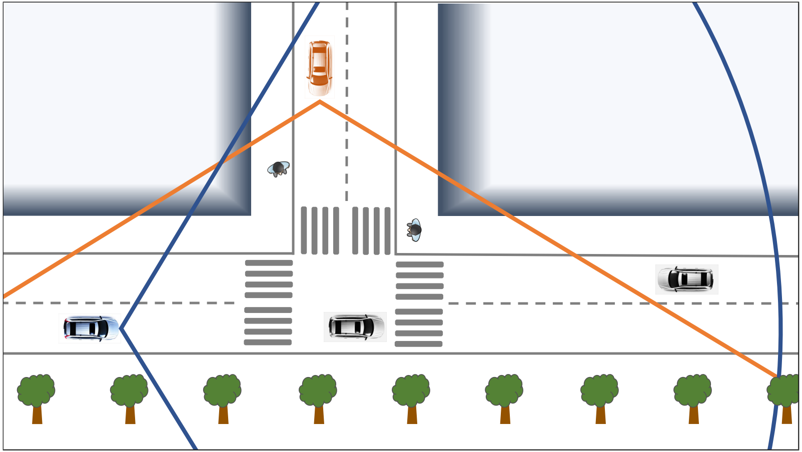
UNREALISTIC SWITCHING: FINAL RESULTS



Final trajectories at time $k = 100$

Physically unrealistic switching does not happen every time that we use labels, but it happens often enough to be an important problem.

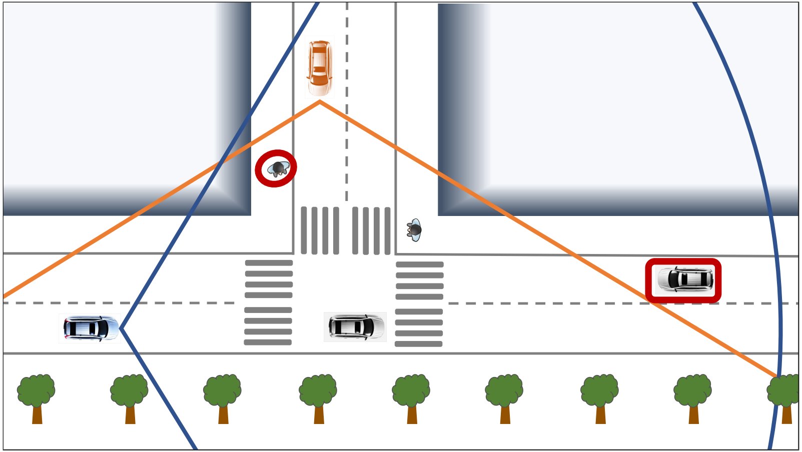
COLLABORATING AUTONOMOUS VEHICLES



Two autonomous vehicles, with partially overlapping surveillance areas

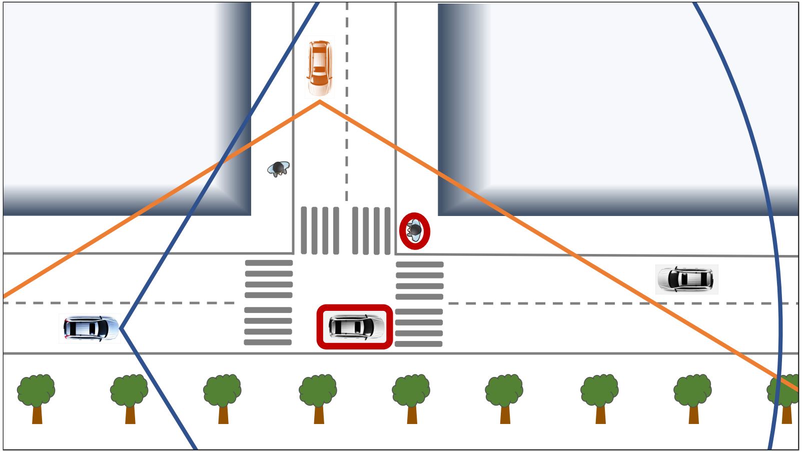
Some objects seen by both vehicles, other objects only seen by one of the vehicles

COLLABORATING AUTONOMOUS VEHICLES



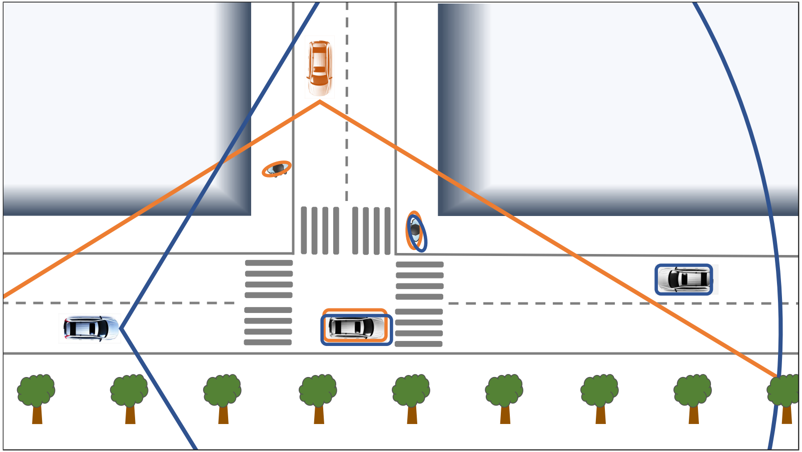
Share information about occluded objects, and objects outside field of view

COLLABORATING AUTONOMOUS VEHICLES



Share information about detected objects, to improve tracking and localisation

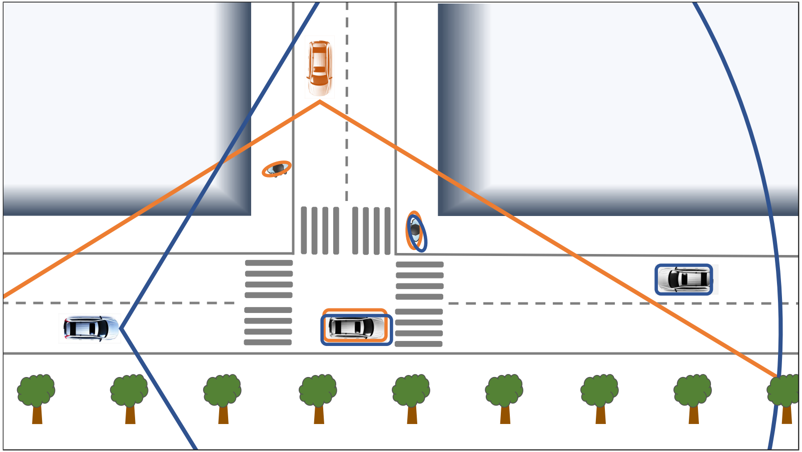
FUSING ESTIMATES WITH DIFFERENT LABELS



Rectangular estimates for vehicles, ellipsoidal estimates for pedestrians

Colors correspond to the autonomous vehicle whose tracker output the estimates

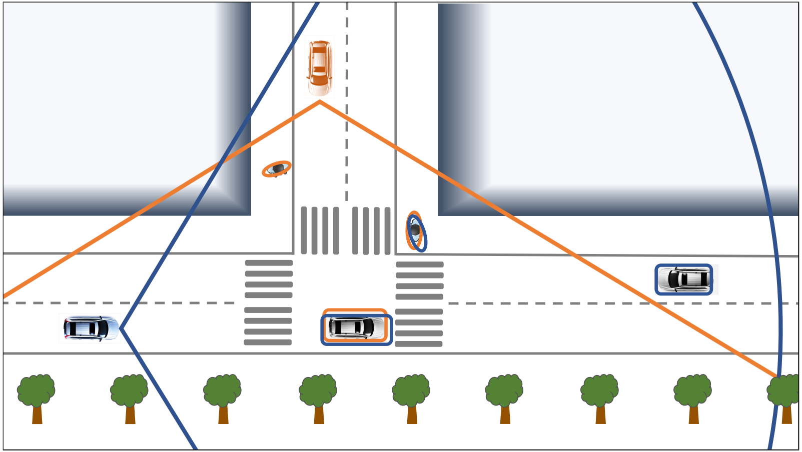
FUSING ESTIMATES WITH DIFFERENT LABELS



Two labelled Bernoulli densities can only be fused if labels are identical.

If labels are different, KL and CS divergences between the two labelled Bernoullis are $= \infty$

FUSING ESTIMATES WHILE IGNORING LABELS



Simple fix: “ignore” the labels while fusing

Problem: we have to “ignore” the very thing we introduced to form trajectories

ALTERNATIVE SOLUTION: SETS OF TRAJECTORIES

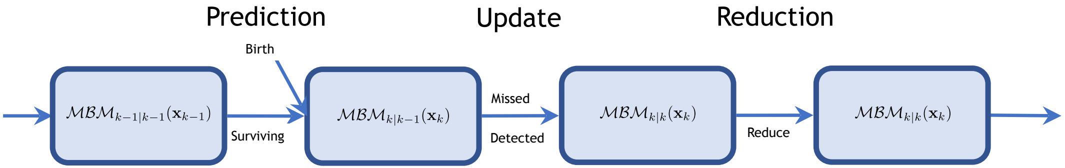
- Using labels is simple, but can be surprisingly problematic.
- We are interested in trajectories
  - Begin at some time step β
  - End at some time step ε, or are ongoing if $\varepsilon = k$
  - State sequence from β to ε, $x_\beta, \dots, x_\varepsilon$.
- Trajectory state defined as $(\beta, \varepsilon, x_{\beta:\varepsilon})$, where $x_{\beta:\varepsilon} = x_\beta, \dots, x_\varepsilon$
- Instead of estimating sets of object states at each time step and stiching them together using labels, **estimate the set of trajectories**
- No problems with implicit labels, gaps, switching or fusion or trajectories.

Summary of conjugate multi-object tracking algorithms

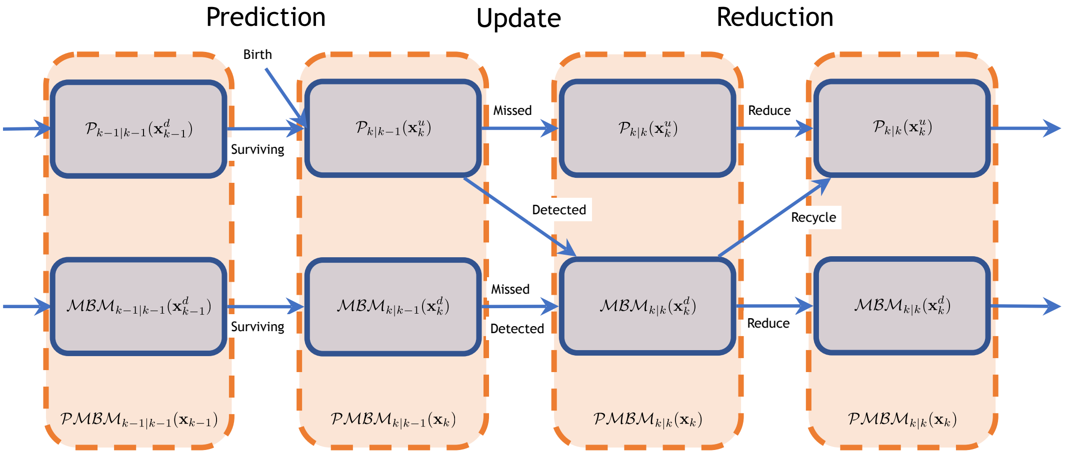
Multi-Object Tracking

Karl Granström

MBM TRACKING ALGORITHM

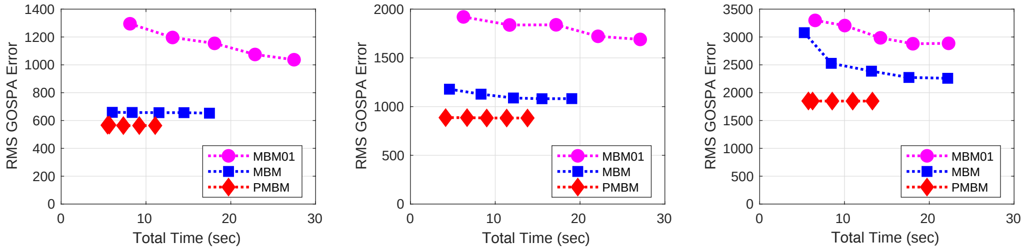


PMBM TRACKING ALGORITHM



PMBM OR MBM?

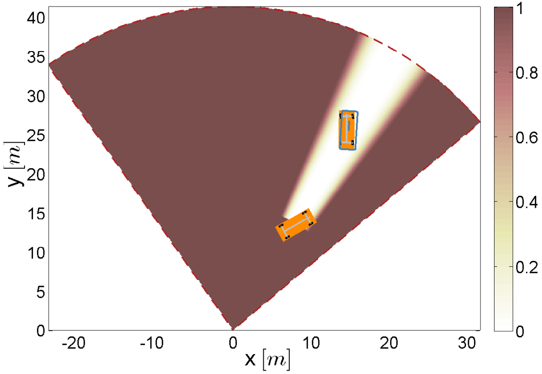
- To compare MBM and PMBM, we measure
  - Computational cost, e.g., average cycle time or total time to process all data
  - Tracking performance (GOSPA): localization error, false objects, missed objects

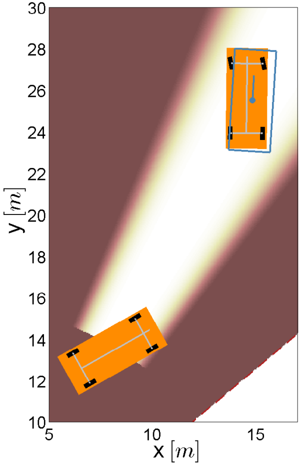


Three different scenarios. Each filter was implemented for different $N_{\max}$

- **We recommend PMBM filters**

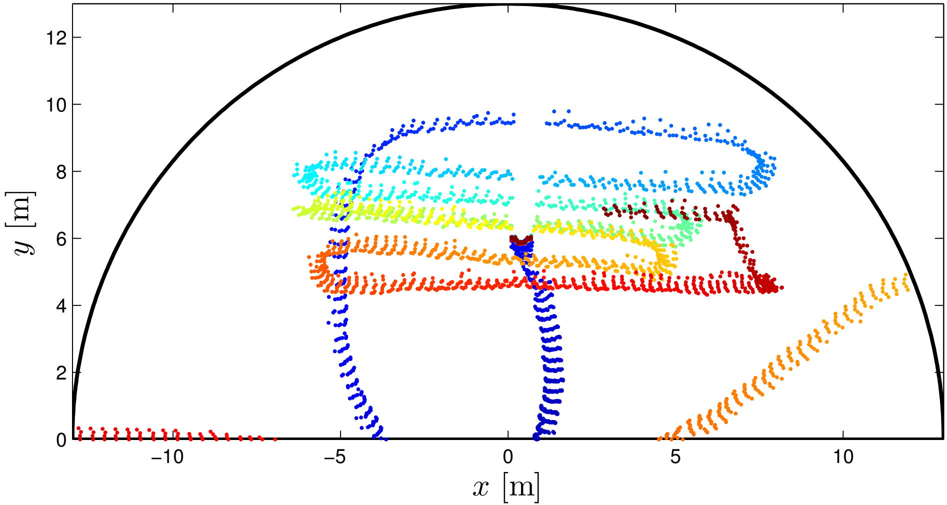
NON-HOMOGENEOUS PROBABILITY OF DETECTION





Track detected objects during occlusions, represent where undetected could be

PEDESTRIAN TRACKING USING 2D LIDAR



Using a non-homogeneous $P^{\mathrm{D}}(x)$ objects can be tracked while occluded

DETECTED PEDESTRIANS

UNDETECTED PPP INTENSITY

MULTI-BERNOULLI MIXTURE MERGING

- Even better performance if MBM merging is used:
  - Approximate $\mathcal{MBM}_{k|k}(\mathbf{x}_k)$ with an $\mathcal{MB}_{k|k}(\mathbf{x}_k)$
  - Track-oriented merging: relatively simple, merge local hypotheses
  - Variational merging: minimizes KL-divergence,

$$KL\left(\mathcal{MBM}_{k|k}(\mathbf{x}_k)||\mathcal{MB}_{k|k}(\mathbf{x}_k)\right) = \int \mathcal{MBM}_{k|k}(\mathbf{x}_k)\log\left(\frac{\mathcal{MBM}_{k|k}(\mathbf{x}_k)}{\mathcal{MB}_{k|k}(\mathbf{x}_k)}\right)\delta\mathbf{x}_k$$

    more complicated but better tracking performance
  - Resulting tracking algorithms called PMB and MB filters
  - Outside scope of course